Nonlinear Analysis: Real World Applications 14 (2013) 829–841

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Complex dynamics in an OLG model of neoclassical growth with endogenous retirement age and public pensions

Luciano Fanti [a,1], Luca Gori [b,*], Mauro Sodini [a,2]

[a] Department of Economics and Management, University of Pisa, Via Cosimo Ridolfi, 10, I–56124 Pisa (PI), Italy

[b] Department of Law, University of Genoa, Via Balbi, 30/19, I–16126 Genoa (GE), Italy

ARTICLE INFO

Article history:
Received 10 August 2011
Accepted 2 August 2012

Keywords:
Health
Old-age labour
OLG model
Perfect foresight
PAYG pensions

ABSTRACT

This study analyses the dynamics of a general equilibrium, overlapping-generations (closed) economy with pay-as-you-go public pensions and tax-financed health investments that affect the retirement time when old. Life of the typical agent is divided between youth (firth period) and old age (second period). The latter period of life is, in turn, divided between work time and retirement time in a proportion contingent on an individual's state of health. We show that: (i) a unique non-trivial steady state exists, and (ii) when the labour income tax rates used to finance health expenditure or public pensions are included in an intermediate range of values, complex dynamics occur when individuals have perfect foresight. This holds because the increase either in the fraction of time spent working when old or disability pensions reduces savings and capital accumulation. In addition, dynamic phenomena such as multiple bubbling structures related to the bifurcation diagram can be observed. Under some general assumptions, we show that the rise in health care expenditure and/or public pensions initially destabilises the steady-state equilibrium and causes complex dynamics but eventually acts as a stabilising device.

1. Introduction

It is recognised that health status of humans and income are significantly correlated and observed to dramatically vary across time and nations (see [1]). The tremendous improvement in lifestyles and standards of living experienced especially in Western countries after World War II, has contributed to make individuals healthier. This fact has accelerated and sustained economic growth in several nations [2], with a resulting increase in both the quantity (longevity) and quality of life of humans, because health spending essentially reduces mortality risks [3,4][3] and fertility, while increasing per capita income [9], even if a non-monotonic relationship that contributes to explain the Demographic Transition is also observed [10–13]. This phenomenon has increased the individual demand for health services, while also requiring the response of governments (especially in developed countries) and international organisations to the question of whether

* Corresponding author. Tel.: +39 010 209 95 03; fax: +39 010 209 55 36.
E-mail addresses: lfanti@ec.unipi.it, fanti.luciano@gmail.com (L. Fanti), luca.gori@unige.it, dr.luca.gori@gmail.com (L. Gori), m.sodini@ec.unipi.it (M. Sodini).

1 Tel.: +39 050 22 16 369; fax: +39 050 22 16 384.

2 Tel.: +39 050 22 16 234; fax: +39 050 22 16 375.

3 At the time of writing, a lot of people continue to suffer from malnutrition in several developing and underdeveloped countries, where also epidemics (e.g., HIV/AIDS) contribute to keep the individual state of health poorer and child mortality higher (see [5]) than in developed countries, HIV/AIDS being one of the main causes of the reversal in the observed positive trend in life expectancy in such countries (see [6,7]). See also [8], which provides estimates of the impact of the tragedy of AIDS for the South African population.

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doi:10.1016/j.nonrwa.2012.08.004

and to what extent health investments and/or disability pensions have to be financed to an increasing number of old age individuals. Therefore, problems concerning the public provision of health services and pension benefits for mature workers are currently high on the political agenda in several industrialised countries.

The study of problems of the interrelationship between health and macroeconomics – notably, income per person – has gained popularity in both the empirical and theoretical literatures,[4] since the state of health of humans may affect economic behaviours: the role played by health on both the ability to work and productivity of work (see [21]); the relationship between adult mortality and private and/or public health spending (see [1,22–26]); the relationship between adult mortality and the accumulation of human capital due to changes in investments in education (see [27]).

The link between health status and labour productivity has been early recognised in the pioneering paper by Grossman [28]. In addition, several empirical studies have found that health plays a relevant role on the labour supply of older people (e.g. [29–33]). However, to the best of our knowledge, the effects of both public health spending and retirement age on the long-run dynamics in a neoclassical growth model has not been so far investigated in a theoretical context. In this paper we aim at filling this gap by using the overlapping-generations (OLG) model à la Diamond [34], extended with the following assumptions: (i) people inelastically supply labour when old in a proportion contingent on their state of health, (ii) an individual's health status when old is improved by the provision of public health expenditure when young, and (iii) public PAYG pensions exist to support old-age people unable to work and then retired (disability pensions).

In order to concentrate on the effects of health spending and pensions on the length of the age of retirement, we avoid including adult mortality in the analysis and the utility effect of health per se. We also neglect to explicitly account for labour/leisure choices (of both young and old people), which can be an important determinant of macroeconomic outcomes. In particular, we assume that the length of the retirement time depends on an individual's health status when old. This means that the age of retirement is chosen neither voluntarily by mature workers nor it is fixed by the government with appropriate laws. When the health status is low, mature workers are allowed to retire and they are entitled to a pension benefit (alternatively, it may be assumed that for the period of ill-health they receive a pay-as-you-go health insurance bonus because they cannot work).

The main finding of the present paper is the following: when the conventional Diamond's model with rational individuals is extended with the three assumptions above mentioned, the dynamics may be oscillatory. Moreover, both periodic and chaotic dynamics seem to be the rule rather than the exception for this simple one-dimensional OLG economy with logarithmic preferences and Cobb–Douglas technology. So far the OLG economic growth literature has shown that complex dynamics typically occur either in two-dimensional model under rational expectations, when the elasticity of substitution between capital and labour in production is sufficiently small (e.g., the case of Leontief technology),[5] or in one-dimensional models when individual have myopic expectations.[6] In the latter case, periodic and/or complex dynamics occur for high values of the inter-temporal elasticity of substitution in the utility function (e.g., [41–44]). In either cases of rational and myopic expectations, therefore, the OLG economy with logarithmic preferences and Cobb–Douglas technology represents a framework not prone to describe periodic or chaotic dynamic events, even if some exceptions do exist [21,45].

In addition, the long-run dynamics described in the present paper shows a multiplicity of "bubbling" phenomena[7] related to the bifurcation diagram [47], i.e., a sequence of period doubling bifurcations is followed a sequence of period halving bifurcations, when either the pension contribution rate or health tax rate is included in an intermediate range of values. However, further increases in the pension contribution rate or the health tax rate eventually reduce economic fluctuations, thus properly working for the global stability of the economy. This twofold role is remarkable from an economic point of view: on the one hand, it contributes to explain the observed business cycles in per capita income—showing that an endogenous deterministic origin of economic cycles may complement its stochastic origin, the latter being at the core of the real business cycle theory (see [48,49]); on the other hand, the health tax rate may even be used to control and eventually suppress periodic or complex dynamics. Thus, the equilibrium dynamics in this simple economy may reconcile the existence of business cycles [50] or monotonic dynamics [51] depending on the configuration of parameters.

The rest of the paper is organised as follows. Section 2 presents the model. Section 3 (resp. 4) analyses the steady-state and dynamic outcomes when the health tax rate (resp. the contribution rate to the pension system) varies. Section 5 concludes.

2. The economy

2.1. Individuals

Consider a general equilibrium OLG closed economy comprised of a continuum of rational and identical individuals of measure one per generation. Population is constant and there is no adult mortality, i.e., an individual is alive at the end of

4 On the empirical side, see [14–17]. On the theoretical side, see [18–20].

5 For instance, Reichlin [35] discusses the Leontief case, while in the Farmer's [36] example with a Constant Elasticity of Substitution (CES) technology, endogenous fluctuations can occur only whether the production function exhibits lower factor substitutability than the Cobb–Douglas function.

6 Note that interesting economic models either with overlapping generations and discrete time or Ramsey-type models with continuous time exist, where nonlinear dynamics, oscillations and deterministic chaos are studied, especially with regards to problems of environmental sustainability and externality (see [37–40]).

7 Pioneering discussions about similar dynamic outcomes can be found in [46,47].

youth with certainty. Life of the typical agent is divided into youth (first period) and old-age (second period), as in [34]. An individual born at the beginning of time t has preferences towards first-period consumption ($c_{1,t}$) and second-period consumption ($c_{2,t+1}$). When young, he/she is endowed with one unit of labour inelastically supplied to firms, while receiving wage income w_t per unit of labour. This income is used for consumption and saving purposes. Moreover, the government collects wage income taxes at the constant rates $0 < \tau < 1$ and $0 < \theta < 1$, with $0 < \tau + \theta < 1$, to *separately* finance health investments and public pensions, respectively. Therefore, the first-period budget constraint of individuals of generation t reads as follows:

$$c_{1,t} + s_t = w_t(1 - \tau - \theta), \tag{1a}$$

where s_t is saving.

The time endowment of an old individual of generation t is divided between work time (d_{t+1}) and retirement time ($1 - d_{t+1}$). He/she expects to receive the market wage w^e_{t+1} when he/she works, and (disability) pensions p^e_{t+1} when he/she retires (see [52,53]) because of ill-health. This amounts to say that the total retirement benefit depends on both the pension entitlement p and length of retirement $1 - d$. The budget constraint at time $t + 1$ of an old person born at time t therefore is:

$$c_{2,t+1} = R^e_{t+1} s_t + d_{t+1} w^e_{t+1}(1 - \theta) + (1 - d_{t+1}) p^e_{t+1}, \tag{1b}$$

where $R^e_{t+1} = 1 + r^e_{t+1}$ is the expected factor of interest from time t to time $t + 1$ on saving. Moreover, Eq. (1b) implies that a tax on mature workers to finance pensions does exist (see [54]).

We assume that the length of the work time when old, d_{t+1}, depends on an individual's state of health when old, which is improved by health expenditure h_t provided when young, i.e. the healthier an old individual, the larger the fraction of time spent working. In a content with exogenous (constant) population (namely, fertility and longevity), the assumption that an individual's state of health when old (and the age of retirement accordingly) is determined only by public health investments provided when young, is general enough to capture the case under which an individual's health status when old depends upon investments in health provided both when young and when old.[8]

The relationship between the length of work time of mature workers and health expenditure is captured by the following strictly increasing (though bounded) function:

$$d_{t+1} = d(h_t), \tag{2}$$

where $d(0) = d_0 \geq 0$, $\lim_{h\to\infty} d(h) = d_1$ and $0 \leq d_0 < d_1 < 1$.

Preferences of the individual representative of generation t over consumption when young and consumption when old are described by the logarithmic utility function:

$$U_t = \ln(c_{1,t}) + \beta \ln(c_{2,t+1}), \tag{3}$$

where $0 < \beta < 1$ is the degree of individual (im)patience to consume over the life cycle. He/she chooses how much to save out of his/her disposable income to maximise Eq. (3) subject to Eqs. (1a) and (1b), where actual and expected factor prices, the coefficient d_{t+1}, the tax rates and the expected pension benefit p^e_{t+1} are taken as given. Therefore, optimal savings for an individual are given by:

$$s_t = \frac{\beta\, w_t(1 - \tau - \theta)}{1 + \beta} - \frac{1}{(1 + \beta)R^e_{t+1}}[d_{t+1} w^e_{t+1}(1 - \theta) + (1 - d_{t+1}) p^e_{t+1}]. \tag{4}$$

2.2. Firms

At time t, identical and competitive firms produce a homogeneous good, Y_t, by combining capital and labour, K_t and L_t, respectively. The (aggregate) constant returns to scale Cobb–Douglas technology is $Y_t = AK_t^\alpha L_t^{1-\alpha}$, where $A > 0$ is a scale parameter and $0 < \alpha < 1$ the output elasticity of capital. The labour supply is $L_t = \bar{L}(1 + d_t)$, where $\bar{L}$ is the constant number of (young and old) workers in each cohort (see [21,55])[9]; then, we set $\bar{L} = 1$ without loss of generality. Therefore, output per efficient worker (y_t) as a function of capital per efficient worker (k_t) is

$$y_t = Ak_t^\alpha, \tag{5}$$

where $y_t := Y_t/L_t$ and $k_t := K_t/L_t$ are output and capital per efficient labour.

By assuming that capital totally depreciates at the end of every period and output is sold at the unit price, firms maximise profit by taking factor prices as given, so that perfect competition guarantees that factor inputs are paid their marginal products, that is:

$$R_t = \alpha A k_t^{\alpha-1}, \tag{6a}$$

$$w_t = (1 - \alpha) A k_t^\alpha. \tag{6b}$$

8 See Footnote 10 for a discussion on the difference in assuming that an individual's state of health when old depends on the public provision of health investments when young and when old in a model with exogenous or endogenous population.

9 See also [54,56].

2.3. Government

The government separately finances health investments and unfunded pensions at a balanced budget by levying labour income taxes.

Health expenditure at time t $(N_t h_t)$ is constrained by the amount of tax receipts $(N_t \tau\, w_t)$, where $0 < \tau < 1$ is the health tax rate and N_t is the number of young people in period t. Therefore, the health budget per young person in period t reads as follows[10]:

$$h_t = \tau\, w_t, \tag{7a}$$

see [1,23,45].

PAYG pensions $P_t = (1 - d_t)p_t N_{t-1}$ are also used to redistribute across generations, where p_t is the pension expenditure per pensioner in period t weighted by the retirement time of the old that belong to generation $t - 1$. Such an expenditure is financed by levying a constant wage income tax rate $0 < \theta < 1$ on both young-age workers and old-age workers in every period, so that $N_t\theta\, w_t + N_{t-1}d_t\theta\, w_t$ represents the amount of tax receipts. Knowing $N_t = N_{t-1} = 1$, the time-t per pensioner budget constraint of the government is given by the following accounting rule:

$$(1 - d_t)p_t = \theta\, w_t(1 + d_t). \tag{7b}$$

2.4. General equilibrium

Exploiting the one-period forward pension accounting rule Eq. (7b) to substitute out for the pension expenditure, $(1 - d_{t+1})p^e_{t+1}$, into Eq. (4) and using Eq. (2), saving can be rewritten as follows:

$$s_t = \frac{\beta}{1+\beta} w_t(1 - \tau - \theta) - \frac{w^e_{t+1}}{(1+\beta)R^e_{t+1}}[\theta + d(h_t)]. \tag{8}$$

Given the government budget constraints Eqs. (7a) and (7b), market-clearing in the capital market yields the equilibrium condition:

$$k_{t+1}[1 + d(h_t)] = s_t. \tag{9}$$

Since $h_t = \tau\, w_t$ and w_t depends on k_t through Eq. (6b), then we can define $D(k_t) := d(\tau(1-\alpha)Ak_t^{\alpha}) = d(h_t)$.[11] Now, let individuals have perfect foresight, i.e. $R^e_{t+1} = \alpha A k^{\alpha-1}_{t+1}$ and $w^e_{t+1} = (1-\alpha)Ak^{\alpha}_{t+1}$. Then, by using Eqs. (8) and (9) capital accumulation is described by the following first order nonlinear difference equation:

$$k_{t+1} = J(k_t) := \frac{Iy(k_t)}{F + MD(k_t)}, \tag{10}$$

where $I := \beta\alpha(1-\alpha)(1-\tau-\theta) > 0$, $F := \alpha(1+\beta) + \theta(1-\alpha) > 0$, $M := 1 + \alpha\beta > 0$, $F < M$, and $y(k_t) = Ak_t^{\alpha}$ from Eq. (5).

3. Dynamics

This section studies the existence and stability properties of the steady states of the map defined in Eq. (10).

Proposition 1. (1) *$k = 0$ is a locally unstable steady state.* (2) *The infinity is not an attractor.*

Proof. It is easy to verify that $k = 0$ is a stationary solution of Eq. (10). From the hypotheses stated below Eq. (2), we have that for $k \to 0$

$$J(k) \sim \frac{IA}{G}k^{\alpha}.$$

Then, it follows that $k = 0$ is locally unstable. This proves point (1). In addition,

$$\lim_{k\to\infty} J(k) = +\infty, \qquad \lim_{k\to\infty} \frac{J(k)}{k} = 0.$$

This proves point (2). □

Proposition 2. *A unique non-trivial steady state $k^* > 0$ of the dynamic system described by Eq. (10) does exist.*

[10] Things may be different if: (i) population (fertility and longevity) is endogenous, and (ii) an individual's health status when old depends on health investments provided both when young and when old. In fact, the government budget constraint equation (7a) modifies to: $h_t(1 + \frac{\pi_{t-1}}{n_{t-1}}) = \tau\, w_t$, where n_{t-1} (resp. π_{t-1}) is the rate of fertility (resp. longevity) at time $t-1$. Another interesting extension would be the assumption that an individual's health status when old depends on public health investments provided when young and when old, and the government collects wage income taxes on both young workers and old workers accordingly. The budget constraint equation (7a), therefore, modifies to $h_t = \tau\, w_t(1 + d_t)$ and then through Eq. (1) one obtains $d(h_t) = d(\tau\, w_t(1 + d_t))$. These extensions are left for future research.

[11] Therefore, the retirement age is endogenous as it depends on health expenditure, which in turn depends on capital accumulation.

Proof. Positive fixed points of the map are determined as interior solutions to $k = J(k)$, that can be rearranged as follows: $Z_1(k) = Z_2(k)$, where $Z_1(k) := k^{1-\alpha}$ and $Z_2(k) := \frac{IA}{F+MD(k)}$. Since $Z_1(k)$ and $Z_2(k)$ are continuous functions and: (i) $Z_1(0) = 0$, $Z_1'(k) = (1-\alpha)\, k^{-\alpha} > 0$ for any $k > 0$, $\lim_{k\to+\infty} Z_1(k) = +\infty$, and (ii) $Z_2(0) = \frac{IA}{F+Md_0} = \frac{IA}{G} > 0$, where $G := F + Md_0 > 0$, $Z_2'(k) = \frac{-MIAD'(k)}{[F+MD(k)]^2} < 0$ for any $k > 0$ since $D'(k) > 0$, then $Z_1(k) = Z_2(k)$ only once at k^* for any $k > 0$. □

From Propositions 1 and 2 it follows that the map Eq. (10) has the following properties: $J(k) > k$ for any $0 < k < k^*$ (that is, the graph of J lies above the 45° line) and $J(k) < k$ for any $k > k^*$ (that is, the graph of J lies below the 45° line).

Now, let $\varepsilon_d := D'(k^*)\frac{k^*}{D(k^*)}$ and $\varepsilon_y := y'(k^*)\frac{k^*}{y(k^*)}$ be the elasticity of the supply of labour when old and the elasticity of GDP per efficient worker with respect to the stock of capital per efficient worker evaluated at k^*, respectively. Then, the following proposition holds.

Proposition 3 (*Non-Monotonic Local Dynamics*)**.** *If*

$$\varepsilon_d > \varepsilon_y, \tag{11}$$

and

$$D(k^*) > \hat{D}(k^*) := \frac{\varepsilon_y}{\varepsilon_d - \varepsilon_y} \cdot \frac{F}{M}, \tag{12}$$

then the (local) dynamics described by Eq. (10) are non-monotonic and characterised by oscillations.

Proof. By differentiating $J(k)$ with respect to k and evaluating it at k^*, we get:

$$J'(k^*) = \frac{IA[F\varepsilon_y - M(\varepsilon_d - \varepsilon_y)D(k^*)]}{(k^*)^{1-\alpha}[F + MD(k^*)]^2}. \tag{13}$$

Since $\operatorname{sgn}(J'(k^*)) = \operatorname{sgn}(F\varepsilon_y - M(\varepsilon_d - \varepsilon_y)D(k^*))$, then: (1) for any $\varepsilon_d < \varepsilon_y$, $J'(k^*) > 0$ always holds; (2) for any $\varepsilon_d > \varepsilon_y$, $J'(k^*) > 0$ when $D(k^*) < \hat{D}(k^*)$ and $J'(k^*) < 0$ when $D(k^*) > \hat{D}(k^*)$. □

It is important to stress that when conditions (11) and (12) are fulfilled, the dynamics around k^* can be either non-monotonic and convergent (damped oscillations) or non-convergent. This is shown below by using a particular functional form of the function D. In particular, in order to show: (i) the possibility of long-run non-monotonic (and/or complex) dynamics, and (ii) how the main long-run macroeconomic variables react to a change in either the size of the pension system (θ) or the public health system (τ), we now introduce the following functional form of d (see [24,27]) that satisfies the general properties listed below Eq. (2), that is:

$$d_{t+1} = d(h_t) = \frac{d_0 + d_1\,\Delta h_t^{\delta}}{1 + \Delta h_t^{\delta}}, \tag{14}$$

where $\delta, \Delta > 0$, $0 \le d_0 < 1$, $d_0 < d_1 < 1$, $d(0) = d_0 \ge 0$, $d'(h) = \frac{\delta\,\Delta h^{\delta-1}(d_1-d_0)}{(1+\Delta h^{\delta})^2} > 0$, $\lim_{h\to\infty} d(h) = d_1 \le 1$, $d''(h) < 0$ if $\delta \le 1$ and $d''(h) \gtrless 0$ for any $h \lessgtr h_T := \left[\frac{\delta-1}{(1+\delta)\Delta}\right]^{\frac{1}{\delta}}$ if $\delta > 1$.

Eq. (14) captures several features of the ability to work when old of the typical agent as a function of the health measure h. In particular, it encompasses (i) the monotonic concave function used in the numerical examples by Chakraborty [1] and Leung and Wang [25] when $\delta = \Delta = 1$ and $d_0 = 0$, while also preserving a positive exogenously given level of old age working period regardless of whether public health spending exists; (ii) the S-shaped function when $\delta > 1$ (i.e., threshold effects of public health investments exist) used in the numerical experiments by de la Croix and Ponthière [24] and Blackburn and Cipriani [27].

The relationship between health status and income (which is a proxy of the public health spending in this model), has been shown to be S-shaped by Ecob and Davey Smith [57],[12] by arguing that "these indices of morbidity, both self-reported and measured, are approximately linearly related to the logarithm of income, in all except very high and low incomes (this means that increasing income is associated with better health, but that there are diminishing returns at higher levels of income)". (p. 693). Moreover, Strauss and Thomas [58] contrast an index that captures nutritional status and health (the Body Mass Index) with the logarithm of wage income, and find an S-shaped relationship for Brazil [58, Fig. 3, p. 774].

Now, by using Eq. (14) as a particular functional form of d, the map equation (10) becomes the following:

$$J(k) = \frac{Pk^{\alpha}(1 + Qk^{\alpha\delta})}{1 + QBk^{\alpha\delta}}, \tag{15}$$

[12] See [57, Figs. 1 and 2, pp. 698–700], where some indices of morbidity as a function of the logarithm of income for England, Wales and Scotland in 1984 and 1985 are presented.

where $Q := \Delta(\tau(1-\alpha)A)^{\delta} > 0$, $B := 1 + \frac{(d_1-d_0)(1+\alpha\beta)}{\alpha(1+\beta)+\theta(1-\alpha)+(1+\alpha\beta)d_0} = 1 + \frac{(d_1-d_0)M}{G} > 1$ and $P := \frac{\beta\alpha(1-\alpha)(1-\tau-\theta)A}{\alpha(1+\beta)+\theta(1-\alpha)+(1+\alpha\beta)d_0} = \frac{IA}{G}$. From Eq. (15), we note that the parameter δ enters only Q, and B and P are both decreasing functions of θ, with P being a decreasing function of τ.

In order to study the dynamic behaviour of the map equation (15), it is important to inquire about the monotonic properties of J.[13] By simple calculations we find that:

$$\operatorname{sgn}(J'(k)) = \operatorname{sgn}(BQ^2k^{2\alpha} + Q[\delta + 1 + (1-\delta)B]\,k^{\alpha} + 1). \tag{16}$$

If $\delta < 1$, then $J'(k) > 0$ for any $k > 0$. If $\delta > 1$, then from Eq. (16), it is easy to recognise that the expression of $J'(k)$ is a quadratic equation in k^{α} and the corresponding discriminant is the following:

$$\Psi := (B-1)(B(\delta-1)^2 - (\delta+1)^2). \tag{17}$$

From straightforward calculation we have that

$$\Psi < 0 \Leftrightarrow B < \left(\frac{\delta+1}{\delta-1}\right)^2, \tag{18}$$

which in turn implies that $\Psi < 0$ if and only if

$$\delta < \delta^{\circ} := \frac{\sqrt{(1+\alpha\beta)\frac{d_1-d_0+G}{G}} + 1}{\sqrt{(1+\alpha\beta)\frac{d_1-d_0+G}{G}} - 1} > 1. \tag{19}$$

Then, from Eq. (19) we have the following proposition.

Proposition 4. *Let J be the map defined in Eq. (15). If $\delta < \delta^{\circ}$, then k^* is the global attractor of the map for positive initial conditions.*

In the case described by Proposition 4 the dynamics of map J are monotonic and convergent to the steady state. This is in line with the result obtained in the standard Diamond's model.

In contrast, by considering the case $\Psi > 0$, that is $\delta > \delta^{\circ}$, we find that:

$$B > \left(\frac{\delta+1}{\delta-1}\right)^2 > \frac{\delta+1}{\delta-1} > 1, \tag{20}$$

from which it follows that

$$\delta + 1 + (1-\delta)B < 0. \tag{21}$$

Thus, when $\delta > \delta^{\circ}$, by applying Descartes' rule of sign we find that two positive critical points $k_{\max}$ (local maximum) and $k_{\min}$ (local minimum) exist, with $k_{\min} > k_{\max}$, whose coordinates are respectively given by:

$$k_{\max} := \left(\frac{B(\delta-1) - (1+\delta) - \sqrt{(B-1)(B(\delta-1)^2 - (\delta+1)^2)}}{2BQ}\right)^{\frac{1}{\alpha\delta}}. \tag{22}$$

$$k_{\min} := \left(\frac{B(\delta-1) - (1+\delta) + \sqrt{(B-1)(B(\delta-1)^2 - (\delta+1)^2)}}{2BQ}\right)^{\frac{1}{\alpha\delta}}. \tag{23}$$

Then, the map defined in Eq. (15) is bimodal. However, if $k^* < k_{\max}$ or $k^* > k_{\min}$, then k^* continues to be the unique global attractor of the dynamic system and, if trajectories start out from an initial condition sufficiently close to k^*, then they are monotonic. In contrast, if $k_{\max} < k^* < k_{\min}$, then the dynamics can be dramatically different. However, in this case (see [61]) trajectories with a positive initial stock of capital fall in the forward invariant interval $[J(k_{\max}),\ J(k_{\min})]$.

Amongst the parameters of the model, in what follows we concentrate on the study of the dynamic effects of changes in the health tax rate τ and the contribution rate to the pension system θ (Section 4), given the importance of them as policy parameters.

We now study the dynamics of map (15) by focusing on the role played by the health tax rate τ. First, we note that τ does not enter the discriminant equation (17). This implies that by assuming a parameter constellation such that the critical points $k_{\max}$ (local maximum) and $k_{\min}$ (local minimum) exist, they continue to exist when τ varies. Second, we note that τ

13 In general, for the map defined in Eq. (15) it is difficult to obtain analytical results about the number of attracting cycles since the Schwarzian derivative (see [59]) may change sign and no limitation on the first derivative of the map exists (see [60] for a case in which such limitation allows to have a uniqueness result).

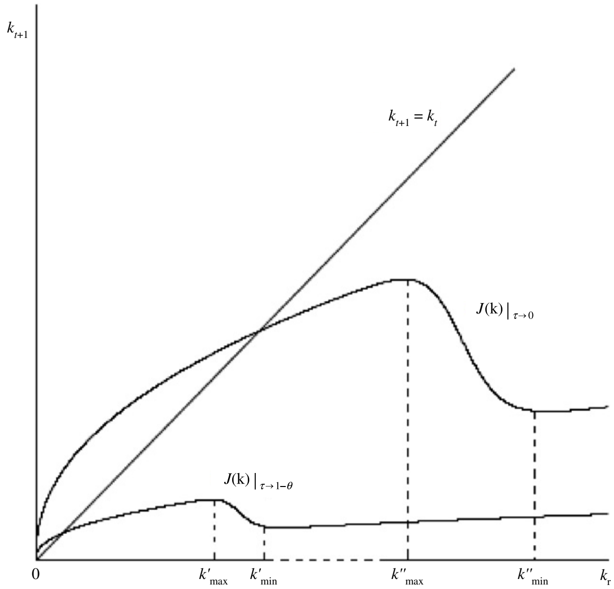

Fig. 1. ($\delta > \delta°$). Alteration in the shape of the graph of J (Eq. (15)) when $\tau \to 0$ and when $\tau \to 1-\theta$.

belongs to the interval $[0,\ 1-\theta)$. In particular, (1) when $\tau \to 1-\theta$ map (15) boils down into $J(k) = 0$. This, in turn, implies that $k = 0$ is superstable. It is important to note that $k_{max} > 0$ and $k_{min} > 0$ even if $\tau \to 1-\theta$, while (2) when $\tau = 0$ the critical points k_{max} and k_{min} are moved towards $+\infty$. Point (2) implies that a threshold value $\tau_1 > 0$ exists such that map (15) admits a globally stable fixed point for any $\tau < \tau_1$: the critical points k_{max} and k_{min} are located on the right of the fixed point k^* and both $J(k_{max}) < k_{max}$ and $J(k_{min}) < k_{min}$. On the other hand, by starting from a value of the health tax rate such that $\tau \to 1-\theta$, we note that the graph of the map moves towards the abscissa line. Then, a threshold value $\tau_2 > 0$ exists such that the critical points k_{max} and k_{min} are located on the right of the fixed point k^* and both $J(k_{max}) < k_{max}$ and $J(k_{min}) < k_{min}$. This implies that the graph of the map crosses the diagonal in its initial increasing branch; then, a globally stable fixed point exists for any $\tau > \tau_2$.[14] Fig. 1 depicts in a stylised way the alteration in the shape of the graph of the map when $\tau \to 0$ and $\tau \to 1-\theta$. Notice alto that an alteration in the graph of the map when τ varies does not necessarily imply the existence of complex dynamics. It is possible that the critical points k_{max} and k_{min} are located on the right of k^* for any $\tau \in [0,\ 1-\theta)$ or, alternatively, even if a range of τ such that $k_{max} < k^* < k_{min}$ exists, then $J'(k^*) > -1$ for any $\tau \in [0,\ 1-\theta)$.

However, it is possible to refine the knowledge of the dynamic behaviour of the map defined in Eq. (15) by introducing some hypotheses about the configuration of parameters, the results of which are summarised in the following proposition.

Proposition 5. *Let $\delta > \delta°$ and A be sufficiently large. Then, there exists at least $\underline{\tau} \in [0,\ 1-\theta)$ (resp. $\bar{\tau} \in [0,\ 1-\theta)$) such that $k_{min} = k^*$ (resp. $k_{max} = k^*$). If $J'(k_{flex}) < -1$ for any $\tau \in (\underline{\tau}, \bar{\tau})$, where k_{flex} is the unique inflection point of J in the interval (k_{max}, k_{min}), then there exists $\underline{\tau} < \tau° < \tau^{°°} < \bar{\tau}$ such that map (15):*

(1) *admits a locally asymptotically stable fixed point k^* for any $\tau \in (\underline{\tau}, \tau°)$;*
(2) *generically[15] undergoes a flip bifurcation when $\tau = \tau°$, and an attracting two-period cycle around the fixed point k^* is born;*
(3) *generically undergoes a flip bifurcation when $\tau = \tau^{°°}$, and an attracting two-period cycle around the fixed point k^* is born;*
(4) *admits a locally asymptotically stable fixed point k^* for any $\tau \in (\tau^{°°}, \bar{\tau})$.*

Proof. Consider a sufficiently large value of A such that $J(k_{min}) > k_{min}$. This can always be assumed because the map defined in Eq. (15) is increasing and unbounded with A. From the study of the second derivative of J, it follows that a unique maximum point k_{flex} of J' exists in the interval (k_{max}, k_{min}). Then, by the properties of J there exist $\underline{\tau}, \bar{\tau} \in [0,\ 1-\theta)$, with $\bar{\tau} > \underline{\tau}$, such that $k_{min} = k^*$ and $k_{max} = k^*$, respectively. By the continuity of $J'(k^*)$, considered as a function of τ, and by the hypothesis that $J'(k_{flex}) < -1$ for any $\tau \in (\underline{\tau}, \bar{\tau})$, we have the existence of $\tau°, \tau^{°°}$, with $\tau^{°°} > \tau°$, such that $J'(k^*)|_{\tau=\tau°} = -1$ and $J'(k^*)|_{\tau=\tau^{°°}} = -1$. By the shape of J, the stability properties of cycles hold. □

Remark 1. In general, the study of map J defined in Eq. (15) shows more intricate bifurcation sequences, when τ varies, than those usually classified in the literature on bimodal maps (see [62], which describes the skeleton of the bifurcation pattern

14 We note that τ_1 (resp. τ_2) is the highest (resp. lowest) value of τ, *when it exists*, such that $J'(k^*) = -1$. Otherwise, τ_1 (resp. τ_2) coincides with $1-\theta$ (resp. 0).

15 For some parameter values it is possible that the condition on the mixed derivative of higher order (see [59]) is not fulfilled.

of a cubic map, and Hommes [60], which classifies the dynamics of a family of bimodal maps, with first order derivative bounded from above, when its graph is vertically shifted). This is due to the fact that τ induces deep changes in the shape of the graph of J. In particular, by starting from $k_{\min} = k^*$ and $J'(k_{\text{flex}}) < -1$, it is possible to observe that $J'(k_{\text{flex}}) > -1$ as τ raises, so that no flip bifurcation occurs.[16] Hence, the need to assume that $J'(k_{\text{flex}}) < -1$ for any $\tau \in (\underline{\tau}, \overline{\tau})$. Although this is a strong assumption, it is easily verified, because k_{flex} can be solved in closed-form.

Proposition 5 allows us to get information on the values of τ for which the fixed point k^* is "close enough" to the critical points $k_{\max}$ and $k_{\min}$. Indeed, this leaves a question of what can happen from a dynamic point of view when τ is included in an intermediate range of values.

In order to clarify the interesting dynamic properties of the system defined in Eq. (15), we now resort to some numerical experiments by taking the following configuration of parameters: $\alpha = 0.45$ (which is an average between the values usually referred to developed countries, i.e. $\alpha = 0.36$, see e.g., [63,64], and those usually used for developing countries, i.e. $\alpha = 0.5$, see Purdue University's Global Trade Analysis Project 2005 database—GTAP), $\beta = 0.6$ (see [65]), $\theta = 0.3$ (see, e.g., [66,67]), $d_0 = 0$, $d_1 = 0.9$, $\delta = 50$, and $\Delta = 50$. Then, we let health tax rate τ vary as the bifurcation parameter, given its importance as a policy variable. Note that the choice of $\delta = 50$[17] amounts to assume that health investments have a stronger effect in improving an individual's state of health when a certain threshold value of health investments is achieved, while becoming scarcely effective when the ability to work is close to its saturating value.[18] The simplest possible bifurcation scenario is obtained when a flip bifurcation emerges and it is followed by a reverse flip bifurcation without any further dynamic events, as it can be seen by looking at Fig. 2(a) where $A = 8.5$. By increasing the value of the production scale parameter to $A = 8.7$, Fig. 2(b) shows that a second period doubling bifurcation and a subsequent period halving occur. For larger values of A, the number of period doubling and period halving bifurcations increases, as depicted in Fig. 2(c) ($A = 8.8$) and 2(d) ($A = 8.9$), where an apparently chaotic attracting set emerges (see also Fig. 3, where the Lyapunov exponent against τ is plotted the configuration of parameters corresponding to Fig. 2(d)). Fig. 2(a)–(d) are characterised by the existence of a global attractor for polar values of τ, and cyclical or more complex behaviours for an intermediate range of values of τ.[19] The dynamic scenario drastically changes for larger values of A. The bifurcation diagram for τ plotted in Fig. 2(e) ($A = 10$) and 2(f) ($A = 14$) show several ranges of the bifurcation parameter where both a stable periodic and apparently chaotic behaviour alternate several times. In particular, Fig. 2(f) depicts the case where a window where the steady state τ is a global attractor exists between two different "bubbles". This last phenomenon is due to the fact that when τ varies we observe both horizontal and vertical alterations in the shape of the graph of the map, such that $(k_{\max}, J(k_{\max}))$ and $(k_{\min}, J(k_{\min}))$ cross the 45° line several times.

4. Dynamics when θ varies

In this section we study how the long-run and dynamic outcomes of the system defined by Eq. (15) change when the contribution rate to the pension system (θ) varies. We also present some comparative static exercises to study how a change in θ affect steady state outcomes in a Chakraborty-type economy with tax-financed health expenditure. With regard to the steady state, we now perform numerical experiments to show how the main macroeconomic variables react when θ varies for a given level of τ. The configuration of parameters is the following: $A = 22,\ \alpha = 0.45,\ \beta = 0.6,\ \tau = 0.06,\ d_0 = 0,\ d_1 = 0.9,\ \delta = 50$, and $\Delta = 1$. To this purpose, Table 1 illustrates Fig. 4 with regards to the effects of a rise in θ on the steady state stock of capital per efficient worker, while also reporting the steady state values of per capita health expenditure, h^*, the length of time spent working when old, d^*, the level of per capita GDP, $Y^* = A(k^*)^\alpha(1+d^*)$, the ratio of per capita health spending to per capita GDP, h^*/Y^*, and the ratio of the per capita pension expenditure to per capita GDP, $\frac{(1-d^*)p^*}{Y^*}$.
Then, the following results hold.

Result 1. *The steady-state stock of capital per efficient worker (k^*), the per capita health expenditure (h^*), the length of the work time when old (d^*), and the per capita GDP (Y^*) monotonically reduce as θ rises.*

Result 2. *The ratio of per capita health expenditure to per capita GDP at the steady state (h^*/Y^*) and the ratio of capita pension expenditure to per capita GDP ($\frac{(1-d^*)p^*}{Y^*}$) monotonically increase as θ rises.*

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16 As an example, for the following parameter set: $A = 14.1$, $\alpha = 0.45$, $\beta = 0.6$, $\tau = 0.58$, $\theta = 0.3$, $d_0 = 0$, $d_1 = 0.9$, $\delta = 50$ and $\Delta = 50$, we have that $k^* = k_{\min} \cong 0.0256$ and $J'(k_{\text{flex}}) \cong -1.524$. Nevertheless if we let τ increase, no bifurcation occurs due to the strong "flattening" of the graph of the map.

17 Note that if $d(h)$ in Eq. (14) is a concave function (i.e. $\delta \leq 1$), the condition $\varepsilon_d > \varepsilon_y$ cannot hold and, hence, non-monotonic trajectories can never be observed in such a case.

18 As an example, think of the existence of threshold effects in the accumulation of knowledge required for new medical advances and discoveries in the treatment of diseases (e.g. vaccines) to be effective: the public health expenditure to finance new research projects may be high and apparently useless as long as a certain degree of knowledge is achieved. Beyond such a threshold, however, a "jump" effect exists that allows to trigger and bring the beneficial effects of the new discoveries to light, to make them efficient, usable and operative across population and eventually transformed into better health for older people.

19 Note that similar phenomena have been studied and classified for analogous (bimodal) maps by using the kneading theory (see, e.g., [68]).

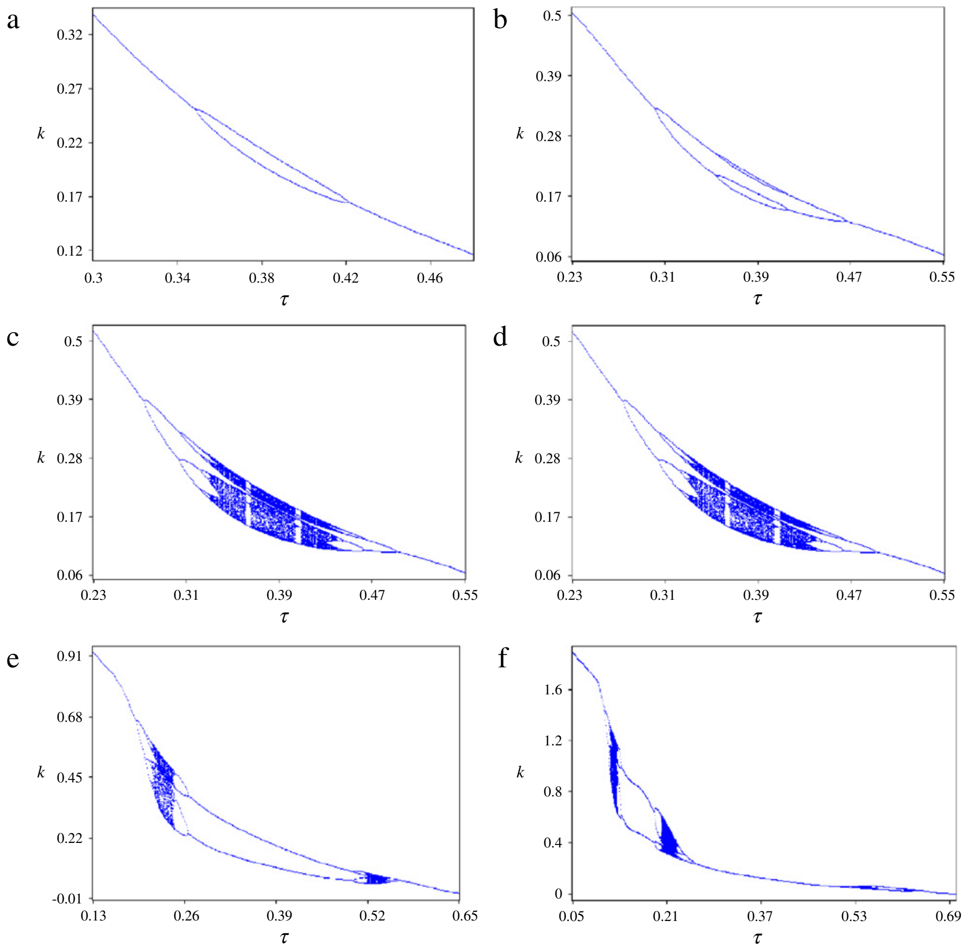

Fig. 2. Parameter set: $\alpha = 0.45$, $\beta = 0.6$, $\theta = 0.3$, $d_0 = 0$, $d_1 = 0.9$, $\delta = 50$, and $\Delta = 50$. Bifurcation diagram with respect to τ for different values of A. (a) $A = 8.5$: one period doubling bifurcation followed by one period halving bifurcation. (b) $A = 8.7$: an example of finitely many period doubling and period halving bifurcations. (c) $A = 8.8$ and (d) $A = 8.9$: apparently infinitely many period doubling and period halving bifurcations which generate a chaotic set. (e) $A = 10$: stable periodic and apparently chaotic behaviour alternate one time. (f) $A = 14$: stable periodic and apparently chaotic behaviour alternate several times.

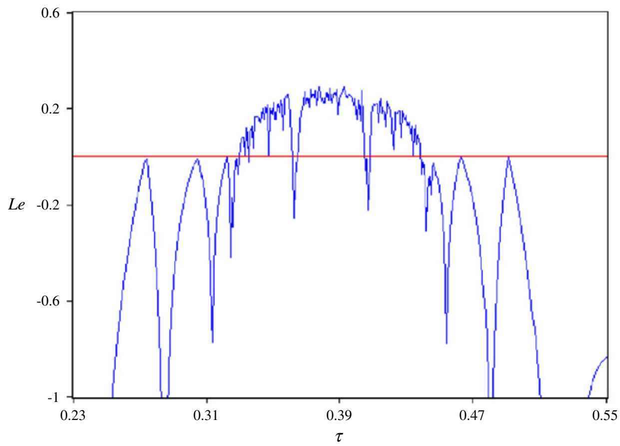

Fig. 3. Lyapunov exponent plotted against τ in the case corresponding to Fig. 2(d) ($A = 8.9$). The (positive) value of the Lyapunov exponent (Le) implies sensitive dependence on initial conditions of the system for some ranges of τ.[20]

[20] As is well known a positive value of the Lyapunov exponent does not necessarily imply the existence of a chaotic set. Nevertheless, for some parameter values the Li and Yorke [69] theorem can be applied to show the existence of a three-period cycle.

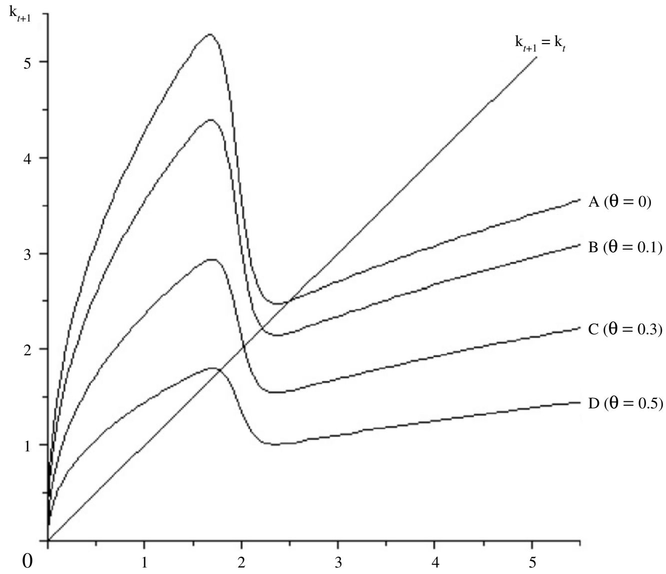

Fig. 4. Phase map Eq. (15) when θ raises ($\tau = 0.06$).

Table 1
Steady-state macroeconomic variables when θ varies.

θ	0	0.05	0.1	0.15	0.25	0.3	0.5
k^*	2.5	2.31	2.21	2.15	2.06	2.02	1.77
h^*	1.09	1.05	1.03	1.02	1	0.98	0.94
d^*	0.89	0.85	0.78	0.70	0.51	0.42	0.04
Y^*	62	59	56	52	46	42	29
h^*/Y^*	0.017	0.017	0.018	0.019	0.022	0.023	0.031
$\frac{(1-d^*)p^*}{Y^*}$	0	0.02	0.05	0.08	0.137	0.16	0.275

The economic intuition behind Results 1 and 2 is the following. An increase in PAYG pensions causes an expected negative effect on capital accumulation and wages, and this in turn reduces health expenditure. Although the reduction in health spending is small and the ratio between health spending to GDP slightly increases,[21] the worsening in the state of health status and the corresponding reduction in the ability to work, is large: for example, while in the absence of public pensions ($\theta = 0$) mature workers spend 89% of their second period of life to work, when pension benefits are close to the values observed in several European countries ($\theta = 0.15$, see [67]), the work time when old reduces to 70% of the whole time endowment. When the contribution rate further increases to 25%–30% of wage income, as predicted by several economists for the near future, the time spent working when old dramatically reduces. Thus, the higher the size of disability pensions, the lower the individual state of health because capital accumulation per efficient worker and per capita GDP reduce. The reduction in capital accumulation causes a reduction in wages earned by both the young and the old in the long run. The reduction in the former causes, in turn, a reduction in health expenditure which is eventually transformed into bad health. Therefore, a rise in the contribution rate to the pension system requires an increases in the health tax rate to keep health expenditure (and the individual health status) unchanged. The negative effect on the (neoclassical) economic growth is therefore due to a twofold channel: (i) a crowding out effect of PAYG pensions on saving and capital accumulation, and (ii) a negative effect on the labour supply of mature workers, because of a sort of crowding out effect of PAYG pensions on health spending. This mechanism, therefore, resembles a rather vicious circle: the higher the size of the PAYG system, the larger the number of pensioners.

With regard to dynamic outcomes, the classification of bifurcation patterns when θ varies is more difficult to be handled from an analytical point of view, because θ enters the expression of the discriminant Eq. (17). This implies that the graph of map J can either be monotonic or bimodal as long as θ changes. In particular, the map is monotonically increasing if and only if the discriminant in Eq. (17) is negative. By solving with respect to θ, we have that:

$$\Psi < 0 \Leftrightarrow \theta > \theta^\circ := \frac{1}{1-\alpha}\left[\frac{(1+\alpha\beta)(d_1-d_0)}{\left(\frac{\delta+1}{\delta-1}\right)^2 - 1} - \alpha(1+\beta) - (1+\alpha\beta)d_0\right]. \tag{24}$$

[21] Note that the ratio of both per capita health spending to per capita GDP and pension expenditure to per capita GDP are close to the actual values observed for some industrialised countries (see World Bank [70], World Health Statistics [71]), going from 1.7% to 2.2% and from 0% to 13.7%, respectively, when θ raises from zero up to 0.25.

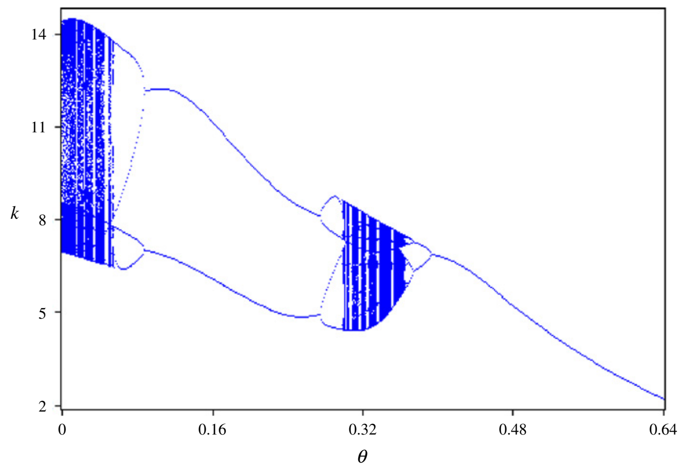

Fig. 5. Bifurcation diagram for θ. Parameter set: $A = 33$, $\alpha = 0.45$, $\beta = 0.6$, $\tau = 0.022$, $d_0 = 0$, $d_1 = 0.9$, $\delta = 59$, and $\Delta = 1$.

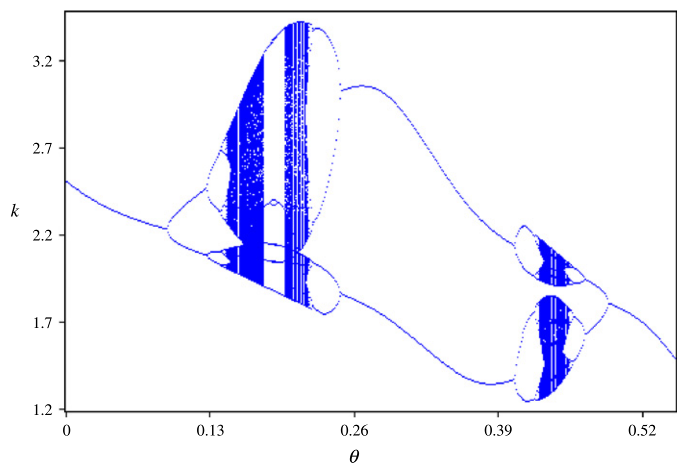

Fig. 6. Bifurcation diagram for θ ($\tau = 0.06$).

Then, by the study of Eq. (24) the following proposition follows.

Proposition 6. (1) *If $\theta^\circ < 0$, then for any given positive initial condition k_0, the dynamics are monotonic for any $\theta \in [0,\ 1-\tau)$.* (2) *If $0 < \theta^\circ < 1-\tau$, then for any given positive initial condition k_0, the dynamics are monotonic for any $\theta \in (\theta^\circ,\ 1-\tau)$.*

From Proposition 6, it follows that for $\theta \to 1-\tau$ the map admits a unique locally asymptotically stable fixed point k^*. In case (2) of Proposition 6, however, things are much more complicated when $\theta \to 0$ since any of the possible dynamic regimes described in Section 3 can be observed for any $\tau > 0$. In particular, different from the case $\tau \to 0$, small values of θ do not necessarily induce the existence of a global attractor (see Fig. 5).

Focusing on case (2) of Proposition 6, we note that large values of θ play a stabilising role. From a bimodal map with nonlinear dynamics when $\theta = 0$, we get an increasing map where the fixed point represents the global attractor of the system when $\theta > \theta^\circ$, *irrespective of the (positive) initial value of the stock of capital per efficient worker*. This result can be extended (at least locally) for any θ°.

By considering a parameter constellation characterised by nonlinear dynamics when $\theta = 0$, the map boils down into $J(k) = 0$ when $\theta \to 1-\tau$. Then, a threshold value $\theta^{\circ\circ}$ exists such that the fixed point is locally asymptotically stable for any $\theta > \theta^{\circ\circ}$. This means that if an economy starts with an initial condition close enough to the long-run equilibrium, the dynamics are convergent with monotonic trajectories.

The bifurcation diagram for θ depicted in Fig. 6 shows (the other parameters are the same as those used in Table 1) that for an economically meaningful range of values of θ, a stable periodic and an apparently chaotic behaviour alternate one times with a stable two-period cycle between the two "bubbles".

5. Conclusions

Health is an important determinant of economic growth and development of nations [72–74], as it directly affects mortality rates of humans. As stressed by Weil [17, p. 1265]: "People in poor countries are, on average, much less healthy than their counterparts in rich countries". This phenomenon has had dramatic effects on macroeconomic variables in underdeveloped, developing and developed countries across history and then it has stimulated economists to understand the reasons why some countries are rich and some countries are poor, as well as the reasons why the quantity and quality of life is different between them [75,6].

A burgeoning theoretical literature based on models with overlapping generations deals with the effects on economic growth of public and private health expenditure in models with endogenous lifetime [27,1,22,23,45]. In this class of models, however, less attention has been paid to the study of economies where an individual's state of health affects both the ability and productivity to work [21]. This study has analysed the long-run dynamics of an overlapping generations model with capital accumulation and tax-financed public pensions, by assuming that the age of retirement of an old individual is contingent on his/her state of health, which is in turn improved by the public provision of health care services when young. If an individual's health status when old is good, an agent works and earns a wage income. If an individual's health status when old is bad (ill-health), an agent cannot work and he/she is entitled to a pension benefit (disability pensions).

We have found that periodic and/or complex dynamics can occur when individuals have perfect foresight when the size of the public health system (for any given level of public pensions) or the pension system (for any given level of health expenditure) is included in an intermediate range of values. In addition, multiple "bubbling" phenomena can also be observed. Since problems concerning retirement age, disability pensions and health spending currently figure prominently on both the pension and public health system reform agendas in several developed and developing countries, we believe that our analysis may be useful to understand the effects of such policies on the dynamics of capital in a theoretical model.

As a possible extension, one can take demographic variables explicitly into analysis, so that problems about endogenous fertility under either weak altruism towards children [76–79] or pure altruism towards children [80–82], and endogenous mortality (private and/or public investments in health) can be added. Alternatively, the study of possible complex dynamics in a general equilibrium model with infectious disease transmission à la Chakraborty et al. [83] may be of interest.

Acknowledgments

The authors gratefully acknowledge Gian-Italo Bischi, Giancarlo Gandolfo, Laura Gardini, John Barkley Rosser, Iryna Sushko, Pietro Dindo and seminar participants at 6th Workshop MDEF 2010 (*Modelli Dinamici in Economia a Finanza*), held on September 23rd–25th, 2010 at the University of Urbino, Italy, for stimulating discussions and very helpful comments and suggestions. Mauro Sodini acknowledges that this work has been performed within the activity of the PRIN project "Local interactions and global dynamics in economics and finance: models and tools", MIUR, Italy. The authors are also indebted to an anonymous reviewer for insightful comments that have permitted to substantially improve the quality of the paper. The usual disclaimer applies. Please note that the published version of the present paper has significantly changed with respect to the working paper version by Fanti and Gori [84] entitled "*Complex equilibrium dynamics in a simple OLG model of neoclassical growth with endogenous retirement age and public pensions*".

References

[1] S. Chakraborty, Endogenous lifetime and economic growth, Journal of Economic Theory 116 (2004) 119–137.
[2] R.W. Fogel, The Escape from Hunger and Premature Death, Cambridge University Press, New York, NY, 2004.
[3] B.T. Evans, C. Pritchard, Cancer survival rates and GDP expenditure on health: a comparison of England and Wales and the USA, Denmark, Netherlands, Finland, France, Germany, Italy, Spain and Switzerland in the 1990s, Public Health 114 (2000) 336–339.
[4] F. Cairncross, Economics: age, health and wealth, Nature 448 (2007) 875–876.
[5] T. Fioroni, Child mortality and fertility: public vs. private education, Journal of Population Economics 23 (2010) 73–97.
[6] G.S. Becker, T.J. Philipson, R.R. Soares, The quantity and quality of life and the evolution of world inequality, American Economic Review 95 (2005) 277–291.
[7] D.M. Cutler, A.S. Deaton, A. Lleras-Muney, The determinants of mortality, The Journal of Economic Perspectives 20 (2006) 97–120.
[8] A. Young, The gift of the dying: the tragedy of AIDS and the welfare of future African generations, Quarterly Journal of Economics 120 (2005) 243–266.
[9] M. Cervellati, U. Sunde, Human capital formation, life expectancy, and the process of development, American Economic Review 95 (2005) 1653–1672.
[10] P. Lorentzen, J. McMillan, R. Wacziarg, Death and development, Journal of Economic Growth 13 (2008) 81–124.
[11] O. Galor, From stagnation to growth: unified growth theory, in: P. Aghion, S. Durlauf (Eds.), Handbook of Economic Growth, Elsevier, Amsterdam, 2005 (Chapter 4).
[12] O. Galor, Comparative economic development: insights from unified growth theory, International Economic Review 51 (2010) 1–44.
[13] M. Cervellati, U. Sunde, Life expectancy and economic growth: the role of the demographic transition, Journal of Economic Growth 16 (2011) 99–133.
[14] D.M. Cutler, M.B. McClellan, J.P. Newhouse, D. Remler, Are medical prices declining? Evidence from heart attack treatments, Quarterly Journal of Economics 113 (1998) 991–1024.
[15] D.M. Cutler, M.B. McClellan, Is technological change in medicine worth it? Health Affairs 20 (2001) 11–29.
[16] D.M. Cutler, Your Money or Your Life: Strong Medicine for America's Health Care System, Oxford University Press, New York, NY, 2004.
[17] D.N. Weil, Accounting for the effect of health on economic growth, Quarterly Journal of Economics 122 (2007) 1265–1306.
[18] I. Ehrlich, H. Chuma, A model of the demand for longevity and the value of life extension, Journal of Political Economy 98 (1990) 761–782.
[19] I. Ehrlich, Y. Yin, Explaining diversities in age-specific life expectancies and values of life saving: a numerical analysis, Journal of Risk and Uncertainty 31 (2005) 129–162.
[20] R.E. Hall, C.I. Jones, The value of life and the rise in health spending, Quarterly Journal of Economics 122 (2007) 39–72.

[21] L. Fanti, L. Gori, Public health spending, old-age productivity and economic growth: chaotic cycles under perfect foresight, Journal of Economic Behavior & Organization 78 (2011) 137–151.
[22] S. Chakraborty, M. Das, Mortality, human capital and persistent inequality, Journal of Economic Growth 116 (2005) 119–137.
[23] J. Bhattacharya, X. Qiao, Public and private expenditures on health in a growth model, Journal of Economic Dynamics and Control 31 (2007) 2519–2535.
[24] D. de la Croix, G. Ponthière, On the golden rule of capital accumulation under endogenous longevity, Mathematical Social Sciences 59 (2010) 227–238.
[25] M.C.M. Leung, Y. Wang, Endogenous health care, life expectancy and economic growth, Pacific Economic Review 15 (2010) 11–31.
[26] L. Fanti, L. Gori, Public expenditure on health and private old-age insurance in an OLG growth model with endogenous fertility: chaotic dynamics under perfect foresight, Computational Economics (forthcoming). http://dx.doi.org/10.1007/s10614-012-9326-y.
[27] K. Blackburn, G.P. Cipriani, A model of longevity, fertility and growth, Journal of Economic Dynamics and Control 26 (2002) 187–204.
[28] M. Grossman, On the concept of health capital and the demand for health, Journal of Political Economy 80 (1972) 223–255.
[29] T.N. Chirikos, The relationship between health and labor market status, Annual Review of Public Health 14 (1993) 293–312.
[30] J. Currie, B.C. Madrian, Health, health insurance and the labor market, in: O. Ashenfelter, D. Card (Eds.), Handbook of Labor Economics, Elsevier, 1999, pp. 3309–3416 (Chapter 50).
[31] M. Campolieti, Disability and the labor force participation of older men in Canada, Labour Economics 9 (2002) 405–432.
[32] L. Cai, G. Kalb, Health status and labour force participation: evidence from Australia, Journal of Health Economics 15 (2006) 241–261.
[33] R. Disney, C. Emmerson, M. Wakefield, Ill health and retirement in Britain: a panel data-based analysis, Journal of Health Economics 25 (2006) 621–649.
[34] P.A. Diamond, National debt in a neoclassical growth model, American Economic Review 55 (1965) 1126–1150.
[35] P. Reichlin, Equilibrium cycles in an overlapping generations economy with production, Journal of Economic Theory 40 (1986) 89–102.
[36] R.E.A. Farmer, Deficits and cycles, Journal of Economic Theory 40 (1986) 77–86.
[37] J. Zhang, Environmental sustainability, nonlinear dynamics and chaos, Journal of Economic Theory 14 (1999) 489–500.
[38] A. Antoci, L. Brugnano, M. Galeotti, Sustainability, indeterminacy and oscillations in a growth model with environmental assets, Nonlinear Analysis: Real World Applications 5 (2004) 571–587.
[39] A. Antoci, M. Sodini, Indeterminacy, bifurcations and chaos in an overlapping generations model with negative environmental externalities, Chaos, Solitons & Fractals 42 (2009) 1439–1450.
[40] H.J. Chen, M.C. Li, Environmental tax policy, habit formation and nonlinear dynamics, Nonlinear Analysis: Real World Applications 12 (2011) 246–253.
[41] P. Michel, D. de la Croix, Myopic and perfect foresight in the OLG model, Economics Letters 67 (2000) 53–60.
[42] D. de la Croix, P. Michel, A Theory of Economic Growth. Dynamics and Policy in Overlapping Generations, Cambridge University Press, Cambridge, 2002.
[43] H.J. Chen, M.C. Li, Y.J. Lin, Chaotic dynamics in an overlapping generations model with myopic and adaptive expectations, Journal of Economic Behavior and Organization 67 (2008) 48–56.
[44] L. Fanti, L. Spataro, Poverty traps and intergenerational transfers, International Tax and Public Finance 15 (2008) 693–711.
[45] L. Fanti, L. Gori, Endogenous lifetime in an overlapping generations small open economy, FinanzArchiv/Public Finance Analysis 68 (2012) 1–32.
[46] M. Bier, T.C. Bountis, Remerging Feigenbaum trees in dynamical systems, Physics Letters A 104 (1984) 239–244.
[47] L. Stone, Period-doubling reversals and chaos in simple ecological models, Nature 365 (1993) 617–620.
[48] F.E. Kydland, E.C. Prescott, Time to build and aggregate fluctuations, Econometrica 50 (1982) 1345–1370.
[49] J.B. Long, C.I. Plosser, Real business cycles, Journal of Political Economy 91 (1983) 39–69.
[50] J.M. Grandmont, On endogenous competitive business cycles, Econometrica 53 (1985) 995–1045.
[51] O. Galor, H.E. Ryder, Existence, uniqueness, and stability of equilibrium in an overlapping generations model with productive capital, Journal of Economic Theory 49 (1989) 360–375.
[52] S.C. Hu, Social security, the supply of labor, and capital accumulation, American Economic Review 69 (1979) 274–283.
[53] A. Momota, A retirement decision in the presence of a social security system, Journal of Macroeconomics 25 (2003) 73–86.
[54] P. Michel, P. Pestieau, Social security and early retirement in an overlapping-generations growth model, CORE Discussion Paper No. 9951, 1999.
[55] L. Gori, M. Sodini, Nonlinear dynamics in an OLG growth model with young and old age labour supply: the role of public health expenditure, Computational Economics 38 (2011) 261–275.
[56] B. Crettez, P. Le Maitre, Optimal age of retirement and population growth, Journal of Population Economics 15 (2002) 737–755.
[57] R. Ecob, G. Davey Smith, Income and health: what is the nature of the relationship? Social Science & Medicine 48 (1999) 693–705.
[58] J. Strauss, D. Thomas, Health, nutrition, and economic development, Journal of Economic Literature 36 (1998) 766–817.
[59] R.L. Devaney, An Introduction to Chaotic Dynamical Systems, second ed., Addison-Wesley, Redwood City, CA, 1989.
[60] C.H. Hommes, Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demands, Journal of Economic Behavior and Organization 24 (1994) 315–335.
[61] C. Mira, Chaotic Dynamics: From the One-Dimensional Endomorphism to the Two-Dimensional Diffeomorphism, World Scientific, Singapore, 1987.
[62] H. Skjolding, B. Branner-Jorgensen, P.L. Christiansen, H.E. Jensen, Bifurcations in discrete dynamical systems with cubic maps, SIAM Journal of Applied Mathematics 43 (1983) 520–534.
[63] D. Gollin, Getting income shares right, Journal of Political Economy 110 (2002) 458–474.
[64] P.J. Kehoe, F. Perri, International business cycles with endogenous incomplete markets, Econometrica 70 (2002) 907–928.
[65] J. Žamac, Pension design when fertility fluctuates: the role of education and capital mobility, Journal of Public Economics 91 (2007) 619–639.
[66] M. Feldstein, Structural reform of social security, The Journal of Economic Perspectives 19 (2005) 33–55.
[67] E. Liikanen, Population ageing, pension savings and the financial markets, Bank of International Settlement Review 53 (2007) 1–6.
[68] R.S. MacKay, C. Tresser, Boundary of topological chaos for bimodal maps of the interval, Journal of the London Mathematical Society 37 (1988) 164–181.
[69] T.Y. Li, J.A. Yorke, Period three implies chaos, American Mathematical Monthly 82 (1975) 985–992.
[70] World Bank, World development indicators database, World Bank, Washington, DC, 2009.
[71] World Health Statistics, Part II, global health indicators, France, 2010.
[72] J. Zhang, J. Zhang, R. Lee, Mortality decline and long-run economic growth, Journal of Public Economics 80 (2001) 485–507.
[73] J. Zhang, J. Zhang, R. Lee, Rising longevity, education, savings, and growth, Journal of Development Economics 70 (2003) 83–101.
[74] D. Acemoglu, S. Johnson, Disease and development: the effect of life expectancy on economic growth, Journal of Political Economy 115 (2007) 925–985.
[75] D. Acemoglu, S. Johnson, J.A. Robinson, The colonial origins of comparative development: an empirical investigation, American Economic Review 91 (2001) 1369–1401.
[76] Z. Eckstein, W.I. Wolpin, Endogenous fertility and optimal population size, Journal of Public Economics 27 (1985) 93–106.
[77] Z. Eckstein, S. Stern, K.I. Wolpin, Fertility choice, land, and the Malthusian hypothesis, International Economic Review 29 (1988) 353–361.
[78] O. Galor, D.N. Weil, The gender gap, fertility, and growth, American Economic Review 86 (1996) 374–387.
[79] J. Zhang, J. Zhang, Social security, intergenerational transfers, and endogenous growth, Canadian Journal of Economics 31 (1998) 1225–1241.
[80] R.J. Barro, Are government bonds net wealth? Journal of Political Economy 82 (1974) 1095–1117.
[81] G.S. Becker, R.J. Barro, A reformulation of the economic theory of fertility, Quarterly Journal of Economics 103 (1988) 1–25.
[82] R.J. Barro, G.S. Becker, Fertility choice in a model of economic growth, Econometrica 57 (1989) 481–501.
[83] S. Chakraborty, C. Papageorgiou, F. Pérez Sebastián, Diseases, infection dynamics, and development, Journal of Monetary Economics 57 (2010) 859–872.
[84] L. Fanti, L. Gori, Complex equilibrium dynamics in a simple OLG model of neoclassical growth with endogenous retirement age and public pensions, MPRA Working Paper No. 23694, 2010. Available at: http://mpra.ub.uni-muenchen.de/23694/.