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6.042 Problem Set 1

Problem 1 *(Collaborators: None)*

Let P(n) represent the following proposition: $\sum_{n=0}^{\infty} i^3 = (\sum_{n=1}^{\infty} i)^2$

P(n) can be rewritten as

$(n^2(n+1)^2)/4 = (n*(n+1)/2)^2$

Base case:

n = 0

P(n) holds for this base case

Inductive Step:

Prove that P(n) implies P(n+1)

$((n+1)^2(n+2)^2)/4 = ((n+1)*(n+2)/2)^2$

$((n+1)^2(n+2)^2)/4 = (n+1)*(n+1)*(n+2)*(n+2)/4 = (n^4+6n^3+13n^2+12n+4)$

$((n+1)*(n+2)/2)^2 = ((n+1)*(n+2)/2)^2 = ((n^2+3n+2)/2)^2 = (n^4+6n^3+13n^2+12n+4)$

Thus, $((n+1)^2(n+2)^2)/4 = ((n+1)*(n+2)/2)^2$.

Therefore, P(n) implies P(n+1) and that means that P(n) holds by Induction. QED

Problem 2 *(Collaborators: None)*

Part 2(a)

Assertion: $x = \emptyset$

$(x = \emptyset) := \forall z.(z \text{ NOT} \in x)$

Part 2(b)

Assertion: $x \subseteq y$

$(x \subseteq y) := \forall z(z \in y \text{ IF } z \in x)$

Part 2(c)

Assertion: $x = y \cap z$

$(x = y \cap z) := \forall a(a \in x \text{ IFF } (a \in y \text{ AND } a \in z)$

Part 2(d)

Assertion: $|x| = 1$

$(|x| = 1) := \forall a, b.((a = b \text{ IF } (a \in x \text{ AND } b \in x) \text{ AND NOT}(x = \emptyset))$

Problem 3 *(Collaborators: None)*

For the sake of contradiction, assume that there is a nonempty set P such that

$P ::= x >= 50$

This means that x cannot be represented as a linear combination of 7, 11, and 13. If x cannot be represented as a sum of nonnegative integer multiples of 7, 11, and 13, then neither can x - 7. By WOP, C contains a least number p. $p >= 51$ because $x = 50$ can be represented as a linear combo of 7, 11, and 13. Therefore, x cannot be greater than 56 because if $m = 57$, then $m - 7$ would be in C, which is a contradiction since x is the least element of P.

50 = (13 * 3) + (11)

51 = (11 * 4) + (7)

52 = (13 * 4)

53 = (11) + (7 * 6)

54 = (13 * 2) + (7 * 4)

55 = (11 * 5)

56 = (7 * 8)

QED

Problem 4 *(Collaborators: Andy Kaspers)*

Proof by contradiction

Suppose $\sqrt{2}$ is rational.

In that case, there exists a non-empty set $S : m \in |$ *the product of* $(\sqrt{2}-1)$ *and* m *is a nonnegative integer*

Then, by the WOP, there must exist a smallest element q that is the smallest integer such that $(\sqrt{2}-1)*q$ is a nonnegative integer.

Let p = $(\sqrt{2}-1)*q$ **where** $0 < p < q$ **and both q and p are positive integers.**

$((\sqrt{2}-1)/(\sqrt{2}-1))*(p/(\sqrt{2}-1)) = q$

$((\sqrt{2}-1)*p) = q$

We know that since q and p are positive integers and $q > p$, that $((\sqrt{2}-1)*p)$ is a positive integer.

However, $p < q$ and we said above that q is the smallest possible element such that $(\sqrt{2}-1)*q$ is a nonnegative integer.

Yet here $((\sqrt{2}-1)*p)$ is a nonnegative integer and $p > q$. This is a contradiction!

By proof of contradiction, we know that $\sqrt{2}$ is irrational. QED

Problem 5 *(Collaborators: Julian Hamelberg)*

Part 5(a)

Let P(n) represent the following proposition: "from any state, the algorithm terminates after at most $1 + \log_2 s$ steps.

base case:

P(1) = 1 step, which holds true

For P(n+1), if n + 1 is even, it will take P(n+0.5) + 1 steps.

So, $1 + \log_2 (n + 1)/2 + 1 = 2 + \log_2 n + 1 - 1 = 1 + \log_2 n + 1$

For P(n+1), if n + 1 is odd, it will take P(n/2) + 1 steps.

So, $1 + \log_2 n/2 + 1 = 2 + \log_2 n - 1 = 1 + \log_2 n$ which is less than $1 + \log_2 (n + 1)$

Therefore, by Strong Induction, P(n+1) is true anytime $n > 0$. QED

Part 5(b)

Let P(n) represent the following proposition: $rs + a = xy$ is an invariant of the procedure outlined in Problem 1-5

The following is a proof that P(n) is true.

Proof by cases:

Case 1: s is even

r becomes 2r, s becomes s/2, a remains a

$2r * s/2 + a = rs + a = xy$

Case 1: s is odd

r becomes 2r, s becomes (s-1) / 2 and a becomes a+r

$2r * (s - 1)/2 + (a + r) = rs - r + a + r = rs + a = xy$

These cases are exhaustive and in all cases, $rs + a = xy$. Therefore P(n) holds. QED

Part 5(c)

In every state, rs + a = xy. In every state, the algorithm terminates and s quickly becomes 0. There will be a state where r*0 + a = xy. Therefore, we can conclude that the algorithm computes the product xy.