Integrals with algebraic weights,

$$I_n = \int_a^b x^n (x-a)^{\mu-1}(b-x)^{\nu-1}\,\mathrm{d}x.$$

Using G&R 3.196.3:

$$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}\,\mathrm{d}x = (b-a)^{\mu+\nu-1}B(\mu,\nu).$$

Recursion relation. Consider

$$J_n = \int_a^b x^n \frac{\mathrm{d}}{\mathrm{d}x}\left[(x-a)^{\mu}(b-x)^{\nu}\right]\,\mathrm{d}x.$$

Integrating by parts

$$\begin{aligned}
J_n &= x^n (x-a)^{\mu}(b-x)^{\nu}\big|_a^b \\
&\quad - n\int_a^b x^{n-1}(x-a)^{\mu}(b-x)^{\nu}\,\mathrm{d}x, \\
&= n\int_a^b \left(x^{n+1} - (a+b)x^n + abx^{n-1}\right)(x-a)^{\mu-1}(b-x)^{\nu-1}\,\mathrm{d}x, \\
&= nI_{n+1} - n(a+b)I_n + nabI_{n-1}.
\end{aligned}$$

Evaluating the derivative,

$$\begin{aligned}
J_n &= \int_a^b x^n\left[b\mu + a\nu - (\mu+\nu)x\right](x-a)^{\mu-1}(b-x)^{\nu-1}\,\mathrm{d}x, \\
&= (b\mu + a\nu)I_n - (\mu+\nu)I_{n+1}.
\end{aligned}$$

Equating,

$$(n+\mu+\nu)I_{n+1} = \left[n(a+b) + b\mu + a\nu\right]I_n - nabI_{n-1}.$$

Recursion is seeded using,

$$\begin{aligned}
\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[(x-a)^{\mu}(b-x)^{\nu}\right]\,\mathrm{d}x = 0, \\
= (b\mu + a\nu)I_0 - (\mu+\nu)I_1,
\end{aligned}$$

giving

$$\begin{aligned}
I_0 &= (b-a)^{\mu+\nu-1}B(\mu,\nu), \\
I_1 &= \frac{b\mu + a\nu}{\mu+\nu}I_0.
\end{aligned}$$