Problem Set 2

This problem set is due **at 11:59pm** on **Thursday, September 19, 2019.**

Please make note of the following instructions:

- Your solution must be submitted as a PDF file. While we allow handwritten solutions, we STRONGLY SUGGEST that you **typeset on LaTeX**, using the template available in the course materials. Each submitted solution should start with your name, the course number, the problem number, your recitation section, the date, and the names of any students with whom you collaborated.

Reading Assignment

Sections 2.5–2.7 and 3.0–3.4

Problem 1-1. Induction [10 points]

Prove using induction that for $n \in \mathbb{N}$:

$$\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2. \tag{1}$$

Hint: Recall that $1 + 2 + \cdots + n = n(n+1)/2$ (Theorem 3.2.1 in the textbook).

Problem 1-2. Sets and Predicates [10 points]

A *formula of set theory* is a predicate formula that only uses the predicate "$x \in y$." The domain of discourse is the collection of sets, and "$x \in y$" is interpreted to mean the set x is one of the elements in the set y.

For example, since x and y are the same set iff they have the same members, here's how we can express equality of x and y with a formula of set theory:

$$(x = y) ::= \ \forall z. \, (z \in x \ \text{IFF} \ \ z \in y).$$

Express each of the following assertions about sets by a formula of set theory, where each answer may use abbreviations introduced earlier (so it is now legal to use "=" because we just defined it).

(a) $x = \emptyset$.

(b) $x \subseteq y$. (x is a subset of y that might equal y.)

(c) $x = y \cap z$.

(d) $|x| = 1$. (x is a set of size 1.)

Problem 1-3. Well Ordering Principle [10 points]

Use the Well Ordering Principle to prove that any integer greater than or equal to 50 can be represented as the sum of nonnegative integer multiples of 7, 11, and 13.

Hint: Verify that integers betwen 50 and 56, inclusive, are sums of nonnegative integer multiples of 7, 11, and 13.

Problem 1-4. Well Ordering Principle and $\sqrt{2}$ [5 points]

Here is a different proof that $\sqrt{2}$ is irrational, taken from the American Mathematical Monthly, v.116, #1, Jan. 2009, p.69:

Proof. Suppose for the sake of contradiction that $\sqrt{2}$ is rational, and choose the least integer $q > 0$ such that $\left(\sqrt{2} - 1\right) q$ is a nonnegative integer. Let $q' ::= \left(\sqrt{2} - 1\right) q$. Clearly $0 < q' < q$. But an easy computation shows that $\left(\sqrt{2} - 1\right) q'$ is a nonnegative integer, contradicting the minimality of q. □

This proof was written for an audience of college teachers, and at this point it is too concise for this class. Write out a careful version of this proof that relies on the Well Ordering Principle and includes an explanation of each step.

Hint: What set are you using with WOP? How do you know it is nonempty?

Problem 1-5. Invariants and Strong Induction [15 points]

We'll show that the following simple algorithm (state machine) efficiently computes the product of two nonnegative integers, x and y. (It takes advantage of the fact that multiplying or dividing an integer by 2 is especially fast on most computers, requiring just a single bit-shift operation.) The set of **states** is $\mathbb{N}^3 = \{(r, s, a) \mid r, s, a \in \mathbb{N}\}$, i.e., the set of triples of nonnegative integers. The **start state** is $(x, y, 0)$. Finally, the state transitions are as follows:

$$(r, s, a) \mapsto \begin{cases} (2r, s/2, a) & \text{for even } s > 0, \\ (2r, (s-1)/2, a + r) & \text{for odd } s > 0, \\ \text{nothing} & \text{for } s = 0 \end{cases}$$

(a) Prove by strong induction on s that from any state (r, s, a), this algorithm terminates after at most $1 + \log_2 s$ steps (if $s > 0$). So the algorithm finishes, and it does so quickly. *Hint:* Recall that $\log_2(s/2) = (\log_2 s) - 1$.

(b) Prove that $rs + a = xy$ is an invariant of this procedure.

(c) Conclude that this algorithm indeed computes the product xy.