

# A UNIFYING CARTESIAN CUBICAL TYPE THEORY

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**ABSTRACT.** This note presents a univalent type theory based on cartesian cubical sets. The difference from earlier work on similar models is that it depends neither on diagonal cofibrations nor on connections or reversals. In the presence of these additional structures, our notion of fibration coincides with that of the existing cartesian and De Morgan cubical set models. This work can therefore be seen as a generalization of both models which also clarifies the connection between them.

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## 1. INTRODUCTION

Cubical type theory and constructive cubical set models of type theory have been developed in three flavors, which vary principally in their choice of structure on the interval. The pioneering work of [BCH14, BCH18] uses a substructural (monoidal) interval, while the later work of [AFH18, ABC<sup>+</sup>17] use a structural (cartesian) interval, and [CCHM18] uses a structural (cartesian) interval with additional De Morgan structure (connections and reversals). Different interval structures lead to different

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definitions of fibrations and thus different definitions of the Kan operations for the standard type formers; in particular, [AFH18, ABC<sup>+</sup>17] use so-called diagonal cofibrations in order to model univalent universes. However, many of the constructions follow roughly the same logic, especially in the cartesian variants. For the purpose of proving theorems about the theories and their models, we would like to view them as instances of a more general construction. In this note, we do so for the cartesian cubical type theories,<sup>1</sup> presenting a type theory which is a subsystem of both but has sufficient structure to accommodate all the usual type formers, including the universe and **Glue** types used to prove the univalence axiom.

The main idea is to use *weak Kan composition*, a relaxation of the Kan operation used in [AFH18, ABC<sup>+</sup>17]. When the type theory is extended with diagonal cofibrations, this weak composition is inter-derivable with the ordinary composition of [AFH18, ABC<sup>+</sup>17]; when it is extended with connections and reversals, it is inter-derivable with the composition operations used in [CCHM18].<sup>2</sup>

From the perspective of practical implementation and usability, the type theory we present is greatly inferior to the theories it generalizes: equalities that are strict in the specialized theories here only hold up to paths, so additional path algebra becomes necessary to implement composition at the various types. Our goal is rather to present a theory with which the mathematical properties of the various cartesian cubical type theories and models can be studied simultaneously.

**Preliminaries.** We assume a discrete countably infinite set of dimension variables  $(i, j, k, \dots)$ . The interval  $\mathbb{I}$  is described by the following grammar; we write “ $\dots$ ” to stress that the definition of the type theory works for any extension of the interval grammar (e.g., with connections), so long as the quantifier elimination property stated below is satisfied.

$$r, s, t ::= 0 \mid 1 \mid i \mid \dots$$

Given a term  $u$ , we write  $u(r/i)$  for  $u$  with  $r$  substituted for  $i$ . Dimension substitutions are structural, i.e., the interval is cartesian.

The face lattice  $\mathbb{F}$  is described by the following grammar.

$$\varphi, \psi ::= \perp \mid \top \mid (r = 0) \mid (r = 1) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \dots$$

Following [ABC<sup>+</sup>17], we call elements of  $\mathbb{F}$  “cofibrations”, the intuition being that they form the generating cofibrations of a model structure. The definitional equality on cofibrations can be any decidable equivalence relation containing the distributive lattice laws and the equation  $(r = 0) \wedge (r = 1) = \perp$ .<sup>3</sup> We require a quantifier elimination property per [CCHM18]: for any  $\varphi$  varying in a dimension  $i$ , there must

<sup>1</sup>We intentionally blur the distinction between the type theory and its intended model. Although the presentation of this note is type theoretical, it can also be seen as presenting a semantics in the internal language of the topos of cartesian cubical sets following [OP16]. Furthermore, as the interval is tiny in cartesian cubical sets we can construct a universe internally following [LOPS18, Remark 6.2] as is done in [ABC<sup>+</sup>17].

<sup>2</sup>Substructural cubical type theory sadly remains out of reach. There is a natural way to formulate weak composition in the [BCH14] model, but it is not clear to us whether this is equivalent to the usual composition, and the definitions of the Kan operations at many type formers presented here would violate the variable freshness requirements necessary in that setting.

<sup>3</sup>Following [AFH18, ABC<sup>+</sup>17], it is possible to avoid relying on  $\wedge$  and non-trivial equality of cofibrations except in the definition of the identity type (Section 3.5). We include them only for convenience.

exist a cofibration  $\forall i.\varphi$  such that, for any  $\psi$  not mentioning  $i$ , we have  $\psi \leq \varphi$  iff  $\psi \leq \forall i.\varphi$ .

We use the same notations for context restrictions, partial elements, systems and boundary conditions as in [CCHM18].

## 2. WEAKENING THE CARTESIAN KAN OPERATIONS

The key operation is **weak** cartesian Kan composition:

$$\frac{\Gamma, i : \mathbb{I} \vdash A \quad \Gamma \vdash r : \mathbb{I} \quad \Gamma \vdash s : \mathbb{I} \quad \Gamma \vdash \varphi : \mathbb{F} \quad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \quad \Gamma \vdash u_0 : A(r/i)[\varphi \mapsto u(r/i)]}{\Gamma \vdash \mathbf{wcom}_i^{r \rightarrow s} A [\varphi \mapsto u] u_0 : A(s/i)[\varphi \mapsto u(s/i)]}$$

Note that  $i$  is bound in both  $A$  and  $u$ . This is a “weakening” of the cartesian Kan operations of [AFH18, ABC<sup>+</sup>17] in that we don’t strictly require the equality  $\mathbf{wcom}_i^{r \rightarrow r} A [\varphi \mapsto u] u_0 = u_0$ . Instead, a second operation enforces that this holds up to a line that is constant on  $\varphi$ .

$$\frac{\Gamma, i : \mathbb{I} \vdash A \quad \Gamma \vdash r : \mathbb{I} \quad \Gamma \vdash t : \mathbb{I} \quad \Gamma \vdash \varphi : \mathbb{F} \quad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \quad \Gamma \vdash u_0 : A(r/i)[\varphi \mapsto u(r/i)]}{\Gamma \vdash \mathbf{wcom}_i^{r;t} A [\varphi \mapsto u] u_0 : A(r/i)[\varphi \mapsto u(r/i)]}$$

We add two additional constraints to make this term into a line connecting the trivial composition and cap.

$$\begin{aligned} \Gamma, (t = 0) &\vdash \mathbf{wcom}_i^{r;t} A [\varphi \mapsto u] u_0 = \mathbf{wcom}_i^{r \rightarrow r} A [\varphi \mapsto u] u_0 : A(r/i) \\ \Gamma, (t = 1) &\vdash \mathbf{wcom}_i^{r;t} A [\varphi \mapsto u] u_0 = u_0 : A(r/i) \end{aligned}$$

In particular, we can abstract to form a path as follows.

$$\langle j \rangle \mathbf{wcom}_i^{r;j} A [\varphi \mapsto u] u_0 : \mathbf{Path} \ A(r/i) \ (\mathbf{wcom}_i^{r \rightarrow r} A [\varphi \mapsto u] u_0) \ u_0$$

We refer to  $\mathbf{wcom}$  as the “cap path” (with the intuition that  $\mathbf{wcom}$  is a path that fixes the cap of an open box). Geometrically it seems very natural to require this strictly: transporting/coercing an element from  $r$  to  $r$  should intuitively not do anything. So it is a bit surprising that requiring this weakly gives us the flexibility needed to generalize the existing models.

We say that a type is *fibrant* if it can be equipped with  $\mathbf{wcom}$  and  $\mathbf{wcom}$  operations. Note that being fibrant is a *structure*, not only a property of a type.

We will first compare the weak composition operation with the Kan operations of the existing systems, showing that it is inter-derivable with strict cartesian composition in the presence of diagonal cofibrations and with [CCHM18] composition in the presence of connections and reversals. (It is sufficient here to speak of logical equivalence, as all these variants of composition are uniquely characterized up to path equality.) This is a refinement of the observation in [ABC<sup>+</sup>17, §3.4] that [CCHM18] and cartesian composition are equivalent in the presence of both extensions. The rest of the note is devoted to proving that  $\mathbf{wcom}$  and  $\mathbf{wcom}$  are definable for all the type formers of cubical type theory:  $\mathbf{N}$ ,  $\mathbf{Path}$ ,  $\Sigma$ ,  $\Pi$ ,  $\mathbf{Id}$ ,  $\mathbf{Glue}$ ,  $\mathcal{U}$  and  $\mathbb{S}^1$ . This hence means that all of these types are fibrant in the model.

**2.1. Strict composition from weak composition.** When terms  $r, s : \mathbb{I}$  are such that  $(r = s)$  is a cofibration, we can “correct” weak composition from  $r$  to  $s$  to derive strict composition from  $r$  to  $s$ . Given an open box, abbreviate the outputs of weak composition as follows.

$$w := \text{wcom}_i^{r \rightarrow s} A [\varphi \mapsto u] u_0$$

$$\underline{w}^j := \underline{\text{wcom}}_i^{r:j} A [\varphi \mapsto u] u_0$$

We define the strict composition by adjusting the  $(r = s)$  face of  $w$  with  $\underline{w}^j$ .

$$\text{com}_i^{r \rightarrow s} A [\varphi \mapsto u] u_0 := \text{wcom}_j^{0 \rightarrow 1} A(s/i) [\varphi \mapsto u(s/i), (r = s) \mapsto \underline{w}^j] w$$

In particular, strict composition  $\text{com}^{\varepsilon \rightarrow s}$  and  $\text{com}^{r \rightarrow \varepsilon}$  are derivable for  $\varepsilon \in \{0, 1\}$ . (Note that the weak compositions  $\text{wcom}^{0 \rightarrow 1}$  and  $\text{wcom}^{1 \rightarrow 0}$  are already strict, as the cap condition is vacuous.) When  $(r = s)$  is a cofibration for all  $r, s$  (i.e., we have *diagonal cofibrations*), as it is in [AFH18, ABC<sup>+</sup>17], weak composition and strict composition are inter-derivable. (This is already observed for coercion in [ABC<sup>+</sup>17, Section 2.7].)

**2.2. Comparison with the cartesian Kan operations.** As observed above, if the language of cofibrations includes diagonal equations  $(r = s)$ , then we can derive strict composition  $\text{com}^{r \rightarrow s}$  from weak composition. Of course, we can always go in the reverse direction, deriving weak composition from strict composition.

$$\text{wcom}_i^{r \rightarrow s} A [\varphi \mapsto u] u_0 := \text{com}_i^{r \rightarrow s} A [\varphi \mapsto u] u_0$$

$$\underline{\text{wcom}}_i^{r:-} A [\varphi \mapsto u] u_0 := \text{com}_i^{r \rightarrow r} A [\varphi \mapsto u] u_0$$

With diagonal cofibrations, a type is therefore Kan in our sense if and only if it is Kan in the sense of [ABC<sup>+</sup>17]. The Kan operation in [ABC<sup>+</sup>17] is inter-derivable with the decomposed  $\text{hcom}^{r \rightarrow s}$  and  $\text{coe}^{r \rightarrow s}$  used in [AFH18], so this pair is also equivalent to weak composition.

**2.3. Comparison with the CCHM Kan operations.** As shown in [CCHM18], one can derive a strict composition  $\text{com}^{0 \rightarrow 1}$  from  $\text{com}^{1 \rightarrow 0}$  using reversals. From there, we can derive weak composition as follows. Let  $i : \mathbb{I} \vdash A$ ,  $r : \mathbb{I}$ ,  $\varphi : \mathbb{F}$ ,  $\varphi, i : \mathbb{I} \vdash u : A$ , and  $u_0 : A(r/i) [\varphi \mapsto u(r/i)]$  be given. First, we define the following term, which composes from  $A(r/i)$  to  $A(t \wedge r/i)$  for any  $t : \mathbb{I}$ .

$$q^t := \text{com}_i^{1 \rightarrow 0} A((t \vee i) \wedge r/i) \left[ \begin{array}{cc} \varphi & \mapsto u((t \vee i) \wedge r/i) \\ (t = 1) & \mapsto u_0 \end{array} \right] u_0$$

We can then define weak composition to  $s : \mathbb{I}$ .

$$\text{wcom}_i^{r \rightarrow s} A [\varphi \mapsto u] u_0 := \text{com}_i^{0 \rightarrow 1} A(i \wedge s/i) \left[ \begin{array}{cc} \varphi & \mapsto u(i \wedge s/i) \\ (0 = 1) & \mapsto \perp \end{array} \right] q^0$$

The cap path at  $t : \mathbb{I}$  is defined as follows.

$$\underline{\text{wcom}}_i^{r:t} A [\varphi \mapsto u] u_0 := \text{com}_i^{0 \rightarrow 1} A((t \vee i) \wedge r/i) \left[ \begin{array}{cc} \varphi & \mapsto u((t \vee i) \wedge r/i) \\ (t = 1) & \mapsto u_0 \end{array} \right] q^t$$

The same argument shows that weak composition is equivalent to a pair of operations  $\text{comp}^{0 \rightarrow 1}$  and  $\text{comp}^{1 \rightarrow 0}$  in cubical type theory with connections but not reversals (i.e., based on the distributive lattice or “Dedekind” cube category). As in the cartesian case, the composition used in [CCHM18] is in turn inter-derivable with a decomposed pair of operations  $\text{hcomp}^{0 \rightarrow 1}$  and  $\text{trans}^{0 \rightarrow 1}$  [CHM18].

**2.4. Homogeneous composition and coercion.** As with strict cartesian composition, weak composition can be decomposed into weak homogeneous composition and coercion. The former is the special case of composition where the type line  $i : \mathbb{I} \vdash A$  is constant; the latter, where the boundary  $\varphi$  is empty.

$$\begin{aligned} \text{whcom}_i^{r \rightarrow s} A [\varphi \mapsto u] u_0 &:= \text{wcom}_i^{r \rightarrow s} A [\varphi \mapsto u] u_0 \\ \underline{\text{whcom}}_i^{r;t} A [\varphi \mapsto u] u_0 &:= \underline{\text{wcom}}_i^{r;t} A [\varphi \mapsto u] u_0 \\ \text{wcoe}_i^{r \rightarrow s} A u_0 &:= \text{wcom}_i^{r \rightarrow s} A [] u_0 \\ \underline{\text{wcoe}}_i^{r;t} A u_0 &:= \underline{\text{wcom}}_i^{r;t} A [] u_0 \end{aligned}$$

Note that we can derive strict homogeneous composition  $\text{hcom}^{\varepsilon \rightarrow s}$  and  $\text{hcom}^{r \rightarrow \varepsilon}$  from these just as in Section 2.1 (and likewise for coercion); this will be convenient in the following.

Given weak homogeneous composition and coercion for a given type, we can derive general weak composition. Given the usual arguments to a weak composition, define the following auxiliary terms.

$$\begin{aligned} h &:= \text{whcom}_i^{r \rightarrow s} A(s/i) [\varphi \mapsto \text{wcoe}_i^{i \rightarrow s} A u] (\text{wcoe}_i^{r \rightarrow s} A u) \\ \underline{h} &:= \underline{\text{whcom}}_i^{r;t} A(r/i) [\varphi \mapsto \text{wcoe}_i^{i \rightarrow r} A u] (\text{wcoe}_i^{r \rightarrow r} A u) \end{aligned}$$

We define the weak composition as follows.

$$\text{wcom}_i^{r \rightarrow s} A [\varphi \mapsto u] u_0 := \text{hcom}_j^{0 \rightarrow 1} A(s/i) [\varphi \mapsto \underline{\text{wcoe}}_i^{s;j} A u] h$$

For the cap path, we define the following.

$$\begin{aligned} c &:= \text{hcom}_j^{0 \rightarrow 1} A(r/i) [\varphi \mapsto \underline{\text{wcoe}}_i^{r;j} A u] \underline{h} \\ d^k &:= \text{hcom}_j^{k \rightarrow 1} A(r/i) [\varphi \mapsto \underline{\text{wcoe}}_i^{r;j} A u] (\underline{\text{wcoe}}_i^{r;k} A u) \end{aligned}$$

So that

$$\begin{aligned} &\underline{\text{wcom}}_i^{r;t} A [\varphi \mapsto u] u_0 = \\ &\text{whcom}_k^{0 \rightarrow 1} A(r/i) [\varphi \mapsto u(r/i), (t = 0) \mapsto c, (t = 1) \mapsto d^k] c \end{aligned}$$

Because of the observations in this section, the weak Kan operations can be seen as sitting in between the Kan operations of [AFH18, ABC<sup>+</sup>17] and [CCHM18], clarifying the close relationship between the two models. We will now show that the weak Kan operations are closed under all of the type formers of cubical type theory and hence provide a model of univalent type theory with a simple higher inductive type.

### 3. BASIC TYPE FORMERS

We now explain  $\text{wcom}_i^{r \rightarrow s} A [\varphi \mapsto u] u_0$  and  $\underline{\text{wcom}}_i^{r;t} A [\varphi \mapsto u] u_0$  by cases on the type  $A$  for natural numbers, Path-types,  $\Sigma$ -types,  $\Pi$ -types and Id-types.

**3.1. Natural numbers.** For the (strict) natural numbers we take

$$\begin{aligned} \text{wcom}_i^{r \rightarrow s} \mathbf{N} [\varphi \mapsto 0] 0 &= 0 \\ \text{wcom}_i^{r \rightarrow s} \mathbf{N} [\varphi \mapsto S \ n] (S \ n_0) &= S(\text{wcom}_i^{r \rightarrow s} \mathbf{N} [\varphi \mapsto n] n_0) \end{aligned}$$

together with

$$\begin{aligned} \underline{\text{wcom}}_i^{r;t} \mathbf{N} [\varphi \mapsto 0] 0 &= 0 \\ \underline{\text{wcom}}_i^{r;t} \mathbf{N} [\varphi \mapsto S \ n] (S \ n_0) &= S(\underline{\text{wcom}}_i^{r;t} \mathbf{N} [\varphi \mapsto n] n_0) \end{aligned}$$

**3.2. Dependent paths.** Let  $\Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash A$ ,  $\Gamma, i : \mathbb{I} \vdash v : A(0/j)$  and  $\Gamma, i : \mathbb{I} \vdash w : A(1/j)$ . We define weak composition in the **Path** type and its cap path as follows.

$$\begin{aligned} \text{wcom}_i^{r \rightarrow s} (\text{Path}^j \ A \ v \ w) [\varphi \mapsto u] u_0 &= \\ \langle j \rangle \text{wcom}_i^{r \rightarrow s} A [\varphi \mapsto u \ j, (j = 0) \mapsto v, (j = 1) \mapsto w] (u_0 \ j) \end{aligned}$$

$$\begin{aligned} \underline{\text{wcom}}_i^{r;t} (\text{Path}^j \ A \ v \ w) [\varphi \mapsto u] u_0 &= \\ \langle j \rangle \underline{\text{wcom}}_i^{r;t} A [\varphi \mapsto u \ j, (j = 0) \mapsto v, (j = 1) \mapsto w] (u_0 \ j) \end{aligned}$$

In this case, the definition of weak composition is exactly that used for strict composition in existing cubical type theories.

**3.3. Dependent pairs.** Let  $\Gamma, i : \mathbb{I} \vdash A$  and  $\Gamma, i : \mathbb{I}, x : A \vdash B$ . We first define the composite and cap path for the first components of the open box.

$$\begin{aligned} w_A^i &:= \text{wcom}_i^{r \rightarrow i} A [\varphi \mapsto u.1] u_0.1 \\ \underline{w}_A^j &:= \underline{\text{wcom}}_i^{r;j} A [\varphi \mapsto u.1] u_0.1 \end{aligned}$$

To define the composite of the second components, we first need to adjust the type of the cap. For this, we use a strict composition from 1 to  $k$ , which is derivable from weak composition in  $B$  per Section 2.1.

$$b^k := \text{com}_j^{1 \rightarrow k} B(r/i)(\underline{w}_A^j/x) [\varphi \mapsto u.2(r/i)] u_0.2$$

The  $(0/k)$  instance is the cap of our composition in  $B$ .

$$\begin{aligned} w_B &:= \text{wcom}_i^{r \rightarrow s} B(w_A^i/x) [\varphi \mapsto u.2] b^0 \\ \underline{w}_B^j &:= \underline{\text{wcom}}_i^{r;j} B(w_A^i/x) [\varphi \mapsto u.2] b^0 \end{aligned}$$

Composition in the pair type is then defined to be the pair of compositions in  $A$  and  $B$ .

$$\text{wcom}_i^{r \rightarrow s} ((x : A) \times B) [\varphi \mapsto u] u_0 = (w_A^s, w_B)$$

For the cap path, we first combine the cap path  $\underline{w}_B^j$  for the composition in  $B$  with the path  $b^t$  that relates  $b^0$  to  $u_0.2$  over  $\underline{w}_A^t$ .

$$c^t := \text{wcom}_j^{1 \rightarrow 0} B(r/i)(\underline{w}_A^t/x) [\varphi \mapsto u.2(r/i), (t = 0) \mapsto \underline{w}_B^j, (t = 1) \mapsto u_0.2] b^t$$

We then let

$$\underline{\text{wcom}}_i^{r;t} ((x : A) \times B) [\varphi \mapsto u] u_0 = (\underline{w}_A^t, c^t)$$

**3.4. Dependent functions.** Let  $\Gamma, i : \mathbb{I} \vdash A$  and  $\Gamma, i : \mathbb{I}, x : A \vdash B$ . For any  $x : A(s/i)$  we define its coercion backward over  $A$  as follows.

$$\begin{aligned} w_A^i &:= \text{wcoe}_i^{s \rightarrow i} A x \\ \underline{w}_A^j &:= \underline{\text{wcoe}}_i^{s;j} A x \end{aligned}$$

For any  $x : A(s/i)$ , we can define a term in  $B(s/i)(w_A^s/x)$  using composition in  $B$ .

$$\begin{aligned} w_B^i &:= \text{wcom}_i^{r \rightarrow i} B(w_A^i/x) [\varphi \mapsto u w_A^i] (u_0 w_A^r) \\ \underline{w}_B^j &:= \underline{\text{wcom}}_i^{r;j} B(w_A^i/x) [\varphi \mapsto u w_A^i] (u_0 w_A^r) \end{aligned}$$

Finally, for any  $k : \mathbb{I}$ ,  $x : A(s/i)$  and  $y : B(s/i)(\underline{w}_A^k/x) [\varphi \mapsto u(s/i) \underline{w}_A^k]$ , we can adjust its type and boundary using strict composition in  $B$  to obtain a term in  $B(s/i) [\varphi \mapsto u(s/i) x]$ .

$$b^k := \text{com}_j^{k \rightarrow 1} B(s/i)(\underline{w}_A^j/x) [\varphi \mapsto u(s/i) \underline{w}_A^j] y$$

We now define the composition in the function type.

$$\text{wcom}_i^{r \rightarrow s} ((x : A) \rightarrow B) [\varphi \mapsto u] u_0 = \lambda(x : A(s/i)). b^0(w_B^s/y)$$

For the cap path, we use a composition in  $B(r/i)$  to combine  $\underline{w}_A^t$  and  $\underline{w}_B^j$  while maintaining the required behavior on  $\varphi$ .

$$\underline{\text{wcom}}_i^{r;t} ((x : A) \rightarrow B) [\varphi \mapsto u] u_0 = \lambda(x : A(r/i)).$$

$$\text{wcom}_j^{1 \rightarrow 0} B(r/i) [\varphi \mapsto u(r/i) x, (t = 0) \mapsto b^0(\underline{w}_B^j/y), (t = 1) \mapsto u_0 x] (b^t(u_0 \underline{w}_A^t/y))$$

**3.5. Identity types.** We adapt the identity types presented in [ABC<sup>+</sup>17, §2.16], which originated in [CCHM18, §9] based on discussions with Swan [Swa18, §6]. Let  $\Gamma, i : \mathbb{I} \vdash A$ ,  $\Gamma, i : \mathbb{I} \vdash v : A$  and  $\Gamma, i : \mathbb{I} \vdash w : A$  be given. We define the weak composition as

$$\begin{aligned} \text{wcom}_i^{r \rightarrow s} (\text{Id } A v w) [\varphi \mapsto (\psi, p)] (\psi_0, p_0) &= (\varphi \wedge \psi(s/i), \alpha) \\ \underline{\text{wcom}}_i^{r;t} (\text{Id } A v w) [\varphi \mapsto (\psi, p)] (\psi_0, p_0) &= (\varphi \wedge \psi(r/i) \vee ((t = 1) \wedge \psi_0), \underline{\alpha}) \end{aligned}$$

where

$$\begin{aligned} \alpha &:= \text{wcom}_i^{r \rightarrow s} (\text{Path } A v w) [\varphi \mapsto p] p_0 \\ \underline{\alpha} &:= \underline{\text{wcom}}_i^{r;t} (\text{Path } A v w) [\varphi \mapsto p] p_0 \end{aligned}$$

The definition in [ABC<sup>+</sup>17] of the J eliminator for this type does not use diagonal cofibrations, and it only uses compositions which are available to us strictly by Section 2.1. It can therefore be repeated in this setting unchanged.

#### 4. UNIVALENT FIBRANT UNIVERSES

We use the following internal definition of equivalence.

$$\begin{aligned} A \simeq B &:= (e : A \rightarrow B) \times \text{IsEquiv } e \\ \text{IsEquiv } e &:= (x : B) \rightarrow \text{IsContr } (\text{Fiber } e x) \\ \text{IsContr } C &:= (x : C) \times ((y : C) \rightarrow \text{Path } C y x) \\ \text{Fiber } e x &:= (y : A) \times \text{Path } B (e y) x \end{aligned}$$

We write  $1_A$  for the identity equivalence. For simplicity, we write  $e u$  for  $e.1 u$ .

**4.1. Glue types.** The typing rules for Glue types are the same as in [CCHM18] and are given in Figure 1.

$\frac{\Gamma \vdash A \quad \Gamma, \varphi \vdash T \quad \Gamma, \varphi \vdash e : \simeq T \ A}{\Gamma \vdash \text{Glue}[\varphi \mapsto (T, e)] \ A}$	$\frac{\Gamma \vdash b : \text{Glue}[\varphi \mapsto (T, e)] \ A}{\Gamma \vdash \text{unglue}[\varphi \mapsto e] \ b : A[\varphi \mapsto e \ b]}$
$\frac{\Gamma, \varphi \vdash e : T \simeq A \quad \Gamma, \varphi \vdash u : T \quad \Gamma \vdash a : A[\varphi \mapsto e \ u]}{\Gamma \vdash \text{glue}[\varphi \mapsto u] \ a : \text{Glue}[\varphi \mapsto (T, e)] \ A}$	
$\frac{\Gamma \vdash T \quad \Gamma \vdash e : T \simeq A}{\Gamma, \varphi \vdash \text{Glue}[\varphi \mapsto (T, e)] \ A = T}$	$\frac{\Gamma \vdash u : T \quad \Gamma \vdash e : T \simeq A}{\Gamma, \varphi \vdash \text{glue}[\varphi \mapsto u] \ (e \ u) = u : T}$
$\frac{\Gamma \vdash b : \text{Glue}[\varphi \mapsto (T, e)] \ A}{\Gamma \vdash b = \text{glue}[\varphi \mapsto b] \ (\text{unglue}[\varphi \mapsto e] \ b) : \text{Glue}[\varphi \mapsto (T, e)] \ A}$	
$\frac{\Gamma, \varphi \vdash e : T \simeq A \quad \Gamma, \varphi \vdash u : T \quad \Gamma \vdash a : A[\varphi \mapsto e \ u]}{\Gamma \vdash \text{unglue}[\varphi \mapsto e] \ (\text{glue}[\varphi \mapsto u] \ a) = a : A}$	

FIGURE 1. Typing rules for Glue types

**4.2. Fibrancy of Glue types.** We omit the ambient context  $\Gamma$  and assume given the following.

$$i : \mathbb{I} \vdash A \quad i : \mathbb{I} \vdash \varphi : \mathbb{F} \quad i : \mathbb{I}, \varphi \vdash T \quad i : \mathbb{I}, \varphi \vdash e : T \simeq A$$

We abbreviate  $i : \mathbb{I} \vdash G := \text{Glue}[\varphi \mapsto (T, e)] \ A$ . Suppose we have arguments to a weak composition as follows.

$$\vdash r : \mathbb{I} \quad \vdash \psi : \mathbb{F} \quad \psi, i : \mathbb{I} \vdash u : G \quad \vdash u_0 : G(r/i)[\psi \mapsto u(r/i)]$$

For **wcom** we additionally assume  $s : \mathbb{I}$ ; for **wcom** we assume  $t : \mathbb{I}$ . We begin by giving the definitions of **wcom** and **wcom** in full, after which we demonstrate that the definitions are type-correct and satisfy the required boundary conditions.

**Definition 1** (Composition for Glue types). The construction of the weak composition for  $G$  is performed in several steps.

$$w_T^i := \text{wcom}_i^{r \rightarrow i} T [\psi \mapsto u] u_0 \tag{1}$$

$$\underline{w}_T^j := \underline{\text{wcom}}_i^{r:j} T [\psi \mapsto u] u_0 \tag{1}$$

$$a_0 := \text{unglue}[\varphi(r/i) \mapsto e(r/i)] u_0 \tag{2a}$$

$$a := \text{unglue}[\varphi \mapsto e] u \tag{2b}$$

$$\tilde{a}^j := \text{com}_j^{1 \rightarrow j} A(r/i) [\psi \mapsto a_0, \forall i. \varphi \mapsto e(r/i) \ \underline{w}_T^j] a_0 \tag{3}$$

$$w_A := \text{wcom}_i^{r \rightarrow s} A [\psi \mapsto a, \forall i. \varphi \mapsto e \ w_T^i] \tilde{a}^0 \tag{4}$$

$$(C_1, C_2) := e(s/i).2 w_A \tag{5}$$

$$R := \text{wcom}_k^{1 \rightarrow 0} (\text{Fiber } e(s/i) \ w_A) \left[ \begin{array}{l} \psi \mapsto C_2(u(s/i), \langle \_ \rangle w_A) k \\ \forall i. \varphi \mapsto C_2(w_T^s, \langle \_ \rangle w_A) k \end{array} \right] C_1 \tag{6}$$

$$w'_A := \text{wcom}_k^{1 \rightarrow 0} A(s/i) [\psi \mapsto a(s/i), \varphi(s/i) \mapsto R.2 k] w_A \tag{7}$$



As the final step, we define the composition as follows.

$$\mathbf{wcom}_i^{r \rightarrow s} G [\varphi \mapsto u] u_0 = \mathbf{glue} [\varphi(s/i) \mapsto R.1] w'_A$$

The definition of the cap path follows a similar structure.

$$\underline{w}_A^j := \mathbf{wcom}_i^{r:j} A [\psi \mapsto a, \forall i. \varphi \mapsto e w_T^i] \tilde{a}^0 \quad (4a)$$

$$d := \mathbf{wcom}_j^{1 \rightarrow 0} A(r/i) [(t=0) \mapsto \underline{w}_A^j, (t=1) \vee \psi \vee \forall i. \varphi \mapsto \tilde{a}^t] \tilde{a}^t \quad (4b)$$

$$(\underline{C}_1, \underline{C}_2) := e(r/i).2 d \quad (5)$$

$$\underline{R} := \mathbf{wcom}_k^{1 \rightarrow 0} (\text{Fiber } e(r/i) d) \left[ \begin{array}{ll} \psi & \mapsto \underline{C}_2 (u(r/i), \langle \_ \rangle d) k \\ \forall i. \varphi & \mapsto \underline{C}_2 (\underline{w}_T^t, \langle \_ \rangle d) k \\ (t=1) & \mapsto \underline{C}_2 (u_0, \langle \_ \rangle d) k \end{array} \right] \underline{C}_1 \quad (6)$$

$$\underline{w}'_A := \mathbf{wcom}_k^{1 \rightarrow 0} A(r/i) [\psi \mapsto a(r/i), \varphi(r/i) \mapsto \underline{R}.2 k, (t=1) \mapsto a_0] d \quad (7)$$

$$\mathbf{wcom}_i^{r:t} G [\varphi \mapsto u] u_0 = \mathbf{glue} [\varphi(r/i) \mapsto \underline{R}.1] \underline{w}'_A$$

**Theorem 2** (Correctness of composition for Glue types). *The definitions of the composition and cap path in  $G$  are well-formed and have the correct boundary.*

*Proof.* As with the definition, we proceed in steps.

**Steps 1 and 1.** We have  $i : \mathbb{I}, \varphi \vdash u : T$  and  $\varphi(r/i) \vdash u_0 : T(r/i)$ , so on  $\forall i. \varphi$  we can form the weak composition and cap path in  $T$ .

$$i : \mathbb{I}, \forall i. \varphi \vdash w_T^i := \mathbf{wcom}_i^{r \rightarrow i} T [\psi \mapsto u] u_0 : T[\psi \mapsto u]$$

$$j : \mathbb{I}, \forall i. \varphi \vdash \underline{w}_T^j := \mathbf{wcom}_i^{r:j} T [\psi \mapsto u] u_0 : T[\psi \mapsto u(r/i)]$$

**Steps 2a and 2b.** We unglue  $u_0$  and  $u$  to obtain a cap and tube in  $A$ .

$$\vdash a_0 := \mathbf{unglue} [\varphi(r/i) \mapsto e(r/i)] u_0 : A(r/i)[\varphi(r/i) \mapsto e(r/i) u_0]$$

$$i : \mathbb{I}, \psi \vdash a := \mathbf{unglue} [\varphi \mapsto e] u : A[\varphi \mapsto e u]$$

This satisfies

$$\psi \vdash a(r/i) = a_0 : A(r/i)$$

by the adjacency conditions for  $u$  and  $u_0$ .

**Step 3.** We adjust the cap  $a_0$  with a strict composition in  $A(r/i)$  so that it agrees with  $e(r/i) w_T^r$ —rather than  $e(r/i) u_0$ —on  $\forall i. \varphi$ .

$$j : \mathbb{I} \vdash \tilde{a}^j := \mathbf{com}_j^{1 \rightarrow j} A(r/i) [\psi \mapsto a_0, \forall i. \varphi \mapsto e(r/i) \underline{w}_T^j] a_0 : A(r/i)$$

**Step 4.** We now compute the composition of  $\tilde{a}^0$  and the tube  $a$  in  $A$  (in direction  $i$ ), with an additional face on  $\forall i. \varphi$  given by the image of the composition in  $T$  under  $e$ .

$$\vdash w_A := \mathbf{wcom}_i^{r \rightarrow s} A [\psi \mapsto a, \forall i. \varphi \mapsto e w_T^i] \tilde{a}^0 : A(s/i)$$

This is well-formed thanks to the following equations.

$$i : \mathbb{I}, \psi, \forall i. \varphi \vdash a = e u = e w_T^i \quad (\text{Steps 2b, 1})$$

$$\psi \vdash a(r/i) = a_0 = \tilde{a}^0 \quad (\text{Steps 2, 3})$$

$$\forall i. \varphi \vdash (e w_T^i)(r/i) = e(r/i) w_T^r = \tilde{a}^0 \quad (\text{Step 3})$$

**Step 5.** Recall that  $\varphi(s/i) \vdash e(s/i) : T(s/i) \simeq A(s/i)$  and that the fiber type of  $e(s/i)$  over  $w_A$  is

$$\varphi(s/i) \vdash \mathbf{Fiber} \, e(s/i) \, w_A := (x : T(s/i)) \times \mathbf{Path} \, A(s/i) \, (e(s/i) \, x) \, w_A$$

We extract a proof that this type is contractible from the equivalence  $e(s/i)$ .

$$\varphi(s/i) \vdash (C_1, C_2) := e(s/i).2 \, w_A : \mathbf{lsContr} \, (\mathbf{Fiber} \, e(s/i) \, w_A)$$

Here the first component  $C_1 : \mathbf{Fiber} \, e(s/i) \, w_A$  is the center of contraction, while  $C_2$  has type  $(x : \mathbf{Fiber} \, e(s/i) \, w_A) \rightarrow \mathbf{Path} \, (\mathbf{Fiber} \, e(s/i) \, w_A) \, x \, C_1$ .

**Step 6.** Using  $(C_1, C_2)$ , we construct an element of  $\mathbf{Fiber} \, e(s/i) \, w_A$  taking values of our choice on  $\psi$  and  $\forall i. \varphi$ .

$$\varphi(s/i) \vdash R := \mathbf{wcom}_k^{1 \rightarrow 0} \, (\mathbf{Fiber} \, e(s/i) \, w_A) \left[ \begin{array}{l} \psi \mapsto C_2(u(s/i), \langle \_ \rangle w_A) \, k \\ \forall i. \varphi \mapsto C_2(w_T^s, \langle \_ \rangle w_A) \, k \end{array} \right] C_1$$

To see that the pairs  $(u(s/i), \langle \_ \rangle w_A)$  and  $(w_T^s, \langle \_ \rangle w_A)$  are well-typed elements of the fiber, observe the following.

$$\begin{aligned} \varphi(s/i), \psi \vdash e(s/i) \, u(s/i) &= a = w_A : A(s/i) && (\text{Steps 2b, 4}) \\ \varphi(s/i), \forall i. \varphi \vdash e(s/i) \, w_T^s &= w_A : A(s/i) && (\text{Step 4}) \end{aligned}$$

The two faces agree on their intersection because  $\varphi(s/i), \psi, \forall i. \varphi \vdash w_T^s = u(s/i)$  by definition of  $w_T^s$ . The faces agree with the cap by virtue of the type of  $C_2$ .

Note that the two components of  $R$ , which have the types  $\varphi(s/i) \vdash R.1 : T(s/i)$  and  $\varphi(s/i) \vdash R.2 : \mathbf{Path} \, A(s/i) \, (e(s/i) \, R.1) \, w_A$ , satisfy the following.

$$\begin{aligned} \varphi(s/i), \psi \vdash R.1 &= u(s/i) : T(s/i) \\ \varphi(s/i), \forall i. \varphi \vdash R.1 &= w_T^s : T(s/i) \\ \varphi(s/i), \psi \vdash R.2 &= \langle \_ \rangle w_A : \mathbf{Path} \, A(s/i) \, (e(s/i) \, u(s/i)) \, w_A \\ \varphi(s/i), \forall i. \varphi \vdash R.2 &= \langle \_ \rangle w_A : \mathbf{Path} \, A(s/i) \, (e(s/i) \, w_T^s) \, w_A \end{aligned}$$

**Step 7.** We use the second component of  $R$  to build a term in  $A(s/i)[\varphi \mapsto a(s/i)]$  which is in the image of  $e(s/i)$  on  $\varphi(s/i)$ .

$$\vdash w'_A := \mathbf{wcom}_k^{1 \rightarrow 0} \, A(s/i) \, [\psi \mapsto a(s/i), \varphi(s/i) \mapsto R.2 \, k] \, w_A : A(s/i)$$

To see that this composition is well-formed, we verify the following.

$$\begin{aligned} \psi \vdash a(s/i) &= w_A : A(s/i) && (\text{Step 4}) \\ \varphi(s/i) \vdash R.2 \, 1 &= w_A : A(s/i) && (\text{type of } R.2) \\ k : \mathbb{I}, \psi, \varphi(s/i) \vdash a(s/i) &= (e \, u)(s/i) = \\ &= e(s/i) \, w_T^s = w_A = R.2 \, k : A(s/i) && (\text{Steps 2b, 1, 4, 6}) \end{aligned}$$

**Definition of wcom.** Finally, we combine all the pieces we have computed.

$$\vdash \text{wcom}_i^{r \rightarrow s} G [\varphi \mapsto u] u_0 = \text{glue} [\varphi(s/i) \mapsto R.1] w'_A : G(s/i)$$

To see that this term is well-formed, note the following.

$$\varphi(s/i) \vdash e(s/i) R.1 = R.2 \ 0 = w'_A : A(s/i)$$

The first equality holds because  $R.2 : \text{Path } A(s/i) (e(s/i) R.1) w_A$ , the second by definition of  $w_A$ . As required by the typing rule for **wcom**, we have the following.

$$\psi \vdash \text{glue} [\varphi(s/i) \mapsto R.1] w'_A = \text{glue} [\varphi(s/i) \mapsto u(s/i)] a(s/i) = u(i/s) : G(s/i)$$

The first equation holds by Steps 6 and 7, the second by the  $\eta$ -rule for **Glue**. Finally, in order for the definition of **wcom** in  $G$  to be satisfactory, we need it to agree with **wcom** in  $T$  on  $\forall i.\varphi$ .

$$\forall i.\varphi \vdash \text{glue} [\varphi(s/i) \mapsto R.1] w'_A = R.1 = w_T^s = \text{wcom}_i^{r \rightarrow s} T [\psi \mapsto u] u_0 : T(s/i)$$

Here the first equation holds because  $\forall i.\varphi \leq \varphi(s/i)$ , the second is by Step 6, and the third is by definition.

**Step 4a-4b.** We now begin constructing the cap path. Where we refer to terms constructed in the course of defining **wcom**, we mean the case where  $s$  is instantiated with  $r$ . We begin by defining the cap path corresponding to  $w_A$ .

$$\vdash \underline{w}_A^j := \text{wcom}_i^{r:j} A [\psi \mapsto a, \forall i.\varphi \mapsto e \ w_T^i] \tilde{a}^0 : A(r/i)$$

This is well-typed for the same reasons that  $w_A$  is well-typed, and has endpoints  $\underline{w}_A^0 = w_A$  and  $\underline{w}_A^1 = \tilde{a}^0$ . In the second part of this step, we extend that path to connect to  $a_0$ .

$$\vdash d := \text{wcom}_j^{1 \rightarrow 0} A(r/i) [(t=0) \mapsto \underline{w}_A^j, (t=1) \vee \psi \vee \forall i.\varphi \mapsto \tilde{a}^t] \tilde{a}^t : A(r/i)$$

This term is well-typed thanks to the following equations.

$$\begin{aligned} j : \mathbb{I}, (t=0), \psi \vdash \underline{w}_A^j &= a(r/i) = a_0 = \tilde{a}^t : A(r/i) && (\text{Steps 4a, 2, 3}) \\ j : \mathbb{I}, (t=0), \forall i.\varphi \vdash \underline{w}_A^j &= e(r/i) \ w_T^r = e(r/i) \ \underline{w}_T^t = \tilde{a}^t : A(r/i) && (\text{Steps 4a, 3}) \\ (t=0) \vdash \underline{w}_A^j(j/1) &= \tilde{a}^t : A(r/i) && (\text{Step 4a}) \end{aligned}$$

The output satisfies the following.

$$\begin{aligned} (t=0) \vdash d &= w_A : A(r/i) \\ (t=1) \vdash d &= \tilde{a}^t = a_0 : A(r/i) \\ \psi \vdash d &= \tilde{a}^t = a_0 : A(r/i) \\ \forall i.\varphi \vdash d &= \tilde{a}^t = e(r/i) \ \underline{w}_T^t : A(r/i) \end{aligned}$$

**Step 5.** This step mimics the structure of Step 5, replacing the fiber over  $w_A$  with the fiber over  $d$ .

$$\varphi(r/i) \vdash (\underline{C}_1, \underline{C}_2) := e(r/i).2 \ d : \text{IsContr} (\text{Fiber } e(r/i) \ d)$$

Note that we have the following boundary equation.

$$\varphi(r/i), (t=0) \vdash (\underline{C}_1, \underline{C}_2) = (C_1, C_2) : \text{IsContr} (\text{Fiber } e(r/i) \ w_A)$$

**Step 6.** In this step we construct an element of  $\text{Fiber } e(r/i) d$  to mediate between  $R : \text{Fiber } e(r/i) w_A$  and  $(u_0, \langle \_ \rangle a_0)$  along  $t$ .

$$\varphi(r/i) \vdash \underline{R} := \text{wcom}_k^{1 \rightarrow 0} (\text{Fiber } e(r/i) d) \left[ \begin{array}{ll} \psi & \mapsto \underline{C}_2 (u(r/i), \langle \_ \rangle d) k \\ \forall i. \varphi & \mapsto \underline{C}_2 (\underline{w}_T^t, \langle \_ \rangle d) k \\ (t = 1) & \mapsto \underline{C}_2 (u_0, \langle \_ \rangle d) k \end{array} \right] \underline{C}_1$$

Note that this composition has an additional  $(t = 1)$  face in comparison with  $R$ . To see that it is well-typed, we first check that the pairs in each tube are elements of the fiber type.

$$\varphi(r/i), \psi \vdash e(r/i) u(r/i) = a(r/i) = a_0 = d : A(r/i) \quad (\text{Steps 2, } \underline{4b})$$

$$\varphi(r/i), \forall i. \varphi \vdash e(r/i) \underline{w}_T^t = d : A(r/i) \quad (\text{Step } \underline{4b})$$

$$\varphi(r/i), (t = 1) \vdash e(r/i) u_0 = a_0 = d : A(r/i) \quad (\text{Steps 2, } \underline{4b})$$

Second, we check that their first components satisfy the necessary adjacency conditions.

$$\varphi(r/i), \psi, \forall i. \varphi \vdash u(r/i) = \underline{w}_T^t : T(r/i) \quad (\text{Step } \underline{1})$$

$$\varphi(r/i), \psi, (t = 1) \vdash u(r/i) = u_0 : T(r/i) \quad (\text{Premise})$$

$$\varphi(r/i), \forall i. \varphi, (t = 1) \vdash \underline{w}_T^t = u_0 : T(r/i) \quad (\text{Step } \underline{1})$$

Thus  $\underline{R}$  is well-typed. We observe that we have the following boundary equation.

$$(t = 0) \vdash \underline{R} = R : \text{Fiber } e(r/i) w_A$$

This follows from the boundary equation on  $(\underline{C}_1, \underline{C}_2)$ , the equation  $\underline{w}_T^0 = w_T^r$ , and the fact that the  $(t = 1)$  face is irrelevant when  $(t = 0)$ .

**Step 7.** Now, we construct an element of  $A(r/i)$  to connect  $w'_A$  and  $a_0$ .

$$\underline{w}'_A := \text{wcom}_k^{1 \rightarrow 0} A(r/i) [\psi \mapsto a(r/i), \varphi(r/i) \mapsto \underline{R}.2 k, (t = 1) \mapsto a_0] d$$

To see that this is well-typed, we check the following adjacency conditions.

$$k : \mathbb{I}, \psi, \varphi(r/i) \vdash a(r/i) = a_0 = d = \underline{R}.2 k : A(r/i) \quad (\text{Steps 2, } \underline{4b}, \underline{6})$$

$$k : \mathbb{I}, \psi, (t = 1) \vdash a(r/i) = a_0 \quad (\text{Step } \underline{2})$$

$$k : \mathbb{I}, \varphi(r/i), (t = 1) \vdash \underline{R}.2 k = d = a_0 \quad (\text{Steps } \underline{6}, \underline{4b})$$

$$\psi \vdash a(r/i) = a_0 = d \quad (\text{Steps 2, } \underline{4b})$$

$$\varphi(r/i) \vdash \underline{R}.2 1 = d \quad (\text{type of } \underline{R})$$

$$(t = 1) \vdash a_0 = d \quad (\text{Step } \underline{4b})$$

We observe that we have the following boundary equation, which follows from the corresponding boundary equation for  $\underline{R}$  and the fact that the  $(t = 1)$  faces drops out when  $(t = 0)$ .

$$(t = 0) \vdash \underline{w}'_A = w'_A : A(r/i)$$

**Definition of  $\text{wcom}$ .** We combine the preceding definitions as in the construction of  $\text{wcom}$ .

$$\vdash \text{wcom}_i^{r;t} G [\varphi \mapsto u] u_0 = \text{glue} [\varphi(r/i) \mapsto \underline{R}.1] \underline{w}'_A : G(r/i)$$

This is well-typed thanks to the following equation.

$$\varphi(r/i) \vdash e(r/i) \underline{R}.1 = \underline{R}.2 0 = \underline{w}'_A : A(r/i)$$

To see that it has the correct boundary, we observe the following.

$$\begin{aligned} \psi &\vdash \text{glue} [\varphi(r/i) \mapsto \underline{R}.1] \underline{w}'_A = \text{glue} [\varphi(r/i) \mapsto u(r/i)] a(r/i) = u(r/i) : G(r/i) \\ (t=0) &\vdash \text{glue} [\varphi(r/i) \mapsto \underline{R}.1] \underline{w}'_A = \text{glue} [\varphi(r/i) \mapsto R.1] w'_A : G(r/i) \\ (t=1) &\vdash \text{glue} [\varphi(r/i) \mapsto \underline{R}.1] \underline{w}'_A = \text{glue} [\varphi(r/i) \mapsto u_0] a_0 = u_0 : G(r/i) \end{aligned}$$

Each of these follows from boundary equations ensured by the definitions in Steps 6 and 7, and the  $\eta$ -rule for Glue for the first and third. Finally, we need to check that the cap path for  $G$  agrees with the cap path for  $T$  on  $\forall i.\varphi$ . In that case we have the following.

$$\forall i.\varphi \vdash \text{glue} [\varphi(r/i) \mapsto \underline{R}.1] \underline{w}'_A = \underline{R}.1 = \underline{w}_T^t = u(r/i) : T(r/i)$$

This follows from Steps 6 and 1.  $\square$

**4.3. Fibrancy of the universe.** To define weak composition in the universe, we follow a similar strategy to [ABC<sup>+</sup>17, Section 2.12]. As usual, an extra correction step is necessary, here to mediate between  $\text{wcoe}_i^{r \rightarrow r} A(-)$  and the identity function on  $A$ .

**Lemma 3.** *For any  $r, s : \mathbb{I}$  and line  $i : \mathbb{I} \vdash A$  the map*

$$\text{wcoe}_i^{r \rightarrow s} A(-) : A(r/i) \rightarrow A(s/i)$$

*is an equivalence. Writing  $\widetilde{\text{wcoe}}_i^{r \rightarrow s} A : A(r/i) \simeq A(s/i)$  for this equivalence, there is moreover a term*

$$\widetilde{\text{wcoe}}_i^{r;t} A : A(r/i) \simeq A(r/i)[(t=0) \mapsto \widetilde{\text{wcoe}}_i^{r \rightarrow r} A, (t=1) \mapsto 1_{A(r/i)}]$$

*Proof.* First, we coerce the proof that the identity function on  $A$  is an equivalence to show that  $\text{wcoe}_i^{r;k} A(-)$  is an equivalence for any  $k$ .

$$c^k := \text{coe}_i^{1 \rightarrow k} (\text{IsEquiv} (\text{wcoe}_i^{r;i} A(-))) 1_{A.2} : \text{IsEquiv} (\text{wcoe}_i^{r;k} A(-))$$

In particular,  $\text{wcoe}_i^{r \rightarrow r} A(-)$  (i.e.  $c^0$ ) is an equivalence, and we can coerce this to prove that  $\text{wcoe}_i^{r \rightarrow s} A(-)$  is an equivalence.

$$e := \text{wcoe}_i^{r \rightarrow s} (\text{IsEquiv} (\text{wcoe}_i^{r \rightarrow i} A(-))) c^0 : \text{IsEquiv} (\text{wcoe}_i^{r \rightarrow s} A(-))$$

By combining the cap path for this coercion with  $c^k$ , we can show that the resulting equivalence is connected by a path to  $1_{A(r/i)}$  when  $s$  is  $r$ .

$$\underline{e}^j := \text{wcoe}_i^{r;j} (\text{IsEquiv} (\text{wcoe}_i^{r \rightarrow i} A(-))) c^0 : \text{IsEquiv} (\text{wcoe}_i^{r \rightarrow r} A(-))$$

$$\underline{e}' := \text{whcom}_j^{1 \rightarrow 0} (\text{IsEquiv} (\text{wcoe}_i^{r;t} A(-))) [(t=0) \mapsto \underline{e}^j, (t=1) \mapsto 1_{A.2}] c^t$$

Finally, we define

$$\widetilde{\text{wcoe}}_i^{r \rightarrow s} A := (\text{wcoe}_i^{r \rightarrow s} A(-), e)$$

$$\widetilde{\text{wcoe}}_i^{r;t} A := (\text{wcoe}_i^{r;t} A(-), \underline{e}')$$

$\square$

With this equivalence in hand, it is simple to define composition in the universe using a Glue type.

$$\text{wcom}_i^{r \rightarrow s} \mathcal{U} [\varphi \mapsto T] B = \text{Glue} [\varphi \mapsto (T(s/i), \widetilde{\text{wcoe}}_i^{s \rightarrow r} T)] B$$

$$\text{wcom}_i^{r;t} \mathcal{U} [\varphi \mapsto T] B = \text{Glue} [\varphi \mapsto (T(r/i), \widetilde{\text{wcoe}}_i^{r;t} T), (t=1) \mapsto (B, 1_B)] B$$

**4.4. The univalence theorem.** The standard arguments for deriving univalence from **Glue** types transfer straightforwardly to this weaker setting. A direct proof of univalence in cartesian cubical type theory (using **V** types, which are a special case of **Glue** types) can be found in [Red18, prelude.univalence]. One may see by inspection that this proof does not rely on diagonal cofibrations or compositions that are not available strictly via Section 2.1. This is essentially an inlined version of the proof of Corollary 10 in [CCHM18].

## 5. HIGHER INDUCTIVE TYPES

We confirm that the circle, a simple example of a higher inductive type, is definable in the weaker setting. Of course, the circle is an exceedingly simple special case, so this is not definitive evidence that higher inductive types are definable in the same generality as in [CHM18, CH19]. In particular, weak composition considerably complicates cases like the pushout where “endpoint correction” is necessary for coercion of path constructors. However, we do not foresee any essential issues.

**5.1. The circle.** As in existing cubical type theories, a composition in the circle type reduces to a homogeneous composition. As we have both weak homogeneous composition and its cap path we add these as constructors to  $\mathbb{S}^1$ .

$$\begin{aligned} \text{wcom}_i^{r \rightarrow s} \mathbb{S}^1 [\varphi \mapsto u] u_0 &= \text{whcom}_i^{r \rightarrow s} \mathbb{S}^1 [\varphi \mapsto u] u_0 \\ \underline{\text{wcom}}_i^{r;t} \mathbb{S}^1 [\varphi \mapsto u] u_0 &= \underline{\text{whcom}}_i^{r;t} \mathbb{S}^1 [\varphi \mapsto u] u_0 \end{aligned}$$

Suppose we have  $x : \mathbb{S}^1 \vdash P$ ,  $\vdash b : P(\text{base}/x)$ , and  $\vdash \ell : \text{Path}^i P(\text{loop } i/x) b b$ . We define the reduction behavior of the eliminator on the values of the circle.

$$\begin{aligned} \mathbb{S}^1\text{-elim}_{x.P} b \ell (\text{base}) &= b \\ \mathbb{S}^1\text{-elim}_{x.P} b \ell (\text{loop } i) &= \ell i \end{aligned}$$

Now suppose we have  $r : \mathbb{I}$ ,  $\varphi : \mathbb{F}$ ,  $\varphi, i : \mathbb{I} \vdash u : \mathbb{S}^1$ , and  $\vdash u_0 : \mathbb{S}^1[\varphi \mapsto u(r/i)]$ . To reduce  $\mathbb{S}^1\text{-elim}_{x.P} b \ell$  on  $\text{whcom}_i^{r \rightarrow s} \mathbb{S}^1 [\varphi \mapsto u] u_0$  for  $s : \mathbb{I}$  and  $\underline{\text{whcom}}_i^{r;t} \mathbb{S}^1 [\varphi \mapsto u] u_0$  for  $t : \mathbb{I}$ , we first introduce some auxiliary definitions, roughly following the shape of composition in the dependent pair type.

$$\begin{aligned} i : \mathbb{I} \vdash c^i &:= \text{whcom}_i^{r \rightarrow i} \mathbb{S}^1 [\varphi \mapsto u] u_0 \\ j : \mathbb{I} \vdash \underline{c}^j &:= \underline{\text{whcom}}_i^{r \rightarrow j} \mathbb{S}^1 [\varphi \mapsto u] u_0 \\ i : \mathbb{I}, \varphi \vdash a &:= \mathbb{S}^1\text{-elim}_{x.P} b \ell u : P(u/x) \\ \vdash a_0 &:= \mathbb{S}^1\text{-elim}_{x.P} b \ell u_0 : P(u_0/x) \\ k : \mathbb{I} \vdash \tilde{a}^k &:= \text{com}_j^{1 \rightarrow k} P(\underline{c}^j/x) [\varphi \mapsto a(r/i)] a_0 : P(c^r/x) \\ \vdash w &:= \text{wcom}_i^{r \rightarrow s} P(s^i/x) [\varphi \mapsto a] \tilde{a}^0 : P(c^s/x) \\ j : \mathbb{I} \vdash \underline{w}^j &:= \underline{\text{wcom}}_i^{r;t} P(c^r/x) [\varphi \mapsto a] \tilde{a}^0 : P(c^r/x) \\ \vdash \underline{w}' &:= \text{wcom}_j^{1 \rightarrow 0} P(\underline{c}^t/x) [\varphi \mapsto a(r/i), (t = 0) \mapsto \underline{w}^j, (t = 1) \mapsto a_0] \tilde{a}^k \end{aligned}$$

The reductions are then defined as follows.

$$\begin{aligned} \mathbb{S}^1\text{-elim}_{x.P} b \ell (\text{whcom}_i^{r \rightarrow s} \mathbb{S}^1 [\varphi \mapsto u] u_0) &= w \\ \mathbb{S}^1\text{-elim}_{x.P} b \ell (\underline{\text{whcom}}_i^{r;t} \mathbb{S}^1 [\varphi \mapsto u] u_0) &= \underline{w}' \end{aligned}$$

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