RESISTIVE MAGNETO-ROTATIONAL INSTABILITY IN MASSIVE PROTOPLANETARY DISKS

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ABSTRACT

Subject headings:

1. INTRODUCTION

2. LOCAL DISK MODEL

We study the local stability of an inviscid, self-gravitating and magnetized barotropic fluid disk orbiting a central mass with potential $\Phi_*(r,z)$, where (r,φ,z) are cylindrical co-ordinates from the central mass. We use the shearing box approximation (Goldreich & Lynden-Bell 1965b) and consider a small patch of the disk at $r=r_0$. The local frame rotates at angular velocity $\Omega_0 = \Omega(r_0,0)$ about the central mass, where $r\Omega^2 = \partial\Phi_*/\partial r$. We also define $S=-r\partial\Omega/\partial r$ as the local shear rate and $\Omega_z^2 \equiv \partial^2\Phi_*/\partial z^2$ as the square of the local vertical frequency.

A Cartesian co-ordinate system (x,y,z) is set up in this local frame, corresponding to the radial, azimuthal and vertical directions of the global disk, respectively. The shearing box fluid equations read

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0, \tag{1}$$

$$\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + 2\Omega_0\hat{\boldsymbol{z}}\times\boldsymbol{v} = -\frac{1}{\rho}\nabla\Pi + \frac{1}{\rho\mu_0}\boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla\Phi, \tag{2}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B} - \eta\nabla\times\boldsymbol{B}), \tag{3}$$

where ρ is the density field; $\boldsymbol{v}$ is the total velocity in the local frame; $\boldsymbol{B}$ is the magnetic field which satisfies $\nabla\cdot\boldsymbol{B}=0$; $\Pi \equiv P + |\boldsymbol{B}|^2/2\mu_0$ is the total pressure (μ_0 is the vacuum permeability) and $P=P(\rho)$ is the gas pressure given by a barotropic equation of state chosen later.

The total potential is $\Phi = \Phi_{\rm ext} + \Phi_d$, where

$$\Phi_{\rm ext}(x,z) = -\Omega_0 S_0 x^2 + \frac{1}{2}\Omega_{z0}^2 z^2 \tag{4}$$

is the effective external potential (central plus centrifugal) in the shearing box approximation, where $S_0 \equiv S(r_0,0)$ and $\Omega_{z0} \equiv \Omega_z(r_0,0)$; and the gas potential Φ_d statisfies Poisson's equation

$$\nabla^2\Phi_d = 4\pi G\rho, \tag{5}$$

where G is the gravitational constant. For clarity, hereafter we drop the subscript 0 on the frequencies.

2.1. *Self-gravitating equilibria*

The unperturbed disk is steady and described by $\rho = \rho(z)$, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ where B is a constant, and velocity field $\boldsymbol{v} = -Sx\hat{\boldsymbol{y}}$. We have assumed a thin disk and neglected the radial component of the self-gravitational force in the unperturbed disk.

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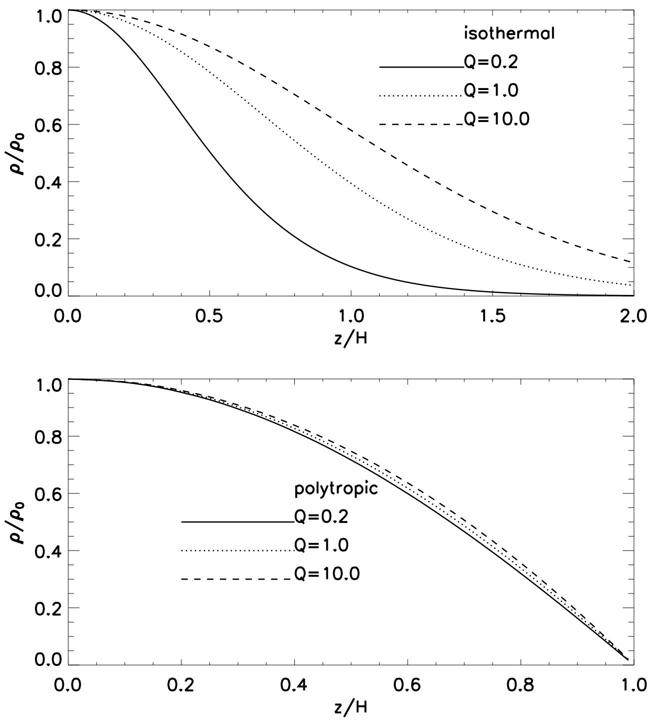

Fig. 1.— caption

The equilibrium density field is obtained by solving

$$0 = \frac{1}{\rho}\frac{dP}{dz} + \Omega_z^2 z + \frac{d\Phi_d}{dz}, \tag{6}$$

$$\frac{d^2\Phi_d}{dz^2} = 4\pi G\rho. \tag{7}$$

We consider (i) isothermal disks with $P = c_{s0}^2\rho$; (ii) polytropic disks with $P = K\rho^2$ where $K = c_{s0}^2/2\rho_0$ and $\rho_0 \equiv \rho(0)$. In both cases c_{s0} is the midplane sound speed. For the polytropic disk we define the disk thickness as H such that $\rho(H)=0$. In the isothermal case we define a characterisic disk thickness as $H = c_{s0}/\Omega$, which is the pressure scale-height in the non-self-gravitating limit.

We solve for $\hat{\rho} \equiv \rho/\rho_0$ subject to $\hat{\rho}=1$ and $d\hat{\rho}/dz = 0$ at $z=0$. This is done numerically for isothermal disks and analytically for the polytropic disk (see Appendix A). Examples of density profiles are shown in Fig. 1.

2.2. *Resistivity profile*

Following Fleming & Stone (2003), we adopt the resistivity profile

$$\eta(z) = \sqrt{2}\eta_0 \left[\exp\left(-g_+\right) + \exp\left(-g_-\right)\right]^{-1/2}, \quad (8)$$

where

$$g_\pm(z) = \frac{\Sigma_\pm(z) - \Sigma_0}{\Sigma_*}, \quad (9)$$

$$\Sigma_\pm(z) = \int_{\pm z}^{\infty} \rho(z')dz', \quad (10)$$

and $\Sigma_0 \equiv \Sigma_\pm(0)$, so that $g_\pm(0) = 0$ and $\eta_0 = \eta(0)$. The constant Σ_* is chosen such that

$$\cosh\left(\frac{\Sigma_0}{\Sigma_*}\right) = \left[\frac{\eta_0}{\eta(\infty)}\right]^2, \quad (11)$$

and we define $\eta_0/\eta(\infty) \equiv A_\eta$ as the conductivity boost factor from the midplane to the disk surface. We remark that once the equilibrium ρ and $d\rho/dz$ are obtained from Eq. 6 — 7, the integration for Eq. 10 can be done implicitly by using the Poisson equation.

2.3. *System parameters*

The strength of self-gravity is parametrized by

$$Q \equiv \frac{\Omega^2}{4\pi G\rho_0} \quad (12)$$

(Mamatsashvili & Rice 2010). The plasma β measures the inverse strength of the magnetic field,

$$\beta \equiv \frac{c_{s0}^2}{v_{A0}^2} = \frac{c_{s0}^2\mu_0\rho_0}{B^2}, \quad (13)$$

where v_{A0} is the midplane Alfven speed. The strength of resistivity is measured by the midplane Elsasser number

$$\Lambda \equiv \frac{v_{A0}^2}{\eta_0\Omega} = \frac{f^2 R_m}{\beta}, \quad (14)$$

where the second equality defines the magnetic Reynolds number R_m and $f \equiv c_{s0}/H\Omega$ is a numerical factor of order unity.

3. LINEAR PROBLEM

We consider axisymmetric Eulerian perturbations to the above equilibrium in the form $\mathrm{Re}[\delta\rho(z)\exp\mathrm{i}(k_x x + \sigma t)]$ and similarly for other fluid variables. Here, k_x is a constant radial wavenumber and $\sigma = -(\omega + \mathrm{i}\gamma)$ is a complex frequency, where $-\omega$ is the real mode frequency and γ is the growth rate. Hereafter, we surpress the exponential factor, as well as the real part notation. We further write

$$z = \hat{z}H, \quad k_x = k_x/H, \quad \sigma = \hat{\sigma}\Omega, \quad \delta\boldsymbol{v} = c_{s0}\delta\hat{\boldsymbol{v}}, \quad (15)$$

$$\delta\boldsymbol{B} = B\delta\hat{\boldsymbol{B}}, \quad \delta\rho = \rho\hat{W}/\hat{c}_s^2, \quad \delta\Phi = c_{s0}^2\delta\hat{\Phi}, \quad (16)$$

where we have introduced the non-dimensional enthalpy perturbation $\hat{W}$ and the sound-speed $\hat{c}_s = c_{s0}^{-1}\sqrt{dP/d\rho}$ can be obtained from the equation of state. The background frequencies are written $S = \hat{S}\Omega$ and $\Omega_z = \hat{\Omega}_z\Omega$ and the resistivity as $\eta = \hat{\eta}H^2\Omega$.

We will now drop the ˆ notation. Henceforth it is understood that all variables have been appropriately normalized. The linearized continuity equation is

$$\frac{\mathrm{i}\sigma}{fc_s^2}W + \mathrm{i}k_x\delta v_x + (\ln\rho)'\delta v_z + \delta v_z' = 0, \quad (17)$$

where $'$ denotes d/dz. The linearized equations of motion are

$$\mathrm{i}\sigma\delta v_x - 2\delta v_y = f\left[v_A^2\delta B_x' - \mathrm{i}k_x\left(\tilde{W} + v_A^2\delta B_z\right)\right], \quad (18)$$

$$\mathrm{i}\sigma\delta v_y + \frac{\kappa^2}{2}\delta v_x = fv_A^2\delta B_y', \quad (19)$$

$$\mathrm{i}\sigma\delta v_z = -f\tilde{W}', \quad (20)$$

where the Alfven speed $v_A = (\beta\rho)^{-1/2}$, the effective enthalpy perturbation $\tilde{W} = W + \delta\Phi$ and the epicycle frequency $\kappa = \sqrt{2(2-S)}$. The linearized induction equation is

$$\mathrm{i}\bar{\sigma}\delta B_x = f\delta v_x' + \eta\delta B_x'' + \eta'\delta B_x' - \mathrm{i}k_x\eta'\delta B_z, \quad (21)$$

$$\mathrm{i}\bar{\sigma}\delta B_y = f\delta v_y' - S\delta B_x + \eta\delta B_y'' + \eta'\delta B_y', \quad (22)$$

$$\mathrm{i}\bar{\sigma}\delta B_z = -\mathrm{i}fk_x\delta v_x + \eta\delta B_z'', \quad (23)$$

where $\mathrm{i}\bar{\sigma} = \mathrm{i}\sigma + \eta k_x^2$, and the divergence-free condition is $\mathrm{i}k_x\delta B_x + \delta B_z' = 0$. Finally, the linearized Poisson equation is

$$\delta\Phi'' - k_x^2\delta\Phi = \frac{\rho}{c_s^2 f^2 Q}W. \quad (24)$$

We now eliminate $\delta\boldsymbol{B}$ and δv_z between the linearized equations to obtain a system of ordinary differential equations for $\boldsymbol{U} = (\delta v_x, \delta v_y, W, \delta\Phi)$. We detail this technical procedure in Appendix B. The schematic problem is

$$L_{11}\delta v_x + L_{12}\delta v_y + L_{13}W + L_{14}\delta\Phi = 0, \quad (25)$$

$$L_{21}\delta v_x + L_{22}\delta v_y + L_{23}W + L_{24}\delta\Phi = 0, \quad (26)$$

$$L_{31}\delta v_x \qquad\qquad + L_{33}W + L_{34}\delta\Phi = 0, \quad (27)$$

$$L_{43}W + L_{44}\delta\Phi = 0, \quad (28)$$

where the differential operators L_{1j}, L_{2j} and L_{3j} can be read off Eq. B8, Eq. B11 and Eq. B13 respectively, and L_{4j} from Eq. 24.

Note that $L_{31} \propto k_x$, so that if $k_x = 0$ then Eq. 25—26 are decoupled from Eq. 27—28.

3.1. *Boundary conditions*

We assume $\boldsymbol{U}$ is an even function of z, so that $d\boldsymbol{U}/dz = 0$ at $z = 0$. This implies that $\delta B_x = \delta B_y = 0$ at the midplane, consistent with a highly resistive dead zone. At the upper disk boundary $z = Z_s$ we assume $\delta v_z = \delta B_x = \delta B_y = 0$, and

$$\delta\Phi'(Z_s) + k_x\delta\Phi(Z_s) = 0 \quad (29)$$

for the potential perturbation (see Goldreich & Lynden-Bell 1965a).

3.2. *Numerical procedure*

We use a pseudo-spectral approach to solve the set of linearized equations. Let

$$\boldsymbol{U}(z) = \sum_{k=1}^{N_z}\boldsymbol{U}_k\psi_k(z/Z_s), \quad (30)$$

where $\psi_k = T_{2(k-1)}$, and T_l is a Chebyshev polynomial of the first kind of order l (Abramowitz & Stegun 1965). Note that we only use even Chebyshev polynomials which automatically satisifies the symmetry condition at $z = 0$.

Henceforth we only consider $z \geq 0$. The pseudo-spectral coefficients $\boldsymbol{U}_n$ are obtained by demanding the set of linear equations to be satisfied at N_z collocation points along the vertical direction. We choose these points as the extrema of $T_{2(N_z-1)}$ plus end points.

This procedure discretizes the linear equations to a matrix equation,

$$\boldsymbol{M}\boldsymbol{w} = 0, \tag{31}$$

where $\boldsymbol{M}$ is a $4N_z \times 4N_z$ matrix representing the L_{ij} and upper disk boundary conditions, and $\boldsymbol{w}$ is a vector storing the pseudo-spectral coefficients. Starting with an initial guess for σ, non-trivial solutions to Eq. 31 are obtained by varying σ using Newton-Raphson iteration such that $\det\boldsymbol{M} = 0$ (details can be found in Lin 2012).

3.3. *Diagnostics*

APPENDIX

ANALYTIC EQUILIBRIUM FOR THE POLYTROPIC DISK

For a polytropic disk with $P = K\rho^2$ the dimensional equilibrium equation to be solved is

$$0 = c_{s0}^2 \frac{d^2}{dz^2}\left(\frac{\rho}{\rho_0}\right) + \Omega_z^2 + \frac{\Omega^2}{Q}\left(\frac{\rho}{\rho_0}\right), \tag{A1}$$

which is obtained by combining Eq. 6 and 7 with the above equation of state. The solution is

$$\frac{\rho}{\rho_0} = \left(1 + \frac{\Omega_z^2}{\Omega^2}Q\right)\cos\left(az\right) - \frac{\Omega_z^2}{\Omega^2}Q, \tag{A2}$$

where

$$a^2 \equiv \frac{\Omega^2}{Qc_{s0}^2}. \tag{A3}$$

The polytropic disk thickness is

$$H = \frac{c_{s0}}{\Omega}\sqrt{Q}\arccos\left(\frac{\Omega_z^2 Q}{\Omega^2 + \Omega_z^2 Q}\right). \tag{A4}$$

REDUCTION TO LINEAR HYDRODYNAMICS

Our task here is to remove the magnetic field and vertical velocity perturbations from the non-dimensional linearized equations. Let us first define operators

$$D_0 = 1, \quad D_1 = \frac{\rho'}{\rho} + \frac{d}{dz}, \quad D_2 = \frac{\rho''}{\rho} + \frac{2\rho'}{\rho}\frac{d}{dz} + \frac{d^2}{dz^2}, \tag{B1}$$

and

$$\hat{D}_0 = \eta D_0, \quad \hat{D}_1 = \eta' D_0 + \eta D_1, \quad \hat{D}_2 = \eta'' D_0 + 2\eta' D_1 + \eta D_2. \tag{B2}$$

Denoting the n^{th} vertical derivative as $^{(n)}$, the equations of motion give

$$\delta B_x^{(n)} = \frac{\beta\rho}{f} D_{n-1}\left(\mathrm{i}\sigma\delta v_x - 2\delta v_y + \mathrm{i}fk_x\tilde{W}\right) + \mathrm{i}k_x\delta B_z^{(n-1)}, \tag{B3}$$

$$\delta B_y^{(n)} = \frac{\beta\rho}{f} D_{n-1}\left(\mathrm{i}\sigma\delta v_y + \frac{\kappa^2}{2}\delta v_x\right), \tag{B4}$$

for $n \geq 1$. Differentiating the divergence-free condition for the magnetic field gives

$$\mathrm{i}k_x\delta B_x' + \delta B_z'' = 0. \tag{B5}$$

We insert the expression for $\delta B_x'$ from Eq. B3 and the expressoin for $\delta B_z''$ from the z component of the linearized induction equation (Eq. 23) to obtain

$$-\sigma\delta B_z^{(n)} = fk_x\delta v_x^{(n)} + \frac{k_x\beta\rho}{f}\hat{D}_n\left(\mathrm{i}\sigma\delta v_x - 2\delta v_y + \mathrm{i}fk_x\tilde{W}\right). \tag{B6}$$

Eliminating δB_x

Inserting the above expressions for $\delta B_x''$, $\delta B_x'$ (Eq. B3) and $\delta B_z'$ (Eq. B6) into the right-hand-side of the x-induction equation (Eq. 21) gives

$$\mathrm{i}\sigma\frac{f}{\beta\rho}\delta B_x = \frac{f^2}{\beta\rho}\delta v_x' + \hat{D}_1\left(\mathrm{i}\sigma\delta v_x - 2\delta v_y + \mathrm{i}fk_x\tilde{W}\right). \tag{B7}$$

($\bar{\sigma} \neq 0$ has been assumed to obtain this.) We multiply this expression by ρ, differentiate with respect to z and eliminate the resulting $\delta B_x'$ using Eq. B3, to obtain

$$0 = \frac{f^2}{\beta\rho}\delta v_x'' - \frac{f^2}{\beta\rho}k_x^2\delta v_x + \left(\hat{D}_2 - k_x^2\hat{D}_0 - \mathrm{i}\sigma D_0\right)\left(\mathrm{i}\sigma\delta v_x - 2\delta v_y + \mathrm{i}fk_x\tilde{W}\right). \tag{B8}$$

Eliminating δB_y

We follow a similar procedure as above to remove δB_y. We first use the divergence-free condition to eliminate δB_x from the right-hand-side of the y-induction equation (Eq. 22),

$$\mathrm{i}\bar{\sigma}\delta B_y = f\delta v_y' - \mathrm{i}S\frac{\delta B_z'}{k_x} + \eta\delta B_y'' + \eta'\delta B_y'. \tag{B9}$$

We can now insert expressions for the magnetic field derivatives using Eq. B4 and B6 to obtain an expression for δB_y,

$$\mathrm{i}\bar{\sigma}\delta B_y = f\delta v_y' + \frac{\mathrm{i}S}{\sigma}\left[f\delta v_x' + \frac{\beta\rho}{f}\hat{D}_1\left(\mathrm{i}\sigma\delta v_x - 2\delta v_y + \mathrm{i}fk_x\tilde{W}\right)\right] + \frac{\beta\rho}{f}\hat{D}_1\left(\mathrm{i}\sigma\delta v_y + \frac{\kappa^2}{2}\delta v_x\right). \tag{B10}$$

We differentiate this expression with respect to z, then eliminate δB_y and $\delta B_y'$ from the left-hand-side of the resulting expression using Eq. B10 and Eq. B4, respectively. We obtain

$$\begin{aligned} 0 =& \frac{f^2}{\beta\rho}\left(\delta v_y'' - \frac{\bar{\sigma}'}{\bar{\sigma}}\delta v_y'\right) + \bar{\sigma}\sigma D_0\delta v_y + \frac{\mathrm{i}S}{\sigma}\frac{f^2}{\beta\rho}\left(\delta v_x'' - \frac{\bar{\sigma}'}{\bar{\sigma}}\delta v_x'\right) - \frac{\mathrm{i}\bar{\sigma}\kappa^2}{2}D_0\delta v_x \\ &+ \left(\hat{D}_2 - \frac{\bar{\sigma}'}{\bar{\sigma}}\hat{D}_1\right)\left[\mathrm{i}\left(\sigma - \frac{2S}{\sigma}\right)\delta v_y + \left(\frac{\kappa^2}{2} - S\right)\delta v_x - \frac{Sfk_x}{\sigma}\tilde{W}\right]. \end{aligned} \tag{B11}$$

Elminating δv_z

The vertical velocity perturbation is

$$\delta v_z = \frac{\mathrm{i}f}{\sigma}\tilde{W}'. \tag{B12}$$

Inserting this into the linearized continuity equation (Eq. 17) and using the Poisson equation, we obtain

$$0 = W'' + (\ln\rho)'\,W' + \frac{1}{c_s^2 f^2}\left(\frac{\rho}{Q} + \sigma^2\right)W + (\ln\rho)'\,\delta\Phi' + k_x^2\delta\Phi + \frac{\sigma k_x}{f}\delta v_x. \tag{B13}$$

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