MODEL STRUCTURES ON CUBICAL SETS

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Abstract. This note presents a model structure on locally cartesian closed categories satisfying various conditions inspired by cubical set models of univalent type theory. A special case is a model structure on cartesian cubical sets, generalizing prior work by Sattler to the setting with no connections without assuming that the diagonal on the interval is a cofibration.

Contents

1. A Model Structure

This note contains a categorical treatment of the generalized notion of uniform Kan fibrations for cartesian cubical sets of Cavallo and Mörtberg [CM19]. As this notion of fibration coincides with the one of Cohen et al. [Coh+18] and Orton and Pitts [OP16] in the presence of connections we recover the model structure of Sattler [Sat17] when the category also has connections. Furthermore, as we recover the fibrations of Angiuli et al. [Ang+17] when the diagonal on the interval is a cofibration we also recover the model structure on cartesian cubical sets sketched by Coquand [Coq18] based on Sattler's model structure, and the more recent one of Awodey [Awo19]. This generalization hence clarifies the relationship between some of the various model structures on different cubical set categories.

1.1. The Weak Left Lifting Property.

We define an abstract formulation of weak composition based on lifting problems. We prove some useful results regarding the interaction of this notion with mapping cylinders and the Leibniz adjunction, and some other useful lemmas.

Definition 1. Let $m\colon A \to B$ and $f\colon X \to Y$. We say m has the *weak left lifting property* against f if for every commutative square, as in the solid lines below, there is a diagonal map, as in the dotted line below, such that the lower triangle commutes

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strictly, and the upper triangle commutes up to a homotopy $h\colon j \circ m \sim a$ such that $f \circ h$ is constant.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\sim", near start] & X \arrow[d, "f"] \\ B \arrow[r, "b"] \arrow[ur, dotted] & Y \end{tikzcd}$$

We will refer to squares as above as *lifting problems* of m against f, and refer to pairs (j, h) as above as *weak fillers.*

We will sometimes refer to the usual left/right lifting property as the *strict* left-/right lifting property to distinguish it from the weak left lifting property. Given $\varepsilon \in \{0, 1\}$ we write $\delta_{A\varepsilon} : A \to \mathbb{I} \times A$ for the map $a \mapsto (\varepsilon, a)$. For simplicity we also write δ_ε for $\delta_{1\varepsilon}$.

Definition 2. Let $m\colon A \to B$. We define the *mapping cylinder factorisation* to be the maps $A \xrightarrow{L(m)} \mathsf{Cyl}(m) \xrightarrow{R(m)} B$, defined as follows. We first define $\mathsf{Cyl}(m)$ as the pushout of δ_{A0} and m, and then define $L(m)$ and $R(m)$ as the unique maps making the diagrams below commute where ι_0 and ι_1 are the pushout inclusions.

$$\begin{tikzcd} & A \arrow[d, "\delta_{A1}"] \arrow[dd, bend left=60, "L(m)"] \\ A \arrow[r, "\delta_{A0}"] \arrow[d, "m"] & \mathbb{I} \times A \arrow[d, "\iota_0"] \\ B \arrow[r, "\iota_1"] & \mathsf{Cyl}(m) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\delta_{A0}"] \arrow[d, "m"] & \mathbb{I} \times A \arrow[d, "\iota_0"] \arrow[dr, bend left, "\pi_1"] & \\ B \arrow[r, "\iota_1"] \arrow[drr, bend right, "1_B"] & \mathsf{Cyl}(m) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "R(m)"] & A \arrow[d, "m"] \\ & & B \end{tikzcd}$$

Observe that $m = R(m) \circ L(m)$. We also have the following lemma.

Lemma 3. *$L(m)$ is a pullback of $\delta_1 : 1 \to \mathbb{I}$.*

Proof. By disjointness of endpoints. $\square$

Lemma 4. *Let $m\colon A \to B$ and $f\colon X \to Y$. If $L(m)$ has the left lifting property against f, then m has the weak left lifting property against f.*

Proof. Note that m is a retract of $L(m)$ "up to homotopy" in the sense that we have the diagram below, where the left hand square only commutes up to a homotopy from $\iota_1 \circ m$ to $L(m)$, whose underlying map is given by $\iota_0\colon \mathbb{I} \times A \to \mathsf{Cyl}(m)$.

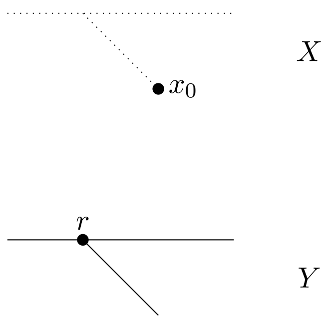
$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "m"] \arrow[dr, phantom, "\sim"] & A \arrow[r, "1_A"] \arrow[d, "L(m)"] & A \arrow[d, "m"] \\ B \arrow[r, "\iota_1"] \arrow[rr, bend right, "1_B"'] & \mathsf{Cyl}(m) \arrow[r, "R(m)"] & B \end{tikzcd}$$

We give a variant of the standard proof that the class of maps with the left lifting property against f is closed under retracts.

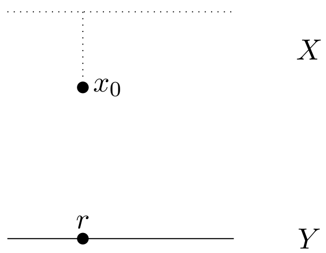
Given a lifting problem of m against f, we paste it to the right hand square above, to get a lifting problem of $L(m)$ against f. Let $j\colon \mathsf{Cyl}(m) \to X$ be a diagonal filler. Then $j' := j \circ \iota_1$ provides a diagonal filler of the original lifting problem, with $j \circ \iota_0$ witnessing the commutativity of the upper triangle up to homotopy. $\square$

We can give some geometric intuition to mapping cylinders and Lemma 4 by considering a special case that will be useful later. We take the map $A \to B$ to be a point of the interval, i.e. a map $r\colon 1 \to \mathbb{I}$. We can visualise $\mathsf{Cyl}(r)$ as a T-shape, where the top of the vertical bar (the 0-endpoint of one copy of the interval) is glued to the horizontal bar at a point r. $L(r)\colon 1 \to \mathsf{Cyl}(r)$ is then the point at the base of the T, i.e. the 1-endpoint of the vertical bar. Then if $L(r)$ has the left lifting property against a map f, then r has the weak left lifting property, where the path over the horizontal bar of the T provides the diagonal filler, and the path over the vertical bar provides the homotopy witnessing the commutativity of the upper triangle.

In general, to have the left lifting property against $L(r)$ tells us that given a T-shape in Y, as in the solid lines below and a point x_0 in X above the base of the T, one can find a T-shape in X over the one in Y, as in the dotted lines below.



If we want to show the map $X \to Y$ has the weak right lifting property against $r\colon 1 \to \mathbb{I}$, we can derive it from f having the right lifting property against $L(r)$ via the special case of where the T-shape in Y factors through the map that "collapses" the vertical bar to the point r, as below.



The filler is provided by the horizontal bar, and the homotopy by vertical bar of the dotted T in X.

Lemma 5. *Suppose we are given maps $m\colon A \to B$, $l\colon C \to D$ and $f\colon X \to Y$ together with a lifting problem of $m \hat{\times} l$ against f, as below.*

$$\begin{array}{ccc} A \times D +_{A \times C} B \times C & \xrightarrow{p} & X \\ \big\downarrow{\scriptstyle m\hat{\times} l} & & \big\downarrow{\scriptstyle f} \\ B \times D & \xrightarrow{q} & Y \end{array} \tag{1}$$

Then weak fillers (j', h') of the corresponding lifting problem of m against $\hat{\hom}(l, f)$ correspond precisely to weak fillers (j, h) of (1) that satisfy the additional requirement that the restriction of the homotopy $h\colon \mathbb{I} \times (A \times D +_{A \times C} B \times C) \to X$ to $B \times C$

is constant (and in particular the restrictions of $j \circ (m \hat{\times} l)$ and p to $B \times C$ are strictly equal).

Proof. A lengthy but standard argument using the adjunction between products and exponentials. □

Lemma 6. *Suppose we are given maps $m \colon A \to B$ and $f \colon X \to Y$. If δ_0 has the weak left lifting property against $\widehat{\hom}(m, f)$ and m has the weak left lifting property against f, then m has the strict left lifting property against f.*

Proof. Suppose we are given a lifting problem, as in the solid lines below. Let j be a weak filler, as in the dotted line below.

(2)
$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "m"'] & X \arrow[d, "f"] \\
B \arrow[r, "b"'] \arrow[ur, dotted, "j"'] \arrow[ur, phantom, shift left=3, "\sim"] & Y
\end{tikzcd}$$

Let $h \colon \mathbb{I} \times A \to X$ be the underlying map of the homotopy from $j \circ m$ to a. Note that we can define a lifting problem of $\delta_0 \hat{\times} m$ against f as, in the right hand square of the diagram below. By Lemma 5 and the fact that δ_0 has the weak left lifting property against $\widehat{\hom}(m, f)$, there is a diagonal filler j' of the square satisfying the additional property that $j' \circ (\delta_0 \hat{\times} m) \circ \iota_1 = h$.

$$\begin{tikzcd}
B +_A \mathbb{I} \times A \arrow[r, "{[j,h]}"] \arrow[d, "\delta_0 \hat{\times} m"'] & X \arrow[d, "f"] \\
\mathbb{I} \times B \arrow[r, "b \circ \pi_1"'] \arrow[ur, dotted, "j'"'] \arrow[ur, phantom, shift left=3, "\sim"] & Y
\end{tikzcd}$$

The latter means that this square fits into the following diagram where the extended upper triangle commutes strictly, making $j' \circ \delta_{B1}$ into a strict filler of (2).

$$\begin{tikzcd}
A \arrow[r, "\iota_1 \circ \delta_{A1}"] \arrow[d, "m"] \arrow[rr, bend left, "a"] & B +_A \mathbb{I} \times A \arrow[r, "{[j,h]}"] & X \arrow[d, "f"] \\
B \arrow[r, "\delta_{B1}"'] \arrow[urr, dotted, "j' \circ \delta_{B1}"] \arrow[rr, bend right, "b"'] & \mathbb{I} \times B \arrow[r, "b \circ \pi_1"'] \arrow[ur, dotted, "j'"] & Y
\end{tikzcd}$$

□

Lemma 7. *Suppose we are given maps $m \colon A \to B$ and $f \colon X \to Y$. Suppose that m has the weak left lifting property against f and that δ_{A1} and $\delta_{A1} \hat{\times} [\delta_0, \delta_1]$ have the strict left lifting property against f. Then $L(m)$ has the strict left lifting property against f.*

Proof. Suppose we are given a lifting problem of $L(m)$ against f. Expanding out the definition of $L(m)$, we can write this as the solid lines in the diagram below.

$$\begin{tikzcd}
& A \arrow[r, "x"] \arrow[d, "\delta_{A1}"] & X \arrow[dd, "f"] \\
A \arrow[r, "\delta_{A0}"] \arrow[d, "m"] & \mathbb{I} \times A \arrow[d, "\iota_0"] \arrow[ur, dotted, "j"] & \\
B \arrow[r, "\iota_1"] \arrow[uurr, dotted, "j'"] & \mathsf{Cyl}(m) \arrow[r, "y"] \arrow[ul, phantom, "\ulcorner", very near start] & Y
\end{tikzcd}$$

First, using the left lifting property of δ_{A1} against f, we obtain a diagonal lift j, which gives us a lifting problem of m against f.

Next we use the fact that m has the weak left lifting property against f to obtain the map $j' \colon B \to X$ such that the lower triangle strictly commutes and we have a homotopy $h \colon j' \circ m \sim j \circ \delta_{A0}$.

We will now show how to replace j with a map j'' such that $j'' \circ \delta_{A1} = x$ and $j' \circ m = j'' \circ \delta_{A0}$.

We now define a lifting problem of $\delta_{A1} \hat{\times} [\delta_0, \delta_1]$ as in the solid lines of the square below.

$$\begin{tikzcd}
\mathbb{I} \times A +_{A+A} (\mathbb{I} \times A + \mathbb{I} \times A) \arrow[rr, "{[j,[h,x \circ \pi_1]]}"] \arrow[d, "{\delta_{A1} \hat{\times} [\delta_0,\delta_1]}"'] & & X \arrow[d, "f"] \\
\mathbb{I} \times (\mathbb{I} \times A) \arrow[r, "\iota_0 \circ \pi_1"'] \arrow[urr, dotted, "k"] & \mathsf{Cyl}(m) \arrow[r, "y"'] & Y
\end{tikzcd}$$

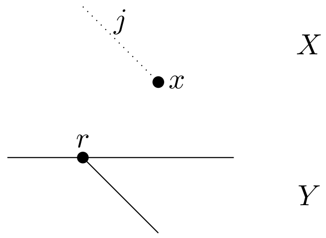
We define $j'' \colon \mathbb{I} \times A \to X$ to be $k \circ (\delta_0 \times \mathbb{I} \times A)$. By pasting the upper triangle of the lifting diagram to the commutative square below, we get the equality $j'' \circ \delta_{A0} = h \circ \delta_{A0}$ as well as one of the required equalities, $j'' \circ \delta_{A1} = x$.

$$\begin{tikzcd}
A + A \arrow[r, "\delta_{A0} + \delta_{A0}"] \arrow[d, "{[\delta_{A0},\delta_{A1}]}"'] & \mathbb{I} \times A + \mathbb{I} \times A \arrow[r, hook, "\iota_1"] & \mathbb{I} \times A +_{A+A} (\mathbb{I} \times A + \mathbb{I} \times A) \arrow[d, "{\delta_{A1} \hat{\times} [\delta_0,\delta_1]}"] \\
\mathbb{I} \times A \arrow[rr, "\delta_0 \times \mathbb{I} \times A"'] & & \mathbb{I} \times (\mathbb{I} \times A)
\end{tikzcd}$$

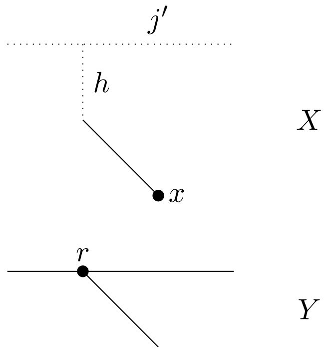
However, h is a homotopy from $j' \circ m$ to $j \circ \delta_{A0}$, so $h \circ \delta_{A0} = j' \circ m$. We thereby obtain the other required equality $j'' \circ \delta_{A0} = j' \circ m$.

Since we have ensured $j'' \circ \delta_{A0} = j' \circ m$, we can apply the universal property of the pushout to get a map $l \colon \mathsf{Cyl}(m) \to X$ such that $l \circ \iota_0 = j''$ and $l \circ \iota_1 = j'$. Then l is a diagonal filler of the original lifting problem. $\square$

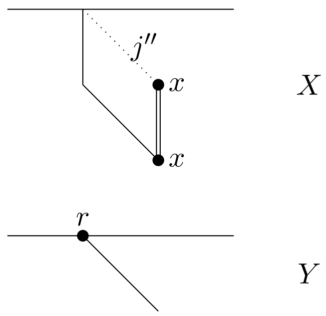
We illustrate this with the example from earlier where the map m is given as $r \colon 1 \to \mathbb{I}$. This time we are given the solid lines in the diagram below, together with the point x. Requiring that f has the right lifting property against δ_1 tells us that we can lift the path given by the vertical bar of the T-shape to X, as in the dotted line below.



The next step is to use the fact that m has the weak left lifting property against f to produce a path j' over the horizontal bar of the T. The homotopy witnessing the commutativity of the upper triangle gives us a path h from the point of j' over r to the endpoint of the path j, which lies entirely in the fiber of r.



The final step is to first generate an open box with left hand side given by h, right hand side the constant path on x, and base given by j. Since f has the right lifting property against $\delta_1 \hat{\times} [\delta_0, \delta_1]$, we can find a filler of the open box, and in particular get a path j'' as illustrated below.



This completes the construction of a T shape in X over the one in Y.

1.2. **Review of Lifting Problems over a Codomain Fibration.** We recall the following definitions and theorems from [Swa18a] and [Swa18b]. We fix a locally cartesian closed category $\mathcal{C}$.

Definition 8. Let m be a map in a slice category $\mathcal{C}/I$ and let f be a map in another slice category $\mathcal{C}/J$ as illustrated below.

$$\begin{tikzcd} A \arrow[rr, "m"] \arrow[dr] & & B \arrow[dl] \\ & I & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & J & \end{tikzcd}$$

A *family of lifting problems of m against f* consists of an object K, together with maps $\sigma\colon K \to I$ and $\tau\colon K \to J$ and a lifting problem of $\sigma^*(m)$ against $\tau^*(f)$ in $\mathcal{C}/K$.

A family of lifting problems K, σ, τ, p, q is *universal* if for any other family of lifting problems $K', \sigma', \tau', p', q'$, there is a unique map $t\colon K' \to K$ such that $\sigma' = t \circ \sigma$, $\tau' = t \circ \tau$, $p' = t^*(p)$ and $q' = t^*(q)$.

We will also refer to universal families of lifting problems simply as universal lifting problems.

Proposition 9. *Universal lifting problems exist.*

Proof. This is an instance of a general result for locally small Grothendieck fibrations (see [Swa18a, Section 3]). For later reference we remark that the indexing object K can be explicitly described as $\sum_{i:I} \sum_{j:J} \sum_{\beta : B_i \to Y_j} \prod_{a:A_i} X_{\beta(m(a))}$, with horizontal maps in the lifting problem given by evaluation. □

Proposition 10. *Every family of lifting problems has a diagonal filler if and only if the universal family of lifting problems has a diagonal filler.*

Proof. See [Swa18a, Proposition 3.2.5]. □

Let $\delta_0, \delta_1\colon 1 \to \mathbb{I}$ be an interval object in $\mathcal{C}$. For each slice category $\mathcal{C}/I$ write $\mathbb{I}_I$ for the object of $\mathcal{C}/I$ given by projection $\mathbb{I} \times I \to I$ and observe that δ_0 and δ_1 make $\mathbb{I}_I$ into an interval object in $\mathcal{C}/I$.

We observe that the above proposition also applies to weak lifting problems, in the following sense.

Proposition 11. *Every family of lifting problems K, σ, τ, p, q has a weak filler (relative to $\mathbb{I}_K$) if and only if this is the case for the universal family of lifting problems.*

Proof. Since the interval object and products are preserved by reindexing, so are homotopies. Hence this follows from the definition of universal lifting problem, as for proposition 10. □

We recall the notion of cofibrantly generated fibred awfs (for the special case of codomain fibrations).

Definition 12. A *fibred algebraic weak factorisation system* or *fibred awfs* consists of an algebraic weak factorisation system (L_J, R_J) on each slice category $\mathcal{C}/J$ preserved by reindexing (up to isomorphism).

A fibred awfs is *cofibrantly generated* if there exists a map m in some slice category $\mathcal{C}/I$ such that for each J and each map f in $\mathcal{C}/J$, R_J algebra structures on f correspond precisely to diagonal fillers of the universal lifting problem of m against f.

Theorem 13. *Suppose that the category $\mathcal{C}$ has finite colimits and disjoint coproducts. Let m be a map in some slice category $\mathcal{C}/I$. Suppose that either of the two conditions below are satisfied.*

(1) $\mathcal{C}$ is an (internal) category of presheaves and m is a locally decidable monomorphism.
(2) $\mathcal{C}$ is a ΠW-pretopos (e.g. $\mathcal{C}$ is a topos with natural number object) and it satisfies the axiom weakly initial set of covers (WISC).

Then the fibred awfs cofibrantly generated by m exists.

Proof. See [Swa18b, Corollary 6.12 and Theorem 6.14]. □

Lemma 14. *Let $(L_J, R_J)_{J \in \mathcal{C}}$ be a fibred awfs in a category $\mathcal{C}$, cofibrantly generated by a map $m \colon A \to B$ in $\mathcal{C}/I$.*

(1) If $\mathcal{C}$ is a topos and m is a monomorphism, then every left map is a monomorphism.
(2) If $\mathcal{C}$ is a category of presheaves and m is a locally decidable monomorphism, then every left map is a locally decidable monomorphism.

Proof. Let $J \in \mathcal{C}$. We have an underlying wfs $(\mathcal{L}_J, \mathcal{R}_J)$ using the classes of left maps and right maps for the awfs (L_J, R_J). We can construct a second wfs $(\mathcal{M}, \mathcal{S})$ on $\mathcal{C}/J$ as follows. If condition 1 holds, we take $\mathcal{M}$ to be the class of monomorphisms, and if condition 2 holds, we take $\mathcal{M}$ to be locally decidable monomorphisms. In both cases the factorisation can be constructed using partial map classifiers.

Let $f \colon X \to Y$ belong to the class $\mathcal{S}$. Let K be an object of $\mathcal{C}$ and suppose we are given maps $\sigma \colon K \to I$ and $\tau \colon K \to J$. Since monomorphisms and locally decidable monomorphisms are both preserved by pullback and $\sum$, we know that $\sum_\tau \sigma^*(m)$ belongs to the class $\mathcal{M}$, and so has the left lifting property against f. Since this holds for every choice of K, τ and σ, we may deduce that f belongs to $\mathcal{R}_J$. That is, we have shown $\mathcal{S} \subseteq \mathcal{R}_J$. It follows that $\mathcal{L}_J \subseteq \mathcal{M}$, which is what we needed to show. □

We also recall the notion of strongly fibred awfs.

Definition 15. We say a fibred awfs $(L_J, R_J)_{J \in \mathcal{C}}$ is *strongly fibred* if for each J, the awfs (L_J, R_J) preserves pullbacks. That is, whenever the square on the left below is a pullback, so is the square on the right below (with both squares lying in the slice category $\mathcal{C}/J$, and where K_J is the object part of the factorisation system).

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} K_J f' \arrow[r] \arrow[d, "R_J f'"] \arrow[dr, phantom, "\lrcorner", very near start] & K_J f \arrow[d, "R_J f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

Proposition 16. *Suppose $(L_J, R_J)_{J \in \mathcal{C}}$ is strongly fibred. Then the class of left maps in each slice category is stable under pullback.*

Proof. A map $m \colon A \to B$ in the slice category over J is a left map if and only if there is a filler in the lifting problem below.

$$\begin{tikzcd} A \arrow[r, "L_J m"] \arrow[d, "m"] & K_J m \arrow[d, "R_J m"] \\ B \arrow[r, "1_B"] \arrow[ur, dotted] & B \end{tikzcd}$$

However, if $m' \colon A' \to B'$ is a pullback of m along a map $f \colon B' \to B$, then pulling back the diagram above along f, gives us a diagonal filler in the lifting problem below.

$$\begin{tikzcd} A' \arrow[r, "L_J m'"] \arrow[d, "m'"] & K_J m' \arrow[d, "R_J m'"] \\ B' \arrow[r, "1_{B'}"] \arrow[ur, dotted] & B' \end{tikzcd}$$

But this tells us that m' is also a left map. □

We can generate strongly fibred awfs's using the following lemma.

Lemma 17. *Suppose that (L, R) is the fibred awfs cofibrantly generated by a map m into the terminal object of a slice category $\mathcal{C}/I$. Then (L, R) is strongly fibred.*

Proof. See [Swa18a, Corollary 7.5.5]. $\square$

1.3. **Cofibrations and Trivial Fibrations.** Let $\top \colon \Phi_{\mathsf{true}} \rightarrowtail \Phi$ be the classifier for cofibrant propositions.

We will use the following notational conventions for working in a slice category $\mathcal{C}/B$. The terminal object of $\mathcal{C}/B$ is the identity on B, which we will write as 1_B, as usual.

Definition 18. Let $m \colon A \to B$ be a map in a slice category $\mathcal{C}/I$. We say m is a *generating cofibration* if either of the equivalent conditions below holds.

(1) $\sum_I m$ is a pullback of $\top$.
(2) m is a pullback of $I^*(\top) \colon I^*(A) \to I^*(B)$ in $\mathcal{C}/I$.

Proposition 19. *Generating cofibrations are closed under pullbacks and binary unions. Every isomorphism is a generating cofibration.*

Proof. Closure under pullback follows from the definition. For closure under binary unions, we use the join operator $\vee \colon \Phi \times \Phi \to \Phi$. For isomorphisms, we use the fact that true is a cofibrant proposition. $\square$

Proposition 20. *Fix a slice category $\mathcal{C}/I$.*

(1) If m and l are generating cofibrations, then $m \,\hat{\times}_I\, l$ is a generating cofibration.
(2) $[\delta_{I0}, \delta_{I1}]$ is a generating cofibration.

Proof. $[\delta_{I0}, \delta_{I1}]$ and $m \,\hat{\times}_I\, l$ are both binary unions of generating cofibrations. $\square$

Lemma 21. *Let $m \colon A \to B$ be a generating cofibration in a slice category $\mathcal{C}/I$. Then m is of the form $\sum_B m'$ where $m' \colon A \to 1_B$ is a generating cofibration in $\mathcal{C}/B$.*

Proof. We take m' to have the same underlying map as m, and observe that this ensures $\sum_B m' = m$. From the definition of generating cofibration we see that $\sum_B$ reflects generating cofibrations, and so m' is a generating cofibration. $\square$

Proposition 22. *Let $f \colon X \to Y$ be a map in a slice category $\mathcal{C}/J$. The following are equivalent.*

(1) f has the fibred right lifting property against $\top$, viewed as a map $\Phi_{\mathsf{true}} \to 1_\Phi$ in $\mathcal{C}/\Phi$.
(2) f has the fibred right lifting property against generating cofibrations of the form $A \to 1_B$ in slice categories $\mathcal{C}/B$.
(3) f has the fibred right lifting property against every generating cofibration.
(4) f has the right lifting property against every generating cofibration in $\mathcal{C}/J$.

Definition 23. If a map $f \colon X \to Y$ in a slice category $\mathcal{C}/J$ satisfies one, and so all, of the equivalent conditions in Proposition 22 we say f is a *trivial fibration*.

We say a map m in a slice category $\mathcal{C}/I$ is a cofibration if it has the fibred left lifting property against every trivial fibration.

We note that as a special case of Lemma 17 if either of the conditions in Theorem 13 holds, then cofibrations and trivial fibrations form part of a strongly fibred awfs cofibrantly generated by the map $\top \colon \Phi_{\mathsf{true}} \to \Phi$, viewed as a map $\Phi_{\mathsf{true}} \to 1_\Phi$ in the slice category $\mathcal{C}/\Phi$.

1.4. **The Trivial Cofibrations-Fibrations Awfs.** We give some further notational conventions for working in a slice category. The cartesian product in $\mathcal{C}/B$ is given by pullback, which we write as $\times_B$, as usual. We will also write the internal Leibniz product as $\hat{\times}_B$. We will write $\mathbb{I}_B$ for the projection $\mathbb{I} \times B \to B$, and note that this forms an interval object with endpoints $\delta_{Bi} \colon 1_B \to \mathbb{I}_B$ defined as $\delta_i \times B$ for $i = 0, 1$. We write $\hat{\hom}_B(m, f)$ for the Leibniz exponential from m to f computed in B (i.e. using the exponentials in $\mathcal{C}/B$). We observe that all of this structure is stable under reindexing, up to isomorphism.

We define the fibred awfs of trivial cofibrations and fibrations to be the one cofibrantly generated by the following family of maps over $\Phi \times \mathbb{I}$. We write Δ for the map $1_{\Phi \times \mathbb{I}} \to \mathbb{I}_{\Phi \times \mathbb{I}}$ whose underlying map in $\mathcal{C}$ is $\lambda(\varphi, i).(i, \varphi, i)$. Write $\mathbb{T}$ for the object of $\mathcal{C}/\Phi \times \mathbb{I}$ given by $\top \times \mathbb{I} : \Phi_{\mathsf{true}} \times \mathbb{I} \to \Phi \times \mathbb{I}$. Write T for the map $\mathbb{T} \to 1_{\Phi \times \mathbb{I}}$. The generating family of trivial cofibrations is as follows, where $+$, $\times$, $\hat{\times}$, and Cyl are all computed in $\mathcal{C}/\Phi \times \mathbb{I}$.

$$\begin{tikzcd} 1 +_{\mathbb{T}} (\mathsf{Cyl}(\Delta) \times \mathbb{T}) \arrow[rr, "L(\Delta) \hat{\times} T"] \arrow[dr] & & \mathsf{Cyl}(\Delta) \arrow[dl] \\ & \Phi \times \mathbb{I} & \end{tikzcd}$$

The existence of the awfs cofibrantly generated by this map follows from Theorem 13 if either of the necessary conditions holds.

Definition 24. We refer to maps belonging to the right hand side of the fibred awfs, or equivalently maps with the fibred right lifting property against the generating family of trivial cofibrations as *fibrations*.

We refer to maps belonging to the left hand side of the fibred awfs, or equivalently maps with the fibred left lifting property against all fibrations as *trivial cofibrations*.

Proposition 25. *Let f be a map in a slice category $\mathcal{C}/I$. Then f is a fibration if and only if for every map $\tau \colon B \to I$, every generating cofibration $m \colon A \rightarrowtail 1_B$, and every map $r \colon 1_B \to \mathbb{I}_B$ in $\mathcal{C}/B$, $L(r) \hat{\times}_B m$ has the left lifting property against $\tau^*(f)$.*

Proof. We observe that pairs consisting of a generating cofibration $m \colon A \to 1_B$ and map $r \colon 1_B \to \mathbb{I}_B$ correspond precisely to maps $\sigma \colon B \to \Phi \times \mathbb{I}$ such that $\sigma^*(T) = m$ and $\sigma^*(\Delta) = r$. Hence families of lifting problems of $L(\Delta) \hat{\times}_{\Phi \times \mathbb{I}} T$ against f indexed by B are precisely lifting problems of $L(r) \hat{\times}_B m$ against $\tau^*(f)$ in $\mathcal{C}/B$. $\square$

Lemma 26. *Let $f \colon X \to Y$ be a fibration in a slice category $\mathcal{C}/J$. Then for any generating cofibration $m \colon A \rightarrowtail B$ and map $r \colon 1_J \to \mathbb{I}_J$ in $\mathcal{C}/J$, f has the right lifting property against $L(r) \hat{\times}_B m$.*

Proof. Using the Leibniz adjunction, this is equivalent to showing that m has the left lifting property against $\hat{\hom}_J(L(r), f)$. Write $b \colon B \to J$ for the map underlying B. By Lemma 21 we can view m as $\sum_b m'$ where $m' \colon A \to 1_B$ is a generating cofibration in $\mathcal{C}/B$. Using the adjunction $\sum_b \dashv b^*$, the above is equivalent to m' having the left lifting property against $\hat{\hom}_B(L(b^*(r)), b^*(f))$. Using the Leibniz adjunction again, this is the same as $L(b^*(r)) \hat{\times}_B m'$ having the left lifting property against $b^*(f)$, which it does by Proposition 25, noticing that $b^*(r) \colon 1_B \to \mathbb{I}_B$, m' is a generating cofibration from A to 1_B, and that b^* preserves fibrations. $\square$

Lemma 27. *Let $f\colon X \to Y$ be a fibration in a slice category $\mathcal{C}/J$. Then for any generating cofibration $m\colon A \rightarrowtail B$ and map $r\colon 1_J \to \mathbb{I}_J$ in $\mathcal{C}/J$, $r \mathbin{\hat{\times}_J} m$ has the weak left lifting property against f. Furthermore, if r is a generating cofibration, then $r \mathbin{\hat{\times}_J} m$ has the strict left lifting property against f.*

Proof. By Lemma 26, we know that $L(r)$ has the left lifting property against $\hat{\hom}_J(m, f)$. We deduce that r has the weak left lifting property against $\hat{\hom}_J(m, f)$ by Lemma 4, and so $r \mathbin{\hat{\times}_J} m$ has the weak left lifting property against f by Lemma 5.

Now assume that r is a generating cofibration. We need to show that $r \mathbin{\hat{\times}_J} m$ has the left lifting property against f. By Lemma 6 and the above, it suffices to show that δ_{J0} has the weak left lifting property against $\hat{\hom}_J(r \mathbin{\hat{\times}_J} m, f)$. However, since r is generating cofibration, so is $r \mathbin{\hat{\times}_J} m$, and so this follows from the observation above that each r has the weak left lifting property against $\hat{\hom}_J(m, f)$, applied with $r := \delta_{J0}$. □

We now prove a version of Lemma 26 for weak lifting. Unfortunately, the short proof using the Leibniz adjunction that we used before doesn't quite work for the weak lifting property, so we need a little more work.

Lemma 28. *Let f be a map in a slice category $\mathcal{C}/B$. Suppose that for every map $\tau\colon C \to B$, every generating cofibration $m\colon A \rightarrowtail 1_C$ and map $r\colon 1_C \to \mathbb{I}_C$ in $\mathcal{C}/C$, r has the weak left lifting property against $\hat{\hom}_C(m, \tau^*(f))$. Then for every generating cofibration $m\colon A \to C$ in $\mathcal{C}/B$ and $r\colon 1_B \to \mathbb{I}_B$, r has the weak left lifting property against $\hat{\hom}_B(m, f)$.*

Proof. By Lemma 5, we need to show that $r \mathbin{\hat{\times}_B} m$ has the weak left lifting property against f together with the extra condition that the restriction of the upper triangle homotopy to the $\mathbb{I}_B \times_B A$ component of $C +_A \mathbb{I}_B \times_B A$ is constant. We recall from Lemma 21 that $m = \sum_\tau m'$ for some generating cofibration $m'\colon A \to 1_B$ in $\mathcal{C}/B$. The functors $r \mathbin{\hat{\times}_B} \sum_\tau -$ and $\sum_\tau(\tau^*(r) \mathbin{\hat{\times}_C} -)$ are naturally isomorphic, since both are left adjoint to $\tau^*(\hat{\hom}_B(r, -)) \cong \hat{\hom}_C(\tau^*(r), \tau^*(-))$. It follows that we have a canonical isomorphism between $\sum_\tau(\tau^*(r) \mathbin{\hat{\times}_B} m')$ and $r \mathbin{\hat{\times}_I} m$. Using the adjunction $\sum_\tau \dashv \tau^*$, we see that diagonal maps in a lifting problem of $\sum_\tau(\tau^*(r) \mathbin{\hat{\times}_B} m')$ against f correspond to the diagonal maps in a lifting problem of $\tau^*(r) \mathbin{\hat{\times}_B} m'$ against $\tau^*(f)$, which has a weak filler by assumption and Lemma 5. It is clear that the strict commutativity of the lower triangles are equivalent, but we need to check that we can transfer the homotopy witnessing commutativity of the upper triangle across the adjunction, and that it is constant on the $\mathbb{I}_B \times_B A$ component. However, this follows from the fact that $\sum_\tau$ preserves product with the interval, in the sense that there is a natural isomorphism between the functors $\sum_\tau(\mathbb{I}_C \times_C -)$ and $\mathbb{I}_B \times_B \sum_\tau -$, which is easy to check by expanding out definitions and standard arguments with pullbacks, or by noticing that both are left adjoint to the isomorphic functors $\tau^*(\hom_B(\mathbb{I}_B, -)) \cong \hom_C(\mathbb{I}_C, \tau^*(-))$. □

Lemma 29. *Let f be a map in a slice category $\mathcal{C}/I$. Then f is a fibration if and only if for every map $\tau\colon B \to I$, every generating cofibration $m\colon A \rightarrowtail 1_B$ and map $r\colon 1_B \to \mathbb{I}_B$ in $\mathcal{C}/B$, r has the weak left lifting property against $\hat{\hom}_B(m, \tau^*(f))$.*

Proof. Suppose first that f is a fibration. Let $\tau\colon B \to I$, and suppose we are given a generating cofibration $m\colon A \rightarrowtail 1_B$, and map $r\colon 1_B \to \mathbb{I}_B$. Since f is a fibration, we know that $L(r)$ has the left lifting property against $\hat{\hom}_B(m, \tau^*(f))$

by Proposition 25. By Lemma 4, we deduce that r has the weak left lifting property against $\hat{\hom}_B(m, \tau^*(f))$.

We now show the converse.

We first note that we can strengthen the hypothesis we are given as follows. We are told that given any $\tau\colon B \to I$ and any generating cofibration into the terminal object $m\colon A \rightarrowtail 1_B$ and every map $r\colon 1_B \to \mathbb{I}_B$ in $\mathcal{C}/B$, r has the weak left lifting property against $\hat{\hom}_B(m, \tau^*(f))$. We will show that in fact we can take m to be a generating cofibration with any codomain, say $m\colon A \to C$. By Lemma 28 it suffices to show that for every $\tau'\colon C \to B$ and any generating cofibration $m\colon A \to 1_C$, any map $r\colon 1_C \to \mathbb{I}_C$ has the weak left lifting property against $\hat{\hom}_C(m, \tau'^*(\tau^*(f)))$. However, this is just the hypothesis applied to the map $\tau' \circ \tau\colon C \to I$.

By Proposition 25, to show f is a fibration it suffices to show that $L(r)$ has the left lifting property against $\hat{\hom}_B(m, \tau^*(f))$ for $\tau\colon B \to I$, $r\colon 1_B \to \mathbb{I}_B$ and a generating cofibration $m\colon A \to 1_B$. We now fix such a choice of B, τ, r and m. We aim to apply Lemma 7.

We first show that δ_{B1} has the strict left lifting property against $\hat{\hom}_B(m', \tau^*(f))$ for any generating cofibration $m' : A \rightarrowtail C$ in $\mathcal{C}/B$. By the Leibniz adjunction, this amounts to showing $\delta_{B1} \mathbin{\hat{\times}_B} m'$ has the left lifting property against $\tau^*(f)$. By Lemma 5, we know $\delta_{B1} \mathbin{\hat{\times}_B} m'$ has the weak left lifting property against $\tau^*(f)$, so by Lemma 6, it suffices to show that δ_{B0} has the weak left lifting property against $\hat{\hom}_B(\delta_{B1} \mathbin{\hat{\times}_B} m', \tau^*(f))$. But $\delta_{B1} \mathbin{\hat{\times}_B} m'$ is a generating cofibration by Proposition 20, so it does by the strengthened version of the hypothesis.

To apply Lemma 7, we must check that δ_{B1} and $\delta_{B1} \mathbin{\hat{\times}_B} [\delta_{B0}, \delta_{B1}]$ have the left lifting property against $\hat{\hom}_B(m, \tau^*(f))$. The first of these follows directly from the claim above. For the second, it suffices by the Leibniz adjunction to show that δ_{B1} has the left lifting property against $\hat{\hom}_B([\delta_{B0}, \delta_{B1}] \mathbin{\hat{\times}_B} m, \tau^*(f))$, which again follows from the claim. □

Theorem 30. *Let f be a map in $\mathcal{C}$. Then f is a fibration if and only if it possesses* wcom *and* $\underline{\mathsf{wcom}}$ *operators.*

Proof. We saw in Lemma 29 that f is a fibration if and only if it has the weak right lifting property against $r \mathbin{\hat{\times}_B} m$ for any object B, map $r\colon 1_B \to \mathbb{I}_B$ and generating cofibration $m\colon A \to 1_B$ such that homotopy witnessing commutativity of the upper triangle is constant on the $\mathbb{I} \times_B A$ component of $1_B +_A \mathbb{I} \times_B A$. However, this is true for all B, r and m if and only if it is true for the universal lifting problem of $\Delta \mathbin{\hat{\times}_{\Phi \times \mathbb{I}}} T$ against f. Expanding out the explicit description of universal lifting problem, we see that this is equivalent to the existence of wcom and $\underline{\mathsf{wcom}}$ operators. □

1.5. **The Model Structure.** We will use the following theorem due to Sattler to generate the model structure.

Theorem 31 (Sattler)**.** *Suppose that we are given weak factorisation systems (C, F^t) and (C^t, F) where $C^t \subseteq C$. Then these form part of a model structure if the following conditions are satisfied.*

1. *The span property holds.*
2. *Trivial fibrations satisfy 2-out-of-3 relative to fibrations.*
3. *Fibrations and trivial fibrations extend along trivial cofibration.*
4. *The wfs (C^t, F) satisfies the Frobenius property.*

Proof. See [Sat17, Theorem 2.8]. □

Lemma 32. *For any object I in $\mathcal{C}$, the functor $- \hat{\times}_I [\delta_{I0}, \delta_{I1}]$ preserves cofibrations and the functor $\hat{\hom}_I([\delta_{I0}, \delta_{I1}], -)$ preserves trivial fibrations. Let f be a fibration in $\mathcal{C}/I$. Then,*

(1) $\hat{\hom}_I(\delta_{Ii}, f)$ *is a trivial fibration for $i = 0, 1$.*
(2) $\hat{\hom}_I([\delta_{I0}, \delta_{I1}], f)$ *is a fibration.*

Proof. To show $m \hat{\times}_I [\delta_{I0}, \delta_{I1}]$ is a cofibration for every cofibration m is equivalent, by the Leibniz adjunction, to showing $\hat{\hom}_I([\delta_{I0}, \delta_{I1}], f)$ is a trivial fibration for every trivial fibration f. Applying the Leibniz adjunction again, it suffices to show that $m \hat{\times}_I [\delta_{I0}, \delta_{I1}]$ is a generating cofibration for all generating cofibrations m, which follows from Proposition 20.

Now let f be a fibration in $\mathcal{C}/I$. To show $\hat{\hom}_I(\delta_{Ii}, f)$ is a trivial fibration is equivalent to showing that $\delta_{Ii} \hat{\times}_I m$ has the left lifting property against f for all generating cofibrations $m\colon A \to B$ in $\mathcal{C}/I$. This follows from Lemma 27 as δ_{Ii} is a generating cofibration.

We next show $\hat{\hom}_I([\delta_{I0}, \delta_{I1}], f)$ is a fibration whenever f is. By Proposition 25 we need to show for all $\tau\colon B \to I$, $r\colon 1_B \to \mathbb{I}_B$ and generating cofibrations $m\colon A \to 1_B$ in $\mathcal{C}/B$, that $L(r) \hat{\times}_B m$ has the left lifting property against $\tau^*(\hat{\hom}_I([\delta_{I0}, \delta_{I1}], f))$. By the isomorphism $\tau^*(\hat{\hom}_I([\delta_{I0}, \delta_{I1}], f) \cong \hat{\hom}_B([\delta_{B0}, \delta_{B1}], \tau^*(f))$, this is equivalent to showing that $L(r) \hat{\times}_B m \hat{\times}_B [\delta_{B0}, \delta_{B1}]$ has the left lifting property against $\tau^*(f)$. But $m \hat{\times}_B [\delta_{B0}, \delta_{B1}]$ is a generating cofibration by Proposition 20, and $\tau^*(f)$ is a fibration in $\mathcal{C}/B$, so this follows from Lemma 26. □

Lemma 33. *The span property holds for (C, F^t) and (C^t, F) in the sense defined by Sattler in [Sat17, Definition 2.2].*

That is, in any diagram as below,

$$\begin{tikzcd} & A \arrow[dl, "l"'] \arrow[dr, "m"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

if l and m are trivial cofibrations, and f is a fibration, then f is a trivial fibration.

Proof. Sattler's proof of [Sat17, Corollary 4.9] holds here more or less as stated, but for completeness, we write out the details.

We will first show that f is a strong codeformation retract. That is, we will find $g\colon Y \to X$ such that $f \circ g = 1_Y$ together with a homotopy $h\colon g \circ f \sim 1_X$ such that $f \circ h$ is a constant homotopy.

First, using the left lifting property of m against f, we can construct g such that $f \circ g = 1_Y$ and $g \circ m = l$. We next construct the homotopy $h\colon g \circ f \sim 1_X$. For convenience, we will view h as a map $X \to \mathsf{Path}(f)$ (where $\mathsf{Path}(f)$ is defined to be the pullback $Y \times_{Y^{\mathbb{I}}} X^{\mathbb{I}}$). We consider the lifting problem given by the solid lines below.

$$\begin{tikzcd} A \arrow[r, "l"] \arrow[d, "l"] & X \arrow[r] & \mathsf{Path}(f) \arrow[d] \\ X \arrow[rr, "\langle g\circ f, 1_X\rangle"'] \arrow[urr, dotted, "h"] & & X \times_Y X \end{tikzcd}$$

Note that the map $\mathsf{Path}(f) \to X \times_Y X$ (given by endpoint projections) is a fibration, since it is a pullback of $\hat{\hom}([\delta_0, \delta_1], f)$ along the canonical map $X \times_Y X \to Y^{\mathbb{I}} \times_{Y^2}$

X^2, which is a fibration by Lemma 32. Hence there is a diagonal filler h, as in the dotted line above.

It follows that we can exhibit f as a retract of the endpoint projection map $e_0 \colon \mathsf{Path}(f) \to X$ as illustrated below.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"] & \mathsf{Path}(f) \arrow[r, "e_1"] \arrow[d, "e_0"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Now note that each endpoint projection $e_i \colon \mathsf{Path}(f) \to X$ is a trivial fibration, since it is a pullback of the map $\hat{\hom}(\delta_i, f)$, which is a trivial fibration by Lemma 32. Since we have shown f is a retract of $e_0 \colon \mathsf{Path}(f) \to X$, it is also a trivial fibration, as required. $\square$

Lemma 34. *Trivial fibrations satisfy 2-out-of-3 amongst fibrations.*

Proof. Again Sattler's proof [Sat17, Lemma 4.5] holds as stated, but we will write out the details for completeness.

Suppose we are given fibrations f, g and h as below.

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

First suppose that f and g are trivial fibrations. Since the class of right maps in a wfs is closed under composition, h is also a trivial fibration.

Next suppose that f and h are trivial fibrations. Note that f has a section $s \colon Y \to X$, since it has the right lifting property against the cofibration $0 \to Y$. However, we can now show that g is a trivial fibration, by exhibiting it as a retract of the trivial fibration h, as below.

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[d, "g"] & X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "g"] \\ Z \arrow[r, "1_Z"] & Z \arrow[r, "1_Z"] & Z \end{tikzcd}$$

Finally we suppose that h and g are trivial fibrations and show that f is a trivial fibration. Note that f can be written as a retract of the map $\langle h^{\mathbb{I}}, f \circ e_0 \rangle \colon X^{\mathbb{I}} \to Z^{\mathbb{I}} \times_Z Y$. However, we can decompose $\langle h^{\mathbb{I}}, f \circ e_0 \rangle$ as three trivial fibrations.

$$X^{\mathbb{I}} \longrightarrow Y^{\mathbb{I}} \times_Y X \longrightarrow (Z^{\mathbb{I}} \times_Z Y) \times_Z X \longrightarrow Z^{\mathbb{I}} \times_Z Y$$

The first is $\hat{\hom}(\delta_1, f)$, which is a trivial fibration by Lemma 32 together with the assumption that f is a fibration. The second map, whose codomain is the pullback of $e_1 \circ \pi_0$ and h, is given by pulling back $\hat{\hom}([\delta_0, \delta_1], g)$ along the map $\langle \pi_0 \circ \pi_0, \langle \pi_1 \circ \pi_0, f \circ \pi_1 \rangle \rangle \colon (Z^{\mathbb{I}} \times_Z Y) \times_Z X \to Z^{\mathbb{I}} \times_{Z^2} Y^2$. This is a trivial fibration, since $\hat{\hom}([\delta_0, \delta_1], g)$ is by Lemma 32 together with the assumption that g is a trivial fibration. The final map is given by pulling back h, which is a trivial fibration by assumption. We have shown f is a retract of a trivial fibration, and so it is a trivial fibration. $\square$

We can now prove the main theorem.

Theorem 35. *Let $\mathcal{C}$ be a locally cartesian closed category with finite colimits and disjoint coproducts, and let $\top\colon \Phi_{\mathsf{true}} \rightarrowtail \Phi$ be a monomorphism such that pullbacks of $\top$ include the maps $0 \to 1$ and $1 \to 1$ and are closed under binary union.*

Suppose further that every fibration is U-small for some universe of small fibrations where the underlying object U is fibrant, and that $\mathcal{C}$ satisfies one of the conditions required to apply theorem 13.

Let $\delta_0, \delta_1\colon 1 \to \mathbb{I}$ be an interval object with δ_i pullbacks of $\top$ for $i = 0, 1$. Let (C, F^t) be the awfs defined in section 1.3 and let (C^t, F) be the wfs defined in section 1.4 (restricted to $\mathcal{C}/1$). Then C and F form the cofibrations and fibrations of a (uniquely determined) model structure on $\mathcal{C}$.

Proof. We check the conditions required to apply Theorem 31.

We first check that we do in fact have $C^t \subseteq C$. It suffices to show that the generating trivial cofibration $L(\Delta)\hat{\times}T$ is a generating cofibration as a map in the slice category $\mathcal{C}/\Phi \times \mathbb{I}$, since this implies that every trivial fibration is a fibration, which then also implies every trivial cofibration is a cofibration. However, $L(\Delta)$ is a pullback of δ_1 by Lemma 3, and so is a generating cofibration. Then $L(\Delta)\hat{\times}T$ is a binary union of generating cofibrations, and so a generating cofibration, as required.

We checked the span property in Lemma 33 and that trivial fibrations satisfy 2-out-of-3 amongst fibrations in Lemma 34.

To show trivial fibrations extend along trivial cofibrations, we note that the awfs of cofibrations and trivial fibrations is strongly fibred by lemma 17, and so cofibrations are stable under pullback by proposition 16. Furthermore, every cofibration is a monomorphism by lemma 14. Hence the proof used by Sattler for [Sat17, Lemma 3.9] ("Joyal's trick") holds here as stated, and in fact shows that trivial fibrations extend along all cofibrations.

We can deduce that fibrations extend along trivial cofibrations using that every fibration is U-small for some fibrant universe U, as Sattler explains in [Sat17, Remark 7.6].

Finally, to show that (C^t, F) is Frobenius, recall that this is equivalent to showing that dependent products preserve fibrations (see e.g. [GS17]). However, we have already shown that our definition of fibration is preserved by dependent product. □

Note that the theorem has the following special case.

Corollary 36. *Suppose that every set belongs to an inaccessible set and that $\mathcal{C}$ is a category of presheaves. Suppose that C is a class of locally decidable monomorphisms containing all isomorphisms and closed under pullbacks, composition, finite unions and colimits. Let $\delta_0, \delta_1\colon 1 \to \mathbb{I}$ be disjoint elements of C, and suppose that every fibration is U-small for some universe of small fibrations where the underlying object U is fibrant. Then there is a model structure on $\mathcal{C}$ where the class of cofibrations is equal to C and the class of fibrations is as defined in section 1.4.*

Proof. See [GS17, Lemma 9.7] for the construction of Φ [1]. The rest is a straightforward application of Theorem 35. □

1.6. **Characterisation of the Model Structure.** In this section we aim to give a natural characterisation of the model structure that we defined before. We will use the notion of Hurewicz fibration, below.

[1]When Φ is constructed in this way, we additionally have proposition extensionality and closure of generating cofibrations under composition, that we do not assume in general.

Proposition 37. *Let $\delta_0, \delta_1 \colon 1 \to \mathbb{I}$ be an interval object, and let $f \colon X \to Y$. Then the following are equivalent.*

(1) f has the fibred right lifting property against δ_0 where both are viewed as maps in $\mathcal{C}/1$.

(2) f has the enriched right lifting property against δ_0 when $\mathcal{C}$ is viewed as a category enriched over itself with cartesian monoidal product.

(3) f has the right lifting property against $\delta_0 \times B$ for all objects B.

Definition 38. If f satisfies one (and so all) of the equivalent conditions in Proposition 37 we say it is an $\mathbb{I}$-*Hurewicz fibration*.

We aim towards Theorem 41, which states that our model structure is the one with the largest class of fibrations such that every fibration is a Hurewicz fibration, or equivalently, the one with the smallest class of trivial cofibrations containing δ_{A0} for each object A.

Throughout we fix a model structure (C, W, F) on $\mathcal{C}$ such that C is as given, and every element of F is an $\mathbb{I}$-Hurewicz fibration. Define F^t to be $F \cap W$ and C^t to be $C \cap W$. We observe that every element of F is an $\mathbb{I}$-Hurewicz fibration if and only if $\langle \delta_0, 1_B \rangle \colon B \to \mathbb{I} \times B$ is a trivial cofibration for all B.

Lemma 39. *Let B be any object, and let $r \colon B \to \mathbb{I}$. Then $\langle r, 1_B \rangle \colon B \to \mathbb{I} \times B$ is a weak equivalence.*

Proof. Using that $\langle \delta_0, 1_B \rangle$ is a trivial cofibration and applying 3-for-2 twice in the diagram below.

$$\begin{tikzcd}
B \arrow[r, "{\langle r,1_B\rangle}"] \arrow[dr, "1_B"'] & \mathbb{I}\times B \arrow[d, "\pi_1"] & B \arrow[l, "{\langle\delta_0,1_B\rangle}"'] \arrow[dl, "1_B"] \\
 & B &
\end{tikzcd}$$

□

We observe in passing that it follows from Lemma 39 that $\langle \delta_1, 1_B \rangle \colon B \to \mathbb{I} \times B$ is a trivial cofibration for all B.

Lemma 40. *Let B be any object, and let $r \colon B \to \mathbb{I}$. Then $L_B(r) \colon B \to \mathsf{Cyl}_B(r)$ is a trivial cofibration.*

Proof. Expanding out the definition of mapping cylinder in the slice category $\mathcal{C}/B$ we get the following diagram.

$$\begin{tikzcd}
 & & B \arrow[dl, "{\langle\delta_1,1_B\rangle}"'] \arrow[ddl, bend left, "L_B(r)"] \\
B \arrow[r, "{\langle\delta_0,1_B\rangle}"] \arrow[d, "{\langle r,1_B\rangle}"] & \mathbb{I}\times B \arrow[d, "\iota_1"] & \\
\mathbb{I}\times B \arrow[r, "\iota_0"] & \mathsf{Cyl}_B(r) &
\end{tikzcd}$$

Then ι_0 is a pushout of a trivial cofibration, and so a trivial cofibration. By Lemma 39, $\langle r, 1_B \rangle$ is a weak equivalence. Hence ι_1 is also a weak equivalence. $\langle \delta_1, 1_B \rangle$ is a trivial cofibration, and so by 3-for-2 again, $L_B(r)$ is a weak equivalence. Since it is a pullback of δ_1, it is a cofibration and therefore a trivial cofibration, as required. □

We now state and prove the main theorem of this section.

Theorem 41. *Let $\mathcal{C}$ be a locally cartesian closed category with finite colimits and disjoint coproducts, and let $\top \colon \Phi_{\mathsf{true}} \rightarrowtail \Phi$ be a monomorphism such that pullbacks of $\top$ include the maps $0 \to 1$ and $1 \to 1$ and are closed under binary union. Let F^t be the class of maps with the fibred right lifting property against every pullback of $\top$.*

Suppose further that every map is U-small for some universe of small fibrations where the underlying object U is fibrant, and that $\mathcal{C}$ satisfies one of the conditions required to apply theorem 13.

Let $\delta_0, \delta_1 \colon 1 \to \mathbb{I}$ be an interval object with δ_i pullbacks of $\top$ for $i = 0, 1$.

Then the class of fibrations F defined in section 1.4 (restricted to $\mathcal{C}/1$) is the largest class of maps subject to the following conditions.

1. *Every element of F is an $\mathbb{I}$-Hurewicz fibration.*
2. *F^t and F form the trivial fibrations and fibrations of a (uniquely determined) model structure.*

Proof. We need to show that for every object B, every cofibration $A \to B$, and every map $r \colon B \to \mathbb{I}$, $\sum_B (L_B(r) \hat{\times}_B m)$ belongs to C^t. Unfolding the definition of pushout product in $\mathcal{C}/B$ and simplifying, we have the diagram below. In particular we note that the left hand map is the pullback of $L_B(m)$ along $m \colon A \to B$, and so since mapping cylinder is stable under reindexing it is equal to $L_A(r \circ m)$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "L_A(r\circ m)"] & B \arrow[d] \arrow[ddr, bend left, "L_B(r)"] & \\
\mathsf{Cyl}_A(r \circ m) \arrow[r] \arrow[drr, bend right] & \cdot \arrow[dr, "r \hat{\times}_B m"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & \mathsf{Cyl}_B(r)
\end{tikzcd}$$

The leftmost and rightmost maps are both trivial cofibrations by Lemma 40. It follows that $r \hat{\times}_B m$ is a weak equivalence. Since it is a binary union of cofibrations, it is therefore a trivial cofibration, as required. □

Note that the theorem has the following special case.

Corollary 42. *Suppose that every set belongs to an inaccessible set and that $\mathcal{C}$ is a category of presheaves. Suppose that C is a class of locally decidable monomorphisms containing all isomorphisms and closed under pullbacks, composition, finite unions and colimits. Let $\delta_0, \delta_1 \colon 1 \to \mathbb{I}$ be disjoint elements of C, and that every fibration is U-small for some universe of small fibrations where the underlying object U is fibrant. Then there is a (unique) model structure on $\mathcal{C}$ where the class of cofibrations is equal to C and the class of fibrations is the largest class F forming part of a model structure such that every element of F is an $\mathbb{I}$-Hurewicz fibration.*

We make a few final remarks about the model structure. This definition is related to a class of model structures developed by Cisinski [Cis02]. In some ways our construction is less general. In particular the class of trivial cofibrations is as small as possible, and Sattler has shown that in some of the variants of cubical sets, this leads to model structures that are not Quillen equivalent to topological spaces. We also required that the functorial cylinder is given by cartesian product with an interval, rather than the usual, more general definition. On the other hand it is also important to point out that in a few ways our construction is more general. Following Gambino and Sattler [GS17; Sat17], our results are valid within a constructive metatheory, and we are not restricted in taking all monomorphisms to be cofibrations, but can instead take the class of cofibrations to be any class of monomorphisms satisfying suitable conditions. We have improved further on Sattler's result by eliminating the requirement that the interval object has connections. We have also eliminated any accessibility requirements, including the condition that we are given a cocomplete category. The additional requirements of completeness and cocompleteness are sometimes included in the definition of model structure, so we note that many examples of toposes are complete and cocomplete, including all Grothendieck toposes.

We also remark that one obtains not just a model structure, but an algebraic model structure in the sense of Riehl [Rie11]. Finally, note also that the algebraic model structure is fibred over the codomain fibration, in the sense that we have an algebraic model structure on each slice category, that both awfs's are preserved by pullback, and that for each $\sigma\colon J \to K$, the adjunctions $\sum_\sigma \dashv \sigma^*$ and $\sigma^* \dashv \prod_\sigma$ are both Quillen adjunctions.

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