Portfolio Optimization: MAD vs. Markowitz

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We look at investment portfolio optimization. We create portfolios consisting of five stocks and a six-month bond by randomly selecting the stocks from the S&P 500. We take the data from July 1, 2004 to December 31, 2004 and use the Markowitz minimum variance model as well as the Mean Absolute Deviation model to determine the allocation of funds to each asset in each of the portfolios. We then compare the returns of the portfolios from January 3, 2005 to June 30, 2005 using a series of parametric and non-parametric tests.

1. Introduction

Portfolio optimization is a key idea in investing. Markowitz's (1952) paper "Portfolio Selection" sparked further interest in developing a mathematical approach to optimizing multi-asset portfolios. After many years of research, Markowitz, along with Sharpe and Miller, won the Nobel Prize in Economics in 1990 for their theory of portfolio optimization. His model used the relationship between mean returns and variance of the returns to find a minimum variance point in the *feasible region* – the set of all points which correspond to possible portfolios using every possible weighting scheme. This minimum variance point is the point on the *efficient frontier*, the upper portion of the minimum variance set, where the variance is minimized for a given mean return. The variance corresponds to $\sigma_p^2 = E[(R_p - r_p)^2]$, where R_p is the random portfolio return, r_p is the average return, and E denotes the expectation operator. The computation calls for the use of the covariance matrix, which for large portfolios can become quite cumbersome or inefficient.

An alternative to the Minimum-Variance (M-V) model is the Mean-Absolute Deviation (MAD) model, proposed by Konno and Yamazaki (1992). The M-V model assumes normality of stock returns, which is not the case; however the MAD model does not make this assumption. The MAD model also minimizes a measure of risk, where the measure in this case is the mean absolute deviation. For a larger mean absolute deviation, the risk is increased. Mean absolute deviation corresponds to

$MAD_p = E[|R_p - r_p|]$, which is theoretically the same as the variance in M-V when the returns are normally distributed. MAD is easier to compute than Markowitz because it eliminates the need for a covariance matrix.

In this paper, we empirically compare the MAD and M-V optimization models. First we describe our experiment, as well as how the portfolios were created. Next we mathematically formulate the M-V and MAD models and consider their respective solutions. We then use the two models to compute the actual returns on each portfolio and statistically compare these results. Finally we present concluding remarks.

2. Methodology

For our studies we decided to use the S&P 500, which is a stock exchange consisting of 500 of the top American companies. On Yahoo Finance, these 500 stocks are listed alphabetically with 50 stocks per page. We randomly picked 15 numbers from 1 to 50 and the stocks that corresponded to that number on each page were the stocks we used. For each portfolio, groupings of five stocks were made based on the order in which they were shown on the webpage (alphabetically). Each group made up a portfolio to which we added an identical six-month bond, giving us a total of 30 portfolios. Using historical data from July 1, 2004 through December 31, 2004, we computed the average rate of return for each stock within the portfolios. This information then helped us to determine a reasonable target rate of return, ρ. Using the MAD and M-V methods described in Sections 3 and 4, we determined what percentage of our money to put in each stock of the portfolio. We then found what the six-month bond rate was for January 3, 2005. This was used when we computed the tangent fund (Sections 3 and 4) and determined what percentage of our fund to put in the six-month bond and what percentage to put in the five stocks. To simplify calculations, we assumed no transaction fees and no short-selling of the bond. We then assumed that we were an investor and wanted to invest for the six-month period from January 3, 2005 through June 30, 2005. Using the information from the historical data for each portfolio, we invested based on the proportions given by our calculations of MAD and M-V and their respective tangent

funds. This then gave us a return rate for each portfolio using each method and a basis for comparison.

3. Markowitz (Mean-variance model)

A. Model

The Markowitz model, below, minimizes the variance of a given portfolio. It assumes that portfolios can be completely characterized by their mean return and variance (or risk). The portfolios solved for by this program map out the efficient frontier (see Figure 1).

$$\text{Minimize } E[\sum_{j=1}^{n} x_j R_j - \sum_{j=1}^{n} r_j x_j]^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \qquad (3.1)$$

$$\text{Subject to } \sum_{j=1}^{n} r_j x_j = \rho M_o \qquad (3.2)$$

$$\sum_{j=1}^{n} x_j = M_0 \qquad (3.3)$$

$$x_j \le u_j$$
$$j = 1,2...,n \qquad (3.4)$$

We are minimizing the variance in (3.1), where σ_{ij} is the covariance between assets i and j, x_i is the amount invested in asset i, x_j is the amount invested in asset j, and n is the number of assets in each portfolio. The constraints require that in (3.2) the total sum of the returns of each asset times the amount invested in that asset is equal to the minimum rate of return the investor wants times the total amount of money being invested, where r_j is the average daily return of asset j, ρ is the mimimal rate of return required by the investor which is not portfolio dependent, M_o is the total amount of money being invested which is constant, and u_j is the maximum amount the investor wishes to place in a single stock.

In (3.3), the constraint says that the sum of the amount invested in each asset has to equal the total fund being invested. (3.4) requires that the amount invested in each asset is less than or equal to the maximum amount the investor wants invested in each

asset. Notice that we do not require that x_j is greater than or equal to 0, rather we want to allow short selling which is what $x_j < 0$ signifies.

B. Solving Markowitz

Using Lagrangian multipliers μ and λ we can find a solution of the Markowitz problem. The formulation of the Lagrangian is:

$$L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} - \lambda\left(\sum_{i=1}^{n} w_i \overline{r}_i - \overline{r}\right) - \mu\left(\sum_{i=1}^{n} w_i - 1\right),$$

where $w_i = \dfrac{x_i}{M_0}$. Here, we changed variables from x_i to w_i for simplicity, but the solution does not depend on M_o. It follows that the upper bound on x_j is still u_j and there is no lower bound since we are allowing short-selling. Note that λ corresponds to constraint (3.2) and μ corresponds to (3.3) of the Markowitz formula.

After differentiating the Lagrangian with respect to w_i and setting the derivatives equal to zero, we get the following generalization: For a portfolio with n weights, two Lagrangian multipliers λ and μ, and mean rate of return $\overline{r}$, we have:

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda \overline{r}_i - \mu = 0 \quad \text{for } i=1,2,\ldots,n \qquad (3.5)$$

$$\sum_{i=1}^{n} w_i \overline{r}_i = \overline{r}$$

$$\sum_{i=1}^{n} w_i = 1$$

Note: μ is not unrestricted in sign.

The solution to these equations produce a set of weights for an efficient portfolio with a mean rate of return $\overline{r}$, which is equivalent to ρM_o in equations (3.1)-(3.4). While there is not a closed form solution to (3.5) in linear algebra, it can be solved using numerical analysis techniques to solve systems of linear equations, such as LU matrix factorization.

Markowitz, as formulated in (3.1)-(3.4), has a solution that can be found mathematically, as opposed to numerically. Using linear algebra we can solve Markowitz by solving $w^T \Sigma w$, where Σ is the covariance matrix and w is a vector of weights, which is equivalent to minimizing the variance.

The two fund theorem states that: *Two efficient funds can be established so that any efficient portfolio can be duplicated, in terms of mean and variance, as a combination of these two. In other words, all investors seeking efficient portfolios need only invest in combinations of these two funds* (Luenberger, 163). To find all values of $\bar{r}$ we can simply solve for two solutions and create combinations of these. We choose to find these two solutions by letting $\lambda=0$, $\mu=1$ and $\lambda=1$, $\mu=0$ and solving (3.5). Any combination of these two weight vectors w_1 and w_2 respectively map out the efficient frontier.

In our experiment we include a risk-free asset in each portfolio. The one-fund theorem states: *There is a single fund F of risky assets such that any efficient portfolio can be constructed as a combination of the fund F and the risk-free asset* (Luenberger, 167). A risk-free asset has a return that is known with certainty. Using the one-fund theorem we solved for a *tangent fund*, the optimal combination of risky assets and the risk-free bond, which included the six-month bond rate (r_f). The motivation behind the tangent fund is to find the line that is tangent to your original feasible region composed of risky assets (Figure 1). The one-fund theorem follows directly from this concept

(Luenberger 166). This tangent fund is solved by maximizing $tan\ \theta = \dfrac{\sum\limits_{i=1}^{n} x_i (\bar{r}_i - r_f)}{(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} \sigma_{ij} x_i x_j)^{1/2}}$

(3.6) and solving for a linear combination of the two weight vectors found using the two-fund theorem. This linear combination is given by $w_2 - r_f * w_1$ (Luenberger, 168). The tangent fund gives us a normalized vector of weights along with a value for the return of the portfolio (r_F) using the normalized vector. Finding this normalized vector of weights involves setting the derivative of $tan\ \theta$ with respect to each w_k equal to zero. Doing this we get the following equation:

$$\sum_{i=1}^{n} \sigma_{ki} \lambda w_i = \bar{r}_k - r_f, \qquad k{=}1,2,\ldots,n \quad (3.7)$$

where λ is a constant that is unknown. Then substituting $v_i = \lambda w_i$ for each i, (3.7) becomes:

$$\sum_{i=1}^{n} \sigma_{ki} v_i = \bar{r}_k - r_f, \qquad k{=}1,2,\ldots,n. \quad (3.8)$$

Finally, we solve for the v_i's in (3.8) and normalize to find the resulting vector of weights, w_i's:

$$w_i = \frac{v_i}{\sum_{k=1}^{n} v_k} \quad \text{(Luenberger, 168)}.$$

The next step was to solve for α, where α is the proportion of money we invest in the six-month bond and *1-α* is the amount we invest in the stocks. Using the following formula: $\rho = r_f \alpha + (1{-}\alpha)\, r_F$ (3.7), one can easily solve for α, where $r_F = \sum_{j=1}^{n} x_j r_j$.

From here we used r_f and α to determine the return for only the bond in the portfolio. Using the average rate of return, r_j, for each stock from the new data and the final normalized vector of weights, we found the returns, $x_j r_j$, for each stock. This was then summed up and multiplied by (*1-α*). From here, the daily return for the portfolio was easily calculated by taking the return for the bond and adding the summed returns for the stocks. In our calculations we consistently used the daily returns, so to solve for the six-month return, we simply multiplied by 124 (the number of market days in our six-month period). These steps were then used to calculate the return rates for each of the portfolios.

C. Example

Using Microsoft Excel, we downloaded the historical data for each of our 150 stocks. The stocks were made into 30 portfolios and the average returns for each stock within the portfolios were calculated using Excel. For solving the Markowitz problem, we used a statistical software, R, that is a nice package for linear algebra manipulations. Using R, we scanned in the daily returns for the stocks in each portfolio and wrote a

program which took in the daily returns, computed the covariance matrix, inverted it, calculated the average returns, and solved for two vectors of weights, w_1 and w_2. These vectors of weights are two solutions to the Markowitz problem. Only one of these is the overall minimum variance point, which is needed to apply the one-fund theorem.

Portfolio 1 consists of the stocks Albertson's Inc., Alberto-Culver Company, Aetna Inc., Anadarko Petroleum Corp., and Allegheny Technologies Inc. The six-month bond rate for each portfolio is 2.63% and our minimal daily rate of return required by the investor is .1061%, which is the same for all portfolios. Using R, the normalized vector

of weights was $\begin{bmatrix} -2.3407577 \\ -1.8414845 \\ 3.9644492 \\ 1.0734978 \\ .1442952 \end{bmatrix}$ and r_F was .01583944. Using equation (3.6), the solution

for α is .94531, making ($1-\alpha$) .05469. This tells us that we should invest 94.531% of our total fund in the six-month bond and 5.469% in our stocks. A possible explanation of why we would place 94.531% in our bond is that this particular grouping of stocks has a high risk. Placing only $\approx$ 5% of the fund in the stocks provides a buffer against possible stock price fluctuations. Gathering the average daily returns of the new data gave us -.00086, -.00075, .002541, .002445, .001208 for the five stocks respectively. Using this, our weight vector, and ($1-\alpha$), we found that the daily return for these stocks was .00089. Adding this to our α multiplied by the bond rate gave us .1085% as a daily return.

4. MAD

A. Model

Even though both models minimize a measure of risk, the MAD model attempts to reduce the mean absolute deviation as opposed to the variance. The model below also maps out an efficient frontier, but unlike M-V the efficient frontier is not a smooth curve.

$$\text{Minimize } \frac{\sum\limits_{t=1}^{T}\left|\sum\limits_{j=1}^{n} a_{jt}x_j\right|}{T} \qquad (4.1)$$

$$\text{Subject to } \sum\limits_{j=1}^{n} r_j x_j \geq \rho M_0 \qquad (4.2)$$

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$$\sum_{j=1}^{n} x_j = M_0 \qquad\qquad (4.3)$$

$$x_j \le u_j$$
$$j = 1,2,...,n \qquad\qquad (4.4)$$

We are minimizing the mean absolute deviation in (4.1), where a_{jt} is the daily returns minus the average returns $(r_j - \bar{r}_j)$ for asset j for each time t, where T is the time horizon. Equations (4.2) through (4.4) are the same constraints as in the M-V model. We must note that the MAD formulation does not have an analytical solution, and so it must be approximated numerically.

B. Implementation

Using Microsoft Excel and the historical downloaded data, we created a column for each asset's a_{jt}, and set each weight, x_j, at an initial value of 20,000 (assuming we had 100,000 in our total fund to make calculations easier). Any variety of computational software packages would have worked for solving MAD, but we knew how to use Excel, which is why Excel was chosen to numerically solve the MAD problem. Using the Excel add-in called Solver, we were able to numerically minimize this value by changing our x_j's and using constraints (4.2) through (4.4). After minimizing the value of (4.1), we then needed to solve for the tangent fund. We found no existing formula for solving the tangent fund in MAD, so we created (4.5) by mimicking the tangent fund for Markowitz. We found our tangent fund from

$$\tan \theta = \frac{\sum_{i=1}^{n} x_i (\bar{r}_i - r_f)}{\sum_{t=1}^{T} \left| \sum_{j=1}^{n} a_{jt} x_j \right| / T} \qquad (4.5).$$

Using this tangent fund formula, we want to maximize $\tan \theta$ to find the weights of the tangent fund portfolio. To do this we needed to maximize the numerator divided by our minimized value (4.1) from above because the denominator of (4.5) is the mean absolute deviation (4.1) that we previously minimized. It is important to note that our original minimization problem was to merely find an initial set of weights which we then used as a preliminary guess allowing the weights to change while maximizing the tangent fund.

Using the normalized vector of weights and r_F, we then solved for α in the same manner as in Section 3, and then we calculated the six-month rate of return for each portfolio. Here we are simply mimicking the tangent fund from Markowitz. Something to note is when solving for the tangent fund in MAD, the efficient frontier is not smooth. This means that when we maximize the tangent fund, we are finding a tangent point, which may not be the unique fund as it was in M-V. Changing the original guesses for x_j may change the answer for the tangent fund in MAD. To keep our calculation consistent, we began with the same initial guess of 20,000 for each x_j. Thus, it is important to note that there is no theoretical justification for this tangent fund approach, and as a possible pitfall there may exist an infinite number of tangent funds.

C. Example

We will again use Portfolio 1 as an example. Using the same bond rate and ρ as before, we minimized (4.1) and found our x vector of weights to be:

$$\begin{bmatrix} 18112.29 \\ 20229.71 \\ 258381.71 \\ 34010.76 \\ 2265.536 \end{bmatrix}$$

From there we maximized the tangent fund (4.5) and found our final normalized vector of weights to be:

$$\begin{bmatrix} -.2152522 \\ -.3596484 \\ 1.136764 \\ .4165535 \\ .02158306 \end{bmatrix}$$

and our r_F to be .004234018. Using equation (3.7), the solution for α is .787737, making (*1-α*) .212263. This tells us we should invest 78.7737% in our six-month bond and 21.2263% in our stocks. Using the average daily returns for the new data (these are the same from the example in Section 3, part C) along with our weight vector and (*1-α*), we

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found that the daily return for these stocks was .000931. Adding this to our α multiplied by the bond rate gave us a .1093% daily return.

5. Statistical Analysis of Results

After running both the M-V and MAD models on each of the 30 portfolios, we took the six-month return rates and compared them between portfolios. Table 1 shows the return rates for each portfolio. The average return over all the portfolios for each method yields .0423% using the M-V and .0410% using MAD. As you can see from Table 1, the M-V method gives a higher average return for 16 of the 30 portfolios.

To determine if the averages for each method were significantly different, we performed a paired t-test using the statistical program Statistix. The t-test results in Table 2 show that there is no significant difference between the average returns of M-V and MAD. We wanted to see if there was a large difference between the magnitudes of the returns for both models and compute a binomial probability using this information. We found that MAD was more extreme than M-V in 15 of the 30 portfolios, which is not significant. We would expect that you have an equal likelihood of M-V being more extreme as MAD and using the binomial formula, we found that there is a 14.45% chance that in exactly 15 of 30 portfolios MAD is more extreme. MAD is more extreme or as extreme as M-V 57.22% of the time according to the sign test.

In Statistix we created a variable that was equal to MAD divided by M-V because the ratio should be approximately one if the two are the same. Testing our null hypothesis of $\mu=1$ against the alternative that $\mu>1$, we are trying to see if MAD is bigger than M-V or if they are equal. At the 10% significance level, we reject our null hypothesis (p=.0868) in favor of the alternative that MAD returns are larger in value than M-V. Although this is the kind of evidence we are looking for, it needs to be taken with some caution because our study is only 30 small scale portfolios, and the strength of this finding does not necessarily mean that it will hold true in all cases.

6. Concluding Remarks

In this paper, we have shown two different methods to approaching a portfolio optimization problem. The M-V approach was solved by implementing the two-fund

theorem along with the tangent fund. As we stated earlier, the M-V method assumes normality of stock returns, which is usually not the case, so we then looked at the numerical solution of the MAD approach. We modified the one-fund theorem to mimic the tangent fund.

Through our experience working with these 30 portfolios, we found that neither M-V nor MAD tend to produce returns that are better than the other. Since there is little statistically significant difference between the returns using both methods at the 10% level, and no significant difference at the 5% level, we can conclude that with our small portfolios, MAD is the less complicated method to use. As the size of a portfolio increases, MAD becomes increasingly quicker than inverting a large scale covariance matrix. Since both return an answer that is not significantly different, in general it is acceptable to substitute MAD calculations for the M-V method for small-scale portfolios. Possible further research can be done of large portfolios to determine if this principle holds true.
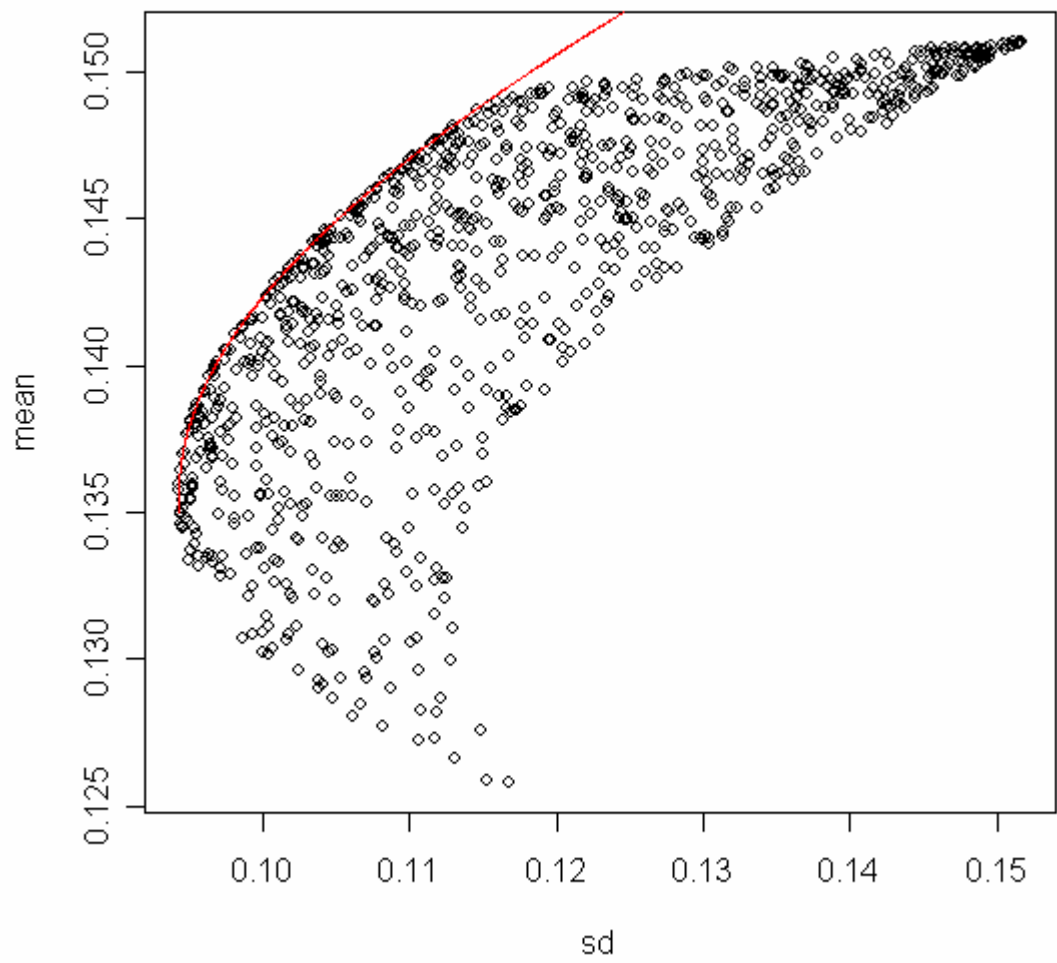
Figure 1. The Markowitz feasible region, with the efficient frontier shown in red.

Portfolio	Markowitz % Return(daily)	MAD % Return(daily)
1	0.1085	0.1093
2	-0.0382	-0.0349
3	-0.0053	-0.0289
4	0.0342	0.0229
5	0.0102	0.0178
6	0.0700	0.0869
7	0.0022	0.0179
8	0.0208	0.0324
9	0.0779	0.0623
10	0.0891	0.0749
11	0.0554	0.0654
12	0.0964	0.1480
13	0.0438	0.0368
14	0.0690	0.0198
15	0.0513	0.0674
16	0.0459	0.0495
17	0.0953	0.0776
18	0.0594	0.0751
19	0.0477	0.0243
20	0.0633	0.0653
21	0.0648	0.0803
22	0.0278	0.0208
23	0.0393	0.0288
24	0.0231	0.0161
25	0.1007	0.0869
26	0.0632	0.0582
27	-0.1839	-0.1993
28	0.0235	0.0215
29	0.0444	0.0637
30	0.0678	0.0635

Table 1. The 6 month return rates for each portfolio. The highlighted cells show where the return for the portfolio is the highest.

```
Paired T Test for MAD - Markowitz

Null Hypothesis: difference =  0
Alternative Hyp: difference <> 0

Mean          -1.24E-03
Std Error      3.34E-03
Mean - H0     -1.24E-03
Lower 95% CI -8.08E-03
Upper 95% CI  5.60E-03
T                -0.37
DF                  29
P               0.7128


Cases Included 30    Missing Cases 0
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Table 2.

Acknowledgements

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