Problem Set 1

This problem set is due **at 11:59pm** on **Thursday, September 12, 2019.**

Please make note of the following instructions:

- Your solutions must be submitted to Gradescope as a single PDF file. While we allow handwritten solutions, we **strongly suggest** that you **typeset in LaTeX**, using the template available in the course materials. **Each problem** should be accompanied by a **collaboration statement**: the students you collaborated with (and any external resources you consulted) when solving the problem. Do not leave this blank—instead write "collaborators: none" if appropriate.

Reading Assignment

Chapter 1 & Sections 2.0-2.4

Problem 1-1. Equivalence [10 points]

Use a truth table to prove that the propositional formulas

$$P \text{ OR } Q \text{ OR } R$$

and

$$(P \text{ AND NOT}(Q)) \text{ OR } (Q \text{ AND NOT}(R)) \text{ OR } (R \text{ AND NOT}(P)) \text{ OR } (P \text{ AND } Q \text{ AND } R).$$

are equivalent. Your table should have enough detail that each column can be verified at a glance from previously computed columns. This makes it easy for your audience to follow and verify your computations.

Problem 1-2. Irrationality [10 points]

Prove by contradiction that the number $x = \sqrt{3} + \sqrt{2}$ is irrational.

Hint: One possible argument begins by showing that $1/x = \sqrt{3} - \sqrt{2}$. Use this to "prove" that $\sqrt{2}$ is rational.

Problem 1-3. Logical Operators [10 points]

We've defined a half-dozen operators that may appear in propositional formulas: AND, OR, NOT, IMPLIES, XOR, and IFF. This problem will show that, in fact, AND, OR, and NOT are enough to do the job on their own. That is because each of the operators is equivalent to a simple formula using only these three operators. For example, A IMPLIES B is equivalent to NOT(A) OR B. So all occurences of IMPLIES in a formula can be replaced using just NOT and OR.

(a) Write formulas using only AND, OR, NOT that are equivalent to each of A IFF B and A XOR B, and explain why. Conclude that every propositional formula is equivalent to an AND-OR-NOT formula.

(b) Explain why you don't even need AND.

(c) Explain how to get by with the single operator NAND where A NAND B is equivalent by definition to NOT(A AND B).

Problem 1-4. Cases [10 points]

Define the function

$$f(x) := 2\,|x+2| - |x-3| - |x+4|$$

for real numbers x. Carefully prove that

$$-7 \le f(x) \le 3 \quad \text{for all } x \in \mathbb{R},$$

using a **proof by cases** based on the value of x.

Hint: $|x+4|$ equals $x+4$ when $x \ge -4$ and equals $-(x+4)$ when $x \le -4$. You should have 4 cases.

Problem 1-5. Predicates [10 points]

Translate the following predicates and propositions into predicate logic. For each, specify the domain of discourse[1]. In addition to logic symbols, you may build predicates using arithmetic, relational symbols and constants. For example, the statement "n is an odd number" could be translated as $\exists m.\, 2m+1 = n$ where the domain of discourse is $\mathbb{Z}$, the set of integers. Another example is "p is a prime number," could be

$$(p > 1) \text{ AND } \text{ NOT } (\exists m \exists n (m > 1 \text{ AND } n > 1 \text{ AND } mn = p))$$

Let prime(p) be an abbreviation for this formula which you may use in further formulas below.

(a) (Sum of 3 Cubes) The number 42 is the sum of 3 cubes. (The cubes are allowed to be negative.)[2]

(b) (Silliness) Every multiple of 7 is prime.[3]

[1] The *domain of discourse* is the set of possible values of your variables. In other words, if the domain of discourse is specified to be a set D, statements such as $\forall x$ and $\exists x$ are shorthand for $\forall x \in D$ and $\exists x \in D$, and predicates $P(x)$ need to make sense for every $x \in D$. See Section 1.3.7 for more detail. Common choices include the "natural" numbers a.k.a. the nonnegative integers $\mathbb{N} = \{0, 1, 2, \ldots\}$, the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$.

[2] This longstanding conjecture was solved this week, by Andrew Booker and Andrew Sutherland here at MIT! Don't actually find the cubes—just translate the proposition.

[3] This statement is clearly false, but you're not being asked to prove or disprove it—just to express it in predicate notation.

(c) (7-smoothness) n has no prime factors greater than 7.

(d) (Twin Prime Conjecture) There are infinitely many pairs of primes that differ by 2. *Hint:* Express the equivalent idea that no matter how large an integer you pick, there is a pair of twin primes greater than that.