Problem Set 6

This problem set is due **at 11:59pm** on **Thursday, October 24, 2019**

Reading Assignment

Sections 7.2 and 11.1–11.10

Problem 6-1. Counting

Answer the following questions with a number or a simple formula involving factorials and binomial coefficients. Briefly explain your answers.

(a) How many ways are there to order the 26 letters of the alphabet so that no two of the vowels `a`, `e`, `i`, `o`, `u` appear consecutively and the last letter in the ordering is not a vowel?

Hint: Every vowel appears to the left of a consonant.

(b) How many ways are there to order the 26 letters of the alphabet so that there are *at least two* consonants immediately following each vowel?

(c) In how many different ways can $2n$ students be paired up?

(d) Two n-digit sequences of digits $0,1,\ldots,9$ are said to be of the *same type* if the digits of one are a permutation of the digits of the other. For $n = 8$, for example, the sequences `03088929` and `00238899` are the same type. How many types of n-digit sequences are there?

Problem 6-2. Pigeonhole Principle

(a) Let R be an 82×4 rectangular matrix each of whose entries are colored red, white or blue. Explain why at least two of the 82 rows in R must have identical color patterns.

(b) Conclude that R contains four points with the same color that form the corners of a rectangle.

(c) Now show that the conclusion from part (b) holds even when R has only 19 rows.

Hint: How many ways are there to pick two positions in a row of length four and color them the same?

Problem 6-3. Inclusion Exclusion

Fearing post-regrade exam complaints, the TAs decide to create a ten digit numeric password blocking students from demanding more points on their midterms. Since the students are smart, the TAs do not use any passwords containing the sequences "6042", "18062", or "38476", the course 6 undergraduate office room number.

(a) How many 10-digit passwords can the TAs pick that don't contain forbidden sequences if each number $0, 1, \ldots, 9$ can only be used once (i.e. without replacement)?

(b) How many 10-digit passwords can the TAs pick that don't contain forbidden sequences if each number $0, 1, \ldots, 9$ can be chosen any number of times (i.e. with replacement)?

Problem 6-4. Combinatorial Identities

Prove the following identities using **combinatorial proofs**. In other words, choose a set S and prove separately that each side of the identity counts $|S|$.

(a)

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$$

Hint: Imagine that in our n-person class, we wish to choose a subset of students to form the Cookie Committee, who will decide once and for all whether Chocolate Chip or Sugar cookies are superior. One member of the committee must be chosen as the Cookie Captain. How many committees like this are possible?

(b)

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$