Problem Set 5

This problem set is due **at 11:59pm** on **Thursday, October 17**.

Reading Assignment

Sections 7.2, 11.1–11.4, and 11.10

Problem 5-1. Pokerer

Poker is a card game where certain 5-card *hands* are better than others. These hands are simply 5-element subsets of a standard deck of 52 cards, where each card has one of 13 possible *ranks* and one of 4 possible *suits*: the 13 ranks are 2, 3, ..., 10, Jack, Queen, King, and Ace, and the 4 suits are ♠ (Spades), ♡ (Hearts), ♣ (Clubs), and ♢ (Diamonds). Each of the $13 \times 4 = 52$ combinations appears once in the deck.

There is much already known about Poker hands, so local card shark Joe Kerr invented a new game, Pokerer, where each player gets 6 cards instead of 5. He likewise introduced new kinds of 6-card hands; help Joe compute how (un)common they are! For each part, count the number of 6-card hands that satisfy the specified condition, and justify your answer.

You may leave your numerical answers unevaluated: arithmetic, factorials, and binomial coefficients are fine to leave in. For example, the total number of 6-card hands may be expressed as $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47/6!$ (by choosing 6 cards in order and then using the division rule), or equivalently as the binomial coefficient $\binom{52}{6}$ (the number of 6-element subsets of a 52-element set, which we showed equals $52!/(6! \cdot 46!)$ in Recitation 8).

(a) A *fuller house* is a hand with four cards of one rank and two cards of a different rank. How many possible fuller houses are there?

(b) A *three pair* is a hand where two cards have one rank, two cards have a different rank, and the last two cards have some third rank. How many three pairs are there?

(c) An *imperial flush* is a hand where all six cards have the same suit and all ranks are consecutive. The ranks are ordered according to the list above. Additionally, and Ace can come before 2, or after King, but not both. Here are examples of imperial flushes:

$$\{A\heartsuit, 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit\}$$

$$\{9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit\}$$

$$\{6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, J\spadesuit\}$$

However, this is NOT an imperial flush, because an Ace cannot be both high and low at the same time:

$$\{Q\spadesuit, K\spadesuit, A\spadesuit, 2\spadesuit, 3\spadesuit, 4\spadesuit\}$$

How many imperial flushes are there?

(d) A *basically-a-flush* is a hand where *at least* five of the cards in the hand have the same suit. How many basically-a-flushes are there?

Problem 5-2. Pigeonhole Principle

Suppose $n+1$ numbers are selected from $\{1, 2, 3, \ldots, 2n\}$. Using the Pigeonhole Principle, show that there must be two distinct selected numbers whose quotient is a power of two. You should clearly describe what your pigeons and pigeonholes are, as well your rule for assigning pigeons to the pigeonholes.

Hint: Factor each number into the product of an odd number and a power of 2. How many possibilities are there for that odd number?

Problem 5-3. Stars and Bars

In Problem 3 of Recitation 8, we counted the number of ways to select 12 donuts from 5 possible flavors, when we only care about *how many* of each flavor are chosen. We solved this by constructing a bijection to the number of distinct 16-bit sequences with 12 zeros and 4 ones. This technique, sometimes known as "stars and bars", is quite versatile! Let's explore it further.

(a) Show that the number of ways to select n donuts from k possible flavors is $\binom{n+k-1}{k-1}$, by constructing a bijection to a certain set of binary sequences, similar to Problem 3 of Recitation 8. Be sure to explain why your mapping really is a bijection.

(b) How many ways are there to select 12 donuts from 5 flavors, if we are required to select at least one donut in each flavor? Show that the answer is $\binom{11}{4}$ by constructing a bijection using binary sequences, similar to part (a). Be sure to explain why your mapping really is a bijection.

(c) How many ways are there to select 12 donuts from 5 flavors, if we require an even number of donuts for every flavor in our selection?

You may use the stars and bars formula from part (a).

(d) How many ways are there to select 12 donuts from 5 flavors, if we require at least three distinct flavors in our selection?

You may use the stars and bars formula from part (a).

Hint: How many selections have at most 2 flavors?