Unifying Cubical Models of Univalent Type Theory

Evan Cavallo

School of Computer Science, Carnegie Mellon University, USA
ecavallo@cs.cmu.edu

Anders Mörtberg

School of Computer Science, Carnegie Mellon University, USA
Department of Mathematics, Stockholm University, Sweden
anders.mortberg@math.su.se

Andrew W Swan

Institute for Logic, Language and Computation, University of Amsterdam, Netherlands
a.w.swan@uva.nl

Abstract

We present a new constructive model of univalent type theory based on cubical sets. Unlike prior work on cubical models, ours depends neither on diagonal cofibrations nor connections. This is made possible by weakening the notion of fibration from the cartesian cubical set model, so that it is not necessary to assume that the diagonal on the interval is a cofibration. We have formally verified in `Agda` that these fibrations are closed under the type formers of cubical type theory and that the model satisfies the univalence axiom. By applying the construction in the presence of diagonal cofibrations or connections and reversals, we recover the existing cartesian and De Morgan cubical set models as special cases. Generalizing earlier work of Sattler for cubical sets with connections, we also obtain a Quillen model structure.

2012 ACM Subject Classification Theory of computation → Constructive mathematics; Theory of computation → Type theory

Keywords and phrases Cubical Set Models, Cubical Type Theory, Homotopy Type Theory, Univalent Foundations.

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Funding *Evan Cavallo*: This author was supported by the Air Force Office of Scientific Research through MURI grant FA9550-15-1-0053. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the AFOSR.

Acknowledgements We are grateful to Carlo Angiuli, Thierry Coquand, Kuen-Bang Hou (Favonia), Robert Harper, Daniel R. Licata, Andrew Pitts and Jon Sterling for helpful comments and remarks on earlier versions of this work. The first two authors are also thankful to Mathieu Anel and Steve Awodey for their illuminating lectures on homotopical algebra in the CMU HoTT seminar.

1 Introduction

Cubical set models provide a constructive justification for Voevodsky's univalence axiom and higher inductive types, as introduced in Homotopy Type Theory and Univalent Foundations (HoTT/UF) [38]. In this paper we develop a general axiomatization encompassing many existing cubical set models, allowing us to better understand the relationship between them and prove results about the entire class of models simultaneously.

The first model of HoTT/UF was developed by Voevodsky using Kan simplicial sets [26] and relies crucially on classical logic [9]. A major source of open problems in HoTT/UF has been the quest for constructive models; besides recent progress on a constructive variation of the Kan simplicial set model [23], the most fruitful approaches have been based on cubical sets. This was pioneered by the Bezem, Coquand and Huber (BCH) model [7, 8], which uses presheaves on the *symmetric monoidal cube category.* These cubical sets have degeneracy and face maps, but it is not possible to take the diagonal face of a square. An important feature of cubical sets, relative to simplicial sets, is that the product of representable cubical sets is again representable. This makes it possible to represent n-dimensional terms as ordinary terms in a context of n variables, each ranging over the interval object $\mathbb{I}$. The lack of diagonals in the BCH model corresponds to a lack of contraction for these contexts; the BCH model is *substructural.* This complicates giving a type-theoretic presentation; more fundamentally, it is unclear how to formulate and construct higher inductive types.

A natural approach, then, is to instead allow diagonals and study *cartesian cubical sets*, which model structural interval contexts. The base category here has a compact description as the free finite product category on an interval object [4, 29]. Cartesian cubical sets are hence better-suited as a basis for cubical type theory, and they are known to support higher inductive types. However, constructing univalent universes was an open problem for many years. The difficulties in modeling univalent universes motivated Cohen, Coquand, Huber and Mörtberg (CCHM) [15] to consider a cube category with even more structure, namely connections ($\wedge$ and $\vee$) and an involutive reversal operation ($\neg$) satisfying the axioms of a De Morgan algebra. Using these additional operations, they gave the first cubical set model of univalent type theory with higher inductive types, as well as the first cubical type theory. It was later observed by Orton and Pitts (OP) [28] that the CCHM constructions do not require the full structure of a De Morgan algebra; a so-called "connection algebra" suffices. As a special case, there is a cubical category where the connection algebra is the free bounded distributive lattice. We call the resulting presheaf category *Dedekind cubical sets*, following Awodey, as the number of elements of $\mathrm{Hom}(\mathbb{I}^n, \mathbb{I})$ are the Dedekind numbers [5]. Angiuli, Favonia, and Harper (AFH) [3] showed that that a model of HoTT/UF could also be developed in cartesian cubical sets *without* connections or reversals; their computational model was then adapted to an Orton-Pitts style construction by Angiuli et al. (ABCFHL) [2].

In short, a wide variety of cube categories give rise to models of univalent type theory. Moreover, the underlying cube category is not the only parameter: one must also formulate *Kan composition*, i.e., choose a class of *fibrations.* Kan composition, a cubical analogue of the lifting condition in Kan simplicial sets, ensures that Path types induce a notion of equality. A representative special case of composition is *coercion.* Given a type A that depends on a dimension variable $i : \mathbb{I}$, coercion establishes a relationship between the elements of $A(r/i)$ and $A(s/i)$ for various $r, s : \mathbb{I}$. The nature of this relationship varies from model to model. In CCHM, the simplest case, coercion provides a map $\mathsf{coe}^{0 \rightarrow 1}_{i.A} : A(0/i) \to A(1/i)$. In AFH, on the other hand, there is an operation $\mathsf{coe}^{r \rightarrow s}_{i.A} : A(r/i) \to A(s/i)$ for *every* $r, s : \mathbb{I}$, together with an equation $\mathsf{coe}^{r \rightarrow r}_{i.A}\ a = a : A(r/i)$. Other model constructions use intermediate points between these two extremes. For example, OP include $0 \to 1$ and $1 \to 0$. A more expressive cube category can compensate for a more limited form of coercion; in CCHM, coercions $\varepsilon \to s$ and $r \to \varepsilon$ for $\varepsilon : \{0, 1\}$ are derivable from the primitive $0 \to 1$ coercion.

In its general form, Kan composition coerces a cube while preserving some part of its boundary, a generalization necessary in order to derive coercion for Path types. The choice of allowable boundary shapes is a third parameter; from the model categorical perspective, it corresponds to a choice of *generating cofibrations.* In CCHM cubical sets, a boundary is specified by a collection of (conjunctions of) faces of the form $(r = 0)$ or $(r = 1)$. For cartesian cubes, AFH took the crucial step of also including $(r = s)$ boundary constraints, corresponding to diagonal faces of cubes. Model categorically, this corresponds to including the diagonal on the interval as a generating cofibration, i.e. to assume *diagonal cofibrations.*

	Diagonals	Additional structure	Kan operations	Diagonal cofibrations
BCH			$0 \to r$, $1 \to r$	
CCHM	✓	$\wedge$, $\vee$, $\neg$ (De Morgan)	$0 \to 1$	
Dedekind	✓	$\wedge$, $\vee$ (distributive lattice)	$0 \to 1$, $1 \to 0$	
OP	✓	$\wedge$, $\vee$ (connection algebra)	$0 \to 1$, $1 \to 0$	
AFH/ABCFHL	✓		$r \to s$	✓

Table 1 Varieties of cubical models of HoTT/UF.

We collect the existing cubical set models in Table 1. As a general rule, these constructions can still be conducted in a setting with additional structure. For example, both the CCHM and ABCFHL model constructions can both be carried out in cubical sets with connections, reversals, and diagonal cofibrations. (The exception is BCH, which apparently relies crucially on the *absence* of diagonal maps.) The constructions produce the same notions of fibration where they are mutually applicable, as is observed for the CCHM and ABCFHL models in [2, Sec. 3.4]. What is lacking, however, is a *single* construction that applies in all cases.

Contributions

Our main contribution is a unification of the structural cubical models (i.e., all but BCH) as instances of a single construction. This is achieved by axiomatizing a class of models in the internal language style of Orton and Pitts [28], based on a "weak" variation of cartesian Kan composition. This notion of fibration specializes to the AFH definition in the presence of diagonal cofibrations (Section 2.3.1) and to the CCHM definition in the presence of connections and reversals (Section 2.3.2). The "weak" fibrations are closed under basic type formers (Section 2.4), `Glue` types (Section 2.5), and fibrant univalent universes (Section 2.6), thus give rise to a model of HoTT/UF. Furthermore, we obtain algebraic weak factorization systems of *cofibrations and trivial fibrations* (Section 3.2) and of *trivial cofibrations and fibrations* (Section 3.3). Finally, we verify that a theorem of Sattler [32, Thm. 2.8] applies, allowing us to obtain a model structure (Section 3.4) from the factorization systems.

2 A general axiomatization

Following Orton and Pitts [28], we construct models of cubical type theory from locally cartesian closed categories $\mathcal{C}$: we describe a collection of axioms in the internal language of such categories, then use the language as a tool to show that any category satisfying the axioms induces a class of fibrations closed under various type formers. Rather than relying on an impredicative universe of propositions, as Orton and Pitts do, we follow Licata, Orton, Pitts and Spitters (LOPS) [27] and work in a predicative theory. We use `Agda` [1] extended with postulates for function extensionality and uniqueness of identity proofs to simulate the internal type theory of a locally cartesian closed category.[1]

We adopt `Agda`'s (ultimately Nuprl's) syntax here, writing $(x : A) \to B\ x$ for dependent and $A \to B$ for non-dependent functions. We assume a non-cumulative hierarchy of universes $\mathcal{U}_0 : \mathcal{U}_1 : \ldots$; here, we leave levels implicit and write $\mathcal{U}$ for simplicity, but they are explicit

[1] The formalization and additional material can be found at `https://github.com/mortberg/gen-cart`. For a summary of where all of the results in the paper can be found, see `https://github.com/mortberg/gen-cart/blob/master/agda/unifying-summary.agda`.

in the formalization. Among `Agda`'s inductive types, we need identity types (written $u = v$ and with a single constructor `refl`), an empty type $\bot : \mathcal{U}$, and sum types $A \uplus B$ (with constructors inl and inr). We write $\Sigma(x : A), B\ x$ for dependent and $A \times B$ for non-dependent product types. Following HoTT/UF, we define the type of (homotopy) propositions as $\mathsf{hProp} \triangleq \Sigma(A : \mathcal{U}), (x\, y : A) \to x = y$. We assume a propositional truncation operation $\|-\| : \mathcal{U} \to \mathsf{hProp}$ universally approximating any type as an hProp. We then define disjunction $P \vee Q$ of propositions P and Q as the propositional truncation $\|P \uplus Q\|$. The negation of a type $\neg A$ is defined as $A \to \bot$; this is always a proposition.

This type theory can be interpreted in any presheaf topos [25], in particular the various cubical and simplicial set categories, assuming enough Grothendieck universes. The standard example throughout the paper is the category of cartesian cubical sets.

2.1 The interval and Path types

The axiomatic requirements on $\mathcal{C}$ begin with an interval type $\mathbb{I} : \mathcal{U}$ with endpoints $\mathtt{0} : \mathbb{I}$ and $\mathtt{1} : \mathbb{I}$. We require $\mathbb{I}$ to be connected ($\mathbf{ax}_1$) and $\mathtt{0}, \mathtt{1}$ to be distinct ($\mathbf{ax}_2$).

$$\mathbf{ax}_1 : (P : \mathbb{I} \to \mathcal{U}) \to ((i : \mathbb{I}) \to P\ i \uplus \neg(P\ i)) \to ((i : \mathbb{I}) \to P\ i) \uplus ((i : \mathbb{I}) \to \neg(P\ i))$$
$$\mathbf{ax}_2 : \neg(\mathtt{0} = \mathtt{1})$$

Given $A : \mathbb{I} \to \mathcal{U}$, we define the type of *paths* in A as $\mathsf{Path}(A) \triangleq (i : \mathbb{I}) \to A\ i$. Given $a : A\ 0$ and $b : A\ 1$, we write $a \sim b \triangleq \Sigma(p : \mathsf{Path}(A)), (p\ 0 = a) \times (p\ 1 = b)$. Given $p : a \sim b$ and $r : \mathbb{I}$, we write $p\ @\ r$ for the application of $\mathsf{fst}\, p$ to r, which satisfies $p\ @\ 0 = a$ and $p\ @\ 1 = b$.

2.2 Cofibrant propositions

Next, we assume a universe à la Tarski of generating cofibrant propositions $\Phi : \mathcal{U}$ supporting the following operations. We write $[\_] : \Phi \to \mathsf{hProp}$ for the decoding function and stipulate that it interprets the code constructors appropriately.

$$(\_ \approx 0) : \mathbb{I} \to \Phi \qquad \mathbf{ax}_3 : (i : \mathbb{I}) \to [\,(i \approx 0)\,] = (i = 0)$$
$$(\_ \approx 1) : \mathbb{I} \to \Phi \qquad \mathbf{ax}_4 : (i : \mathbb{I}) \to [\,(i \approx 1)\,] = (i = 1)$$
$$\vee : \Phi \to \Phi \to \Phi \qquad \mathbf{ax}_5 : (\varphi\, \psi : \Phi) \to [\,\varphi \vee \psi\,] = [\,\varphi\,] \vee [\,\psi\,]$$

Note that we have two bottom elements, $(0 \approx 1)$ and $(1 \approx 0)$. The decoding of these imply each other, but we need not assume they are equal. The same holds for the two top elements $(0 \approx 0)$ and $(1 \approx 1)$. Note that for all $A : \mathcal{U}$, we have $\mathsf{elim}_\bot : [\,(0 \approx 1)\,] \to A$ by $\mathbf{ax}_2$.

▶ Remark 1. If $\mathcal{C}$ is a topos, we can take Φ to be the subobject classifier Ω. To obtain a constructive presheaf model, we can instead take Φ to be the subobject of Ω of sieves with decidable image at each stage. However, the axiomatization of Φ does not presume the existence of a subobject classifier; nor does it require that inter-derivable cofibrations are equal. This is similar to the approach taken in [2, 27], where $\Phi \triangleq \Sigma(A : \mathcal{U}), \mathsf{cof}\ A$ is specified by a predicate $\mathsf{cof} : \mathcal{U} \to \mathcal{U}$ on types. However, our variation requires that Φ is a *small* type, which is needed to construct identity types while preserving universe level.

A *partial element* of A is a term $f : [\,\varphi\,] \to A$. Given such a partial element f and an element $x : A$, we define the *extension* relation $f \nearrow x \triangleq (u : [\,\varphi\,]) \to f\ u = x$, so that $f \nearrow x$ is the type of proofs that the partial element f extends to the total element x. Following [15], we write $A[\varphi \mapsto f] \triangleq \Sigma(x : A), f \nearrow x$ for the type of all elements of A extending f. Given a partial path $f : [\,\varphi\,] \to \mathsf{Path}(A)$ and $r : \mathbb{I}$, we write $f \cdot r \triangleq \lambda u.f\ u\ r : [\,\varphi\,] \to A\ r$.

This completes the basic set of axioms, which will suffice to interpret the Σ-, Π-, Path types and basic datatypes. We defer the introduction of two final axioms to Section 2.5, where we will need them to interpret (strict) Glue types.

2.3 Fibration structures

Using the interval and the universe of cofibrant propositions, we can now define our notion of fibration structure, a weaker variation on the fibration structures used in [2, 3].

▶ **Definition 2** (Weak composition). *Given $r : \mathbb{I}$, $A : \mathbb{I} \to \mathcal{U}$, $\varphi : \Phi$, $f : [\varphi] \to \mathsf{Path}(A)$ and $x_0 : (A\ r)[\varphi \mapsto f \cdot i]$, a weak composition structure is given by two operations*

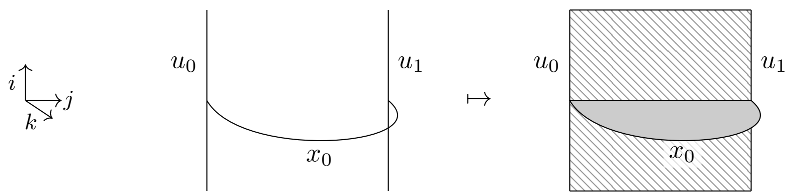
$$\mathsf{wcom} : (s : \mathbb{I}) \to (A\ s)[\varphi \mapsto f \cdot s] \qquad\qquad \underline{\mathsf{wcom}} : \mathsf{fst}\,(\mathsf{wcom}\ r) \sim \mathsf{fst}\,x_0$$

satisfying $(i : \mathbb{I}) \to f \cdot r \nearrow \underline{\mathsf{wcom}} \mathbin{@} i$. We write $\mathsf{WComp}\ r\ A\ \varphi\ f\ x_0$ for the type of such weak composition structures, i.e.,

$$\mathsf{WComp}\ r\ A\ \varphi\ f\ x_0 \triangleq \Sigma\ (\mathsf{wcom} : \ldots), \Sigma(\underline{\mathsf{wcom}} : \ldots), (i : \mathbb{I}) \to f \cdot r \nearrow \underline{\mathsf{wcom}} \mathbin{@} i$$

In contrast with [2, 3], we do not require that the equality $\mathsf{wcom}\ r\ A\ \varphi\ f\ x_0\ r = x_0$ holds strictly. Instead, the $\underline{\mathsf{wcom}}$ operation enforces the equation up to a path constant on φ. We say that $\mathsf{wcom}\ r\ A\ \varphi\ f\ x_0\ s$ composes $r \to s$ in A, and refer to f as the *tube* and x_0 as the *cap* of the composition. We refer to $\underline{\mathsf{wcom}}$ as the "cap path", as it relates $\mathsf{wcom}\ r\ A\ \varphi\ f\ x_0\ r$ to the cap x_0.

▶ **Example 3.** We can illustrate the above choice of terminology with the following example. The composition problem is given by the tube u_0 and u_1 at $(j \approx 0)$ and $(j \approx 1)$ together with a cap x_0 at $(i \approx r)$. The composition from r to i is the interior of the square on the right, while the cap path is the gray path connecting the composition at r to x_0.



▶ **Definition 4** (Weak fibrations and fibration structures). *A weak fibration (A, α) over $\Gamma : \mathcal{U}$ is a family $A : \Gamma \to \mathcal{U}$ equipped with a fibration structure $\alpha : \mathsf{isFib}\ A$, where*

$$\begin{aligned}\mathsf{isFib}\ A \triangleq\ & (r : \mathbb{I})(p : \mathbb{I} \to \Gamma)(\varphi : \Phi)(f : [\varphi] \to (i : \mathbb{I}) \to A(p\ i))(x_0 : A(p\ r)[\varphi \mapsto f \cdot r]) \\ & \to \mathsf{WComp}\ r\ (A \circ p)\ \varphi\ f\ x_0\end{aligned}$$

We write $\mathsf{Fib}\ \Gamma \triangleq \Sigma(A : \Gamma \to \mathcal{U}), \mathsf{isFib}\ A$ for the type of weak fibrations over Γ. As in [28, Def. 5.8], we obtain a *category with families* (CwF) [21] where the families over $\Gamma : \mathcal{U}$ are $(A, \alpha) : \mathsf{Fib}\ \Gamma$ and elements of such a family are dependent functions in $(x : \Gamma) \to A\ x$. Given $P : \mathsf{Fib}\ \Gamma$ and $\sigma : \Delta \to \Gamma$, we write $P[\sigma] : \mathsf{Fib}\ \Delta$ for the reindexing of P along σ.

▶ Remark 5. When discussing the model structure in Section 3.4, we will use the term *fibration* for the usual external notion of a map that has the right lifting property against trivial cofibrations. Whenever this overloading of terminology might be confusing we use the terms *weak fibration* and *fibration structure* when referring to the internal notions.

Given $\alpha : \mathsf{isFib}\ A$, $s : \mathbb{I}$ and r, p, φ, f and x_0 as in Definition 4, we introduce the following more readable notation for the composites provided by α.

$$\mathsf{wcom}_{\alpha}^{r \to s}\, p\, [\varphi \mapsto f]\, x_0 \triangleq \mathsf{fst}\,(\mathsf{fst}\,(\alpha\ r\ p\ \varphi\ f\ x_0)\ s) : A\ (p\ s)$$
$$\underline{\mathsf{wcom}}_{\alpha}^{r}\, p\, [\varphi \mapsto f]\, x_0 \triangleq \mathsf{fst}\,(\mathsf{snd}\,(\alpha\ r\ p\ \varphi\ f\ x_0)) : (\mathsf{wcom}_{\alpha}^{r \to r}\, p\, [\varphi \mapsto f]\, x_0) \sim \mathsf{fst}\, x_0$$

Given $\varphi, \psi : \Phi$, we follow [15] and write $[\varphi \mapsto f, \psi \mapsto g] : [\,\varphi \vee \psi\,] \to A$ for the union of partial elements $f : [\,\varphi\,] \to A$ and $g : [\,\psi\,] \to A$ that agree where they are both defined, i.e. such that $\forall (u : [\,\varphi\,])\ (v : [\,\psi\,]). f\ u = g\ v$. This generalizes directly to $[\varphi_1 \mapsto f_1, ..., \varphi_n \mapsto f_n]$.

We say that a proposition $A : \mathsf{hProp}$ is *cofibrant* if it is logically equivalent to the decoding of a generating cofibrant proposition, i.e. $\texttt{isCofProp}\ A \triangleq \Sigma(\varphi : \Phi), A \leftrightarrow [\,\varphi\,]$. When $r, s : \mathbb{I}$ are such that $(r = s)$ is cofibrant, we will be able to "improve" weak composition $r \to s$ to obtain a strict composition that is exactly equal to its cap when $r = s$.

▶ **Definition 6** (Strict composition). *Given $r : \mathbb{I}$, $A : \mathbb{I} \to \mathcal{U}$, $\varphi : \Phi$, $f : [\,\varphi\,] \to \mathsf{Path}(A)$ and $x_0 : (A\ r)[\varphi \mapsto f \cdot i]$, a strict composition structure is given by an operation*

$$\mathsf{scom} : (s : \mathbb{I}) \to \texttt{isCofProp}(r = s) \to (A\ s)[\varphi \mapsto f \cdot s]$$

satisfying $\mathsf{fst}\,(\mathsf{scom}\ r\ c) = \mathsf{fst}\, x_0$ *for all* $c : \texttt{isCofProp}(r = r)$.

We will leave the argument $\texttt{isCofProp}(r = s)$ implicit. Writing $\mathsf{SComp}\ r\ A\ \varphi\ f\ x_0$ for the type of strict composition operations on A, we define strict fibrations as follows.

▶ **Definition 7** (Strict fibrations). *A* strict fibration (A, α) *over* $\Gamma : \mathcal{U}$ *is a family* $A : \Gamma \to \mathcal{U}$ *equipped with a strict fibration structure* $\alpha : \mathsf{isSFib}\ A$*, where*

$$\mathsf{isSFib}\ A \triangleq (r : \mathbb{I})(p : \mathbb{I} \to \Gamma)(\varphi : \Phi)(f : [\,\varphi\,] \to (i : \mathbb{I}) \to A(p\ i))(x_0 : A(p\ r)[\varphi \mapsto f \cdot r])$$
$$\to \mathsf{SComp}\ r\ (A\ \circ\ p)\ \varphi\ f\ x_0$$

▶ **Lemma 8** (Strictification). *Given $\Gamma : \mathcal{U}$ and $A : \Gamma \to \mathcal{U}$, there is a map $\mathsf{isFib}\ A \to \mathsf{isSFib}\ A$.*

Proof. Given $\alpha : \mathsf{isFib}\ A$ and r, p, φ, f and x_0 as in Definition 7, let

$$w \triangleq \mathsf{wcom}_{\alpha}^{r \to s}\, p\, [\varphi \mapsto f]\, x_0 \qquad \underline{w} \triangleq \underline{\mathsf{wcom}}_{\alpha}^{r}\, p\, [\varphi \mapsto f]\, x_0$$

Given $s : \mathbb{I}$, we define the following term that corrects the $(r = s)$ face of w using $\underline{w}$.

$$\mathsf{scom}\ s \triangleq \mathsf{wcom}_{\alpha}^{0 \to 1}\, (\lambda\_.p\ s)\, [\varphi \mapsto \lambda u\ \_.f\ u\ s, (r = s) \mapsto \lambda\_\ i.\underline{w}\ @\ i]\, w$$ ◀

In particular, as $(r = \varepsilon)$ and $(\varepsilon = r)$ are always cofibrant for $\varepsilon : \{0, 1\}$, we have strict composition operations $\varepsilon \to r$ and $r \to \varepsilon$ in any fibration. Defining $\overline{0} \triangleq 1$ and $\overline{1} \triangleq 0$, we note that the weak compositions $\varepsilon \to \overline{\varepsilon}$ are already strict, as the cap condition is vacuous.

2.3.1 AFH fibrations

We now compare our definition of fibration to that of existing *cartesian* cubical type theories and models. A key feature of these is the use of diagonal cofibrations, which correspond to an operation $(\_ \approx \_) : \mathbb{I} \to \mathbb{I} \to \Phi$ decoding as follows.

$$\mathbf{ax}_{\Delta} : (r\ s : \mathbb{I}) \to [\,(r \approx s)\,] = (r = s)$$

The form of fibration used in these models was originally proposed by Coquand [16], but it was initially unclear how to model univalent universes. AFH observed that the problems could be dealt with by introducing diagonal cofibrations, and used them to give a complete computational semantics of univalent type theory (we hence refer to these as "AFH fibrations"). These ideas were then adapted in ABCFHL to give an Orton-Pitts style model construction.

▶ **Definition 9** (AFH composition). *Given* $r : \mathbb{I}$, $A : \mathbb{I} \to \mathcal{U}$, $\varphi : \Phi$, $f : [\,\varphi\,] \to \mathsf{Path}(A)$ *and* $x_0 : (A\ r)[\varphi \mapsto f \cdot i]$, *an* AFH composition structure *is given by* $\mathsf{com} : (s : \mathbb{I}) \to (A\ s)[\varphi \mapsto f \cdot s]$ *satisfying* $\mathsf{fst}\,(\mathsf{com}\ r) = \mathsf{fst}\,x_0$. *We write* $\mathsf{AFHComp}\ r\ A\ \varphi\ f\ x_0$ *for the type of such AFH composition structures, and write*

$$\begin{aligned}\mathsf{isAFHFib}\ A \triangleq (r : \mathbb{I})(p : \mathbb{I} \to \Gamma)(\varphi : \Phi)(f : [\,\varphi\,] \to (i : \mathbb{I}) \to A(p\ i))\\ (x_0 : A(p\ r)[\varphi \mapsto f \cdot r]) \to \mathsf{AFHComp}\ r\ (A\ \circ\ p)\ \varphi\ f\ x_0\end{aligned}$$

When $\mathsf{isAFHFib}$ is taken as the definition of fibration, it seems that diagonal cofibrations are crucial to construct fibrant univalent universes of fibrant types. Specifically, they are needed to ensure that composition in Glue/V types and the universe satisfies the strict cap condition. In the presence of diagonal cofibrations, our definition of fibration coincides with $\mathsf{isAFHFib}$.

▶ **Theorem 10.** *Given* $\Gamma : \mathcal{U}$ *and* $A : \Gamma \to \mathcal{U}$, *we have* $\mathsf{isAFHFib}\ A$ *iff we have* $\mathsf{isFib}\ A$.[2]

Proof. Any AFH composition structure induces a weak composition structure, as any equality can be turned into a path. For the converse direction, apply Lemma 8 with $\mathbf{ax}_\Delta$. ◀

▶ Remark 11. Awodey [6] has formulated a categorical notion of *unbiased fibrations* and shown that this coincides with AFH fibrations; it thus also coincides with weak composition in the presence of diagonal cofibrations.

2.3.2 CCHM fibrations

Next, we compare with the CCHM definition of fibration. Following Orton and Pitts [28], we assume operations $\sqcap, \sqcup : \mathbb{I} \to \mathbb{I} \to \mathbb{I}$ satisfying the axioms of a *connection algebra*.

$$\begin{aligned}&\mathbf{ax}_\sqcap : (r : \mathbb{I}) \to (0 \sqcap r = 0 = r \sqcap 0) \wedge (1 \sqcap r = r = r \sqcap 1)\\ &\mathbf{ax}_\sqcup : (r : \mathbb{I}) \to (0 \sqcup r = r = r \sqcup 0) \wedge (1 \sqcup r = 1 = r \sqcup 1)\end{aligned}$$

▶ Remark 12. A connection algebra is weaker than the De Morgan algebra used in CCHM: there is no reversal $\neg : \mathbb{I} \to \mathbb{I}$ and the connections need not form a distributive lattice. Thus, Orton and Pitts [28] obtain a construction that applies to both CCHM and Dedekind cubical sets, compensating for the lack of reversals by parametrizing the composition operation by $\varepsilon : \{0,1\}$. Following Orton and Pitts, we continue to call this "CCHM composition" despite the superficial difference from the operation defined in [15].

▶ **Definition 13** (CCHM composition). *Given* $\varepsilon : \{0,1\}$, $A : \mathbb{I} \to \mathcal{U}$, $\varphi : \Phi$, $f : [\,\varphi\,] \to \mathsf{Path}(A)$ *and* $x_0 : (A\ \varepsilon)[\varphi \mapsto f \cdot i]$, *a* CCHM composition structure *is a term* $\mathsf{com} : (A\ \overline{\varepsilon})[\varphi \mapsto f \cdot \overline{\varepsilon}]$. *We write* $\mathsf{CCHMComp}\ \varepsilon\ A\ \varphi\ f\ x_0$ *for the type of such CCHM composition structures, and*

$$\begin{aligned}\mathsf{isCCHMFib}\ A \triangleq (\varepsilon : \{0,1\})(p : \mathbb{I} \to \Gamma)(\varphi : \Phi)(f : [\,\varphi\,] \to (i : \mathbb{I}) \to A(p\ i))\\ (x_0 : A(p\ \varepsilon)[\varphi \mapsto f \cdot r]) \to \mathsf{CCHMComp}\ \varepsilon\ (A\ \circ\ p)\ \varphi\ f\ x_0\end{aligned}$$

A key result in CCHM is that connections and composition $0 \to 1$ suffice to derive composition $0 \to r$ (i.e. Kan filling). The following result shows that we can in fact derive all of the cartesian composition operations, except for the strict equality for $r \to r$. This clarifies the relationship between CCHM and AFH composition. As CCHM only requires compositions $\varepsilon \to \overline{\varepsilon}$, diagonal cofibrations are not needed for Glue types and the universe.

[2] This is already observed for weak coercion in [2, Sec. 2.7].

▶ **Theorem 14.** *Given* $\Gamma : \mathcal{U}$ *and* $A : \Gamma \to \mathcal{U}$*, we have* $\mathsf{isCCHMFib}\ A$ *iff we have* $\mathsf{isFib}\ A$*.*

Proof. We can go from $\mathsf{isFib}\ A$ to $\mathsf{isCCHMFib}\ A$ by simply instantiating r with ε and s with $\overline{\varepsilon}$. For the other direction, let r, p, φ, f and x_0 be as in Definition 4. First, we define the following term, which composes from $A\ (p\ r)$ to $A\ (p\ (j \wedge r))$ for any $j : \mathbb{I}$.

$$q\ j \triangleq \mathsf{com}_{\alpha}^{1 \to 0}\ (\lambda i.p\ ((j \vee i) \wedge r)) \begin{bmatrix} \varphi & \mapsto \lambda u\ i.f\ u\ ((j \vee i) \wedge r) \\ (j = 1) & \mapsto \lambda\_\ \_.x_0 \end{bmatrix} x_0$$

We can then define weak composition to $s : \mathbb{I}$.

$$\mathsf{wcom}\ s \triangleq \mathsf{com}_{\alpha}^{0 \to 1}\ (\lambda i.p\ (i \wedge s))\ [\varphi \mapsto \lambda u\ i.f\ u\ (i \wedge s), (0 \approx 1) \mapsto \mathsf{elim}_{\perp}]\ (q\ 0)$$

The cap path is defined as follows.

$$\underline{\mathsf{wcom}} \triangleq \lambda(j : \mathbb{I}).\mathsf{com}_{\alpha}^{0 \to 1}\ (\lambda i.p\ ((j \vee i) \wedge r)) \begin{bmatrix} \varphi & \mapsto \lambda u\ i.f\ u\ ((j \vee i) \wedge r) \\ (j = 1) & \mapsto \lambda\_\ \_.x_0 \end{bmatrix} (q\ j)$$ ◀

2.4 Fibration structures for basic type formers

The collection of fibrations is closed under all of the basic type formers of cubical type theory: Σ-, Π-, Path types and any basic datatypes that $\mathcal{C}$ supports. The arguments are very similar to those of [2, 3], but additional adjustments are necessary to compensate for the new weakness. We include the proof for Σ-types in order to illustrate this in detail.

▶ **Theorem 15** (Fibrant Σ-types)**.** *Given* $\Gamma : \mathcal{U}$*,* $A : \Gamma \to \mathcal{U}$*,* $B : (\Sigma(x : \Gamma), A\ x) \to \mathcal{U}$*, we have*

$$\mathsf{isFib}_{\Sigma} : \mathsf{isFib}\ A \to \mathsf{isFib}\ B \to \mathsf{isFib}\ (\Sigma\ A\ B)$$

where $(\Sigma\ A\ B)\ x \triangleq \Sigma(a : A\ x), B\ (x, a)$*.*

Proof. Let $\alpha : \mathsf{isFib}\ A$ and $\beta : \mathsf{isFib}\ B$ and r, p, φ, f and x_0 be as in Definition 4. We first define the composite and cap path for the first components of the open box.

$$w_A\ i \triangleq \mathsf{wcom}_{\alpha}^{r \to i}\ p\ [\varphi \mapsto \lambda u\ j.\,\mathsf{fst}\ (f\ u\ j)]\ (\mathsf{fst}\ x_0)$$
$$\underline{w}_A \triangleq \underline{\mathsf{wcom}}_{\alpha}^{r}\ p\ [\varphi \mapsto \lambda u\ j.\,\mathsf{fst}\ (f\ u\ j)]\ (\mathsf{fst}\ x_0)$$

To define the composite of the second components, we first adjust the type of the cap. For this, we use a strict composition $1 \to k$ in B, which is derivable from β per Lemma 8.

$$b\ k \triangleq \mathsf{scom}_{\beta}^{1 \to k}\ (\lambda j.(p\ r, \underline{w}_A\ @\ j))\ [\varphi \mapsto \lambda u\ \_.\,\mathsf{snd}\ (f\ u\ r)]\ (\mathsf{snd}\ x_0)$$

When k is 0, this is the corrected cap of our composition in B.

$$w_B \triangleq \mathsf{wcom}_{\beta}^{r \to s}\ (\lambda i.(p\ i, w_A\ i))\ [\varphi \mapsto \lambda u\ i.\,\mathsf{snd}\ (f\ u\ i)]\ (b\ 0)$$
$$\underline{w}_B \triangleq \underline{\mathsf{wcom}}_{\beta}^{r}\ (\lambda i.(p\ i, w_A\ i))\ [\varphi \mapsto \lambda u\ i.\,\mathsf{snd}\ (f\ u\ i)]\ (b\ 0)$$

Composition in the pair type is then defined to be the pair $\mathsf{wcom}\ s \triangleq (w_A\ s, w_B)$. For the cap path, we combine the cap path $\underline{w}_B$ for the composition in B with the path b that relates $b\ 0$ to $\mathsf{snd}\, x_0$ over $\underline{w}_A$.

$$c\ t \triangleq \mathsf{wcom}_{\beta}^{1 \to 0}\ (\lambda j.(p\ r, \underline{w}_A\ @\ j)) \begin{bmatrix} \varphi & \mapsto \lambda u\ \_.\,\mathsf{snd}\ (f\ u\ r) \\ (t = 0) & \mapsto \lambda\_\ j.\underline{w}_B\ @\ j \\ (t = 1) & \mapsto \lambda\_\ \_.\,\mathsf{snd}\, x_0 \end{bmatrix} (b\ t)$$

We then let $\underline{\mathsf{wcom}} \triangleq \lambda(t : \mathbb{I}).(\underline{w}_A\ @\ t, c\ t)$. ◀

The case for Π-types is similar to that of Σ-types: the proof roughly follows that of strict composition, but additional composites have to be inserted to mediate between composites and their caps. The proofs for Path types and natural numbers are essentially identical to those of [2, 3]. We omit the details here, but the interested reader may consult [13, Sec. 3] or our Agda formalization. It is also straightforward to verify that these definitions are stable under reindexing, so that we obtain a CwF that supports Σ-, Π- and Path types. This CwF also supports natural numbers if $\mathcal{C}$ has a natural numbers object.

2.5 Glueing

Glue types were introduced in [15, Sec. 6] to unify the proofs that the universe of fibrant types is fibrant and univalent. This construction also occurs implicitly in the proof that the universe is univalent in the Kan simplicial set model [26, Thm. 3.4.1]. The construction of these types in the internal language was described in detail by Orton and Pitts [28, Sec. 6]. In this section we only briefly sketch their construction; apart from the proof of Theorem 17, there are no major differences.

▶ **Definition 16** (Glueing). *Given $\varphi : \Phi$, $A : [\,\varphi\,] \to \mathcal{U}$, $B : \mathcal{U}$ and $f : (x : [\,\varphi\,]) \to A\ x \to B$, we define* Glue $\varphi\ A\ B\ f : \mathcal{U}$ *as follows.*

$$\mathsf{Glue}\ \varphi\ A\ B\ f \triangleq \Sigma(a : (x : [\,\varphi\,]) \to A\ x), \Sigma(b : B), (x : [\,\varphi\,]) \to f\ x\ (a\ x) = b$$

Elements of this type are thus pairs (a, b) where a is a partial element of A and b is an element of B such that f applied to a extends to b. When φ is $\top$, the Glue type is isomorphic to A. The Glue operator lifts to a fiberwise operation on families of types, which we also call Glue. To prove that it takes fibrations to fibrations, however, we must also require that f is an equivalence. There are various ways to express this; we follow Voevodsky and say that f is an equivalence when its fibers are contractible [38, 39]. We write $A \simeq B$ for the type of equivalences between A and B.

▶ **Theorem 17** (Fibrant Glue types). *Given $\Gamma : \mathcal{U}$, $\varphi : \Gamma \to \Phi$, $A : (x : \Gamma) \to [\,\varphi\ x\,] \to \mathcal{U}$, $B : \Gamma \to \mathcal{U}$ and $f : (x : \Gamma)\ (v : [\,\varphi\ x\,]) \to A\ x\ v \to B\ x$. If f has the structure of an equivalence then there is a function* $\mathsf{isFib}_{\mathsf{Glue}} : \mathsf{isFib}\ A \to \mathsf{isFib}\ B \to \mathsf{isFib}\ (\mathsf{Glue}\ \varphi\ A\ B\ f)$.

The proof of this theorem is a variation of the one of [2]; as with Σ-types, some additional compositions are needed to compensate for the weakness. We refer the interested reader to the detailed type theoretic presentation in [13, Sec. 4.2] and to the Agda formalization.

Note that the fibrancy of these types does not require any additional axioms. However, they are weaker than the Glue types of [15]: they are not strictly equal to A when φ is $\top$, only isomorphic. In order to prove univalence and fibrancy of the universe, we first need to strictify. Writing $A \cong B$ for the type of isomorphisms between A and B, we require the following *strictness axiom* ($\mathbf{ax}_9$ in [28]).

$$\begin{aligned}\mathbf{ax}_6 : (\varphi : \Phi)\,(A : [\,\varphi\,] \to \mathcal{U})\,(B : \mathcal{U})\,(s : (u : [\,\varphi\,]) \to A\ u \cong B) \to\\ \Sigma(B' : \mathcal{U}), \Sigma(s' : B' \cong B), (u : [\,\varphi\,]) \to (A\ u, s\ u) = (B', s')\end{aligned}$$

Using this axiom, we can perform the same construction as in [28, Def. 6.1] and obtain a type SGlue $\varphi\ A\ B\ f$ that satisfies the desired equation strictly and is isomorphic to Glue $\varphi\ A\ B\ f$. We then transport the weak fibration structure from Glue to SGlue along this isomorphism. However, the weak composition operation that we obtain this way will not necessarily reduce to the composition operation of A when φ is $\top$. In order to correct this, we assume an operation $\forall : (\mathbb{I} \to \Phi) \to \Phi$ satisfying the following.

$$\mathbf{ax}_7 : (\varphi : \mathbb{I} \to \Phi) \to [\,\forall\varphi\,] = (i : \mathbb{I}) \to [\,\varphi\, i\,]$$

Using this axiom, we can perform the same "alignment" as in [28, Thm. 6.13] and obtain a weak fibration structure for SGlue that reduces that of A when φ is $\top$.

2.5.1 Univalence

Voevodsky's univalence axiom states that the canonical map $\mathsf{idtoequiv} : (A \sim B) \to (A \simeq B)$ is an equivalence. This formulation of univalence assumes a universe of (fibrant) types. As we have not yet constructed a universe, we instead define a variation of univalence that uses a primitive notion of lines between types. For $\Gamma : \mathcal{U}$ and $A, B : \mathsf{Fib}\,\Gamma$, we define

$$A \sim_{\mathcal{U}} B \triangleq \Sigma(P : \mathsf{Fib}\,(\Gamma \times \mathbb{I})), P[(\mathsf{id}, 0)] = A \times P[(\mathsf{id}, 1)] = B$$

▶ **Theorem 18** (Univalence for $\sim_{\mathcal{U}}$). *We have* $(A \sim_{\mathcal{U}} B) \simeq (\mathsf{fst}\, A \simeq \mathsf{fst}\, B)$.

Proof. This is equivalent[3] to the existence of a term $\mathsf{ua} : A \simeq B \to A \sim_{\mathcal{U}} B$ such that $\mathsf{idtoequiv} \circ \mathsf{ua} = \mathsf{id}$. The ua term follows directly from SGlue in the standard way [28, Thm. 7.2]. The inverse condition can be proven by unfolding the algorithm for weak composition in SGlue, in analogy with [28, Thm. 7.3]. ◀

This model hence satisfies this variation of the univalence axiom. Following [27], we may also construct a universe and prove the standard formulation of the univalence axiom.

2.6 Fibrant univalent universes

The universe construction of LOPS [27] can be performed in a modal extension of type theory called *crisp type theory.* Andrea Vezzosi has developed an extension of `Agda` with the crisp modality called `Agda-♭`. However, this was only recently incorporated into the standard version of `Agda`, so we have not formally verified the content of this section.

A key component in the LOPS universe construction is a special feature of the interval in the various cubical set categories: it is *tiny*, i.e. exponentiation by it has a right adjoint. This is *not* true for Δ^1, so the following theorem does not apply to Kan simplicial sets.

▶ **Theorem 19** (Universe construction). *If $\mathbb{I}$ is tiny, then we can construct a universe* U *with a fibration* El *that is classifying in the sense of [27, Thm. 5.2].*

Proof. We need to check that the assumptions of [27, Thm. 5.2] are satisfied. First of all, the arguments of isFib and WComp can be rearranged to match [27, Def. 2.2]. We then need to check that axioms (1)–(4) in [27] hold. The first two are function extensionality and uniqueness of identity proofs, which we are assuming. The other two are disjointness of endpoints and that $\bot$ is a cofibrant proposition, both of which follow from $\mathbf{ax}_2$. ◀

We next need to show that this universe has a weak fibration structure, is closed under all of the type formers of cubical type theory, and satisfies the univalence axiom. This has been formalized in `Agda-♭` for AFH fibrations in [2], and we do not expect any difficulty doing the

[3] This was originally pointed out by Daniel R. Licata in `https://groups.google.com/forum/#!msg/homotopytypetheory/j2KBIvDw53s/YTDK4D0NFQAJ`.

same here, the only difference being the strictness of the cap equation. For a type theoretic proof that the universe is fibrant and univalent using the fibration structures in this paper, see [13, Sec. 4.3 and 4.4].

3 Model structures on cubical sets

We will now prove that our definition of fibration structures forms part of a Quillen model structure. This helps to clarify the relation between our definition and already established and well known definitions in homotopical algebra. We assume the reader is familiar with standard concepts in homotopical algebra such as model structures, algebraic weak factorization systems (awfs's), and the Leibniz adjunction. See e.g. [31] for these definitions.

Further details, including proofs of these results, are available in [14]. We have also defined the two factorization systems in `Agda` by postulating the existence of W*-types with reductions* [36], a simple class of (extensional) higher inductive types.

We will use some extra notational conventions for this section. We write $\delta_i : 1 \to \mathbb{I}$ for $i : \{0, 1\}$ for the endpoint inclusions. We use the subscript B when working with objects in a slice category $\mathcal{C}/B$. In particular, we have an interval object $\mathbb{I}_B$ defined as the projection $\mathbb{I} \times B \to B$, with obvious endpoint maps $\delta_{Bi} : 1_B \to \mathbb{I}_B$.

3.1 Cofibrantly generated awfs's

To construct a model structure, we first need to define two weak factorization systems, one for *cofibrations and trivial fibrations* and one for *trivial cofibrations and fibrations.* In both cases, we will use the following definitions and theorems from [36] and [34].

▶ **Definition 20** ([36, Def. 6.1])**.** *Let m be a map in a slice category $\mathcal{C}/I$ and let f be a map in another slice category $\mathcal{C}/J$. A* family of lifting problems of m against f *consists of an object K, together with maps $\sigma\colon K \to I$ and $\tau\colon K \to J$ and a lifting problem of $\sigma^*(m)$ against $\tau^*(f)$ in $\mathcal{C}/K$.*

We say m has the fibered left lifting property against f *and f has the* fibered right lifting property against m *if every family of lifting problems has a diagonal filler.*

A family of lifting problems K, σ, τ, p, q is universal *if for any other family of lifting problems $K', \sigma', \tau', p', q'$, there is a unique map $t\colon K' \to K$ such that $\sigma' = t \circ \sigma$, $\tau' = t \circ \tau$, $p' = t^*(p)$ and $q' = t^*(q)$.*

▶ **Proposition 21** ([34, Prop. 3.2.4], [36, Def. 6.2])**.** *Universal lifting problems exist.*

▶ **Proposition 22** ([34, Prop. 3.2.5])**.** *f has the* fibered right lifting property against m *iff the universal lifting problem has a filler.*

▶ **Definition 23.** *A* fibered algebraic weak factorization system *or* fibered awfs *consists of an algebraic weak factorization system (L_J, R_J) on each slice category $\mathcal{C}/J$ preserved by reindexing (up to isomorphism).*

A fibered awfs is cofibrantly generated *if there exists a map m in some slice category $\mathcal{C}/I$ such that for each J and each map f in $\mathcal{C}/J$, R_J algebra structures on f correspond precisely to diagonal fillers of the universal lifting problem of m against f.*

The following theorem will allow us to construct the two weak factorization systems of the model structure.

▶ **Theorem 24.** *Let m be a map in some slice category $\mathcal{C}/I$. The fibered awfs cofibrantly generated by m exists if either of the two conditions below are satisfied.*

1. *$\mathcal{C}$ is an internal category of presheaves in a locally cartesian closed category with finite colimits, disjoint sums and W-types, and m is a locally decidable monomorphism.*
2. *$\mathcal{C}$ is a ΠW-pretopos (e.g. $\mathcal{C}$ is a topos with natural number object), and it satisfies the axiom weakly initial set of covers (WISC).*

Proof. If (1) holds, apply [36, Thm. 6.14], and if (2) holds, apply [36, Cor. 6.12]. ◀

3.2 Cofibration and trivial fibration awfs

We can view the cofibrant propositions $[\,-\,] : \Phi \to \mathsf{hProp}$ as a monomorphism $\top\colon \Phi_{\mathsf{true}} \rightarrowtail \Phi$, where $\Phi_{\mathsf{true}} \triangleq \Sigma(\varphi : \Phi), [\,\varphi\,] = \top$.

▶ **Definition 25** (Generating cofibrations)**.** *Let $m\colon A \to B$ be a map in a slice category $\mathcal{C}/I$. We say m is a* generating cofibration *if either of the equivalent conditions below holds.*
1. *$\sum_I m$ is a pullback of $\top$.*
2. *m is a pullback of $I^*(\top)\colon I^*(\Phi_{\mathsf{true}}) \to I^*(\Phi)$ in $\mathcal{C}/I$.*

▶ **Proposition 26.** *Generating cofibrations are closed under pullbacks and binary unions. Every isomorphism is a generating cofibration.*

▶ **Proposition 27.** *Let $f\colon X \to Y$ be a map in a slice $\mathcal{C}/J$. The following are equivalent.*
1. *f has the fibered right lifting property against $\top$, viewed as a map $\Phi_{\mathsf{true}} \to 1_\Phi$ in $\mathcal{C}/\Phi$.*
2. *f has the fibered right lifting property against generating cofibrations of the form $A \to 1_B$ in slice categories $\mathcal{C}/B$.*
3. *f has the fibered right lifting property against every generating cofibration.*
4. *f has the right lifting property against every generating cofibration in $\mathcal{C}/J$.*

▶ **Definition 28** (Trivial fibrations and cofibrations)**.** *If a map $f\colon X \to Y$ in a slice category $\mathcal{C}/J$ satisfies one, and so all, of the equivalent conditions in Proposition 27 we say that f is a* trivial fibration*. A map m in a slice category $\mathcal{C}/I$ is a* cofibration *if it has the fibered left lifting property against every trivial fibration.*

When working in `Agda` we found it helpful to use an alternative definition of trivial fibration following [15, Sec. 5.1]. We say that a type $A : \mathcal{U}$ is *contractible* if the type $\mathsf{SContr}\ A$ is inhabited, where we define $\mathsf{SContr}\ A \triangleq (\varphi : \Phi) \to (t : [\varphi] \to A) \to A[\varphi \mapsto t]$. We define a map $f : X \to Y$ to be a trivial fibration if every fiber is contractible.

If m and $\mathcal{C}$ satisfy the necessary conditions to apply Theorem 24 then there is an awfs (C, F^t) where the class underlying F^t is precisely the class of trivial fibrations. We refer to maps in the class underlying C as *cofibrations*.

3.3 Trivial cofibration and fibration awfs

We now give a more abstract characterization of weak fibrations (Definition 4) and define an awfs where the right maps are weak fibrations. Following Gambino and Sattler [24], we use the Leibniz adjunction to describe fibrations, writing $\hat{\times}_B$ and $\hat{\hom}_B(-,-)$ for the Leibniz product and exponential constructed in a slice category $\mathcal{C}/B$. We also use the following notion of *weak lifting property.* This definition (although not the name) has been used before in homotopical algebra by Dold [20] and also by Reedy [30]. Note however that the definition of fibration considered by Dold is weaker than the one here, as one may see from Lemma 8.

▶ **Definition 29** (Weak left lifting property). *Let $m : A \to B$ and $f : X \to Y$. We say m has the* weak left lifting property *against f if for every commutative square, as in the solid lines below, there is a diagonal map, as in the dotted line below, such that the lower triangle commutes strictly, and the upper triangle commutes up to a homotopy $h : j \circ m \sim a$ such that $f \circ h$ is constant. We refer to such diagonal maps as* weak fillers.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\sim", pos=0.3] & X \arrow[d, "f"] \\ B \arrow[r, "b"] \arrow[ur, dotted] & Y \end{tikzcd}$$

▶ **Theorem 30.** *A map $f : X \to Y$ is a weak fibration if and only if for every object B, every map $r : 1_B \to \mathbb{I}_B$ and generating cofibration $m : A \to 1_B$ in $\mathcal{C}/B$, r has the weak left lifting property against $\hat{\hom}_B(m, f)$.*

Proof. Working in $\mathcal{C}/B$, r has the weak left lifting property against $\hat{\hom}_B(m, f)$ iff every lifting problem of $r \mathbin{\hat{\times}_B} m$ against f has a weak filler satisfying the additional condition of being strict on A. This holds for all B, r and m and every choice of lifting problem iff it holds for the universal lifting problem of $\Delta \mathbin{\hat{\times}_{\mathbb{I}\times\Phi}} \top$ against f, where Δ is the map $1_{\mathbb{I}\times\Phi} \to \mathbb{I}_{\mathbb{I}\times\Phi}$ in $\mathcal{C}/(\mathbb{I} \times \Phi)$ defined as the diagonal map $\mathbb{I} \times \Phi \to \mathbb{I} \times \mathbb{I} \times \Phi$. Such fillers of the universal lifting problem correspond precisely to WComp terms. ◀

In order to obtain an awfs, we show that the above is equivalent to an alternative definition using the mapping cylinder factorization, which we recall is defined as below.

▶ **Definition 31** (Mapping cylinder factorization). *Let $m : A \to B$. We define the* mapping cylinder factorization *to be the maps $A \xrightarrow{L(m)} \mathsf{Cyl}(m) \xrightarrow{R(m)} B$, defined as follows. We first define $\mathsf{Cyl}(m)$ as the pushout of δ_{A0} and m, writing $\iota_0 : \mathbb{I} \times A \to \mathsf{Cyl}(m)$ and $\iota_1 : B \to \mathsf{Cyl}(m)$ for the pushout inclusions. We define $L(m)$ to be $\iota_0 \circ \delta_{A1}$ and define $R(m)$ to be the unique map such that $R(m) \circ \iota_0 = m \circ \pi_1$ and $R(m) \circ \iota_1 = 1_B$.*

▶ **Theorem 32.** *Let f be a map in $\mathcal{C}$. Then f is a weak fibration if and only if it has the fibered right lifting property against the map $L_{\mathbb{I}\times\Phi}(\Delta) \mathbin{\hat{\times}_{\mathbb{I}\times\Phi}} \top$ in the slice category $\mathcal{C}/(\mathbb{I} \times \Phi)$.*

Using this alternative definition, we can apply Theorem 24 to obtain an awfs (C^t, F) where F is precisely the class of weak fibrations. We refer to maps in C^t as *trivial cofibrations*.

3.4 The model structure

Now that we have defined the awfs's (C, F^t) and (C^t, F), we use Sattler's [32, Thm. 2.8] in order to obtain a model structure on $\mathcal{C}$.

▶ **Lemma 33.** *The awfs's (C, F^t) and (C^t, F) have the following key properties.*

1. *The functor $\hat{\hom}(\delta_i, -)$ maps fibrations to trivial fibrations.*
2. *The functor $\hat{\hom}([\delta_0, \delta_1], -)$ preserves fibrations and trivial fibrations.*
3. *Every cofibration is a monomorphism.*
4. *Cofibrations are stable under pullback.*

▶ **Theorem 34.** *Suppose that $\mathcal{C}$ satisfies axioms $\mathbf{ax}_1$–$\mathbf{ax}_5$ and that every fibration is U-small for some universe of small fibrations where the underlying object U is fibrant, and that $\mathcal{C}$ and Φ satisfy one of the conditions required to apply Theorem 24.*

Let (C, F^t) be the awfs defined in Section 3.2 and let (C^t, F) be the awfs defined in Section 3.3 (restricted to $\mathcal{C}/1$). Then C and F form the cofibrations and fibrations of a (uniquely determined) model structure on $\mathcal{C}$.

Proof. By Sattler's [32, Thm. 2.8] it suffices to check the following conditions.

1. The span property holds.
2. Trivial fibrations satisfy 2-out-of-3 relative to fibrations.
3. Fibrations and trivial fibrations extend along trivial cofibrations.
4. The wfs (C^t, F) satisfies the Frobenius property.

Conditions (1) and (2) follow from the key properties (1) and (2) in Lemma 33 by essentially the same arguments used by Sattler in [32, Sec. 4].

Trivial fibrations extend along all cofibrations, by the same argument used by Sattler in [32, Lem. 3.9] together with the key properties (3) and (4) in Lemma 33.

As Sattler remarks in [32, Rem. 7.6], to show fibrations extend along trivial cofibrations it suffices to show every fibration belongs to a universe U where the underlying object is fibrant, which we assumed.

Finally, (C^t, F) is Frobenius by the existence of fibration structures on Π-types and the adjunction between pullback and dependent product. ◀

In particular, if $\mathbf{ax}_6$ and $\mathbf{ax}_7$ hold and $\mathbb{I}$ is tiny, we can use the construction of U from Section 2.6 together with the proof of fibrancy in [13, Sec. 4.3].

The model structure obtained this way is "minimal" in the following sense [14, Sec. 1.6].

▶ **Theorem 35.** *The class C^t is as small as possible subject to the following two conditions.*

1. *For every object B, the map $\delta_{B0} : B \to B \times \mathbb{I}$ belongs to C^t.*
2. *C and C^t form the cofibrations and trivial cofibrations of a model structure.*

4 Identity types and higher inductive types

We have formalized three constructions of identity types in `Agda`, each of which requires additional assumptions. The first follows [15, Sec. 9.1]; this requires a dominance on Φ and extensionality for cofibrant propositions. The second approach uses the (C, F^t) factorization system following [33], while the third approach uses the (C^t, F) factorization system following [12, 11]. These rely on W-types with reductions to obtain the factorization systems. We refer the interested reader to the `Agda` formalization for details.

A crucial component for modeling universes closed under higher inductive types is the decomposition of composition into *homogeneous* composition and coercion [12, 18]. A type $A : \Gamma \to \mathcal{U}$ supports weak homogeneous composition if all of its fibers support weak composition, i.e. for all $(x : \Gamma)$ the type $A\ x$ has a weak composition structure. Supporting weak coercion corresponds to having weak composition only in the case when φ is $\bot$ (i.e., the tube is empty). We have formalized that a type has weak composition if and only if it has weak homogeneous composition and weak coercion. This makes it possible for us to follow the same approach as in [12, 18] to model higher inductive types. We refer the reader to [13, Sec. 5.1] for the construction of a circle type in this setting.

5 Conclusions

We have proved that any locally cartesian closed category $\mathcal{C}$ with $\mathbb{I}$ and Φ satisfying $\mathbf{ax}_1$–$\mathbf{ax}_7$ and where $\mathbb{I}$ is tiny provides a constructive model of HoTT/UF. Examples of such categories are CCHM and Dedekind cubical sets as proven in [28, Sec. 8], and cartesian cubical sets as proven in [2, Sec. 3.2]. Our conditions hold for cubical assemblies [37] and also apply to new variants of cubical assemblies based on cartesian cubes rather than Dedekind cubes.

Our construction of a model structure also applies to all of the above examples. As observed by Sattler [32, Cor. 8.5], the LOPS construction of a universe does not apply for simplicial sets because the interval is not tiny, but one can still obtain a model structure using the non-constructive theorem that the definition of Kan fibration here is equivalent to the classical definition using horn inclusions.

From the perspective of practical implementation and usability, the type theory corresponding to this model is inferior to the type theories it generalizes: equalities that are strict in the specialized type theories here only hold up to paths, so additional path algebra is necessary to implement composition at the various types. The objective is rather to present a theory with which the mathematical properties of the various type theories and models can be studied simultaneously.

Future work

Now that we have given a unified construction for the various cubical models, the natural next step is to use it to establish relationships between its various instantiations. One option is to prove homotopy canonicity for the type theory using categorical gluing as in [19]. This would show that closed terms of natural number type written in weak cartesian type theory evaluate to the same numeral in any of the existing cubical type theories.

The construction may also be useful for uniformly analyzing the model structures induced by different choices of cube category and generating cofibrations. Sattler has observed [17] that the CCHM and ABCFHL constructions give model structures that are *not* Quillen equivalent to spaces. However, the question is open for Dedekind cubes. One might also investigate the relationships *between* the various cubical model structures.

Finally, the program of unification remains unfinished, as the BCH model is not an instance of our construction. Indeed, our approach seems ill-suited to BCH, as it crucially involves the diagonal $(r = s)$ of compositions $r \to s$. It is unclear to us whether BCH can be naturally accommodated; it may simply be a fundamentally different construction.

5.1 Related work

As the notion of fibration defined in this paper coincides with the one of Orton and Pitts [28] in the presence of a connection algebra, and this is equivalent to the Gambino-Sattler definition [24], we recover the model structure of Sattler [32] when the category also has connections. Another presentation of this model structure on CCHM and Dedekind cubical sets can be found in Boulier's Ph.D. thesis [10], formalized in the `Coq` proof assistant. Since an equivalent definition of fibration was used by Van den Berg and Frumin in [22], when our model structure exists we can recover theirs by restricting to fibrant objects. However, our proof does not apply to their main example of the effective topos because it is unknown how to construct a universe satisfying $\mathbf{ax}_6$ in this setting (see [35, Thm. 5.7]).

Furthermore, as we recover AFH fibrations when we assume diagonal cofibrations, we also recover the model structure on cartesian cubical sets sketched by Coquand based on Sattler's model structure [17]. Awodey [4] uses a variation of composition $0 \to r$ and $1 \to r$ to construct an awfs on cartesian cubical sets, but it is unclear whether this is sufficient to obtain a model structure. Awodey has recently [6] introduced a notion of "unbiased fibrations" that are equivalent to AFH fibrations, so the resulting model structure is also a special case of ours when we assume diagonal cofibrations. Our generalization hence clarifies the relationship between some of the various model structures on different cubical set categories.

References

1 Agda development team. *Agda 2.6.0.1 documentation*, 2018. URL: `https://agda.readthedocs.io/en/v2.6.0.1/`.

2 Carlo Angiuli, Guillaume Brunerie, Thierry Coquand, Kuen-Bang Hou (Favonia), Robert Harper, and Daniel R. Licata. Syntax and Models of Cartesian Cubical Type Theory. Draft available at `https://github.com/dlicata335/cart-cube/blob/master/cart-cube.pdf`, 2017.

3 Carlo Angiuli, Kuen-Bang Hou (Favonia), and Robert Harper. Cartesian cubical computational type theory: Constructive reasoning with paths and equalities. In *27th EACSL Annual Conference on Computer Science Logic, CSL 2018, September 4-7, 2018, Birmingham, UK*, pages 6:1–6:17, 2018. `doi:10.4230/LIPIcs.CSL.2018.6`.

4 Steve Awodey. A cubical model of homotopy type theory. *Annals of Pure and Applied Logic*, 169(12):1270 – 1294, 2018. Logic Colloquium 2015. URL: `http://www.sciencedirect.com/science/article/pii/S0168007218300861`, `doi:https://doi.org/10.1016/j.apal.2018.08.002`.

5 Steve Awodey. Recent Work in Homotopy Type Theory: Modal, Algebraic, Synthetic, and Cubical, March 2018. Talk given at MURI Team Meeting at CMU, available at `https://ncatlab.org/homotopytypetheory/files/awodeyMURI18.pdf`.

6 Steve Awodey. A Quillen model structure on the category of cartesian cubical sets. Preprint available at `https://github.com/awodey/math/blob/master/QMS/qms.pdf`, 2019.

7 Marc Bezem, Thierry Coquand, and Simon Huber. A Model of Type Theory in Cubical Sets. In *19th International Conference on Types for Proofs and Programs (TYPES 2013)*, volume 26 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 107–128, 2014. URL: `http://drops.dagstuhl.de/opus/volltexte/2014/4628`, `doi:http://dx.doi.org/10.4230/LIPIcs.TYPES.2013.107`.

8 Marc Bezem, Thierry Coquand, and Simon Huber. The univalence axiom in cubical sets. *J. Autom. Reasoning*, 63(2):159–171, 2019. `doi:10.1007/s10817-018-9472-6`.

9 Marc Bezem, Thierry Coquand, and Erik Parmann. Non-constructivity in Kan simplicial sets. In Thorsten Altenkirch, editor, *13th International Conference on Typed Lambda Calculi and Applications (TLCA 2015)*, volume 38 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 92–106, Dagstuhl, Germany, 2015. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. URL: `http://drops.dagstuhl.de/opus/volltexte/2015/5157`, `doi:http://dx.doi.org/10.4230/LIPIcs.TLCA.2015.92`.

10 Simon Boulier. *Extending Type Theory with Syntactic Models.* PhD thesis, Université Bretagne Loire, 2018.

11 Evan Cavallo. Stable factorization from a fibred algebraic weak factorization system. Preprint arXiv:1910.03121 [math.CT], 2019.

12 Evan Cavallo and Robert Harper. Higher Inductive Types in Cubical Computational Type Theory. *Proc. ACM Program. Lang.*, 3(POPL):1:1–1:27, January 2019. `doi:10.1145/3290314`.

13 Evan Cavallo and Anders Mörtberg. A Unifying Cartesian Cubical Type Theory. Preprint available at `https://github.com/mortberg/gen-cart/blob/master/unifying-cartesian.pdf`, 2019.

14 Evan Cavallo, Anders Mörtberg, and Andrew Swan. Model structures on cubical sets. Preprint available at `https://github.com/mortberg/gen-cart/blob/master/modelstructure.pdf`, 2019.

15 Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical Type Theory: A Constructive Interpretation of the Univalence Axiom. In *Types for Proofs and Programs (TYPES 2015)*, volume 69 of *LIPIcs*, pages 5:1–5:34, 2018. `doi:10.4230/LIPIcs.TYPES.2015.5`.

16 Thierry Coquand. Variation on cubical sets. Unpublished note available at `www.cse.chalmers.se/~coquand/diag1.pdf`, 2014.

17 Thierry Coquand. Quillen model structure. Link and discussion available at `https://groups.google.com/forum/#!msg/homotopytypetheory/RQkLWZ_83kQ/tAyb3zYTBQAJ`, 2018.

18 Thierry Coquand, Simon Huber, and Anders Mörtberg. On Higher Inductive Types in Cubical Type Theory. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, pages 255–264. ACM, 2018. URL: `http://doi.acm.org/10.1145/3209108.3209197`, `doi:10.1145/3209108.3209197`.

19 Thierry Coquand, Simon Huber, and Christian Sattler. Homotopy Canonicity for Cubical Type Theory. In Herman Geuvers, editor, *4th International Conference on Formal Structures for Computation and Deduction, FSCD 2019, June 24-30, 2019, Dortmund, Germany.*, volume 131 of *LIPIcs*, pages 11:1–11:23. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2019. URL: `http://drops.dagstuhl.de/opus/volltexte/2019/10518`, `doi:10.4230/LIPIcs.FSCD.2019.11`.

20 Albrecht Dold and Rene Thom. Quasifaserungen und unendliche symmetrische produkte. *Annals of Mathematics*, 67(2):239–281, 1958. URL: `http://www.jstor.org/stable/1970005`, `doi:10.2307/1970005`.

21 Peter Dybjer. Internal type theory. In *Lecture Notes in Computer Science*, pages 120–134. Springer Verlag, Berlin, Heidelberg, New York, 1996. `doi:10.1007/3-540-61780-9_66`.

22 Dan Frumin and Benno Van den Berg. A homotopy-theoretic model of function extensionality in the effective topos. *Mathematical Structures in Computer Science*, 29(4):588–614, 2019. `doi:10.1017/S0960129518000142`.

23 Nicola Gambino and Simon Henry. Towards a constructive simplicial model of Univalent Foundations. Preprint arXiv:1905.06281 [math.CT], 2019.

24 Nicola Gambino and Christian Sattler. The Frobenius condition, right properness, and uniform fibrations. *Journal of Pure and Applied Algebra*, 221(12):3027 – 3068, 2017. `doi:10.1016/j.jpaa.2017.02.013`.

25 Martin Hofmann. Syntax and semantics of dependent types. In A.M. Pitts and P. Dybjer, editors, *Semantics and logics of computation*, volume 14 of *Publ. Newton Inst.*, pages 79–130. Cambridge University Press, Cambridge, 1997. Papers from the Summer School held at the University of Cambridge, Cambridge, September 1995. `doi:10.1017/CBO9780511526619.004`.

26 Chris Kapulkin and Peter LeFanu Lumsdaine. The simplicial model of univalent foundations (after Voevodsky). Preprint arXiv:1211.2851v4 [math.LO], November 2012.

27 Daniel R. Licata, Ian Orton, Andrew M. Pitts, and Bas Spitters. Internal Universes in Models of Homotopy Type Theory. In *FSCD*, volume 108 of *LIPIcs*, pages 22:1–22:17. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2018. `doi:10.4230/LIPIcs.FSCD.2018.22`.

28 Ian Orton and Andrew M. Pitts. Axioms for Modelling Cubical Type Theory in a Topos. *Logical Methods in Computer Science*, Volume 14, Issue 4, December 2018. URL: `https://lmcs.episciences.org/5028`, `doi:10.23638/LMCS-14(4:23)2018`.

29 Jason Parker. Duality between Cubes and Bipointed Sets. Master's thesis, Carnegie Mellon University, 2014.

30 C. L. Reedy. Homotopy theory of model categories. Unpublished note available at `http://www-math.mit.edu/~psh/reedy.pdf`, 1974.

31 Emily Riehl. *Categorical Homotopy Theory*. New Mathematical Monographs. Cambridge University Press, 2014. `doi:10.1017/CBO9781107261457`.

32 Christian Sattler. The Equivalence Extension Property and Model Structures. Preprint arXiv:1704.06911v1 [math.CT], 2017.

33 Andrew Swan. Identity types in algebraic model structures and cubical sets. Preprint arXiv:1808.00915 [math.CT], August 2018.

34 Andrew Swan. Lifting problems in Grothendieck fibrations. Preprint arXiv:1802.06718 [math.CT], February 2018.

35 Andrew Swan. Separating Path and Identity Types in Presheaf Models of Univalent Type Theory. Preprint arXiv:1808.00920 [math.LO], 2018.

36 Andrew Swan. W-types with reductions and the small object argument. Preprint arXiv:1802.07588 [math.CT], February 2018.

37 Taichi Uemura. Cubical Assemblies and Independence of the Propositional Resizing Axiom. Preprint arXiv:1803.06649 [cs.LO], 2018.

38 The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `http://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

39 Vladimir Voevodsky. An experimental library of formalized mathematics based on the univalent foundations. *Mathematical Structures in Computer Science*, 25:1278–1294, 2015. `doi:10.1017/S0960129514000577`.