
Revisiting Stochastic Extragradient

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Abstract

We fix a fundamental issue in the stochastic extragradient method by providing a new sampling strategy that is motivated by approximating implicit updates. Since the existing stochastic extragradient algorithm, called Mirror-Prox, of (Juditsky et al., 2011) diverges on a simple bilinear problem when the domain is not bounded, we prove guarantees for solving variational inequality that go beyond existing settings. Furthermore, we illustrate numerically that the proposed variant converges faster than many other methods on bilinear saddle-point problems. We also discuss how extragradient can be applied to training Generative Adversarial Networks (GANs) and how it compares to other methods. Our experiments on GANs demonstrate that the introduced approach may make the training faster in terms of data passes, while its higher iteration complexity makes the advantage smaller.

1 Introduction

Algorithmic machine learning has for a long time been centered around minimization of a single function. A lot of works are still targeting solving empirical risk minimization and new results touch upon methods as old as gradient descent itself.

However, as the gap between lower bounds and available minimization algorithms is shrinking, the focus is shifting towards more challenging problems such as variational inequality, where a significant number of

long-unresolved questions is remaining. This problem has a rich history with applications in economics and computer science, but the arising applications provide new desiderata on algorithm properties. In particular, due to high dimensionality and large scale of the corresponding problems, we shall consider the impact of having a *stochastic* objective. In particular, recently invented generative adversarial neural networks (Goodfellow et al., 2014) are often trained using schemes that resemble primal-dual and variational inequality methods, which we shall discuss in detail later.

Variational inequality can be seen as an extension of the necessary first-order optimality condition for minimization problem, which is also sufficient in the convex case. When the operator involved in its formulation is monotone and is equal to the gradient of a function, this corresponds to convex minimization.

Formally, the problem that we consider is that of finding a point x^* satisfying

$$g(x) - g(x^*) + \langle F(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in \mathbb{R}^d, \quad (1)$$

where $g: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous convex function and $F: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a monotone operator. Some application of interest are not covered by the monotonicity framework, but, unfortunately, little is known about variational inequality and even saddle point problems when monotonicity is missing. Thus, we stick to this assumption and rather try to model oscillations arising in some problems by considering particularly unstable (Gidel et al., 2019b; Chavdarova et al., 2019) bilinear minimax problems.

Of particular interest to us is the situation where $F(x)$ is the expectation with respect to random variable ξ of the random operator $F(x; \xi)$. This formulation has two aspects. First, one can model data distribution, especially when a large dataset is available and the problem is that of minimizing empirical loss. Second, ξ can be a random variable sampled by one of the GAN networks, called *generator*. In any case, throughout

the work we assume that we sample unbiased estimates $F(\cdot; \xi)$ of $F(\cdot)$ such that $\mathbb{E}_\xi F(\cdot; \xi) = F(\cdot)$.

Let us explicitly mention that a special case of (1) is constrained saddle point optimization,

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y),$$

where \mathcal{X} and \mathcal{Y} are some convex sets and f is a smooth function. While this example looks deceptively simple, simultaneous gradient descent-ascent is known to diverge on this problem (Goodfellow, 2016) even when f is convex-concave. In particular, the objective $f(x, y) = x^\top y$ leads to geometrical divergence for any nontrivial initialization (Daskalakis et al., 2018). See (Mishchenko and Richtárik, 2019) for more applications of the convex-concave saddle point problem in machine learning and (Gidel et al., 2019a) for extra discussion on variational inequality and its relation to GANs.

1.1 Related work

The extragradient method was first proposed by (Korpelevich, 1977). Since then there have been developed a number of its extensions, most famous of which is the Mirror-Prox method (Nemirovski, 2004) that uses mirror descent update. At each iteration, the standard extragradient method is trying to approximate the implicit update, which is known to be much more stable. Assuming the operator is Lipschitz, it is enough to compute the operator twice to do the approximation accurately enough. We base our intuition upon this property and we shall discuss it in detail later in the paper.

While extragradient uses future information, i.e., information from one gradient step ahead, past information can also help to stabilize convergence. In particular, *Optimistic mirror descent* (OMD), first proposed by (Rakhlin and Sridharan, 2013) for convex-concave zero-sum games, has been analyzed in a number of works (Mokhtari et al., 2019; Daskalakis and Panageas, 2019; Gidel et al., 2019a) and it was applied to GAN training in (Daskalakis et al., 2018). The rates that we prove in this work for stochastic extragradient match the best known results for OMD, but are given under more general assumptions. Moreover, the method of (Gidel et al., 2019a) diverges on bilinear problems.

Many other techniques also allow to improve stability and achieve convergence for monotone operators in the particular case of saddle point problems. For instance, *alternating* gradient descent-ascent does not, in general, converge to a solution (Gidel et al., 2019b), the negative momentum trick proposed in (Gidel et al., 2019b) can fix this.

We note that our work is not the first to consider a variant of stochastic extragradient. A stochastic version of the Mirror-Prox method (Nemirovski, 2004) was analyzed in (Juditsky et al., 2011) under pretty restrictive assumptions. While deterministic extragradient approximates implicit update, the authors of (Juditsky et al., 2011) chose to sample two different instances of the stochastic operator, which leads to a poor approximation of stochastic implicit update unless the variance is tiny. It was observed in (Chavdarova et al., 2019) that this approach leads to terrible practical performance, dubious convergence guarantees and divergence on bilinear problems. All later variants of stochastic extragradient, that we are aware of, consider the same update model.

Surprisingly, a variant of extragradient was also rediscovered by practitioners (Metz et al., 2016) as a way to stabilize training of GANs. The main difference of the method of (Metz et al., 2016) to what we consider is in applying extra steps only on one of two neural networks. In addition, (Metz et al., 2016) proposed to use more than one extra step and claim that in on specific problems 5 steps is a good trade-off between results quality and computation.

(Chavdarova et al., 2019) showed that the methods of (Juditsky et al., 2011) and (Gidel et al., 2019a) diverge on stochastic bilinear saddle point problem. As a fix, they proposed a stochastic extragradient method with variance reduction (SVRE), which achieves a linear rate $\mathcal{O}((n + \frac{L}{\mu}) \log \frac{1}{\varepsilon})$. However, their theory works only for saddle point problems and it does not cover the case without strong monotonicity, so it is less general than ours.

1.2 Theoretical background

Here we provide several technical assumptions that are standard for variational inequality.

Assumption 1. *Operator $F: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is monotone, that is $\langle F(x) - F(y), x - y \rangle \geq 0$ for all $x, y \in \mathbb{R}^d$. In stochastic case, we assume that $F(x; \xi)$ is monotone almost surely.*

The monotonicity assumption is an extension of the notion of convexity and is quite standard in the literature. There are several versions of pseudo-monotonicity, but without it the variational inequality problem becomes extremely hard to solve.

Assumption 2. *Operator $F(\cdot; \xi)$ is almost-surely L -Lipschitz, that is for all $x, y \in \mathbb{R}^d$*

$$\|F(x; \xi) - F(y; \xi)\| \leq L \|x - y\|. \quad (2)$$

In addition to operator monotonicity, we ask for convexity and some regularity properties of $g(\cdot)$ as given

Algorithm 1 Same-Sample Stochastic Extragradient Method for Variational Inequality.

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1: Parameters:  $x^0 \in \mathcal{K}$ , stepsize  $\eta > 0$ 
2: for  $t = 0, 1, 2, \dots$  do
3:   Sample  $\xi^t$ 
4:    $y^t = \text{prox}_{\eta g}(x^t - \eta F(x^t; \xi^t))$ 
5:    $x^{t+1} = \text{prox}_{\eta g}(x^t - \eta F(y^t; \xi^t))$ 
6: end for
```

Algorithm 2 The extragradient method for min-max problems.

Require: Stepsizes η_1, η_2 , initial vectors x^0, y^0

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1: for  $t = 0, 1, \dots$  do
2:    $u^t = x^t - \eta_1 \nabla_x f(x^t, y^t)$ 
3:    $v^t = y^t + \eta_1 \nabla_y f(x^t, y^t)$ 
4:    $x^{t+1} = x^t - \eta_2 \nabla_x f(u^t, v^t)$ 
5:    $y^{t+1} = y^t + \eta_2 \nabla_y f(u^t, v^t)$ 
6: end for
```

below.

Assumption 3. Function $g: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous and μ -strongly convex for $\mu \geq 0$, i.e., for all $x, y \in \mathbb{R}^d$ and any $h \in \partial g(y)$

$$g(x) - g(y) - \langle h, x - y \rangle \geq \frac{\mu}{2} \|x - y\|^2.$$

If $\mu = 0$, then g is just convex.

Even in simple minimization problems, the classical theoretical analysis of stochastic methods ask for uniformly bounded variance, an assumption rarely satisfied in practice. Recent developments of the theory for SGD have removed this assumption, but we are not aware of any results in more general settings. Thus, it is one of our contributions is to relax the uniform variance bound the one below.

Assumption 4. In the strongly convex case, we assume that F has bounded variance at the optimum, i.e.,

$$\mathbb{E}\|F(x^*; \xi) - F(x^*)\|^2 \leq \sigma^2.$$

Depending on the assumptions, we will either work with the variance at the optimum or with a merit function, which involves the variance of a bounded set.

2 Theory

It is known that implicit updates are more stable when solving variational inequality and sometimes it is argued that the main goal of algorithmic design is to approximate those (Mokhtari et al., 2019). From that perspective, the current stochastic extragradient,

which was suggested in (Juditsky et al., 2011), does not make much sense. Since it uses two independent samples, it will rarely approximate the implicit update, so it is rather not surprisingly that it fails on bilinear problems.

To better explain this phenomenon, below we show that extragradient efficiently approximates implicit update.

Theorem 1. Let F be an L -Lipschitz operator and define $y \stackrel{\text{def}}{=} \text{prox}_{\eta g}(x - \eta F(x))$, $z \stackrel{\text{def}}{=} \text{prox}_{\eta g}(x - \eta F(y))$, $w \stackrel{\text{def}}{=} \text{prox}_{\eta g}(x - \eta F(w))$, where $\eta > 0$ is any stepsize. Then,

$$\|w - z\| \leq \eta^2 L^2 \|w - x\|.$$

The right-hand side in Theorem 1 serves as a measure of stationarity and decreases as x gets closer to the problem's solution. The essential part of the bound is that the error is of order $O(\eta^2)$ rather than $O(\eta)$. This allows the approximation to be better than simple gradient update making it possible for the method to solve variational inequality. One can also mention that having extra factor of ηL is beneficial only when $\eta < 1/L$, which provides a good intuition on why extragradient uses smaller stepsizes than gradient.

However, when the stochastic update is used, this result is not applicable directly. If two different samples of the operator are used, $F(\cdot; \xi^t)$ and $F(\cdot; \xi^{t+1/2})$, as is done in stochastic Mirror-Prox (Juditsky et al., 2011), then the update does not seem to approximate implicit update of any operator. This is why we propose in this work to use the same sample, ξ^t , when computing y^t and x^{t+1} , see Algorithm 1. Equipped with our update, we are always approximating the implicit update of stochastic operator $F(\cdot; \xi^t)$ and our theoretical results suggest that this is the right approach.

2.1 Stochastic variational inequality

Our first goal is to show that our stochastic version of the extragradient method converges for strongly monotone variational inequality. The next theorem provides the rate that we obtained.

Theorem 2. Assume that g is a μ -strongly convex function, operator $F(\cdot; \xi)$ is almost surely monotone and L -Lipschitz, and that its variance at the optimum x^* is bounded by constant, $\mathbb{E}\|F(x^*; \xi) - F(x^*)\|^2 \leq \sigma^2$. Then, for any $\eta \leq 1/(2L)$

$$\mathbb{E}\|x^t - x^*\|^2 \leq (1 - 2\eta\mu/3)^t \|x^0 - x^*\|^2 + 3\eta\sigma^2/\mu.$$

In the case where at the optimum the noise is zero, we recover a slight generalization of linear convergence of

extragradient (Tseng, 1995). This is also similar to the rate proved for optimistic mirror descent in (Gidel et al., 2019a), however we do not ask for uniform bounds on the variance. Therefore, we believe that this result is significantly more general.

Theorem 3. *Let g be a convex function, $F(\cdot; \xi)$ be monotone and L -Lipschitz almost surely. Then, the iterates of Algorithm 1 with stepsize $\eta = \mathcal{O}(1/(\sqrt{t}L))$ satisfy for any set \mathcal{X} and $x \in \mathcal{X}$*

$$\begin{aligned} & \mathbb{E}[g(\hat{x}^t) - g(x) + \langle F(x), \hat{x}^t - x \rangle] \\ & \leq \frac{1}{\sqrt{t}L} \sup_{x \in \mathcal{X}} \left\{ \frac{L^2}{2} \|x^0 - x\|^2 + \sigma_x^2 \right\}. \end{aligned}$$

where $\hat{x}^t = \frac{1}{t} \sum_{k=0}^t y^k$ and $\sigma_x^2 \stackrel{\text{def}}{=} \mathbb{E}\|F(x) - F(x; \xi)\|^2$, i.e., σ_x^2 is the variance of F at point x .

The left-hand side in the bound above is a merit function that has been used in variational inequality literature (Nesterov, 2007). This result is more general than the one obtained in (Gidel et al., 2019a), where the authors require for the same rate bounded variance and even $\mathbb{E}\|F(x; \xi)\|^2 \leq M < \infty$ uniformly over x .

In fact, the claim that we prove in the appendix is a bit more general than the one presented in the previous theorem. If we know that σ_x is sufficiently small on a bounded set \mathcal{X} , then we can get a $\mathcal{O}(1/t + \sup_{x \in \mathcal{X}} \sigma_x)$ rate, i.e., fast convergence to a neighborhood.

2.2 Adversarial bilinear problems

The work (Gidel et al., 2019b) argues that a good illustration of method's stability can be obtained when considering minimax bilinear problems, which is given by

$$\min_x \max_y f(x, y) = x^\top \mathbf{B} y + a^\top x + b^\top y,$$

where \mathbf{B} is a full rank square matrix. One can show that if there exists a Nash equilibrium point, then $f(x, y) = (x - x^*)^\top \mathbf{B} (y - y^*) + \text{const}$ for some pair (x^*, y^*) ¹. This problem is particularly interesting because simple gradient descent-ascent diverges geometrically when solving it,

Theorem 4. *Let f be bilinear with a full-rank matrix \mathbf{B} and apply Algorithm 2 to it. Choose any η_1 and η_2 such that $\eta_2 < 1/\sigma_{\max}(\mathbf{B})$ and $\eta_1 \eta_2 < 2/\sigma_{\max}(\mathbf{B})^2$, then the rate is*

$$\|x^t - x^*\|^2 + \|y^t - y^*\|^2 \leq \rho^{2t} (\|x^0 - x^*\|^2 + \|y^0 - y^*\|^2),$$

¹If a does not belong to the column space of B or b does not belong to the column space of B^\top , the unconstrained minimax problem admits no equilibrium. Otherwise, if we introduce \tilde{a}, \tilde{b} such that $a = -By^*$ and $b = -B^\top x^*$, we have $(x - x^*)^\top B(y - y^*) = x^\top By + a^\top x + b^\top y + (x^*)^\top By^*$.

where $\rho \stackrel{\text{def}}{=} \max \{(1 - \eta_1 \eta_2 \sigma_{\max}(\mathbf{B})^2)^2 + \eta_2^2 \sigma_{\max}(\mathbf{B})^2, (1 - \eta_1 \eta_2 \sigma_{\min}(\mathbf{B})^2)^2 + \eta_2^2 \sigma_{\min}(\mathbf{B})^2\}$.

The conditions for η_1 and η_2 in Theorem 4 are necessary, but not sufficient. To guarantee convergence, one needs to have $\rho < 1$ and below we provide two such examples.

Corollary 1. *Under the same assumption as in Theorem 4, consider two choices of stepsizes:*

1. if $\eta_1 = \eta_2 = 1/(\sqrt{2}\sigma_{\max}(\mathbf{B}))$ we get
$$\begin{aligned} & \|x^t - x^*\|^2 + \|y^t - y^*\|^2 \\ & \leq \left(1 - \frac{\sigma_{\min}(\mathbf{B})^2}{6\sigma_{\max}(\mathbf{B})^2}\right)^{2t} (\|x^0 - x^*\|^2 + \|y^0 - y^*\|^2), \end{aligned}$$
2. if $\sigma_{\min}(\mathbf{B}) > 0$, and $\eta_1 = \kappa/(\sqrt{2}\sigma_{\max}(\mathbf{B})^2)$, $\eta_2 = 1/(\sqrt{2}\kappa\sigma_{\max}(\mathbf{B})^2)$ with $\kappa \stackrel{\text{def}}{=} \sigma_{\min}^2(\mathbf{B})/\sigma_{\max}^2(\mathbf{B})$, then the rate is
$$\begin{aligned} & \|x^t - x^*\|^2 + \|y^t - y^*\|^2 \\ & \leq \left(1 - \frac{\sigma_{\min}(\mathbf{B})^2}{4\sigma_{\max}(\mathbf{B})^2}\right)^{2t} (\|x^0 - x^*\|^2 + \|y^0 - y^*\|^2). \end{aligned}$$

If we denote $\kappa \stackrel{\text{def}}{=} \frac{\sigma_{\min}^2(\mathbf{B})}{\sigma_{\max}^2(\mathbf{B})}$ as in (Mokhtari et al., 2019), then the complexity in both cases is $O(\kappa \log \frac{1}{\varepsilon})$. However, we provide this result for potentially different stepsizes to obtain new insights about how they should be chosen. One can see, in particular, that choosing a huge η_1 is possible if η_2 is chosen small, but not vice versa.

3 Nonconvex extragradient

Since the objective of neural networks is not convex, it is desirable to have a guarantee for convergence that would not assume operator monotonicity. Alas, there is almost no theory even for nonconvex minimax problems and full gradient updates as even the notion of stationarity becomes tricky. Therefore, in this section we only discuss the method performance when minimizing loss function.

Formally, the problem that we consider here is

$$\min_x \mathbb{E}_\xi f(x; \xi), \quad (3)$$

where f is a smooth bounded from below and potentially nonconvex function. To show convergence, we need the following standard assumption.

Assumption 5. *There exists a constant $\sigma > 0$ such that for all x it holds*

$$\mathbb{E}\|\nabla f(x; \xi) - \nabla f(x)\|^2 \leq \sigma^2.$$

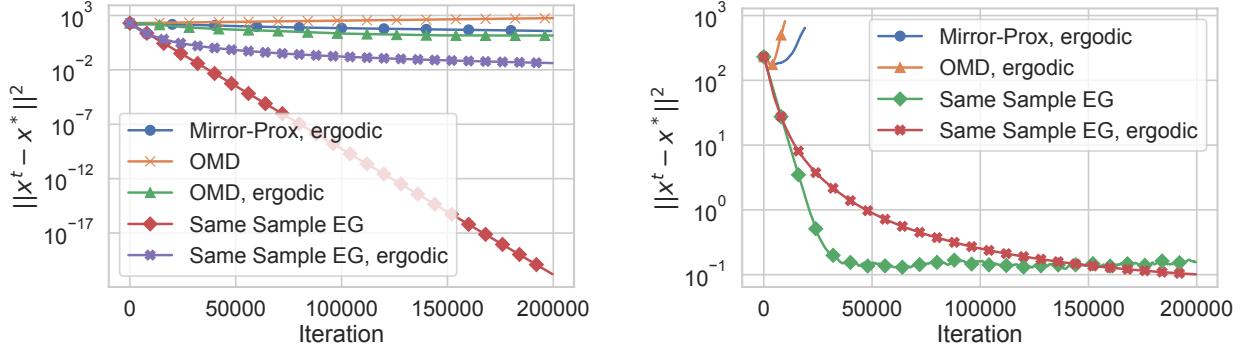


Figure 1: Left: comparison of using independent samples and averaging as suggested by (Juditsky et al., 2011) and the same sample as proposed in this work. The problem here is the sum of randomly sampled matrices $\min_x \max_y \sum_{i=1}^n x^\top \mathbf{B}_i y$. Since at point (x^*, y^*) the noise is equal 0, the convergence of Algorithm 1 is linear unlike the slow rates of (Juditsky et al., 2011) and (Gidel et al., 2019a). 'EGm' is the version with negative momentum (Gidel et al., 2019b) equal $\beta = -0.3$. Right: bilinear example with linear terms.

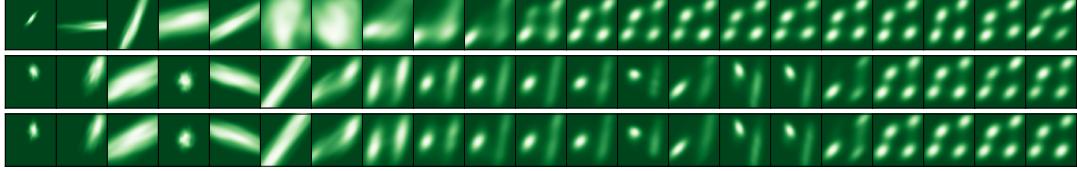


Figure 2: Top line: extragradient with the same sample. Middle line: gradient descent-ascent. Bottom line: extragradient with different samples. Since the same seed was used for all methods, the former two methods performed extremely similarly, although when zooming it should be clear that their results are slightly different.

Then, we are able to show that the method converges to a local minimum.

Theorem 5. Choose $\eta \leq \frac{1}{4L}$ and apply extragradient to (3). Then, its iterates satisfy

$$\mathbb{E}\|\nabla f(\hat{x}^t)\|^2 \leq \frac{5}{\eta t}(f(x^0) - f^*) + 11\eta L\sigma^2,$$

where \hat{x}^t is sampled uniformly from $\{x^0, \dots, x^{t-1}\}$ and $f^* = \inf_x f(x)$.

Corollary 2. If we choose $\eta = \Theta(1/(L\sqrt{t}))$, then the rate is $O((f(x^0) - f^*)/\sqrt{t} + \sigma^2/\sqrt{t})$, which is the same as the rate of SGD under our assumptions.

The statement of the theorem almost coincides with that of SGD, see for instance (Ghadimi and Lan, 2013). This suggests that extragradient in most cases should not be seen as an alternative to SGD. We also provide a simple experiment with training Resnet-18 (He et al., 2016) on Cifar10 (Krizhevsky and Hinton, 2009) in Appendix B.2, which gives a similar message.

4 Experiments

4.1 Bilinear minimax

In this experiment, we generated a matrix with entries from standard normal distribution and dimensions 200. Since we did not observe much difference when changing the matrix size, we provide only one run in Figure 1. The results are very encouraging and show the superiority of the proposed approach on this problem. We provide two cases, with zero noise at the optimum and non-zero noise. In the latter case, only our method did not diverge.

When the noise at the optimum is zero, this is mostly like deterministic case for our method, but for the rest it is a difficult problem. On the other hand, when the noise is not equal 0 at the solution, the ergodic convergence of our method is faster, just as predicted by Theorem 3.

4.2 Generating mixture of Gaussians

Here we compare gradient descent-ascent as well as Mirror-Prox to our method on the task of learning mixture of 4 Gaussians. We provide the evolution of the

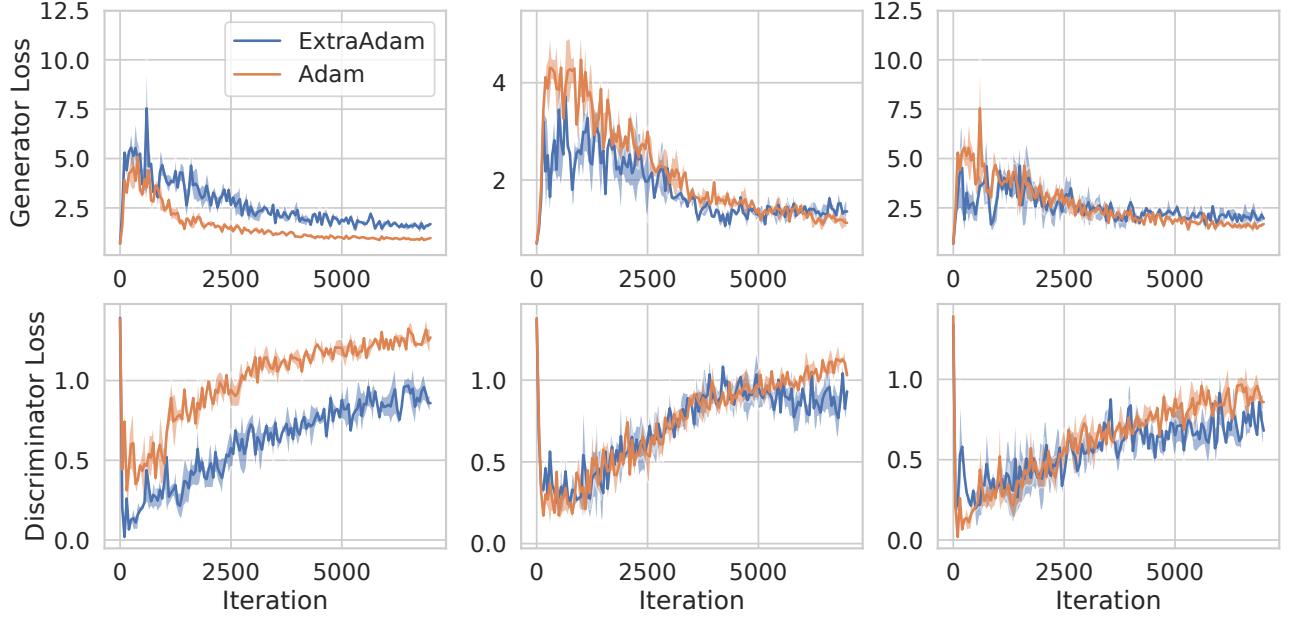


Figure 3: The columns differ in step sizes for generator and discriminator: 1) $(10^{-4}, 10^{-4})$, 2) $(5 \cdot 10^{-5}, 5 \cdot 10^{-5})$, 3) $(10^{-4}, 5 \cdot 10^{-5})$. In the top row, we show the generator loss and in the bottom row that of discriminator.

process in Figure 2, although we note that the process is rather unstable and all results should be taken with a grain of salt.

To our surprise, negative momentum was rarely helpful and even positive momentum sometimes was giving significant improvement. We suspect that this is due to the different roles of generator and discriminator, but leave further exploration for future work.

The details of the experiment are as follows. For generator we use neural net with 2 hidden layers of size 16 and tanh activation function and output layer with size 2 and no activation function, which represents coordinates in 2D. Generator uses standard Gaussian vector of size 16 as an input. For discriminator we use neural net with input layer of size 2, which takes a point from 2D, 2 hidden layers of size 16 and tanh activation function and output layer with size 1 and sigmoid activation function, which represents probability of input point to be sampled from data distribution. We choose the same stepsize $5 \cdot 10^{-3}$ for all methods, which is close to maximal possible stepsize under which the methods rarely diverge.

4.3 Comparison of Adam and ExtraAdam

Unfortunately, pure extragradient did not perform extremely well on big datasets, so for the Fashion MNIST and Celeba experiments we used adaptive stepsizes as in Adam (Kingma and Ba, 2014).

In the first set of experiments, we compared the perfor-

mance of ExtraAdam (Gidel et al., 2019a) and Adam in a Conditional GAN (Mirza and Osindero, 2014) setup on Fashion MNIST (Xiao et al., 2017) dataset. The generator and discriminator were simple feedforward networks (detailed architectures description in Table 1). Optimizers were run with mini-batch size of 64 samples, no weight decay and $\beta_1 = 0.5, \beta_2 = 0.999$. One iteration of ExtraAdam was counted as two due to a double gradient calculation. The results are depicted in Figure 3. One can see that extragradient is slower because of the need to compute twice more gradients.

We suspect that Adam is faster partially due to that the problem's structure is something more specific than just a variational inequality. One validation of this guess is that in (Gidel et al., 2019b), the networks were trained with negative momentum only on discriminator, while generator was trained with constant momentum +0.5. Another reason we make this conjecture is that in (Metz et al., 2016) there was proposed a method that can be seen as a variant of extragradient, in which parameters of only one network requires extra steps.

In the second experiment, following (Chavdarova et al., 2019), we trained Self Attention GAN (Zhang et al., 2018). We note that the loss was generally an ambiguous metric of method comparison, so we provide the Inception score (Salimans et al., 2016) in Figure 4 as performance measure for image synthesis. We used Pytorch implementation provided by this GitHub repository. Besides, samples generated after training

Generator	Discriminator
<i>Input:</i> $z \in \mathbb{R}^{100} \sim \mathcal{N}(0, I)$	<i>Input:</i> $x \in \mathbb{R}^{1 \times 28 \times 28}$
Embedding layer for the label	Embedding layer for the label
Linear ($110 \rightarrow 256$)	Linear ($794 \rightarrow 1024$)
LeakyReLU (negative slope: 0.2)	LeakyReLU (negative slope: 0.2)
Linear ($256 \rightarrow 512$)	Dropout ($p=0.3$)
LeakyReLU (negative slope: 0.2)	Linear ($1024 \rightarrow 512$)
Linear ($512 \rightarrow 1024$)	LeakyReLU (negative slope: 0.2)
LeakyReLU (negative slope: 0.2)	Dropout ($p=0.3$)
Linear ($1024 \rightarrow 784$)	Linear ($512 \rightarrow 256$)
$\text{Tanh}(\cdot)$	LeakyReLU (negative slope: 0.2)
	Dropout ($p=0.3$)
	Linear ($1024 \rightarrow 784$)
	$\text{Sigmoid}(\cdot)$

Table 1: Architectures used for our experiments on *Fashion MNIST*.

for two epochs are provided in Figure 9 in the appendix.

The work (Gidel et al., 2019b) suggests using negative momentum to improve game dynamics and achieve faster convergence of the iterates. We consider using two types of momentum together: β_1 in the first step and β_2 in the second, i.e., we use $y^t = x^t - \eta_1 F(x^t; \xi^t) + \beta_1(x^t - x^{t-1})$ and $x^{t+1} = x^t - \eta_2 F(y^t; \xi^t) + \beta_2(y^t - x^{t-1})$. Detailed investigation on bilinear problems shows that β_1 can be chosen to be positive and β_2 should rather be negative. Intuitively, positive β_1 allows the method to look further ahead, while negative β_2 compensates for inaccuracy in the approximation of implicit update. In Appendix A.1, we discuss it in more details.

4.4 Discussion

The bilinear example is very clear and the results that we obtained showed enough stability. However, the message from training GANs is very vague due to their well-known instability. We did not observe a significant impact of negative momentum on convergence speed or stability, but at the same time we mentioned that setting first momentum to 0 in Adam is important for the extra update to have impact. We believe that the bilinear problem in this situation is the best way to make conclusion, but we still aim to obtain new methods for GANs in future.

It is also worth mentioning that the actual loss functions used in GANs are typically nonsmooth due to the choice of loss functions. For instance, the popular WGAN formulation (Arjovsky et al., 2017) includes hinge loss. On top of that, neural networks themselves have nonsmooth activations such as ReLU and its variants. Therefore, it is an interesting direction to understand what happens when the assumptions typical to variational inequalities are violated.

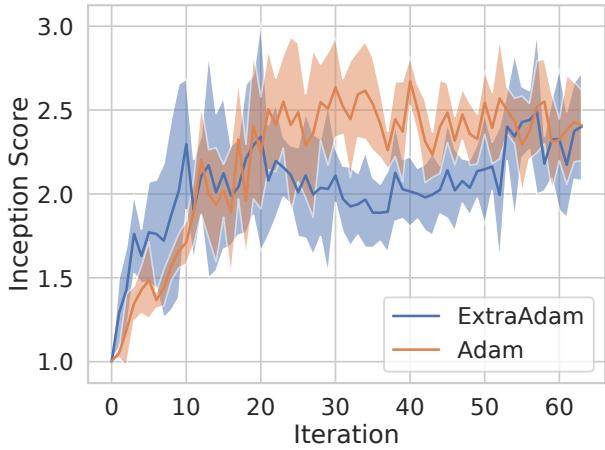
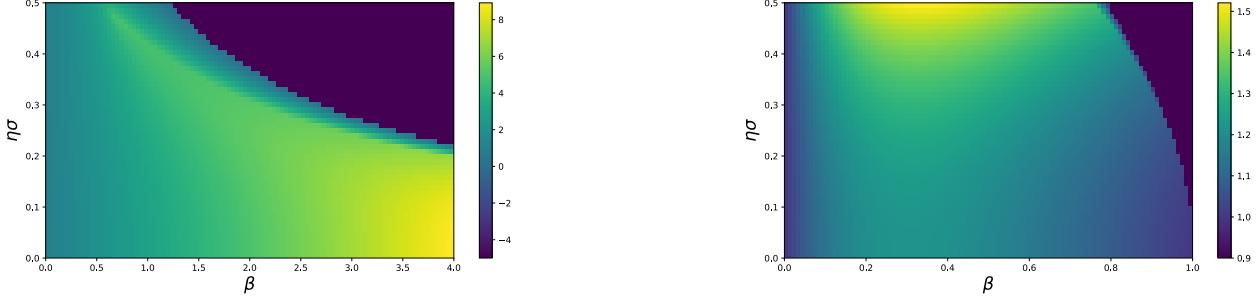


Figure 4: Inception score computed every 50 iterations during the training process on *CelebA* dataset for 2 epochs.



(a) $\eta_1 = \eta_2$, $\beta_2 = 0$, $\beta = \beta_1$ is the x -axis, $\eta\sigma_i$ is the y -axis. The optimal value of β_1 depends on $\eta\sigma_i$ and only for small values is significantly bigger 0. The dark area is where the method diverges.

(b) $\eta_1 = \eta_2$, $\beta_1 = 0$, $\beta = -\beta_2$ (negative momentum) is the x -axis, $\eta\sigma_i$ is the y -axis. The optimal value of β_2 is always very close to -0.3 . The dark area is where the method diverges.

Figure 5: Values of the spectral radius of the extragradient momentum matrix (5) for bilinear problems for different values of $\eta\sigma$ and β . The heat values is the multiplicative speed up from using $\beta > 0$ compared to $\beta = 0$, which we define as the ratio $\frac{\rho(\mathbf{T}(\eta\sigma, \beta))}{\rho(\mathbf{T}(\eta\sigma, 0))}$, where $\rho(\mathbf{A})$ is the spectral radius of a matrix \mathbf{A} for any \mathbf{A} and $\mathbf{T}(\eta\sigma, \beta)$ is the value of matrix in the update under given $\eta\sigma$ and β , see (5) in Appendix A.1.

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