Digital Control

CSE421

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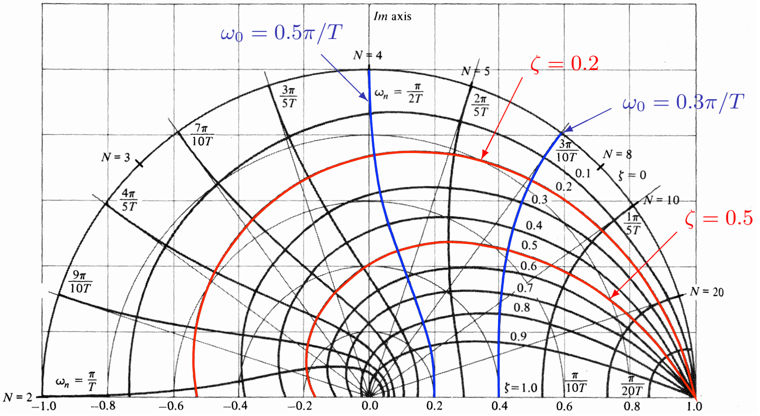
goo.gl/GHZZio

Lecture 5: **Step response and pole locations**



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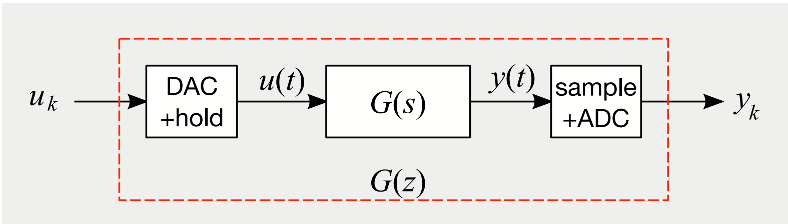
Review

- Definition of z-transform: $U(z) = \mathcal{Z}\{u_k\} = \sum_{k=0}^{\infty} u_k z^{-k}$

- Discrete transfer function: $\dfrac{Y(z)}{U(z)} = G(z) = \mathcal{Z}\{g_k\}, \quad g_k =$ pulse response

- Construct a discrete model of a continuous sampled-data system $G(s)$ . . .



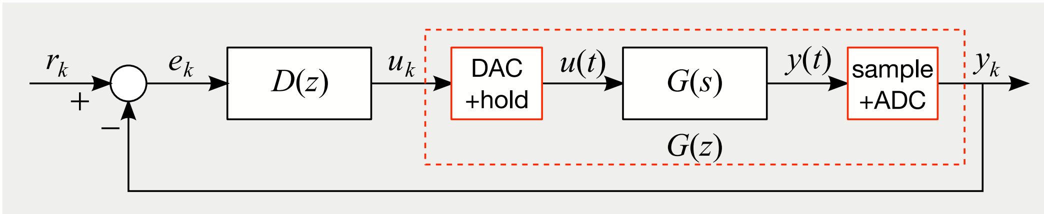
. . . by computing the pulse response g_k and transforming to get $G(z)$:

$$\boxed{G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}}$$

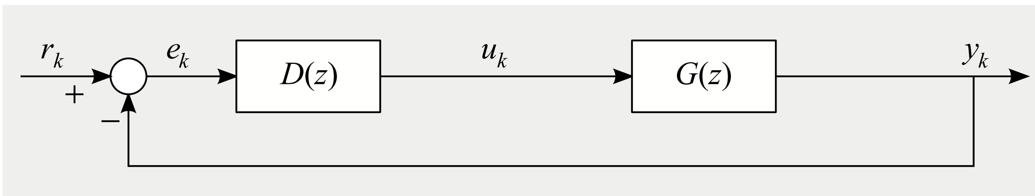
- Output response: $Y(z) = G(z)U(z) \iff y_k = g_k * u_k$

Review

Analyse/design a discrete controller $D(z)$:



by considering the purely discrete time system:



Closed loop system tranfer function: $\dfrac{Y(z)}{R(z)} = \dfrac{G(z)D(z)}{1 + G(z)D(z)}$

How do the closed loop poles relate to $\rightarrow$ stability?

$\rightarrow$ performance?

Response of 2nd order system

Consider the z-transform of a sinusoid multiplied by a an exponential signal:

$$y(t) = e^{-at}\cos(bt)\,\mathcal{U}(t) \qquad\qquad (\mathcal{U}(t) = \text{unit step})$$

- ⋆ sample: $y(kT) = r^k \cos(k\theta)\,\mathcal{U}(kT)$ with $r = e^{-aT}$ & $\theta = bT$

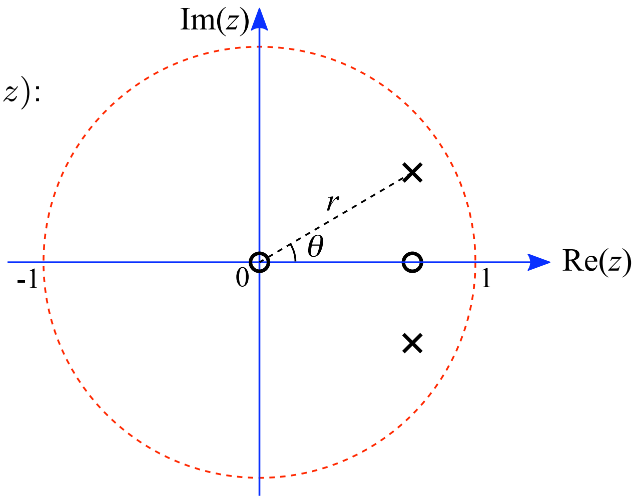
- ⋆ transform: $$Y(z) = \frac{1}{2}\frac{z}{(z - re^{j\theta})} + \frac{1}{2}\frac{z}{(z - re^{-j\theta})}$$
$$= \frac{z(z - r\cos\theta)}{(z - re^{j\theta})(z - re^{-j\theta})}$$

- ⋆ e.g. y_k is the pulse response of $G(z)$:

$$G(z) = \frac{z(z - r\cos\theta)}{(z - re^{j\theta})(z - re^{-j\theta})}$$

poles: $\begin{cases} z = re^{j\theta} \\ z = re^{-j\theta} \end{cases}$

zeros: $\begin{cases} z = 0 \\ z = r\cos\theta \end{cases}$



Response of 2nd order system

Responses for varying r:

- $r < 1$

  $\Downarrow$

  exponentially decaying envelope

- $r = 1$

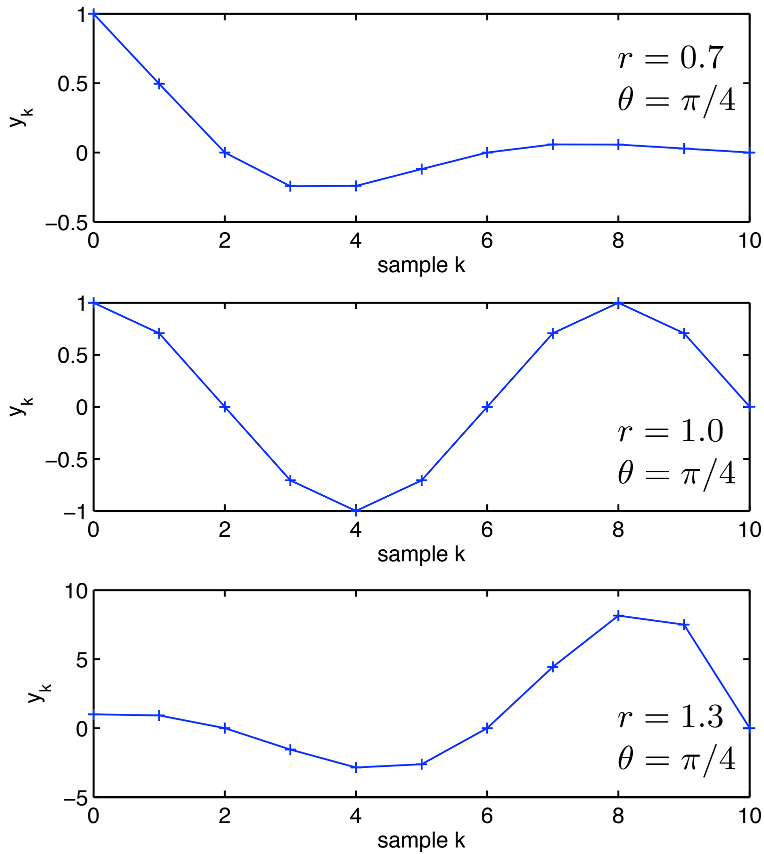
  $\Downarrow$

  sinusoidal response with $2\pi/\theta$ samples per period

- $r > 1$

  $\Downarrow$

  exponentially increasing envelope



Response of 2nd order system

Responses for varying θ:

- $\theta = 0$

  $\Downarrow$

  decaying exponential

- $\theta = \pi/2$

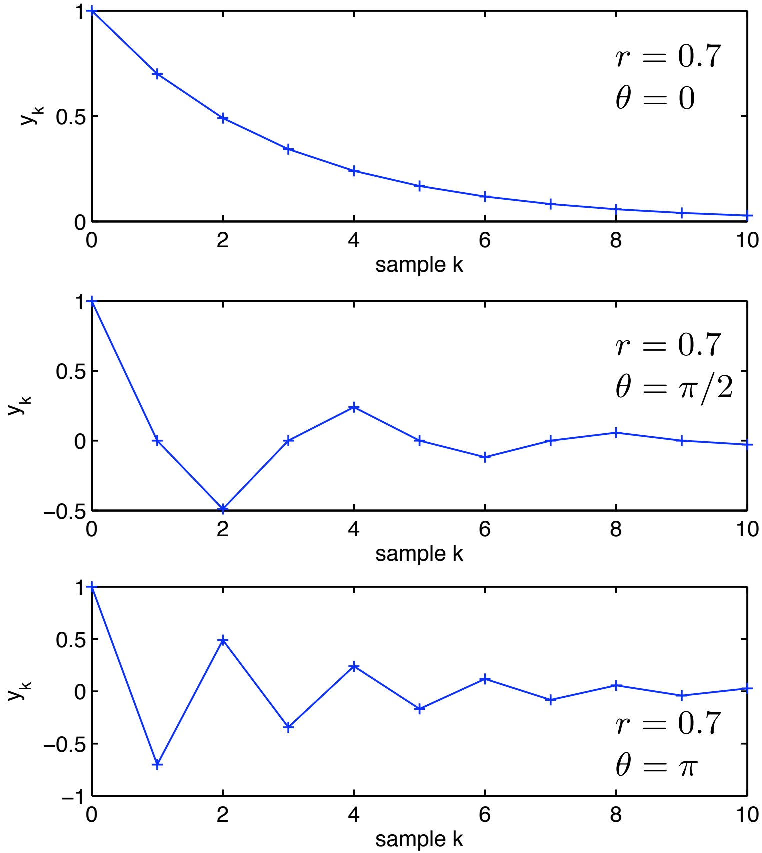
  $\Downarrow$

  $2\pi/\theta = 4$ samples per period

- $\theta = \pi$

  $\Downarrow$

  2 samples per period



Response of 2nd order system

Some special cases:

- for $\theta = 0$, $Y(z)$ simplifies to:

$$Y(z) = \frac{z}{z - r}$$

$\Longrightarrow$ exponentially decaying response

- when $\theta = 0$ and $r = 1$:

$$Y(z) = \frac{z}{z - 1}$$

$\Longrightarrow$ unit step

- when $r = 0$:

$$Y(z) = 1$$

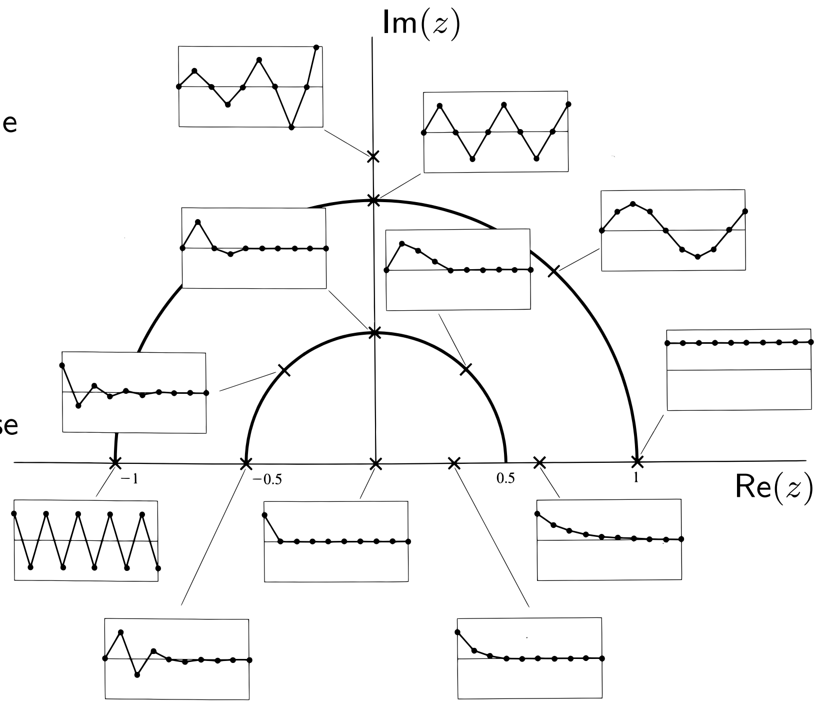
$\Longrightarrow$ unit pulse

- when $\theta = 0$ and $-1 < r < 0$:

samples of alternating signs

Pole positions in the z-plane

- Poles inside the unit circle
  are stable
- Poles outside the unit circle
  are unstable
- Poles on the unit circle
  are oscillatory
- Real poles at $0 < z < 1$
  give exponential response
- Higher frequency of
  oscillation for larger θ
- Lower apparent damping
  for larger θ and r



Relationship with s-plane poles

If $F(s)$ has a pole at $s = a$
then $F(z)$ has a pole at $z = e^{aT}$

$\uparrow$

consistent with $z = e^{sT}$

What about transfer functions?

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$\downarrow$

If $G(s)$ has poles $s = a_i$
then $G(z)$ has poles $z = e^{a_i T}$

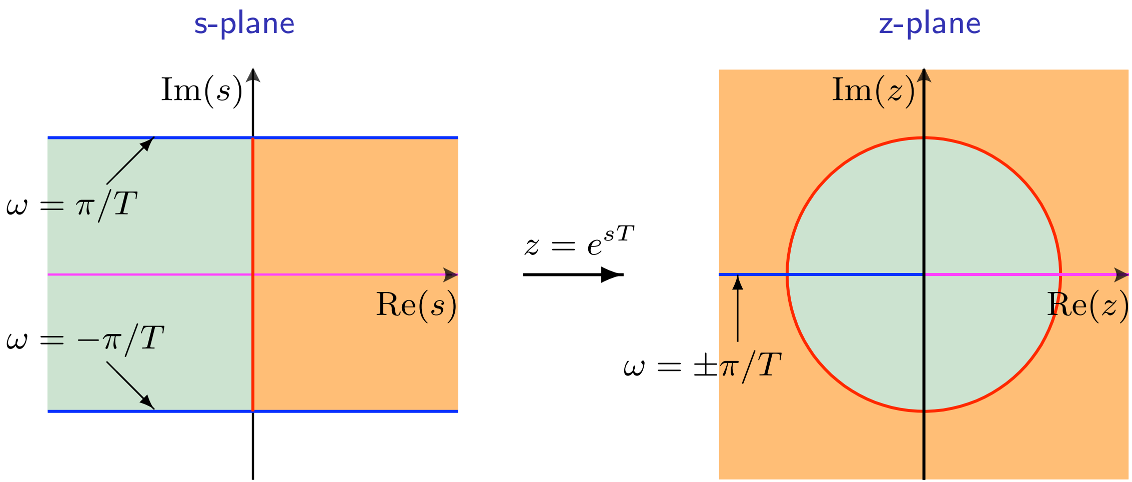
but the zeros are unrelated

$\mathcal{F}(s)$	$f(kT)$	$F(z)$
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{b-1}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$
$\frac{b}{(s+a)^2+b^2}$	$e^{-akT}\sin bkT$	$\frac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$

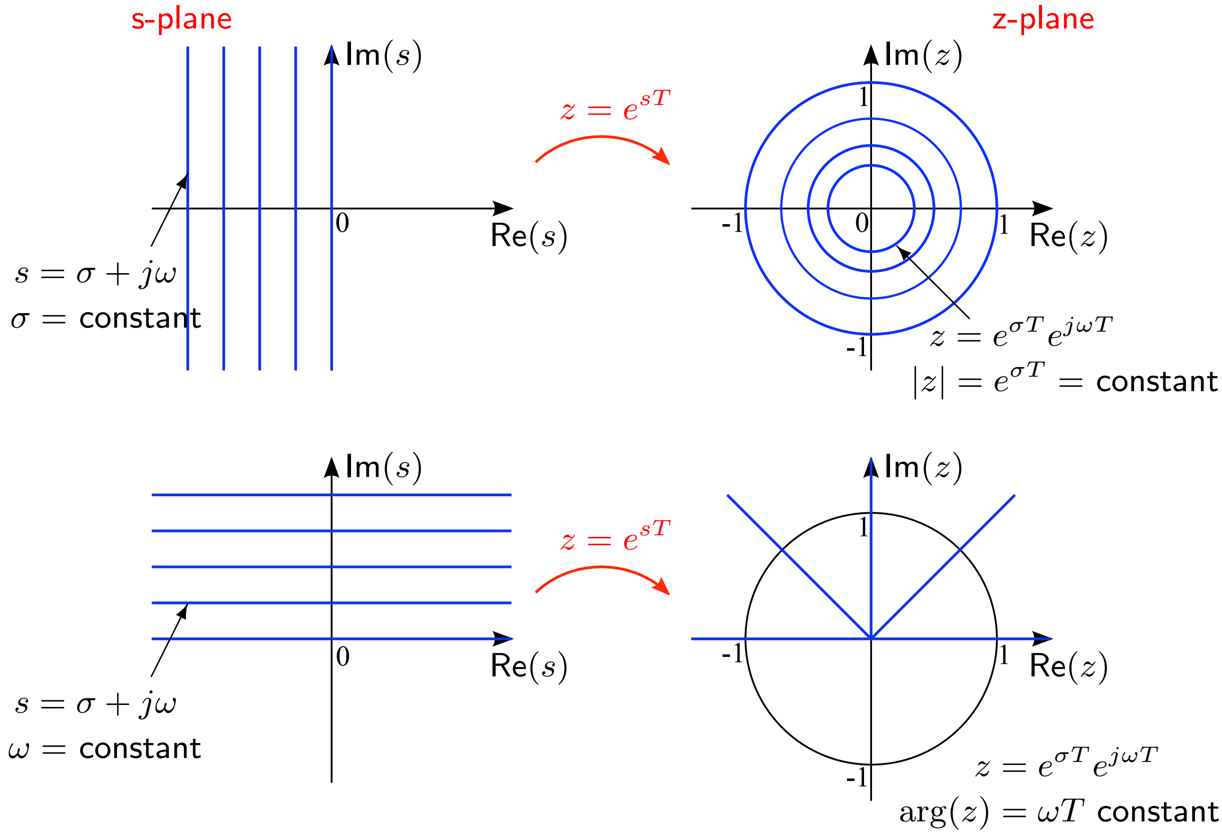
The mapping from s-plane to z-plane

Locus of $s = \sigma + j\omega$ under the mapping $z = e^{sT}$:

- $\star$ imaginary axis ($s = j\omega$, $\sigma = 0$) $\longrightarrow$ unit circle ($|z| = 1$)
- $\star$ left-half plane ($\sigma < 0$) $\longrightarrow$ inside of unit circle ($|z| < 1$)
- $\star$ right-half plane ($\sigma > 0$) $\longrightarrow$ outside of unit circle ($|z| > 1$)
- $\star$ region of s-plane within the Nyquist rate ($|\omega| < \pi/T$) $\longrightarrow$ entire z-plane



The mapping from s-plane to z-plane



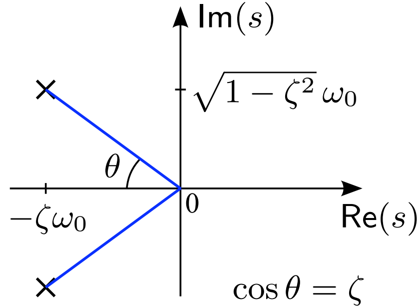
The mapping from s-plane to z-plane

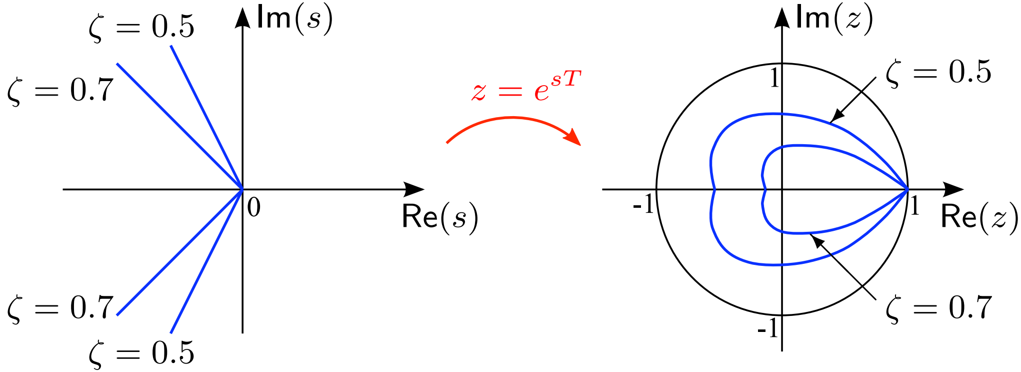
Pole locations for constant damping ratio $\zeta < 1$

$$s^2 + \zeta\omega_0 s + \omega_0^2 = 0$$

$$\Downarrow$$

$$s = -\zeta\omega_0 \pm j\sqrt{1-\zeta^2}\,\omega_0$$

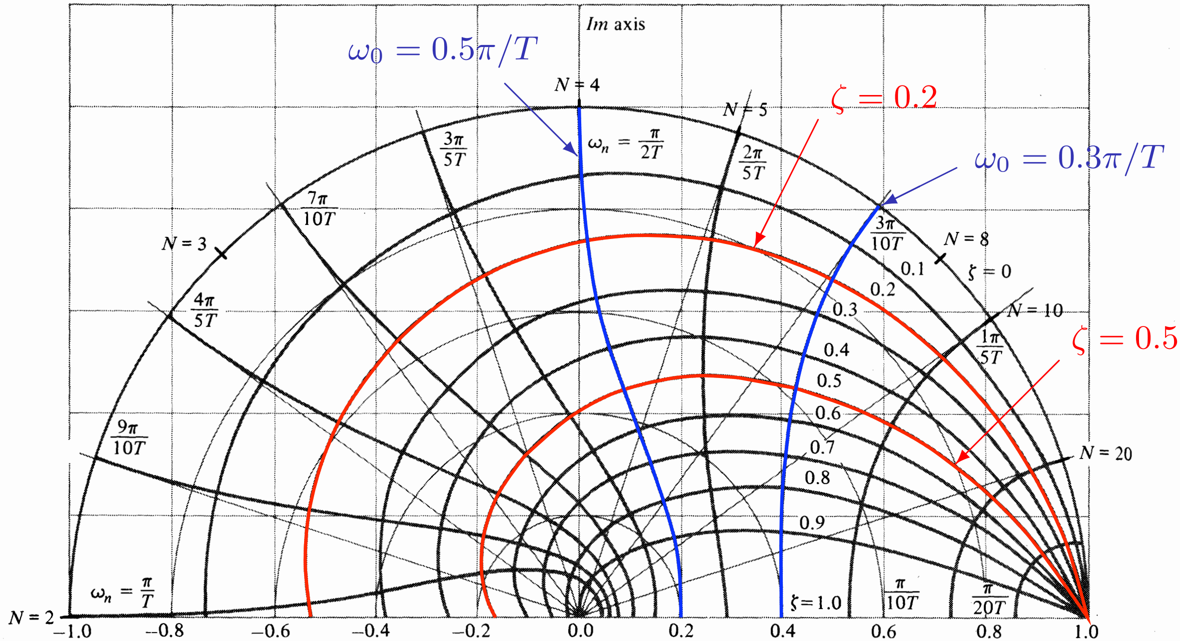




$s = -\zeta\omega_0 + j\sqrt{1-\zeta^2}\,\omega_0$: $\zeta =$ constant

$$z = e^{-\zeta\omega_0 T} e^{-j\sqrt{1-\zeta^2}\,\omega_0 T}$$

The mapping from s-plane to z-plane



z = plane loci of roots of constant ζ and ω_n

$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

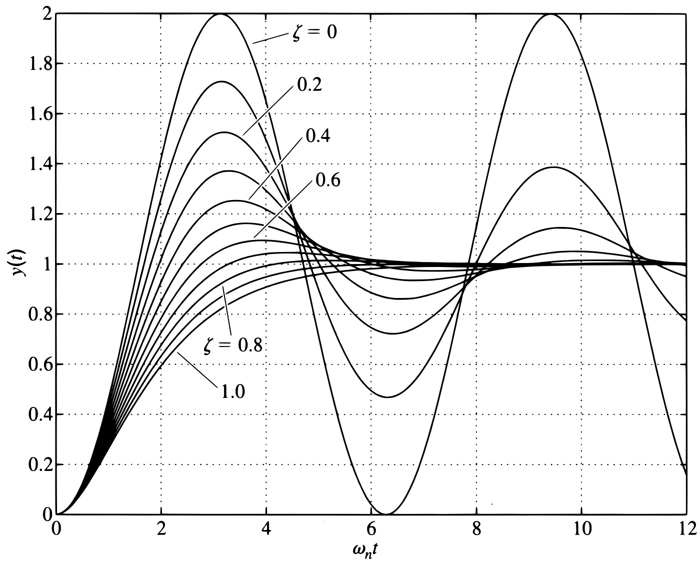
$z = e^{Ts}$

T = sampling period

The mapping from s-plane to z-plane

System specifications

Second order step responses (e.g. see HLT)

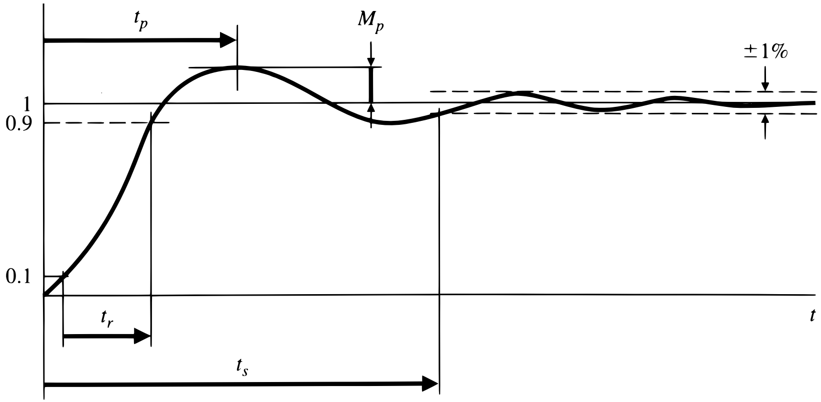


Design criteria based on step response:

- ⋆ Damping ratio ζ in range $0.5 – 0.9$ [application-dependent]
- ⋆ Natural frequency ω_0 as large as possible [for fastest response]

System specifications

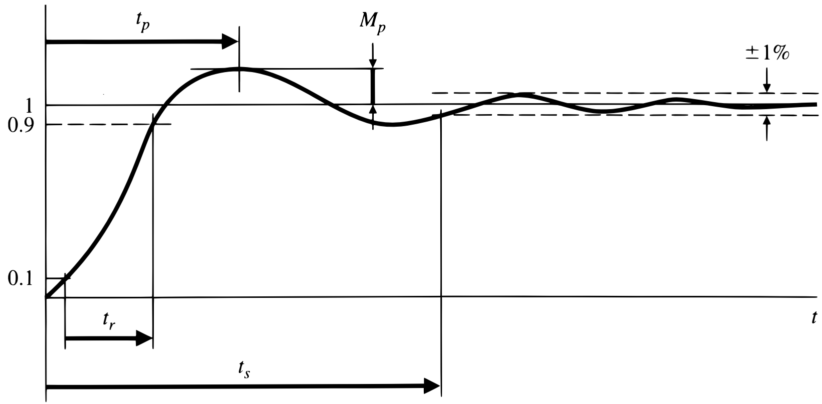
Typical specifications for the step response:



- ⋆ Rise time ($10\% \to 90\%$): $t_r \approx 1.8/\omega_0$
- ⋆ Peak overshoot: $M_p \approx e^{-\pi\zeta/\sqrt{1-\zeta^2}}$
- ⋆ Settling time (to 1%): $t_s = 4.6/(\zeta\omega_0)$
- ⋆ Steady state error to unit step: e_{ss}
- ⋆ Phase margin: $\phi_{PM} \approx 100\zeta$

System specifications

Typical specifications for the step response:



t_r, M_p $\longrightarrow$ ζ, ω_0 $\longrightarrow$ locations of dominant poles

t_s $\longrightarrow$ radius of poles: $|z| < 0.01^{T/t_s}$

e_{ss} $\longrightarrow$ final value theorem: $e_{ss} = \lim_{z \to 1} (z-1)E(z)$

System specifications

Example – A continuous system with transfer function

$$G(s) = \frac{1}{s(10s+1)}$$

is controlled by a discrete control system with a ZOH

The closed loop system is required to have:

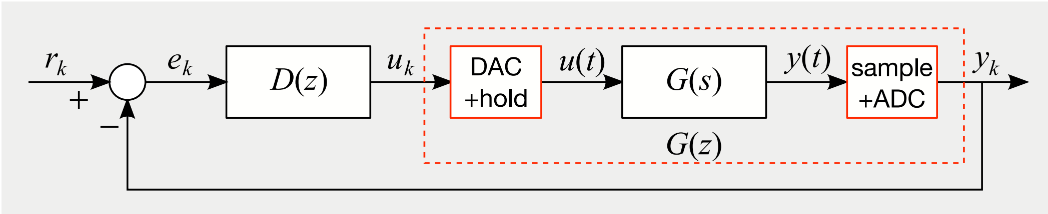
- step response overshoot: $M_p < 16\%$
- step response settling time (1%): $t_s < 10\,\mathrm{s}$
- steady state error to unit ramp: $e_{ss} < 1$

Check these specifications if $T = 1\,\mathrm{s}$ and the controller is

$$u_k = -0.5u_{k-1} + 13(e_k - 0.88e_{k-1})$$

System specifications

1. (a) Find the pulse transfer function of $G(s)$ plus the ZOH



$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{(z-1)}{z}\mathcal{Z}\left\{\frac{0.1}{s^2(s+0.1)}\right\}$$

e.g. look up $\mathcal{Z}\{a/s^2(s+a)\}$ in tables:

$$\begin{aligned} G(z) &= \frac{(z-1)}{z}\,\frac{z\Big((0.1 - 1 + e^{-0.1})z + (1 - e^{-0.1} - 0.1e^{-0.1})\Big)}{0.1(z-1)^2(z - e^{-0.1})} \\ &= \frac{0.0484(z + 0.9672)}{(z-1)(z-0.9048)} \end{aligned}$$

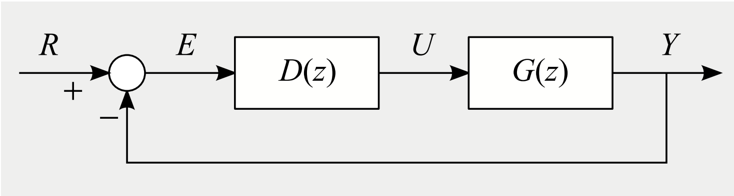
(b) Find the controller transfer function (using z = shift operator):

$$\frac{U(z)}{E(z)} = D(z) = 13\frac{(1 - 0.88z^{-1})}{(1 + 0.5z^{-1})} = 13\frac{(z-0.88)}{(z+0.5)}$$

System specifications

2. Check the steady state error e_{ss} when r_k = unit ramp

$$e_{ss} = \lim_{k\to\infty} e_k = \lim_{z\to 1}(z-1)E(z)$$



$$\frac{E(z)}{R(z)} = \frac{1}{1+D(z)G(z)}$$

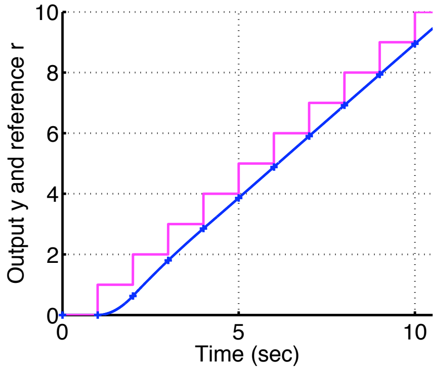
$$R(z) = \frac{Tz}{(z-1)^2}$$

so $$e_{ss} = \lim_{z\to 1}\left\{(z-1)\frac{Tz}{(z-1)^2}\frac{1}{1+D(z)G(z)}\right\} = \lim_{z\to 1}\frac{T}{(z-1)D(z)G(z)}$$

$$= \lim_{z\to 1}\frac{T}{(z-1)\dfrac{0.0484(z+0.9672)}{(z-1)(z-0.9048)}D(1)}$$

$$= \frac{1-0.9048}{0.0484(1+0.9672)D(1)} = 0.96$$

$\Longrightarrow$ $e_{ss} < 1$ (as required)



System specifications

3. Step response: overshoot $M_p < 16\% \implies \zeta > 0.5$

   settling time $t_s < 10 \implies |z| < 0.01^{1/10} = 0.63$

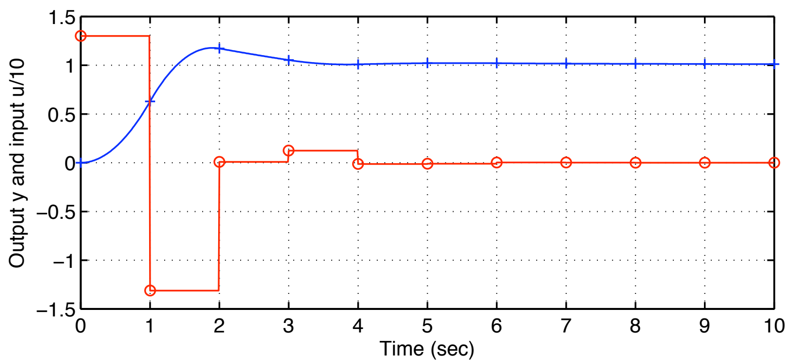
The closed loop poles are the roots of $1 + D(z)G(z) = 0$, i.e.

$$1 + 13\frac{(z-0.88)}{(z+0.5)}\,\frac{0.0484(z+0.9672)}{(z-1)(z-0.9048)} = 0$$

$$\implies z = 0.88,\ -0.050 \pm j0.304$$

But the pole at $z = 0.88$ is cancelled by controller zero at $z = 0.88$, and

$$z = -0.050 \pm j0.304 = re^{\pm j\theta} \implies \begin{cases} r = 0.31, & \theta = 1.73 \\ \zeta = 0.56 \end{cases}$$



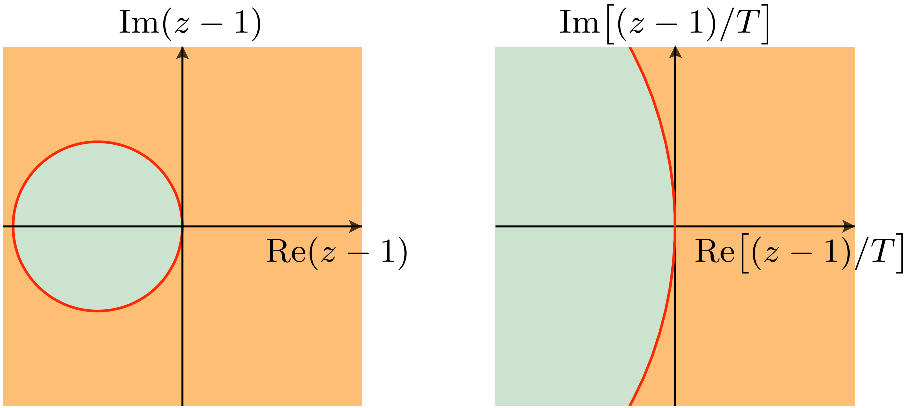
all specs satisfied!

Fast sampling revisited

For small T:

$$z = e^{sT} = 1 + sT + (sT)^2/2 + \cdots \approx 1 + sT \quad \Longrightarrow \quad s \approx \frac{z-1}{T}$$

Hence the image of the unit circle under the map from z to s-plane becomes



z-plane loci of constant ζ & ω_0

$\approx$ s-plane loci near $z = 1$

but the dominant poles lie near $z = 1 \ldots$

. . . so the discrete response tends to the continuous response as $T \to 0$

Summary

- Dependence of system pulse response on pole locations

- For a sampled data system with a ZOH:

  if $s = a_i$ is a pole of $G(s)$, then $z = e^{a_i T}$ is a pole of $G(z)$

- Locus of $s = \sigma + j\omega$ under the mapping $z = e^{sT}$:

  ⋆ the left half plane ($\sigma < 0$) maps to the unit disk ($|z| < 1$)

  ⋆ s-plane poles with damping ratio ζ, natural frequency ω_0 map to z-plane poles with:

$$|z| = e^{-\zeta\omega_0 T}$$

$$\arg(z) = \sqrt{1-\zeta^2}\,\omega_0 T$$

- Design specifications (rise time, settling time, overshoot)
  imply constraints on locations of dominant poles