Mean-Variance vs. Mean-Absolute Deviation: A Performance Comparison of Portfolio Optimization Models

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Abstract. This paper provides an introduction to Modern Portfolio Theory (MPT) and compares two fundamental models: Markowitz's Mean-Variance model and Konno's Mean-Absolute Deviation model. Generally speaking, MPT reduces the complexities of the investment universe into two categories: risk and return. We implemented the aforementioned models using Python as an array of software packages for solving complex programming problems was readily available. Using historical data from the S&P 500, we compared the computed optimal portfolios and find that Mean Absolute Deviation model outperforms the Mean-Variance model.

1 Introduction

Harry Markowitz is widely recognized for his groundbreaking contributions in financial economics. His seminal paper, Portfolio Selection, published in 1952, laid the foundation for Modern Portfolio Theory by analyzing risk as a major determinant for optimal investment holdings. Markowitz formulated the portfolio problem as a trade-off between the mean and variance of a portfolio. His use of mean-variance analysis involved holding constant expected return and minimizing risk or equivalently holding constant variance and maximizing expected return. This allowed Markowitz to construct an efficient frontier from which an investor could select an efficient portfolio dependent on their individual risk-return preferences [1]. While Markowitzs theory on mean-variance analysis has received wide acclaim, in practice it had been not been used extensively due to early computational difficulties associated with large-scale quadratic programming [4].

In 1992, Hiroshi Konno and Hiroaki Yamazaki provided an alternative to Markowitzs Mean-Variance model through their proposed Mean-Absolute Deviation (MAD) model. As its name suggests, the MAD model defines risk by using the mean absolute deviation. By construction, the MAD model was computationally easier to solve, as the portfolio problem is a linear program, as opposed to the quadratic program associated with the Mean-Variance model. At the time, this allowed Konno to solve optimization problems involving over 1,000 stocks within a fraction of the time of the mean-variance model [4].

Advancements in computing have removed many of the concerns surrounding large-scale portfolio optimization. Dismissing computation time, we can focus solely on performance measures which we define based on portfolio return and volatility. In this paper, we provide an empirical comparison of the Mean-Variance and MAD models for varying portfolio sizes. We begin with a review of concepts in probability which are essential to the models. We then formally introduce the Mean-Variance and Mean-Absolute Deviation models. Finally, we describe their implementation through python and provide a discussion of the results.

2 Probability Theory

In this section, we provide a review of relevant probability theory based on the concepts considered in Markowitz's Portfolio Selection [2]. We refer the reader to introductory probability texts for further explanations and proofs of subsequent results.

Let X be a random variable. For the sake of simplicity, let X take on a finite number of outcomes $x_1, x_2, ..., x_n$ with corresponding probability $p_1, p_2, ..., p_n$. Two useful measures of a random variable are its expected value (or mean), and variance. The expected value of X is defined as

$$E[X] = p_1x_1 + p_2x_2 + ... + p_nx_n$$

Variance is commonly used to measure the dispersion of a random variable. It is the squared deviation of a random variable from its mean and is defined by

$$Var[X] = p_1(x_1 - E[X])^2 + p_2(x_2 - E[X])^2 + ... + p_n(x_n - E[X])^2$$

Two useful measures of dispersion derived from variance are standard deviation: $\sigma = \sqrt{Var[X]}$, and correlation coefficient $p = \sigma/E[X]$.

Let $X_1, X_2, ..., X_n$ be random variables with corresponding weights $a_1, a_2, ..., a_n$. The following hold: If X is a sum of weighted random variables, then X is a random variable.

$$X = a_1X_1 + a_2X_2 + ... + a_nX_n \tag{1}$$

The expectation of a weighted sum is equal to the weighted sum of expectations.

$$E[X] = a_1E[X_1] + a_2E[X_2] + ... + a_nE[X_n] \tag{2}$$

The formula for the variance of a weighted sum is not as trivial. We first define the covariance of X_i and X_j by

$$\sigma_{ij} = E\{[X_i - E[X_i][X_j - E[X_j]\} \tag{3}$$

Equivalently, (3) can be expressed by the product of the correlation coefficient and the respective standard deviation of X_i and X_j

$$\sigma_{ij} \equiv p_{ij}\sigma_i\sigma_j \tag{4}$$

Using the results of (3) and (4), we define the variance of a weighted sum by

$$Var[X] = \sum_{i=1}^{n} {a_i}^2 Var(X_i) + 2\sum_{i=1}^{n}\sum_{i\,j}^{n} \sigma_{ij}a_ia_j \tag{5}$$

Note that the covariance of a random variable and itself is equal to the variance. Thus, the variance simplifies to

$$Var[X] = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}w_iw_j \tag{6}$$

3 Relevant Definitions

The subsequent models will make use of the following:

$$
\begin{aligned}
n =\ & \text{number of assets in a portfolio.} \\
\vec{w} = [w_1\ w_2\ ...\ w_n]^T =\ & w_i \text{ is the weight of stock i for i = 1, ..., n.} \\
R(\vec{w}) =\ & \text{portfolio rate of return.} \\
V(\vec{w}) = E[(R(\vec{w}) - E[R(\vec{w})])^2] =\ & \text{variance of the portfolio rate of return.} \\
\sigma(\vec{w}) = \sqrt{V(\vec{w})} =\ & L_2 \text{ risk function (standard deviation)} \\
K(\vec{w}) = E[\left|R(\vec{w}) - E[R(\vec{w})]\right|] =\ & L_1 \text{ risk function (absolute deviation)} \\
\gamma =\ & \text{rate of return required by the investor.} \\
I =\ & \text{n} \times \text{1 identity matrix} \\
(E[R(\vec{w})] - \text{risk free rate})/\sigma(\vec{w}) =\ & \text{Sharpe Ratio (Note: we define the risk} \\
& \text{free rate using the 3 Month Treasury Bill)}
\end{aligned}
$$

4 Model Framework

4.1 Mean-Variance Model

The original formulation of the Mean-Variance model minimizes risk subject to some required rate of return. Markowitz defines risk using the L_2 risk function (standard deviation). Standard deviation is equal to the quadratic deviations from the mean, thus the model is formulated as a convex quadratic programming problem. The Mean-Variance model can be constructed as follows:

$$
\begin{aligned}
\text{Minimize } & \sigma(\vec{w}) \\
\text{Subject to } & E[R(\vec{w})] = \gamma \\
& \vec{w}^T I \quad = 1 \\
& \vec{w} \geq 0
\end{aligned}
\tag{7}
$$

The Mean-Variance model is valid under 2 assumptions:
1. Portfolio returns are multivariate normally distributed.
2. Investors are risk averse – for a given return, the investor strictly prefers a portfolio with less risk.

The portfolios generated by this model map out the Efficient Frontier – an upper boundary for which feasible portfolios have minimum risk for a given rate of return [3]. The graph of an efficient frontier is displayed in Figure 1.

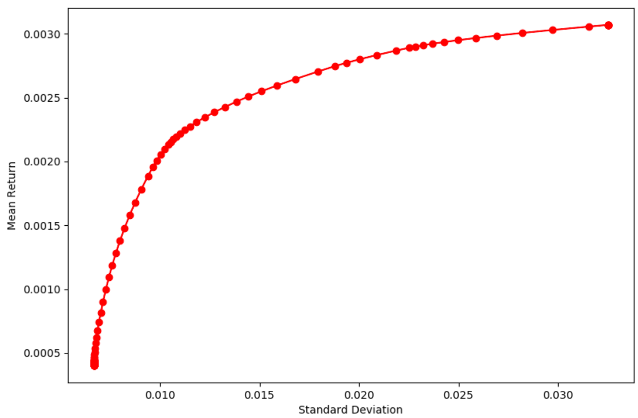

Fig. 1: The Efficient Frontier

4.2 Mean-Absolute Deviation Model

In 1992, Konno introduced an alternative to the Mean-Variance model in which risk is defined by the L_1 risk (absolute deviation) function [4]. Before introducing the model, let us show show a simple relation between the MAD and Mean-Variance models.

Proposition 1. *Suppose $R(\vec{w})$ is normally distributed with mean $E[R(\vec{w})] = r(\vec{w})$ and variance $V(\vec{w}) = \sigma^2(\vec{w})$, then:*

$$K(\vec{w}) = \sqrt{\frac{2}{\pi}}\sigma(\vec{w}) \tag{8}$$

Proof.

$$K(\vec{w}) = \frac{1}{\sqrt{2\pi}\sigma(\vec{w})}\int_{-\infty}^{+\infty} |z - r(\vec{w})| exp\{\frac{(z-r(\vec{w}))^2}{2\sigma^2(\vec{w})}\}dz \tag{9}$$

$$= \frac{2}{\sqrt{2\pi}\sigma(\vec{w})}\int_{0}^{+\infty} t\ exp\{-\frac{t^2}{2\sigma^2(\vec{w})}\}dt \tag{10}$$

$$= \sqrt{\frac{2}{\pi}}\sigma(\vec{w}) \quad \blacksquare \tag{11}$$

For normally distributed $R(\vec{w})$, (8) implies that the Mean-Absolute Deviation model generates the same optimal portfolio as the corresponding Mean-Variance model [5].

Here we provide the motivation for a comparison under the assumption of normality. However, in practice this assumption is not often satisfied, implying the models are no longer equivalent.

Proposition 2. *The L_1 risk measure is less than or equal to the L_2 risk measure. That is, $K(\vec{w}) \leq \sigma(\overline{w})$.*

Proof. By Jensen's inequality [6] we have

$$E[\left|R(\overline{w}) - E[R(\overline{w})]\right|]^2 \leq E[\left|R(\overline{w}) - E[R(\overline{w})]\right|^2] \tag{12}$$

$$E[\left|R(\overline{w}) - E[R(\overline{w})]\right|]^2 \leq Var[R(\overline{w})] \tag{13}$$

Since both sides are strictly positive and the square root is an increasing monotone function, we have

$$\sqrt{E[\left|R(\overline{w}) - E[R(\overline{w})]\right|]^2} \leq \sqrt{Var[R(\overline{w})]} \tag{14}$$

$$K(\vec{w}) \leq \sigma(\overline{w}) \quad \blacksquare \tag{15}$$

Based on the result from (15), we hypothesize that for equal expected return, the MAD model will generate lower risk. Equivalently, for the equal risk we expect the MAD model to have higher expected return. The MAD model can be constructed as follows:

$$\begin{array}{ll} \text{Minimize} & K(\vec{w}) \\ \text{Subject to} & E[R(\vec{w})] = \gamma \\ & \vec{w}^T I \quad = 1 \\ & \vec{w} \geq 0 \end{array} \tag{16}$$

5 Implementation

For our research, we selected stocks from the S&P 500 as it is widely recognized as a leading indicator of large-cap U.S. equities and provides a diverse selection of commonly traded stocks. Using the interactive Python Environment, Jupyter, we implemented the Mean-Variance and Mean-Absolute Deviation models. Our program utilizes an array of software packages used for matrix creation and manipulation, and for solving complex programming problems.

We constructed portfolios with the top 75, 150 and 200 performing stocks from 01/01/2016 to 01/01/2017 (See Appendix). Obtaining and selecting data proved challenging. The calculation of the covariance matrix required that all selected stocks were listed and traded in the given time period. To fix this, we wrote a program that generated a list of stocks and their corresponding rate of return, such that all stocks were publicly listed and traded in the same time interval. Furthermore, the program sorted stocks based on the highest mean daily return. The resulting list was used to construct optimal portfolios in the Mean-Variance and Mean-Absolute Deviation models.

6 Results

Through Zipline and the Interactive Development Environment provided by Quantopian, we back-tested our program subject to realistic factors such as market liquidity, trade volume, transaction cost, and slippage. We performed the back-test from 01/01/2017 to 25/11/2017, with an initial capital amount of $1,000,000. Additionally, we compared the models subject to monthly and weekly re-balancing – the process of readjusting portfolio weights. The results are presented below:

Measures	**Model (# of stocks)**					
	MV (200)	**MAD (200)**	**MV (150)**	**MAD (150)**	**MV (75)**	**MAD (75)**
Daily Expected Return	0.0923%	0.0955%	0.0936%	0.0962%	0.1071%	0.1076%
Total Expected Return (227 days)	20.95%	21.68%	21.25%	21.84%	24.31%	24.43%
Expected Risk	0.00721	0.00089	0.00744	0.0098	0.00980	0.00112
Total Return (227 days) Monthly Rebalancing	10.53%	11.26%	12.61%	12.66%	1.24%	2.69%
Sharpe Ratio (227 days) Monthly Rebalancing	1.37	1.50	1.49	1.51	0.18	0.34
Total Return (227 days) Weekly Rebalancing	9.81%	9.54%	13.12%	14.94%	1.82%	1.94%
Sharpe Ratio (227 days) Weekly Rebalancing	1.31	1.30	1.57	1.80	0.25	0.26

The optimal portfolios generated in our program are significantly different and also perform differently. The composition and historical performance (Zipline back-tests) of each portfolio is displayed in the Appendix. Based on the generated results, we find that the MAD model provided a higher return and a higher Sharpe Ratio in 5 out of 6 cases. We could not find a pattern between monthly and weekly re-balancing schedules.

7 Conclusion

This paper explores two classical portfolio optimization models. While the models are formulated in a similar manner, their choice of risk measure results in fundamental differences. Based on the result of (15) we hypothesized that the MAD model would yield a greater rate of return for equal risk as the Mean-Variance model. Our results align with our hypothesis. We find that the MAD model outperforms the Mean-Variance model and provides a larger risk adjusted return (Sharpe Ratio) in 5 out of 6 cases. Furthermore, the assumption of normally distributed portfolio returns is unlikely to be satisfied. Thus, we conclude that the MAD model is a more robust optimization technique subject to real-market conditions.

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Appendix

The following tables provide a list of the selected stocks for our program.

Selected Stock List (75)

'AMD'	'ALK'	'CSRA'
'NVDA'	'PWR'	'DFS'
'CFG'	'PNC'	'BBT'
'MU'	'STT'	'LUK'
'ZION'	'PFG'	'PBCT'
'UAL'	'RJF'	'AMAT'
'LNC'	'C'	'FTI'
'RF'	'BBY'	'KMX'
'CHK'	'UNM'	'TXT'
'STX'	'MET'	'RHI'
'CMA'	'SYF'	'MPC'
'BAC'	'NAVI'	'XLNX'
'KEY'	'JPM'	'APC'
'URI'	'COF'	'TIF'
'MS'	'CF'	'DE'
'GS'	'ADSK'	'JWN'
'AAL'	'CSX'	'ADI'
'FITB'	'KSS'	'USB'
'SCHW'	'NTRS'	'PH'
'WMB'	'VLO'	'FCX'
'WDC'	'DAL'	'TWX'
'HBAN'	'BWA'	'PKG'
'ETFC'	'STI'	'AMP'
'NTAP'	'WRK'	'DVN'
'PRU'	'MTB'	'FL'

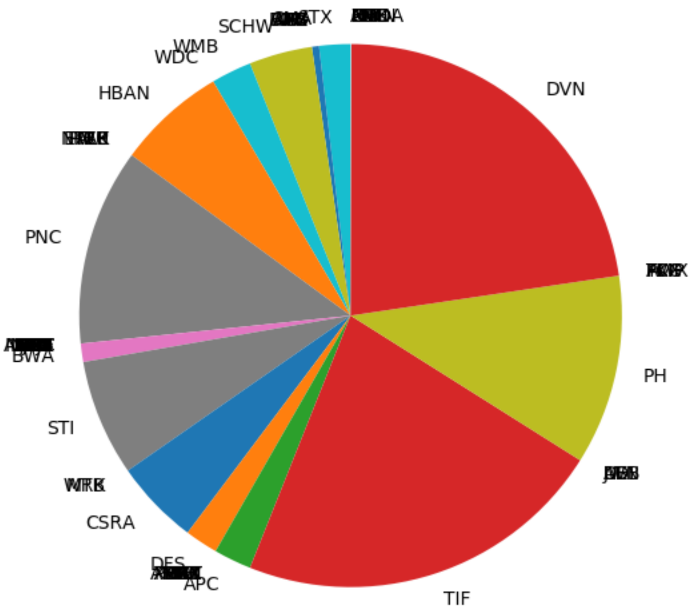
Selected Stock List (150)

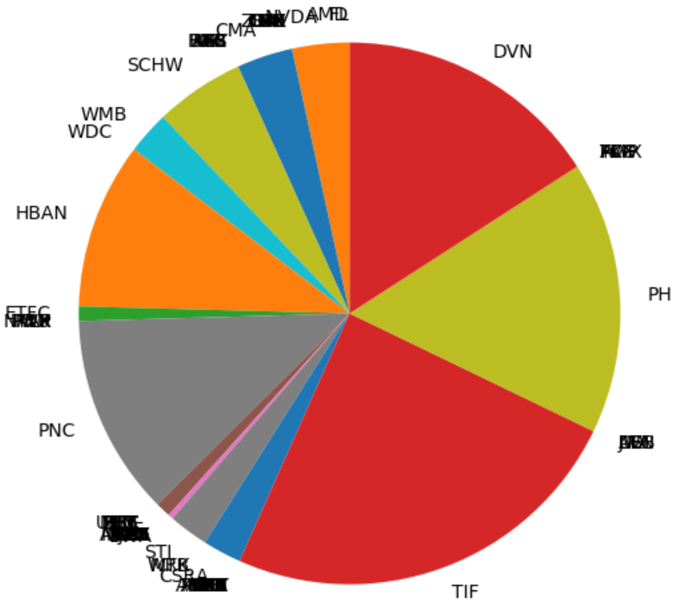
'AMD'	'ALK'	'CSRA'	'MCHP'	'OKE'	'TMK'
'NVDA'	'PWR'	'DFS'	'LRCX'	'IVZ'	'HLT'
'CFG'	'PNC'	'BBT'	'NFLX'	'EBAY'	'WFC'
'MU'	'STT'	'LUK'	'NSC'	'GT'	'RE'
'ZION'	'PFG'	'PBCT'	'JNPR'	'MRO'	'CCL'
'UAL'	'RJF'	'AMAT'	'SWKS'	'AXP'	'LYB'
'LNC'	'C'	'FTI'	'LUV'	'RCL'	'ROK'
'RF'	'BBY'	'KMX'	'IP'	'TEL'	'JBHT'
'CHK'	'UNM'	'TXT'	'INCY'	'AKAM'	'HST'
'STX'	'MET'	'RHI'	'CBG'	'BEN'	'MOS'
'CMA'	'SYF'	'MPC'	'MSI'	'NUE'	'PXD'
'BAC'	'NAVI'	'XLNX'	'PCAR'	'CHTR'	'TXN'
'KEY'	'JPM'	'APC'	'FFIV'	'GD'	'DRI'
'URI'	'COF'	'TIF'	'AIG'	'EOG'	'UNP'
'MS'	'CF'	'DE'	'BK'	'BA'	'MLM'
'GS'	'ADSK'	'JWN'	'FMC'	'MAR'	'IR'
'AAL'	'CSX'	'ADI'	'QCOM'	'HP'	'APH'
'FITB'	'KSS'	'USB'	'HPE'	'COP'	'APA'
'SCHW'	'NTRS'	'PH'	'MGM'	'AAPL'	'XEC'
'WMB'	'VLO'	'FCX'	'CAT'	'HPQ'	'NOV'
'WDC'	'DAL'	'TWX'	'GM'	'HOG'	'ADS'
'HBAN'	'BWA'	'PKG'	'HAL'	'MSFT'	'ALGN'
'ETFC'	'STI'	'AMP'	'IDXX'	'GLW'	'GRMN'
'NTAP'	'WRK'	'DVN'	'HRS'	'FDX'	'SIG'
'PRU'	'MTB'	'FL'	'CMI'	'CME'	'BIIB'

Selected Stock List (200)

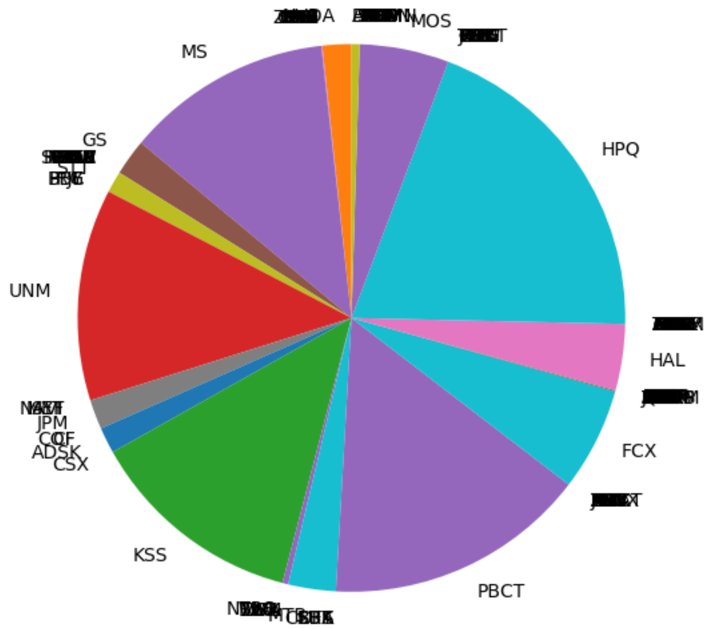
'AMD'	'ALK'	'CSRA'	'MCHP'	'OKE'	'TMK'	'AVGO'	'BLK'
'NVDA'	'PWR'	'DFS'	'LRCX'	'IVZ'	'HLT'	'ITW'	'DLPH'
'CFG'	'PNC'	'BBT'	'NFLX'	'EBAY'	'WFC'	'SYMC'	'BRK-B'
'MU'	'STT'	'LUK'	'NSC'	'GT'	'RE'	'FLIR'	'UNH'
'ZION'	'PFG'	'PBCT'	'JNPR'	'MRO'	'CCL'	'JEC'	'DOV'
'UAL'	'RJF'	'AMAT'	'SWKS'	'AXP'	'LYB'	'CTAS'	'DGX'
'LNC'	'C'	'FTI'	'LUV'	'RCL'	'ROK'	'XL'	'HES'
'RF'	'BBY'	'KMX'	'IP'	'TEL'	'JBHT'	'SNI'	'M'
'CHK'	'UNM'	'TXT'	'INCY'	'AKAM'	'HST'	'CBS'	'COL'
'STX'	'MET'	'RHI'	'CBG'	'BEN'	'MOS'	'MA'	'APD'
'CMA'	'SYF'	'MPC'	'MSI'	'NUE'	'PXD'	'L'	'WHR'
'BAC'	'NAVI'	'XLNX'	'PCAR'	'CHTR'	'TXN'	'CELG'	'CTXS'
'KEY'	'JPM'	'APC'	'FFIV'	'GD'	'DRI'	'HUM'	'GPS'
'URI'	'COF'	'TIF'	'AIG'	'EOG'	'UNP'	'ETN'	'INTC'
'MS'	'CF'	'DE'	'BK'	'BA'	'MLM'	'ROST'	'PSX'
'GS'	'ADSK'	'JWN'	'FMC'	'MAR'	'IR'	'EMN'	'ICE'
'AAL'	'CSX'	'ADI'	'QCOM'	'HP'	'APH'	'HOLX'	'GOOGL'
'FITB'	'KSS'	'USB'	'HPE'	'COP'	'APA'	'CXO'	'HIG'
'SCHW'	'NTRS'	'PH'	'MGM'	'AAPL'	'XEC'	'KMI'	'ANTM'
'WMB'	'VLO'	'FCX'	'CAT'	'HPQ'	'NOV'	'CVX'	'DISCA'
'WDC'	'DAL'	'TWX'	'GM'	'HOG'	'ADS'	'WU'	'EMR'
'HBAN'	'BWA'	'PKG'	'HAL'	'MSFT'	'ALGN'	'MTD'	'KLAC'
'ETFC'	'STI'	'AMP'	'IDXX'	'GLW'	'GRMN'	'DISH'	'SNA'
'NTAP'	'WRK'	'DVN'	'HRS'	'FDX'	'SIG'	'PCLN'	'XYL'
'PRU'	'MTB'	'FL'	'CMI'	'CME'	'BIIB'	'DISCK'	'ADP'

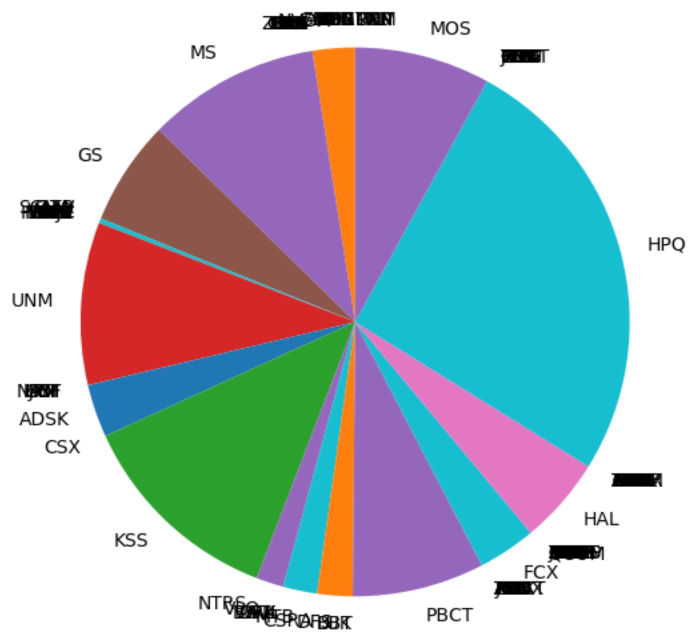
The subsequent pie charts represent the optimal portfolios generated by the Mean-Absolute Deviation and Mean-Variance models respectively for 75 stocks.





The subsequent pie charts represent the optimal portfolios generated by the Mean-Absolute Deviation and Mean-Variance models respectively for 150 stocks.





The subsequent pie charts represent the optimal portfolios generated by the Mean-Absolute Deviation and Mean-Variance models respectively for 200 stocks.