Problem Set 3

This problem set is due **at 11:59pm** on **Thursday, September 26, 2019.**

Reading Assignment

Sections 5.1–5.4 (skip 5.2.3), 6.1–6.2.1

Problem 3-1. Build-Up Error [5 points]

We attempt to prove the following false claim about graphs:

Claim. *For every $n \in \mathbb{Z}^+$, if graph G has n vertices and $n-1$ edges, then G has no cycles.*

Proof. We proceed by induction on the number of vertices.

Inductive Hypothesis: Let $P(n)$ be the proposition that if a graph has n vertices and $n-1$ edges, then it has no cycles.

Base Case: $P(1)$ is true since the singleton graph has no cycles.

Inductive Step: Assume $P(n)$ is true, and consider any graph G with n vertices and $n-1$ edges. By the assumption $P(n)$, G must not have any cycles.

Now, we want to show that $P(n+1)$ is true, so let's construct a new graph G' on $n+1$ vertices by adding a new vertex v to G. In addition, we can assume that G' has n edges, because otherwise $P(n+1)$ will be vacuously true for the graph G'.

Since G' has n edges and G has $n-1$ edges, it must be that v has degree 1 in G'. However, vertices in a cycle must have degree at least 2, so v cannot be part of any cycles. We also know that the original graph G has no cycles, so any new cycle in G' must involve v.

We conclude that G' has no cycles, so the induction is complete. $\square$

(a) Provide a counterexample to show that the claim is false.

(b) Identify the logical error in the induction proof.

Problem 3-2. Rankings in Round-Robin Tournaments [10 points]

In a round-robin tournament, every two distinct players play against each other just once. For a round-robin tournament with no tied games, a record of who beat whom can be described with a ***tournament digraph***, where the vertices correspond to players and there is an edge (x, y) iff x beat y in their game.

A ***partial ranking*** is a simple path in the graph, and a ***full ranking*** is one that includes all the players. So in a (partial or full) ranking, each player won the game against the next ranked player, but may very well have lost their games against players ranked later—whoever does the ranking may have a lot of room to play favorites.

(a) Give an example of a tournament digraph with more than one full ranking.

(b) Show that if a partial ranking R does not include player v, then it is possible to make a longer partial ranking by inserting v somewhere into sequence R (without otherwise rearranging R).

(c) Use the previous part to prove by induction that every tournament digraph has a full ranking. As usual, be sure to carefully explain your induction hypothesis, base case(s), and inductive step. *Note*: Theorem 6.2.1 provides a different proof of this fact.

Problem 3-3. Matching [15 points]

At the Massive-Cheezits Insta-Chute, first-year students compete not only for grades, but for prizes! At the end of the semester, the instructor of each course gives a box of Cheezits to a high performing first-year student. However, the professors agree that no two professors should give the same student the prized snack, in order to spread their rewards as well as possible.

(a) Explain how to model this prize-selection process as a bipartite matching problem. What do your left vertices L, right vertices R, and edges E represent? Explain why it's possible to hand out prizes in the desired way *if and only if* your graph has a matching that includes all left vertices L.

(b) The credit limit ensures that no first-year student takes more than 5 classes. Additionally, it's discovered that every class has at least 8 first-years enrolled. Use Hall's Theorem[1] to show that this is enough to guarantee there is a proper winner selection.

Hint: For $S \subset L$, what can we say about the number of edges connecting S to $N(S)$ in terms of $|S|$ and $|N(S)|$?

(c) Say that a bipartite graph $G = (L, R, E)$ is *degree constrained* if $\deg(u) \geq \deg(v)$ for every $u \in L$ and $v \in R$. Briefly show how your argument from the previous part can be extended to prove this more general claim: Every degree-constrained graph has a matching that covers L.

(d) Be careful: all degree-constrained graphs have a perfect matching, but so do some graphs that are *not* degree constrained! Find a bipartite graph that has a perfect matching but is not degree constrained.

[1] Theorem 5.2.5: a matching exists that includes all of L **iff** no subset of L is a *bottleneck*, i.e., has strictly fewer neighbors than members.

Problem 3-4. Walks and Cycles [10 points]

Since you can go back and forth on an edge in a simple graph, every vertex with positive degree is on an even length closed walk. So even length closed walks don't tell you much about even length cycles. The situation with odd-length closed walks is more interesting.

(a) [**Corrected 9/22/19**] Give an example of a simple graph in which every vertex is on a unique[2] even-length cycle and at least one odd-length cycle.

Hint: Four vertices are enough.

(b) Give an example of a simple graph in which every vertex is on a unique odd-length cycle and no vertex is on an even-length cycle.

(c) If a directed graph has an odd-length closed walk, prove that it must have an odd-length cycle. *Hint:* WOP

Problem 3-5. Coloring [10 points]

A basic example of a simple graph with chromatic number n is the complete graph on n vertices, that is $\chi(K_n) = n$. This implies that any graph with K_n as a subgraph must have chromatic number at least n. It's a common misconception to think that, conversely, graphs with high chromatic number must contain a large complete subgraph. In this problem we exhibit a simple example countering this misconception, namely a graph with chromatic number four that contains no *triangle*—length three cycle—and hence no subgraph isomorphic to K_n for $n \geq 3$. Namely, let G be the 11-vertex graph of Figure 1. The reader can verify that G is triangle-free.

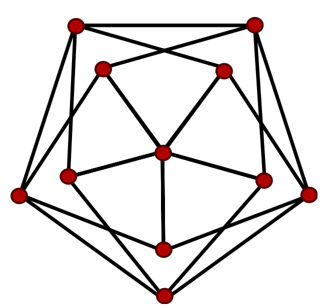

Figure 1: Graph G with no triangles and $\chi(G) = 4$.

(a) Show that G is 4-colorable.

(b) Prove that G can't be colored with 3 colors.

[2]For this problem, reversing the order of a cycle or choosing a different starting point on the cycle do not change the cycle. For example, the cycle through some five vertices a, b, c, d, e (and then back to a) is no different from the cycle through b, c, d, e, a or the cycle through a, e, d, c, b.