Problem Set 8

This problem set is due **at 11:59pm** on **Tuesday, November 12**. (You have extra time for this problem set!)

Reading Assignment

Chapters 16–18

Problem 8-1. Mutual Independence

Independently flip three fair coins (with "fair" meaning "equally likely to come up with a head or a tail"), and define six random variables H_1, H_2, H_3, C, M, S as follows:

- For $i = 1, 2, 3$, H_i is the indicator variable for a head occurring on the ith flip. So H_i is 1 if the ith coin is heads, and H_i is 0 if the ith coin is tails.
- $C = H_1 + H_2 + H_3$ is the number of heads flipped.
- M is the indicator variable for the event that all three coins match, $[H_1 = H_2 = H_3]$.
- S is the indicator variable for the event that the number of heads (C) is odd.

(a) Show that none of these six variables is independent of C.
Hint: Consider the case when $C = 3$.

(b) Show that M and S are pairwise independent.

(c) Show that H_1, H_2, H_3, and S are 3-wise independent, but not mutually independent.

Problem 8-2. PDFs and CDFs

(a) Let X be a Bernoulli distribution that is 1 with probability $\frac{1}{3}$ and 0 otherwise. Let Y be the result of a fair die roll. Given that random variables X and Y are independent, describe the probability density function (pdf) of $Z = Y \cdot (1 + X)$.

(b) Let random variable D indicate the value rolled by a particular *unfair* 10-sided die: the possible rolls are $\{1, 2, \ldots, 10\}$ as usual, but for integers $1 \leq n \leq 9$, $\text{PDF}_D(n) = 2(\frac{1}{3})^n$. Describe the CDF of D, and compute the probability of rolling a 10.

Problem 8-3. Independence

There is a subject—naturally not *Math for Computer Science*—in which 10% of the assigned problems contain errors. If you ask a Teaching Assistant (TA) whether a problem has an error, then they will answer correctly 80% of the time, regardless of whether or not a problem has an error. If you ask a lecturer, they will identify whether or not there is an error with only 75% accuracy. For each problem, assume that the correctness of TA's and lecturer's responses are independent of each other.

We formulate this as an experiment of choosing one problem randomly and asking a particular TA and lecturer about it. Define the following events:

$$\begin{aligned} E &::= [\text{the problem has an error}], \\ T &::= [\text{the TA says the problem has an error}], \\ L &::= [\text{the lecturer says the problem has an error}]. \end{aligned}$$

(a) Translate the description above into a precise set of equations involving conditional probabilities among the events E, T and L.

(b) Suppose you have doubts about a problem and ask a TA about it, and they tell you that the problem is correct. To double-check, you ask a lecturer, who says that the problem has an error. What is the probability that there is an error in the problem?

(c) Is event T independent of event L? First, give an argument based on intuition, and then verify your intuition with calculations.

Problem 8-4. Linearity of Expectation

The *complete graph* K_r is a simple, undirected graph with r vertices and where every pair of vertices is connected by an edge, meaning there are $\binom{r}{2}$ edges.

An *r-clique* in a graph is defined to be a *subgraph* that forms a copy of the complete graph K_r.

Let G be the complete graph on 10 vertices, K_{10}. There are $\binom{10}{5} = 252$ 5-cliques in K_{10}, because every subset of five vertices induces a subgraph of G that is a copy of K_5.

In this problem, we will prove that there exists a way to assign a color to each *edge* of G, either red or blue, in such a way that none of these 5-cliques are monochromatic. (We say a subgraph is *monochromatic* if every edge within the subgraph has the same color.)

For example, here's an edge 2-coloring of K_5 with no monochromatic 3-cliques:

To prove a similar result for K_{10} and 5-cliques, however, it may be quite challenging to construct and verify such solutions by hand.

Hence, we present a non-constructive approach to this problem. We'll color the edges of G randomly: each edge gets assigned red or blue with 50% probability, and all the edges are colored *mutually independently*. Said differently, each of the $2^{\binom{10}{2}}$ possible edge 2-colorings of G occur with equal probability.

(a) Let C be some particular 5-clique in G. What is the probability that C is monochromatic?

(b) What is the expected number of monochromatic 5-cliques in G? *Hint:* Apply linearity of expectation to the appropriate indicator variables.

(c) Conclude that there must exist an edge coloring of G that contains no monochromatic 5-cliques. *Hint:* Proof by contradiction. If not, what could we say about the answer to part (b)?

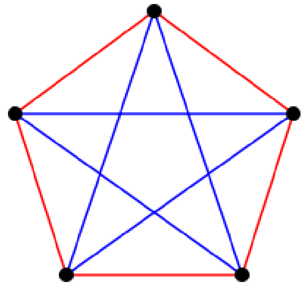

Figure 1: Edge 2-coloring of K_5 with no monochromatic 3-cliques