

# The mpFormulaPy Library and Toolbox Manual

Dietrich Hadler  
Helge Hadler  
Thomas Hadler

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Original authors of the mpmath documentation: Frederik Johanson and mpmath contributors

Original authors of the Python documentation: the Python development Team

Subsequent modifications: Dietrich Hadler, Helge Hadler and Thomas Hadler

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# Preface

The mpFormulaPy library provides a comprehensive set of real and complex functions in multiprecision arithmetic. For a subset of functions there is also support for decimal and interval arithmetic.

Additional planned functionality includes integration in Microsoft Excel (Windows) and LibreOffice Calc (Windows, Mac OSX, GNU/Linux), with multiprecision support for the numerical functions of these spreadsheet programs.

mpFormulaPy is based on the Python runtime and the mpmath library, both of which implement multiprecision arithmetic.

The current version is 0.1, alpha1 (pre-release), and much of the planned functionality is still missing.

This manual is divided in various parts, which reflect different levels of confidence regarding the accuracy of the results.

## Part II: Functions with Error Bounds.

Functions in this part come optionally with a guaranteed error bound, which can (in principal) be made arbitrarily small. The set of functions includes all real and complex functions that are included in Microsoft Excel and OpenOffice/LibreOffice Calc, including the financial functions.

## Part III: Special Functions

Functions in this part include a rich set of real and complex functions with an emphasis on mathematical physics. These functions are available in arbitrary precision, but without guaranteed error bounds.

## Part IV: Numerical Calculus

Functions in this part include root-finding and optimization, differentiation and integration, sums, products and limit.

## Part V: Application Examples

This is work in progress, with an emphasis on statistical distribution functions.

The use of these function in various environments is described in some detail in the appendices:

Appendix A describes the interfaces to a number of popular flavors of Python, including CPython (Windows, Mac OSX, GNU/Linux), Jython (Windows, Mac OSX, GNU/Linux), IronPython (Windows), QPython (Android) and Pythonista (iOS).

Appendix B describes the interfaces to popular spreadsheet programs: LibreOffice Calc (Windows, Mac OSX, GNU/Linux) and Microsoft Excel (Windows, Mac OSX).

Windows-specific interfaces to languages with CLR support (e.g. C#, F#, Visual Basic, Matlab) or COM support (e.g. VBA, JScript, Perl, R) are described in appendices C and D.

If you want to re-build or change the library and/or toolbox, have a look at appendix E.

Finally, the mpFormulaPy Library and Toolbox would not exist without the many authors and contributors of the underlying libraries. They are acknowledged in appendix F.

Dietrich Hadler

Helge Hadler

Thomas Hadler

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# **Part I**

## **Getting Started**

# Chapter 1

## Introduction

### 1.1 Overview: Features and Setup

#### 1.1.1 Features

The mpFormulaPy distribution consists of two parts: the mpFormulaPy Library and the mpFormulaPy Toolbox.

#### 1.1.2 The mpFormulaPy Library

The mpFormulaPy Library is a collection of numerical functions and procedures in multiprecision arithmetic. It is intended to be usable on multiple platforms (i.e. platforms supported by a recent version of Python) and is provided in the form of source code in Python.

The following numerical types are supported:

- The conventional double (64 bit) precision binary floating point type (double in C).
- The mpf arbitrary precision binary floating point type of the mpmath library.
- The mpi arbitrary precision interval arithmetic binary floating point type of the mpmath library.
- The mpc arbitrary precision complex binary floating point type of the mpmath library.
- The mpci arbitrary precision complex interval arithmetic binary floating point type of the mpmath library.
- The long arbitrary precision integer type of the Python library.
- The Fraction arbitrary precision rational type of the Python library.
- The Decimal arbitrary precision decimal floating point type of the Python library.

All of these types are available as real and complex scalars, vectors, and matrices.

The mpFormulaPy Library is based on mpmath [Johansson \*et al.\* \(2013\)](#), and the standard Python Library, including fractions.py and decimal.py.

### 1.1.3 The mpFormulaPy Toolbox

The mpFormulaPy Toolbox provides a setup with the Ironpython compiler 2.7.4 for the Windows platform with multiple interfaces:

- A .NET Framework 4.0 interface: arithmetic functions, operators and procedures are accessible in a familiar syntax. Both 32 bit and 64 bit versions are provided. This interface makes the numerical routines available to all languages with .NET Framework support, including VB.NET, C#, JScript 2010, F#, MS C++ (CLI), IronPython and Matlab.
- A COM (Component Object Model) interface: multiprecision arithmetic functions and procedures, with arithmetic operators emulated as properties. Both 32 bit and 64 bit versions are provided. This interface makes the numerical routines available to all languages with COM support, including VBScript, JScript (Windows Script Host), Visual Basic for Applications, Visual Basic 6.0, OpenOffice Basic, Lua, Ruby, PHP CLI, Perl, Python, R (Statistical System) and Mathematica.
- A Names Pipes and Command Line interface: this is designed to make sure that the calling application and the routines in the library are executed in separate processes, greatly enhancing stability.

In addition, the mpFormulaPy Toolbox offers the following features:

- A compact IDE for VB.NET and C# (no need to install Visual Studio). The IDE provides a Code Editor, Windows Forms Designer, Debugger, Profiler, Unit Tester and Microsoft Visual Studio compatible project files. Based on a trimmed version of Sharp Develop ([Krüger et al., 2013](#)).
- A Microsoft Excel (versions 2000 to 2013) interface: multiprecision arithmetic functions are provided for use in spreadsheet cells, using Excel's XLL interface. It is also possible to write user-defined functions and procedures in multiprecision arithmetic (provided as add-ins written in VB.NET or C#), which run even if macro security is set to "disable all macros". Based on Excel-DNA ([van Drimmelen, 2013](#)).
- Full access for VB.NET and C# to the object model of MS Excel (including Intellisense support in the Code Editor). All Microsoft Excel versions from 2000 to 2013, both 32 bit and 64 bit (only 2010 and 2013), are supported, and no other software (like PIAs) is needed. Based on NetOffice ([Lange, 2012](#)).
- An OpenOffice Calc (versions 2.1 or later, incl. Apache OpenOffice Calc or LibreOffice Calc) interface: multiprecision arithmetic functions are provided for use in spreadsheet cells, using the OpenOffice Basic interface. It is also possible to write user-defined functions and procedures in multiprecision arithmetic (provided as add-ins written in VB.NET or C#).

### 1.1.4 System Requirement

This mpFormulaPy Toolbox has the following system requirement:

- Microsoft Windows with Microsoft .NET Framework version 4.x (Full).

This mpFormulaPy Toolbox can take advantage of the following software:

- Adobe Reader 8 or later, for use with mpFormulaPy’s interactive help-system.
- Microsoft Office 2000 or later.
- OpenOffice (versions 2.1 or later), or Apache OpenOffice or LibreOffice.

### 1.1.5 Installation

The mpFormulaPy Library and Toolbox can be downloaded from

<http://mpFormula.github.io/Py/>.

Unzip the downloaded file in a directory for which you have write-access.

## 1.2 License

The mpFormulaPy Library and Toolbox is free software. It is licensed under the GNU Lesser General Public License (LGPL), Version 3 (see appendix [G.1.3](#)). The manual for the mpFormulaPy Toolbox (this document) is licensed under the GNU Free Documentation License, Version 1.3 (see appendix [G.1.5](#)).

## 1.3 No Warranty

There is no warranty. See the GNU Lesser General Public License, Version 3 (see appendix [G.1.3](#)) for details.

## 1.4 Related Software

The mpFormulaC Library and Toolbox provides fast multiprecision routines written in C, with interfaces to CPython, R, .NET and COM. It can be downloaded from  
<http://mpFormula.github.io/C/>.

# Chapter 2

## Tutorials

### 2.1 Why multi-precision arithmetic?

An introduction to the problems of rounding errors and catastrophic cancellation can be found in [Goldberg \(1991\)](#). Excellent reference texts are [Higham \(2002\)](#) and [Higham \(2009\)](#).

In the following paragraphs we will give a few examples of how widely used programs like MS Excel or Libreoffice Calc can give wrong results due to the fact that they are using double precision arithmetic and not multi-precision arithmetic

#### 2.1.1 Example 1: Sums

Sums are often calculated exactly if all summands have an exact representation. If this is not the case, results can be unpredictable. In MS Excel, the formula

```
=SUM(10000000000,-16000000000,6000000000)
```

will give the correct result 0, but the analogous formula

```
=SUM(1E+40,-1.6E+40,6E+39)
```

returns 1.20893E+24 instead of the correct result 0.

#### 2.1.2 Example 2: Standard Deviation

Like sums, variances and standard deviations are often calculated exactly if all arguments have an exact representation. If this is not the case, results can again be unpredictable. In MS Excel, the formula

```
=VAR(1E+30,1E+30,1E+30)
```

returns 2.97106E+28 instead of the correct result 0, which should be the obvious results since all arguments are the same.

#### 2.1.3 Example 3: Overflow and underflow

In many situations where the final result is representable in double precision, some of the interim results cause overflow or underflow. A popular example is the function  $f(x, y) = \sqrt{x^2 + y^2}$ . With  $x = 3 \cdot 10^{300}$  and  $y = 4 \cdot 10^{300}$  the result  $f(x, y) = 5 \cdot 10^{300}$  is representable in double precision, but the (naive) calculation will overflow.

### 2.1.4 Example 4: Polynomials

Consider the following example [Cuyt et al. \(2001\)](#):

For  $a = 77617$  and  $b = 33096$ , calculate

$$Y = 333.75b^6 + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8 + \frac{a}{2b} \quad (2.1.1)$$

The correct result is  $Y = -54767/66192 = -0.827396\dots$

### 2.1.5 Example 5: Trigonometric Functions

Trigonometric functions are sensitive to small perturbations.

In double precision and binary floating point arithmetic, the tangent of  $x = 1.57079632679489$  is calculated as  $\tan(x) = 1.48752 \cdot 10^{14}$ , whereas the correct result is  $\tan(x) = 1.51075 \cdot 10^{14}$ . This amounts to an absolute error of  $2.32287 \cdot 10^{12}$  and a relative error of 1.54%.

There are also limits on the range of arguments, e.g.  $\sin(10^8)$  returns the value  $0.931639\dots$  (with an relative error of  $-6.22776 \cdot 10^{-13}$ ), whereas  $\sin(10^9)$  returns an invalid result (the exact result is  $0.545843\dots$ )

### 2.1.6 Example 6: Logarithms and Exponential Functions

Consider the following example ([Ghazi et al., 2010](#)) :

Determine 10 decimal digits of the constant

$$Y = 173746a + 94228b - 78487c, \quad \text{where} \quad (2.1.2)$$

$$a = \sin(10^{22}), b = \ln(17.1), c = \exp(0.42). \quad (2.1.3)$$

The expected result is  $Y = -1.341818958 \cdot 10^{-12}$ .

### 2.1.7 Example 7: Linear Algebra

#### 2.1.7.1 Linear Solver

The following example is from [Hofschuster & Krämer \(2004\)](#):

We want to solve the (ill-conditioned) system of linear equations  $Ax = b$  with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 64919121 & -159018721 \\ 41869520.5 & -102558961 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.1.4)$$

The correct solution is  $x_1 = 205117922$ ,  $x_2 = 83739041$ .

To solve this  $2 \times 2$  system numerically we first use the well known formulas

$$x_1 = \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}}, \quad x_2 = \frac{-a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, \quad (2.1.5)$$

Calculating this directly in double precision gives the following wrong result:

$x_1 = 102558961$ ,  $x_2 = 41869520.5$

## 2.2 Graphics using Latex

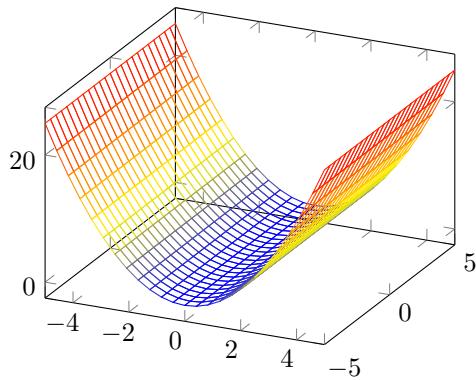
pgfplots ([Feuersänger, 2014](#)) - A TeX package to draw normal and/or logarithmic plots directly in TeX in two and three dimensions with a user-friendly interface and pgfplotstable - a TeX package to round and format numerical tables. Examples in manuals and/or on web site.

<http://pgfplots.net/>.

<http://pgfplots.sourceforge.net/>.

<http://pgfplots.sourceforge.net/gallery.html>.

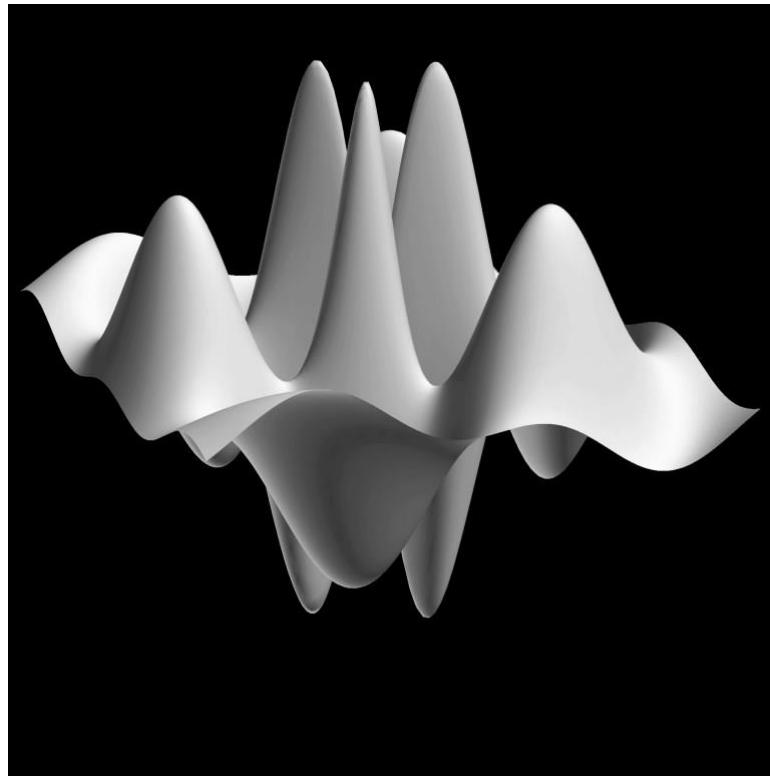
[https://www.sharelatex.com/learn/Pgfplots\\_package](https://www.sharelatex.com/learn/Pgfplots_package).



**Figure 2.1:** A pdf plot

## 2.3 Graphics using .NET Framework

The mpformulaPy toolbox offers a facility for producing 3D charts. The following pages give a few examples.



**Figure 2.2:** plot of a 2-dimensional function

The corresponding code is:

---

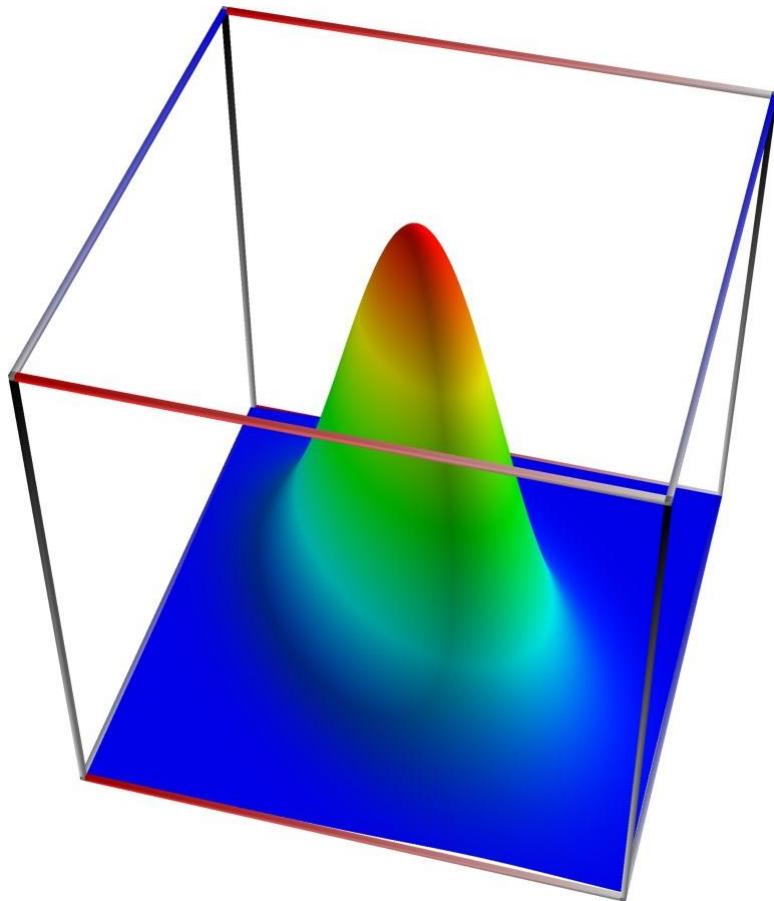
```
const double two_pi = 2 * Math.PI;
double r2 = x * x + z * z;
double r = Math.Sqrt(r2);
double theta = Math.Atan2(z, x);
result = Math.Exp(-r2) * Math.Sin(two_pi * r) * Math.Cos(3 * theta);
```

---

### 2.3.1 Surface plots for bivariate real functions

The bivariate normal distribution has the following density:

$$g(x, y; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}} \quad (2.3.1)$$



**Figure 2.3:** Surface plot of the probability density function of the bivariate normal distribution with  $\rho = -0.5$

The corresponding code is:

---

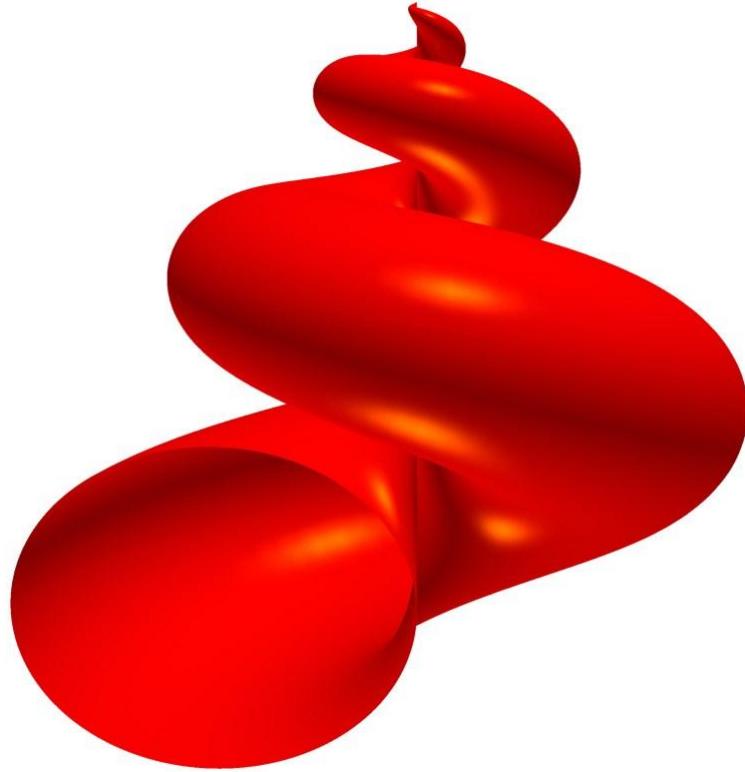
```
const double two_pi = 2 * Math.PI;
double rho = -0.5;
double r2 = 1.0 - rho*rho;
double f = 1 / (two_pi * Math.Sqrt(r2));
double e = -(x*x - 2*rho*x*z + z*z)/(2*r2);
result = f * Math.Exp(e);
```

---

## 2.3.2 3D Plots of parametric functions

### 2.3.2.1 3D Plot of a Seashell

This is a plot of a seashell.



**Figure 2.4:** 3D plot of a parametric function: Seashell.  $umin = 0$ ;  $umax = 6 * \text{Math.PI}$ ;  $vmin = 0$ ;  $vmax = 6 * \text{Math.PI}$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .

The parametrization is:

---

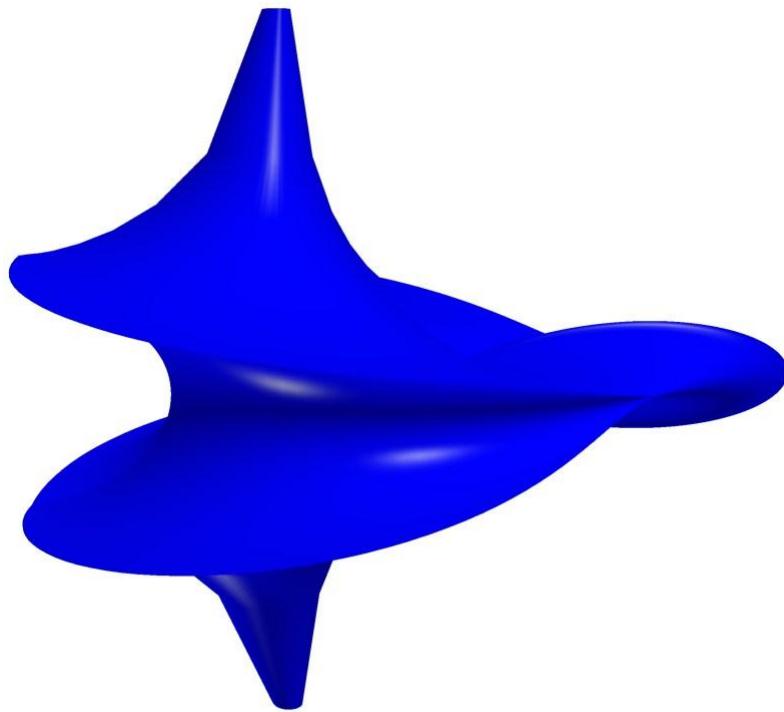
```
double a = Math.Exp(u / (6.0 * Math.PI));
double b = Math.Cos(v / 2.0);

x = 2.0 * (1.0 - a) * Math.Cos(u) * b * b;
z = 2.0 * (-1.0 + a) * Math.Sin(u) * b * b;
y = 1.0 - a * a - Math.Sin(v) * (1.0 - a);
```

---

### 2.3.2.2 3D Plot of Kuen's surface

This is a plot of Kuen's surface.



**Figure 2.5:** 3D plot of a parametric function: Kuen's surface.  $u_{\min} = -4.5$ ;  $u_{\max} = 4.5$ ;  $v_{\min} = 0.01$ ;  $v_{\max} = 3.14$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .

The parametrization is:

---

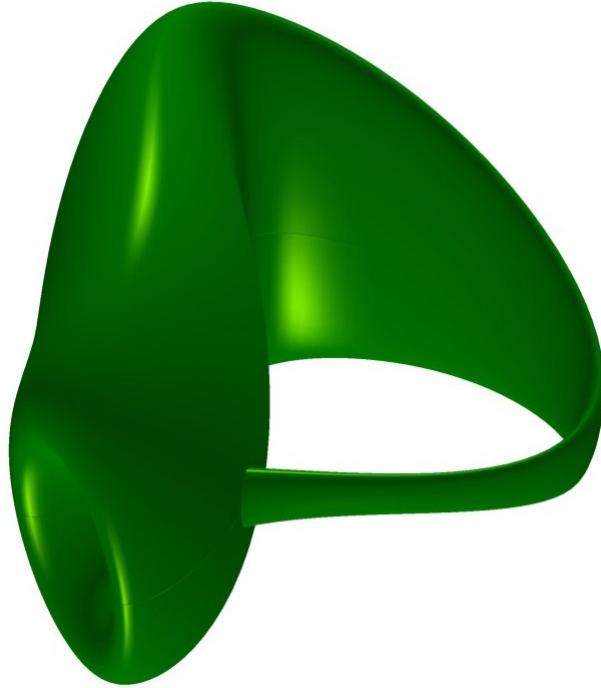
```
double a = 1.0 * Math.Sin(v);
double b = 1.0 + u * u * a * a;

x = 2.0 * a * (Math.Cos(u) + u * Math.Sin(u)) / b;
z = 2.0 * a * (Math.Sin(u) - u * Math.Cos(u)) / b;
y = Math.Log(Math.Tan(v/2.0)) + 2.0 * Math.Cos(v) / b;
```

---

### 2.3.2.3 3D Plot of Klein's Bottle

This is a plot of Klein's Bottle.



**Figure 2.6:** 3D plot of a parametric function: Klein's Bottle.  $u_{\min} = 0.0$ ;  $u_{\max} = 3.14$ ;  $v_{\min} = 0.0$ ;  $v_{\max} = 6.28$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .

The following parametrization is due to Robert Israel (with some rearrangements):

---

```

double a = Math.Cos(u);
double b = Math.Sin(u);
double c = Math.Cos(v);
double a2 = a * a;
double a4 = a2 * a2;

x = -(2.0/15.0) * a * (3*c + b*(-30 + a4*(90 - 60*a2) + 5*a*c));
z = -(1.0/15.0) * b*b * (c*b* (3 - 48*a4 + 5*a*b*(1 - 16*a4)) - 60);
y = (2.0/15.0) * (3 + 5*a*b) * Math.Sin(v);

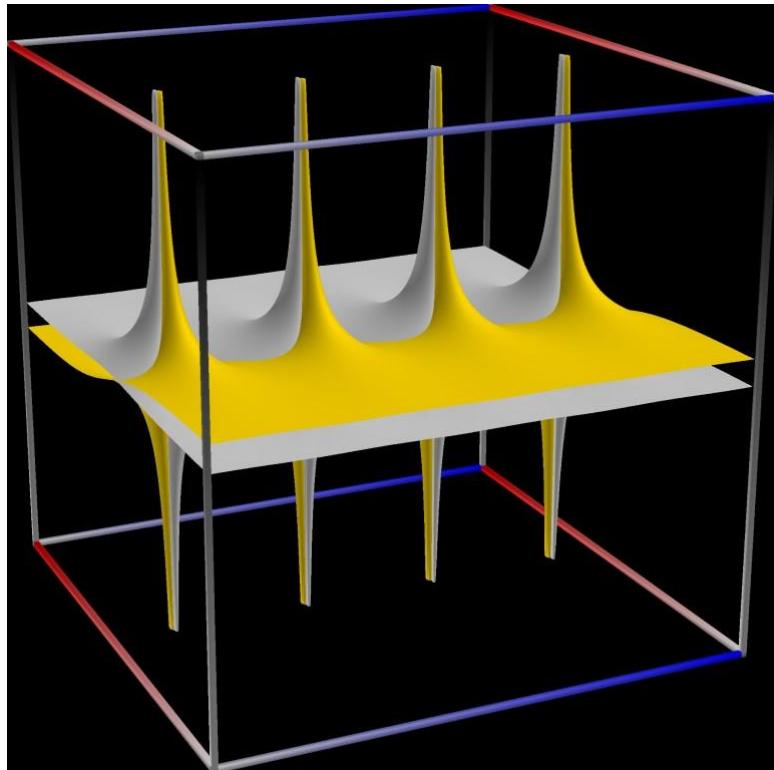
```

---

### 2.3.3 Surface plots of complex functions

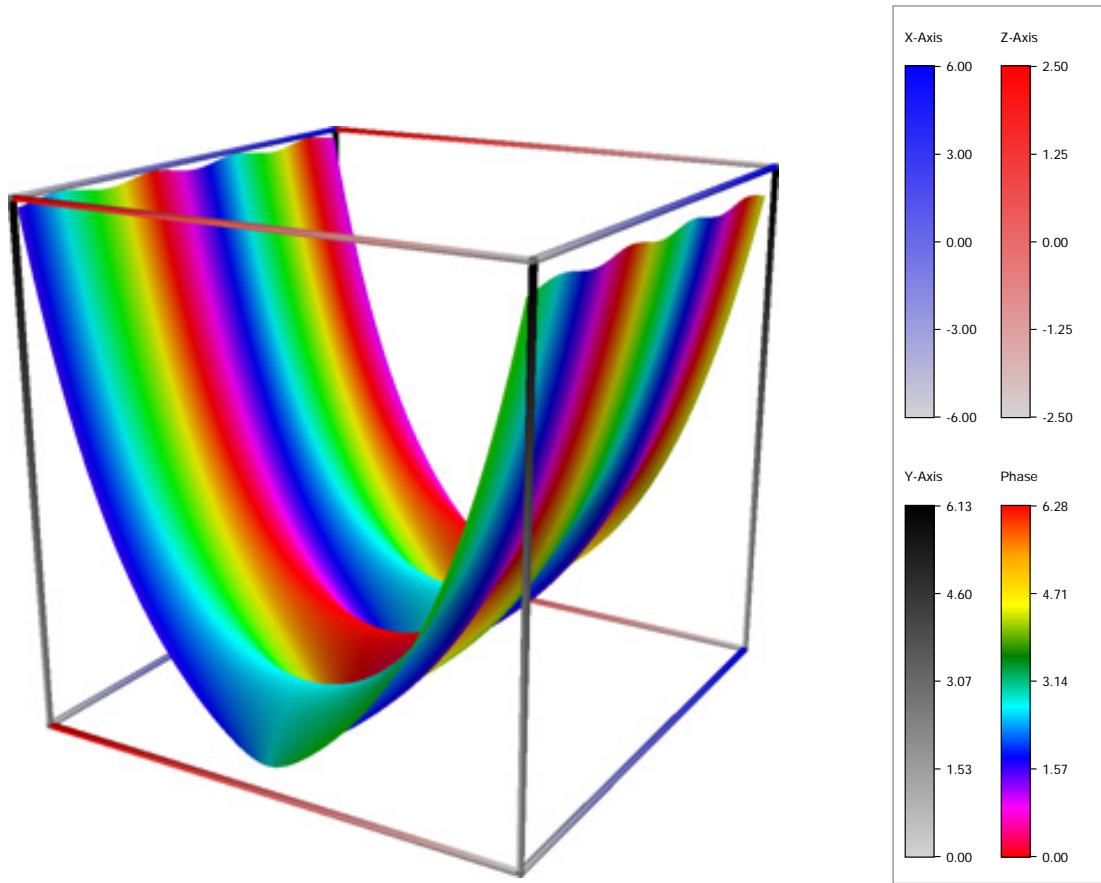
It is straight forward to produce surface plots of complex functions; these are available in two forms:

As plots of the real and imaginary component:



**Figure 2.7:** Surface plot of the real ("silver") and imaginary ("gold") component of  $z = \tan(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $-10 \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .

As plots of the absolute value with the phase color-coded:



**Figure 2.8:** Surface plot of the magnitude and phase (color-coded) of  $z = \sin(x+iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $-10 \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .

## 2.4 Eval, Options, Tables and Charts

The following functions provide quick access to function evaluations and charts:

---

### Function **Eval(*Expression* As String)** As String

---

The function **Eval** returns the result of the evaluation of an arithmetic expression, containing number and functions, but no variables.

**Parameter:**

*Expression*: an arithmetic expression.

---

### Function **Options(*BaseOptions* As String)** As String

---

The function **Options** returns an identifier for a set of calculation options.

**Parameter:**

*BaseOptions*: an identifier for a set of base calculation options.

---

### Function **Table(*TableRef* As String)** As Range

---

The function **Table** returns an identifier for a set of calculation options.

**Parameter:**

*TableRef*: a reference for a table.

---

### Function **Chart(*Data* As Range)** As String

---

The function **Chart** returns an identifier for an XML Chart.

**Parameter:**

*Data*: a reference for a data table.

## **Part II**

# **Functions with Error Bounds**

# Chapter 3

## Basic Usage

In interactive code examples that follow, it will be assumed that all items in the mpFormulaPy namespace have been imported:

---

```
>>> from mpFormulaPy import *
```

---

Importing everything can be convenient, especially when using mpFormulaPy interactively, but be careful when mixing mpFormulaPy with other libraries! To avoid inadvertently overriding other functions or objects, explicitly import only the needed objects, or use the mpFormulaPy or mp.namespaces:

---

```
from mpFormulaPy import sin, cos
sin(1), cos(1)
import mpFormulaPy
mpFormulaPy.sin(1), mpFormulaPy.cos(1)
from mpFormulaPy import mp # mp context object -- to be explained
mp.sin(1), mp.cos(1)>>> from mpFormulaPy import *
```

---

### 3.1 Number types

Mpmath provides the following numerical types:

Class	Description
mpf	Real float
mpc	Complex float
matrix	Matrix

Currently missing: decimals. The MPD reference is [Krah \(2012\)](#)

The following section will provide a very short introduction to the types mpf and mpc. Intervals and matrices are described further in the documentation chapters on interval arithmetic and matrices / linear algebra.

The mpf type is analogous to Python's built-in float. It holds a real number or one of the special values inf (positive infinity), -inf (negative infinity) and nan (not-a-number, indicating an indeterminate result). You can create mpf instances from strings, integers, floats, and other mpf instances:

---

```
>>> mpf(4)
```

---

```
mpf('4.0')
>>> mpf(2.5)
mpf('2.5')
>>> mpf("1.25e6")
mpf('1250000.0')
>>> mpf(mpf(2))
mpf('2.0')
>>> mpf("inf")
mpf('+inf')
```

---

The mpc type represents a complex number in rectangular form as a pair of mpf instances. It can be constructed from a Python complex, a real number, or a pair of real numbers:

---

```
>>> mpc(2,3)
mpc(real='2.0', imag='3.0')
>>> mpc(complex(2,3)).imag
mpf('3.0')
```

---

You can mix mpf and mpc instances with each other and with Python numbers:

---

```
>>> mpf(3) + 2*mpf('2.5') + 1.0
mpf('9.0')
>>> mp.dps = 15 # Set precision (see below)
>>> mpc(1j)**0.5
mpc(real='0.70710678118654757', imag='0.70710678118654757')
```

---

### 3.1.1 Setting the precision

Mpmath uses a global working precision; it does not keep track of the precision or accuracy of individual numbers. Performing an arithmetic operation or calling mpf() rounds the result to the current working precision. The working precision is controlled by a context object called mp, which has the following default states:

---

```
>>> from mpFormulaPy import *
>>> mp.dps
25
>>> mp.prec
86
>>> mp.trap_complex
False
>>>
```

---

The term prec denotes the binary precision (measured in bits) while dps (short for decimal places) is the decimal precision. Binary and decimal precision are related roughly according to the formula prec = 3.33\*dps. For example, it takes a precision of roughly 333 bits to hold an approximation of pi that is accurate to 100 decimal places (actually slightly more than 333 bits is used).

Changing either precision property of the mp object automatically updates the other; usually you just want to change the dps value:

---

```
>>> mp.dps = 100
>>> mp.dps
```

---

```
100  
>>> mp.prec  
336
```

When the precision has been set, all mpf operations are carried out at that precision:

The precision of complex arithmetic is also controlled by the mp object:

```
>>> mp.dps = 10  
>>> mpc(1,2) / 3  
mpc(real='0.3333333333321', imag='0.6666666666642')
```

There is no restriction on the magnitude of numbers. An mpf can for example hold an approximation of a large Mersenne prime:

```
>>> mp.dps = 15  
>>> print mpf(2)**32582657 - 1  
1.24575026015369e+9808357
```

Or why not 1 googolplex:

The (binary) exponent is stored exactly and is independent of the precision.

### 3.1.2 Temporarily changing the precision

It is often useful to change the precision during only part of a calculation. A way to temporarily increase the precision and then restore it is as follows:

```
>>> mp.prec += 2  
>>> # do_something()  
>>> mp.prec == 2
```

As of Python 2.5, the `with` statement along with the `mpFormulaPy` functions `workprec`, `workdps`, `extraprec` and `extradps` can be used to temporarily change precision in a more safe manner:

```
>>> from __future__ import with_statement
>>> with workdps(20):
...     print mpf(1)/7
...     with extradps(10):
...         print mpf(1)/7
...
0.14285714285714285714
0.142857142857142857142857142857
>>> mp.dps
```

---

15

The with statement ensures that the precision gets reset when exiting the block, even in the case that an exception is raised. (The effect of the with statement can be emulated in Python 2.4 by using a try/finally block.)

The workprec family of functions can also be used as function decorators:

---

```
>>> @workdps(6)
... def f():
...     return mpf(1)/3
...
>>> f()
mpf('0.3333331346511841')
```

---

Some functions accept the prec and dps keyword arguments and this will override the global working precision. Note that this will not affect the precision at which the result is printed, so to get all digits, you must either use increase precision afterward when printing or use nstr/nprint:

---

```
>>> mp.dps = 15
>>> print exp(1)
2.71828182845905
>>> print exp(1, dps=50) # Extra digits won't be printed
2.71828182845905
>>> nprint(exp(1, dps=50), 50)
2.7182818284590452353602874713526624977572470937
```

---

Finally, instead of using the global context object mp, you can create custom contexts and work with methods of those instances instead of global functions. The working precision will be local to each context object:

---

```
>>> mp2 = mp.clone()
>>> mp.dps = 10
>>> mp2.dps = 20
>>> print mp.mpf(1) / 3
0.3333333333
>>> print mp2.mpf(1) / 3
0.3333333333333333
```

---

Note: the ability to create multiple contexts is a new feature that is only partially implemented. Not all mpFormulaPy functions are yet available as context-local methods. In the present version, you are likely to encounter bugs if you try mixing different contexts.

### 3.1.3 Providing correct input

Note that when creating a new mpf, the value will at most be as accurate as the input. Be careful when mixing mpFormulaPy numbers with Python floats. When working at high precision, fractional mpf values should be created from strings or integers:

---

```
>>> mp.dps = 30
>>> mpf(10.9) # bad
mpf('10.9000000000000003552713678800501')
>>> mpf('10.9') # good
```

---

```
mpf('10.89999999999999999999999999999999997')
>>> mpf(109) / mpf(10) # also good
mpf('10.8999999999999999999999999999999999997')
>>> mp.dps = 15
```

---

(Binary fractions such as 0.5, 1.5, 0.75, 0.125, etc, are generally safe as input, however, since those can be represented exactly by Python floats.)

### 3.1.4 Printing

By default, the `repr()` of a number includes its type signature. This way `eval` can be used to recreate a number from its string representation:

---

```
>>> eval(repr(mpf(2.5)))
mpf('2.5')
```

---

Prettier output can be obtained by using `str()` or `print`, which hide the `mpf` and `mpc` signatures and also suppress rounding artifacts in the last few digits:

---

```
>>> mpf("3.14159")
mpf('3.141599999999999')
>>> print mpf("3.14159")
3.14159
>>> print mpc(1j)**0.5
(0.707106781186548 + 0.707106781186548j)
```

---

Setting the `mp.pretty` option will use the `str()`-style output for `repr()` as well:

---

```
>>> mp.pretty = True
>>> mpf(0.6)
0.6
>>> mp.pretty = False
>>> mpf(0.6)
mpf('0.5999999999999998')
```

---

The number of digits with which numbers are printed by default is determined by the working precision. To specify the number of digits to show without changing the working precision, use `mpFormulaPy.nstr()` and `mpFormulaPy.nprint()`:

---

```
>>> a = mpf(1) / 6
>>> a
mpf('0.1666666666666666')
>>> nstr(a, 8)
'0.16666667'
>>> nprint(a, 8)
0.16666667
>>> nstr(a, 50)
'0.1666666666666665741480812812369549646973609924316'
```

---

### 3.1.5 Contexts

High-level code in mpFormulaPy is implemented as methods on a 'context object'. The context implements arithmetic, type conversions and other fundamental operations. The context also holds settings such as precision, and stores cache data. A few different contexts (with a mostly compatible interface) are provided so that the high-level algorithms can be used with different implementations of the underlying arithmetic, allowing different features and speed/accuracy tradeoffs. Currently, mpFormulaPy provides the following contexts:

Arbitrary-precision arithmetic (mp)

A faster Cython-based version of mp (used by default in Sage, and currently only available there)

Arbitrary-precision interval arithmetic (iv)

Double-precision arithmetic using Python's builtin float and complex types (fp)

Most global functions in the global mpFormulaPy namespace are actually methods of the mp context. This fact is usually transparent to the user, but sometimes shows up in the form of an initial parameter called 'ctx' visible in the help for the function:

---

```
>>> import mpFormulaPy
>>> help(mpFormulaPy.fsum)
Help on method fsum in module mpFormulaPy.ctx_mp_python:
fsum(ctx, terms, absolute=False, squared=False) method of
    mpFormulaPy.ctx_mp.MPContext ins
Calculates a sum containing a finite number of terms (for infinite
series, see :func:`~mpFormulaPy.nsum`). The terms will be converted to
...
```

---

The following operations are equivalent:

---

```
>>> mpFormulaPy.mp.dps = 15; mpFormulaPy.mp.pretty = False
>>> mpFormulaPy.fsum([1,2,3])
mpf('6.0')
>>> mpFormulaPy.mp.fsum([1,2,3])
mpf('6.0')
```

---

The corresponding operation using the fp context:

---

```
>>> mpFormulaPy.fp.fsum([1,2,3])
6.0
```

---

### 3.1.6 Common interface

ctx.mpf creates a real number:

---

```
>>> from mpFormulaPy import mp, fp
>>> mp.mpf(3)
mpf('3.0')
>>> fp.mpf(3)
3.0
```

---

ctx.mpc creates a complex number:

---

```
>>> mp.mpc(2,3)
```

---

```
mpc(real='2.0', imag='3.0')
>>> fp.mpc(2,3)
(2+3j)
```

---

ctx.matrix creates a matrix:

---

```
>>> mp.matrix([[1,0],[0,1]])
matrix(
[[1.0, 0.0],
 [0.0, 1.0]])
>>> _[0,0]
mpf('1.0')
>>> fp.matrix([[1,0],[0,1]])
matrix(
[[1.0, 0.0],
 [0.0, 1.0]])
>>> _[0,0]
1.0
```

---

ctx.prec holds the current precision (in bits):

---

```
>>> mp.prec
53
>>> fp.prec
53
```

---

ctx.dps holds the current precision (in digits):

---

```
>>> mp.dps
15
>>> fp.dps
15
```

---

ctx.pretty controls whether objects should be pretty-printed automatically by repr(). Prettyprinting for mp numbers is disabled by default so that they can clearly be distinguished from Python numbers and so that eval(repr(x)) == x works:

---

```
>>> mp.mpf(3)
mpf('3.0')
>>> mpf = mp.mpf
>>> eval(repr(mp.mpf(3)))
mpf('3.0')
>>> mp.pretty = True
>>> mp.mpf(3)
3.0
>>> fp.matrix([[1,0],[0,1]])
matrix(
[[1.0, 0.0],
 [0.0, 1.0]])
>>> fp.pretty = True
>>> fp.matrix([[1,0],[0,1]])
[1.0 0.0]
[0.0 1.0]
```

---

```
>>> fp.pretty = False
>>> mp.pretty = False
```

---

### 3.1.7 Arbitrary-precision floating-point (mp)

The mp context is what most users probably want to use most of the time, as it supports the most functions, is most well-tested, and is implemented with a high level of optimization. Nearly all examples in this documentation use mp functions.

See Basic usage for a description of basic usage.

### 3.1.8 Arbitrary-precision interval arithmetic (iv)

The iv.mpf type represents a closed interval  $[a, b]$ ; that is, the set  $\{x : a \leq x \leq b\}$ , where  $a$  and  $b$  are arbitrary-precision floating-point values, possibly  $\pm\infty$ . The iv.mpc type represents a rectangular complex interval  $[a, b] + [c, d]i$ ; that is, the set  $\{z = x + iy : a \leq x \leq b \wedge c \leq y \leq d\}$ .

Interval arithmetic provides rigorous error tracking. If  $f$  is a mathematical function and  $\hat{f}$  is its interval arithmetic version, then the basic guarantee of interval arithmetic is that  $f(v) \subseteq \hat{f}(v)$  for any input interval  $v$ . Put differently, if an interval represents the known uncertainty for a fixed number, any sequence of interval operations will produce an interval that contains what would be the result of applying the same sequence of operations to the exact number. The principal drawbacks of interval arithmetic are speed (iv arithmetic is typically at least two times slower than mp arithmetic) and that it sometimes provides far too pessimistic bounds.

Note: The support for interval arithmetic in mpFormulaPy is still experimental, and many functions do not yet properly support intervals. Please use this feature with caution.

Intervals can be created from single numbers (treated as zero-width intervals) or pairs of endpoint numbers. Strings are treated as exact decimal numbers. Note that a Python float like 0.1 generally does not represent the same number as its literal; use '0.1' instead:

---

```
>>> from mpFormulaPy import iv
>>> iv.dps = 15; iv.pretty = False
>>> iv.mpf(3)
mpi('3.0', '3.0')
>>> print iv.mpf(3)
[3.0, 3.0]
>>> iv.pretty = True
>>> iv.mpf([2,3])
[2.0, 3.0]
>>> iv.mpf(0.1) # probably not intended
[0.100000000000000555, 0.100000000000000555]
>>> iv.mpf('0.1') # good, gives a containing interval
[0.0999999999999991673, 0.100000000000000555]
>>> iv.mpf(['0.1', '0.2'])
[0.0999999999999991673, 0.200000000000000111]
```

---

The fact that '0.1' results in an interval of nonzero width indicates that 1/10 cannot be represented using binary floating-point numbers at this precision level (in fact, it cannot be represented exactly at any precision).

Intervals may be infinite or half-infinite:

---

```
>>> print 1 / iv.mpf([2, 'inf'])
[0.0, 0.5]
```

---

The equality testing operators `==` and `!=` check whether their operands are identical as intervals; that is, have the same endpoints. The ordering operators `<`, `<=`, `>` and `>=` permit inequality testing using triple-valued logic: a guaranteed inequality returns `True` or `False` while an indeterminate inequality returns `None`:

---

```
>>> iv.mpf([1,2]) == iv.mpf([1,2])
True
>>> iv.mpf([1,2]) != iv.mpf([1,2])
False
>>> iv.mpf([1,2]) <= 2
True
>>> iv.mpf([1,2]) > 0
True
>>> iv.mpf([1,2]) < 1
False
>>> iv.mpf([1,2]) < 2 # returns None
>>> iv.mpf([2,2]) < 2
False
>>> iv.mpf([1,2]) <= iv.mpf([2,3])
True
>>> iv.mpf([1,2]) < iv.mpf([2,3]) # returns None
>>> iv.mpf([1,2]) < iv.mpf([-1,0])
False
```

---

The `in` operator tests whether a number or interval is contained in another interval:

---

```
>>> iv.mpf([0,2]) in iv.mpf([0,10])
True
>>> 3 in iv.mpf(['-inf', 0])
False
```

---

Intervals have the properties `.a`, `.b` (endpoints), `.mid`, and `.delta` (width):

---

```
>>> x = iv.mpf([2, 5])
>>> x.a
[2.0, 2.0]
>>> x.b
[5.0, 5.0]
>>> x.mid
[3.5, 3.5]
>>> x.delta
[3.0, 3.0]
```

---

Some transcendental functions are supported:

---

```
>>> iv.dps = 15
>>> mp.dps = 15
>>> iv.mpf([0.5,1.5]) ** iv.mpf([0.5, 1.5])
```

```
[0.35355339059327373086, 1.837117307087383633]
>>> iv.exp(0)
[1.0, 1.0]
>>> iv.exp([-inf,'inf'])
[0.0, +inf]
>>>
>>> iv.exp([-inf,0])
[0.0, 1.0]
>>> iv.exp([0,'inf'])
[1.0, +inf]
>>> iv.exp([0,1])
[1.0, 2.7182818284590455349]
>>>
>>> iv.log(1)
[0.0, 0.0]
>>> iv.log([0,1])
[-inf, 0.0]
>>> iv.log([0,'inf'])
[-inf, +inf]
>>> iv.log(2)
[0.69314718055994528623, 0.69314718055994539725]
>>>
>>> iv.sin([100,'inf'])
[-1.0, 1.0]
>>> iv.cos([-0.1,'0.1'])
[0.99500416527802570954, 1.0]
```

---

Interval arithmetic is useful for proving inequalities involving irrational numbers. Naive use of mp arithmetic may result in wrong conclusions, such as the following:

```
>>> mp.dps = 25
>>> x = mp.exp(mp.pi*mp.sqrt(163))
>>> y = mp.mpf(640320**3+744)
>>> print x
262537412640768744.0000001
>>> print y
262537412640768744.0
>>> x > y
True
```

---

But the correct result is  $e^{\pi\sqrt{163}} < 262537412640768744$ , as can be seen by increasing the precision:

```
>>> mp.dps = 50
>>> print mp.exp(mp.pi*mp.sqrt(163))
262537412640768743.99999999999925007259719818568888
```

---

With interval arithmetic, the comparison returns None until the precision is large enough for  $x - y$  to have a definite sign:

```
>>> iv.dps = 15
>>> iv.exp(iv.pi*iv.sqrt(163)) > (640320**3+744)
>>> iv.dps = 30
>>> iv.exp(iv.pi*iv.sqrt(163)) > (640320**3+744)
```

---

```
>>> iv.dps = 60
>>> iv.exp(iv.pi*iv.sqrt(163)) > (640320**3+744)
False
>>> iv.dps = 15
```

---

### 3.1.9 Fast low-precision arithmetic (fp)

Although mpFormulaPy is generally designed for arbitrary-precision arithmetic, many of the high-level algorithms work perfectly well with ordinary Python float and complex numbers, which use hardware double precision (on most systems, this corresponds to 53 bits of precision).

Whereas the global functions (which are methods of the mp object) always convert inputs to mpFormulaPy numbers, the fp object instead converts them to float or complex, and in some cases employs basic functions optimized for double precision. When large amounts of function evaluations (numerical integration, plotting, etc) are required, and when fp arithmetic provides sufficient accuracy, this can give a significant speedup over mp arithmetic.

To take advantage of this feature, simply use the fp prefix, i.e. write fp.func instead of func or mp.func:

---

```
>>> u = fp.erfc(2.5)
>>> print u
0.000406952017445
>>> type(u)
<type 'float'>
>>> mp.dps = 15
>>> print mp.erfc(2.5)
0.000406952017444959
>>> fp.matrix([[1,2],[3,4]]) ** 2
matrix(
[[7.0, 10.0],
[15.0, 22.0]])
>>>
>>> type(_[0,0])
<type 'float'>
>>> print fp.quad(fp.sin, [0, fp.pi]) # numerical integration
2.0
```

---

The fp context wraps Python's math and cmath modules for elementary functions. It supports both real and complex numbers and automatically generates complex results for real inputs (math raises an exception):

---

```
>>> fp.sqrt(5)
2.23606797749979
>>> fp.sqrt(-5)
2.23606797749979j
>>> fp.sin(10)
-0.5440211108893698
>>> fp.power(-1, 0.25)
(0.7071067811865476+0.7071067811865475j)
>>> (-1) ** 0.25
Traceback (most recent call last):
```

```
...
ValueError: negative number cannot be raised to a fractional power
```

The prec and dps attributes can be changed (for interface compatibility with the mp context) but this has no effect:

```
>>> fp.prec
53
>>> fp.dps
15
>>> fp.prec = 80
>>> fp.prec
53
>>> fp.dps
15
```

Due to intermediate rounding and cancellation errors, results computed with fp arithmetic may be much less accurate than those computed with mp using an equivalent precision (mp.prec = 53), since the latter often uses increased internal precision. The accuracy is highly problem-dependent: for some functions, fp almost always gives 14-15 correct digits; for others, results can be accurate to only 2-3 digits or even completely wrong. The recommended use for fp is therefore to speed up large-scale computations where accuracy can be verified in advance on a subset of the input set, or where results can be verified afterwards.

## 3.2 Precision and representation issues

Most of the time, using mpFormulaPy is simply a matter of setting the desired precision and entering a formula. For verification purposes, a quite (but not always!) reliable technique is to calculate the same thing a second time at a higher precision and verifying that the results agree.

To perform more advanced calculations, it is important to have some understanding of how mpFormulaPy works internally and what the possible sources of error are. This section gives an overview of arbitrary-precision binary floating-point arithmetic and some concepts from numerical analysis.

The following concepts are important to understand:

The main sources of numerical errors are rounding and cancellation, which are due to the use of finite-precision arithmetic, and truncation or approximation errors, which are due to approximating infinite sequences or continuous functions by a finite number of samples.

Errors propagate between calculations. A small error in the input may result in a large error in the output.

Most numerical algorithms for complex problems (e.g. integrals, derivatives) give wrong answers for sufficiently ill-behaved input. Sometimes virtually the only way to get a wrong answer is to design some very contrived input, but at other times the chance of accidentally obtaining a wrong result even for reasonable-looking input is quite high.

Like any complex numerical software, mpFormulaPy has implementation bugs. You should be reasonably suspicious about any results computed by mpFormulaPy, even those it claims to be able to compute correctly! If possible, verify results analytically, try different algorithms, and cross-compare with other software.

### 3.2.1 Precision, error and tolerance

The following terms are common in this documentation:

Precision (or working precision) is the precision at which floating-point arithmetic operations are performed.

Error is the difference between a computed approximation and the exact result.

Accuracy is the inverse of error.

Tolerance is the maximum error (or minimum accuracy) desired in a result.

Error and accuracy can be measured either directly, or logarithmically in bits or digits. Specifically, if a  $\hat{y}$  is an approximation for  $y$ , then

$$(\text{Direct}) \text{ absolute error} = |\hat{y} - y|$$

$$(\text{Direct}) \text{ relative error} = |\hat{y} - y| |y|^{-1}$$

$$(\text{Direct}) \text{ absolute accuracy} = |\hat{y} - y|^{-1}$$

$$(\text{Direct}) \text{ relative accuracy} = |\hat{y} - y|^{-1} |y|$$

$$(\text{Logarithmic}) \text{ absolute error} = \log_b |\hat{y} - y|$$

$$(\text{Logarithmic}) \text{ relative error} = \log_b |\hat{y} - y| - \log_b |y|$$

$$(\text{Logarithmic}) \text{ absolute accuracy} = -\log_b |\hat{y} - y|$$

$$(\text{Logarithmic}) \text{ relative accuracy} = -\log_b |\hat{y} - y| - \log_b |y|$$

where  $b = 2$  and  $b = 10$  for bits and digits respectively. Note that:

The logarithmic error roughly equals the position of the first incorrect bit or digit.

The logarithmic accuracy roughly equals the number of correct bits or digits in the result.

These definitions also hold for complex numbers, using  $|a + bi| = \sqrt{a^2 + b^2}$ .

Full accuracy means that the accuracy of a result at least equals prec-1, i.e. it is correct except possibly for the last bit.

### 3.2.2 Representation of numbers

Mpmath uses binary arithmetic. A binary floating-point number is a number of the form  $man \times 2^{exp}$  where both man (the mantissa) and exp (the exponent) are integers. Some examples of floating-point numbers are given in the following table.

Number	Mantissa	Exponent
3	3	0
10	5	1
-16	-1	4
1.25	5	-2

The representation as defined so far is not unique; one can always multiply the mantissa by 2 and subtract 1 from the exponent with no change in the numerical value. In mpFormulaPy, numbers are always normalized so that man is an odd number, with the exception of zero which is always taken to have man = exp = 0. With these conventions, every representable number has a unique representation. (Mpmath does not currently distinguish between positive and negative zero.)

Simple mathematical operations are now easy to define. Due to uniqueness, equality testing of two numbers simply amounts to separately checking equality of the mantissas and the exponents. Multiplication of nonzero numbers is straightforward:  $(m2^e) \times (n2^f) = (mn) \times 2^{e+f}$ . Addition is a bit more involved: we first need to multiply the mantissa of one of the operands by a suitable power of 2 to obtain equal exponents.

More technically, mpFormulaPy represents a floating-point number as a 4-tuple (sign, man, exp, bc) where sign is 0 or 1 (indicating positive vs negative) and the mantissa is nonnegative; bc (bitcount) is the size of the absolute value of the mantissa as measured in bits. Though redundant, keeping a separate sign field and explicitly keeping track of the bitcount significantly speeds up arithmetic (the bitcount, especially, is frequently needed but slow to compute from scratch due to the lack of a Python built-in function for the purpose).

Contrary to popular belief, floating-point numbers do not come with an inherent 'small uncertainty', although floating-point arithmetic generally is inexact. Every binary floating-point number is an exact rational number. With arbitrary-precision integers used for the mantissa and exponent, floating-point numbers can be added, subtracted and multiplied exactly. In particular, integers and integer multiples of  $1/2$ ,  $1/4$ ,  $1/8$ ,  $1/16$ , etc. can be represented, added and multiplied exactly in binary floating-point arithmetic.

Floating-point arithmetic is generally approximate because the size of the mantissa must be limited for efficiency reasons. The maximum allowed width (bitcount) of the mantissa is called the precision or prec for short. Sums and products of floating-point numbers are exact as long as

the absolute value of the mantissa is smaller than  $2^{prec}$ . As soon as the mantissa becomes larger than this, it is truncated to contain at most prec bits (the exponent is incremented accordingly to preserve the magnitude of the number), and this operation introduces a rounding error. Division is also generally inexact; although we can add and multiply exactly by setting the precision high enough, no precision is high enough to represent for example  $1/3$  exactly (the same obviously applies for roots, trigonometric functions, etc).

The special numbers `+inf`, `-inf` and `nan` are represented internally by a zero mantissa and a nonzero exponent.

Mpmath uses arbitrary precision integers for both the mantissa and the exponent, so numbers can be as large in magnitude as permitted by the computer's memory. Some care may be necessary when working with extremely large numbers. Although standard arithmetic operators are safe, it is for example futile to attempt to compute the exponential function of  $10^{100000}$ . Mpmath does not complain when asked to perform such a calculation, but instead chugs away on the problem to the best of its ability, assuming that computer resources are infinite. In the worst case, this will be slow and allocate a huge amount of memory; if entirely impossible Python will at some point raise `OverflowError`: long int too large to convert to int.

For further details on how the arithmetic is implemented, refer to the `mpFormulaPy` source code. The basic arithmetic operations are found in the `libmp` directory; many functions there are commented extensively.

### 3.2.3 Decimal issues

Mpmath uses binary arithmetic internally, while most interaction with the user is done via the decimal number system. Translating between binary and decimal numbers is a somewhat subtle matter; many Python novices run into the following 'bug' (addressed in the General Python FAQ):

---

```
>>> 1.2 - 1.0
0.1999999999999996
```

---

Decimal fractions fall into the category of numbers that generally cannot be represented exactly in binary floating-point form. For example, none of the numbers  $0.1$ ,  $0.01$ ,  $0.001$  has an exact representation as a binary floating-point number. Although `mpFormulaPy` can approximate decimal fractions with any accuracy, it does not solve this problem for all uses; users who need exact decimal fractions should look at the `decimal` module in Python's standard library (or perhaps use `fractions`, which are much faster).

With `prec` bits of precision, an arbitrary number can be approximated relatively to within  $2^{-prec}$ , or within  $10^{-dps}$  for `dps` decimal digits. The equivalent values for `prec` and `dps` are therefore related proportionally via the factor  $C = \log(1)/\log(2)$ , or roughly 3.32. For example, the standard (binary) precision in `mpFormulaPy` is 53 bits, which corresponds to a decimal precision of 15.95 digits.

More precisely, `mpFormulaPy` uses the following formulas to translate between `prec` and `dps`:

---

```
dps(prec) = max(1, int(round(int(prec) / C - 1)))
prec(dps) = max(1, int(round((int(dps) + 1) * C)))
```

---

Note that the `dps` is set 1 decimal digit lower than the corresponding binary precision. This is done to hide minor rounding errors and artifacts resulting from binary-decimal conversion. As a result, `mpFormulaPy` interprets 53 bits as giving 15 digits of decimal precision, not 16.

The `dps` value controls the number of digits to display when printing numbers with `str()`, while the decimal precision used by `repr()` is set two or three digits higher. For example, with 15 `dps` we have:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15
>>> str(pi)
'3.14159265358979'
>>> repr(+pi)
"mpf('3.1415926535897931')"
```

---

The extra digits in the output from `repr` ensure that `x == eval(repr(x))` holds, i.e. that numbers can be converted to strings and back losslessly.

It should be noted that precision and accuracy do not always correlate when translating between binary and decimal. As a simple example, the number 0.1 has a decimal precision of 1 digit but is an infinitely accurate representation of 1/10. Conversely, the number  $2^{-50}$  has a binary representation with 1 bit of precision that is infinitely accurate; the same number can actually be represented exactly as a decimal, but doing so requires 35 significant digits:

---

```
0.000000000000000088817841970012523233890533447265625
```

---

All binary floating-point numbers can be represented exactly as decimals (possibly requiring many digits), but the converse is false.

### 3.2.4 Correctness guarantees

Basic arithmetic operations (with the `mp` context) are always performed with correct rounding. Results that can be represented exactly are guaranteed to be exact, and results from single inexact operations are guaranteed to be the best possible rounded values. For higher-level operations, `mpFormulaPy` does not generally guarantee correct rounding. In general, `mpFormulaPy` only guarantees that it will use at least the user-set precision to perform a given calculation. The user may have to manually set the working precision higher than the desired accuracy for the result, possibly much higher.

Functions for evaluation of transcendental functions, linear algebra operations, numerical integration, etc., usually automatically increase the working precision and use a stricter tolerance to give a correctly rounded result with high probability: for example, at 50 bits the temporary precision might be set to 70 bits and the tolerance might be set to 60 bits. It can often be assumed that such functions return values that have full accuracy, given inputs that are exact (or sufficiently precise approximations of exact values), but the user must exercise judgement about whether to trust `mpFormulaPy`.

The level of rigor in `mpFormulaPy` covers the entire spectrum from 'always correct by design' through 'nearly always correct' and 'handling the most common errors' to 'just computing blindly and hoping for the best'. Of course, a long-term development goal is to successively increase the rigor where possible. The following list might give an idea of the current state.

Operations that are correctly rounded:

Addition, subtraction and multiplication of real and complex numbers. Division and square roots of real numbers.

Powers of real numbers, assuming sufficiently small integer exponents (huge powers are rounded in the right direction, but possibly farther than necessary).

Conversion from decimal to binary, for reasonably sized numbers (roughly  $10^{-100}$  between and  $10^{100}$ ).

Typically, transcendental functions for exact input-output pairs.

Operations that should be fully accurate (however, the current implementation may be based on a heuristic error analysis):

Radix conversion (large or small numbers).

Mathematical constants like  $\pi$ .

Both real and imaginary parts of exp, cos, sin, cosh, sinh, log.

Other elementary functions (the largest of the real and imaginary part).

The gamma and log-gamma functions (the largest of the real and the imaginary part; both, when close to real axis).

Some functions based on hypergeometric series (the largest of the real and imaginary part).

Correctness of root-finding, numerical integration, etc. largely depends on the well-behavedness of the input functions. Specific limitations are sometimes noted in the respective sections of the documentation.

### 3.2.5 Double precision emulation

On most systems, Python's float type represents an IEEE 754 double precision number, with a precision of 53 bits and rounding-to-nearest. With default precision (`mp.prec = 53`), the `mpf` type roughly emulates the behavior of the float type. Sources of incompatibility include the following:

In hardware floating-point arithmetic, the size of the exponent is restricted to a fixed range: regular Python floats have a range between roughly  $10^{-300}$  and  $10^{300}$ ). Mpmath does not emulate overflow or underflow when exponents fall outside this range.

On some systems, Python uses 80-bit (extended double) registers for floating-point operations. Due to double rounding, this makes the float type less accurate. This problem is only known to occur with Python versions compiled with GCC on 32-bit systems.

Machine floats very close to the exponent limit round subnormally, meaning that they lose accuracy (Python may raise an exception instead of rounding a float subnormally).

Mpmath is able to produce more accurate results for transcendental functions.

## 3.3 Conversion of formatted numbers

### 3.3.1 Conversions between Roman and Arabic Numbers

---

`WorksheetFunction.ROMAN(Number As mpNum, Form As Integer) As String`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.ROMAN` returns Converts an arabic numeral to roman, as text.

**Parameters:**

**Number:** The Arabic numeral you want converted.

**Form:** A number from 0 to 4 specifying the type of roman numeral you want. The roman numeral style ranges from Classic to Simplified, becoming more concise as the value of form increases..

---

`WorksheetFunction.ARABIC(Number As mpNum) As String`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.ARABIC` returns Converts an roman numeral to arabic, as text.

**Parameter:**

**Number:** The roman numeral you want converted.

### 3.3.2 Conversions from Binary

---

`WorksheetFunction.BIN2DEC(Number As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.BIN2DEC` returns Converts a binary number to decimal.

**Parameter:**

**Number:** The binary number you want to convert. Number cannot contain more than 10 characters (10 bits). The most significant bit of number is the sign bit. The remaining 9 bits are magnitude bits. Negative numbers are represented using two's-complement notation.

---

`WorksheetFunction.BIN2HEX(Number As mpNum, Places As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.BIN2HEX` returns Converts a binary number to decimal.

**Parameters:**

**Number:** The binary number you want to convert. Number cannot contain more than 10 characters (10 bits). The most significant bit of number is the sign bit. The remaining 9 bits are magnitude bits. Negative numbers are represented using two's-complement notation.

**Places:** The number of characters to use. If places is omitted, BIN2HEX uses the minimum number of characters necessary. Places is useful for padding the return value with leading 0s (zeros).

---

WorksheetFunction.**BIN2OCT**(**Number** As mpNum, **Places** As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.BIN2OCT returns Converts a binary number to octal.

**Parameters:**

**Number:** The binary number you want to convert. Number cannot contain more than 10 characters (10 bits). The most significant bit of number is the sign bit. The remaining 9 bits are magnitude bits. Negative numbers are represented using two's-complement notation.

**Places:** The number of characters to use. If places is omitted, BIN2OCT uses the minimum number of characters necessary. Places is useful for padding the return value with leading 0s (zeros).

### 3.3.3 Conversions from Decimal

---

WorksheetFunction.**DEC2BIN**(**Number** As mpNum, **Places** As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.DEC2BIN returns Converts a decimal number to binary.

**Parameters:**

**Number:** The decimal integer you want to convert. If number is negative, valid place values are ignored and DEC2BIN returns a 10-character (10-bit) binary number in which the most significant bit is the sign bit. The remaining 9 bits are magnitude bits. Negative numbers are represented using two's-complement notation.

**Places:** The number of characters to use. If places is omitted, DEC2BIN uses the minimum number of characters necessary. Places is useful for padding the return value with leading 0s (zeros).

---

WorksheetFunction.**DEC2HEX**(**Number** As mpNum, **Places** As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.DEC2HEX returns Converts a decimal number to hexadecimal.

**Parameters:**

**Number:** The decimal integer you want to convert. If number is negative, places is ignored and DEC2HEX returns a 10-character (40-bit) hexadecimal number in which the most significant bit is the sign bit. The remaining 39 bits are magnitude bits. Negative numbers are represented using two's-complement notation.

**Places:** The number of characters to use. If places is omitted, DEC2HEX uses the minimum number of characters necessary. Places is useful for padding the return value with leading 0s (zeros).

---

WorksheetFunction.**DEC2OCT**(**Number** As mpNum, **Places** As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.DEC2OCT returns Converts a decimal number to octal.

**Parameters:**

**Number:** The decimal integer you want to convert. If number is negative, places is ignored and DEC2OCT returns a 10-character (30-bit) octal number in which the most significant bit is the sign bit. The remaining 29 bits are magnitude bits. Negative numbers are represented using two's-complement notation.

**Places:** The number of characters to use. If places is omitted, DEC2OCT uses the minimum number of characters necessary. Places is useful for padding the return value with leading 0s (zeros).

### 3.3.4 Conversions from Hexadecimal

---

WorksheetFunction.**HEX2BIN**(**Number** As mpNum, **Places** As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.HEX2BIN returns Converts a hexadecimal number to binary.

**Parameters:**

**Number:** The hexadecimal number you want to convert. Number cannot contain more than 10 characters. The most significant bit of number is the sign bit (40th bit from the right). The remaining 9 bits are magnitude bits. Negative numbers are represented using two's-complement notation.

**Places:** The number of characters to use. If places is omitted, HEX2BIN uses the minimum number of characters necessary. Places is useful for padding the return value with leading 0s (zeros).

---

WorksheetFunction.**HEX2DEC**(**Number** As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.HEX2DEC returns Converts a hexadecimal number to decimal.

**Parameter:**

**Number:** The hexadecimal number you want to convert. Number cannot contain more than 10 characters (40 bits). The most significant bit of number is the sign bit. The remaining 39 bits are magnitude bits. Negative numbers are represented using two's-complement notation

---

WorksheetFunction.**HEX2OCT**(**Number** As mpNum, **Places** As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.HEX2OCT returns Converts a hexadecimal number to octal.

**Parameters:**

**Number:** The hexadecimal number you want to convert. Number cannot contain more than 10 characters. The most significant bit of number is the sign bit. The remaining 39 bits are magnitude bits. Negative numbers are represented using two's-complement notation.

**Places:** The number of characters to use. If places is omitted, HEX2OCT uses the minimum number of characters necessary. Places is useful for padding the return value with leading 0s (zeros).

### 3.3.5 Conversions from Octal

---

WorksheetFunction.**OCT2BIN**(*Number As mpNum, Places As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.OCT2BIN returns Converts a octal number to binary.

**Parameters:**

*Number*: The octal number you want to convert. Number may not contain more than 10 characters. The most significant bit of number is the sign bit. The remaining 29 bits are magnitude bits. Negative numbers are represented using two's-complement notation.

*Places*: The number of characters to use. If places is omitted, OCT2BIN uses the minimum number of characters necessary. Places is useful for padding the return value with leading 0s (zeros).

---

WorksheetFunction.**OCT2DEC**(*Number As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.OCT2DEC returns Converts an octal number to decimal.

**Parameter:**

*Number*: The octal number you want to convert. Number may not contain more than 10 octal characters (30 bits). The most significant bit of number is the sign bit. The remaining 29 bits are magnitude bits. Negative numbers are represented using two's-complement notation.

---

WorksheetFunction.**OCT2HEX**(*Number As mpNum, Places As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.OCT2HEX returns Converts an octal number to hexadecimal.

**Parameters:**

*Number*: The octal number you want to convert. Number may not contain more than 10 octal characters (30 bits). The most significant bit of number is the sign bit. The remaining 29 bits are magnitude bits. Negative numbers are represented using two's-complement notation.

*Places*: The number of characters to use. If places is omitted, OCT2HEX uses the minimum number of characters necessary. Places is useful for padding the return value with leading 0s (zeros).

### 3.3.6 Conversion to and from a Given Base

---

WorksheetFunction.**BASE**(*Number As mpNum, Radix As mpNum, MinLength As mpNum*) As mpNum

---

The function WorksheetFunction.BASE returns converts a number into a text representation with the given radix (base).

**Parameters:**

*Number*: The number that you want to convert. Must be an integer greater than or equal to 0 and less than  $2^{53}$ .

*Radix*: The base radix that you want to convert the number into. Must be an integer greater than or equal to 2 and less than or equal to 36.

*MinLength*: The minimum length of the returned string. Must be an integer greater than or equal to 0.

---

WorksheetFunction.**DECIMAL**( *Text As String*, *Radix As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.DECIMAL returns Converts a text representation of a number in a given base into a decimal number.

**Parameters:**

*Text*: The string length of Text must be less than or equal to 255 characters.

*Radix*: Radix must be an integer greater than or equal to 2 (binary, or base 2) and less than or equal to 36 (base 36).

## 3.4 Conversion and printing

### 3.4.1 convert()

`mpFormulaPy.mpFormulaify(x, strings=True)`

Converts x to an mpf or mpc. If x is of type mpf, mpc, int, float, complex, the conversion will be performed losslessly.

If x is a string, the result will be rounded to the present working precision. Strings representing fractions or complex numbers are permitted.

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> convert(3.5)
mpf('3.5')
>>> convert('2.1')
mpf('2.100000000000001')
>>> convert('3/4')
mpf('0.75')
>>> convert('2+3j')
mpc(real='2.0', imag='3.0')
```

---

### 3.4.2 nstr()

`mpFormulaPy.nstr(x, n=6, **kwargs)`

Convert an mpf or mpc to a decimal string literal with n significant digits. The small default value for n is chosen to make this function useful for printing collections of numbers (lists, matrices, etc.).

If x is a list or tuple, nstr() is applied recursively to each element. For unrecognized classes, nstr() simply returns str(x).

The companion function nprint() prints the result instead of returning it.

---

```
>>> from mpFormulaPy import *
>>> nstr([+pi, ldexp(1,-500)])
'[3.14159, 3.05494e-151]'
>>> nprint([+pi, ldexp(1,-500)])
[3.14159, 3.05494e-151]
```

---

## 3.5 Rounding

### 3.5.1 Nearest integer: Round( $x$ )

---

WorksheetFunction.ROUND(***Number*** As mpNum, ***Digits*** As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.ROUND returns a number rounded to a specified number of digits

**Parameters:**

*Number*: A real number you want to round.

*Digits*: The number of digits to which you want to round. Negative rounds to the left of the decimal point; zero to the nearest integer.

ROUND rounds to the nearest representable integer, rounding halfway cases away from zero (as in the roundTiesToAway mode of IEEE 754-2008).

The returned indicator value is zero when the result is exact, positive when it is greater than the original value of op, and negative when it is smaller. More precisely, the returned value is 0 when op is an integer representable in rop, 1 or -1 when op is an integer that is not representable in rop, 2 or -2 when op is not an integer.

Note that mpfr\_round is different from mpfr\_rint called with the rounding to nearest mode (where halfway cases are rounded to an even integer or significand). Note also that no double rounding is performed; for instance, 10.5 (1010.1 in binary) is rounded by mpfr\_rint with rounding to nearest to 12 (1100 in binary) in 2-bit precision, because the two enclosing numbers representable on two bits are 8 and 12, and the closest is 12. (If one first rounded to an integer, one would round 10.5 to 10 with even rounding, and then 10 would be rounded to 8 again with even rounding.)

### 3.5.2 Next higher or equal integer: Ceil( $x$ )

---

WorksheetFunction.CEILING(***Number*** As mpNum, ***Significance*** As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.CEILING returns a number rounded up to the nearest multiple of significance

**Parameters:**

*Number*: A real number you want to round.

*Significance*: The multiple to which you want to round.

---

WorksheetFunction.CEILING.PRECISE(***Number*** As mpNum, ***Significance*** As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.CEILING.PRECISE returns a number rounded up to the nearest multiple of significance

**Parameters:**

*Number*: A real number you want to round.

*Significance*: The multiple to which you want to round.

---

WorksheetFunction.**CEILING.MATH**(*Number As mpNum*, *Significance As mpNum*) As mp-Num

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.CEILING.MATH returns a number rounded up to the nearest multiple of significance

**Parameters:**

*Number*: A real number you want to round.

*Significance*: The multiple to which you want to round.

CEILING rounds to the next higher or equal representable integer.

---

Function **ceil(x As mpNum)** As mpNum

---

The function ceil returns a number down to the nearest integer.

**Parameter:**

*x*: A real number.

Computes the ceiling of *x*,  $\lceil x \rceil$ , defined as the smallest integer greater than or equal to *x*:

---

```
>>> from mpFormulaPy import *
>>> mp.pretty = False
>>> ceil(3.5)
mpf('4.0')
```

---

The ceiling function is defined for complex numbers and acts on the real and imaginary parts separately:

---

```
>>> ceil(3.25+4.75j)
mpc(real='4.0', imag='5.0')
```

---

See notes about rounding for floor().

### 3.5.3 Next lower or equal integer: Floor(*x*)

---

WorksheetFunction.**FLOOR**(*Number As mpNum*, *Significance As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.FLOOR returns a number rounded down to the nearest multiple of significance

**Parameters:**

*Number*: A real number you want to round.

*Significance*: The multiple to which you want to round. Number and Significance must either both be positive or both negative

---

WorksheetFunction.**FLOOR.PRECISE**(*Number As mpNum*, *Significance As mpNum*) As mp-Num

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.FLOOR.PRECISE` returns a number rounded down to the nearest integer or to the nearest multiple of significance

#### Parameters:

*Number*: A real number you want to round.

*Significance*: The multiple to which you want to round.

`WorksheetFunction.FLOOR.MATH(Number As mpNum, Significance As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.FLOOR.MATH` returns a number rounded down to the nearest multiple of significance

#### Parameters:

*Number*: A real number you want to round.

*Significance*: The multiple to which you want to round.

`FLOOR` rounds to the next lower or equal representable integer.

**Function `floor(x As mpNum) As mpNum`**

The function `floor` returns a number down to the nearest integer.

#### Parameter:

*x*: A real number.

Computes the floor of *x*,  $\lfloor x \rfloor$ , defined as the largest integer less than or equal to *x*:

```
>>> from mpFormulaPy import *
>>> mp.pretty = False
>>> floor(3.5)
mpf('3.0')
```

Note: `floor()`, `ceil()` and `nint()` return a floating-point number, not a Python `int`. If it is too large to be represented exactly at the present working precision, the result will be rounded, not necessarily in the direction implied by the mathematical definition of the function.

To avoid rounding, use `prec=0`:

```
>>> mp.dps = 15
>>> print(int(floor(10**30+1)))
1000000000000000000019884624838656
>>> print(int(floor(10**30+1, prec=0)))
100000000000000000000000000000000000000000000000000000000000000001
```

The `floor` function is defined for complex numbers and acts on the real and imaginary parts separately:

```
>>> floor(3.25+4.75j)
mpc(real='3.0', imag='4.0')
```

### 3.5.4 Next integer, rounded toward zero: Trunc( $x$ )

---

WorksheetFunction.**TRUNC**(*Number* As mpNum, *Digits* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.TRUNC returns a number truncated to an integer by removing the decimal, or fractional, part of a number

**Parameters:**

*Number*: A real number you want to round.

*Digits*: A number specifying the precision of te truncation, 0 if omitted.

TRUNC rounds to the next representable integer toward zero.

### 3.5.5 EVEN( $x$ )

---

WorksheetFunction.**EVEN**(*x* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.EVEN returns the rounded value of  $x$ . Rounds a positive number up and a negative number down to the nearest even integer.

**Parameter:**

$x$ : A real number.

Returns number rounded up to the nearest even integer. You can use this function for processing items that come in twos. For example, a packing crate accepts rows of one or two items. The crate is full when the number of items, rounded up to the nearest two, matches the crate's capacity.

### 3.5.6 ODD( $x$ )

---

WorksheetFunction.**ODD**(*x* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.ODD returns the rounded value of  $x$ . Rounds a positive number up and a negative number down to the nearest odd integer.

**Parameter:**

$x$ : A real number.

Returns number rounded to the nearest odd integer.

### 3.5.7 Nearest integer

---

WorksheetFunction.**INT**(*x* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.INT returns a number down to the nearest integer.

**Parameter:**

*x*: A real number.

Rounds a number down to the nearest integer.

---

#### Function **nint(x As mpNum) As mpNum**

---

The function `nint` returns a number down to the nearest integer.

**Parameter:**

*x*: A real number.

Evaluates the nearest integer function,  $\text{nint}(x)$ . This gives the nearest integer to  $x$ ; on a tie, it gives the nearest even integer:

---

```
>>> from mpFormulaPy import *
>>> mp.pretty = False
>>> nint(3.2)
mpf('3.0')
>>> nint(3.8)
mpf('4.0')
>>> nint(3.5)
mpf('4.0')
>>> nint(4.5)
mpf('4.0')
```

---

The nearest integer function is defined for complex numbers and acts on the real and imaginary parts separately:

---

```
>>> nint(3.25+4.75j)
mpc(real='3.0', imag='5.0')
```

---

See notes about rounding for `floor()`.

### 3.5.8 Fractional Part

---

#### Function **frac(x As mpNum) As mpNum**

---

The function `frac` returns the fractional part of  $x$ .

**Parameter:**

*x*: A colpmex or real number.

Gives the fractional part of  $x$ , defined as  $\text{frac}(x) = x - \lfloor x \rfloor$  (see `floor()`). In effect, this computes  $x$  modulo 1, or  $x + n$  where  $n \in \mathbb{Z}$  is such that  $x + n \in [0, 1]$ :

---

```
>>> from mpFormulaPy import *
>>> mp.pretty = False
>>> frac(1.25)
mpf('0.25')
>>> frac(3)
mpf('0.0')
>>> frac(-1.25)
mpf('0.75')
```

---

For a complex number, the fractional part function applies to the real and imaginary parts separately:

---

```
>>> frac(2.25+3.75j)
mpc(real='0.25', imag='0.75')
```

---

Plotted, the fractional part function gives a sawtooth wave. The Fourier series coefficients have a simple form:

---

```
>>> mp.dps = 15
>>> nprint(fourier(lambda x: frac(x)-0.5, [0,1], 4))
[[0.0, 0.0, 0.0, 0.0, 0.0], [0.0, -0.31831, -0.159155, -0.106103, -0.0795775])
>>> nprint([-1/(pi*k) for k in range(1,5)])
[-0.31831, -0.159155, -0.106103, -0.0795775]
```

---

Note: The fractional part is sometimes defined as a symmetric function, i.e. returning  $-\text{frac}(-x)$  if  $x < 0$ . This convention is used, for instance, by Mathematica's FractionalPart.

### 3.5.9 ROUNDDOWN(*Number*, *Digits*)

---

WorksheetFunction.**ROUNDDOWN**(*Number* As mpNum, *Digits* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ROUNDDOWN returns a number rounded down, toward zero.

**Parameters:**

*Number*: A real number you want to round.

*Digits*: A number specifying the precision of te truncation, 0 if omitted.

Rounds a number down, toward zero.

### 3.5.10 ROUNDUP(*Number*, *Digits*)

---

WorksheetFunction.**ROUNDUP**(*Number* As mpNum, *Digits* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ROUNDUP returns a number rounded down, away from zero.

**Parameters:**

*Number*: A real number you want to round.

*Digits*: A number specifying the precision of te truncation, 0 if omitted.

Rounds a number up, away from 0 (zero).

### 3.5.11 MROUND(*Number*, *Multiple*)

---

WorksheetFunction.**MROUND**(*Number* As mpNum, *Multiple* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.MROUND returns a number rounded to the desired multiple.

**Parameters:**

*Number*: A real number you want to round.

*Multiple*: The multiple to which you want to round.

MROUND rounds up, away from zero, if the remainder of dividing number by multiple is greater than or equal to half the value of multiple.

### 3.5.12 QUOTIENT(*x, y*)

---

WorksheetFunction.**QUOTIENT**(*x As mpNum, y As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.QUOTIENT returns the integer portion of a division.

**Parameters:**

*x*: A real number

*y*: A real number

Returns the integer portion of a division. Use this function when you want to discard the remainder of a division.

## 3.6 Components of Real and Complex Numbers

### 3.6.1 Number generated from Significand and Exponent: Ldexp( $x, y$ )

---

Function **ldexp**( $x$  As *mpNum*,  $y$  As *mpNum*) As *mpNum*

---

The function **ldexp** returns  $x \cdot 2^y$

**Parameters:**

$x$ : A real number.

$y$ : A real number.

Returns the result of multiplying  $x$  (the significand) by 2 raised to the power of  $y$  (the exponent):

$$\text{Ldexp}(x, y) = x \cdot 2^y.$$

`mpFormulaPy.ldexp(x, n)`

Computes  $x2^n$  efficiently. No rounding is performed. The argument  $x$  must be a real floating-point number (or possible to convert into one) and  $n$  must be a Python int.

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> ldexp(1, 10)
mpf('1024.0')
>>> ldexp(1, -3)
mpf('0.125')
```

---

### 3.6.2 Significand and Exponent: Frexp( $x$ )

---

Function **frexp**( $x$  As *mpNum*) As *mpNumList*

---

The function **frexp** returns simultaneously significand and exponent of  $x$

**Parameter:**

$x$ : A real number.

Set  $\text{exp}$  (formally, the value pointed to by  $\text{exp}$ ) and  $y$  such that  $0.5 \leq |y| < 1$  and  $y \times 2^{\text{exp}}$  equals  $x$  rounded to the precision of  $y$ , using the given rounding mode. If  $x$  is zero, then  $y$  is set to a zero of the same sign and  $\text{exp}$  is set to 0. If  $x$  is NaN or an infinity, then  $y$  is set to the same value and  $\text{exp}$  is undefined.

`mpFormulaPy.frexp(x, n)`

Given a real number  $x$ , returns  $(y, n)$  with  $y \in [0.5, 1)$ ,  $n$  a Python integer, and such that  $x = y2^n$ . No rounding is performed.

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> frexp(7.5)
(mpf('0.9375'), 3)
```

---

### 3.6.3 Building a Complex Number from Real Components

---

WorksheetFunction.**COMPLEX**(*x* As mpReal, *y* As mpReal) As String

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.COMPLEX returns a complex number  $z$  build from the real components  $x$  and  $y$ , as string.

**Parameters:**

*x*: A real number.

*y*: A real number.

---

Function **mpc**(*x* As mpNum, *y* As mpNum) As mpNum

---

The function mpc returns a complex number  $z$  build from the real components  $x$  and  $y$  as  $z = x + iy$ .

**Parameters:**

*x*: A real number.

*y*: A real number.

### 3.6.4 Representations of Complex Numbers

---

Function **polar**(*z* As mpNum) As mpNum

---

The function polar returns Returns the polar representation of the complex number  $z$ .

**Parameter:**

*z*: A complex or real number.

Returns the polar representation of the complex number  $z$  as a pair  $(r, \phi)$  such that  $z = re^{i\phi}$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> polar(-2)
(2.0, 3.14159265358979)
>>> polar(3-4j)
(5.0, -0.927295218001612)
```

---



---

Function **rect**(*x* As mpNum, *y* As mpNum) As mpNum

---

The function rect returns the complex number represented by polar coordinates  $(r, \phi)$ .

**Parameters:**

*x*: A real number.

*y*: A real number.

Returns the complex number represented by polar coordinates  $(r, \phi)$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> chop(rect(2, pi))
```

---

```
-2.0
>>> rect(sqrt(2), -pi/4)
(1.0 - 1.0j)
```

---

### 3.6.5 Real Component

---

WorksheetFunction.**IMREAL**(*z As String*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.IMREAL returns the real component  $x$  of  $z = x + iy$ .

**Parameter:**

*z*: A String representing a complex number.

---

Function **re**(*z As mpNum*) As mpNum

---

The function re returns the real part of  $x$ ,  $\Re(x)$ .

**Parameter:**

*z*: A complex number.

Unlike x.real, re() converts  $x$  to a mpFormulaPy number:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> re(3)
mpf('3.0')
>>> re(-1+4j)
mpf('-1.0')
```

---

### 3.6.6 Imaginary Component

---

WorksheetFunction.**IMAGINARY**(*z As String*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.IMAGINARY returns the imaginary component  $y$  of  $z = x + iy$ .

**Parameter:**

*z*: A String representing a complex number.

---

Function **im**(*z As mpNum*) As mpNum

---

The function im returns the imaginary part of  $x$ ,  $\Im(x)$ .

**Parameter:**

*z*: A complex number.

Unlike x.imag, im() converts  $x$  to a mpFormulaPy number:

---

```
>>> from mpFormulaPy import *
```

---

```
>>> mp.dps = 15; mp.pretty = False
>>> im(3)
mpf('0.0')
>>> im(-1+4j)
mpf('4.0')
```

---

### 3.6.7 Absolute Value

---

**WorksheetFunction.ABS(*x* As *mpNum*) As *mpNum***

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.ABS** returns the absolute value of  $x$ ,  $|x| = \sqrt{x^2}$ .

**Parameter:**

*x*: A real number.

---

**WorksheetFunction.IMABS(*z* As *String*) As *mpReal***

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.IMABS** returns the absolute value of  $z = x + iy$

**Parameter:**

*z*: A String representing a complex number.

The absolute value of  $z = x + iy$  is calculated as

$$|z| = \sqrt{x^2 + y^2}. \quad (3.6.1)$$

---

**Function **abs**(*z* As *mpNum*) As *mpNum***

---

The function **abs** returns the absolute value of  $z = x + iy$

**Parameter:**

*z*: A real or complex number.

---

**Function **fabs**(*z* As *mpNum*) As *mpNum***

---

The function **fabs** returns the absolute value of  $z = x + iy$

**Parameter:**

*z*: A real or complex number.

Returns the absolute value of  $x$ ,  $|x|$ . Unlike **abs()**, **fabs()** converts non-mpFormulaPy numbers (such as **int**) into mpFormulaPy numbers:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> fabs(3)
mpf('3.0')
```

---

```
>>> fabs(-3)
mpf('3.0')
>>> fabs(3+4j)
mpf('5.0')
```

---

### 3.6.8 Argument

---

WorksheetFunction.**IMARGUMENT**(*z As String*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.IMARGUMENT returns the argument of  $z = x + iy$

**Parameter:**

*z*: A String representing a complex number.

The argument  $\theta$  of  $z = x + iy$ , is defined such that

$$z = x + iy = |x + iy|e^\theta = |x + iy|(\cos(\theta) + i \sin(\theta)). \quad (3.6.2)$$

cplxArg(*z*) is calculated as

$$\text{cplxArg}(z) = \arctan\left(\frac{y}{x}\right) = \theta, \text{ where } \theta \in (-\pi; \pi]. \quad (3.6.3)$$


---

Function **arg**(*z As mpNum*) As mpNum

---

The function arg returns the argument of  $z = x + iy$

**Parameter:**

*z*: A complex number.

---

Function **phase**(*z As mpNum*) As mpNum

---

The function phase returns the argument of  $z = x + iy$

**Parameter:**

*z*: A complex number.

Computes the complex argument (phase) of *x*, defined as the signed angle between the positive real axis and in the complex plane:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> arg(3)
0.0
>>> arg(3+3j)
0.785398163397448
>>> arg(3j)
1.5707963267949
>>> arg(-3)
3.14159265358979
```

---

```
>>> arg(-3j)
-1.5707963267949
```

---

The angle is defined to satisfy  $-\pi < \arg(x) \leq \pi$  and with the sign convention that a nonnegative imaginary part results in a nonnegative argument.

The value returned by `arg()` is an `mpf` instance.

### 3.6.9 Sign

---

`WorksheetFunction.SIGN(x As mpNum) As mpNum`

---

NOT YET IMPLEMENTED

---

The function `WorksheetFunction.SIGN` returns the value of the sign of  $x$ ,  $\text{sign}(x)$ .

**Parameter:**

$x$ : A real number.

---

Function `sign(x As mpNum) As mpNum`

---

The function `sign` returns the value of the sign of  $x$ ,  $\text{sign}(x)$ .

**Parameter:**

$x$ : A real or complex number.

The sign of  $x$  is defined as  $\text{sign}(x) = x/|x|$  (with the special case  $\text{sign}(0) = 0$ ):

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> sign(10)
mpf('1.0')
>>> sign(-10)
mpf('-1.0')
>>> sign(0)
mpf('0.0')
```

---

Note that the `sign` function is also defined for complex numbers, for which it gives the projection onto the unit circle:

---

```
>>> mp.dps = 15; mp.pretty = True
>>> sign(1+j)
(0.707106781186547 + 0.707106781186547j)
```

---

### 3.6.10 Conjugate

---

`WorksheetFunction.IMCONJUGATE(z As String) As String`

---

NOT YET IMPLEMENTED

---

The function `WorksheetFunction.IMCONJUGATE` returns the conjugate of  $z$ ,  $\bar{z} = x - iy$

**Parameter:**

*z*: A String representing a complex number.

---

**Function `conj(z As mpNum) As mpNum`**

---

The function `conj` returns the complex conjugate of  $z$ ,  $\bar{z}$

**Parameter:**

*z*: A complex number.

Unlike `x.conjugate()`, `im()` converts *x* to a `mpFormulaPy` number:

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> conj(3)
mpf('3.0')
>>> conj(-1+4j)
mpc(real='-1.0', imag='-4.0')
```

---

## 3.7 Arithmetic operations

See also `mpFormulaPy.sqrt()`, `mpFormulaPy.exp()` etc., listed in Powers and logarithms

### 3.7.1 Addition and Sum

---

Operator `+`

---

Number.Function **.Plus(a As mpNum, b As mpNum) As mpNum**

---

The binary operator `+` is used to return the sum of the 2 operands  $a$  and  $b$ , and assign the result to  $c$ :  $c = a + b$ .

For languages not supporting operator overloading, the function `.Plus` can be used to achieve the same:  $c = a$ .`Plus(b)`

The operator `+` returns the sum of  $z_1$  and  $z_2$ .

---

Function **fadd(x As mpNum, y As mpNum, Keywords As String) As mpNum**

---

The function `fadd` returns the sum of the numbers  $x$  and  $y$ , giving a floating-point result, optionally using a custom precision and rounding mode..

**Parameters:**

`x`: A complex number.

`y`: A complex number.

*Keywords*: `prec`, `dps`, `exact`, `rounding`.

`mpFormulaPy.fadd(ctx, x, y, **kwargs)`

Adds the numbers  $x$  and  $y$ , giving a floating-point result, optionally using a custom precision and rounding mode.

The default precision is the working precision of the context. You can specify a custom precision in bits by passing the `prec` keyword argument, or by providing an equivalent decimal precision with the `dps` keyword argument. If the precision is set to `+inf`, or if the flag `exact=True` is passed, an exact addition with no rounding is performed.

When the precision is finite, the optional rounding keyword argument specifies the direction of rounding. Valid options are '`n`' for nearest (default), '`f`' for floor, '`c`' for ceiling, '`d`' for down, '`u`' for up.

**Examples**

Using `fadd()` with precision and rounding control:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> fadd(2, 1e-20)
mpf('2.0')
>>> fadd(2, 1e-20, rounding='u')
mpf('2.000000000000004')
>>> nprint(fadd(2, 1e-20, prec=100), 25)
2.0000000000000000000000000000001
>>> nprint(fadd(2, 1e-20, dps=15), 25)
2.0
```

```
>>> nprint(fadd(2, 1e-20, dps=25), 25)
2.00000000000000000000000000000001
>>> nprint(fadd(2, 1e-20, exact=True), 25)
2.00000000000000000000000000000001
```

Exact addition avoids cancellation errors, enforcing familiar laws of numbers such as  $x + y - x = y$ , which do not hold in floating-point arithmetic with finite precision:

```
>>> x, y = mpf(2), mpf('1e-1000')
>>> print(x + y - x)
0.0
>>> print(fadd(x, y, prec=inf) - x)
1.0e-1000
>>> print(fadd(x, y, exact=True) - x)
1.0e-1000
```

Exact addition can be inefficient and may be impossible to perform with large magnitude differences:

```
>>> fadd(1, '1e-10000000000000000000000000000000', prec=inf)
Traceback (most recent call last):
...
OverflowError: the exact result does not fit in memory
```

### 3.7.2 Sums and Series

**WorksheetFunction.IMSUM(z As String()) As String**

## NOT YET IMPLEMENTED

The function `WorksheetFunction.IMSUM` returns the sum of up to 255 complex numbers.

## Parameter:

*z*: An array of Strings representing an array of complex numbers.

The function **IMSUM**( $z_1, z_2$ ) returns the sum of  $z_1$  and  $z_2$ :

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2). \quad (3.7.1)$$

Function **fsum**(*terms* As mpNum, *Keywords* As String) As mpNum

The function `fsum` returns the sum of the numbers `x` and `y`, giving a floating-point result, optionally using a custom precision and rounding mode..

### Parameters:

*terms*: a finite number of terms.

*Keywords:* absolute=False, squared=False.

```
mpFormulaPy.fsum(terms, absolute=False, squared=False)
```

Calculates a sum containing a finite number of terms (for infinite series, see nsum()). The terms

will be converted to mpFormulaPy numbers. For  $\text{len}(\text{terms}) \geq 2$ , this function is generally faster and produces more accurate results than the builtin Python function `sum()`.

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> fsum([1, 2, 0.5, 7])
mpf('10.5')
```

---

With `squared=True` each term is squared, and with `absolute=True` the absolute value of each term is used.

---

`WorksheetFunction.SUMX2MY2(X As mpNum[], Y As mpNum[]) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.SUMX2MY2` returns the sum of the difference of squares of corresponding values in two arrays.

**Parameters:**

`X`: A matrix of real numbers.

`Y`: A matrix of real numbers.

Returns the sum of the difference of squares of corresponding values in two arrays, which need have the same number of rows  $R$  and same number of columns  $C$ . The equation for the sum of the difference of squares is:

$$\text{SUMX2MY2} = \sum_{i=1}^R \sum_{j=1}^C (X_{i,j}^2 - Y_{i,j}^2) \quad (3.7.2)$$

---

`WorksheetFunction.SUMX2PY2(X As mpNum[], Y As mpNum[]) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.SUMX2PY2` returns the sum of the sum of squares of corresponding values in two arrays.

**Parameters:**

`X`: A matrix of real numbers.

`Y`: A matrix of real numbers.

Returns the sum of the sum of squares of corresponding values in two arrays, which need have the same number of rows  $R$  and same number of columns  $C$ . The equation for the sum of the sum of squares is:

$$\text{SUMX2PY2} = \sum_{i=1}^R \sum_{j=1}^C (X_{i,j}^2 + Y_{i,j}^2) \quad (3.7.3)$$

---

`WorksheetFunction.SUMXMY2(X As mpNum[], Y As mpNum[]) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.SUMXMY2` returns the sum of squares of differences of corresponding values in two arrays.

**Parameters:**

`X`: A matrix of real numbers.

$Y$ : A matrix of real numbers.

Returns the sum of squares of differences of corresponding values in two arrays, which need have the same number of rows  $R$  and same number of columns  $C$ . The equation for the sum of squared differences is:

$$\text{SUMXMY2} = \sum_{i=1}^R \sum_{j=1}^C (X_{i,j} - Y_{i,j})^2 \quad (3.7.4)$$

WorksheetFunction.**SUMSQ**( $X$  As mpNumList) As mpNum

**NOT YET IMPLEMENTED**

The function WorksheetFunction.SUMSQ returns the sum of the sum of the squares of up to 255 given arrays.

**Parameter:**

$X$ : A list of up to 255 given arrays.

Returns the sum of the sum of the squares of up to 255 given arrays. The arrays  $A, B, \dots$  do not need have the same number of rows or same number of columns. The equation for SUMSQ is:

$$\text{SUMSQ} = \sum A_{i,j}^2 + \sum B_{i,j}^2 + \dots \quad (3.7.5)$$

where the summation is over all entries of  $A, B, \dots$

WorksheetFunction.**SUMPRODUCT**( $X$  As mpNumList) As mpNum

**NOT YET IMPLEMENTED**

The function WorksheetFunction.SUMPRODUCT returns the product of corresponding components in up to 255 given arrays.

**Parameter:**

$X$ : A list of up to 255 given arrays.

Multiplies corresponding components in up to 255 given arrays, and returns the sum of those products. The arrays  $A, B, \dots$  all need have the same number of rows  $R$  and same number of columns  $C$ . The equation for the sum of the products is:

$$\text{SUMPRODUCT} = \sum_{i=1}^R \sum_{j=1}^C (A_{i,j} \times B_{i,j} \times \dots) \quad (3.7.6)$$

WorksheetFunction.**SERIESSUM**( $x$  As mpNum,  $n$  As Integer,  $m$  As Integer,  $a$  As mpNum[]) As mpNum

The function WorksheetFunction.SERIESSUM returns the sum of a (finite) power series.

**Parameters:**

$x$ : The input value to the power series.

$n$ : The initial power to which you want to raise  $x$ .

$m$ : The step by which to increase  $n$  for each term in the series..

$a$ : A set of  $j$  coefficients by which each successive power of  $x$  is multiplied.

The sum of a power series is calculated based on this formula:

$$\text{SERIESSUM} = a_1 x^n + a_2 x^{n+m} + a_3 x^{n+2m} + \dots + a_j x^{n+(j-1)m}. \quad (3.7.7)$$

The number of values in coefficients determines the number of terms in the power series. For example, if there are three values in coefficients, then there will be three terms in the power series.

### 3.7.3 Subtraction

---

Operator –

---

Function **.Minus(a As mpNum, b As mpNum) As mpNum**

---

The binary operator – is used to return the difference of the 2 operands  $a$  and  $b$ , and assign the result to  $c$ :  $c = a - b$ .

For languages not supporting operator overloading, the function **.Minus** can be used to achieve the same:  $c = a.Minus(b)$

---

WorksheetFunction.**IMSUB(z1 As String, z2 As String) As String**

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.IMSUB returns the difference of  $z_1$  and  $z_2$

**Parameters:**

$z_1$ : A Strings representing a complex number.  
 $z_2$ : A Strings representing a complex number.

The function IMSUB( $z_1, z_2$ ) returns the difference of  $z_1$  and  $z_2$ :

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2). \quad (3.7.8)$$

---

Function **fsub(x As mpNum, y As mpNum, Keywords As String) As mpNum**

---

The function **fsub** returns the sum of the numbers  $x$  and  $y$ , giving a floating-point result, optionally using a custom precision and rounding mode..

**Parameters:**

$x$ : A complex number.  
 $y$ : A complex number.

**Keywords:** prec, dps, exact, rounding.

mpFormulaPy.fsub(ctx, x, y, \*\*kwargs)

Subtracts the numbers  $x$  and  $y$ , giving a floating-point result, optionally using a custom precision and rounding mode.

See the documentation of fadd() for a detailed description of how to specify precision and rounding.

Examples Using fsub() with precision and rounding control:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> fsub(2, 1e-20)
mpf('2.0')
>>> fsub(2, 1e-20, rounding='d')
mpf('1.999999999999998')
>>> nprint(fsub(2, 1e-20, prec=100), 25)
```

```
1.99999999999999999999999  
>>> nprint(fsub(2, 1e-20, dps=15), 25)  
2.0  
>>> nprint(fsub(2, 1e-20, dps=25), 25)  
1.99999999999999999999999  
>>> nprint(fsub(2, 1e-20, exact=True), 25)  
1.99999999999999999999999
```

Exact subtraction avoids cancellation errors, enforcing familiar laws of numbers such as  $x+y-x = y$ , which don't hold in floating-point arithmetic with finite precision:

```
>>> x, y = mpf(2), mpf('1e1000')
>>> print(x - y + y)
0.0
>>> print(fsub(x, y, prec=inf) + y)
2.0
>>> print(fsub(x, y, exact=True) + y)
2.0
```

Exact subtraction can be inefficient and may be impossible to perform with large magnitude differences:

### 3.7.4 Negation

Function **fneg**(*x As mpNum, Keywords As String*) As mpNum

The function **fneg** returns the sum of the numbers x and y, giving a floating-point result, optionally using a custom precision and rounding mode..

## Parameters:

$x$ : A complex number.

**Keywords:** prec, dps, exact, rounding.

```
mpFormulaPy.fneg(ctx, x, **kwargs)
```

Negates the number  $x$ , giving a floating-point result, optionally using a custom precision and rounding mode.

See the documentation of `fadd()` for a detailed description of how to specify precision and rounding.

## Examples

An mpFormulaPy number is returned:

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> fneg(2.5)
mpf('-2.5')
>>> fneg(-5+2j)
```

```
mpc(real='5.0', imag='-2.0')
```

Precise control over rounding is possible:

```
>>> x = fadd(2, 1e-100, exact=True)
>>> fneg(x)
mpf('-2.0')
>>> fneg(x, rounding='f')
mpf('-2.0000000000000004')
```

Negating with and without roundoff:

### 3.7.5 Multiplication

## Operator \*

Function .Times(a As mpNum, b As mpNum) As mpNum

Function .TimesMat(**a** As mpNum, **b** As Integer) As mpNum

```
Function .DotProd(a As mpNum, b As Integer) As mpNum
```

Function .LSH(**a** As mpNum, **b** As Integer) As mpNum

The binary operator `*` is used to return the product of the 2 operands `a` and `b`, and assign the result to `c`: `c = a * b`.

For languages not supporting operator overloading, the function `.Times` can be used to achieve the same: `c = a.Times(b)`

Function **fmul**(*x* As *mpNum*, *y* As *mpNum*, **Keywords** As String) As *mpNum*

The function `fmul` returns the sum of the numbers `x` and `y`, giving a floating-point result, optionally using a custom precision and rounding mode..

## Parameters:

$x$ : A complex number.

$y$ : A complex number.

*Keywords:* prec, dps, exact, rounding.

---

```
mpFormulaPy.fmul(ctx, x, y, **kwargs)
```

Multiplies the numbers x and y, giving a floating-point result, optionally using a custom precision and rounding mode.

See the documentation of fadd() for a detailed description of how to specify precision and rounding.

### Examples

The result is an mpFormulaPy number:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> fmul(2, 5.0)
mpf('10.0')
>>> fmul(0.5j, 0.5)
mpc(real='0.0', imag='0.25')
```

---

Avoiding roundoff:

---

```
>>> x, y = 10**10+1, 10**15+1
>>> print(x*y)
10000000001000010000000001
>>> print(mpf(x) * mpf(y))
1.0000000001e+25
>>> print(int(mpf(x) * mpf(y)))
10000000001000011026399232
>>> print(int(fmul(x, y)))
10000000001000011026399232
>>> print(int(fmul(x, y, dps=25)))
10000000001000010000000001
>>> print(int(fmul(x, y, exact=True)))
10000000001000010000000001
```

---

Exact multiplication with complex numbers can be inefficient and may be impossible to perform with large magnitude differences between real and imaginary parts:

---

```
>>> x = 1+2j
>>> y = mpc(2, '1e-10000000000000000000000000')
>>> fmul(x, y)
mpc(real='2.0', imag='4.0')
>>> fmul(x, y, rounding='u')
mpc(real='2.0', imag='4.0000000000000009')
>>> fmul(x, y, exact=True)
Traceback (most recent call last):
...
OverflowError: the exact result does not fit in memory
```

---

### 3.7.6 Products

---

WorksheetFunction.**PRODUCT**(*X As mpNumList*) As mpNum

---

NOT YET IMPLEMENTED

---

The function `WorksheetFunction.PRODUCT` returns the product of all the numbers of up to 255 given arrays.

**Parameter:**

`X`: A list of up to 255 given arrays.

The `PRODUCT` function multiplies all the numbers of up to 255 given arrays and returns the product. The arrays  $A, B, \dots$  do not need have the same number of rows or same number of columns. The equation for `PRODUCT` is:

$$\text{PRODUCT} = \prod A_{i,j} \times \prod B_{i,j} \times \dots \quad (3.7.9)$$

where the multiplication is over all entries of  $A, B, \dots$

`WorksheetFunction.IMPRODUCT(z As String[])` As String

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.IMPRODUCT` returns the product of up to 255 complex numbers.

**Parameter:**

`z`: An array of Strings representing an array of complex numbers.

The function `cplxMul(z1, z2)` returns the product of  $z1$  and  $z2$ :

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \quad (3.7.10)$$

**Function `fprod(factors As mpNum, Keywords As String)` As mpNum**

The function `fprod` returns a product containing a finite number of factors

**Parameters:**

`factors`: a finite number of factors

`Keywords`: prec, dps, exact, rounding.

`mpFormulaPy.fprod(factors)`

Calculates a product containing a finite number of factors (for infinite products, see `nprod()`). The factors will be converted to `mpFormulaPy` numbers.

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> fprod([1, 2, 0.5, 7])
mpf('7.0')
```

**Function `fdot(factors As mpNum, Keywords As String)` As mpNum**

The function `fdot` returns a product containing a finite number of factors

**Parameters:**

`factors`: a finite number of factors

`Keywords`: prec, dps, exact, rounding.

`mpFormulaPy.fdot(A, B=None, conjugate=False)`

Computes the dot product of the iterables  $A$  and  $B$ ,

$$\sum_{k=0} A_k B_k \quad (3.7.11)$$

Alternatively, `fdot()` accepts a single iterable of pairs. In other words, `fdot(A,B)` and `fdot(zip(A,B))` are equivalent. The elements are automatically converted to mpFormulaPy numbers.

With `conjugate=True`, the elements in the second vector will be conjugated:

$$\sum_{k=0} A_k \overline{B_k} \quad (3.7.12)$$

## Examples

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> A = [2, 1.5, 3]
>>> B = [1, -1, 2]
>>> fdot(A, B)
mpf('6.5')
>>> list(zip(A, B))
[(2, 1), (1.5, -1), (3, 2)]
>>> fdot(_)
mpf('6.5')
>>> A = [2, 1.5, 3j]
>>> B = [1+j, 3, -1-j]
>>> fdot(A, B)
mpc(real='9.5', imag='-1.0')
>>> fdot(A, B, conjugate=True)
mpc(real='3.5', imag='5.0')
```

---

### 3.7.7 Multiplication by multiples of 2 (LSH)

---

`WorksheetFunction.BITLSHIFT(n As Integer, k As Integer) As Integer`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.BITLSHIFT` returns the product of  $n$  and  $2^k$ .

**Parameters:**

- n*: An Integer.
- k*: An Integer.

The function `BITLSHIFT(n, k)` returns the product of  $n$  and  $2^k$ :

$$\text{intLSH}(n, k) = n \times 2^k. \quad (3.7.13)$$

This operation can also be defined as a left shift by  $k$  bits.

### 3.7.8 Division

---

Operator /

---

Function .Div(**a** As mpNum, **b** As mpNum) As mpNum

Function .RSH(**a** As mpNum, **b** As Integer) As mpNum

---

The binary operator / is used to return the quotient of the 2 operands *a* and *b*, and assign the result to *c*:  $c = a / b$ .

For languages not supporting operator overloading, the function .Div can be used to achieve the same:  $c = a.\text{Div}(b)$

The function .DivInt can be used if the second operand is an integer:  $c = a.\text{DivInt}(b)$

---

WorksheetFunction.IMDIV(**z1** As String, **z2** As String) As String

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.INDIV returns the quotient of *z1* and *z2*

**Parameters:**

*z1*: A Strings representing a complex number.

*z2*: A Strings representing a complex number.

The function cplxDiv(*z1*, *z2*) returns the quotient of *z1* and *z2*:

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \quad (3.7.14)$$

---

Function **fdiv**(**x** As mpNum, **y** As mpNum, **Keywords** As String) As mpNum

---

The function fdiv returns the sum of the numbers *x* and *y*, giving a floating-point result, optionally using a custom precision and rounding mode..

**Parameters:**

*x*: A complex number.

*y*: A complex number.

**Keywords**: prec, dps, exact, rounding.

mpFormulaPy.fdiv(ctx, *x*, *y*, \*\*kwargs)

Divides the numbers *x* and *y*, giving a floating-point result, optionally using a custom precision and rounding mode.

See the documentation of fadd() for a detailed description of how to specify precision and rounding.

**Examples**

The result is an mpFormulaPy number:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> fdiv(3, 2)
mpf('1.5')
>>> fdiv(2, 3)
mpf('0.6666666666666663')
```

---

```
>>> fdiv(2+4j, 0.5)
mpc(real='4.0', imag='8.0')
```

---

The rounding direction and precision can be controlled:

---

```
>>> fdiv(2, 3, dps=3) # Should be accurate to at least 3 digits
mpf('0.6666259765625')
>>> fdiv(2, 3, rounding='d')
mpf('0.6666666666666663')
>>> fdiv(2, 3, prec=60)
mpf('0.6666666666666667')
>>> fdiv(2, 3, rounding='u')
mpf('0.6666666666666674')
```

---

Checking the error of a division by performing it at higher precision:

---

```
>>> fdiv(2, 3) - fdiv(2, 3, prec=100)
mpf('-3.7007434154172148e-17')
```

---

Unlike fadd(), fmul(), etc., exact division is not allowed since the quotient of two floating-point numbers generally does not have an exact floating-point representation. (In the future this might be changed to allow the case where the division is actually exact.)

---

```
>>> fdiv(2, 3, exact=True)
Traceback (most recent call last):
...
ValueError: division is not an exact operation
```

---

### 3.7.9 Division by multiples of 2 (RSH)

---

WorksheetFunction.BITRSHIFT(*n* As Integer, *k* As Integer) As Integer

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.BITRSHIFT returns the quotient of *n* and  $2^k$ .

**Parameters:**

- n*: An Integer.
- k*: An Integer.

The function BITRSHIFT(*n, k*) returns the quotient of *n* and  $2^k$ :

$$\text{intRSH}(n, k) = n \div 2^k. \quad (3.7.15)$$

This operation can also be defined as a right shift by *k* bits.

### 3.7.10 Modulo

---

Operator **Mod** (VB.NET)

---

Operator **%** (Python)

---

Function **.Mod(a As mpNum, b As mpNum) As mpNum**

---

The binary operator `mod` is used to return the modulo of the 2 operands  $a$  and  $b$ , and assign the result to  $c$ :  $c = a \bmod b$ .

For languages not supporting operator overloading, the function `.Mod` can be used to achieve the same:  $c = a.\text{Mod}(b)$

---

**WorksheetFunction.MOD(*x As mpReal, y As mpReal*) As mpReal**

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.MOD` returns the remainder of  $x/y$

**Parameters:**

*x*: A real number.

*y*: A real number.

Returns the value of  $x - ny$ ,  $n = \lfloor x/y \rfloor$ , i.e. rounded according to the direction *rnd*, where  $n$  is the integer quotient of  $x$  divided by  $y$ , rounded toward zero.

---

**Function fmod(*x As mpReal, y As mpReal*) As mpReal**

---

The function `fmod` returns the remainder of  $x/y$

**Parameters:**

*x*: A real number.

*y*: A real number.

Converts  $x$  and  $y$  to mpFormulaPy numbers and returns  $x \bmod y$ . For mpFormulaPy numbers, this is equivalent to `x % y`.

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> fmod(100, pi)
2.61062773871641
```

You can use `fmod()` to compute fractional parts of numbers:

```
>>> fmod(10.25, 1)
0.25
```

---

### 3.7.11 Power

---

Operator `^` (VB.NET)

Operator `**` (Python)

---

**Function .Pow(*a As mpNum, b As mpNum*) As mpNum**

---

The binary operator `^` is used to return  $a$  raised to the power of  $b$ , and assign the result to  $c$ :  $c = a ^ b$ .

For languages not supporting operator overloading, the function `.Pow` can be used to achieve the same:  $c = a.\text{Pow}(b)$

## 3.8 Logical Operators

### 3.8.1 Bitwise AND

---

WorksheetFunction.BITAND(*n1* As Integer, *n2* As Integer) As Integer

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.BITAND returns  $n_1$  bitwise-and  $n_2$ .

**Parameters:**

*n1*: An Integer.

*n2*: An Integer.

### 3.8.2 Bitwise Inclusive OR

---

WorksheetFunction.BITOR(*n1* As Integer, *n2* As Integer) As Integer

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.BITOR returns  $n_1$  bitwise-inclusive-or  $n_2$ .

**Parameters:**

*n1*: An Integer.

*n2*: An Integer.

### 3.8.3 Bitwise Exclusive OR

---

WorksheetFunction.BITXOR(*n1* As Integer, *n2* As Integer) As Integer

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.BITXOR returns  $n_1$  bitwise-exclusive-or  $n_2$ .

**Parameters:**

*n1*: An Integer.

*n2*: An Integer.

## 3.9 Comparison Operators and Sorting

### 3.9.1 Equal

---

Operator `=` (VB.NET)

Operator `==` (C#)

---

Function `.EQ(a As mpNum, b As mpNum) As Boolean`

---

The binary logical operator `=` returns TRUE if  $a = b$  and FALSE otherwise, e.g.:

`if (a = b) then`

For languages not supporting operator overloading, the function `.EQ` can be used to achieve the same, e.g.:

`if a.EQ(b) then`

### 3.9.2 Greater or equal

---

Operator `>=`

---

Function `.GE(a As mpNum, b As mpNum) As Boolean`

---

The binary logical operator `>=` returns TRUE if  $a \geq b$  and FALSE otherwise, e.g.:

`if (a >= b) then`

For languages not supporting operator overloading, the function `.GE` can be used to achieve the same, e.g.:

`if a.GE(b) then`

### 3.9.3 Greater than

---

Operator `>`

---

Function `.GT(a As mpNum, b As mpNum) As Boolean`

---

The binary logical operator `>` returns TRUE if  $a > b$  and FALSE otherwise, e.g.:

`if (a > b) then`

For languages not supporting operator overloading, the function `.GT` can be used to achieve the same, e.g.:

`if a.GT(b) then`

### 3.9.4 Less or equal

---

Operator `<=`

---

Function `.LE(a As mpNum, b As mpNum) As Boolean`

---

The binary logical operator `<=` returns TRUE if  $a \leq b$  and FALSE otherwise, e.g.:

`if (a <= b) then`

For languages not supporting operator overloading, the function `.LE` can be used to achieve the same, e.g.:

`if a.LE(b) then`

### 3.9.5 Less than

---

Operator <

---

Function **.LT(a As mpNum, b As mpNum) As Boolean**

---

The binary logical operator > returns TRUE if  $a < b$  and FALSE otherwise, e.g.:

if ( $a < b$ ) then

For languages not supporting operator overloading, the function .LT can be used to achieve the same, e.g.:

if a.LT(b) then

### 3.9.6 Not equal

---

Operator <> (VB.NET)

---

Operator != (C#)

---

Function **.NE(a As mpNum, b As mpNum) As Boolean**

---

The binary logical operator <> returns TRUE if  $a \neq b$  and FALSE otherwise, e.g.:

if ( $a <> b$ ) then

For languages not supporting operator overloading, the function .NE can be used to achieve the same, e.g.:

if a.NE(b) then

### 3.9.7 Tolerances and approximate comparisons

---

Function **chop(x As mpNum, Keywords As String) As mpNum**

---

The function chop returns Chops off small real or imaginary parts, or converts numbers close to zero to exact zeros

#### Parameters:

*x*: A real or complex number.

*Keywords*: tol=None

mpFormulaPy.chop(*x*, tol=None)

Chops off small real or imaginary parts, or converts numbers close to zero to exact zeros. The input can be a single number or an iterable:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> chop(5+1e-10j, tol=1e-9)
mpf('5.0')
>>> nprint(chop([1.0, 1e-20, 3+1e-18j, -4, 2]))
[1.0, 0.0, 3.0, -4.0, 2.0]
```

---

The tolerance defaults to 100\*eps.

---

Function **almosteq(s As mpNum, t As mpNum, Keywords As String) As mpNum**

---

The function `almosteq` returns Determine whether the difference between  $s$  and  $t$  is smaller than a given epsilon, either relatively or absolutely.

#### Parameters:

`s`: A real or complex number.

`t`: A real or complex number.

**Keywords:** `rel_eps=None`, `abs_eps=None`

`mpFormulaPy.almosteq(s, t, rel_eps=None, abs_eps=None)`

Determine whether the difference between  $s$  and  $t$  is smaller than a given epsilon, either relatively or absolutely.

Both a maximum relative difference and a maximum difference ('epsilons') may be specified. The absolute difference is defined as  $|s - t|$  and the relative difference is defined as  $|s - t|/\max(|s|, |t|)$ .

If only one epsilon is given, both are set to the same value. If none is given, both epsilons are set to  $2^{-p+m}$  where  $p$  is the current working precision and  $m$  is a small integer. The default setting typically allows `almosteq()` to be used to check for mathematical equality in the presence of small rounding errors.

#### Examples

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15
>>> almosteq(3.141592653589793, 3.141592653589790)
True
>>> almosteq(3.141592653589793, 3.141592653589700)
False
>>> almosteq(3.141592653589793, 3.141592653589700, 1e-10)
True
>>> almosteq(1e-20, 2e-20)
True
>>> almosteq(1e-20, 2e-20, rel_eps=0, abs_eps=0)
False
```

---



---

`WorksheetFunction.GESTEP(Number As mpNum, Step As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.GESTEP` returns 1 if  $\text{number} \geq \text{step}$ ; returns 0 (zero) otherwise. Use this function to filter a set of values. For example, by summing several `GESTEP` functions you calculate the count of values that exceed a threshold.

#### Parameters:

`Number`: The value to test against step.

`Step`: The threshold value. If you omit a value for step, `GESTEP` uses zero.

---

`WorksheetFunction.DELTA(Number1 As mpNum, Number2 As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.DELTA` returns 1 if `Number1 = Number2`; returns 0 otherwise.

**Parameters:**

*Number1*: The first number to compare.

*Number2*: The second number to compare.

Use this function to filter a set of values. For example, by summing several `DELTA` functions you calculate the count of equal pairs. This function is also known as the Kronecker Delta function.

## 3.10 Properties of numbers

### 3.10.1 Testing for special values

---

**WorksheetFunction.ISNUMBER(*x* As *mpNum*) As Boolean**

---

NOT YET IMPLEMENTED

---

The function **WorksheetFunction.ISNUMBER** returns TRUE if *x* is an ordinary number (i.e. neither NaN nor an infinity), and FALSE otherwise.

**Parameter:**

*x*: A real number.

---

**Function isnormal(*Number1* As *mpNum*) As Boolean**

---

The function **isnormal** returns Determine whether *x* is 'normal' in the sense of floating-point representation; that is, return False if *x* is zero, an infinity or NaN; otherwise return True. By extension, a complex number *x* is considered 'normal' if its magnitude is normal

**Parameter:**

*Number1*: A real or complex number.

`mpFormulaPy.isnormal(x)`

Determine whether *x* is 'normal' in the sense of floating-point representation; that is, return False if *x* is zero, an infinity or NaN; otherwise return True. By extension, a complex number *x* is considered 'normal' if its magnitude is normal:

---

```
>>> from mpFormulaPy import *
>>> isnormal(3)
True
>>> isnormal(0)
False
>>> isnormal(inf); isnormal(-inf); isnormal(nan)
False
False
False
>>> isnormal(0+0j)
False
>>> isnormal(0+3j)
True
>>> isnormal(mpcc(2,nan))
False
```

---

**Function isfinite(*Number1* As *mpNum*) As Boolean**

---

The function **isfinite** returns Return True if *x* is a finite number, i.e. neither an infinity or a NaN

**Parameter:**

*Number1*: A real or complex number.

`mpFormulaPy.isfinite(x)`

Return True if *x* is a finite number, i.e. neither an infinity or a NaN.

---

```
>>> from mpFormulaPy import *
```

```
>>> isfinite(inf)
False
>>> isfinite(-inf)
False
>>> isfinite(3)
True
>>> isfinite(nan)
False
>>> isfinite(3+4j)
True
>>> isfinite(mpc(3,inf))
False
>>> isfinite(mpc(nan,3))
False
```

---

**Function `isinf`(*Number1* As *mpNum*) As Boolean**

The function `isinf` returns True if the absolute value of x is infinite; otherwise return False

**Parameter:**

*Number1*: A real or complex number.

`mpFormulaPy.isinf(x)`

Return True if the absolute value of x is infinite; otherwise return False:

```
>>> from mpFormulaPy import *
>>> isinf(inf)
True
>>> isinf(-inf)
True
>>> isinf(3)
False
>>> isinf(3+4j)
False
>>> isinf(mpc(3,inf))
True
>>> isinf(mpc(inf,3))
True
```

---

**Function `isnan`(*Number1* As *mpNum*) As Boolean**

The function `isnan` returns Return True if x is a NaN (not-a-number), or for a complex number, whether either the real or complex part is NaN; otherwise return False

**Parameter:**

*Number1*: A real or complex number.

`mpFormulaPy.isnan(x)`

Return True if x is a NaN (not-a-number), or for a complex number, whether either the real or complex part is NaN; otherwise return False:

```
>>> from mpFormulaPy import *
>>> isnan(3.14)
```

```
False
>>> isnan(nan)
True
>>> isnan(mpc(3.14,2.72))
False
>>> isnan(mpc(3.14,nan))
True
```

---

### 3.10.2 Testing for integers

---

Function **isint(x As mpNum, Kewords As String)** As Boolean

---

The function **isint** returns Return True if x is integer-valued; otherwise return False.

**Parameters:**

x: A real number.

Kewords: gaussian=False.

mpFormulaPy.isint(x, gaussian=False)

Return True if x is integer-valued; otherwise return False:

---

```
>>> from mpFormulaPy import *
>>> isint(3)
True
>>> isint(mpf(3))
True
>>> isint(3.2)
False
>>> isint(inf)
False
```

---

Optionally, Gaussian integers can be checked for:

---

```
>>> isint(3+0j)
True
>>> isint(3+2j)
False
>>> isint(3+2j, gaussian=True)
True
```

---



---

WorksheetFunction.**ISEVEN(x As mpNum)** As Boolean

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.ISEVEN returns TRUE if n is an even integer, and FALSE otherwise.

**Parameter:**

x: A real number.

---

WorksheetFunction.**ISODD(x As mpNum)** As Boolean

---

NOT YET IMPLEMENTED

---

The function `WorksheetFunction.ISODD` returns TRUE if  $n$  is an odd integer, and FALSE otherwise.

**Parameter:**

$x$ : A real number.

### 3.10.3 Approximating magnitude and precision

---

#### Function `mag(x As mpNum) As mpNum`

---

The function `mag` returns Quick logarithmic magnitude estimate of a number.

**Parameter:**

$x$ : A real number.

`mpFormulaPy.mag(x)`

Quick logarithmic magnitude estimate of a number. Returns an integer or infinity  $m$  such that  $|x| \leq 2^m$ . It is not guaranteed that  $m$  is an optimal bound, but it will never be too large by more than 2 (and probably not more than 1).

#### Examples

---

```
>>> from mpFormulaPy import *
>>> mp.pretty = True
>>> mag(10), mag(10.0), mag(mpf(10)), int(ceil(log(10,2)))
(4, 4, 4, 4)
>>> mag(10j), mag(10+10j)
(4, 5)
>>> mag(0.01), int(ceil(log(0.01,2)))
(-6, -6)
>>> mag(0), mag(inf), mag(-inf), mag(nan)
(-inf, +inf, +inf, nan)
```

---

#### Function `.nint_distance(x As mpNum) As mpNum`

---

The function `.nint_distance` returns Return  $(n, d)$  where  $n$  is the nearest integer to  $x$  and  $d$  is an estimate of  $\log_2(|x - n|)$ .

**Parameter:**

$x$ : A real number.

`mpFormulaPy.nint_distance(x)`

Return  $(n, d)$  where  $n$  is the nearest integer to  $x$  and  $d$  is an estimate of  $\log_2(|x - n|)$ . If  $d < 0$ ,  $-d$  gives the precision (measured in bits) lost to cancellation when computing  $x - n$ .

```
>>> from mpFormulaPy import *
>>> n, d = nint_distance(5)
>>> print(n); print(d)
5
-inf
>>> n, d = nint_distance(mpf(5))
>>> print(n); print(d)
```

```
5
-inf
>>> n, d = nint_distance(mpf(5.00000001))
>>> print(n); print(d)
5
-26
>>> n, d = nint_distance(mpf(4.99999999))
>>> print(n); print(d)
5
-26
>>> n, d = nint_distance(mpc(5,10))
>>> print(n); print(d)
5
4
>>> n, d = nint_distance(mpc(5,0.000001))
>>> print(n); print(d)
5
-19
```

---

## 3.11 Number generation

### 3.11.1 Random numbers

---

#### WorksheetFunction.RAND() As mpNum

---

The function WorksheetFunction.RAND returns an evenly distributed random real number greater than or equal to 0 and less than 1.

Returns an evenly distributed random real number greater than or equal to 0 and less than 1. To generate a random real number between a and b, use: RAND()\*(b-a)+a.

---

#### WorksheetFunction.RANDBETWEEN(*Bottom* As mpNum, *Top* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.RANDBETWEEN returns a random integer number between the numbers you specify.

**Parameters:**

*Bottom*: The smallest integer RANDBETWEEN will return.

*Top*: The largest integer RANDBETWEEN will return.

---

#### Function **rand()** As mpNum

---

The function **rand** returns Returns an mpf with value chosen randomly from [0, 1). The number of randomly generated bits in the mantissa is equal to the working precision.

mpFormulaPy.rand()

Returns an mpf with value chosen randomly from [0, 1). The number of randomly generated bits in the mantissa is equal to the working precision.

### 3.11.2 Fractions

---

#### Function **fraction(*p* As mpNum, *q* As mpNum)** As mpNum

---

The function **fraction** returns Given Python integers (*p*, *q*), returns a lazy mpf representing the fraction *p/q*. The value is updated with the precision.

**Parameters:**

*p*: an integer.

*q*: an integer.

mpFormulaPy.fraction(*p*, *q*)

Given Python integers (*p*, *q*), returns a lazy mpf representing the fraction *p/q*. The value is updated with the precision.

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15
>>> a = fraction(1,100)
>>> b = mpf(1)/100
>>> print(a); print(b)
0.01
0.01
```

### 3.11.3 Ranges

Function **arange**(*a* As mpNum, *b* As mpNum, *h* As mpNum) As mpNum

The function `arange` returns This is a generalized version of Python's `range()` function that accepts fractional endpoints and step sizes and returns a list of `mpf` instance.

## Parameters:

*a*: a real number.

*b*: a real number.

$h$ : a real number.

mpFormulaPy.arange(\*args)

This is a generalized version of Python's range() function that accepts fractional endpoints and step sizes and returns a list of mpf instances. Like range(), arange() can be called with 1, 2 or 3 arguments:

`arange(b): [0, 1, 2, ..., x].`

arange(a, b):  $[a, a + 1, a + 2, \dots, x]$

`arange(a, b, h): [a, a + h, a + 2h, ..., x]`

where  $b - 1 \leq x < b$  (in the third case,  $b - h \leq x < b$ ).

Like Python's `range()`, the endpoint is not included. To produce ranges where the endpoint is included, `linspace()` is more convenient.

## Examples

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> arange(4)
[mpf('0.0'), mpf('1.0'), mpf('2.0'), mpf('3.0')]
>>> arange(1, 2, 0.25)
[mpf('1.0'), mpf('1.25'), mpf('1.5'), mpf('1.75')]
>>> arange(1, -1, -0.75)
[mpf('1.0'), mpf('0.25'), mpf('-0.5')]
```

Function ***linspace(a As mpNum, b As mpNum, h As mpNum, Keywords As String)*** As mpNum

The function `linspace` returns This is a generalized version of Python's `range()` function that accepts fractional endpoints and step sizes and returns a list of `mpf` instance.

### Parameters:

*a*: a real number.

*b*: a real number.

*h*: a real number.

**Keywords:** endpoint=True.

```
mpFormulaPy.linspace(*args, **kwargs)
```

`linspace(a, b, n)` returns a list of  $n$  evenly spaced samples from  $a$  to  $b$ . The syntax `linspace(mpi(a,b), n)` is also valid.

This function is often more convenient than `arange()` for partitioning an interval into subintervals, since the endpoint is included:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> linspace(1, 4, 4)
[mpf('1.0'), mpf('2.0'), mpf('3.0'), mpf('4.0')]
```

---

You may also provide the keyword argument `endpoint=False`:

---

```
>>> linspace(1, 4, 4, endpoint=False)
[mpf('1.0'), mpf('1.75'), mpf('2.5'), mpf('3.25')]
```

---

## 3.12 Matrices

### 3.12.1 Basic methods

---

`WorksheetFunction.MUNIT(n As Integer) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.MUNIT` returns the unity matrix of dimension *n*.

**Parameter:**

*n*: An integer specifying the dimension of the matrix.

---

**Function `matrix(data As Object, Keywords As String) As mpNum`**

---

The function `matrix` returns This is a generalized version of Python's `range()` function that accepts fractional endpoints and step sizes and returns a list of `mpf` instance.

**Parameters:**

*data*: an object specifying the matrix.

*Keywords*: random, random-symmetric, random-complex, random-hermitian, zeros, ones, eye, row-vector, col-vector, diagonal.

Matrices in `mpFormulaPy` are implemented using dictionaries. Only non-zero values are stored, so it is cheap to represent sparse matrices.

The most basic way to create one is to use the `matrix` class directly. You can create an empty matrix specifying the dimensions:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> matrix(2)
matrix(
[['0.0', '0.0'],
['0.0', '0.0']])
>>> matrix(2, 3)
matrix(
[['0.0', '0.0', '0.0'],
['0.0', '0.0', '0.0']])
```

---

Calling `matrix` with one dimension will create a square matrix.

To access the dimensions of a matrix, use the `rows` or `cols` keyword:

---

```
>>> A = matrix(3, 2)
>>> A
matrix(
[['0.0', '0.0'],
['0.0', '0.0'],
['0.0', '0.0']])
>>> A.rows
3
>>> A.cols
2
```

---

You can also change the dimension of an existing matrix. This will set the new elements to 0. If the new dimension is smaller than before, the concerning elements are discarded:

---

```
>>> A.rows = 2
>>> A
matrix(
[[0.0, 0.0],
[0.0, 0.0]])
```

---

Internally convert is applied every time an element is set. This is done using the syntax  $A[\text{row},\text{column}]$ , counting from 0:

---

```
>>> A = matrix(2)
>>> A[1,1] = 1 + 1j
>>> print A
[0.0 0.0]
[0.0 (1.0 + 1.0j)]
```

---

A more comfortable way to create a matrix lets you use nested lists:

---

```
>>> matrix([[1, 2], [3, 4]])
matrix(
[[1.0, 2.0],
[3.0, 4.0]])
```

---

Interval matrices can be used to perform linear algebra operations with rigorous error tracking:

---

```
>>> a = iv.matrix([[0.1, 0.3, 1.0],
... [7.1, 5.5, 4.8],
... [3.2, 4.4, 5.6]])
>>>
>>> b = iv.matrix([4, 0.6, 0.5])
>>> c = iv.lu_solve(a, b)
>>> print c
[ [5.2582327113062393041, 5.2582327113062749951]
[ [-13.155049396267856583, -13.155049396267821167]
[ [7.4206915477497212555, 7.4206915477497310922]
>>> print a*c
[ [3.999999999999866773, 4.000000000000133227]
[ [0.59999999999972430942, 0.60000000000027142733]
[ [0.4999999999982236432, 0.50000000000018474111]]
```

---

Convenient functions are available for creating various standard matrices:

---

```
>>> zeros(2)
matrix(
[[0.0, 0.0],
[0.0, 0.0]])
>>> ones(2)
matrix(
[[1.0, 1.0],
[1.0, 1.0]])
>>> diag([1, 2, 3]) # diagonal matrix
matrix(
[[1.0, 0.0, 0.0],
```

---

---

```
[0.0, 2.0, 0.0],
[0.0, 0.0, 3.0]]))
>>> eye(2) # identity matrix
matrix(
[[1.0, 0.0],
[0.0, 1.0]])
```

---

You can even create random matrices:

---

```
>>> randmatrix(2)
matrix(
[[0.53491598236191806, 0.57195669543302752],
[0.85589992269513615, 0.82444367501382143]])
```

---

### 3.12.2 Vectors

Vectors may also be represented by the matrix class (with rows = 1 or cols = 1). For vectors there are some things which make life easier. A column vector can be created using a flat list, a row vectors using an almost flat nested list:

---

```
>>> matrix([1, 2, 3])
matrix(
[[1.0],
[2.0],
[3.0]])
>>> matrix([[1, 2, 3]])
matrix(
[[1.0, 2.0, 3.0]])
```

---

Optionally vectors can be accessed like lists, using only a single index:

---

```
>>> x = matrix([1, 2, 3])
>>> x[1]
mpf('2.0')
>>> x[1,0]
mpf('2.0')
```

---

### 3.12.3 Other

Like you probably expected, matrices can be printed:

---

```
>>> print randmatrix(3)
[ 0.782963853573023 0.802057689719883 0.427895717335467]
[0.0541876859348597 0.708243266653103 0.615134039977379]
[ 0.856151514955773 0.544759264818486 0.686210904770947]
```

---

Use nstr or nprint to specify the number of digits to print:

---

```
>>> nprint(randmatrix(5), 3)
[2.07e-1 1.66e-1 5.06e-1 1.89e-1 8.29e-1]
[6.62e-1 6.55e-1 4.47e-1 4.82e-1 2.06e-2]
```

---

---

```
[4.33e-1 7.75e-1 6.93e-2 2.86e-1 5.71e-1]
[1.01e-1 2.53e-1 6.13e-1 3.32e-1 2.59e-1]
[1.56e-1 7.27e-2 6.05e-1 6.67e-2 2.79e-1]
```

---

As matrices are mutable, you will need to copy them sometimes:

---

```
>>> A = matrix(2)
>>> A
matrix(
[[0.0, 0.0],
 [0.0, 0.0]])
>>> B = A.copy()
>>> B[0,0] = 1
>>> B
matrix(
[[1.0, 0.0],
 [0.0, 0.0]])
>>> A
matrix(
[[0.0, 0.0],
 [0.0, 0.0]])
```

---

Finally, it is possible to convert a matrix to a nested list. This is very useful, as most Python libraries involving matrices or arrays (namely NumPy or SymPy) support this format:

---

```
>>> B.tolist()
[[mpf('1.0'), mpf('0.0')], [mpf('0.0'), mpf('0.0')]]
```

---

### 3.12.4 Transposition

Matrix transposition is straightforward:

---

```
>>> A = ones(2, 3)
>>> A
matrix(
[[1.0, 1.0, 1.0],
 [1.0, 1.0, 1.0]])
>>> A.T
matrix(
[[1.0, 1.0],
 [1.0, 1.0],
 [1.0, 1.0]])
```

---

### 3.12.5 Matrix Properties

Add a note on matrix properties:

Rows, Cols, T etc.

### 3.12.6 Addition

---

Operator +

---

**matrix.Plus(a As mpNum, b As mpNum) As mpNum**

---

The binary operator + is used to return the sum of the 2 operands  $a$  and  $b$ , and assign the result to  $c$ :  $c = a + b$ .

For languages not supporting operator overloading, the function .Plus can be used to achieve the same:  $c = a.Plus(b)$

The operator + returns the sum of  $z_1$  and  $z_2$ .

---

**Function MatrixAdd(x As mpNum, y As mpNum, Keywords As String) As mpNum**

---

The function MatrixAdd returns the sum of the numbers  $x$  and  $y$ , giving a floating-point result, optionally using a custom precision and rounding mode..

**Parameters:**

$x$ : A complex number.

$y$ : A complex number.

**Keywords:** prec, dps, exact, rounding.

mpFormulaPy.fadd(ctx, x, y, \*\*kwargs)

Adds the matrices  $x$  and  $y$ , giving a floating-point result, optionally using a custom precision and rounding mode.

You can add and subtract matrices of compatible dimensions:

---

```
>>> A = matrix([[1, 2], [3, 4]])
>>> B = matrix([[-2, 4], [5, 9]])
>>> A + B
matrix(
[[ '-1.0', '6.0'],
['8.0', '13.0']])
>>> A - B
matrix(
[[ '3.0', '-2.0'],
['-2.0', '-5.0']])
>>> A + ones(3)
Traceback (most recent call last):
File "<stdin>", line 1, in <module>
File "...", line 238, in __add__
raise ValueError('incompatible dimensions for addition')
ValueError: incompatible dimensions for addition
```

---

It is possible to multiply or add matrices and scalars. In the latter case the operation will be done element-wise:

---

```
>>> A * 2
matrix(
[[ '2.0', '4.0'],
['6.0', '8.0']])
>>> A / 4
```

---

```

matrix(
[['0.25', '0.5'],
['0.75', '1.0']])
>>> A = 1
matrix(
[['0.0', '1.0'],
['2.0', '3.0']])

```

---

### 3.12.7 Multiplication

---

Operator \*

---



---

**matrix.TimesMat(a As mpNum, b As Integer) As mpNum**

---

The binary operator \* is used to return the product of the 2 operands  $a$  and  $b$ , and assign the result to  $c$ :  $c = a * b$ .

For languages not supporting operator overloading, the function .Times can be used to achieve the same:  $c = a.Times(b)$

---

**WorksheetFunction.MMULT(X As mpNum[], Y As mpNum[]) As mpNum[]**

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.MMULT returns the matrix product of two arrays  $X$  and  $Y$ . The result is an array with the same number of rows as  $X$  and the same number of columns as  $Y$ .

**Parameters:**

$X$ : A matrix of real numbers.

$Y$ : A matrix of real numbers.

---

**Function MatrixMul(x As mpNum, y As mpNum, *Keywords* As String) As mpNum**

---

The function MatrixMul returns the sum of the numbers  $x$  and  $y$ , giving a floating-point result, optionally using a custom precision and rounding mode..

**Parameters:**

$x$ : A complex number.

$y$ : A complex number.

**Keywords:** prec, dps, exact, rounding.

mpFormulaPy.fmul(ctx, x, y, \*\*kwargs)

Multiplies the numbers  $x$  and  $y$ , giving a floating-point result, optionally using a custom precision and rounding mode.

See the documentation of fadd() for a detailed description of how to specify precision and rounding. You can perform matrix multiplication, if the dimensions are compatible:

---

```

>>> A * B
matrix(
[['8.0', '22.0'],
['14.0', '48.0']])
>>> matrix([[1, 2, 3]]) * matrix([-6, 7, -2])
matrix(

```

---

```
[['2.0']])
```

---

You can raise powers of square matrices:

---

```
>>> A**2
matrix(
[['7.0', '10.0'],
 ['15.0', '22.0']])
```

---

# Chapter 4

## Elementary Functions

### 4.1 Constants

#### 4.1.1 Mathematical constants

Mpmath supports arbitrary-precision computation of various common (and less common) mathematical constants. These constants are implemented as lazy objects that can evaluate to any precision. Whenever the objects are used as function arguments or as operands in arithmetic operations, they automagically evaluate to the current working precision. A lazy number can be converted to a regular mpf using the unary + operator, or by calling it as a function:

```
>>> from mpFormulaPy import *
>>> mp.dps = 15
>>> pi
<pi: 3.14159~>
>>> 2*pi
mpf('6.2831853071795862')
>>> +pi
mpf('3.1415926535897931')
>>> pi()
mpf('3.1415926535897931')
>>> mp.dps = 40
>>> pi
<pi: 3.14159~>
>>> 2*pi
mpf('6.283185307179586476925286766559005768394338')
>>> +pi
mpf('3.141592653589793238462643383279502884197169')
>>> pi()
mpf('3.141592653589793238462643383279502884197169')
```

---

#### WorksheetFunction.PI() As mpNum

---

The function WorksheetFunction.PI returns the value of  $\pi = 3.1415926535897932\dots$ .

---

#### Function pi() As mpNum

---

The function `pi` returns pi: 3.14159...

#### Function `degree()` As mpNum

The function `degree` returns  $\text{degree} = 1 \text{ deg} = \pi / 180$ : 0.0174533...

#### Function `e()` As mpNum

The function `e` returns the base of the natural logarithm,  $e = \exp(1)$ : 2.71828...

#### Function `phi()` As mpNum

The function `phi` returns Golden ratio phi: 1.61803...

#### Function `euler()` As mpNum

The function `euler` returns Euler's constant: 0.577216...

#### Function `catalan()` As mpNum

The function `catalan` returns Catalan's constant: 0.915966...

#### Function `apery()` As mpNum

The function `apery` returns Apery's constant: 1.20206...

#### Function `khinchin()` As mpNum

The function `khinchin` returns Khinchin's constant: 2.68545...

#### Function `glaisher()` As mpNum

The function `glaisher` returns Glaisher's constant: 1.28243...

#### Function `mertens()` As mpNum

The function `mertens` returns Mertens' constant: 0.261497...

#### Function `twinprime()` As mpNum

The function `twinprime` returns Twin prime constant: 0.660162...

### 4.1.2 Special values

The predefined objects `j` (imaginary unit), `inf` (positive infinity) and `nan` (not-a-number) are shortcuts to `mpc` and `mpf` instances with these fixed values.

#### Function `inf()` As mpNum

The function `inf` returns the value of the representation of  $+\infty$  in the current precision.

---

#### Function `nan()` As `mpNum`

---

The function `nan` returns the value of the representation of Not a Number (NaN) in the current precision.

## 4.2 Exponential and Logarithmic Functions

### 4.2.1 Exponential Function $e^z = \exp(z)$

---

WorksheetFunction.**EXP**(*x* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.EXP returns the value of the exponential function,  $\exp(x) = e^x = \exp(x)$ .

**Parameter:**

*x*: A real number.

---

WorksheetFunction.**IMEXP**(*z* As String) As String

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.IMEXP returns the complex exponential of *z*, as a String representing a complex number.

**Parameter:**

*z*: A String representing a complex number.

The function IMEXP(*z*) returns the complex exponential function of *z*:

$$\exp(z) = e^x \cos(y) + ie^x \sin(y). \quad (4.2.1)$$


---

Function **exp**(*z* As mpNum) As mpNum

---

The function exp returns the complex exponential of *z*

**Parameter:**

*z*: A complex number.

Computes the exponential function,

$$\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \quad (4.2.2)$$

For complex numbers, the exponential function also satisfies

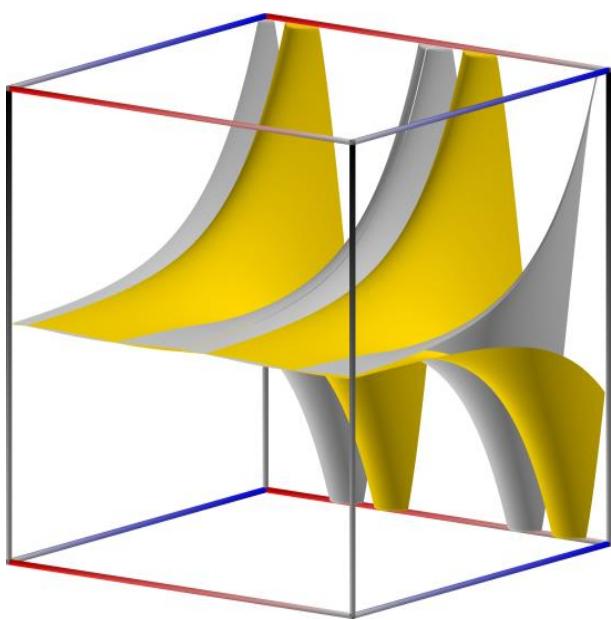
$$\exp(x + yi) = e^x(\cos(y) + i \sin(y)). \quad (4.2.3)$$

Basic examples

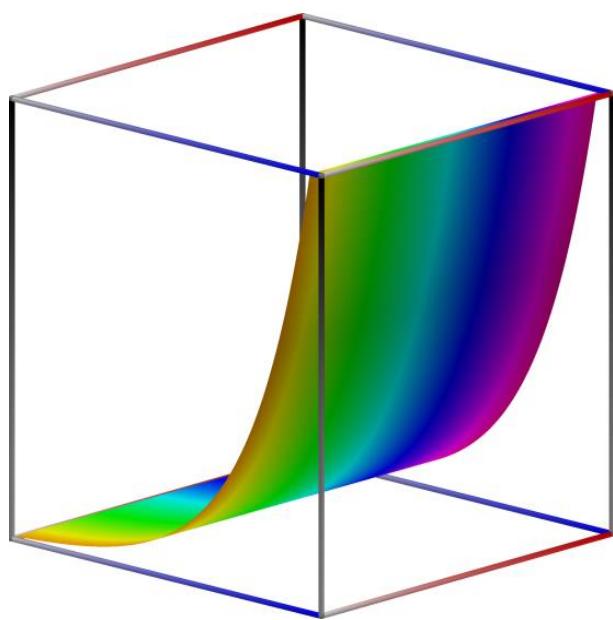
Some values of the exponential function:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> exp(0)
1.0
>>> exp(1)
2.718281828459045235360287
>>> exp(-1)
0.3678794411714423215955238
```



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.1:** Surface plots of  $z = \exp(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

---

```
>>> exp(inf)
+inf
>>> exp(-inf)
0.0
```

---

Arguments can be arbitrarily large:

---

```
>>> exp(10000)
8.806818225662921587261496e+4342
>>> exp(-10000)
1.135483865314736098540939e-4343
```

---

Evaluation is supported for interval arguments via `mpFormulaPy.iv.exp()`:

---

```
>>> iv.dps = 25; iv.pretty = True
>>> iv.exp([-inf,0])
[0.0, 1.0]
>>> iv.exp([0,1])
[1.0, 2.71828182845904523536028749558]
```

---

The exponential function can be evaluated efficiently to arbitrary precision:

---

```
>>> mp.dps = 10000
>>> exp(pi)
23.140692632779269005729...8984304016040616
```

---

Functional properties

Numerical verification of Euler's identity for the complex exponential function:

---

```
>>> mp.dps = 15
>>> exp(j*pi)+1
(0.0 + 1.22464679914735e-16j)
>>> chop(exp(j*pi)+1)
0.0
```

---

This recovers the coefficients (reciprocal factorials) in the Maclaurin series expansion of  $\exp$ :

---

```
>>> nprint(taylor(exp, 0, 5))
[1.0, 1.0, 0.5, 0.166667, 0.0416667, 0.00833333]
```

---

The exponential function is its own derivative and antiderivative:

---

```
>>> exp(pi)
23.1406926327793
>>> diff(exp, pi)
23.1406926327793
>>> quad(exp, [-inf, pi])
23.1406926327793
```

---

The exponential function can be evaluated using various methods, including direct summation of the series, limits, and solving the defining differential equation:

---

```
>>> nsum(lambda k: pi**k/fac(k), [0,inf])
23.1406926327793
>>> limit(lambda k: (1+pi/k)**k, inf)
23.1406926327793
>>> odefun(lambda t, x: x, 0, 1)(pi)
23.1406926327793
```

---

#### 4.2.1.1 $\exp(jx)$

---

**Function  $\exp(jz)$  As  $mpNum$**

---

The function  $\exp(jz)$  returns  $10^z$

**Parameter:**

$z$ : A complex number.

Convenience function for computing  $e^{ix}$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> expj(0)
(1.0 + 0.0j)
>>> expj(-1)
(0.5403023058681397174009366 - 0.8414709848078965066525023j)
>>> expj(j)
(0.3678794411714423215955238 + 0.0j)
>>> expj(1+j)
(0.1987661103464129406288032 + 0.3095598756531121984439128j)
```

### 4.2.1.2 expjpi(z)

---

Function **expjpi(z As mpNum) As mpNum**

---

The function `expjpi` returns  $10^z$

**Parameter:**

*z*: A complex number.

Convenience function for computing  $e^{i\pi z}$ . Evaluation is accurate near zeros (see also `cospi()`, `sinpi()`):

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> expjpi(0)
(1.0 + 0.0j)
>>> expjpi(1)
(-1.0 + 0.0j)
>>> expjpi(0.5)
(0.0 + 1.0j)
>>> expjpi(-1)
(-1.0 + 0.0j)
>>> expjpi(j)
(0.04321391826377224977441774 + 0.0j)
>>> expjpi(1+j)
(-0.04321391826377224977441774 + 0.0j)
```

---

### 4.2.1.3 expm1(x)

---

Function **expm1(z As mpNum) As mpNum**

---

The function `expm1` returns  $10^z$

**Parameter:**

*z*: A complex number.

Convenience function for computing  $e^x - 1$  accurately for small *x*.

Unlike the expression `exp(x) - 1`, `expm1(x)` does not suffer from potentially catastrophic cancellation:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> exp(1e-10)-1; print(expm1(1e-10))
1.00000008274037e-10
1.00000000005e-10
>>> exp(1e-20)-1; print(expm1(1e-20))
0.0
1.0e-20
>>> 1/(exp(1e-20)-1)
Traceback (most recent call last):
...
...
```

```
ZeroDivisionError
>>> 1/expm1(1e-20)
1.0e+20
```

---

Evaluation works for extremely tiny values:

---

```
>>> expm1(0)
0.0
>>> expm1('1e-10000000')
1.0e-10000000
```

---

## 4.2.2 Exponential Function $10^z = \exp_{10}(z)$

---

**Function `exp10(z As mpNum) As mpNum`**

---

The function `exp10` returns  $10^z$

**Parameter:**

$z$ : A complex number.

The function `exp10(z)` returns  $10^z = \exp_{10}(z) = \exp(z \cdot \ln(10))$ .

## 4.2.3 Exponential Function $2^z = \exp_2(z)$

---

**Function `exp2(z As mpNum) As mpNum`**

---

The function `exp2` returns  $2^z$

**Parameter:**

$z$ : A complex number.

The function `cplxExp2(z)` returns  $2^z = \exp_2(z) = \exp(z \cdot \ln(2))$ .

#### 4.2.4 Natural logarithm $\ln(x) = \log_e(x)$

`WorksheetFunction.LN(x As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.LN` returns the value of the natural logarithm  $\ln(x) = \log_e(x)$ .

**Parameter:**

`x`: A real number.

`WorksheetFunction.ILog(z As String) As String`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.ILog` returns the complex natural logarithm of  $z$ , as a String representing a complex number.

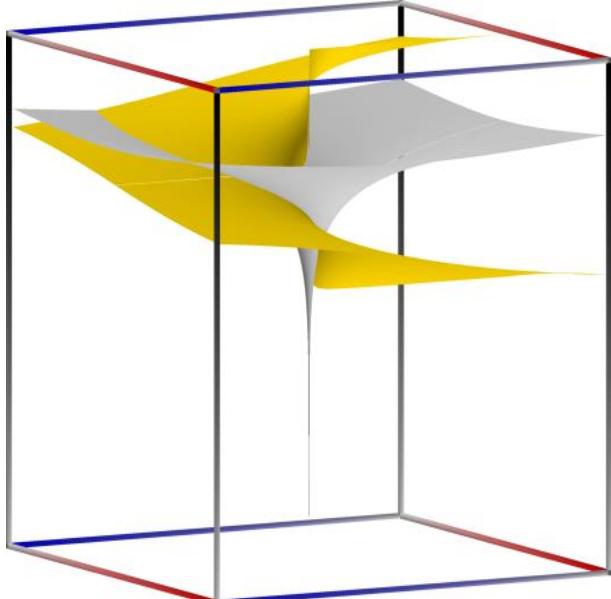
**Parameter:**

`z`: A String representing a complex number.

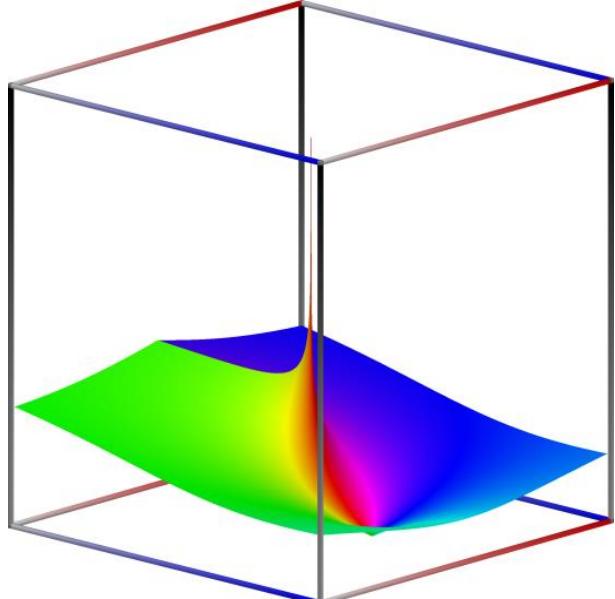
The function `cplxLn(z)` returns the complex natural logarithm of  $z$ :

$$\ln(z) = \log_e(z) = \ln(r) + i\theta, \quad (4.2.4)$$

where  $r = \sqrt{x^2 + y^2}$ , and  $\theta = \arctan(y/x)$ .



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.2:** Surface plots of  $z = \log(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

#### 4.2.4.1 `log(x, b=None)`

`WorksheetFunction.LOG(x As mpNum, b As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.LOG` returns the value of the logarithm to base  $b$ :  $\log_b(x) = \log_b(x)$ .

**Parameters:**

*x*: A real number.

*b*: A real number.

`Function Logb(x As mpNum, b As mpNum) As mpNum`

The function `Logb` returns the value of the logarithm to base  $b$ :  $\log_b(x) = \log_b(x)$ .

**Parameters:**

*x*: A real number.

*b*: A real number.

`Function log(z As mpNum, base As mpNum) As mpNum`

The function `log` returns the complex natural logarithm of  $z$

**Parameters:**

*z*: A complex number.

*base*: the base of the logarithm. A real number.

`Function ln(z As mpNum) As mpNum`

The function `ln` returns the complex natural logarithm of  $z$

**Parameter:**

*z*: A complex number.

Computes the base- $b$  logarithm of  $x$ ,  $\log_b(x)$ . If  $b$  is unspecified, `log()` computes the natural (base  $e$ ) logarithm and is equivalent to `ln()`. In general, the base  $b$  logarithm is defined in terms of the natural logarithm as  $\log_b(x) = \ln(x)/\ln(b)$ .

By convention, we take  $\log(0) = -\infty$ .

The natural logarithm is real if  $x > 0$  and complex if  $x < 0$  or if  $x$  is complex. The principal branch of the complex logarithm is used, meaning that  $\Im(\ln(x)) = -\pi < \arg(x) \leq \pi$ .

Examples Some basic values and limits:

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> log(1)
0.0
>>> log(2)
0.693147180559945
>>> log(1000,10)
3.0
>>> log(4, 16)
0.5
>>> log(j)
```

---

```
(0.0 + 1.5707963267949j)
>>> log(-1)
(0.0 + 3.14159265358979j)
>>> log(0)
-inf
>>> log(inf)
+inf
```

---

The natural logarithm is the antiderivative of  $1/x$ :

---

```
>>> quad(lambda x: 1/x, [1, 5])
1.6094379124341
>>> log(5)
1.6094379124341
>>> diff(log, 10)
0.1
```

---

The Taylor series expansion of the natural logarithm around  $x = 1$  has coefficients  $(-1)^{n+1}/n$ :

---

```
>>> nprint(taylor(log, 1, 7))
[0.0, 1.0, -0.5, 0.333333, -0.25, 0.2, -0.166667, 0.142857]
```

---

`log()` supports arbitrary precision evaluation:

---

```
>>> mp.dps = 50
>>> log(pi)
1.1447298858494001741434273513530587116472948129153
>>> log(pi, pi**3)
0.3333333333333333333333333333333333333333333333333333333333
>>> mp.dps = 25
>>> log(3+4j)
(1.609437912434100374600759 + 0.9272952180016122324285125j)
```

---

#### 4.2.5 Common (decadic) logarithm $\log_{10}(z)$

The function `log10(z)` returns the complex natural logarithm of  $z$ :

$$\log_{10}(z) = \ln(z)/\ln(10). \quad (4.2.5)$$

`log10(x)` is equivalent to `log(x, 10)`.

---

`WorksheetFunction.LOG10(x As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.LOG10` returns the value of the decadic logarithm  $\log_{10}(x) = \log_{10}(x)$ .

**Parameter:**

`x`: A real number.

---

`WorksheetFunction.IMLOG10(z As String) As String`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.IMLOG10` returns  $\log_{10}(z)$ , as a String representing a complex number.

**Parameter:**

`z`: A String representing a complex number.

---

**Function `log10(z As mpNum)` As mpNum**

---

The function `log10` returns  $\log_{10}(z)$

**Parameter:**

`z`: A complex number.

#### 4.2.6 Binary logarithm $\log_2(z)$

---

`WorksheetFunction.IMLOG2(z As String)` As String

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.IMLOG2` returns  $\log_2(z)$ , as a String representing a complex number.

**Parameter:**

`z`: A String representing a complex number.

The function `cplxLn(z)` returns the complex natural logarithm of  $z$ :

$$\log_2(z) = \ln(z)/\ln(2). \quad (4.2.6)$$

---

**Function `log2(z As mpNum)` As mpNum**

---

The function `log2` returns  $\log_2(z)$

**Parameter:**

`z`: A complex number.

#### 4.2.7 Auxiliary Function $\ln(1 + x)$

---

**Function `lnp1(x As mpNum)` As mpNum**

---

The function `lnp1` returns the value of the function  $\ln(1 + x)$ .

**Parameter:**

`x`: A real number.

## 4.3 Roots and Power Functions

### 4.3.1 Square: $z^2$

---

Function **square(z As mpNum) As mpNum**

---

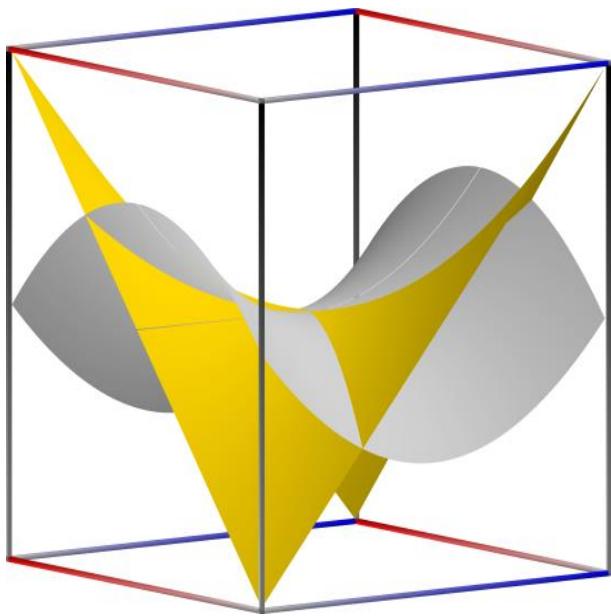
The function **square** returns the square of  $z$ .

**Parameter:**

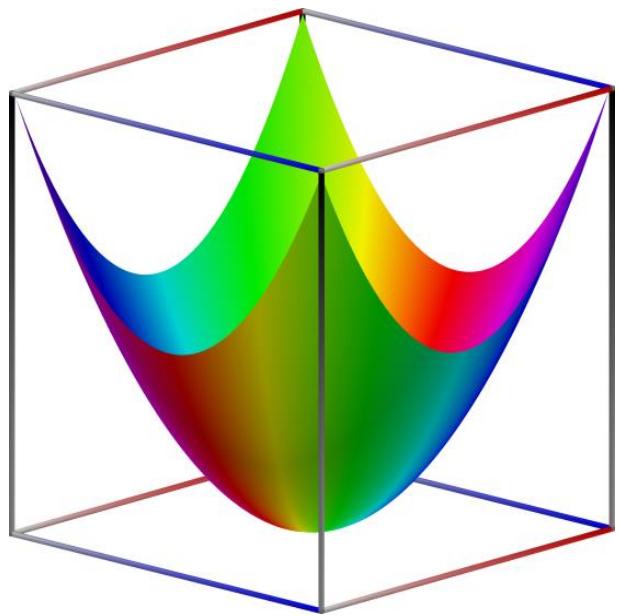
$z$ : A complex number.

The function **cplxSqr(z1, z2)** returns the square of  $z$ :

$$z^2 = x^2 - y^2 + i(2xy). \quad (4.3.1)$$



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.3:** Surface plots of  $z = (x + iy)^2$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

### 4.3.2 Power Function

---

WorksheetFunction.**POWER**(*x* As mpNum, *y* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.POWER returns the value of  $x^y$ ,  $y \in \mathbb{R}$ .

**Parameters:**

*x*: A real number.

*y*: A real number.

---

WorksheetFunction.**IMPOWER**(*z* As String, *k* As Integer) As String

---

**NOT YET IMPLEMENTED**

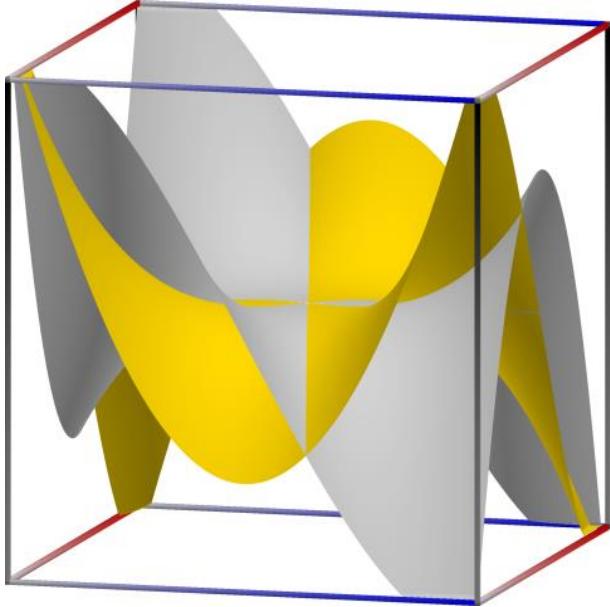
---

The function WorksheetFunction.IMPOWER returns an integer power of  $z$ , as a String representing a complex number.

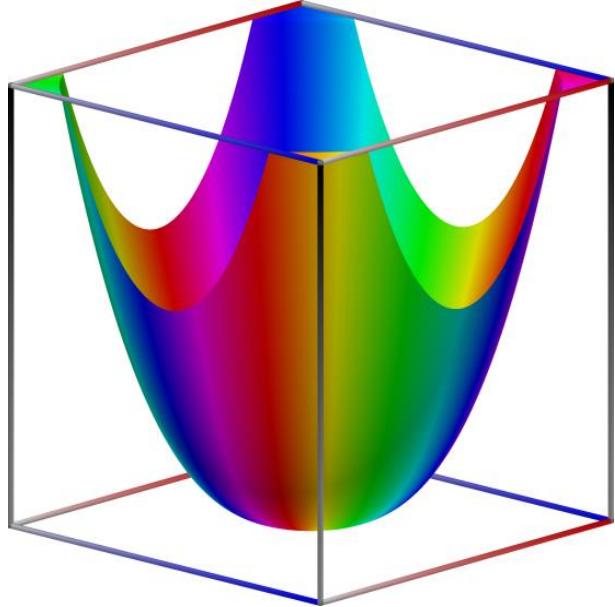
**Parameters:**

*z*: A String representing a complex number.

*k*: An integer.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.4:** Surface plots of  $z = (x + iy)^3$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

---

Function **power**(*z1* As mpNum, *z2* As mpNum) As mpNum

---

The function power returns an complex power of *z*

**Parameters:**

*z1*: A complex number.

*z2*: A complex number.

Converts *x* and *y* to mpFormulaPy numbers and evaluates  $x^y = \exp(y \log(x))$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 30; mp.pretty = True
>>> power(2, 0.5)
1.41421356237309504880168872421
```

---

This shows the leading few digits of a large Mersenne prime (performing the exact calculation  $2^{**43112609}-1$  and displaying the result in Python would be very slow):

---

```
>>> power(2, 43112609)-1
3.16470269330255923143453723949e+12978188
```

---

The function `cplxPower(z, k)` returns an integer power of *z*:

$$z^k = r^k \cos(k\theta) + i(r^k \sin(k\theta)), \quad k \in \mathbb{Z}, \quad (4.3.2)$$

where  $r = \sqrt{x^2 + y^2}$ , and  $\theta = \arctan(y/x)$ .

The function `cplxPowR(z, k)` returns a real power of *z*:

$$z^a = r^a \cos(a\theta) + i(r^a \sin(a\theta)), \quad a \in \mathbb{R}, \quad (4.3.3)$$

where  $r = \sqrt{x^2 + y^2}$ , and  $\theta = \arctan(y/x)$ .

The function `cplxPowC(z, k)` returns a complex power of *z1*:

$$z_1^{z_2} = \exp(\ln(z_1)z_2), \quad z_1, z_2 \in \mathbb{C}. \quad (4.3.4)$$

### 4.3.2.1 `powm1(x, y)`

---

#### Function `powm1(z As mpNum, k As mpNum) As mpNum`

---

The function `powm1` returns an integer power of *z*

**Parameters:**

*z*: A complex number.

*k*: A complex number.

Convenience function for computing  $x^x - 1$  accurately when  $x^y$  is very close to 1. This avoids potentially catastrophic cancellation:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> power(0.99999995, 1e-10) - 1
0.0
>>> powm1(0.99999995, 1e-10)
-5.00000012791934e-18
```

---

Powers exactly equal to 1, and only those powers, yield 0 exactly:

```
>>> powm1(-j, 4)
(0.0 + 0.0j)
>>> powm1(3, 0)
0.0
>>> powm1(fadd(-1, 1e-100, exact=True), 4)
-4.0e-100
```

---

Evaluation works for extremely tiny  $y$ :

```
>>> powm1(2, '1e-100000')
6.93147180559945e-100001
>>> powm1(j, '1e-1000')
(-1.23370055013617e-2000 + 1.5707963267949e-1000j)
```

---

### 4.3.3 Square Root: $\sqrt{z}$

---

WorksheetFunction.**SQRT**(*x* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.SQRT returns the absolute value of the square root of *x*,  $\sqrt{x}$ .

**Parameter:**

*x*: A real number.

---

WorksheetFunction.**IMSQRT**(*z* As String) As String

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.IMSQRT returns the square root of *z*, as a String representing a complex number.

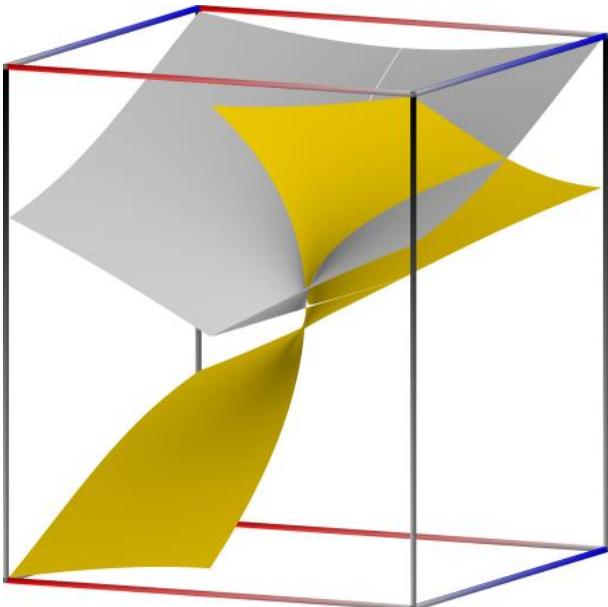
**Parameter:**

*z*: A String representing a complex number.

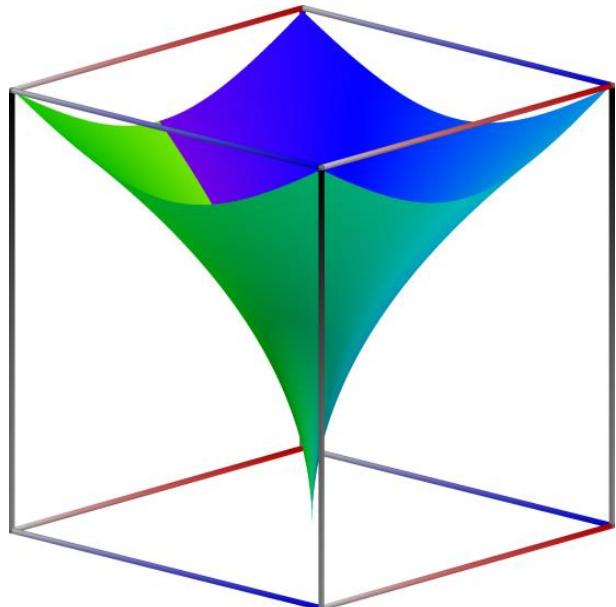
The function cplxSqrt(*z*) returns the square root of *z*:

$$\sqrt{z} = \sqrt{r} \cos\left(\frac{1}{2}\theta\right) + i\sqrt{r} \sin\left(\frac{1}{2}\theta\right), \quad (4.3.5)$$

where  $r = \sqrt{x^2 + y^2}$ , and  $\theta = \arctan(y/x)$ .



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.5:** Surface plots of  $z = \sqrt{x+iy}$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis). *z* values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

**Function `sqrt(z As mpNum) As mpNum`**

---

The function `sqrt` returns the square root of  $z$

**Parameter:**

$z$ : A complex number.

`sqrt(x)` gives the principal square root of  $x$ ,  $\sqrt{x}$ . For positive real numbers, the principal root is simply the positive square root. For arbitrary complex numbers, the principal square root is defined to satisfy  $\sqrt{x} = \exp(\log(x)/2)$ . The function thus has a branch cut along the negative half real axis. For all mpFormulaPy numbers  $x$ , calling `sqrt(x)` is equivalent to performing  $x^{**0.5}$ .

Examples

Basic examples and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> sqrt(10)
3.16227766016838
>>> sqrt(100)
10.0
>>> sqrt(-4)
(0.0 + 2.0j)
>>> sqrt(1+1j)
(1.09868411346781 + 0.455089860562227j)
>>> sqrt(inf)
+inf
```

---

Square root evaluation is fast at huge precision:

---

```
>>> mp.dps = 50000
>>> a = sqrt(3)
>>> str(a)[-10:]
'9329332814'
```

---

mpFormulaPy.iv.sqrt() supports interval arguments:

---

```
>>> iv.dps = 15; iv.pretty = True
>>> iv.sqrt([16,100])
[4.0, 10.0]
>>> iv.sqrt(2)
[1.4142135623730949234, 1.4142135623730951455]
>>> iv.sqrt(2) ** 2
[1.99999999999999559, 2.0000000000000004441]
```

---

#### 4.3.4 Auxiliary Function $\sqrt{x^2 + y^2}$

Computes the Euclidean norm of the vector  $(x, y)$ , equal to  $\sqrt{x^2 + y^2}$ . Both  $x$  and  $y$  must be real.

---

**Function `hypot(x As mpNum, y As mpNum) As mpNum`**

---

The function `hypot` returns the value of  $\sqrt{x^2 + y^2}$ .

**Parameters:**

*x*: A real number.  
*y*: A real number.

**4.3.5 Cube Root:**  $\sqrt[3]{x}, n = 2, 3, \dots$ **Function `cbrt(z As mpNum) As mpNum`**

The function `cbrt` returns the square root of *z*

**Parameter:**

*z*: A complex number.

`cbrt(x)` computes the cube root of *x*,  $x^{1/3}$ . This function is faster and more accurate than raising to a floating-point fraction:

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> 125**(mpf(1)/3)
mpf('4.999999999999991')
>>> cbrt(125)
mpf('5.0')
```

Every nonzero complex number has three cube roots. This function returns the cube root defined by  $\exp(\log(x)/3)$ , where the principal branch of the natural logarithm is used. Note that this does not give a real cube root for negative real numbers:

```
>>> mp.pretty = True
>>> cbrt(-1)
(0.5 + 0.866025403784439j)
```

**4.3.6 Nth Root:**  $\sqrt[n]{z}, n = 2, 3, \dots$ **Function `root(z As mpNum, n As mpNum) As mpNum`**

The function `root` returns the value of the *n*<sup>th</sup> root of *x*,  $\sqrt[n]{x}, n = 2, 3, \dots$ .

**Parameters:**

*z*: A complex number.  
*n*: An integer.

The *n*<sup>th</sup> root of *z* is defined as:

$$\sqrt[n]{z} = z^{1/n} = \sqrt[n]{r} \exp\left(\frac{i\theta}{n}\right), \quad n \in \mathbb{N}, \quad (4.3.6)$$

where  $r = \sqrt{x^2 + y^2}$ , and  $\theta = \arctan(y/x)$ . This is the principal root if  $-\pi < \theta \leq \pi$ . The other roots are given by

$$\sqrt[n]{z} = \sqrt[n]{r} \exp\left(\frac{i(\theta + 2\pi k)}{n}\right), \quad k = 1, 2, \dots, n - 1. \quad (4.3.7)$$

**Function `nthroot(n As mpNum, y As mpNum) As mpNum`**

The function `nthroot` returns the value of the  $n^{th}$  root of  $x$ ,  $\sqrt[n]{x}$ ,  $n = 2, 3, \dots$

### Parameters:

`n`: An integer.

`y`: A real number.

This is an alternative version for `root` (see above).

Every complex number  $z \neq 0$  has  $n$  distinct  $n$ -th roots, which are equidistant points on a circle with radius  $|z|^{1/n}$ , centered around the origin. A specific root may be selected using the optional index  $k$ . The roots are indexed counterclockwise, starting with  $k = 0$  for the root closest to the positive real half-axis.

The  $k = 0$  root is the so-called principal  $n$ -th root, often denoted by  $\sqrt[n]{z}$  or  $z^{1/n}$ , and also given by  $\exp(\log(z)/n)$ . If  $z$  is a positive real number, the principal root is just the unique positive  $n$ -th root of  $z$ . Under some circumstances, non-principal real roots exist: for positive real  $z$ ,  $n$  even, there is a negative root given by  $k = n/2$ ; for negative real  $z$ ,  $n$  odd, there is a negative root given by  $k = (n - 1)/2$ .

To obtain all roots with a simple expression, use

---

`[root(z, n, k) for k in range(n)]`.

---

An important special case, `root(1, n, k)` returns the  $k$ -th  $n$ -th root of unity,  $\zeta_k = e^{2\pi ik/n}$ . Alternatively, `unitroots()` provides a slightly more convenient way to obtain the roots of unity, including the option to compute only the primitive roots of unity.

Both  $k$  and  $n$  should be integers;  $k$  outside of `range(n)` will be reduced modulo  $n$ . If  $n$  is negative,  $x^{-1/n} = 1/x^{1/n}$  (or the equivalent reciprocal for a non-principal root with  $k \neq 0$ ) is computed.

`root()` is implemented to use Newton's method for small  $n$ . At high precision, this makes  $x^{1/n}$  not much more expensive than the regular exponentiation,  $x^n$ . For very large  $n$ , `nthroot()` falls back to use the exponential function.

### Examples

`nthroot()`/`root()` is faster and more accurate than raising to a floating-point fraction:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> 16807 ** (mpf(1)/5)
mpf('7.000000000000009')
>>> root(16807, 5)
mpf('7.0')
>>> nthroot(16807, 5) # Alias
mpf('7.0')
```

---

A high-precision root:

---

```
>>> mp.dps = 50; mp.pretty = True
>>> nthroot(10, 5)
1.584893192461113485202101373391507013269442133825
>>> nthroot(10, 5) ** 5
10.0
```

---

Computing principal and non-principal square and cube roots:

---

```
>>> mp.dps = 15
>>> root(10, 2)
3.16227766016838
>>> root(10, 2, 1)
-3.16227766016838
>>> root(-10, 3)
(1.07721734501594 + 1.86579517236206j)
>>> root(-10, 3, 1)
-2.15443469003188
>>> root(-10, 3, 2)
(1.07721734501594 - 1.86579517236206j)
```

---

All the 7th roots of a complex number:

---

```
>>> for r in [root(3+4j, 7, k) for k in range(7)]:
...     print("%s %s" % (r, r**7))
...
(1.24747270589553 + 0.166227124177353j) (3.0 + 4.0j)
(0.647824911301003 + 1.07895435170559j) (3.0 + 4.0j)
(-0.439648254723098 + 1.17920694574172j) (3.0 + 4.0j)
(-1.19605731775069 + 0.391492658196305j) (3.0 + 4.0j)
(-1.05181082538903 - 0.691023585965793j) (3.0 + 4.0j)
(-0.115529328478668 - 1.25318497558335j) (3.0 + 4.0j)
(0.907748109144957 - 0.871672518271819j) (3.0 + 4.0j)
```

---

Cube roots of unity:

---

```
>>> for k in range(3): print(root(1, 3, k))
...
1.0
(-0.5 + 0.866025403784439j)
(-0.5 - 0.866025403784439j)
```

---

Some exact high order roots:

---

```
>>> root(75**210, 105)
5625.0
>>> root(1, 128, 96)
(0.0 - 1.0j)
>>> root(4**128, 128, 96)
(0.0 - 4.0j)
```

---

#### 4.3.6.1 unitroots(n, primitive=False)

`unitroots(n)` returns  $\zeta_0, \zeta_1, \dots, \zeta_{n-1}$ , all the distinct  $n$ -th roots of unity, as a list. If the option `primitive=True` is passed, only the primitive roots are returned.

Every  $n$ -th root of unity satisfies  $(\zeta_k)^n = 1$ . There are distinct roots for each  $n$  ( $\zeta_k$  and  $\zeta_j$  are the same when  $k = j \pmod n$ ), which form a regular polygon with vertices on the unit circle. They are ordered counterclockwise with increasing  $k$ , starting with  $\zeta_0 = 1$ .

Examples

The roots of unity up to  $n = 4$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> nprint(unitroots(1))
[1.0]
>>> nprint(unitroots(2))
[1.0, -1.0]
>>> nprint(unitroots(3))
[1.0, (-0.5 + 0.866025j), (-0.5 - 0.866025j)]
>>> nprint(unitroots(4))
[1.0, (0.0 + 1.0j), -1.0, (0.0 - 1.0j)]
```

---

Roots of unity form a geometric series that sums to 0:

---

```
>>> mp.dps = 50
>>> chop(fsum(unitroots(25)))
0.0
```

---

Primitive roots up to  $n = 4$ :

---

```
>>> mp.dps = 15
>>> nprint(unitroots(1, primitive=True))
[1.0]
>>> nprint(unitroots(2, primitive=True))
[-1.0]
>>> nprint(unitroots(3, primitive=True))
[(-0.5 + 0.866025j), (-0.5 - 0.866025j)]
>>> nprint(unitroots(4, primitive=True))
[(0.0 + 1.0j), (0.0 - 1.0j)]
```

---

There are only four primitive 12th roots:

---

```
>>> nprint(unitroots(12, primitive=True))
[(0.866025 + 0.5j), (-0.866025 + 0.5j), (-0.866025 - 0.5j), (0.866025 - 0.5j)]
```

---

The  $n$ -th roots of unity form a group, the cyclic group of order  $n$ . Any primitive root  $r$  is a generator for this group, meaning that  $r^0, r^1, \dots, r^{n-1}$  gives the whole set of unit roots (in some permuted order):

---

```
>>> for r in unitroots(6): print(r)
...
1.0
(0.5 + 0.866025403784439j)
(-0.5 + 0.866025403784439j)
-1.0
(-0.5 - 0.866025403784439j)
(0.5 - 0.866025403784439j)
>>> r = unitroots(6, primitive=True)[1]
>>> for k in range(6): print(chop(r**k))
...
1.0
(0.5 - 0.866025403784439j)
```

---

```
(-0.5 - 0.866025403784439j)
-1.0
(-0.5 + 0.866025403784438j)
(0.5 + 0.866025403784438j)
```

---

The number of primitive roots equals the Euler totient function  $\phi(n)$ :

```
>>> [len(unitroots(n, primitive=True)) for n in range(1,20)]
[1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, 6, 8, 8, 16, 6, 18]
```

---

## 4.4 Trigonometric Functions

### 4.4.1 Trigonometric functions: overview

Except where otherwise noted, the trigonometric functions take a radian angle as input and the inverse trigonometric functions return radian angles.

The ordinary trigonometric functions are single-valued functions defined everywhere in the complex plane (except at the poles of tan, sec, csc, and cot). They are defined generally via the exponential function, e.g.

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}) \quad (4.4.1)$$

The inverse trigonometric functions are multivalued, thus requiring branch cuts, and are generally real-valued only on a part of the real line. Definitions and branch cuts are given in the documentation of each function. The branch cut conventions used by mpFormulaPy are essentially the same as those found in most standard mathematical software, such as Mathematica and Python's own cmath library (as of Python 2.6; earlier Python versions implement some functions erroneously).

### 4.4.2 Conversion between Degrees and radians

---

`WorksheetFunction.DEGREES(x As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.DEGREES` returns the value of  $x$  converted to degrees, with the input  $x$  in radians.

**Parameter:**

$x$ : A real number. The following formula is used:

$$\text{Degrees}(x) = x \cdot \frac{180}{\pi}, \quad (4.4.2)$$

---

`Function degrees(x As mpNum) As mpNum`

---

The function `degrees` returns the value of  $x$  converted to degrees, with the input  $x$  in radians.

**Parameter:**

$x$ : A real number.

Converts the radian angle  $x$  to a degree angle:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> degrees(pi/3)
60.0
```

---

`WorksheetFunction.RADIANS(x As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.RADIANS` returns the value of  $x$  converted to radians, with the input  $x$  in degrees.

**Parameter:**

*x*: A real number.

The following formula is used:

$$\text{Radians}(x) = x \cdot \frac{\pi}{180}. \quad (4.4.3)$$

**Function `radians(x As mpNum) As mpNum`**

The function `radians` returns the value of *x* converted to radians, with the input *x* in degrees.

**Parameter:**

*x*: A real number.

Converts the degree angle *x* to radians:

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> radians(60)
1.0471975511966
```

### 4.4.3 SQRTPI

**WorksheetFunction.SQRTPI(*x* As mpNum) As mpNum**

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.SQRTPI` returns the value of  $\sqrt{n \cdot \pi}$ .

**Parameter:**

*x*: A real number.

#### 4.4.4 Sine: $\sin(z)$

`WorksheetFunction.SIN(x As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.SIN` returns the value of the sine of  $x$ , with  $x$  in radians.

**Parameter:**

$x$ : A real number.

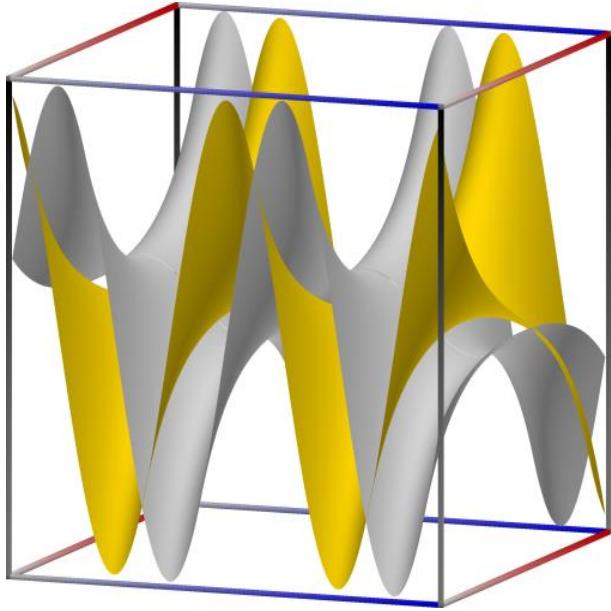
`WorksheetFunction.IMSIN(z As String) As String`

**NOT YET IMPLEMENTED**

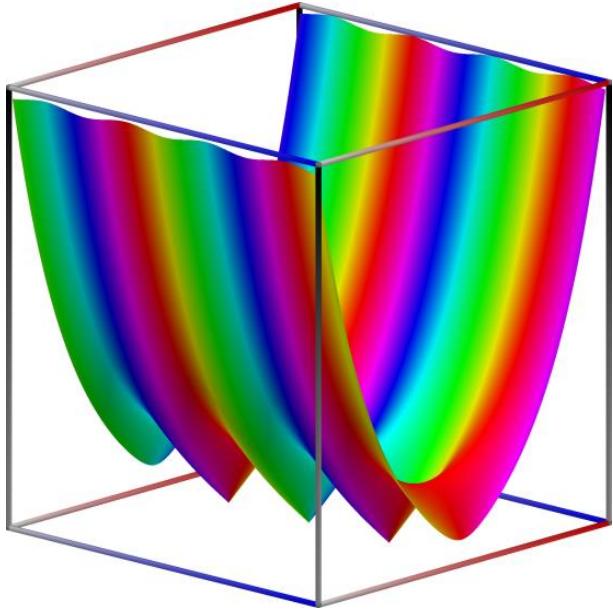
The function `WorksheetFunction.IMSIN` returns complex sine of  $z$ , as a String representing a complex number.

**Parameter:**

$z$ : A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.6:** Surface plots of  $z = \sin(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

**Function  $\sin(z As mpNum) As mpNum$**

The function `sin` returns complex sine of  $z$

**Parameter:**

$z$ : A complex number.

The function `cplxSin(z)` returns the complex sine of  $z$ :

$$\sin(z) = \sin(x) \cosh(y) + i \cos(x) \sinh(y). \quad (4.4.4)$$

Computes the sine of  $x$ ,  $\sin(x)$ .

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> sin(pi/3)
0.8660254037844386467637232
>>> sin(100000001)
0.1975887055794968911438743
>>> sin(2+3j)
(9.1544991469114295734673 - 4.168906959966564350754813j)
>>> sin(inf)
nan
>>> nprint(chop(taylor(sin, 0, 6)))
[0.0, 1.0, 0.0, -0.166667, 0.0, 0.00833333, 0.0]
```

---

#### 4.4.5 Cosine: $\cos(z)$

---

WorksheetFunction.**COS**(*x* As *mpNum*) As *mpNum*

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.COS returns the value of the cosine of *x*, with *x* in radians.

**Parameter:**

*x*: A real number.

---

WorksheetFunction.**IMCOS**(*z* As *String*) As *String*

---

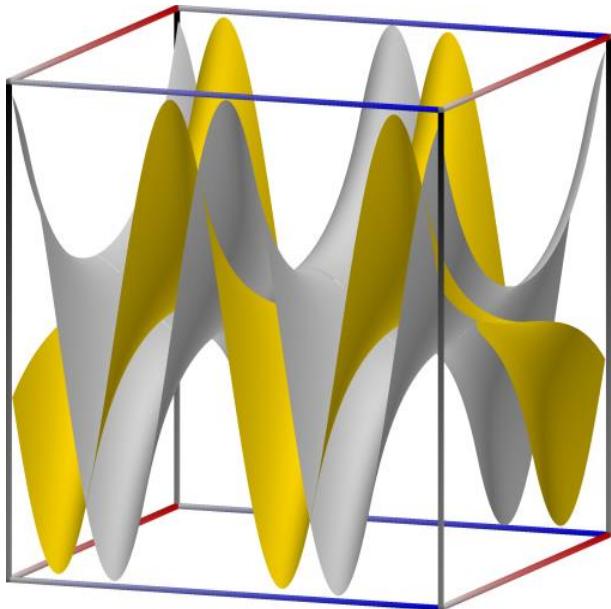
**NOT YET IMPLEMENTED**

---

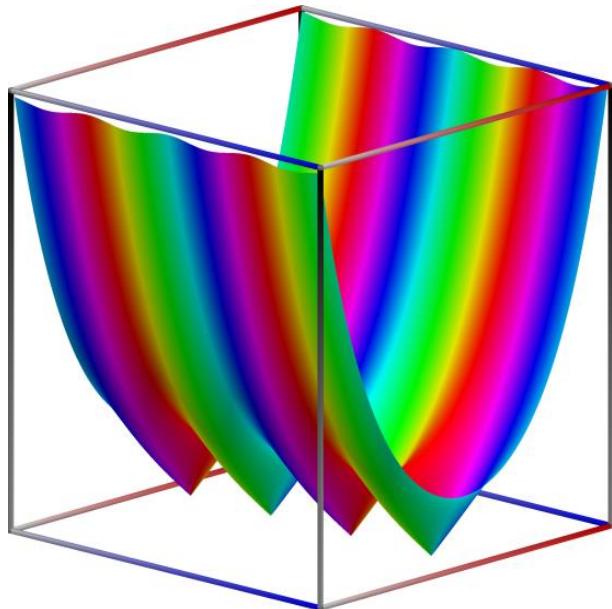
The function WorksheetFunction.IMCOS returns complex cosine of *z*, as a String representing a complex number.

**Parameter:**

*z*: A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.7:** Surface plots of  $z = \cos(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis). *z* values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

---

Function **cos**(*z* As *mpNum*) As *mpNum*

---

The function **cos** returns complex cosine of *z*

**Parameter:**

*z*: A complex number.

The function `cplxCos(z)` returns the complex cosine of  $z$ :

$$\cos(z) = \cos(x) \cosh(y) - i \sin(x) \sinh(y). \quad (4.4.5)$$

Computes the cosine of  $x$ ,  $\cos(x)$ .

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> cos(pi/3)
0.5
>>> cos(100000001)
-0.9802850113244713353133243
>>> cos(2+3j)
(-4.189625690968807230132555 - 9.109227893755336597979197j)
>>> cos(inf)
nan
>>> nprint(chop(taylor(cos, 0, 6)))
[1.0, 0.0, -0.5, 0.0, 0.0416667, 0.0, -0.00138889]
```

---

#### 4.4.6 Tangent: $\tan(z)$

WorksheetFunction.TAN(*x* As mpNum) As mpNum

**NOT YET IMPLEMENTED**

The function WorksheetFunction.TAN returns the value of the tangent of *x*, with *x* in radians.

**Parameter:**

*x*: A real number.

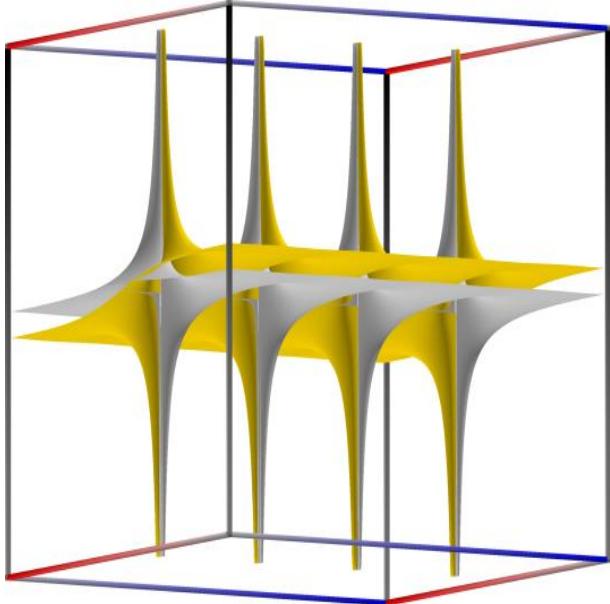
WorksheetFunction.IMTAN(*z* As String) As String

**NOT YET IMPLEMENTED**

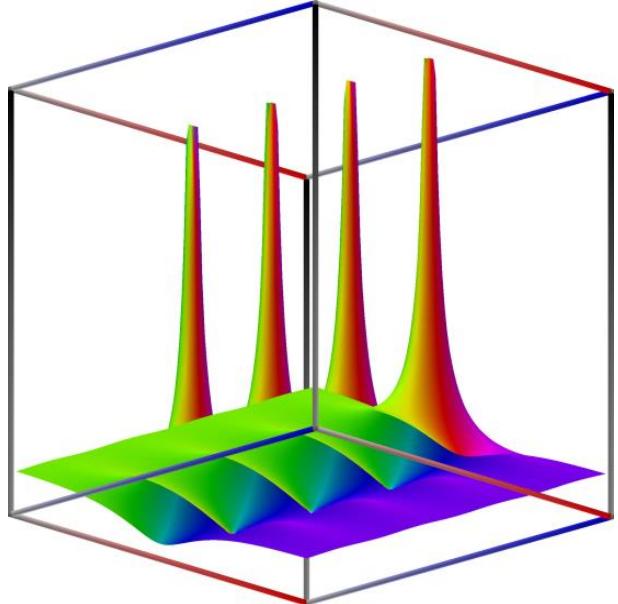
The function WorksheetFunction.IMTAN returns complex tangent of *z*, as a String representing a complex number.

**Parameter:**

*z*: A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.8:** Surface plots of  $z = \tan(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis). *z* values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

Function **tan(*z* As mpNum) As mpNum**

The function tan returns complex tangent of *z*

**Parameter:**

*z*: A complex number.

The function `cplxTan(z)` returns the tangent tangent of  $z$ :

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{\sin(2x) + i \sinh(2y)}{\cos(2x) + i \cosh(2y)} \quad (4.4.6)$$

Computes the tangent of  $x$ ,  $\tan(x) = \frac{\sin(x)}{\cos(x)}$ . The tangent function is singular at  $x = (n + \frac{1}{2})\pi$ , but  $\tan(x)$  always returns a finite result since  $(n + \frac{1}{2})\pi$  cannot be represented exactly using floating-point arithmetic.

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> tan(pi/3)
1.732050807568877293527446
>>> tan(100000001)
-0.2015625081449864533091058
>>> tan(2+3j)
(-0.003764025641504248292751221 + 1.003238627353609801446359j)
>>> tan(inf)
nan
>>> nprint(chop(taylor(tan, 0, 6)))
[0.0, 1.0, 0.0, 0.333333, 0.0, 0.133333, 0.0]
```

---

#### 4.4.7 Secant: $\sec(z) = 1/\cos(z)$

`WorksheetFunction.SEC(x As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.SEC` returns the value of the secant of  $x$ , with  $x$  in radians.

**Parameter:**

$x$ : A real number.

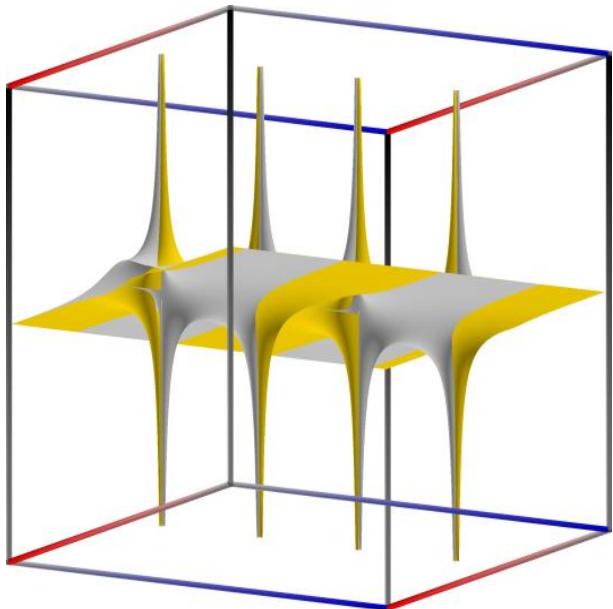
`WorksheetFunction.IMSEC(z As String) As String`

**NOT YET IMPLEMENTED**

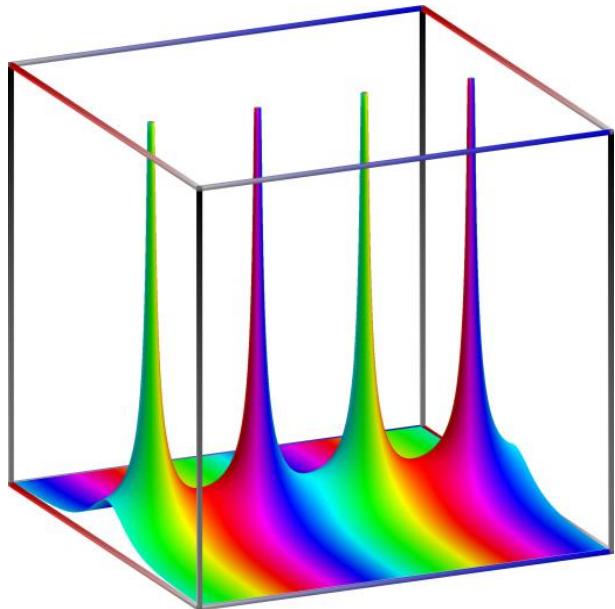
The function `WorksheetFunction.IMSEC` returns the complex secant of  $z$ , as a String representing a complex number.

**Parameter:**

$z$ : A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.9:** Surface plots of  $z = \sec(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

**Function `sec(z As mpNum) As mpNum`**

The function `sec` returns the complex secant of  $z$

**Parameter:**

$z$ : A complex number.

The function `cplxSec(z)` returns the complex secant of  $z$ :

$$\sec(z) = 1/\cos(z). \quad (4.4.7)$$

Computes the secant of  $x$ ,  $\sec(x) = \frac{1}{\cos(x)}$ . The secant function is singular at  $x = (n + \frac{1}{2})\pi$ , but `sec(x)` always returns a finite result since  $(n + \frac{1}{2})\pi$  cannot be represented exactly using floating-point arithmetic.

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> sec(pi/3)
2.0
>>> sec(10000001)
-1.184723164360392819100265
>>> sec(2+3j)
(-0.04167496441114427004834991 + 0.0906111371962375965296612j)
>>> sec(inf)
nan
>>> nprint(chop(taylor(sec, 0, 6)))
[1.0, 0.0, 0.5, 0.0, 0.208333, 0.0, 0.0847222]
```

---

#### 4.4.8 Cosecant: $\csc(z) = 1/\sin(z)$

`WorksheetFunction.CSC(x As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.CSC` returns the value of the cosecant of  $x$ , with  $x$  in radians.

**Parameter:**

$x$ : A real number.

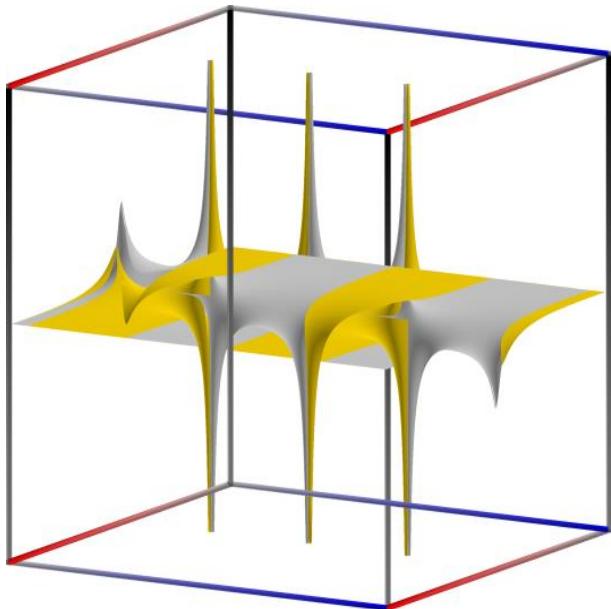
`WorksheetFunction.IMCSC(z As String) As String`

**NOT YET IMPLEMENTED**

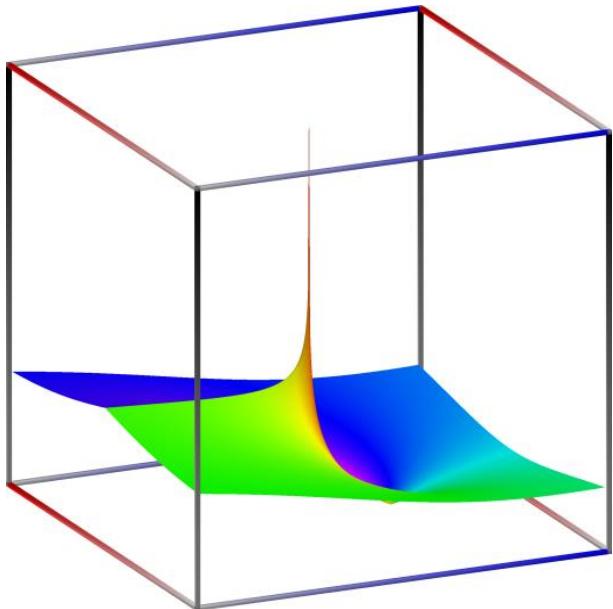
The function `WorksheetFunction.IMCSC` returns the complex cosecant of  $z$ , as a String representing a complex number.

**Parameter:**

$z$ : A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.10:** Surface plots of  $z = \csc(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

**Function `csc(z As mpNum) As mpNum`**

The function `csc` returns the complex cosecant of  $z$

**Parameter:**

$z$ : A complex number.

The function `cplxCsc(z)` returns the complex cosecant of  $z$ :

$$\sec(z) = 1/\sin(z). \quad (4.4.8)$$

Computes the cosecant of  $x$ ,  $\csc(x) = \frac{1}{\sin(x)}$ . This cosecant function is singular at  $x = n\pi$ , but with the exception of the point  $x = 0$ , `csc(x)` returns a finite result since  $n\pi$  cannot be represented exactly using floating-point arithmetic.

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> csc(pi/3)
1.154700538379251529018298
>>> csc(10000001)
-1.864910497503629858938891
>>> csc(2+3j)
(0.09047320975320743980579048 + 0.04120098628857412646300981j)
>>> csc(inf)
nan
```

---

#### 4.4.9 Cotangent: $\cot(z) = 1/\tan(z)$

`WorksheetFunction.COT(x As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.COT` returns the value of the cotangent of  $x$ , with  $x$  in radians.

**Parameter:**

$x$ : A real number.

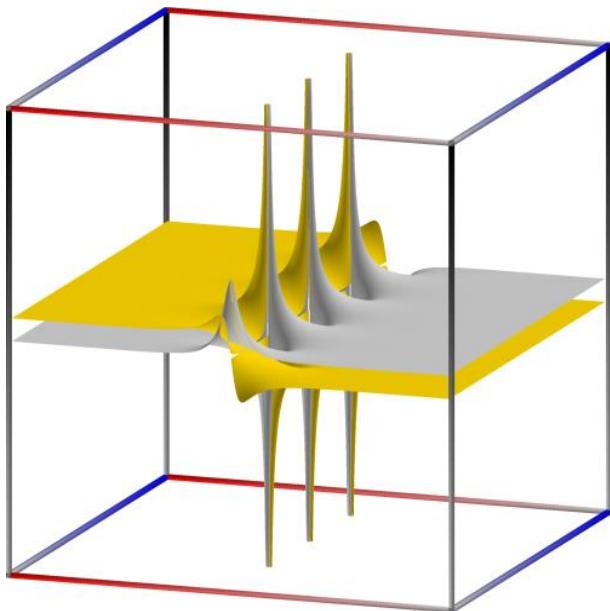
`WorksheetFunction.IMPOT(z As String) As String`

**NOT YET IMPLEMENTED**

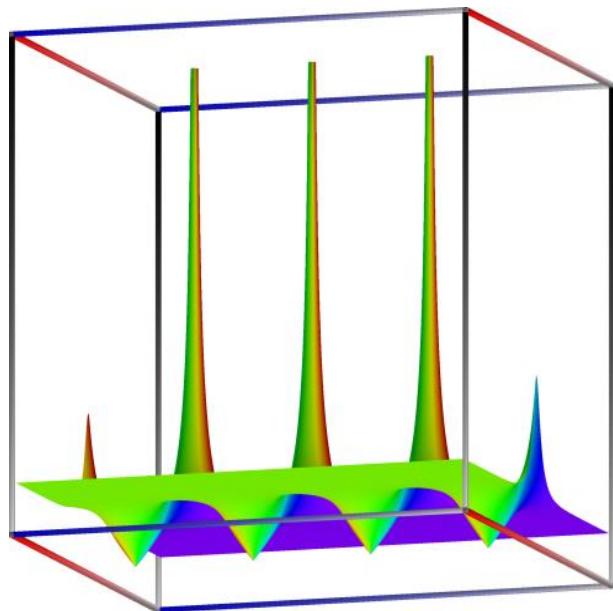
The function `WorksheetFunction.IMPOT` returns the complex cotangent of  $z$ , as a String representing a complex number.

**Parameter:**

$z$ : A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.11:** Surface plots of  $z = \cot(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

**Function `cot(z As mpNum) As mpNum`**

The function `cot` returns the complex cotangent of  $z$

**Parameter:**

$z$ : A complex number.

The function `cplxCot(z)` returns the complex cotangent of  $z$ :

$$\cot(z) = \frac{\cos(z)}{\sin(z)} = \frac{\sin(2x) - i \sinh(2y)}{\cosh(2y) - i \cos(2x)} \quad (4.4.9)$$

Computes the cotangent of  $x$ ,  $\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$ . The cotangent function is singular at  $x = n\pi$ , but with the exception of the point  $x = 0$ , `csc(x)` returns a finite result since  $n\pi$  cannot be represented exactly using floating-point arithmetic.

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> cot(pi/3)
0.5773502691896257645091488
>>> cot(10000001)
1.574131876209625656003562
>>> cot(2+3j)
(-0.003739710376336956660117409 - 0.9967577965693583104609688j)
>>> cot(inf)
nan
```

---

## 4.4.10 Sinc function

### 4.4.10.1 `sinc(x)`

`sinc(x)` computes the unnormalized sinc function, defined as

$$\text{sinc}(x) = \begin{cases} \sin(x)/x & \text{for } x \neq 0 \\ 1 & \text{for } x = 0. \end{cases} \quad (4.4.10)$$

See `sincpi()` for the normalized sinc function.

Simple values and limits include

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> sinc(0)
1.0
>>> sinc(1)
0.841470984807897
>>> sinc(inf)
0.0
```

---

The integral of the sinc function is the sine integral Si:

---

```
>>> quad(sinc, [0, 1])
0.946083070367183
>>> si(1)
0.946083070367183
```

---

### 4.4.10.2 `sincpi(x)`

`sincpi(x)` computes the normalized sinc function, defined as

$$\text{sinc}_\pi(x) = \begin{cases} \sin(\pi x)/(\pi x) & \text{for } x \neq 0 \\ 1 & \text{for } x = 0. \end{cases} \quad (4.4.11)$$

Equivalently, we have  $\text{sinc}_\pi(x) = \text{sinc}(\pi x)$ . The normalization entails that the function integrates to unity over the entire real line:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> quadosc(sincpi, [-inf, inf], period=2.0)
1.0
```

---

Like, `sinpi()`, `sincpi()` is evaluated accurately at its roots:

---

```
>>> sincpi(10)
0.0
```

---

## 4.4.11 Trigonometric functions with modified argument

### 4.4.11.1 `cospi(x)`

Computes  $\cos(\pi x)$ , more accurately than the expression  $\cos(\pi * x)$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> cospi(10**10), cos(pi*(10**10))
(1.0, 0.999999999997493)
>>> cospi(10**10+0.5), cos(pi*(10**10+0.5))
(0.0, 1.59960492420134e-6)
```

---

#### 4.4.11.2 sinpi(x)

Computes  $\sin(\pi x)$ , more accurately than the expression  $\sin(\pi \cdot x)$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> sinpi(10**10), sin(pi*(10**10))
(0.0, -2.23936276195592e-6)
```

---

## 4.5 Hyperbolic Functions

### 4.5.1 Hyperbolic Sine: $\sinh(z)$

---

WorksheetFunction.**SINH**(*x As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.SINH returns the value of the hyperbolic sine of  $x$ , with  $x$  in radians.

**Parameter:**

*x*: A real number.

---

WorksheetFunction.**IMSINH**(*z As String*) As String

---

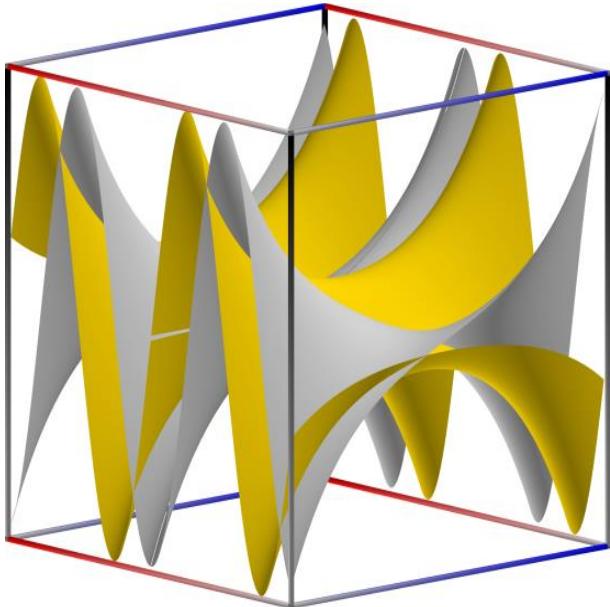
**NOT YET IMPLEMENTED**

---

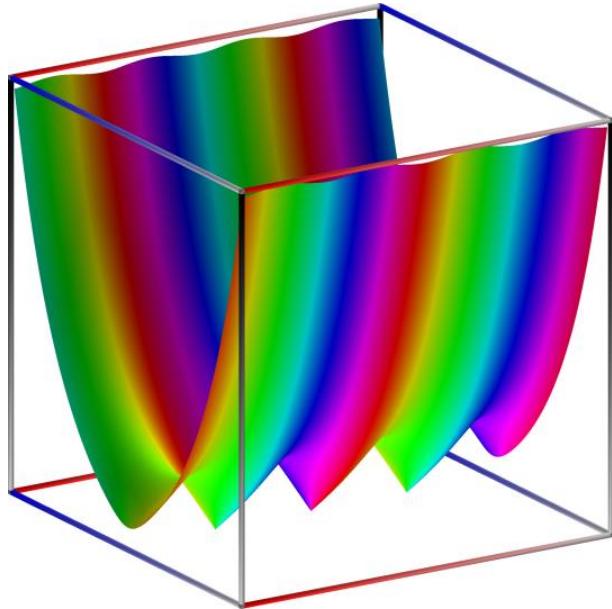
The function WorksheetFunction.IMSINH returns the complex hyperbolic sine of  $z$ , as a String representing a complex number.

**Parameter:**

*z*: A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.12:** Surface plots of  $z = \sinh(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

---

Function **sinh**(*z As mpNum*) As mpNum

---

The function sinh returns the complex hyperbolic sine of  $z$

**Parameter:**

$z$ : A complex number.

The function `cplxSinh(z)` returns the complex hyperbolic sine of  $z$ :

$$\sinh(z) = \sinh(x) \cos(y) + i \cosh(x) \sin(y). \quad (4.5.1)$$

Computes the hyperbolic sine of  $x$ ,  $\sinh(x) = (e^x - e^{-x})/2$ . Values and limits include:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> sinh(0)
0.0
>>> sinh(1)
1.175201193643801456882382
>>> sinh(-inf), sinh(+inf)
(-inf, +inf)
```

---

Generalized to complex numbers, the hyperbolic sine is essentially a sine with a rotation  $i$  applied to the argument; more precisely,  $\sinh(x) = -i \sin(ix)$ :

---

```
>>> sinh(2+3j)
(-3.590564589985779952012565 + 0.5309210862485198052670401j)
>>> j*sin(3-2j)
(-3.590564589985779952012565 + 0.5309210862485198052670401j)
```

---

### 4.5.2 Hyperbolic Cosine: $\cosh(z)$

---

WorksheetFunction.**COSH**(*x* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.COSH returns the value of the hyperbolic cosine of *x*, with *x* in radians.

**Parameter:**

*x*: A real number.

---

WorksheetFunction.**IMCOSH**(*z* As String) As String

---

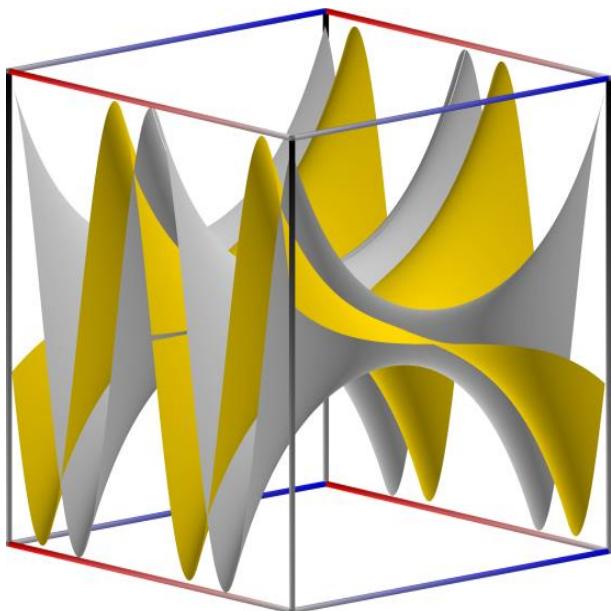
**NOT YET IMPLEMENTED**

---

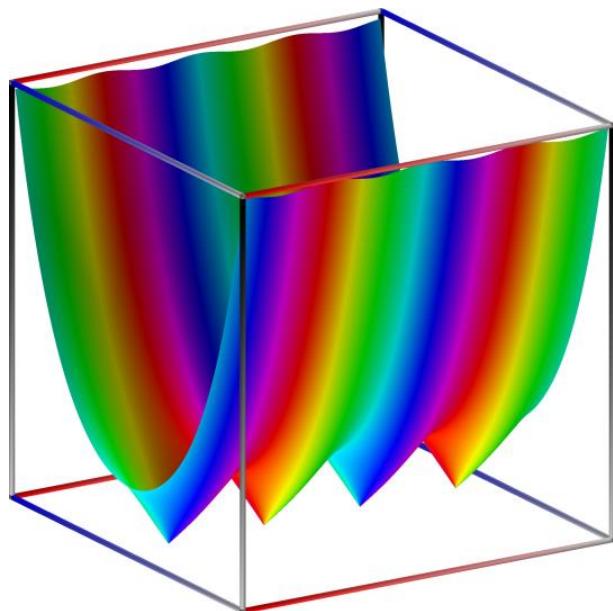
The function WorksheetFunction.IMCOSH returns the complex hyperbolic cosine of *z*, as a String representing a complex number.

**Parameter:**

*z*: A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.13:** Surface plots of  $z = \cosh(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

---

Function **cosh**(*z* As mpNum) As mpNum

---

The function **cosh** returns the complex hyperbolic cosine of *z*

**Parameter:**

$z$ : A complex number.

The function `cplxCosh(z)` returns the complex hyperbolic cosine of  $z$ :

$$\cosh(z) = \cosh(x) \cos(y) + i \sinh(x) \sin(y). \quad (4.5.2)$$

Computes the hyperbolic cosine of  $x$ ,  $\cosh(x) = (e^x + e^{-x})/2$ . Values and limits include:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> cosh(0)
1.0
>>> cosh(1)
1.543080634815243778477906
>>> cosh(-inf), cosh(+inf)
(+inf, +inf)
```

---

Generalized to complex numbers, the hyperbolic cosine is equivalent to a cosine with the argument rotated in the imaginary direction, or  $\cosh(x) = \cos(ix)$ :

---

```
>>> cosh(2+3j)
(-3.724545504915322565473971 + 0.5118225699873846088344638j)
>>> cos(3-2j)
(-3.724545504915322565473971 + 0.5118225699873846088344638j)
```

---

### 4.5.3 Hyperbolic Tangent: $\tanh(z)$

---

WorksheetFunction.TANH(*x* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.TANH returns the value of the hyperbolic cosine of *x*, with *x* in radians.

**Parameter:**

*x*: A real number.

---

WorksheetFunction.IMTANH(*z* As String) As String

---

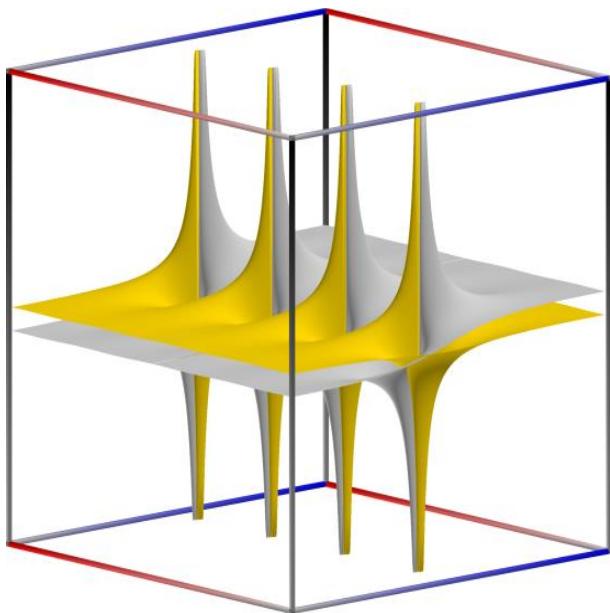
**NOT YET IMPLEMENTED**

---

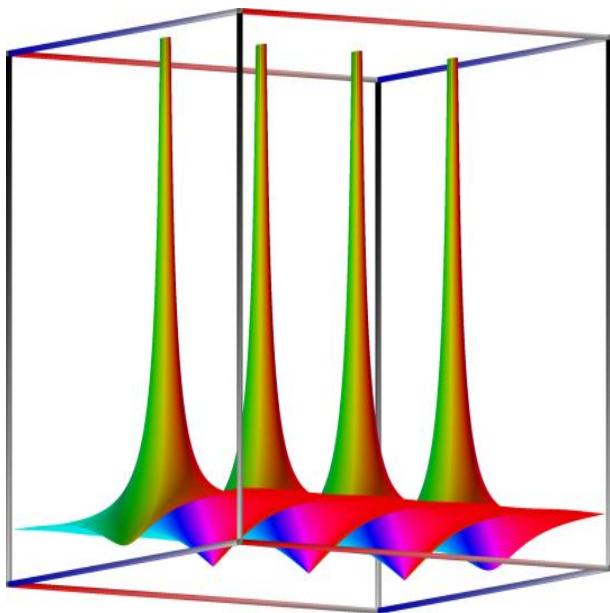
The function WorksheetFunction.IMTANH returns the complex hyperbolic tangent of *z*, as a String representing a complex number.

**Parameter:**

*z*: A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.14:** Surface plots of  $z = \tanh(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

---

Function **tanh**(*z* As mpNum) As mpNum

---

The function **tanh** returns the complex hyperbolic tangent of *z*

**Parameter:**

$z$ : A complex number.

The function `cplxTanh(z)` returns the complex hyperbolic tangent of  $z$ :

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{\sinh(2x) + i \sin(2y)}{\cosh(2x) + i \cos(2y)} \quad (4.5.3)$$

Computes the hyperbolic tangent of  $x$ ,  $\tanh(x) = \sinh(x)/\cosh(x)$ . Values and limits include:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> tanh(0)
0.0
>>> tanh(1)
0.7615941559557648881194583
>>> tanh(-inf), tanh(inf)
(-1.0, 1.0)
```

---

Generalized to complex numbers, the hyperbolic tangent is essentially a tangent with a rotation  $i$  applied to the argument; more precisely,  $\tanh(x) = -i \tan(ix)$ :

---

```
>>> tanh(2+3j)
(0.9653858790221331242784803 - 0.009884375038322493720314034j)
>>> j*tan(3-2j)
(0.9653858790221331242784803 - 0.009884375038322493720314034j)
```

---

#### 4.5.4 Hyperbolic Secant: $\text{sech}(x) = 1/\cosh(z)$

WorksheetFunction.**SECH**(*x* As mpNum) As mpNum

**NOT YET IMPLEMENTED**

The function WorksheetFunction.SECH returns the value of the hyperbolic cosecant of *x*, with *x* in radians.

**Parameter:**

*x*: A real number.

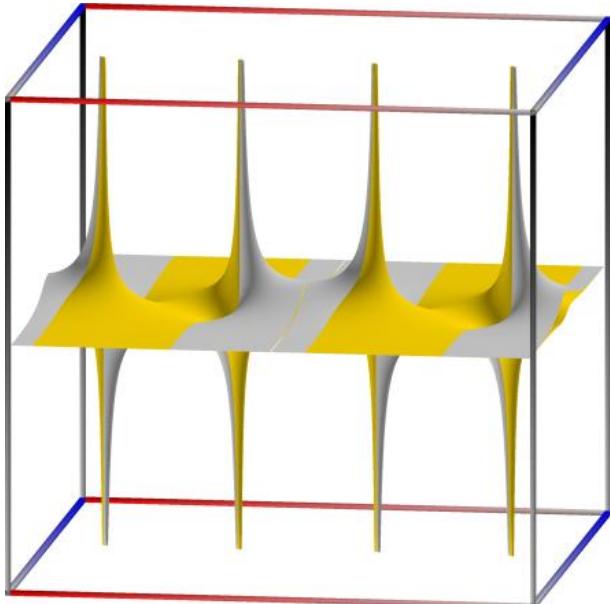
WorksheetFunction.**IMSECH**(*z* As String) As String

**NOT YET IMPLEMENTED**

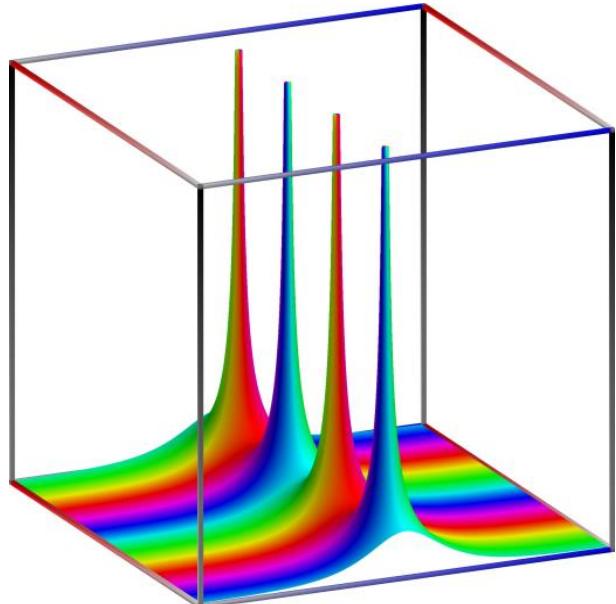
The function WorksheetFunction.IMSECH returns the complex hyperbolic secant of *z*, as a String representing a complex number.

**Parameter:**

*z*: A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.15:** Surface plots of  $z = \text{sech}(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis). *z* values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

Function **sech**(*z* As mpNum) As mpNum

The function **sech** returns the complex hyperbolic secant of *z*

**Parameter:**

$z$ : A complex number.

The function `cplxSech(z)` returns the complex hyperbolic secant of  $z$ :

$$\operatorname{sech}(z) = 1/\cosh(z). \quad (4.5.4)$$

#### 4.5.5 Hyperbolic Cosecant: $\text{csch}(x) = 1/\sinh(z)$

`WorksheetFunction.CSCH(x As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.CSCH` returns the value of the hyperbolic cosecant of  $x$ , with  $x$  in radians.

**Parameter:**

$x$ : A real number.

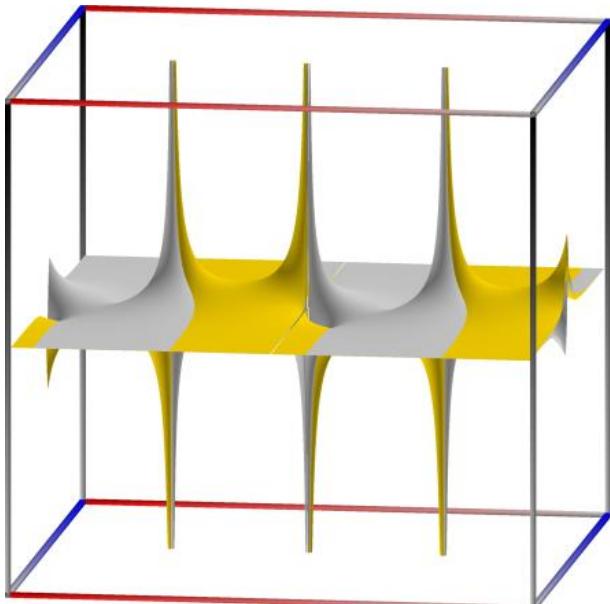
`WorksheetFunction.IMCSCH(z As String) As String`

**NOT YET IMPLEMENTED**

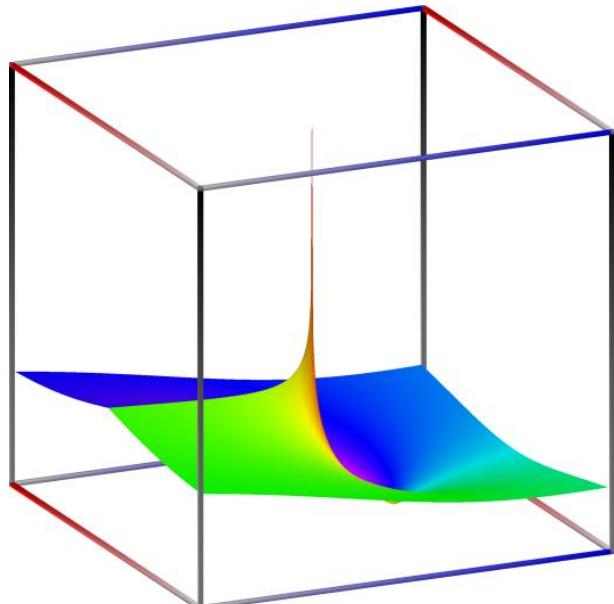
The function `WorksheetFunction.IMCSCH` returns the complex hyperbolic cosecant of  $z$ , as a String representing a complex number.

**Parameter:**

$z$ : A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ . wrong, missing

**Figure 4.16:** Surface plots of  $z = \text{csch}(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

`Function csch(z As mpNum) As mpNum`

The function `csch` returns the complex hyperbolic cosecant of  $z$

**Parameter:**

$z$ : A complex number.

The function `csch(z)` returns the complex hyperbolic cosecant of  $z$ :

$$\operatorname{csch}(z) = 1 / \sinh(z). \quad (4.5.5)$$

#### 4.5.6 Hyperbolic Cotangent: $\coth(x) = 1/\tanh(z)$

WorksheetFunction.**COTH**(*x* As mpNum) As mpNum

**NOT YET IMPLEMENTED**

The function WorksheetFunction.COTH returns the value of the hyperbolic cotangent of *x*, with *x* in radians.

**Parameter:**

*x*: A real number.

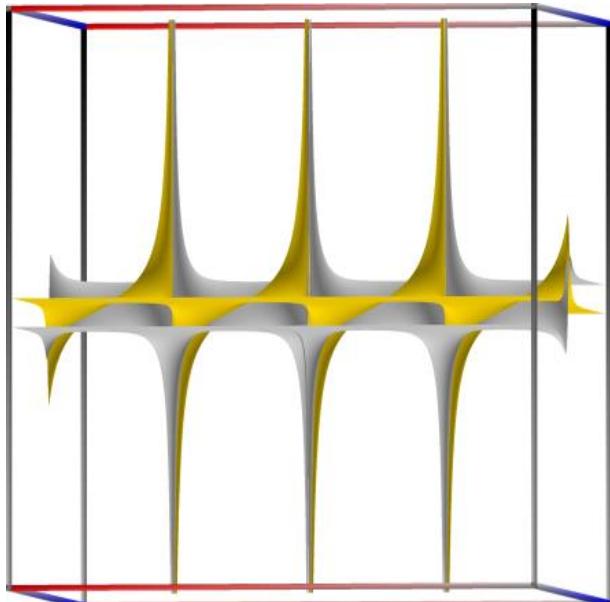
WorksheetFunction.**IMCOTH**(*z* As String) As String

**NOT YET IMPLEMENTED**

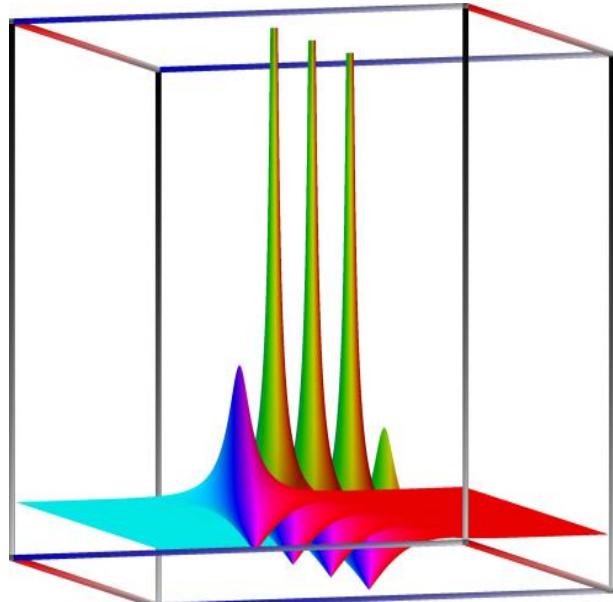
The function WorksheetFunction.IMCOTH returns the complex hyperbolic cotangent of *z*, as a String representing a complex number.

**Parameter:**

*z*: A String representing a complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.17:** Surface plots of  $z = \text{csch}(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis). *z* values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

Function **coth**(*z* As mpNum) As mpNum

The function coth returns the complex hyperbolic cotangent of *z*

**Parameter:**

$z$ : A complex number.

The function  $\coth(z)$  returns the complex hyperbolic cotangent of  $z$ :

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)} = \frac{\sinh(2x) - i \sin(2y)}{\cosh(2x) - i \cos(2y)} \quad (4.5.6)$$

## 4.6 Inverse Trigonometric Functions

The formulas in section follow [Olver et al. \(2010\)](#), equations 4.23.34 - 4.23.38 for sections [4.6.1](#) - [4.6.3](#), equation 4.23.9 for section [4.6.5](#), and [Abramowitz & Stegun. \(1970\)](#), equations 4.6.14 - 4.6.19 for sections [4.7.1](#) - [4.7.4](#).

### 4.6.1 Arcsine: $\text{asin}(z)$

---

`WorksheetFunction.ASIN(x As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.ASIN` returns the value of the arc-sine of  $x$  in radians.

**Parameter:**

$x$ : A real number.

---

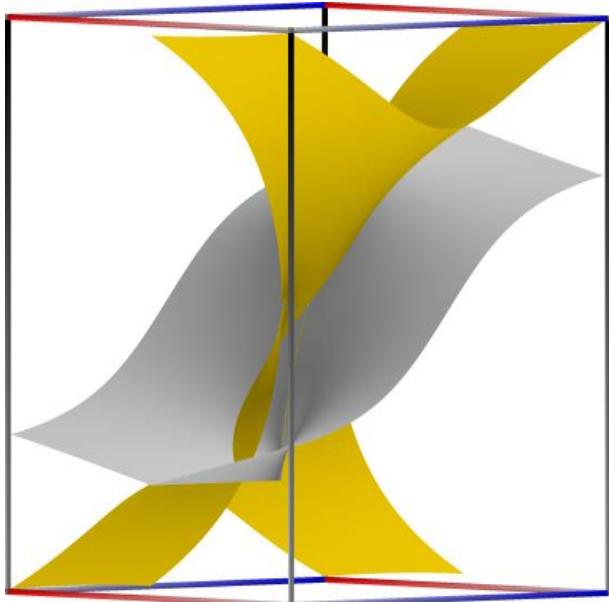
**Function  $\text{asin}(z \text{ As mpNum}) \text{ As mpNum}$**

---

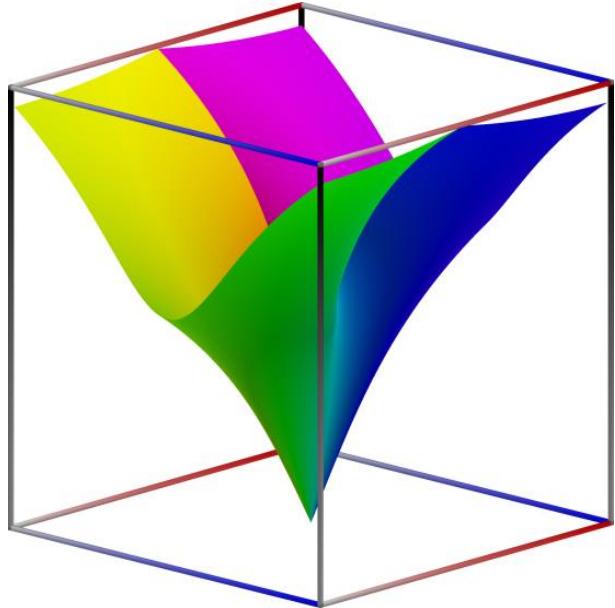
The function `asin` returns the inverse complex sine of  $z$

**Parameter:**

$z$ : A complex number.



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.18:** Surface plots of  $z = \text{asin}(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section [2.3.3](#) for more information about charts for complex functions.

The function `cplxASin(z)` returns the inverse complex sine of  $z = x + iy$ :

$$\arcsin(z) = \arcsin(\beta) + i \ln \left( \alpha + \sqrt{\alpha^2 - 1} \right), \quad \text{where} \quad (4.6.1)$$

$$\alpha = \frac{1}{2} \sqrt{(x+1)^2 + y^2} + \frac{1}{2} \sqrt{(x-1)^2 + y^2}, \quad (4.6.2)$$

$$\beta = \frac{1}{2} \sqrt{(x+1)^2 + y^2} - \frac{1}{2} \sqrt{(x-1)^2 + y^2}, \quad (4.6.3)$$

and  $x \in [-1, 1]$ .

Computes the inverse sine or arcsine of  $x$ ,  $\sin^{-1}(x)$ . Since  $-1 \leq \cos(x) \leq 1$  for real  $x$ , the inverse sine is real-valued only for  $-1 < x < 1$ . On this interval, `asin()` is defined to be a monotonically decreasing function assuming values between  $-\pi/2$  and  $\pi/2$ .

Basic values are:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> asin(-1)
-1.570796326794896619231322
>>> asin(0)
0.0
>>> asin(1)
1.570796326794896619231322
>>> nprint(chop(taylor(asin, 0, 6)))
[0.0, 1.0, 0.0, 0.166667, 0.0, 0.075, 0.0]
```

---

`asin()` is defined so as to be a proper inverse function of  $\sin(\theta)$  for  $-\pi/2 < \theta < \pi/2$ . We have  $\sin(\sin^{-1}(x) = x)$  for all  $x$ , but  $\sin^{-1}(\sin(x)) = x$  only for  $-\pi/2 \leq \Re[x] < \pi/2$ :

---

```
>>> for x in [1, 10, -1, 1+3j, -2+3j]:
... print("%s %s" % (chop(sin(asin(x))), asin(sin(x))))
...
1.0 1.0
10.0 -0.5752220392306202846120698
-1.0 -1.0
(1.0 + 3.0j) (1.0 + 3.0j)
(-2.0 + 3.0j) (-1.141592653589793238462643 - 3.0j)
```

---

The inverse sine has two branch points:  $x = \pm 1$ . `asin()` places the branch cuts along the line segments  $(-\infty, -1)$  and  $(+1, +\infty)$ . In general,

$$\sin^{-1}(x) = -i \log \left( ix + \sqrt{1 - x^2} \right) \quad (4.6.4)$$

where the principal-branch log and square root are implied.

## 4.6.2 Arcosine: $\text{acos}(z)$

`WorksheetFunction.ACOS(x As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.ACOS` returns the value of the arc-cosine of  $x$  in radians.

**Parameter:**

$x$ : A real number.

**Function  $\text{acos}(z \text{ As mpNum}) \text{ As mpNum}$**

The function `acos` returns the inverse complex cosine of  $z$

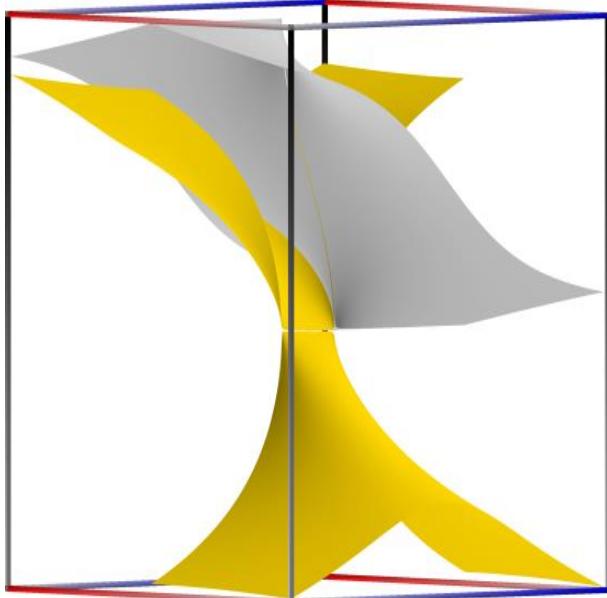
**Parameter:**

$z$ : A complex number.

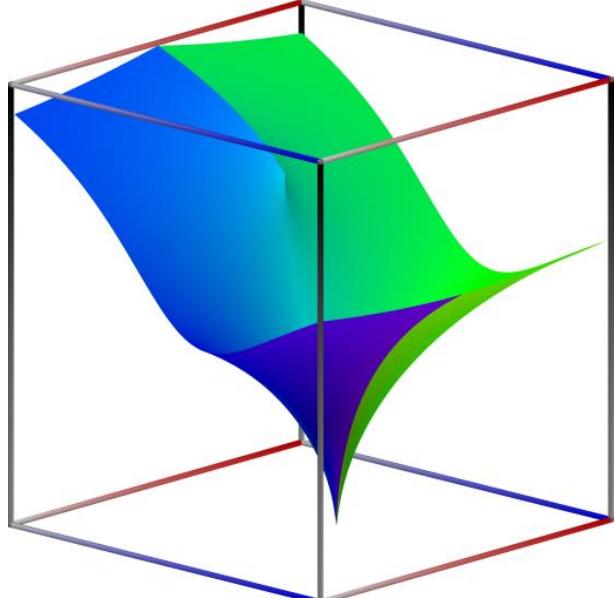
The function `cplxACos(z)` returns the inverse complex cosine of  $z = x + iy$ :

$$\arccos(z) = \arccos(\beta) - i \ln \left( \alpha + \sqrt{\alpha^2 - 1} \right), \quad \text{where} \quad (4.6.5)$$

$\alpha$  and  $\beta$  are defined in equations 4.6.2 and 4.6.3, and  $x \in [-1, 1]$ .



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.19:** Surface plots of  $z = \text{acos}(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

Computes the inverse cosine or arccosine of  $x$ ,  $\cos^{-1}(x)$ . Since  $-1 \leq \cos(x) \leq 1$  for real  $x$ , the inverse cosine is real-valued only for  $-1 < x < 1$ . On this interval,  $\text{acos}()$  is defined to be a monotonically decreasing function assuming values between  $+\pi$  and 0.

Basic values are:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> acos(-1)
3.141592653589793238462643
>>> acos(0)
1.570796326794896619231322
>>> acos(1)
0.0
>>> nprint(chop(taylor(acos, 0, 6)))
[1.5708, -1.0, 0.0, -0.166667, 0.0, -0.075, 0.0]
```

---

`acos()` is defined so as to be a proper inverse function of  $\cos(\theta)$  for  $0 < \theta < \pi$ . We have  $\cos(\cos^{-1}(x) = x)$  for all  $x$ , but  $\cos^{-1}(\cos(x)) = x$  only for  $0 \leq \Re[x] < \pi$ :

---

```
>>> for x in [1, 10, -1, 2+3j, 10+3j]:
...     print("%s %s" % (cos(acos(x)), acos(cos(x))))
...
1.0 1.0
(10.0 + 0.0j) 2.566370614359172953850574
-1.0 1.0
(2.0 + 3.0j) (2.0 + 3.0j)
(10.0 + 3.0j) (2.566370614359172953850574 - 3.0j)
```

---

The inverse cosine has two branch points:  $x = \pm 1$ . `acos()` places the branch cuts along the line segments  $(-\infty, -1)$  and  $(+1, +\infty)$ . In general,

$$\cos^{-1}(x) = \frac{\pi}{2} + i \log \left( ix + \sqrt{1 - x^2} \right) \quad (4.6.6)$$

where the principal-branch log and square root are implied.

### 4.6.3 Arctangent: $\text{atan}(z)$

`WorksheetFunction.ATAN(x As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.ATAN` returns the value of the arc-tangent of  $x$  in radians.

**Parameter:**

$x$ : A real number.

**Function  $\text{atan}(z \text{ As mpNum}) \text{ As mpNum}$**

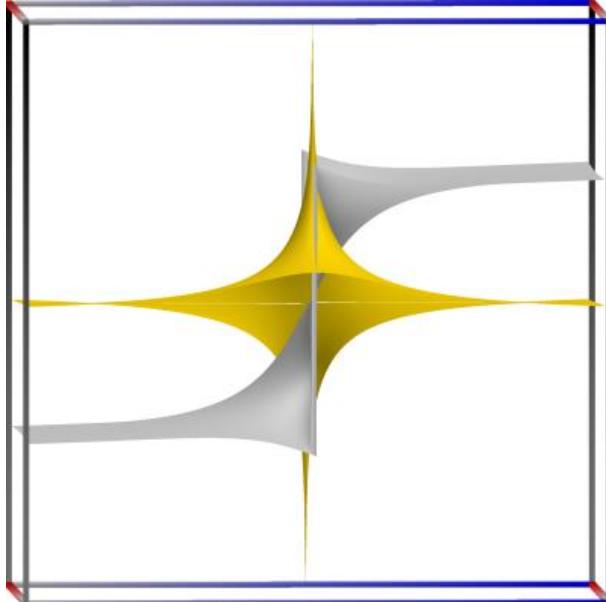
The function `atan` returns the inverse complex tangent of  $z$

**Parameter:**

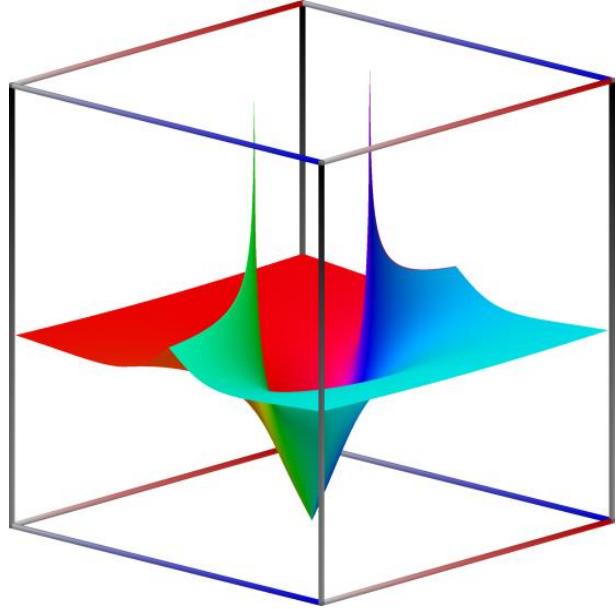
$z$ : A complex number.

The function `cplxATan(z)` returns the inverse complex tangent of  $z = x + iy$ :

$$\arctan(z) = \frac{1}{2} \arctan\left(\frac{2x}{1 - x^2 - y^2}\right) + \frac{1}{4}i \ln\left(\frac{x^2 + (y+1)^2}{x^2 + (y-1)^2}\right), \quad \text{where } |z| < 1. \quad (4.6.7)$$



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.20:** Surface plots of  $z = \text{atan}(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

Computes the inverse tangent or arctangent of  $x$ ,  $\tan^{-1}(x)$ . This is a real-valued function for all real  $x$ , with range  $(-\pi/2, \pi/2)$ .

Basic values are:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> atan(-inf)
-1.570796326794896619231322
>>> atan(-1)
-0.7853981633974483096156609
>>> atan(0)
0.0
>>> atan(1)
0.7853981633974483096156609
>>> atan(inf)
1.570796326794896619231322
>>> nprint(chop(taylor(atan, 0, 6)))
[0.0, 1.0, 0.0, -0.333333, 0.0, 0.2, 0.0]
```

The inverse tangent is often used to compute angles. However, the atan2 function is often better for this as it preserves sign (see atan2()).

$\text{atan}()$  is defined so as to be a proper inverse function of  $\tan(\theta)$  for  $-\pi/2 < \theta < \pi/2$ . We have  $\tan(\text{tan}^{-1}(x) = x)$  for all  $x$ , but  $\text{tan}^{-1}(\tan(x)) = x$  only for  $-\pi/2 \leq \Re[x] < \pi/2$ :

```
>>> mp.dps = 25
>>> for x in [1, 10, -1, 1+3j, -2+3j]:
...     print("%s %s" % (tan(atan(x)), atan(tan(x))))
...
1.0 1.0
10.0 0.5752220392306202846120698
-1.0 -1.0
(1.0 + 3.0j) (1.00000000000000000000000000000001 + 3.0j)
(-2.0 + 3.0j) (1.141592653589793238462644 + 3.0j)
```

The inverse tangent has two branch points:  $x = \pm i$ . `atan()` places the branch cuts along the line segments  $(-i\infty, -i)$  and  $(+i, +i\infty)$ . In general,

$$\tan^{-1}(x) = \frac{i}{2} (\log(1 - ix) - \log(1 + ix)) \quad (4.6.8)$$

where the principal-branch log and square root are implied.

#### 4.6.4 Arc-tangent, version with 2 arguments: atan2( $x, y$ )

---

WorksheetFunction.ATAN2( $x$  As mpNum,  $y$  As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ATAN2 returns the value of the arc-tangent of  $x$  in radians.

**Parameters:**

$x$ : A real number.

$y$ : A real number.

---

Function Atan2( $x$  As mpNum,  $y$  As mpNum) As mpNum

---

The function Atan2 returns the value of the arc-tangent of  $x$  in radians.

**Parameters:**

$x$ : A real number.

$y$ : A real number.

Computes the two-argument arctangent,  $\text{atan}2(y, x)$ , giving the signed angle between the positive  $x$ -axis and the point  $(x, y)$  in the 2D plane. This function is defined for real  $x$  and  $y$  only.

The two-argument arctangent essentially computes  $\text{atan}(y/x)$ , but accounts for the signs of both  $x$  and  $y$  to give the angle for the correct quadrant. The following examples illustrate the difference:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> atan2(1,1), atan(1/1.)
(0.785398163397448, 0.785398163397448)
>>> atan2(1,-1), atan(1/-1.)
(2.35619449019234, -0.785398163397448)
>>> atan2(-1,1), atan(-1/1.)
(-0.785398163397448, -0.785398163397448)
>>> atan2(-1,-1), atan(-1/-1.)
(-2.35619449019234, 0.785398163397448)
```

---

The angle convention is the same as that used for the complex argument; see arg().

### 4.6.5 Arccotangent: $\text{acot}(z)$

---

WorksheetFunction.ACOT(*x* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ACOT returns the value of the arc-cotangent of *x* in radians.

**Parameter:**

*x*: A real number.

---

Function **acot(z As mpNum)** As mpNum

---

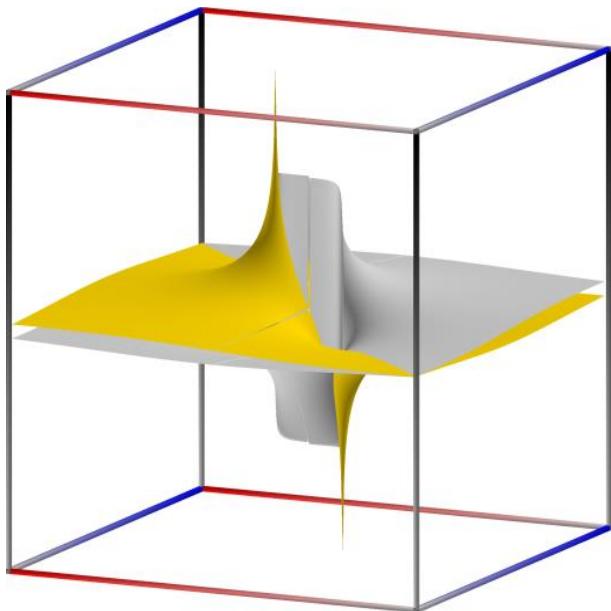
The function acot returns the inverse complex cotangent of *z*

**Parameter:**

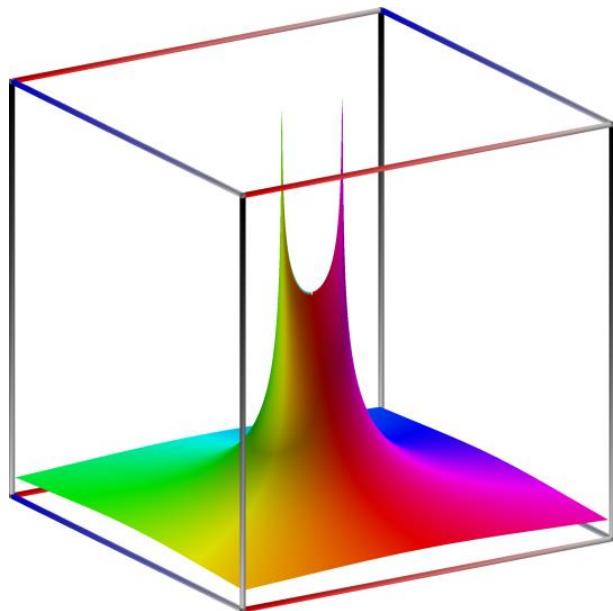
*z*: A complex number.

The function cplxACot(*z*) returns the inverse complex cotangent of *z*:

$$\text{arccot}(z) = \arctan(1/z), \quad z \neq \pm i. \quad (4.6.9)$$



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.21:** Surface plots of  $z = \text{acot}(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

### 4.6.6 asec(x)

Computes the inverse secant of *x*,  $\sec^{-1} = \cos^{-1}(1/x)$ .

**4.6.7 acsc(x)**

Computes the inverse cosecant of  $x$ ,  $\csc^{-1} = \sin^{-1}(1/x)$ .

## 4.7 Inverse Hyperbolic Functions

### 4.7.1 Inverse Hyperbolic Sine: $\text{asinh}(z)$

---

WorksheetFunction.**ASINH**(*x As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ASINH returns the value of the hyperbolic arc-sine of *x* in radians.

**Parameter:**

*x*: A real number.

---

Function **asinh(z As mpNum)** As mpNum

---

The function **asinh** returns the inverse complex hyperbolic sine of *z*

**Parameter:**

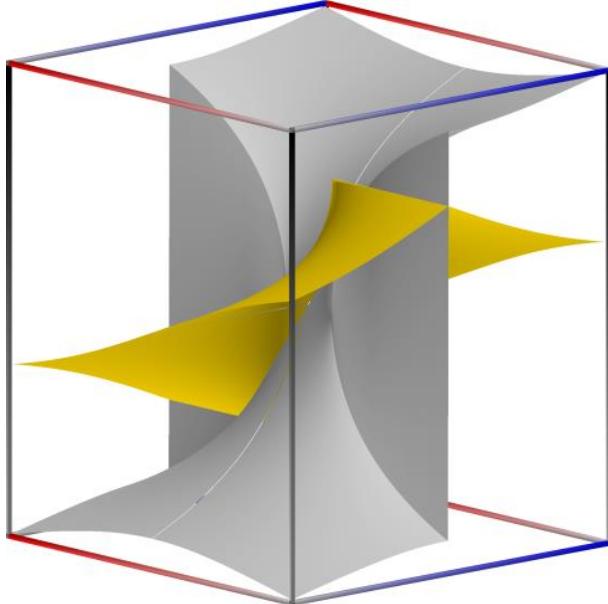
*z*: A complex number.

The function **cplxASinh(z)** returns the inverse complex hyperbolic sine of *z*:

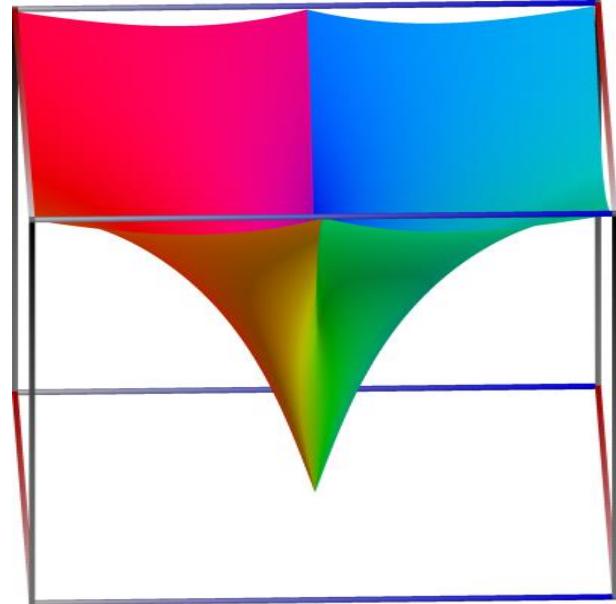
$$\text{arcsinh}(z) = -i \arcsin(iz), \quad (4.7.1)$$

where  $\arcsin(z)$  is defined in section 4.6.1

Computes the inverse hyperbolic sine of *x*,  $\sinh^{-1}(x) = \log(x + \sqrt{1 + x^2})$ .



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.22:** Surface plots of  $z = \text{asinh}(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis). *z* values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

### 4.7.2 Inverse Hyperbolic Cosine: $\text{acosh}(z)$

`WorksheetFunction.ACOSH(x As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.ACOSH` returns the value of the hyperbolic arc-cosine of  $x$  in radians.

**Parameter:**

$x$ : A real number.

**Function  $\text{acosh}(z \text{ As mpNum}) \text{ As mpNum}$**

The function `acosh` returns the inverse complex hyperbolic cosine of  $z$

**Parameter:**

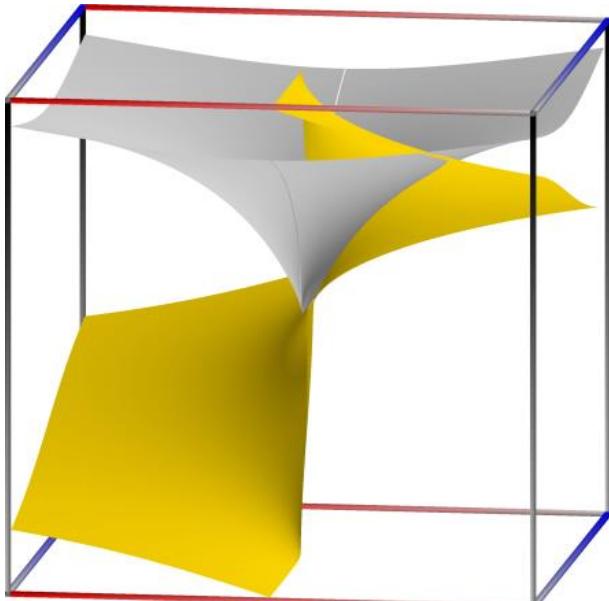
$z$ : A complex number.

The function `cplxACosh(z)` returns the inverse complex hyperbolic cosine of  $z$ :

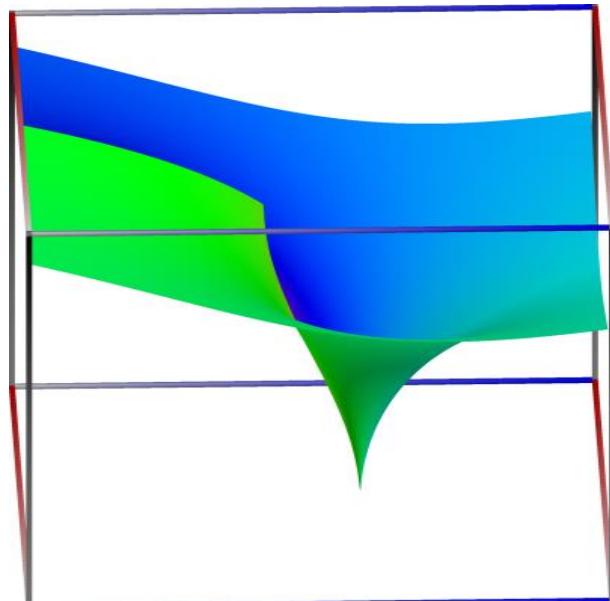
$$\text{arccosh}(z) = \pm i \arccos(z), \quad (4.7.2)$$

where  $\arccos(z)$  is defined in section 4.6.2

Computes the inverse hyperbolic cosine of  $x$ ,  $\cosh^{-1}(x) = \log(x + \sqrt{x+1}\sqrt{x-1})$ .



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.23:** Surface plots of  $z = \text{acosh}(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

### 4.7.3 Inverse Hyperbolic Tangent: $\operatorname{atanh}(z)$

WorksheetFunction.ATANH( $x$  As mpNum) As mpNum

**NOT YET IMPLEMENTED**

The function WorksheetFunction.ATANH returns the value of the hyperbolic arc-tangent of  $x$  in radians.

**Parameter:**

$x$ : A real number.

Function **atanh(z As mpNum)** As mpNum

The function **atanh** returns the inverse complex hyperbolic tangent of  $z$

**Parameter:**

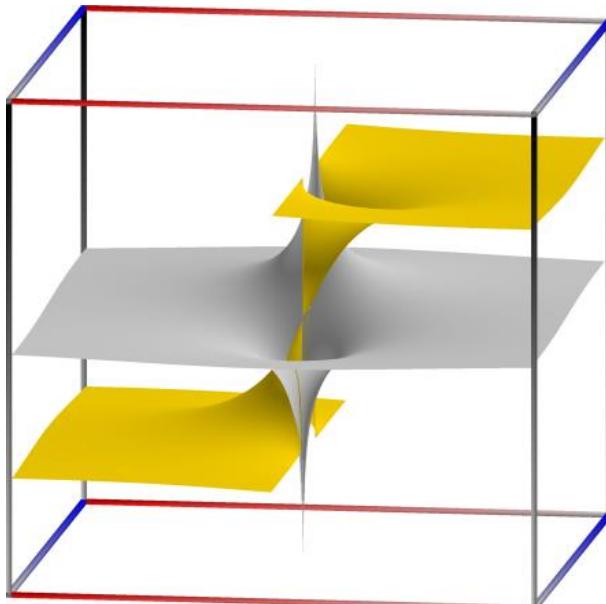
$z$ : A complex number.

The function **cplxATanh(z)** returns the inverse complex hyperbolic tangent of  $z$ :

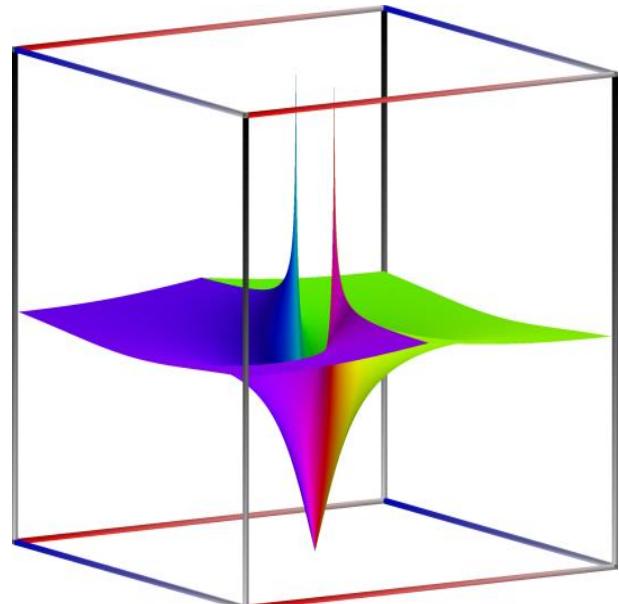
$$\operatorname{arctanh}(z) = -i \operatorname{arctan}(z), \quad (4.7.3)$$

where  $\operatorname{arctan}(z)$  is defined in section 4.6.3

Computes the inverse hyperbolic tangent of  $x$ ,  $\tanh^{-1}(x) = \frac{1}{2}(\log(1+x) - \log(1-x))$ .



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.24:** Surface plots of  $z = \operatorname{atanh}(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

#### 4.7.4 Inverse Hyperbolic Cotangent: $\text{acoth}(z)$

WorksheetFunction.ACOTH( $x$  As mpNum) As mpNum

NOT YET IMPLEMENTED

The function WorksheetFunction.ACOTH returns the value of the hyperbolic arc-cotangent of  $x$  in radians.

**Parameter:**

$x$ : A real number.

Function **acoth(z As mpNum) As mpNum**

The function  $\text{acoth}$  returns the inverse complex hyperbolic cotangent of  $z$

**Parameter:**

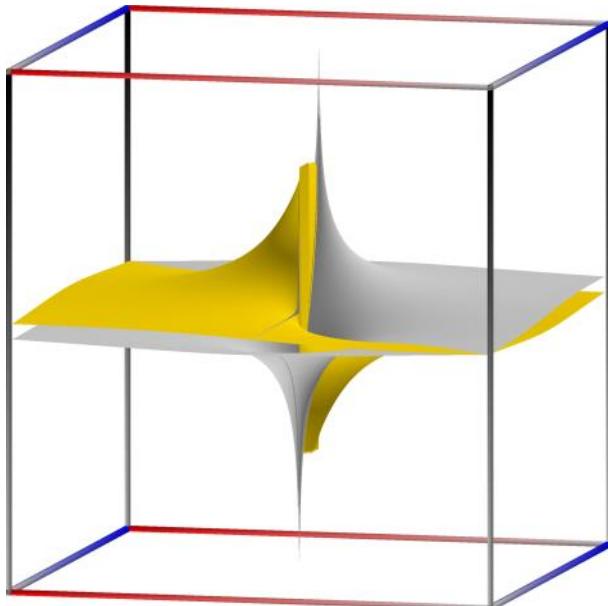
$z$ : A complex number.

The function  $\text{cplxACoth}(z)$  returns the inverse complex hyperbolic cotangent of  $z$ :

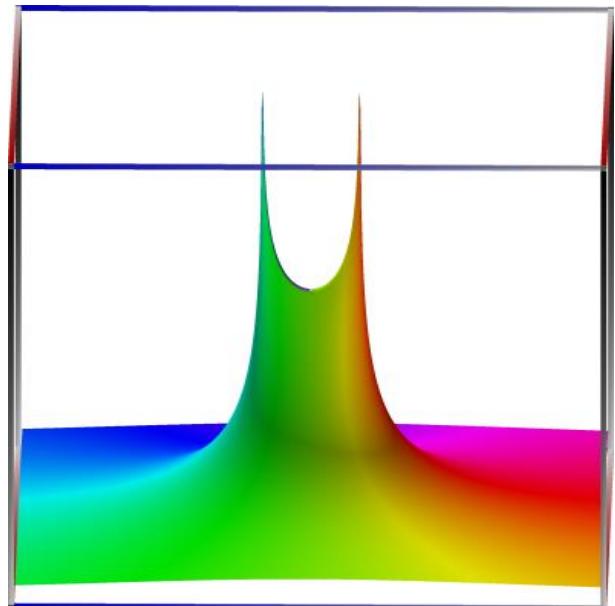
$$\text{arctanh}(z) = i \arctan(iz), \quad (4.7.4)$$

where  $\arctan(z)$  is defined in section 4.6.5

Computes the inverse hyperbolic cotangent of  $x$ ,  $\coth^{-1}(x) = \tanh^{-1}(1/x)$



(a) Real ("silver") and imaginary ("gold") component,  $z_{\min} = -10$ . Camera angles are  $\theta = 135^\circ$  and  $\phi = -12^\circ$ .



(b) Magnitude and phase (color-coded),  $z_{\min} = 0$ . Camera angles are  $\theta = 35^\circ$  and  $\phi = -112^\circ$ .

**Figure 4.25:** Surface plots of  $z = \text{acoth}(x + iy)$ ,  $-3 \leq x \leq 3$  (blue axis),  $-2\pi \leq y \leq 2\pi$  (red axis),  $z_{\min} \leq z \leq 10$  (black axis).  $z$  values are truncated at  $\pm 10$ . There is a branch cut along the negative real axis. Orthographic camera. See section 2.3.3 for more information about charts for complex functions.

**4.7.5 asech(x)**

Computes the inverse hyperbolic secant of  $x$ ,  $\text{sech}^{-1}(x) = \cosh^{-1}(1/x)$

**4.7.6 acsch(x)**

Computes the inverse hyperbolic cosecant of  $x$ ,  $\text{csch}^{-1}(x) = \sinh^{-1}(1/x)$

## 4.8 Elementary Functions of Mathematical Physics

### 4.8.1 Bessel Function $J_\nu(x)$

---

WorksheetFunction.**BESSELJ**(*x As mpNum, ν As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.BESSELJ returns  $J_\nu(z)$ , the Bessel function of the first kind of real order  $\nu$ .

**Parameters:**

*x*: A real number.

*ν*: A real number.

$J_\nu(z)$ , the Bessel function of the first kind of order  $\nu$ , is defined as

$$J_\nu(x) = \left(\frac{1}{2}x\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{(x^2/4)^k}{k!\Gamma(\nu+k+1)} \quad (4.8.1)$$

### 4.8.2 Bessel Function $Y_\nu(x)$

---

WorksheetFunction.**BESSELY**(*x As mpNum, ν As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.BESSELY returns  $Y_\nu(z)$ , the Bessel function of the second kind of order  $\nu$ .

**Parameters:**

*x*: A real number.

*ν*: A real number.

$Y_\nu(z)$ , the Bessel function of the second kind of order  $\nu$ , is defined as

$$Y_\nu(x) = \frac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \quad (4.8.2)$$

### 4.8.3 Bessel Function $I_\nu(x)$

---

WorksheetFunction.**BESSELI**(*x As mpNum, ν As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.BESSELI returns  $J_\nu(z)$ , the Bessel function of the first kind of real order  $\nu$ .

**Parameters:**

*x*: A real number.

*ν*: A real number.

This function returns the modified Bessel function  $I_\nu(z)$  of the first kind of order  $\nu$ , defined as

$$I_\nu(z) = \frac{z}{2} \sum_{j=0}^{\infty} \frac{(z^2/4)^j}{j!\Gamma(\nu+j+1)} \quad (4.8.3)$$

#### 4.8.4 Bessel Function $K_\nu(x)$

---

WorksheetFunction.**BESSELK**(*x As mpNum, ν As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.BESSELK returns  $K_\nu(x)$ , the modified Bessel function of the second kind of order  $\nu$ .

**Parameters:**

*x*: A real number.

*ν*: A real number.

This function returns  $K_\nu(z)$ , the modified Bessel function of the second kind of order  $\nu$ , defined as

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x)) - I_\nu(x)}{\sin(\nu\pi)} \quad (4.8.4)$$

---

WorksheetFunction.**ERF**(*x As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ERF returns the value of the error function.

**Parameter:**

*x*: A real number.

---

WorksheetFunction.**ERF.PRECISE**(*x As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ERF.PRECISE returns the value of the error function.

**Parameter:**

*x*: A real number.

The error function is defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad (4.8.5)$$

#### 4.8.5 Complementary Error Function

---

WorksheetFunction.**ERFC**(*x As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ERFC returns the value of the complementary error function.

**Parameter:**

*x*: A real number.

---

`WorksheetFunction.ERFC.PRECISE(x As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.ERFC.PRECISE` returns the value of the complementary error function.

**Parameter:**

*x*: A real number.

The complementary error function is defined by

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt, \quad (4.8.6)$$

#### 4.8.6 Gamma function $\Gamma(x)$

---

`WorksheetFunction.GAMMA(x As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.GAMMA` returns the gamma function for  $x \neq 0, -1, -2, \dots$ .

**Parameter:**

*x*: A real number.

The gamma function for  $x \neq 0, -1, -2, \dots$  is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad (x > 0), \quad (4.8.7)$$

and by analytic continuation if  $x < 0$ , using the reflection formula

$$\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x). \quad (4.8.8)$$

---

`WorksheetFunction.GAMMALN(x As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.GAMMALN` returns the logarithm of the gamma function.

**Parameter:**

*x*: A real number.

---

`WorksheetFunction.GAMMALN.PRECISE(x As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.GAMMALN.PRECISE` returns the logarithm of the gamma function.

**Parameter:**

*x*: A real number.

### 4.8.7 Beta Function B(a,b)

---

Function **Beta**(*a As mpNum, b As mpNum*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function Beta returns the Beta function.

**Parameters:**

*a*: A real number.

*b*: A real number.

This function computes  $B(a, b)$  for  $a, b \neq 0, -1, -2, \dots$

## 4.9 Factorial and Related Functions

### 4.9.1 Factorial

---

`WorksheetFunction.FACT(n As Integer) As Integer`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.FACT` returns  $n!$ , the factorial of  $n$

**Parameter:**

*n*: An Integer.

### 4.9.2 Double Factorial

---

`WorksheetFunction.FACTDOUBLE(n As Integer) As Integer`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.FACTDOUBLE` returns  $n!!$ , the double factorial of  $n$

**Parameter:**

*n*: An Integer.

### 4.9.3 Binomial Coefficient, Combinations

---

`WorksheetFunction.COMBIN(n As Integer, k As Integer) As Integer`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.COMBIN` returns the binomial coefficient

**Parameters:**

*n*: An Integer.

*k*: An Integer.

Returns the binomial coefficient,  $\binom{n}{k}$ . Negative values of  $n$  are supported, using the identity

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}. \quad (4.9.1)$$

---

`WorksheetFunction.COMBINA(n As Integer, k As Integer) As Integer`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.COMBINA` returns the binomial coefficient

**Parameters:**

*n*: An Integer.

*k*: An Integer.

#### 4.9.4 Multinomial

---

**WorksheetFunction.MULTINOMIAL(*a*[]) As mpReal**) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.MULTINOMIAL** returns the multinomial

**Parameter:**

*a*[]: An array of integers.

Returns the multinomial, defined as the ratio of the factorial of a sum of values to the product of factorials:

$$\text{MULTINOMIAL}(a_1, a_2, \dots, a_n) = \frac{(a_1 + a_2 + \dots + a_n)!}{a_1! a_2! \dots a_n!} \quad (4.9.2)$$

#### 4.9.5 Permutations

---

**WorksheetFunction.PERMUT(*n* As Integer, *k* As Integer) As Integer**

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.PERMUT** returns the number of permutations for a given number *k* of objects that can be selected from *n* objects.

**Parameters:**

*n*: An Integer.

*k*: An Integer.

**WorksheetFunction.PERMUTATIONA(*n* As Integer, *k* As Integer) As Integer**

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.PERMUTATIONA** returns the number of permutations for a given number *k* of objects that can be selected from *n* objects.

**Parameters:**

*n*: An Integer.

*k*: An Integer.

Returns the number of permutations for a given number *k* of objects that can be selected from *n* objects. A permutation is any set or subset of objects or events where internal order is significant.

$$\text{PERMUT}(n, k) = \frac{n!}{(n - k)!} \quad (4.9.3)$$

#### 4.9.6 Greatest Common Divisor (GCD)

---

**WorksheetFunction.GCD(*n1* As Integer, *n2* As Integer) As Integer**

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.GCD** returns the greatest common divisor of *n*<sub>1</sub> and *n*<sub>2</sub>

**Parameters:**

*n1*: An Integer.

*n2*: An Integer.

The result is always positive even if one or both input operands are negative. Except if both inputs are zero; then this function defines  $\text{intGcd}(0, 0) = 0$ .

#### 4.9.7 Least Common Multiple (LCM)

---

WorksheetFunction.**LCM**(*n1* As Integer, *n2* As Integer) As Integer

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.LCM returns the least common multiple of  $n_1$  and  $n_2$ .

**Parameters:**

*n1*: An Integer.

*n2*: An Integer.

Returns the least common multiple of  $n_1$  and  $n_2$ . The returned value is always positive, irrespective of the signs of  $n_1$  and  $n_2$ . The returned value will be zero if either  $n_1$  or  $n_2$  is zero.

# Chapter 5

## Linear Algebra

### 5.1 Multiple Linear Regression

#### 5.1.1 Determinant

---

WorksheetFunction.**MDETERM**(*X As mpNum[]*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.MDETERM returns the matrix determinant of a numeric array *X* with an equal number of rows and columns.

**Parameter:**

*X*: A matrix of real numbers.

#### 5.1.2 Inverse

---

WorksheetFunction.**MINVERSE**(*X As mpNum[]*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.MINVERSE returns the inverse matrix for the matrix stored in the numeric array *X* with an equal number of rows and columns.

**Parameter:**

*X*: A matrix of real numbers.

Negative powers will calculate the inverse:

---

```
>>> A**-1
matrix(
[[ '-2.0', '1.0'],
 ['1.5', '-0.5']])
>>> nprint(A * A**-1, 3)
[ 1.0 1.08e-19]
[-2.17e-19 1.0]
```

---

### 5.1.3 LinEst

---

`WorksheetFunction.LINEST(Y As mpNum[], X As mpNum[], Const As Boolean, Stats As Boolean) As mpNumList`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.LINEST` returns information obtained by performing multiple liner regression.

**Parameters:**

*Y*: An array of real numbers.

*X*: An array of real numbers.

*Const*: A logical value.

*Stats*: A logical value.

The `LINEST` function calculates the statistics for a line by using the "least squares" method to calculate a straight line that best fits your data, and then returns an array that describes the line.

You can also combine `LINEST` with other functions to calculate the statistics for other types of models that are linear in the unknown parameters, including polynomial, logarithmic, exponential, and power series. Because this function returns an array of values, it must be entered as an array formula. Instructions follow the examples in this article.

This function can also be used to perform multiple linear regression.

This function needs a detailed explanation.

### 5.1.4 TREND

---

`WorksheetFunction.TREND(Y As mpNum[], X As mpNum[], NewX As mpNum[], Const As Boolean) As mpNumList`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.TREND` returns values along a linear trend.

**Parameters:**

*Y*: An array of real numbers.

*X*: An array of real numbers.

*NewX*: An array of real numbers.

*Const*: A logical value.

Returns values along a linear trend. Fits a straight line (using the method of least squares) to the arrays *Y* and *X*. Returns the y-values along that line for the array of *NewX* that you specify. For information about how Microsoft Excel fits a line to data, see `LINEST`. You can use `TREND` for polynomial curve fitting by regressing against the same variable raised to different powers. For example, suppose column A contains y-values and column B contains x-values. You can enter  $x^2$  in column C,  $x^3$  in column D, and so on, and then regress columns B through D against column A.

Formulas that return arrays must be entered as array formulas.

## 5.2 Exponential Growth Curves

### 5.2.1 LogEst

---

WorksheetFunction.**LOGEST**(*Y* As mpNum[], *X* As mpNum[], *Const* As Boolean, *Stats* As Boolean) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.LOGEST returns an exponential curve that fits your data and returns an array of values that describes the curve.

**Parameters:**

*Y*: An array of real numbers.

*X*: An array of real numbers.

*Const*: A logical value.

*Stats*: A logical value.

In regression analysis, calculates an exponential curve that fits your data and returns an array of values that describes the curve. Because this function returns an array of values, it must be entered as an array formula.

This function does not perform a non-linear estimation.

### 5.2.2 Growth

---

WorksheetFunction.**GROWTH**(*Y* As mpNum[], *X* As mpNum[], *NewX* As mpNum[], *Const* As Boolean) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.GROWTH returns predicted exponential growth by using existing data.

**Parameters:**

*Y*: An array of real numbers.

*X*: An array of real numbers.

*NewX*: An array of real numbers.

*Const*: A logical value.

GROWTH returns the *y*-values for a series of new *x*-values that you specify by using existing *x*-values and *y*-values. You can also use the GROWTH worksheet function to fit an exponential curve to existing *x*-values and *y*-values.

## 5.3 Norms

Sometimes you need to know how 'large' a matrix or vector is. Due to their multidimensional nature it is not possible to compare them, but there are several functions to map a matrix or a vector to a positive real number, the so called norms.

---

### Function **norm**(*Y As mpNum[]*, *Keywords As String*) As mpNumList

---

The function **norm** returns the entrywise  $p$ -norm of an iterable  $x$ , i.e. the vector norm.

**Parameters:**

*Y*: An array of real numbers.

*Keywords*:  $p=2$ .

`norm(ctx, x, p=2)` Gives the entrywise  $p$ -norm of an iterable  $x$ , i.e. the vector norm

$$\left( \sum_k |x_k|^p \right)^{1/p}, \quad (5.3.1)$$

for any given  $1 \leq p \leq \infty$ .

Special cases:

If  $x$  is not iterable, this just returns `absmax(x)`.

$p=1$  gives the sum of absolute values.

$p=2$  is the standard Euclidean vector norm.

$p=\infty$  gives the magnitude of the largest element.

For  $x$  a matrix,  $p=2$  is the Frobenius norm. For operator matrix norms, use `mnorm()` instead.

You can use the string ' $\infty$ ' as well as `float('inf')` or `mpf('inf')` to specify the infinity norm.

**Examples**

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> x = matrix([-10, 2, 100])
>>> norm(x, 1)
mpf('112.0')
>>> norm(x, 2)
mpf('100.5186549850325')
>>> norm(x, inf)
mpf('100.0')
```

---



---

### Function **mnorm**(*A As mpNum[]*, *Keywords As String*) As mpNumList

---

The function **mnorm** returns the matrix (operator)  $p$ -norm of  $A$ . Currently  $p=1$  and  $p=\infty$  are supported.

**Parameters:**

*A*: An array of real numbers.

*Keywords*:  $p=1$ .

`mnorm(ctx, A, p=1)`

Gives the matrix (operator)  $p$ -norm of  $A$ . Currently  $p=1$  and  $p=\infty$  are supported:

$p=1$  gives the 1-norm (maximal column sum)

`p=inf` gives the  $\infty$ -norm (maximal row sum). You can use the string '`inf`' as well as `float('inf')` or `mpf('inf')`

`p=2` (not implemented) for a square matrix is the usual spectral matrix norm, i.e. the largest singular value.

`p='F'` (or '`F`', '`fro`', '`Frobenius`', '`frobenius`') gives the Frobenius norm, which is the elementwise 2-norm. The Frobenius norm is an approximation of the spectral norm and satisfies

$$\frac{1}{\sqrt{\text{rank}(A)}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F. \quad (5.3.2)$$

The Frobenius norm lacks some mathematical properties that might be expected of a norm. For general elementwise  $p$ -norms, use `norm()` instead.

### Examples

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = False
>>> A = matrix([[1, -1000], [100, 50]])
>>> mnorm(A, 1)
mpf('1050.0')
>>> mnorm(A, inf)
mpf('1001.0')
>>> mnorm(A, 'F')
mpf('1006.2310867787777')
```

---

## 5.4 Decompositions

Function **cholesky(A As mpNum[], Keywords As String)** As mpNum

The function **cholesky** returns the Cholesky decomposition of a symmetric positive-definite matrix  $A$ .

### Parameters:

**A:** A symmetric matrix.

**Keywords:** tol=None.

**cholesky(ctx, A, tol=None)**

Cholesky decomposition of a symmetric positive-definite matrix  $A$ . Returns a lower triangular matrix  $L$  such that  $A = L \times L^T$ . More generally, for a complex Hermitian positive-definite matrix, a Cholesky decomposition satisfying  $A = L \times L^H$  is returned.

The Cholesky decomposition can be used to solve linear equation systems twice as efficiently as LU decomposition, or to test whether  $A$  is positive-definite.

The optional parameter tol determines the tolerance for verifying positive-definiteness.

### Examples

Cholesky decomposition of a positive-definite symmetric matrix:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> A = eye(3) + hilbert(3)
>>> nprint(A)
[ 2.0 0.5 0.333333]
[ 0.5 1.33333 0.25]
[0.333333 0.25 1.2]
>>> L = cholesky(A)
>>> nprint(L)
[ 1.41421 0.0 0.0]
[0.353553 1.09924 0.0]
[0.235702 0.15162 1.05899]
>>> chop(A - L*L.T)
[0.0 0.0 0.0]
[0.0 0.0 0.0]
[0.0 0.0 0.0]
```

Cholesky decomposition of a Hermitian matrix:

```
>>> A = eye(3) + matrix([[0,0.25j,-0.5j],[-0.25j,0,0],[0.5j,0,0]])
>>> L = cholesky(A)
>>> nprint(L)
[ 1.0 0.0 0.0]
[(0.0 - 0.25j) (0.968246 + 0.0j) 0.0]
[ (0.0 + 0.5j) (0.129099 + 0.0j) (0.856349 + 0.0j)]
>>> chop(A - L*L.H)
[0.0 0.0 0.0]
[0.0 0.0 0.0]
[0.0 0.0 0.0]
```

Attempted Cholesky decomposition of a matrix that is not positive definite:

```
>>> A = -eye(3) + hilbert(3)
>>> L = cholesky(A)
Traceback (most recent call last):
...
ValueError: matrix is not positive-definite
```

## 5.5 Linear Equations

---

Function **lu\_solve(A As mpNum[], b As mpNum[], Keywords As String)** As mpNum

---

The function lu\_solve returns solves a linear equation system using a LU decomposition.

**Parameters:**

*A*: A symmetric matrix.

*b*: A symmetric matrix.

*Keywords*: tol=None.

You can for example solve the linear equation system using a LU decomposition:

```
x + 2*y = -10
3*x + 4*y = 10
```

---

using lu\_solve:

```
>>> A = matrix([[1, 2], [3, 4]])
>>> b = matrix([-10, 10])
>>> x = lu_solve(A, b)
>>> x
matrix(
[['30.0'],
 ['-20.0']])
```

---

Function **residual(A As mpNum[], b As mpNum[], x As mpNum[], Keywords As String)** As mpNum

---

The function residual returns the residual  $\|Ax - b\|$ .

**Parameters:**

*A*: A square matrix.

*b*: A vector.

*x*: A vector.

*Keywords*: tol=None.

Calculates the residual  $\|Ax - b\|$ :

```
>>> residual(A, x, b)
matrix(
[['3.46944695195361e-18'],
 ['3.46944695195361e-18']])
>>> str(eps)
'2.22044604925031e-16'
```

---

As you can see, the solution is quite accurate. The error is caused by the inaccuracy of the internal floating point arithmetic. Though, it is even smaller than the current machine epsilon, which basically means you can trust the result.

If you need more speed, use NumPy, or use fp instead mp matrices and methods:

```
>>> A = fp.matrix([[1, 2], [3, 4]])
>>> b = fp.matrix([-10, 10])
>>> fp.lu_solve(A, b)
matrix()
```

```
[['30.0'],
 ['-20.0']])
```

---

lu\_solve accepts overdetermined systems. It is usually not possible to solve such systems, so the residual is minimized instead. Internally this is done using Cholesky decomposition to compute a least squares approximation. This means that lu\_solve will square the errors. If you cannot afford this, use qr\_solve instead. It is twice as slow but more accurate, and it calculates the residual automatically.

## 5.6 Matrix Factorization

Function **lu(A As mpNum[], Keywords As String)** As mpNum

The function lu returns an explicit LU factorization of a matrix, returning P, L, U

**Parameters:**

A: A square matrix.

*Keywords:* tol=None.

The function lu computes an explicit LU factorization of a matrix:

```
>>> P, L, U = lu(matrix([[0,2,3],[4,5,6],[7,8,9]]))
>>> print P
[0.0 0.0 1.0]
[1.0 0.0 0.0]
[0.0 1.0 0.0]
>>> print L
[ 1.0 0.0 0.0]
[ 0.0 1.0 0.0]
[0.571428571428571 0.214285714285714 1.0]
>>> print U
[7.0 8.0 9.0]
[0.0 2.0 3.0]
[0.0 0.0 0.214285714285714]
>>> print P.T*L*U
[0.0 2.0 3.0]
[4.0 5.0 6.0]
[7.0 8.0 9.0]
```

Function **qr(A As mpNum[], Keywords As String)** As mpNum

The function qr returns an explicit QR factorization of a matrix, returning Q, R

**Parameters:**

A: A square matrix.

*Keywords:* tol=None.

Examples:

```
>>> A = matrix([[1, 2], [3, 4], [1, 1]])
>>> Q, R = qr(A)
>>> print Q
[-0.301511344577764 0.861640436855329 0.408248290463863]
[-0.904534033733291 -0.123091490979333 -0.408248290463863]
[-0.301511344577764 -0.492365963917331 0.816496580927726]
>>> print R
[-3.3166247903554 -4.52267016866645]
[ 0.0 0.738548945875996]
[ 0.0 0.0]
>>> print Q * R
[1.0 2.0]
[3.0 4.0]
[1.0 1.0]
```

```
>>> print chop(Q.T * Q)
[1.0 0.0 0.0]
[0.0 1.0 0.0]
[0.0 0.0 1.0]
```

---

# Chapter 6

## Distribution Functions

### 6.1 Introduction to Distribution Functions

The following references have been used in this chapter:

This is a citation [Walck \(2007\)](#), and some more.

This is a citation [Van Hauwermeiren & Vose \(2009\)](#), and some more.

This is a citation [Rinne \(2008\)](#), and some more.

This is a citation [Johnson \*et al.\* \(1994.\)](#), and some more.

This is a citation [Johnson \*et al.\* \(1995.\)](#), and some more.

See also [Monahan \(2011\)](#)

See also [Lange \(2010\)](#)

See also [Chernick \(2008\)](#)

See also [Cheney & Kincaid \(2008\)](#)

#### 6.1.1 Continuous Distribution Functions

Continuous random number distributions are defined by a probability density function,  $p(x)$ , such that the probability of  $x$  occurring in the infinitesimal range  $x$  to  $x + dx$  is  $p dx$ . The cumulative distribution function for the lower tail  $P(x)$  gives the probability of a variate taking a value less than  $x$ , and the cumulative distribution function for the upper tail  $Q(x)$  gives the probability of a variate taking a value greater than  $x$ .

The upper and lower cumulative distribution functions are related by  $P(x) + Q(x) = 1$  and satisfy  $0 \leq P(x) \leq 1, 0 \leq Q(x) \leq 1$ . The inverse cumulative distributions,  $x = P^{-1}(P)$  and  $x = Q^{-1}(Q)$  give the values of  $x$  which correspond to a specific value of  $P$  or  $Q$ . They can be used to find confidence limits from probability values.

#### 6.1.2 Discrete Distribution Functions

For discrete distributions the probability of sampling the integer value  $k$  is given by  $p(k)$ . The cumulative distribution for the lower tail  $P(k)$  of a discrete distribution is defined as the sum over the allowed range of the distribution less than or equal to  $k$ . The cumulative distribution for the upper tail of a discrete distribution  $Q(k)$  is defined as the sum of probabilities for all values greater than  $k$ . These two definitions satisfy the identity  $P(k) + Q(k) = 1$ . If the range of the distribution is 1 to  $n$  inclusive then  $P(n) = 1$ ,  $Q(n) = 0$  while  $P(1) = p(1)$ ,  $Q(1) = 1 - p(1)$ .

### 6.1.3 Commonly Used Function Types

#### 6.1.3.1 Functions returning pdf, CDF, and related information

These functions have the form `?Dist(x; [Parameters], OutputString)`. Here  
 “?” is a placeholder for the name of the distribution,  
 “*x*” is the value for which we want to calculate the pdf, CDF etc,  
 “[Parameters;]” denote any parameters (like degrees of freedom) of the distribution, and  
 “OutputString” specifies the computed results which will be returned. This can be any of the following:

- **pdf**: the probability density function
- **P**: the cumulative distribution function (CDF)
- **Q**: the complement of cumulative distribution function (CDF)
- **logpdf**: the logarithm of the probability density function
- **logP**: the logarithm of the cumulative distribution function (CDF)
- **logQ**: the logarithm of the complement of cumulative distribution function (CDF)
- **h**: hazard function
- **H**: cumulative hazard function

As an example, for Student’s t-distribution, a “T” is used to specify the name of the distribution, and there is just one distribution parameter,  $\nu$ , the degrees of freedom. Therefore, the function has the form

`TDist(x As nmNum;  $\nu$  As mpNum, OutputString As String) As mpNumList`,

and an actual call to the function, requesting the pdf, CDF, and the complement of the CDF for  $x = 2.3$  and  $\nu = 22$  could be

---

```
Result = TDist(2.3, 22, "pdf + P + Q")
mp.Print Result
```

---

which produces the output

```
pdf: 0.434234342343434
P: 0.943453463453453
Q: 0.054564564564236
```

### 6.1.3.2 Functions returning Quantiles

These functions have the form `?DistInv(Prob; [Parameters;], OutputString)`. Here  
 “?” is a placeholder for the name of the distribution,  
 “Prob” sets the target values for  $P$  and  $Q$ ,  
 “[Parameters;]” denote any parameters (like degrees of freedom) of the distribution, and  
 “OutputString” specifies the computed results which will be returned. This can be any of the following:

- **PInv:** the inverse of the cumulative distribution function (CDF). For discrete distribution, this will be outwardly rounded
- **QInv:** the inverse of the complement of the cumulative distribution function (CDF). For discrete distribution, this will be outwardly rounded
- **P:** the value of the cumulative distribution function (CDF), which has actually been achieved
- **Q:** the value of the complement of the cumulative distribution function (CDF), which has actually been achieved

As an example, for Student’s t-distribution, a “T” is used to specify the name of the distribution, and there is just one distribution parameter,  $\nu$ , the degrees of freedom. Therefore, the function has the form

`TDistInv(Prob As mpNum;  $\nu$  As mpNum, OutputString As String) As mpNumList`,

and an actual call to the function, requesting the inverse of the complement of the CDF for  $Prob = 0.01$  and  $\nu = 22$  could be

---

```
Result = TDistInv(0,01, 22, "QInv")
mp.Print Result
```

---

which produces the output

`QInv: 2.943453463453453`

### 6.1.3.3 Functions returning moments and related information

These functions have the form `?DistInfo([Parameters;], OutputString)`. Here “?” is a placeholder for the name of the distribution, “[Parameters;]” denote any parameters (like degrees of freedom) of the distribution, and “OutputString” specifies the computed results which will be returned. This can be any of the following:

- **range**: Returns the valid range of the random variable over distribution dist.
- **support**:
- **mode**: Returns the mode of the distribution dist. This function may return a domain\_error if the distribution does not have a defined mode.
- **median**: Returns the median of the distribution dist.
- **mean**: Returns the mean of the distribution dist. This function may return a domain\_error if the distribution does not have a defined mean (for example the Cauchy distribution).
- **stdev**: Returns the standard deviation of distribution dist. This function may return a domain\_error if the distribution does not have a defined standard deviation.
- **variance**: Returns the variance of the distribution dist. This function may return a domain\_error if the distribution does not have a defined variance.
- **skewness**: Returns the skewness of the distribution dist. This function may return a domain\_error if the distribution does not have a defined skewness.
- **kurtosis**: Returns the ‘proper’ kurtosis (normalized fourth moment) of the distribution dist.
- **kurtosis excess**: Returns the kurtosis excess of the distribution dist. kurtosis excess = kurtosis - 3

As an example, for Student’s t-distribution, a “T” is used to specify the name of the distribution, and there is just one distribution parameter,  $\nu$ , the degrees of freedom. Therefore, the function has the form

`TDistInfo( $\nu$  As mpNum, OutputString As String) As mpNumList,`

and an actual call to the function, requesting the mean, variance, skewness and kurtosis with  $\nu = 22$  could be

---

```
Result = TDistInfo(22, "mean + variance + skewness + kurtosis")
mp.Print Result
```

---

which produces the output

```
mean: 0.434234342343434
variance: 0.943453463453453
skewness: 0.054564564564236
kurtosis: 0.6054564564564236
```

#### 6.1.3.4 Functions returning Sample Size estimates

These functions have the form `?SampleSize(Alpha; Beta; ModifiedNoncentrality; [Parameters;], OutputString)`. Here

`"?"` is a placeholder for the name of the distribution,

`"Alpha"` specifies the confidence level (or Type I error),

`"Beta"` specifies the Type I error (or  $1 - \text{Power}$ ),

`"ModifiedNoncentrality"` specifies the (modified) noncentrality parameter of the distribution in a form which does not depend on sample size (which may require a modification compared to the conventional form for stating the noncentrality parameter),

`"[Parameters;]"` denote any additional parameters of the distribution (if any) which are not a function of the sample size, and

`"OutputString"` specifies the computed results which will be returned. This can be any of the following:

- **ExactN**: returns an "exact", i.e. typically non-integer sample size estimate
- **UpperN**: upper integer sample size estimate
- **LowerN**: lower integer sample size estimate
- **UpperNPower**: actual power when using UpperN
- **LowerNPower**: actual power when using LowerN

As an example, for the noncentral t-distribution, the prefix `"NoncentralT"` is used to specify the name of the distribution. The distribution parameter  $\nu$ , the degrees of freedom, which depends on the sample size, and is therefore not included in the parameter list of this function. The modified noncentrality parameter is called  $\tilde{\rho} = \Delta/\sigma$ . Therefore, the function has the form

---

`NoncentralTSampleSize(α As mpNum, β As mpNum, ρ̃ As mpNum, OutputString As String) As mpNumList`

and an actual call to the function, requesting an upper sample size estimate (and actual power) for  $\alpha = 0.95$ ,  $\beta = 0.1$ , and  $\tilde{\rho} = \Delta/\sigma = 0.6$  would be

---

```
Result = NoncentralTSampleSize(0.95, 0.1, 0.6, "UpperN + UpperNPower")
mp.Print Result
```

---

which produces the output

```
UpperN: 26
UpperNPower: 0.92435435
```

### 6.1.3.5 Functions related to noncentrality parameters

These functions have the form `?Noncentrality(Alpha; Noncentrality; [Parameters;], OutputString)`. Here

”?” is a placeholder for the name of the distribution,  
 ”Alpha” specifies the confidence level (or Type I error),  
 ”Noncentrality” specifies the noncentrality parameter of the distribution,  
 ”[Parameters;]” denote any additional parameters of the distribution, and  
 ”OutputString” specifies the computed results which will be returned. This can be any of the following:

- **UpperCI**: upper confidence interval
- **LowerCI**: lower confidence interval
- **TwoSidedCI**: two-sided confidence interval

As an example, for the noncentral t-distribution, the prefix ”NoncentralT” is used to specify the name of the distribution. The noncentrality parameter is  $\delta$ , and the other distribution parameter is  $\nu$ , the degrees of freedom. Therefore, the function has the form

`NoncentralTNoncentrality(α As mpNum, δ As mpNum, ν As mpNum, OutputString As String) As mpNumList`

and an actual call to the function, requesting an upper confidence interval for  $\delta$  with  $\alpha = 0.95$ ,  $\delta = 0.6$  and  $\nu = 22$  would be

---

```
Result = NoncentralTNoncentrality(0.95, 0.6, 22, "UpperCI")
mp.Print Result
```

---

which produces the output

`UpperCI: 0.7546534`

### 6.1.3.6 Functions returning Random numbers

These functions have the form `?DistRan(Size; [Parameters]; Generator, OutputString)`. Here  
 “?” is a placeholder for the name of the distribution,  
 “Size” specifies the size of the random sample,  
 “[Parameters;]” denote any parameters (like degrees of freedom) of the distribution, and  
 “Generator” specifies the pseudo random generator which will be used to produce the random sample,  
 “OutputString” specifies the computed results which will be returned. This can be any of the following:

- **Unsorted:** produces unsorted output
- **Ascending:** output sorted in ascending order
- **Descending:** output sorted in descending order
- **Histogram( $k$ ):** output grouped in histogram format, with  $k$  buckets
- **HistogramCDF( $k$ ):** cumulated output grouped in histogram format, with  $k$  buckets

As an example, for Student’s t-distribution, a “T” is used to specify the name of the distribution, and there is just one distribution parameter,  $\nu$ , the degrees of freedom. Therefore, the function has the form

`TDistRan(Size As Integer;  $\nu$  As mpNum, Generator As String, OutputString As String) As mpNumList,`

and an actual call to the function, requesting a random sample of  $Size = 10000$  of a t-distribution with  $\nu = 22$ , using the default pseudo-random number generator, sorting output in ascending order could be

---

```
Result = TDistRan(10000, 22, "Default", "Ascending")
mp.Plot Result
```

---

which produces the output

`QInv: 2.943453463453453`

## 6.2 Beta-Distribution

### 6.2.1 Definition

If  $X_1$  and  $X_2$  are independent random variables following  $\chi^2$ -distribution with  $2a$  and  $2b$  degrees of freedom respectively, then the distribution of the ratio  $\frac{X_1}{X_1+X_2}$  is said to follow a Beta-distribution with  $a$  and  $b$  degrees of freedom.

See [Tretter & Walster \(1979\)](#)

### 6.2.2 Density and CDF

---

Function **BetaDist**(*x As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNumList

**NOT YET IMPLEMENTED**

---

The function **BetaDist** returns pdf, CDF and related information for the central Beta-distribution

**Parameters:**

*x*: A real number

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section [6.1.3.1](#) for the options for *Output*. Algorithms and formulas are given in sections [6.2.2.1](#) and [6.2.2.2](#).

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**BETADIST**(*x As mpReal, a As mpNum, b As mpNum*) As mpReal

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.BETADIST** returns the CDF and of the central Beta-distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

---

WorksheetFunction.**BETA.DIST**(*x As mpReal, a As mpNum, b As mpNum, Cumulative As Boolean*) As mpReal

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.BETA.DIST** returns the CDF and of the central Beta-distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

### 6.2.2.1 Density

The pdf of a variable following a central Beta-distribution with  $a$  and  $b$  degrees of freedom is given by

$$f_{\text{Beta}}(a, b, x) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} \quad (6.2.1)$$

where  $B(a, b)$  denotes the beta function (see section 4.8.7).

### 6.2.2.2 CDF: General formulas

The cdf of a variable following a central Beta-distribution with  $a$  and  $b$  degrees of freedom is given by

$$\Pr [X \leq x] = F_{\text{Beta}}(a, b, x) = \int_0^x f_{\text{Beta}}(a, b, t) dt \quad (6.2.2)$$

### 6.2.2.3 Exact cdf as continued fraction

The following representation as continued fraction is used (Peizer 1968, .1428 and 1452):

$$I(a, b, x) = \binom{n}{a} p^{b-1} q^a \frac{1}{(1 + u_1/(v_1 + u_2/(v_2 + u_3/(v_3 + \dots)))), \quad \text{where} \quad (6.2.3)}$$

$$p = (1 - x), \quad q = x, \quad n = a + b - 1, \quad u_1 = \frac{-(b-1)q}{p}, \quad u_{2j} = \frac{j(n+j)q}{p},$$

$$u_{2j+1} = \frac{-(a+j)(b-j-1)q}{p}, \quad v_j = a + j, \quad j = 1, 2, \dots$$

## 6.2.3 Quantiles

Function **BetaDistInv**(*Prob* As mpNum, *m* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

**NOT YET IMPLEMENTED**

The function **BetaDistInv** returns quantiles and related information for the the central Beta-distribution

#### Parameters:

*Prob*: A real number between 0 and 1.

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*).

The following functions are provided for compatibility with established spreadsheet functions

WorksheetFunction.**BETAINV**(*Prob* As mpReal, *a* As mpNum, *b* As mpNum) As mpReal

The function WorksheetFunction.BETAINV returns the two-tailed inverse of the central Beta-distribution

**Parameters:**

*Prob*: A real number

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

WorksheetFunction.BETA.INV(*Prob* As mpReal, *a* As mpNum, *b* As mpNum) As mpReal

The function WorksheetFunction.BETA.INV returns the left-tailed inverse of the central Beta-distribution

**Parameters:**

*Prob*: A real number

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

## 6.2.4 Properties

Function **BetaDistInfo**(*a* As mpNum, *b* As mpNum, *Output* As String) As mpNumList

NOT YET IMPLEMENTED

The function BetaDistInfo returns moments and related information for the central Beta-distribution

**Parameters:**

*a*: A real number greater 0, representing the degrees of freedom

*b*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.13.4.

### 6.2.4.1 Moments: algorithms and formulas

The raw moments are given by:

$$E^h(W) = \frac{\Gamma(a+h)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+h)} \quad (6.2.4)$$

The raw moments of the power of a beta variable are given by:

$$E^h(W^s) = \frac{\Gamma(a+hs)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+hs)} \quad (6.2.5)$$

### 6.2.4.2 Recurrences

$$I(a, b; x) = 1 - I(b, a; 1 - x) \quad (6.2.6)$$

$$I(a, b; x) = \binom{n}{a} x^a (1-x)^{b-1} + I(a+1, b-1; x) \quad (6.2.7)$$

$$I(a, b; x) = \binom{n}{a} x^a (1-x)^b + I(a+1, b; x) \quad (6.2.8)$$

$$I(a, b+1; x) = \binom{n}{a} x^a (1-x)^b + I(a, b; x) \quad (6.2.9)$$

$$I(a, b; x) = \binom{n}{a+b} x^a (1-x)^b \frac{a}{a+b-x} + I(a+1, b+1; x) \quad (6.2.10)$$

$$I(a, b; x) = F\left(2a, 2b, \frac{nx}{m-mx}\right) \quad (6.2.11)$$

## 6.2.5 Random Numbers

---

**Function BetaDistRandom(*Size* As mpNum, *a* As mpNum, *b* As mpNum, *Generator* As String, *Output* As String) As mpNumList**

---

**NOT YET IMPLEMENTED**

---

The function BetaDistRandom returns random numbers following a central Beta-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

### 6.2.5.1 Random Numbers: algorithms and formulas

In order to obtain random numbers from a Beta distribution we first single out a few special cases. For  $p = 1$  and/or  $q = 1$  we may easily solve the equation  $F(x) = \xi$  where  $F(x)$  is the cumulative function and  $\xi$  a uniform random number between zero and one. In these cases

$$\begin{aligned} p = 1 \Rightarrow x &= 1 - \xi^{1/q} \\ q = 1 \Rightarrow x &= \xi^{1/q} \end{aligned}$$

For  $p$  and  $q$  half-integers we may use the relation to the chi-square distribution by forming the ratio  $\frac{y_m}{y_m+y_n}$  with  $y_m$  and  $y_n$  two independent random numbers from chi-square distributions with  $m = 2p$  and  $n = 2q$  degrees of freedom, respectively.

Yet another way of obtaining random numbers from a Beta distribution valid when  $p$  and  $q$  are both integers is to take the  $l^{th}$  out of  $k$  ( $1 \leq l \leq k$ ) independent uniform random numbers between zero and one (sorted in ascending order). Doing this we obtain a Beta distribution with parameters  $p = l$  and  $q = k + 1 - l$ . Conversely, if we want to generate random numbers from a Beta distribution with integer parameters  $p$  and  $q$  we could use this technique with  $l = p$  and

$k = p + q - 1$ . This last technique implies that for low integer values of  $p$  and  $q$  simple code may be used, e.g. for  $p = 2$  and  $q = 1$  we may simply take  $\max(\xi_1, \xi_2)$  i.e. the maximum of two uniform random numbers ([Walck, 2007](#)).

## 6.3 Binomial Distribution

These functions return PMF and CDF of the (discrete) binomial distribution with number of trials  $n \geq 0$  and success probability  $0 \leq p \leq 1$ .

### 6.3.1 Density and CDF

---

Function **BinomialDist**(*x As mpNum, n As mpNum, p As mpNum, Output As String*) As mp-NumList

---

**NOT YET IMPLEMENTED**

---

The function **BinomialDist** returns pdf, CDF and related information for the central Binomial-distribution

**Parameters:**

*x*: The number of successes in trials.  
*n*: The number of independent trials.  
*p*: The probability of success on each trial  
*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.3.1.1 and 6.3.1.2.

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**BINOMDIST**(*x As mpNum, n As mpNum, p As mpNum, Cumulative As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.BINOMDIST** returns pdf, CDF, and related information of the central Binomial-distribution

**Parameters:**

*x*: The number of successes in trials.  
*n*: The number of independent trials.  
*p*: The probability of success on each trial

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

---

WorksheetFunction.**BINOM.DIST**(*x As mpNum, n As mpNum, p As mpNum, Cumulative As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.BINOM.DIST** returns the CDF and pdf of the central Binomial-distribution

**Parameters:**

*x*: The number of successes in trials.

*n*: The number of independent trials.

*p*: The probability of success on each trial

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

WorksheetFunction.BINOM.DIST.RANGE(*n As mpNum, p As mpNum, x1 As mpNum, x2 As mpNum*) As mpReal

NOT YET IMPLEMENTED

The function WorksheetFunction.BINOM.DIST.RANGE returns the probability that the number of successful trials will fall between x1 and x2

#### Parameters:

*n*: The number of independent trials.

*p*: The probability of success on each trial

*x1*: The number x1 of successes in trials.

*x2*: The number x2 of successes in trials.

##### 6.3.1.1 Density

$$f_{\text{Bin}}(n, k; p) = \binom{n}{k} p^k (1-p)^{n-k} = f_{\text{Beta}}(k+1, n-k+1, p)/(n+1) \quad (6.3.1)$$

##### 6.3.1.2 CDF

$$F_{\text{Bin}}(n, k; p) = I_{1-p}(n-k, k+1) = ibeta(n-k, k+1, 1-p) \quad (6.3.2)$$

#### 6.3.2 Quantiles

Function BinomialDistInv(*Prob As mpNum, n As mpNum, p As mpNum, Output As String*) As mpNumList

NOT YET IMPLEMENTED

The function BinomialDistInv returns quantiles and related information for the the central binomial distribution

#### Parameters:

*Prob*: A real number between 0 and 1.

*n*: The number of Bernoulli trials.

*p*: The probability of a success on each trial.

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*.

The following functions are provided for compatibility with established spreadsheet functions

WorksheetFunction.CRITBINOM(*n As mpNum, p As mpNum, Alpha As mpReal*) As mpReal

NOT YET IMPLEMENTED

The function `WorksheetFunction.CRITBINOM` returns the smallest value for which the cumulative binomial distribution is greater than or equal to a criterion value.

**Parameters:**

*n*: The number of Bernoulli trials.

*p*: The probability of a success on each trial.

*Alpha*: The criterion value.

`WorksheetFunction.BINOM.INV(n As mpNum, p As mpNum, Alpha As mpReal) As mpReal`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.BINOM.INV` returns the smallest value for which the cumulative binomial distribution is greater than or equal to a criterion value.

**Parameters:**

*n*: The number of Bernoulli trials.

*p*: The probability of a success on each trial.

*Alpha*: The criterion value.

### 6.3.3 Properties

`Function BinomialDistInfo(n As mpNum, p As mpNum, Output As String) As mpNumList`

**NOT YET IMPLEMENTED**

The function `BinomialDistInfo` returns moments and related information for the central Binomial-distribution

**Parameters:**

*n*: The number of Bernoulli trials.

*p*: The probability of a success on each trial.

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.13.4.

#### 6.3.3.1 Moments: algorithms and formulas

$$\mu'_r = \sum_{i=0}^r \binom{n}{i} \left( \sum_{j=0}^i \binom{i}{j} (-1)^j (i-j)^r \right) \quad (6.3.3)$$

$$\mu_1 = np \quad (6.3.4)$$

$$\mu_2 = np(1-p) = npq \quad (6.3.5)$$

$$\mu_3 = npq(q-p) \quad (6.3.6)$$

$$\mu_4 = 3(npq)^3 + npq(1-6pq) \quad (6.3.7)$$

### 6.3.4 Random Numbers

---

Function **BinomialDistRandom**(*Size* As mpNum, *n* As mpNum, *p* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function BinomialDistRandom returns random numbers following a central Binomial-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*n*: The number of Bernoulli trials.

*p*: The probability of a success on each trial.

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

#### 6.3.4.1 Random Numbers: algorithms and formulas

In order to obtain random numbers from a Binomial distribution we first single out a few special cases. For  $p = 1$  and/or  $q = 1$  we may easily solve the equation  $F(x) = \xi$  where  $F(x)$  is the cumulative function and  $\xi$  a uniform random number between zero and one. In these cases

$$\begin{aligned} p = 1 \Rightarrow x &= 1 - \xi^{1/q} \\ q = 1 \Rightarrow x &= \xi^{1/p} \end{aligned}$$

For  $p$  and  $q$  half-integers we may use the relation to the chi-square distribution by forming the ratio  $\frac{y_m}{y_m + y_n}$  with  $y_m$  and  $y_n$  two independent random numbers from chi-square distributions with  $m = 2p$  and  $n = 2q$  degrees of freedom, respectively.

Yet another way of obtaining random numbers from a Beta distribution valid when  $p$  and  $q$  are both integers is to take the  $l^{th}$  out of  $k$  ( $1 \leq l \leq k$ ) independent uniform random numbers between zero and one (sorted in ascending order). Doing this we obtain a Beta distribution with parameters  $p = l$  and  $q = k + 1 - l$ . Conversely, if we want to generate random numbers from a Beta distribution with integer parameters  $p$  and  $q$  we could use this technique with  $l = p$  and  $k = p + q - 1$ . This last technique implies that for low integer values of  $p$  and  $q$  simple code may be used, e.g. for  $p = 2$  and  $q = 1$  we may simply take  $\max(\xi_1, \xi_2)$  i.e. the maximum of two uniform random numbers (Walck, 2007).

## 6.4 Chi-Square Distribution

### 6.4.1 Definition

Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed random variables each following a normal distribution with mean zero and unit variance. Then  $\chi^2 = \sum_{j=1}^n X_j$  is said to follow a  $\chi^2$ -distribution with  $n$  degrees of freedom.

### 6.4.2 Density and CDF

---

Function **CDist**(*x* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function CDist returns pdf, CDF and related information for the central  $\chi^2$ -distribution

**Parameters:**

*x*: A real number

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.4.2.1 and 6.4.2.2.

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**CHIDIST**(*x* As mpReal, *deg\_freedom* As mpReal, *Tails* As Integer) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.CHIDIST returns the CDF and of the central  $\chi^2$ -distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

*Tails*: Specifies the number of distribution tails to return. If tails = 1, TDIST returns the one-tailed distribution. If tails = 2, TDIST returns the two-tailed distribution.

---

WorksheetFunction.**CHISQDIST**(*x* As mpReal, *deg\_freedom* As mpReal, *Tails* As Integer) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.CHISQDIST returns the CDF and of the central  $\chi^2$ -distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

*Tails*: Specifies the number of distribution tails to return. If tails = 1, TDIST returns the one-tailed distribution. If tails = 2, TDIST returns the two-tailed distribution.

---

WorksheetFunction.**CHISQ.DIST**(*x* As mpReal, *deg\_freedom* As mpReal, *Cumulative* As Boolean) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.CHISQ.DIST returns the CDF and of the central  $\chi^2$ -distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

---

WorksheetFunction.**CHISQ.DIST.RT**(*x* As mpReal, *deg\_freedom* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.CHISQ.DIST.RT returns the complement of the CDF and of the central  $\chi^2$ -distribution

**Parameters:**

*x*: A real number

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

---

WorksheetFunction.**CHISQ.DIST.2T**(*x* As mpReal, *deg\_freedom* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.CHISQ.DIST.2T returns the two-sided CDF of the central  $\chi^2$ -distribution

**Parameters:**

*x*: A real number

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

### 6.4.2.1 Density

The density of a central chi-square variable with  $n$  degrees of freedom is given by

$$f_{\chi^2}(n, x) = \frac{1}{2^{n/2}\Gamma(n/2)}x^{(n-2)/2}e^{-x/2}. \quad (6.4.1)$$

### 6.4.2.2 CDF: General formulas

The cdf of a central chi-square variable with  $n$  degrees of freedom is given by

$$\Pr [\chi^2 \leq x] = F_{\chi^2}(n, x) = \int_0^x f_{\chi^2}(n, t)dt \quad (6.4.2)$$

### 6.4.2.3 CDF: Continued fraction

For real  $n > 0$ , the CDF can be calculated using continued fraction ([Peizer & Pratt, 1968](#)).

If  $(n - 1) \leq x$  let  $1 - F_{\chi^2}(n, x)$  be a right tail chi square probability. Then

$$1 - F_{\chi^2}(n, x) = f_{\chi^2}(n, x) \frac{1}{(1 + u_1/(v_1 + u_2/(v_2 + u_3/(v_3 + \dots))))} \quad (6.4.3)$$

where  $M = \frac{1}{2}x$ ,  $b = \frac{1}{2}n$ ,  $u_{2j-1} = j - b$ ,  $v_{2j-1} = M$ ,  $u_{2j} = j$ ,  $v_{2j} = 1$ ,  $j = 1, 2, \dots$   
If  $(n - 1) > x$  let  $F_{\chi^2}(n, x)$  be a left tail chi square probability. Then

$$F_{\chi^2}(n, x) = f_{\chi^2}(n, x) \frac{m}{b} \frac{1}{(1 + u_1/(v_1 + u_2/(v_2 + u_3/(v_3 + \dots))))} \quad (6.4.4)$$

where  $M = \frac{1}{2}x$ ,  $b = \frac{1}{2}n$ ,  $u_1 = -M$ ,  $u_{2j} = jM$ ,  $u_{2j+1} = -(b+j)M$ ,  $v_j = b+j$ ,  $j = 1, 2, \dots$

#### 6.4.2.4 CDF (central case): Finite sum

The cdf can be expressed as a finite sum if  $n$  is an integer:

$$F_{\chi^2}(n, x) = 1 + 2\Phi(-\sqrt{x}) + 2\phi(\sqrt{x}) \sum_{r=1}^{(n-1)/2} \frac{\sqrt{x}^{2r-1}}{1 \cdot 3 \cdot 5 \dots (2r-1)}, \quad \text{for } n \text{ odd}, \quad (6.4.5)$$

$$F_{\chi^2}(n, x) = e^{-x/2} \left( 1 + \sum_{r=1}^{(n-2)/2} \frac{x^r}{2 \cdot 4 \cdot 6 \dots (2r)} \right), \quad \text{for } n \text{ even}, \quad (6.4.6)$$

where  $\phi(\cdot)$  denotes the pdf of the normal distribution (see section 6.11.2.1) and  $\Phi(\cdot)$  denotes the cdf of the normal distribution (see section 6.11.2.2).

### 6.4.3 Quantiles

Function **CDistInv**(*Prob* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

NOT YET IMPLEMENTED

The function CDistInv returns quantiles and related information for the the central  $\chi^2$ -distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*).

The following functions are provided for compatibility with established spreadsheet functions

WorksheetFunction.**CHIINV**(*x* As mpReal, *deg\_freedom* As mpReal) As mpReal

NOT YET IMPLEMENTED

The function WorksheetFunction.CHIINV returns the two-tailed inverse of the central  $\chi^2$ -distribution

**Parameters:**

*x*: A real number

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

WorksheetFunction.**CHISQ.INV**(*x* As mpReal, *deg\_freedom* As mpReal) As mpReal

NOT YET IMPLEMENTED

The function `WorksheetFunction.CHISQ.INV` returns the left-tailed inverse of the central  $\chi^2$ -distribution

**Parameters:**

*x*: A real number

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

`WorksheetFunction.CHISQ.INV.RT(x As mpReal, deg_freedom As mpReal) As mpReal`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.CHISQ.INV.RT` returns the two-tailed inverse of the central  $\chi^2$ -distribution

**Parameters:**

*x*: A real number

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

`WorksheetFunction.CHISQINV(x As mpReal, deg_freedom As mpReal) As mpReal`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.CHISQINV` returns the two-tailed inverse of the central  $\chi^2$ -distribution

**Parameters:**

*x*: A real number

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

#### 6.4.3.1 Quantiles (central case): algorithms and formulas

Let  $z_\alpha$  and  $\chi_{n,\alpha}^2$  be the  $\alpha$ -quantiles of the standard normal distribution and central chi-square distribution with  $n$  the degrees of freedom. For  $n = 1$  and  $n = 2$ , the following closed form expressions can be used:

$$\chi_{1,\alpha}^2 = z_\alpha^2, \quad \chi_{2,\alpha}^2 = 2 \log(1 - \alpha) \quad (6.4.7)$$

At the extreme left tail of the distribution, for small  $x$ , the CDF of a  $\chi^2$  variable with  $n$  degrees of freedom can be approximated by the density of a  $\chi^2$  variable with  $n + 2$  degrees of freedom:

$$F_{\chi^2}(n, x) \approx 2f_{\chi^2}(n + 2, x).$$

The density of a  $\chi^2$  variable with  $n + 2$  degrees of freedom can be inverted in closed form using the Lambert  $W$  function, which leads to the following approximation:

$$\chi_{n,\alpha}^2 \approx f_{\chi^2}^{-1}(n + 2, \alpha) = -2W(t)/a, \quad \text{where} \quad (6.4.8)$$

$$a = \frac{1}{(n + 2)/2 - 1}, \quad k = \ln(\Gamma((n + 2)/2)), \quad d = a - \ln(1 - \alpha) + k, \quad t = -ae^{p+d}$$

This approximation is used for  $|t| < 0.1$ , and the Lambert  $W$  function is approximated as

$$W(x) \approx x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 - \frac{125}{24}x^5.$$

Otherwise, the quantile is approximated by inverting a formula proposed by [Canal \(2005\)](#):

$$\chi_{n,\alpha}^2 \approx n \left( \frac{1}{2} + \frac{t}{2} - \frac{3}{2t} \right)^6, \quad \text{where} \quad (6.4.9)$$

$$t = \left( -5 + 2L + 2\sqrt{13 - 5L + L^2} \right)^{1/3}, \quad L = 6 \left( m + s \left( az_\alpha^2 + z_\alpha - a \right) \right)$$

$$m = \frac{5}{6} - \frac{1}{9n} - \frac{7}{648n^2} - \frac{25}{2187n^3}, \quad s^2 = \frac{1}{18n} + \frac{1}{162n^2} - \frac{37}{11664n^3}, \quad a = \frac{1}{162\sqrt{2n^3}}.$$

These approximations are then used as a starting point for a Newton iteration.

## 6.4.4 Properties

---

Function **CDistInfo**(*n* As mpNum, **Output** As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **CDistInfo** returns moments and related information for the central  $\chi^2$ -distribution

**Parameters:**

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.4.4.

### 6.4.4.1 Median (central case)

The median is given approximately by

$$k - \frac{2}{3} + \frac{4}{27k} - \frac{8}{729k^2}. \quad (6.4.10)$$

### 6.4.4.2 Moments and Cumulants (central case)

The cumulants are given by

$$\kappa_{r+1} = 2^r r! n \quad (6.4.11)$$

The  $k^{th}$  null-moment of the  $r^{th}$  root of a chi-square variable is given by:

$$E(X^{k/r}) = \frac{2^{k/r}\Gamma(n/2) + k/r}{\Gamma(n/2)} \quad (6.4.12)$$

where  $\Gamma(\cdot)$  is the Gamma function (see section 4.8.6.)

The first 4 cumulants of cube root central  $\chi^2$ ,  $\chi^{2/3}$ , are given by (Aty, 1954)

$$\kappa_1^*(n, 0) = 1 - \frac{2}{9n} + \frac{80}{3^7 n^3} + \frac{176}{3^9 n^4} + o(n^{-4}) \quad (6.4.13)$$

$$\kappa_2^*(n, 0) = \frac{2}{9n} - \frac{104}{3^7 n^3} - \frac{160}{3^8 n^4} + o(n^{-4}) \quad (6.4.14)$$

$$\kappa_3^*(n, 0) = -\frac{32}{3^6 n^3} - \frac{256}{3^8 n^4} + o(n^{-4}) \quad (6.4.15)$$

$$\kappa_4^*(n, 0) = -\frac{16}{3^6 n^3} - \frac{256}{3^8 n^4} + o(n^{-4}) \quad (6.4.16)$$

#### 6.4.4.3 Recurrence Relations (central case)

The following recurrence relations hold for the pdf and CDF:

$$f_{\chi^2}(n+2, x) = \frac{x}{n} f_{\chi^2}(n, x) \quad (6.4.17)$$

$$F_{\chi^2}(n, x) - F_{\chi^2}(n+2, x) = 2f_{\chi^2}(n+2, x) \quad (6.4.18)$$

#### 6.4.5 Random Numbers

---

Function **CDistRan**(*Size As mpNum, n As mpNum, Generator As String, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function CDistRan returns random numbers following a central  $\chi^2$ -distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*n*: A real number greater 0, representing the degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in section 6.4.5.

As we saw above the sum of *n* independent standard normal random variables gave a chi-square distribution with *n* degrees of freedom. This may be used as a technique to produce pseudorandom numbers from a chi-square distribution. This required a generator for standard normal random numbers and may be quite slow. However, if we make use of the Box-Muller transformation in order to obtain the standard normal random numbers we may simplify the calculations. Adding *n* such squared random numbers implies that

$$\begin{aligned} y_{2k} &= -2 \ln(\xi_1 \cdot \xi_2 \cdot \dots \cdot \xi_k) \\ y_{2k+1} &= -2 \ln(\xi_1 \cdot \xi_2 \cdot \dots \cdot \xi_k) - 2 \ln(\xi_{k+1}) [\cos(2\pi\xi_{k+2})]^2 \end{aligned}$$

for *k* a positive integer will be distributed as chi-square variable with even or odd number of degrees of freedom. In this manner a lot of unnecessary operations are avoided. Since the chi-square distribution is a special case of the Gamma distribution we may also use a generator for this distribution.

#### 6.4.6 Wishart Matrix

See [Gleser \(1976\)](#)

## 6.5 Exponential Distribution

These functions return PDF, CDF, and ICDF of the exponential distribution with location  $a$ , rate  $\alpha > 0$ , and the support interval  $(a, +\infty)$ :

### 6.5.1 Density and CDF

---

Function **ExponentialDist**(*x As mpNum, lambda As mpNum, Output As String*) As mpNumList  
**NOT YET IMPLEMENTED**

---

The function **ExponentialDist** returns pdf, CDF and related information for the central Exponential distribution

**Parameters:**

*x*: The value of the distribution.

*lambda*: The parameter of the distribution.

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.2.2.1 and 6.2.2.2.

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**EXPONDIST**(*x As mpNum, lambda As mpNum, Cumulative As Boolean*) As mpReal  
**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.EXPONDIST** returns pdf, CDF, and related information of the central Binomial-distribution

**Parameters:**

*x*: The value of the distribution.

*lambda*: The parameter of the distribution.

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

---

WorksheetFunction.**EXPON.DIST**(*x As mpNum, lambda As mpNum, Cumulative As Boolean*) As mpReal  
**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.EXPON.DIST** returns the CDF and pdf of the central Binomial-distribution

**Parameters:**

*x*: The value of the distribution.

*lambda*: The parameter of the distribution.

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

### 6.5.1.1 Density

$$f(x) = \alpha \exp(-\alpha(x - a)) \quad (6.5.1)$$

### 6.5.1.2 CDF

$$F(x) = 1 - \exp(-\alpha(x - a)) = \text{expm1}(-\alpha(x - a)) \quad (6.5.2)$$

## 6.5.2 Quantiles

---

Function **ExponentialDistInv**(*Prob* As mpNum, *lambda* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function ExponentialDistInv returns quantiles and related information for the central Exponential distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*lambda*: The number of Bernoulli trials.

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*.

$$F^{-1}(y) = a - \ln1p(-y)/\alpha \quad (6.5.3)$$

## 6.5.3 Properties

---

Function **ExponentialDistInfo**(*lambda* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function ExponentialDistInfo returns moments and related information for the central *t*-distribution

**Parameters:**

*lambda*: A real number greater 0, representing the parameter of the distribution

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.13.4.

### 6.5.3.1 Moments and cumulants

The mean or expected value of an exponentially distributed random variable X with rate parameter  $\lambda$  is given by

$$E[X] = \frac{1}{\lambda} \quad (6.5.4)$$

The variance of X is given by

$$E[X^2] = \frac{1}{\lambda^2} \quad (6.5.5)$$

so the standard deviation is equal to the mean.

The moments of  $X$ , for  $n = 1, 2, \dots$ , are given by

$$E[X^n] = \frac{n!}{\lambda^n} \quad (6.5.6)$$

## 6.5.4 Random Numbers

---

Function **ExponentialDistRandom**(*Size As mpNum*, *lambda As mpNum*, *Generator As String*, *Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function ExponentialDistRandom returns random numbers following a central Beta-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*lambda*: A real number greater 0, representing the numerator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in section 6.6.5.

### 6.5.4.1 Random Numbers: algorithms and formulas

Random numbers can be generated using the inversion formula.

## 6.6 Fisher's F-Distribution

### 6.6.1 Definition

If  $X_1$  and  $X_2$  are independent random variables following  $\chi^2$ -distribution with  $m$  and  $n$  degrees of freedom respectively, then the distribution of the ratio  $F = \frac{X_1/m}{X_2/n}$  is said to follow a F-distribution with  $m$  and  $n$  degrees of freedom.

### 6.6.2 Density and CDF

---

Function **FDist**(*x* As mpNum, *m* As mpNum, *n* As mpNum, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function FDist returns pdf, CDF and related information for the central  $F$ -distribution

**Parameters:**

*x*: A real number

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.6.2.1 and 6.6.2.2.

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**FDIST**(*x* As mpReal, *m* As mpNum, *n* As mpNum) As mpReal

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.FDIST returns the CDF and of the central  $F$ -distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

---

WorksheetFunction.**F.DIST**(*x* As mpReal, *m* As mpNum, *n* As mpNum, **Cumulative** As Boolean) As mpReal

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.F.DIST returns the CDF and of the central  $F$ -distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, F.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

**WorksheetFunction.F.DIST.RT(*x As mpReal, m As mpNum, n As mpNum*) As mpReal**

**NOT YET IMPLEMENTED**

The function **WorksheetFunction.F.DIST.RT** returns the complement of the CDF and of the central *F*-distribution

#### Parameters:

*x*: A real number

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

#### 6.6.2.1 Density

The density of a variable following a central F-distribution with *m* and *n* degrees of freedom is given by

$$f_F(m, n, x) = \frac{m^{m/2} n^{n/2}}{B(m/2, n/2)} x^{(m-2)/2} (n + mx)^{-(m+n)/2} \quad (6.6.1)$$

#### 6.6.2.2 CDF: General formulas

The cdf of a variable following a central F-distribution with *m* and *n* degrees of freedom is given by

$$\Pr[X \leq x] = F_F(m, n, x) = \int_0^x f(m, n, t) dt \quad (6.6.2)$$

#### 6.6.2.3 CDF (central): finite series

The cdf can be expressed as a finite sum if *m* is an integer, and *n* is a positive real number:

$$1 - F_F(m, n, x) = a_m + b_m(c_1 + c_3 + \dots + c_{m-2}), \quad \text{for } m \text{ odd}, \quad (6.6.3)$$

$$\text{where } a_m = 2T(n, -z_m); b_m = 2t(n, z_m) \cdot z_m; z_m = \sqrt{mx}; \quad (6.6.4)$$

$$1 - F_F(m, n, x) = d_m(c_0 + c_2 + \dots + c_{m-2}), \quad \text{for } m \text{ even}, \quad (6.6.5)$$

$$\text{where } d_m = (1 - u_m)^{n/2} \quad (6.6.6)$$

$$\text{and } u_m = mx/(mx + n), \quad c_0 = c_1 = 1, \quad c_k = c_{k-2}u_m \cdot (n + k - 2)/k \quad (6.6.7)$$

#### 6.6.3 Quantiles

**Function FDistInv(*Prob As mpNum, m As mpNum, n As mpNum*) As mpNumList**

**NOT YET IMPLEMENTED**

The function **FDistInv** returns quantiles and related information for the the central *t*-distribution

#### Parameters:

*Prob*: A real number between 0 and 1.

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom  
Output? String? A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*).

The following functions are provided for compatibility with established spreadsheet functions

---

**WorksheetFunction.FINV(*x* As mpReal, *m* As mpNum, *n* As mpNum) As mpReal**

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.FINV returns the two-tailed inverse of the central *t*-distribution

**Parameters:**

*x*: A real number

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

---

**WorksheetFunction.F.INV(*x* As mpReal, *m* As mpNum, *n* As mpNum) As mpReal**

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.F.INV returns the left-tailed inverse of the central *t*-distribution

**Parameters:**

*x*: A real number

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

---

**WorksheetFunction.F.INV.RT(*x* As mpReal, *m* As mpNum, *n* As mpNum) As mpReal**

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.F.INV.RT returns the right-tailed inverse of the central *t*-distribution

**Parameters:**

*x*: A real number

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

### 6.6.3.1 Quantiles (central case): algorithms and formulas

Let *p* be a right tail probability,  $z_\alpha$  the  $\alpha$ -quantile of the standard normal distribution, and *n* the degrees of freedom. Depending on *n* and *p*, one proceeds as follows:

$$F_{1,n,\alpha} = t_{n,\alpha}^2 \quad (6.6.8)$$

$$F_{2,n,\alpha} = \frac{n(1-x)}{2x}, \quad \text{where } x = \frac{2\log(1-\alpha)}{n} \quad (6.6.9)$$

### 6.6.3.2 Cornish-Fisher expansion

Otherwise, the Cornish-Fisher expansion of  $\frac{1}{2}\log(F_{m,n})$  is used, as given by Sahai & Thompson (1974), including cumulants through order eight.

$$F_{m,n,\alpha} \approx e^{2w}, \quad \text{where} \quad (6.6.10)$$

$$s = \frac{1}{m} + \frac{1}{n}, \quad d = \frac{1}{m} - \frac{1}{n}, \quad r = \sqrt{s/2}, \quad \text{and} \quad (6.6.11)$$

$$\begin{aligned} w = & zr - \frac{d(z^2 + 2)}{6} + \frac{rs(z^3 + 3z)}{24} + \frac{rd^2(z^3 + 11z)}{72s} - \frac{ds(z^4 + 9z^2 + 8)}{120} + \frac{d^3(3z^4 + 7z^2 - 16)}{3240s} \\ & + \frac{rs^2(z^5 + 20z^3 + 15z)}{1920} + \frac{rd^2(z^5 + 44z^3 + 183z)}{2880} + \frac{d^4(9z^5 + 284z^3 - 1513z)}{155520s^2} \\ & + \frac{ds^2(4z^6 - 25z^4 - 177z^2 + 192)}{20160} + \frac{d^3(4z^6 + 101z^4 + 177z^2 - 480)}{90720} \\ & + \frac{d^5(12z^6 + 513z^4 + 841z^2 - 2560)}{1632960s^2} - \frac{rs^3(z^7 + 7z^5 + 7z^3 + 105z)}{21504} \\ & + \frac{rd^2s(801z^7 + 10511z^5 + 30151z^3 + 62241z)}{4838400} - \frac{rd^4(477z^7 + 4507z^5 - 82933z^3 - 264363z)}{43545600s} \\ & + \frac{rd^6(3753z^7 + 55383z^5 - 368897z^3 - 1213927z)}{1175731200s^3} \end{aligned}$$

### 6.6.3.3 Box-Davis Expansion

From [Box \(1949\)](#) and [Davis \(1971\)](#) we derive the following approximation: let  $u_\alpha$  be the  $\alpha$  percentage point of a chi-square-distribution with  $m$  degrees of freedom

$$F_{m,n,\alpha} \approx \frac{n}{m} \frac{1 - e^{-X}}{e^{-X}}, \quad \text{where} \quad (6.6.12)$$

$$\mu = n + \frac{1}{2}m - 1, \quad m_1 = m, \quad m_k = m_{k-1}(m + 2k - 2), \quad P_1 = u/m, \quad P_k = P_{k-1} + u^k/m_k$$

$$P_{22} = \frac{-8u^4(m+3)}{m_2m_4} + \frac{8u^3}{m_2m_3} + \frac{6u^2}{mm_2} + \frac{2u}{m^2}$$

$$P_{42} = \frac{-16u^6(m+5)}{m_2m_6} - \frac{4u^5(m-4)}{m_2m_5} + \frac{2u^4(3m+14)}{m_2m_4} + \frac{2u^3(3m+10)}{m_2m_3} + \frac{6u^2}{mm_2} + \frac{2u}{m^2}$$

$$\begin{aligned} P_{222} = & \frac{32u^6(7m^2 + 62m + 120)}{m_2^2m_6} - \frac{32u^5(2m^2 + 37m + 96)}{m_2^2m_5} - \frac{8u^4(23m^2 + 124m + 132)}{m_2^2m_4} \\ & - \frac{8u^3(m-10)}{m_1m_2m_3} + \frac{28u^2}{m^2m_2} + \frac{4u}{m^3} \end{aligned}$$

$$\omega_2 = \frac{m(m^2 - 4)}{48\mu^2}, \quad \omega_4 = \frac{m(3m^4 - 40m^2 + 112)}{1920\mu^4}, \quad \omega_6 = \frac{m(3m^6 - 84m^3 + 784m^2 - 1984)}{16128\mu^6},$$

$$s_2 = \omega_2 P_2, \quad s_4 = \omega_4 P_4 + \frac{1}{2}\omega_2^2 P_{22}, \quad s_6 = \omega_6 P_6 + \omega_4 \omega_2 P_{42} + \omega_2^3 P_{222}$$

$$X = (u + 2(s_2 + s_4 + s_6))/\mu$$

### 6.6.3.4 Confidence Interval

See [Guenther \(1977a\)](#)

## 6.6.4 Properties

---

Function **FDistInfo**(*m* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function FDistInfo returns moments and related information for the central *t*-distribution

**Parameters:**

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.6.4.

### 6.6.4.1 Recurrence relations (central case)

Let the density  $g_{m,n}$  be that of  $m/n$  times an  $F_{m,n}$  random variable. Let  $G_{m,n}(y)$  be its distribution function. Then the following recurrence relations hold (Chattamvelli & Jones, 1995)

$$n [G_{m,n+2}(y) - G_{m-2,n+2}(y)] = -2g_{m,n}(y) \quad (6.6.13)$$

$$m(1+y)g_{m+2,n}(y) + y(m+n)g_{m,n}(y). \quad (6.6.14)$$

$$n(1+y)g_{m,n+2}(y) = (m+n)g_{m,n}(y). \quad (6.6.15)$$

$$mg_{m+2,n-2}(y) = (n-2)y g_{m,n}(y). \quad (6.6.16)$$

From equations 6.6.13 to 6.6.16 we obtain

$$\begin{aligned} [(m+2)(1+y)]G_{m+4,n}(y) &= [(m+2)(1+y) + y(m+n)]G_{m+2,n}(y) \\ &\quad - y(m+n)G_{m,n}(y) \end{aligned} \quad (6.6.17)$$

$$n(1+y)[G_{m,n+2}(y) - G_{m+2,n+2}(y)] = (m+n)[G_{m,n}(y) - G_{m+2,n}(y)] \quad (6.6.18)$$

$$(m+2)[G_{m+2,n}(y) - G_{m+4,n-2}(y)] = (n-2)[G_{m,n}(y) - G_{m+2,n}(y)] \quad (6.6.19)$$

### 6.6.4.2 Relations to other distributions (central case)

$$F_F(m, n; x) = 1 - F_F\left(n, m; \frac{1}{x}\right) \quad (6.6.20)$$

$$F_F(m, n; x) = F_B\left(n/2, m/2; \frac{mx}{mx+n}\right) \quad (6.6.21)$$

where  $F_B(\cdot)$  denotes the cdf of the central Beta-distribution (see section 6.2.2.2).

## 6.6.5 Random Numbers

---

Function **FDistRan**(*Size* As mpNum, *m* As mpNum, *n* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function FDistRan returns random numbers following a central *F*-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in section 6.6.5.

#### 6.6.5.1 Random Numbers: algorithms and formulas

Following the definition the quantity  $F = \frac{y_m/m}{y_n/n}$  where  $y_n$  and  $y_m$  are two variables distributed according to the chi-square distribution with  $n$  and  $m$  degrees of freedom respectively follows the F-distribution. We may thus use this relation inserting random numbers from chi-square distributions (see section ...).

## 6.7 Gamma (and Erlang) Distribution

These functions return PDF, CDF, and ICDF of the gamma distribution with shape  $a > 0$ , scale  $b > 0$ , and the support interval  $(0, +\infty)$ .

A gamma distribution with shape  $a \in \mathbb{N}$  is called Erlang distribution.

### 6.7.1 Density and CDF

---

Function **GammaDist**(*x As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNum-List

---

**NOT YET IMPLEMENTED**

---

The function **GammaDist** returns pdf, CDF and related information for the central Gamma-distribution

**Parameters:**

*x*: A real number

*a*: A real number greater 0, a parameter to the distribution

*b*: A real number greater 0, a parameter to the distribution

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.2.2.1 and 6.2.2.2.

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**GAMMADIST**(*x As mpReal, a As mpNum, b As mpNum, Cumulative As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.GAMMADIST** returns the CDF and of the central Gamma-distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*a*: A real number greater 0, a parameter to the distribution

*b*: A real number greater 0, a parameter to the distribution

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, **GAMMA.DIST** returns the cumulative distribution function; if FALSE, it returns the probability density function.

---

WorksheetFunction.**GAMMA.DIST**(*x As mpReal, a As mpNum, b As mpNum, Cumulative As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.GAMMA.DIST** returns the CDF and of the central Gamma-distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

- a*: A real number greater 0, a parameter to the distribution  
*b*: A real number greater 0, a parameter to the distribution

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, GAMMA.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function.

### 6.7.1.1 Density

$$f(x; a, b) = \frac{x^{a-1} e^{-x/b}}{\Gamma(a)b^a} \quad (6.7.1)$$

### 6.7.1.2 CDF: General formulas

$$F(x; a, b) = P(a, x/b) = igammap(a, x/b) \quad (6.7.2)$$

## 6.7.2 Quantiles

---

Function **GammaDistInv**(*Prob* As mpNum, *m* As mpNum, *n* As mpNum) As mpNumList

---

NOT YET IMPLEMENTED

---

The function GammaDistInv returns quantiles and related information for the the central Gamma-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*m*: A real number greater 0, a parameter to the distribution

*n*: A real number greater 0, a parameter to the distributionOutput? String? A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*).

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**GAMMAINV**(*Prob* As mpReal, *a* As mpNum, *b* As mpNum) As mpReal

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.GAMMAINV returns the two-tailed inverse of the central Gamma-distribution

**Parameters:**

*Prob*: A real number

*a*: A real number greater 0, a parameter to the distribution

*b*: A real number greater 0, a parameter to the distribution

---

WorksheetFunction.**GAMMA.INV**(*Prob* As mpReal, *a* As mpNum, *b* As mpNum) As mpReal

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.GAMMA.INV returns the left-tailed inverse of the central Gamma-distribution

**Parameters:**

*Prob*: A real number

*a*: A real number greater 0, a parameter to the distribution

*b*: A real number greater 0, a parameter to the distribution

$$F^{-1}(y) = b \cdot igammaInv(a, y) \quad (6.7.3)$$

### 6.7.3 Properties

---

Function **GammaDistInfo**(*a* As mpNum, *b* As mpNum) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **GammaDistInfo** returns moments and related information for the central Gamma-distribution

**Parameters:**

*a*: A real number greater 0, representing the degrees of freedom

*b*: A real number greater 0, representing the degrees of freedom  
Output? String? A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.13.4.

#### 6.7.3.1 Moments

The algebraic moments are given by (Wolfram)

$$\mu'_r = \frac{b^r \Gamma(a + r)}{\Gamma(a)} \quad (6.7.4)$$

### 6.7.4 Random Numbers

---

Function **GammaDistRandom**(*Size* As mpNum, *a* As mpNum, *b* As mpNum, **Generator** As String, **Output** As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **GammaDistRandom** returns random numbers following a central Beta-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: A real number greater 0, a parameter to the distribution

*b*: A real number greater 0, a parameter to the distribution

**Generator**: A string describing the random generator

**Output**: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

#### 6.7.4.1 Random Numbers: algorithms and formulas

In the case of an Erlangian distribution ( $b$  a positive integer) we obtain a random number by adding  $b$  independent random numbers from an exponential distribution i.e.

$$x = -\ln(\xi_1 \cdot \xi_2 \cdot \dots \cdot \xi_b)/a$$

where all the  $\xi_i$  are uniform random numbers in the interval from zero to one. Note that care must be taken if  $b$  is large in which case the product of uniform random numbers may become zero due to machine precision. In such cases simply divide the product in pieces and add the logarithms afterwards.

#### 6.7.4.2 General case

In a more general case we use the so called Johnk's algorithm

1. Denote the integer part of  $b$  with  $i$  and the fractional part with  $f$  and put  $r = 0$ . Let  $\xi$  denote uniform random numbers in the interval from zero to one.
2. If  $i > 0$  then put  $r = -\ln(\xi_1 \cdot \xi_2 \cdot \dots \cdot \xi_i)$ .
3. If  $f = 0$  then go to 7.
4. Calculate  $w_1 = \xi_{i+1}^{1/f}$  and  $w_2 = \xi_{i+2}^{1/(1-f)}$ .
5. If  $w_1 + w_2 > 1$  then go back to iv.
6. Put  $r = r - \ln(\xi_{i+3}) \cdot \frac{w_1}{w_1 + w_2}$ .
7. Quit with  $r = r/a$ .

## 6.8 Hypergeometric Distribution

See [Upton \(1982\)](#), [Harkness & Katz \(1964\)](#)

See [Ling & Pratt \(1984\)](#)

See [Knüsel & Michalk \(1987\)](#)

See also [Conlon & Thomas \(1993\)](#)

See also [Casagrande \*et al.\* \(1978\)](#)

### 6.8.1 Definition

These functions return PMF and CDF of the (discrete) hypergeometric distribution; the PMF gives the probability that among  $n$  randomly chosen samples from a container with  $n_1$  type1 objects and  $n_2$  type2 objects there are exactly  $k$  type1 objects.

### 6.8.2 Density and CDF

---

Function **HypergeometricDist**(*x As mpNum, n As mpNum, M As mpNum, N As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **HypergeometricDist** returns pdf, CDF and related information for the central hypergeometric distribution

**Parameters:**

*x*: The number of successes in the sample.

*n*: The size of the sample.

*M*: The number of successes in the population

*N*: The population size

*Output*: A string describing the output choices

See section [6.1.3.1](#) for the options for *Output*. Algorithms and formulas are given in sections [6.2.2.1](#) and [6.2.2.2](#).

The following functions are provided for compatibility with established spreadsheet functions

---

**WorksheetFunction.HYPGEOMDIST**(*x As mpNum, n As mpNum, M As mpNum, N As mpNum, Cumulative As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.HYPGEOMDIST** returns pdf, CDF, and related information of the central hypergeometric distribution

**Parameters:**

*x*: The number of successes in the sample.

*n*: The size of the sample.

*M*: The number of successes in the population

*N*: The population size

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

---

WorksheetFunction.HYPGEOM.DIST(*x As mpNum, n As mpNum, M As mpNum, N As mpNum, Cumulative As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.HYPGEOM.DIST returns the CDF and pdf of the central hypergeometric distribution

#### Parameters:

*x*: The number of successes in the sample.

*n*: The size of the sample.

*M*: The number of successes in the population

*N*: The population size

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

#### 6.8.2.1 Density

$$f(k) = \frac{\binom{n_1}{k} \binom{n_2}{n-k}}{\binom{n_1+n_2}{n}}, \quad (n, n_1, n_2 \geq 0; n \leq n_1 + n_2). \quad (6.8.1)$$

*f(k)* is computed with the R trick [39], which replaces the binomial coefficients by binomial PMFs with  $p = n/(n_1 + n_2)$ .

#### 6.8.2.2 CDF

There is no explicit formula for the CDF, it is calculated as  $\sum f(i)$ , using the lower tail if  $k < nn_1/(n_1+n_2)$  and the upper tail otherwise with one value of the PMF and the recurrence formulas:

$$f(k+1) = \frac{(n_1 - k)(n - k)}{(k+1)(n_2 - n + k + 1)} f(k) \quad (6.8.2)$$

$$f(k-1) = \frac{k(n_2 - n + k)}{(n_1 - k + 1)(n - k + 1)} f(k) \quad (6.8.3)$$

#### 6.8.3 Quantiles

---

Function HypergeometricDistInv(*Prob As mpNum, n As mpNum, M As mpNum, N As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function HypergeometricDistInv returns quantiles and related information for the the central hypergeometric distribution

#### Parameters:

*Prob*: A real number between 0 and 1.

*n*: The size of the sample.

*M*: The number of successes in the population

*N*: The population size

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*).

## 6.8.4 Properties

---

Function **HypergeometricDistInfo**(*n* As mpNum, *M* As mpNum, *N* As mpNum, **Output** As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function HypergeometricDistInfo returns moments and related information for the central hypergeometric distribution

**Parameters:**

*n*: The size of the sample.

*M*: The number of successes in the population

*N*: The population size

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.13.4.

### 6.8.4.1 Moments

$$\mu_1 = nP \quad (6.8.4)$$

$$\mu_2 = nPQ \frac{N-n}{N-1} \quad (6.8.5)$$

$$\mu_3 = nPQ(Q-P) \frac{(N-n)(N-2n)}{(N-1)(N-2)} \quad (6.8.6)$$

$$\kappa_4 = \frac{6nP^2Q^2(N-n)}{N-1} \frac{n(N-n)(5N-6)-N(N-1)}{(N-2)(N-3)} \quad (6.8.7)$$

## 6.8.5 Random Numbers

---

Function **HypergeometricDistRandom**(**Size** As mpNum, *n* As mpNum, *M* As mpNum, *N* As mpNum, **Generator** As String, **Output** As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function HypergeometricDistRandom returns random numbers following a central hypergeometric distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*n*: The size of the sample.

*M*: The number of successes in the population

*N*: The population size

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

## 6.9 Lognormal Distribution

### 6.9.1 Definition

These functions return PDF, CDF, and ICDF of the lognormal distribution with location  $a$ , scale  $b > 0$ , and the support interval  $(0, +\infty)$ :

A log-normal (or lognormal) distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed. Thus, if the random variable is log-normally distributed, then has a normal distribution. Likewise, if has a normal distribution, then has a log-normal distribution. A random variable which is log-normally distributed takes only positive real values.

In a log-normal distribution  $X$ , the parameters denoted  $\mu$  and  $\sigma$  are, respectively, the mean and standard deviation of the variable's natural logarithm (by definition, the variable's logarithm is normally distributed), which means

$$X = e^{\mu + \sigma Z} \quad (6.9.1)$$

with  $Z$  a standard normal variable.

This relationship is true regardless of the base of the logarithmic or exponential function. If  $\log_a(Y)$  is normally distributed, then so is  $\log_b(Y)$ , for any two positive numbers  $a, b \neq 1$ . Likewise, if  $e^X$  is log-normally distributed, then so is  $a^X$ , where  $a$  is a positive number  $\neq 1$ .

On a logarithmic scale,  $\mu$  and  $\sigma$  can be called the location parameter and the scale parameter, respectively.

In contrast, the mean, standard deviation, and variance of the non-logarithmized sample values are respectively denoted  $m$ , *s.d.*, and  $v$  in this article. The two sets of parameters can be related as

$$\mu = \ln \left( \frac{m^2}{\sqrt{v + m^2}} \right), \quad \sigma = \sqrt{\ln \left( 1 + \frac{v}{m^2} \right)} \quad (6.9.2)$$

### 6.9.2 Density and CDF

---

Function **LogNormalDist**(*x As mpNum, mean As mpNum, stdev As mpNum, Output As String*)  
As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function LogNormalDist returns pdf, CDF and related information for the Lognormal-distribution

#### Parameters:

*x*: A real number

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.2.2.1 and 6.2.2.2.

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**LOGNORMDIST**(*x As mpReal, mean As mpNum, stdev As mpNum*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.LOGNORMDIST returns the CDF and of the Lognormal-distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

---

WorksheetFunction.**LOGNORM.DIST**(*x As mpReal, mean As mpNum, stdev As mpNum, Cumulative As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.LOGNORM.DIST returns the CDF and of the Lognormal-distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

### 6.9.2.1 Density

$$f(x) = \frac{1}{bx\sqrt{2\pi}} \exp\left(-\frac{(\ln(x) - a)^2}{2b^2}\right) \quad (6.9.3)$$

### 6.9.2.2 CDF

$$F(x) = \frac{1}{2} \left( 1 + \operatorname{erf}\left(\frac{\ln(x) - a}{b\sqrt{2}}\right) \right) \quad (6.9.4)$$

## 6.9.3 Quantiles

---

Function **LognormalDistInv**(*Prob As mpNum, mean As mpNum, stdev As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function LognormalDistInv returns quantiles and related information for the the Lognormal-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*).

The following functions are provided for compatibility with established spreadsheet functions

---

**WorksheetFunction.LOGINV(*Prob* As mpReal, *mean* As mpNum, *stdev* As mpNum) As mpReal**  
**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.LOGINV returns the two-tailed inverse of the Lognormal-distribution

**Parameters:**

*Prob*: A real number

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

---

**WorksheetFunction.LOGNORM.INV(*Prob* As mpReal, *mean* As mpNum, *stdev* As mpNum) As mpReal**  
**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.LOGNORM.INV returns the left-tailed inverse of the Lognormal-distribution

**Parameters:**

*Prob*: A real number

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

### 6.9.3.1 Quantiles: algorithms and formulas

$$F^{-1}(y) = \exp(a + b \cdot \text{normstdinv}(y)) \quad (6.9.5)$$

## 6.9.4 Properties

---

**Function LognormalDistInfo(*mean* As mpNum, *stdev* As mpNum, *Output* As String) As mp-NumList**  
**NOT YET IMPLEMENTED**

---

The function LognormalDistInfo returns moments and related information for the central Lognormal-distribution

**Parameters:**

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.13.4.

### 6.9.4.1 Moments: algorithms and formulas

Algebraic moments of the log-normal distribution are given by

$$\mu'_k = e^{k\mu + k^2\sigma^2/2} \quad (6.9.6)$$

### 6.9.5 Random Numbers

---

Function **LognormalRandom**(*Size* As mpNum, *mean* As mpNum, *stdev* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

NOT YET IMPLEMENTED

---

The function LognormalRandom returns random numbers following a central Beta-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in section 6.6.5.

#### 6.9.5.1 Random Numbers: algorithms and formulas

The most straightforward way of achieving random numbers from a log-normal distribution is to generate a random number  $u$  from a normal distribution with mean  $\mu$  and standard deviation  $\sigma$  and construct  $r = e^u$ .

## 6.10 Negative Binomial Distribution

These functions return PMF and CDF of the (discrete) negative binomial distribution with target for number of successful trials  $r > 0$  and success probability  $0 \leq p \leq 1$ .

If  $r = n$  is a positive integer the name Pascal distribution is used, and for  $r = 1$  it is called geometric distribution.

See [Ong & Lee \(1979\)](#) for information on the noncentral negative binomial distribution

### 6.10.1 Density and CDF

---

Function **NegativeBinomialDist**(*x As mpNum, r As mpNum, p As mpNum, Output As String*)  
As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function NegativeBinomialDist returns pdf, CDF and related information for the central negative binomial distribution

**Parameters:**

- x*: The number of failures in trials.
- r*: The threshold number of successes.
- p*: The probability of a success
- Output*: A string describing the output choices

See section [6.1.3.1](#) for the options for *Output*. Algorithms and formulas are given in sections [6.3.1.1](#) and [6.3.1.2](#).

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**NEGBINOMDIST**(*x As mpNum, r As mpNum, p As mpNum, Cumulative As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.NEGBINOMDIST returns pdf, CDF, and related information of the central negative binomial distribution

**Parameters:**

- x*: The number of failures in trials.
- r*: The threshold number of successes.
- p*: The probability of a success

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

---

WorksheetFunction.**NEGBINOM.DIST**(*x As mpNum, r As mpNum, p As mpNum, Cumulative As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.NEGBINOM.DIST returns the CDF and pdf of the central negative binomial distribution

**Parameters:***x*: The number of failures in trials.*r*: The threshold number of successes.*p*: The probability of a success

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

**6.10.1.1 Density**

$$f_{\text{NegBin}}(r, k; p) = \frac{\Gamma(k+r)}{k!\Gamma(r)} p^r (1-p)^k = \frac{p}{r+k} f_{\text{Beta}}(r, k+1, p) \quad (6.10.1)$$

**6.10.1.2 CDF**

$$F_{\text{NegBin}}(r, k; p) = I_{1-p}(r, k+1) = ibeta(r, k+1, 1-p) \quad (6.10.2)$$

**6.10.2 Quantiles**


---

Function **NegativeBinomialDistInv(*Prob* As mpNum, *r* As mpNum, *p* As mpNum, **Output** As String) As mpNumList**

---

**NOT YET IMPLEMENTED**

---

The function NegativeBinomialDistInv returns quantiles and related information for the the central binomial-distribution

**Parameters:***Prob*: A real number between 0 and 1.*r*: The threshold number of successes.*p*: The probability of a success*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*.

**6.10.3 Properties**


---

Function **NegativeBinomialDistInfo(*r* As mpNum, *p* As mpNum, **Output** As String) As mpNumList**

---

**NOT YET IMPLEMENTED**

---

The function NegativeBinomialDistInfo returns moments and related information for the central Binomial-distribution

**Parameters:***r*: The threshold number of successes.*p*: The probability of a success*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.13.4.

### 6.10.3.1 Moments: algorithms and formulas

$$\mu_1 = np \quad (6.10.3)$$

$$\mu_2 = np(1 - p) = npq \quad (6.10.4)$$

$$\mu_3 = npq(q + p) \quad (6.10.5)$$

$$\mu_4 = npq(3npq + 6pq + 1) \quad (6.10.6)$$

### 6.10.3.2 Recurrence relations

The following recurrence relations hold:

$$f_{\text{NegBin}}(r, k + 1; p) = \frac{(r + k)(1 - p)}{k + 1} f_{\text{NegBin}}(r, k; p) \quad (6.10.7)$$

$$f_{\text{NegBin}}(r, k - 1; p) = \frac{k}{(r + k - 1)(1 - p)} f_{\text{NegBin}}(r, k; p) \quad (6.10.8)$$

## 6.10.4 Random Numbers

Function **NegativeBinomialDistRandom**(*Size* As mpNum, *r* As mpNum, *p* As mpNum, *Generator* As String, *Output* As String) As mpNumList

**NOT YET IMPLEMENTED**

The function NegativeBinomialDistRandom returns random numbers following a central Binomial-distribution

#### Parameters:

*Size*: A positive integer up to  $10^7$

*r*: The threshold number of successes.

*p*: The probability of a success

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

### 6.10.4.1 Random Numbers: algorithms and formulas

Random numbers from a negative binomial distribution can be obtained using the algorithms outline for the beta distribution.

## 6.11 Normal Distribution

### 6.11.1 Definition

A random variable is said to follow a normal distribution with mean  $\mu$  and variance  $\sigma^2$ , if its pdf is given by 6.11.1. It is said to follow a standardized normal distribution if its pdf is given by 6.11.2.

### 6.11.2 Density and CDF

---

Function **NDist**(*x* As mpNum, *mean* As mpNum, *stdev* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function NDist returns pdf, CDF and related information for the normal-distribution

**Parameters:**

*x*: A real number

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.2.2.1 and 6.2.2.2.

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**NORMDIST**(*x* As mpReal, *mean* As mpNum, *stdev* As mpNum) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.NORMDIST returns the CDF and of the Lognormal-distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

---

WorksheetFunction.**NORM.DIST**(*x* As mpReal, *mean* As mpNum, *stdev* As mpNum, *Cumulative* As Boolean) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.NORM.DIST returns the CDF and of the Lognormal-distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

**WorksheetFunction.NORMSDIST(*x* As mpReal) As mpReal**

**NOT YET IMPLEMENTED**

The function **WorksheetFunction.NORMSDIST** returns the CDF and of the standard normal distribution

**Parameter:**

*x*: A real number. The numeric value at which to evaluate the distribution

**WorksheetFunction.NORM.S.DIST(*x* As mpReal, *Cumulative* As Boolean) As mpReal**

**NOT YET IMPLEMENTED**

The function **WorksheetFunction.NORM.S.DIST** returns the CDF and of the standard normal distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, NORM.S.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

**WorksheetFunction.GAUSS(*x* As mpReal) As mpReal**

**NOT YET IMPLEMENTED**

The function **WorksheetFunction.GAUSS** returns the CDF of the standard normal distribution

**Parameter:**

*x*: A real number. The numeric value at which to evaluate the distribution

**WorksheetFunction.PHI(*x* As mpReal) As mpReal**

**NOT YET IMPLEMENTED**

The function **WorksheetFunction.PHI** returns the pdf of the standard normal distribution

**Parameter:**

*x*: A real number. The numeric value at which to evaluate the distribution

### 6.11.2.1 Density

This functions returns the pdf of the normal distribution with mean  $\mu$  and variance  $\sigma^2$ , which is given by

$$f_N(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \quad (6.11.1)$$

The pdf of the standardized normal distribution with mean 0 and variance 1 is given by

$$\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}, \quad (6.11.2)$$

These two functions are related by

$$f_N(x; \mu, \sigma^2) = \frac{1}{\sigma} \phi\left(\frac{x - \mu}{\sigma}\right), \text{ and } \phi(u) = \sigma f_N(\mu + \sigma u) \quad (6.11.3)$$

### 6.11.2.2 CDF

This function returns the cdf of the normal distribution with mean  $\mu$  and variance  $\sigma^2$ , which is given by

$$F_N(x; \mu, \sigma^2) = \int_{-\infty}^x f_N(v) dv \quad (6.11.4)$$

The cdf of the standardized normal distribution with mean 0 and variance 1 is given by

$$\Phi(u) = \int_{-\infty}^u \phi(w) dw \quad (6.11.5)$$

These two functions are related by

$$F_N(x; \mu, \sigma^2) = \Phi\left(\frac{x - \mu}{\sigma}\right), \text{ and } \Phi(u) = F_N(\mu + \sigma u) \quad (6.11.6)$$

### 6.11.3 Quantiles

These functions return the quantile of the normal distribution with mean  $\mu$  and variance  $\sigma^2$ ,  $F_N^{-1}(\alpha; \mu, \sigma^2)$ , or the standardized normal distribution with mean 0 and variance 1,  $\Phi^{-1}(\alpha)$ .

**Function NDistInv(*Prob* As mpNum, *mean* As mpNum, *stdev* As mpNum, *Output* As String)  
As mpNumList**

**NOT YET IMPLEMENTED**

The function NDistInv returns quantiles and related information for the Lognormal-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*.

The following functions are provided for compatibility with established spreadsheet functions

**WorksheetFunction.NORMINV(*Prob* As mpReal, *mean* As mpNum, *stdev* As mpNum) As mpReal**

**NOT YET IMPLEMENTED**

The function WorksheetFunction.NORMINV returns the two-tailed inverse of the normal distribution

**Parameters:**

*Prob*: A real number

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

---

WorksheetFunction.**NORM.INV**(*Prob* As mpReal, *mean* As mpNum, *stdev* As mpNum) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.NORM.INV returns the left-tailed inverse of the normal distribution

**Parameters:**

*Prob*: A real number

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

---

WorksheetFunction.**NORMSINV**(*Prob* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.NORMSINV returns the two-tailed inverse of the standardized normal distribution

**Parameter:**

*Prob*: A real number

---

WorksheetFunction.**NORM.S.INV**(*Prob* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.NORM.S.INV returns the left-tailed inverse of the standardized normal distribution

**Parameter:**

*Prob*: A real number

#### 6.11.3.1 Quantiles: algorithms and formulas

$$F^{-1}(y) = \exp(a + b \cdot \text{normstdinv}(y)) \quad (6.11.7)$$

#### 6.11.4 Properties

---

Function **NormalDistInfo**(*mean* As mpNum, *stdev* As mpNum, *Output* As String) As mpNum-List

---

**NOT YET IMPLEMENTED**

---

The function NormalDistInfo returns moments and related information for the central Lognormal-distribution

**Parameters:**

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.13.4.

#### 6.11.4.1 Moments: algorithms and formulas

$$\begin{aligned}\kappa_1 &= \mu \\ \kappa_2 &= \sigma^2 \\ \kappa_r &= 0 \text{ for } r \geq 3.\end{aligned}$$

#### 6.11.4.2 Differential Equation

Let  $Z^{(m)}$  denote the  $m^{th}$  derivative of  $Z(x)$ . Then (Abramowitz & Stegun., 1970)

$$Z^{(1)} = -xZ(x) \quad (6.11.8)$$

$$Z^{(m+2)} + xZ^{(m+1)} + (m+1)Z^{(m)} = 0 \quad (6.11.9)$$

### 6.11.5 Random Numbers

Function **NormalRandom**(*Size As mpNum*, *mean As mpNum*, *stdev As mpNum*, *Generator As String*, *Output As String*) As mpNumList

**NOT YET IMPLEMENTED**

The function NormalRandom returns random numbers following a central Beta-distribution

#### Parameters:

*Size*: A positive integer up to  $10^7$

*mean*: A real number greater 0, representing the mean of the distribution

*stdev*: A real number greater 0, representing the standard deviation of the distribution

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in section 6.6.5.

#### 6.11.5.1 Random Numbers: algorithms and formulas

Let  $Z_1 \sim Re(0; 1)$ ,  $Z_2 \sim Re(0, 1)$  be independent random variables. Then

$X_1 = \sqrt{-2 \ln Z_1} \cos(2\pi Z_2)$  and  $X_2 = \sqrt{-2 \ln Z_1} \sin(2\pi Z_2)$  are  $\sim No(0; 1)$ .

It is also possible to directly use  $\Phi^{-1}(\alpha)$ .

## 6.12 Poisson Distribution

### 6.12.1 Definition

The Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time and/or space if these events occur with a known average rate and independently of the time since the last event. The following functions return PMF and CDF of the Poisson distribution with mean  $\mu \geq 0$ .

### 6.12.2 Density and CDF

---

Function **PoissonDist**(*x As mpNum, lambda As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function PoissonDist returns pdf, CDF and related information for the Poisson distribution

**Parameters:**

*x*: A real number

*lambda*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.4.2.1 and 6.4.2.2.

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.POISSON(*x As mpReal, deg\_freedom As mpReal, Tails As Integer*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.POISSON returns the CDF and of the Poisson distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

*Tails*: Specifies the number of distribution tails to return. If tails = 1, TDIST returns the one-tailed distribution. If tails = 2, TDIST returns the two-tailed distribution.

---

WorksheetFunction.POISSON.DIST(*x As mpReal, deg\_freedom As mpReal, Cumulative As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.POISSON.DIST returns the CDF and of the Poisson distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, POISSON.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

### 6.12.2.1 Density

$$f(k) = \frac{\mu^k}{k!} e^{-\mu} = sfcIgprefix(1 + k, \mu) \quad (6.12.1)$$

### 6.12.2.2 CDF

$$F(k) = e^{-\mu} \sum_{i=0}^k \frac{\mu^i}{i!} = igammaq(1 + k, \mu) \quad (6.12.2)$$

### 6.12.3 Quantiles

---

Function **PoissonDistInv**(*Prob* As mpNum, *lambda* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function PoissonDistInv returns quantiles and related information for the the Poisson distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*lambda*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given in section 6.12.3.1.

### 6.12.3.1 Quantiles: algorithms and formulas

The algorithms follow the one for the chisquare distribution.

### 6.12.4 Properties

---

Function **PoissonDistInfo**(*lambda* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function PoissonDistInfo returns moments and related information for the Poisson distribution

**Parameters:**

*lambda*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.4.4.

### 6.12.4.1 Moments and Cumulants

The momemnts and cumulants are given by

$$\kappa_r = \lambda \quad (6.12.3)$$

$$\mu_1 = \mu_2 = \mu_3 = \lambda \quad (6.12.4)$$

$$\mu_4 = 3\lambda^2 + \lambda \quad (6.12.5)$$

### 6.12.5 Random Numbers

---

Function **PoissonDistRan**(*Size* As mpNum, *lambda* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

NOT YET IMPLEMENTED

---

The function PoissonDistRan returns random numbers following a Poisson distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*lambda*: A real number greater 0, representing the degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in section 6.4.5.

## 6.13 Student's t-Distribution

### 6.13.1 Definition

If  $X$  is a random variable following a normal distribution with mean zero and variance unity and  $\chi^2$  is a random variable following an independent  $\chi^2$ -distribution with  $n$  degrees of freedom, then the distribution of the ratio  $\frac{X}{\sqrt{\chi^2/n}}$  is called Student's t-distribution with  $n$  degrees of freedom

### 6.13.2 Density and CDF

---

Function **TDist**(*x* As mpNum, *n* As mpNum, **Output** As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function TDist returns pdf, CDF and related information for the central *t*-distribution

**Parameters:**

*x*: A real number

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.13.2.1 and 6.13.2.2.

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**TDIST**(*x* As mpReal, **deg\_freedom** As mpReal, **Tails** As Integer) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.TDIST returns the CDF and of the central *t*-distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

*Tails*: Specifies the number of distribution tails to return. If tails = 1, TDIST returns the one-tailed distribution. If tails = 2, TDIST returns the two-tailed distribution.

---

WorksheetFunction.**T.DIST**(*x* As mpReal, **deg\_freedom** As mpReal, **Cumulative** As Boolean) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.T.DIST returns the CDF and of the central *t*-distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

**WorksheetFunction.T.DIST.RT(*x As mpReal, deg\_freedom As mpReal*) As mpReal**

**NOT YET IMPLEMENTED**

The function **WorksheetFunction.T.DIST.RT** returns the complement of the CDF and of the central *t*-distribution

**Parameters:**

*x*: A real number

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

**WorksheetFunction.T.DIST.2T(*x As mpReal, deg\_freedom As mpReal*) As mpReal**

**NOT YET IMPLEMENTED**

The function **WorksheetFunction.T.DIST.2T** returns the two-sided CDF of the central *t*-distribution

**Parameters:**

*x*: A real number

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

### 6.13.2.1 Density

The density of a variable following a central Student's t-distribution with *n* degrees of freedom is given by

$$f_t(n, x) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)} \left(\frac{n}{n+x^2}\right)^{(n+1)/2} \quad (6.13.1)$$

where  $\Gamma(\cdot)$  denotes the Gamma function (see section 4.8.6.)

### 6.13.2.2 CDF: General formulas

The cdf of a variable following a central t-distribution with *n* degrees of freedom is defined as

$$\Pr[X \leq x] = F_t(n, x) = \int_0^x f_t(n, t) dt \quad (6.13.2)$$

The cdf of the central t-distribution is calculated for any positive degrees of freedom *n* using the relationships

$$2F_t(n, x) = F_F(1, n; x^2), \quad x \leq 0 \quad (6.13.3)$$

$$F_t(n, x) - F_t(n, -x) = F_F(1, n; x^2), \quad x \geq 0 \quad (6.13.4)$$

$$F_t(n, x) = 1 - F_t(n, -x) \quad (6.13.5)$$

where  $F_F(1, n, x^2)$  denotes the cdf of the central *F*-distribution with 1 and *n* of freedom (see section 6.6.2.2).

### 6.13.2.3 CDF (central): Finite sum

The cdf can be expressed as a finite sum if *n* is an integer:

$$F_t(n, x) = \frac{1}{2} + z_n + (c_1 + c_3 + \cdots + c_{n-2}), \quad \text{for } n \text{ odd}, \quad (6.13.6)$$

$$\text{where } z_n = \frac{1}{\pi} \arctan\left(\frac{x}{\sqrt{n}}\right); a_n = \frac{1}{\sqrt{n\pi}}; b_n = \frac{n}{n+x^2}; c_k = c_{k-2}b_n(1 - 1/k) \quad (6.13.7)$$

$$F_t(n, x) = \frac{1}{2} + (c_0 + c_2 + \cdots + c_{n-2}), \quad \text{for } n \text{ even,} \quad (6.13.8)$$

$$\text{where } d_n = \frac{1}{2\sqrt{n+x^2}}; b_n = \frac{n}{n+x^2}; c_0 = xd_n; c_k = c_{k-2}b_n(1 - 1/k) \quad (6.13.9)$$

### 6.13.3 Quantiles

---

Function **TDistInv**(*Prob* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **TDistInv** returns quantiles and related information for the the central *t*-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*.

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**TINV**(*x* As mpReal, *deg\_freedom* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.TINV** returns the two-tailed inverse of the central *t*-distribution

**Parameters:**

*x*: A real number

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

---

WorksheetFunction.**T.INV**(*x* As mpReal, *deg\_freedom* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.T.INV** returns the left-tailed inverse of the central *t*-distribution

**Parameters:**

*x*: A real number

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

---

WorksheetFunction.**T.INV.2T**(*x* As mpReal, *deg\_freedom* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.T.INV.2T** returns the two-tailed inverse of the central *t*-distribution

**Parameters:**

*x*: A real number

*deg\_freedom*: An integer greater 0, indicating the degrees of freedom

### 6.13.3.1 Quantiles: algorithms and formulas

Let  $p$  be a right tail probability,  $z_\alpha$  the  $\alpha$ -quantile of the standard normal distribution, and  $n$  the degrees of freedom. Depending on  $n$  and  $p$ , one proceeds as follows:

$$t_{1,\alpha} = \tan(x), \quad \text{where } x = ((1 - \alpha) - 0.5)\pi. \quad (6.13.10)$$

$$t_{2,\alpha} = \sqrt{2x/(1-x)}, \quad \text{where } x = (2\alpha - 1)^2. \quad (6.13.11)$$

$$t_{4,\alpha} = 2\sqrt{\cos(\arccos(x)/3)/x - 1}, \quad \text{where } x = 2\sqrt{\alpha(1-\alpha)}. \quad (6.13.12)$$

Otherwise, the quantile is approximated as (Peizer & Pratt, 1968)

$$t_{n,\alpha} \approx 2\sqrt{n \exp(z_\alpha^2/d^2) - 1}, \quad \text{where } d = \frac{n - 2/3 + 1/(10n)}{\sqrt{n - 5/6}} \quad (6.13.13)$$

## 6.13.4 Properties

Function **TDistInfo**(*n* As mpNum, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

The function **TDistInfo** returns moments and related information for the central  $t$ -distribution

#### Parameters:

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.13.4.

### 6.13.4.1 Moments: algorithms and formulas

The algebraic moments (defined for  $n > r$ ) are given by

$$\mu'_r = \left(\frac{1}{2}n\right)^{r/2} \frac{\Gamma\left(\frac{1}{2}(n-r)\right)}{\Gamma\left(\frac{1}{2}n\right)}. \quad (6.13.14)$$

### 6.13.4.2 Derivatives (central)

$$f_t(n; x) = \frac{n}{x} \left[ F_t\left(n+2; x\sqrt{1+2/n}\right) - F_t(n; x) \right], \quad x \neq 0 \quad (6.13.15)$$

### 6.13.4.3 Relationships to other distributions (central)

$$F_t(n, x) = F_F\left(n, n; [n + 2x^2 + 2x\sqrt{n+x^2}] / n\right) \quad (6.13.16)$$

$$F_t(n, x) = F_B\left(\frac{1}{2}n, \frac{1}{2}n; \frac{1}{2}(x+1)/\sqrt{n+x^2}\right) \quad (6.13.17)$$

where  $F_F(\cdot)$  denotes the cdf of the central  $F$ -distribution (see section 6.6.2.2) and  $F_B(\cdot)$  denotes the cdf of the central Beta-distribution (see section 6.2.2.2).

### 6.13.5 Random Numbers

---

Function **TDistRan**(*Size* As *mpNum*, *n* As *mpNum*, *Generator* As String, *Output* As String) As *mpNumList*

---

**NOT YET IMPLEMENTED**

---

The function **TDistRan** returns random numbers following a central *t*-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*n*: A real number greater 0, representing the degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in section 6.13.5.

#### 6.13.5.1 Random Numbers: algorithms and formulas

Following the definition we may define a random number *t* from a *t*-distribution, using random numbers from a normal and a chi-square distribution, as  $t = \frac{z}{\sqrt{y_n/n}}$ , where *z* is a standard normal and *y<sub>n</sub>* a chi-squared variable with *n* degrees of freedom. To obtain random numbers from these distributions see the appropriate sections.

### 6.13.6 Behrens-Fisher Problem

See [Golhar \(1972\)](#)

## 6.14 Weibull Distribution

These functions return PDF, CDF, and ICDF of the Weibull distribution with shape parameter  $a$  and scale  $b > 0$  and the support interval  $(0, +\infty)$ :

### 6.14.1 Density and CDF

---

Function **WeibullDist**(*x As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNum-List

---

**NOT YET IMPLEMENTED**

---

The function WeibullDist returns pdf, CDF and related information for the Weibull distribution

**Parameters:**

*x*: A real number

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.2.2.1 and 6.2.2.2.

The following functions are provided for compatibility with established spreadsheet functions

---

WorksheetFunction.**WEIBULL**(*x As mpReal, a As mpNum, b As mpNum*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.WEIBULL returns the CDF and of the Weibull distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

---

WorksheetFunction.**WEIBULL.DIST**(*x As mpReal, a As mpNum, b As mpNum, Cumulative As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.WEIBULL.DIST returns the CDF and of the Weibull distribution

**Parameters:**

*x*: A real number. The numeric value at which to evaluate the distribution

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Cumulative* : A logical value that determines the form of the function. If cumulative is TRUE, T.DIST returns the cumulative distribution function; if FALSE, it returns the probability density function

### 6.14.1.1 Density

$$f(x) = \frac{x}{b^2} \exp\left(-\frac{x^2}{2b^2}\right) \exp(-(x/b)^a) \quad (6.14.1)$$

### 6.14.1.2 CDF

$$F(x) = 1 - \exp(-(x/b)^a) = -\text{expm1}(-(x/b)^a) \quad (6.14.2)$$

## 6.14.2 Quantiles

---

Function **WeibullDistInv**(*Prob* As mpNum, *a* As mpNum, *b* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function WeibullDistInv returns quantiles and related information for the the central Beta-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*).

$$F^{-1}(y) = b(-\ln 1p(-y))^{1/a} \quad (6.14.3)$$

## 6.14.3 Properties

---

Function **WeibullDistInfo**(*a* As mpNum, *b* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function WeibullDistInfo returns moments and related information for the central Beta-distribution

**Parameters:**

*a*: A real number greater 0, representing the degrees of freedom

*b*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.13.4.

### 6.14.3.1 Moments: algorithms and formulas

$$\mu'_r = \sum_{j=0}^r \binom{r}{j} \Gamma\left(\frac{r-j}{c} + 1\right) b^{r-j} \quad (6.14.4)$$

$$\mu_1 = b \Gamma\left(\frac{1}{c} + 1\right) \quad (6.14.5)$$

$$\mu_2 = b^2 \left[ \Gamma \left( \frac{1}{c} + 1 \right) \Gamma^2 \left( \frac{1}{c} + 1 \right) \right] \quad (6.14.6)$$

See [Rinne \(2008\)](#) for further details.

#### 6.14.4 Random Numbers

---

Function **WeibullDistRandom**(*Size* As mpNum, *a* As mpNum, *b* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function WeibullDistRandom returns random numbers following a central Beta-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section [6.1.3.6](#) for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in section [6.6.5](#).

## 6.15 Bernoulli Distribution

The Bernoulli distribution is a discrete distribution of the outcome of a single trial with only two results, 0 (failure) or 1 (success), with a probability of success  $p$ . The Bernoulli distribution is the simplest building block on which other discrete distributions of sequences of independent Bernoulli trials can be based. The Bernoulli is the binomial distribution ( $(k = 1, p)$ ) with only one trial.

### 6.15.1 Density and CDF

---

Function **BernoulliDistBoost(*k* As mpNum, *p* As mpNum, *Output* As String) As mpNumList**

**NOT YET IMPLEMENTED**

---

The function **BernoulliDistBoost** returns pdf, CDF and related information for the central  $t$ -distribution

**Parameters:**

*k*: A real number, 0 or 1

*p*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.15.1.1 and 6.15.1.2.

#### 6.15.1.1 Density

$$f(x) = \begin{cases} q = 1 - p & \text{for } k = 0 \\ p & \text{for } k = 1. \end{cases} \quad (6.15.1)$$

#### 6.15.1.2 CDF

$$F(x) = \begin{cases} 0 & \text{for } k = 0 \\ q & \text{for } k = 0 \\ 1 & \text{for } k = 1. \end{cases} \quad (6.15.2)$$

### 6.15.2 Quantiles

---

Function **BernoulliDistInvBoost(*Prob* As mpNum, *p* As mpNum, *Output* As String) As mpNumList**

**NOT YET IMPLEMENTED**

---

The function **BernoulliDistInvBoost** returns quantiles and related information for the the central  $t$ -distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*p*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*).

### 6.15.2.1 Quantiles: Algorithm

Using the relation:  $\text{cdf} = 1 - p$  for  $k = 0$ , else 1.

## 6.15.3 Properties

---

Function **BernoulliDistInfoBoost(*p* As mpNum, *Output* As String) As mpNumList**

---

**NOT YET IMPLEMENTED**

---

The function BernoulliDistInfoBoost returns moments and related information for the central  $t$ -distribution

**Parameters:**

*p*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.15.3.

### 6.15.3.1 Moments: algorithms and formulas

$$\mu_r' = \sum_{i=0}^{r-1} \binom{r}{i} (-1)^i p^{i+1} + (-p)^r \quad (6.15.3)$$

$$\mu_1 = p \quad (6.15.4)$$

$$\mu_2 = pq \quad (6.15.5)$$

$$\mu_3 = pq(1 - 2p) \quad (6.15.6)$$

$$\mu_4 = pq(1 - 3pq) \quad (6.15.7)$$

## 6.15.4 Random Numbers

---

Function **BernoulliDistRandomBoost(*Size* As mpNum, *p* As mpNum, *Generator* As String, *Output* As String) As mpNumList**

---

**NOT YET IMPLEMENTED**

---

The function BernoulliDistRandomBoost returns random numbers following a central Binomial-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*p*: The probability of a success on each trial.

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

## 6.16 Cauchy Distribution

### 6.16.1 Density and CDF

---

Function **CauchyDistBoost**(*x* As mpNum, *a* As mpNum, *b* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **CauchyDistBoost** returns pdf, CDF and related information for the Cauchy distribution

**Parameters:**

*x*: A real number

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*.

#### 6.16.1.1 Density

$$f(x) = \frac{1}{\pi(1 + ((x - a)/b)^2)} \quad (6.16.1)$$

#### 6.16.1.2 CDF

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x - a}{b}\right) \quad (6.16.2)$$

## 6.16.2 Quantiles

---

Function **CauchyDistInvBoost**(*Prob* As mpNum, *a* As mpNum, *b* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **CauchyDistInvBoost** returns quantiles and related information for the Cauchy distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

$$F^{-1}(y) = \begin{cases} a - b/\tan(\pi y), & y < 0.5, \\ a, & y = 0.5, \\ a - b/\tan(\pi(1 - y)) & y > 0.5. \end{cases} \quad (6.16.3)$$

### 6.16.3 Properties

---

Function **CauchyDistInfoBoost**(*a* As mpNum, *b* As mpNum, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function **CauchyDistInfoBoost** returns moments and related information for the Cauchy distribution

**Parameters:**

*a*: A real number greater 0, representing the degrees of freedom

*b*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*.

All the usual non-member accessor functions that are generic to all distributions are supported: Cumulative Distribution Function, Probability Density Function, Quantile, Hazard Function, Cumulative Hazard Function, mean, median, mode, variance, standard deviation, skewness, kurtosis, kurtosis\_excess, range and support. Note however that the Cauchy distribution does not have a mean, standard deviation, etc. See mathematically undefined function to control whether these should fail to compile with a BOOST\_STATIC\_ASSERTION\_FAILURE, which is the default. Alternately, the functions mean, standard deviation, variance, skewness, kurtosis and kurtosis\_excess will all return a domain\_error if called.

### 6.16.4 Random Numbers

---

Function **CauchyDistRandomBoost**(**Size** As mpNum, *a* As mpNum, *b* As mpNum, **Generator** As String, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function **CauchyDistRandomBoost** returns random numbers following a Cauchy distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in below.

## 6.17 Extreme Value (or Gumbel) Distribution

These functions return PDF, CDF, and ICDF of the Extreme Value Type I distribution with location  $a$ , scale  $b > 0$ , and the support interval  $(-\infty, +\infty)$ :

### 6.17.1 Density and CDF

---

Function **ExtremevalueDistBoost**(*x As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **ExtremevalueDistBoost** returns pdf, CDF and related information for the Extreme Value distribution

**Parameters:**

*x*: A real number

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.17.1.1 and 6.17.1.2.

#### 6.17.1.1 Density

$$f(x) = \frac{e^{-(x-a)/b}}{b} e^{e^{-(x-a)/b}} \quad (6.17.1)$$

#### 6.17.1.2 CDF

$$F(x) = e^{e^{-(x-a)/b}} \quad (6.17.2)$$

### 6.17.2 Quantiles

---

Function **ExtremevalueDistInvBoost**(*Prob As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **ExtremevalueDistInvBoost** returns quantiles and related information for the the Extreme Value distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

$$F^{-1}(y) = a - \ln(-\ln(y)) \quad (6.17.3)$$

### 6.17.3 Properties

---

Function **ExtremevalueDistInfoBoost**(*a* As mpNum, *b* As mpNum, **Output** As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function ExtremevalueDistInfoBoost returns moments and related information for the Extreme Value distribution

**Parameters:**

*a*: A real number greater 0, representing the degrees of freedom

*b*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*.

### 6.17.4 Random Numbers

---

Function **ExtremevalueDistRandomBoost**(**Size** As mpNum, *a* As mpNum, *b* As mpNum, **Generator** As String, **Output** As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function ExtremevalueDistRandomBoost returns random numbers following a Extreme Value distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in below.

## 6.18 Geometric Distribution

Geometric distribution: it is used when there are exactly two mutually exclusive outcomes of a Bernoulli trial: these outcomes are labelled "success" and "failure". For Bernoulli trials each with success fraction  $p$ , the geometric distribution gives the probability of observing  $k$  trials (failures, events, occurrences, or arrivals) before the first success.

### 6.18.1 Density and CDF

---

Function **GeometricDistBoost(*k* As mpNum, *p* As mpNum, *Output* As String) As mpNumList**

**NOT YET IMPLEMENTED**

---

The function **GeometricDistBoost** returns pdf, CDF and related information for the Geometric distribution

**Parameters:**

*k*: A real number

*p*: A real number greater 0, representing the numerator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.18.1.1 and 6.18.1.2.

#### 6.18.1.1 Density

$$f(k; p) = p(1 - p)^k \quad (6.18.1)$$

#### 6.18.1.2 CDF

$$F(k; p) = 1 - (1 - p)^{k+1} \quad (6.18.2)$$

### 6.18.2 Quantiles

---

Function **GeometricDistInvBoost(*Prob* As mpNum, *p* As mpNum, *Output* As String) As mpNumList**

**NOT YET IMPLEMENTED**

---

The function **GeometricDistInvBoost** returns quantiles and related information for the Geometric distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*p*: A real number greater 0, representing the numerator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

$$F^{-1}(x; p) = \frac{\log1p(-x)}{\log1p(-p)} - 1 \quad (6.18.3)$$

### 6.18.3 Properties

---

Function **GeometricDistInfoBoost**(*p* As mpNum, **Output** As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function GeometricDistInfoBoost returns moments and related information for the Geometric distribution

**Parameters:**

*p*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*.

### 6.18.4 Random Numbers

---

Function **GeometricDistRandomBoost**(*Size* As mpNum, *p* As mpNum, **Generator** As String, **Output** As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function GeometricDistRandomBoost returns random numbers following a Geometric distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*p*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in below.

## 6.19 Inverse Chi Squared Distribution

### 6.19.1 Definition

The inverse-chi-squared distribution (or inverted-chi-square distribution[1] ) is the probability distribution of a random variable whose multiplicative inverse (reciprocal) has a chi-squared distribution. It is also often defined as the distribution of a random variable whose reciprocal divided by its degrees of freedom is a chi-squared distribution. That is, if  $X$  has the chi-squared distribution with  $\nu$  degrees of freedom, then according to the first definition,  $1/X$  has the inverse-chi-squared distribution with  $\nu$  degrees of freedom; while according to the second definition,  $\nu/X$  has the inverse-chi-squared distribution with  $\nu$  degrees of freedom.

The inverse-chi-squared distribution is a special case of a inverse-gamma distribution with  $\nu$  (degrees of freedom), shape ( $\alpha$ ) and scale ( $\beta$ ), where  $\alpha = \nu/2$  and  $\beta = 1/2$ .

### 6.19.2 Density and CDF

---

Function **InverseChiSquaredDistBoost**(*x* As *mpNum*, *n* As *mpNum*, *Output* As *String*) As *mpNumList*

---

**NOT YET IMPLEMENTED**

---

The function **InverseChiSquaredDistBoost** returns pdf, CDF and related information for the inverse-chi-squared -distribution

**Parameters:**

*x*: A real number

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*.

#### 6.19.2.1 Density

The first definition yields a probability density function given by

$$f(x; \nu) = \frac{2^{-\nu/2}}{\Gamma(\nu/2)} x^{-\nu/2-1} e^{-1/(2x)} \quad (6.19.1)$$

while the second definition yields the density function

$$f(x; \nu) = \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} x^{-\nu/2-1} e^{-\nu/(2x)} \quad (6.19.2)$$

In both cases,  $x > 0$  and  $\nu$  is the degrees of freedom parameter. Further,  $\Gamma$  is the gamma function. Both definitions are special cases of the scaled-inverse-chi-squared distribution. For the first definition the variance of the distribution is  $\sigma^2 = 1/\nu$ , while for the second definition  $\sigma^2 = 1$ .

#### 6.19.2.2 CDF

$$F(x; \nu) = \frac{1}{\Gamma(\nu/2)} \Gamma\left(\frac{\nu}{2}, \frac{1}{2x}\right) \quad (6.19.3)$$

### 6.19.3 Quantiles

---

Function **InverseChiSquaredDistInvBoost**(*Prob* As mpNum, *n* As mpNum, *Output* As String)  
As mpNumList

**NOT YET IMPLEMENTED**

---

The function `InverseChiSquaredDistInvBoost` returns quantiles and related information for the inverse-chi-squared distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*).

$$F^{-1}(prob; \nu) = \beta // gamma - q - inv(\alpha, p) \quad (6.19.4)$$

### 6.19.4 Properties

---

Function **InverseChiSquaredDistInfoBoost**(*n* As mpNum, *Output* As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function `InverseChiSquaredDistInfoBoost` returns moments and related information for the inverse-chi-squared distribution

**Parameters:**

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 6.19.4.1 Moments and Cumulants

$$\mu_1 = \frac{\nu}{\nu - 2} \text{ for } \nu > 2. \quad (6.19.5)$$

### 6.19.5 Random Numbers

---

Function **InverseChiSquaredDistRanBoost**(*Size* As mpNum, *n* As mpNum, *Generator* As String, *Output* As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function `InverseChiSquaredDistRanBoost` returns numbers following a inverse-chi-squared distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*n*: A real number greater 0, representing the degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

## 6.20 Inverse Gamma Distribution

### 6.20.1 Definition

In probability theory and statistics, the inverse gamma distribution is a two-parameter family of continuous probability distributions on the positive real line, which is the distribution of the reciprocal of a variable distributed according to the gamma distribution. Perhaps the chief use of the inverse gamma distribution is in Bayesian statistics, where the distribution arises as the marginal posterior distribution for the unknown variance of a normal distribution if an uninformative prior is used; and as an analytically tractable conjugate prior if an informative prior is required.

However, it is common among Bayesians to consider an alternative parametrization of the normal distribution in terms of the precision, defined as the reciprocal of the variance, which allows the gamma distribution to be used directly as a conjugate prior. Other Bayesians prefer to parametrize the inverse gamma distribution differently, as a scaled inverse chi-squared distribution

### 6.20.2 Density and CDF

---

Function **InverseGammaDistBoost**(*x* As mpNum, *a* As mpNum, *b* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **InverseGammaDistBoost** returns pdf, CDF and related information for the inverse gamma distribution

**Parameters:**

*x*: A real number

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.20.2.1 and 6.20.2.2.

#### 6.20.2.1 Density

The inverse gamma distribution's probability density function is defined over the support  $x > 0$

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right) \quad (6.20.1)$$

with shape parameter  $\alpha$  and scale parameter  $\beta$ .

#### 6.20.2.2 CDF

The cumulative distribution function is the regularized gamma function

$$F(x; \alpha, \beta) = \frac{\Gamma(\alpha, \beta/x)}{\Gamma(\alpha)} = Q\left(\alpha, -\frac{\beta}{x}\right) \quad (6.20.2)$$

where the numerator is the upper incomplete gamma function and the denominator is the gamma function. Many math packages allow you to compute  $Q$ , the regularized gamma function, directly.

### 6.20.3 Quantiles

---

Function **InverseGammaDistInvBoost**(*Prob* As mpNum, *m* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function InverseGammaDistInvBoost returns quantiles and related information for the inverse gamma distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

$$F^{-1}(prob; \nu) = \beta // gamma - q - inv(\alpha, p) \quad (6.20.3)$$

### 6.20.4 Properties

---

Function **InverseGammaDistInfoBoost**(*a* As mpNum, *b* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function InverseGammaDistInfoBoost returns moments and related information for the inverse gamma distribution

**Parameters:**

*a*: A real number greater 0, representing the degrees of freedom

*b*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 6.20.4.1 Moments and Cumulants

$$\mu_1 = \frac{\nu}{\nu - 2} \text{ for } \nu > 2. \quad (6.20.4)$$

### 6.20.5 Random Numbers

---

Function **InverseGammaDistRanBoost**(*Size* As mpNum, *a* As mpNum, *b* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function InverseGammaDistRanBoost returns random numbers following a inverse gamma distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section [6.1.3.6](#) for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

## 6.21 Inverse Gaussian (or Wald) Distribution

### 6.21.1 Definition

In probability theory, the inverse Gaussian distribution (also known as the Wald distribution) is a two-parameter family of continuous probability distributions with mean  $\mu$  and shape parameter  $\lambda$  and support on  $(0, \infty)$ . As  $\lambda$  tends to infinity, the inverse Gaussian distribution becomes more like a normal (Gaussian) distribution. The inverse Gaussian distribution has several properties analogous to a Gaussian distribution. The name can be misleading: it is an "inverse" only in that its cumulant generating function (logarithm of the characteristic function) is the inverse of the cumulant generating function of a Gaussian random variable.

While the Gaussian describes a Brownian Motion's level at a fixed time, the inverse Gaussian describes the distribution of the time a Brownian Motion with positive drift takes to reach a fixed positive level.

See also [http://en.wikipedia.org/wiki/Inverse\\_Gaussian\\_distribution](http://en.wikipedia.org/wiki/Inverse_Gaussian_distribution).

### 6.21.2 Density and CDF

---

Function **InverseGaussianDistBoost**(*x* As mpNum, *mu* As mpNum, *lambda* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **InverseGaussianDistBoost** returns pdf, CDF and related information for the inverse Gaussian distribution

**Parameters:**

*x*: A real number

*mu*: A real number greater 0, representing the numerator degrees of freedom

*lambda*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.21.2.1 and 6.21.2.2.

#### 6.21.2.1 Density

$$f(x; \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(\frac{-\lambda(x - \mu)^2}{2\mu^2 x}\right) \quad (6.21.1)$$

#### 6.21.2.2 CDF

$$F(x; \mu, \lambda) = \Phi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right) + \exp\left(\frac{2\lambda}{\mu}\right) \Phi\left(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} + 1\right)\right) \quad (6.21.2)$$

### 6.21.3 Quantiles

---

Function **InverseGaussianDistInvBoost**(*Prob* As mpNum, *mu* As mpNum, *lambda* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function `InverseGaussianDistInvBoost` returns quantiles and related information for the the inverse Gaussian distribution

#### Parameters:

*Prob*: A real number between 0 and 1.

*mu*: A real number greater 0, representing the numerator degrees of freedom

*lambda*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

$$F^{-1}(prob; \nu) = \beta // gamma - q - inv(\alpha, p) \quad (6.21.3)$$

### 6.21.4 Properties

---

Function `InverseGaussianDistInfoBoost(mu As mpNum, lambda As mpNum, Output As String) As mpNumList`

**NOT YET IMPLEMENTED**

---

The function `InverseGaussianDistInfoBoost` returns moments and related information for the inverse Gaussian distribution

#### Parameters:

*mu*: A real number greater 0, representing the degrees of freedom

*lambda*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 6.21.4.1 Moments and Cumulants

$$\mu_1 = \frac{\nu}{\nu - 2} \text{ for } \nu > 2. \quad (6.21.4)$$

### 6.21.5 Random Numbers

---

Function `InverseGaussianDistRanBoost(Size As mpNum, mu As mpNum, lambda As mpNum, Generator As String, Output As String) As mpNumList`

**NOT YET IMPLEMENTED**

---

The function `InverseGaussianDistRanBoost` returns random numbers following a inverse Gaussian distribution

#### Parameters:

*Size*: A positive integer up to  $10^7$

*mu*: A real number greater 0, representing the numerator degrees of freedom

*lambda*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

## 6.22 Laplace Distribution

These functions return PDF, CDF, and ICDF of the Laplace distribution with location  $a$ , scale  $b > 0$ , and the support interval  $(-\infty, +\infty)$ :

### 6.22.1 Density and CDF

---

Function **LaplaceDistBoost**(*x As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **LaplaceDistBoost** returns pdf, CDF and related information for the Laplace distribution

**Parameters:**

*x*: A real number

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.22.1.1 and 6.22.1.2.

#### 6.22.1.1 Density

$$f(x) = \exp(-|x - a|/b)/(2b) \quad (6.22.1)$$

#### 6.22.1.2 CDF

$$F(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}\text{expm1}\left(-\frac{x-a}{b}\right) & x \geq a \\ \frac{1}{2}\exp\left(-\frac{x-a}{b}\right) & x < a. \end{cases} \quad (6.22.2)$$

## 6.22.2 Quantiles

---

Function **LaplaceDistInvBoost**(*Prob As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **LaplaceDistInvBoost** returns quantiles and related information for the the Laplace distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

$$F^{-1}(y) = \begin{cases} a + b \ln(2y), & y < 0.5, \\ a - b \ln(2(1-y)) & y > 0.5. \end{cases} \quad (6.22.3)$$

### 6.22.3 Properties

---

Function **LaplaceDistInfoBoost**(*a* As mpNum, *b* As mpNum, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function **LaplaceDistInfoBoost** returns moments and related information for the Laplace distribution

**Parameters:**

*a*: A real number greater 0, representing the degrees of freedom

*b*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*.

### 6.22.4 Random Numbers

---

Function **LaplaceDistRanBoost**(**Size** As mpNum, *a* As mpNum, *b* As mpNum, **Generator** As String, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function **LaplaceDistRanBoost** returns random numbers following a Laplace distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in below.

## 6.23 Logistic Distribution

### 6.23.1 Definition

These functions return PDF, CDF, and ICDF of the logistic distribution with location  $a$ , scale  $b > 0$ , and the support interval  $(-\infty, +\infty)$ :

### 6.23.2 Density and CDF

---

Function **LogisticDistBoost**(*x As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function `LogisticDistBoost` returns pdf, CDF and related information for the Logistic distribution

**Parameters:**

*x*: A real number

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.23.2.1 and 6.23.2.2.

#### 6.23.2.1 Density

$$f(x) = \frac{1}{b} \frac{\exp\left(-\frac{x-a}{b}\right)}{\left(1 + \exp\left(-\frac{x-a}{b}\right)\right)^2} \quad (6.23.1)$$

#### 6.23.2.2 CDF

$$F(x) = \frac{1}{1 + \exp\left(-\frac{x-a}{b}\right)} \quad (6.23.2)$$

### 6.23.3 Quantiles

---

Function **LogisticDistInvBoost**(*Prob As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function `LogisticDistInvBoost` returns quantiles and related information for the the Logistic distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

$$F^{-1}(y) = a - b \ln((1-y)/y) \quad (6.23.3)$$

### 6.23.4 Properties

---

Function **LogisticDistInfoBoost**(*a* As mpNum, *b* As mpNum, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function LogisticDistInfoBoost returns moments and related information for the Logistic distribution

**Parameters:**

*a*: A real number greater 0, representing the degrees of freedom

*b*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*.

### 6.23.5 Random Numbers

---

Function **LogisticDistRanBoost**(**Size** As mpNum, *a* As mpNum, *b* As mpNum, **Generator** As String, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function LogisticDistRanBoost returns random numbers following a Logistic distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in below.

## 6.24 Pareto Distribution

### 6.24.1 Definition

These functions return PDF, CDF, and ICDF of the Pareto distribution with minimum (real) value  $k > 0$ , shape  $a > 0$ , and the support interval  $(k, +\infty)$  : This is a reference: [Wikipedia contributors \(2013\)](#)

### 6.24.2 Density and CDF

---

Function **ParetoDistBoost**(*x As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function ParetoDistBoost returns pdf, CDF and related information for the Pareto distribution

**Parameters:**

*x*: A real number

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section [6.1.3.1](#) for the options for *Output*. Algorithms and formulas are given in sections [6.24.2.1](#) and [6.24.2.2](#).

#### 6.24.2.1 Density

$$f(x) = \frac{a}{x} \left(\frac{k}{x}\right)^a \quad (6.24.1)$$

#### 6.24.2.2 CDF

$$F(x) = 1 - \left(\frac{k}{x}\right)^a = -\text{powm1}(k/x, a) \quad (6.24.2)$$

### 6.24.3 Quantiles

---

Function **ParetoDistInvBoost**(*Prob As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function ParetoDistInvBoost returns quantiles and related information for the the Pareto distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section [6.1.3.2](#) for the options for *Prob* and *Output*). Algorithms and formulas are given below.

$$F^{-1}(y) = k(1 - y)^{-1/a} \quad (6.24.3)$$

#### 6.24.4 Properties

---

Function **ParetoDistInfoBoost**(*a* As mpNum, *b* As mpNum, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function ParetoDistInfoBoost returns moments and related information for the Pareto distribution

**Parameters:**

*a*: A real number greater 0, representing the degrees of freedom

*b*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*.

#### 6.24.5 Random Numbers

---

Function **ParetoDistRanBoost**(**Size** As mpNum, *a* As mpNum, *b* As mpNum, **Generator** As String, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function ParetoDistRanBoost returns random numbers following a Pareto distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: A real number greater 0, representing the numerator degrees of freedom

*b*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in below.

## 6.25 Raleigh Distribution

### 6.25.1 Definition

These functions return PDF, CDF, and ICDF of the Rayleigh distribution with scale  $b > 0$  and the support interval  $(0, +\infty)$ :

### 6.25.2 Density and CDF

---

Function **RaleighDistBoost**(*x* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function RaleighDistBoost returns pdf, CDF and related information for the Raleigh distribution

**Parameters:**

*x*: A real number

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.25.2.1 and 6.25.2.2.

#### 6.25.2.1 Density

$$f(x) = \frac{x}{b^2} \exp\left(-\frac{x^2}{2b^2}\right) \quad (6.25.1)$$

#### 6.25.2.2 CDF

$$F(x) = 1 - \exp\left(-\frac{x^2}{2b^2}\right) = -\text{expm1}\left(-\frac{x^2}{2b^2}\right) \quad (6.25.2)$$

### 6.25.3 Quantiles

---

Function **RaleighDistInvBoost**(*Prob* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function RaleighDistInvBoost returns quantiles and related information for the Raleigh distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are below.

$$F^{-1}(y) = b\sqrt{-2 \cdot \ln 1p(-y)} \quad (6.25.3)$$

### 6.25.4 Properties

---

Function **RaleighDistInfoBoost**(*n* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function RaleighDistInfoBoost returns moments and related information for the Raleigh distribution

**Parameters:**

*n*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 6.25.4.1 Moments and Cumulants

$$\mu_1 = s\sqrt{\pi/2} \quad (6.25.4)$$

### 6.25.5 Random Numbers

---

Function **RaleighDistRanBoost**(*Size* As mpNum, *n* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function RaleighDistRanBoost returns random numbers following a Raleigh distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*n*: A real number greater 0, representing the degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

## 6.26 Triangular Distribution

### 6.26.1 Definition

These functions return PDF, CDF, and ICDF of the triangular distribution on the support interval  $[a, b]$  with finite  $a < b$  and mode  $c$ ,  $a \leq c \leq b$ .

### 6.26.2 Density and CDF

---

Function **TriangularDistBoost**(*x As mpNum, a As mpNum, b As mpNum, c As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **TriangularDistBoost** returns pdf, CDF and related information for the triangular distribution

**Parameters:**

*x*: A real number.

*a*: The left border parameter.

*b*: The right border parameter.

*c*: The mode parameter.

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.26.2.1 and 6.26.2.2.

#### 6.26.2.1 Density

$$f(x) = \begin{cases} 0 & x < a \\ \frac{2(x-a)}{(b-a)(c-a)} & a \leq x < c \\ \frac{2}{b-a} & x = c \\ \frac{2(b-x)}{(b-a)(b-c)} & c < x \leq b \\ 0 & x > b \end{cases} \quad (6.26.1)$$

#### 6.26.2.2 CDF

$$F(x) = \begin{cases} 0 & x < a \\ \frac{(x-a)^2}{(b-a)(c-a)} & a \leq x < c \\ \frac{c-a}{b-a} & x = c \\ 1 - \frac{(b-x)^2}{(b-a)(b-c)} & c < x \leq b \\ 1 & x > b \end{cases} \quad (6.26.2)$$

### 6.26.3 Quantiles

---

Function **TriangularDistInvBoost**(*Prob As mpNum, a As mpNum, b As mpNum, c As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function `TriangularDistInvBoost` returns quantiles and related information for the triangular distribution

#### Parameters:

*Prob*: A real number between 0 and 1.

*a*: The left border parameter.

*b*: The right border parameter.

*c*: The mode parameter.

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*).

$$F^{-1}(y) = \begin{cases} a + \sqrt{(b-a)(c-a)y} & y < t \\ c & y = t \\ b - \sqrt{(b-a)(b-c)(1-y)} & y > t \end{cases} \quad (6.26.3)$$

where  $t = (c-a)/(b-a)$ .

### 6.26.4 Properties

---

Function **TriangularDistInfoBoost**(*a* As mpNum, *b* As mpNum, *c* As mpNum, *Output* As String)  
As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function `TriangularDistInfoBoost` returns moments and related information for the triangular distribution

#### Parameters:

*a*: The left border parameter.

*b*: The right border parameter.

*c*: The mode parameter.

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 6.26.4.1 Moments

$$\mu_1 = \frac{a+b+c}{3} \quad (6.26.4)$$

$$\mu_2 = \frac{a^2 + b^2 + c^2 - ab - ac - bc}{18} \quad (6.26.5)$$

$$\gamma_1 = \frac{\sqrt{2}(a+b-2c)(2a-b-c)(a-2b+c)}{5(a^2+b^2+c^2-ab-ac-bc)^{3/2}} \quad (6.26.6)$$

$$\gamma_2 = -\frac{3}{5} \quad (6.26.7)$$

### 6.26.5 Random Numbers

---

Function **TriangularDistRanBoost**(*Size* As mpNum, *a* As mpNum, *b* As mpNum, *c* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

NOT YET IMPLEMENTED

---

The function TriangularDistRanBoost returns random numbers following a triangular distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: The left border parameter.

*b*: The right border parameter.

*c*: The mode parameter.

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

## 6.27 Uniform Distribution

### 6.27.1 Definition

These functions return PDF, CDF, and ICDF of the uniform distribution on the support interval  $[a, b]$  with finite  $a < b$ :

### 6.27.2 Density and CDF

---

Function **UniformDistBoost**(*x As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **UniformDistBoost** returns pdf, CDF and related information for the uniform distribution

**Parameters:**

*x*: A real number

*a*: The left border parameter.

*b*: The right border parameter.

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.27.2.1 and 6.27.2.2.

#### 6.27.2.1 Density

$$f(x) = \frac{1}{b-a} \quad (6.27.1)$$

#### 6.27.2.2 CDF

$$F(x) = \frac{x-a}{b-a} \quad (6.27.2)$$

### 6.27.3 Quantiles

---

Function **UniformDistInvBoost**(*Prob As mpNum, a As mpNum, b As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **UniformDistInvBoost** returns quantiles and related information for the the uniform distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*a*: The left border parameter.

*b*: The right border parameter.

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

$$F^{-1}(y) = a + y(b - a) \quad (6.27.3)$$

## 6.27.4 Properties

---

Function **UniformDistInfoBoost**(*a* As mpNum, *b* As mpNum, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function UniformDistInfoBoost returns moments and related information for the uniform distribution

**Parameters:**

*a*: A real number greater 0, representing the degrees of freedom

*b*: A real number greater 0, representing the degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*.

### 6.27.4.1 Moments

$$\mu_1 = \frac{a + b}{2} \quad (6.27.4)$$

$$\mu_2 = \frac{(b - a)^2}{12} \quad (6.27.5)$$

$$\gamma_1 = 0 \quad (6.27.6)$$

$$\gamma_2 = -\frac{6}{5} \quad (6.27.7)$$

## 6.27.5 Random Numbers

---

Function **UniformDistRanBoost**(*Size* As mpNum, *a* As mpNum, *b* As mpNum, **Generator** As String, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

---

The function UniformDistRanBoost returns random numbers following a uniform distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: A real number greater 0, representing the degrees of freedom

*b*: A real number greater 0, representing the degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given in below.

# Chapter 7

## Statistical Functions

### 7.1 Frequencies and Percentages

#### 7.1.1 Count, CountA

---

WorksheetFunction.**COUNT**(*x As mpNum[]*) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.COUNT returns the number of cells that contain numbers, and counts numbers within the list of arguments.

**Parameter:**

*x*: An array of real numbers.

Use the COUNT function to get the number of entries in a number field that is in a range or array of numbers.

---

WorksheetFunction.**COUNTA**(*x As mpNum[]*) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.COUNTA returns the number of cells that are not empty in a range.

**Parameter:**

*x*: An array of real numbers.

The COUNTA function counts the number of cells that are not empty in a range (range: Two or more cells on a sheet. The cells in a range can be adjacent or nonadjacent.).

---

WorksheetFunction.**COUNTBLANK**(*x As mpNum[]*) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.COUNTBLANK returns the number of empty cells in a specified range of cells.

**Parameter:**

*x*: An array of real numbers.

---

WorksheetFunction.**COUNTIF**(*x* As mpNum[], *Criteria* As String) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.COUNTIF returns the number of cells within a range that meet a single criterion that you specify.

**Parameters:**

*x*: An array of real numbers.

*Criteria*: A String specifying the criteria.

For example, you can count all the cells that start with a certain letter, or you can count all the cells that contain a number that is larger or smaller than a number you specify. For example, suppose you have a worksheet that contains a list of tasks in column A, and the first name of the person assigned to each task in column B. You can use the COUNTIF function to count how many times a person's name appears in column B and, in that way, determine how many tasks are assigned to that person.

---

WorksheetFunction.**COUNTIFS**(*x* As mpNumList, *Criteria* As String[]) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.COUNTIFS returns the number of cells within multiple ranges that meet all criteria that you specify.

**Parameters:**

*x*: An array of real numbers.

*Criteria*: An array of strings specifying the criteria.

Reference to Klemens ([Klemens, 2008](#)).

See [Ogita et al. \(2005\)](#)

## 7.1.2 Frequency and 1D Histogram

---

WorksheetFunction.**FREQUENCY**(*DataArray* As mpNum[], *BinsArray* As mpNum[]) As mp-Num

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.FREQUENCY returns a vertical array of numbers, calculating how often values occur within a range of values.

**Parameters:**

*DataArray*: An array of a set of values for which you want to count frequencies.

*BinsArray*: An array of intervals into which you want to group the values in *DataArray*

For example, use FREQUENCY to count the number of test scores that fall within ranges of scores. Because FREQUENCY returns an array, it must be entered as an array formula.

If *DataArray* contains no values, FREQUENCY returns an array of zeros. If *BinsArray* contains no values, FREQUENCY returns the number of elements in *DataArray*.

FREQUENCY is entered as an array formula after you select a range of adjacent cells into which you want the returned distribution to appear. The number of elements in the returned array is one more than the number of elements in *BinsArray*. The extra element in the returned array

returns the count of any values above the highest interval. For example, when counting three ranges of values (intervals) that are entered into three cells, be sure to enter FREQUENCY into four cells for the results. The extra cell returns the number of values in **DataArray** that are greater than the third interval value. FREQUENCY ignores blank cells and text.

---

WorksheetFunction.**Histogram**(*DataArray* As mpNum[], *BinsArray* As mpNum[]) As mpNum

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.Histogram returns a vertical array of numbers, calculating how often values occur within a range of values.

**Parameters:**

*DataArray*: An array of a set of values for which you want to count frequencies.

*BinsArray*: An array of intervals into which you want to group the values in **DataArray**

This section covers Histogram, as implemented in Excel Toolpak

## 7.2 Transformations

### 7.2.1 Linear Transformations (CONVERT)

`WorksheetFunction.CONVERT(Number As mpNum, FromUnit As String, ToUnit As String)  
As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.CONVERT` returns a number converted from one measurement system to another.

#### Parameters:

*Number*: The value in *FromUnit* to convert.

*FromUnit*: The units for *Number*.

*ToUnit*: The units for the result.

For example, `CONVERT` can translate a table of distances in miles to a table of distances in kilometers.

If the input data types are incorrect, `CONVERT` returns the #VALUE! error value.

If the unit does not exist, `CONVERT` returns the #N/A error value.

If the unit does not support an abbreviated unit prefix, `CONVERT` returns the #N/A error value.

If the units are in different groups, `CONVERT` returns the #N/A error value.

Unit names and prefixes are case-sensitive. `CONVERT` accepts the following text values (in quotation marks) for *FromUnit* and *ToUnit*:

#### 7.2.1.1 Weight and mass

Gram "g"

Slug "sg"

Pound mass (avoirdupois) "lbm"

U (atomic mass unit) "u"

Ounce mass (avoirdupois) "ozm"

#### 7.2.1.2 Distance

Meter "m"

Statute mile "mi"

Nautical mile "Nmi"

Inch "in"

Foot "ft"

Yard "yd"

Angstrom "ang"

Pica "pica"

#### 7.2.1.3 Time

Year "yr"

Day "day"

Hour "hr"

Minute "mn"

Second "sec"

#### 7.2.1.4 Pressure

Pascal "Pa" (or "p")

Atmosphere "atm" (or "at")

mm of Mercury "mmHg"

#### 7.2.1.5 Force

Newton "N"

Dyne "dyn" (or "dy")

Pound force "lbf"

#### 7.2.1.6 Energy

Joule "J"

Erg "e"

Thermodynamic calorie "c"

IT calorie "cal"

Electron volt "eV" (or "ev")

Horsepower-hour "HPh" (or "hh")

Watt-hour "Wh" (or "wh")

Foot-pound "flb"

BTU "BTU" (or "btu")

#### 7.2.1.7 Power

Horsepower "HP" (or "h")

Watt "W" (or "w")

#### 7.2.1.8 Magnetism

Tesla "T"

Gauss "ga"

#### 7.2.1.9 Temperature

Degree Celsius "C" (or "cel")

Degree Fahrenheit "F" (or "fah")

Kelvin "K" (or "kel")

#### 7.2.1.10 Liquid measure

Teaspoon "tsp"

Tablespoon "tbs"

Fluid ounce "oz"

Cup "cup"

U.S. pint "pt" (or "us\_pt")

U.K. pint "uk\_pt"

Quart "qt"

Gallon "gal"  
Liter "l" (or "lt")

### 7.2.1.11 Prefix (Multiplier)

exa 1E+18 "E"  
peta 1E+15 "P"  
tera 1E+12 "T"  
giga 1E+09 "G"  
mega 1E+06 "M"  
kilo 1E+03 "k"  
hecto 1E+02 "h"  
deka 1E+01 "e"  
deci 1E-01 "d"  
centi 1E-02 "c"  
milli 1E-03 "m"  
micro 1E-06 "u"  
nano 1E-09 "n"  
pico 1E-12 "p"  
femto 1E-15 "f"  
atto 1E-18 "a"

## 7.2.2 Standardization

---

`WorksheetFunction.STANDARDIZE(Number As mpNum, Mean As mpNum, StDev As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.STANDARDIZE` returns a normalized value  $Z$  from a distribution with mean  $\mu$  and standard deviation  $\sigma$ .

**Parameters:**

*Number*: The value you want to normalize.

*Mean*: The arithmetic mean  $\mu$  of the distribution.

*StDev*: The standard deviation  $\sigma$  of the distribution.

The equation for the normalized value  $Z$  is:  $Z = \frac{x - \mu}{\sigma}$ .

$$Z = \frac{x - \mu}{\sigma}. \quad (7.2.1)$$

## 7.2.3 Trimming and Winsorizing

---

`WorksheetFunction.TRIMMEAN(X As mpNum[], Percent As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.TRIMMEAN` returns the mean of the interior of a data set.

**Parameters:**

*X*: The array or range of values to trim and average.

*Percent*: The fractional number of data points to exclude from the calculation.

Calculates the mean taken by excluding a percentage of data points from the top and bottom tails of a data set. You can use this function when you wish to exclude outlying data from your analysis. For example, if *Percent* = 0.2, 4 points are trimmed from a data set of 20 points ( $20 \times 0.2$ ): 2 from the top and 2 from the bottom of the set.

If *Percent* < 0 or *Percent* > 1, TRIMMEAN returns the #NUM! error value. TRIMMEAN rounds the number of excluded data points down to the nearest multiple of 2. If *Percent* = 0.1, 10 percent of 30 data points equals 3 points. For symmetry, TRIMMEAN excludes a single value from the top and bottom of the data set.

## 7.3 Sums, Means, Moments and Cumulants

### 7.3.1 Sum

---

WorksheetFunction.**SUM**(*x* As mpNum $[]$ ) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.SUM returns the sum of the numbers in an array

**Parameter:**

*x*: An array of real numbers.

---

WorksheetFunction.**SUMA**(*x* As mpNumList) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.SUMA returns the sum of the numbers in a list of arguments

**Parameter:**

*x*: An array of real numbers.

Text and FALSE in arguments are evaluated as zero; TRUE is evaluated as 1.

---

WorksheetFunction.**SUMIF**(*x* As mpNum $[]$ , *Criteria* As String) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.SUMIF returns the sum of cells within a range that meet a single criterion that you specify.

**Parameters:**

*x*: An array of real numbers.

*Criteria*: A String specifying the criteria.

The SUMIF function You use the SUMIF function to sum the values in a range that meets criteria that you specify. For example, suppose that in a column that contains numbers, you want to sum only the values that are larger than 5. You can use the following formula:

=SUMIF(B2:B25, ">5")

---

WorksheetFunction.**SUMIFS**(*x* As mpNumList, *Criteria* As String $[]$ ) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.SUMIFS returns the sum of cells within multiple ranges that meet all criteria that you specify.

**Parameters:**

*x*: An array of real numbers.

*Criteria*: An array of strings specifying the criteria.

### 7.3.2 Arithmetic Mean

`WorksheetFunction.AVERAGE(x As mpNum[])` As mpNum

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.AVERAGE` returns the average (arithmetic mean) of the numbers in an array

**Parameter:**

*x*: An array of real numbers.

`WorksheetFunction.AVERAGEA(x As mpNumList)` As mpNum

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.AVERAGEA` returns the average (arithmetic mean) of the numbers in a list of arguments

**Parameter:**

*x*: An array of real numbers.

Text and FALSE in arguments are evaluated as zero; TRUE is evaluated as 1.

`WorksheetFunction.AVERAGEIF(x As mpNum[], Criteria As String)` As mpNum

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.AVERAGEIF` returns the average (arithmetic mean) of cells within a range that meet a single criterion that you specify.

**Parameters:**

*x*: An array of real numbers.

*Criteria*: A String specifying the criteria.

The SUMIF function You use the SUMIF function to sum the values in a range that meets criteria that you specify. For example, suppose that in a column that contains numbers, you want to sum only the values that are larger than 5. You can use the following formula:

=AVERAGEIF(B2:B25, ">5")

`WorksheetFunction.AVERAGEIFS(x As mpNumList, Criteria As String[])` As mpNum

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.AVERAGEIFS` returns the average (arithmetic mean) of cells within multiple ranges that meet all criteria that you specify.

**Parameters:**

*x*: An array of real numbers.

*Criteria*: An array of strings specifying the criteria.

### 7.3.3 Geometric Mean

---

WorksheetFunction.**GEOMEAN**(*x* As mpNum $\text{[]}$ ) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.GEOMEAN returns the geometric mean of an array or range of positive data.

**Parameter:**

*x*: An array of real numbers.

For example, you can use GEOMEAN to calculate average growth rate given compound interest with variable rates. If any data point  $\leq 0$ , GEOMEAN returns the #NUM! error value. The equation for the geometric mean is:

$$GM = \sqrt[n]{x_1 \times x_2 \times x_3, \dots, \times x_n} \quad (7.3.1)$$

### 7.3.4 Harmonic Mean

---

WorksheetFunction.**HARMEAN**(*x* As mpNum $\text{[]}$ ) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.HARMEAN returns the harmonic mean of an array or range of positive data.

**Parameter:**

*x*: An array of real numbers.

The harmonic mean is the reciprocal of the arithmetic mean of reciprocals. If any data point  $\leq 0$ , HARMEAN returns the #NUM! error value. The equation for the harmonic mean is:

$$\frac{1}{HM} = \frac{1}{n} \sum_{i=1}^n \frac{1}{x_i} \quad (7.3.2)$$

### 7.3.5 Variance

---

WorksheetFunction.**VAR**(*x* As mpNum $\text{[]}$ ) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.VAR returns the sample variance  $s^2$  of an array or range of numerical data, ignoring non-numeric entries.

**Parameter:**

*x*: An array of real numbers.

The sample variance  $s^2$  of a sample of size  $n$  is calculated using the formula

$$s^2 = \frac{\sum(x - \bar{x})^2}{(n - 1)} \quad (7.3.3)$$

where  $\bar{x}$  denotes the arithmetic mean of the sample.

---

**WorksheetFunction.VAR.S(x As mpNum[])** As mpNum

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.VAR.S** returns the sample variance  $s^2$  of an array or range of numerical data, ignoring non-numeric entries.

**Parameter:**

*x*: An array of real numbers.

The sample variance  $s^2$  of a sample of size  $n$  is calculated as in equation 7.3.3.

---

**WorksheetFunction.VARA(x As mpNum[])** As mpNum

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.VARA** returns the sample variance  $s^2$  of an array or range of data, including text entries and FALSE as 0 and TRUE as 1.

**Parameter:**

*x*: An array of real numbers.

The sample variance  $s^2$  of a sample of size  $n$  is calculated as in equation 7.3.3.

---

**WorksheetFunction.VARP(x As mpNum[])** As mpNum

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.VARP** returns the population variance  $S^2$  of an array or range of numerical data, ignoring non-numeric entries.

**Parameter:**

*x*: An array of real numbers.

The population variance  $S^2$  of a population of size  $n$  is calculated using the formula

$$S^2 = \frac{\sum(x - \bar{x})^2}{n} \quad (7.3.4)$$

where  $\bar{x}$  denotes the arithmetic mean of the population.

---

**WorksheetFunction.VAR.P(x As mpNum[])** As mpNum

---

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.VAR.P** returns the population variance  $S^2$  of an array or range of numerical data, ignoring non-numeric entries.

**Parameter:**

*x*: An array of real numbers.

The population variance  $S^2$  of a population of size  $n$  is calculated as in equation 7.3.4.

---

**WorksheetFunction.VARPA(x As mpNum[])** As mpNum

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.VARPA` returns the population variance  $S^2$  of an array or range of data, including text entries and FALSE as 0 and TRUE as 1.

**Parameter:**

*x*: An array of real numbers.

The population variance  $S^2$  of a population of size  $n$  is calculated as in equation 7.3.4.

### 7.3.6 Standard Deviation

`WorksheetFunction.STDEV(x As mpNum[])` As mpNum

NOT YET IMPLEMENTED

The function `WorksheetFunction.STDEV` returns the sample standard deviation  $s$  of an array or range of numerical data, ignoring non-numeric entries.

**Parameter:**

*x*: An array of real numbers.

The sample standard deviation  $s$  of a sample of size  $n$  is calculated using the formula

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{(n - 1)}} \quad (7.3.5)$$

where  $\bar{x}$  denotes the arithmetic mean of the sample.

`WorksheetFunction.STDEV.S(x As mpNum[])` As mpNum

NOT YET IMPLEMENTED

The function `WorksheetFunction.STDEV.S` returns the sample standard deviation  $s$  of an array or range of numerical data, ignoring non-numeric entries.

**Parameter:**

*x*: An array of real numbers.

The sample standard deviation  $s$  of a sample of size  $n$  is calculated as in equation 7.3.3.

`WorksheetFunction.STDEVA(x As mpNum[])` As mpNum

NOT YET IMPLEMENTED

The function `WorksheetFunction.STDEVA` returns the sample standard deviation  $s$  of an array or range of data, including text entries and FALSE as 0 and TRUE as 1.

**Parameter:**

*x*: An array of real numbers.

The sample standard deviation  $s$  of a sample of size  $n$  is calculated as in equation 7.3.3.

`WorksheetFunction.STDEVP(x As mpNum[])` As mpNum

NOT YET IMPLEMENTED

The function `WorksheetFunction.STDEVP` returns the population standard deviation  $S$  of an array or range of numerical data, ignoring non-numeric entries.

**Parameter:**

*x*: An array of real numbers.

The population standard deviation  $S$  of a population of size  $n$  is calculated using the formula

$$S = \sqrt{\frac{\sum(x - \bar{x})^2}{n}} \quad (7.3.6)$$

where  $\bar{x}$  denotes the arithmetic mean of the population.

**WorksheetFunction.STDEV.P(*x* As mpNum[])** As mpNum

**NOT YET IMPLEMENTED**

The function **WorksheetFunction.STDEV.P** returns the population standard deviation  $S$  of an array or range of numerical data, ignoring non-numeric entries.

**Parameter:**

*x*: An array of real numbers.

The population standard deviation  $S$  of a population of size  $n$  is calculated as in equation 7.3.4.

**WorksheetFunction.STDEVPA(*x* As mpNum[])** As mpNum

**NOT YET IMPLEMENTED**

The function **WorksheetFunction.STDEVPA** returns the population standard deviation  $S$  of an array or range of data, including text entries and FALSE as 0 and TRUE as 1.

**Parameter:**

*x*: An array of real numbers.

The population standard deviation  $S$  of a population of size  $n$  is calculated as in equation 7.3.4.

### 7.3.7 Average Deviation

**WorksheetFunction.AVEDEV(*x* As mpNum[])** As mpNum

**NOT YET IMPLEMENTED**

The function **WorksheetFunction.AVEDEV** returns the average of the absolute deviations of data points from their mean.

**Parameter:**

*x*: An array of real numbers.

AVEDEV is a measure of the variability in a data set. The equation for AVEDEV is:

$$\text{AVEDEV} = \frac{1}{n} \sum |x - \bar{x}| \quad (7.3.7)$$

where  $\bar{x}$  denotes the arithmetic mean of the sample.

### 7.3.8 Sum of Squares of Deviations (DEVSQ)

**WorksheetFunction.DEVSQ(*x* As mpNum[])** As mpNum

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.DEVSQ` returns the sum of squares of deviations of data points from their mean.

**Parameter:**

*x*: An array of real numbers.

`DEVSQ` is a measure of the variability in a data set. The equation for `DEVSQ` is:

$$\text{DEVSQ} = \sum_{i=1}^n (x_i - \bar{x})^2 \quad (7.3.8)$$

where  $\bar{x}$  denotes the arithmetic mean of the sample.

### 7.3.9 Skewness

---

`WorksheetFunction.SKEW(x As mpNum[])` As mpNum

---

NOT YET IMPLEMENTED

---

The function `WorksheetFunction.SKEW` returns the skewness of a sample.

**Parameter:**

*x*: An array of real numbers.

Skewness characterizes the degree of asymmetry of a distribution around its mean. Positive skewness indicates a distribution with an asymmetric tail extending toward more positive values. Negative skewness indicates a distribution with an asymmetric tail extending toward more negative values. The equation for `SKEW` is:

$$\text{SKEW} = \frac{n}{(n-1)(n-2)s^3} \sum_{i=1}^n (x_i - \bar{x})^3 \quad (7.3.9)$$

where  $\bar{x}$  denotes the arithmetic mean and  $s$  denotes the standard deviation of the sample.

---

`WorksheetFunction.SKEW.P(x As mpNum[])` As mpNum

---

NOT YET IMPLEMENTED

---

The function `WorksheetFunction.SKEW.P` returns the skewness of a population.

**Parameter:**

*x*: An array of real numbers.

Skewness characterizes the degree of asymmetry of a distribution around its mean. Positive skewness indicates a distribution with an asymmetric tail extending toward more positive values. Negative skewness indicates a distribution with an asymmetric tail extending toward more negative values. The equation for `SKEW.P` is:

$$\text{SKEW.P} = \frac{1}{n\sigma^3} \sum_{i=1}^n (x_i - \bar{x})^3 \quad (7.3.10)$$

where  $\bar{x}$  denotes the arithmetic mean and  $\sigma$  denotes the standard deviation of the population.

### 7.3.10 Kurtosis

---

WorksheetFunction.KURT(*x* As mpNum $[]$ ) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.KURT returns the kurtosis of a sample.

**Parameter:**

*x*: An array of real numbers.

Kurtosis characterizes the relative peakedness or flatness of a distribution compared with the normal distribution. Positive kurtosis indicates a relatively peaked distribution. Negative kurtosis indicates a relatively flat distribution. Kurtosis is defined as:

$$\text{KURT} = \left[ \frac{n(n+1)}{(n-1)(n-2)(n-3)s^4} \sum_{i=1}^n (x_i - \bar{x})^4 \right] - \frac{3(n-1)^2}{(n-2)(n-3)} \quad (7.3.11)$$

where  $\bar{x}$  denotes the arithmetic mean and  $s$  denotes the standard deviation of the sample.

## 7.4 Min, Max, Median, Percentiles

### 7.4.1 Minimum

---

WorksheetFunction.**MIN**(*x* As *mpNum[]*) As *mpNum*

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.MIN returns the smallest number in an array, ignoring non-numerical values,

**Parameter:**

*x*: An array of real numbers.

---

WorksheetFunction.**MINA**(*x* As *mpNumList*) As *mpNum*

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.MINA returns the smallest number in an array; Text and FALSE in arguments are evaluated as zero; TRUE is evaluated as 1.

**Parameter:**

*x*: An array of real numbers.

### 7.4.2 Maximum

---

WorksheetFunction.**MAX**(*x* As *mpNum[]*) As *mpNum*

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.MAX returns the largest number in an array, ignoring non-numerical values,

**Parameter:**

*x*: An array of real numbers.

---

WorksheetFunction.**MAXA**(*x* As *mpNumList*) As *mpNum*

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.MAXA returns the largest number in an array; Text and FALSE in arguments are evaluated as zero; TRUE is evaluated as 1.

**Parameter:**

*x*: An array of real numbers.

### 7.4.3 Median

---

WorksheetFunction.**MEDIAN**(*x* As *mpNum[]*) As *mpNum*

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.MEDIAN returns the median in an array, ignoring non-numerical values.

**Parameter:**

*x*: An array of real numbers.

The median is the number in the middle of a set of numbers.

#### 7.4.4 Mode

---

WorksheetFunction.**MODE**(*x* As mpNum $[]$ ) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.MODE returns the most frequently occurring, or repetitive, value in an array or range of data.

**Parameter:**

*x*: An array of real numbers.

---

WorksheetFunction.**MODE.SNGL**(*x* As mpNum $[]$ ) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.MODE.SNGL returns the most frequently occurring, or repetitive, value in an array or range of data.

**Parameter:**

*x*: An array of real numbers.

---

WorksheetFunction.**MODE.MULT**(*x* As mpNum $[]$ ) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.MODE.MULT returns a vertical array of the most frequently occurring, or repetitive values in an array or range of data.

**Parameter:**

*x*: An array of real numbers.

For horizontal arrays, use TRANSPOSE(MODE.MULT(number1,number2,...)). This will return more than one result if there are multiple modes. Because this function returns an array of values, it must be entered as an array formula.

#### 7.4.5 *K*-th Largest Number

---

WorksheetFunction.**LARGE**(*x* As mpNum $[]$ , *k* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.LARGE returns the *k*<sup>th</sup> largest value in a data set.

**Parameters:**

*x*: The array or range of data for which you want to determine the *k*<sup>th</sup> largest value.

*k*: The position (from the largest) in the array or cell range of data to return.

You can use this function to select a value based on its relative standing. For example, you can use LARGE to return the highest, runner-up, or third-place score.

### 7.4.6 K-th Smallest Number

---

WorksheetFunction.**SMALL**(*x As mpNum[]*, *k As mpNum*) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.SMALL returns the *k*<sup>th</sup> smallest value in a data set.

**Parameters:**

- x*: The array or range of data for which you want to determine the *k*<sup>th</sup> smallest value.
- k*: The position (from the smallest) in the array or cell range of data to return.

### 7.4.7 Percentile

---

WorksheetFunction.**PERCENTILE**(*x As mpNum[]*, *k As mpNum*) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.PERCENTILE returns the k-th percentile of values in a data set as a percentage (0..1, inclusive) of the data set.

**Parameters:**

- x*: The array or range of data with numeric values that defines relative standing.
- k*: The percentile value in the range 0..1, inclusive.

---

WorksheetFunction.**PERCENTILE.INC**(*x As mpNum[]*, *k As mpNum*) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.PERCENTILE.INC returns the k-th percentile of values in a data set as a percentage (0..1, inclusive) of the data set.

**Parameters:**

- x*: The array or range of data with numeric values that defines relative standing.
- k*: The percentile value in the range 0..1, inclusive.

---

WorksheetFunction.**PERCENTILE.EXC**(*x As mpNum[]*, *k As mpNum*) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.PERCENTILE.EXC returns the k-th percentile of values in a data set as a percentage (0..1, exclusive) of the data set.

**Parameters:**

- x*: The array or range of data with numeric values that defines relative standing.
- k*: The percentile value in the range 0..1, exclusive.

You can use this function to establish a threshold of acceptance. For example, you can decide to examine candidates who score above the 90th percentile.

### 7.4.8 PercentRank

---

WorksheetFunction.**PERCENTRANK**(*Data* As mpNum[], *x* As mpNum, *digits* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.PERCENTRANK returns the rank of a value in a data set as a percentage (0..1, inclusive) of the data set.

**Parameters:**

*Data*: The array or range of data for which you want to determine the  $k^{th}$  smallest value.

*x*: The value for which you want to know the rank.

*digits*: A value that identifies the number of significant digits for the returned percentage value. If omitted, three digits (0.xxx) are used.

---

WorksheetFunction.**PERCENTRANK.INC**(*Data* As mpNum[], *x* As mpNum, *digits* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.PERCENTRANK.INC returns the rank of a value in a data set as a percentage (0..1, inclusive) of the data set.

**Parameters:**

*Data*: The array or range of data for which you want to determine the  $k^{th}$  smallest value.

*x*: The value for which you want to know the rank.

*digits*: A value that identifies the number of significant digits for the returned percentage value. If omitted, three digits (0.xxx) are used.

---

WorksheetFunction.**PERCENTRANK.EXC**(*Data* As mpNum[], *x* As mpNum, *digits* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.PERCENTRANK.EXC returns the rank of a value in a data set as a percentage (0..1, exclusive) of the data set.

**Parameters:**

*Data*: The array or range of data for which you want to determine the  $k^{th}$  smallest value.

*x*: The value for which you want to know the rank.

*digits*: A value that identifies the number of significant digits for the returned percentage value. If omitted, three digits (0.xxx) are used.

These function can be used to evaluate the relative standing of a value within a data set. For example, you can use PERCENTRANK to evaluate the standing of an aptitude test score among all scores for the test.

### 7.4.9 Quartile

---

WorksheetFunction.**QUARTILE**(*x* As mpNum[], *Quart* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.QUARTILE` returns the quartile of a data set, based on percentile values from 0..1, inclusive.

**Parameters:**

*x*: The array or cell range of numeric values for which you want the quartile value.

*Quart*: Indicates which quartile to return.

---

`WorksheetFunction.QUARTILE.INC(x As mpNum[], Quart As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.QUARTILE.INC` returns quartile of a data set, based on percentile values from 0..1, inclusive.

**Parameters:**

*x*: The array or cell range of numeric values for which you want the quartile value.

*Quart*: Indicates which quartile to return.

---

`WorksheetFunction.QUARTILE.EXC(x As mpNum[], Quart As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.QUARTILE.EXC` returns quartile of a data set, based on percentile values from 0..1, exclusive.

**Parameters:**

*x*: The array or cell range of numeric values for which you want the quartile value.

*Quart*: Indicates which quartile to return.

**Quart**: Indicates which value to return:

0: Minimum value (not for QUARTILE.EXC)

1: First quartile (25th percentile)

2: Median value (50th percentile)

3: Third quartile (75th percentile)

4: Maximum value (not for QUARTILE.EXC)

## 7.4.10 Rank

---

`WorksheetFunction.RANK(x As mpNum, Data As mpNum[], order As mpNum) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.RANK` returns the rank of a number in a list of numbers.

**Parameters:**

*x*: The number whose rank you want to find.

*Data*: The array or cell range of numeric values for which you want the rank value.

*order*: A number specifying how to rank number.

This function gives duplicate numbers the same rank. However, the presence of duplicate numbers affects the ranks of subsequent numbers. For example, in a list of integers sorted in ascending order, if the number 10 appears twice and has a rank of 5, then 11 would have a rank of 7 (no number would have a rank of 6).

**WorksheetFunction.RANK.EQ(*x* As mpNum, *Data* As mpNum[], *order* As mpNum) As mpNum**

**NOT YET IMPLEMENTED**

The function WorksheetFunction.RANK.EQ returns the rank of a number in a list of numbers.

**Parameters:**

*x*: The number whose rank you want to find.

*Data*: The array or cell range of numeric values for which you want the rank value.

*order*: A number specifying how to rank number.

This function gives duplicate numbers the same rank. However, the presence of duplicate numbers affects the ranks of subsequent numbers. For example, in a list of integers sorted in ascending order, if the number 10 appears twice and has a rank of 5, then 11 would have a rank of 7 (no number would have a rank of 6).

**WorksheetFunction.RANK.AVG(*x* As mpNum, *Data* As mpNum[], *order* As mpNum) As mpNum**

**NOT YET IMPLEMENTED**

The function WorksheetFunction.RANK.AVG returns the (average) rank of a number in a list of numbers.

**Parameters:**

*x*: The number whose rank you want to find.

*Data*: The array or cell range of numeric values for which you want the rank value.

*order*: A number specifying how to rank number.

This function returns the rank of a number in a list of numbers: its size relative to other values in the list; if more than one value has the same rank, the average rank is returned.

**Order**: A number specifying how to rank number:

0: ranks number as if ref were a list sorted in descending order.

1: ranks number as if ref were a list sorted in ascending order.

#### 7.4.11 Prob

**WorksheetFunction.PROB(*XRange* As mpNum[], *ProbRange* As mpNum[], *LowerLimit* As mpNum, *UpperLimit* As mpNum) As mpNum**

**NOT YET IMPLEMENTED**

The function WorksheetFunction.PROB returns the probability that values in a range are between two limits.

**Parameters:**

*XRange*: The range of numeric values of *x* with which there are associated probabilities.

*ProbRange*: A set of probabilities associated with values in *XRange*.

*LowerLimit*: The lower bound on the value for which you want a probability.

*UpperLimit*: The optional upper bound on the value for which you want a probability.

If *UpperLimit* is not supplied, returns the probability that values in *XRange* are equal to *LowerLimit*.

## 7.5 Summary Tables of Aggregates

### 7.5.1 SUBTOTAL

---

`WorksheetFunction.SUBTOTAL(FunctionNum As Integer, Data As mpNumList) As mpNum`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.SUBTOTAL` returns a subtotal in a list or database.

**Parameters:**

*FunctionNum*: The number that specifies which function to use in calculating subtotals within a list.

*Data*: An array of real numbers.

It is generally easier to create a list with subtotals by using the Subtotal command in the Outline group on the Data tab in the Excel desktop application. Once the subtotal list is created, you can modify it by editing the `SUBTOTAL` function.

**FunctionNum**: The number 1 to 11 (includes hidden values) or 101 to 111 (ignores hidden values) that specifies which function to use in calculating subtotals within a list.

1 (101): AVERAGE

2 (102): COUNT

3 (103): COUNTA

4 (104): MAX

5 (105): MIN

6 (106): PRODUCT

7 (107): STDEV

8 (108): STDEVP

9 (109): SUM

10 (110): VAR

11 (111): VARP

### 7.5.2 AGGREGATE

---

`WorksheetFunction.AGGREGATE(FunctionNum As Integer, Options As Integer, Data As mpNumList, k As Integer) As mpNum`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.AGGREGATE` returns a subtotal in a list or database.

**Parameters:**

*FunctionNum*: The number that specifies which function to use in calculating subtotals within a list.

*Options*: A numerical value that determines which values to ignore in the evaluation range for the function

*Data*: An array of real numbers.

*k*: A selection parameter required for the certain functions.

The `AGGREGATE` function can apply different aggregate functions to a list or database with the option to ignore hidden rows and error values.

**FunctionNum** Required. A number 1 to 19 that specifies which function to use:

- 1: AVERAGE
- 2: COUNT
- 3: COUNTA
- 4: MAX
- 5: MIN
- 6: PRODUCT
- 7: STDEV.S
- 8: STDEV.P
- 9: SUM
- 10: VAR.S
- 11: VAR.P
- 12: MEDIAN
- 13: MODE.SNGL
- 14: LARGE
- 15: SMALL
- 16: PERCENTILE.INC
- 17: QUARTILE.INC
- 18: PERCENTILE.EXC
- 19: QUARTILE.EXC

**Options** Required. A numerical value that determines which values to ignore in the evaluation range for the function:

- 0: or omitted Ignore nested SUBTOTAL and AGGREGATE functions
- 1: Ignore hidden rows, nested SUBTOTAL and AGGREGATE functions
- 2: Ignore error values, nested SUBTOTAL and AGGREGATE functions
- 3: Ignore hidden rows, error values, nested SUBTOTAL and AGGREGATE functions
- 4: Ignore nothing
- 5: Ignore hidden rows
- 6: Ignore error values
- 7: Ignore hidden rows and error values

**Data1** Required. The first numeric argument for functions that take multiple numeric arguments for which you want the aggregate value.

**k**: Required for the following functions:

- LARGE(array,k)
- SMALL(array,k)
- PERCENTILE.INC(array,k)
- QUARTILE.INC(array,quart)
- PERCENTILE.EXC(array,k)
- QUARTILE.EXC(array,quart)

## 7.6 Confidence intervals and tests

### 7.6.1 Confidence Interval for the mean, with known variance

---

WorksheetFunction.**CONFIDENCE**(*Alpha* As mpNum, *SteDev* As mpNum, *N* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.CONFIDENCE returns the confidence interval for a population mean.

**Parameters:**

*Alpha*: The significance level used to compute the confidence level. The confidence level equals  $100*(1 - \alpha)\%$ , or in other words, an alpha of 0.05 indicates a 95 percent confidence level.

*SteDev*: The population standard deviation for the data range and is assumed to be known.

*N*: The sample size.

---

WorksheetFunction.**CONFIDENCE.NORM**(*Alpha* As mpNum, *SteDev* As mpNum, *N* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.CONFIDENCE.NORM returns the confidence interval for a population mean.

**Parameters:**

*Alpha*: The significance level used to compute the confidence level. The confidence level equals  $100*(1 - \alpha)\%$ , or in other words, an alpha of 0.05 indicates a 95 percent confidence level.

*SteDev*: The population standard deviation for the data range and is assumed to be known.

*N*: The sample size.

The confidence interval is calculated as

$$\bar{x} \pm z_{\alpha} \left( \frac{\sigma}{\sqrt{n}} \right) \quad (7.6.1)$$

### 7.6.2 Confidence Interval for the mean, with unknown variance

---

WorksheetFunction.**CONFIDENCE.T**(*Alpha* As mpNum, *SteDev* As mpNum, *N* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.CONFIDENCE.T returns the confidence interval for a population mean.

**Parameters:**

*Alpha*: The significance level used to compute the confidence level. The confidence level equals  $100*(1 - \alpha)\%$ , or in other words, an alpha of 0.05 indicates a 95 percent confidence level.

*SteDev*: The population standard deviation for the data range

*N*: The sample size.

The confidence interval is calculated as

$$\bar{x} \pm t_\alpha \left( \frac{\sigma}{\sqrt{n}} \right) \quad (7.6.2)$$

### 7.6.3 Gauss z-Tests

---

`WorksheetFunction.ZTEST(X As mpNum[], Mean As mpNum, Sigma As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.ZTEST` returns the one-tailed P-value of a z-test.

**Parameters:**

*X*: The array or range of data against which to test Mean.

*Mean*: The value to test.

*Sigma*: The population (known) standard deviation. If omitted, the sample standard deviation is used.

---

`WorksheetFunction.Z.TEST(X As mpNum[], Mean As mpNum, Sigma As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.Z.TEST` returns the one-tailed P-value of a z-test.

**Parameters:**

*X*: The array or range of data against which to test Mean.

*Mean*: The value to test.

*Sigma*: The population (known) standard deviation. If omitted, the sample standard deviation is used.

$$ZTEST(X, \mu_0, \sigma) = 1 - \Phi \left( \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \right) \quad (7.6.3)$$

where  $\bar{x}$  denotes the mean of  $X$  and  $n$  denotes the sample size of  $X$ .

Placeholder for z-test for 2 samples

### 7.6.4 Student's t-Test, 2 samples

---

`WorksheetFunction.TTEST(X As mpNum[], Y As mpNum[], Tails As Integer, Type As mpNum) As mpNum`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.TTEST` returns the probability associated with a Student's t-Test.

**Parameters:**

*X*: the first data set.

*Y*: the second data set.

*Tails*: specifies the number of distribution tails. If tails = 1, TTEST uses the one-tailed distribution. If tails = 2, TTEST uses the two-tailed distribution.

*Type*: the kind of t-Test to perform. type = 1 paired, type = 2 unpaired, equal variance (homoscedastic), type = 3 unpaired, unequal variance (heteroscedastic).

---

WorksheetFunction.**T.TEST**(*X As mpNum[]*, *Y As mpNum[]*, *Tails As Integer*, *Type As mpNum*)  
As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.T.TEST returns the probability associated with a Student's t-Test.

**Parameters:**

*X*: the first data set.

*Y*: the second data set.

*Tails*: specifies the number of distribution tails. If tails = 1, TTEST uses the one-tailed distribution. If tails = 2, TTEST uses the two-tailed distribution.

*Type*: the kind of t-Test to perform. type = 1 paired, type = 2 unpaired, equal variance (homoscedastic), type = 3 unpaired, unequal variance (heteroscedastic).

### 7.6.5 F-Test (variances of 2 independent samples)

---

WorksheetFunction.**FTEST**(*X As mpNum[]*, *Y As mpNum[]*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.FTEST returns the two-tailed probability that the variances in array1 and array2 are not significantly different.

**Parameters:**

*X*: the first data set.

*Y*: the second data set.

---

WorksheetFunction.**F.TEST**(*X As mpNum[]*, *Y As mpNum[]*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.F.TEST returns the two-tailed probability that the variances in array1 and array2 are not significantly different.

**Parameters:**

*X*: the first data set.

*Y*: the second data set.

### 7.6.6 Anova: Single Factor

---

Function **ANOVA1**(*X As mpNum[]*, *Y As mpNum[]*) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function ANOVA1 returns the two-tailed probability that the variances in array1 and array2 are not significantly different.

**Parameters:**

*X*: the first data set.

*Y*: the second data set.

### 7.6.7 Anova: Two Factors (with or without replication)

---

Function **ANOVA2**(*X* As mpNum[], *Y* As mpNum[]) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function ANOVA2 returns the two-tailed probability that the variances in array1 and array2 are not significantly different.

**Parameters:**

*X*: the first data set.

*Y*: the second data set.

### 7.6.8 Chi-Square-Test (Homogeneity)

---

WorksheetFunction.**CHITEST**(*A* As mpNum[], *E* As mpNum[]) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.CHITEST returns the probability that a value of the  $\chi^2$  statistic at least as high as the value calculated by the above formula could have happened by chance under the assumption of independence.

**Parameters:**

*A*: The range of data that contains observations to test against expected values.

*E*: The range of data that contains the ratio of the product of row totals and column totals to the grand total.

---

WorksheetFunction.**CHISQ.TEST**(*A* As mpNum[], *E* As mpNum[]) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.CHISQ.TEST returns the probability that a value of the  $\chi^2$  statistic at least as high as the value calculated by the above formula could have happened by chance under the assumption of independence.

**Parameters:**

*A*: The range of data that contains observations to test against expected values.

*E*: The range of data that contains the ratio of the product of row totals and column totals to the grand total.

The  $\chi^2$  test first calculates a  $\chi^2$  statistic using the formula:

$$\chi^2 = \sum_{i=1}^r \sum_{j=1}^c \frac{(A_{ij} - E_{ij})^2}{E_{ij}}, \quad (7.6.4)$$

where:

$A_{ij}$  = actual frequency in the *i*-th row, *j*-th column

$E_{ij}$  = expected frequency in the *i*-th row, *j*-th column

*r* = number of rows

*c* = number of columns

The test then uses the  $\chi^2$  distribution with an appropriate number of degrees of freedom,  $df$ . If  $r > 1$  and  $c > 1$ , then  $df = (r - 1)(c - 1)$ . If  $r = 1$  and  $c > 1$ , then  $df = c - 1$  or if  $r > 1$  and  $c = 1$ , then  $df = r - 1$ .  $r = c = 1$  is not allowed.

## 7.7 Covariance and Correlation

### 7.7.1 Covariance

---

`WorksheetFunction.COVAR(X As mpNum[], Y As mpNum[]) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.COVAR` returns the sample covariance  $\text{cov}(X, Y)$

**Parameters:**

$X$ : An array of real numbers.

$Y$ : An array of real numbers.

The sample covariance  $\text{cov}(X, Y)$  of a sample of size  $n$  is calculated using the formula

$$\text{cov}(X, Y) = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{(n - 1)} \quad (7.7.1)$$

where  $\bar{x}$  and  $\bar{y}$  denote the arithmetic means of the samples  $X$  and  $Y$ .

---

`WorksheetFunction.COVARIANCE.S(X As mpNum[], Y As mpNum[]) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.COVARIANCE.S` returns the sample covariance  $\text{cov}(X, Y)$

**Parameters:**

$X$ : An array of real numbers.

$Y$ : An array of real numbers.

The sample covariance  $\text{cov}(X, Y)$  of a sample of size  $n$  is calculated as in equation 7.7.1

---

`WorksheetFunction.COVARIANCE.P(X As mpNum[], Y As mpNum[]) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.COVARIANCE.P` returns the sample covariance  $\text{cov}(X, Y)$

**Parameters:**

$X$ : An array of real numbers.

$Y$ : An array of real numbers.

The population covariance  $\text{COV}(X, Y)$  of a population of size  $n$  is calculated using the formula

$$\text{COV}(X, Y) = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{n} \quad (7.7.2)$$

where  $\bar{x}$  and  $\bar{y}$  denote the arithmetic means of the populations  $X$  and  $Y$ .

### 7.7.2 Correlation

---

`WorksheetFunction.CORREL(X As mpNum[], Y As mpNum[]) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.CORREL` returns the Pearson product moment correlation coefficient  $r = \text{corr}(X, Y)$

**Parameters:**

$X$ : An array of real numbers.

$Y$ : An array of real numbers.

The Pearson product moment correlation coefficient  $r = \text{corr}(X, Y)$  of a sample of size  $n$  is calculated using the formula

$$r = \text{corr}(X, Y) = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2}} \quad (7.7.3)$$

where  $\bar{x}$  and  $\bar{y}$  denote the arithmetic means of the samples  $X$  and  $Y$ .

`WorksheetFunction.PEARSON( $X$  As mpNum[],  $Y$  As mpNum[])` As mpNum

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.PEARSON` returns the Pearson product moment correlation coefficient  $r = \text{corr}(X, Y)$

**Parameters:**

$X$ : An array of real numbers.

$Y$ : An array of real numbers.

The sample covariance  $\text{cov}(X, Y)$  of a sample of size  $n$  is calculated as in equation 7.7.3

`WorksheetFunction.RSQ( $X$  As mpNum[],  $Y$  As mpNum[])` As mpNum

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.RSQ` returns  $r^2$ , the square of the Pearson product moment correlation coefficient  $r$ , with  $r = \text{corr}(X, Y)$

**Parameters:**

$X$ : An array of real numbers.

$Y$ : An array of real numbers.

$r$  is calculated as in equation 7.7.3

### 7.7.3 Fisher's z-transform

`WorksheetFunction.FISHER( $X$  As mpNum)` As mpNum

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.FISHER` returns Fisher's z-transform

**Parameter:**

$X$ : Areal numbers

The Fisher  $z$ -transform is defined by

$$Z(a) = \frac{1}{2} \log \left( \frac{1+a}{1-a} \right) = \text{atanh}(a) \quad (7.7.4)$$

WorksheetFunction.**FISHERINV(X As mpNum) As mpNum**

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.FISHERINV returns Fisher's inverse z-transform

**Parameter:**

X: Areal numbers

The inverse Fisher  $z$ -transform is defined by

$$Z^{-1}(a) = \frac{e^{2a} - 1}{e^{2a} + 1} = \tanh(a) \quad (7.7.5)$$

## 7.8 Linear Regression

### 7.8.1 INTERCEPT

---

`WorksheetFunction.INTERCEPT(X As mpNum[]], Y As mpNum[]]) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.INTERCEPT` returns the point at which a line will intersect the y-axis by using linear regression.

**Parameters:**

$X$ : An array of real numbers.

$Y$ : An array of real numbers.

The equation for the intercept of the regression line,  $a$ , is:

$$a = \bar{y} - b\bar{x}, \text{ where } b = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}, \quad (7.8.1)$$

and  $\bar{x}$  and  $\bar{y}$  denote the arithmetic means of the samples  $X$  and  $Y$ .

### 7.8.2 SLOPE

---

`WorksheetFunction.SLOPE(X As mpNum[]], Y As mpNum[]]) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.SLOPE` returns the slope of the linear regression line through data points in  $X$  and  $Y$ .

**Parameters:**

$X$ : An array of real numbers.

$Y$ : An array of real numbers.

The slope is the vertical distance divided by the horizontal distance between any two points on the line, which is the rate of change along the regression line. The equation for the slope of the regression line is:

$$b = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}, \quad (7.8.2)$$

and  $\bar{x}$  and  $\bar{y}$  denote the arithmetic means of the samples  $X$  and  $Y$ .

### 7.8.3 Forecast

---

`WorksheetFunction.FORECAST(X As mpNum[]], Y As mpNum[]]) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.FORECAST` returns an  $y_0$ -value for a given  $x_0$ -value, using linear regression.

**Parameters:**

$X$ : An array of real numbers.

$Y$ : An array of real numbers.

Calculates an  $y_0$ -value for a given  $x_0$ -value, using linear regression. The equation for FORECAST is

$$y_0 = f(x_0) = a + bx_0, \text{ where } a = \bar{y} - b\bar{x}, \text{ and } b = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}, \quad (7.8.3)$$

and  $\bar{x}$  and  $\bar{y}$  denote the arithmetic means of the samples  $X$  and  $Y$ .

## 7.8.4 SteYX

---

`WorksheetFunction.STEYX(X As mpNum[], Y As mpNum[]) As mpNum`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.STEYX` returns the standard error of the predicted  $y$ -value for each  $x$  in the regression.

**Parameters:**

$X$ : An array of real numbers.

$Y$ : An array of real numbers.

The standard error is a measure of the amount of error in the prediction of  $y$  for an individual  $x$ . The equation for `STEYX` is

$$\text{STEYX} = \sqrt{\frac{1}{n-2} \left[ \sum_{i=1}^n (y_i - \bar{y})^2 - \frac{[\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})]^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right]}, \quad (7.8.4)$$

and  $\bar{x}$  and  $\bar{y}$  denote the arithmetic means of the samples  $X$  and  $Y$ .

## 7.9 Database related functions

The Database related functions share the following arguments:

**Table:** The range of cells that makes up the list or database. A database is a list of related data in which rows of related information are records, and columns of data are fields. The first row of the list contains labels for each column.

**Field:** Indicates which column is used in the function. Enter the column label enclosed between double quotation marks, such as "Age" or "Yield," or a number (without quotation marks) that represents the position of the column within the list: 1 for the first column, 2 for the second column, and so on.

**Criteria:** The range of cells that contains the conditions that you specify. You can use any range for the criteria argument, as long as it includes at least one column label and at least one cell below the column label in which you specify a condition for the column.

### 7.9.1 DGET

---

WorksheetFunction.**DGET**(*Table* As mpNum[], *Field* As String, *Criteria* As String) As mpNum

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.DGET returns a single value from a column of a list or database that matches conditions that you specify.

**Parameters:**

*Table:* An array of real numbers, using Strings as headers.

*Field:* Indicates which column is used in the function.

*Criteria:* A String containing the criteria.

### 7.9.2 DPRODUCT

---

WorksheetFunction.**DPRODUCT**(*Table* As mpNum[], *Field* As String, *Criteria* As String) As mpNum

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.DPRODUCT returns the product of the values from a column of a list or database that matches conditions that you specify.

**Parameters:**

*Table:* An array of real numbers, using Strings as headers.

*Field:* Indicates which column is used in the function.

*Criteria:* A String containing the criteria.

### 7.9.3 DCOUNT, DCOUNTA

---

WorksheetFunction.**DCOUNT**(*Table* As mpNum[], *Field* As String, *Criteria* As String) As mpNum

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.DCOUNT` returns the number of cells that contain numbers in a field (column) of records in a list or database that match conditions that you specify.

**Parameters:**

*Table*: An array of real numbers, using Strings as headers.

*Field*: Indicates which column is used in the function.

*Criteria*: A String containing the criteria.

If *field* is omitted, `DCOUNT` counts all records in the database that match the criteria.

`WorksheetFunction.DCOUNTA(Table As mpNum[], Field As String, Criteria As String) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.DCOUNTA` returns the number of nonblank cells that contain numbers in a field (column) of records in a list or database that match conditions that you specify.

**Parameters:**

*Table*: An array of real numbers, using Strings as headers.

*Field*: Indicates which column is used in the function.

*Criteria*: A String containing the criteria.

If *field* is omitted, `DCOUNTA` counts all records in the database that match the criteria.

## 7.9.4 DSum

`WorksheetFunction.DSUM(Table As mpNum[], Field As String, Criteria As String) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.DSUM` returns the sum of cells that contain numbers in a field (column) of records in a list or database that match conditions that you specify.

**Parameters:**

*Table*: An array of real numbers, using Strings as headers.

*Field*: Indicates which column is used in the function.

*Criteria*: A String containing the criteria.

## 7.9.5 DAverage

`WorksheetFunction.DAVERAGE(Table As mpNum[], Field As String, Criteria As String) As mpNum`

**NOT YET IMPLEMENTED**

The function `WorksheetFunction.DAVERAGE` returns the arithmetic mean of cells that contain numbers in a field (column) of records in a list or database that match conditions that you specify.

**Parameters:**

*Table*: An array of real numbers, using Strings as headers.

*Field*: Indicates which column is used in the function.

*Criteria*: A String containing the criteria.

## 7.9.6 Variance

---

WorksheetFunction.DVAR(**Table** As mpNum[], **Field** As String, **Criteria** As String) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.DVAR returns the sample variance of cells that contain numbers in a field (column) of records in a list or database that match conditions that you specify.

**Parameters:**

*Table*: An array of real numbers, using Strings as headers.

*Field*: Indicates which column is used in the function.

*Criteria*: A String containing the criteria.

---

WorksheetFunction.DVARP(**Table** As mpNum[], **Field** As String, **Criteria** As String) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.DVARP returns the population variance of cells that contain numbers in a field (column) of records in a list or database that match conditions that you specify.

**Parameters:**

*Table*: An array of real numbers, using Strings as headers.

*Field*: Indicates which column is used in the function.

*Criteria*: A String containing the criteria.

## 7.9.7 Standard Deviation

---

WorksheetFunction.DSTDEV(**Table** As mpNum[], **Field** As String, **Criteria** As String) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.DSTDEV returns the sample standard deviation of cells that contain numbers in a field (column) of records in a list or database that match conditions that you specify.

**Parameters:**

*Table*: An array of real numbers, using Strings as headers.

*Field*: Indicates which column is used in the function.

*Criteria*: A String containing the criteria.

---

WorksheetFunction.DSTDEVP(**Table** As mpNum[], **Field** As String, **Criteria** As String) As mp-  
Num

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.DSTDEVP returns the population standard deviation of cells that contain numbers in a field (column) of records in a list or database that match conditions that

you specify.

**Parameters:**

*Table*: An array of real numbers, using Strings as headers.

*Field*: Indicates which column is used in the function.

*Criteria*: A String containing the criteria.

### 7.9.8 Minimum

---

WorksheetFunction.DMIN(***Table As mpNum[]***, ***Field As String***, ***Criteria As String***) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.DMIN returns the smallest number in a field (column) of records in a list or database that match conditions that you specify.

**Parameters:**

*Table*: An array of real numbers, using Strings as headers.

*Field*: Indicates which column is used in the function.

*Criteria*: A String containing the criteria.

### 7.9.9 Maximum

---

WorksheetFunction.DMAX(***Table As mpNum[]***, ***Field As String***, ***Criteria As String***) As mpNum

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.DMAX returns the largest number in a field (column) of records in a list or database that match conditions that you specify.

**Parameters:**

*Table*: An array of real numbers, using Strings as headers.

*Field*: Indicates which column is used in the function.

*Criteria*: A String containing the criteria.

# Chapter 8

## Date, Time and Financial Functions

Reference text [Benninga \(2008\)](#)

Reference text [Benninga \(2010\)](#)

Reference text [Day \(2010\)](#)

### 8.1 Date and Time: Conversions from Serial Number

Microsoft Excel stores dates as sequential serial numbers so they can be used in calculations. By default, January 1, 1900 is serial number 1, and January 1, 2008 is serial number 39448 because it is 39,448 days after January 1, 1900.

#### 8.1.1 Serial Number to Second

---

`WorksheetFunction.SECOND(Timevalue As Variant) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.SECOND` returns the seconds of a time value. The second is given as an integer in the range 0 (zero) to 59.

**Parameter:**

*Timevalue*: The time that contains the seconds you want to find.

Note: Times may be entered as text strings within quotation marks (for example, "6:45 PM"), as decimal numbers (for example, 0.78125, which represents 6:45 PM), or as results of other formulas or functions (for example, `TIMEVALUE("6:45 PM")`).

#### 8.1.2 Serial Number to Minute

---

`WorksheetFunction.MINUTE(Timevalue As Variant) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.MINUTE` returns the minutes of a time value. The minute is given as an integer, ranging from 0 to 59.

**Parameter:**

*Timevalue*: The time that contains the minute you want to find.

Note: Times may be entered as text strings within quotation marks (for example, "6:45 PM"), as decimal numbers (for example, 0.78125, which represents 6:45 PM), or as results of other formulas or functions (for example, TIMEVALUE("6:45 PM")).

### 8.1.3 Serial Number to Hour

---

`WorksheetFunction.HOUR(Timevalue As Variant) As mpReal`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.HOUR` returns the hour of a time value. The hour is given as an integer, ranging from 0 (12:00 A.M.) to 23 (11:00 P.M.).

**Parameter:**

*Timevalue*: The time that contains the hour you want to find.

Note: Times may be entered as text strings within quotation marks (for example, "6:45 PM"), as decimal numbers (for example, 0.78125, which represents 6:45 PM), or as results of other formulas or functions (for example, TIMEVALUE("6:45 PM")).

### 8.1.4 Serial Number to Day of the Month

---

`WorksheetFunction.DAY(Datevalue As Date) As mpReal`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.DAY` returns the day of a date, represented by a serial number. The day is given as an integer ranging from 1 to 31.

**Parameter:**

*Datevalue*: The date of the day you are trying to find.

Note: Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use DATE(2008,5,23) for the 23rd day of May, 2008.

### 8.1.5 Number of days between two dates

---

`WorksheetFunction.DAYS(EndDatevalue As Date, StartDatevalue As Date) As mpReal`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.DAYS` returns the number of days between two dates. *EndDateValue* and *StartDateValue* are the two dates between which you want to know the number of days.

**Parameters:**

*EndDatevalue*: The end of the time interval.

*StartDatevalue*: The start of the time interval.

### 8.1.6 Serial Number to Month

---

`WorksheetFunction.MONTH(Datevalue As Date) As mpReal`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.MONTH` returns the month of a date represented by a serial number. The month is given as an integer, ranging from 1 (January) to 12 (December).

**Parameter:**

*Datevalue*: The date of the month you are trying to find.

Note: Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use `DATE(2008,5,23)` for the 23rd day of May, 2008.

### 8.1.7 Serial Number to Year

---

`WorksheetFunction.YEAR(Datevalue As Date) As mpReal`

---

NOT YET IMPLEMENTED

---

The function `WorksheetFunction.YEAR` returns the year corresponding to a date. The year is returned as an integer in the range 1900-9999.

**Parameter:**

*Datevalue*: The date of the year you want to find.

Note: Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use `DATE(2008,5,23)` for the 23rd day of May, 2008.

### 8.1.8 Serial Number to a Day of the Week

---

`WorksheetFunction.WEEKDAY(Datevalue As Date, ReturnType As Integer) As mpReal`

---

NOT YET IMPLEMENTED

---

The function `WorksheetFunction.WEEKDAY` returns the day of the week corresponding to a date. The day is given as an integer, ranging from 1 (Sunday) to 7 (Saturday), by default.

**Parameters:**

*Datevalue*: A sequential number that represents the date of the day you are trying to find.

*ReturnType*: A number that determines the type of return value.

Note: Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use `DATE(2008,5,23)` for the 23rd day of May, 2008.

### 8.1.9 Serial Number to Calendar Week

---

`WorksheetFunction.WEEKNUM(Datevalue As Date, ReturnType As Integer) As mpReal`

---

NOT YET IMPLEMENTED

---

The function `WorksheetFunction.WEEKNUM` returns the week number of a specific date.

**Parameters:**

*Datevalue*: A date within the week.

*ReturnType*: A number that determines on which day the week begins. The default is 1.

`WorksheetFunction.WEEKNUM-ADD(Datevalue As Date, ReturnType As Integer) As mpReal`

---

NOT YET IMPLEMENTED

---

The function `WorksheetFunction.WEEKNUM-ADD` returns the week number of a specific date.

**Parameters:**

*Datevalue*: A date within the week.

*ReturnType*: A number that determines on which day the week begins. The default is 1.

---

`WorksheetFunction.ISOWEEKNUM(Datevalue As Date, ReturnType As Integer) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.ISOWEEKNUM` returns the week number of a specific date.

**Parameters:**

*Datevalue*: A date within the week.

*ReturnType*: A number that determines on which day the week begins. The default is 1.

Returns the week number of a specific date. For example, the week containing January 1 is the first week of the year, and is numbered week 1.

There are two systems used for these functions:

System 1 The week containing January 1 is the first week of the year, and is numbered week 1.

System 2 The week containing the first Thursday of the year is the first week of the year, and is numbered as week 1.

Note: Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use DATE(2008,5,23) for the 23rd day of May, 2008.

## 8.2 Date and Time: Conversions to Serial Number

Microsoft Excel stores dates as sequential serial numbers so they can be used in calculations. By default, January 1, 1900 is serial number 1, and January 1, 2008 is serial number 39448 because it is 39,448 days after January 1, 1900.

### 8.2.1 Serial Number of a particular Date

---

`WorksheetFunction.DATE(Year As mpReal, Month As mpReal, Day As mpReal) As Date`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.DATE` returns the sequential serial number that represents a particular date.

**Parameters:**

*Year*: A number that determines the Year (1900-9999).

*Month*: A number that determines the Month (1-12).

*Day*: A number that determines the day of the month (1-31).

The `DATE` function returns the sequential serial number that represents a particular date. For example, the formula `DATE(2008,7,8)` returns 39637, the serial number that represents 8th of July, 2008.

The `DATE` function is most useful in situations where the year, month, and day are supplied by formulas or cell references. For example, you might have a worksheet that contains dates in a format that Excel does not recognize, such as YYYYMMDD. You can use the `DATE` function in conjunction with other functions to convert the dates to a serial number that Excel recognizes.

### 8.2.2 Serial Number of Easter Sunday

---

`WorksheetFunction.EASTERSUNDAY(Year As mpReal) As Date`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.EASTERSUNDAY` returns the date of Easter Sunday in a given year.

**Parameter:**

*Year*: an integer between 1583 and 9956 or between 0 and 99, specifying the year.

Example: `EASTERSUNDAY(2008)` returns the date 23rd March 2008, which is the date of Easter Sunday in 2008.

### 8.2.3 Date as Text to Serial Number

---

`WorksheetFunction.DATEVALUE(DateText As String) As Date`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.DATEVALUE` returns a serial number that Excel recognizes as a date

**Parameter:**

*DateText*: Text that represents a date in an Excel date format

The DATEVALUE function converts a date that is stored as text to a serial number that Excel recognizes as a date. For example, the formula =DATEVALUE("1/1/2008") returns 39448, the serial number of the date 1/1/2008.

For example, "1/30/2008" or "30-Jan-2008" are text strings within quotation marks that represent dates. Using the default date system in Microsoft Excel for Windows, the DateText argument must represent a date between January 1, 1900 and December 31, 9999.

## 8.2.4 Serial Number of Months before or after Start Date

---

WorksheetFunction.EDATE(*StartDate* As Date, *Months* As Integer) As Date

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.EDATE returns a serial number that Excel recognizes as a date

**Parameters:**

*StartDate*: A date that represents the start date.

*Months*: The number of months before or after StartDate. A positive value for months yields a future date; a negative value yields a past date.

Returns the serial number that represents the date that is the indicated number of months before or after a specified date (the StartDate). Use EDATE to calculate maturity dates or due dates that fall on the same day of the month as the date of issue.

Note: Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use DATE(2008,5,23) for the 23rd day of May, 2008. Problems can occur if dates are entered as text.

## 8.2.5 Serial Number of the last day of the months

---

WorksheetFunction.EOMONTH(*StartDate* As Date, *Months* As Integer) As Date

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.EOMONTH returns a serial number that Excel recognizes as a date

**Parameters:**

*StartDate*: A date that represents the start date.

*Months*: The number of months before or after StartDate. A positive value for months yields a future date; a negative value yields a past date.

Returns the serial number for the last day of the month that is the indicated number of months before or after StartDate. Use EOMONTH to calculate maturity dates or due dates that fall on the last day of the month.

Note: Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use DATE(2008,5,23) for the 23rd day of May, 2008. Problems can occur if dates are entered as text.

### 8.2.6 Serial Number of the current date and time

---

WorksheetFunction.**NOW()** As Date

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.NOW returns the serial number of the current date and time.

### 8.2.7 Serial Number of a particular Time

---

WorksheetFunction.**TIME(*Hour* As mpReal, *Minute* As mpReal, *Second* As mpReal)** As Date

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.TIME returns the decimal number for a particular time. If the cell format was General before the function was entered, the result is formatted as a date.

**Parameters:**

*Hour*: A number from 0 (zero) to 32767 representing the hour. Any value greater than 23 will be divided by 24 and the remainder will be treated as the hour value. For example, TIME(27,0,0) = TIME(3,0,0) = .125 or 3:00 AM.

*Minute*: A number from 0 to 32767 representing the minute. Any value greater than 59 will be converted to hours and minutes. For example, TIME(0,750,0) = TIME(12,30,0) = .520833 or 12:30 PM.

*Second*: A number from 0 to 32767 representing the second. Any value greater than 59 will be converted to hours, minutes, and seconds. For example, TIME(0,0,2000) = TIME(0,33,22) = .023148 or 12:33:20 AM.

The decimal number returned by TIME is a value ranging from 0 (zero) to 0.99999999, representing the times from 0:00:00 (12:00:00 AM) to 23:59:59 (11:59:59 P.M.).

### 8.2.8 Time as Text to Serial Number

---

WorksheetFunction.**TIMEVALUE(*TimeText* As String)** As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.TIMEVALUE returns the decimal number of the time represented by a text string. The decimal number is a value ranging from 0 (zero) to 0.99999999, representing the times from 0:00:00 (12:00:00 AM) to 23:59:59 (11:59:59 P.M.).

**Parameter:**

*TimeText*: A text string that represents a time in any one of the Microsoft Excel time formats.

For example, "6:45 PM" and "18:45" text strings within quotation marks that represent time.

### 8.2.9 Serial Number of today's Date

---

WorksheetFunction.**TODAY()** As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.TODAY returns the serial number of the current date.

### 8.2.10 Serial Number of Date +/- n Workdays

---

WorksheetFunction.**WORKDAY**(*StartDate* As Date, *Days* As Integer, *Holidays* As DateList) As Date

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.WORKDAY returns a number that represents a date that is the indicated number of working days before or after a date (the starting date).

**Parameters:**

*StartDate*: A date that represents the start date.

*Days*: The number of nonweekend and nonholiday days before or after StartDate. A positive value for days yields a future date; a negative value yields a past date.

*Holidays*: An optional list of one or more dates to exclude from the working calendar

Note: Working days exclude weekends and any dates identified as holidays. Use WORKDAY to exclude weekends or holidays when you calculate invoice due dates, expected delivery times, or the number of days of work performed.

To calculate the serial number of the date before or after a specified number of workdays by using parameters to indicate which and how many days are weekend days, use the WORKDAY.INTL function.

*Holidays*: Optional. An optional list of one or more dates to exclude from the working calendar, such as state and federal holidays and floating holidays. The list can be either a range of cells that contain the dates or an array constant (array: Used to build single formulas that produce multiple results or that operate on a group of arguments that are arranged in rows and columns. An array range shares a common formula; an array constant is a group of constants used as an argument.) of the serial numbers that represent the dates.

### 8.2.11 Serial Number of Date +/- n Workdays, international

---

WorksheetFunction.**WORKDAY.INTL**(*StartDate* As Date, *Days* As Integer, *Weekend* As Integer, *Holidays* As DateList) As Date

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.WORKDAY.INTL returns a number that represents a date that is the indicated number of working days before or after a date (the starting date).

**Parameters:**

*StartDate*: The start date, truncated to integer.

*Days*: The number of nonweekend and nonholiday days before or after StartDate. A positive value for days yields a future date; a negative value yields a past date. Day-offset is truncated to an integer.

*Weekend*: Indicates the days of the week that are weekend days and are not considered working days.

*Holidays*: An optional list of one or more dates to exclude from the working calendar

Note: Weekend parameters indicate which and how many days are weekend days. Weekend days and any days that are specified as holidays are not considered as workdays

*Weekend*: Optional. Indicates the days of the week that are weekend days and are not considered working days. Weekend is a weekend number or string that specifies when weekends occur.

Weekend string values are seven characters long and each character in the string represents a day of the week, starting with Monday. 1 represents a non-workday and 0 represents a workday. Only the characters 1 and 0 are permitted in the string. 1111111 is an invalid string.

For example, 0000011 would result in a weekend that is Saturday and Sunday.

Holidays shall be a range of cells that contain the dates, or an array constant of the serial values that represent those dates. The ordering of dates or serial values in holidays can be arbitrary.

## 8.3 Date and Time: Calculations

---

`WorksheetFunction.DAYS360(StartDate As Date, EndDate As Date, Method As Boolean) As Date`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.DAYS360` returns the number of days between two dates based on a 360-day year (twelve 30-day months), which is used in some accounting calculations

**Parameters:**

*StartDate*: A date that represents the start date.

*EndDate*: A date that represents the end date

*Method*: A logical value that specifies whether to use the U.S. or European method in the calculation

Use this function to help compute payments if your accounting system is based on twelve 30-day months

*StartDate*, *EndDate*: The two dates between which you want to know the number of days. If *StartDate* occurs after *EndDate*, the `DAYS360` function returns a negative number. Dates should be entered by using the `DATE` function, or derived from the results of other formulas or functions. For example, use `DATE(2008,5,23)` to return the 23rd day of May, 2008.

**Method:**

`FALSE` or omitted: U.S. (NASD) method. If the starting date is the last day of a month, it becomes equal to the 30th day of the same month. If the ending date is the last day of a month and the starting date is earlier than the 30th day of a month, the ending date becomes equal to the 1st day of the next month; otherwise the ending date becomes equal to the 30th day of the same month.

`TRUE`: European method. Starting dates and ending dates that occur on the 31st day of a month become equal to the 30th day of the same month.

### 8.3.1 Number of whole workdays between two dates

---

`WorksheetFunction.NETWORKDAYS(StartDate As Date, EndDate As Date, Holidays As DateList) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.NETWORKDAYS` returns the number of whole working days between *StartDate* and *EndDate*.

**Parameters:**

*StartDate*: A date that represents the start date.

*EndDate*: A date that represents the end date

*Holidays*: An optional range of one or more dates to exclude from the working calendar, such as state and federal holidays and floating holidays

Returns the number of whole working days between *StartDate* and *EndDate*. Working days exclude weekends and any dates identified in *holidays*. Use `NETWORKDAYS` to calculate employee benefits that accrue based on the number of days worked during a specific term.

To calculate whole workdays between two dates by using parameters to indicate which and how many days are weekend days, use the NETWORKDAYS.INTL function.

Holidays: Optional. An optional range of one or more dates to exclude from the working calendar, such as state and federal holidays and floating holidays. The list can be either a range of cells that contains the dates or an array constant (array: Used to build single formulas that produce multiple results or that operate on a group of arguments that are arranged in rows and columns. An array range shares a common formula; an array constant is a group of constants used as an argument.) of the serial numbers that represent the dates.

Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use DATE(2008,5,23) for the 23rd day of May, 2008.

### 8.3.2 Number of whole workdays between two dates, international

---

WorksheetFunction.NETWORKDAYS.INTL(*StartDate As Date, EndDate As Date, Weekend As String*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.NETWORKDAYS.INTL returns the number of whole working days between StartDate and EndDate.

#### Parameters:

*StartDate*: A date that represents the start date.

*EndDate*: A date that represents the end date

*Weekend*: Indicates the days of the week that are weekend days and are not included in the number of whole working days between StartDate and EndDate. Holidays? DateList? An optional range of one or more dates to exclude from the working calendar, such as state and federal holidays and floating holidays

Working days exclude weekends and any dates identified in holidays. Use NETWORKDAYS to calculate employee benefits that accrue based on the number of days worked during a specific term.

*Weekend*: Optional. Indicates the days of the week that are weekend days and are not included in the number of whole working days between StartDate and EndDate. Weekend is a weekend number or string that specifies when weekends occur. Weekend string values are seven characters long and each character in the string represents a day of the week, starting with Monday. 1 represents a non-workday and 0 represents a workday. Only the characters 1 and 0 are permitted in the string. Using 1111111 will always return 0.

For example, 0000011 would result in a weekend that is Saturday and Sunday.

Holidays: Optional. An optional set of one or more dates that are to be excluded from the working day calendar. holidays shall be a range of cells that contain the dates, or an array constant of the serial values that represent those dates. The ordering of dates or serial values in holidays can be arbitrary.

Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use DATE(2008,5,23) for the 23rd day of May, 2008.

### 8.3.3 Year Fraction representing whole days between 2 Dates

---

WorksheetFunction.**YEARFRAC**(*StartDate* As Date, *EndDate* As Date, *Basis* As DateList) As  
mpReal

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.YEARFRAC returns the fraction of the year represented by the number of whole days between two dates.

**Parameters:**

*StartDate*: A date that represents the start date.

*EndDate*: A date that represents the end date

*Basis*: The type of day count basis to use. Basis Day count basis

Use the YEARFRAC worksheet function to identify the proportion of a whole year's benefits or obligations to assign to a specific term.

*StartDate*: A date that represents the start date.

*EndDate*: A date that represents the end date.

*Basis*:

0 or omitted: US (NASD) 30/360

1: Actual/actual

2: Actual/360

3: Actual/365

4: European 30/360

Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use DATE(2008,5,23) for the 23rd day of May, 2008.

## 8.4 Coupons

The Coupons functions share the following arguments and terminology:

*SettlementDate*: The security's settlement date. The security settlement date is the date after the issue date when the security is traded to the buyer.

*MaturityDate*: The security's maturity date. The maturity date is the date when the security expires.

*Frequency*: The number of coupon payments per year. For annual payments, frequency = 1; for semiannual, frequency = 2; for quarterly, frequency = 4.

*Yield*: The security's annual yield.

*Coupon*: Coupon payments per year.

*Basis*: Optional. The type of day count basis to use:

0 or omitted: US (NASD) 30/360

1: Actual/actual

2: Actual/360

3: Actual/365

4: European 30/360

For example, suppose a 30-year bond is issued on January 1, 2008, and is purchased by a buyer six months later. For example, suppose a 30-year bond is issued on January 1, 2008, and is purchased by a buyer six months later. The issue date would be January 1, 2008, the settlement date would be July 1, 2008, and the maturity date is January 1, 2038, 30 years after the January 1, 2008 issue date.

Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use DATE(2008,5,23) for the 23rd day of May, 2008.

### 8.4.1 Days from Beginning to Settlement Date

---

WorksheetFunction.COUPDAYBS(*SettlementDate* As Date, *MaturityDate* As Date, *Frequency* As mpReal, *Basis* As Integer) As mpReal

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.COUPDAYBS returns the number of days from the beginning of the coupon period to the settlement date.

#### Parameters:

*SettlementDate*: The security's settlement date.

*MaturityDate*: The security's maturity date.

*Frequency*: The number of coupon payments per year.

*Basis*: The type of day count basis to use

### 8.4.2 Days in Coupon Period containing the Settlement Date

---

WorksheetFunction.**COUPDAYS**(*SettlementDate* As Date, *MaturityDate* As Date, *Frequency* As mpReal, *Basis* As Integer) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.COUPDAYS returns the number of days in the coupon period that contains the settlement date.

**Parameters:**

*SettlementDate*: The security's settlement date.

*MaturityDate*: The security's maturity date.

*Frequency*: The number of coupon payments per year.

*Basis*: The type of day count basis to use

### 8.4.3 Days from Settlement Date to next Coupon Date

---

WorksheetFunction.**COUPDAYSNC**(*SettlementDate* As Date, *MaturityDate* As Date, *Frequency* As mpReal, *Basis* As Integer) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.COUPDAYSNC returns the number of days from the settlement date to the next coupon date.

**Parameters:**

*SettlementDate*: The security's settlement date.

*MaturityDate*: The security's maturity date.

*Frequency*: The number of coupon payments per year.

*Basis*: The type of day count basis to use

### 8.4.4 Next Coupon Date after the Settlement Date

---

WorksheetFunction.**COUPNCD**(*SettlementDate* As Date, *MaturityDate* As Date, *Frequency* As mpReal, *Basis* As Integer) As Date

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.COUPNCD returns a number that represents the next coupon date after the settlement date.

**Parameters:**

*SettlementDate*: The security's settlement date.

*MaturityDate*: The security's maturity date.

*Frequency*: The number of coupon payments per year.

*Basis*: The type of day count basis to use

### 8.4.5 Coupons payable between Settlement and Maturity Date

---

**WorksheetFunction.COUPNUM(*SettlementDate* As Date, *MaturityDate* As Date, *Frequency* As mpReal, *Basis* As Integer) As mpReal**

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.COUPNUM returns the number of coupons payable between the settlement date and maturity date, rounded up to the nearest whole coupon.

**Parameters:**

*SettlementDate*: The security's settlement date.

*MaturityDate*: The security's maturity date.

*Frequency*: The number of coupon payments per year.

*Basis*: The type of day count basis to use

### 8.4.6 Previous Coupon Date before the Settlement Date

---

**WorksheetFunction.COUPPCD(*SettlementDate* As Date, *MaturityDate* As Date, *Frequency* As mpReal, *Basis* As Integer) As Date**

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.COUPPCD returns a number that represents the previous coupon date before the settlement date.

**Parameters:**

*SettlementDate*: The security's settlement date.

*MaturityDate*: The security's maturity date.

*Frequency*: The number of coupon payments per year.

*Basis*: The type of day count basis to use

### 8.4.7 Macauley Duration for an assumed par Value of 100

---

**WorksheetFunction.DURATION(*SettlementDate* As Date, *MaturityDate* As Date, *Coupon* As Integer, *Yield* As Integer, *Frequency* As mpReal, *Basis* As Integer) As mpReal**

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.DURATION returns the Macauley duration for an assumed par value of \$100.

**Parameters:**

*SettlementDate*: The security's settlement date.

*MaturityDate*: The security's maturity date.

*Coupon*: Coupon payments per year

*Yield*: The security's annual yield.

*Frequency*: The number of coupon payments per year.

*Basis*: The type of day count basis to use

Duration is defined as the weighted average of the present value of the cash flows and is used as a measure of a bond price's response to changes in yield.

### 8.4.8 Modified Macauley Duration

**WorksheetFunction.MDURATION(*SettlementDate* As Date, *MaturityDate* As Date, *Coupon* As Integer, *Yield* As Integer, *Frequency* As mpReal, *Basis* As Integer) As mpReal**

NOT YET IMPLEMENTED

The function WorksheetFunction.MDURATION returns the modified Macauley duration for a security with an assumed par value of \$100.

**Parameters:**

*SettlementDate*: The security's settlement date.

*MaturityDate*: The security's maturity date.

*Coupon*: Coupon payments per year

*Yield*: The security's annual yield.

*Frequency*: The number of coupon payments per year.

*Basis*: The type of day count basis to use

The Modified Duration is defined as follows:

$$\text{MDURATION} = \text{DURATION} \div \left( 1 + \frac{\text{Market yield}}{\text{Coupon payments per year}} \right) \quad (8.4.1)$$

## 8.5 Securities

The Securities functions share the following arguments and terminology:

*IssueDate*: The security's issue date.

*FirstInterestDate*: The security's first interest date.

*SettlementDate*: The security's settlement date. The security settlement date is the date after the issue date when the security is traded to the buyer.

*Rate*: The security's annual coupon rate.

*Par*: The security's par value. If you omit par, ACCRINT uses \$1,000.

*Frequency*: The number of coupon payments per year. For annual payments, frequency = 1; for semiannual, frequency = 2; for quarterly, frequency = 4.

*Basis*: Optional. The type of day count basis to use:

0 or omitted: US (NASD) 30/360

1: Actual/actual

2: Actual/360

3: Actual/365

4: European 30/360

*CalcMethod*: Optional. A logical value that specifies the way to calculate the total accrued interest when the date of settlement is later than the date of *FirstInterest*.

A value of TRUE (1) returns the total accrued interest from issue to settlement.

A value of FALSE (0) returns the accrued interest from *FirstInterest* to settlement. If you do not enter the argument, it defaults to TRUE.

Dates should be entered by using the DATE function, or as results of other formulas or functions. For example, use DATE(2008,5,23) for the 23rd day of May, 2008.

### 8.5.1 Accrued Interest

---

WorksheetFunction.**ACCRINT**(*Issue As Date*, *First\_interest As Date*, *Settlement As Date*, *Rate As Integer*, *Par As mpReal*, *Frequency As Integer*, *Basis As Integer*, *Calc\_method As Boolean*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ACCRINT returns the accrued interest for a security that pays periodic interest.

#### Parameters:

*Issue*: The security's issue date.

*First\_interest*: The security's first interest date.

*Settlement*: The security's settlement date.

*Rate*: The security's annual coupon rate.

*Par*: The security's par value

*Frequency*: The number of coupon payments per year

*Basis*: The type of day count basis to use

*Calc\_method*: The type calculation to use

ACCRINT is calculated as follows:

$$\text{ACCRINT} = \text{par} \times \frac{\text{rate}}{\text{frequency}} \times \sum_{i=1}^{NC} \frac{A_i}{NL_i}, \text{ where} \quad (8.5.1)$$

$A_i$  is the number of accrued days for the  $i$ th quasi-coupon period within odd period.

$NC$  is the number of quasi-coupon periods that fit in odd period. If this number contains a fraction, raise it to the next whole number.

$NL_i$  is the normal length in days of the  $i$ th quasi-coupon period within odd period.

### 8.5.2 Accrued Interest at Maturity

---

WorksheetFunction.**ACCRINTM**(*Issue As Date*, **Settlement As Date**, **Rate As Integer**, **Par As mpReal**, **Basis As Integer**) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ACCHRINTM returns the accrued interest for a security that pays interest at maturity.

**Parameters:**

*Issue*: The security's issue date.

*Settlement*: The security's settlement date.

*Rate*: The security's annual coupon rate.

*Par*: The security's par value

*Basis*: The type of day count basis to use.

ACCHRINTM is calculated as follows:

$$\text{ACCHRINTM} = \text{par} \times \text{rate} \times \frac{A}{D}, \text{ where} \quad (8.5.2)$$

$A$  is the number of accrued days counted according to a monthly basis. For interest at maturity items, the number of days from the issue date to the maturity date is used.

$D$  is the Annual Year Basis.

### 8.5.3 Discount Rate for a Security

---

WorksheetFunction.**DISC**(**Settlement As Date**, **Maturity As Date**, **Pr As Integer**, **Redemption As mpReal**, **Basis As Integer**) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.DISC returns the discount rate for a security.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Pr*: The security's price per \$100 face value.

*Redemption*: The security's redemption value per \$100 face value.

*Basis*: The type of day count basis to use.

DISC is calculated as follows:

$$\text{DISC} = \frac{\text{redemption} - \text{par}}{\text{par}} \times \frac{B}{DSM}, \text{ where} \quad (8.5.3)$$

$B$  is the number of days in a year, depending on the year basis, and  $DSM$  is the number of days between settlement and maturity.

#### 8.5.4 Interest Rate for a fully invested Security

---

WorksheetFunction.**INTRATE**(*Settlement As Date, Maturity As Date, Investment As mpReal, Redemption As mpReal, Basis As Integer*) As mpReal

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.INTRATE returns the interest rate for a fully invested security.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Investment*: The amount invested in the security.

*Redemption*: The amount to be received at maturity.

*Basis*: The type of day count basis to use.

INTRATE is calculated as follows:

$$\text{INTRATE} = \frac{\text{redemption} - \text{investment}}{\text{investment}} \times \frac{B}{DSM}, \text{ where} \quad (8.5.4)$$

$B$  is the number of days in a year, depending on the year basis, and  $DSM$  is the number of days between settlement and maturity.

#### 8.5.5 Price of a Security having an odd first Period

---

WorksheetFunction.**ODDFPRICE**(*Settlement As Date, Maturity As Date, Issue As Date, First\_Coupon As Date, Rate As mpReal, Yld As mpReal, Redemption As mpReal, Frequency As Integer*) As mpReal

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.ODDFPRICE returns the price per \$100 face value of a security having an odd (short or long) first period.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Issue*: The security's maturity date.

*First\_Coupon*: The security's first coupon date.

*Rate*: The security's interest rate.

*Yld*: The security's annual yield.

*Redemption*: The security's redemption value per \$100 face value.

*Frequency*: The number of coupon payments per year

NOTE: The parameter Basis is missing from this list (Latex issue).

ODDFPRICE is calculated as follows:

Odd short first coupon:

$$\begin{aligned} \text{ODDFPRICE} = & \frac{\text{Redemption}}{(1 + \text{YF})^{N-1+\text{DSC}/E}} + \frac{100 \times \text{RF} \times \text{DFC}/E}{(1 + \text{YF})^{\text{DSC}/E}} \\ & + \sum_{k=2}^N \frac{100 \times \text{RF}}{(1 + \text{YF})^{k-1+\text{DSC}/E}} - 100 \times \text{RF} \times \frac{A}{E} \end{aligned} \quad (8.5.5)$$

$A$  = number of days from the beginning of the coupon period to the settlement date (accrued days).

$\text{DSC}$  = number of days from the settlement to the next coupon date.

$\text{DFC}$  = number of days from the beginning of the odd first coupon to the first coupon date.

$E$  = number of days in the coupon period.

$N$  = number of coupons payable between the settlement date and the redemption date. (If this number contains a fraction, it is raised to the next whole number.)

Odd long first coupon:

$$\begin{aligned} \text{ODDFPRICE} = & \frac{\text{Redemption}}{(1 + \text{YF})^{N-1+\text{DSC}/E}} + \frac{100 \times \text{RF} \times \text{DFC}/E}{(1 + \text{YF})^{\text{DSC}/E}} \\ & + \sum_{k=2}^N \frac{100 \times \text{RF}}{(1 + \text{YF})^{k-1+\text{DSC}/E}} - 100 \times \text{RF} \times \frac{A}{E} \end{aligned} \quad (8.5.6)$$

$A_i$  = number of days from the beginning of the  $i$ th, or last, quasi-coupon period within odd period.

$DC_i$  = number of days from dated date (or issue date) to first quasi-coupon ( $i = 1$ ) or number of days in quasi-coupon ( $i = 2, \dots, i = NC$ ).

$\text{DSC}$  = number of days from settlement to next coupon date.

$E$  = number of days in coupon period.

$N$  = number of coupons payable between the first real coupon date and redemption date. (If this number contains a fraction, it is raised to the next whole number.)

$NC$  = number of quasi-coupon periods that fit in odd period. (If this number contains a fraction, it is raised to the next whole number.)

$NL_i$  = normal length in days of the full  $i$ th, or last, quasi-coupon period within odd period.

$N_q$  = number of whole quasi-coupon periods between settlement date and first coupon.

## 8.5.6 Yield of a Security that has an odd first Period

---

WorksheetFunction.ODDFYIELD(**Settlement As Date, Maturity As Date, Issue As Date, First\_Coupon As Date, Rate As mpReal, Pr As mpReal, Redemption As mpReal, Frequency As Integer**) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ODDFYIELD returns the yield of a security that has an odd (short or long) first period.

### Parameters:

**Settlement:** The security's settlement date.

*Maturity*: The security's maturity date.

*Issue*: The security's maturity date.

*First\_Coupon*: The security's first coupon date.

*Rate*: The security's interest rate.

*Pr*: The security's price.

*Redemption*: The security's redemption value per \$100 face value.

*Frequency*: The number of coupon payments per year

NOTE: The parameter Basis is missing from this list (Latex issue).

Returns the yield of a security that has an odd (short or long) first period.

Excel uses an iterative technique to calculate ODDFYIELD. This function uses the Newton method based on the formula used for the function ODDFPRICE. The yield is changed through 100 iterations until the estimated price with the given yield is close to the price. See ODDFPRICE for the formula that ODDFYIELD uses.

### 8.5.7 Price of a Security having an odd last Coupon

---

WorksheetFunction.**ODDLPRICE**(*Settlement As Date*, *Maturity As Date*, *Issue As Date*, *Last\_interest As Date*, *Rate As mpReal*, *Yld As mpReal*, *Redemption As mpReal*, *Frequency As Integer*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ODDLPRICE returns the price per \$100 face value of a security having an odd (short or long) last coupon period.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Issue*: The security's maturity date.

*Last\_interest*: The security's last coupon date.

*Rate*: The security's interest rate.

*Yld*: The security's price.

*Redemption*: The security's redemption value per \$100 face value.

*Frequency*: The number of coupon payments per year

NOTE: The parameter Basis is missing from this list (Latex issue).

Returns the price per \$100 face value of a security having an odd (short or long) last coupon period.

### 8.5.8 Yield of a Security that has an odd last Period

---

WorksheetFunction.**ODDLYIELD**(*Settlement As Date*, *Maturity As Date*, *Last\_interest As Date*, *Rate As mpReal*, *Pr As mpReal*, *Redemption As mpReal*, *Frequency As Integer*, *Basis As Integer*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.ODDLYIELD returns the yield of a security that has an odd (short or long) last coupon period.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Last\_interest*: The security's last coupon date.

*Rate*: The security's interest rate.

*Pr*: The security's price.

*Redemption*: The security's redemption value per \$100 face value.

*Frequency*: The number of coupon payments per year

*Basis*: The type of day count basis to use.

ODDLYIELD is calculated as follows:

$$\text{ODDLYIELD} = \frac{\text{Redemption} + SDC \times 100RF - par + SA \times 100RF}{par + SA \times 100RF} \times \frac{\text{Frequency}}{SDSC} \quad (8.5.7)$$

$$SDC = \sum_{j=1}^{NC} \frac{DC_i}{NL_i}; \quad SDSC = \sum_{j=1}^{NC} \frac{DSC_i}{NL_i}; \quad SA = \sum_{j=1}^{NC} \frac{A_i}{NL_i}; \quad (8.5.8)$$

where:

$A_i$  = number of accrued days for the ith, or last, quasi-coupon period within odd period counting forward from last interest date before redemption.

$DC_i$  = number of days counted in the ith, or last, quasi-coupon period as delimited by the length of the actual coupon period.

$NC$  = number of quasi-coupon periods that fit in odd period; if this number contains a fraction it will be raised to the next whole number.

$NL_i$  = normal length in days of the ith, or last, quasi-coupon period within odd coupon period.

### 8.5.9 Price of a Security that pays periodic Interest

---

WorksheetFunction.PRICE(**Settlement** As Date, **Maturity** As Date, **Rate** As mpReal, **Yld** As mpReal, **Redemption** As mpReal, **Frequency** As Integer, **Basis** As Integer) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.PRICE returns the price per \$100 face value of a security that pays periodic interest.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Rate*: The security's interest rate.

*Yld*: The security's annual yield.

*Redemption*: The security's redemption value per \$100 face value.

*Frequency*: The number of coupon payments per year

*Basis*: The type of day count basis to use.

Returns the price per \$100 face value of a security that pays periodic interest. PRICE is calculated as follows:

$$\begin{aligned} \text{PRICE} &= \frac{\text{Redemption}}{(1 + YF)^{N-1+DSC/E}} \\ &+ \sum_{k=2}^N \frac{100 \times RF}{(1 + YF)^{k-1+DSC/E}} - 100 \times RF \times \frac{A}{E} \end{aligned} \quad (8.5.9)$$

where:

*DSC* = number of days from settlement to next coupon date.

*E* = number of days in coupon period in which the settlement date falls.

*N* = number of coupons payable between settlement date and redemption date.

*A* = number of days from beginning of coupon period to settlement date.

### 8.5.10 Price of a discounted Security

---

WorksheetFunction.**PRICEDISC**(*Settlement As Date*, *Maturity As Date*, *Discount As mpReal*, *Redemption As mpReal*, *Basis As Integer*) As mpReal

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.PRICEDISC returns Returns the price per \$100 face value of a discounted security.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Discount*: The security's interest rate.

*Redemption*: The security's redemption value per \$100 face value.

*Basis*: The type of day count basis to use.

PRICEDISC is calculated as follows:

$$\text{PRICEDISC} = \text{redemption} - \text{discount} \times \text{redemption} \times \frac{DSM}{B} \quad (8.5.10)$$

where:

*B* = number of days in year, depending on year basis.

*DSM* = number of days from settlement to maturity.

### 8.5.11 Price of a Security that pays Interest at Maturity

---

WorksheetFunction.**PRICEMAT**(*Settlement As Date*, *Maturity As Date*, *Issue As Date*, *Rate As mpReal*, *Yld As mpReal*, *Basis As Integer*) As mpReal

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.PRICEMAT returns the price per \$100 face value of a security that pays interest at maturity.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Issue*: The security's issue date.

*Rate*: The security's interest rate.

*Yld*: The security's annual yield.

*Basis*: The type of day count basis to use.

PRICEMAT is calculated as follows:

$$\text{PRICEMAT} = \frac{100 + (\text{DIM} \times \text{Rate} \times 100/B)}{1 + (\text{DSM} \times \text{yld}/B)} - \frac{\text{A} \times \text{Rate} \times 100}{B} \quad (8.5.11)$$

where:

*B* = number of days in year, depending on year basis.

*DSM* = number of days from settlement to maturity.

*DIM* = number of days from issue to maturity.

*A* = number of days from issue to settlement.

### 8.5.12 Amount received at Maturity for a fully invested Security

---

WorksheetFunction.RECEIVED(**Settlement As Date, Maturity As Date, Investment As mpReal, Discount As mpReal, Basis As Integer**) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.RECEIVED returns the amount received at maturity for a fully invested security.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Investment*: The amount invested in the security.

*Discount*: The security's discount rate.

*Basis*: The type of day count basis to use.

RECEIVED is calculated as follows:

$$\text{RECEIVED} = \frac{\text{investment}}{1 - (\text{DIM} \times \text{discount}/B)} \quad (8.5.12)$$

where:

*B* = number of days in year, depending on year basis.

*DIM* = number of days from issue to maturity.

### 8.5.13 Yield on a Security that pays periodic Interest

---

WorksheetFunction.YIELD(**Settlement As Date, Maturity As Date, Rate As mpReal, Pr As mpReal, Redemption As mpReal, Frequency As Integer, Basis As Integer**) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.YIELD returns the yield on a security that pays periodic interest.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Rate*: The security's interest rate.

*Pr*: The security's price per \$100 face value.

*Redemption*: The security's redemption value per \$100 face value.

*Frequency*: The number of coupon payments per year

*Basis*: The type of day count basis to use.

YIELD is calculated as follows:

$$\text{YIELD} = \frac{(\text{Redemption}/100) + \text{RF} - S_1}{S_1} \frac{\text{Frequency} \times E}{\text{DSR}}; \quad S_1 = \frac{\text{Par}}{100} + \frac{A}{E} \times \text{RF} \quad (8.5.13)$$

where:

*A* = number of days from the beginning of the coupon period to the settlement date (accrued days).

*DSR* = number of days from the settlement date to the redemption date.

*E* = number of days in the coupon period.

### 8.5.14 Annual Yield for a discounted Security

---

WorksheetFunction.**YIELDDISC**(*Settlement As Date*, *Maturity As Date*, *Pr As mpReal*, *Redemption As mpReal*, *Basis As Integer*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.YIELDDISC returns the annual yield for a discounted security.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Pr*: The security's price per \$100 face value.

*Redemption*: The security's redemption value per \$100 face value.

*Basis*: The type of day count basis to use.

Returns the annual yield for a discounted security.

### 8.5.15 Annual Yield of a Security that pays Interest at Maturity

---

WorksheetFunction.**YIELDMAT**(*Settlement As Date*, *Maturity As Date*, *Issue As Date*, *Rate As mpReal*, *Pr As mpReal*, *Basis As Integer*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.YIELDMAT returns the price per \$100 face value of a security that pays interest at maturity.

**Parameters:**

*Settlement*: The security's settlement date.

*Maturity*: The security's maturity date.

*Issue*: The security's issue date.

*Rate*: The security's interest rate.

*Pr*: The security's price per \$100 face value.

*Basis*: The type of day count basis to use.

Returns the annual yield of a security that pays interest at maturity.

## 8.6 Treasury Bills

### 8.6.1 Bond-equivalent Yield for a Treasury bill

---

WorksheetFunction.**TBILLEQ**(*Settlement As Date*, *Maturity As Date*, *Discount As mpReal*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.TBILLEQ returns the bond-equivalent yield for a Treasury bill.

**Parameters:**

*Settlement*: The Treasury bill's settlement date.

*Maturity*: The Treasury bill's maturity date.

*Discount*: The Treasury bill's discount rate.

TBILLEQ is calculated as

$$\text{TBILLEQ} = \frac{365 \times \text{Rate}}{360 - \text{Rate} \times \text{DSM}} \quad (8.6.1)$$

where:

DSM = number of days between settlement and maturity computed according to the 360 days per year basis.

### 8.6.2 Price for a Treasury bill

---

WorksheetFunction.**TBILLPRICE**(*Settlement As Date*, *Maturity As Date*, *Discount As mpReal*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.TBILLPRICE returns the price per \$100 face value for a Treasury bill.

**Parameters:**

*Settlement*: The Treasury bill's settlement date.

*Maturity*: The Treasury bill's maturity date.

*Discount*: The Treasury bill's discount rate.

Returns the price per \$100 face value for a Treasury bill. TBILLPRICE is calculated as

$$\text{TBILLPRICE} = 100 \times \left(1 - \frac{\text{Discount} \times \text{DSM}}{360}\right) \quad (8.6.2)$$

where:

DSM = number of days from settlement to maturity, excluding any maturity date that is more than one calendar year after the settlement date.

### 8.6.3 Yield for a Treasury bill

---

WorksheetFunction.**TBILLYIELD**(*Settlement As Date*, *Maturity As Date*, *Pr As mpReal*) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.TBILLYIELD` returns the yield for a Treasury bill.

**Parameters:**

*Settlement*: The Treasury bill's settlement date.

*Maturity*: The Treasury bill's maturity date.

*Pr*: The Treasury bill's price per \$face value.

Returns the yield for a Treasury bill. `TBILLYIELD` is calculated as

$$\text{TBILLYIELD} = \frac{100 - \text{Price}}{\text{Price}} \frac{360}{\text{DSM}} \quad (8.6.3)$$

where:

*DSM* = number of days from settlement to maturity, excluding any maturity date that is more than one calendar year after the settlement date.

## 8.7 Depreciation Functions

The depreciation functions are used in accounting to calculate the amount of monetary value a fixed asset loses over a period of time. These functions share the following arguments and terminology:

*Cost*: Initial cost of asset

*Salvage*: Required. The value at the end of the depreciation (sometimes called the salvage value of the asset). This value can be 0.

*Life*: The number of periods over which the asset is being depreciated (sometimes called the useful life of the asset).

*Period* . The period for which you want to calculate the depreciation. Period must use the same units as life.

*Factor*: Optional. The rate at which the balance declines. If factor is omitted, it is assumed to be 2 (the double-declining balance method).

*TD*: Total depreciation from prior periods.

### 8.7.1 Depreciation of an Asset

---

WorksheetFunction.DDB(***Cost As mpReal, Salvage As mpReal, Life As mpReal, Period As mpReal, Factor As mpReal***) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.DDB returns the depreciation of an asset for a specified period using the double-declining balance method or some other method you specify.

**Parameters:**

*Cost*: The initial cost of the asset.

*Salvage*: The salvage value at the end of the life of the asset.

*Life*: The number of periods over which the asset is being depreciated.

*Period*: The period for which you want to calculate the depreciation.

*Factor*: The rate at which the balance declines.

The double-declining balance method computes depreciation at an accelerated rate. Depreciation is highest in the first period and decreases in successive periods. DDB uses the following formula to calculate depreciation for a period:

$$\text{DDB} = \text{Min}((\text{Cost} - \text{TD}) * (\text{Factor}/\text{Life}), (\text{Cost} - \text{Salvage} - \text{TD})) \quad (8.7.1)$$

Change factor if you do not want to use the double-declining balance method. Use the VDB function if you want to switch to the straight-line depreciation method when depreciation is greater than the declining balance calculation.

### 8.7.2 Straight-Line Depreciation of an Asset

---

WorksheetFunction.SLN(***Cost As mpReal, Salvage As mpReal, Life As mpReal***) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.SLN returns the straight-line depreciation of an asset for a single period

**Parameters:**

*Cost*: The initial cost of the asset.

*Salvage*: The salvage value at the end of the life of the asset.

*Life*: The number of periods over which the asset is being depreciated.

Returns the straight-line depreciation of an asset for a single period

### 8.7.3 Sum-of-Years' Digits Depreciation of an Asset

---

WorksheetFunction.**SYD**(*Cost* As mpReal, *Salvage* As mpReal, *Life* As mpReal, *Period* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.SYD returns the sum-of-years' digits depreciation of an asset for a specified period.

**Parameters:**

*Cost*: The initial cost of the asset.

*Salvage*: The salvage value at the end of the life of the asset.

*Life*: The number of periods over which the asset is being depreciated.

*Period*: The period for which you want to calculate the depreciation.

SYD is calculated as follows:

$$\text{SYD} = \frac{(Cost - Salvage) \times 2(Life - Period + 1)}{Life(Life + 1)} \quad (8.7.2)$$

### 8.7.4 Fixed Declining Balance Method

---

WorksheetFunction.**DB**(*Cost* As mpReal, *Salvage* As mpReal, *Life* As mpReal, *Period* As mpReal, *Month* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.DB returns the depreciation of an asset for a specified period using the fixed-declining balance method.

**Parameters:**

*Cost*: The initial cost of the asset.

*Salvage*: The salvage value at the end of the life of the asset.

*Life*: The number of periods over which the asset is being depreciated.

*Period*: The period for which you want to calculate the depreciation.

*Month*: The number of months in the first year.

The fixed-declining balance method computes depreciation at a fixed rate. DB uses the following formulas to calculate depreciation for a period:

$$\text{DB} = (Cost - TD) \times Rate, \quad (8.7.3)$$

where  $Rate = 1 - (Salvage/Cost)^{1/Life}$ , rounded to three decimal places

Depreciation for the first and last periods is a special case.

For the first period, DB uses this formula:

$$\text{DB} = \text{Cost} \times \text{Rate} \times \text{Month}/12. \quad (8.7.4)$$

For the last period, DB uses this formula:

$$\text{DB} = ((\text{Cost} - \text{TD}) \times \text{Rate} \times (12 - \text{Month}))/12. \quad (8.7.5)$$

## 8.7.5 Variable Declining Balance

---

**WorksheetFunction.VDB(*Cost* As mpReal, *Salvage* As mpReal, *Life* As mpReal, *Start\_Period* As mpReal, *End\_Period* As mpReal, *Factor* As mpReal, *No\_switch* As mpReal) As mpReal**

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.VDB returns the depreciation of an asset for any period you specify, including partial periods, using the double-declining balance method or some other method you specify. VDB stands for variable declining balance.

**Parameters:**

*Cost*: The initial cost of the asset.

*Salvage*: The salvage value at the end of the life of the asset.

*Life*: The number of periods over which the asset is being depreciated.

*Start\_Period*: The period for which you want to calculate the depreciation.

*End\_Period*: The ending period for which you want to calculate the depreciation. EndPeriod must use the same units as life.

*Factor*: The rate at which the balance declines.

*No\_switch*: A logical value specifying whether to switch to straight-line depreciation when depreciation is greater than the declining balance calculation.

If NoSwitch is TRUE, Microsoft Excel does not switch to straight-line depreciation even when the depreciation is greater than the declining balance calculation.

If NoSwitch is FALSE or omitted, Excel switches to straight-line depreciation when depreciation is greater than the declining balance calculation.

## 8.7.6 Depreciation for each accounting period

---

**WorksheetFunction.AMORLINC(*Cost* As mpReal, *Date\_Purchased* As Date, *First\_Period* As mpReal, *Salvage* As mpReal, *Period* As mpReal, *Rate* As mpReal, *Basis* As mpReal) As mpReal**

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.AMORLINC returns the depreciation for each accounting period.

**Parameters:**

*Cost*: The initial cost of the asset.

*Date\_Purchased*: The date the asset is purchased.

*First\_Period*: The date of the end of the first period.

*Salvage*: The salvage value at the end of the life of the asset.

*Period*: The period.

*Rate*: The rate of depreciation.

*Basis*: Year Basis: 0 for 360 days, 1 for actual, 3 for 365 days.

This function is provided for the French accounting system. If an asset is purchased in the middle of the accounting period, the prorated depreciation is taken into account.

### 8.7.7 Depreciation using a depreciation coefficient

---

WorksheetFunction.**AMORDEGRC**(*Cost* As mpReal, *Date\_Purchased* As Date, *First\_Period* As mpReal, *Salvage* As mpReal, *Period* As mpReal, *Rate* As mpReal, *Basis* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.AMORDEGRC returns the prorated linear depreciation of an asset for each accounting period.

**Parameters:**

*Cost*: The initial cost of the asset.

*Date\_Purchased*: The date the asset is purchased.

*First\_Period*: The date of the end of the first period.

*Salvage*: The salvage value at the end of the life of the asset.

*Period*: The period.

*Rate*: The rate of depreciation.

*Basis*: Year Basis: 0 for 360 days, 1 for actual, 3 for 365 days.

This function is provided for the French accounting system. If an asset is purchased in the middle of the accounting period, the prorated depreciation is taken into account. The function is similar to AMORLINC, except that a depreciation coefficient is applied in the calculation depending on the life of the assets.

## 8.8 Annuity Functions

An annuity is a series of payments that represents either the return on an investment or the amortization of a loan. Negative numbers represent monies paid out, like contributions to savings or loan payments. Positive numbers represent monies received, like dividends. The Annuity Functions share the following arguments and terminology:

*Rate*: Interest rate per period, must use the same unit for Period as used for Nper.

*Nper*: Total number of payment periods in the annuity.

*PMT*: Payment to be made each period

*PV*: Present value (or lump sum) that a series of payments to be paid in the future is worth now.

*FV*: Optional. Value of the annuity after the final payment has been made (if omitted, 0 is assumed, which is the usual future value of a loan).

*Type*: Optional. Number indicating when payments are due: 0 if payments are due at the end of the payment period and 1 if payments are due at the beginning of the period, if omitted, 0 is assumed.

In general, the routines solve for one financial argument in terms of the others. If rate is not 0, then:

$$PV \times (1 + Rate)^{Nper} + PMT(1 + Rate \times Type) \times \left( \frac{(1 + Rate)^{Nper} - 1}{Rate} \right) + FV = 0. \quad (8.8.1)$$

If *Rate* is 0, then

$$(PMT \times Nper) + PV + FV = 0. \quad (8.8.2)$$

### 8.8.1 Future Value

---

**WorksheetFunction.FV(*Rate* As mpReal, *Nper* As mpReal, *Pmt* As mpReal, *PV* As mpReal, *Type* As mpReal) As mpReal**

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.FV returns the future value of an annuity based on periodic fixed payments and a fixed interest rate.

**Parameters:**

*Rate*: The the interest rate per period.

*Nper*: The total number of payment periods in the investment.

*Pmt*: The payment made each period.

*PV*: The present value.

*Type*: a value representing the timing of payment.

The future value is calculated as

$$FV = -PV(1 + r)^n + PMT \left( \frac{(1 + r)^n - 1}{r} \right) \quad (8.8.3)$$

### 8.8.2 Present Value

---

**WorksheetFunction.PV(*Rate* As mpReal, *Nper* As mpReal, *Pmt* As mpReal, *FV* As mpReal, *Type* As mpReal) As mpReal**

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.PV` returns the present value of an annuity based on periodic fixed payments to be paid in the future at a fixed interest rate.

#### Parameters:

*Rate*: The interest rate per period.

*Nper*: The total number of payment periods in the investment.

*Pmt*: The payment made each period.

*FV*: The future value.

*Type*: a value representing the timing of payment.

The present value is the total amount that a series of future payments is worth now. For example, when you borrow money, the loan amount is the present value to the lender.

The present value is calculated as

$$PV = - \left( FV + PMT \left( \frac{(1+r)^n - 1}{r} \right) \right) (1+r)^{-n} \quad (8.8.4)$$

### 8.8.3 Payment

---

`WorksheetFunction.PMT(Rate As mpReal, Nper As mpReal, PV As mpReal, FV As mpReal, Type As mpReal) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.PMT` returns the payment for a loan based on constant payments and a constant interest rate.

#### Parameters:

*Rate*: The interest rate per period.

*Nper*: The total number of payment periods in the investment.

*PV*: The present value.

*FV*: The future value.

*Type*: a value representing the timing of payment.

The payment is calculated as

$$PMT = - (FV + PV(1+r)^n) \times \left( \frac{r}{(1+r)^n - 1} \right) \quad (8.8.5)$$

### 8.8.4 Number of periods

---

`WorksheetFunction.NPER(Rate As mpReal, Pmt As mpReal, PV As mpReal, FV As mpReal, Type As mpReal) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.NPER` returns the number of periods for an investment based on periodic, constant payments and a constant interest rate.

#### Parameters:

*Rate*: The interest rate per period.

*Pmt*: The made each period.

*PV*: The present value.

*FV*: The future value.

*Type*: a value representing the timing of payment.

The number of periods is calculated as

$$n = \frac{1}{\ln(1+r)} \ln \left( \frac{(PMT/r) - FV}{(PMT/r) + PV} \right) \quad (8.8.6)$$

### 8.8.5 Number of periods required

---

WorksheetFunction.**PDURATION**(*Rate* As mpReal, *PV* As mpReal, *FV* As mpReal) As mpReal

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.PDURATION returns the number of periods required by an investment to reach a specified value.

**Parameters:**

*Rate*: The interest rate per period.

*PV*: The present value.

*FV*: The future value.

PDURATION is calculated as

$$PDURATION = \frac{\ln(FV) - \ln(PV)}{\ln(1+r)} \quad (8.8.7)$$

### 8.8.6 Interest Rate

---

WorksheetFunction.**RATE**(*Nper* As mpReal, *Pmt* As mpReal, *PV* As mpReal, *FV* As mpReal, *Type* As mpReal) As mpReal

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.RATE returns the interest rate per period of an annuity

**Parameters:**

*Nper*: The the interest rate per period.

*Pmt*: The made each period.

*PV*: The present value.

*FV*: The future value.

*Type*: a value representing the timing of payment.

RATE is calculated by iteration and can have zero or more solutions. If the successive results of RATE do not converge to within 0.0000001 after 20 iterations, RATE returns the #NUM! error value.

An iterative scheme is used to solve

$$f(r) = FV + PV(1+r)^n + PMT \left( \frac{(1+r)^n - 1}{r} \right) = 0 \quad (8.8.8)$$

### 8.8.7 Interest Payment

---

`WorksheetFunction.IPMT(Rate As mpReal, Per As mpReal, Nper As mpReal, Pmt As mpReal, FV As mpReal, Type As mpReal) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.IPMT` returns the interest payment for a given period for an investment based on periodic, constant payments and a constant interest rate.

**Parameters:**

*Rate*: The interest rate per period.

*Per*: The period for which you want to find the interest and must be in the range 1 to NPer

*Nper*: The total number of payment periods in the investment.

*Pmt*: The payment made each period.

*FV*: The future value.

*Type*: a value representing the timing of payment.

The Interest Payment is calculated as

$$IPMT = - \left( (1 + r)^{i-1} (PMT + PV \times r) \right) \quad (8.8.9)$$

### 8.8.8 Principal Payment

---

`WorksheetFunction.PPMT(Rate As mpReal, Per As mpReal, Nper As mpReal, PV As mpReal, FV As mpReal, Type As mpReal) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.PPMT` returns the payment on the principal for a given period for an investment based on periodic, constant payments and a constant interest rate.

**Parameters:**

*Rate*: The the interest rate per period.

*Per*: The period for which you want to find the interest and must be in the range 1 to NPer

*Nper*: The total number of payment periods in the investment.

*PV*: The payment made each period.

*FV*: The future value.

*Type*: a value representing the timing of payment.

Returns the payment on the principal for a given period for an investment based on periodic, constant payments and a constant interest rate.

The Principal Payment is calculated as

$$PPMT = PMT - IPMT \quad (8.8.10)$$

### 8.8.9 Cumulative Interest Paid

---

`WorksheetFunction.CUMIPMT(Rate As mpReal, Nper As mpReal, PV As mpReal, StartPeriod As mpReal, EndPeriod As mpReal, Type As mpReal) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.CUMIPMT` returns the cumulative interest paid on a loan between `StartPeriod` and `EndPeriod`.

**Parameters:**

*Rate*: The interest rate per period.

*Nper*: The total number of payment periods in the investment.

*PV*: The payment made each period.

*StartPeriod*: The first period in the calculation.

*EndPeriod*: the last period in the calculation

*Type*: a value representing the timing of payment.

### 8.8.10 Cumulative Principal Paid

---

`WorksheetFunction.CUMPRINC(Rate As mpReal, Nper As mpReal, PV As mpReal, StartPeriod As mpReal, EndPeriod As mpReal, Type As mpReal) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.CUMPRINC` returns the effective annual interest rate, given the nominal annual interest rate and the number of compounding periods per year.

**Parameters:**

*Rate*: The interest rate per period.

*Nper*: The total number of payment periods in the investment.

*PV*: The payment made each period.

*StartPeriod*: The first period in the calculation.

*EndPeriod*: the last period in the calculation

*Type*: a value representing the timing of payment.

### 8.8.11 Effective Annual Interest Rate

---

`WorksheetFunction.EFFECT(NominalRate As mpReal, Npery As mpReal) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.EFFECT` returns the effective annual interest rate.

**Parameters:**

*NominalRate*: The nominal interest rate per period.

*Npery*: The number of compounding periods per year

### 8.8.12 Nominal Annual Interest Rate

---

`WorksheetFunction.NOMINAL(EffectiveRate As mpReal, Npery As mpReal) As mpReal`

---

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.NOMINAL` returns the nominal annual interest rate, given the effective rate and the number of compounding periods per year.

**Parameters:**

*EffectiveRate*: The nominal interest rate per period.

*Npery*: The number of compounding periods per year

The relationship between NOMINAL and EFFECT is shown in the following equation:

$$EFFECT = \left(1 + \frac{NominalRate}{Npery}\right)^{Npery} - 1. \quad (8.8.11)$$

### 8.8.13 FV Schedule, variable Compound Interest Rates

---

`WorksheetFunction.FVSCHEDULE(Principal As mpReal, Schedule As mpNum) As mpReal`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.FVSCHEDULE` returns the future value of an initial principal after applying a series of compound interest rates. Use `FVSCHEDULE` to calculate the future value of an investment with a variable or adjustable rate.

**Parameters:**

*Principal*: The present value.

*Schedule*: An array of interest rates to apply

### 8.8.14 Interest paid during a specific Period of an Investment

---

`WorksheetFunction.ISPMT(Rate As mpReal, Per As mpReal, Nper As mpReal, PV As mpReal) As mpReal`

**NOT YET IMPLEMENTED**

---

The function `WorksheetFunction.ISPMT` returns the interest paid during a specific period of an investment.

**Parameters:**

*Rate*: The interest rate per period.

*Per*: The period for which you want to find the interest and must be in the range 1 to NPer

*Nper*: The total number of payment periods in the investment.

*PV*: The present value.

## 8.9 Cash-Flow Functions

The cash-flow functions perform financial calculations based on a series of periodic payments and receipts. As with the annuity functions, negative numbers represent payments and positive numbers represent receipts. However, unlike the annuity functions, the cash-flow functions allow to list varying amounts for the payments or receipts over the course of a loan or investment. Payments and receipts can even be mixed up within the cash-flow series. The cash-flow functions share the following arguments and terminology:

*Values()*: array of cash-flow values; the array must contain at least one negative value (a payment) and one positive value (a receipt).

*Rate*: Discount rate over the length of the period, expressed as a decimal.

*FinanceRate*: Interest rate paid as the cost of financing.

*ReinvestRate*: Interest rate received on gains from cash reinvestment.

[*Guess*]: Optional value as estimate of return; if omitted, Guess is 0.1 (10 percent).

### 8.9.1 Internal Rate of Return

---

WorksheetFunction.IRR(*Values* As mpNum, *Guess* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.IRR returns the effective annual interest rate.

**Parameters:**

*Values*: An array which contains numbers for which the internal rate of return is calculated.

*Guess*: An initial guess for the IRR, 0.1 if omitted

Returns the internal rate of return for a series of cash flows represented by the numbers in values. These cash flows do not have to be even, as they would be for an annuity. However, the cash flows must occur at regular intervals, such as monthly or annually. The internal rate of return is the interest rate received for an investment consisting of payments (negative values) and income (positive values) that occur at regular periods.

Microsoft Excel uses an iterative technique for calculating IRR. Starting with guess, IRR cycles through the calculation until the result is accurate within 0.00001 percent. If IRR can't find a result that works after 20 tries, the #NUM! error value is returned.

In most cases you do not need to provide guess for the IRR calculation. If guess is omitted, it is assumed to be 0.1 (10 percent).

IRR is closely related to NPV, the net present value function. The rate of return calculated by IRR is the interest rate corresponding to a 0 (zero) net present value.

### 8.9.2 Calc: Rate of Return

---

WorksheetFunction.RRI(*Nper* As mpReal, *PV* As mpReal, *FV* As mpReal) As mpReal

---

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.RRI returns an equivalent interest rate for the growth of an investment

**Parameters:**

*Nper*: The total number of periods for the investment.

*PV*: The present value for the investment.

*PV*: The future value for the investment.

RRI returns the interest rate given nper (the number of periods), pv (present value), and fv (future value), calculated by using the following equation:

$$RRI = \frac{FV^{1/N_{per}}}{PV} - 1. \quad (8.9.1)$$

### 8.9.3 Modified Internal Rate of Return

---

WorksheetFunction.MIRR(**Values** As mpNum[], **FinanceRate** As mpReal, **ReinvestRate** As mpReal) As mpReal

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.MIRR returns the modified internal rate of return for a series of periodic cash flows. MIRR considers both the cost of the investment and the interest received on reinvestment of cash.

**Parameters:**

**Values**: An array that contains numbers that represent a series of payments (negative) and income (positive) at regular periods.

**FinanceRate**: The interest rate paid on the money used in the cash flows.

**ReinvestRate**: The interest rate received on the money used in the cash flows.

MIRR uses the order of values to interpret the order of cash flows. Be sure to enter your payment and income values in the sequence you want and with the correct signs (positive values for cash received, negative values for cash paid).

If n is the number of cash flows in values, frate is the FinanceRate, and rrate is the ReinvestRate, then the formula for MIRR is:

$$MIRR = \left( \frac{-NPV(rrate, values[positive]) \times (1 + rrate)^n}{NPV(frate, values[negative]) \times (1 + frate)} \right)^{1/(n-1)} - 1. \quad (8.9.2)$$

### 8.9.4 Net Present Value

---

WorksheetFunction.NPV(**Rate** As mpReal, **Values** As mpNum[]) As mpReal

---

NOT YET IMPLEMENTED

---

The function WorksheetFunction.NPV returns the net present value of an investment based on a series of periodic cash flows and a discount rate.

**Parameters:**

**Rate**: The total number of periods for the investment.

**Values**: An array that contains numbers that represent a series of payments (negative) and income (positive) at regular periods.

Calculates the net present value of an investment by using a discount rate and a series of future payments (negative values) and income (positive values).

The NPV investment begins one period before the date of the value1 cash flow and ends with the last cash flow in the list. The NPV calculation is based on future cash flows. If your first cash flow occurs at the beginning of the first period, the first value must be added to the NPV result, not included in the values arguments. For more information, see the examples below.

If n is the number of cash flows in the list of values, the formula for NPV is:

$$NPV = \sum_{i=1}^n \frac{values_i}{(1 + rate)^i} \quad (8.9.3)$$

NPV is similar to the PV function (present value). The primary difference between PV and NPV is that PV allows cash flows to begin either at the end or at the beginning of the period. Unlike the variable NPV cash flow values, PV cash flows must be constant throughout the investment. For information about annuities and financial functions, see PV.

NPV is also related to the IRR function (internal rate of return). IRR is the rate for which NPV equals zero:  $NPV(IRR(...), ...) = 0$ .

### 8.9.5 Internal Rate of Return, non-periodic Schedule

---

**WorksheetFunction.XIRR(*Values As mpNum[]*, *Dates As mpReal*, *Guess As mpReal*) As mpReal**

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.XIRR** returns the modified internal rate of return for a series of periodic cash flows. MIRR considers both the cost of the investment and the interest received on reinvestment of cash.

**Parameters:**

*Values*: An array that contains cash flows that correspond to a schedule of payments in *Dates*.

*Dates*: A schedule of payment dates that correspond to the cash flow payments

*Guess*: An initial guess for XIRR.

Returns the internal rate of return for a schedule of cash flows that is not necessarily periodic. To calculate the internal rate of return for a series of periodic cash flows, use the IRR function. XIRR is closely related to XNPV, the net present value function. The rate of return calculated by XIRR is the interest rate corresponding to  $XNPV = 0$ .

Excel uses an iterative technique for calculating XIRR. Using a changing rate (starting with guess), XIRR cycles through the calculation until the result is accurate within 0.000001 percent. If XIRR can't find a result that works after 100 tries, the #NUM! error value is returned. The rate is changed until:

$$\sum_{i=1}^N \frac{P_i}{(1 + rate)^{(d_i - d_1)/365}} = 0, \quad (8.9.4)$$

where:

$d_i$  = the ith, or last, payment date,

$d_1$  = the 0th payment date,

$P_i$  = the ith, or last, payment.

### 8.9.6 Net Present Value, non-periodic Schedule

---

**WorksheetFunction.XNPV(*Rate As mpReal*, *Values As mpNum[]*, *Dates As mpReal*) As mpReal**

**NOT YET IMPLEMENTED**

---

The function **WorksheetFunction.XNPV** returns the modified internal rate of return for a series of periodic cash flows. MIRR considers both the cost of the investment and the interest received on reinvestment of cash.

**Parameters:**

*Rate*: The discount rate to apply to the cash flows.

*Values*: An array that contains cash flows that correspond to a schedule of payments in *Dates*.

*Dates*: A schedule of payment dates that correspond to the cash flow payments

Returns the net present value for a schedule of cash flows that is not necessarily periodic. To calculate the net present value for a series of cash flows that is periodic, use the NPV function. XNPV is calculated as follows:

$$XNPV = \sum_{i=1}^N \frac{P_i}{(1 + rate)^{(d_i - d_1)/365}}, \quad (8.9.5)$$

where:

$d_i$  = the ith, or last, payment date,

$d_1$  = the 0th payment date,

$P_i$  = the ith, or last, payment.

## 8.10 Conversion

### 8.10.1 Price as a fraction into a price as decimal

---

WorksheetFunction.**DOLLARDE**(*FractionalDollar* As mpReal, *Fraction* As mpReal) As mpReal

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.DOLLARDE returns a dollar price expressed as a decimal number, converted from a dollar price expressed as an integer part and a fraction part.

**Parameters:**

*FractionalDollar*: A number expressed as a fraction.

*Fraction*: The integer to use in the denominator of the fraction

Converts a dollar price expressed as an integer part and a fraction part, such as 1.02, into a dollar price expressed as a decimal number. Fractional dollar numbers are sometimes used for security prices.

The fraction part of the value is divided by an integer that you specify. For example, if you want your price to be expressed to a precision of 1/16 of a dollar, you divide the fraction part by 16. In this case, 1.02 represents \$1.125 ( $\$1 + 2/16 = \$1.125$ ).

### 8.10.2 Price as a decimal into a price as fraction

---

WorksheetFunction.**DOLLARFR**(*DecimalDollar* As mpReal, *Fraction* As mpReal) As mpReal

**NOT YET IMPLEMENTED**

---

The function WorksheetFunction.DOLLARFR returns a dollar price expressed as a fraction, converted from a dollar price expressed as a decimal number.

**Parameters:**

*DecimalDollar*: A decimal number.

*Fraction*: The integer to use in the denominator of the fraction

Converts a dollar price expressed as a decimal number into a dollar price expressed as a fraction. Use DOLLARFR to convert decimal numbers to fractional dollar numbers, such as securities prices.

# **Part III**

## **Special Functions**

# Chapter 9

# Factorials and gamma functions

Factorials and factorial-like sums and products are basic tools of combinatorics and number theory. Much like the exponential function is fundamental to differential equations and analysis in general, the factorial function (and its extension to complex numbers, the gamma function) is fundamental to difference equations and functional equations.

A large selection of factorial-like functions is implemented in mpFormulaPy. All functions support complex arguments, and arguments may be arbitrarily large. Results are numerical approximations, so to compute exact values a high enough precision must be set manually:

The gamma and polygamma functions are closely related to Zeta functions, L-series and polylogarithms. See also q-functions for q-analogs of factorial-like functions.

## 9.1 Factorials

### 9.1.1 Factorial

Function **Factorial(z As mpNum)** As mpNum

The function Factorial returns the factorial,  $x!$ .

### Parameter:

*z*: A real or complex number.

Function `fac(z As mpNum)` As mpNum

The function `fac` returns the factorial  $x!$

#### Parameter:

$z$ : A real or complex number.

Computes the factorial,  $x!$ . For integers  $n > 0$ , we have  $n! = 1 \cdot 2 \cdots (n-1) \cdot n$  and more generally the factorial is defined for real or complex  $x$  by  $x! = \Gamma(x+1)$ .

Examples

Basic values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> for k in range(6):
...     print("%s %s" % (k, fac(k)))
...
0 1.0
1 1.0
2 2.0
3 6.0
4 24.0
5 120.0
>>> fac(inf)
+inf
>>> fac(0.5), sqrt(pi)/2
(0.886226925452758, 0.886226925452758)
```

---

`fac()` supports evaluation for astronomically large values:

---

```
>>> fac(10**30)
6.22311232304258e+29565705518096748172348871081098
```

---

## 9.1.2 Double factorial

---

Function **fac2( $z$  As  $mpNum$ ) As  $mpNum$**

---

The function `fac2` returns the double factorial  $x!!$ .

**Parameter:**

$z$ : A real or complex number.

Computes the double factorial  $x!!$ , defined for integers  $x > 0$  by

$$x!! = \begin{cases} 1 \cdot 3 \cdots (x-2) \cdot x & \text{for } x \text{ odd} \\ 2 \cdot 4 \cdots (x-2) \cdot x & \text{for } x \text{ even} \end{cases} \quad (9.1.1)$$

and more generally by [1]

$$x!! = 2^{x/2} \left(\frac{\pi}{2}\right)^{(\cos(\pi x)-1)/4} \Gamma\left(\frac{x}{2} + 1\right) \quad (9.1.2)$$

Examples

The integer sequence of double factorials begins:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> nprint([fac2(n) for n in range(10)])
[1.0, 1.0, 2.0, 3.0, 8.0, 15.0, 48.0, 105.0, 384.0, 945.0]
```

---

With the exception of the poles at negative even integers, fac2() supports evaluation for arbitrary complex arguments. The recurrence formula is valid generally:

---

```
>>> fac2(pi+2j)
(-1.3697207890154e-12 + 3.93665300979176e-12j)
>>> (pi+2j)*fac2(pi-2+2j)
(-1.3697207890154e-12 + 3.93665300979176e-12j)
```

---

## 9.2 Binomial coefficient

Function **binomial(*n* As mpNum, *k* As mpNum)** As mpNum

The function `binomial` returns the binomial coefficient.

**Parameters:**

*n*: A real or complex number.

*k*: A real or complex number.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \quad (9.2.1)$$

The binomial coefficient gives the number of ways that *k* items can be chosen from a set of *n* items. More generally, the binomial coefficient is a well-defined function of arbitrary real or complex *n* and *k*, via the gamma function.

Examples

Generate Pascal's triangle:

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> for n in range(5):
...     nprint([binomial(n,k) for k in range(n+1)])
...
[1.0]
[1.0, 1.0]
[1.0, 2.0, 1.0]
[1.0, 3.0, 3.0, 1.0]
[1.0, 4.0, 6.0, 4.0, 1.0]
```

`binomial()` supports large arguments:

```
>>> binomial(10**20, 10**20-5)
8.3333333333333e+97
>>> binomial(10**20, 10**10)
2.60784095465201e+104342944813
```

## 9.3 Pochhammer symbol, Rising and falling factorials

### 9.3.1 Relative Pochhammer symbol

---

Function **RelativePochhammerMpMath(*a* As mpNum, *x* As mpNum) As mpNum**

**NOT YET IMPLEMENTED**

---

The function **RelativePochhammerMpMath** returns the relative Pochhammer symbol.

**Parameters:**

*a*: An integer.

*x*: An integer.

The relative Pochhammer symbol defined as

$$\text{poch1}(a, x) = \frac{(a)_x - 1}{x}, \quad (9.3.1)$$

accurate even for small *x*. If  $|x|$  is small, cancellation errors are avoided by using an expansion by Fields and Luke with generalized Bernoulli polynomials. For  $x = 0$  the value  $\psi(a)$  is returned, otherwise the result is calculated from the definition.

In mathematics, the Pochhammer symbol introduced by Leo August Pochhammer is the notation  $(x)_n$ , where *n* is a non-negative integer. Depending on the context the Pochhammer symbol may represent either the rising factorial or the falling factorial as defined below. Care needs to be taken to check which interpretation is being used in any particular article.

### 9.3.2 Rising factorial

---

Function **rf(*x* As mpNum, *n* As mpNum) As mpNum**

The function **rf** returns the rising factorial.

**Parameters:**

*x*: A real or complex number.

*n*: A real or complex number.

Computes the rising factorial,

$$x^{(n)} = x(x + 1) \cdots (x + n - 1) = \frac{\Gamma(x + n)}{\Gamma(x)} \quad (9.3.2)$$

where the rightmost expression is valid for nonintegral *n*.

**Examples**

For integral *n*, the rising factorial is a polynomial:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> for n in range(5):
...     nprint(taylor(lambda x: rf(x,n), 0, n))
...
[1.0]
[0.0, 1.0]
[0.0, 1.0, 1.0]
[0.0, 2.0, 3.0, 1.0]
```

---

[0.0, 6.0, 11.0, 6.0, 1.0]

---

Evaluation is supported for arbitrary arguments:

---

```
>>> rf(2+3j, 5.5)
(-7202.03920483347 - 3777.58810701527j)
```

---

### 9.3.3 Falling factorial

---

Function **ff(x As mpNum, n As mpNum) As mpNum**

---

The function ff returns the falling factorial.

**Parameters:**

*x*: A real or complex number.

*n*: A real or complex number.

The falling factorial is defined as,

$$x_{(n)} = x(x - 1) \cdots (x - n + 1) = \frac{\Gamma(x + 1)}{\Gamma(x - n + 1)} \quad (9.3.3)$$

where the rightmost expression is valid for nonintegral *n*.

Examples

For integral *n*, the falling factorial is a polynomial:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> for n in range(5):
...     nprint(taylor(lambda x: ff(x,n), 0, n))
...
[1.0]
[0.0, 1.0]
[0.0, -1.0, 1.0]
[0.0, 2.0, -3.0, 1.0]
[0.0, -6.0, 11.0, -6.0, 1.0]
```

---

Evaluation is supported for arbitrary arguments:

---

```
>>> ff(2+3j, 5.5)
(-720.41085888203 + 316.101124983878j)
```

---

## 9.4 Super- and hyperfactorials

### 9.4.1 Superfactorial

Function **superfac(z As mpNum)** As mpNum

The function `superfac` returns the superfactorial.

**Parameter:**

*z*: A real or complex number.

The superfactorial is defined as the product of consecutive factorials:

$$\text{sf}(n) = \prod_{k=1}^n k! \quad (9.4.1)$$

For general complex *z*,  $\text{sf}(n)$  is defined in terms of the Barnes G-function (see `barnesg()`).

Examples

The first few superfactorials are (OEIS A000178):

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> for n in range(10):
...     print("%s %s" % (n, superfac(n)))
...
0 1.0
1 1.0
2 2.0
3 12.0
4 288.0
5 34560.0
6 24883200.0
7 125411328000.0
8 5.05658474496e+15
9 1.83493347225108e+21
```

Superfactorials grow very rapidly:

```
>>> superfac(1000)
3.24570818422368e+1177245
>>> superfac(10**10)
2.61398543581249e+467427913956904067453
```

Evaluation is supported for arbitrary arguments:

```
>>> mp.dps = 25
>>> superfac(pi)
17.20051550121297985285333
>>> superfac(2+3j)
(-0.005915485633199789627466468 + 0.008156449464604044948738263j)
>>> diff(superfac, 1)
0.2645072034016070205673056
```

## 9.4.2 Hyperfactorial

---

Function **hyperfac(z As mpNum) As mpNum**

---

The function `hyperfac` returns the hyperfactorial.

**Parameter:**

*z*: A real or complex number.

The hyperfactorial is defined for integers as the product

$$H(n) = \prod_{k=1}^n k^k \quad (9.4.2)$$

The hyperfactorial satisfies the recurrence formula  $H(z) = z^z H(z - 1)$ . It can be defined more generally in terms of the Barnes G-function (see `barnesg()`) and the gamma function by the formula.

$$H(z) = \frac{\Gamma(z + 1)^z}{G(z)}. \quad (9.4.3)$$

The extension to complex numbers can also be done via the integral representation

$$H(z) = (2\pi)^{-z/2} \exp \left[ \binom{z+1}{2} + \int_0^z \log(t!) dt \right]. \quad (9.4.4)$$

Examples

The rapidly-growing sequence of hyperfactorials begins (OEIS A002109):

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> for n in range(10):
...     print("%s %s" % (n, hyperfac(n)))
...
0 1.0
1 1.0
2 4.0
3 108.0
4 27648.0
5 86400000.0
6 4031078400000.0
7 3.3197663987712e+18
8 5.56964379417266e+25
9 2.15779412229419e+34
```

---

Some even larger hyperfactorials are:

---

```
>>> hyperfac(1000)
5.46458120882585e+1392926
>>> hyperfac(10**10)
4.60408207642219e+489142638002418704309
```

---

Evaluation is supported for arbitrary arguments:

---

```
>>> hyperfac(0.5)
0.880449235173423
```

---

```
>>> diff(hyperfac, 1)
0.581061466795327
>>> hyperfac(pi)
205.211134637462
>>> hyperfac(-10+1j)
(3.01144471378225e+46 - 2.45285242480185e+46j)
```

---

### 9.4.3 Barnes G-function

---

Function **barnesg(z As mpNum) As mpNum**

---

The function **barnesg** returns the Barnes G-function.

**Parameter:**

*z*: A real or complex number.

The Barnes G-function generalizes the superfactorial (**superfac()**) and by extension also the hyperfactorial (**hyperfac()**) to the complex numbers in an analogous way to how the gamma function generalizes the ordinary factorial.

The Barnes G-function may be defined in terms of a Weierstrass product:

$$G(z + 1) = (2\pi)^{z/2} e^{[z(z+1)+\gamma z^2]/2} \prod_{n=1}^{\infty} \left[ \left(1 + \frac{z}{n}\right)^n e^{-z+z^2/(2n)} \right] \quad (9.4.5)$$

For positive integers *n*, we have have relation to superfactorials  $G(n) = sf(n-2) = 0! \cdot 1! \cdots (n-2)!$ .  
REF: Whittaker & Watson, A Course of Modern Analysis, Cambridge University Press, 4th edition (1927), p.264s

Examples

Some elementary values and limits of the Barnes G-function:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> barnesg(1), barnesg(2), barnesg(3)
(1.0, 1.0, 1.0)
>>> barnesg(4)
2.0
>>> barnesg(5)
12.0
>>> barnesg(6)
288.0
>>> barnesg(7)
34560.0
>>> barnesg(8)
24883200.0
>>> barnesg(inf)
+inf
>>> barnesg(0), barnesg(-1), barnesg(-2)
(0.0, 0.0, 0.0)
```

---

## 9.5 Gamma functions

### 9.5.1 Gamma function

---

Function **gamma(z As mpNum) As mpNum**

---

The function `gamma` returns the gamma function,  $\Gamma(x)$ .

**Parameter:**

`z`: A real or complex number.

The gamma function is a shifted version of the ordinary factorial, satisfying  $\Gamma(n) = (n - 1)!$  for integers  $n > 0$ . More generally, it is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad (9.5.1)$$

for any real or complex  $x$  with  $\Re(x) > 0$  and for  $\Re(x) < 0$  by analytic continuation.

Examples

Basic values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> for k in range(1, 6):
...     print("%s %s" % (k, gamma(k)))
...
1 1.0
2 1.0
3 2.0
4 6.0
5 24.0
>>> gamma(inf)
+inf
>>> gamma(0)
Traceback (most recent call last):
...
ValueError: gamma function pole
```

---

`gamma()` supports arbitrary-precision evaluation and complex arguments:

---

```
>>> mp.dps = 50
>>> gamma(sqrt(3))
0.91510229697308632046045539308226554038315280564184
>>> mp.dps = 25
>>> gamma(2j)
(0.009902440080927490985955066 - 0.07595200133501806872408048j)
```

---

Arguments can also be large. Note that the gamma function grows very quickly:

---

```
>>> mp.dps = 15
>>> gamma(10**20)
1.9328495143101e+1956570551809674817225
```

---

## 9.5.2 Reciprocal of the gamma function

---

### Function `rgamma(z As mpNum) As mpNum`

---

The function `rgamma` returns the reciprocal of the gamma function,  $1/\Gamma(z)$ .

#### Parameter:

*z*: A real or complex number.

This function evaluates to zero at the poles of the gamma function,  $z = 0, -1, -2, \dots$

Examples

Basic values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> rgamma(1)
1.0
>>> rgamma(4)
0.16666666666666666666666666666667
>>> rgamma(0); rgamma(-1)
0.0
0.0
>>> rgamma(1000)
2.485168143266784862783596e-2565
>>> rgamma(inf)
0.0
```

---

## 9.5.3 The product / quotient of gamma functions

---

### Function `gammaprod(a As mpNum, b As mpNum) As mpNum`

---

The function `gammaprod` returns the product / quotient of gamma functions.

#### Parameters:

*a*: A real or complex iterables.

*b*: A real or complex iterables.

Given iterables *a* and *b*, `gammaprod(a, b)` computes the product / quotient of gamma functions:

$$\frac{\Gamma(a_0)\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_0)\Gamma(b_1)\cdots\Gamma(b_p)} \quad (9.5.2)$$

Unlike direct calls to `gamma()`, `gammaprod()` considers the entire product as a limit and evaluates this limit properly if any of the numerator or denominator arguments are nonpositive integers such that poles of the gamma function are encountered. That is, `gammaprod()` evaluates

$$\lim_{\epsilon \rightarrow 0} \frac{\Gamma(a_0 + \epsilon)\Gamma(a_1 + \epsilon)\cdots\Gamma(a_p + \epsilon)}{\Gamma(b_0 + \epsilon)\Gamma(b_1 + \epsilon)\cdots\Gamma(b_p + \epsilon)} \quad (9.5.3)$$

In particular:

If there are equally many poles in the numerator and the denominator, the limit is a rational number times the remaining, regular part of the product.

If there are more poles in the numerator, `gammaprod()` returns `+inf`.

If there are more poles in the denominator, gammaprod() returns 0.

Examples

The reciprocal gamma function  $1/\Gamma(x)$  evaluated at 0:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15
>>> gammaprod([], [0])
0.0
```

---

A limit:

---

```
>>> gammaprod([-4], [-3])
-0.25
>>> limit(lambda x: gamma(x-1)/gamma(x), -3, direction=1)
-0.25
>>> limit(lambda x: gamma(x-1)/gamma(x), -3, direction=-1)
-0.25
```

---

## 9.5.4 The log-gamma function

---

Function **loggamma(z As mpNum)** As mpNum

---

The function loggamma returns the principal branch of the log-gamma function,  $\ln \Gamma(z)$ .

**Parameter:**

*z*: A real or complex number.

Unlike  $\ln(\Gamma(z))$ , which has infinitely many complex branch cuts, the principal log-gamma function only has a single branch cut along the negative half-axis. The principal branch continuously matches the asymptotic Stirling expansion

$$\ln \Gamma(z) \approx \frac{\ln(2\pi)}{2} + \left(z - \frac{1}{2}\right) \ln(z) - z + O(z^{-1}) \quad (9.5.4)$$

The real parts of both functions agree, but their imaginary parts generally differ by  $2n\pi$  for some  $n \in \mathbb{Z}$ . They coincide for  $z \in \mathbb{R}, z > 0$ .

Computationally, it is advantageous to use loggamma() instead of gamma() for extremely large arguments.

Examples

Comparing with :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> loggamma('13.2'); log(gamma('13.2'))
20.49400419456603678498394
20.49400419456603678498394
>>> loggamma(3+4j)
(-1.756626784603784110530604 + 4.742664438034657928194889j)
>>> log(gamma(3+4j))
(-1.756626784603784110530604 - 1.540520869144928548730397j)
>>> log(gamma(3+4j)) + 2*pi*j
(-1.756626784603784110530604 + 4.742664438034657928194889j)
```

---

Note the imaginary parts for negative arguments:

---

```
>>> loggamma(-0.5); loggamma(-1.5); loggamma(-2.5)
(1.265512123484645396488946 - 3.141592653589793238462643j)
(0.8600470153764810145109327 - 6.283185307179586476925287j)
(-0.05624371649767405067259453 - 9.42477796076937971538793j)
```

---

### 9.5.5 Generalized incomplete gamma function

---

Function **gammainc(z As mpNum, a As mpNum, b As mpNum, Keywords As String) As mpNum**

---

The function gammainc returns the incomplete gamma function with integration limits  $[a, b]$ .

**Parameters:**

*z*: A real or complex number.

*a*: A real or complex number (default = 0).

*b*: A real or complex number (default = inf).

*Keywords*: regularized=False.

The (generalized) incomplete gamma function with integration limits  $[a, b]$  is defined as:

$$\Gamma(z, a, b) = \int_a^b t^{z-1} e^{-t} dt \quad (9.5.5)$$

The generalized incomplete gamma function reduces to the following special cases when one or both endpoints are fixed:

$\Gamma(z, 0, \infty)$  is the standard ('complete') gamma function,  $\Gamma(z)$ , available directly as the mpFormulaPy function gamma().

$\Gamma(z, a, \infty)$  is the 'upper' incomplete gamma function,  $\Gamma(z, a)$ .

$\Gamma(z, 0, b)$  is the 'lower' incomplete gamma function,  $\gamma(z, a)$ .

Of course, we have  $\Gamma(z, a, \infty) + \Gamma(z, 0, b) = \Gamma(z)$  for all  $z$  and  $x$ .

Note however that some authors reverse the order of the arguments when defining the lower and upper incomplete gamma function, so one should be careful to get the correct definition.

If also given the keyword argument regularized=True, gammainc() computes the 'regularized' incomplete gamma function

$$P(z, a, b) = \frac{\Gamma(z, a, b)}{\Gamma(z)}. \quad (9.5.6)$$

Examples

We can compare with numerical quadrature to verify that gammainc() computes the integral in the definition:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> gammainc(2+3j, 4, 10)
(0.00977212668627705160602312 - 0.0770637306312989892451977j)
>>> quad(lambda t: t**(2+3j-1) * exp(-t), [4, 10])
(0.00977212668627705160602312 - 0.0770637306312989892451977j)
```

---

Evaluation for arbitrarily large arguments:

---

```
>>> gammainc(10, 100)
4.083660630910611272288592e-26
>>> gammainc(10, 10000000000000000)
5.290402449901174752972486e-4342944819032375
>>> gammainc(3+4j, 1000000+1000000j)
(-1.257913707524362408877881e-434284 + 2.556691003883483531962095e-434284j)
```

---

## 9.5.6 Derivative of the normalised incomplete gamma function

---

Function **GammaPDerivativeMpMath(*a* As mpNum, *x* As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function **GammaPDerivativeMpMath** returns the partial derivative with respect to *x* of the incomplete gamma function  $P(a, x)$ .

**Parameters:**

*a*: A real number.

*x*: A real number.

The partial derivative with respect to *x* of the incomplete gamma function  $P(a, x)$  is defined as:

$$\frac{\partial}{\partial x} P(a, x) = \frac{e^{-x} x^{a-1}}{\Gamma(a)}. \quad (9.5.7)$$

## 9.5.7 Normalised incomplete gamma functions

---

Boost references are [Temme \(1979\)](#) and [Temme \(1994\)](#)

---

Function **GammaPMpMath(*a* As mpNum, *x* As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function **GammaPMpMath** returns the normalised incomplete gamma function  $P(a, x)$ .

**Parameters:**

*a*: A real number.

*x*: A real number.

The normalised incomplete gamma function  $P(a, x)$  is defined as

$$P(a, x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt \quad (9.5.8)$$

for  $a \geq 0$  and  $x \geq 0$ .

---

Function **GammaQMpMath(*a* As mpNum, *x* As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function **GammaQMpMath** returns the normalised incomplete gamma function  $Q(a, x)$ .

**Parameters:**

*a*: A real number.

*x*: A real number.

The normalised incomplete gamma function  $Q(a, x)$  is defined as

$$Q(a, x) = \frac{1}{\Gamma(a)} \int_x^{\infty} t^{a-1} e^{-t} dt \quad (9.5.9)$$

for  $a \geq 0$  and  $x \geq 0$ .

### 9.5.8 Non-Normalised incomplete gamma functions

---

Function **NonNormalisedGammaPMpMath(*a* As mpNum, *x* As mpNum) As mpNum**

---

NOT YET IMPLEMENTED

---

The function NonNormalisedGammaPMpMath returns the non-normalised incomplete gamma function  $\Gamma(a, x)$ .

**Parameters:**

*a*: A real number.

*x*: A real number.

The non-normalised incomplete gamma function  $\Gamma(a, x)$  is defined as

$$\Gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt \quad (9.5.10)$$

for  $a \geq 0$  and  $x \geq 0$ .

---

Function **NonNormalisedGammaQMpMath(*a* As mpNum, *x* As mpNum) As mpNum**

---

NOT YET IMPLEMENTED

---

The function NonNormalisedGammaQMpMath returns the non-normalised incomplete gamma function  $\gamma(a, x)$ .

**Parameters:**

*a*: A real number.

*x*: A real number.

The non-normalised incomplete gamma function  $\gamma(a, x)$  is defined as

$$\gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt \quad (9.5.11)$$

for  $a \geq 0$  and  $x \geq 0$ .

Note: in Boost, the functions are referred to as TgammaLower and TgammaUpper.

### 9.5.9 Tricomi's entire incomplete gamma function

---

Function **TricomiGammaMpMath(*a* As mpNum, *x* As mpNum) As mpNum**

---

NOT YET IMPLEMENTED

---

The function **TricomiGammaMpMath** returns Tricomi's entire incomplete gamma function  $\gamma^*(a, x)$ .

**Parameters:**

*a*: A real number.

*x*: A real number.

This routine returns Tricomi's incomplete gamma function  $\gamma^*$ , defined as

$$\gamma^*(a, x) = e^{-x} \frac{M(1, a + 1, x)}{\Gamma(a + 1)} \quad (9.5.12)$$

Special cases are  $\gamma^*(0, x) = 1$ ,  $\gamma^*(a, 0) = 1/\Gamma(a + 1)$ , and  $\gamma^*(-n, x) = x^n$ , if  $-n$  is a negative integer. Otherwise there are the following relations to the other incomplete functions:

$$\gamma^*(a, x) = \frac{x^{-a}}{\Gamma(a)} \gamma(a, x) = x^{-a} P(a, x). \quad (9.5.13)$$

## 9.5.10 Inverse normalised incomplete gamma functions

---

Function **GammaPinvMpMath(*a* As mpNum, *p* As mpNum)** As mpNum

---

NOT YET IMPLEMENTED

---

The function **GammaPinvMpMath** returns the inverse of the normalised incomplete gamma function  $P(a, x)$ .

**Parameters:**

*a*: A real number.

*p*: A real number.

This function returns the inverse normalised incomplete gamma function, i.e. it calculates *x* with  $P(a, x) = p$ . The input parameters are  $a > 0$ ,  $p \geq 0$ , and  $p + q = 1$ .

---

Function **GammaQinvMpMath(*a* As mpNum, *q* As mpNum)** As mpNum

---

NOT YET IMPLEMENTED

---

The function **GammaQinvMpMath** returns the inverse of the normalised incomplete gamma function  $Q(a, x)$ .

**Parameters:**

*a*: A real number.

*q*: A real number.

## 9.6 Polygamma functions and harmonic numbers

### 9.6.1 Polygamma function

---

Function **polygamma(*m* As mpNum, *z* As mpNum)** As mpNum

---

The function **polygamma** returns the polygamma function of order *m* of *z*,  $\psi^{(m)}(z)$ .

**Parameters:**

*m*: A real or complex number.

*z*: A real or complex number.

Special cases are known as the digamma function ( $\psi^{(0)}(z)$ ), the trigamma function ( $\psi^{(1)}(z)$ ), etc. The polygamma functions are defined as the logarithmic derivatives of the gamma function:

$$\psi^{(m)}(z) = \left( \frac{d}{dz} \right)^{m+1} \log \Gamma(z). \quad (9.6.1)$$

In particular,  $\psi^{(0)}(z) = \Gamma'(z)/\Gamma(z)$ . In the present implementation of **psi()**, the order *m* must be a nonnegative integer, while the argument *z* may be an arbitrary complex number (with exception for the polygamma function's poles at *z* = 0, -1, -2, ...).

Examples

For various rational arguments, the polygamma function reduces to a combination of standard mathematical constants:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> polygamma(0, 1), -euler
(-0.5772156649015328606065121, -0.5772156649015328606065121)
>>> polygamma(1, '1/4'), pi**2+8*catalan
(17.19732915450711073927132, 17.19732915450711073927132)
>>> polygamma(2, '1/2'), -14*apery
(-16.82879664423431999559633, -16.82879664423431999559633)
```

---

Evaluation for a complex argument:

---

```
>>> polygamma(2, -1-2j)
(0.03902435405364952654838445 + 0.1574325240413029954685366j)
```

---

Evaluation is supported for large orders and/or large arguments :

---

```
>>> psi(3, 10**100)
2.0e-300
>>> psi(250, 10**30+10**20*j)
(-1.293142504363642687204865e-7010 + 3.232856260909107391513108e-7018j)
```

---



---

Function **psi(*m* As mpNum, *z* As mpNum)** As mpNum

---

The function **psi** returns the polygamma function of order *m* of *z*,  $\psi^{(m)}(z)$ .

**Parameters:**

*m*: A real or complex number.

*z*: A real or complex number.

A shortcut for polygamma(m,z).

## 9.6.2 Digamma function

---

Function **digamma(z As mpNum) As mpNum**

---

The function digamma returns the digamma function.

**Parameter:**

*z*: A real or complex number.

A shortcut for psi(0,z).

## 9.6.3 Harmonic numbers

---

Function **harmonic(n As mpNum) As mpNum**

---

The function harmonic returns a floating-point approximation of the *n*-th harmonic number  $H(n)$ .

**Parameter:**

*n*: An real or complex number.

If *n* is an integer, harmonic(*n*) gives a floating-point approximation of the *n*-th harmonic number  $H(n)$ , defined as

$$H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \quad (9.6.2)$$

The first few harmonic numbers are:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> for n in range(8):
...     print("%s %s" % (n, harmonic(n)))
...
0 0.0
1 1.0
2 1.5
3 1.8333333333333
4 2.0833333333333
5 2.2833333333333
6 2.45
7 2.59285714285714
```

---

harmonic() supports arbitrary precision evaluation:

---

```
>>> mp.dps = 50
>>> harmonic(11)
3.0198773448773448773448773448773448773448773
>>> harmonic(pi)
1.8727388590273302654363491032336134987519132374152
```

---

## 9.7 Beta Functions

### 9.7.1 Beta function $B(a, b)$

#### 9.7.2 Beta function

---

Function **beta(x As mpNum, y As mpNum) As mpNum**

---

The function `beta` returns the beta function,  $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ .

**Parameters:**

*x*: A real or complex number.

*y*: A real or complex number.

Computes the beta function,  $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ . The beta function is also commonly defined by the integral representation

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \quad (9.7.1)$$

Examples

`beta()` supports complex numbers and arbitrary precision evaluation:

---

```
>>> beta(1, 2+j)
(0.4 - 0.2j)
>>> mp.dps = 25
>>> beta(j, 0.5)
(1.079424249270925780135675 - 1.410032405664160838288752j)
>>> mp.dps = 50
>>> beta(pi, e)
0.037890298781212201348153837138927165984170287886464
```

---

### 9.7.3 Logarithm of $B(a, b)$

---

Function **LnBetaMpMath(a As mpNum, b As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function `LnBetaMpMath` returns the logarithm of the beta function  $\ln B(a, b)$  with  $a, b \neq 0, -1, -2, \dots$

**Parameters:**

*a*: A real number.

*b*: A real number.

### 9.7.4 Generalized incomplete beta function

---

Function **betainc(a As mpNum, b As mpNum, x1 As mpNum, x2 As mpNum, Keywords As String) As mpNum**

---

The function `betainc` returns the generalized incomplete beta function.

**Parameters:**

*a*: A real or complex number.

*b*: A real or complex number.

*x1*: A real or complex number (default = 0).

*x2*: A real or complex number (default = 1).

*Keywords*: regularized=False.

The generalized incomplete beta function is defined as,

$$I_{x_1}^{x_2}(a, b) = \int_{x_1}^{x_2} t^{a-1}(1-t)^{b-1} dt \quad (9.7.2)$$

When  $x_1 = 0, x_2 = 1$ , this reduces to the ordinary (complete) beta function  $B(a, b)$ ; see beta().

With the keyword argument regularized=True, betainc() computes the regularized incomplete beta function  $I_{x_1}^{x_2}(a, b)/B(a, b)$ . This is the cumulative distribution of the beta distribution with parameters *a*, *b*.

Examples

Verifying that betainc() computes the integral in the definition:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> x,y,a,b = 3, 4, 0, 6
>>> betainc(x, y, a, b)
-4010.4
>>> quad(lambda t: t**(x-1) * (1-t)**(y-1), [a, b])
-4010.4
```

---

The arguments may be arbitrary complex numbers:

---

```
>>> betainc(0.75, 1-4j, 0, 2+3j)
(0.2241657956955709603655887 + 0.3619619242700451992411724j)
```

---

With regularization:

---

```
>>> betainc(1, 2, 0, 0.25, regularized=True)
0.4375
>>> betainc(pi, e, 0, 1, regularized=True) # Complete
1.0
```

---

## 9.7.5 Non-Normalised incomplete beta functions

The algorithm is implemented as in [DiDonato & Morris \(1986\)](#)

---

Function **IBetaNonNormalizedMpMath(*a* As mpNum, *b* As mpNum, *x* As mpNum) As mpNum**  
**NOT YET IMPLEMENTED**

---

The function IBetaNonNormalizedMpMath returns the non-normalised incomplete beta function.

**Parameters:**

*a*: A real number.

*b*: A real number.

*x*: A real number.

This function returns the non-normalised incomplete beta function  $B_x(a, b)$  for  $a > 0$ ,  $b > 0$ , and  $0 \leq x \leq 1$ :

$$B_x(a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt. \quad (9.7.3)$$

There are some special cases

$$B_0(a, b) = 0, \quad B_1(a, b) = B(a, b), \quad B_x(a, 1) = \frac{x^a}{a}, \quad B_x(1, b) = \frac{1 - (1-x)^b}{b}, \quad (9.7.4)$$

and the relation  $B_{1-x}(a, b) = B(a, b) - B_x(b, a)$ , which is used if  $x > a/(a+b)$ .

When  $a \leq 0$  or  $b \leq 0$ , the Gauss hypergeometric function  ${}_2F_1(\cdot)$  is applied: If  $a \neq 0$  is not a negative integer, the result is

$$B_x(a, b) = \frac{x^a}{a} {}_2F_1(a, 1-b, a+1, x), \quad -a \notin \mathbb{N} \quad (9.7.5)$$

else if  $b \neq 0$  is not a negative integer, the result is

$$B_x(a, b) = B(a, b) - \frac{(1-x)^b x^a}{b} {}_2F_1(1, a+b, b+1, 1-x), \quad -b \notin \mathbb{N}. \quad (9.7.6)$$

## 9.7.6 Normalised incomplete beta functions

---

Function **IBetaMpMath**(*a* As mpNum, *b* As mpNum, *x* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function **IBetaMpMath** returns the normalised incomplete beta function.

### Parameters:

*a*: A real number.

*b*: A real number.

*x*: A real number.

This function returns the normalised incomplete beta function  $I_x(a, b)$  for  $a > 0$ ,  $b > 0$ , and  $0 \leq x \leq 1$ :

$$I_x(a, b) = \frac{B_x(a, b)}{B(a, b)}, \quad B_x(a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt. \quad (9.7.7)$$

There are some special cases

$$I_0(a, b) = 0, \quad I_1(a, b) = 1, \quad I_x(a, 1) = x^a, \quad (9.7.8)$$

and the symmetry relation  $I_x(a, b) = 1 - I_{1-x}(b, a)$ , which is used for  $x > a/(a+b)$ .

# Chapter 10

## Exponential integrals and error functions

Exponential integrals give closed-form solutions to a large class of commonly occurring transcendental integrals that cannot be evaluated using elementary functions. Integrals of this type include those with an integrand of the form  $t^a e^t$  or  $e^{-x^2}$ , the latter giving rise to the Gaussian (or normal) probability distribution.

All functions in this section can be reduced to the incomplete gamma function. The incomplete gamma function, in turn, can be expressed using hypergeometric functions (see Hypergeometric functions).

### 10.1 Exponential integrals

#### 10.1.1 Exponential integral Ei

---

Function **ei(z As mpNum)** As mpNum

---

The function `ei` returns the exponential integral.

**Parameter:**

`z`: A real or complex number.

The exponential integral is defined as

$$\text{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} dt \quad (10.1.1)$$

When the integration range includes  $t = 0$ , the exponential integral is interpreted as providing the Cauchy principal value.

For real  $x$ , the Ei-function behaves roughly like  $\text{Ei}(x) \approx \exp(x) + \log(|x|)$ .

The Ei-function is related to the more general family of exponential integral functions denoted by  $E_n$ , which are available as `expint()`.

`ei()` supports complex arguments and arbitrary precision evaluation:

---

```
>>> mp.dps = 50
>>> ei(pi)
10.928374389331410348638445906907535171566338835056
>>> mp.dps = 25
>>> ei(3+4j)
```

---

(-4.154091651642689822535359 + 4.294418620024357476985535j)

---

### 10.1.2 Exponential integral E1

---

**Function e1(z As mpNum) As mpNum**

---

The function e1 returns the exponential integral  $E_1(x)$ .

**Parameter:**

*z*: A real or complex number.

The exponential integral  $E_1(x)$  is defined as

$$E_1(x) = \int_z^\infty \frac{e^t}{t} dt \quad (10.1.2)$$

This is equivalent to expint() with  $n = 1$ .

The E1-function is essentially the same as the Ei-function (ei()) with negated argument, except for an imaginary branch cut term:

---

```
>>> e1(2.5)
0.02491491787026973549562801
>>> -ei(-2.5)
0.02491491787026973549562801
>>> e1(-2.5)
(-7.073765894578600711923552 - 3.141592653589793238462643j)
>>> -ei(2.5)
-7.073765894578600711923552
```

---

### 10.1.3 Generalized exponential integral En

---

**Function expint(n As mpNum, z As mpNum) As mpNum**

---

The function expint returns the generalized exponential integral or En-function.

**Parameters:**

*n*: A real or complex number.

*z*: A real or complex number.

The generalized exponential integral or En-function is defined as

$$E_n(x) = \int_1^\infty \frac{e^{-zt}}{t^n} dt \quad (10.1.3)$$

where *n* and *z* may both be complex numbers. The case with *n* is also given by e1().

**Examples**

Evaluation at real and complex arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> expint(1, 6.25)
0.0002704758872637179088496194
>>> expint(-3, 2+3j)
```

---

```
(0.00299658467335472929656159 + 0.06100816202125885450319632j)
>>> expint(2+3j, 4-5j)
(0.001803529474663565056945248 - 0.002235061547756185403349091j)
```

---

### 10.1.4 Generalized Exponential Integrals $E_p$

---

Function **GeneralizedExponentialIntegralEpMpMath**(*x As mpNum, p As mpNum*) As mpNum  
**NOT YET IMPLEMENTED**

---

The function **GeneralizedExponentialIntegralEpMpMath** returns the generalized exponential integrals  $E_n(x)$  of real order  $p$ .

**Parameters:**

*x*: A real number.  
*p*: A real number.

This function returns the generalized exponential integrals  $E_n(x)$  of real order  $p \in \mathbb{R}$

$$E_p(x) = x^{p-1} \int_x^\infty \frac{e^{-t}}{t^p} dt = \int_1^\infty \frac{e^{-xt}}{t^p} dt \quad (10.1.4)$$

with  $x > 0$  for  $p \leq 1$ , and  $x \geq 0$  for  $p > 1$ .

## 10.2 Logarithmic integral

### 10.2.1 logarithmic integral li

Function **li(z As mpNum) As mpNum**

The function **li** returns the logarithmic integral.

**Parameter:**

**z:** A real or complex number.

The logarithmic integral or li-function  $\text{li}(x)$  is defined by

$$\text{li}(x) = \int_0^x \frac{1}{\log(t)} dt \quad (10.2.1)$$

The logarithmic integral has a singularity at  $x = 1$ .

Alternatively, **li(x, offset=True)** computes the offset logarithmic integral (used in number theory)

$$\text{Li}(x) = \int_2^x \frac{1}{\log(t)} dt \quad (10.2.2)$$

These two functions are related via the simple identity  $\text{Li}(x) = \text{li}(x) - \text{li}(2)$ .

The logarithmic integral should also not be confused with the polylogarithm (also denoted by **Li**), which is implemented as **polylog()**.

**Examples**

Some basic values and limits:

```
>>> from mpFormulaPy import *
>>> mp.dps = 30; mp.pretty = True
>>> li(0)
0.0
>>> li(1)
-inf
>>> li(1)
-inf
>>> li(2)
1.04516378011749278484458888919
>>> findroot(li, 2)
1.45136923488338105028396848589
>>> li(inf)
+inf
>>> li(2, offset=True)
0.0
>>> li(1, offset=True)
-inf
>>> li(0, offset=True)
-1.04516378011749278484458888919
```

The logarithmic integral can be evaluated for arbitrary complex arguments:

```
>>> mp.dps = 20
>>> li(3+4j)
(3.1343755504645775265 + 2.6769247817778742392j)
```

## 10.3 Trigonometric integrals

### 10.3.1 cosine integral ci

---

Function **ci(z As mpNum) As mpNum**

---

The function ci returns the cosine integral.

**Parameter:**

*z*: A real or complex number.

The cosine integral is defined as

$$\text{Ci}(x) = \int_x^{\infty} \frac{\cos(t)}{t} dt = \gamma + \log(x) + \int_0^x \frac{\cos(t) - 1}{t} dt \quad (10.3.1)$$

Examples

Some values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> ci(0)
-inf
>>> ci(1)
0.3374039229009681346626462
>>> ci(pi)
0.07366791204642548599010096
>>> ci(inf)
0.0
>>> ci(-inf)
(0.0 + 3.141592653589793238462643j)
>>> ci(2+3j)
(1.408292501520849518759125 - 2.983617742029605093121118j)
```

---

### 10.3.2 sine integral si

---

Function **si(z As mpNum) As mpNum**

---

The function si returns the sine integral.

**Parameter:**

*z*: A real or complex number.

The sine integral is defined as

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt \quad (10.3.2)$$

The sine integral is thus the antiderivative of the sinc function (see sinc()).

Examples

Some values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
```

```
>>> si(0)
0.0
>>> si(1)
0.9460830703671830149413533
>>> si(-1)
-0.9460830703671830149413533
>>> si(pi)
1.851937051982466170361053
>>> si(inf)
1.570796326794896619231322
>>> si(-inf)
-1.570796326794896619231322
>>> si(2+3j)
(4.547513889562289219853204 + 1.399196580646054789459839j)
```

---

## 10.4 Hyperbolic integrals

### 10.4.1 hyperbolic cosine integral chi

---

Function **chi(z As mpNum) As mpNum**

---

The function **chi** returns the hyperbolic cosine integral.

**Parameter:**

**z:** A real or complex number.

The hyperbolic cosine integral, in analogy with the cosine integral (see **ci()**), is defined as

$$\text{Chi}(x) = \int_x^{\infty} \frac{\cosh(t)}{t} dt = \gamma + \log(x) + \int_0^x \frac{\cosh(t) - 1}{t} dt \quad (10.4.1)$$

Some values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> chi(0)
-inf
>>> chi(1)
0.8378669409802082408946786
>>> chi(inf)
+inf
>>> findroot(chi, 0.5)
0.5238225713898644064509583
>>> chi(2+3j)
(-0.1683628683277204662429321 + 2.625115880451325002151688j)
```

---

### 10.4.2 hyperbolic sine integral shi

---

Function **shi(z As mpNum) As mpNum**

---

The function **shi** returns the hyperbolic sine integral.

**Parameter:**

**z:** A real or complex number.

Computes the hyperbolic sine integral, defined in analogy with the sine integral (see **si()**) as

$$\text{Shi}(x) = \int_0^x \frac{\sinh(t)}{t} dt \quad (10.4.2)$$

Some values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> shi(0)
0.0
>>> shi(1)
1.057250875375728514571842
>>> shi(-1)
```

---

---

```
-1.057250875375728514571842
>>> shi(inf)
+inf
>>> shi(2+3j)
(-0.1931890762719198291678095 + 2.645432555362369624818525j)
```

---

## 10.5 Error functions

### 10.5.1 Error Function

---

Function **erf(z As mpNum) As mpNum**

---

The function `erf` returns the error function,  $\text{erf}(x)$ .

**Parameter:**

`z`: A real or complex number.

The error function is the normalized antiderivative of the Gaussian function  $\exp(-t^2)$ . More precisely,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \quad (10.5.1)$$

Basic examples

Simple values and limits include:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> erf(0)
0.0
>>> erf(1)
0.842700792949715
>>> erf(-1)
-0.842700792949715
>>> erf(inf)
1.0
>>> erf(-inf)
-1.0
```

---

`erf()` implements arbitrary-precision evaluation and supports complex numbers:

---

```
>>> mp.dps = 50
>>> erf(0.5)
0.52049987781304653768274665389196452873645157575796
>>> mp.dps = 25
>>> erf(1+j)
(1.316151281697947644880271 + 0.1904534692378346862841089j)
```

---

See also `erfc()`, which is more accurate for large  $x$ , and `erfi()` which gives the antiderivative of  $\exp(t^2)$ . The Fresnel integrals `fresnels()` and `fresnelc()` are also related to the error function

### 10.5.2 Complementary Error Function

---

#### Function `erfc(z As mpNum) As mpNum`

---

The function `erfc` returns the complementary error function,  $\text{erfc}(x) = 1 - \text{erf}(x)$ .

**Parameter:**

`z`: A real or complex number.

This function avoids cancellation that occurs when naively computing the complementary error function as  $1 - \text{erf}(x)$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> 1 - erf(10)
0.0
>>> erfc(10)
2.08848758376254e-45
```

---

`erfc()` works accurately even for ludicrously large arguments:

---

```
>>> erfc(10**10)
4.3504398860243e-43429448190325182776
```

---

Complex arguments are supported:

---

```
>>> erfc(500+50j)
(1.19739830969552e-107492 + 1.46072418957528e-107491j)
```

---

### 10.5.3 Imaginary Error Function

---

#### Function `erfi(z As mpNum) As mpNum`

---

The function `erfi` returns the imaginary error function,  $\text{erfi}(x)$ .

**Parameter:**

`z`: A real or complex number.

The imaginary error function is defined in analogy with the error function, but with a positive sign in the integrand:

$$\text{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(t^2) dt \quad (10.5.2)$$

Whereas the error function rapidly converges to 1 as grows, the imaginary error function rapidly diverges to infinity. The functions are related as  $\text{erfi}(x) = -i \text{erf}(ix)$  for all complex numbers  $x$ .

Examples

Basic values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> erfi(0)
0.0
```

---

---

```
>>> erfi(1)
1.65042575879754
>>> erfi(-1)
-1.65042575879754
>>> erfi(inf)
+inf
>>> erfi(-inf)
-inf
```

---

Large arguments are supported:

---

```
>>> erfi(1000)
1.71130938718796e+434291
>>> erfi(10**10)
7.3167287567024e+43429448190325182754
>>> erfi(-10**10)
-7.3167287567024e+43429448190325182754
>>> erfi(1000-500j)
(2.49895233563961e+325717 + 2.6846779342253e+325717j)
>>> erfi(100000j)
(0.0 + 1.0j)
>>> erfi(-100000j)
(0.0 - 1.0j)
```

---

Complex arguments are supported:

---

```
>>> erfc(500+50j)
(1.19739830969552e-107492 + 1.46072418957528e-107491j)
```

---

#### 10.5.4 Inverse Error Function

---

**Function `erfinv(x As mpNum) As mpNum`**

---

The function `erfinv` returns the inverse error function,  $\text{erfinv}(x)$ .

**Parameter:**

*x*: A real number.

The inverse error function satisfies  $\text{erf}(\text{erfinv}(x)) = \text{erfinv}(\text{erf}(x)) = x$ . This function is defined only for  $-1 < x < 1$ .

Examples

Special values include:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> erfinv(0)
0.0
>>> erfinv(1)
+inf
>>> erfinv(-1)
-inf
```

---

`erfinv()` supports arbitrary-precision evaluation:

---

```
>>> mp.dps = 50
>>> x = erf(2)
>>> x
0.99532226501895273416206925636725292861089179704006
>>> erfinv(x)
2.0
```

---

## 10.6 The normal distribution

### 10.6.1 The normal probability density function

---

Function **npdf(*x* As mpNum, *mu* As mpNum, *sigma* As mpNum)** As mpNum

---

The function `npdf` returns the normal probability density function.

**Parameters:**

*x*: A real number.

*mu*: A real number.

*sigma*: A real number.

`npdf(x, mu=0, sigma=1)` evaluates the probability density function of a normal distribution with mean value  $\mu$  and variance  $\sigma^2$ .

Elementary properties of the probability distribution can be verified using numerical integration:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> quad(npdf, [-inf, inf])
1.0
>>> quad(lambda x: npdf(x, 3), [3, inf])
0.5
>>> quad(lambda x: npdf(x, 3, 2), [3, inf])
0.5
```

---

### 10.6.2 The normal cumulative distribution function

---

Function **ncdf(*x* As mpNum, *mu* As mpNum, *sigma* As mpNum)** As mpNum

---

The function `ncdf` returns the normal cumulative distribution function.

**Parameters:**

*x*: A real number.

*mu*: A real number.

*sigma*: A real number.

`ncdf(x, mu=0, sigma=1)` evaluates the cumulative distribution function of a normal distribution with mean value  $\mu$  and variance  $\sigma^2$ .

See also `npdf()`, which gives the probability density.

Elementary properties include:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> ncdf(pi, mu=pi)
0.5
>>> ncdf(-inf)
0.0
>>> ncdf(+inf)
1.0
```

---

## 10.7 Fresnel integrals

### 10.7.1 Fresnel sine integral

---

Function **fresnels(z As mpNum) As mpNum**

---

The function `fresnels` returns the Fresnel sine integral.

**Parameter:**

*z*: A real or complex number.

The Fresnel sine integral is defined as

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \quad (10.7.1)$$

Note that some sources define this function without the normalization factor  $\pi/2$ .

Examples

Some basic values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> fresnels(0)
0.0
>>> fresnels(inf)
0.5
>>> fresnels(-inf)
-0.5
>>> fresnels(1)
0.4382591473903547660767567
>>> fresnels(1+2j)
(36.72546488399143842838788 + 15.58775110440458732748279j)
```

---

### 10.7.2 Fresnel cosine integral

---

Function **fresnelc(z As mpNum) As mpNum**

---

The function `fresnelc` returns the Fresnel cosine integral.

**Parameter:**

*z*: A real or complex number.

The Fresnel cosine integral is defined as

$$C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \quad (10.7.2)$$

Note that some sources define this function without the normalization factor  $\pi/2$ .

Examples

Some basic values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
```

```
>>> fresnelc(0)
0.0
>>> fresnelc(inf)
0.5
>>> fresnelc(-inf)
-0.5
>>> fresnelc(1)
0.7798934003768228294742064
>>> fresnelc(1+2j)
(16.08787137412548041729489 - 36.22568799288165021578758j)
```

---

## 10.8 Other Special Functions

### 10.8.1 Lambert W function

---

Function **lambertw(z As mpNum, Keywords As String)** As mpNum

---

The function `lambertw` returns the Lambert W function.

**Parameters:**

*z*: A real or complex number.

*Keywords*: *k*=0.

The Lambert W function  $W(z)$  is defined as the inverse function of  $w \exp(w)$ . In other words, the value of  $W(z)$  is such that  $z = W(z) \exp(W(z))$  for any complex number  $z$ .

The Lambert W function is a multivalued function with infinitely many branches  $W_k(z)$ , indexed by  $k \in \mathbb{Z}$ . Each branch gives a different solution  $w$  of the equation  $z = w \exp(w)$ . All branches are supported by `lambertw()`:

`lambertw(z)` gives the principal solution (branch 0).

`lambertw(z, k)` gives the solution on branch  $k$ .

The Lambert W function has two partially real branches: the principal branch ( $k = 0$ ) is real for real  $z > -1/e$ , and the branch  $k = -1$  is real for  $-1/e < z < 0$ . All branches except  $k = 0$  have a logarithmic singularity at  $z = 0$ .

The definition, implementation and choice of branches is based on [Corless et al. \(1996\)](#).

**Basic examples**

The Lambert W function is the inverse of  $w \exp(w)$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> w = lambertw(1)
>>> w
0.5671432904097838729999687
>>> w*exp(w)
1.0
```

---

Any branch gives a valid inverse:

---

```
>>> w = lambertw(1, k=3)
>>> w
(-2.853581755409037807206819 + 17.11353553941214591260783j)
>>> w = lambertw(1, k=25)
>>> w
(-5.047020464221569709378686 + 155.4763860949415867162066j)
>>> chop(w*exp(w))
1.0
```

---

### 10.8.2 Arithmetic-geometric mean

---

Function **agm(*a* As mpNum, *b* As mpNum)** As mpNum

---

The function **agm** returns the arithmetic-geometric mean of *a* and *b*.

**Parameters:**

*a*: A real or complex number.

*b*: A real or complex number.

`agm(a, b)` computes the arithmetic-geometric mean of *a* and *b*, defined as the limit of the following iteration:

$$a_0 = a; \quad b_0 = b; \quad a_{n+1} = \frac{1}{2}(a_n + b_n); \quad b_{n+1} = \sqrt{a_n b_n}. \quad (10.8.1)$$

This function can be called with a single argument, computing  $\text{agm}(a, 1) = \text{agm}(1, a)$ .

Examples

It is a well-known theorem that the geometric mean of two distinct positive numbers is less than the arithmetic mean. It follows that the arithmetic-geometric mean lies between the two means:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> a = mpf(3)
>>> b = mpf(4)
>>> sqrt(a*b)
3.46410161513775
>>> agm(a,b)
3.48202767635957
>>> (a+b)/2
3.5
```

---

The arithmetic-geometric mean can also be computed for complex numbers:

---

```
>>> agm(3, 2+j)
(2.51055133276184 + 0.547394054060638j)
```

---

A formula for  $\Gamma(1/4)$ :

---

```
>>> gamma(0.25)
3.62560990822191
>>> sqrt(2*sqrt(2*pi**3)/agm(1,sqrt(2)))
3.62560990822191
```

---

# Chapter 11

## Bessel functions and related functions

The functions in this section arise as solutions to various differential equations in physics, typically describing wavelike oscillatory behavior or a combination of oscillation and exponential decay or growth. Mathematically, they are special cases of the confluent hypergeometric functions  ${}_0F_1$ ,  ${}_1F_1$  and  ${}_1F_2$  (see Hypergeometric functions).

### 11.1 Bessel functions

#### 11.1.1 Exponentially scaled Bessel function $I_{\nu,e}(x)$

---

Function **BesselleMpMath(x As mpNum, ν As mpNum) As mpNum**

---

NOT YET IMPLEMENTED

---

The function **BesselleMpMath** returns  $I_{\nu,e}(x) = I_{\nu}(x) \exp(-|x|)$ , the exponentially scaled modified Bessel function  $I_{\nu}(z)$  of the first kind of order  $\nu$ ,  $x \geq 0$  if  $\nu$  is not an integer.

**Parameters:**

$x$ : A real number.

$\nu$ : A real number.

#### 11.1.2 Exponentially scaled Bessel function $K_{\nu,e}(x)$

---

Function **BesselKeMpMath(x As mpNum, ν As mpNum) As mpNum**

---

NOT YET IMPLEMENTED

---

The function **BesselKeMpMath** returns  $K_{\nu,e}(x) = K_{\nu}(x) \exp(x)$ , the exponentially scaled modified Bessel function  $K_{\nu}(z)$  of the first kind of order  $\nu$ ,  $x > 0$ .

**Parameters:**

$x$ : A real number.

$\nu$ : A real number.

#### 11.1.3 Bessel function of the first kind

---

Function **besselj(n As mpNum, x As mpNum, Keywords As String) As mpNum**

---

The function **besselj** returns the Bessel function of the first kind  $J_n(x)$ .

**Parameters:**

*n*: A real or complex number.

*x*: A real or complex number.

*Keywords*: derivative=0.

Bessel functions of the first kind are defined as solutions of the differential equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \quad (11.1.1)$$

which appears, among other things, when solving the radial part of Laplace's equation in cylindrical coordinates. This equation has two solutions for given *n*, where the  $J_n$ -function is the solution that is nonsingular at  $x = 0$ . For positive integer *n*,  $J_n(x)$  behaves roughly like a sine (odd *n*) or cosine (even *n*) multiplied by a magnitude factor that decays slowly as  $x \rightarrow \pm\infty$ .

Generally,  $J_n$  is a special case of the hypergeometric function  ${}_0F_1$ :

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} {}_0F_1 \left( n+1, -\frac{x^2}{4} \right) \quad (11.1.2)$$

With derivative = *m* ≠ 0, the *m*-th derivative

$$\frac{d^m}{dx^m} J_n(x) \quad (11.1.3)$$

is computed.

Evaluation is supported for arbitrary arguments, and at arbitrary precision:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> besselj(2, 1000)
-0.024777229528606
>>> besselj(4, 0.75)
0.000801070086542314
>>> besselj(2, 1000j)
(-2.48071721019185e+432 + 6.41567059811949e-437j)
>>> mp.dps = 25
>>> besselj(0.75j, 3+4j)
(-2.778118364828153309919653 - 1.5863603889018621585533j)
>>> mp.dps = 50
>>> besselj(1, pi)
0.28461534317975275734531059968613140570981118184947
```

---

Arguments may be large:

---

```
>>> mp.dps = 25
>>> besselj(0, 10000)
-0.007096160353388801477265164
>>> besselj(0, 10**10)
0.000002175591750246891726859055
>>> besselj(2, 10**100)
7.337048736538615712436929e-51
>>> besselj(2, 10**5*j)
(-3.540725411970948860173735e+43426 + 4.4949812409615803110051e-43433j)
```

---

Derivatives of any order can be computed (negative orders correspond to integration):

---

```
>>> mp.dps = 25
>>> besselj(0, 7.5, 1)
-0.1352484275797055051822405
>>> diff(lambda x: besselj(0,x), 7.5)
-0.1352484275797055051822405
>>> besselj(0, 7.5, 10)
-0.1377811164763244890135677
>>> diff(lambda x: besselj(0,x), 7.5, 10)
-0.1377811164763244890135677
>>> besselj(0,7.5,-1) - besselj(0,3.5,-1)
-0.1241343240399987693521378
>>> quad(j0, [3.5, 7.5])
-0.1241343240399987693521378
```

---



---

**Function *j0(x As mpNum) As mpNum***

---

The function *j0* returns the Bessel function  $J_0(x)$ .

**Parameter:**

*x*: A real or complex number.

Computes the Bessel function  $J_0(x)$ . See *besselj()*.

---

**Function *j1(x As mpNum) As mpNum***

---

The function *j1* returns the Bessel function  $J_1(x)$ .

**Parameter:**

*x*: A real or complex number.

Computes the Bessel function  $J_1(x)$ . See *besselj()*.

### 11.1.4 Bessel function of the second kind

---

**Function *bessely(n As mpNum, x As mpNum, Keywords As String) As mpNum***

---

The function *bessely* returns the Bessel function of the second kind  $Y_n(x)$ .

**Parameters:**

*n*: A real or complex number.

*x*: A real or complex number.

*Keywords*: derivative=0.

The Bessel function of the second kind are defined as

$$Y_n(x) = \frac{J_n(x) \cos(\pi n) - J_{-n}(x)}{\sin(\pi n)} \quad (11.1.4)$$

For  $n$  an integer, this formula should be understood as a limit. With derivative =  $m \neq 0$ , the  $m$ -th derivative

$$\frac{d^m}{dx^m} Y_n(x) \quad (11.1.5)$$

is computed.

Evaluation is supported for arbitrary arguments, and at arbitrary precision:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> bessely(0,0), bessely(1,0), bessely(2,0)
(-inf, -inf, -inf)
>>> bessely(1, pi)
0.3588729167767189594679827
>>> bessely(0.5, 3+4j)
(9.242861436961450520325216 - 3.085042824915332562522402j)
```

---

Arguments may be large:

---

```
>>> bessely(0, 10000)
0.00364780555898660588668872
>>> bessely(2.5, 10**50)
-4.8952500412050989295774e-26
>>> bessely(2.5, -10**50)
(0.0 + 4.8952500412050989295774e-26j)
```

---

Derivatives and antiderivatives of any order can be computed:

---

```
>>> bessely(2, 3.5, 1)
0.3842618820422660066089231
>>> diff(lambda x: bessely(2, x), 3.5)
0.3842618820422660066089231
>>> bessely(0.5, 3.5, 1)
-0.2066598304156764337900417
>>> diff(lambda x: bessely(0.5, x), 3.5)
-0.2066598304156764337900417
>>> diff(lambda x: bessely(2, x), 0.5, 10)
-208173867409.5547350101511
>>> bessely(2, 0.5, 10)
-208173867409.5547350101511
>>> bessely(2, 100.5, 100)
0.02668487547301372334849043
>>> quad(lambda x: bessely(2,x), [1,3])
-1.377046859093181969213262
>>> bessely(2,3,-1) - bessely(2,1,-1)
-1.377046859093181969213262
```

---

### 11.1.5 Modified Bessel function of the first kind

---

Function **besseli**(*n* As mpNum, *x* As mpNum, **Keywords** As String) As mpNum

---

The function **besseli** returns the modified Bessel function of the first kind  $J_n(x)$ .

**Parameters:**

*n*: A real or complex number.

*x*: A real or complex number.

**Keywords**: derivative=0.

The modified Bessel function of the first kind are defined as

$$I_n(x) = i^{-n} J_n(ix) \quad (11.1.6)$$

With derivative =  $m \neq 0$ , the  $m$ -th derivative

$$\frac{d^m}{dx^m} I_n(x) \quad (11.1.7)$$

is computed.

Evaluation is supported for arbitrary arguments, and at arbitrary precision:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> besseli(0,0)
1.0
>>> besseli(1,0)
0.0
>>> besseli(0,1)
1.266065877752008335598245
>>> besseli(3.5, 2+3j)
(-0.2904369752642538144289025 - 0.4469098397654815837307006j)
```

---

Arguments may be large:

---

```
>>> besseli(2, 1000)
2.480717210191852440616782e+432
>>> besseli(2, 10**10)
4.299602851624027900335391e+4342944813
>>> besseli(2, 6000+10000j)
(-2.114650753239580827144204e+2603 + 4.385040221241629041351886e+2602j)
```

---

Derivatives and antiderivatives of any order can be computed:

---

```
>>> mp.dps = 25
>>> besseli(2, 7.5, 1)
195.8229038931399062565883
>>> diff(lambda x: besseli(2,x), 7.5)
195.8229038931399062565883
>>> besseli(2, 7.5, 10)
153.3296508971734525525176
>>> diff(lambda x: besseli(2,x), 7.5, 10)
153.3296508971734525525176
>>> besseli(2,7.5,-1) - besseli(2,3.5,-1)
202.5043900051930141956876
>>> quad(lambda x: besseli(2,x), [3.5, 7.5])
202.5043900051930141956876
```

---

## 11.1.6 Modified Bessel function of the second kind

---

Function **besselk**(*n* As *mpNum*, *x* As *mpNum*) As *mpNum*

---

The function **besselk** returns the modified Bessel function of the second kind  $K_n(x)$ .

**Parameters:**

*n*: A real or complex number.

*x*: A real or complex number.

The modified Bessel function of the second kind are defined as

$$K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin(\pi n)} \quad (11.1.8)$$

For *n* an integer, this formula should be understood as a limit.

Evaluation is supported for arbitrary arguments, and at arbitrary precision:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> besselk(0,1)
0.4210244382407083333356274
>>> besselk(0, -1)
(0.4210244382407083333356274 - 3.97746326050642263725661j)
>>> besselk(3.5, 2+3j)
(-0.02090732889633760668464128 + 0.2464022641351420167819697j)
>>> besselk(2+3j, 0.5)
(0.9615816021726349402626083 + 0.1918250181801757416908224j)
```

---

Arguments may be large:

---

```
>>> besselk(0, 100)
4.656628229175902018939005e-45
>>> besselk(1, 10**6)
4.131967049321725588398296e-434298
>>> besselk(1, 10**6*j)
(0.001140348428252385844876706 - 0.0005200017201681152909000961j)
>>> besselk(4.5, fmul(10**50, j, exact=True))
(1.561034538142413947789221e-26 + 1.243554598118700063281496e-25j)
```

---

## 11.2 Bessel function zeros

### 11.2.1 Zeros of the Bessel function of the first kind

---

Function **besseljzero(v As mpNum, m As mpNum, Keywords As String) As mpNum**

---

The function `besseljzero` returns the  $m$ -th positive zero of the Bessel function of the first kind

**Parameters:**

*v*: A real or complex number.

*m*: A real or complex number.

*Keywords*: derivative=0.

For a real order  $\nu \geq 0$  and a positive integer  $m$ , returns  $j_{\nu,m}$ , the  $m$ -th positive zero of the Bessel function of the first kind  $J_\nu(z)$ (see `besselj()`). Alternatively, with derivative=1, gives the first nonnegative simple zero  $j'_{\nu,m}$  of  $J'_\nu(z)$ .

The indexing convention is that used by Abramowitz & Stegun and the DLMF. Note the special case  $j'_{0,1} = 0$ , while all other zeros are positive. In effect, only simple zeros are counted (all zeros of Bessel functions are simple except possibly  $z = 0$ ) and becomes a monotonic function of both  $\nu$  and  $m$ .

The zeros are interlaced according to the inequalities

$$j'_{\nu,k} < j_{\nu,k} < j'_{\nu,k+1} \quad (11.2.1)$$

$$j_{\nu,1} < j_{\nu+1,2} < j_{\nu,2} < j'_{\nu,k+1} < j_{\nu+1,2} < j_{\nu,3} \quad (11.2.2)$$

Initial zeros of the Bessel functions  $J_0(z), J_1(z), J_2(z)$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> besseljzero(0,1); besseljzero(0,2); besseljzero(0,3)
2.404825557695772768621632
5.520078110286310649596604
8.653727912911012216954199
>>> besseljzero(1,1); besseljzero(1,2); besseljzero(1,3)
3.831705970207512315614436
7.01558666981561875353705
10.17346813506272207718571
>>> besseljzero(2,1); besseljzero(2,2); besseljzero(2,3)
5.135622301840682556301402
8.417244140399864857783614
11.61984117214905942709415
```

---

Initial zeros of  $J'_0(z), J'_1(z), J'_2(z)$ :

---

```
>>> besseljzero(0,1,1); besseljzero(0,2,1); besseljzero(0,3,1)
0.0
3.831705970207512315614436
7.01558666981561875353705
>>> besseljzero(1,1,1); besseljzero(1,2,1); besseljzero(1,3,1)
1.84118378134065930264363
5.331442773525032636884016
8.536316366346285834358961
>>> besseljzero(2,1,1); besseljzero(2,2,1); besseljzero(2,3,1)
```

---

3.054236928227140322755932  
 6.706133194158459146634394  
 9.969467823087595793179143

---

Zeros with large index:

---

```
>>> besseljzero(0,100); besseljzero(0,1000); besseljzero(0,10000)
313.3742660775278447196902
3140.807295225078628895545
31415.14114171350798533666
>>> besseljzero(5,100); besseljzero(5,1000); besseljzero(5,10000)
321.1893195676003157339222
3148.657306813047523500494
31422.9947255486291798943
>>> besseljzero(0,100,1); besseljzero(0,1000,1); besseljzero(0,10000,1)
311.8018681873704508125112
3139.236339643802482833973
31413.57032947022399485808
```

---

## 11.2.2 Zeros of the Bessel function of the second kind

---

Function **besselyzero(v As mpNum, m As mpNum, Keywords As String) As mpNum**

---

The function besselyzero returns the m-th positive zero of the Bessel function of the second kind

**Parameters:**

*v*: A real or complex number.

*m*: A real or complex number.

*Keywords*: derivative=0.

For a real order  $\nu > 0$  and a positive integer  $m$ , returns  $y_{\nu,m}$ , the m-th positive zero of the Bessel function of the second kind  $Y_\nu(z)$ (see besselj()). Alternatively, with derivative=1, gives the first nonnegative simple zero  $y'_{\nu,m}$  of  $Y'_\nu(z)$ .

The zeros are interlaced according to the inequalities

$$y'_{\nu,k} < y_{\nu,k} < y'_{\nu,k+1} \quad (11.2.3)$$

$$y_{\nu,1} < y_{\nu+1,2} < y_{\nu,2} < y'_{\nu,k+1} < y_{\nu+1,2} < y_{\nu,3} \quad (11.2.4)$$

Initial zeros of the Bessel functions  $Y_0(z), Y_1(z), Y_2(z)$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> besselyzero(0,1); besselyzero(0,2); besselyzero(0,3)
0.8935769662791675215848871
3.957678419314857868375677
7.086051060301772697623625
>>> besselyzero(1,1); besselyzero(1,2); besselyzero(1,3)
2.197141326031017035149034
5.429681040794135132772005
8.596005868331168926429606
>>> besselyzero(2,1); besselyzero(2,2); besselyzero(2,3)
3.384241767149593472701426
```

```
6.793807513268267538291167  
10.02347797936003797850539
```

---

Initial zeros of  $Y'_0(z)$ ,  $Y'_1(z)$ ,  $Y'_2(z)$ ::

```
>>> besselyzero(0,1,1); besselyzero(0,2,1); besselyzero(0,3,1)  
2.197141326031017035149034  
5.429681040794135132772005  
8.596005868331168926429606  
>>> besselyzero(1,1,1); besselyzero(1,2,1); besselyzero(1,3,1)  
3.683022856585177699898967  
6.941499953654175655751944  
10.12340465543661307978775  
>>> besselyzero(2,1,1); besselyzero(2,2,1); besselyzero(2,3,1)  
5.002582931446063945200176  
8.350724701413079526349714  
11.57419546521764654624265
```

---

Zeros with large index:

```
>>> besselyzero(0,100); besselyzero(0,1000); besselyzero(0,10000)  
311.8034717601871549333419  
3139.236498918198006794026  
31413.57034538691205229188  
>>> besselyzero(5,100); besselyzero(5,1000); besselyzero(5,10000)  
319.6183338562782156235062  
3147.086508524556404473186  
31421.42392920214673402828  
>>> besselyzero(0,100,1); besselyzero(0,1000,1); besselyzero(0,10000,1)  
313.3726705426359345050449  
3140.807136030340213610065  
31415.14112579761578220175
```

---

## 11.3 Hankel functions

### 11.3.1 Hankel function of the first kind

---

Function **hankel1(*n* As mpNum, *x* As mpNum) As mpNum**

---

The function hankel1 returns the Hankel function of the first kind

**Parameters:**

*n*: A real or complex number.

*x*: A real or complex number.

The Hankel function of the first kind is the complex combination of Bessel functions given by

$$H_n^{(1)}(x) = J_n(x) + iY_n(x). \quad (11.3.1)$$

The Hankel function is generally complex-valued:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> hankel1(2, pi)
(0.4854339326315091097054957 - 0.0999007139290278787734903j)
>>> hankel1(3.5, pi)
(0.2340002029630507922628888 - 0.6419643823412927142424049j)
```

---

### 11.3.2 Hankel function of the second kind

---

Function **hankel2(*n* As mpNum, *x* As mpNum) As mpNum**

---

The function hankel2 returns the Hankel function of the second kind

**Parameters:**

*n*: A real or complex number.

*x*: A real or complex number.

The Hankel function of the second kind is the complex combination of Bessel functions given by

$$H_n^{(2)}(x) = J_n(x) - iY_n(x). \quad (11.3.2)$$

The Hankel function is generally complex-valued:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> hankel2(2, pi)
(0.4854339326315091097054957 + 0.0999007139290278787734903j)
>>> hankel2(3.5, pi)
(0.2340002029630507922628888 + 0.6419643823412927142424049j)
```

---

## 11.4 Kelvin functions

### 11.4.1 Kelvin function ber

---

Function **ber(*n* As mpNum, *z* As mpNum) As mpNum**

---

The function **ber** returns the Kelvin function ber

**Parameters:**

*n*: A real or complex number.

*z*: A real or complex number.

The Kelvin function ber returns for real arguments the real part of the Bessel J function of a rotated argument

$$J_n(xe^{3\pi i/4}) = \text{ber}_n(x) + i\text{bei}_n(x). \quad (11.4.1)$$

The imaginary part is given by **bei()**.

Verifying the defining relation:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> n, x = 2, 3.5
>>> ber(n,x)
1.442338852571888752631129
>>> bei(n,x)
-0.948359035324558320217678
>>> besselj(n, x*root(1,8,3))
(1.442338852571888752631129 - 0.948359035324558320217678j)
```

---

The ber and bei functions are also defined by analytic continuation for complex arguments:

---

```
>>> ber(1+j, 2+3j)
(4.675445984756614424069563 - 15.84901771719130765656316j)
>>> bei(1+j, 2+3j)
(15.83886679193707699364398 + 4.684053288183046528703611j)
```

---

### 11.4.2 Kelvin function bei

---

Function **bei(*n* As mpNum, *z* As mpNum) As mpNum**

---

The function **bei** returns the Kelvin function bei

**Parameters:**

*n*: A real or complex number.

*z*: A real or complex number.

The Kelvin function bei returns for real arguments the imaginary part of the Bessel J function of a rotated argument. See **ber()**.

### 11.4.3 Kelvin function ker

---

Function **ker(*n* As mpNum, *z* As mpNum) As mpNum**

---

The function `ker` returns the Kelvin function `ker`

**Parameters:**

`n`: A real or complex number.

`z`: A real or complex number.

The Kelvin function `ker` returns for real arguments the real part of the (rescaled) Bessel K function of a rotated argument

$$e^{-\pi i/2} K_n(xe^{3\pi i/4}) = \text{ker}_n(x) + i\text{kei}_n(x). \quad (11.4.2)$$

The imaginary part is given by `kei()`.

Verifying the defining relation:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> n, x = 2, 4.5
>>> ker(n,x)
0.02542895201906369640249801
>>> kei(n,x)
-0.02074960467222823237055351
>>> exp(-n*pi*j/2) * besselk(n, x*root(1,8,1))
(0.02542895201906369640249801 - 0.02074960467222823237055351j)
```

---

The `ker` and `kei` functions are also defined by analytic continuation for complex arguments:

---

```
>>> ker(1+j, 3+4j)
(1.586084268115490421090533 - 2.939717517906339193598719j)
>>> kei(1+j, 3+4j)
(-2.940403256319453402690132 - 1.585621643835618941044855j)
```

---

#### 11.4.4 Kelvin function `kei`

---

Function `kei(n As mpNum, z As mpNum) As mpNum`

---

The function `kei` returns the Kelvin function `kei`

**Parameters:**

`n`: A real or complex number.

`z`: A real or complex number.

The Kelvin function `kei` returns for real arguments the imaginary part of the (rescaled) Bessel K function of a rotated argument. See `ker()`.

## 11.5 Struve Functions

The Struve functions  $\mathbf{H}_\nu(x)$  and the modified Struve functions  $\mathbf{L}_\nu(x)$  have the power series expansions (see Abramowitz & Stegun. (1970) [1, 12.1.3 and 12.2.1]):

$$\mathbf{H}_\nu(x) = \left(\frac{1}{2}x\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}x\right)^{2k}}{\Gamma\left(k + \frac{3}{2}\right) \Gamma\left(k + \nu + \frac{3}{2}\right)} \quad (11.5.1)$$

$$\mathbf{L}_\nu(x) = \left(\frac{1}{2}x\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^{2k}}{\Gamma\left(k + \frac{3}{2}\right) \Gamma\left(k + \nu + \frac{3}{2}\right)} \quad (11.5.2)$$

### 11.5.1 Struve function H

---

Function **struveh(*n* As mpNum, *z* As mpNum)** As mpNum

---

The function **struveh** returns the Struve function H

**Parameters:**

*n*: A real or complex number.

*z*: A real or complex number.

The Struve function H is defined as

$$\mathbf{H}_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma\left(k + \frac{3}{2}\right) \Gamma\left(k + n + \frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2k+n+1} \quad (11.5.3)$$

which is a solution to the Struve differential equation

$$z^2 f''(z) + z f'(z) + (z^2 - n^2) f(z) = \frac{2z^{n+1}}{\pi(2n-1)!!} \quad (11.5.4)$$

Examples Evaluation for arbitrary real and complex arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> struveh(0, 3.5)
0.3608207733778295024977797
>>> struveh(-1, 10)
-0.255212719726956768034732
>>> struveh(1, -100.5)
0.5819566816797362287502246
>>> struveh(2.5, 1000000000000000)
3153915652525200060.308937
>>> struveh(2.5, -1000000000000000)
(0.0 - 3153915652525200060.308937j)
>>> struveh(1+j, 1000000+4000000j)
(-3.066421087689197632388731e+1737173 - 1.596619701076529803290973e+1737173j)
```

---

### 11.5.2 Modified Struve function L

---

Function **struvel(*n* As mpNum, *z* As mpNum) As mpNum**

---

The function **struvel** returns the modified Struve function L

**Parameters:**

*n*: A real or complex number.

*z*: A real or complex number.

The modified Struve function  $L_n(z)$  is defined as

$$L_n(z) = -ie^{-n\pi i/2} H_n(iz) \quad (11.5.5)$$

which solves to the modified Struve differential equation

$$z^2 f''(z) + z f'(z) + (z^2 + n^2) f(z) = \frac{2z^{n+1}}{\pi(2n-1)!!} \quad (11.5.6)$$

Examples

Evaluation for arbitrary real and complex arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> struvel(0, 3.5)
7.180846515103737996249972
>>> struvel(-1, 10)
2670.994904980850550721511
>>> struvel(1, -100.5)
1.757089288053346261497686e+42
>>> struvel(2.5, 100000000000000)
4.160893281017115450519948e+4342944819025
>>> struvel(2.5, -100000000000000)
(0.0 - 4.160893281017115450519948e+4342944819025j)
>>> struvel(1+j, 700j)
(-0.1721150049480079451246076 + 0.1240770953126831093464055j)
>>> struvel(1+j, 1000000+4000000j)
(-2.973341637511505389128708e+434290 - 5.164633059729968297147448e+434290j)
```

---

## 11.6 Anger-Weber functions

### 11.6.1 Anger function J

---

Function **angerj(*v* As mpNum, *z* As mpNum)** As mpNum

---

The function `angerj` returns the Anger function J

**Parameters:**

- v*: A real or complex number.
- z*: A real or complex number.

The Anger function  $\mathbf{J}_\nu(z)$  is defined as

$$\mathbf{J}_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu t - z \sin(t)) dt \quad (11.6.1)$$

which is an entire function of both the parameter  $\nu$  and the argument  $z$ . It solves the inhomogeneous Bessel differential equation

$$f''(z) + \frac{1}{z} f'(z) + \left(1 - \frac{\nu^2}{z^2}\right) f(z) = \frac{(z - \nu)}{\pi z^2} \sin(\pi\nu). \quad (11.6.2)$$

Examples

Evaluation for real and complex parameter and argument:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> angerj(2,3)
0.4860912605858910769078311
>>> angerj(-3+4j, 2+5j)
(-5033.358320403384472395612 + 585.8011892476145118551756j)
>>> angerj(3.25, 1e6j)
(4.630743639715893346570743e+434290 - 1.117960409887505906848456e+434291j)
>>> angerj(-1.5, 1e6)
0.0002795719747073879393087011
```

---

### 11.6.2 Weber function E

---

Function **webere(*v* As mpNum, *z* As mpNum)** As mpNum

---

The function `webere` returns the Weber function E

**Parameters:**

- v*: A real or complex number.
- z*: A real or complex number.

The Weber function  $\mathbf{E}_\nu(z)$  is defined as

$$\mathbf{E}_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(\nu t - z \sin(t)) dt \quad (11.6.3)$$

which is an entire function of both the parameter  $\nu$  and the argument  $z$ . It solves the inhomogeneous Bessel differential equation

$$f''(z) + \frac{1}{z} f'(z) + \left(1 - \frac{\nu^2}{z^2}\right) f(z) = \frac{1}{\pi z^2} (z + \nu + (z - \nu) \cos(\pi\nu)). \quad (11.6.4)$$

Examples

Evaluation for real and complex parameter and argument:

---

```
>>> from mpmath import *
>>> mp.dps = 25; mp.pretty = True
>>> webere(2,3)
-0.1057668973099018425662646
>>> webere(-3+4j, 2+5j)
(-585.8081418209852019290498 - 5033.314488899926921597203j)
>>> webere(3.25, 1e6j)
(-1.117960409887505906848456e+434291 - 4.630743639715893346570743e+434290j)
>>> webere(3.25, 1e6)
-0.00002812518265894315604914453
```

---

## 11.7 Lommel functions

### 11.7.1 First Lommel function s

---

Function **lommels1(*u* As mpNum, *v* As mpNum, *z* As mpNum) As mpNum**

---

The function lommels1 returns the First Lommel functions s

**Parameters:**

*u*: A real or complex number.

*v*: A real or complex number.

*z*: A real or complex number.

The Lommel function  $s_{\mu,\nu}$  or  $s_{\mu,\nu}^{(1)}$  is defined as

$$s_{\mu,\nu} = \frac{z^{\mu+1}}{(\mu - \nu + 1)(\mu + \nu + 1)} {}_1F_2 \left( 1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; -\frac{z^2}{4} \right) \quad (11.7.1)$$

which solves the inhomogeneous Bessel equation

$$z^2 f''(z) + z f'(z) + (z^2 - \nu^2) f(z) = z^{\mu+1}. \quad (11.7.2)$$

A second solution is given by lommels2().

An integral representation:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> u,v,z = 0.25, 0.125, mpf(0.75)
>>> lommels1(u,v,z)
0.4276243877565150372999126
>>> (bessely(v,z)*quad(lambda t: t**u*besselj(v,t), [0,z]) - \
... besselj(v,z)*quad(lambda t: t**u*bessely(v,t), [0,z]))*(pi/2
0.4276243877565150372999126
```

---

### 11.7.2 Second Lommel function S

---

Function **lommels2(*u* As mpNum, *v* As mpNum, *z* As mpNum) As mpNum**

---

The function lommels2 returns the Second Lommel functions S

**Parameters:**

*u*: A real or complex number.

*v*: A real or complex number.

*z*: A real or complex number.

The second Lommel function or  $S_{\mu,\nu}$  or  $s_{\mu,\nu}^{(2)}$  is defined as

$$\begin{aligned} S_{\mu,\nu}(z) &= s_{\mu,\nu}(z) + 2^{\mu-1} \Gamma\left(\frac{1}{2}(\mu - \nu + 1)\right) \Gamma\left(\frac{1}{2}(\mu + \nu + 1)\right) \\ &\times \left[ \sin\left(\frac{1}{2}(\mu - \nu)\pi\right) J_\nu(z) - \cos\left(\frac{1}{2}(\mu - \nu)\pi\right) Y_\nu(z) \right] \end{aligned} \quad (11.7.3)$$

which solves the same differential equation as lommels1().

Verifying the differential equation:

```
>>> f = lambda z: lommels2(u,v,z)
>>> z**2*diff(f,z,2) + z*diff(f,z) + (z**2-v**2)*f(z)
0.6495190528383289850727924
>>> z**(u+1)
0.6495190528383289850727924
```

---

## 11.8 Airy and Scorer functions

### 11.8.1 Airy function Ai

---

Function **airyai(z As mpNum, Keywords As String)** As mpNum

---

The function **airyai** returns the Airy function Ai

**Parameters:**

*z*: A real or complex number.

*Keywords*: derivative=0.

The Airy function  $\text{Ai}(z)$  is the solution of the Airy differential equation  $f''(z) - zf(z) = 0$  with initial conditions

$$\text{Ai}(0) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}; \quad \text{Ai}'(0) = \frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)} \quad (11.8.1)$$

Other common ways of defining the Ai-function include integrals such as

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt, \quad x \in \mathbb{R} \quad (11.8.2)$$

$$\text{Ai}(z) = \frac{\sqrt{3}}{2\pi} \int_0^\infty \exp\left(-\frac{t^3}{3} - \frac{z^3}{3t^3}\right) dt. \quad (11.8.3)$$

The Airy function  $\text{Ai}(x)$  can also be defined as

$$\text{Ai}(x) = \frac{1}{\pi} \sqrt{\frac{x}{3}} K_{1/3}(z), \quad (x > 0) \quad (11.8.4)$$

$$\text{Ai}(x) = \frac{1}{3^{2/3}\Gamma(2/3)}, \quad (x = 0) \quad (11.8.5)$$

$$\text{Ai}(x) = \frac{1}{2}\sqrt{-x} \left( J_{1/3}(z) - \frac{1}{\sqrt{3}} Y_{1/3}(z) \right), \quad (x < 0) \quad (11.8.6)$$

The Ai-function is an entire function with a turning point, behaving roughly like a slowly decaying sine wave for  $z < 0$  and like a rapidly decreasing exponential for  $z > 0$ . A second solution of the Airy differential equation is given by Bi(z)(see **airybi()**).

Optionally, with derivative=alpha, **airyai()** can compute the  $\alpha$ -th order fractional derivative with respect to  $z$ .

For  $\alpha = n = 1, 2, 3, \dots$  this gives the derivative  $\text{Ai}^n(z)$ , and for  $\alpha = -n = -1, -2, -3, \dots$  this gives the  $n$ -fold iterated integral

$$f_0(z) = \text{Ai}(z); \quad f_n(z) = \int_0^z f_{n-1}(t) dt. \quad (11.8.7)$$

The Ai-function has infinitely many zeros, all located along the negative half of the real axis. They can be computed with **airyaizero()**.

Limits and values include:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> airyai(0); 1/(power(3,'2/3')*gamma('2/3'))
0.3550280538878172392600632
0.3550280538878172392600632
>>> airyai(1)
0.1352924163128814155241474
>>> airyai(-1)
0.5355608832923521187995166
>>> airyai(inf); airyai(-inf)
0.0
0.0
```

---

Evaluation is supported for large magnitudes of the argument:

```
>>> airyai(-100)
0.1767533932395528780908311
>>> airyai(100)
2.634482152088184489550553e-291
>>> airyai(50+50j)
(-5.31790195707456404099817e-68 - 1.163588003770709748720107e-67j)
>>> airyai(-50+50j)
(1.041242537363167632587245e+158 + 3.347525544923600321838281e+157j)
>>> airyai(10**10)
1.162235978298741779953693e-289529654602171
>>> airyai(-10**10)
0.0001736206448152818510510181
>>> w = airyai(10**10*(1+j))
>>> w.real
5.711508683721355528322567e-186339621747698
>>> w.imag
1.867245506962312577848166e-186339621747697
```

---

### 11.8.2 Airy function Bi

---

**Function `airybi(z As mpNum, Keywords As String) As mpNum`**

---

The function `airybi` returns the Airy function Bi

**Parameters:**

`z`: A real or complex number.

`Keywords`: derivative=0.

The Airy function  $\text{Bi}(z)$  is the solution of the Airy differential equation  $f''(z) - zf(z) = 0$  with initial conditions

$$\text{Bi}(0) = \frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)}; \quad \text{Bi}'(0) = \frac{1}{3^{1/6}\Gamma\left(\frac{1}{3}\right)} \quad (11.8.8)$$

The Airy function  $\text{Bi}(x)$ , can also be defined as

$$\text{Bi}(x) = \sqrt{x} \left( \frac{2}{\sqrt{3}} I_{1/3}(z) + \frac{1}{\pi} K_{1/3}(z) \right), \quad (x > 0) \quad (11.8.9)$$

$$\text{Bi}(x) = \frac{1}{3^{1/6} \Gamma(2/3)}, \quad (x = 0) \quad (11.8.10)$$

$$\text{Bi}(x) = -\frac{1}{2} \sqrt{-x} \left( \frac{1}{\sqrt{3}} J_{1/3}(z) + Y_{1/3}(z) \right), \quad (x < 0) \quad (11.8.11)$$

Like the Ai-function (see `airyai()`), the Bi-function is oscillatory for  $z < 0$ , but it grows rather than decreases for  $z > 0$ .

Optionally, as for `airyai()`, derivatives, integrals and fractional derivatives can be computed with the derivative parameter.

The Bi-function has infinitely many zeros along the negative half-axis, as well as complex zeros, which can all be computed with `airybizer()`.

Limits and values include:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> airybi(0); 1/(power(3,'1/6')*gamma('2/3'))
0.6149266274460007351509224
0.6149266274460007351509224
>>> airybi(1)
1.207423594952871259436379
>>> airybi(-1)
0.10399738949694461188869
>>> airybi(inf); airybi(-inf)
+inf
0.0
```

---

Evaluation is supported for large magnitudes of the argument:

---

```
>>> airybi(-100)
0.02427388768016013160566747
>>> airybi(100)
6.041223996670201399005265e+288
>>> airybi(50+50j)
(-5.322076267321435669290334e+63 + 1.478450291165243789749427e+65j)
>>> airybi(-50+50j)
(-3.347525544923600321838281e+157 + 1.041242537363167632587245e+158j)
>>> airybi(10**10)
1.369385787943539818688433e+289529654602165
>>> airybi(-10**10)
0.001775656141692932747610973
>>> w = airybi(10**10*(1+j))
>>> w.real
-6.559955931096196875845858e+186339621747689
>>> w.imag
-6.822462726981357180929024e+186339621747690
```

---

### 11.8.3 Zeros of the Airy function Ai

---

Function **airyaizero(k As mpNum, Keywords As String)** As mpNum

---

The function airyaizero returns the  $k$ -th zero of the Airy Ai-function

**Parameters:**

*k*: An integer.

**Keywords:** derivative=0.

Gives the  $k$ -th zero of the Airy Ai-function, i.e. the  $k$ -th number ordered by magnitude for which  $\text{Ai}(a_k) = 0$ .

Optionally, with derivative=1, the corresponding zero  $a'_k$  of the derivative function, i.e.  $\text{Ai}'(a'_k) = 0$ , is computed.

Examples

Some values of :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> airyaizero(1)
-2.338107410459767038489197
>>> airyaizero(2)
-4.087949444130970616636989
>>> airyaizero(3)
-5.520559828095551059129856
>>> airyaizero(1000)
-281.0315196125215528353364
```

---

### 11.8.4 Zeros of the Airy function Bi

---

Function **airybizero(k As mpNum, Keywords As String)** As mpNum

---

The function airybizero returns the  $k$ -th zero of the Airy Bi-function

**Parameters:**

*k*: An integer.

**Keywords:** derivative=0, complex=0.

With complex=False, gives the  $k$ -th real zero of the Airy Bi-function, i.e. the  $k$ -th number ordered by magnitude for which  $\text{Bi}(b_k) = 0$ .

With complex=True, gives the  $k$ -th complex zero in the upper half plane  $\beta_k$ . Also the conjugate  $\bar{\beta}_k$  is a zero.

Optionally, with derivative=1, the corresponding zero  $b'_k$  or  $\beta'_k$  of the derivative function, i.e.  $\text{Bi}'(b'_k) = 0$  or  $\text{Bi}'(\beta'_k) = 0$ , is computed.

Examples

Some values of :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> airybizero(1)
-1.17371322270912792491998
>>> airybizero(2)
```

---

---

```
-3.271093302836352715680228
>>> airybizer0(3)
-4.830737841662015932667709
>>> airybizer0(1000)
-280.9378112034152401578834
```

---

### 11.8.5 Scorer function Gi

---

Function **scorergi(z As mpNum) As mpNum**

---

The function **scorergi** returns the Scorer function Gi

**Parameter:**

*z*: A real or complex number.

Evaluates the Scorer function

$$Gi(z) = Ai(z) \int_0^z Bi(t)dt + Bi(z) \int_z^\infty Ai(t)dt \quad (11.8.12)$$

which gives a particular solution to the inhomogeneous Airy differential equation  $f''(z) - zf(z) = 1 - \pi$ . Another particular solution is given by the Scorer Hi-function (**scorerhi()**). The two functions are related as  $Gi(z) + Hi(z) = Bi(z)$ .

Examples

Some values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> scorergi(0); 1/(power(3, '7/6')*gamma('2/3'))
0.2049755424820002450503075
0.2049755424820002450503075
>>> diff(scorergi, 0); 1/(power(3, '5/6')*gamma('1/3'))
0.1494294524512754526382746
0.1494294524512754526382746
>>> scorergi(+inf); scorergi(-inf)
0.0
0.0
>>> scorergi(1)
0.2352184398104379375986902
>>> scorergi(-1)
-0.1166722172960152826494198
```

---

### 11.8.6 Scorer function Hi

---

Function **scorerhi(z As mpNum) As mpNum**

---

The function **scorerhi** returns the Scorer function Gi

**Parameter:**

*z*: A real or complex number.

Evaluates the second Scorer function

$$\text{Hi}(z) = \text{Bi}(z) \int_{-\infty}^z \text{Ai}(t) dt - \text{Ai}(z) \int_{-\infty}^z \text{Bi}(t) dt \quad (11.8.13)$$

which gives a particular solution to the inhomogeneous Airy differential equation  $f''(z) - zf(z) = 1 - \pi$ . See also `scorerhi()`.

Examples

Some values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> scorerhi(0); 2/(power(3,'7/6')*gamma('2/3'))
0.4099510849640004901006149
0.4099510849640004901006149
>>> diff(scorerhi,0); 2/(power(3,'5/6')*gamma('1/3'))
0.2988589049025509052765491
0.2988589049025509052765491
>>> scorerhi(+inf); scorerhi(-inf)
+inf
0.0
>>> scorerhi(1)
0.9722051551424333218376886
>>> scorerhi(-1)
0.2206696067929598945381098
```

---

## 11.9 Coulomb wave functions

### 11.9.1 Regular Coulomb wave function

---

Function **coulombf(*l* As mpNum, *eta* As mpNum, *z* As mpNum)** As mpNum

---

The function `coulombf` returns the regular Coulomb wave function

**Parameters:**

*l*: A real or complex number.

*eta*: A real or complex number.

*z*: A real or complex number.

Calculates the regular Coulomb wave function

$$F_l(\eta, z) = C_l(\eta) z^{l+1} e^{iz} {}_1F_1(l + 1 - i\eta, 2l + 2, 2iz) \quad (11.9.1)$$

where the normalization constant  $C_l(\eta)$  is as calculated by `coulombc()`. This function solves the differential equation

$$f''(z) + \left(1 - \frac{2\eta}{z} - \frac{l(l+1)}{z^2}\right) f(z) = 0. \quad (11.9.2)$$

A second linearly independent solution is given by the irregular Coulomb wave function  $G_l(\eta, z)$  (see `coulombg()`) and thus the general solution is

$$f(z) = C_1 F_l(\eta, z) + C_2 G_l(\eta, z) \quad (11.9.3)$$

for arbitrary constants  $C_1, C_2$ . Physically, the Coulomb wave functions give the radial solution to the Schrodinger equation for a point particle in a  $1/z$  potential;  $z$  is then the radius and  $l, \eta$  are quantum numbers.

The Coulomb wave functions with real parameters are defined in Abramowitz & Stegun, section 14. However, all parameters are permitted to be complex in this implementation (see references). Evaluation is supported for arbitrary magnitudes of :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> coulombf(2, 1.5, 3.5)
0.4080998961088761187426445
>>> coulombf(-2, 1.5, 3.5)
0.7103040849492536747533465
>>> coulombf(2, 1.5, '1e-10')
4.143324917492256448770769e-33
>>> coulombf(2, 1.5, 1000)
0.4482623140325567050716179
>>> coulombf(2, 1.5, 10**10)
-0.066804196437694360046619
```

---

Some test case with complex parameters, taken from Michel [2]:

---

```
>>> mp.dps = 15
>>> coulombf(1+0.1j, 50+50j, 100.156)
(-1.02107292320897e+15 - 2.83675545731519e+15j)
```

---

```
>>> coulombg(1+0.1j, 50+50j, 100.156)
(2.83675545731519e+15 - 1.02107292320897e+15j)
>>> coulombf(1e-5j, 10+1e-5j, 0.1+1e-6j)
(4.30566371247811e-14 - 9.03347835361657e-19j)
>>> coulombg(1e-5j, 10+1e-5j, 0.1+1e-6j)
(778709182061.134 + 18418936.2660553j)
```

---

## 11.9.2 Irregular Coulomb wave function

---

Function **coulombg(*l* As mpNum, *eta* As mpNum, *z* As mpNum) As mpNum**

---

The function **coulombg** returns the irregular Coulomb wave function

**Parameters:**

*l*: A real or complex number.

*eta*: A real or complex number.

*z*: A real or complex number.

Calculates the irregular Coulomb wave function

$$G_l(\eta, z) = \frac{F_l(\eta, z) \cos(\chi) - F_{-l-1}(\eta, z)}{\sin(\chi)} \quad (11.9.4)$$

where

$$\chi = \sigma_l - \sigma_{l-1} - (l + 1/2)\pi \quad (11.9.5)$$

and

$$\sigma_l(\eta) = (\ln \Gamma(1 + l + i\eta) - \ln \Gamma(1 + l - i\eta))/(2i) \quad (11.9.6)$$

See **coulombf()** for additional information.

Evaluation is supported for arbitrary magnitudes of :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> coulombg(-2, 1.5, 3.5)
1.380011900612186346255524
>>> coulombg(2, 1.5, 3.5)
1.919153700722748795245926
>>> coulombg(-2, 1.5, '1e-10')
201126715824.7329115106793
>>> coulombg(-2, 1.5, 1000)
0.1802071520691149410425512
>>> coulombg(-2, 1.5, 10**10)
0.652103020061678070929794
```

---

## 11.9.3 Normalizing Gamow constant

---

Function **coulombc(*l* As mpNum, *eta* As mpNum) As mpNum**

---

The function **coulombc** returns the normalizing Gamow constant for Coulomb wave functions

**Parameters:**

*l*: A real or complex number.

*eta*: A real or complex number.

The normalizing Gamow constant for Coulomb wave functions is defined as

$$C_l(\eta) = 2^l \exp(-\pi\eta/2 + [\ln \Gamma(1 + l + i\eta) + \ln \Gamma(1 + l - i\eta)]/2 - \ln \Gamma(2l + 2)) \quad (11.9.7)$$

where the log gamma function with continuous imaginary part away from the negative half axis (see `loggamma()`) is implied.

This function is used internally for the calculation of Coulomb wave functions, and automatically cached to make multiple evaluations with fixed *l*, *η* fast.

## 11.10 Parabolic cylinder functions

### 11.10.1 Parabolic cylinder function D

---

Function **pcfD(*n* As mpNum, *z* As mpNum) As mpNum**

---

The function pcfD returns the parabolic cylinder function D

**Parameters:**

*n*: A real or complex number.

*z*: A real or complex number.

Gives the parabolic cylinder function in Whittaker's notation  $D_n(z) = U(-n-1/2, z)$  (see pcfu()). It solves the differential equation

$$y'' + \left(n + \frac{1}{2} - \frac{1}{4}z^2\right)y = 0. \quad (11.10.1)$$

and can be represented in terms of Hermite polynomials (see hermite()) as

$$D_n(z) = 2^{-n/2} e^{-z^2/4} H_n\left(\frac{z}{\sqrt{2}}\right) \quad (11.10.2)$$

Examples

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> pcfD(0,0); pcfD(1,0); pcfD(2,0); pcfD(3,0)
1.0
0.0
-1.0
0.0
>>> pcfD(4,0); pcfD(-3,0)
3.0
0.6266570686577501256039413
>>> pcfD('1/2', 2+3j)
(-5.363331161232920734849056 - 3.858877821790010714163487j)
>>> pcfD(2, -10)
1.374906442631438038871515e-9
```

---

### 11.10.2 Parabolic cylinder function U

---

Function **pcfU(*a* As mpNum, *z* As mpNum) As mpNum**

---

The function pcfU returns the parabolic cylinder function U

**Parameters:**

*a*: A real or complex number.

*z*: A real or complex number.

Gives the parabolic cylinder function  $U(a, z)$ , which may be defined for  $\Re(z) > 0$  in terms of the confluent U-function (see hyperu()) by

$$U(a, z) = 2^{-\frac{1}{4} - \frac{a}{2}} e^{-\frac{1}{4}z^2} U\left(\frac{a}{2} + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}z^2\right) \quad (11.10.3)$$

or, for arbitrary  $z$ ,

$$e^{-\frac{1}{4}z^2} U(a, z) = U(a, 0) {}_1F_1\left(-\frac{a}{2} + \frac{1}{4}, \frac{1}{2}, -\frac{1}{2}z^2\right) + U'(a, 0) {}_1F_1\left(-\frac{a}{2} + \frac{3}{4}, \frac{3}{2}, -\frac{1}{2}z^2\right) \quad (11.10.4)$$

Examples

Connection to other functions:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> z = mpf(3)
>>> pcfu(0.5,z)
0.03210358129311151450551963
>>> sqrt(pi/2)*exp(z**2/4)*erfc(z/sqrt(2))
0.03210358129311151450551963
>>> pcfu(0.5,-z)
23.75012332835297233711255
>>> sqrt(pi/2)*exp(z**2/4)*erfc(-z/sqrt(2))
23.75012332835297233711255
>>> pcfu(0.5,-z)
23.75012332835297233711255
>>> sqrt(pi/2)*exp(z**2/4)*erfc(-z/sqrt(2))
23.75012332835297233711255
```

---

### 11.10.3 Parabolic cylinder function V

---

Function **pcfV**(*a* As mpNum, *z* As mpNum) As mpNum

---

The function pcfv returns the parabolic cylinder function V

**Parameters:**

*a*: A real or complex number.

*z*: A real or complex number.

Gives the parabolic cylinder function  $V(a, z)$ , which can be represented in terms of pcfu() as

$$V(a, z) = \frac{\Gamma(a + \frac{1}{2})(U(a, -z) - \sin(\pi a)U(a, z))}{\pi} \quad (11.10.5)$$

Examples

Wronskian relation between  $U$  and  $V$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> a, z = 2, 3
>>> pcfu(a,z)*diff(pcfv,(a,z),(0,1))-diff(pcfu,(a,z),(0,1))*pcfV(a,z)
0.7978845608028653558798921
>>> sqrt(2/pi)
0.7978845608028653558798921
```

---

---

```
>>> a, z = 2.5, 3
>>> pcfu(a,z)*diff(pcfv,(a,z),(0,1))-diff(pcfu,(a,z),(0,1))*pcf(v,a,z)
0.7978845608028653558798921
>>> a, z = 0.25, -1
>>> pcfu(a,z)*diff(pcfv,(a,z),(0,1))-diff(pcfu,(a,z),(0,1))*pcf(v,a,z)
0.7978845608028653558798921
>>> a, z = 2+1j, 2+3j
>>> chop(pcfu(a,z)*diff(pcfv,(a,z),(0,1))-diff(pcfu,(a,z),(0,1))*pcf(v,a,z))
0.7978845608028653558798921
```

---

## 11.10.4 Parabolic cylinder function W

---

**Function `pcfW(a As mpNum, z As mpNum)` As mpNum**

---

The function `pcfW` returns Computes the parabolic cylinder function W

**Parameters:**

*a*: A real or complex number.

*z*: A real or complex number.

Gives the parabolic cylinder function  $W(a, z)$  defined in (DLMF 12.14).

Examples

Value at the origin:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> a = mpf(0.25)
>>> pcfw(a,0)
0.9722833245718180765617104
>>> power(2,-0.75)*sqrt(abs(gamma(0.25+0.5j*a)/gamma(0.75+0.5j*a)))
0.9722833245718180765617104
>>> diff(pcfw,(a,0),(0,1))
-0.5142533944210078966003624
>>> -power(2,-0.25)*sqrt(abs(gamma(0.75+0.5j*a)/gamma(0.25+0.5j*a)))
-0.5142533944210078966003624
```

---

# Chapter 12

## Orthogonal polynomials

An orthogonal polynomial sequence is a sequence of polynomials  $P_0(x), P_1(x), \dots$  of degree  $0, 1, \dots$ , which are mutually orthogonal in the sense that

$$\int_S P_n(x)P_m(x)w(x) = \begin{cases} c_n \neq 0 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \quad (12.0.1)$$

where  $S$  is some domain (e.g. an interval  $[a, b] \in \mathbb{R}$ ) and  $w(x)$  is a fixed weight function. A sequence of orthogonal polynomials is determined completely by  $w$ ,  $S$ , and a normalization convention (e.g.  $c_n = 1$ ). Applications of orthogonal polynomials include function approximation and solution of differential equations.

Orthogonal polynomials are sometimes defined using the differential equations they satisfy (as functions of  $x$ ) or the recurrence relations they satisfy with respect to the order  $n$ . Other ways of defining orthogonal polynomials include differentiation formulas and generating functions. The standard orthogonal polynomials can also be represented as hypergeometric series (see Hypergeometric functions), more specifically using the Gauss hypergeometric function  ${}_2F_1$  in most cases. The following functions are generally implemented using hypergeometric functions since this is computationally efficient and easily generalizes.

For more information, see the Wikipedia article on orthogonal polynomials.

### 12.1 Legendre functions

#### 12.1.1 Legendre polynomial

---

Function **legendre**(*n* As mpNum, *x* As mpNum) As mpNum

---

The function `legendre` returns the Legendre polynomial  $P_n(x)$

**Parameters:**

*n*: A real or complex number.

*x*: A real or complex number.

The Legendre polynomials are given by the formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (12.1.1)$$

Alternatively, they can be computed recursively using

$$P_0(x) = 1; \quad P_1(x) = x; \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad (12.1.2)$$

A third definition is in terms of the hypergeometric function  ${}_2F_1$ , whereby they can be generalized to arbitrary  $n$ :

$$P_n(x) = {}_2F_1\left(-n, n+1, 1, \frac{1-x}{2}\right) \quad (12.1.3)$$

The Legendre polynomials assume fixed values at the points  $x = -1$  and  $x = 1$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> nprint([legendre(n, 1) for n in range(6)])
[1.0, 1.0, 1.0, 1.0, 1.0, 1.0]
>>> nprint([legendre(n, -1) for n in range(6)])
[1.0, -1.0, 1.0, -1.0, 1.0, -1.0]
```

---

### 12.1.2 Associated Legendre function of the first kind

---

**Function `legenp(n As mpNum, m As mpNum, z As mpNum, Keywords As String) As mpNum`**

---

The function `legenp` returns the (associated) Legendre function of the first kind of degree  $n$  and order  $m$ ,  $P_n^m(z)$ .

**Parameters:**

$n$ : A real or complex number.

$m$ : A real or complex number.

$z$ : A real or complex number.

**Keywords:** type=2.

Calculates the (associated) Legendre function of the first kind of degree  $n$  and order  $m$ ,  $P_n^m(z)$ . Taking  $m = 0$  gives the ordinary Legendre function of the first kind,  $P_n(z)$ . The parameters may be complex numbers.

In terms of the Gauss hypergeometric function, the (associated) Legendre function is defined as

$$P_n^m = \frac{1}{\Gamma(1-m)} \frac{(1+z)^{m/2}}{(1-z)^{m/2}} {}_2F_1\left(-n, n+1, 1-m, \frac{1-z}{2}\right). \quad (12.1.4)$$

With type=3 instead of type=2, the alternative definition

$$\hat{P}_n^m = \frac{1}{\Gamma(1-m)} \frac{(1+z)^{m/2}}{(z-1)^{m/2}} {}_2F_1\left(-n, n+1, 1-m, \frac{1-z}{2}\right). \quad (12.1.5)$$

is used. These functions correspond respectively to `LegendreP[n,m,2,z]` and `LegendreP[n,m,3,z]` in Mathematica.

The general solution of the (associated) Legendre differential equation

$$(1-z^2)f''(z) - 2zf'(z) + \left(n(n+1) - \frac{m^2}{1-z^2}\right)f(z) = 0 \quad (12.1.6)$$

is given by  $C_1 P_n^m(z) + C_2 Q_n^m(z)$  for arbitrary constants  $C_1, C_2$ , where  $Q_n^m(z)$  is a Legendre function of the second kind as implemented by `legenq()`.

### Examples

Evaluation for arbitrary parameters and arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> legenp(2, 0, 10); legendre(2, 10)
149.5
149.5
>>> legenp(-2, 0.5, 2.5)
(1.972260393822275434196053 - 1.972260393822275434196053j)
>>> legenp(2+3j, 1-j, -0.5+4j)
(-3.335677248386698208736542 - 5.663270217461022307645625j)
>>> chop(legenp(3, 2, -1.5, type=2))
28.125
>>> chop(legenp(3, 2, -1.5, type=3))
-28.125
```

---

### 12.1.3 Associated Legendre function of the second kind

---

Function **legenq(*n* As mpNum, *m* As mpNum, *z* As mpNum, **Keywords** As String) As mpNum**

---

The function `legenq` returns the (associated) Legendre function of the second kind of degree *n* and order *m*,  $Q_n^m(z)$ .

**Parameters:**

*n*: A real or complex number.

*m*: A real or complex number.

*z*: A real or complex number.

**Keywords:** type=2.

Calculates the (associated) Legendre function of the second kind of degree *n* and order *m*,  $Q_n^m(z)$ . Taking *m* = 0 gives the ordinary Legendre function of the second kind,  $Q_n(z)$ . The parameters may complex numbers.

The Legendre functions of the second kind give a second set of solutions to the (associated) Legendre differential equation. (See `legenp()`.) Unlike the Legendre functions of the first kind, they are not polynomials of *z* for integer *n, m* but rational or logarithmic functions with poles at *z* = ±1.

There are various ways to define Legendre functions of the second kind, giving rise to different complex structure. A version can be selected using the `type` keyword argument. The `type=2` and `type=3` functions are given respectively by

$$Q_n^m(z) = \frac{\pi}{2 \sin(\pi m)} \left( \cos(\pi m) P_n^m(z) - \frac{\Gamma(1+m+n)}{\Gamma(1-m+n)} P_n^{-m}(z) \right) \quad (12.1.7)$$

$$\hat{Q}_n^m(z) = \frac{\pi}{2 \sin(\pi m)} e^{\pi i m} \left( \hat{P}_n^m(z) - \frac{\Gamma(1+m+n)}{\Gamma(1-m+n)} \hat{P}_n^{-m}(z) \right) \quad (12.1.8)$$

where  $P$  and  $\hat{P}$  are the `type=2` and `type=3` Legendre functions of the first kind. The formulas above should be understood as limits when *m* is an integer.

These functions correspond to LegendreQ[n,m,2,z] (or LegendreQ[n,m,z]) and LegendreQ[n,m,3,z] in Mathematica. The type=3 function is essentially the same as the function defined in Abramowitz & Stegun (eq. 8.1.3) but with  $(z+1)^{m/2}(z-1)^{m/2}$  instead of  $(z^2-1)^{m/2}$ , giving slightly different branches.

### Examples

Evaluation for arbitrary parameters and arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> legenq(2, 0, 0.5)
-0.8186632680417568557122028
>>> legenq(-1.5, -2, 2.5)
(0.6655964618250228714288277 + 0.3937692045497259717762649j)
>>> legenq(2-j, 3+4j, -6+5j)
(-10001.95256487468541686564 - 6011.691337610097577791134j)
```

---

## 12.1.4 Spherical harmonics

---

Function **spherharm(*l* As mpNum, *m* As mpNum, *theta* As mpNum, *phi* As mpNum) As mpNum**

---

The function spherharm returns the spherical harmonic  $Y_l^m(\theta, \phi)$

### Parameters:

*l*: A real or complex number.

*m*: A real or complex number.

*theta*: A real or complex number.

*phi*: A real or complex number.

Evaluates the spherical harmonic  $Y_l^m(\theta, \phi)$ ,

$$Y_l^m(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos(\theta)) e^{im\phi} \quad (12.1.9)$$

where  $P_l^m$  is an associated Legendre function (see legenp()).

Here  $\theta \in [0, \pi]$  denotes the polar coordinate (ranging from the north pole to the south pole) and  $\phi \in [0, 2\pi]$  denotes the azimuthal coordinate on a sphere. Care should be used since many different conventions for spherical coordinate variables are used.

Usually spherical harmonics are considered for  $l \in \mathbb{N}, m \in \mathbb{Z}, |m| \leq l$ . More generally,  $l, m, \theta, \phi$  are permitted to be complex numbers.

Some low-order spherical harmonics with reference values:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> theta = pi/4
>>> phi = pi/3
>>> spherharm(0,0,theta,phi); 0.5*sqrt(1/pi)*expj(0)
(0.2820947917738781434740397 + 0.0j)
(0.2820947917738781434740397 + 0.0j)
>>> spherharm(1,-1,theta,phi); 0.5*sqrt(3/(2*pi))*expj(-phi)*sin(theta)
(0.1221506279757299803965962 - 0.2115710938304086076055298j)
```

```
(0.1221506279757299803965962 - 0.2115710938304086076055298j)
>>> spherharm(1,0,theta,phi); 0.5*sqrt(3/pi)*cos(theta)*expj(0)
(0.3454941494713354792652446 + 0.0j)
(0.3454941494713354792652446 + 0.0j)
>>> spherharm(1,1,theta,phi); -0.5*sqrt(3/(2*pi))*expj(phi)*sin(theta)
(-0.1221506279757299803965962 - 0.2115710938304086076055298j)
(-0.1221506279757299803965962 - 0.2115710938304086076055298j)
```

---

## 12.2 Chebyshev polynomials

### 12.2.1 Chebyshev polynomial of the first kind

---

Function **chebyt(*n* As mpNum, *x* As mpNum)** As mpNum

---

The function chebyt returns the Chebyshev polynomial of the first kind  $T_n(x)$

**Parameters:**

*n*: A real or complex number.

*x*: A real or complex number.

The Chebyshev polynomial of the first kind  $T_n(x)$  are defined by the identity

$$T_n(\cos(x)) = \cos(nx). \quad (12.2.1)$$

The  $T_n(x)$  are orthogonal on the interval  $(-1, 1)$ , with respect to the weight function  $w(x) = (1 - x^2)^{-1/2}$ .

For  $0 \leq n \leq 64$  the function evaluates  $T_n(x)$  with the standard recurrence formulas [1, 22.7.4]:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x). \end{aligned} \quad (12.2.2)$$

If  $n > 64$  the following trigonometric and hyperbolic identities [1, 22.3.15]:

$$T_n(x) = \cos(n \arccos(x)), \quad |x| < 1 \quad (12.2.3)$$

$$T_n(x) = \cosh(n \operatorname{arccosh}(x)), \quad |x| > 1 \quad (12.2.4)$$

are used, and the special cases  $|x| = 1$  are handled separately. If  $n < 0$  the function result is  $T_n(x) = T_{-n}(x)$ .

The Chebyshev polynomials of the first kind are a special case of the Jacobi polynomials, and by extension of the hypergeometric function  ${}_2F_1$ . They can thus also be evaluated for nonintegral  $n$ . The Chebyshev polynomials of the first kind are orthogonal on the interval  $[-1, 1]$  with respect to the weight function  $w(x) = 1/\sqrt{1 - x^2}$ :

---

```
>>> f = lambda x: chebyt(m,x)*chebyt(n,x)/sqrt(1-x**2)
>>> m, n = 3, 4
>>> nprint(quad(f, [-1, 1]),1)
0.0
>>> m, n = 4, 4
>>> quad(f, [-1, 1])
1.57079632596448
```

---

### 12.2.2 Chebyshev polynomial of the second kind

---

Function **chebyu(*n* As mpNum, *x* As mpNum)** As mpNum

---

The function chebyu returns the Chebyshev polynomial of the second kind  $U_n(x)$

**Parameters:**

$n$ : A real or complex number.

$x$ : A real or complex number.

The Chebyshev polynomial of the second kind  $U_n(x)$  are defined by the identity

$$U_n(\cos(x)) = \frac{\sin((n+1)x)}{\sin(x)} \quad (12.2.5)$$

The  $U_n(x)$  are orthogonal on the interval  $(-1, 1)$ , with respect to the weight function  $w(x) = (1 - x^2)^{1/2}$ .

For  $0 \leq n \leq 64$  the function evaluates  $U_n(x)$  with the standard recurrence formulas [1, 22.7.4]:

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x). \end{aligned} \quad (12.2.6)$$

If  $n > 64$  the following trigonometric and hyperbolic identities [1, 22.3.15]:

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sin(\arccos(x))}, \quad |x| < 1 \quad (12.2.7)$$

$$U_n(x) = \frac{\sin((n+1)\operatorname{arccosh}(x))}{\sin(\operatorname{arccosh}(x))}, \quad |x| > 1 \quad (12.2.8)$$

are used, and the special cases  $|x| = 1$  are handled separately. If  $n < 0$  the function result are  $U_{-1}(x) = 0$  and  $U_n(x) = -U_{-n-2}(x)$ .

The Chebyshev polynomials of the second kind are a special case of the Jacobi polynomials, and by extension of the hypergeometric function  ${}_2F_1$ . They can thus also be evaluated for nonintegral  $n$ .

The Chebyshev polynomials of the first kind are orthogonal on the interval  $[-1, 1]$  with respect to the weight function  $w(x) = \sqrt{1 - x^2}$ :

---

```
>>> f = lambda x: chebyu(m,x)*chebyu(n,x)*sqrt(1-x**2)
>>> m, n = 3, 4
>>> quad(f, [-1, 1])
0.0
>>> m, n = 4, 4
>>> quad(f, [-1, 1])
1.5707963267949
```

---

## 12.3 Jacobi and Gegenbauer polynomials

### 12.3.1 Jacobi polynomial

Function **jacobi(*n* As mpNum, *a* As mpNum, *b* As mpNum, *z* As mpNum)** As mpNum

The function **jacobi** returns the Jacobi polynomial  $P_n^{(a,b)}$

**Parameters:**

*n*: A real or complex number.

*a*: A real or complex number.

*b*: A real or complex number.

*z*: A real or complex number.

These functions return  $P_n^{(a,b)}(x)$ , the Jacobi polynomial of degree  $n \geq 0$  with parameters  $(a, b)$ .  $a, b$  should be greater than  $-1$ , and  $a + b$  must not be an integer less than  $-1$ . Jacobi polynomials are orthogonal on the interval  $(-1, 1)$ , with respect to the weight function  $w(x) = (1-x)^a(1+x)^b$ , if  $a, b$  are greater than  $-1$ . The cases  $n \leq 1$  are computed with the explicit formulas

$$P_0^{(a,b)} = 1, \quad 2P_1^{(a,b)} = (a - b) + (a + b + 2)x, \quad (12.3.1)$$

and for  $n > 1$  there are the somewhat complicated recurrence relations from [30] (18.9.1) and (18.9.2):

$$\begin{aligned} P_{n+1}^{(a,b)} &= (A_n x + B_n) P_n^{(a,b)} - C_n P_{n+1}^{(a,b)} \\ A_n &= \frac{(2n+a+b+1)(2n+a+b+2)}{2(n+1)(n+a+b+1)} \\ B_n &= \frac{(a^2 - b^2)(2n+a+b+1)}{2(n+1)(n+a+b+1)(2n+a+b)} \\ C_n &= \frac{(n+a)(n+b)(2n+a+b+2)}{(n+1)(n+a+b+1)(2n+a+b)}. \end{aligned} \quad (12.3.2)$$

**jacobi(*n*, *a*, *b*, *x*)** evaluates the Jacobi polynomial  $P_n^{(a,b)}$ . The Jacobi polynomials are a special case of the hypergeometric function  ${}_2F_1$  given by:

$$P_n^{(a,b)} = \binom{n+a}{n} {}_2F_1\left(-n, 1+a+b+n, a+1, \frac{1-x}{2}\right). \quad (12.3.3)$$

Note that this definition generalizes to nonintegral values of  $n$ . When  $n$  is an integer, the hypergeometric series terminates after a finite number of terms, giving a polynomial in  $x$ .

Evaluation of Jacobi polynomials

A special evaluation is  $P_n^{(a,b)} = \binom{n+a}{n}$ :

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> jacobi(4, 0.5, 0.25, 1)
2.4609375
>>> binomial(4+0.5, 4)
2.4609375
```

### 12.3.2 Zernike Radial Polynomials

---

Function **ZernikeRadialMpMath(*n* As mpNum, *m* As mpNum, *x* As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function **ZernikeRadialMpMath** returns the Zernike radial polynomials  $R_n^m(r)$ .

**Parameters:**

*n*: An Integer.

*m*: An Integer.

*x*: A real number.

This function returns the Zernike radial polynomials  $R_n^m(r)$  with  $r \geq 0$  and  $n \geq m \geq 0$ ,  $n - m$  even, and zero otherwise. The  $R_n^m(r)$  are special cases of the Jacobi Polynomials

$$R_n^m(r) = (-1)^{(n-m)/2} r^m P_{(n-m)/2}^{(m,0)}(1 - 2r^2). \quad (12.3.4)$$

### 12.3.3 Gegenbauer polynomial

---

Function **gegenbauer(*n* As mpNum, *a* As mpNum, *z* As mpNum) As mpNum**

---

The function **gegenbauer** returns the Gegenbauer polynomial  $C_n^{(a)}(z)$

**Parameters:**

*n*: A real or complex number.

*a*: A real or complex number.

*z*: A real or complex number.

Evaluates the Gegenbauer polynomial, or ultraspherical polynomial,

$$C_n^{(a)}(z) = \binom{n+2a-1}{n} {}_2F_1\left(-n, n+2a; a + \frac{1}{2}, \frac{1}{2}(1-z)\right). \quad (12.3.5)$$

When *n* is a nonnegative integer, this formula gives a polynomial in *z* of degree *n*, but all parameters are permitted to be complex numbers. With *a* = 1/2, the Gegenbauer polynomial reduces to a Legendre polynomial.

These functions return  $C_n^{(a)}(x)$ , the Gegenbauer (ultraspherical) polynomial of degree *n* with parameter *a*. The degree *n* must be non-negative; *a* should be  $> -1/2$ . The Gegenbauer polynomials are orthogonal on the interval  $(-1, 1)$ , with respect to the weight function  $w(x) = (1 - x^2)^{a-1/2}$ . If *a* ≠ 0 the function uses the standard recurrence formulas [1, 22.7.3]:

$$\begin{aligned} C_0^{(a)}(x) &= 1 \\ C_1^{(a)}(x) &= 2ax \\ nC_n^{(a)}(x) &= 2(n+a-1)x C_{n-1}^{(a)}(x) - (n+2a-2) C_{n-2}^{(a)}(x). \end{aligned} \quad (12.3.6)$$

For *a* = 0 the result can be expressed in Chebyshev polynomials:

$$C_0^{(0)}(x) = 1, \quad C_n^{(0)}(x) = 2/n T_n(x). \quad (12.3.7)$$

Evaluation for arbitrary arguments:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> gegenbauer(3, 0.5, -10)
-2485.0
>>> gegenbauer(1000, 10, 100)
3.012757178975667428359374e+2322
>>> gegenbauer(2+3j, -0.75, -1000j)
(-5038991.358609026523401901 + 9414549.285447104177860806j)
```

---

## 12.4 Hermite and Laguerre polynomials

### 12.4.1 Hermite polynomials

Function **hermite(*n* As mpNum, *z* As mpNum) As mpNum**

The function `hermite` returns the Hermite polynomial  $H_n(z)$

**Parameters:**

*n*: A real or complex number.

*z*: A real or complex number.

Evaluates the Hermite polynomial  $H_n(z)$ , which may be defined using the recurrence

$$H_0(z) = 1; \quad H_1(z) = 2z; \quad H_{n+1} = 2zH_n(z) - 2nH_{n-1}(z). \quad (12.4.1)$$

The  $H_n$  are orthogonal on the interval  $(-\infty, \infty)$ , with respect to the weight function  $w(x) = e^{-x^2}$ . They are computed with the standard recurrence formulas [1, 22.7.13]:

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_n(x) &= 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x). \end{aligned} \quad (12.4.2)$$

The Hermite polynomials are orthogonal on  $(-\infty, \infty)$  with respect to the weight  $e^{-z^2}$ . More generally, allowing arbitrary complex values of  $n$ , the Hermite function  $H_n(z)$  is defined as

$$H_n(z) = (2z)^n {}_2F_0\left(-\frac{n}{2}, \frac{1-n}{2}, -\frac{1}{z^2}\right) \quad (12.4.3)$$

for  $\Re z > 0$ , or generally

$$H_n(z) = 2^n \sqrt{\pi} \left( \frac{1}{\Gamma(\frac{1-n}{2})} {}_1F_1\left(-\frac{n}{2}, \frac{1}{2}, z^2\right) - \frac{2z}{\Gamma(-\frac{n}{2})} {}_1F_1\left(-\frac{1-n}{2}, \frac{3}{2}, z^2\right) \right) \quad (12.4.4)$$

Evaluation for arbitrary arguments:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> hermite(0, 10)
1.0
>>> hermite(1, 10); hermite(2, 10)
20.0
398.0
>>> hermite(10000, 2)
4.950440066552087387515653e+19334
>>> hermite(3, -10**8)
-7999999999999998800000000.0
>>> hermite(-3, -10**8)
1.675159751729877682920301e+4342944819032534
>>> hermite(2+3j, -1+2j)
(-0.07652130602993513389421901 - 0.1084662449961914580276007j)
```

### 12.4.2 Laguerre polynomials

Function **laguerre(*n* As mpNum, *a* As mpNum, *z* As mpNum) As mpNum**

The function laguerre returns the generalized Laguerre polynomial  $L_n^{\alpha}(z)$

**Parameters:**

*n*: A real or complex number.

*a*: A real or complex number.

*z*: A real or complex number.

These functions return  $L_n^{(a)}(x)$ , the generalized Laguerre polynomials of degree  $n \geq 0$  with parameter  $a; x \geq 0$  and  $a > -1$  are the standard ranges. These polynomials are orthogonal on the interval  $(0, \infty)$ , with respect to the weight function  $w(x) = e^{-x}x^a$ .

If  $x < 0$  and  $a > -1$  the function tries to avoid inaccuracies and computes the result with KummerâŽs confluent hypergeometric function, see Abramowitz and Stegun[1, 22.5.34]

$$L_n^{(a)}(x) = \binom{n+a}{n} M(-n, a+1, x), \quad (12.4.5)$$

otherwise the standard recurrence formulas are used:

$$\begin{aligned} L_0^{(a)}(x) &= 1 \\ L_1^{(a)}(x) &= -x + 1 + a \\ nL_n^{(a)}(x) &= (2n + a - 1 - x)L_{n-1}^{(a)}(x) - (n + a - 1)L_{n-2}^{(a)}(x). \end{aligned} \quad (12.4.6)$$

Gives the generalized (associated) Laguerre polynomial, defined by

$$L_n^{\alpha}(z) = \frac{\Gamma(n + b + 1)}{\Gamma(b + 1)\Gamma(n + 1)} {}_1F_1(-n, a + 1, z). \quad (12.4.7)$$

With  $a = 0$  and  $n$  a nonnegative integer, this reduces to an ordinary Laguerre polynomial, the sequence of which begins

$$L_0(z) = 1, \quad L_1(z) = 1 - z, \quad L_2(z) = z^2 - 2z + 1, \dots \quad (12.4.8)$$

The Laguerre polynomials are orthogonal with respect to the weight  $z^a e^{-z}$  on  $[0, \infty)$ .

Evaluation for arbitrary arguments:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> laguerre(5, 0, 0.25)
0.037263997395833333333333333
>>> laguerre(1+j, 0.5, 2+3j)
(4.474921610704496808379097 - 11.02058050372068958069241j)
>>> laguerre(2, 0, 10000)
49980001.0
>>> laguerre(2.5, 0, 10000)
-9.327764910194842158583189e+4328
```

### 12.4.3 Laguerre Polynomials

---

Function **LaguerreLMpMath(*n* As mpNum, *x* As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function `LaguerreLMpMath` returns  $L_n(x)$ , the Laguerre polynomials of degree  $n \geq 0$ .

**Parameters:**

*n*: An Integer.

*x*: A real number.

This function returns  $L_n(x)$ , the Laguerre polynomials of degree  $n \geq 0$ . The Laguerre polynomials are just special cases of the generalized Laguerre polynomials

$$L_n(x) = L_n^{(0)}(x). \quad (12.4.9)$$

### 12.4.4 Associated Laguerre Polynomials

---

Function **AssociatedLaguerreMpMath(*n* As mpNum, *m* As mpNum, *x* As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function `AssociatedLaguerreMpMath` returns  $L_n^m(x)$ , the associated Laguerre polynomials of degree  $n \geq 0$  and order  $m \geq 0$ .

**Parameters:**

*n*: An Integer.

*m*: An Integer.

*x*: A real number.

This function returns  $L_n^m(x)$ , the associated Laguerre polynomials of degree  $n \geq 0$  and order  $m \geq 0$ , defined as

$$L_n^m(x) = (-1)^m \frac{d^m}{dx^m} L_{n+m}(x). \quad (12.4.10)$$

The  $L_n^m(x)$  are computed using the generalized Laguerre polynomials

$$L_n^m(x) = L_n^{(m)}(x). \quad (12.4.11)$$

# Chapter 13

## Hypergeometric functions

The functions listed in Exponential integrals and error functions, Bessel functions and related functions and Orthogonal polynomials, and many other functions as well, are merely particular instances of the generalized hypergeometric function  ${}_pF_q$ . The functions listed in the following section enable efficient direct evaluation of the underlying hypergeometric series, as well as linear combinations, limits with respect to parameters, and analytic continuations thereof. Extensions to twodimensional series are also provided. See also the basic or q-analog of the hypergeometric series in q-functions.

For convenience, most of the hypergeometric series of low order are provided as standalone functions. They can equivalently be evaluated using `hyper()`. As will be demonstrated in the respective docstrings, all the `hyp#f#` functions implement analytic continuations and/or asymptotic expansions with respect to the argument , thereby permitting evaluation for anywhere in the complex plane. Functions of higher degree can be computed via `hyper()`, but generally only in rapidly convergent instances.

Most hypergeometric and hypergeometric-derived functions accept optional keyword arguments to specify options for `hypercomb()` or `hyper()`. Some useful options are `maxprec`, `maxterms`, `zeroprec`, `accurate_small`, `hmag`, `force_series`, `asymp_tol` and `eliminate`. These options give control over what to do in case of slow convergence, extreme loss of accuracy or evaluation at zeros (these two cases cannot generally be distinguished from each other automatically), and singular parameter combinations.

For alternative implementations, see e.g. [Pearson \(2009\)](#), [Muller \(2001\)](#) or [Forrey \(1997\)](#).

## 13.1 Confluent Hypergeometric Limit Function

### 13.1.1 Confluent Hypergeometric Limit Function

Function **hyp0f1(a As mpNum, z As mpNum) As mpNum**

The function `hyp0f1` returns the hypergeometric function  ${}_0F_1$

**Parameters:**

- a*: A real or complex number.
- z*: A real or complex number.

Gives the hypergeometric function  ${}_0F_1$ , sometimes known as the confluent limit function, defined as

$${}_0F_1(a, z) = \sum_{k=0}^{\infty} \frac{1}{(a)_k} \frac{z^k}{k!}. \quad (13.1.1)$$

This function satisfies the differential equation  $zf''(z) + af'(z) = f(z)$ , and is related to the Bessel function of the first kind (see `besselj()`).

This functions returns the confluent hypergeometric limit function  ${}_0F_1(b; x)$ , defined for  $b \neq 0, -1, -2, -3, \dots$ , by the series

$${}_0F_1(b; x) = {}_0F_1(-; b; x) = \sum_{n=0}^{\infty} \frac{x^n}{(b)_n n!} \quad (13.1.2)$$

where  $(a)_k$  is the Pochammer symbol (see section 9.3)

The function is calculated by treating  ${}_0F_1(b; 0) = 1$  as special case, and otherwise using the following relations to Bessel functions:

$${}_0F_1(b; x) = \Gamma(b)(+x)^{(1-b)/2} I_{b-1}(2\sqrt{+x}), \quad x > 0, \quad (13.1.3)$$

$${}_0F_1(b; x) = \Gamma(b)(-x)^{(1-b)/2} J_{b-1}(2\sqrt{-x}), \quad x < 0, \quad (13.1.4)$$

`hyp0f1(a,z)` is equivalent to `hyper([], [a], z)`; see documentation for `hyper()` for more information.

Examples

Evaluation for arbitrary arguments:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> hyp0f1(2, 0.25)
1.130318207984970054415392
>>> hyp0f1((1,2), 1234567)
6.27287187546220705604627e+964
>>> hyp0f1(3+4j, 1000000j)
(3.905169561300910030267132e+606 + 3.807708544441684513934213e+606j)
```

Evaluation is supported for arbitrarily large values of *z*, using asymptotic expansions:

```
>>> hyp0f1(1, 10**50)
2.131705322874965310390701e+8685889638065036553022565
>>> hyp0f1(1, -10**50)
1.115945364792025420300208e-13
```

Verifying the differential equation:

---

```
>>> a = 2.5
>>> f = lambda z: hyp0f1(a,z)
>>> for z in [0, 10, 3+4j]:
... chop(z*diff(f,z,2) + a*diff(f,z) - f(z))
...
0.0
0.0
0.0
```

---

### 13.1.2 Regularized Confluent Hypergeometric Limit Function

---

Function **Hypergeometric0F1RegularizedMpMath(*b* As mpNum, *x* As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function **Hypergeometric0F1RegularizedMpMath** returns the regularized confluent hypergeometric limit function  ${}_0\tilde{F}_1(b; x)$ .

**Parameters:**

*b*: A real number.

*x*: A real number.

The regularized confluent hypergeometric limit function  ${}_0\tilde{F}_1(b; x)$  for unrestricted *b* is defined by [30, 15.1.2]

$${}_0\tilde{F}_1(b; x) = \frac{1}{\Gamma(b)} {}_0F_1(b; x), \quad (b \neq 0, -1, -2, \dots) \quad (13.1.5)$$

and by the corresponding limit if  $b = 0, -1, -2, \dots, = -n$ , with the value

$${}_0\tilde{F}_1(-n; x) = x^{n+1} {}_0F_1(n+2; x), \quad n \in \mathbb{N} \quad (13.1.6)$$

where  $\Gamma(\cdot)$  is the Gamma function (see section 4.8.6).

## 13.2 Kummer's Confluent Hypergeometric Functions and related functions

### 13.2.1 Kummer's Confluent Hypergeometric Function of the first kind

---

Function **hyp1f1(a As mpNum, b As mpNum, z As mpNum) As mpNum**

---

The function `hyp1f1` returns the confluent hypergeometric function of the first kind  ${}_1F_1(a, b; z)$

**Parameters:**

*a*: A real or complex number.

*b*: A real or complex number.

*z*: A real or complex number.

The confluent hypergeometric function of the first kind is defined as

$${}_1F_1(a, b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \cdot \frac{z^k}{k!} \quad (13.2.1)$$

also known as Kummer's function and sometimes denoted by  $M(a, b; z)$ . This function gives one solution to the confluent (Kummer's) differential equation

$$zf''(z) + (b - z)f'(z) - af(z) = 0. \quad (13.2.2)$$

A second solution is given by the *U* function; see `hyperu()`. Solutions are also given in an alternate form by the Whittaker functions (`whitm()`, `whitw()`).

This function returns the Kummer's confluent hypergeometric function  ${}_1F_1(a, b; x)$ , defined by the series

$${}_1F_1(a, b; z) = M(a, b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \cdot \frac{z^n}{n!} \quad (13.2.3)$$

where  $(a)_k$  is the Pochammer symbol (see section 9.3)

The function has the following integral representation

$${}_1F_1(a, b; z) = B(a, b - a)^{-1} \int_0^1 e^{zt} t^{a-1} (1 - t)^{b-a-1} dt, \quad \Re b > \Re a > 0 \quad (13.2.4)$$

The following closed-form approximation based on a Laplace approximation has been derived by [Butler & Wood \(2002\)](#):

$${}_1F_1(a, b; z) \approx \frac{G_1(a, b; z)}{G_1(a, b; 0)}, \quad \text{where} \quad (13.2.5)$$

$$G_1(a, b, c; z) = w^{-1/2} y^a (1 - y)^{b-a} e^x y,$$

$$w = a(1 - y)^2 + (b - a)y^2$$

$$y = [(x - b) + \sqrt{(x - b)^2 + 4ax}] / 2x, \text{ if } x \neq 0, \text{ and } y = a/b \text{ otherwise.}$$

`hyp1f1(a,b,z)` is equivalent to `hyper([a],[b],z)`; see documentation for `hyper()` for more information. Parameters may be complex:

```
>>> hyp1f1(2+3j, -1+j, 10j)
(261.8977905181045142673351 + 160.8930312845682213562172j)
```

Arbitrarily large values of are supported:

```
>>> hyp1f1(3, 4, 10**20)
3.890569218254486878220752e+43429448190325182745
>>> hyp1f1(3, 4, -10**20)
6.0e-60
>>> hyp1f1(3, 4, 10**20*j)
(-1.935753855797342532571597e-20 - 2.291911213325184901239155e-20j)
```

Verifying the differential equation:

```
>>> a, b = 1.5, 2
>>> f = lambda z: hyp1f1(a,b,z)
>>> for z in [0, -10, 3, 3+4j]:
... chop(z*diff(f,z,2) + (b-z)*diff(f,z) - a*f(z))
...
0.0
0.0
0.0
0.0
```

### 13.2.2 Kummer's Regularized Confluent Hypergeometric Function

Function **Hypergeometric1F1RegularizedMpMath**(*a* As mpNum, *b* As mpNum, *z* As mpNum)

**NOT YET IMPLEMENTED**

The function **Hypergeometric1F1RegularizedMpMath** returns Kummer's regularized confluent hypergeometric function  ${}_1F_1(a, b; z)$ .

**Parameters:**

*a*: A real number.

*b*: A real number.

*z*: A real number.

This function returns the Kummer's regularized confluent hypergeometric function  $\tilde{F}_1(a, b; z)$  for unrestricted *b*, defined by [30, 15.1.2]

$${}_1\tilde{F}_1(a, b; z) = \frac{1}{\Gamma(b)} {}_1F_1(a; b; z) = M(a, b; z) = \frac{1}{\Gamma(b)} M(a; b; z), \quad (b \neq 0, -1, -2, \dots) \quad (13.2.6)$$

and by the corresponding limit if  $b = 0, -1, -2, \dots, = -n$ , with the value

$${}_1\tilde{F}_1(a, b; z) = \frac{(a)_{n+1}}{(n+1)!} x^{n+1} {}_1F_1(a+n+1; n+2; z), \quad n \in \mathbb{N} \quad (13.2.7)$$

where  $\Gamma(\cdot)$  is the Gamma function (see section 4.8.6) and  $(a)_k$  is the Pochammer symbol (see section 9.3)

The function has the following integral representation

$${}_1\tilde{F}_1(a, b; -m; z) = \frac{1}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt, \quad \Re b > \Re a > 0 \quad (13.2.8)$$

### 13.2.3 Kummer's Confluent Hypergeometric Function of the second kind

---

Function **hyperu**(*a* As mpNum, *b* As mpNum, *z* As mpNum) As mpNum

---

The function `hyperu` returns the Tricomi confluent hypergeometric function  $U$

**Parameters:**

- a*: A real or complex number.
- b*: A real or complex number.
- z*: A real or complex number.

The Kummer or confluent hypergeometric function of the second kind is also known as the Tricomi confluent hypergeometric function,  $U$ . This function gives a second linearly independent solution to the confluent hypergeometric differential equation (the first is provided by  ${}_1F_1$  - see `hyp1f1()`).

This functions returns the Tricomi's confluent hypergeometric function  $U(a, b; x)$  for  $x > 0$  and  $b \neq 0, \pm 1 \pm 2, \dots$ , defined by

$$U(a, b; x) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} M(a, b; c; z) + \frac{\Gamma(1-b)}{\Gamma(a)} x^{1-b} M(1+a-b, 2-b; x) \quad (13.2.9)$$

where  $\Gamma(\cdot)$  is the Gamma function (see section 4.8.6).

Examples

Evaluation for arbitrary complex arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> hyperu(2,3,4)
0.0625
>>> hyperu(0.25, 5, 1000)
0.1779949416140579573763523
>>> hyperu(0.25, 5, -1000)
(0.1256256609322773150118907 - 0.1256256609322773150118907j)
```

---

### 13.2.4 Hypergeometric Function 2F0

---

Function **hyp2f0**(*a* As mpNum, *b* As mpNum, *z* As mpNum) As mpNum

---

The function `hyp2f0` returns the hypergeometric function  ${}_2F_0$

**Parameters:**

- a*: A real or complex number.
- b*: A real or complex number.
- z*: A real or complex number.

The hypergeometric function  ${}_2F_0$  is defined formally by the series

$${}_2F_0(a, b; z) = \sum_{n=0}^{\infty} (a)_n (b)_n \frac{z^n}{n!} \quad (13.2.10)$$

This series usually does not converge. For small enough  $z$ , it can be viewed as an asymptotic series that may be summed directly with an appropriate truncation. When this is not the case, `hyp2f0()` gives a regularized sum, or equivalently, it uses a representation in terms of the hypergeometric U function [1]. The series also converges when either  $a$  or  $b$  is a nonpositive integer, as it then terminates into a polynomial after  $-a$  or  $-b$  terms.

### Examples

Evaluation is supported for arbitrary complex arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> hyp2f0((2,3), 1.25, -100)
0.07095851870980052763312791
>>> hyp2f0((2,3), 1.25, 100)
(-0.03254379032170590665041131 + 0.07269254613282301012735797j)
>>> hyp2f0(-0.75, 1-j, 4j)
(-0.3579987031082732264862155 - 3.052951783922142735255881j)
```

---

Even with real arguments, the regularized value of  ${}_2F_0$  is often complex-valued, but the imaginary part decreases exponentially as  $z \rightarrow 0$ . In the following example, the first call uses complex evaluation while the second has a small enough  $z$  to evaluate using the direct series and thus the returned value is strictly real (this should be taken to indicate that the imaginary part is less than  $\text{eps}$ ):

---

```
>>> mp.dps = 15
>>> hyp2f0(1.5, 0.5, 0.05)
(1.04166637647907 + 8.34584913683906e-8j)
>>> hyp2f0(1.5, 0.5, 0.0005)
1.00037535207621
```

---

The imaginary part can be retrieved by increasing the working precision:

---

```
>>> mp.dps = 80
>>> nprint(hyp2f0(1.5, 0.5, 0.009).imag)
1.23828e-46
```

---

## 13.3 Whittaker functions M and W

### 13.3.1 Whittaker function M

---

Function **whitm**(*k* As mpNum, *m* As mpNum, *z* As mpNum) As mpNum

---

The function `whitm` returns the Whittaker function  $M$

**Parameters:**

*k*: A real or complex number.

*m*: A real or complex number.

*z*: A real or complex number.

The Whittaker function  $M(k, m, z)$  gives a solution to the Whittaker differential equation

$$\frac{d^2}{dz^2} + \left( -\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2} \right) f = 0. \quad (13.3.1)$$

A second solution is given by `whitw()`.

The Whittaker functions are defined in Abramowitz & Stegun, section 13. They are alternate forms of the confluent hypergeometric functions  ${}_1F_1$  and  $U$ :

$$M(k, m, z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+m} {}_1F_1\left(\frac{1}{2} + m - k, 1 + 2m, z\right) \quad (13.3.2)$$

$$W(k, m, z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+m} U\left(\frac{1}{2} + m - k, 1 + 2m, z\right) \quad (13.3.3)$$

Examples

Evaluation for arbitrary real and complex arguments is supported:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> whitm(1, 1, 1)
0.7302596799460411820509668
>>> whitm(1, 1, -1)
(0.0 - 1.417977827655098025684246j)
>>> whitm(j, j/2, 2+3j)
(3.245477713363581112736478 - 0.822879187542699127327782j)
>>> whitm(2, 3, 100000)
4.303985255686378497193063e+21707
```

---

### 13.3.2 Whittaker function W

---

Function **whitw**(*k* As mpNum, *m* As mpNum, *z* As mpNum) As mpNum

---

The function `whitw` returns the Whittaker function  $W$

**Parameters:**

*k*: A real or complex number.

*m*: A real or complex number.

*z*: A real or complex number.

The Whittaker function  $W(k, m, z)$  gives a solution to the Whittaker differential equation. See `whitw()`.

### Examples

Evaluation for arbitrary real and complex arguments is supported:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> whitw(1, 1, 1)
1.19532063107581155661012
>>> whitw(1, 1, -1)
(-0.9424875979222187313924639 - 0.2607738054097702293308689j)
>>> whitw(j, j/2, 2+3j)
(0.1782899315111033879430369 - 0.01609578360403649340169406j)
>>> whitw(2, 3, 100000)
1.887705114889527446891274e-21705
>>> whitw(-1, -1, 100)
1.905250692824046162462058e-24
```

---

## 13.4 Gauss Hypergeometric Function

### 13.4.1 Gauss Hypergeometric Function

---

Function **hyp2f1(a As mpNum, b As mpNum, c As mpNum, z As mpNum)** As mpNum

---

The function hyp2f1 returns the square of  $z$ .

**Parameters:**

- a*: A real or complex number.
- b*: A real or complex number.
- c*: A real or complex number.
- z*: A real or complex number.

The Gauss hypergeometric function  ${}_2F_1$  (often simply referred to as *the* hypergeometric function), defined for  $|z| < 1$  by the series

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{z^k}{k!} \quad (13.4.1)$$

and for  $|z| \geq 1$  by analytic continuation, with a branch cut on  $1, \infty$  when necessary.

Special cases of this function include many of the orthogonal polynomials as well as the incomplete beta function and other functions. Properties of the Gauss hypergeometric function are documented comprehensively in many references, for example Abramowitz & Stegun, section 15.

The implementation supports the analytic continuation as well as evaluation close to the unit circle where  $|z| \approx 1$ . The syntax hyp2f1(a,b,c,z) is equivalent to hyper([a,b],[c],z).

This function returns the Gauss hypergeometric function  ${}_2F_1(a, b; c; x)$ , defined for  $|x| < 1$  by the series

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{x^k}{k!} \quad (13.4.2)$$

where  $(a)_k$  is the Pochhammer symbol (see section 9.3)

The function has the following integral representation

$${}_2F_1(a, b; c; z) = B(a, c-a)^{-1} \int_0^1 \frac{t^{b-1} (1-t)^{c-b-1}}{(1-zt)^a} dt, \quad \Re c > \Re b > 0 \quad (13.4.3)$$

where  $B(\cdot, \cdot)$  is the Beta function (see section 4.8.7)

The following closed-form approximation based on a Laplace approximation has been derived by Butler & Wood (2002):

$${}_2F_1(a, b; c; z) \approx \frac{G_2(a, b, c; z)}{G_2(a, b, c; 0)}, \quad \text{where} \quad (13.4.4)$$

$$G_2(a, b, c; z) = w^{-1/2} y^a (1-y)^{c-a} (1-xy)^{-b},$$

$$w = a(1-y)^2 + (c-a)y^2 - bx^2y^2(1-y)^2/(1-xy)^2$$

$$y = [\tau + \sqrt{\tau^2 - 4ax(c-b)}]/[2x(b-c)], \text{ if } x \neq 0, \text{ and } y = a/c \text{ otherwise.}$$

$$\tau = x(b-a) - c.$$

## Examples

Evaluation with inside, outside and on the unit circle, for fixed parameters:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> hyp2f1(2, (1,2), 4, 0.75)
1.303703703703703703704
>>> hyp2f1(2, (1,2), 4, -1.75)
0.7431290566046919177853916
>>> hyp2f1(2, (1,2), 4, 1.75)
(1.418075801749271137026239 - 1.114976146679907015775102j)
>>> hyp2f1(2, (1,2), 4, 1)
1.6
>>> hyp2f1(2, (1,2), 4, -1)
0.8235498012182875315037882
>>> hyp2f1(2, (1,2), 4, j)
(0.9144026291433065674259078 + 0.2050415770437884900574923j)
>>> hyp2f1(2, (1,2), 4, 2+j)
(0.9274013540258103029011549 + 0.7455257875808100868984496j)
>>> hyp2f1(2, (1,2), 4, 0.25j)
(0.9931169055799728251931672 + 0.06154836525312066938147793j)
```

---

Evaluation with complex parameter values:

---

```
>>> hyp2f1(1+j, 0.75, 10j, 1+5j)
(0.8834833319713479923389638 + 0.7053886880648105068343509j)
```

---

Evaluation with  $z = 1$ :

---

```
>>> hyp2f1(-2.5, 3.5, 1.5, 1)
0.0
>>> hyp2f1(-2.5, 3, 4, 1)
0.06926406926406926406926407
>>> hyp2f1(2, 3, 4, 1)
+inf
```

---

Arbitrarily large values of are supported:

---

```
>>> hyp2f1((-1,3), 1.75, 4, '1e100')
(7.883714220959876246415651e+32 + 1.365499358305579597618785e+33j)
>>> hyp2f1((-1,3), 1.75, 4, '1e1000000')
(7.883714220959876246415651e+333332 + 1.365499358305579597618785e+333333j)
>>> hyp2f1((-1,3), 1.75, 4, '1e1000000j')
(1.365499358305579597618785e+333333 - 7.883714220959876246415651e+333332j)
```

---

Verifying the differential equation:

---

```
>>> f = lambda z: hyp2f1(a,b,c,z)
>>> chop(z*(1-z)*diff(f,z,2) + (c-(a+b+1)*z)*diff(f,z) - a*b*f(z))
0.0
```

---

### 13.4.2 Gauss Regularized Hypergeometric Function

---

Function **Hypergeometric2F1RegularizedMpMath**(*a* As mpNum, *b* As mpNum, *c* As mpNum, *z* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function **Hypergeometric2F1RegularizedMpMath** returns the regularized Gauss hypergeometric function  ${}_2F_1(a, b; c; z)$ .

**Parameters:**

*a*: A real number.

*b*: A real number.

*c*: A real number.

*z*: A real number.

The regularized Gauss hypergeometric function  ${}_2\tilde{F}_1(a, b; c; z)$  for unrestricted *c*, is defined by [30, 15.1.2]

$${}_2\tilde{F}_1(a, b; c; z) = \frac{1}{\Gamma(c)} {}_2F_1(a, b; c; z) = \mathbf{F}(a, b; c; z), \quad (c \neq 0, -1, -2, \dots) \quad (13.4.5)$$

and by the corresponding limit if  $c = 0, -1, -2, \dots, = -m$ , with the value

$${}_2\tilde{F}_1(a, b; -m; z) = \frac{(a)_{m+1} (b)_{m+1}}{(m+1)!} x^{m+1} {}_2F_1(a+m+1, a+m+1; m+2; z) \quad (13.4.6)$$

where  $\Gamma(\cdot)$  is the Gamma function (see section 4.8.6) and  $(a)_k$  is the Pochammer symbol (see section 9.3)

The function has the following integral representation

$${}_2\tilde{F}_1(a, b; c; z) = \frac{1}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1} (1-t)^{c-b-1}}{(1-zt)^a} dt, \quad \Re c > \Re b > 0 \quad (13.4.7)$$

## 13.5 Additional Hypergeometric Functions

### 13.5.1 Hypergeometric Function 1F2

---

Function **hyp1f2(a1 As mpNum, b1 As mpNum, b2 As mpNum, z As mpNum) As mpNum**

---

The function hyp1f2 returns the the hypergeometric function  ${}_1F_2(a_1; b_1, b_2; z)$

**Parameters:**

*a1*: A real or complex number.

*b1*: A real or complex number.

*b2*: A real or complex number.

*z*: A real or complex number.

Gives the hypergeometric function  ${}_1F_2(a_1; b_1, b_2; z)$ . The call hyp1f2(a1,b1,b2,z) is equivalent to hyper([a1],[b1,b2],z).

Evaluation works for complex and arbitrarily large arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> a, b, c = 1.5, (-1,3), 2.25
>>> hyp1f2(a, b, c, 10**20)
-1.159388148811981535941434e+8685889639
>>> hyp1f2(a, b, c, -10**20)
-12.60262607892655945795907
>>> hyp1f2(a, b, c, 10**20*j)
(4.237220401382240876065501e+6141851464 - 2.950930337531768015892987e+6141851464j)
>>> hyp1f2(2+3j, -2j, 0.5j, 10-20j)
(135881.9905586966432662004 - 86681.95885418079535738828j)
```

---

### 13.5.2 Hypergeometric Function 2F2

---

Function **hyp2f2(a1 As mpNum, a2 As mpNum, b1 As mpNum, b2 As mpNum, z As mpNum) As mpNum**

---

The function hyp2f2 returns the hypergeometric function  ${}_2F_2(a_1, a_2; b_1, b_2; z)$ .

**Parameters:**

*a1*: A real or complex number.

*a2*: A real or complex number.

*b1*: A real or complex number.

*b2*: A real or complex number.

*z*: A real or complex number.

Gives the hypergeometric function  ${}_2F_2(a_1, a_2; b_1, b_2; z)$ . The call hyp2f2(a1,a2,b1,b2,z) is equivalent to hyper([a1,a2],[b1,b2],z).

Evaluation works for complex and arbitrarily large arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> a, b, c, d = 1.5, (-1,3), 2.25, 4
```

---

---

```
>>> hyp2f2(a, b, c, d, 10**20)
-5.275758229007902299823821e+43429448190325182663
>>> hyp2f2(a, b, c, d, -10**20)
2561445.079983207701073448
>>> hyp2f2(a, b, c, d, 10**20*j)
(2218276.509664121194836667 - 1280722.539991603850462856j)
>>> hyp2f2(2+3j, -2j, 0.5j, 4j, 10-20j)
(80500.68321405666957342788 - 20346.82752982813540993502j)
```

---

### 13.5.3 Hypergeometric Function 2F3

---

Function **hyp2f3(*a1* As mpNum, *a2* As mpNum, *b1* As mpNum, *b2* As mpNum, *b3* As mpNum, *z* As mpNum) As mpNum**

---

The function hyp2f3 returns the hypergeometric function  ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ .

**Parameters:**

*a1*: A real or complex number.  
*a2*: A real or complex number.  
*b1*: A real or complex number.  
*b2*: A real or complex number.  
*b3*: A real or complex number.  
*z*: A real or complex number.

Gives the hypergeometric function  ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ . The call hyp2f3(*a1,a2,b1,b2,b3,z*) is equivalent to hyper([*a1,a2*],[*b1,b2,b3*],*z*).

Evaluation works for complex and arbitrarily large arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> a1,a2,b1,b2,b3 = 1.5, (-1,3), 2.25, 4, (1,5)
>>> hyp2f3(a1,a2,b1,b2,b3,10**20)
-4.169178177065714963568963e+8685889590
>>> hyp2f3(a1,a2,b1,b2,b3,-10**20)
7064472.587757755088178629
>>> hyp2f3(a1,a2,b1,b2,b3,10**20*j)
(-5.163368465314934589818543e+6141851415 + 1.783578125755972803440364e+6141851416j)
>>> hyp2f3(2+3j, -2j, 0.5j, 4j, -1-j, 10-20j)
(-2280.938956687033150740228 + 13620.97336609573659199632j)
>>> hyp2f3(2+3j, -2j, 0.5j, 4j, -1-j, 10000000-20000000j)
(4.849835186175096516193e+3504 - 3.365981529122220091353633e+3504j)
```

---

### 13.5.4 Hypergeometric Function 3F2

---

Function **hyp3f2(*a1* As mpNum, *a2* As mpNum, *a3* As mpNum, *b1* As mpNum, *b2* As mpNum, *z* As mpNum) As mpNum**

---

The function hyp3f2 returns hypergeometric function  ${}_3F_2$ .

**Parameters:**

- $a_1$ : A real or complex number.
- $a_2$ : A real or complex number.
- $a_3$ : A real or complex number.
- $b_1$ : A real or complex number.
- $b_2$ : A real or complex number.
- $z$ : A real or complex number.

Gives the hypergeometric function  ${}_3F_2$ , defined for  $|z| < 1$  as

$${}_3F_2(a_1, a_2, a_3; b_1, b_2; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k (a_3)_k}{(b_1)_k (b_2)_k} \frac{z^k}{k!}, \quad (13.5.1)$$

and for  $|z| \geq 1$  by analytic continuation. The analytic structure of this function is similar to that of  ${}_2F_1$ , generally with a singularity at  $z = 1$  and a branch cut on  $(1, \infty)$ .

Evaluation is supported inside, on, and outside the circle of convergence  $|z| = 1$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> hyp3f2(1,2,3,4,5,0.25)
1.083533123380934241548707
>>> hyp3f2(1,2+2j,3,4,5,-10+10j)
(0.1574651066006004632914361 - 0.03194209021885226400892963j)
>>> hyp3f2(1,2,3,4,5,-10)
0.3071141169208772603266489
>>> hyp3f2(1,2,3,4,5,10)
(-0.4857045320523947050581423 - 0.5988311440454888436888028j)
>>> hyp3f2(0.25,1,1,2,1.5,1)
1.157370995096772047567631
>>> (8-pi-2*ln2)/3
1.157370995096772047567631
>>> hyp3f2(1+j,0.5j,2,1,-2j,-1)
(1.74518490615029486475959 + 0.1454701525056682297614029j)
>>> hyp3f2(1+j,0.5j,2,1,-2j,sqrt(j))
(0.9829816481834277511138055 - 0.4059040020276937085081127j)
>>> hyp3f2(-3,2,1,-5,4,1)
1.41
>>> hyp3f2(-3,2,1,-5,4,2)
2.12
```

---

Evaluation very close to the unit circle:

---

```
>>> hyp3f2(1,2,3,4,5,'1.0001')
(1.564877796743282766872279 - 3.76821518787438186031973e-11j)
>>> hyp3f2(1,2,3,4,5,'1+0.0001j')
(1.564747153061671573212831 + 0.000130575750366084557648482j)
>>> hyp3f2(1,2,3,4,5,'0.9999')
1.564616644881686134983664
>>> hyp3f2(1,2,3,4,5,'-0.9999')
0.7823896253461678060196207
```

---

## 13.6 Generalized hypergeometric functions

### 13.6.1 Generalized hypergeometric function $pFq$

---

Function **hyper(as As mpNum, bs As mpNum, z As mpNum) As mpNum**

---

The function `hyper` returns the generalized hypergeometric function  $pF_q$

**Parameters:**

*as*: list of real or complex numbers.

*bs*: list of real or complex numbers.

*z*: A real or complex number.

Evaluates the generalized hypergeometric function

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n (b_2)_n \dots (b_q)_n} \frac{z^n}{n!}, \quad (13.6.1)$$

where  $(x)_n$  denotes the rising factorial (see `rf()`).

The parameters lists *a\_s* and *b\_s* may contain integers, real numbers, complex numbers, as well as exact fractions given in the form of tuples  $(p, q)$ . `hyper()` is optimized to handle integers and fractions more efficiently than arbitrary floating-point parameters (since rational parameters are by far the most common).

The parameters can be any combination of integers, fractions, floats and complex numbers:

---

```
>>> a, b, c, d, e = 1, (-1,2), pi, 3+4j, (2,3)
>>> x = 0.2j
>>> hyper([a,b],[c,d,e],x)
(0.9923571616434024810831887 - 0.005753848733883879742993122j)
>>> b, e = -0.5, mpf(2)/3
>>> fn = lambda n: rf(a,n)*rf(b,n)/rf(c,n)/rf(d,n)/rf(e,n)*x**n/fac(n)
>>> nsum(fn, [0, inf])
(0.9923571616434024810831887 - 0.005753848733883879742993122j)
```

---

If any  $b_k$  is a nonpositive integer, the function is undefined (unless the series terminates before the division by zero occurs):

---

```
>>> hyper([1,1,1,-3],[-2,5],1)
Traceback (most recent call last):
...
ZeroDivisionError: pole in hypergeometric series
>>> hyper([1,1,1,-1],[-2,5],1)
1.1
```

---

Except for polynomial cases, the radius of convergence  $R$  of the hypergeometric series is either  $R = \infty$  (if  $p \leq q$ ),  $R = 1$  (if  $p = q + 1$ ), or  $R = 0$  (if  $p > q + 1$ ).

The analytic continuations of the functions with  $p = q + 1$ , i.e.  ${}_2F_1$ ,  ${}_3F_2$ ,  ${}_4F_3$ , etc, are all implemented and therefore these functions can be evaluated for  $|z| \geq 1$ . The shortcuts `hyp2f1()`, `hyp3f2()` are available to handle the most common cases (see their documentation), but functions of higher degree are also supported via `hyper()`:

---

```
>>> hyper([1,2,3,4], [5,6,7], 1) # 4F3 at finite-valued branch point
1.141783505526870731311423
```

---

```
>>> hyper([4,5,6,7], [1,2,3], 1) # 4F3 at pole
+inf
>>> hyper([1,2,3,4,5], [6,7,8,9], 10) # 5F4
(1.543998916527972259717257 - 0.5876309929580408028816365j)
>>> hyper([1,2,3,4,5,6], [7,8,9,10,11], 1j) # 6F5
(0.9996565821853579063502466 + 0.0129721075905630604445669j)
```

---

Please note that, as currently implemented, evaluation of  ${}_pF_{p-1}$  with  $p \geq 3$  may be slow or inaccurate when  $|z - 1|$  is small, for some parameter values.

When  $p > q + 1$ , `hyper` computes the (iterated) Borel sum of the divergent series. For  ${}_2F_0$  the Borel sum has an analytic solution and can be computed efficiently (see `hyp2f0()`). For higher degrees, the functions is evaluated first by attempting to sum it directly as an asymptotic series (this only works for tiny  $|z|$ ), and then by evaluating the Borel regularized sum using numerical integration. Except for special parameter combinations, this can be extremely slow.

---

```
>>> hyper([1,1], [], 0.5) # regularization of 2F0
(1.340965419580146562086448 + 0.8503366631752726568782447j)
>>> hyper([1,1,1,1], [1], 0.5) # regularization of 4F1
(1.108287213689475145830699 + 0.5327107430640678181200491j)
```

---

With the following magnitude of argument, the asymptotic series for  ${}_3F_1$  gives only a few digits. Using Borel summation, `hyper` can produce a value with full accuracy:

---

```
>>> mp.dps = 15
>>> hyper([2,0.5,4], [5.25], '0.08', force_series=True)
Traceback (most recent call last):
...
NoConvergence: Hypergeometric series converges too slowly. Try increasing maxterms.
>>> hyper([2,0.5,4], [5.25], '0.08', asymp_tol=1e-4)
1.0725535790737
>>> hyper([2,0.5,4], [5.25], '0.08')
(1.07269542893559 + 5.54668863216891e-5j)
>>> hyper([2,0.5,4], [5.25], '-0.08', asymp_tol=1e-4)
0.946344925484879
>>> hyper([2,0.5,4], [5.25], '-0.08')
0.946312503737771
>>> mp.dps = 25
>>> hyper([2,0.5,4], [5.25], '-0.08')
0.9463125037377662296700858
```

---

Note that with the positive  $z$  value, there is a complex part in the correct result, which falls below the tolerance of the asymptotic series.

### 13.6.2 Weighted combination of hypergeometric functions

---

Function **hypercomb**(*f* As *mpFunction*, *params* As *mpNum*, *z* As *mpNum*, **Keywords** As String ) As *mpNum*

---

The function `hypercomb` returns a weighted combination of hypergeometric functions

**Parameters:**

*f*: a real or function.

*params*: list of real or complex numbers.

*z*: A real or complex number.

*Keywords*: discardknownzeros=True.

Computes a weighted combination of hypergeometric functions

$$\sum_{r=1}^N \left[ \prod_{k=1}^{l_r} (w_{r,k})^{c_{r,k}} \frac{\prod_{k=1}^{m_r} \Gamma(\alpha_{r,k})}{\prod_{k=1}^{n_r} \Gamma(\beta_{r,k})} {}^{p_r}F_{q_r}(a_{r,1}, \dots, a_{r,p}; b_{r,1}, \dots, b_{r,q}; z_r) \right] \quad (13.6.2)$$

Typically the parameters are linear combinations of a small set of base parameters; `hypercomb()` permits computing a correct value in the case that some of the  $\alpha$ ,  $\beta$ ,  $b$  turn out to be nonpositive integers, or if division by zero occurs for some  $w^c$ , assuming that there are opposing singularities that cancel out. The limit is computed by evaluating the function with the base parameters perturbed, at a higher working precision.

The first argument should be a function that takes the perturbable base parameters `params` as input and returns tuples (`w`, `c`, `alpha`, `beta`, `a`, `b`, `z`), where the coefficients `w`, `c`, gamma factors `alpha`, `beta`, and hypergeometric coefficients `a`, `b` each should be lists of numbers, and `z` should be a single number.

Examples

The following evaluates

$$(a - 1) \frac{\Gamma(a - 3)}{\Gamma(a - 4)} {}_1F_1(a, a - 1, z) = e^z (a - 4)(a + z - 1) \quad (13.6.3)$$

with  $a = 1, z = 3$ . There is a zero factor, two gamma function poles, and the  ${}_1F_1$  function is singular; all singularities cancel out to give a finite value:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> hypercomb(lambda a: [([a-1],[1],[a-3],[a-4],[a],[a-1],3)], [1])
-180.769832308689
>>> -9*exp(3)
-180.769832308689
```

---

## 13.7 Meijer G-function

Function **meijerg**(*as* As mpNum, *bs* As mpNum, *z* As mpNum, **Keywords** As String) As mpNum

The function meijerg returns the Meijer G-function

**Parameters:**

*as*: list of real or complex numbers.

*bs*: list of real or complex numbers.

*z*: A real or complex number.

**Keywords**: r=1, series=1.

Evaluates the Meijer G-function, defined as

$$G_{p,q}^{m,n}\left(z; r \middle| \begin{matrix} a_1, \dots, a_n, a_{n+1}, \dots, a_p \\ b_1, \dots, b_m, b_{m+1}, \dots, b_q \end{matrix}\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=n+1}^p \Gamma(a_j + s) \prod_{j=m+1}^q \Gamma(1 - b_j - s)} z^{-s/r} ds \quad (13.7.1)$$

for an appropriate choice of the contour  $L$  (see references).

There are  $p$  elements  $a_j$ . The argument *a\_s* should be a pair of lists, the first containing the  $n$  elements  $a_1, \dots, a_n$  and the second containing the  $p - n$  elements  $a_{n+1}, \dots, a_p$ .

There are  $q$  elements  $b_j$ . The argument *b\_s* should be a pair of lists, the first containing the  $m$  elements  $b_1, \dots, b_m$  and the second containing the  $q - m$  elements  $b_{m+1}, \dots, b_q$ .

The implicit tuple  $(m, n, p, q)$  constitutes the order or degree of the Meijer G-function, and is determined by the lengths of the coefficient vectors. Confusingly, the indices in this tuple appear in a different order from the coefficients, but this notation is standard. The many examples given below should hopefully clear up any potential confusion.

The Meijer G-function is evaluated as a combination of hypergeometric series. There are two versions of the function, which can be selected with the optional *series* argument.

*series*=1 uses a sum of  $m {}_pF_{q-1}$  functions of  $z$

*series*=2 uses a sum of  $n {}_qF_{p-1}$  functions of  $1/z$

The default series is chosen based on the degree and  $|z|$  in order to be consistent with Mathematica's. This definition of the Meijer G-function has a discontinuity at  $|z| = 1$  for some orders, which can be avoided by explicitly specifying a series.

Keyword arguments are forwarded to hypercomb().

Many standard functions are special cases of the Meijer G-function (possibly rescaled and/or with branch cut corrections). We define some test parameters:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> a = mpf(0.75)
>>> b = mpf(1.5)
>>> z = mpf(2.25)
```

The exponential function:

$$e^z = G_{0,1}^{1,0}\left(-z \middle| \begin{matrix} 1 \\ 1 \end{matrix}\right) \quad (13.7.2)$$

---

```
>>> meijerg([],[], [[0],[]], -z)
9.487735836358525720550369
>>> exp(z)
9.487735836358525720550369
```

---

The natural logarithm

$$\log(1+z) = G_{2,2}^{1,2}\left(-z \middle| \begin{matrix} a_1, a_2 \\ b_1, b_2 \end{matrix}\right) \quad (13.7.3)$$


---

```
>>> meijerg([[1,1],[]], [[1],[0]], z)
1.178654996341646117219023
>>> log(1+z)
1.178654996341646117219023
```

---

A rational function

$$\frac{z}{z+1} = G_{2,2}^{1,2}\left(-z \middle| \begin{matrix} a_1, a_2 \\ b_1, b_2 \end{matrix}\right) \quad (13.7.4)$$


---

```
>>> meijerg([[1,1],[]], [[1],[1]], z)
0.6923076923076923076923077
>>> z/(z+1)
0.6923076923076923076923077
```

---

The sine and cosine functions:

$$\frac{1}{\sqrt{\pi}} \sin(2\sqrt{z}) = G_{0,2}^{1,0}\left(z \middle| \begin{matrix} b_1, b_2 \end{matrix}\right) \quad (13.7.5)$$

$$\frac{1}{\sqrt{\pi}} \cos(2\sqrt{z}) = G_{0,2}^{1,0}\left(z \middle| \begin{matrix} b_1, b_2 \end{matrix}\right) \quad (13.7.6)$$


---

```
>>> meijerg([],[], [[0.5],[0]], (z/2)**2)
0.4389807929218676682296453
>>> sin(z)/sqrt(pi)
0.4389807929218676682296453
>>> meijerg([],[], [[0],[0.5]], (z/2)**2)
-0.3544090145996275423331762
>>> cos(z)/sqrt(pi)
-0.3544090145996275423331762
```

---

Bessel functions:

$$J_\alpha(2\sqrt{z}) = G_{0,2}^{1,0}\left(z \middle| \begin{matrix} b_1, b_2 \end{matrix}\right) \quad (13.7.7)$$

$$Y_\alpha(2\sqrt{z}) = G_{1,3}^{2,0}\left(z \middle| \begin{matrix} a_1 \\ b_1, b_2, b_3 \end{matrix}\right) \quad (13.7.8)$$

$$(-z)^{\alpha/2} z^{-\alpha/2} I_\alpha(2\sqrt{z}) = G_{0,2}^{2,0}\left(-z \middle| \begin{matrix} b_1, b_2 \end{matrix}\right) \quad (13.7.9)$$

$$2K_\alpha(2\sqrt{z}) = G_{0,2}^{2,0}\left(z \middle| \begin{matrix} b_1, b_2 \end{matrix}\right) \quad (13.7.10)$$

As the example with the Bessel I function shows, a branch factor is required for some arguments when inverting the square root.

```

>>> meijerg([],[], [[a/2], [-a/2]], (z/2)**2)
0.5059425789597154858527264
>>> besselj(a,z)
0.5059425789597154858527264
>>> meijerg([], [(-a-1)/2], [[a/2, -a/2], [(-a-1)/2]], (z/2)**2)
0.1853868950066556941442559
>>> bessely(a, z)
0.1853868950066556941442559
>>> meijerg([],[], [[a/2], [-a/2]], -(z/2)**2)
(0.8685913322427653875717476 + 2.096964974460199200551738j)
>>> (-z)**(a/2) / z**(a/2) * besseli(a, z)
(0.8685913322427653875717476 + 2.096964974460199200551738j)
>>> 0.5*meijerg([],[], [[a/2, -a/2], []], (z/2)**2)
0.09334163695597828403796071
>>> besselk(a,z)
0.09334163695597828403796071

```

Error functions:

$$\sqrt{\pi} z^{2(\alpha-1)} \operatorname{erfc}(z) = G_{1,2}^{2,0} \left( z, \frac{1}{2} \middle| \begin{matrix} a_1 \\ b_1, b_2 \end{matrix} \right) \quad (13.7.11)$$

```

>>> meijerg([], [a], [[a-1, a-0.5], []], z, 0.5)
0.00172839843123091957468712
>>> sqrt(pi) * z**(2*a-2) * erfc(z)
0.00172839843123091957468712

```

A Meijer G-function of higher degree, (1,1,2,3):

```

>>> meijerg([[a], [b]], [[a], [b, a-1]], z)
1.55984467443050210115617
>>> sin((b-a)*pi)/pi*(exp(z)-1)*z***(a-1)
1.55984467443050210115617

```

A Meijer G-function of still higher degree, (4,1,2,4), that can be expanded as a messy combination of exponential integrals:

```

>>> meijerg([[a], [2*b-a]], [[b, a, b-0.5, -1-a+2*b], []], z)
0.3323667133658557271898061
>>> chop(4*(a-b+1)*sqrt(pi)*gamma(2*b-2*a)*z**a*\n... expint(2*b-2*a, -2*sqrt(-z))*expint(2*b-2*a, 2*sqrt(-z)))
0.3323667133658557271898061

```

In the following case, different series give different values:

```

>>> chop(meijerg([[1], [0.25]], [[3], [0.5]], -2))
-0.06417628097442437076207337
>>> meijerg([[1], [0.25]], [[3], [0.5]], -2, series=1)
0.1428699426155117511873047
>>> chop(meijerg([[1], [0.25]], [[3], [0.5]], -2, series=2))
-0.06417628097442437076207337

```

## 13.8 Bilateral hypergeometric series

---

Function **bihyper(as As mpNum, bs As mpNum, z As mpNum, Keywords As String) As mpNum**

---

The function bihyper returns the bilateral hypergeometric series

**Parameters:**

*as*: list of real or complex numbers.

*bs*: list of real or complex numbers.

*z*: A real or complex number.

*Keywords*: r=1, series=1.

Evaluates the bilateral hypergeometric series

$${}_A H_B(a_1; \dots, a_A; b_1, \dots, b_B) = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n \dots (a_A)_n}{(b_1)_n \dots (b_B)_n} z^n \quad (13.8.1)$$

where, for direct convergence,  $A = B$  and  $|z| = 1$ , although a regularized sum exists more generally by considering the bilateral series as a sum of two ordinary hypergeometric functions. In order for the series to make sense, none of the parameters may be integers.

Examples

The value of  ${}_2H_2$  at  $z = 1$  is given by Dougall's formula:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> a,b,c,d = 0.5, 1.5, 2.25, 3.25
>>> bihyper([a,b],[c,d],1)
-14.49118026212345786148847
>>> gammaproduct([c,d,1-a,1-b,c+d-a-b-1],[c-a,d-a,c-b,d-b])
-14.49118026212345786148847
```

---

The regularized function  ${}_1H_0$  can be expressed as the sum of one  ${}_2F_0$  function and one  ${}_1F_1$  function:

---

```
>>> a = mpf(0.25)
>>> z = mpf(0.75)
>>> bihyper([a], [], z)
(0.2454393389657273841385582 + 0.2454393389657273841385582j)
>>> hyper([a,1], [], z) + (hyper([1], [1-a], -1/z)-1)
(0.2454393389657273841385582 + 0.2454393389657273841385582j)
>>> hyper([a,1], [], z) + hyper([1], [2-a], -1/z)/z/(a-1)
(0.2454393389657273841385582 + 0.2454393389657273841385582j)
```

---

## 13.9 Hypergeometric functions of two variables

### 13.9.1 Generalized 2D hypergeometric series

---

Function **hyper2d(*a* As mpNum, *b* As mpNum, *x* As mpNum, *y* As mpNum) As mpNum**

---

The function `hyper2d` returns the sum the generalized 2D hypergeometric series

**Parameters:**

*a*: A real or complex number.

*b*: A real or complex number.

*x*: A real or complex number.

*y*: A real or complex number.

The sum of the generalized 2D hypergeometric series is calculated as

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{P((a), m, n)}{Q((b))} \frac{x^m y^n}{m!, n!} \quad (13.9.1)$$

where  $(a) = (a_1, \dots, a_r)$ ,  $(b) = (b_1, \dots, b_r)$  and where  $P$  and  $Q$  are products of rising factorials such as  $(a_j)_n$  or  $(a_j)_{m+n}$ .  $P$  and  $Q$  are specified in the form of dicts, with the  $m$  and  $n$  dependence as keys and parameter lists as values. The supported rising factorials are given in the following table (note that only a few are supported in  $Q$ ):

Key	Rising factorial	$Q$
”m”	$(a_j)_m$	Yes
”n”	$(a_j)_n$	Yes
”m+n”	$(a_j)_{m+n}$	Yes
”m-n”	$(a_j)_{m-n}$	No
”n-m”	$(a_j)_{n-m}$	No
”2m+n”	$(a_j)_{2m+n}$	No
”2m-n”	$(a_j)_{2m-n}$	No
”2n-m”	$(a_j)_{2n-m}$	No

For example, the Appell F1 and F4 functions

$$F_1 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_m (c)_n}{(d)_{m+n}} \frac{x^m y^n}{m!, n!} \quad (13.9.2)$$

$$F_4 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(d)_{m+n}} \frac{x^m y^n}{m!, n!} \quad (13.9.3)$$

can be represented respectively as

---

`hyper2d({'m+n': [a], 'm': [b], 'n': [c]}, {'m+n': [d]}, x, y)`

---

`hyper2d({'m+n': [a, b]}, {'m': [c], 'n': [d]}, x, y)`

---

More generally, `hyper2d()` can evaluate any of the 34 distinct convergent secondorder (generalized Gaussian) hypergeometric series enumerated by Horn, as well as the Kampe de Feriet function. The series is computed by rewriting it so that the inner series (i.e. the series containing  $n$  and  $y$ ) has the form of an ordinary generalized hypergeometric series and thereby can be evaluated

efficiently using `hyper()`. If possible, manually swapping  $x$  and  $y$  and the corresponding parameters can sometimes give better results.

#### Examples

Two separable cases: a product of two geometric series, and a product of two Gaussian hypergeometric functions:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> x, y = mpf(0.25), mpf(0.5)
>>> hyper2d({'m':1, 'n':1}, {}, x,y)
2.66666666666666666666666666667
>>> 1/(1-x)/(1-y)
2.66666666666666666666666666667
>>> hyper2d({'m':[1,2], 'n':[3,4]}, {'m':[5], 'n':[6]}, x,y)
4.164358531238938319669856
>>> hyp2f1(1,2,5,x)*hyp2f1(3,4,6,y)
4.164358531238938319669856
```

---

### 13.9.2 Appell F1 hypergeometric function

---

Function `appellf1(a As mpNum, b1 As mpNum, b2 As mpNum, c As mpNum, x As mpNum, y As mpNum) As mpNum`

---

The function `appellf1` returns the Appell F1 hypergeometric function of two variables.

#### Parameters:

- a*: A real or complex number.
- b1*: A real or complex number.
- b2*: A real or complex number.
- c*: A real or complex number.
- x*: A real or complex number.
- y*: A real or complex number.

Gives the Appell F1 hypergeometric function of two variables,

$$F_1(a_1, b_1, b_2, c, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c)_{m+n}} \frac{x^m y^n}{m!, n!} \quad (13.9.4)$$

This series is only generally convergent when  $|x| < 1$  and  $|y| < 1$ , although `appellf1()` can evaluate an analytic continuation with respect to either variable, and sometimes both.

#### Examples

Evaluation is supported for real and complex parameters:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> appellf1(1,0,0.5,1,0.5,0.25)
1.154700538379251529018298
>>> appellf1(1,1+j,0.5,1,0.5,0.5j)
(1.138403860350148085179415 + 1.510544741058517621110615j)
```

---

### 13.9.3 Appell F2 hypergeometric function

---

Function **appellf2(*a* As mpNum, *b1* As mpNum, *b2* As mpNum, *c1* As mpNum, *c2* As mpNum, *x* As mpNum, *y* As mpNum) As mpNum**

---

The function appellf2 returns the Appell F2 hypergeometric function of two variables.

**Parameters:**

- a*: A real or complex number.
- b1*: A real or complex number.
- b2*: A real or complex number.
- c1*: A real or complex number.
- c2*: A real or complex number.
- x*: A real or complex number.
- y*: A real or complex number.

Gives the Appell F2 hypergeometric function of two variables

$$F_2(a_1, b_1, b_2, c_1, c_2, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} \frac{x^m y^n}{m! n!} \quad (13.9.5)$$

The series is generally absolutely convergent for  $|x| + |y| < 1$ .

Examples

Evaluation is supported for real and complex parameters:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> appellf2(1,2,3,4,5,0.25,0.125)
1.257417193533135344785602
>>> appellf2(1,-3,-4,2,3,2,3)
-42.8
>>> appellf2(0.5,0.25,-0.25,2,3,0.25j,0.25)
(0.9880539519421899867041719 + 0.01497616165031102661476978j)
>>> chop(appelf2(1,1+j,1-j,3j,-3j,0.25,0.25))
1.201311219287411337955192
>>> appellf2(1,1,1,4,6,0.125,16)
(-0.09455532250274744282125152 - 0.7647282253046207836769297j)
```

---

### 13.9.4 Appell F3 hypergeometric function

---

Function **appelf3(*a1* As mpNum, *a2* As mpNum, *b1* As mpNum, *b2* As mpNum, *c* As mpNum, *x* As mpNum, *y* As mpNum) As mpNum**

---

The function appellf3 returns the Appell F3 hypergeometric function of two variables.

**Parameters:**

- a1*: A real or complex number.
- a2*: A real or complex number.
- b1*: A real or complex number.
- b2*: A real or complex number.
- c*: A real or complex number.

*x*: A real or complex number.

*y*: A real or complex number.

Gives the Appell F3 hypergeometric function of two variables

$$F_3(a_1, a_2, b_1, b_2, c, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n}{(c)_{m+n}} \frac{x^m y^n}{m!, n!} \quad (13.9.6)$$

The series is generally absolutely convergent for  $|x| < 1, |y| < 1$ .

Examples

Evaluation for various parameters and variables:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> appelf3(1,2,3,4,5,0.5,0.25)
2.221557778107438938158705
>>> appelf3(1,2,3,4,5,6,0); hyp2f1(1,3,5,6)
(-0.5189554589089861284537389 - 0.1454441043328607980769742j)
(-0.5189554589089861284537389 - 0.1454441043328607980769742j)
>>> appelf3(1,-2,-3,1,1,4,6)
-17.4
>>> appelf3(1,2,-3,1,1,4,6)
(17.7876136773677356641825 + 19.54768762233649126154534j)
>>> appelf3(1,2,-3,1,1,6,4)
(85.02054175067929402953645 + 148.4402528821177305173599j)
>>> chop(appelf3(1+j,2,1-j,2,3,0.25,0.25))
1.719992169545200286696007
```

---

### 13.9.5 Appell F4 hypergeometric function

---

Function **appelf4**(*a* As *mpNum*, *b* As *mpNum*, *c1* As *mpNum*, *c2* As *mpNum*, *x* As *mpNum*, *y* As *mpNum*) As *mpNum*

---

The function **appelf4** returns the Appell F4 hypergeometric function of two variables.

**Parameters:**

*a*: A real or complex number.

*b*: A real or complex number.

*c1*: A real or complex number.

*c2*: A real or complex number.

*x*: A real or complex number.

*y*: A real or complex number.

Gives the Appell F4 hypergeometric function of two variables

$$F_4(a, b, c_1, c_2, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (a_2)_n (b)_{m+n}}{(c_1)_m (c_2)_n} \frac{x^m y^n}{m!, n!} \quad (13.9.7)$$

The series is generally absolutely convergent for  $\sqrt{|x|} + \sqrt{|y|} < 1$ .

Examples

Evaluation for various parameters and variables:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> appellf4(1,1,2,2,0.25,0.125)
1.286182069079718313546608
>>> appellf4(-2,-3,4,5,4,5)
34.8
>>> appellf4(5,4,2,3,0.25j,-0.125j)
(-0.2585967215437846642163352 + 2.436102233553582711818743j)
```

---

# Chapter 14

## Elliptic functions

Elliptic functions historically comprise the elliptic integrals and their inverses, and originate from the problem of computing the arc length of an ellipse. From a more modern point of view, an elliptic function is defined as a doubly periodic function, i.e. a function which satisfies

$$f(z + 2\omega_1) = f(z + 2\omega_2) = f(z) \quad (14.0.1)$$

for some half-periods  $\omega_1, \omega_2$  with  $\Im[\omega_1/\omega_2] > 0$ . The canonical elliptic functions are the Jacobi elliptic functions. More broadly, this section includes quasi-doubly periodic functions (such as the Jacobi theta functions) and other functions useful in the study of elliptic functions.

Many different conventions for the arguments of elliptic functions are in use. It is even standard to use different parametrizations for different functions in the same text or software (and mpFormulaPy is no exception). The usual parameters are the elliptic nome  $q$ , which usually must satisfy  $|q| < 1$ ; the elliptic parameter  $m$  (an arbitrary complex number); the elliptic modulus  $k$  (an arbitrary complex number); and the half-period ratio  $\tau$ , which usually must satisfy  $\Im[\tau] > 0$ . These quantities can be expressed in terms of each other using the following relations:

$$m = k^2; \quad \tau = i \frac{K(1-m)}{K(m)}; \quad q = e^{i\pi\tau}; \quad k = \frac{\vartheta_2^4(q)}{\vartheta_2^4(q)} \quad (14.0.2)$$

In addition, an alternative definition is used for the nome in number theory, which we here denote by  $\bar{q}$ :

$$\bar{q} = q^2 = e^{2i\pi\tau} \quad (14.0.3)$$

### 14.1 Elliptic arguments

For convenience, mpFormulaPy provides functions to convert between the various parameters (`qfrom()`, `mfrom()`, `kfrom()`, `taufrom()`, `qbarfrom()`).

---

#### Function `qfrom(Keywords As String) As mpNum`

---

The function `qfrom` returns the elliptic nome  $q$ .

**Parameter:**

*Keywords:* `m=x`; `k=x`; `tau=x`; `qbar=x`.

Returns the elliptic nome  $q$ , given any of  $m, k, \tau, \bar{q}$ :

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> qfrom(q=0.25)
0.25
>>> qfrom(m=mfrom(q=0.25))
0.25
>>> qfrom(k=kfrom(q=0.25))
0.25
>>> qfrom(tau=taufrom(q=0.25))
(0.25 + 0.0j)
>>> qfrom(qbar=qbarfrom(q=0.25))
0.25
```

---



---

### Function **qbarfrom(*Keywords As String*) As mpNum**

---

The function **qbarfrom** returns the number-theoretic nome  $\bar{q}$ .

**Parameter:**

*Keywords:* m=x; k=x; tau=x; q=x.

Returns the number-theoretic nome  $\bar{q}$ , given any of  $q, m, k, \tau$ :

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> qbarfrom(qbar=0.25)
0.25
>>> qbarfrom(q=qfrom(qbar=0.25))
0.25
>>> qbarfrom(m=extraprec(20)(mfrom)(qbar=0.25)) # ill-conditioned
0.25
>>> qbarfrom(k=extraprec(20)(kfrom)(qbar=0.25)) # ill-conditioned
0.25
>>> qbarfrom(tau=taufrom(qbar=0.25))
(0.25 + 0.0j)
```

---



---

### Function **mfrom(*Keywords As String*) As mpNum**

---

The function **mfrom** returns the elliptic parameter  $m$ .

**Parameter:**

*Keywords:* k=x; tau=x; q=x; qbar=x.

Returns the elliptic parameter  $m$ , given any of  $q, k, \tau, \bar{q}$ :

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> mfrom(m=0.25)
0.25
>>> mfrom(q=qfrom(m=0.25))
0.25
```

---

```
>>> mfrom(k=kfrom(m=0.25))
0.25
>>> mfrom(tau=taufrom(m=0.25))
(0.25 + 0.0j)
>>> mfrom(qbar=qbarfrom(m=0.25))
0.25
```

---



---

### Function **kfrom**(*Keywords As String*) As mpNum

---

The function kfrom returns the elliptic modulus  $k$ .

**Parameter:**

*Keywords:* m=x; tau=x; q=x; qbar=x.

Returns the elliptic modulus  $k$ , given any of  $q, m, \tau, \bar{q}$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> kfrom(k=0.25)
0.25
>>> kfrom(m=mfrom(k=0.25))
0.25
>>> kfrom(q=qfrom(k=0.25))
0.25
>>> kfrom(tau=taufrom(k=0.25))
(0.25 + 0.0j)
>>> kfrom(qbar=qbarfrom(k=0.25))
0.25
```

---



---

### Function **taufrom**(*Keywords As String*) As mpNum

---

The function taufrom returns the elliptic half-period ratio  $\tau$ .

**Parameter:**

*Keywords:* m=x; k=x; q=x; qbar=x.

Returns the elliptic half-period ratio  $\tau$ , given any of  $q, m, k, \bar{q}$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> taufrom(tau=0.5j)
(0.0 + 0.5j)
>>> taufrom(q=qfrom(tau=0.5j))
(0.0 + 0.5j)
>>> taufrom(m=mfrom(tau=0.5j))
(0.0 + 0.5j)
>>> taufrom(k=kfrom(tau=0.5j))
(0.0 + 0.5j)
>>> taufrom(qbar=qbarfrom(tau=0.5j))
(0.0 + 0.5j)
```

---

## 14.2 Legendre elliptic integrals

### 14.2.1 Complete elliptic integral of the first kind

---

Function **ellipk(*m* As mpNum)** As mpNum

---

The function **ellipk** returns the complete elliptic integral of the first kind,  $K(m)$ .

**Parameter:**

*m*: A real or complex number.

The complete elliptic integral of the first kind,  $K(m)$  is defined by

$$K(m) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - m \sin^2 t}} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, m\right) \quad (14.2.1)$$

Note that the argument is the parameter  $m = k^2$ , not the modulus  $k$  which is sometimes used.  
Values and limits include:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> ellipk(0)
1.570796326794896619231322
>>> ellipk(inf)
(0.0 + 0.0j)
>>> ellipk(-inf)
0.0
>>> ellipk(1)
+inf
>>> ellipk(-1)
1.31102877714605990523242
>>> ellipk(2)
(1.31102877714605990523242 - 1.31102877714605990523242j)
```

---

Evaluation is supported for arbitrary complex *m*:

---

```
>>> ellipk(3+4j)
(0.9111955638049650086562171 + 0.6313342832413452438845091j)
```

---

### 14.2.2 Incomplete elliptic integral of the first kind

---

Function **ellipf(*phi* As mpNum, *m* As mpNum)** As mpNum

---

The function **ellipf** returns the Legendre incomplete elliptic integral of the first kind  $F(\phi, m)$ .

**Parameters:**

*phi*: A real or complex number.

*m*: A real or complex number.

The Legendre incomplete elliptic integral of the first kind is defined as

$$F(\phi, m) = \int_0^\phi \frac{dt}{\sqrt{1 - m \sin^2 t}} \quad (14.2.2)$$

or equivalently

$$F(\phi, m) = \int_0^{\sin \phi} \frac{dt}{\sqrt{1-t^2}\sqrt{1-mt^2}} \quad (14.2.3)$$

The function reduces to a complete elliptic integral of the first kind (see `ellipk()`) when  $\phi = \pi/2$ ; that is,  $F(\pi/2) = K(m)$ .

In the defining integral, it is assumed that the principal branch of the square root is taken and that the path of integration avoids crossing any branch cuts. Outside  $-\pi/2 \leq \Re(\phi) \leq \pi/2$ , the function extends quasi-periodically as

$$F(\phi + n\pi, m) = 2nK(m) + F(\phi, m), n \in \mathbb{Z}. \quad (14.2.4)$$

Basic values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> ellipf(0,1)
0.0
>>> ellipf(0,0)
0.0
>>> ellipf(1,0); ellipf(2+3j,0)
1.0
(2.0 + 3.0j)
>>> ellipf(1,1); log(sec(1)+tan(1))
1.226191170883517070813061
1.226191170883517070813061
>>> ellipf(pi/2, -0.5); ellipk(-0.5)
1.415737208425956198892166
1.415737208425956198892166
>>> ellipf(pi/2+eps, 1); ellipf(-pi/2-eps, 1)
+inf
+inf
>>> ellipf(1.5, 1)
3.340677542798311003320813
```

---

Evaluation is supported for arbitrary complex  $m$ :

---

```
>>> ellipf(3j, 0.5)
(0.0 + 1.713602407841590234804143j)
>>> ellipf(3+4j, 5-6j)
(1.269131241950351323305741 - 0.3561052815014558335412538j)
>>> z,m = 2+3j, 1.25
>>> k = 1011
>>> ellipf(z+pi*k,m); ellipf(z,m) + 2*k*ellipk(m)
(4086.184383622179764082821 - 3003.003538923749396546871j)
(4086.184383622179764082821 - 3003.003538923749396546871j)
```

---

### 14.2.3 Complete elliptic integral of the second kind

---

Function **ellipe(*m* As mpNum) As mpNum**

---

The function `ellipe` returns the Legendre complete elliptic integral of the second kind  $E(m)$ .

**Parameter:**

*m*: A real or complex number.

The Legendre complete elliptic integral of the second kind is defined by

$$E(m) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - m \sin^2 t}} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}, 1, m\right) \quad (14.2.5)$$


---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> ellipe(0)
1.570796326794896619231322
>>> ellipe(1)
1.0
>>> ellipe(-1)
1.910098894513856008952381
>>> ellipe(2)
(0.5990701173677961037199612 + 0.5990701173677961037199612j)
>>> ellipe(inf)
(0.0 + +infj)
>>> ellipe(-inf)
+inf
```

---

Evaluation is supported for arbitrary complex *m*:

```
>>> ellipe(0.5+0.25j)
(1.360868682163129682716687 - 0.1238733442561786843557315j)
>>> ellipe(3+4j)
(1.499553520933346954333612 - 1.577879007912758274533309j)
```

---

### 14.2.4 Incomplete elliptic integral of the second kind

---

Function **ellipef(*phi* As mpNum, *m* As mpNum) As mpNum**

---

The function `ellipef` returns the Legendre incomplete elliptic integral of the second kind  $E(\phi, m)$ .

**Parameters:**

*phi*: A real or complex number.

*m*: A real or complex number.

The incomplete elliptic integral of the second kind

$$E(\phi, m) = \int_0^\phi \frac{dt}{\sqrt{1 - m \sin^2 t}} = \int_0^{\sin \phi} \frac{\sqrt{1 - mt^2}}{\sqrt{1 - t^2}} dt \quad (14.2.6)$$

The incomplete integral reduces to a complete integral when  $\phi = \pi/2$ ; that is,  $E(\pi/2, m) = E(m)$ .

In the defining integral, it is assumed that the principal branch of the square root is taken and that the path of integration avoids crossing any branch cuts. Outside  $-\pi/2 \leq \Re(\phi) \leq \pi/2$ , the function extends quasi-periodically as

$$E(\phi + n\pi, m) = 2nE(m) + F(\phi, m), n \in \mathbb{Z}. \quad (14.2.7)$$

Basic values and limits:

---

```
>>> ellipe(0,1)
0.0
>>> ellipe(0,0)
0.0
>>> ellipe(1,0)
1.0
>>> ellipe(2+3j,0)
(2.0 + 3.0j)
>>> ellipe(1,1); sin(1)
0.8414709848078965066525023
0.8414709848078965066525023
>>> ellipe(pi/2, -0.5); ellipe(-0.5)
1.751771275694817862026502
1.751771275694817862026502
>>> ellipe(pi/2, 1); ellipe(-pi/2, 1)
1.0
-1.0
>>> ellipe(1.5, 1)
0.9974949866040544309417234
```

---

Evaluation is supported for arbitrary complex  $m$ :

---

```
>>> ellipe(0.5+0.25j)
>>> ellipe(3j, 0.5)
(0.0 + 7.551991234890371873502105j)
>>> ellipe(3+4j, 5-6j)
(24.15299022574220502424466 + 75.2503670480325997418156j)
>>> k = 35
>>> z,m = 2+3j, 1.25
>>> ellipe(z+pi*k,m); ellipe(z,m) + 2*k*ellipe(m)
(48.30138799412005235090766 + 17.47255216721987688224357j)
(48.30138799412005235090766 + 17.47255216721987688224357j)
```

---

### 14.2.5 Complete elliptic integral of the third kind

---

Function **ellippi(*n* As mpNum, *m* As mpNum) As mpNum**

---

The function **ellippi** returns the complete elliptic integral of the third kind  $\Pi(n, m)$ .

**Parameters:**

*n*: A real or complex number.

*m*: A real or complex number.

The complete elliptic integral of the third kind is defined as

$$\Pi(n, m) = \Pi(n; \pi/2, m).$$

Basic values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> ellippi(0,-5); ellipk(-5)
0.9555039270640439337379334
0.9555039270640439337379334
>>> ellippi(inf,2)
0.0
>>> ellippi(2,inf)
0.0
>>> abs(ellippi(1,5))
+inf
>>> abs(ellippi(0.25,1))
+inf
```

---

### 14.2.6 Incomplete elliptic integral of the third kind

---

Function **ellippif(*n* As mpNum, *phi* As mpNum, *m* As mpNum) As mpNum**

---

The function **ellippif** returns the Legendre incomplete elliptic integral of the third kind  $\Pi(n; \phi, m)$ .

**Parameters:**

*n*: A real or complex number.

*phi*: A real or complex number.

*m*: A real or complex number.

The Legendre incomplete elliptic integral of the third kind is defined as

$$\Pi(n; \phi, m) = \int_0^\phi \frac{dt}{(1 - n \sin^2 t) \sqrt{1 - m \sin^2 t}} = \int_0^{\sin \phi} \frac{dt}{(1 - nt^2) \sqrt{1 - t^2} \sqrt{1 - mt^2}} \quad (14.2.8)$$

In the defining integral, it is assumed that the principal branch of the square root is taken and that the path of integration avoids crossing any branch cuts. Outside  $-\pi/2 \leq \Re(\phi) \leq \pi/2$ , the function extends quasi-periodically as

$$\Pi(n, \phi + k\pi, m) = 2k\Pi(n, m) + \Pi(n, \phi, m), k \in \mathbb{Z}. \quad (14.2.9)$$

Basic values and limits:

---

```
>>> ellippi(0.25,-0.5); ellippi(0.25,pi/2,-0.5)
1.622944760954741603710555
1.622944760954741603710555
>>> ellippi(1,0,1)
0.0
>>> ellippi(inf,0,1)
0.0
>>> ellippi(0,0.25,0.5); ellipf(0.25,0.5)
0.2513040086544925794134591
0.2513040086544925794134591
>>> ellippi(1,1,1); (log(sec(1)+tan(1))+sec(1)*tan(1))/2
2.054332933256248668692452
```

```
2.054332933256248668692452
>>> ellippi(0.25, 53*pi/2, 0.75); 53*ellippi(0.25,0.75)
135.240868757890840755058
135.240868757890840755058
>>> ellippi(0.5,pi/4,0.5); 2*ellipe(pi/4,0.5)-1/sqrt(3)
0.9190227391656969903987269
0.9190227391656969903987269
```

---

Complex arguments are supported:

```
>>> ellippi(0.5, 5+6j-2*pi, -7-8j)
(-0.3612856620076747660410167 + 0.5217735339984807829755815j)
```

---

## 14.3 Carlson symmetric elliptic integrals

The Carlson style elliptic integrals are a complete alternative group to the classical Legendre style integrals. They are symmetric and the numerical calculation is usually performed by duplication as described in [Carlson & Gustafson \(1994\)](#) and [Carlson \(1995\)](#).

### 14.3.1 Symmetric elliptic integral of the first kind, RF

---

Function **elliprf(x As mpNum, y As mpNum, z As mpNum) As mpNum**

---

The function `elliprf` returns the Carlson symmetric elliptic integral of the first kind.

**Parameters:**

- x*: A real or complex number.
- y*: A real or complex number.
- z*: A real or complex number.

The Carlson symmetric elliptic integral of the first kind is given by

$$R_F(x, y, z) = \frac{1}{2} \int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}} \quad (14.3.1)$$

which is defined for  $x, y, z \in (-\infty, 0)$ , and with at most one of  $x, y, z$  being zero.

For real  $x, y, z \geq 0$ , the principal square root is taken in the integrand. For complex  $x, y, z$ , the principal square root is taken as  $t \rightarrow \infty$  and as  $t \rightarrow 0$  non-principal branches are chosen as necessary so as to make the integrand continuous.

Basic values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> elliprf(0,1,1); pi/2
1.570796326794896619231322
1.570796326794896619231322
>>> elliprf(0,1,inf)
0.0
>>> elliprf(1,1,1)
1.0
>>> elliprf(2,2,2)**2
0.5
>>> elliprf(1,0,0); elliprf(0,0,1); elliprf(0,1,0); elliprf(0,0,0)
+inf
+inf
+inf
+inf
```

---

With the following arguments, the square root in the integrand becomes discontinuous at  $t = 1/2$  if the principal branch is used. To obtain the right value,  $-\sqrt{r}$  must be taken instead of  $\sqrt{r}$  on  $t \in (0, 1/2)$ :

---

```
>>> x,y,z = j-1,j,0
>>> elliprf(x,y,z)
(0.7961258658423391329305694 - 1.213856669836495986430094j)
```

---

```
>>> -q(f, [0,0.5]) + q(f, [0.5,inf])
(0.7961258658423391329305694 - 1.213856669836495986430094j)
```

---

### 14.3.2 Degenerate Carlson symmetric elliptic integral of the first kind, RC

---

Function **elliprc**(*x* As mpNum, *y* As mpNum, **Keywords** As String) As mpNum

---

The function **elliprc** returns the degenerate Carlson symmetric elliptic integral of the first kind.

**Parameters:**

*x*: A real or complex number.

*y*: A real or complex number.

**Keywords**: pv=True.

The degenerate Carlson symmetric elliptic integral of the first kind is given by

$$R_C(x, y) = R_F(x, y, y) = \frac{1}{2} \int_0^\infty \frac{dt}{(t+y)\sqrt{(t+x)}} \quad (14.3.2)$$

If  $y \in (-\infty, 0)$ , either a value defined by continuity, or with pv=True the Cauchy principal value, can be computed.

If  $x \geq 0, y > 0$ , the value can be expressed in terms of elementary functions as

$$R_C(x, y) = \begin{cases} \frac{1}{\sqrt{y-x}} \cos^{-1} \left( \sqrt{x/y} \right), & x < y \\ \frac{1}{\sqrt{y}}, & x = y \\ \frac{1}{\sqrt{y-x}} \cosh^{-1} \left( \sqrt{x/y} \right), & x > y \end{cases} \quad (14.3.3)$$

Examples

Some special values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> elliprc(1,2)*4; elliprc(0,1)*2; +pi
3.141592653589793238462643
3.141592653589793238462643
3.141592653589793238462643
>>> elliprc(1,0)
+inf
>>> elliprc(5,5)**2
0.2
>>> elliprc(1,inf); elliprc(inf,1); elliprc(inf,inf)
0.0
0.0
0.0
```

---

Comparing with numerical integration:

---

```
>>> q = extradps(25)(quad)
>>> elliprc(2, -3, pv=True)
0.3333969101113672670749334
```

---

```
>>> elliprc(2, -3, pv=False)
(0.3333969101113672670749334 + 0.7024814731040726393156375j)
>>> 0.5*q(lambda t: 1/(sqrt(t+2)*(t-3)), [0,3-j,6,inf])
(0.3333969101113672670749334 + 0.7024814731040726393156375j)
```

---

### 14.3.3 Symmetric elliptic integral of the third kind, RJ

---

Function **elliprj(*x As mpNum, y As mpNum, z As mpNum, p As mpNum*) As mpNum**

---

The function `elliprj` returns the Carlson symmetric elliptic integral of the third kind.

**Parameters:**

- x*: A real or complex number.
- y*: A real or complex number.
- z*: A real or complex number.
- p*: A real or complex number.

The Carlson symmetric elliptic integral of the third kind is given by

$$R_J(x, y, z, p) = \frac{3}{2} \int_0^{\infty} \frac{dt}{(t+p)\sqrt{(t+x)(t+y)(t+z)}} \quad (14.3.4)$$

Like `elliprf()`, the branch of the square root in the integrand is defined so as to be continuous along the path of integration for complex values of the arguments.

**Examples**

Some values and limits:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> elliprj(1,1,1,1)
1.0
>>> elliprj(2,2,2,2); 1/(2*sqrt(2))
0.3535533905932737622004222
0.3535533905932737622004222
>>> elliprj(0,1,2,2)
1.067937989667395702268688
>>> 3*(2*gamma('5/4')**2-pi**2/gamma('1/4')**2)/(sqrt(2*pi))
1.067937989667395702268688
>>> elliprj(0,1,1,2); 3*pi*(2-sqrt(2))/4
1.380226776765915172432054
1.380226776765915172432054
>>> elliprj(1,3,2,0); elliprj(0,1,1,0); elliprj(0,0,0,0)
+inf
+inf
+inf
>>> elliprj(1,inf,1,0); elliprj(1,1,1,inf)
0.0
0.0
>>> chop(elliprj(1+j, 1-j, 1, 1))
0.8505007163686739432927844
```

---

Comparing with numerical integration:

---

```
>>> elliprj(1,2,3,4)
0.2398480997495677621758617
>>> f = lambda t: 1/((t+4)*sqrt((t+1)*(t+2)*(t+3)))
>>> 1.5*quad(f, [0,inf])
0.2398480997495677621758617
>>> elliprj(1,2+1j,3,4-2j)
(0.216888906014633498739952 + 0.04081912627366673332369512j)
>>> f = lambda t: 1/((t+4-2j)*sqrt((t+1)*(t+2+1j)*(t+3)))
>>> 1.5*quad(f, [0,inf])
(0.216888906014633498739952 + 0.04081912627366673332369511j)
```

---

#### 14.3.4 Symmetric elliptic integral of the second kind, RD

---

Function **elliprd**(*x As mpNum, y As mpNum, z As mpNum*) As mpNum

---

The function **elliprd** returns the Carlson symmetric elliptic integral of the second kind.

**Parameters:**

*x*: A real or complex number.

*y*: A real or complex number.

*z*: A real or complex number.

Evaluates the degenerate Carlson symmetric elliptic integral of the third kind or Carlson elliptic integral of the second kind  $R_D(x, y, z) = R_j(x, y, z, z)$ .

See **elliprj()** for additional information.

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> elliprd(1,2,3)
0.2904602810289906442326534
>>> elliprj(1,2,3,3)
0.2904602810289906442326534
```

---

#### 14.3.5 Completely symmetric elliptic integral of the second kind, RG

---

Function **elliprg**(*x As mpNum, y As mpNum, z As mpNum*) As mpNum

---

The function **elliprg** returns the Carlson completely symmetric elliptic integral of the second kind.

**Parameters:**

*x*: A real or complex number.

*y*: A real or complex number.

*z*: A real or complex number.

The Carlson completely symmetric elliptic integral of the second kind is defined as

$$R_G(x, y, z) = \frac{1}{4} \int_0^\infty \frac{t}{\sqrt{(t+x)(t+y)(t+z)}} \left( \frac{x}{t+x} + \frac{y}{t+y} + \frac{z}{t+z} \right) dt \quad (14.3.5)$$

Evaluation for real and complex arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> elliprg(0,1,1)*4; +pi
3.141592653589793238462643
3.141592653589793238462643
>>> elliprg(0,0.5,1)
0.6753219405238377512600874
>>> chop(elliprg(1+j, 1-j, 2))
1.172431327676416604532822
```

---

## 14.4 Jacobi theta functions

---

Function **jtheta(*n* As mpNum, *z* As mpNum, *q* As mpNum, **Keywords** As String) As mpNum**

---

The function jtheta returns the Jacobi theta function  $\vartheta_n(z, q)$ .

**Parameters:**

*n*: An integer, where  $n = 1, 2, 3, 4$ .

*z*: A real or complex number.

*q*: A real or complex number.

**Keywords**: derivative=0.

The Jacobi theta function  $\vartheta_n(z, q)$ , where  $n = 1, 2, 3, 4$ , is defined by the infinite series:

$$\vartheta_1(z, q) = 2q^{1/4} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n+1)z \quad (14.4.1)$$

$$\vartheta_2(z, q) = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} \cos(2n+1)z \quad (14.4.2)$$

$$\vartheta_3(z, q) = 1 + 2 \sum_{n=0}^{\infty} q^{n^2} \cos(2nz) \quad (14.4.3)$$

$$\vartheta_4(z, q) = 1 + 2 \sum_{n=0}^{\infty} (-1)^n q^{n^2} \cos(2nz) \quad (14.4.4)$$

The theta functions are functions of two variables:

*z* is the argument, an arbitrary real or complex number

*q* is the nome, which must be a real or complex number in the unit disk (i.e.  $|q| < 1$ ). For  $|q| \ll 1$ , the series converge very quickly, so the Jacobi theta functions can efficiently be evaluated to high precision.

The compact notations  $\vartheta_n(q) = \vartheta_n(0, q)$  and  $\vartheta_n = \vartheta_n(0, q)$  are also frequently encountered. Finally, Jacobi theta functions are frequently considered as functions of the half-period ratio  $\tau$  and then usually denoted by  $\vartheta_n(z|\tau)$ .

Optionally, jtheta(*n*, *z*, *q*, derivative=*d*) with  $d > 0$  computes a *d*-th derivative with respect to *z*. Examples and basic properties

Considered as functions of *z*, the Jacobi theta functions may be viewed as generalizations of the ordinary trigonometric functions cos and sin. They are periodic functions:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> jtheta(1, 0.25, '0.2')
0.2945120798627300045053104
>>> jtheta(1, 0.25 + 2*pi, '0.2')
0.2945120798627300045053104
```

---

jtheta() supports arbitrary precision evaluation and complex arguments:

```
>>> mp.dps = 50
>>> jtheta(4, sqrt(2), 0.5)
2.0549510717571539127004115835148878097035750653737
>>> mp.dps = 25
```

```
>>> jtheta(4, 1+2j, (1+j)/5)
(7.180331760146805926356634 - 1.634292858119162417301683j)
```

---

## 14.5 Jacobi elliptic functions

These procedures return the Jacobi elliptic functions  $\text{sn}$ ,  $\text{cn}$ ,  $\text{dn}$  for argument  $x$  and complementary parameter  $m_c$ . A convenient implicit definition of the functions is

$$x = \int_0^{\text{sn}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad \text{sn}^2 + \text{cn}^2 = 1, \quad k^2\text{sn}^2 + \text{cn}^2 = 1 \quad (14.5.1)$$

with  $k^2 = 1 - m_c$ . There are a lot of equivalent definition of the Jacobi elliptic functions, e.g. with the Jacobi amplitude function (see e.g. [Olver et al. \(2010\)](#) [30, 22.16.11/12])

$$\begin{aligned}\text{sn}(x, k) &= \sin(\text{am}(x, k)), \\ \text{cn}(x, k) &= \cos(\text{am}(x, k)),\end{aligned}$$

or with Jacobi theta functions (cf. [\[Olver et al. \(2010\), 22.2\]](#)).

**Function `ellipfun(kind As String, u As mpNum, m As mpNum, Keywords As String) As mpNum`**

The function `ellipfun` returns any of the Jacobi elliptic functions.

### Parameters:

*kind*: A function identifier.

*u*: A real or complex number.

*m*: A real or complex number.

*Keywords*: q=None, k=None, tau=None.

Computes any of the Jacobi elliptic functions, defined in terms of Jacobi theta functions as

$$\text{sn}(u, m) = \frac{\vartheta_3(0, q)\vartheta_1(t, q)}{\vartheta_2(0, q)\vartheta_4(t, q)} \quad (14.5.2)$$

$$\text{cn}(u, m) = \frac{\vartheta_4(0, q)\vartheta_2(t, q)}{\vartheta_2(0, q)\vartheta_4(t, q)} \quad (14.5.3)$$

$$\text{dn}(u, m) = \frac{\vartheta_4(0, q)\vartheta_3(t, q)}{\vartheta_3(0, q)\vartheta_4(t, q)} \quad (14.5.4)$$

or more generally computes a ratio of two such functions. Here  $t = u/\vartheta_3(0, q)^2$ , and  $q = q(m)$  denotes the nome (see `nome()`). Optionally, you can specify the nome directly instead of by passing `q=jvalue`, or you can directly specify the elliptic parameter with `k=jvalue`.

The first argument should be a two-character string specifying the function using any combination of 's', 'c', 'd', 'n'. These letters respectively denote the basic functions  $\text{sn}(u, m)$ ,  $\text{cn}(u, m)$ ,  $\text{dn}(u, m)$ , and 1. The identifier specifies the ratio of two such functions. For example, 'ns' identifies the function

$$\text{cd}(u, m) = \frac{1}{\text{sn}(u, m)} \quad (14.5.5)$$

and 'cd' identifies the function

$$\text{ns}(u, m) = \frac{\text{cn}(u, m)}{\text{dn}(u, m)} \quad (14.5.6)$$

If called with only the first argument, a function object evaluating the chosen function for given arguments is returned.

Examples

Basic evaluation

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> ellipfun('cd', 3.5, 0.5)
-0.9891101840595543931308394
>>> ellipfun('cd', 3.5, q=0.25)
0.07111979240214668158441418
```

---

## 14.6 Klein j-invariant

---

Function **kleinj(*tau*** As mpNum) As mpNum

---

The function `kleinj` returns the Klein j-invariant.

**Parameter:**

*tau*: A real or complex number.

The Klein j-invariant is a modular function defined for  $\tau$  in the upper half-plane as

$$J(\tau) = \frac{g_2^3(\tau)}{g_2^3(\tau) - 27g_3^2(\tau)} \quad (14.6.1)$$

where  $g_2$  and  $g_3$  are the modular invariants of the Weierstrass elliptic function,

$$g_2(\tau) = 60 \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} (m\tau + n)^{-4} \quad (14.6.2)$$

$$g_3(\tau) = 140 \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} (m\tau + n)^{-6} \quad (14.6.3)$$

An alternative, common notation is that of the j-function  $j(\tau) = 1728J(\tau)$ .

Examples

Verifying the functional equation  $J(\tau) = J(\tau + 1) = J(-\tau^{-1})$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> tau = 0.625+0.75*j
>>> tau = 0.625+0.75*j
>>> kleinj(tau)
(-0.1507492166511182267125242 + 0.07595948379084571927228948j)
>>> kleinj(tau+1)
(-0.1507492166511182267125242 + 0.07595948379084571927228948j)
>>> kleinj(-1/tau)
(-0.1507492166511182267125242 + 0.07595948379084571927228946j)
```

---

# Chapter 15

## Zeta functions, L-series and polylogarithms

This section includes the Riemann zeta functions and associated functions pertaining to analytic number theory.

### 15.1 Riemann and Hurwitz zeta functions

---

Function **zeta(s As mpNum, Keywords As String)** As mpNum

---

The function **zeta** returns the Riemann zeta function

**Parameters:**

**s:** A real or complex number.

**Keywords:** derivative=0.

---

Function **hurwitz(s As mpNum, a As mpNum, Keywords As String)** As mpNum

---

The function **hurwitz** returns the Hurwitz zeta function

**Parameters:**

**s:** A real or complex number.

**a:** A real or complex number.

**Keywords:** derivative=0.

Computes the Riemann zeta function or the Hurwitz zeta function.

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots \quad (15.1.1)$$

or, with  $a \neq 1$ , the more general Hurwitz zeta function

$$\zeta(s, a) = \sum_{k=0}^{\infty} \frac{1}{(a+k)^s}. \quad (15.1.2)$$

Optionally, **zeta(s, a, n)** computes the  $n$ -th derivative with respect to  $s$ ,

$$\zeta^{(n)}(s, a) = (-1)^n \sum_{k=0}^{\infty} \frac{\log^n(a+k)}{(a+k)^s}. \quad (15.1.3)$$

Although these series only converge for  $\Re(s) > 1$ , the Riemann and Hurwitz zeta functions are defined through analytic continuation for arbitrary complex  $s \neq 1$  ( $s = 1$  is a pole).

The implementation uses three algorithms: the Borwein algorithm for the Riemann zeta function when  $s$  is close to the real line; the Riemann-Siegel formula for the Riemann zeta function when  $s$  is large imaginary, and Euler-Maclaurin summation in all other cases. The reflection formula for  $\Re(s) < 0$  is implemented in some cases. The algorithm can be chosen with method = 'borwein', method='riemann-siegel' or method = 'euler-maclaurin'.

The parameter  $a$  is usually a rational number  $a = p/q$ , and may be specified as such by passing an integer tuple  $(p, q)$ . Evaluation is supported for arbitrary complex  $a$ , but may be slow and/or inaccurate when  $\Re(s) < 0$  for nonrational  $a$  or when computing derivatives.

### Examples

Some values of the Riemann zeta function:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> zeta(2); pi**2 / 6
1.644934066848226436472415
1.644934066848226436472415
>>> zeta(0)
-0.5
>>> zeta(-1)
-0.0833333333333333333333333
>>> zeta(-2)
0.0
```

---

Evaluation is supported for complex  $s$  and  $a$ :

---

```
>>> zeta(-3+4j)
(-0.03373057338827757067584698 + 0.2774499251557093745297677j)
>>> zeta(2+3j, -1+j)
(389.6841230140842816370741 + 295.2674610150305334025962j)
```

---

Some values of the Hurwitz zeta function:

---

```
>>> zeta(2, 3); -5./4 + pi**2/6
0.3949340668482264364724152
0.3949340668482264364724152
>>> zeta(2, (3,4)); pi**2 - 8*catalan
2.541879647671606498397663
2.541879647671606498397663
```

---

## 15.2 Dirichlet L-series

### 15.2.1 Dirichlet eta function

---

Function **altzeta(s As mpNum) As mpNum**

---

The function `altzeta` returns the Dirichlet eta function,  $\eta(s)$

**Parameter:**

*s*: A real or complex number.

The Dirichlet eta function,  $\eta(s)$  is also known as the alternating zeta function. This function is defined in analogy with the Riemann zeta function as providing the sum of the alternating series

$$\eta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots \quad (15.2.1)$$

The eta function, unlike the Riemann zeta function, is an entire function, having a finite value for all complex *s*. The special case  $\eta(1) = \log(2)$  gives the value of the alternating harmonic series.

The alternating zeta function may be expressed using the Riemann zeta function as

$\eta(s) = (1 - 2^{1-s})\zeta(s)$ . It can also be expressed in terms of the Hurwitz zeta function, for example using `dirichlet()` (see documentation for that function).

Examples

Some special values are:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> altzeta(1)
0.693147180559945
>>> altzeta(0)
0.5
>>> altzeta(-1)
0.25
>>> altzeta(-2)
0.0
```

---

### 15.2.2 Dirichlet $\eta(s) - 1$

---

Function **DirichletEtam1MpMath(x As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function `DirichletEtam1MpMath` returns the Dirichlet function  $\eta(s) - 1$ .

**Parameter:**

*x*: A real number.

This function returns the Dirichlet function  $\eta(s) - 1$ .

### 15.2.3 Dirichlet Beta Function

---

Function **DirichletBetaMpMath(s As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function `DirichletBetaMpMath` returns the Dirichlet function  $\beta(s)$ .

**Parameter:**

*s*: A real number.

This function returns the Dirichlet function  $\beta(s)$ , defined for  $s > 0$  as

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^s} = 2^{-s}\Phi\left(-1, s, \frac{1}{2}\right) \quad (15.2.2)$$

### 15.2.4 Dirichlet Lambda Function

---

Function **DirichletLambdaMpMath(s As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function `DirichletLambdaMpMath` returns the Dirichlet function  $\beta(s)$ .

**Parameter:**

*s*: A real number.

This function returns the Dirichlet function  $\lambda(s)$ , defined for  $s > 0$  as

$$\lambda(s) = \sum_{n=0}^{\infty} (2n+1)^s \quad (15.2.3)$$

and by analytic continuation for  $s < 1$ . The function is calculated as

$$\lambda(s) = (1 - 2^{-s})\eta(s). \quad (15.2.4)$$

### 15.2.5 Dirichlet L-function

---

Function **dirichlet(s As mpNum, chi As mpNum, Keywords As mpNum) As mpNum**

---

The function `dirichlet` returns the Dirichlet L-function

**Parameters:**

*s*: A real or complex number.

*chi*: A periodic sequence.

*Keywords*: derivative=0.

The Dirichlet L-function is defined as

$$L(s, \chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s} \quad (15.2.5)$$

where  $\chi$  is a periodic sequence of length  $q$  which should be supplied in the form of a list  $[\chi(0), \chi(1), \dots, \chi(q-1)]$ . Strictly,  $\chi$  should be a Dirichlet character, but any periodic sequence will work.

For example, `dirichlet(s, [1])` gives the ordinary Riemann zeta function and `dirichlet(s, [-1,1])` gives the alternating zeta function (Dirichlet eta function).

Also the derivative with respect to (currently only a first derivative) can be evaluated.

Examples

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> dirichlet(3, [1]); zeta(3)
1.202056903159594285399738
1.202056903159594285399738
>>> dirichlet(1, [1])
+inf
```

---

## 15.3 Stieltjes constants

Function **stieltjes(*n* As mpNum, *a* As mpNum) As mpNum**

The function `stieltjes` returns the  $n$ -th Stieltjes constant

**Parameters:**

*n*: A real or complex number.

*a*: A real or complex number.

For a nonnegative integer  $n$ , `stieltjes(n)` computes the  $n$ -th Stieltjes constant  $\gamma_n$ , defined as the  $n$ -th coefficient in the Laurent series expansion of the Riemann zeta function around the pole at  $s = 1$ . That is, we have:

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n \quad (15.3.1)$$

More generally, `stieltjes(n, a)` gives the corresponding coefficient  $\gamma_n(a)$  for the Hurwitz zeta function  $\zeta(s, a)$  (with  $\gamma_n = \gamma_n(1)$ ).

`stieltjes()` numerically evaluates the integral in the following representation due to Ainsworth, Howell and Coffey [1], [2]:

$$\gamma_n(a) = \frac{\log^n a}{2a} \frac{\log^{n+1}(a)}{n+1} + \frac{2}{a} \Re \int_0^\infty \frac{(x/a - i) \log^n(a - ix)}{(1 + x^2/a^2)(e^{2\pi x} - 1)} dx \quad (15.3.2)$$

Examples

The zeroth Stieltjes constant is just Euler's constant  $\gamma$ :

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> stieltjes(0)
0.577215664901533
```

Some more values are:

```
>>> stieltjes(1)
-0.0728158454836767
>>> stieltjes(10)
0.000205332814909065
>>> stieltjes(30)
0.00355772885557316
>>> stieltjes(1000)
-1.57095384420474e+486
>>> stieltjes(2000)
2.680424678918e+1109
>>> stieltjes(1, 2.5)
-0.23747539175716
```

## 15.4 Zeta function zeros

These functions are used for the study of the Riemann zeta function in the critical strip.

---

### Function `zetazero(n As mpNum, Keywords As String) As mpNum`

---

The function `zetazero` returns the  $n$ -th nontrivial zero of  $\zeta(s)$  on the critical line

**Parameters:**

$n$ : An integer.

*Keywords*: `verbose=False`.

Computes the  $n$ -th nontrivial zero of  $\zeta(s)$  on the critical line, i.e. returns an approximation of the  $n$ -th largest complex number  $s = \frac{1}{2} + ti$  for which  $\zeta(s) = 0$ .

Equivalently, the imaginary part  $t$  is a zero of the Z-function (`siegelz()`).

Examples

The first few zeros:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> zetazero(1)
(0.5 + 14.13472514173469379045725j)
>>> zetazero(2)
(0.5 + 21.02203963877155499262848j)
>>> zetazero(20)
(0.5 + 77.14484006887480537268266j)
```

---

Verifying that the values are zeros:

---

```
>>> for n in range(1,5):
...     s = zetazero(n)
...     chop(zeta(s)), chop(siegelz(s.imag))
...
(0.0, 0.0)
(0.0, 0.0)
(0.0, 0.0)
(0.0, 0.0)
```

---



---

### Function `nzeros(t As mpNum) As mpNum`

---

The function `nzeros` returns the number of zeros of the Riemann zeta function in  $(0, 1) \times (0, t)$ , usually denoted by  $N(t)$ .

**Parameter:**

$t$ : An integer.

Examples

The first zero has imaginary part between 14 and 15:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> nzeros(14)
0
>>> nzeros(15)
```

```
1
>>> zetazero(1)
(0.5 + 14.1347251417347j)
```

---

## 15.5 Riemann-Siegel Z function and related functions

### 15.5.1 Riemann-Siegel Z

---

Function **siegelz(*t* As mpNum)** As mpNum

---

The function `siegelz` returns the Riemann-Siegel Z function

**Parameter:**

*t*: A real or complex number.

The Riemann-Siegel Z function is defined as

$$Z(t) = e^{i\theta(t)} \zeta(1/2 + it) \quad (15.5.1)$$

where  $\zeta(s)$  is the Riemann zeta function (`zeta()`) and where  $\theta(t)$  denotes the Riemann-Siegel theta function (see `siegeltheta()`).

Evaluation is supported for real and complex arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> siegelz(1)
-0.7363054628673177346778998
>>> siegelz(3+4j)
(-0.1852895764366314976003936 - 0.2773099198055652246992479j)
```

---

The first four derivatives are supported, using the optional derivative keyword argument:

---

```
>>> siegelz(1234567, derivative=3)
56.89689348495089294249178
>>> diff(siegelz, 1234567, n=3)
56.89689348495089294249178
```

---

### 15.5.2 Riemann-Siegel theta function

---

Function **siegeltheta(*t* As mpNum)** As mpNum

---

The function `siegeltheta` returns the Riemann-Siegel theta function

**Parameter:**

*t*: A real or complex number.

The Riemann-Siegel theta function is defined as

$$\theta(t) = \frac{\log \Gamma(\frac{1+2it}{4}) - \log \Gamma(\frac{1-2it}{4})}{2i} - \frac{\log \pi}{2} t. \quad (15.5.2)$$

The Riemann-Siegel theta function is important in providing the phase factor for the Zfunction (see `siegelz()`). Evaluation is supported for real and complex arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> siegeltheta(0)
0.0
```

---

---

```
>>> siegeltheta(inf)
+inf
>>> siegeltheta(-inf)
-inf
>>> siegeltheta(1)
-1.767547952812290388302216
>>> siegeltheta(10+0.25j)
(-3.068638039426838572528867 + 0.05804937947429712998395177j)
```

---

Arbitrary derivatives may be computed with derivative = k

---

```
>>> siegeltheta(1234, derivative=2)
0.0004051864079114053109473741
>>> diff(siegeltheta, 1234, n=2)
0.0004051864079114053109473741
```

---

### 15.5.3 Gram point (Riemann-Siegel Z function)

---

Function **grampoint(*n* As mpNum) As mpNum**

---

The function **grampoint** returns the *n*-th Gram point  $g_n$ , defined as the solution to the equation  $\theta(g_n) = \pi n$  where  $\theta(t)$  is the Riemann-Siegel theta function

**Parameter:**

*n*: A real or complex number.

The first few Gram points are:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> grampoint(0)
17.84559954041086081682634
>>> grampoint(1)
23.17028270124630927899664
>>> grampoint(2)
27.67018221781633796093849
>>> grampoint(3)
31.71797995476405317955149
```

---

### 15.5.4 Backlunds function

---

Function **backlunds(*t* As mpNum) As mpNum**

---

The function **backlunds** returns the function  $S(t) = \arg\zeta\left(\frac{1}{2} + it\right)/\pi$ .

**Parameter:**

*t*: A real or complex number.

See Titchmarsh Section 9.3 for details of the definition.

Examples

---

```
>>> from mpFormulaPy import *
```

```
>>> mp.dps = 15; mp.pretty = True
>>> backlunds(217.3)
0.16302205431184
```

## 15.6 Lerch transcendent and related functions

Function **lerchphi(z As mpNum, s As mpNum, a As mpNum) As mpNum**

The function `lerchphi` returns the Lerch transcendent

**Parameters:**

*z*: A real or complex number.

*s*: A real or complex number.

*a*: A real or complex number.

The Lerch transcendent, defined for  $|z| < 1$  and  $\Re a > 0$ , is given by

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s} \quad (15.6.1)$$

and generally by the recurrence  $\Phi(z, s, a) = z\Phi(z, s, a+1) + a^{-s}$  along with the integral representation valid for  $\Re a > 0$

$$\Phi(z, s, a) = \frac{1}{2a^s} + \int_0^\infty \frac{z^t}{(a+t)^s} dt - 2 \int_0^\infty \frac{\sin(t \log z) - s \arctan(t/a)}{(a^2 + t^2)^{s/2} (e^{2\pi t} - 1)} dt. \quad (15.6.2)$$

The Lerch transcendent generalizes the Hurwitz zeta function `zeta()` ( $z = 1$ ) and the polylogarithm `polylog()` ( $a = 1$ ).

**Examples**

Several evaluations in terms of simpler functions:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> lerchphi(-1,2,0.5); 4*catalan
3.663862376708876060218414
3.663862376708876060218414
>>> diff(lerchphi, (-1,-2,1), (0,1,0)); 7*zeta(3)/(4*pi**2)
0.2131391994087528954617607
0.2131391994087528954617607
>>> lerchphi(-4,1,1); log(5)/4
0.4023594781085250936501898
0.4023594781085250936501898
>>> lerchphi(-3+2j,1,0.5); 2*atanh(sqrt(-3+2j))/sqrt(-3+2j)
(1.142423447120257137774002 + 0.2118232380980201350495795j)
(1.142423447120257137774002 + 0.2118232380980201350495795j)
```

Evaluation works for complex arguments and  $|z| \geq 1$ :

```
>>> lerchphi(1+2j, 3-j, 4+2j)
(0.002025009957009908600539469 + 0.003327897536813558807438089j)
>>> lerchphi(-2,2,-2.5)
-12.28676272353094275265944
>>> lerchphi(10,10,10)
(-4.462130727102185701817349e-11 + 1.575172198981096218823481e-12j)
>>> lerchphi(10,10,-10.5)
(112658784011940.5605789002 + 498113185.5756221777743631j)
```

### 15.6.1 Fermi-Dirac integrals of integer order

---

Function **FermiDiracIntMpMath(x As mpNum, n As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function FermiDiracIntMpMath returns the complete Fermi-Dirac integrals  $F_n(x)$  of integer order.

**Parameters:**

*x*: A real number.

*n*: An Integer.

This function returns the complete Fermi-Dirac integrals  $F_n(x)$  of integer order. They are defined for real orders  $s > -1$  by

$$F_s(x) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{t^s}{e^{tx} + 1} dt \quad (15.6.3)$$

and by analytic continuation for  $s \leq -1$  using polylogarithms

$$F_s(x) = -\text{Li}_{s+1}(-e^x) = e^x \Phi(-e^x, s+1, 1). \quad (15.6.4)$$

### 15.6.2 Fermi-Dirac integral $F_{-1/2}(x)$

---

Function **FermiDiracPHalfMpMath(s As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function FermiDiracPHalfMpMath returns the complete Fermi-Dirac integral  $F_{-1/2}(x)$ .

**Parameter:**

*s*: A real number.

### 15.6.3 Fermi-Dirac integral $F_{1/2}(x)$

---

Function **FermiDiracHalfMpMath(s As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function FermiDiracHalfMpMath returns the complete Fermi-Dirac integral  $F_{1/2}(x)$ .

**Parameter:**

*s*: A real number.

### 15.6.4 Fermi-Dirac integral $F_{3/2}(x)$

---

Function **FermiDirac3HalfMpMath(s As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function FermiDirac3HalfMpMath returns the complete Fermi-Dirac integral  $F_{3/2}(x)$ .

**Parameter:**

*s*: A real number.

### 15.6.5 Legendre Chi-Function

---

Function **LegendreChiMpMath(s As mpNum, x As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function **LegendreChiMpMath** returns the Legendre Chi-Function function  $\chi_s(x)$ .

**Parameters:**

*s*: A real number.

*x*: A real number.

This function calculates the Legendre Chi-Function function  $\chi_s(x)$  defined for  $s \geq 0, |x| \leq 1$  by

$$\chi_s(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)^s}. \quad (15.6.5)$$

The function can be expressed as

$$\chi_s(x) = 2^{-s} x \Phi\left(x^2, s, \frac{1}{2}\right) = \frac{1}{2} (\text{Li}_s(x) - \text{Li}_s(-x)). \quad (15.6.6)$$

For large  $s > 22.8$  the function adds up to three terms of the sum, for  $s = 0$  or  $s = 1$  the *Li<sub>s</sub>* relation is used, otherwise the result is computed with Lerch's transcendent.

### 15.6.6 Inverse Tangent Integral

---

Function **InverseTangentMpMath(x As mpNum) As mpNum**

---

**NOT YET IMPLEMENTED**

---

The function **InverseTangentMpMath** returns the inverse-tangent integral.

**Parameter:**

*x*: A real number.

This function returns the inverse-tangent integral

$$\text{Ti}_2(x) = \int_0^x \frac{\arctan(t)}{t} dt. = \frac{1}{4} x \Phi\left(-x^2, 2, \frac{1}{2}\right) \quad (15.6.7)$$

For  $x > 1$  the relation

$$\text{Ti}_2(x) = \text{Ti}_2\left(\frac{1}{x}\right) + \frac{\pi}{2} \ln(x) \quad (15.6.8)$$

is used, and for  $x < 0$  the result is  $\text{Ti}_2(x) = -\text{Ti}_2(-x)$ .

## 15.7 Polylogarithms and Clausen functions

### 15.7.1 Polylogarithm

---

Function **polylog(s As mpNum, z As mpNum)** As mpNum

---

The function polylog returns the polylogarithm

**Parameters:**

*s*: A real or complex number.

*z*: A real or complex number.

The polylogarithm is defined by the sum

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}. \quad (15.7.1)$$

This series is convergent only for  $|z| < 1$ , so elsewhere the analytic continuation is implied.

The polylogarithm should not be confused with the logarithmic integral (also denoted by Li or li), which is implemented as li().

Examples

The polylogarithm satisfies a huge number of functional identities. A sample of polylogarithm evaluations is shown below:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> polylog(1,0.5), log(2)
(0.693147180559945, 0.693147180559945)
>>> polylog(2,0.5), (pi**2-6*log(2)**2)/12
(0.582240526465012, 0.582240526465012)
>>> polylog(2,-phi), -log(phi)**2-pi**2/10
(-1.21852526068613, -1.21852526068613)
>>> polylog(3,0.5), 7*zeta(3)/8-pi**2*log(2)/12+log(2)**3/6
(0.53721319360804, 0.53721319360804)
```

---

polylog() can evaluate the analytic continuation of the polylogarithm when *s* is an integer:

---

```
>>> polylog(2, 10)
(0.536301287357863 - 7.23378441241546j)
>>> polylog(2, -10)
-4.1982778868581
>>> polylog(2, 10j)
(-3.05968879432873 + 3.71678149306807j)
>>> polylog(-2, 10)
-0.150891632373114
>>> polylog(-2, -10)
0.067618332081142
>>> polylog(-2, 10j)
(0.0384353698579347 + 0.0912451798066779j)
```

---

### 15.7.2 Dilogarithm Function

---

Function **dilog**(*x* As mpNum) As mpNum

---

The function `dilog` returns the dilogarithm function  $\text{Li}_2(x)$ .

**Parameter:**

*x*: A real number.

The dilogarithm function is defined as

$$\text{dilog}(x) = \Re \text{Li}_2(x) = -\Re \int_0^x \frac{\ln(1-t)}{t} dt. \quad (15.7.2)$$

Note that there is some confusion about the naming: some authors and/or computer algebra systems use  $\text{dilog}(x) = \text{Li}_2(1-x)$  and then call  $\text{Li}_2(x)$  Spence function/integral or similar.

### 15.7.3 Debye Functions

---

Function **DebyeMpMath**(*n* As mpNum, *x* As mpNum) As mpNum

---

**NOT YET IMPLEMENTED**

---

The function `DebyeMpMath` returns the Debye function of order *n*.

**Parameters:**

*n*: An Integer.

*x*: A real number.

This routine returns the Debye functions

$$D_n(x) = \frac{n}{x^n} \int_0^x \frac{t^n}{e^t - 1} dt \quad (n > 0, x \geq 0). \quad (15.7.3)$$

$$D_k(x) = \frac{k}{x^{k+1}} \left[ (-1)^k k! \zeta(k+1) + \sum_{m=0}^k (-1)^{k-m+1} \frac{k!}{m!} x^m \text{Li}_{k-m+1}(e^x) \right] - \frac{k}{k+1}, \quad (15.7.4)$$

where  $\text{Li}_s$  denotes the polylogarithm (see [Dubinov & Dubinova \(2008\)](#)).

### 15.7.4 Clausen sine function

---

Function **clsinlog**(*s* As mpNum, *z* As mpNum) As mpNum

---

The function `clsinlog` returns the Clausen sine function

**Parameters:**

*s*: A real or complex number.

*z*: A real or complex number.

The Clausen sine function is defined formally by the series

$$\text{Cl}_s(z) = \sum_{k=1}^{\infty} \frac{\sin(kz)}{k^s}. \quad (15.7.5)$$

The special case  $\text{Cl}_2(z)$  (i.e. `clsin(2,z)`) is the classical 'Clausen function'. More generally, the Clausen function is defined for complex  $s$  and  $z$ , even when the series does not converge. The Clausen function is related to the polylogarithm (`polylog()`) as

$$\text{Cl}_s(z) = \frac{1}{2i} [\text{Li}_s(e^{iz}) - \text{Li}_s(e^{-iz})] \quad (15.7.6)$$

$$\text{Cl}_s(z) = \Im[\text{Li}_s(e^{iz})], \quad (s, z \in \mathbb{R}), \quad (15.7.7)$$

and this representation can be taken to provide the analytic continuation of the series. The complementary function `clcos()` gives the corresponding cosine sum.

Examples

Evaluation for arbitrarily chosen  $s$  and  $z$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> s, z = 3, 4
>>> clsin(s, z); nsum(lambda k: sin(z*k)/k**s, [1,inf])
-0.6533010136329338746275795
-0.6533010136329338746275795
```

---

The classical Clausen function  $\text{Cl}_s(z)$  gives the value of the integral  $\int_0^\theta -\ln(2\sin(x/2))dx$  for  $0 < \theta < 2\pi$ :

---

```
>>> cl2 = lambda t: clsin(2, t)
>>> cl2(3.5)
-0.2465045302347694216534255
>>> -quad(lambda x: ln(2*sin(0.5*x)), [0, 3.5])
-0.2465045302347694216534255
```

---

## 15.7.5 Clausen cosine function

---

**Function `clcos(s As mpNum, z As mpNum) As mpNum`**

---

The function `clcos` returns the Clausen cosine function

**Parameters:**

$s$ : A real or complex number.

$z$ : A real or complex number.

The Clausen cosine function is defined formally by the series

$$\widetilde{\text{Cl}}_s(z) = \sum_{k=1}^{\infty} \frac{\cos(kz)}{k^s}. \quad (15.7.8)$$

This function is complementary to the Clausen sine function `clsin()`. In terms of the polylogarithm,

$$\widetilde{\text{Cl}}_s(z) = \frac{1}{2} [\text{Li}_s(e^{iz}) - \text{Li}_s(e^{-iz})] \quad (15.7.9)$$

$$\widetilde{\text{Cl}}_s(z) = \Re[\text{Li}_s(e^{iz})], \quad (s, z \in \mathbb{R}), \quad (15.7.10)$$

Examples

Evaluation for arbitrarily chosen  $s$  and  $z$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> s, z = 3, 4
>>> clcos(s, z); nsum(lambda k: cos(z*k)/k**s, [1,inf])
-0.6518926267198991308332759
-0.6518926267198991308332759
```

---

### 15.7.6 Polyexponential function

---

Function **polyexp(s As mpNum, z As mpNum) As mpNum**

---

The function `polyexp` returns the polyexponential function

**Parameters:**

`s`: A real or complex number.

`z`: A real or complex number.

The polyexponential function is defined for arbitrary complex  $s, z$  by the series

$$E_s(z) = \sum_{k=1}^{\infty} \frac{k^s}{k!} z^k. \quad (15.7.11)$$

$E_s(z)$  is constructed from the exponential function analogously to how the polylogarithm is constructed from the ordinary logarithm; as a function of  $s$  (with  $z$  fixed),  $E_s$  is an L-series. It is an entire function of both  $s$  and  $z$ .

The polyexponential function provides a generalization of the Bell polynomials  $B_n(x)$  (see `bell()`) to noninteger orders  $n$ . In terms of the Bell polynomials,

$$E_s(z) = e^z B_s(z) - \text{sinc}(\pi s). \quad (15.7.12)$$

Note that  $B_n(x)$  and  $e^{-x} E_n(x)$  are identical if  $n$  is a nonzero integer, but not otherwise. In particular, they differ at  $n = 0$ .

Examples

Evaluating a series:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> nsum(lambda k: sqrt(k)/fac(k), [1,inf])
2.101755547733791780315904
>>> polyexp(0.5,1)
2.101755547733791780315904
```

---

Evaluation for arbitrary arguments:

---

```
>>> polyexp(-3-4j, 2.5+2j)
(2.351660261190434618268706 + 1.202966666673054671364215j)
```

---

## 15.8 Zeta function variants

### 15.8.1 Prime zeta function

---

Function **primezeta**(*s* As mpNum) As mpNum

---

The function primezeta returns the prime zeta function.

**Parameter:**

*s*: A real or complex number.

The prime zeta function is defined in analogy with the Riemann zeta function (zeta()) as

$$P(s) = \sum_p \frac{1}{p^s} \quad (15.8.1)$$

where the sum is taken over all prime numbers  $p$ . Although this sum only converges for  $\Re(s) > 1$ , the function is defined by analytic continuation in the half-plane  $\Re(s) > 0$ .

Examples

Arbitrary-precision evaluation for real and complex arguments is supported:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 30; mp.pretty = True
>>> primezeta(2)
0.452247420041065498506543364832
>>> primezeta(pi)
0.15483752698840284272036497397
>>> mp.dps = 50
>>> primezeta(3)
0.17476263929944353642311331466570670097541212192615
>>> mp.dps = 20
>>> primezeta(3+4j)
(-0.12085382601645763295 - 0.013370403397787023602j)
```

---

The analytic continuation to  $0 < \Re(s) \leq 1$  is implemented. In this strip the function exhibits very complex behavior; on the unit interval, it has poles at  $1/n$  for every squarefree integer  $n$ :

---

```
>>> primezeta(0.5) # Pole at s = 1/2
(-inf + 3.1415926535897932385j)
>>> primezeta(0.25)
(-1.0416106801757269036 + 0.52359877559829887308j)
>>> primezeta(0.5+10j)
(0.54892423556409790529 + 0.45626803423487934264j)
```

---

### 15.8.2 Secondary zeta function

---

Function **secondzeta**(*s* As mpNum, **Keywords** As String) As mpNum

---

The function secondzeta returns the secondary zeta function

**Parameters:**

*s*: A real or complex number.

*Keywords:* a=0.015, error=False.

The secondary zeta function  $Z(s)$  is defined for  $\Re(s) > 1$  by

$$Z(s) = \sum_{n=1}^{\infty} \frac{1}{\tau_n^s} \quad (15.8.2)$$

where  $\frac{1}{2} + i\tau_n$  runs through the zeros of  $\zeta(s)$  with imaginary part positive.

$Z(s)$  extends to a meromorphic function on  $\mathbb{C}$  with a double pole at  $s = 1$  and simple poles at the points  $-2n$  for  $n = 0, 1, 2, \dots$

Examples

---

```
>>> from mpFormulaPy import *
>>> mp.pretty = True; mp.dps = 15
>>> secondzeta(2)
0.023104993115419
>>> xi = lambda s: 0.5*s*(s-1)*pi**(-0.5*s)*gamma(0.5*s)*zeta(s)
>>> Xi = lambda t: xi(0.5+t*j)
>>> -0.5*diff(Xi,0,n=2)/Xi(0)
(0.023104993115419 + 0.0j)
```

---

We may ask for an approximate error value:

---

```
>>> secondzeta(0.5+100j, error=True)
((-0.216272011276718 - 0.844952708937228j), 2.22044604925031e-16)
```

---

# Chapter 16

## Number-theoretical, combinatorial and integer functions

For factorial-type functions, including binomial coefficients, double factorials, etc., see the separate section Factorials and gamma functions.

### 16.1 Fibonacci numbers

---

#### Function **fibonacci(*n* As mpNum, *Keywords* As String)** As mpNum

---

The function fibonacci returns the  $n$ -th Fibonacci number,  $F(n)$

**Parameters:**

*n*: A real or complex number.

*Keywords*: derivative=0.

---

#### Function **fib(*n* As mpNum, *Keywords* As String)** As mpNum

---

The function fib returns the  $n$ -th Fibonacci number,  $F(n)$

**Parameters:**

*n*: A real or complex number.

*Keywords*: derivative=0.

The Fibonacci numbers are defined by the recurrence  $F(n) = F(n-1) + F(n-2)$  with the initial values  $F(0) = 0$ ,  $F(1) = 1$ . fibonacci() extends this definition to arbitrary real and complex arguments using the formula

$$F(z) = \frac{\phi^z - \cos(\pi z)\phi^{-z}}{\sqrt{5}} \quad (16.1.1)$$

where  $\phi$  is the golden ratio. fibonacci() also uses this continuous formula to compute  $F(n)$  for extremely large  $n$ , where calculating the exact integer would be wasteful.

For convenience, fib() is available as an alias for fibonacci().

Basic examples

Some small Fibonacci numbers are:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
```

```
>>> for i in range(10):
...     print(fibonacci(i))
...
0.0
1.0
1.0
2.0
3.0
5.0
8.0
13.0
21.0
34.0
>>> fibonacci(50)
12586269025.0
```

---

`fibonacci()` can compute approximate Fibonacci numbers of stupendous size:

---

```
>>> mp.dps = 15
>>> fibonacci(10**25)
3.49052338550226e+2089876402499787337692720
```

---

The extended Fibonacci function is an analytic function. The property  $F(z) = F(z-1) + F(z-2)$  holds for arbitrary  $z$ :

---

```
>>> mp.dps = 15
>>> fib(pi)
2.1170270579161
>>> fib(pi-1) + fib(pi-2)
2.1170270579161
>>> fib(3+4j)
(-5248.51130728372 - 14195.962288353j)
>>> fib(2+4j) + fib(1+4j)
(-5248.51130728372 - 14195.962288353j)
```

---

## 16.2 Bernoulli numbers and polynomials

### 16.2.1 Bernoulli numbers

---

Function **bernoulli(*n* As mpNum) As mpNum**

---

The function `bernoulli` returns the  $n$ th Bernoulli number,  $B_n$ , for any integer  $n > 0$

**Parameter:**

*n*: An integer

The Bernoulli numbers  $B_n$  are defined by their generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi. \quad (16.2.1)$$

If  $n < 0$  or if  $n > 2$  is odd, the result is 0, and  $B_1 = -1/2$ . If  $n \leq 120$  the function value is taken from a pre-calculated table. For large  $n$  the asymptotic approximation [30, 24.11.1]

$$(-1)^{n+1} B_{2n} \approx \frac{2(2n)!}{(2\pi)^{2n}}, \quad (16.2.2)$$

gives an asymptotic recursion formula

$$B_{2n+2} \approx -\frac{(2n+1)(2n+2)}{(2\pi)^2} B_{2n}, \quad (16.2.3)$$

which is used for computing  $B_n$  for  $120 < n \leq 2312$  from a pre-calculated table of values  $B_{32k+128}$  ( $0 \leq k \leq 68$ ). The average iteration count is 4, and the maximum relative error of 4.5 eps occurs for  $n = 878$ .

Computes the  $n$ th Bernoulli number,  $B_n$ , for any integer  $n > 0$ . The Bernoulli numbers are rational numbers, but this function returns a floating-point approximation. To obtain an exact fraction, use `bernfrac()` instead.

For small  $n$  ( $n < 3000$ ) `bernoulli()` uses a recurrence formula due to Ramanujan. All results in this range are cached, so sequential computation of small Bernoulli numbers is guaranteed to be fast. For larger  $n$ ,  $B_n$  is evaluated in terms of the Riemann zeta function

Examples

Numerical values of the first few Bernoulli numbers:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> for n in range(15):
...     print("%s %s" % (n, bernoulli(n)))
...
0 1.0
1 -0.5
2 0.166666666666667
3 0.0
4 -0.0333333333333333
5 0.0
6 0.0238095238095238
7 0.0
8 -0.0333333333333333
9 0.0
```

---

```
10 0.0757575757575758
11 0.0
12 -0.253113553113553
13 0.0
14 1.166666666666667
```

---

Bernoulli numbers can be approximated with arbitrary precision:

---

```
>>> mp.dps = 50
>>> bernoulli(100)
-2.8382249570693706959264156336481764738284680928013e+78
```

---

Arbitrarily large are supported:

---

```
>>> mp.dps = 15
>>> bernoulli(10**20 + 2)
3.09136296657021e+1876752564973863312327
```

---



---

### Function **bernfrac(*n* As mpNum) As mpNum**

---

The function `bernfrac` returns a tuple of integers  $(p, q)$  such that  $p/q = B_n$  exactly, where  $B_n$  denotes the  $n$ -th Bernoulli number.

**Parameter:**

*n*: An integer

The fraction is always reduced to lowest terms. Note that for  $n > 1$  and  $n$  odd,  $B_n = 0$ , and  $(0, 1)$  is returned.

`bernoulli()` computes a floating-point approximation directly, without computing the exact fraction first. This is much faster for large  $n$ .

`bernfrac()` works by computing the value of  $B_n$  numerically and then using the von Staudt-Clausesen theorem [1] to reconstruct the exact fraction. For large  $n$ , this is significantly faster than computing  $B_1, B_2, \dots, B_n$  recursively with exact arithmetic.

The implementation has been tested for  $n = 10^m$  up to  $m = 6$ . In practice, `bernfrac()` appears to be about three times slower than the specialized program `calcbn.exe` [2]

Examples

The first few Bernoulli numbers are exactly:

---

```
>>> from mpFormulaPy import *
>>> for n in range(15):
...     p, q = bernfrac(n)
...     print("%s %s/%s" % (n, p, q))
...
0 1/1
1 -1/2
2 1/6
3 0/1
4 -1/30
5 0/1
```

```

6 1/42
7 0/1
8 -1/30
9 0/1
10 5/66
11 0/1
12 -691/2730
13 0/1
14 7/6

```

---

This function works for arbitrarily large  $n$ :

```

>>> p, q = bernfrac(10**4)
>>> print(q)
2338224387510
>>> print(len(str(p)))
27692
>>> mp.dps = 15
>>> print(mpf(p) / q)
-9.04942396360948e+27677
>>> print(bernoulli(10**4))
-9.04942396360948e+27677

```

---

## 16.2.2 Bernoulli polynomials

---

**Function `bernpoly(n As mpNum, z As mpNum) As mpNum`**

---

The function `bernpoly` returns the Bernoulli polynomial  $B_n(z)$

**Parameters:**

*n*: A real or complex number.

*z*: A real or complex number.

The Bernoulli polynomials  $B_n(x)$  of degree  $n \geq 0$  are defined by the generating function [30, 24.2.3]

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi. \quad (16.2.4)$$

or the simple explicit representation [30, 24.2.5]

$$B_n(x) = \sum_{n=0}^{\infty} \binom{n}{k} B_k(x) x^{n-k}. \quad (16.2.5)$$

The first few Bernoulli polynomials are:

---

```

>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> for n in range(6):
... nprint(chop(taylor(lambda x: bernpoly(n,x), 0, n)))
...
[1.0]
[-0.5, 1.0]

```

```
[0.166667, -1.0, 1.0]
[0.0, 0.5, -1.5, 1.0]
[-0.0333333, 0.0, 1.0, -2.0, 1.0]
[0.0, -0.166667, 0.0, 1.66667, -2.5, 1.0]
```

---

Evaluation is accurate for large and small  $z$ :

```
>>> mp.dps = 25
>>> bernpoly(100, 0.5)
2.838224957069370695926416e+78
>>> bernpoly(1000, 10.5)
5.318704469415522036482914e+1769
```

---

## 16.3 Euler numbers and polynomials

### 16.3.1 Euler numbers

---

Function **eulernum(*n* As mpNum)** As mpNum

---

The function eulernum returns the  $n$ -th Euler number

**Parameter:**

*n*: An integer

The  $n$ -th Euler number is defined as the  $n$ -th derivative of  $\text{sech}(t) = 1/\cosh(t)$  evaluated at  $t = 0$ . Equivalently, the Euler numbers give the coefficients of the Taylor series

$$\text{sech}(t) = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n. \quad (16.3.1)$$

The Euler numbers are closely related to Bernoulli numbers and Bernoulli polynomials. They can also be evaluated in terms of Euler polynomials (see eulerpoly()) as  $E_n = 2^n E_n(1/2)$ .

Examples

Euler numbers grow very rapidly. eulernum() efficiently computes numerical approximations for large indices:

---

```
>>> eulernum(50)
-6.053285248188621896314384e+54
>>> eulernum(1000)
3.887561841253070615257336e+2371
>>> eulernum(10**20)
4.346791453661149089338186e+1936958564106659551331
```

---

Pass exact=True to obtain exact values of Euler numbers as integers:

---

```
>>> print(eulernum(50, exact=True))
-6053285248188621896314383785111649088103498225146815121
>>> print(eulernum(200, exact=True) % 10**10)
1925859625
>>> eulernum(1001, exact=True)
0
```

---

### 16.3.2 Euler polynomials

---

Function **eulerpoly(*n* As mpNum, *z* As mpNum)** As mpNum

---

The function eulerpoly returns the Euler polynomial  $E_n(z)$

**Parameters:**

*n*: A real or complex number.

*z*: A real or complex number.

The Euler polynomial  $E_n(z)$  is defined by the generating function representation

$$\frac{2e^{zt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!}. \quad (16.3.2)$$

The Euler polynomials may also be represented in terms of Bernoulli polynomials (see `bernpoly()`) using various formulas, for example

$$En(z) = \frac{2}{n+1} \left( B_n(z) - 2^{n+2} B_n(z/2) \right) \quad (16.3.3)$$

Special values include the Euler numbers  $E_n = 2^{n+1} E_n(1/2)$  (see `eulernum()`).

Examples

Evaluation for arbitrary  $z$ :

---

```
>>> eulerpoly(2,3)
6.0
>>> eulerpoly(5,4)
423.5
>>> eulerpoly(35, 11111111112)
3.994957561486776072734601e+351
>>> eulerpoly(4, 10+20j)
(-47990.0 - 235980.0j)
>>> eulerpoly(2, '-3.5e-5')
0.000035001225
>>> eulerpoly(3, 0.5)
0.0
>>> eulerpoly(55, -10**80)
-1.0e+4400
>>> eulerpoly(5, -inf)
-inf
>>> eulerpoly(6, -inf)
+inf
```

---

## 16.4 Bell numbers and polynomials

Function **bell(*n* As mpNum, *x* As mpNum) As mpNum**

The function **bell** returns the Bell polynomial  $B_n(x)$

**Parameters:**

*n*: A non-negative integer.

*x*: A real or complex number.

The Bell polynomial  $B_n(x)$  are feinde for  $n > 0$ . The first few are

$$B_0(x) = 1; \quad B_1(x) = x; \quad B_2(x) = x^2 + x; \quad B_3(x) = x^3 + 3x^2 + x. \quad (16.4.1)$$

If  $x = 1$  or **bell()** is called with only one argument, it gives the  $n$ -th Bell number  $B_n$ , which is the number of partitions of a set with  $n$  elements. By setting the precision to at least  $\log_{10} B_n$  digits, **bell()** provides fast calculation of exact Bell numbers.

In general, **bell()** computes

$$B_n(x) = e^{-x} (\text{sinc}(\pi n) + E_n(x)) \quad (16.4.2)$$

where  $E_n(x)$  is the generalized exponential function implemented by **polyexp()**. This is an extension of Dobinski's formula [1], where the modification is the sinc term ensuring that  $B_n(x)$  is continuous in  $n$ ; **bell()** can thus be evaluated, differentiated, etc for arbitrary complex arguments.

Examples

Simple evaluations:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> bell(0, 2.5)
1.0
>>> bell(1, 2.5)
2.5
>>> bell(2, 2.5)
8.75
```

Evaluation for arbitrary complex arguments:

```
>>> bell(5.75+1j, 2-3j)
(-10767.71345136587098445143 - 15449.55065599872579097221j)
```

## 16.5 Stirling numbers

### 16.5.1 Stirling number of the first kind

---

Function **stirling1**(*n* As mpNum, *k* As mpNum, **Keywords** As String) As mpNum

---

The function **stirling1** returns the Stirling number of the first kind  $s(n, k)$

**Parameters:**

*n*: A real or complex number.

*k*: A real or complex number.

**Keywords**: exact=False.

The Stirling number of the first kind  $s(n, k)$  is defined by

$$x(x - 1)(x - 2) \cdots (x - n + 1) = \sum_{k=0}^n s(n, k)x^k. \quad (16.5.1)$$

The value is computed using an integer recurrence. The implementation is not optimized for approximating large values quickly.

Examples

Comparing with the generating function:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> taylor(lambda x: ff(x, 5), 0, 5)
[0.0, 24.0, -50.0, 35.0, -10.0, 1.0]
>>> [stirling1(5, k) for k in range(6)]
[0.0, 24.0, -50.0, 35.0, -10.0, 1.0]
```

---

Pass exact=True to obtain exact values of Stirling numbers as integers:

---

```
>>> stirling1(42, 5)
-2.864498971768501633736628e+50
>>> print stirling1(42, 5, exact=True)
-286449897176850163373662803014001546235808317440000
```

---

### 16.5.2 Stirling number of the second kind

---

Function **stirling2**(*n* As mpNum, *k* As mpNum, **Keywords** As String) As mpNum

---

The function **stirling2** returns the Stirling number of the second kind  $s(n, k)$

**Parameters:**

*n*: A real or complex number.

*k*: A real or complex number.

**Keywords**: exact=False.

The Stirling number of the second kind  $S(n, k)$  is defined by

$$x^n = \sum_{k=0}^n S(n, k)x(x - 1)(x - 2) \cdots (x - k + 1). \quad (16.5.2)$$

The value is computed using integer arithmetic to evaluate a power sum. The implementation is not optimized for approximating large values quickly.

Examples

Comparing with the generating function:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> taylor(lambda x: sum(stirling2(5,k) * ff(x,k) for k in range(
[0.0, 0.0, 0.0, 0.0, 0.0, 1.0]
```

---

Pass exact=True to obtain exact values of Stirling numbers as integers:

---

```
>>> stirling2(52, 10)
2.641822121003543906807485e+45
>>> print stirling2(52, 10, exact=True)
2641822121003543906807485307053638921722527655
```

---

## 16.6 Prime counting functions

### 16.6.1 Exact prime counting function

---

#### Function **primepi(*x* As mpNum) As mpNum**

---

The function primepi returns the prime counting function

**Parameter:**

*x*: A real number

The prime counting function,  $\pi(x)$ , gives the number of primes less than or equal to  $x$ . The argument  $x$  may be fractional.

The prime counting function is very expensive to evaluate precisely for large  $x$ , and the present implementation is not optimized in any way. For numerical approximation of the prime counting function, it is better to use primepi2() or riemannr().

Some values of the prime counting function:

---

```
>>> from mpFormulaPy import *
>>> [primepi(k) for k in range(20)]
[0, 0, 1, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 6, 6, 7, 7, 8]
>>> primepi(3.5)
2
>>> primepi(100000)
9592
```

---

### 16.6.2 Prime counting function interval

---

#### Function **primepi2(*x* As mpNum) As mpNum**

---

The function primepi2 returns an interval (as an mpi instance) providing bounds for the value of the prime counting function  $\pi(x)$

**Parameter:**

*x*: A real number

For small  $x$ , primepi2() returns an exact interval based on the output of primepi(). For  $x > 2656$ , a loose interval based on Schoenfeld's inequality

$$|\pi(x)| - \text{li}(x) < \frac{\sqrt{x} \log x}{8\pi} \quad (16.6.1)$$

is returned. This estimate is rigorous assuming the truth of the Riemann hypothesis, and can be computed very quickly.

Examples

Exact values of the prime counting function for small  $x$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> iv.dps = 15; iv.pretty = True
>>> primepi2(10)
[4.0, 4.0]
```

---

```
>>> primepi2(100)
[25.0, 25.0]
>>> primepi2(1000)
[168.0, 168.0]
```

---

Loose intervals are generated for moderately large  $x$ :

---

```
>>> primepi2(10000), primepi(10000)
([1209.0, 1283.0], 1229)
>>> primepi2(50000), primepi(50000)
([5070.0, 5263.0], 5133)
```

---

As  $x$  increases, the absolute error gets worse while the relative error improves. The exact value of  $\pi(10^{23})$  is 1925320391606803968923, and primepi2() gives 9 significant digits:

---

```
>>> p = primepi2(10**23)
>>> p
[1.9253203909477020467e+21, 1.925320392280406229e+21]
>>> mpf(p.delta) / mpf(p.a)
6.9219865355293e-10
```

---

A more precise, nonrigorous estimate for  $\pi(x)$  can be obtained using the Riemann R function (riemannr()). For large enough  $x$ , the value returned by primepi2() essentially amounts to a small perturbation of the value returned by riemannr():

---

```
>>> primepi2(10**100)
[4.3619719871407024816e+97, 4.3619719871407032404e+97]
>>> riemannr(10**100)
4.3619719871407e+97
```

---

### 16.6.3 Riemann R function

---

**Function `riemannr(x As mpNum)` As mpNum**

---

The function `riemannr` returns the Riemann R function, a smooth approximation of the prime counting function  $\pi(x)$

**Parameter:**

$x$ : A real number

The Riemann R function gives a fast numerical approximation useful e.g. to roughly estimate the number of primes in a given interval (see `primepi()`).

The Riemann R function is computed using the rapidly convergent Gram series,

$$R(x) = 1 + \sum_{k=1}^{\infty} \frac{\log^k x}{kk!\zeta(k+1)}. \quad (16.6.2)$$

From the Gram series, one sees that the Riemann R function is a well-defined analytic function (except for a branch cut along the negative real half-axis); it can be evaluated for arbitrary real or complex arguments.

The Riemann R function gives a very accurate approximation of the prime counting function. For example, it is wrong by at most 2 for  $x < 1000$ , and for  $x = 10^9$  differs from the exact value

of  $\pi(x)$  by 79, or less than two parts in a million. It is about 10 times more accurate than the logarithmic integral estimate (see `li()`), which however is even faster to evaluate. It is orders of magnitude more accurate than the extremely fast  $x/\log x$  estimate.

For small arguments, the Riemann R function almost exactly gives the prime counting function if rounded to the nearest integer:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> primepi(50), riemannr(50)
(15, 14.9757023241462)
>>> max(abs(primepi(n)-int(round(riemannr(n)))) for n in range(100))
1
>>> max(abs(primepi(n)-int(round(riemannr(n)))) for n in range(300))
2
```

---

The Riemann R function can be evaluated for arguments far too large for exact determination of  $\pi(x)$  to be computationally feasible with any presently known algorithm:

---

```
>>> riemannr(10**30)
1.46923988977204e+28
>>> riemannr(10**100)
4.3619719871407e+97
>>> riemannr(10**1000)
4.3448325764012e+996
```

---

Evaluation is supported for arbitrary arguments and at arbitrary precision:

---

```
>>> mp.dps = 30
>>> riemannr(7.5)
3.72934743264966261918857135136
>>> riemannr(-4+2j)
(-0.551002208155486427591793957644 + 2.16966398138119450043195899
```

---

## 16.7 Miscellaneous functions

### 16.7.1 Cyclotomic polynomials

---

Function **cyclotomic(*n* As mpNum, *x* As mpNum) As mpNum**

---

The function `cyclotomic` returns the cyclotomic polynomial  $\Phi_n(x)$

**Parameters:**

*n*: A real or complex number.

*x*: A real or complex number.

The cyclotomic polynomial  $\Phi_n(x)$  is defined by

$$\Phi_n(x) = \prod_{\zeta} (x - \zeta) \quad (16.7.1)$$

where  $\zeta$  ranges over all primitive  $n$ -th roots of unity (see `unitroots()`). An equivalent representation, used for computation, is

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)} \quad (16.7.2)$$

where  $\mu(m)$  denotes the Moebius function. The cyclotomic polynomials are integer polynomials, the first of which can be written explicitly as

$$\Phi_0(x) = 1; \quad \Phi_1(x) = x - 1; \quad \Phi_2(x) = x + 1; \quad \Phi_3(x) = x^3 + x^2 + 1 \quad (16.7.3)$$

$$\Phi_4(x) = x^2 + 1; \quad \Phi_5(x) = x^4 + x^3 + x^2 + x + 1; \quad \Phi_6(x) = x^2 - x + 1. \quad (16.7.4)$$

The coefficients of low-order cyclotomic polynomials can be recovered using Taylor expansion:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> for n in range(9):
... p = chop(taylor(lambda x: cyclotomic(n,x), 0, 10))
... print("%s %s" % (n, nstr(p[:10+1-p[::-1].index(1)])))
...
0 [1.0]
1 [-1.0, 1.0]
2 [1.0, 1.0]
3 [1.0, 1.0, 1.0]
4 [1.0, 0.0, 1.0]
5 [1.0, 1.0, 1.0, 1.0, 1.0]
6 [1.0, -1.0, 1.0]
7 [1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0]
8 [1.0, 0.0, 0.0, 0.0, 1.0]
```

---

### 16.7.2 von Mangoldt function

---

Function **mangoldt(*n* As mpNum) As mpNum**

---

The function `mangoldt` returns the von Mangoldt function

**Parameter:**

*n*: An integer

The von Mangoldt function is defined as  $\Lambda(n) = \log p$  if  $n = p^k$  is a power of a prime, and  $\Lambda(n) = 0$  otherwise.

Examples

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> [mangoldt(n) for n in range(-2,3)]
[0.0, 0.0, 0.0, 0.0, 0.6931471805599453094172321]
>>> mangoldt(6)
0.0
>>> mangoldt(7)
1.945910149055313305105353
>>> mangoldt(8)
0.6931471805599453094172321
>>> fsum(mangoldt(n) for n in range(101))
94.04531122935739224600493
>>> fsum(mangoldt(n) for n in range(10001))
10013.39669326311478372032
```

---

# Chapter 17

## q-functions

### 17.1 q-Pochhammer symbol

---

Function **qp(a As mpNum, q As mpNum, n As mpNum) As mpNum**

---

The function **qp** returns the q-Pochhammer symbol (or q-rising factorial)

**Parameters:**

*a*: A real or complex number.

*q*: A real or complex number.

*n*: An integer.

The q-Pochhammer symbol (or q-rising factorial) is defined as

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \quad (17.1.1)$$

where  $n = \infty$  is permitted if  $|q| < 1$ . Called with two arguments,  $\text{qp}(a, q)$  computes  $(a; q)_\infty$ ; with a single argument,  $\text{qp}(q)$  computes  $(q; q)_\infty$ . The special case

$$\phi(q) = (q; q)_\infty = \prod_{k=1}^{\infty} (1 - q^k) = \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2-k)/2} \quad (17.1.2)$$

is also known as the Euler function, or (up to a factor  $q^{-1/24}$ ) the Dedekind eta function.

Examples

If  $n$  is a positive integer, the function amounts to a finite product:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> qp(2,3,5)
-725305.0
>>> fprod(1-2*3**k for k in range(5))
-725305.0
>>> qp(2,3,0)
1.0
```

---

Complex arguments are allowed:

---

```
>>> qp(2-1j, 0.75j)
(0.4628842231660149089976379 + 4.481821753552703090628793j)
```

---

## 17.2 q-gamma and factorial

### 17.2.1 q-gamma

---

Function **qgamma(z As mpNum, q As mpNum)** As mpNum

---

The function `qgamma` returns the q-gamma function

**Parameters:**

*z*: A real or complex number.

*q*: A real or complex number.

The q-gamma function is defined as

$$\Gamma_q(z) = \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z}. \quad (17.2.1)$$

Examples

Evaluation for real and complex arguments:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> qgamma(4,0.75)
4.046875
>>> qgamma(6,6)
121226245.0
>>> qgamma(3+4j, 0.5j)
(0.1663082382255199834630088 + 0.01952474576025952984418217j)
```

---

### 17.2.2 q-factorial

---

Function **qfac(z As mpNum, q As mpNum)** As mpNum

---

The function `qfac` returns the q-factorial

**Parameters:**

*z*: A real or complex number.

*q*: A real or complex number.

The q-factorial is defined as

$$[n]_q! = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}) \quad (17.2.2)$$

or more generally

$$[z]_q! = \frac{(q; q)_z}{(1 - q)^z} \quad (17.2.3)$$

Examples

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> qfac(0,0)
1.0
```

---

```
>>> qfac(4,3)
2080.0
>>> qfac(5,6)
121226245.0
>>> qfac(1+1j, 2+1j)
(0.4370556551322672478613695 + 0.2609739839216039203708921j)
```

## 17.3 Hypergeometric q-series

Function **qhyper(as As mpNum, bs As mpNum, q As mpNum, z As mpNum) As mpNum**

The function `qhyper` returns the hypergeometric q-series

**Parameters:**

*as*: A real or complex number.

*bs*: A real or complex number.

*q*: A real or complex number.

*z*: A real or complex number.

The basic hypergeometric series or hypergeometric q-series is defined as

$$\text{qhyper}(A, B, q, z) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n, \dots, (a_r; q)_n}{(b_1; q)_n, \dots, (b_s; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} \frac{z^n}{(q; q)_n} \quad (17.3.1)$$

where  $(a; q)_n$  denotes the q-Pochhammer symbol (see `qp()`).

Examples

Evaluation works for real and complex arguments:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> qhyper([0.5], [2.25], 0.25, 4)
-0.1975849091263356009534385
>>> qhyper([0.5], [2.25], 0.25-0.25j, 4)
(2.806330244925716649839237 + 3.568997623337943121769938j)
>>> qhyper([1+j], [2,3+0.5j], 0.25, 3+4j)
(9.112885171773400017270226 - 1.272756997166375050700388j)
```

# Chapter 18

## Matrix functions

### 18.1 Matrix exponential

---

Function **expm(A As mpNum, Keywords As String)** As mpNum

---

The function `expm` returns the matrix exponential of a square matrix  $A$

**Parameters:**

$A$ : A real or complex matrix.

**Keywords:** method='taylor'.

The matrix exponential of a square matrix  $A$  is defined by the power series

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots \quad (18.1.1)$$

With `method='taylor'`, the matrix exponential is computed using the Taylor series. With `method='pade'`, Pade approximants are used instead.

**Examples**

Basic examples:

---

```
>>> from mpFormulaPy import *
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> expm(zeros(3))
[1.0  0.0  0.0]
[0.0  1.0  0.0]
[0.0  0.0  1.0]
>>> expm(eye(3))
[2.71828182845905      0.0      0.0]
[          0.0  2.71828182845905      0.0]
[          0.0          0.0  2.71828182845905]
>>> expm([[1,1,0],[1,0,1],[0,1,0]])
[ 3.86814500615414 2.26812870852145 0.841130841230196]
[ 2.26812870852145 2.44114713886289 1.42699786729125]
[0.841130841230196 1.42699786729125 1.6000162976327]
>>> expm([[1,1,0],[1,0,1],[0,1,0]], method='pade')
[ 3.86814500615414 2.26812870852145 0.841130841230196]
[ 2.26812870852145 2.44114713886289 1.42699786729125]
```

```
[0.841130841230196 1.42699786729125 1.6000162976327]
>>> expm([[1+j, 0], [1+j,1]])
[(1.46869393991589 + 2.28735528717884j) 0.0]
[ (1.03776739863568 + 3.536943175722j) (2.71828182845905 + 0.0j)]
```

---

Matrices with large entries are allowed:

```
>>> expm(matrix([[1,2],[2,3]])**25)
[5.65024064048415e+2050488462815550 9.14228140091932e+2050488462815550]
[9.14228140091932e+2050488462815550 1.47925220414035e+2050488462815551]
```

---

The identity  $\exp(A + B) = \exp(A) \exp(B)$  does not hold for noncommuting matrices:

```
>>> A = hilbert(3)
>>> B = A + eye(3)
>>> chop(mnorm(A*B - B*A))
0.0
>>> chop(mnorm(expm(A+B) - expm(A)*expm(B)))
0.0
>>> B = A + ones(3)
>>> mnorm(A*B - B*A)
1.8
>>> mnorm(expm(A+B) - expm(A)*expm(B))
42.0927851137247
```

---

## 18.2 Matrix cosine

---

Function **cosm(A As mpNum) As mpNum**

---

The function **cosm** returns the matrix cosine of a square matrix  $A$

**Parameter:**

$A$ : A real or complex matrix.

The cosine of a square matrix  $A$  is defined in analogy with the matrix exponential.

Examples:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> X = eye(3)
>>> cosm(X)
[0.54030230586814      0.0      0.0]
[      0.0  0.54030230586814      0.0]
[      0.0      0.0  0.54030230586814]
>>> X = hilbert(3)
>>> cosm(X)
[ 0.424403834569555 -0.316643413047167 -0.221474945949293]
[-0.316643413047167  0.820646708837824 -0.127183694770039]
[-0.221474945949293 -0.127183694770039  0.909236687217541]
>>> X = matrix([[1+j,-2],[0,-j]])
>>> cosm(X)
[(0.833730025131149 - 0.988897705762865j) (1.07485840848393 - 0.17192140544213j)]
[                                         0.0          (1.54308063481524 + 0.0j)]
```

---

## 18.3 Matrix sine

---

Function **sinm(A As mpNum) As mpNum**

---

The function **sinm** returns the matrix sine of a square matrix  $A$

**Parameter:**

$A$ : A real or complex matrix.

The sine of a square matrix  $A$  is defined in analogy with the matrix exponential.

Examples:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> X = eye(3)
>>> sinm(X)
[0.841470984807897      0.0      0.0]
[      0.0  0.841470984807897      0.0]
[      0.0      0.0  0.841470984807897]
>>> X = hilbert(3)
>>> sinm(X)
[0.711608512150994  0.339783913247439  0.220742837314741]
[0.339783913247439  0.244113865695532  0.187231271174372]
[0.220742837314741  0.187231271174372  0.155816730769635]
>>> X = matrix([[1+j,-2],[0,-j]])
>>> sinm(X)
[(1.29845758141598 + 0.634963914784736j) (-1.96751511930922 + 0.314700021761367j)]
[                                0.0          (0.0 - 1.1752011936438j)]
```

---

## 18.4 Matrix square root

Function **sqrtm(A As mpNum, Keywords As String) As mpNum**

The function `sqrtm` returns a square root of a square matrix  $A$

**Parameters:**

$A$ : A real or complex matrix.

*Keywords*: mayrotate=2.

A square root of the square matrix  $A$  is a matrix  $B = A^{1/2}$  such that  $B^2 = A$ . The square root of a matrix, if it exists, is not unique

**Examples:**

Square roots of some simple matrices:

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> sqrtm([[1,0], [0,1]])
[1.0  0.0]
[0.0  1.0]
>>> sqrtm([[0,0], [0,0]])
[0.0  0.0]
[0.0  0.0]
>>> sqrtm([[2,0],[0,1]])
[1.4142135623731 0.0]
[      0.0  1.0]
>>> sqrtm([[1,1],[1,0]])
[(0.920442065259926 - 0.21728689675164j) (0.568864481005783 + 0.351577584254143j)]
[(0.568864481005783 + 0.351577584254143j) (0.351577584254143 - 0.568864481005783j)]
>>> sqrtm([[1,0],[0,1]])
[1.0  0.0]
[0.0  1.0]
>>> sqrtm([[-1,0],[0,1]])
[(0.0 - 1.0j)      0.0]
[      0.0 (1.0 + 0.0j)]
>>> sqrtm([[j,0],[0,j]])
[(0.707106781186547 + 0.707106781186547j)           0.0]
[                           0.0 (0.707106781186547 + 0.707106781186547j)]
```

A square root of a rotation matrix, giving the corresponding half-angle rotation matrix:

```
>>> t1 = 0.75
>>> t2 = t1 * 0.5
>>> A1 = matrix([[cos(t1), -sin(t1)], [sin(t1), cos(t1)]])
>>> A2 = matrix([[cos(t2), -sin(t2)], [sin(t2), cos(t2)]])
>>> sqrtm(A1)
[0.930507621912314 -0.366272529086048]
[0.366272529086048 0.930507621912314]
>>> A2
[0.930507621912314 -0.366272529086048]
[0.366272529086048 0.930507621912314]
```

The identity  $(A^2)^{1/2}$  does not necessarily hold:

---

```
>>> A = matrix([[4,1,4],[7,8,9],[10,2,11]])
>>> sqrtm(A**2)
[ 4.0 1.0 4.0]
[ 7.0 8.0 9.0]
[10.0 2.0 11.0]
>>> sqrtm(A)**2
[ 4.0 1.0 4.0]
[ 7.0 8.0 9.0]
[10.0 2.0 11.0]
>>> A = matrix([[-4,1,4],[7,-8,9],[10,2,11]])
>>> sqrtm(A**2)
[ 7.43715112194995 -0.324127569985474 1.8481718827526]
[-0.251549715716942 9.32699765900402 2.48221180985147]
[ 4.11609388833616 0.775751877098258 13.017955697342]
>>> chop(sqrtm(A)**2)
[-4.0 1.0 4.0]
[ 7.0 -8.0 9.0]
[10.0 2.0 11.0]
```

---

For some matrices, a square root does not exist:

---

```
>>> sqrtm([[0,1], [0,0]])
Traceback (most recent call last):
...
ZeroDivisionError: matrix is numerically singular
```

---

Two examples from the documentation for Matlab's sqrtm:

---

```
>>> mp.dps = 15; mp.pretty = True
>>> sqrtm([[7,10],[15,22]])
[1.56669890360128 1.74077655955698]
[2.61116483933547 4.17786374293675]
>>>
>>> X = matrix(\ 
... [[5,-4,1,0,0], 
... [-4,6,-4,1,0], 
... [1,-4,6,-4,1], 
... [0,1,-4,6,-4], 
... [0,0,1,-4,5]]) 
>>> Y = matrix(\ 
... [[2,-1,-0,-0,-0], 
... [-1,2,-1,0,-0], 
... [0,-1,2,-1,0], 
... [-0,0,-1,2,-1], 
... [-0,-0,-0,-1,2]]) 
>>> mnorm(sqrtm(X) - Y)
4.53155328326114e-19
```

---

## 18.5 Matrix logarithm

Function **logm(A As mpNum) As mpNum**

The function `logm` returns the matrix logarithm of a square matrix  $A$

**Parameter:**

$A$ : A real or complex matrix.

A logarithm of the square matrix  $A$  is a matrix  $B$  such that  $\exp(B) = A$ . The logarithm of a matrix, if it exists, is not unique.

**Examples:**

Logarithms of some simple matrices:

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> X = eye(3)
>>> logm(X)
[[0.0 0.0 0.0]
 [0.0 0.0 0.0]
 [0.0 0.0 0.0]
>>> logm(2*X)
[[0.693147180559945 0.0 0.0]
 [0.0 0.693147180559945 0.0]
 [0.0 0.0 0.693147180559945]
>>> logm(expm(X))
[[1.0 0.0 0.0]
 [0.0 1.0 0.0]
 [0.0 0.0 1.0]]
```

A logarithm of a complex matrix:

```
>>> X = matrix([[2+j, 1, 3], [1-j, 1-2*j, 1], [-4, -5, j]])
>>> B = logm(X)
>>> nprint(B)
[[0.808757 + 0.107759j] (2.20752 + 0.202762j) (1.07376 - 0.773874j)]
[[0.905709 - 0.107795j] (0.0287395 - 0.824993j) (0.111619 + 0.514272j)]
[[-0.930151 + 0.399512j] (-2.06266 - 0.674397j) (0.791552 + 0.519839j)]
>>> chop(expm(B))
[[2.0 + 1.0j] 1.0 3.0]
[[1.0 - 1.0j] (1.0 - 2.0j) 1.0]
[-4.0 -5.0 (0.0 + 1.0j)]
```

A matrix  $X$  close to the identity matrix, for which  $\log(\exp(X)) = \exp(\log(X)) = X$  holds:

```
>>> X = eye(3) + hilbert(3)/4
>>> X
[[1.25 0.125 0.0833333333333333]
 [0.125 1.08333333333333 0.0625]
 [0.0833333333333333 0.0625 1.05]
>>> logm(expm(X))
[[1.25 0.125 0.0833333333333333]
 [0.125 1.08333333333333 0.0625]]
```

---

```
[0.0833333333333333 0.0625 1.05]
>>> expm(logm(X))
[ 1.25 0.125 0.0833333333333333]
[ 0.125 1.083333333333333 0.0625]
[0.0833333333333333 0.0625 1.05]
```

---

A logarithm of a rotation matrix, giving back the angle of the rotation:

---

```
>>> t = 3.7
>>> A = matrix([[cos(t), sin(t)], [-sin(t), cos(t)]])
>>> chop(logm(A))
[ 0.0 -2.58318530717959]
[2.58318530717959 0.0]
>>> (2*pi-t)
2.58318530717959
```

---

For some matrices, a logarithm does not exist:

---

```
>>> logm([[1,0], [0,0]])
Traceback (most recent call last):
...
ZeroDivisionError: matrix is numerically singular
```

---

Logarithm of a matrix with large entries:

---

```
>>> logm(hilbert(3) * 10**20).apply(re)
[ 45.5597513593433 1.27721006042799 0.317662687717978]
[ 1.27721006042799 42.5222778973542 2.24003708791604]
[0.317662687717978 2.24003708791604 42.395212822267]
```

---

## 18.6 Matrix power

Function **powm**(*A* As mpNum, *r* As mpNum) As mpNum

The function **powm** returns  $A^r = \exp(A \log r)$  for a matrix *A* and complex number *r*

### Parameters:

*A*: A real or complex matrix.

*r*: A real or complex number.

Computes  $A^r = \exp(A \log r)$  for a matrix *A* and complex number *r*.

### Examples

Powers and inverse powers of a matrix:

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> A = matrix([[4,1,4],[7,8,9],[10,2,11]])
>>> powm(A, 2)
[ 63.0 20.0 69.0]
[174.0 89.0 199.0]
[164.0 48.0 179.0]
>>> chop(powm(powm(A, 4), 1/4.))
[ 4.0 1.0 4.0]
[ 7.0 8.0 9.0]
[10.0 2.0 11.0]
>>> powm(extraprec(20)(powm)(A, -4), -1/4.)
[ 4.0 1.0 4.0]
[ 7.0 8.0 9.0]
[10.0 2.0 11.0]
>>> chop(powm(powm(A, 1+0.5j), 1/(1+0.5j)))
[ 4.0 1.0 4.0]
[ 7.0 8.0 9.0]
[10.0 2.0 11.0]
>>> powm(extraprec(5)(powm)(A, -1.5), -1/(1.5))
[ 4.0 1.0 4.0]
[ 7.0 8.0 9.0]
[10.0 2.0 11.0]
```

A Fibonacci-generating matrix:

```
>>> powm([[1,1],[1,0]], 10)
[89.0 55.0]
[55.0 34.0]
>>> fib(10)
55.0
>>> powm([[1,1],[1,0]], 6.5)
[(16.5166626964253 - 0.0121089837381789j) (10.2078589271083 + 0.0195927472575932j)]
[(10.2078589271083 + 0.0195927472575932j) (6.30880376931698 - 0.0317017309957721j)]
>>> (phi**6.5 - (1-phi)**6.5)/sqrt(5)
(10.2078589271083 - 0.0195927472575932j)
>>> powm([[1,1],[1,0]], 6.2)
[ (14.3076953002666 - 0.008222855781077j) (8.81733464837593 + 0.0133048601383712j)]
```

```
[(8.81733464837593 + 0.0133048601383712j) (5.49036065189071 - 0.0215277159194482j)]  
>>> (phi**6.2 - (1-phi)**6.2)/sqrt(5)  
(8.81733464837593 - 0.0133048601383712j)
```

---

# Chapter 19

## Eigensystems and related Decompositions

### 19.1 Singular value decomposition

---

Function **svd(A As mpNum, Keywords As String)** As mpNum

---

The function **svd** returns the singular value decomposition of matrix A

**Parameters:**

A: A real or complex number.

*Keywords*: compute\_uv = True.

The routines **svd\_r** and **svd\_c** compute the singular value decomposition of a real or complex matrix A. **svd** is an unified interface calling either **svd\_r** or **svd\_c** depending on whether A is real or complex.

Given A, two orthogonal ( $A$  real) or unitary ( $A$  complex) matrices  $U$  and  $V$  are calculated such that

$$A = USV; \quad U'U = 1; \quad VV' = 1, \quad (19.1.1)$$

where  $S$  is a suitable shaped matrix whose off-diagonal elements are zero. Here ' denotes the hermitian transpose (i.e. transposition and complex conjugation). The diagonal elements of  $S$  are the singular values of  $A$ , i.e. the square roots of the eigenvalues of  $A'A$  or  $AA'$ .

Examples:

---

```
>>> from mpFormulaPy import mp
>>> A = mp.matrix([[2, -2, -1], [3, 4, -2], [-2, -2, 0]])
>>> S = mp.svd_r(A, compute_uv = False)
>>> print S
[6.0]
[3.0]
[1.0]
>>> U, S, V = mp.svd_r(A)
>>> print mp.chop(A - U * mp.diag(S) * V)
[0.0 0.0 0.0]
[0.0 0.0 0.0]
[0.0 0.0 0.0]
```

---

## 19.2 The Schur decomposition

---

Function **schur**(*A* As mpNum) As mpNum

---

The function **schur** returns the Schur decomposition of a square matrix *A*

**Parameter:**

*A*: A real or complex matrix.

This function computes the Schur decomposition of a square matrix *A*. Given *A*, a unitary matrix *Q* is determined such that

$$Q' A Q = R; \quad Q' Q = Q Q' = 1, \quad (19.2.1)$$

where *R* is an upper right triangular matrix. Here ' denotes the hermitian transpose (i.e. transposition and conjugation).

Examples:

---

```
>>> from mpFormulaPy import mp
>>> A = mp.matrix([[3, -1, 2], [2, 5, -5], [-2, -3, 7]])
>>> Q, R = mp.schur(A)
>>> mp.nprint(R, 3)
[2.0 0.417 -2.53]
[0.0 4.0 -4.74]
[0.0 0.0 9.0]
>>> print(mp.chop(A - Q * R * Q.transpose_conj()))
[0.0 0.0 0.0]
[0.0 0.0 0.0]
[0.0 0.0 0.0]
```

---

## 19.3 The eigenvalue problem

Function **eig(A As mpNum, Keywords As String) As mpNum**

The function **eig** returns the solution of the (ordinary) eigenvalue problem for a real or complex square matrix  $A$

**Parameters:**

$A$ : A real or complex number.

*Keywords*: left = False, right = False.

The routine **eig** solves the (ordinary) eigenvalue problem for a real or complex square matrix  $A$ . Given  $A$ , a vector  $E$  and matrices  $ER$  and  $EL$  are calculated such that

```
A ER[:,i] = E[i] ER[:,i]
EL[i,:] A = EL[i,:]*E[i]
```

$E$  contains the eigenvalues of  $A$ . The columns of  $ER$  contain the right eigenvectors of  $A$  whereas the rows of  $EL$  contain the left eigenvectors.

### Examples

```
>>> from mpFormulaPy import mp
>>> A = mp.matrix([[3, -1, 2], [2, 5, -5], [-2, -3, 7]])
>>> E, ER = mp.eig(A)
>>> print(mp.chop(A * ER[:,0] - E[0] * ER[:,0]))
[0.0]
[0.0]
[0.0]
>>> E, EL, ER = mp.eig(A, left = True, right = True)
>>> E, EL, ER = mp.eig_sort(E, EL, ER)
>>> mp.nprint(E)
[2.0, 4.0, 9.0]
>>> print(mp.chop(A * ER[:,0] - E[0] * ER[:,0]))
[0.0]
[0.0]
[0.0]
>>> print(mp.chop(EL[0,:] * A - EL[0,:] * E[0]))
[0.0 0.0 0.0]
```

## 19.4 The symmetric eigenvalue problem

Function **eigh(A As mpNum, Keywords As String) As mpNum**

The function **eigh** returns the solution of the (ordinary) eigenvalue problem for a real symmetric or complex hermitian square matrix  $A$

**Parameters:**

$A$ : A real or complex number.

*Keywords*: `eigvals_only = False`.

The routines `eigsy` and `eighe` solve the (ordinary) eigenvalue problem for a real symmetric or complex hermitian square matrix  $A$ . `eigh` is an unified interface for these two functions calling either `eigsy` or `eighe` depending on whether  $A$  is real or complex.

Given  $A$ , an orthogonal ( $A$  real) or unitary matrix  $Q$  ( $A$  complex) is calculated which diagonalizes  $A$ :

$$Q' A Q = \text{diag}(E); \quad Q' Q = Q' Q = 1. \quad (19.4.1)$$

Here  $\text{diag}(E)$  is a diagonal matrix whose diagonal is  $E$ .  $'$  denotes the hermitian transpose (i.e. ordinary transposition and complex conjugation).

The columns of  $Q$  are the eigenvectors of  $A$  and  $E$  contains the eigenvalues:

`A Q[:, i] = E[i] Q[:, i]`

Examples:

```
>>> from mpFormulaPy import mp
>>> A = mp.matrix([[3, 2], [2, 0]])
>>> E = mp.eigsy(A, eigvals_only = True)
>>> print E
[-1.0]
[ 4.0]
>>> A = mp.matrix([[1, 2], [2, 3]])           # alternative: E, Q = mp.eigh(A)
>>> E, Q = mp.eigsy(A)
>>> print mp.chop(A * Q[:,0] - E[0] * Q[:,0])
[0.0]
[0.0]
>>> A = mp.matrix([[1, 2 + 5j], [2 - 5j, 3]])   # alternative: E, Q = mp.eigh(A)
>>> E, Q = mp.eighe(A)
>>> print mp.chop(A * Q[:,0] - E[0] * Q[:,0])
[0.0]
[0.0]
```

# **Part IV**

## **Numerical Calculus**

# Chapter 20

## Polynomials

See also `taylor()` and `chebyfit()` for approximation of functions by polynomials.

### 20.1 Polynomial evaluation

---

Function **`polyval(coeffs As mpNum, x As mpNum, Keywords As String) As mpNum`**

---

The function `polyval` returns a polynomial

**Parameters:**

*coeffs*: A list of coefficients (real or complex numbers).

*x*: A real or complex number.

*Keywords*: `derivative=False`.

Given coefficients  $[c_n, \dots, c_2, c_1, c_0]$  and a number  $x$ , `polyval()` evaluates the polynomial

$$P(x) = c_n x^n + \dots + c_2 x^2 + c_1 x + c_0 \quad (20.1.1)$$

If `derivative=True` is set, `polyval()` simultaneously evaluates  $P(x)$  with the derivative,  $P'(x)$ , and returns the tuple  $(P(x), P'(x))$ .

---

```
>>> from mpFormulaPy import *
>>> mp.pretty = True
>>> polyval([3, 0, 2], 0.5)
2.75
>>> polyval([3, 0, 2], 0.5, derivative=True)
(2.75, 3.0)
```

---

The coefficients and the evaluation point may be any combination of real or complex numbers.

## 20.2 Polynomial roots

Function **polyroots(coeffs As mpNum, Keywords As String) As mpNum**

The function polyroots returns all roots (real or complex) of a given polynomial.

**Parameters:**

*coeffs*: A real or complex number.

*Keywords*: maxsteps=50, cleanup=True, extraprec=10, error=False.

The roots are returned as a sorted list, where real roots appear first followed by complex conjugate roots as adjacent elements. The polynomial should be given as a list of coefficients, in the format used by polyval(). The leading coefficient must be nonzero.

With error=True, polyroots() returns a tuple (roots, err) where err is an estimate of the maximum error among the computed roots.

Examples

Finding the three real roots of  $x^3 - x^2 - 14x + 24$ :

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> nprint(polyroots([1,-1,-14,24]), 4)
[-4.0, 2.0, 3.0]
```

Finding the two complex conjugate roots of  $4x^2 + 3x + 2$ , with an error estimate:

```
>>> roots, err = polyroots([4,3,2], error=True)
>>> for r in roots:
...     print(r)
...
(-0.375 + 0.59947894041409j)
(-0.375 - 0.59947894041409j)
>>>
>>> err
2.22044604925031e-16
>>>
>>> polyval([4,3,2], roots[0])
(2.22044604925031e-16 + 0.0j)
>>> polyval([4,3,2], roots[1])
(2.22044604925031e-16 + 0.0j)
```

The following example computes all the 5th roots of unity; that is, the roots of  $x^5 - 1$ :

```
>>> mp.dps = 20
>>> for r in polyroots([1, 0, 0, 0, 0, -1]):
...     print(r)
...
1.0
(-0.8090169943749474241 + 0.58778525229247312917j)
(-0.8090169943749474241 - 0.58778525229247312917j)
(0.3090169943749474241 + 0.95105651629515357212j)
(0.3090169943749474241 - 0.95105651629515357212j)
```

### Precision and conditioning

The roots are computed to the current working precision accuracy. If this accuracy cannot be achieved in *maxsteps* steps, then a *NoConvergence* exception is raised.

The algorithm internally is using the current working precision extended by *extraprec*. If *NoConvergence* was raised, that is caused either by not having enough extra precision to achieve convergence (in which case increasing *extraprec* should fix the problem) or too low *maxsteps* (in which case increasing *maxsteps* should fix the problem), or a combination of both.

The user should always do a convergence study with regards to *extraprec* to ensure accurate results. It is possible to get convergence to a wrong answer with too low *extraprec*.

Provided there are no repeated roots, *polyroots()* can typically compute all roots of an arbitrary polynomial to high precision:

---

```
>>> mp.dps = 60
>>> for r in polyroots([1, 0, -10, 0, 1]):
...     print r
...
-3.14626436994197234232913506571557044551247712918732870123249
-0.317837245195782244725757617296174288373133378433432554879127
0.317837245195782244725757617296174288373133378433432554879127
3.14626436994197234232913506571557044551247712918732870123249
>>>
>>> sqrt(3) + sqrt(2)
3.14626436994197234232913506571557044551247712918732870123249
>>> sqrt(3) - sqrt(2)
0.317837245195782244725757617296174288373133378433432554879127
```

---

### Algorithm

*polyroots()* implements the Durand-Kerner method [1], which uses complex arithmetic to locate all roots simultaneously. The Durand-Kerner method can be viewed as approximately performing simultaneous Newton iteration for all the roots. In particular, the convergence to simple roots is quadratic, just like Newton's method.

Although all roots are internally calculated using complex arithmetic, any root found to have an imaginary part smaller than the estimated numerical error is truncated to a real number (small real parts are also chopped). Real roots are placed first in the returned list, sorted by value. The remaining complex roots are sorted by their real parts so that conjugate roots end up next to each other.

# Chapter 21

## Root-finding and optimization

### 21.1 Root-finding

---

Function **findroot**(*f* As mpNum, *x0* As mpNum, **Keywords** As String) As mpNum

---

The function `findroot` returns a solution to  $f(x) = 0$ , using  $x_0$  as starting point or interval for  $x$ .

#### Parameters:

*f*: A one dimensional function

*x0*: A real or complex number.

**Keywords**: solver=Secant, tol=None, verbose=False, verify=True. Many more, see below.

Multidimensional overdetermined systems are supported. You can specify them using a function or a list of functions.

If the found root does not satisfy  $|f(x)|^2 < \text{tol}$ , an exception is raised (this can be disabled with `verify=False`).

#### Arguments

*f*: one dimensional function

*x0*: starting point, several starting points or interval (depends on solver)

*tol*: the returned solution has an error smaller than this

*verbose*: print additional information for each iteration if true

*verify*: verify the solution and raise a `ValueError` if

*solver*: a generator for *f* and *x0* returning approximative solution and error

*maxsteps*: after how many steps the solver will cancel

*df*: first derivative of *f* (used by some solvers)

*d2f*: second derivative of *f* (used by some solvers)

*multidimensional*: force multidimensional solving

*J*: Jacobian matrix of *f* (used by multidimensional solvers)

*norm*: used vector norm (used by multidimensional solvers)

solver has to be callable with  $(f, x_0, \text{**kwargs})$  and return an generator yielding pairs of approximative solution and estimated error (which is expected to be positive).

You can use the following string aliases: 'secant', 'mnewton', 'halley', 'muller', 'illinois', 'pegasus', 'anderson', 'ridder', 'anewton', 'bisect'. See `mpFormulaPy.calculus.optimization` for their documentation.

## Examples

The function `findroot()` locates a root of a given function using the secant method by default. A simple example use of the secant method is to compute  $\pi$  as the root of  $\sin x$  closest to  $x_0 = 3$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 30; mp.pretty = True
>>> findroot(sin, 3)
3.14159265358979323846264338328
```

---

The secant method can be used to find complex roots of analytic functions, although it must in that case generally be given a nonreal starting value (or else it will never leave the real line):

---

```
>>> mp.dps = 15
>>> findroot(lambda x: x**3 + 2*x + 1, j)
(0.226698825758202 + 1.46771150871022j)
```

---

A nice application is to compute nontrivial roots of the Riemann zeta function with many digits (good initial values are needed for convergence):

---

```
>>> mp.dps = 30
>>> findroot(zeta, 0.5+14j)
(0.5 + 14.1347251417346937904572519836j)
```

---

The secant method can also be used as an optimization algorithm, by passing it a derivative of a function. The following example locates the positive minimum of the gamma function:

---

```
>>> mp.dps = 20
>>> findroot(lambda x: diff(gamma, x), 1)
1.4616321449683623413
```

---

Finally, a useful application is to compute inverse functions, such as the Lambert W function which is the inverse of  $we^w$ , given the first term of the solution's asymptotic expansion as the initial value. In basic cases, this gives identical results to `mpFormulaPy`'s built-in `lambertw` function:

---

```
>>> def lambert(x):
...     return findroot(lambda w: w*exp(w) - x, log(1+x))
...
>>> mp.dps = 15
>>> lambert(1); lambertw(1)
0.567143290409784
0.567143290409784
>>> lambert(1000); lambert(1000)
5.2496028524016
5.2496028524016
```

---

Multidimensional functions are also supported:

---

```
>>> f = [lambda x1, x2: x1**2 + x2,
... lambda x1, x2: 5*x1**2 - 3*x1 + 2*x2 - 3]
>>> findroot(f, (0, 0))
[-0.618033988749895]
[-0.381966011250105]
>>> findroot(f, (10, 10))
[ 1.61803398874989]
[-2.61803398874989]
```

---

You can verify this by solving the system manually.

Please note that the following (more general) syntax also works:

---

```
>>> def f(x1, x2):
...     return x1**2 + x2, 5*x1**2 - 3*x1 + 2*x2 - 3
...
>>> findroot(f, (0, 0))
[-0.618033988749895]
[-0.381966011250105]
```

---

## Multiple roots

For multiple roots all methods of the Newtonian family (including secant) converge slowly. Consider this example:

---

```
>>> f = lambda x: (x - 1)**99
>>> findroot(f, 0.9, verify=False)
0.918073542444929
```

---

Even for a very close starting point the secant method converges very slowly. Use verbose=True to illustrate this.

It is possible to modify Newton's method to make it converge regardless of the root's multiplicity:

---

```
>>> findroot(f, -10, solver="mnewton")
1.0
```

---

This variant uses the first and second derivative of the function, which is not very efficient.

Alternatively you can use an experimental Newtonian solver that keeps track of the speed of convergence and accelerates it using Steffensen's method if necessary:

---

```
>>> findroot(f, -10, solver="anewton", verbose=True)
x: -9.8888888888888888889
error: 0.11111111111111111111111
converging slowly
x: -9.77890011223344556678
error: 0.10998877665544332211
converging slowly
x: -9.67002233332199662166
error: 0.108877778911448945119
converging slowly
accelerating convergence
x: -9.5622443299551077669
error: 0.107778003366888854764
```

---

```
converging slowly
x: 0.999999999999999214
error: 10.562244329955107759
x: 1.0
error: 7.8598304758094664213e-18
ZeroDivisionError: canceled with x = 1.0
1.0
```

---

## Complex roots

For complex roots it is recommended to use Muller's method as it converges very quickly even for real starting points :

```
>>> findroot(lambda x: x**4 + x + 1, (0, 1, 2), solver="muller")
(0.727136084491197 + 0.934099289460529j)
```

---

## Intersection methods

When you need to find a root in a known interval, it is highly recommended to use an intersection-based solver like 'anderson' or 'ridder'. Usually they converge faster and more reliable. They have however problems with multiple roots and usually need a sign change to find a root:

```
>>> findroot(lambda x: x**3, (-1, 1), solver="anderson")
0.0
```

---

Be careful with symmetric functions:

```
>>> findroot(lambda x: x**2, (-1, 1), solver="anderson")
Traceback (most recent call last):
...
ZeroDivisionError
```

---

It fails even for better starting points, because there is no sign change:

```
>>> findroot(lambda x: x**2, (-1, .5), solver='anderson')
Traceback (most recent call last):
...
ValueError: Could not find root within given tolerance. (1 > 2.1684e-19)
Try another starting point or tweak arguments.
```

---

## 21.2 Solvers

### 21.2.1 Secant

1d-solver generating pairs of approximative root and error. Needs starting points  $x_0$  and  $x_1$  close to the root.  $x_1$  defaults to  $x_0 + 0.25$ .

Pro: converges quickly

Contra: converges slowly for multiple roots

### 21.2.2 Newton

1d-solver generating pairs of approximative root and error. Needs starting points  $x_0$  close to the root.

Pro: converges fast. Sometimes more robust than secant with bad second starting point

Contra: converges slowly for multiple roots. Needs first derivative. 2 function evaluations per iteration

### 21.2.3 MNewton

1d-solver generating pairs of approximative root and error. Needs starting point  $x_0$  close to the root. Uses modified Newton's method that converges fast regardless of the multiplicity of the root.

Pro: converges quickly for multiple roots.

Contra: needs first and second derivative of  $f$ . 3 function evaluations per iteration

### 21.2.4 Halley

1d-solver generating pairs of approximative root and error. Needs a starting point  $x_0$  close to the root. Uses Halley's method with cubic convergence rate.

Pro: converges even faster than Newton's method. Useful when computing with many digits

Contra: needs first and second derivative of  $f$ . 3 function evaluations per iteration. Converges slowly for multiple roots

### 21.2.5 Muller

1d-solver generating pairs of approximative root and error. Needs starting points  $x_0$ ,  $x_1$  and  $x_2$  close to the root.  $x_1$  defaults to  $x_0 + 0.25$ ;  $x_2$  to  $x_1 + 0.25$ . Uses Muller's method that converges towards complex roots.

Pro: converges quickly (somewhat faster than secant). Can find complex roots

Contra: converges slowly for multiple roots. May have complex values for real starting points and real roots.

### 21.2.6 Bisection

1d-solver generating pairs of approximative root and error. Uses bisection method to find a root of  $f$  in  $[a, b]$ . Might fail for multiple roots (needs sign change).

Pro: robust and reliable

Contra: converges slowly. Needs sign change

### 21.2.7 Illinois

1d-solver generating pairs of approximative root and error.

Uses Illinois method or similar to find a root of  $f$  in  $[a, b]$ . Might fail for multiple roots (needs sign change). Combines bisect with secant (improved regula falsi).

The only difference between the methods is the scaling factor  $m$ , which is used to ensure convergence (you can choose one using the 'method' keyword):

Illinois method ('illinois'):  $m = 0.5$

Pegasus method ('pegasus'):  $m = fb/(fb + fz)$

Anderson-Bjoerk method ('anderson'):  $m = 1 - fz/fb$  if positive else 0.5

Pro: converges very fast

Contra: has problems with multiple roots. Needs sign change

### 21.2.8 Pegasus

1d-solver generating pairs of approximative root and error.

Uses Pegasus method to find a root of  $f$  in  $[a, b]$ .

Wrapper for illinois to use method='pegasus'.

### 21.2.9 Anderson

1d-solver generating pairs of approximative root and error.

Uses Anderson-Bjoerk method to find a root of  $f$  in  $[a, b]$ .

Wrapper for illinois to use method='anderson'.

### 21.2.10 Ridder

1d-solver generating pairs of approximative root and error. Ridder's method to find a root of  $f$  in  $[a, b]$ . Is told to perform as well as Brent's method while being simpler.

Pro: very fast. Simpler than Brent's method

Contra: two function evaluations per step. Has problems with multiple roots. Needs sign change

### 21.2.11 ANewton

EXPERIMENTAL 1d-solver generating pairs of approximative root and error.

Uses Newton's method modified to use Steffensen's method when convergence is slow (i.e. for multiple roots.)

### 21.2.12 MDNewton

Find the root of a vector function numerically using Newton's method.

$f$  is a vector function representing a nonlinear equation system.

$x_0$  is the starting point close to the root.

$J$  is a function returning the Jacobian matrix for a point.

Supports overdetermined systems.

Use the 'norm' keyword to specify which norm to use. Defaults to max-norm. The function to calculate the Jacobian matrix can be given using the keyword 'J'. Otherwise it will be calculated numerically.

Please note that this method converges only locally. Especially for high-dimensional systems it is not trivial to find a good starting point being close enough to the root.

It is recommended to use a faster, low-precision solver from SciPy [1] or OpenOpt [2] to get an initial guess. Afterwards you can use this method for root-polishing to any precision.

# Chapter 22

## Sums, products, limits and extrapolation

The functions listed here permit approximation of infinite sums, products, and other sequence limits. Use mpFormulaPy.fsum() and mpFormulaPy.fprod() for summation and multiplication of finite sequences.

### 22.1 Summation

#### 22.1.1 One-dimensional Summation

---

Function **nsum(*f* As mpNum, *interval* As mpNum, *Keywords* As String) As mpNum**

---

The function **nsum** returns a one dimensional (possibly infinite) sum.

**Parameters:**

*f*: A one dimensional function

*interval*: A real interval.

*Keywords*: method=r+s, tol=eps, verbose=False, maxterms=10\*dps. Many more, see below.

This function computes the sum

$$S = \sum_{k=a}^b f(k) \tag{22.1.1}$$

where  $(a, b)$  = interval, and where  $a = -\infty$  and/or  $b = \infty$  are allowed, or more generally

$$S = \sum_{k_1=a_1}^{b_1} \cdots \sum_{k_n=a_n}^{b_n} f(k_1, \dots, k_n) \tag{22.1.2}$$

if multiple intervals are given.

Two examples of infinite series that can be summed by nsum(), where the first converges rapidly and the second converges slowly, are:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> nsum(lambda n: 1/fac(n), [0, inf])
2.71828182845905
>>> nsum(lambda n: 1/n**2, [1, inf])
```

[1.64493406684823](#)

When appropriate, `nsum()` applies convergence acceleration to accurately estimate the sums of slowly convergent series. If the series is finite, `nsum()` currently does not attempt to perform any extrapolation, and simply calls `fsum()`.

Multidimensional infinite series are reduced to a single-dimensional series over expanding hypercubes; if both infinite and finite dimensions are present, the finite ranges are moved innermost. For more advanced control over the summation order, use nested calls to `nsum()`, or manually rewrite the sum as a single-dimensional series.

## Options

*tol*: Desired maximum final error. Defaults roughly to the epsilon of the working precision.

*method*: Which summation algorithm to use (described below). Default: 'richardson+shanks'.

*maxterms*: Cancel after at most this many terms. Default:  $10^*dps$ .

*steps*: An iterable giving the number of terms to add between each extrapolation attempt. The default sequence is [10, 20, 30, 40, ...]. For example, if you know that approximately 100 terms will be required, efficiency might be improved by setting this to [100, 10]. Then the first extrapolation will be performed after 100 terms, the second after 110, etc.

*verbose*: Print details about progress.

*ignore*: If enabled, any term that raises `ArithmeticError` or `ValueError` (e.g. through division by zero) is replaced by a zero. This is convenient for lattice sums with a singular term near the origin.

## Methods

Unfortunately, an algorithm that can efficiently sum any infinite series does not exist. `nsum()` implements several different algorithms that each work well in different cases.

The `method` keyword argument selects a method.

The default method is 'r+s', i.e. both Richardson extrapolation and Shanks transformation is attempted. A slower method that handles more cases is 'r+s+e'. For very high precision summation, or if the summation needs to be fast (for example if multiple sums need to be evaluated), it is a good idea to investigate which one method works best and only use that.

'richardson' / 'r':

Uses Richardson extrapolation. Provides useful extrapolation when  $f(k) \sim P(k)/Q(k)$  or when  $f(k) \sim (-1)^k P(k)/Q(k)$  for polynomials  $P$  and  $Q$ .

See `richardson()` for additional information.

'shanks' / 's':

Uses Shanks transformation. Typically provides useful extrapolation when  $f(k) \sim c^k$  or when successive terms alternate signs. Is able to sum some divergent series. See `shanks()` for additional information.

'levin' / 'l':

Uses the Levin transformation. It performs better than the Shanks transformation for logarithmic convergent or alternating divergent series. The 'levin\_variant'-keyword selects the variant. Valid choices are "u", "t", "v" and "all" whereby "all" uses all three u,t and v simultaneously (This is good for performance comparison in conjunction with "verbose=True"). Instead of the Levin

transform one can also use the Sidi-S transform by selecting the method 'sidi'. See `levin()` for additional details.

'alternating' / 'a':

This is the convergence acceleration of alternating series developed by Cohen, Villegas and Zagier. See `cohen_alt()` for additional details.

'euler-maclaurin' / 'e':

Uses the Euler-Maclaurin summation formula to approximate the remainder sum by an integral. This requires high-order numerical derivatives and numerical integration. The advantage of this algorithm is that it works regardless of the decay rate of  $f$ , as long as  $f$  is sufficiently smooth. See `sumem()` for additional information.

'direct' / 'd':

Does not perform any extrapolation. This can be used (and should only be used for) rapidly convergent series. The summation automatically stops when the terms decrease below the target tolerance.

## Basic examples

A finite sum:

---

```
>>> nsum(lambda k: 1/k, [1, 6])
2.45
```

---

Summation of a series going to negative infinity and a doubly infinite series:

---

```
>>> nsum(lambda k: 1/k**2, [-inf, -1])
1.64493406684823
>>> nsum(lambda k: 1/(1+k**2), [-inf, inf])
3.15334809493716
```

---

`nsum()` handles sums of complex numbers:

---

```
>>> nsum(lambda k: (0.5+0.25j)**k, [0, inf])
(1.6 + 0.8j)
```

---

The following sum converges very rapidly, so it is most efficient to sum it by disabling convergence acceleration:

---

```
>>> mp.dps = 1000
>>> a = nsum(lambda k: -(-1)**k * k**2 / fac(2*k), [1, inf], method="direct")
>>> b = (cos(1)+sin(1))/4
>>> abs(a-b) < mpf('1e-998')
True
```

---

## Examples with Richardson extrapolation

Richardson extrapolation works well for sums over rational functions, as well as their alternating counterparts:

---

```
>>> mp.dps = 50
>>> nsum(lambda k: 1 / k**3, [1, inf], method="richardson")
1.2020569031595942853997381615114499907649862923405
>>> zeta(3)
```

---

```

1.2020569031595942853997381615114499907649862923405
>>> nsum(lambda n: (n + 3)/(n**3 + n**2), [1, inf], method="richardson")
2.9348022005446793094172454999380755676568497036204
>>> pi**2/2-2
2.9348022005446793094172454999380755676568497036204
>>> nsum(lambda k: (-1)**k / k**3, [1, inf], method="richardson")
-0.90154267736969571404980362113358749307373971925537
>>> -3*zeta(3)/4
-0.90154267736969571404980362113358749307373971925538

```

---

### Examples with Shanks extrapolation

The Shanks transformation works well for geometric series and typically provides excellent acceleration for Taylor series near the border of their disk of convergence. Here we apply it to a series for  $\log(2)$ , which can be seen as the Taylor series for  $\log(1 + x)$  with  $x = 1$ :

---

```

>>> nsum(lambda k: -(-1)**k/k, [1, inf], method="shanks")
0.69314718055994530941723212145817656807550013436025
>>> log(2)
0.69314718055994530941723212145817656807550013436025

```

---

Here we apply it to a slowly convergent geometric series:

---

```

>>> nsum(lambda k: mpf("0.995")**k, [0, inf], method="shanks")
200.0

```

---

Finally, Shanks' method works very well for alternating series where  $f(k) = (-1)^k g(k)$ , and often does so regardless of the exact decay rate of  $g(k)$ :

---

```

>>> mp.dps = 15
>>> nsum(lambda k: (-1)**(k+1) / k**1.5, [1, inf], method="shanks")
0.765147024625408
>>> (2-sqrt(2))*zeta(1.5)/2
0.765147024625408

```

---

The following slowly convergent alternating series has no known closed-form value. Evaluating the sum a second time at higher precision indicates that the value is probably correct:

---

```

>>> nsum(lambda k: (-1)**k / log(k), [2, inf], method="shanks")
0.924299897222939
>>> mp.dps = 30
>>> nsum(lambda k: (-1)**k / log(k), [2, inf], method="shanks")
0.924299897222938855595957018136

```

---

### Examples with Levin transformation

The following example calculates EulerâŽ's constant as the constant term in the Laurent expansion of  $\zeta(s)$  at  $s=1$ . This sum converges extremely slow because of the logarithmic convergence behaviour of the Dirichlet series for  $\zeta$ .

---

```

>>> mp.dps = 30
>>> z = mp.mpf(10) ** (-10)
>>> a = mp.nsum(lambda n: n**(-(1+z)), [1, mp.inf], method = "levin") - 1 / z
>>> print(mp.chop(a - mp.euler, tol = 1e-10))

```

---

---

0.0

---

Now we sum the zeta function outside its range of convergence (attention: This does not work at the negative integers!):

---

```
>>> mp.dps = 15
>>> w = mp.nsum(lambda n: n ** (2 + 3j), [1, mp.inf], method = "levin", levin_variant = "v")
>>> print(mp.chop(w - mp.zeta(-2-3j)))
0.0
```

---

The next example resummates an asymptotic series expansion of an integral related to the exponential integral.

---

```
>>> mp.dps = 15
>>> z = mp.mpf(10)
>>> # exact = mp.quad(lambda x: mp.exp(-x)/(1+x/z), [0,mp.inf])
>>> exact = z * mp.exp(z) * mp.expint(1,z) # this is the symbolic expression for the integral
>>> w = mp.nsum(lambda n: (-1) ** n * mp.fac(n) * z ** (-n), [0, mp.inf], method = "sidi", levin_variant = "t")
>>> print(mp.chop(w - exact))
0.0
```

---

Following highly divergent asymptotic expansion needs some care. Firstly we need copious amount of working precision. Secondly the stepsize must not be chosen too large, otherwise nsum may miss the point where the Levin transform converges and reach the point where only numerical garbage is produced due to numerical cancellation.

---

```
>>> mp.dps = 15
>>> z = mp.mpf(2)
>>> # exact = mp.quad(lambda x: mp.exp( -x * x / 2 - z * x ** 4), [0,mp.inf]) * 2 / mp.sqrt(2 * mp.pi)
>>> exact = mp.exp(mp.one / (32 * z)) * mp.besselk(mp.one / 4, mp.one / (32 * z)) / (4 * mp.sqrt(z * mp.pi)) # this is the symbolic expression for the integral
>>> w = mp.nsum(lambda n: (-z)**n * mp.fac(4 * n) / (mp.fac(n) * mp.fac(2 * n) * (4 ** n)),
...   [0, mp.inf], method = "levin", levin_variant = "t", workprec = 8*mp.prec, steps = [2] + [1 for x in xrange(1000)])
>>> print(mp.chop(w - exact))
0.0
```

---

The hypergeometric function can also be summed outside its range of convergence:

---

```
>>> mp.dps = 15
>>> z = 2 + 1j
>>> exact = mp.hyp2f1(2 / mp.mpf(3), 4 / mp.mpf(3), 1 / mp.mpf(3), z)
>>> f = lambda n: mp.rf(2 / mp.mpf(3), n) * mp.rf(4 / mp.mpf(3), n) * z**n / (mp.rf(1 / mp.mpf(3), n) * mp.fac(n))
>>> v = mp.nsum(f, [0, mp.inf], method = "levin", steps = [10 for x in xrange(1000)])
>>> print(mp.chop(exact-v))
0.0
```

---

### Examples with Cohen's alternating series resummation

The next example sums the alternating zeta function:

---

```
>>> v = mp.nsum(lambda n: (-1)**(n-1) / n, [1, mp.inf], method = "a")
>>> print(mp.chop(v - mp.log(2)))
0.0
```

---

The derivate of the alternating zeta function outside its range of convergence:

---

```
>>> v = mp.nsum(lambda n: (-1)**n * mp.log(n) * n, [1, mp.inf], method = "a")
>>> print(mp.chop(v - mp.diff(lambda s: mp.altzeta(s), -1)))
0.0
```

---

### Examples with Euler-Maclaurin summation

The sum in the following example has the wrong rate of convergence for either Richardson or Shanks to be effective.

---

```
>>> f = lambda k: log(k)/k**2.5
>>> mp.dps = 15
>>> nsum(f, [1, inf], method="euler-maclaurin")
0.38734195032621
>>> -diff(zeta, 2.5)
0.38734195032621
```

---

Increasing steps improves speed at higher precision:

---

```
>>> f = lambda k: log(k)/k**2.5
>>> mp.dps = 50
>>> nsum(f, [1, inf], method="euler-maclaurin", steps=[250])
0.38734195032620997271199237593105101319948228874688
>>> -diff(zeta, 2.5)
0.38734195032620997271199237593105101319948228874688
```

---

### Divergent series

The Shanks transformation is able to sum some divergent series. In particular, it is often able to sum Taylor series beyond their radius of convergence (this is due to a relation between the Shanks transformation and Pade approximations; see pade() for an alternative way to evaluate divergent Taylor series). Furthermore the Levintransform examples above contain some divergent series resummation.

Here we apply it to  $\log(1 + x)$  far outside the region of convergence:

---

```
>>> mp.dps = 50
>>> nsum(lambda k: -(-9)**k/k, [1, inf], method="shanks")
2.3025850929940456840179914546843642076011014886288
>>> log(10)
2.3025850929940456840179914546843642076011014886288
```

---

A particular type of divergent series that can be summed using the Shanks transformation is geometric series. The result is the same as using the closed-form formula for an infinite geometric series:

---

```
>>> mp.dps = 15
```

```
>>> for n in range(-8, 8):
... if n == 1:
... continue
... print("%s %s %s" % (mpf(n), mpf(1)/(1-n),
... nsum(lambda k: n**k, [0, inf], method="shanks")))
...
-8.0 0.1111111111111111 0.1111111111111111
-7.0 0.125 0.125
-6.0 0.142857142857143 0.142857142857143
-5.0 0.1666666666666667 0.1666666666666667
-4.0 0.2 0.2
-3.0 0.25 0.25
-2.0 0.3333333333333333 0.3333333333333333
-1.0 0.5 0.5
0.0 1.0 1.0
2.0 -1.0 -1.0
3.0 -0.5 -0.5
4.0 -0.3333333333333333 -0.3333333333333333
5.0 -0.25 -0.25
6.0 -0.2 -0.2
7.0 -0.1666666666666667 -0.1666666666666667
```

---

### 22.1.2 Two-dimensional Summation

Function **nsum2d**(*f* As *mpNum*, *interval1* As *mpNum*, *interval2* As *mpNum*, *Keywords* As *String*) As *mpNum*

---

The function **nsum2d** returns a two dimensional (possibly infinite) sum.

#### Parameters:

*f*: A one dimensional function

*interval1*: A real interval.

*interval2*: A real interval.

*Keywords*: method=r+s, tol=eps, verbose=False, maxterms=10\*dps. Many more, see below.

Any combination of finite and infinite ranges is allowed for the summation indices:

```
>>> mp.dps = 15
>>> nsum2d(lambda x,y: x+y, [2,3], [4,5])
28.0
>>> nsum2d(lambda x,y: x/2**y, [1,3], [1,inf])
6.0
>>> nsum2d(lambda x,y: y/2**x, [1,inf], [1,3])
6.0
>>> nsum3d(lambda x,y,z: z/(2**x*2**y), [1,inf], [1,inf], [3,4])
7.0
>>> nsum3d(lambda x,y,z: y/(2**x*2**z), [1,inf], [3,4], [1,inf])
7.0
>>> nsum3d(lambda x,y,z: x/(2**z*2**y), [3,4], [1,inf], [1,inf])
7.0
```

---

Some nice examples of double series with analytic solutions or reductions to single-dimensional series (see [1]):

---

```
>>> nsum2d(lambda m, n: 1/2** (m*n), [1,inf], [1,inf])
1.60669515241529
>>> nsum(lambda n: 1/(2**n-1), [1,inf])
1.60669515241529
>>> nsum2d(lambda i,j: (-1)**(i+j)/(i**2+j**2), [1,inf], [1,inf])
0.278070510848213
>>> pi*(pi-3*ln2)/12
0.278070510848213
>>> nsum2d(lambda i,j: (-1)**(i+j)/(i+j)**2, [1,inf], [1,inf])
0.129319852864168
>>> altzeta(2) - altzeta(1)
0.129319852864168
>>> nsum2d(lambda i,j: (-1)**(i+j)/(i+j)**3, [1,inf], [1,inf])
0.0790756439455825
>>> altzeta(3) - altzeta(2)
0.0790756439455825
>>> nsum2d(lambda m,n: m**2*n/(3**m*(n*3**m+m*3**n)), [1,inf], [1,inf])
0.28125
>>> mpf(9)/32
0.28125
>>> nsum2d(lambda i,j: fac(i-1)*fac(j-1)/fac(i+j), [1,inf], [1,inf], workprec=400)
1.64493406684823
>>> zeta(2)
1.64493406684823
```

---

### 22.1.3 Three-dimensional Summation

---

Function **nsum3d**(*f* As *mpNum*, **interval1** As *mpNum*, **interval2** As *mpNum*, **interval3** As *mpNum*, **Keywords** As String) As *mpNum*

---

The function **nsum3d** returns a three dimensional (possibly infinite) sum .

**Parameters:**

*f*: A one dimensional function

*interval1*: A real interval.

*interval2*: A real interval.

*interval3*: A real interval.

**Keywords**: method=r+s, tol=eps, verbose=False, maxterms=10\*dps. Many more, see below.

A hard example of a multidimensional sum is the Madelung constant in three dimensions (see [2]). The defining sum converges very slowly and only conditionally, so **nsum()** is lucky to obtain an accurate value through convergence acceleration. The second evaluation below uses a much more efficient, rapidly convergent 2D sum:

---

```
>>> nsum3d(lambda x,y,z: (-1)**(x+y+z)/(x*x+y*y+z*z)**0.5,
... [-inf,inf], [-inf,inf], [-inf,inf], ignore=True)
-1.74756459463318
>>> nsum2d(lambda x,y: -12*pi*sech(0.5*pi * \
```

---

```
... sqrt((2*x+1)**2+(2*y+1)**2))**2, [0,inf], [0,inf])
-1.74756459463318
```

---

Another example of a lattice sum in 2D:

---

```
>>> nsum(lambda x,y: (-1)**(x+y) / (x**2+y**2), [-inf,inf],
... [-inf,inf], ignore=True)
-2.1775860903036
>>> -pi*ln2
-2.1775860903036
```

---

An example of an Eisenstein series:

---

```
>>> nsum(lambda m,n: (m+n*1j)**(-4), [-inf,inf], [-inf,inf],
... ignore=True)
(3.1512120021539 + 0.0j)
```

---

## 22.1.4 Euler-Maclaurin formula

---

Function **sumem**(*f* As mpNum, *interval* As mpNum, *Keywords* As String) As mpNum

---

The function **sumem** returns an infinite series of an analytic summand *f* using the Euler-Maclaurin formula

**Parameters:**

*f*: A one dimensional function

*interval*: A real interval.

*Keywords*: tol=None, reject=10, integral=None, adiffs=None, bdiffs=None, verbose=False, error=False, fastabort=False

Uses the Euler-Maclaurin formula to compute an approximation accurate to within tol (which defaults to the present epsilon) of the sum

$$S = \sum_{k=a}^b f(k) \quad (22.1.3)$$

where  $(a, b)$  are given by interval and or may be infinite. The approximation is

$$S \sim \int_a^b f(x)dx + \frac{f(a) + f(b)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (f^{(2k-1)}(b) - f^{(2k-1)}(a)) \quad (22.1.4)$$

The last sum in the Euler-Maclaurin formula is not generally convergent (a notable exception is if *f* is a polynomial, in which case Euler-Maclaurin actually gives an exact result).

The summation is stopped as soon as the quotient between two consecutive terms falls below reject. That is, by default (reject = 10), the summation is continued as long as each term adds at least one decimal.

Although not convergent, convergence to a given tolerance can often be "forced" if  $b = \infty$  by summing up to  $a + N$  and then applying the Euler-Maclaurin formula to the sum over the range  $(a + N + 1, \dots, \infty)$ . This procedure is implemented by nsum().

By default numerical quadrature and differentiation is used. If the symbolic values of the integral and endpoint derivatives are known, it is more efficient to pass the value of the integral explicitly as integral and the derivatives explicitly as adiffs and bdiffs. The derivatives should be given as iterables that yield  $f(a), f'(a), f''(a), \dots$  (and the equivalent for  $b$ ).

### Examples

Summation of an infinite series, with automatic and symbolic integral and derivative values (the second should be much faster):

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 50; mp.pretty = True
>>> sumem(lambda n: 1/n**2, [32, inf])
0.03174336652030209012658168043874142714132886413417
>>> I = mpf(1)/32
>>> D = adiffs=(-1)**n*fac(n+1)*32**(-2-n) for n in range(999)
>>> sumem(lambda n: 1/n**2, [32, inf], integral=I, adiffs=D)
0.03174336652030209012658168043874142714132886413417
```

---

An exact evaluation of a finite polynomial sum:

---

```
>>> sumem(lambda n: n**5-12*n**2+3*n, [-100000, 200000])
10500155000624963999742499550000.0
>>> print(sum(n**5-12*n**2+3*n for n in range(-100000, 200001)))
10500155000624963999742499550000
```

---

### 22.1.5 Abel-Plana formula

---

Function **sumap**(*f* As *mpNum*, *interval* As *mpNum*, *Keywords* As String) As *mpNum*

---

The function **sumap** returns an infinite series of an analytic summand *f* using the Abel-Plana formula.

**Parameters:**

*f*: A one dimensional function

*interval*: A real interval.

*Keywords*: integral=None, error=False

Evaluates an infinite series of an analytic summand *f* using the Abel-Plana formula

$$\sum_{k=0}^{\infty} f(k) = \int_0^{\infty} f(t)dt + \frac{1}{2}f(0) + i \int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt \quad (22.1.5)$$

Unlike the Euler-Maclaurin formula (see **sumem()**), the Abel-Plana formula does not require derivatives. However, it only works when  $|f(it) - f(-it)|$  does not increase too rapidly with *t*.

### Examples

The Abel-Plana formula is particularly useful when the summand decreases like a power of *k*; for example when the sum is a pure zeta function:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> sumap(lambda k: 1/k**2.5, [1,inf])
1.34148725725091717975677
```

```
>>> zeta(2.5)
1.34148725725091717975677
>>> sumap(lambda k: 1/(k+1j)**(2.5+2.5j), [1,inf])
(-3.385361068546473342286084 - 0.7432082105196321803869551j)
>>> zeta(2.5+2.5j, 1+1j)
(-3.385361068546473342286084 - 0.7432082105196321803869551j)
```

---

If the series is alternating, numerical quadrature along the real line is likely to give poor results, so it is better to evaluate the first term symbolically whenever possible:

```
>>> n=3; z=-0.75
>>> I = expint(n,-log(z))
>>> chop(sumap(lambda k: z**k / k**n, [1,inf], integral=I))
-0.6917036036904594510141448
>>> polylog(n,z)
-0.6917036036904594510141448
```

---

## 22.2 Products

Function **nprod(*f* As mpNum, *interval* As mpNum, *Keywords* As String) As mpNum**

The function nprod returns a one dimensional (possibly infinite) product.

### Parameters:

*f*: A one dimensional function

*interval*: A real interval.

*Keywords*: nsum=False.

Computes the product

$$P = \prod_{k=a}^b f(k) \quad (22.2.1)$$

where  $= (a, b)$  interval, and where  $a = -\infty$  and/or  $b = \infty$  are allowed.

By default, nprod() uses the same extrapolation methods as nsum(), except applied to the partial products rather than partial sums, and the same keyword options as for nsum() are supported. If nsum=True, the product is instead computed via nsum() as

$$P = \exp \left( \sum_{k=a}^b \log(f(k)) \right). \quad (22.2.2)$$

This is slower, but can sometimes yield better results. It is also required (and used automatically) when Euler-Maclaurin summation is requested.

### Examples

A simple finite product:

```
>>> from mpFormulaPy import *
>>> mp.dps = 25; mp.pretty = True
>>> nprod(lambda k: k, [1, 4])
24.0
```

A large number of infinite products have known exact values, and can therefore be used as a reference. Most of the following examples are taken from MathWorld [1].

A few infinite products with simple values are:

```
>>> 2*nprod(lambda k: (4*k**2)/(4*k**2-1), [1, inf])
3.141592653589793238462643
>>> nprod(lambda k: (1+1/k)**2/(1+2/k), [1, inf])
2.0
>>> nprod(lambda k: (k**3-1)/(k**3+1), [2, inf])
0.66666666666666666666666666666667
>>> nprod(lambda k: (1-1/k**2), [2, inf])
0.5
```

Next, several more infinite products with more complicated values:

```
>>> nprod(lambda k: exp(1/k**2), [1, inf]); exp(pi**2/6)
5.180668317897115748416626
5.180668317897115748416626
>>> nprod(lambda k: (k**2-1)/(k**2+1), [2, inf]); pi*csch(pi)
```

```

0.2720290549821331629502366
0.2720290549821331629502366
>>> nprod(lambda k: (k**4-1)/(k**4+1), [2, inf])
0.8480540493529003921296502
>>> pi*sinh(pi)/(cosh(sqrt(2)*pi)-cos(sqrt(2)*pi))
0.8480540493529003921296502
>>> nprod(lambda k: (1+1/k+1/k**2)**2/(1+2/k+3/k**2), [1, inf])
1.848936182858244485224927
>>> 3*sqrt(2)*cosh(pi*sqrt(3)/2)**2*csch(pi*sqrt(2))/pi
1.848936182858244485224927
>>> nprod(lambda k: (1-1/k**4), [2, inf]); sinh(pi)/(4*pi)
0.9190194775937444301739244
0.9190194775937444301739244
>>> nprod(lambda k: (1-1/k**6), [2, inf])
0.9826842777421925183244759
>>> (1+cosh(pi*sqrt(3)))/(12*pi**2)
0.9826842777421925183244759
>>> nprod(lambda k: (1+1/k**2), [2, inf]); sinh(pi)/(2*pi)
1.838038955187488860347849
1.838038955187488860347849
>>> nprod(lambda n: (1+1/n)**n * exp(1/(2*n)-1), [1, inf])
1.447255926890365298959138
>>> exp(1+euler/2)/sqrt(2*pi)
1.447255926890365298959138

```

---

The following two products are equivalent and can be evaluated in terms of a Jacobi theta function. Pi can be replaced by any value (as long as convergence is preserved):

```

>>> nprod(lambda k: (1-pi**-k)/(1+pi**-k), [1, inf])
0.3838451207481672404778686
>>> nprod(lambda k: tanh(k*log(pi)/2), [1, inf])
0.3838451207481672404778686
>>> jtheta(4,0,1/pi)
0.3838451207481672404778686

```

---

This product does not have a known closed form value:

```

>>> nprod(lambda k: (1-1/2**k), [1, inf])
0.2887880950866024212788997

```

---

A product taken from  $-\infty$ :

```

>>> nprod(lambda k: 1-k*(-3), [-inf,-2])
0.8093965973662901095786805
>>> cosh(pi*sqrt(3)/2)/(3*pi)
0.8093965973662901095786805

```

---

A doubly infinite product:

```

>>> nprod(lambda k: exp(1/(1+k**2)), [-inf, inf])
23.41432688231864337420035
>>> exp(pi/tanh(pi))
23.41432688231864337420035

```

---

A product requiring the use of Euler-Maclaurin summation to compute an accurate value:

```
>>> nprod(lambda k: (1-1/k**2.5), [2, inf], method="e")
0.696155111336231052898125
```

## 22.3 Limits

Function **limit**(*f* As mpNum, **interval** As mpNum, **Keywords** As String) As mpNum

The function `limit` returns an estimate of the limit.

## Parameters:

### f. A one dimensional function

*interval*: A real interval.

*Keywords:* direction=1, exp=False.

Computes an estimate of the limit

$$\lim_{t \rightarrow x} f(t) \quad (22.3.1)$$

where  $x$  may be finite or infinite.

For finite  $x$ , `limit()` evaluates  $f(x + d/n)$  for consecutive integer values of  $n$ , where the approach direction  $d$  may be specified using the `direction` keyword argument. For infinite  $x$ , `limit()` evaluates values of  $f(\text{sign}(x) \cdot n)$ .

If the approach to the limit is not sufficiently fast to give an accurate estimate directly, `limit()` attempts to find the limit using Richardson extrapolation or the Shanks transformation. You can select between these methods using the `method` keyword (see documentation of `nsum()` for more information).

## Options

The following options are available with essentially the same meaning as for `nsum()`: `tol`, `method`, `maxterms`, `steps`, `verbose`.

If the option `exp=True` is set,  $f$  will be sampled at exponentially spaced points  $n = 2^1, 2^2, 2^3, \dots$  instead of the linearly spaced points  $1, 2, 3, \dots$ . This can sometimes improve the rate of convergence so that `limit()` may return a more accurate answer (and faster). However, do note that this can only be used if  $f$  supports fast and accurate evaluation for arguments that are extremely close to the limit point (or if infinite, very large arguments).

## Examples

## A basic evaluation of a removable singularity:

```
>>> from mpFormulaPy import *
>>> mp.dps = 30; mp.pretty = True
>>> limit(lambda x: (x-sin(x))/x**3, 0)
0.166666666666666666666666666667
```

Computing the exponential function using its limit definition:

```
>>> limit(lambda n: (1+3/n)**n, inf)
20.0855369231876677409285296546
>>> exp(3)
20.0855369231876677409285296546
```

A limit for  $\pi$ :

```
>>> f = lambda n: 2**4*n+1*fac(n)**4/(2*n+1)/fac(2*n)**2
>>> limit(f, inf)
3.14159265358979323846264338328
```

Calculating the coefficient in Stirling's formula:

---

```
>>> limit(lambda n: fac(n) / (sqrt(n)*(n/e)**n), inf)
2.50662827463100050241576528481
>>> sqrt(2*pi)
2.50662827463100050241576528481
```

---

Evaluating Euler's constant  $\gamma$  using the limit representation

$$\gamma = \lim_{n \rightarrow \infty} \left[ \left( \sum_{k=1}^n \frac{1}{k} \right) - \log(n) \right] \quad (22.3.2)$$

which converges notoriously slowly):

---

```
>>> f = lambda n: sum([mpf(1)/k for k in range(1,int(n)+1)]) - log(n)
>>> limit(f, inf)
0.577215664901532860606512090082
>>> +euler
0.577215664901532860606512090082
```

---

With default settings, the following limit converges too slowly to be evaluated accurately. Changing to exponential sampling however gives a perfect result:

---

```
>>> f = lambda x: sqrt(x**3+x**2)/(sqrt(x**3)+x)
>>> limit(f, inf)
0.992831158558330281129249686491
>>> limit(f, inf, exp=True)
1.0
```

---

## 22.4 Extrapolation

The following functions provide a direct interface to extrapolation algorithms. `nsum()` and `limit()` essentially work by calling the following functions with an increasing number of terms until the extrapolated limit is accurate enough.

The following functions may be useful to call directly if the precise number of terms needed to achieve a desired accuracy is known in advance, or if one wishes to study the convergence properties of the algorithms.

### 22.4.1 Richardson's algorithm

---

Function **richardson(*seq* As *mpNum*) As *mpNum***

---

The function `richardson` returns the  $N$ -term Richardson extrapolate for the limit.

**Parameter:**

*seq*: a list of the first  $N$  elements of a slowly convergent infinite sequence.

Given a list *seq* of the first  $N$  elements of a slowly convergent infinite sequence, `richardson()` computes the  $N$ -term Richardson extrapolate for the limit.

`richardson()` returns  $(v, c)$  where  $v$  is the estimated limit and  $c$  is the magnitude of the largest weight used during the computation. The weight provides an estimate of the precision lost to cancellation. Due to cancellation effects, the sequence must be typically be computed at a much higher precision than the target accuracy of the extrapolation.

#### Applicability and issues

The  $N$ -step Richardson extrapolation algorithm used by `richardson()` is described in [1].

Richardson extrapolation only works for a specific type of sequence, namely one converging like partial sums of  $P(1)/Q(1) + P(2)/Q(2) + \dots$  where  $P$  and  $Q$  are polynomials. When the sequence does not converge at such a rate `richardson()` generally produces garbage.

Richardson extrapolation has the advantage of being fast: the  $N$ -term extrapolate requires only  $O(N)$  arithmetic operations, and usually produces an estimate that is accurate to  $O(N)$  digits. Contrast with the Shanks transformation (see `shanks()`), which requires  $O(N^2)$  operations.

`richardson()` is unable to produce an estimate for the approximation error. One way to estimate the error is to perform two extrapolations with slightly different  $N$  and comparing the results. Richardson extrapolation does not work for oscillating sequences. As a simple workaround, `richardson()` detects if the last three elements do not differ monotonically, and in that case applies extrapolation only to the even-index elements.

#### Example

Applying Richardson extrapolation to the Leibniz series for  $\pi$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 30; mp.pretty = True
>>> S = [4*sum(mpf(-1)**n/(2*n+1) for n in range(m))
... for m in range(1,30)]
>>> v, c = richardson(S[:10])
>>> v
3.2126984126984126984126984127
>>> nprint([v-pi, c])
```

---

```
[0.0711058, 2.0]
>>> v, c = richardson(S[:30])
>>> v
3.14159265468624052829954206226
>>> nprint([v-pi, c])
[1.09645e-9, 20833.3]
```

---

## 22.4.2 Shanks' algorithm

---

Function **shanks**(*seq* As mpNum, *Keywords* As String) As mpNum

---

The function **shanks** returns the  $N$ -term Shanks extrapolate for the limit.

**Parameters:**

*seq*: a list of the first  $N$  elements of a slowly convergent infinite sequence.

*Keywords*: table=None, randomized=False.

Given a list *seq* of the first  $N$  elements of a slowly convergent infinite sequence  $(A_k)$ , **shanks()** computes the iterated Shanks transformation  $S(A), S(S(A)), \dots, S^{N/2}(A)$ . The Shanks transformation often provides strong convergence acceleration, especially if the sequence is oscillating.

The iterated Shanks transformation is computed using the Wynn epsilon algorithm (see [1]). **shanks()** returns the full epsilon table generated by Wynn's algorithm, which can be read off as follows:

The table is a list of lists forming a lower triangular matrix, where higher row and column indices correspond to more accurate values.

The columns with even index hold dummy entries (required for the computation) and the columns with odd index hold the actual extrapolates.

The last element in the last row is typically the most accurate estimate of the limit.

The difference to the third last element in the last row provides an estimate of the approximation error.

The magnitude of the second last element provides an estimate of the numerical accuracy lost to cancellation.

For convenience, so the extrapolation is stopped at an odd index so that **shanks(seq)[-1][-1]** always gives an estimate of the limit.

Optionally, an existing table can be passed to **shanks()**. This can be used to efficiently extend a previous computation after new elements have been appended to the sequence. The table will then be updated in-place.

### The Shanks transformation

The Shanks transformation is defined as follows (see [2]): given the input sequence  $(A_0, A_1, \dots)$ , the transformed sequence is given by

$$S(A_k) = \frac{A_{k+1}A_{k-1} - A_k^2}{A_{k+1} + A_{k-1} - 2A_k} \quad (22.4.1)$$

The Shanks transformation gives the exact limit  $A_\infty$  in a single step if  $A_k = A + aq^k$ . Note in particular that it extrapolates the exact sum of a geometric series in a single step.

Applying the Shanks transformation once often improves convergence substantially for an arbitrary sequence, but the optimal effect is obtained by applying it iteratively:  $S(S(A_k)), S(S(S(A_k))), \dots$

Wynn's epsilon algorithm provides an efficient way to generate the table of iterated Shanks transformations. It reduces the computation of each element to essentially a single division, at the cost of requiring dummy elements in the table. See [1] for details.

### Precision issues

Due to cancellation effects, the sequence must be typically be computed at a much higher precision than the target accuracy of the extrapolation.

If the Shanks transformation converges to the exact limit (such as if the sequence is a geometric series), then a division by zero occurs. By default, shanks() handles this case by terminating the iteration and returning the table it has generated so far. With *randomized=True*, it will instead replace the zero by a pseudorandom number close to zero. (TODO: find a better solution to this problem.)

### Examples

We illustrate by applying Shanks transformation to the Leibniz series for  $\pi$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 50
>>> S = [4*sum(mpf(-1)**n/(2*n+1) for n in range(m))
... for m in range(1,30)]
>>>
>>> T = shanks(S[:7])
>>> for row in T:
...     nprint(row)
...
[-0.75]
[1.25, 3.16667]
[-1.75, 3.13333, -28.75]
[2.25, 3.14524, 82.25, 3.14234]
[-2.75, 3.13968, -177.75, 3.14139, -969.937]
[3.25, 3.14271, 327.25, 3.14166, 3515.06, 3.14161]
```

---

The extrapolated accuracy is about 4 digits, and about 4 digits may have been lost due to cancellation:

---

```
>>> L = T[-1]
>>> nprint([abs(L[-1] - pi), abs(L[-1] - L[-3]), abs(L[-2])])
[2.22532e-5, 4.78309e-5, 3515.06]
```

---

Now we extend the computation:

---

```
>>> T = shanks(S[:25], T)
>>> L = T[-1]
>>> nprint([abs(L[-1] - pi), abs(L[-1] - L[-3]), abs(L[-2])])
[3.75527e-19, 1.48478e-19, 2.96014e+17]
```

---

The value for  $\pi$  is now accurate to 18 digits. About 18 digits may also have been lost to cancellation.

Here is an example with a geometric series, where the convergence is immediate (the sum is

exactly 1):

---

```
>>> mp.dps = 15
>>> for row in shanks([0.5, 0.75, 0.875, 0.9375, 0.96875]):
...     nprint(row)
[4.0]
[8.0, 1.0]
```

---

### 22.4.3 Levin's algorithm

---

Function **levin(*Keywords* As String) As Object**

---

The function **levin** returns an object with **sn update** method. Only for use within Python

**Parameter:**

**Keywords:** method='levin', variant='u'.

This interface implements Levin's (nonlinear) sequence transformation for convergence acceleration and summation of divergent series. It performs better than the Shanks/Wynn-epsilon algorithm for logarithmic convergent or alternating divergent series.

Let  $A$  be the series we want to sum:

$$A = \sum_{k=0}^{\infty} a_k. \quad (22.4.2)$$

Attention: all  $a_k$  must be non-zero!

Let  $s_n$  be the partial sums of this series:

$$s_n = \sum_{k=0}^n a_k. \quad (22.4.3)$$

#### Methods

Calling **levin** returns an object with the following methods.

**update(...)** works with the list of individual terms  $a_k$  of  $A$ , and **update\_psum(...)** works with the list of partial sums  $s_k$  of  $A$ :

---

```
v, e = ...update([a_0, a_1, ..., a_k])
v, e = ...update_psum([s_0, s_1, ..., s_k])
```

---

**step(...)** works with the individual terms and **step\_psum(...)** works with the partial sums  $s_k$ :

---

```
v, e = ...step(a_k)
v, e = ...step_psum(s_k)
```

---

$v$  is the current estimate for  $A$ , and  $e$  is an error estimate which is simply the difference between the current estimate and the last estimate. One should not mix **update**, **update\_psum**, **step** and **step\_psum**.

#### A word of caution

One can only hope for good results (i.e. convergence acceleration or resummation) if the  $s_n$  have some well defined asymptotic behavior for large  $n$  and are not erratic or random. Furthermore one usually needs very high working precision because of the numerical cancellation.

If the working precision is insufficient, levin may produce silently numerical garbage. Furthermore even if the Levin-transformation converges, in the general case there is no proof that the result is mathematically sound. Only for very special classes of problems one can prove that the Levin-transformation converges to the expected result (for example Stieltjes-type integrals).

Furthermore the Levintransform is quite expensive (i.e. slow) in comparison to Shanks/Wynn-epsilon, Richardson & co. In summary one can say that the Levin-transformation is powerful but unreliable and that it may need a copious amount of working precision.

The Levin transform has several variants differing in the choice of weights. Some variants are better suited for the possible flavours of convergence behaviour of  $A$  than other variants:

---

```
convergence behaviour levin-u levin-t levin-v shanks/wynn-epsilon
```

logarithmic	+	-	+	-
linear	+	+	+	+
alternating divergent	+	+	+	+

"+" means the variant **is** suitable, "-" means the variant **is not** suitable;  
for comparison the Shanks/Wynn-epsilon transform **is** listed, too.

---

The variant is controlled through the variant keyword (i.e. variant="u", variant="t" or variant="v"). Overall "u" is probably the best choice. Finally it is possible to use the Sidi-S transform instead of the Levin transform by using the keyword method='sidi'. The Sidi-S transform works better than the Levin transformation for some divergent series (see the examples).

Parameters:

---

method	"levin" or "sidi" chooses either the Levin or the Sidi-S transformation
variant	"u", "t" or "v" chooses the weight variant.

---

The Levin transform is also accessible through the nsum interface. method="l" or method="levin" select the normal Levin transform while method="sidi" selects the Sidi-S transform. The variant is in both cases selected through the levin\_variant keyword. The stepsize in nsum() must not be chosen too large, otherwise it will miss the point where the Levin transform converges resulting in numerical overflow/garbage. For highly divergent series a copious amount of working precision must be chosen.

## Examples

First we sum the zeta function:

---

```
>>> from mpFormulaPy import mp
>>> mp.prec = 53
>>> eps = mp.mpf(mp.eps)
>>> with mp.extraprec(2 * mp.prec): # levin needs a high working precision
...     L = mp.levin(method = "levin", variant = "u")
...     S, s, n = [], 0, 1
...     while 1:
...         s += mp.one / (n * n)
...         n += 1
...         S.append(s)
...         v, e = L.update_psum(S)
...         if e < eps:
...             break
```

---

---

```

...         if n > 1000: raise RuntimeError("iteration limit exceeded")
>>> print(mp.chop(v - mp.pi ** 2 / 6))
0.0
>>> w = mp.nsum(lambda n: 1 / (n*n), [1, mp.inf], method = "levin", levin_variant =
    "u")
>>> print(mp.chop(v - w))
0.0

```

---

Now we sum the zeta function outside its range of convergence (attention: This does not work at the negative integers!):

---

```

>>> eps = mp.mpf(mp.eps)
>>> with mp.extrarec(2 * mp.prec): # levin needs a high working precision
...     L = mp.levin(method = "levin", variant = "v")
...     A, n = [], 1
...     while 1:
...         s = mp.mpf(n) ** (2 + 3j)
...         n += 1
...         A.append(s)
...         v, e = L.update(A)
...         if e < eps:
...             break
...     if n > 1000: raise RuntimeError("iteration limit exceeded")
>>> print(mp.chop(v - mp.zeta(-2-3j)))
0.0
>>> w = mp.nsum(lambda n: n ** (2 + 3j), [1, mp.inf], method = "levin", levin_variant =
    "v")
>>> print(mp.chop(v - w))
0.0

```

---

Now we sum the divergent asymptotic expansion of an integral related to the exponential integral (see also [2] p.373). The Sidi-S transform works best here:

---

```

>>> z = mp.mpf(10)
>>> exact = mp.quad(lambda x: mp.exp(-x)/(1+x/z), [0,mp.inf])
>>> # exact = z * mp.exp(z) * mp.expint(1,z) # this is the symbolic expression for
      the integral
>>> eps = mp.mpf(mp.eps)
>>> with mp.extrarec(2 * mp.prec): # high working precisions are mandatory for
      divergent resummation
...     L = mp.levin(method = "sidi", variant = "t")
...     n = 0
...     while 1:
...         s = (-1)**n * mp.fac(n) * z ** (-n)
...         v, e = L.step(s)
...         n += 1
...         if e < eps:
...             break
...     if n > 1000: raise RuntimeError("iteration limit exceeded")
>>> print(mp.chop(v - exact))
0.0
>>> w = mp.nsum(lambda n: (-1) ** n * mp.fac(n) * z ** (-n), [0, mp.inf], method =

```

---

```
"sidi", levin_variant = "t")
>>> print(mp.chop(v - w))
0.0
```

---

Another highly divergent integral is also summable:

---

```
>>> z = mp.mpf(2)
>>> eps = mp.mpf(mp.eps)
>>> exact = mp.quad(lambda x: mp.exp(-x * x / 2 - z * x ** 4), [0,mp.inf]) * 2 /
    mp.sqrt(2 * mp.pi)
>>> # exact = mp.exp(mp.one / (32 * z)) * mp.besselk(mp.one / 4, mp.one / (32 * z)) /
    (4 * mp.sqrt(z * mp.pi)) # this is the symbolic expression for the integral
>>> with mp.extraprec(7 * mp.prec): # we need copious amount of precision to sum this
    highly divergent series
...     L = mp.levin(method = "levin", variant = "t")
...     n, s = 0, 0
...     while 1:
...         s += (-z)**n * mp.fac(4 * n) / (mp.fac(n) * mp.fac(2 * n) * (4 ** n))
...         n += 1
...         v, e = L.step_psum(s)
...         if e < eps:
...             break
...         if n > 1000: raise RuntimeError("iteration limit exceeded")
>>> print(mp.chop(v - exact))
0.0
>>> w = mp.nsum(lambda n: (-z)**n * mp.fac(4 * n) / (mp.fac(n) * mp.fac(2 * n) * (4
    ** n)),
...     [0, mp.inf], method = "levin", levin_variant = "t", workprec = 8*mp.prec, steps
    = [2] + [1 for x in xrange(1000)])
>>> print(mp.chop(v - w))
0.0
```

---

These examples run with 15-20 decimal digits precision. For higher precision the working precision must be raised.

### Examples for nsum

Here we calculate Euler's constant as the constant term in the Laurent expansion of  $\zeta(s)$  at  $s = 1$ . This sum converges extremely slowly because of the logarithmic convergence behaviour of the Dirichlet series for zeta:

---

```
>>> mp.dps = 30
>>> z = mp.mpf(10) ** (-10)
>>> a = mp.nsum(lambda n: n**(-(1+z)), [1, mp.inf], method = "l") - 1 / z
>>> print(mp.chop(a - mp.euler, tol = 1e-10))
0.0
```

---

The Sidi-S transform performs excellently for the alternating series of  $\log(2)$ :

---

```
>>> a = mp.nsum(lambda n: (-1)**(n-1) / n, [1, mp.inf], method = "sidi")
>>> print(mp.chop(a - mp.log(2)))
0.0
```

---

Hypergeometric series can also be summed outside their range of convergence. The stepsize in

`nsum()` must not be chosen too large, otherwise it will miss the point where the Levin transform converges resulting in numerical overflow/garbage:

---

```
>>> z = 2 + 1j
>>> exact = mp.hyp2f1(2 / mp.mpf(3), 4 / mp.mpf(3), 1 / mp.mpf(3), z)
>>> f = lambda n: mp.rf(2 / mp.mpf(3), n) * mp.rf(4 / mp.mpf(3), n) * z**n / (mp.rf(1
    / mp.mpf(3), n) * mp.fac(n))
>>> v = mp.nsum(f, [0, mp.inf], method = "levin", steps = [10 for x in xrange(1000)])
>>> print(mp.chop(exact-v))
0.0
```

---

## 22.4.4 Cohen's algorithm

### Function `cohen_alt()` As Object

The function `cohen_alt` returns an object with `sn` update method. Only for use within Python

This interface implements the convergence acceleration of alternating series as described in H. Cohen, F.R. Villegas, D. Zagier - "Convergence Acceleration of Alternating Series". This series transformation works only well if the individual terms of the series have an alternating sign. It belongs to the class of linear series transformations (in contrast to the Shanks/Wynn-epsilon or Levin transform). This series transformation is also able to sum some types of divergent series. See the paper under which conditions this resummation is mathematical sound.

Let  $A$  be the series we want to sum:

$$A = \sum_{k=0}^{\infty} a_k \quad (22.4.4)$$

Let  $s_n$  be the partial sums of this series:

$$s_n = \sum_{k=0}^n a_k. \quad (22.4.5)$$

### Interface

Calling `cohen_alt` returns an object with the following methods.

Then `update(...)` works with the list of individual terms  $a_k$  and `update_psum(...)` works with the list of partial sums  $s_k$ :

---

```
v, e = ...update([a_0, a_1, ..., a_k])
v, e = ...update_psum([s_0, s_1, ..., s_k])
```

---

$v$  is the current estimate for  $A$ , and  $e$  is an error estimate which is simply the difference between the current estimate and the last estimate.

### Examples

Here we compute the alternating zeta function using `update_psum`:

---

```
>>> from mpFormulaPy import mp
>>> AC = mp.cohen_alt()
>>> S, s, n = [], 0, 1
>>> while 1:
...     s += -((-1) ** n) * mp.one / (n * n)
...     n += 1
```

---

---

```

... S.append(s)
... v, e = AC.update_psum(S)
... if e < mp.eps:
...     break
... if n > 1000: raise RuntimeError("iteration limit exceeded")
>>> print(mp.chop(v - mp.pi ** 2 / 12))
0.0

```

---

Here we compute the product  $\prod_{n=1}^{\infty} \Gamma(1 + 1/(2n - 1))/\Gamma(1 + 1/(2n))$ :

```

>>> A = []
>>> AC = mp.cohen_alt()
>>> n = 1
>>> while 1:
...     A.append(mp.loggamma(1 + mp.one / (2 * n - 1)))
...     A.append(-mp.loggamma(1 + mp.one / (2 * n)))
...     n += 1
...     v, e = AC.update(A)
...     if e < mp.eps:
...         break
...     if n > 1000: raise RuntimeError("iteration limit exceeded")
>>> v = mp.exp(v)
>>> print(mp.chop(v - 1.06215090557106, tol = 1e-12))
0.0

```

---

`cohen_alt` is also accessible through the `nsum()` interface:

```

>>> v = mp.nsum(lambda n: (-1)**(n-1) / n, [1, mp.inf], method = "a")
>>> print(mp.chop(v - mp.log(2)))
0.0
>>> v = mp.nsum(lambda n: (-1)**n / (2 * n + 1), [0, mp.inf], method = "a")
>>> print(mp.chop(v - mp.pi / 4))
0.0
>>> v = mp.nsum(lambda n: (-1)**n * mp.log(n) * n, [1, mp.inf], method = "a")
>>> print(mp.chop(v - mp.diff(lambda s: mp.altzeta(s), -1)))
0.0

```

---

# Chapter 23

## Differentiation

### 23.1 Numerical derivatives

#### 23.1.1 One-dimensional Differentiation

---

Function **diff(*f* As mpNum, *x* As mpNum, *n* As mpNum, **Keywords** As String) As mpNum**

---

The function diff returns the  $n$ -th derivative  $f^{(n)}(x)$ .

**Parameters:**

*f*: A one dimensional function

*x*: A real number.

*n*: An integer, indicating the nth derivative.

**Keywords:** method=step, tol=eps, direction=0. Many more, see below.

Numerically computes the derivative of  $f$ ,  $f'(x)$ , or generally for an integer  $n > 0$ , the  $n$ -th derivative  $f^{(n)}(x)$ . A few basic examples are:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> diff(lambda x: x**2 + x, 1.0)
3.0
>>> diff(lambda x: x**2 + x, 1.0, 2)
2.0
>>> diff(lambda x: x**2 + x, 1.0, 3)
0.0
>>> nprint([diff(exp, 3, n) for n in range(5)]) # exp'(x) = exp(x)
[20.0855, 20.0855, 20.0855, 20.0855, 20.0855]
```

---

Even more generally, given a tuple of arguments  $(x_1, \dots, x_k)$  and order  $(n_1, \dots, n_k)$ , the partial derivative  $f^{(n_1, \dots, n_k)}(x_1, \dots, x_k)$  is evaluated. For example:

---

```
>>> diff(lambda x,y: 3*x*y + 2*y - x, (0.25, 0.5), (0,1))
2.75
>>> diff(lambda x,y: 3*x*y + 2*y - x, (0.25, 0.5), (1,1))
3.0
```

---

#### Options

The following optional keyword arguments are recognized:

*method*: Supported methods are 'step' or 'quad': derivatives may be computed using either a finite difference with a small step size  $h$  (default), or numerical quadrature.

*direction*: Direction of finite difference: can be -1 for a left difference, 0 for a central difference (default), or +1 for a right difference; more generally can be any complex number.

*addprec*: Extra precision for  $h$  used to account for the function's sensitivity to perturbations (default = 10).

*relative*: Choose  $h$  relative to the magnitude of  $x$ , rather than an absolute value; useful for large or tiny  $x$ (default = False).

*h*: As an alternative to addprec and relative, manually select the step size  $h$ .

*singular*: If True, evaluation exactly at the point  $x$  is avoided; this is useful for differentiating functions with removable singularities. Default = False.

*radius*: Radius of integration contour (with method = 'quad'). Default = 0.25. A larger radius typically is faster and more accurate, but it must be chosen so that  $f$  has no singularities within the radius from the evaluation point.

A finite difference requires  $n + 1$  function evaluations and must be performed at  $n + 1$  times the target precision. Accordingly,  $f$  must support fast evaluation at high precision.

With integration, a larger number of function evaluations is required, but not much extra precision is required. For high order derivatives, this method may thus be faster if  $f$  is very expensive to evaluate at high precision.

## Further examples

The direction option is useful for computing left- or right-sided derivatives of nonsmooth functions:

---

```
>>> diff(abs, 0, direction=0)
0.0
>>> diff(abs, 0, direction=1)
1.0
>>> diff(abs, 0, direction=-1)
-1.0
```

---

More generally, if the direction is nonzero, a right difference is computed where the step size is multiplied by sign(direction). For example, with direction=+j, the derivative from the positive imaginary direction will be computed:

---

```
>>> diff(abs, 0, direction=j)
(0.0 - 1.0j)
```

---

With integration, the result may have a small imaginary part even even if the result is purely real:

---

```
>>> diff(sqrt, 1, method="quad")
(0.5 - 4.59...e-26j)
>>> chop(_)
0.5
```

---

Adding precision to obtain an accurate value:

---

```
>>> diff(cos, 1e-30)
```

---

```
0.0
>>> diff(cos, 1e-30, h=0.0001)
-9.9999998328279e-31
>>> diff(cos, 1e-30, addprec=100)
-1.0e-30
```

---

### 23.1.2 Sequence of derivatives

---

Function **diffs**(*f* As *mpNum*, *x* As *mpNum*, *n* As *mpNum*, **Keywords** As String) As *mpNum*

---

The function **diffs** returns the *n*-th derivative  $f^{(n)}(x)$ .

**Parameters:**

*f*: A one dimensional function

*x*: A real number.

*n*: An integer, indicating the *n*th derivative.

**Keywords:** method=step, tol=eps, direction=0. Many more, see below.

Returns a generator that yields the sequence of derivatives

$$f(x), f'(x), f''(x), \dots, f^{(k)}(x), \dots \quad (23.1.1)$$

With *method='step'*, **diffs()** uses only  $O(k)$  function evaluations to generate the first *k* derivatives, rather than the roughly  $O(k^2)$  evaluations required if one calls **diff()** *k* separate times.

With  $n < \infty$ , the generator stops as soon as the *n*-th derivative has been generated. If the exact number of needed derivatives is known in advance, this is further slightly more efficient.

Options are the same as for **diff()**.

#### Examples

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15
>>> nprint(list(diffs(cos, 1, 5)))
[0.540302, -0.841471, -0.540302, 0.841471, 0.540302, -0.841471]
>>> for i, d in zip(range(6), diffs(cos, 1)):
... print("%s %s" % (i, d))
...
0 0.54030230586814
1 -0.841470984807897
2 -0.54030230586814
3 0.841470984807897
4 0.54030230586814
5 -0.841470984807897
```

---

## 23.2 Composition of derivatives

### 23.2.1 Differentiation of products of functions

---

Function **diffsprod**(*factors* As Object) As mpNum

---

The function `diffsprod` returns the result of the differentiation of products of functions.

**Parameter:**

*factors*: a list of  $N$  iterables or generators.

Given a list of  $N$  iterables or generators yielding  $f_k(x), f'_k(x), f''_k(x), \dots$  for  $k = 1, \dots, N$ , generate  $g_k(x), g'_k(x), g''_k(x), \dots$  where  $g(x) = f_1(x)f_2(x) \cdots f_N(x)$ .

At high precision and for large orders, this is typically more efficient than numerical differentiation if the derivatives of each  $f_k(x)$  admit direct computation.

Note: This function does not increase the working precision internally, so guard digits may have to be added externally for full accuracy.

Examples

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> f = lambda x: exp(x)*cos(x)*sin(x)
>>> u = diffs(f, 1)
>>> v = mp.diffprod([diffs(exp,1), diffs(cos,1), diffs(sin,1)])
>>> next(u); next(v)
1.23586333600241
1.23586333600241
>>> next(u); next(v)
0.104658952245596
0.104658952245596
>>> next(u); next(v)
-5.96999877552086
-5.96999877552086
>>> next(u); next(v)
-12.4632923122697
-12.4632923122697
```

---

### 23.2.2 Differentiation of the exponential of functions

---

Function **diffsexp**(*fdiffs* As Object) As mpNum

---

The function `diffsexp` returns the result of the differentiation of the exponential of functions.

**Parameter:**

*fdiffs*: a list of  $N$  iterables or generators.

Given an iterable or generator yielding  $f_k(x), f'_k(x), f''_k(x), \dots$  generate  $g_k(x), g'_k(x), g''_k(x), \dots$  where  $g(x) = \exp(f(x))$ .

At high precision and for large orders, this is typically more efficient than numerical differentiation if the derivatives of  $f(x)$  admit direct computation.

Note: This function does not increase the working precision internally, so guard digits may have to be added externally for full accuracy.

### Examples

The derivatives of the gamma function can be computed using logarithmic differentiation:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>>
>>> def diffs_loggamma(x):
...     yield loggamma(x)
...     i = 0
...     while 1:
...         yield psi(i,x)
...         i += 1
...
>>> u = diffs_exp(diffs_loggamma(3))
>>> v = diffs(gamma, 3)
>>> next(u); next(v)
2.0
2.0
>>> next(u); next(v)
1.84556867019693
1.84556867019693
>>> next(u); next(v)
2.49292999190269
2.49292999190269
>>> next(u); next(v)
3.44996501352367
3.44996501352367
```

---

## 23.3 Fractional derivatives

---

Function **different(*f* As *mpNum*, *x* As *mpNum*, *n* As *mpNum*, *x0* As *mpNum*) As *mpNum***

---

The function **different** returns the Riemann-Liouville differintegral.

#### Parameters:

- f*: A one dimensional function
- x*: A real interval.
- n*: A real interval.
- x0*: A real interval.

Calculates the Riemann-Liouville differintegral, or fractional derivative, defined by

$${}_{x_0} \mathbb{D}_x^n f(x) = \frac{1}{\Gamma(m-n)} \frac{d^m}{dx^m} \int_{x_0}^x (x-t)^{m-n-1} f(t) dt \quad (23.3.1)$$

where *f* is a given (presumably well-behaved) function, *x* is the evaluation point, *n* is the order, and *x0* is the reference point of integration (*m* is an arbitrary parameter selected automatically).

With  $n = 1$ , this is just the standard derivative  $f'(x)$ ; with  $n = 2$ , the second derivative  $f''(x)$ , etc. With  $n = -1$ , it gives  $\int_{x_0}^x f(t)dt$ , with  $n = -1$  it gives  $\int_{x_0}^x \left(\int_{x_0}^x f(u)du\right) dt$ , etc.

As  $n$  is permitted to be any number, this operator generalizes iterated differentiation and iterated integration to a single operator with a continuous order parameter.

### Examples

There is an exact formula for the fractional derivative of a monomial  $x^p$ , which may be used as a reference. For example, the following gives a half-derivative (order 0.5):

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> x = mpf(3); p = 2; n = 0.5
>>> differint(lambda t: t**p, x, n)
7.81764019044672
>>> gamma(p+1)/gamma(p-n+1) * x**(p-n)
7.81764019044672
```

---

Another useful test function is the exponential function, whose integration / differentiation formula easily generalizes to arbitrary order. Here we first compute a third derivative, and then a triply nested integral. (The reference point  $x_0$  is set to  $-\infty$  to avoid nonzero endpoint terms.):

---

```
>>> differint(lambda x: exp(pi*x), -1.5, 3)
0.278538406900792
>>> exp(pi*-1.5) * pi**3
0.278538406900792
>>> differint(lambda x: exp(pi*x), 3.5, -3, -inf)
1922.50563031149
>>> exp(pi*3.5) / pi**3
1922.50563031149
```

---

However, for noninteger  $n$ , the differentiation formula for the exponential function must be modified to give the same result as the Riemann-Liouville differintegral:

---

```
>>> x = mpf(3.5)
>>> c = pi
>>> n = 1+2*j
>>> differint(lambda x: exp(c*x), x, n)
(-123295.005390743 + 140955.117867654j)
>>> x**(-n) * exp(c)**x * (x*c)**n * gammaint(-n, 0, x*c) / gamma(-n)
(-123295.005390743 + 140955.117867654j)
```

---

# Chapter 24

## Numerical integration (quadrature)

### 24.1 Standard quadrature

#### 24.1.1 One-dimensional Integration

---

Function **quad**(*f* As mpNum, *interval* As mpNum, **Keywords** As String) As mpNum

---

The function **quad** returns a one dimensional integral.

**Parameters:**

*f*: A one dimensional function

*interval*: A real interval.

**Keywords**: method=TanhSinh, error=False, verbose=False, maxdegree. Many more, see below.

Computes a single, double or triple integral over a given 1D interval, 2D rectangle, or 3D cuboid.  
A basic example:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> quad(sin, [0, pi])
2.0
```

---

A basic 2D integral:

---

```
>>> f = lambda x, y: cos(x+y/2)
>>> quad2d(f, [-pi/2, pi/2], [0, pi])
4.0
```

---

#### Interval format

The integration range for each dimension may be specified using a list or tuple. Arguments are interpreted as follows:

**quad**(*f*, [*x*1, *x*2]) calculates  $\int_{x_1}^{x_2} f(x) dx$

**quad2d**(*f*, [*x*1, *x*2], [*y*1, *y*2]) calculates  $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) dy dx$

**quad3d**(*f*, [*x*1, *x*2], [*y*1, *y*2], [*z*1, *z*2]) calculates  $\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z) dz dy dx$

Endpoints may be finite or infinite. An interval descriptor may also contain more than two points. In this case, the integration is split into subintervals, between each pair of consecutive points. This

is useful for dealing with mid-interval discontinuities, or integrating over large intervals where the function is irregular or oscillates.

### Options

`quad()` recognizes the following keyword arguments:

*method*: Chooses integration algorithm (described below).

*error*: If set to true, `quad()` returns  $(v, e)$  where  $v$  is the integral and  $e$  is the estimated error.

*maxdegree*: Maximum degree of the quadrature rule to try before quitting.

*verbose*: Print details about progress.

### Algorithms

Mpmath presently implements two integration algorithms: tanh-sinh quadrature and Gauss-Legendre quadrature. These can be selected using `method='tanh-sinh'` or `method='gauss-legendre'`, or by passing the classes `method=TanhSinh`, `method=GaussLegendre`. The functions `quadts()` and `quadgl()` are also available as shortcuts.

Both algorithms have the property that doubling the number of evaluation points roughly doubles the accuracy, so both are ideal for high precision quadrature (hundreds or thousands of digits).

At high precision, computing the nodes and weights for the integration can be expensive (more expensive than computing the function values). To make repeated integrations fast, nodes are automatically cached.

The advantages of the tanh-sinh algorithm are that it tends to handle endpoint singularities well, and that the nodes are cheap to compute on the first run. For these reasons, it is used by `quad()` as the default algorithm.

Gauss-Legendre quadrature often requires fewer function evaluations, and is therefore often faster for repeated use, but the algorithm does not handle endpoint singularities as well and the nodes are more expensive to compute. Gauss-Legendre quadrature can be a better choice if the integrand is smooth and repeated integrations are required (e.g. for multiple integrals).

See the documentation for `TanhSinh` and `GaussLegendre` for additional details.

### Examples of 1D integrals

Intervals may be infinite or half-infinite. The following two examples evaluate the limits of the inverse tangent function ( $\int 1/(1+x^2) = \tan^{-1} x$ ), and the Gaussian integral  $\int_{-\infty}^{\infty} \exp(-x^2) dx = \sqrt{\pi}$ :

---

```
>>> mp.dps = 15
>>> quad(lambda x: 2/(x**2+1), [0, inf])
3.14159265358979
>>> quad(lambda x: exp(-x**2), [-inf, inf])**2
3.14159265358979
```

---

Integrals can typically be resolved to high precision. The following computes 50 digits of  $\pi$  by integrating the area of the half-circle defined by  $x^2 + y^2 \leq 1, -1 < x < 1, y \geq 0$ :

---

```
>>> mp.dps = 50
>>> 2*quad(lambda x: sqrt(1-x**2), [-1, 1])
3.1415926535897932384626433832795028841971693993751
```

---

One can just as well compute 1000 digits (output truncated):

---

```
>>> mp.dps = 1000
>>> 2*quad(lambda x: sqrt(1-x**2), [-1, 1])
3.141592653589793238462643383279502884...216420198
```

---

Complex integrals are supported. The following computes a residue at  $z = 0$  by integrating counterclockwise along the diamond-shaped path from 1 to  $+i$  to  $-1$  to  $-i$  to 1:

---

```
>>> mp.dps = 15
>>> chop(quad(lambda z: 1/z, [1,j,-1,-j,1]))
(0.0 + 6.28318530717959j)
```

---

### 24.1.1.1 Singularities

Both tanh-sinh and Gauss-Legendre quadrature are designed to integrate smooth (infinitely differentiable) functions. Neither algorithm copes well with mid-interval singularities (such as mid-interval discontinuities in  $f(x)$  or  $f'(x)$ ). The best solution is to split the integral into parts:

---

```
>>> mp.dps = 15
>>> quad(lambda x: abs(sin(x)), [0, 2*pi]) # Bad
3.99900894176779
>>> quad(lambda x: abs(sin(x)), [0, pi, 2*pi]) # Good
4.0
```

---

The tanh-sinh rule often works well for integrands having a singularity at one or both endpoints:

---

```
>>> mp.dps = 15
>>> quad(log, [0, 1], method="tanh-sinh") # Good
-1.0
>>> quad(log, [0, 1], method="gauss-legendre") # Bad
-0.999932197413801
```

---

However, the result may still be inaccurate for some functions:

---

```
>>> quad(lambda x: 1/sqrt(x), [0, 1], method="tanh-sinh")
1.99999999946942
```

---

This problem is not due to the quadrature rule per se, but to numerical amplification of errors in the nodes. The problem can be circumvented by temporarily increasing the precision:

---

```
>>> mp.dps = 30
>>> a = quad(lambda x: 1/sqrt(x), [0, 1], method="tanh-sinh")
>>> mp.dps = 15
>>> +a
2.0
```

---

### 24.1.1.2 Highly variable functions

For functions that are smooth (in the sense of being infinitely differentiable) but contain sharp mid-interval peaks or many 'bumps', quad() may fail to provide full accuracy.

For example, with default settings, quad() is able to integrate  $\sin(x)$  accurately over an interval of length 100 but not over length 1000:

---

```
>>> quad(sin, [0, 100]); 1-cos(100) # Good
0.137681127712316
0.137681127712316
>>> quad(sin, [0, 1000]); 1-cos(1000) # Bad
-37.8587612408485
0.437620923709297
```

---

One solution is to break the integration into 10 intervals of length 100:

---

```
>>> quad(sin, linspace(0, 1000, 10)) # Good
0.437620923709297
```

---

Another is to increase the degree of the quadrature:

---

```
>>> quad(sin, [0, 1000], maxdegree=10) # Also good
0.437620923709297
```

---

Whether splitting the interval or increasing the degree is more efficient differs from case to case. Another example is the function  $1/(1+x^2)$ , which has a sharp peak centered around  $x = 0$ :

---

```
>>> f = lambda x: 1/(1+x**2)
>>> quad(f, [-100, 100]) # Bad
3.64804647105268
>>> quad(f, [-100, 100], maxdegree=10) # Good
3.12159332021646
>>> quad(f, [-100, 0, 100]) # Also good
3.12159332021646
```

---

## 24.1.2 Two-dimensional Integration

---

Function **quad2d**(*f* As mpNum, *interval1* As mpNum, *interval2* As mpNum, **Keywords** As String)  
As mpNum

---

The function quad2d returns a two dimensional integral.

### Parameters:

*f*: A one dimensional function

*interval1*: A real interval.

*interval2*: A real interval.

**Keywords**: method=TanhSinh, error=False, verbose=False, maxdegree. Many more, see below.

Here are several nice examples of analytically solvable 2D integrals (taken from MathWorld [1]) that can be evaluated to high precision fairly rapidly by quad():

---

```
>>> mp.dps = 30
>>> f = lambda x, y: (x-1)/((1-x*y)*log(x*y))
>>> quad2d(f, [0, 1], [0, 1])
0.577215664901532860606512090082
>>> +euler
```

---

```

0.577215664901532860606512090082
>>> f = lambda x, y: 1/sqrt(1+x**2+y**2)
>>> quad2d(f, [-1, 1], [-1, 1])
3.17343648530607134219175646705
>>> 4*log(2+sqrt(3))-2*pi/3
3.17343648530607134219175646705
>>> f = lambda x, y: 1/(1-x**2 * y**2)
>>> quad2d(f, [0, 1], [0, 1])
1.23370055013616982735431137498
>>> pi**2 / 8
1.23370055013616982735431137498
>>> quad2d(lambda x, y: 1/(1-x*y), [0, 1], [0, 1])
1.64493406684822643647241516665
>>> pi**2 / 6
1.64493406684822643647241516665

```

---

Multiple integrals may be done over infinite ranges:

---

```

>>> mp.dps = 15
>>> print(quad2d(lambda x,y: exp(-x-y), [0, inf], [1, inf]))
0.367879441171442
>>> print(1/e)
0.367879441171442

```

---

For nonrectangular areas, one can call quad() recursively. For example, we can replicate the earlier example of calculating  $\pi$  by integrating over the unit-circle, and use double quadrature to actually measure the area circle:

---

```

>>> f = lambda x: quad(lambda y: 1, [-sqrt(1-x**2), sqrt(1-x**2)])
>>> quad(f, [-1, 1])
3.14159265358979

```

---

### 24.1.3 Three-dimensional Integration

---

Function **quad3d(*f* As mpNum, *interval1* As mpNum, *interval2* As mpNum, *interval3* As mpNum, *Keywords* As String) As mpNum**

---

The function quad3d returns a three dimensional integral.

**Parameters:**

*f*: A one dimensional function

*interval1*: A real interval.

*interval2*: A real interval.

*interval3*: A real interval.

**Keywords:** method=TanhSinh, error=False, verbose=False, maxdegree. Many more, see below.

Here is a simple triple integral:

---

```

>>> mp.dps = 15
>>> f = lambda x,y,z: x*y/(1+z)
>>> quad3d(f, [0,1], [0,1], [1,2], method="gauss-legendre")
0.101366277027041

```

---

---

```
>>> (log(3)-log(2))/4
0.101366277027041
```

---

### 24.1.4 Oscillatory integrals

---

Function **quadosc(*f* As mpNum, *interval* As mpNum, *Keywords* As String) As mpNum**

---

The function quadosc returns a one dimensional oscillatory integral.

**Parameters:**

*f*: A one dimensional function

*interval*: A real interval.

*Keywords*: omega=None, period=None, zeros=None

Calculates

$$I = \int_a^b f(x)dx \tag{24.1.1}$$

where at least one of *a* and *b* is infinite and where  $f(x) = g(x) \cos(\omega x + \phi)$  for some slowly decreasing function *g*(*x*). With proper input, quadosc() can also handle oscillatory integrals where the oscillation rate is different from a pure sine or cosine wave.

In the standard case when  $|a| < \infty, b = \infty$ , quadosc() works by evaluating the infinite series

$$I = \int_a^{x_1} f(x)dx + \sum_{k=1}^{\infty} \int_{x_k}^{x_{k+1}} f(x)dx \tag{24.1.2}$$

where *x<sub>k</sub>* are consecutive zeros (alternatively some other periodic reference point) of *f*(*x*). Accordingly, quadosc() requires information about the zeros of *f*(*x*). For a periodic function, you can specify the zeros by either providing the angular frequency  $\omega$  (omega) or the period  $2\pi/\omega$ . In general, you can specify the *n*-th zero by providing the zeros arguments. Below is an example of each:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> f = lambda x: sin(3*x)/(x**2+1)
>>> quadosc(f, [0,inf], omega=3)
0.37833007080198
>>> quadosc(f, [0,inf], period=2*pi/3)
0.37833007080198
>>> quadosc(f, [0,inf], zeros=lambda n: pi*n/3)
0.37833007080198
>>> (ei(3)*exp(-3)-exp(3)*ei(-3))/2 # Computed by Mathematica
0.37833007080198
```

---

Note that zeros was specified to multiply *n* by the half-period, not the full period. In theory, it does not matter whether each partial integral is done over a half period or a full period. However, if done over half-periods, the infinite series passed to nsum() becomes an alternating series and this typically makes the extrapolation much more efficient.

Here is an example of an integration over the entire real line, and a half-infinite integration starting at  $-\infty$ :

```
>>> quadosc(lambda x: cos(x)/(1+x**2), [-inf, inf], omega=1)
1.15572734979092
>>> pi/e
1.15572734979092
>>> quadosc(lambda x: cos(x)/x**2, [-inf, -1], period=2*pi)
-0.0844109505595739
>>> cos(1)+sin(1)-pi/2
-0.0844109505595738
```

Of course, the integrand may contain a complex exponential just as well as a real sine or cosine:

```
>>> quadosc(lambda x: exp(3*j*x)/(1+x**2), [-inf, inf], omega=3)
(0.156410688228254 + 0.0j)
>>> pi/e**3
0.156410688228254
>>> quadosc(lambda x: exp(3*j*x)/(2+x+x**2), [-inf, inf], omega=3)
(0.0031748698463794 - 0.0447701735209082j)
>>> 2*pi/sqrt(7)/exp(3*(j+sqrt(7))/2)
(0.0031748698463794 - 0.0447701735209082j)
```

#### 24.1.4.1 Non-periodic functions

If  $f(x) = g(x)h(x)$  for some function  $h(x)$  that is not strictly periodic, omega or period might not work, and it might be necessary to use zeros.

A notable exception can be made for Bessel functions which, though not periodic, are 'asymptotically periodic' in a sufficiently strong sense that the sum extrapolation will work out:

```
>>> quadosc(j0, [0, inf], period=2*pi)
1.0
>>> quadosc(j1, [0, inf], period=2*pi)
1.0
```

More properly, one should provide the exact Bessel function zeros:

```
>>> j0zero = lambda n: findroot(j0, pi*(n-0.25))
>>> quadosc(j0, [0, inf], zeros=j0zero)
1.0
```

For an example where zeros becomes necessary, consider the complete Fresnel integrals

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \sqrt{\frac{\pi}{8}} \quad (24.1.3)$$

Although the integrands do not decrease in magnitude as  $x \rightarrow \infty$ , the integrals are convergent since the oscillation rate increases (causing consecutive periods to asymptotically cancel out). These integrals are virtually impossible to calculate to any kind of accuracy using standard quadrature rules. However, if one provides the correct asymptotic distribution of zeros ( $x_n \sim \sqrt{n}$ ), quadosc() works:

```
>>> mp.dps = 30
>>> f = lambda x: cos(x**2)
>>> quadosc(f, [0, inf], zeros=lambda n:sqrt(pi*n))
```

---

```
0.626657068657750125603941321203
>>> f = lambda x: sin(x**2)
>>> quadosc(f, [0,inf], zeros=lambda n:sqrt(pi*n))
0.626657068657750125603941321203
>>> sqrt(pi/8)
0.626657068657750125603941321203
```

---

(Interestingly, these integrals can still be evaluated if one places some other constant than  $\pi$  in the square root sign.)

In general, if  $f(x) \sim g(x) \cos(h(x))$ , the zeros follow the inverse-function distribution  $h^{-1}(x)$ :

---

```
>>> mp.dps = 15
>>> f = lambda x: sin(exp(x))
>>> quadosc(f, [1,inf], zeros=lambda n: log(n))
-0.25024394235267
>>> pi/2-si(e)
-0.250243942352671
```

---

#### 24.1.4.2 Non-alternating functions

If the integrand oscillates around a positive value, without alternating signs, the extrapolation might fail. A simple trick that sometimes works is to multiply or divide the frequency by 2:

---

```
>>> f = lambda x: 1/x**2+sin(x)/x**4
>>> quadosc(f, [1,inf], omega=1) # Bad
1.28642190869861
>>> quadosc(f, [1,inf], omega=0.5) # Perfect
1.28652953559617
>>> 1+(cos(1)+ci(1)+sin(1))/6
1.28652953559617
```

---

#### Fast decay

`quadosc()` is primarily useful for slowly decaying integrands. If the integrand decreases exponentially or faster, `quad()` will likely handle it without trouble (and generally be much faster than `quadosc()`):

---

```
>>> quadosc(lambda x: cos(x)/exp(x), [0, inf], omega=1)
0.5
>>> quad(lambda x: cos(x)/exp(x), [0, inf])
0.5
```

---

## 24.2 Main Quadrature rules

For a finite interval, a simple linear change of variables is used. Otherwise, the following transformations are used:

$$[a, \infty] : t = \frac{1}{x} + (a - 1) \quad (24.2.1)$$

$$[-\infty, b] : t = (b + 1) - \frac{1}{x} \quad (24.2.2)$$

$$[-\infty, \infty] : t = \frac{x}{\sqrt{1 - x^2}} \quad (24.2.3)$$

### 24.2.1 TanhSinh

Tanh-sinh quadrature is a method for numerical integration introduced by Hidetosi Takahasi and Masatake Mori in 1974. It uses the change of variables

$$x = \tanh\left(\frac{1}{2}\pi \sinh t\right) \quad (24.2.4)$$

to transform an integral on the interval  $x \in (-1, +1)$  to an integral on the entire real line  $t \in (-\infty, \infty)$ . After this transformation, the integrand decays with a double exponential rate, and thus, this method is also known as the double exponential (DE) formula.

For a given step size  $h$ , the integral is approximated by the sum

$$\int_{-1}^1 f(x) dx \approx \sum_{k=-\infty}^{\infty} w_k f(x_k), \quad (24.2.5)$$

with the abscissas

$$x_k = \tanh\left(\frac{1}{2}\pi \sinh t_k\right) \quad (24.2.6)$$

and the weights

$$w_k = \frac{\tanh \frac{1}{2}\pi \cosh t_k}{\cosh^2(\frac{1}{2}\pi \sinh t_k)} \quad (24.2.7)$$

where  $t_k = t_0 + hk$  for a step length  $h \sim 2^{-m}$ .

Like Gaussian quadrature, Tanh-Sinh quadrature is well suited for arbitrary-precision integration, where an accuracy of hundreds or even thousands of digits is desired. The convergence is exponential (in the discretization sense) for sufficiently well-behaved integrands: doubling the number of evaluation points roughly doubles the number of correct digits.

Tanh-Sinh quadrature is less efficient than Gaussian quadrature for smooth integrands, but unlike Gaussian quadrature tends to work equally well with integrands having singularities or infinite derivatives at one or both endpoints of the integration interval. A further advantage is that the abscissas and weights are relatively easy to compute. The cost of calculating abscissa-weight pairs for  $n$ -digit accuracy is roughly  $n^2 \log^2 n$  compared to  $n^3 \log n$  for Gaussian quadrature.

This class implements 'tanh-sinh' or 'doubly exponential' quadrature. This quadrature rule is based on the Euler-Maclaurin integral formula. By performing a change of variables involving nested exponentials / hyperbolic functions (hence the name), the derivatives at the endpoints vanish rapidly. Since the error term in the Euler-Maclaurin formula depends on the derivatives at the endpoints, a simple step sum becomes extremely accurate. In practice, this means that doubling the number of evaluation points roughly doubles the number of accurate digits.

The implementation of the tanh-sinh algorithm is based on the description given in Borwein, Bailey & Girgensohn, "Experimentation in Mathematics - Computational Paths to Discovery", A K Peters, 2003, pages 312-313. In the present implementation, a few improvements have been made:

A more efficient scheme is used to compute nodes (exploiting recurrence for the exponential function). The nodes are computed successively instead of all at once.

We exploit the fact that half of the abscissas at degree are precisely the abscissas from degree . Thus reusing the result from the previous level allows a 2x speedup.

A summary is given by [Bailey et al. \(2004\)](#); [Bailey \(2006\)](#).

See also the external links at [Wikipedia](#).

### 24.2.2 Gauss-Legendre

This class implements Gauss-Legendre quadrature, which is exceptionally efficient for polynomials and polynomial-like (i.e. very smooth) integrands.

The abscissas and weights are given by roots and values of Legendre polynomials, which are the orthogonal polynomials on  $[-1, 1]$  with respect to the unit weight (see `legendre()`).

In this implementation, we take the 'degree' of the quadrature to denote a Gauss-Legendre rule of degree  $3 \cdot 2^m$  (following Borwein, Bailey & Girgensohn). This way we get quadratic, rather than linear, convergence as the degree is incremented.

Comparison to tanh-sinh quadrature:

Is faster for smooth integrands once nodes have been computed

Initial computation of nodes is usually slower

Handles endpoint singularities worse

Handles infinite integration intervals worse

## 24.3 Additional Quadrature rules

### 24.3.1 Gauss-Legendre

$$\int_a^b f(x)dx = \frac{b-a}{2} \sum_{i=1}^n w_i f\left(\frac{1}{2}(b-a)(x_i+1)\right) + R_n, \quad \text{where} \quad (24.3.1)$$

$$w_i = \frac{2}{((1-x^2)P'_n(x_i))^2}, \quad R_n = \frac{f^{(2n)}(x)(b-a)^{2n+1}(n!)^4}{((2n+1)(2n)!)^3} \text{ for } a < x < b, \quad (24.3.2)$$

$P_n$  are the Legendre polynomials, and  $x_i$  is the  $i$ th zero of  $P_n$ .

### 24.3.2 Gauss-Hermite

$$\int_{-\infty}^{\infty} e^{-ax^2} f(x)dx = \frac{1}{\sqrt{a}} \sum_{i=1}^n w_i f\left(\frac{x_i}{\sqrt{a}}\right) + R_n, \quad \text{where} \quad (24.3.3)$$

$$w_i = \frac{2^{n-1} n! \sqrt{\pi}}{(n H_{n-1}(x_i))^2}, \quad R_n = \frac{f^{(2n)}(x)n!\sqrt{\pi}}{2^n (2n)!} \text{ for } -\infty < x < \infty, \quad (24.3.4)$$

$H_n$  are the Hermite polynomials, and  $x_i$  is the  $i$ th zero of  $H_n$ .

### 24.3.3 Gauss-Laguerre

$$\int_0^{\infty} e^{-ax} f(x)dx = \frac{1}{a} \sum_{i=1}^n w_i f\left(\frac{x_i}{a}\right) + R_n, \quad \text{where} \quad (24.3.5)$$

$$w_i = \frac{x_i}{((n+1)L_{n+1}(x_i))^2}, \quad R_n = \frac{f^{(2n)}(x)(n!)^2}{(2n)!} \text{ for } 0 < x < \infty, \quad (24.3.6)$$

$L_n$  are the Laguerre polynomials, and  $x_i$  is the  $i$ th zero of  $L_n$ .

# Chapter 25

## Ordinary differential equations

### 25.1 Solving the ODE initial value problem

---

Function **odefun**(*f* As mpNum, *x0* As mpNum, *y0* As mpNum, *Keywords* As String) As mpNum

---

The function **odefun** returns a function  $y(x) = [y_0(x), y_1(x), \dots, y_n(x)]$  that is a numerical solution of a  $n + 1$ -dimensional first-order ordinary differential equation (ODE) system

**Parameters:**

*f*: A one dimensional function

*x0*: A real number.

*y0*: A real number.

*Keywords*: tol=None, degree=None, method='taylor', verbose=False

The (ODE) system has the form

$$y'_0(x) = F_0(x, [y_0(x), y_1(x), \dots, y_n(x)]) \quad (25.1.1)$$

$$y'_1(x) = F_1(x, [y_0(x), y_1(x), \dots, y_n(x)]) \quad (25.1.2)$$

$$\vdots$$
$$y'_n(x) = F_n(x, [y_0(x), y_1(x), \dots, y_n(x)]) \quad (25.1.3)$$

The derivatives are specified by the vector-valued function  $F$  that evaluates

$$[y'_0, \dots, y'_n] = F(x, [y_0, \dots, y_n]).$$

The initial point  $x_0$  is specified by the scalar argument *x0*, and the initial value

$$y(x_0) = [y_0(x_0), \dots, y_n(x_0)]$$

is specified by the vector argument *y0*.

For convenience, if the system is one-dimensional, you may optionally provide just a scalar value for *y0*. In this case,  $F$  should accept a scalar *y* argument and return a scalar. The solution function *y* will return scalar values instead of length-1 vectors.

Evaluation of the solution function  $y(x)$  is permitted for any  $x > x_0$ .

A high-order ODE can be solved by transforming it into first-order vector form. This transformation is described in standard texts on ODEs. Examples will also be given below.

#### Options, speed and accuracy

By default, **odefun()** uses a high-order Taylor series method. For reasonably wellbehaved problems, the solution will be fully accurate to within the working precision. Note that  $F$  must be possible to evaluate to very high precision for the generation of Taylor series to work.

To get a faster but less accurate solution, you can set a large value for tol (which defaults roughly to eps). If you just want to plot the solution or perform a basic simulation, tol = 0.01 is likely sufficient.

The degree argument controls the degree of the solver (with method='taylor', this is the degree of the Taylor series expansion). A higher degree means that a longer step can be taken before a new local solution must be generated from F, meaning that fewer steps are required to get from  $x_0$  to a given  $x_1$ . On the other hand, a higher degree also means that each local solution becomes more expensive (i.e., more evaluations of F are required per step, and at higher precision).

The optimal setting therefore involves a tradeoff. Generally, decreasing the degree for Taylor series is likely to give faster solution at low precision, while increasing is likely to be better at higher precision.

The function object returned by odefun() caches the solutions at all step points and uses polynomial interpolation between step points. Therefore, once  $y(x_1)$  has been evaluated for some  $x_1$ ,  $y(x)$  can be evaluated very quickly for any  $x_0 \leq x \leq x_1$ , and continuing the evaluation up to  $x_2 > x_1$  is also fast.

### Examples of first-order ODEs

We will solve the standard test problem  $y'(x) = y(x), y(0) = 1$  which has explicit solution  $y(x) = \exp(x)$ :

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> f = odefun(lambda x, y: y, 0, 1)
>>> for x in [0, 1, 2.5]:
...     print((f(x), exp(x)))
...
(1.0, 1.0)
(2.71828182845905, 2.71828182845905)
(12.1824939607035, 12.1824939607035)
```

---

The solution with high precision:

---

```
>>> mp.dps = 50
>>> f = odefun(lambda x, y: y, 0, 1)
>>> f(1)
2.7182818284590452353602874713526624977572470937
>>> exp(1)
2.7182818284590452353602874713526624977572470937
```

---

Using the more general vectorized form, the test problem can be input as (note that f returns a 1-element vector):

---

```
>>> mp.dps = 15
>>> f = odefun(lambda x, y: [y[0]], 0, [1])
>>> f(1)
[2.71828182845905]
```

---

odefun() can solve nonlinear ODEs, which are generally impossible (and at best difficult) to solve analytically. As an example of a nonlinear ODE, we will solve  $y'(x) = x \sin(y(x))$  for

$y(0) = \pi/2$ . An exact solution happens to be known for this problem, and is given by  $y(x) = 2\tan^{-1}(\exp(x^2/2))$ :

---

```
>>> f = odefun(lambda x, y: x*sin(y), 0, pi/2)
>>> for x in [2, 5, 10]:
... print((f(x), 2*atan(exp(mpf(x)**2/2))))
...
(2.87255666284091, 2.87255666284091)
(3.14158520028345, 3.14158520028345)
(3.14159265358979, 3.14159265358979)
```

---

If  $F$  is independent of  $y$ , an ODE can be solved using direct integration. We can therefore obtain a reference solution with quad():

---

```
>>> f = lambda x: (1+x**2)/(1+x**3)
>>> g = odefun(lambda x, y: f(x), pi, 0)
>>> g(2*pi)
0.72128263801696
>>> quad(f, [pi, 2*pi])
0.72128263801696
```

---

### Examples of second-order ODEs

We will solve the harmonic oscillator equation  $y''(x) + y(x) = 0$ . To do this, we introduce the helper functions  $y_0 = y, y_1 = y'_0$  whereby the original equation can be written as  $y'_1 + y'_0 = 0$ . Put together, we get the first-order, two-dimensional vector ODE

$$y'_0 = y_1; \quad y'_1 = -y_0. \quad (25.1.4)$$

To get a well-defined IVP, we need two initial values. With  $y(0) = y_0(0) = 1$  and  $-y'(0) = y_1(0) = 0$ , the problem will of course be solved by  $y(x) = y_0(x) = \cos(x)$  and  $y(x) = y_1(x) = \sin(x)$ . We check this:

---

```
>>> f = odefun(lambda x, y: [-y[1], y[0]], 0, [1, 0])
>>> for x in [0, 1, 2.5, 10]:
... nprint(f(x), 15)
... nprint([cos(x), sin(x)], 15)
... print("----")
...
[1.0, 0.0]
[1.0, 0.0]
----
[0.54030230586814, 0.841470984807897]
[0.54030230586814, 0.841470984807897]
----
[-0.801143615546934, 0.598472144103957]
[-0.801143615546934, 0.598472144103957]
----
[-0.839071529076452, -0.54402111088937]
[-0.839071529076452, -0.54402111088937]
----
```

---

Note that we get both the sine and the cosine solutions simultaneously.

# Chapter 26

## Function approximation

### 26.1 Taylor series

---

Function **taylor(*f* As mpNum, *x* As mpNum, *n* As mpNum, **Keywords** As String) As mpNum**

---

The function `taylor` returns a list of coefficients of a degree-*n* Taylor polynomial around the point *x* of the given function *f*.

**Parameters:**

*f*: A one dimensional function

*x*: A real number.

*n*: A real number.

**Keywords**: method=step, tol=eps, direction=0. Many more, see `diff()`

Examples:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> nprint(chop(taylor(sin, 0, 5)))
[0.0, 1.0, 0.0, -0.166667, 0.0, 0.00833333]
```

---

The coefficients are computed using high-order numerical differentiation. The function must be possible to evaluate to arbitrary precision. See `diff()` for additional details and supported keyword options.

Note that to evaluate the Taylor polynomial as an approximation of *f*, e.g. with `polyval()`, the coefficients must be reversed, and the point of the Taylor expansion must be subtracted from the argument:

---

```
>>> p = taylor(exp, 2.0, 10)
>>> polyval(p[::-1], 2.5 - 2.0)
12.1824939606092
>>> exp(2.5)
12.1824939607035
```

---

## 26.2 Pade approximation

Function **pade(*a* As mpNum, *L* As mpNum, *M* As mpNum)** As mpNum

The function `pade` returns coefficients of a Pade approximation of degree  $(L, M)$  to a function

**Parameters:**

*a*: A list of at least  $L + M + 1$  Taylor coefficients approximating a function  $A(x)$

*L*: An integer, specifying the degree of polynomials  $P$ .

*M*: An integer, specifying the degree of polynomials  $Q$ .

Computes a Pade approximation of degree  $(L, M)$  to a function. Given at least  $L + M + 1$  Taylor coefficients approximating a function  $A(x)$ , `pade()` returns coefficients of polynomials  $P, Q$  satisfying

$$P = \sum_{k=0}^L p_k x^k; \quad Q = \sum_{k=0}^M q_k x^k; \quad Q_0 = 1; \quad A(x)Q(x) = P(x) + O(x^{L+M+1}) \quad (26.2.1)$$

$P(x)/Q(x)$  can provide a good approximation to an analytic function beyond the radius of convergence of its Taylor series (example from G.A. Baker 'Essentials of Pade Approximants' Academic Press, Ch.1A):

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> one = mpf(1)
>>> def f(x):
...     return sqrt((one + 2*x)/(one + x))
...
>>> a = taylor(f, 0, 6)
>>> p, q = pade(a, 3, 3)
>>> x = 10
>>> polyval(p[::-1], x)/polyval(q[::-1], x)
1.38169105566806
>>> f(x)
1.38169855941551
```

## 26.3 Chebyshev approximation

Function **chebyfit(*f* As mpNum, *interval* As mpNum, *N* As mpNum, *Keywords* As String) As mpNum**

The function `chebyfit` returns coefficients of a polynomial of degree  $N - 1$  that approximates the given function  $f$  on the interval  $[a, b]$

### Parameters:

*f*: A one dimensional function

*interval*: A real interval.

*N*: An integer.

*Keywords*: error=False

Computes a polynomial of degree  $N - 1$  that approximates the given function  $f$  on the interval  $[a, b]$ . With `error=True`, `chebyfit()` also returns an accurate estimate of the maximum absolute error; that is, the maximum value of  $|f(x) - P(x)|$  for  $x \in [a, b]$ .

`chebyfit()` uses the Chebyshev approximation formula, which gives a nearly optimal solution: that is, the maximum error of the approximating polynomial is very close to the smallest possible for any polynomial of the same degree.

Chebyshev approximation is very useful if one needs repeated evaluation of an expensive function, such as function defined implicitly by an integral or a differential equation. (For example, it could be used to turn a slow `mpFormulaPy` function into a fast machine-precision version of the same.)

### Examples

Here we use `chebyfit()` to generate a low-degree approximation of  $f(x) = \cos(x)$ , valid on the interval  $[1, 2]$ :

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> poly, err = chebyfit(cos, [1, 2], 5, error=True)
>>> nprint(poly)
[0.00291682, 0.146166, -0.732491, 0.174141, 0.949553]
>>> nprint(err, 12)
1.61351758081e-5
```

The polynomial can be evaluated using `polyval`:

```
>>> nprint(polyval(poly, 1.6), 12)
-0.0291858904138
>>> nprint(cos(1.6), 12)
-0.0291995223013
```

Sampling the true error at 1000 points shows that the error estimate generated by `chebyfit` is remarkably good:

```
>>> error = lambda x: abs(cos(x) - polyval(poly, x))
>>> nprint(max([error(1+n/1000.) for n in range(1000)]), 12)
1.61349954245e-5
```

### Choice of degree

The degree  $N$  can be set arbitrarily high, to obtain an arbitrarily good approximation. As a rule of thumb, an  $N$ -term Chebyshev approximation is good to  $N/(b - a)$  decimal places on a unit interval (although this depends on how well-behaved  $f$  is). The cost grows accordingly: `chebyfit` evaluates the function  $(N^2)/2$  times to compute the coefficients and an additional  $N$  times to estimate the error.

### Possible issues

One should be careful to use a sufficiently high working precision both when calling `chebyfit` and when evaluating the resulting polynomial, as the polynomial is sometimes ill-conditioned. It is for example difficult to reach 15-digit accuracy when evaluating the polynomial using machine precision floats, no matter the theoretical accuracy of the polynomial. (The option to return the coefficients in Chebyshev form should be made available in the future.)

It is important to note the Chebyshev approximation works poorly if  $f$  is not smooth. A function containing singularities, rapid oscillation, etc can be approximated more effectively by multiplying it by a weight function that cancels out the nonsmooth features, or by dividing the interval into several segments.

## 26.4 Fourier series

---

Function **fourier(*f* As mpNum, *interval* As mpNum, *N* As mpNum) As mpNum**

---

The function **fourier** returns two lists of coefficients of the Fourier series of degree  $N$  of the given function on the interval  $[a, b]$ .

**Parameters:**

*f*: A one dimensional function

*interval*: A real interval.

*N*: An integer.

Computes the Fourier series of degree  $N$  of the given function on the interval  $[a, b]$ . More precisely, **fourier()** returns two lists  $(c, s)$  of coefficients (the cosine series and sine series, respectively), such that

$$f(x) \sim \sum_{k=0}^N c_k \cos(kmx) + s_k \sin(kmx) \quad (26.4.1)$$

where  $m = 2\pi/(b - a)$ .

Note that many texts define the first coefficient as  $2c_0$  instead of  $c_0$ . The easiest way to evaluate the computed series correctly is to pass it to **fourierval()**.

### Examples

The function  $f(x) = x$  has a simple Fourier series on the standard interval  $[-\pi, \pi]$ . The cosine coefficients are all zero (because the function has odd symmetry), and the sine coefficients are rational numbers:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> c, s = fourier(lambda x: x, [-pi, pi], 5)
>>> nprint(c)
[0.0, 0.0, 0.0, 0.0, 0.0, 0.0]
>>> nprint(s)
[0.0, 2.0, -1.0, 0.666667, -0.5, 0.4]
```

---

This computes a Fourier series of a nonsymmetric function on a nonstandard interval:

---

```
>>> I = [-1, 1.5]
>>> f = lambda x: x**2 - 4*x + 1
>>> cs = fourier(f, I, 4)
>>> nprint(cs[0])
[0.583333, 1.12479, -1.27552, 0.904708, -0.441296]
>>> nprint(cs[1])
[0.0, -2.6255, 0.580905, 0.219974, -0.540057]
```

---

It is instructive to plot a function along with its truncated Fourier series:

---

```
>>> plot([f, lambda x: fourierval(cs, I, x)], I)
```

---

Fourier series generally converge slowly (and may not converge pointwise). For example, if  $f(x) = \cosh(x)$ , a 10-term Fourier series gives an  $L^2$  error corresponding to 2-digit accuracy:

---

```
>>> I = [-1, 1]
```

```
>>> cs = fourier(cosh, I, 9)
>>> g = lambda x: (cosh(x) - fourierval(cs, I, x))**2
>>> nprint(sqrt(quad(g, I)))
0.00467963
```

---

fourier() uses numerical quadrature. For nonsmooth functions, the accuracy (and speed) can be improved by including all singular points in the interval specification:

```
>>> nprint(fourier(abs, [-1, 1], 0), 10)
([0.5000441648], [0.0])
>>> nprint(fourier(abs, [-1, 0, 1], 0), 10)
([0.5], [0.0])
```

---

---

### Function **fouriereval**(*series* As mpNum, *interval* As mpNum, *x* As mpNum) As mpNum

---

The function fouriereval returns the result of the evaluation of a Fourier series (in the format computed by fourier() for the given interval) at the point *x*.

#### Parameters:

*series*: a pair (*c*, *s*) where *c* is the cosine series and *s* is the sine series

*interval*: A real interval.

*x*: A real number.

The series should be a pair (*c*, *s*) where *c* is the cosine series and *s* is the sine series. The two lists need not have the same length.

# Chapter 27

## Number identification

Most functions in mpFormulaPy are concerned with producing approximations from exact mathematical formulas. It is also useful to consider the inverse problem: given only a decimal approximation for a number, such as 0.7320508075688772935274463, is it possible to find an exact formula?

Subject to certain restrictions, such 'reverse engineering' is indeed possible thanks to the existence of integer relation algorithms. Mpmath implements the PSLQ algorithm (developed by H. Ferguson), which is one such algorithm.

Automated number recognition based on PSLQ is not a silver bullet. Any occurring transcendental constants ( $\pi$ ,  $e$ , etc) must be guessed by the user, and the relation between those constants in the formula must be linear (such as  $x = 3\pi + 4e$ ). More complex formulas can be found by combining PSLQ with functional transformations; however, this is only feasible to a limited extent since the computation time grows exponentially with the number of operations that need to be combined.

The number identification facilities in mpFormulaPy are inspired by the Inverse Symbolic Calculator (ISC). The ISC is more powerful than mpFormulaPy, as it uses a lookup table of millions of precomputed constants (thereby mitigating the problem with exponential complexity).

### 27.1 Constant recognition

---

Function **identify(x As mpNum, *constants* As String, *Keywords* As String)** As String

---

The function identify returns the result of an attempt to find an exact formula for a given real number  $x$

#### Parameters:

*x*: A real number

*constants*: A list of known constants.

*Keywords*: tol=None, maxcoeff=1000, full=False, verbose=False

Given a real number  $x$ , identify( $x$ ) attempts to find an exact formula for  $x$ . This formula is returned as a string. If no match is found, None is returned. With full=True, a list of matching formulas is returned.

As a simple example, identify() will find an algebraic formula for the golden ratio:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
```

---

```
>>> identify(phi)
'((1+sqrt(5))/2)'
```

`identify()` can identify simple algebraic numbers and simple combinations of given base constants, as well as certain basic transformations thereof. More specifically, `identify()` looks for the following:

1. Fractions
2. Quadratic algebraic numbers
3. Rational linear combinations of the base constants
4. Any of the above after first transforming  $x$  into  $f(x)$  where  $f(x)$  is  $1/x$ ,  $\sqrt{x}$ ,  $x^2$ , or  $\exp(x)$ , either directly or with  $x$  or  $f(x)$  multiplied or divided by one of the base constants
5. Products of fractional powers of the base constants and small integers

Base constants can be given as a list of strings representing mpFormulaPy expressions (`identify()` will eval the strings to numerical values and use the original strings for the output), or as a dict of formula:value pairs.

In order not to produce spurious results, `identify()` should be used with high precision; preferably 50 digits or more.

### 27.1.1 Examples

Simple identifications can be performed safely at standard precision. Here the default recognition of rational, algebraic, and exp/log of algebraic numbers is demonstrated:

---

```
>>> mp.dps = 15
>>> identify(0.2222222222222222)
'(2/9)'
>>> identify(1.9662210973805663)
'sqrt(((24+sqrt(48))/8))'
>>> identify(4.1132503787829275)
'exp(sqrt(8)/2)'
>>> identify(0.881373587019543)
'log((2+sqrt(8))/2))'
```

By default, `identify()` does not recognize  $\pi$ . At standard precision it finds a not too useful approximation. At slightly increased precision, this approximation is no longer accurate enough and `identify()` more correctly returns None:

---

```
>>> identify(pi)
'(2**((176/117)*3**((20/117)*5**((35/39)))/(7**((92/117)))'
>>> mp.dps = 30
>>> identify(pi)
>>>
```

Numbers such as  $\pi$ , and simple combinations of user-defined constants, can be identified if they are provided explicitly:

---

```
>>> identify(3*pi-2*e, ['pi', 'e'])
'(3*pi + (-2)*e)'
```

Here is an example using a dict of constants. Note that the constants need not be ‘atomic’; identify() can just as well express the given number in terms of expressions given by formulas:

---

```
>>> identify(pi+e, {'a':pi+2, 'b':2*e})
'((-2) + 1*a + (1/2)*b)',
```

---

Next, we attempt some identifications with a set of base constants. It is necessary to increase the precision a bit.

---

```
>>> mp.dps = 50
>>> base = ['sqrt(2)', 'pi', 'log(2)']
>>> identify(0.25, base)
'(1/4)'
>>> identify(3*pi + 2*sqrt(2) + 5*log(2)/7, base)
'(2*sqrt(2) + 3*pi + (5/7)*log(2))'
>>> identify(exp(pi+2), base)
'exp((2 + 1*pi))'
>>> identify(1/(3+sqrt(2)), base)
'((3/7) + (-1/7)*sqrt(2))'
>>> identify(sqrt(2)/(3*pi+4), base)
'sqrt(2)/(4 + 3*pi)'
>>> identify(5**mpf(1)/3*pi*log(2)**2, base)
'5** (1/3)*pi*log(2)**2'
```

---

An example of an erroneous solution being found when too low precision is used:

---

```
>>> mp.dps = 15
>>> identify(1/(3*pi-4*e+sqrt(8)), ['pi', 'e', 'sqrt(2)])
'((11/25) + (-158/75)*pi + (76/75)*e + (44/15)*sqrt(2))'
>>> mp.dps = 50
>>> identify(1/(3*pi-4*e+sqrt(8)), ['pi', 'e', 'sqrt(2)])
'1/(3*pi + (-4)*e + 2*sqrt(2))'
```

---

## 27.1.2 Finding approximate solutions

The tolerance tol defaults to 3/4 of the working precision. Lowering the tolerance is useful for finding approximate matches. We can for example try to generate approximations for pi:

---

```
>>> mp.dps = 15
>>> identify(pi, tol=1e-2)
'(22/7)'
>>> identify(pi, tol=1e-3)
'(355/113)'
>>> identify(pi, tol=1e-10)
'(5** (339/269))/(2** (64/269)*3** (13/269)*7** (92/269))'
```

---

With full=True, and by supplying a few base constants, identify can generate almost endless lists of approximations for any number (the output below has been truncated to show only the first few):

---

```
>>> for p in identify(pi, ['e', 'catalan'], tol=1e-5, full=True):
...     print(p)
```

---

---

```

...
e/log((6 + (-4/3)*e))
(3**3*5*e*catalan**2)/(2*7**2)
sqrt((( -13) + 1*e + 22*catalan))
log((( -6) + 24*e + 4*catalan)/e)
exp(catalan*(( -1/5) + (8/15)*e))
catalan*(6 + (-6)*e + 15*catalan)
sqrt((5 + 26*e + (-3)*catalan))/e
e*sqrt((( -27) + 2*e + 25*catalan))
log((( -1) + (-11)*e + 59*catalan))
((3/20) + (21/20)*e + (3/20)*catalan)
...

```

---

The numerical values are roughly as close to  $\pi$  as permitted by the specified tolerance:

---

```

>>> e/log(6-4*e/3)
3.14157719846001
>>> 135*e*catalan**2/98
3.14166950419369
>>> sqrt(e-13+22*catalan)
3.14158000062992
>>> log(24*e-6+4*catalan)-1
3.14158791577159

```

---

### 27.1.3 Symbolic processing

The output formula can be evaluated as a Python expression. Note however that if fractions (like '2/3') are present in the formula, Python's eval() may erroneously perform integer division. Note also that the output is not necessarily in the algebraically simplest form:

---

```

>>> identify(sqrt(2))
'(sqrt(8)/2)'

```

---

As a solution to both problems, consider using SymPy's sympify() to convert the formula into a symbolic expression. SymPy can be used to pretty-print or further simplify the formula symbolically:

---

```

>>> from sympy import sympify
>>> sympify(identify(sqrt(2)))
2**(1/2)

```

---

Sometimes identify() can simplify an expression further than a symbolic algorithm:

---

```

>>> from sympy import simplify
>>> x = sympify(' -1/(-3/2+(1/2)*5** (1/2))*(3/2-1/2*5** (1/2))** (1/2)')
>>> x
(3/2 - 5** (1/2)/2)** (-1/2)
>>> x = simplify(x)
>>> x
2/(6 - 2*5** (1/2))** (1/2)
>>> mp.dps = 30
>>> x = sympify(identify(x.evalf(30)))

```

```
>>> x  
1/2 + 5**(1/2)/2
```

---

(In fact, this functionality is available directly in SymPy as the function `nsimplify()`, which is essentially a wrapper for `identify()`.)

#### 27.1.4 Miscellaneous issues and limitations

The input  $x$  must be a real number. All base constants must be positive real numbers and must not be rationals or rational linear combinations of each other.

The worst-case computation time grows quickly with the number of base constants. Already with 3 or 4 base constants, `identify()` may require several seconds to finish. To search for relations among a large number of constants, you should consider using `pslq()` directly.

The extended transformations are applied to  $x$ , not the constants separately. As a result, `identify` will for example be able to recognize  $\exp(2\pi i + 3)$  with  $\pi$  given as a base constant, but not  $2\exp(\pi) + 3$ . It will be able to recognize the latter if  $\exp(\pi)$  is given explicitly as a base constant.

## 27.2 Algebraic identification

---

Function **findpoly**(*x* As *mpNum*, *n* As Integer, **Keywords** As String) As String

---

The function `findpoly` returns the coefficients of an integer polynomial  $P$  of degree at most  $n$  such that  $P(x) \sim 0$

**Parameters:**

*x*: A real number

*n*: max degree of polynomial.

**Keywords:** tol=None, maxcoeff=1000, maxsteps=100, verbose=False

`findpoly(x, n)` returns the coefficients of an integer polynomial  $P$  of degree at most  $n$  such that  $P(x) \sim 0$ . If no polynomial having  $x$  as a root can be found, `findpoly()` returns None.

`findpoly()` works by successively calling `pslq()` with the vectors  $[1, x]$ ,  $[1, x, x^2]$ ,  $[1, x, x^2, x^3]$ , ..., as input. Keyword arguments given to `findpoly()` are forwarded verbatim to `pslq()`. In particular, you can specify a tolerance for  $P(x)$  with `tol` and a maximum permitted coefficient size with `maxcoeff`.

For large values of  $n$ , it is recommended to run `findpoly()` at high precision; preferably 50 digits or more.

### 27.2.1 Examples

By default (degree  $n = 1$ ), `findpoly()` simply finds a linear polynomial with a rational root:

---

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> findpoly(0.7)
[-10, 7]
```

---

The generated coefficient list is valid input to `polyval` and `polyroots`:

---

```
>>> nprint(polyval(findpoly(phi, 2), phi), 1)
-2.0e-16
>>> for r in polyroots(findpoly(phi, 2)):
...     print(r)
...
-0.618033988749895
1.61803398874989
```

---

Numbers of the form  $m + n\sqrt{p}$  for integers  $(m, n, p)$  are solutions to quadratic equations. As we find here,  $1 + \sqrt{2}$  is a root of the polynomial  $x^2 - 2x - 1$ :

---

```
>>> findpoly(1+sqrt(2), 2)
[1, -2, -1]
>>> findroot(lambda x: x**2 - 2*x - 1, 1)
2.4142135623731
```

---

Despite only containing square roots, the following number results in a polynomial of degree 4:

---

```
>>> findpoly(sqrt(2)+sqrt(3), 4)
[1, 0, -10, 0, 1]
```

---

In fact,  $x^4 - 10x^2 + 1$  is the minimal polynomial of  $r = \sqrt{2} + \sqrt{3}$ , meaning that a rational polynomial of lower degree having  $r$  as a root does not exist. Given sufficient precision, `findpoly()` will usually find the correct minimal polynomial of a given algebraic number.

### 27.2.2 Non-algebraic numbers

If `findpoly()` fails to find a polynomial with given coefficient size and tolerance constraints, that means no such polynomial exists.

We can verify that  $\pi$  is not an algebraic number of degree 3 with coefficients less than 1000:

---

```
>>> mp.dps = 15
>>> findpoly(pi, 3)
>>>
```

---

It is always possible to find an algebraic approximation of a number using one (or several) of the following methods:

1. Increasing the permitted degree
2. Allowing larger coefficients
3. Reducing the tolerance

One example of each method is shown below:

---

```
>>> mp.dps = 15
>>> findpoly(pi, 4)
[95, -545, 863, -183, -298]
>>> findpoly(pi, 3, maxcoeff=10000)
[836, -1734, -2658, -457]
>>> findpoly(pi, 3, tol=1e-7)
[-4, 22, -29, -2]
```

---

It is unknown whether Euler's constant is transcendental (or even irrational). We can use `findpoly()` to check that if it is an algebraic number, its minimal polynomial must have degree at least 7 and a coefficient of magnitude at least 1000000:

---

```
>>> mp.dps = 200
>>> findpoly(euler, 6, maxcoeff=10**6, tol=1e-100, maxsteps=1000)
>>>
```

---

Note that the high precision and strict tolerance is necessary for such high-degree runs, since otherwise unwanted low-accuracy approximations will be detected. It may also be necessary to set `maxsteps` high to prevent a premature exit (before the coefficient bound has been reached). Running with `verbose=True` to get an idea what is happening can be useful.

## 27.3 Integer relations (PSLQ)

Function **pslq**(*x As mpNum, Keywords As String*) As mpNum[]

The function **pslq** returns list of integers to approximate a function.

**Parameters:**

*x*: A vector of real numbers  $x = [x_0, x_1, \dots, x_n]$

*Keywords*: tol=None, maxcoeff=1000, full=False, verbose=False

Given a vector of real numbers  $x = [x_0, x_1, \dots, x_n]$ , **pslq**(*x*) uses the PSLQ algorithm to find a list of integers  $[c_0, c_1, \dots, c_n]$  such that

$$|c_1x_1 + c_2x_2 + \dots + c_nx_n| < \text{tol} \quad (27.3.1)$$

and such that  $\max|c_k| < \text{maxcoeff}$ . If no such vector exists, **pslq()** returns None. The tolerance defaults to 3/4 of the working precision.

### 27.3.1 Examples

Find rational approximations for  $\pi$ :

```
>>> from mpFormulaPy import *
>>> mp.dps = 15; mp.pretty = True
>>> pslq([-1, pi], tol=0.01)
[22, 7]
>>> pslq([-1, pi], tol=0.001)
[355, 113]
>>> mpf(22)/7; mpf(355)/113; +pi
3.14285714285714
3.14159292035398
3.14159265358979
```

Pi is not a rational number with denominator less than 1000:

```
>>> pslq([-1, pi])
>>>
```

To within the standard precision, it can however be approximated by at least one rational number with denominator less than  $10^{12}$ :

```
>>> p, q = pslq([-1, pi], maxcoeff=10**12)
>>> print(p); print(q)
238410049439
75888275702
>>> mpf(p)/q
3.14159265358979
```

The PSLQ algorithm can be applied to long vectors. For example, we can investigate the rational (in)dependence of integer square roots:

```
>>> mp.dps = 30
>>> pslq([sqrt(n) for n in range(2, 5+1)])
>>>
```

---

```
>>> pslq([sqrt(n) for n in range(2, 6+1)])
>>>
>>> pslq([sqrt(n) for n in range(2, 8+1)])
[2, 0, 0, 0, 0, 0, -1]
```

---

### 27.3.2 Machin formulas

A famous formula for  $\pi$  is Machin's,

$$\frac{\pi}{4} = 4 \operatorname{acot} 5 - \operatorname{acot} 239 \quad (27.3.2)$$

There are actually infinitely many formulas of this type. Two others are

$$\frac{\pi}{4} = \operatorname{acot} 1 \quad (27.3.3)$$

$$\frac{\pi}{4} = 12 \operatorname{acot} 49 + 32 \operatorname{acot} 57 + 5 \operatorname{acot} 239 + 12 \operatorname{acot} 110443 \quad (27.3.4)$$

We can easily verify the formulas using the PSLQ algorithm:

---

```
>>> mp.dps = 30
>>> pslq([pi/4, acot(1)])
[1, -1]
>>> pslq([pi/4, acot(5), acot(239)])
[1, -4, 1]
>>> pslq([pi/4, acot(49), acot(57), acot(239), acot(110443)])
[1, -12, -32, 5, -12]
```

---

We could try to generate a custom Machin-like formula by running the PSLQ algorithm with a few inverse cotangent values, for example  $\operatorname{acot}(2)$ ,  $\operatorname{acot}(3)$  ...  $\operatorname{acot}(10)$ . Unfortunately, there is a linear dependence among these values, resulting in only that dependence being detected, with a zero coefficient for  $\pi$ :

---

```
>>> pslq([pi] + [acot(n) for n in range(2,11)])
[0, 1, -1, 0, 0, 0, -1, 0, 0, 0]
```

---

We get better luck by removing linearly dependent terms:

---

```
>>> pslq([pi] + [acot(n) for n in range(2,11) if n not in (3, 5)])
[1, -8, 0, 0, 4, 0, 0, 0]
```

---

In other words, we found the following formula:

---

```
>>> 8*acot(2) - 4*acot(7)
3.14159265358979323846264338328
>>> +pi
3.14159265358979323846264338328
```

---

# **Part V**

## **Application Examples**

# Chapter 28

## Examples: Multivariate Special Functions

### 28.1 Functions Of Matrix Arguments

#### 28.1.1 Multivariate Gamma function $\Gamma_p(x)$

---

Function **Gammap**(*p* As mpNum, *x* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function Gammap returns the multivariate gamma function.

**Parameters:**

*p*: An integer greater than 0.

*x*: A real number or an array of real numbers.

The multivariate gamma function is defined by

$$\Gamma_p(x) = \pi^{p(p-1)/4} \prod_{i=1}^p \Gamma\left(x - \frac{1}{2}(i-1)\right), \quad \Re(x) > \frac{1}{2}(m-1) \quad (28.1.1)$$

$$\Gamma_p(s_1, \dots, s_p) = \pi^{p(p-1)/4} \prod_{i=1}^p \Gamma\left(s_i - \frac{1}{2}(i-1)\right), \quad \Re(s_i) > \frac{1}{2}(i-1), \quad j = 1, \dots, m. \quad (28.1.2)$$

$$\Gamma_p(a, \dots, a) = \Gamma_p(a) \quad (28.1.3)$$

where  $\Gamma(\cdot)$  is the usual (scalar) gamma function (see section 4.8.6).

## 28.1.2 Zonal Polynomials

### 28.1.2.1 Definitions

A partition  $\kappa = (k_1, \dots, k_m)$  is a vector of nonnegative integers, listed in nonincreasing order. Also,  $|\kappa|$  denotes  $k_1 + \dots + k_m$ , the weight of  $\kappa$ ;  $\ell(\kappa)$  denotes the number of nonzero  $k_j$ ;  $a + \kappa$  denotes the vector  $(a + k_1, \dots, a + k_m)$ .

The partitional shifted factorial is given by (Olver et al., 2010)

$$[a]_\kappa = \frac{\Gamma_m(a + \kappa)}{\Gamma_m(a)} = \prod_{j=1}^m \left( a - \frac{1}{2}(j-1) \right)_{k_j}, \quad (28.1.4)$$

where  $(a)_k$  is the Pochammer symbol (see section 9.3).

For any partition  $\kappa$ , the zonal polynomial  $Z_\kappa : \mathbf{S} \rightarrow \mathbb{R}$  is defined by the properties (Olver et al., 2010)

$$Z_\kappa(\mathbf{I}) = |\kappa|! 2^{2|\kappa|} [m/2]_\kappa \frac{\prod_{\substack{1 \leq j < l \leq \ell(\kappa) \\ \ell(\kappa)}} (2k_j - 2k_l - j + l)}{\prod_{j=1}^{\ell(\kappa)} (2k_j + \ell(\kappa) - j)!}, \quad \text{and} \quad (28.1.5)$$

$$Z_\kappa(\mathbf{T}) = Z_\kappa(\mathbf{I}) |\mathbf{T}|^{k_m} \int_{\mathbf{O}(m)} \prod_{j=1}^{m-1} |(\mathbf{HTH}^{-1})_j|^{k_j - k_{j+1}} d\mathbf{H} \quad (28.1.6)$$

See Muirhead (1982), pp. 68-72, for the definition and properties of the Haar measure  $d\mathbf{H}$ . Alternative notations for the zonal polynomials are  $C_\kappa(\mathbf{T})$  Muirhead (1982), pp. 227-239.

### 28.1.2.2 Properties

Zonal polynomials have the following properties (Olver et al., 2010):

$$Z_\kappa(0) = \begin{cases} 1, & \kappa = (0, \dots, 0), \\ 0, & \kappa \neq (0, \dots, 0). \end{cases} \quad (28.1.7)$$

$$Z_\kappa(\mathbf{HTH}^{-1}) = Z_\kappa(\mathbf{T}), \quad \mathbf{H} \in \mathbf{O}(m). \quad (28.1.8)$$

Therefore  $Z_\kappa(\mathbf{T})$  is a symmetric polynomial in the eigenvalues of  $\mathbf{T}$ . Also, for  $k = 0, 1, 2, \dots$ ,

$$\sum_{|\kappa|=k} Z_\kappa(\mathbf{T}) = (\text{tr } \mathbf{T})^k. \quad (28.1.9)$$

### 28.1.2.3 Implementation

Gupta & Richards (1979) describe an algorithm for the calculation of zonal polynomials for  $2 \times 2$  and  $3 \times 3$  matrices.

James (1968, 1964); McLaren (1976) describe a general algorithm.

### 28.1.3 Gauss Hypergeometric Function of Matrix Argument

Function **Hypergeometric2F1Matrix(*a* As mpNum, *b* As mpNum, *c* As mpNum, *T* As mpNum)**  
As mpNum

NOT YET IMPLEMENTED

The function `Hypergeometric2F1Matrix` returns the Gauss hypergeometric function for matrix argument.

#### Parameters:

*a*: A real number.

*b*: A real number.

*c*: A real number.

*T*: A real matrix.

The Gauss hypergeometric function for matrix argument,  ${}_2F_1(a, b; c; \mathbf{T})$  is defined by ([Olver et al., 2010](#))

$${}_2F_1(a, b; c; \mathbf{T}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{|\kappa|=k} \frac{[a]_{\kappa} [b]_{\kappa}}{[c]_{\kappa}} Z_{\kappa}(\mathbf{T}) \quad (28.1.10)$$

with  $-c + \frac{1}{2}(j+1) \notin \mathbb{N}$ ,  $1 \leq j \leq m$ ;  $\|\mathbf{T}\| < 1$ . Here  $Z_{\kappa}(\mathbf{T})$  is a zonal polynomial, as described in section [28.1.2](#).

See also [Koev & Edelman \(2006\)](#) for an "exact" calculation.

#### 28.1.3.1 Laplace approximation

The following closed-form approximation based on a Laplace approximation has been derived by [Butler & Wood \(2002\)](#):

$${}_2F_1(a, b; c; X) \approx \frac{c^{pc-p(p+1)/4}}{\sqrt{R_{2,1}}} \times \prod_{i=1}^p \left[ \left( \frac{y_i}{a} \right)^a \left( \frac{1-y_i}{c-a} \right)^{c-a} (1-x_i y_i)^{-b} \right], \quad (28.1.11)$$

where  $X = \text{diag}(x_1, \dots, x_p)$ ,  $S_i = x_i y_i (1 - y_i) / (1 - x_i y_i)$ ,  $\tau_i = x_i(b - a) - c$ ,

$$y_i = \frac{2a}{\sqrt{t_i^2 - 4ax_i(c-b)} - \tau_i}, \text{ and } R_{2,1} = \prod_{i=1}^p \prod_{j=i}^p \left[ \frac{y_i y_j}{a} + \frac{(1-y_i)(1-y_j)}{c-a} - \frac{b}{a(c-a)} S_i S_j \right].$$

### 28.1.4 Confluent Hypergeometric Function for Matrix Argument

Function **Hypergeometric1F1Matrix(*a* As mpNum, *b* As mpNum, *T* As mpNum) As mpNum**

**NOT YET IMPLEMENTED**

The function **Hypergeometric1F1Matrix** returns Kummer's confluent hypergeometric function for matrix argument.

**Parameters:**

*a*: A real number.

*b*: A real number.

*T*: A real matrix.

Kummer's confluent hypergeometric function for matrix argument  ${}_1F_1(a; b; \mathbf{T})$  is defined by the series ([Olver et al., 2010](#))

$${}_1F_1(a; b; \mathbf{T}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{|\kappa|=k} \frac{[a]_{\kappa}}{[b]_{\kappa}} Z_{\kappa}(\mathbf{T}) \quad (28.1.12)$$

with  $-b + \frac{1}{2}(j+1) \notin \mathbb{N}$ ,  $1 \leq j \leq m$ . Here  $Z_{\kappa}(\mathbf{T})$  is a zonal polynomial, as described in section [28.1.2](#).

See also [Koev & Edelman \(2006\)](#) for an "exact" calculation.

#### 28.1.4.1 Laplace approximation

The following closed-form approximation based on a Laplace approximation has been derived by [Butler & Wood \(2002\)](#), with  $X = \text{diag}(x_1, \dots, x_p)$ :

$${}_1F_1(a; b; X) \approx \frac{b^{pb-p(p+1)/4}}{\sqrt{R_{1,1}}} \times \prod_{i=1}^p \left[ \left( \frac{y_i}{a} \right)^a \left( \frac{1-y_i}{b-a} \right)^{b-a} e^{x_i y_i} \right], \quad (28.1.13)$$

where  $y_i = \frac{2a}{b - x_i + \sqrt{(x_i - b)^2 + 4ax_i}}$ , and  $R_{1,1} = \prod_{i=1}^p \prod_{j=i}^p \left[ \frac{y_i y_j}{a} + \frac{(1-y_i)(1-y_j)}{b-a} \right]$ .

### 28.1.5 Confluent Hypergeometric Limit Function for Matrix Argument

Function **Hypergeometric0F1Matrix(*n* As mpNum, *T* As mpNum) As mpNum**

**NOT YET IMPLEMENTED**

The function **Hypergeometric0F1Matrix** returns the confluent hypergeometric limit function for matrix argument.

**Parameters:**

*n*: A real number.

*T*: A real matrix.

This functions returns the confluent hypergeometric limit function for matrix argument  ${}_0F_1(a; X)$ , defined by the series ([Butler & Wood, 2003](#); [Olver \*et al.\*, 2010](#))

$${}_0F_1\left(\frac{1}{2}n; \frac{1}{4}XX^T\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{|\kappa|=k} \frac{1}{\left[\frac{1}{2}n\right]_{\kappa}} Z_{\kappa}(\mathbf{X}) \quad (28.1.14)$$

Here  $Z_{\kappa}(\mathbf{X})$  is a zonal polynomial, as described in section [28.1.2](#).

#### 28.1.5.1 Laplace approximation

The following closed-form approximation based on a Laplace approximation has been derived by [Butler & Wood \(2003\)](#), with  $X = \text{diag}(x_1, \dots, x_p)$ :

$${}_0F_1\left(\frac{1}{2}n; \frac{1}{4}XX^T\right) \approx \frac{1}{\sqrt{R_{0,1}}} \times \prod_{i=1}^p \left[ (1 - y_i)^{n/2} e^{x_i y_i} \right], \quad (28.1.15)$$

$$\text{where } y_i = \frac{2x_i/n}{\sqrt{(2x_i/n)^2 + 1 + 1}}, \text{ and } R_{0,1} = \prod_{i=1}^p \prod_{j=i}^p (1 - y_i^2 y_j^2).$$

# Chapter 29

## Examples: Moments, cumulants, and expansions

### 29.1 Moments and cumulants

The following relations hold between central moments, cumulants, and moments about the origin:  
Central moments from null moments:

$$\mu_n = \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \mu'_j (\mu_1)^{n-j} \quad (29.1.1)$$

Null moments from central moments:

$$\mu'_n = \sum_{r=0}^n \binom{n}{r} (\mu_1)^r \quad (29.1.2)$$

Cumulants from null moments:

$$\kappa_r = \mu'_r - \sum_{j=1}^{r-1} \binom{r-1}{j-1} \mu'_{r-j} \kappa_j \quad (29.1.3)$$

Cumulants from central moments:

$$\kappa_r = \mu_r - \sum_{j=2}^{r-1} \binom{r-1}{j-1} \mu_{r-j} \kappa_j, \quad r > 1, \quad \kappa_1 = \mu_1. \quad (29.1.4)$$

---

Function **CentralMomentsToRawMoments**(*Central* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function **CentralMomentsToRawMoments** returns raw moments calculated from central moments.

**Parameter:**

*Central*: A real vector.

---

Function **RawMomentsToCentralMoments**(*Raw* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function `RawMomentsToCentralMoments` returns central moments calculated from raw moments.

**Parameter:**

*Raw*: A real vector.

---

Function **CentralMomentsToCumulants**(*Central* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function `CentralMomentsToCumulants` returns cumulants calculated from central moments.

**Parameter:**

*Central*: A real vector.

---

Function **CumulantsToCentralMoments**(*Cumulants* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function `CumulantsToCentralMoments` returns central moments calculated from cumulants.

**Parameter:**

*Cumulants*: A real vector.

---

Function **RawMomentsToCumulants**(*Raw* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function `RawMomentsToCumulants` returns cumulants calculated from raw moments.

**Parameter:**

*Raw*: A real vector.

---

Function **CumulantsToRawMoments**(*Cumulants* As mpNum) As mpNum

---

NOT YET IMPLEMENTED

---

The function `CumulantsToRawMoments` returns raw moments calculated from cumulants.

**Parameter:**

*Cumulants*: A real vector.

## 29.2 The Edgeworth expansion

### 29.2.1 Continuous variates

Many distribution function can be approximated by an expansion of the normal distribution, provided that the cumulants of the distribution are known (Lee & Lin, 1992).

$$f(x) = \phi(x) + \sum_{i=1}^{\infty} \sum_{\pi(i)} \omega_{\pi(i)} Z^{(D(\pi(i)))}(x) \quad (29.2.1)$$

$$F(x) = \Phi(x) + \sum_{i=1}^{\infty} \sum_{\pi(i)} \omega_{\pi(i)} Z^{(D(\pi(i))-1)}(x) \quad (29.2.2)$$

where the summation is extended over the partitions

$$\pi(i) = (s_1^{m_1}, \dots, s_k^{m_k}) \text{ such that } s_1 > \dots > s_k \text{ and } i = \sum_{i=1}^k m_i s_i, \quad (29.2.3)$$

$$\omega_{\pi(i)} = \prod_{i=1}^k \frac{\gamma_{s_i}^{m_i}}{m_i!}, \quad \gamma_i = \frac{(-1)^i \kappa_i}{\sqrt{\kappa_2^i} (i+2)!}, \text{ and } D(\pi(i)) = \sum_{i=1}^k m_i (s_i + 2). \quad (29.2.4)$$

Using the first 4 cumulants, we have

$$\Pr \left[ \frac{u - \mu}{\sigma} \right] = \Phi(x) + \phi(x) \left[ \frac{\gamma_1}{6} (1 - x^2) + \frac{\gamma_2}{24} (3x - x^3) + \frac{\gamma_1^2}{72} (15x - 10x^3 + x^5) \right] + O(N^{-3/2}) \quad (29.2.5)$$

#### 29.2.1.1 Wilks, Hotelling's T2, Pillai's V

Example: moments of T2: Davis (1968) , equation 7.13

moments of V: Davis (1970a) Davis 1970, equation 2.11 (and section 4).

Box-omega of V:Davis (1970a), equations 5.2 to 5.4

Box-omega of T2: Davis (1970a), equations 5.6, and Davis (1970b), equations 3.10 to 3.11

### 29.2.2 Lattice (discrete) variates

Deccribe formula for Kendall's tau.

## 29.3 The Cornish-Fisher expansion

Many distribution function can be approximated by an expansion of the normal distribution, provided that the cumulants of the distribution are known. Cornish and Fisher derived an inversion formula ([Fisher & Cornish, 1960](#)), which has been generalized by [Lee & Lin \(1992\)](#). Lee's algorithm is based on the following formula for the adjustment of order  $n^{-k/2}$ :

$$\delta_k = a_k H^{k+1} + \sum_{(\kappa*)} \frac{(-1)^\pi}{j_1! \dots j_m!} (\delta_{k_1} - a_{k_1} H^{k_1+1})^{j_1} \dots (\delta_{k_m} - a_{k_m} H^{k_m+1})^{j_m} H^{\pi-1} \quad (29.3.1)$$

where  $\delta_i$  is the  $i$ th-order adjustment (of magnitude  $O(n^{i/2})$ ),  $a_i = \kappa_i / i! \sigma^i$ , where  $\kappa_i$  is the  $i$ th cumulant, and  $(\kappa*)$  is the set of all partitions of  $k$  having two or more parts. The typical term shown in the summand is due to the partition having  $j_1$  occurrences of  $k_1$  and so on;  $\pi$  is the number of parts in the partition; and the powers of  $H$  are to be multiplied out formally, with  $H^j$  replaced by the Hermite polynomial  $H_j(z)$  (with  $z$  representing the normal deviate) in the final result. A recurrence formula connects the  $\delta_k(H)$ :

$$\delta'_k = a'_k H^{k+1} + \sum_{j=1}^{k-1} \binom{k-1}{j} \delta'_{k-1}(H) (\delta'_j - a'_j H^{j+1}), \quad (29.3.2)$$

where  $\delta'_h(H) = h! \delta_h(H)$ ,  $\delta'_h = h! \delta_h$ ,  $a'_h = h! h_h$  and  $\delta_h(H)$  means an expression like the right-hand side of equation (29.3.1), formally a polynomial in  $H$ , and  $\delta_h$  is a numerical value. In this formula the adjustment of a given order is calculated recursively, the values of lower order adjustments being used in each stage of calculation.

Using the first 4 cumulants, we have

$$u = \mu + \sigma \left[ z + \frac{\gamma_1}{6}(z^2 - 1) + \frac{\gamma_2}{24}(z^3 - 3z) - \frac{\gamma_1^2}{36}(2z^3 - 5z) \right] + O(N^{-3/2}) \quad (29.3.3)$$

## 29.4 Saddlepoint approximations

Suppose the moment generating function (MGF) of the random variable  $X$  is analytic and given by  $M(t)$  for  $t$  in some open neighborhood of zero. Let  $K(t) = \log(M(t))$  be the Cumulant Generating Function(CGF) of  $X$ , and denote by  $K'(t)$ ,  $K''(t)$ ,  $K^{(3)}(t)$  and  $K^{(4)}(t)$  the first, second, third and fourth derivative of  $K(t)$ , respectively. Let  $F(x)$  and  $f(x)$  denote the CDF and pdf of  $X$ , respectively, and  $\Phi(\cdot)$  and  $\phi(\cdot)$  denote the CDF (see section 6.11.2.2) and pdf (see section 6.11.2.1) of the normal distribution, respectively.

Let  $s$  be the solution to the saddlepoint equation

$$K'(s) = x. \quad (29.4.1)$$

which in general needs to be solved numerically (sometimes closed form solutions exist), and define

$$w = w(s) = \operatorname{sgn}(s)\sqrt{2(sK'(s) - K(s))} \quad (29.4.2)$$

$$u = u(s) = s\sqrt{K''(s)} \quad (29.4.3)$$

$$r = r(s) = w(s) + \frac{1}{w(s)} \log \frac{u(s)}{w(s)} \quad (29.4.4)$$

### 29.4.1 First order approximations

Add: formulas for  $s = 0$ .

Jensen (1992) gives the following formula:

$$F(x) \approx \Phi(r) =: P1. \quad (29.4.5)$$

Lugannani & Rice (1980) give the following formula:

$$F(x) \approx \Phi(w) + \phi(w) \left( \frac{1}{w} - \frac{1}{u} \right) =: P3. \quad (29.4.6)$$

### 29.4.2 Second order approximations

Daniels (1954, 1987) gives the following formulas:

$$F(x) \approx P3 + \phi(w) \left( \frac{\kappa_4}{8u} - \frac{5}{24u} \kappa_3^2 - \frac{1}{u^3} - \frac{\kappa_3}{2u^2} + \frac{1}{w^3} \right) =: P4, \quad (29.4.7)$$

$$f(x) \approx \phi(w) \frac{s}{u} \left( 1 + \frac{\kappa_4}{8} - \frac{5}{24} \kappa_3^2 \right), \quad \text{where} \quad (29.4.8)$$

$$\kappa_3 = K^{(3)}(s)/K''(s)^{3/2} \quad \kappa_4 = K^{(4)}(s)/K''(s)^2 \quad (29.4.9)$$

## 29.5 Inverse Saddlepoint approximations

We assume that we have defined a function  $F_s(x)$  to convert from  $x$  to  $s$ , and also an inverse function  $F_x(s)$  to convert from  $s$  to  $x$  (this could be, but does not have to be, equation 29.4.1). To obtain an approximation to the  $\alpha$ -quantile of  $F(x)$ , we start with a reasonable estimate, say,  $x_0$ , and convert this to the corresponding saddlepoint, say  $s_0$ .

Let  $s$  be a saddlepoint, and let  $w(s)$  and  $u(s)$  be functions of  $s$ , which could be (but do not have to be) defined as in equations 29.4.2 and 29.4.3

Let  $z_\alpha$  be the  $\alpha$ -quantile of the standard normal distribution. We wish to solve for  $s$  in the equation

$$h(s) = w(s) + \frac{v(s)}{w(s)} - z_\alpha = 0, \quad \text{where } v(s) = \log \frac{u(s)}{w(s)}, \quad \text{and} \quad (29.5.1)$$

$$h'(s) = w'(s) + \frac{w(s)u'(s) - u(s)w'(s)(v(s) + 1)}{u(s)w(s)^2} \quad (29.5.2)$$

Starting with  $n = 0$ , we compute successive approximations as

$$s_{n+1} = s_n - \frac{h(s_n)}{h'(s_n)} \quad (29.5.3)$$

The initial steps can be run with

$$s_{n+1} = s_n - \frac{w(s_n) - z_\alpha}{w'(s_n)} \quad (29.5.4)$$

since  $w(s)$  is the dominant term in the right hand side of equation 29.5.1.

If  $w(s)$  and  $u(s)$  are defined as in equations 29.4.2 and 29.4.3, then

$$w'(s) = \operatorname{sgn}(s) \frac{sK''(s)}{\sqrt{2sK'(s) - 2K(s)}} = \operatorname{sgn}(s) \frac{sK''(s)}{w(s)} \quad (29.5.5)$$

$$u'(s) = \frac{sK^{(3)}(s) + 2K''(s)}{2\sqrt{K''(s)}} \quad (29.5.6)$$

See also Wang (1995)

## 29.6 The Box-Davis expansion for a class of multivariate distributions

### 29.6.1 Definition

[Davis \(1971\)](#) considers a general class of random variables with absolutely continuous distribution functions  $F(x)$ , whose cumulant generating function may be validly represented by an asymptotic series

$$K(\theta) \sim -\frac{1}{2}f \log(1 - 2i\theta) + \sum_{r=1}^{\infty} \omega_r [(1 - 2i\theta)^{-r} - 1], \quad (29.6.1)$$

corresponding to a limiting chi-squared distribution with  $f$  degrees of freedom.

Examples are given in sections [29.6.4](#), [29.6.5](#) and [29.6.6](#).

A special subset of the above are random variables  $W(0 \leq W \leq 1)$  with the  $h$ th moment

$$\mathbb{E}[W^h] = K \left( \frac{\prod_{j=1}^b y_j^{y_j}}{\prod_{k=1}^a x_k^{x_k}} \right)^h \frac{\prod_{k=1}^a \Gamma[x_k(1+h) + \xi_k]}{\prod_{j=1}^b \Gamma[y_j(1+h) + \eta_j]}, \quad h = 0, 1, \dots, \quad (29.6.2)$$

where  $\Gamma(\cdot)$  denotes the Gamma function (see section [4.8.6](#).),  $K$  is a constant such that  $e[W^0] = 1$ , i.e.

$$K = \frac{\prod_{j=1}^b \Gamma(y_j + \eta_j)}{\prod_{k=1}^a \Gamma(x_k + \xi_k)}, \quad \text{and} \quad \sum_{k=1}^a x_k = \sum_{j=1}^b y_j. \quad (29.6.3)$$

We define

$$f = -2 \left[ \sum_{k=1}^a \xi_k - \sum_{j=1}^b \eta_j - \frac{1}{2}(a - b) \right], \quad (29.6.4)$$

$$\rho = 1 - \frac{1}{f} \left[ \sum_{k=1}^a x_k^{-1} (\xi_k^2 - \xi_k + \frac{1}{6}) - \sum_{j=1}^b y_j^{-1} (\eta_j^2 - \eta_j + \frac{1}{6}) \right], \quad (29.6.5)$$

$$\omega_r = \frac{(-1)^{r+1}}{r(r+1)} \left[ \sum_{k=1}^a \frac{B_{r+1}(\beta_k + \xi_k)}{(\rho x_k)^r} - \sum_{j=1}^b \frac{B_{r+1}(\epsilon_j + \eta_j)}{(\rho y_j)^r} \right], \quad (29.6.6)$$

where  $\rho$  is chosen in such a way that  $\omega_1 = 0$ ,  $\beta_k = (1 - \rho)x_k$ ,  $\epsilon_j = (1 - \rho)y_j$ , and  $B_{r+1}(x)$  denotes the Bernoulli polynomial of degree  $r + 1$  (see section [16.2](#)).

We assume  $x_k = c_k \theta$  and  $y_j = d_j \theta$ , where  $c_k$  and  $d_j$  will be constant and  $\theta$  will vary. Then  $M = -2\rho \log(W)$  follows asymptotically (with  $\theta \rightarrow \infty$ ) a  $\chi^2$ -distribution with  $f$  degrees of freedom, and the pdf, CDF and inverse CDF of  $M$  can be developed into asymptotic expansions including terms of order  $O(\theta^{-r})$ , depending only on  $f$  and  $\omega_i (i = 1 \dots r)$ .

The first comprehensive review of this type of expansion for the CDF is due to [Box \(1949\)](#). The corresponding expansion for the inverse CDF is due to [Davis \(1971\)](#). We refer therefore to this system of expansion as the Box-Davis expansion. For additional information, see [Anderson \(2003\)](#).

Many test-criteria of classical univariate and multivariate statistics have moments of the form given in [29.6.2](#) and allow therefore evaluation of their pdf, CDF and inverse CDF by the Box-Davis expansion. Some of them are reviewed in sections [29.6.4](#) and [29.6.5](#).

For a subset of the random variables  $W$  defined in equation (29.6.2) there exists a power transformation  $U = W^{2/N}$ , such that

$$\mathbb{E}[U^h] = \prod_{j=1}^p \frac{\Gamma(c_j)\Gamma(b_j + h)}{\Gamma(b_j)\Gamma(c_j + h)}, \quad (29.6.7)$$

i.e.  $U$  is distributed as the product of  $p$  independent random variables which have a beta distribution with  $b_j$  and  $c_j - b_j$  degrees of freedom. This allows to alternatively calculate their pdf and CDF by the algorithms given in 29.7. Some examples are reviewed in section 29.6.4.

## 29.6.2 Density and CDF

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Function **BoxDavisDist**(*M* As mpNum, *f* As mpNum, *omega* As mpNum[], *Output* As String)  
As mpNumList

---

**NOT YET IMPLEMENTED**

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The function BoxDavisDist returns pdf, CDF and related information for the Box-Davis-distribution

**Parameters:**

*M*: A real number greater 0, defined as  $M = -2\rho \log(W)$

*f*: A real number greater 0, representing the degrees of freedom for the  $\chi^2$ -expansion

*omega*: An array of real numbers, representing the coefficients  $\omega_i$

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given below.

### 29.6.2.1 CDF: General formulas

Box (1949) suggested the following approximation to a class of multivariate criteria (see also Siotani *et al.* (1985)):

$$F_{Box}(x, n; \omega_i) = F_{\chi^2}(n, x) - 2f_{\chi^2}(n, x) \sum_{i=1}^{\infty} \sum_{\pi(i)} \omega_{\pi(i)} h_{\pi(i)}(x) \quad (29.6.8)$$

where the summation is extended over the partitions

$$\pi(i) = (s_1^{m_1}, \dots, s_k^{m_k}) \text{ such that } s_1 > \dots > s_k \text{ and } i = \sum_{i=1}^k m_i s_k, \quad (29.6.9)$$

$$\text{where } \omega_{\pi(i)} = \prod_{i=1}^k \frac{\omega_{s_i}^{m_i}}{m_i!}, \quad (29.6.10)$$

and  $h_{\pi(i)}(x)$  can be computed as

$$h_i = \frac{\sum_{r=1}^i x^r}{\prod_{j=1}^r f + 2j - 2}, \quad (29.6.11)$$

$$h_{p,q} = h_{p+q} - (h_p + h_q) \quad (29.6.12)$$

$$h_{p,q,r} = h_{p+q+r} - (h_{p+q} + h_{p+r} + h_{q+r}) + (h_p + h_q + h_r) \quad (29.6.13)$$

and the extension for higher orders is obvious. For  $\omega_1 = 0$ , the expansion is given by, including terms of order  $O(\theta^{-7})$ :

$$\begin{aligned} S &= \omega_2 h_2 \\ &+ \omega_3 h_3 \\ &+ \omega_4 h_4 + \frac{1}{2}\omega_2^2(h_4 - 2h_2) \\ &+ \omega_5 h_5 + \omega_3\omega_2(h_5 - h_3 - h_2) \\ &+ \omega_6 h_6 + \omega_4\omega_2(h_6 - h_4 - h_2) + \frac{1}{2}\omega_3^2(h_6 - 2h_3) + \omega_2^3(h_6 - 3h_4 + 3h_2)/6 \\ &+ \omega_7 h_7 + \omega_5\omega_2(h_7 - h_5 - h_2) + \omega_4\omega_3(h_7 - h_4 - h_3) + \frac{1}{2}\omega_3\omega_2^2(h_7 - 2h_5 - h_4 + h_3 + 2h_2) \end{aligned} \quad (29.6.14)$$

For  $\omega_1 \neq 0$ , the expansion is given by, including terms of order  $O(\theta^{-4})$ :

$$\begin{aligned} S &= \omega_1 h_1 \\ &+ \omega_2 h_2 + \frac{1}{2}\omega_1^2(h_2 - 2h_1) \\ &+ \omega_3 h_3 + \omega_2\omega_1(h_3 - h_2 - h_1) + \omega_1^3(h_3 - 3h_2 + 3h_1)/6 \\ &+ \omega_4 h_4 + \omega_3\omega_1(h_4 - h_3 - h_1) + \frac{1}{2}\omega_2\omega_1^2(h_4 - 2h_3 + 2h_1) + \frac{1}{2}\omega_2^2(h_4 - 2h_2) \\ &+ \omega_1^4(h_4 - 4h_3 + 6h_2 - 4h_1) \end{aligned} \quad (29.6.15)$$

### 29.6.3 Quantiles

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Function **BoxDavisDistInv**(*Prob* As mpNum, *f* As mpNum, *omega* As mpNum[], *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

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The function **BoxDavisDistInv** returns quantiles and related information for the Box-Davis-distribution

#### Parameters:

*Prob*: A real number between 0 and 1.

*f*: A real number greater 0, representing the degrees of freedom for the  $\chi^2$ -expansion

*omega*: An array of real numbers, representing the coefficients  $\omega_i$

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below

#### 29.6.3.1 Davis Inversion Formula

Davis (1971) derived the following approximate inversion formula, assuming  $\omega_1 = 0$ :

The expansion 29.6.8 may be inverted to express an arbitrary  $100(1 - \alpha)\%$  point  $x_\alpha$  in terms of the corresponding chi-squared percentile  $u = \chi_{f,\alpha}^2$ .

$$F_{Box}^{-1}(\alpha, n; \omega_i) = u - 2 \sum_{i=1}^{\infty} \sum_{\pi(i)} \omega_{\pi(i)} P_{\pi(i)}(u) \quad (29.6.16)$$

where the summation and  $\omega_{\pi(i)}$  are defined as in equations 29.6.9 and 29.6.10, and the polynomials  $P_{\pi(i)}(u)$  are given by

$$P_{\pi(i)}(u) = \sum_{r=1}^{l_\pi} D_{(r)} h_\pi^{(r)}(u), \quad \text{where} \quad (29.6.17)$$

$$D_1 = 1, \quad D_r = 2 \frac{d}{du} [-(r-1)(1 - (f-2)/u)], \quad D_{(r)} = D_1 D_2 \dots D_r. \quad (29.6.18)$$

In particular,  $P_{[n]} = h_n$  ( $n = 1, 2, \dots$ ), with  $h_\pi$  defined in equations 29.6.11 to 29.6.13

Below the  $P_{\pi(i)}(u)$  for those partitions of integers 2 to 7 which do not contain unity are given:

$$P_{2,2} = -\frac{8u^4(f+3)}{(f_2 f_4)} + \frac{8u^3}{(f_2 f_3)} + \frac{6u^2}{(f f_2)} + \frac{2u}{f^2} \quad (29.6.19)$$

$$P_{3,2} = -\frac{12u^5(f+4)}{(f_2 f_5)} - \frac{2u^4(f-6)}{(f_2 f_4)} + \frac{2u^3(3f+10)}{(f_2 f_3)} + \frac{6u^2}{(f f_2)} + \frac{2u}{f^2} \quad (29.6.20)$$

$$\begin{aligned} P_{4,2} = & -16u^6(f+5)/(f_2 f_6) - 4u^5(f-4)/(f_2 f_5) + 2u^4(3f+14)/(f_2 f_4) \\ & + 2u^3(3f+10)/(f_2 f_3) + 6u^2/(f f_2) + 2u/f^2 \end{aligned} \quad (29.6.21)$$

$$\begin{aligned} P_{3,3} = & -6u^6(3f^2+30f+80)/(f_3 f_6) - 6u^5(f_2+2f-16)/(f_3 f_5) \\ & + 4u^4(f+12)/(f_2 f_4) + 4u^3(3f+8)/(f_2 f_3) + 6u^2/(f f_2) + 2u/f^2 \end{aligned} \quad (29.6.22)$$

$$\begin{aligned} P_{2,2,2} = & 32u^6(7f^2+62f+120)/(f_2^2 f_6) - 32u^5(2f^2+37f+96)/(f_2^2 f_5) \\ & + -8u^4(23f^2+124f+132)/(f_2^2 f_4) - 8u^3(f-10)/(f f_2 f_3) \\ & + 28u^2/(f^2 f_2) + 4u/f^3; \end{aligned} \quad (29.6.23)$$

$$P_{5,2} = -\frac{20u^7(f+6)}{(f_2 f_7)} - \frac{2u^6(3f-10)}{(f_2 f_6)} + \frac{2u}{f^2} + 2 \sum_{r=2}^5 \frac{u^r(3f+4r-2)}{f_2 f_r} \quad (29.6.24)$$

$$\begin{aligned} P_{4,3} = & -24u^7(f_2+12f+40)/(f_3 f_7) - 2u^6(5f_2+18f-80)/(f_3 f_6) \\ & + 2u^5(f_2+42f+176)/(f_3 f_5) + 4u^4(3f+16)/(f_2 f_4) \\ & + 4u^3(3f+8)/(f_2 f_3) + 6u^2/(f f_2) + 2u/f^2; \end{aligned} \quad (29.6.25)$$

$$\begin{aligned} P_{3,2,2} = & 192u^7(2f^3+31f^2+154f+240)/(f_2 f_3 f_7) \\ & + -16u^6(4f^3+153f^2+1106f+2160)/(f_2 f_3 f_6) - 8u^5(35f^3 \\ & + 420f^2+1540f+1632)/(f_2 f_3 f_5) - 4u^4(25f^2+80f+12)/(f_2^2 f_4) \\ & + 4u^3(7f+38)/(f f_2 f_3) + 28u^2/(f^2 f_3) + 4u/f^3; \end{aligned} \quad (29.6.26)$$

## 29.6.4 Special Cases: Products of beta variables

### 29.6.4.1 Test of a given covariance matrix

The modified likelihood criterion for testing the hypothesis that a sample of size  $N$  is drawn from a  $p$ -variate normal population with a given matrix  $\Sigma_0$  is a power of

$$W = e^{\frac{1}{2}pn} |S\Sigma_0^{-1}|^{\frac{1}{2}n} \left( \text{tr}(S\Sigma_0^{-1})/p \right)^{\frac{1}{2}pn} \quad (29.6.27)$$

where  $n = N - 1$  and  $S$  is the sample covariance matrix. We have

$$f = \frac{1}{2}(p-1)(p+2); \quad \rho = 1 - \frac{2p^2 + p + 2}{6pn} \quad (29.6.28)$$

$$\omega_r = \frac{2(-1)^r}{r(r+1)(r+2)\rho^r} \sum_{s=1}^{r+2} \binom{r+2}{s+1} (1-\rho)^{r+1-s} \frac{\delta_s + \frac{1}{2}(s+1)B_s/p^{s-1}}{\binom{\frac{1}{2}s-1}{2}}, \quad (29.6.29)$$

where  $B_s$  are the Bernoulli numbers.

#### 29.6.4.2 Test of a given mean and covariance matrix

The modified likelihood criterion for testing the hypothesis that a sample of size  $N$  is drawn from a  $p$ -variate normal population with a given matrix  $\Sigma_0$  is a power of

$$W = e^{\frac{1}{2}pn} |S\Sigma_0^{-1}|^{\frac{1}{2}n} \left( \text{tr}(S\Sigma_0^{-1})/p \right)^{\frac{1}{2}pn} \quad (29.6.30)$$

where  $n = N - 1$  and  $S$  is the sample covariance matrix. We have

$$f = \frac{1}{2}(p-1)(p+2); \quad \rho = 1 - \frac{2p^2 + p + 2}{6pn} \quad (29.6.31)$$

$$\omega_r = \frac{2(-1)^r}{r(r+1)(r+2)\rho^r} \sum_{s=1}^{r+2} \binom{r+2}{s+1} (1-\rho)^{r+1-s} \frac{\delta_s + \frac{1}{2}(s+1)B_s/p^{s-1}}{\binom{\frac{1}{2}s-1}{2}}, \quad (29.6.32)$$

where  $B_s$  are the Bernoulli numbers.

#### 29.6.4.3 Wilks U (MANOVA)

$$f = pq; \quad \rho = 1 - \frac{p+q+1}{2(N-1)} \quad (29.6.33)$$

$$\omega_r = \frac{(-2)^r}{r(r+1)\mu^r} \sum_{j=0}^{q-1} B_{r+1}((\beta-j)/2) - B_{r+1}((\beta-p-j)/2), \quad \text{where} \quad (29.6.34)$$

$$\beta = \frac{p+q+1}{2}; \quad \mu = N - \frac{p+q+3}{2}. \quad (29.6.35)$$

[Wakaki \(2006\)](#) gives the  $s$ th order cumulant  $\kappa^s$  of  $-\log \Lambda$  as

$$\kappa^s = (-1)^s \left( \psi_q^{(s-1)} \left( \frac{n-p+q}{2} \right) - \psi_q^{(s-1)} \left( \frac{n+q}{2} \right) \right) \quad (29.6.36)$$

#### 29.6.4.4 Test for a correlation matrix being the identity matrix

$Pr[W \leq W_0] = F_{\text{betaprod}}(W, b_i = 1 \dots p-1, c_i = 1 \dots p-1)$ , where  $b(i) = (n-i)/2$ ;  $c(i) = n/2$ ;  $n = N - 1$ ;

Lit.: Anderson (1958), p. 269

### 29.6.4.5 Test for independence of k sets of variates

Given  $N$  observations from a  $p$  variate normal population, suppose that the variates are partitioned into  $k$  groups of sizes  $p_i (i = 1, \dots, k; \sum p_i = p)$ , and it required to test the independence of the groups

$$\omega = \frac{(-1)^{r+1}}{r(r+1)(r+2)\mu^r} \sum_{s=0}^{r+1} \binom{s+1}{r+2} 2^s \delta_s(p) \left[ \beta^{r+1-s} - \left( \beta - \sum_{n=1}^{l-1} p_n \right)^{r+1-s} \right] \quad (29.6.37)$$

$$f = \frac{1}{2} S_2; \quad \rho = 1 - \beta/v; \quad \beta = \frac{2S_3 + 3S_2}{12f}; \quad \mu = v\rho; \quad S_i(\sum_l p_l)^i - \sum_l (p_l)^i. \quad (29.6.38)$$

Under the null hypothesis,  $U$  is distributed as  $U = \prod_{i=2}^q \prod_{j=1}^{p(i)} X_{ij}$ , where the  $X_{ij}$  are independent and  $X_{ij}$  has the density  $\beta|x|, (n - pp_i + 1 - j)/2, p_i/2]$ .

Lit.: Anderson 1984, p.383 and 386

### 29.6.4.6 Test for equality of k independent variances (Bartlett)

$L$  = Bartlett's statistic

$p$  = number of samples

$W = L^k$

Exact distribution, algorithm for equal sample sizes

$\Pr[L \leq L_0] = F_{betaprod}(L^k, b_i = 1 \dots p, c_i = 1 \dots p)$ ,

where  $b(i) = \frac{1}{2}(n-1); c(i) = \frac{1}{2}(n-1) + i/p$ ;

Lit.: Glaser (1980)

Exact distribution, algorithm for unequal sample sizes

$\Pr[L \leq L_0] = F_{betaprod}(L^k, b_i = 1 \dots p, c_i = 1 \dots p)$ ,

where  $b(i) = \beta_i + \frac{1}{2}; c(i) = \gamma_i + \frac{1}{2}$ ;

$N_i$  = size of  $i$ th sample

$n_i = N_i - 1$

$T = gcd(n_1, \dots, n_p)$

$d_i = n_i/T$

$d = d_1 + \dots + d_p$

$N = d - 1$

$\gamma_i = (i-1)/d$

The  $\beta_i$  are defined by the following algorithm:

t=0;

for i=1 to k do begin

for j=1 to di do begin

t=t+1

$\beta_j = (j-1)/di$

end

end

sort( $\beta$ )

Lit.: Glaser (1980), Gupta (1984)

[Nagarsenker \(1984\)](#)

Haarsey, Cyr(1982), Glaser(1980), Dyer(1980)

## 29.6.5 Other Test Criteria concerning Covariance Matrices

### 29.6.5.1 Test of Sphericity (Mauchley)

The likelihood criterion for testing the hypothesis that a sample of size  $N$  is drawn from a  $p$ -variate normal population whose covariance matrix is proportional to a given matrix  $\Sigma_0$  is a power of

$$W = |S\Sigma_0^{-1}|^{\frac{1}{2}n} \left( \text{tr}(S\Sigma_0^{-1})/p \right)^{\frac{1}{2}pn} \quad (29.6.39)$$

where  $n = N - 1$  and  $S$  is the sample covariance matrix. We have

$$f = \frac{1}{2}(p-1)(p+2); \quad \rho = 1 - \frac{2p^2 + p + 2}{6pn} \quad (29.6.40)$$

$$\omega_r = \frac{2(-1)^r}{r(r+1)(r+2)\rho^r} \sum_{s=1}^{r+2} \binom{r+2}{s+1} (1-\rho)^{r+1-s} \frac{\delta_s + \frac{1}{2}(s+1)B_s/p^{s-1}}{\binom{\frac{1}{2}s-1}{2}}, \quad (29.6.41)$$

where  $B_s$  are the Bernoulli numbers.

### 29.6.5.2 Test for equality of $k$ independent covariance matrices (Bartlett)

Let  $p$  be the number of variables,  $k$  the number of groups,  $v_i$  the sample size in groups  $i$ ,  $N = \sum_i v_i$ .

$$\omega = \frac{(-1)^rk}{r(r+1)(r+2)\mu^r} \sum_{s=1}^{r+1} \binom{s+1}{r+2} 2^s \delta_s \gamma_s \beta^{r+1-s} \quad (29.6.42)$$

$$\gamma_s = \frac{1}{k} \sum_{i=1}^k \left( \frac{v}{v_i} \right)^{s-1} - \frac{1}{k^s}; \quad \text{for equal } v_i : \gamma_s = 1 - \frac{1}{k^s} \quad (29.6.43)$$

$$f = \frac{p(p+1)(k-1)}{2}; \quad \rho = \frac{2p^2 + 3p - 1}{6(p+1)(k-1)} \left( -\frac{1}{N} + \sum_{i=1}^k \frac{1}{n_i} \right); \quad (29.6.44)$$

$$v = \frac{N}{k}; \quad \mu + \rho v = \frac{N\rho}{k}; \quad \beta = (1-\rho)v. \quad (29.6.45)$$

Korin (1969), Anderson 1984, p.420

## 29.6.6 The Lawley-Hotelling and Pillai traces

The Lawley-Hotelling generalized  $T_0^2$  and Pillai's  $V$  statistic, defined respectively by

$$T_0^2 = n\text{tr}(AB^{-1}), \quad V = n\text{tr}(A(A+B)^{-1}), \quad (29.6.46)$$

have been suggested as alternatives to Wilk's criterion for testing multivariate linear hypotheses. Here  $A$  and  $B$  are independent  $p \times p$  Wishart matrices on  $q$  and  $n$  degrees of freedom respectively. As  $n \rightarrow \infty$ , both criteria are asymptotically distributed as  $\chi^2_{pq}$ , and their cumulant generating functions have expansions as in equation (29.6.1).

Davis (1968) has found the moments for both criteria and has given recurrence relations for them, together with a formula relating the moments for  $T_0^2$  and  $V$ .

Davis (1970b) has established general formulas for the  $\omega_r$  of both criteria and has given recurrence relations for them, together with a formula relating the  $\omega_r$  for  $T_0^2$  and  $V$

### 29.6.6.1 Pillai's V, Central distribution

Central distribution: Coefficients for Davis expansion:

$$s = 1, \quad k = m + 1, \quad a = 2k + n_1 \quad (29.6.47)$$

$$\omega_1 = mn_1k/(2n_2) \quad (29.6.48)$$

$$2r\omega_r = 2(r - 1)\omega_r - 1 - s(1 - k/n_2)c_{1,r}, \quad r = 2, 3, \dots \quad (29.6.49)$$

$$c_{0,0} = 1; \quad c_{0,r} = c_{r,0} = 0; \quad (r = 1, 2, \dots); \quad c_{r,1} = 0 (r = 2, 3, \dots) \quad (29.6.50)$$

$$\begin{aligned} jc_{j,r} &= [(m - j + 1)(n_1 - j + 1)]c_{j-1,r-1} + [(j(2m + n_1 - 2j + 2) + 2(r - 1))/n_2]c_{j,r-1} \quad (29.6.51) \\ &\quad + [(j + 1)/n_2 - (j + 1)(m - j + 1)/n_2^2]c_{j+1,r-1} - [(mn_1 + 2(r - 2)/n_2)c_{j,r-2} \\ &\quad + (2/n_2) \sum_{i=1}^{r-2} i\omega_i(c_{j,r-i-1} - c_{j,r-i-2}) \end{aligned}$$

### 29.6.6.2 Hotelling's T2, Central distribution

Central distribution: Coefficients for Davis expansion:

$$s = -1, \quad k = -n_1, \quad a = 2n_1 + m + 1 \quad (29.6.52)$$

and follow equations (29.6.48) to (29.6.52)

## 29.7 The Product of Independent Beta Variables

### 29.7.1 Definition

Wilks (1932) introduced the concept of random variables  $U$ , which have moments of the form

$$\mathbb{E}(U^h) = \prod_{j=1}^p \frac{\Gamma(c_j)\Gamma(b_j + h)}{\Gamma(b_j)\Gamma(c_j + h)} \quad (29.7.1)$$

i.e.  $U$  is distributed as the product of  $p$  independent random variables which have a beta distribution with  $b_j$  and  $c_j - b_j$  degrees of freedom. See section 29.6.4 for examples of tests statistics which follow this type of distribution.

### 29.7.2 Density and CDF

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Function **BetaProductDist**(*x As mpNum, p As mpNum, a As mpNum[], b As mpNum[], Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function BetaProductDist returns pdf, CDF and related information for the central BetaProduct-distribution

#### Parameters:

*x*: A real number

*p*: An integer greater 0, representing the number of variates

*a*: An array of real numbers greater 0, representing the numerator degrees of freedom

*b*: An array of real numbers greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 6.2.2.1 and 6.2.2.2.

#### 29.7.2.1 Density: Meijer G function

The Product  $X$  of  $n$  independent betas  $X_i$ , i.e.  $X_i \sim \text{beta}(\gamma_i, \delta_i), i = 1, \dots, n$ , has its density defined on  $(0, 1)$ , expressed as a Meijer G function as follows (Mathai et al., 2010; Pham-Gia, 2008):

$$h(x) = \left( \prod_{j=1}^n \frac{\Gamma(\gamma_j + \delta_j)}{\Gamma(\gamma_j)} \right) G_{2,2}^{1,1} \left( n \left| \begin{matrix} *, n \\ 0, n \end{matrix} \right. \right) \gamma_1 + \delta_1 - 1, \dots, \gamma_n + \delta_n - 1 | \gamma_1 - 1, \dots, \gamma_n - 1 | x \quad (29.7.2)$$

#### 29.7.2.2 Density: Beta-Series

Tang & Gupta (1984, 1986) proposes the following approximation:

$$f_{BetaProd}(x; b_{i=1 \dots p}, c_{i=1 \dots p}) = K_p \sum_{r=0}^{\infty} \sigma_{r,p} B(b_p, f_p + r) I'(x; b_p, f_p + r), \quad \text{where} \quad (29.7.3)$$

### 29.7.2.3 CDF: Beta-Series

Tang & Gupta (1984, 1986) proposes the following approximation:

$$\Pr[W_p \leq x] = F_{BetaProd}(x; b_{i=1 \dots p}, c_{i=1 \dots p}) = K_p \sum_{r=0}^{\infty} \sigma_{r,p} B(b_p, f_p + r) I(x; b_p, f_p + r), \quad \text{where} \quad (29.7.4)$$

$$K_m = \prod_{j=1}^m \frac{\Gamma(c_j)}{\Gamma(b_j)}; \quad f_m = \sum_{j=1}^m c_j - b_j \quad (29.7.5)$$

$$\sigma_{r,k} = \frac{\Gamma(f_{k-1})}{\Gamma(f_k + r)} \sum_{s=0}^r \frac{(c_k - b_{k-1})_s \cdot \sigma_{r-s, k-1}}{s!} \quad (29.7.6)$$

with initial values  $\sigma_{0,1} = 1/\Gamma(f_1)$  and  $\sigma_{r,1} = 0$ .

### 29.7.2.4 CDF: Chi-Square Series

Tang & Gupta (1987) proposes the following approximation:

$$F_{BetaProd}(x; b_{i=1 \dots p}, c_{i=1 \dots p}) = K_p \sum_{r=0}^{\infty} l_r(a) a^{-r} f_{\chi^2}(2f_p + 2r, 2ax), \quad 0 < x < 2\pi, \quad \text{where} \quad (29.7.7)$$

$$l_r(a) = \frac{1}{r} \sum_{r=0}^r k q_k(a) l_{r-k}(a), \quad l_0(a) = 1, \quad (29.7.8)$$

$$q_k(a) = \frac{(-1)^{k+1}}{k(k+1)} \sum_{j=1}^p B_{k+1}(b_j - a) - B_{k+1}(c_j - a) \quad (29.7.9)$$

and  $a$  is a positive constant.

### 29.7.2.5 CDF: Algorithm 3

Nagarsenker & Suniaga (1983) proposes the following approximation:

$$\Pr[W_p \leq \lambda] = I(x; sm + d, \nu_1 + r; \delta) + O(sm^{-3}), \quad \text{where} \quad (29.7.10)$$

$$x = \lambda^{1/s}; \quad a = \frac{\nu_1 - \nu_2}{2\nu_1}; \quad d = \frac{1 - \nu_1}{2}; \quad \nu_r = \sum_{i=1}^p c_i^r - b_i^r \quad (29.7.11)$$

$$s^2 = \frac{-2B_2((1 + \nu_1)/2)}{\sum_{i=1}^p B_3(a + b_i) - B_3(a + c_i)} \quad (29.7.12)$$

and  $\delta^2$  is the noncentrality parameter to be used in the non-central  $\chi^2$ -expansion of the criterion.

## 29.7.3 Quantiles

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Function **BetaProductDistInv**(*Prob* As mpNum, *p* As mpNum, *a* As mpNum[], *b* As mpNum[], *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

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The function **BetaProductDistInv** returns quantiles and related information for the the central BetaProduct-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*p*: An integer greater 0, representing the number of variates

*a*: An array of real numbers greater 0, representing the numerator degrees of freedom

*b*: An array of real numbers greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

$$\Pr[W_p \leq \lambda] = I(x; sm + d, \nu_1 + r; \delta) + O(sm^{-3}), \quad \text{where} \quad (29.7.13)$$

$$x = \lambda^{1/s}; \quad a = \frac{\nu_1 - \nu_2}{2\nu_1}; \quad d = \frac{1 - \nu_1}{2}; \quad \nu_r = \sum_{i=1}^p c_i^r - b_i^r \quad (29.7.14)$$

$$s^2 = \frac{-2B_2((1 + \nu_1)/2)}{\sum_{i=1}^p B_3(a + b_i) - B_3(a + c_i)} \quad (29.7.15)$$

From these formulas an inversion in closed form can be derived, which can be used as a starting point for a Newton iteration.

## 29.7.4 Properties

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Function **BetaProductDistInfo**(*p* As mpNum, *a* As mpNum[], *b* As mpNum[], *Output* As String)  
As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function BetaProductDistInfo returns moments and related information for the central BetaProduct-distribution

**Parameters:**

*p*: An integer greater 0, representing the number of variates

*a*: An array of real numbers greater 0, representing the numerator degrees of freedom

*b*: An array of real numbers greater 0, representing the denominator degrees of freedom

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given in section 6.13.4.

### 29.7.4.1 Moments: algorithms and formulas

The raw moments are given by:

$$\mathbb{E}(W^h) = \prod_{j=1}^p \frac{\Gamma(c_j)\Gamma(b_j + h)}{\Gamma(b_j)\Gamma(c_j + h)} \quad (29.7.16)$$

## 29.7.5 Random Numbers

---

Function **BetaProductDistRandom**(*Size* As mpNum, *p* As mpNum, *a* As mpNum[], *b* As mpNum[], *Generator* As String, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function BetaProductDistRandom returns random numbers following a central BetaProduct-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*p*: An integer greater 0, representing the number of variates

*a*: An array of real numbers greater 0, representing the numerator degrees of freedom

*b*: An array of real numbers greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

### 29.7.5.1 Random Numbers: algorithms and formulas

A random number  $\xi$  from a BetaProduct distribution of  $p$  variates is obtained by generating  $p$  random numbers from a Beta distribution with  $b_i$  and  $c_i$  degrees of freedom, and then assigning their product to  $\xi$ .

# Chapter 30

## Examples: Continuous Distribution Functions

### 30.1 Distribution of the Sample Correlation Coefficient

#### 30.1.1 Definition

Let  $(X, Y)$  have a joint bivariate normal distribution with means  $\mu_x, \mu_y$ , standard deviations  $\sigma_x, \sigma_y$ , respectively, and correlation  $\rho$ . If  $(X_1, Y_1), \dots, (X_N, Y_N)$  denotes a random sample of size  $N$  from  $(X, Y)$ , the sample correlation coefficient  $R$  is defined by

$$R = \frac{\sum_{i=1}^N (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^N (X_i - \bar{X})^2 \sum_{j=1}^N (Y_j - \bar{Y})^2}} \quad (30.1.1)$$

where

$$\bar{X} = \frac{\sum_{i=1}^N X_i}{N}, \quad \bar{Y} = \frac{\sum_{i=1}^N Y_i}{N}. \quad (30.1.2)$$

For given  $N \geq 3$ , the distribution of  $R$  is independent of  $\mu_x, \mu_y, \sigma_x, \sigma_y$ , but depends upon  $\rho$ , where  $-1 < \rho < 1$ . For  $-1 \leq r \leq 1$ , define  $f_R(r, N; \rho)$  to be the probability density function for  $R$ . We denote the cumulative distribution function of  $R$  by ([Odeh, 1986](#))

$$F_R(r, N; \rho) = \Pr[R \leq r] = \int_{-1}^r f_R(x, N; \rho) dx \quad \text{and define } P_N(r, \rho) = 1 - F_R(r, N; \rho) \quad (30.1.3)$$

See also [Subrahmaniam & Subrahmaniam \(1983\)](#)

#### 30.1.2 Density and CDF

---

Function **PearsonRhoDist**(*x As mpNum, N As mpNum, rho As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function PearsonRhoDist returns pdf, CDF and related information for the distribution of the sample correlation coefficient

**Parameters:**

*x*: A real number

*N*: A real number greater 2, representing the sample size

*rho*: A real number greater 0, representing the correlation coefficient

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 30.1.2.1 and 30.1.2.3.

### 30.1.2.1 Density: Hotelling's algorithm

We define

$$x = \rho r, \quad A = \sqrt{1 - \rho^2}, \quad B = \sqrt{1 - x^2}, \quad C = \sqrt{1 - r^2}, \quad U = \frac{\arccos(-x)}{B} \quad (30.1.4)$$

The probability density function for  $R$  is then given by (Hotelling, 1953)

$$f_R(r, N; \rho) = K_1 A^{N-1} C^{N-4} (1-x)^{\frac{3}{2}-N} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; N-\frac{1}{2}; \frac{1}{2} + \frac{1}{2}x\right), \quad (30.1.5)$$

$$\text{where } K_1 = \frac{(N-2)\Gamma(N-1)}{\sqrt{2\pi}\Gamma(N-\frac{1}{2})}, \quad (30.1.6)$$

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; N-\frac{1}{2}; \frac{1}{2} + \frac{1}{2}x\right) = \sum_{i=0}^{\infty} M_i, \quad (30.1.7)$$

$$M_0 = 1, \quad M_i = M_{i-1} \frac{a_i^2}{c_i} \frac{1+x}{2i} \quad (30.1.8)$$

$a_1 = \frac{1}{2}$ ,  $c_1 = N - \frac{1}{2}$ ,  $a_i = a_{i-1} + 1$ ,  $c_i = c_{i-1} + 1$  and  ${}_2F_1(\cdot)$  is the Gaussian hypergeometric function (see section 13.4.1).

### 30.1.2.2 Density :Closed form expressions and recursions

For  $N = 3, 4$  the probability density function can be expressed in closed form (Odeh, 1986):

$$f_R(r, 3; \rho) = \frac{A^2(1+xB)}{\pi B^2 C} \quad (30.1.9)$$

$$f_R(r, 4; \rho) = \frac{AC^3(B^2U + 3x(1+xB))}{\pi B^4} \quad (30.1.10)$$

where  $x, A, B, C$  and  $U$  are defined in (30.1.4). The probability density function  $f_N(r, \rho)$  satisfies the following recurrence formula for  $N \geq 5$  (Hotelling, 1953):

$$f_R(r, N; \rho) = \frac{2N-5}{B^2(N-3)} x A C f_R(r, N-1; \rho) + \frac{N-3}{B^2(N-4)} A^2 C^2 f_R(r, N-2; \rho) \quad (30.1.11)$$

For  $0 \leq x \leq 1$ , eqn (30.1.11) can be used to find a sequence of values for  $f_5(r; \rho), f_6(r; \rho), \dots, f_N(r; \rho)$ . However, for  $-1 < x < 0$  the recurrence formula is numerically unstable, since the two terms on the right-hand of eqn are of opposite sign. In this case  $(1+x) < 1$ , so the series given by eqn (30.1.5) will converge extremely fast and can be used.

### 30.1.2.3 CDF: Hotelling's series

Hotelling (1953) defines  $Q_N(r; \rho) = \Pr[\rho < R < r]$  and develops  $Q_N(r; \rho)$  in the following uniformly convergent series for  $-1 < \rho < r < 1$ .

$$Q_N(r, \rho) = K_1 \sum_{j=0}^{\infty} \frac{(1 \cdot 3 \cdots (2j-1))^2 S_j}{j! 2^{2j} \cdot (2N+1) \cdots (2N+2j-1)}, \quad (30.1.12)$$

where  $K_1$  and  $S_j$  are defined in equations (30.1.6) and (30.1.17), respectively. From the relationships

$$P_N(r, \rho) = 1 - F_R(r, N; \rho) = Q_N(1, \rho) - Q_N(r, \rho) \quad (30.1.13)$$

$$P_N(-1, \rho) = 1 \quad (30.1.14)$$

$$F_R(r, N; \rho) = 1 - P_N(r, \rho) \quad (30.1.15)$$

$$F_R(-r, N; -\rho) = 1 - F_R(r, N; \rho) \quad (30.1.16)$$

we can compute  $F_N(r; \rho)$  for any value of  $r$  and  $\rho$  with  $-1 \leq r \leq 1$ ,  $-1 < \rho < 1$ . Hotelling shows that the error committed by truncating the series at any point is less than  $\frac{2}{1-|\rho|}$  times the last term used. However, it is important to note that the series converges very slowly for small values of  $N$ , and in this case a large number of terms must be computed.

$$S_j = \sum_{k=0}^j \binom{j}{k} (-1)^k \frac{1}{2}(1-\rho^2)^k 2^{j-k} N_k \quad (30.1.17)$$

$$N_k = \sum_{s=0}^{\infty} \frac{\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{3}{2}-k-s) s!} \cdot I\left(\frac{1}{2}(s+1), \frac{1}{2}(n-1), \frac{(r-\rho)^2}{(1-\rho r)^2}\right), \quad (30.1.18)$$

where  $I(a, b; x)$  denotes the Incomplete Beta Function (see section 9.7.6). Hotelling shows that in the evaluation of  $N_k$  for large  $s$  the absolute value of the ratio of the term of order  $(s+1)$  to the term of order  $s$  is bounded by  $|\rho(r-\rho)/(1-\rho r)|$ , so that the series converges rapidly.

### 30.1.2.4 CDF: Guenther's series

Guenther (1977b) writes  $P_N(r; \rho) = \Pr[R > 0] - \Pr[0 < R < r]$  and develops  $\Pr[0 < R < r]$  in an infinite series involving the Incomplete Beta Function denoted by  $I(a, b; x)$ . The result is

$$\Pr[R > 0] = \frac{1}{2} \left[ 1 + \operatorname{sgn}(\rho) \cdot I\left(\frac{1}{2}(N-1), \frac{1}{2}; \rho^2\right) \right] \quad (30.1.19)$$

$$\begin{aligned} \Pr[0 < R < r] &= \sum_{j=0}^{\infty} K_1(j) \cdot I\left(\frac{1}{2}(N-2), \frac{1}{2}(2j+1); r^2\right) \\ &\quad + \sum_{j=0}^{\infty} K_2(j) \cdot I\left(\frac{1}{2}(N-2), j+1; r^2\right) \end{aligned} \quad (30.1.20)$$

where  $K_1(j), K_2(j)$  are defined recursively by

$$\begin{aligned} K_1(0) &= \frac{1}{2} (1-\rho^2)^{\frac{1}{2}(N-1)}, \quad K_1(j) = \frac{2j+N-3}{2j} \rho^2 K_1(j-1) \\ K_2(0) &= \frac{\Gamma(\frac{1}{2}N)}{\sqrt{\pi} \Gamma(\frac{1}{2}(N-1))} \rho (1-\rho^2)^{\frac{1}{2}(N-1)}, \quad K_2(j) = \frac{2j+N-2}{2j+1} \rho^2 K_2(j-1) \end{aligned}$$

Eqns (30.1.19) and (30.1.20) are used together to give  $P_N(r; \rho)$ . Guenther also obtains error bounds for truncating the infinite series given by (30.1.20).

### 30.1.2.5 CDF: Closed form expressions and recursions

For  $N = 3, 4, 5, 6$  the cdf can be expressed in closed form (Odeh, 1986):

$$F_3(r; \rho) = \frac{\arccos(-r)}{\pi} - \frac{\rho CU}{\pi} \quad (30.1.21)$$

$$F_4(r; \rho) = \frac{\arccos(\rho)}{\pi} - \frac{\rho AC^2}{\pi B^2} - \frac{r A^3 U}{\pi B^2} \quad (30.1.22)$$

$$F_5(r; \rho) = \frac{\arccos(-r)}{\pi} - \frac{(x^2 - 3\rho^3 + 2)r A^2 C}{2\pi B^4} + \frac{(\rho^2 - 3 + 2\rho^2 x^2)\rho C^3 U}{2\pi B^4} \quad (30.1.23)$$

$$\begin{aligned} F_6(r; \rho) = & \frac{\arccos(\rho)}{\pi} - \frac{[\rho r^2(2x^2 + 13) - 2\rho(4x^4 + 6x^2 + 5) + \rho^3(11x^2 + 4)]AC^2}{6\pi B^6} \\ & + \frac{2x^2(-2r^2 + 1)r A^5 U}{6\pi B^6} \end{aligned} \quad (30.1.24)$$

where  $x, A, B, C$  and  $U$  are defined in (30.1.4). The cumulative distribution function  $F_N(r, \rho)$  satisfies the following recurrence formula for  $N \geq 7$  (Hotelling, 1953):

$$\begin{aligned} F_N(r; \rho) = & \frac{2(N-4)\rho^2 - N + 5}{(N-3)\rho^2} F_{N-2}(r; \rho) \\ & + \frac{(N-5)A^2}{(N-3)\rho^2} F_{N-4}(r; \rho) \\ & + \frac{(N-4)A^2C^2 - (2N-9)B^2}{(N-4)(N-3)\rho^2 AC} \rho f_{N-1}(r; \rho) \\ & + \frac{(N-4)^2 + (3N(N-8) + 47)\rho^2}{(N-4)^2(N-3)\rho^2} r f_{N-2}(r; \rho) \end{aligned} \quad (30.1.25)$$

For  $N$  odd, the above formula can be used repeatedly to find  $F_7, F_9, \dots, F_N$  starting with values of  $F_3$  and  $F_5$ , for which exact expressions are given by eqns (30.1.21) and (30.1.23) respectively. For  $N$  even, the formula can be used repeatedly to find  $F_8, F_{10}, \dots, F_N$  starting with values of  $F_4$  and  $F_6$ , for which exact expressions are given by equations (30.1.22) and (30.1.24) respectively. However, the formula can only be applied if  $\rho^2 \geq \frac{N-5}{2(N-4)}$ , since if  $\rho^2$  is less than this bound, the first two terms on the right hand side of eqn (30.1.25) are of opposite sign and the formula is numerically unstable.

### 30.1.2.6 CDF: Fisher's Approximation

The widely used Fisher  $z$ -transformation leads to the approximation

$$F_N(r, \rho) \approx \Phi\left(\sqrt{N-3}(Z(r) - Z(\rho))\right), \quad \text{where} \quad (30.1.26)$$

$Z(\cdot)$  is defined in equation (30.1.46), and  $\Phi(\cdot)$  denotes the CDF of the normal distribution (see section 6.11.2.2). This approximation can be improved by using the first 4 cumulants of  $Z(R)$ , resulting in

$$F_N(r, \rho) \approx \Phi(x), \quad \text{where} \quad (30.1.27)$$

$$u = \frac{Z(r) - \kappa_1}{\sqrt{\kappa_2}}, \quad x = u + (u^2 - 1)\frac{\gamma_1}{6} + (u^3 - 3u)\frac{\gamma_2}{24} + (4u^3 - 7u)\frac{\gamma_1^2}{36} \quad (30.1.28)$$

and  $\kappa_1, \kappa_2, \gamma_1$  and  $\gamma_2$  are defined in equations (30.1.48) to (30.1.52).

### 30.1.2.7 CDF: Winterbottom's Approximation

The CDF approximation given by ([Winterbottom, 1980](#)) is:

$$F_N(r, \rho) \approx \Phi(\sqrt{m}Y), \quad \text{where } m = N - 1, \quad (30.1.29)$$

$$\begin{aligned} Y = & -\frac{r}{2m} - \frac{3r + r^3}{12m^2} + \left(1 - \frac{1+r^2}{4m} + \frac{3-11r^4}{96m^2}\right)w \\ & + \frac{3r - 4r^3}{24m}w^2 - \left(\frac{1}{12} - \frac{2+7r^2-6r^4}{48m}\right)w^3 + \frac{3}{160}w^5, \end{aligned} \quad (30.1.30)$$

$w = Z(r) - Z(\rho)$  and  $Z(\cdot)$  is defined in equation (30.1.46).

### 30.1.3 Quantiles

Function **PearsonRhoDistInv**(*Prob* As mpNum, *N* As mpNum, *rho* As mpNum, *Output* As String) As mpNumList

NOT YET IMPLEMENTED

The function PearsonRhoDistInv returns quantiles and related information for the distribution of the sample correlation coefficient

#### Parameters:

*Prob*: A real number between 0 and 1.

*N*: A real number greater 2, representing the sample size

*rho*: A real number greater 0, representing the correlation coefficient

*Output*: A string describing the output choices

For a given value of  $\alpha$ ,  $N$  and  $\rho$  a  $100\alpha$  percentage point of  $R$  is the unique value of  $R$ , say  $r_\alpha$ , which satisfies  $F_N(r_\alpha; \rho) = \alpha$

Let  $u_\alpha = \Phi^{-1}(\alpha)$  be the lower  $100\alpha$  percentage point of the standard normal distribution (see section 6.11.3). Then an approximation to the  $100\alpha$  percentage point  $r_\alpha$  is obtained by

$$r_\alpha \approx Z^{-1} \left( Z(\rho) + \frac{u_\alpha}{\sqrt{N-3}} \right). \quad (30.1.31)$$

$Z(\cdot)$  and  $Z^{-1}(\cdot)$  are defined in equations (30.1.46) and (30.1.47). This approximation can be improved by using the first 4 cumulants of  $Z(R)$ , resulting in

$$r_\alpha \approx Z^{-1}(\kappa_1 + x\sqrt{\kappa_2}) \quad \text{where} \quad (30.1.32)$$

$$x = u_\alpha + (u_\alpha^2 - 1)\frac{\gamma_1}{6} + (u_\alpha^3 - 3u_\alpha)\frac{\gamma_2}{24} + (2u_\alpha^3 - 5u_\alpha)\frac{\gamma_1^2}{36} \quad (30.1.33)$$

and  $\kappa_1$ ,  $\kappa_2$ ,  $\gamma_1$  and  $\gamma_2$  are defined in equations (30.1.48) to (30.1.52).

#### 30.1.3.1 Winterbottom's Quantile Approximation

An approximation to the  $100\alpha$  percentage point  $r_\alpha$  is obtained by ([Winterbottom, 1980](#))

$$r_\alpha \approx Z^{-1}(y), \quad \text{where} \quad (30.1.34)$$

$$y = Z(\rho) + \frac{u_\alpha}{\sqrt{m}} + \frac{\rho}{2m} + \frac{u_\alpha^3 + 3(3 - \rho^2)u_\alpha}{12\sqrt{m^3}} + \frac{4\rho^3 u_\alpha^2 + 15\rho - \rho^3}{24m^2} \\ + \frac{u_\alpha^5 + (-60\rho^4 + 30\rho^2 + 80)u_\alpha^3 + (45\rho^4 - 21\rho^2 + 375)u_\alpha}{480\sqrt{m^5}} \quad (30.1.35)$$

$Z(\cdot)$  and  $Z^{-1}(\cdot)$  are defined in equations (30.1.46) and (30.1.47).

### 30.1.4 Properties

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Function **PearsonRhoDistInfo**(*N As mpNum*, *rho As mpNum*, *Output As String*) As mpNumList  
**NOT YET IMPLEMENTED**

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The function **PearsonRhoDistInfo** returns moments and related information for the distribution of the sample correlation coefficient

**Parameters:**

*N*: A real number greater 2, representing the sample size

*rho*: A real number greater 0, representing the correlation coefficient

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 30.1.4.1 The moments of $R$

The moments of  $R$  are given by [Anderson \(2003\)](#), p. 166:

$$\mathbb{E}(R^{2h+1}) = \frac{(1 - \rho^2)^{n/2}}{\sqrt{\pi}\Gamma(n/2)} \sum_{i=0}^{\infty} \frac{(2\rho)^{2i+1}}{(2i+1)!} \frac{\Gamma^2[(n+1)/2 + i]\Gamma(h+i+3/2)}{\Gamma(n/2+h+i+1)} \quad (30.1.36)$$

$$\mathbb{E}(R^{2h}) = \frac{(1 - \rho^2)^{n/2}}{\sqrt{\pi}\Gamma(n/2)} \sum_{i=0}^{\infty} \frac{(2\rho)^{2i}}{(2i)!} \frac{\Gamma^2[(n+1)/2 + i]\Gamma(h+i+1/2)}{\Gamma(n/2+h+i)} \quad (30.1.37)$$

The first 4 moments of the distribution of  $R$  can also be expressed in terms of hypergeometric functions (see [Johnson et al. \(1995.\)](#) p.553):

$$\mu'_1 = c_n b_1 \rho \quad (30.1.38)$$

$$\mu'_2 = 1 - \frac{b_2(n-2)(1-\rho^2)}{n-1} \quad (30.1.39)$$

$$\mu'_3 = c_n(b_3 - b_1) \left( b_1 \rho - \frac{(n-1)(n-2)}{\rho} \right) \quad (30.1.40)$$

$$\mu'_4 = 1 + \frac{b_2(n-2)(n-4)(1-\rho^2)}{2(n-1)} - \frac{(b_2-1)n(n-2)(n-\rho^2)}{4\rho^2} \quad (30.1.41)$$

with

$$c_n = \frac{2}{n-1} \left[ \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} \right]^2 \quad (30.1.42)$$

$$b_1 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}(n+1); \rho^2\right) \quad (30.1.43)$$

$$b_2 = {}_2F_1\left(1, 1; \frac{1}{2}(n+1); \rho^2\right) \quad (30.1.44)$$

$$b_3 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}(n-1); \rho^2\right) \quad (30.1.45)$$

where  $\Gamma(\cdot)$  is the Gamma function (see section 4.8.6), and  ${}_2F_1(\cdot)$  is the Gaussian hypergeometric function (see section 13.4.1).

### 30.1.4.2 Fisher's z-transform

The Fisher  $z$ -transform is defined by

$$Z(a) = \frac{1}{2} \log\left(\frac{1+a}{1-a}\right) = \operatorname{atanh}(a) \quad (30.1.46)$$

The inverse Fisher  $z$ -transform is defined by

$$Z^{-1}(a) = \frac{e^{2a} - 1}{e^{2a} + 1} = \tanh(a) \quad (30.1.47)$$

### 30.1.4.3 The cumulants of $Z(R)$

Let  $m = N - 1$ . Then the first 4 cumulants of  $Z(R)$  (with  $Z(\cdot)$  as defined in equation 30.1.46) are given by (Hotelling, 1953)

$$\kappa_1 = \frac{1}{2} \log\left(\frac{1+\rho}{1-\rho}\right) + \frac{\rho}{2m} + \frac{5+\rho^2}{4m^2} + \frac{11+2\rho^2+3\rho^4}{8m^3} + O(m^{-4}) \quad (30.1.48)$$

$$\kappa_2 = \frac{1}{m} + \frac{4-\rho^2}{2m^2} + \frac{22-6\rho^2-3\rho^4}{6m^3} + O(m^{-4}) \quad (30.1.49)$$

$$\kappa_3 = \frac{\rho^3}{m^3} + O(m^{-4}) \quad (30.1.50)$$

$$\kappa_4 = \frac{2}{m^3} + O(m^{-4}) \quad (30.1.51)$$

$$\gamma_1 = \frac{\kappa_3}{\sqrt{\kappa_2 \kappa_2}}, \quad \gamma_2 = \frac{\kappa_4}{\kappa_2^2} \quad (30.1.52)$$

## 30.1.5 Random Numbers

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Function **PearsonRhoDistRan**(**Size** As mpNum, **N** As mpNum, **rho** As mpNum, **Generator** As String, **Output** As String) As mpNumList

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**NOT YET IMPLEMENTED**

---

The function PearsonRhoDistRan returns random numbers following the distribution of the sample correlation coefficient

#### Parameters:

**Size**: A positive integer up to  $10^7$

**N**: A real number greater 2, representing the sample size

**rho**: A real number greater 0, representing the correlation coefficient

**Generator**: A string describing the random generator

**Output**: A string describing the output choices

See section 6.1.3.6 for the options for **Size**, **Generator** and **Output**. Algorithms and formulas are given below:

The correlation coefficient  $r$  in samples of size  $N > 2$  from a non-singular bivariate normal population with correlation coefficient  $\rho$  can be represented in the form

$$\tilde{r} = \frac{z + \tilde{\rho}\chi_{N-1}}{\chi_{N-2}} \quad (30.1.53)$$

where  $\tilde{r} = r/\sqrt{1 - r^2}$ ,  $\tilde{\rho} = \rho/\sqrt{1 - \rho^2}$ ,  $z$  is a standardized normal variate and  $z$ ,  $\chi_{N-1}$ , and  $\chi_{N-2}$  are independent (Ruben, 1966). This can be used to develop an approximation where

$$\frac{\sqrt{(2N-5)/2}\tilde{r} - \sqrt{(2N-3)/2}\tilde{\rho}}{\sqrt{1 + \frac{1}{2}(\tilde{r}^2 + \tilde{\rho}^2)}} \quad (30.1.54)$$

is distributed as a standard normal variate. See also Akahira & Torigoe (1998).

### 30.1.6 Confidence limits for rho

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Function **PearsonRhoDistNoncentrality**(*alpha* As mpNum, *rho* As mpNum, *N* As mpNum, *Output* As String) As mpNumList

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**NOT YET IMPLEMENTED**

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The function PearsonRhoDistNoncentrality returns confidence limits for the noncentrality parameter *rhi* and related information for the distribution of the sample correlation coefficient.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*rho*: A real number between -1 and 1, representing the noncentrality parameter.

*N*: A real number greater 2, representing the sample size

*Output*: A string describing the output choices

See section 6.1.3.5 for the options for *alpha*, *Noncentrality* and *Output*. Algorithms and formulas are given below.

For a given value of  $\alpha_1 < \frac{1}{2}$ ,  $N$ , and an observed value of  $R$ , say  $r_0$ , a lower  $100(1-\alpha_1)\%$  confidence limit on  $\rho$  is the unique value of  $\rho$ , say  $\rho_L$ , which satisfies  $F_N(r_0; \rho) = 1 - \alpha_1$ . A one-sided lower  $100(1 - \alpha_1)\%$  confidence interval on  $\rho$  is given by  $\rho_L \leq \rho \leq 1$ .

For  $\alpha_1 < \frac{1}{2}$ , an upper  $100(1 - \alpha_2)\%$  confidence limit on  $\rho$  is the unique value of  $\rho$ , say  $\rho_U$ , which satisfies  $F_N(r_0; \rho) = 1 - \alpha_2$ . A one-sided upper  $100(1 - \alpha_2)\%$  confidence interval on  $\rho$  is given by  $-1 \leq \rho \leq \rho_U$ .

For  $\alpha_1 = 1 - \alpha_2 = \alpha_0$ , an equal-tailed two-sided  $100(1 - 2\alpha_0)\%$  confidence interval on  $\rho$  is given by  $\rho_L \leq \rho \leq \rho_U$ .

#### 30.1.6.1 Fisher's Approximation

To obtain a confidence interval for  $\rho$ , we first compute a confidence interval for  $z(r)$ . The inverse Fisher transformation bring the interval back to the correlation scale.

$$\rho_L \approx Z^{-1} \left( Z(r) - \frac{u_\alpha}{\sqrt{N-3}} \right). \quad (30.1.55)$$

For example, suppose we observe  $r = 0.3$  with a sample size of  $n=50$ , and we wish to obtain a 95% confidence interval for  $\rho$ . The transformed value is  $\text{arctanh}(0.3) = 0.30952$ , so the confidence interval on the transformed scale is  $0.30952 - 1.96/\sqrt{47}$ , or  $(0.023624, 0.595415)$ . Converting back to the correlation scale yields  $(0.024, 0.534)$ .

### 30.1.6.2 Winterbottom's Approximation

Let  $u_\alpha = \Phi^{-1}(\alpha)$  be the lower  $100\alpha$  percentage point of the standard normal distribution (see section 6.11.3) and let  $m = N - 1$ . Then an asymptotic approximation for a lower  $100(1 - \alpha)$  confidence limit on  $\rho$  is given by (Winterbottom, 1980)

$$\rho_L \approx Z^{-1}(y), \quad \text{where} \quad (30.1.56)$$

$$y = Z(r) + \frac{u_\alpha}{\sqrt{m}} - \frac{r}{2m} + \frac{u_\alpha^3 + 3(1+r^2)u_\alpha}{12\sqrt{m^3}} - \frac{4r^3u_\alpha^2 + 5r^3 + 9r}{24m^2} \\ + \frac{u_\alpha^5 + (60r^4 - 30r^2 + 20)u_\alpha^3 + (165r^4 + 30r^2 + 15)u_\alpha}{480\sqrt{m^5}} \quad (30.1.57)$$

$Z(\cdot)$  and  $Z^{-1}(\cdot)$  are defined in equations (30.1.46) and (30.1.47).

### 30.1.7 Sample Size Function

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Function **PearsonRhoDistSampleSize**(*alpha* As mpNum, *beta* As mpNum, *ModifiedNoncentrality* As mpNum, *Output* As String) As mpNumList

---

NOT YET IMPLEMENTED

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The function PearsonRhoDistSampleSize returns sample size estimates and related information for the distribution of the sample correlation coefficient.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*beta*: A real number between 0 and 1, specifies the Type II error (or 1 - Power).

*ModifiedNoncentrality*: A real number greater 0, representing the modified noncentrality parameter.

*Output*: A string describing the output choices

See section 6.1.3.4 for the options for *alpha*, *beta*, *ModifiedNoncentrality* and *Output*. Algorithms and formulas are given below.

We denote by  $N_{Rho}(\alpha, \beta, \tilde{\rho})$  the sample size function of the noncentral  $t$ -distribution for a given confidence level  $\alpha$ , power  $\beta$  and modified noncentrality parameter  $\tilde{\rho}$ . This function determines the minimal sample size  $N$  for given noncentrality parameter, Type I error  $1 - \alpha$ , and Type II error  $1 - \beta$ , where  $\delta = \sqrt{N}\tilde{\rho}$

## 30.2 Distribution of the Sample Multiple Correlation Coefficient

### 30.2.1 Definition

Let  $(X_1, \dots, X_N)$  be a sample of independent vector observations from a  $p$ -variate normal population with mean  $\mu$  and covariance matrix  $\Sigma$ . Define

$$\bar{X} = \frac{1}{N} \sum_{i=1}^N X_i \quad \text{and} \quad \mathbf{A} = \sum_{i=1}^N (X_i - \bar{X})(X_i - \bar{X})'. \quad (30.2.1)$$

Partition  $\mathbf{A}$  as

$$\mathbf{A} = \begin{pmatrix} a_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{A}_{22} \end{pmatrix} \quad (30.2.2)$$

so that  $a_{11}$  is  $1 \times 1$ ,  $\mathbf{a}_{12}$  is  $1 \times (p-1)$ , and  $\mathbf{A}_{22}$  is a  $(p-1) \times (p-1)$  matrix. In terms of these submatrices, the squared sample multiple correlation coefficient is given by

$$R^2 = \frac{\mathbf{a}_{12}\mathbf{A}_{22}^{-1}\mathbf{a}_{21}}{a_{11}} \quad (30.2.3)$$

The sample multiple correlation coefficient is the positive square root of  $R^2$ . The squared population multiple correlation coefficient,  $\rho^2$ , is defined similarly in terms of the submatrices of  $\Sigma$ .

For given  $N-p \geq 1$ , the distribution of  $R^2$  is independent of  $\mu$  and  $\Sigma$ , but depends upon  $\rho^2$ , where  $0 \leq \rho^2 < 1$ . For  $0 \leq x \leq 1$ , define  $f_{R^2}(x; p, N, \rho^2)$  to be the probability density function for  $R^2$ . We denote the cumulative distribution function of  $R^2$  by

$$F_{R^2}(x; p, N, \rho^2) = \Pr[R^2 \leq x] = \int_0^x f_{R^2}(t; p, N, \rho^2) dt. \quad (30.2.4)$$

Note: The univariate version of the noncentral distribution of Wilks  $\Lambda$ : Canonical Correlation (see section 30.13.1.2) is equivalent to  $W = 1 - R^2$ .

### 30.2.2 Density and CDF

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Function **Rho2Dist**(*x As mpNum*, *p As mpNum*, *N As mpNum*, *Rho2 As mpNum*, *Output As String*) As mpNumList

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**NOT YET IMPLEMENTED**

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The function Rho2Dist returns pdf, CDF and related information for the distribution of the squared population multiple correlation coefficient

#### Parameters:

*x*: A real number

*p*: An integer greater 2, representing the number of variates

*N*: A real number greater *p*, representing the sample size

*Rho2*: A real number greater or equal 0, representing the squared sample multiple correlation coefficient

*Output*: A string describing the output choices

### 30.2.2.1 Density: Infinite series

The density function of the multiple sample correlation coefficient is given by (Benton & Krishnamoorthy, 2003; Ding, 1996)

$$f_{R^2}(x; p, N, \rho^2) = \sum_{i=0}^{\infty} f_{\text{NegBin}}\left((N-1)/2, i; 1-\rho^2\right) \times f_{\text{Beta}}\left(x; \frac{1}{2}(p-1) + i, \frac{1}{2}(N-p)\right) \quad (30.2.5)$$

where  $f_{\text{NegBin}}(\cdot)$  denotes the pmf of the negative binomial distribution (see section 6.10.1.1) and  $f_{\text{Beta}}(\cdot)$  denotes the pdf of the Beta distribution (see section 6.2.2.1)

### 30.2.2.2 Density: Representation in terms of hypergeometric functions

The density function of the multiple sample correlation coefficient is given by (Gurland, 1968; Lee, 1971c)

$$f_{R^2}(x; p, n, \rho^2) = \frac{(1-\rho^2)^{n/2} (x)^{(n_1-2)/2} (1-x)^{(n_2-2)/2} {}_2F_1\left(\frac{1}{2}n, \frac{1}{2}n, \frac{1}{2}n_1; \rho^2 x\right)}{B\left(\frac{1}{2}n_1, \frac{1}{2}n_2\right)} \quad (30.2.6)$$

$$f_{R^2}(x; p, N, \rho^2) = f_{\text{Beta}}\left(x; \frac{1}{2}(p-1), \frac{1}{2}(N-p)\right) (1-\rho^2)^{n/2} \times {}_2F_1\left(\frac{1}{2}N, \frac{1}{2}N, \frac{1}{2}p; \rho^2 x\right) \quad (30.2.7)$$

where  $f_{\text{Beta}}(\cdot)$  denotes the pdf of the Beta distribution (see section 6.2.2.1) and  ${}_2F_1(\cdot)$  is the Gaussian hypergeometric function (see section 13.4.1).

### 30.2.2.3 Density: Closed form expressions for odd $p$

For the particular cases of  $p = 3$  and  $p = 5$ , we have for the pdf (Lee, 1971c):

$$f_{R^2}(n, 3; \rho^2) = \frac{(n-3)\sqrt{1-\rho^2} [f_R(n-1, R; \rho) - f_R(n-1, -R; \rho)]}{2(n-2)\rho\sqrt{1-R^2}} \quad (30.2.8)$$

$$\begin{aligned} f_{R^2}(n, 5; \rho^2) &= \frac{(n-5)(1-\rho^2)R [f_R(n-2, R; \rho) + f_R(n-2, -R; \rho)]}{2(n-2)\rho^2(1-R^2)} \\ &\quad - \frac{2(1-\rho^2)(1-R^2)f_{R^2}(n-2, 3; \rho^2)}{2(n-2)\rho^2(1-R^2)} \end{aligned} \quad (30.2.9)$$

where  $f_R(\cdot)$  denotes the pdf of the distribution of the sample correlation coefficient (see section 30.1.2.1). Starting with these formulas, the pdf can be calculated for all odd  $p$ , using the recurrence relation given in equation (30.2.17).

### 30.2.2.4 CDF: Infinite series

The CDF of the multiple sample correlation coefficient is given by (Benton & Krishnamoorthy, 2003; Ding, 1996)

$$F_{R^2}(x; p, N, \rho^2) = \sum_{i=0}^{\infty} f_{\text{NegBin}}\left((N-1)/2, i; 1-\rho^2\right) \times F_{\text{Beta}}\left(x; \frac{1}{2}(p-1) + i, \frac{1}{2}(N-p)\right) \quad (30.2.10)$$

where  $f_{\text{NegBin}}(\cdot)$  denotes the pmf of the negative binomial distribution and  $F_{\text{Beta}}(\cdot)$  denotes the CDF of the Beta distribution (see section 6.2.2.2) where  $f_{\text{NegBin}}(\cdot)$  denotes the pmf of the negative binomial distribution (see section 6.10.1.1) and  $F_{\text{Beta}}(\cdot)$  denotes the CDF of the Beta distribution (see section 6.2.2.2)

### 30.2.2.5 CDF: Series of Gurland

Let  $R^2$  be the sample multiple correlation coefficient between  $X$  (consisting of  $p - 1$  variates) and  $Y$ , based on  $N$  observations. If both  $X$  and  $Y$  are random variates, then distribution of  $F$  is given by (Gurland & Milton, 1970; Lee, 1972b),

$$\Pr[F \leq x] = F_{R^2}(x; a, b, \rho^2) = \frac{b^m}{a^{n/2}} \sum_{j=0}^{\infty} c_j I(m+j, k, y) \quad \text{where} \quad (30.2.11)$$

$$\begin{aligned} z &= F(p-1), \quad y = \frac{z}{z + (N-p)}, \quad a = \frac{1}{1 - \rho^2}, \quad n = (N-p)/2, \quad m = (p-1)/2 \\ c_0 &= 1, \quad c_j = \frac{c_{j-1}(-n/2 - j + 1)\rho^2}{j} \end{aligned}$$

### 30.2.2.6 CDF: Closed form expressions for odd $p$

For the particular cases of  $p = 3$  and  $p = 5$ , we have for the CDF (Lee, 1971c):

$$F_{R^2}(n, 3; \rho^2) = F_{R^2}(n, 1; \rho^2) - \frac{\sqrt{(1 - \rho^2)(1 - R^2)} [f_R(n-1, R; \rho) - f_R(n-1, -R; \rho)]}{(n-2)\rho} \quad (30.2.12)$$

$$\begin{aligned} F_{R^2}(n, 5; \rho^2) &= F_{R^2}(n, 3; \rho^2) - \frac{(1 - \rho^2)R [f_R(n-2, R; \rho) - f_R(n-2, -R; \rho)]}{(n-2)\rho} \\ &\quad - \frac{(1 - \rho^2) [F_{R^2}(n-2, 3; \rho^2) - F_{R^2}(n-2, 1; \rho^2)]}{(n-2)\rho} \end{aligned} \quad (30.2.13)$$

where  $F_R(\cdot)$  denotes the CDF of the sample correlation coefficient (see section 30.1.2.3). Starting with these formulas, the CDF can be calculated for all odd  $p$ , using the recurrence relation given in equation (30.2.18).

### 30.2.2.7 CDF: Approximation by non-central F

Lee (1971c) suggests the following approximation, based on the noncentral  $F$  distribution:

$$F_{R^2}(x; n_1, n_2, \rho^2) \approx F_{F'}(y; \nu, n_2, \lambda), \quad \text{where} \quad (30.2.14)$$

$$\begin{aligned} \gamma &= 1/(1 - \rho^2), \\ A_j &= (n_1 + n_2)(\gamma^j - 1) + n_1, \quad j = 1, 2, 3, \\ G &= (A_2 - \sqrt{A_2^2 - A_1 A_3})/A_1, \\ \lambda &= \rho^2 \gamma \sqrt{\gamma(n_1 + n_2)n_2}/G^2, \\ \nu &= (A_2/G^2) - 2\lambda, \\ y &= x/(1-x) \times n_2/(\nu \cdot G), \end{aligned}$$

and  $F_{F'}(\cdot; \nu, n_2, \lambda)$  denotes the CDF of the noncentral  $F$  distribution with  $\nu$  and  $n_2$  degrees of freedom and noncentrality parameter  $\lambda$ . For performance reasons, the noncentral  $F$  distribution is evaluated using the (fast and highly accurate) saddlepoint approximation given in section 30.10.2.7, rather than any of the "exact" algorithms.

### 30.2.2.8 CDF: 2-moment approximation by central F

$$F_{F'}(x; n_1, n_2, \rho^2) \approx F_F(x/c; m_1, n_2), \quad (30.2.15)$$

where  $c = A_1/n_1$ ,  $m_1 = A_1^2/A_2$ ,  $A_1 = (n_1 + n_2)(\gamma - 1) + n_1$ ,  $A_2 = (n_1 + n_2)(\gamma^2 - 1) + n_1$ ,  $\gamma = 1/(1 - \rho^2)$ , and  $F_F(\cdot; m_1, n_2)$  denotes the CDF of a central  $F$  distribution with  $m_1$  and  $n_2$  degrees of freedom (see 6.6.2.2).

### 30.2.3 Quantiles

---

Function **Rho2DistInv**(*Prob* As mpNum, *p* As mpNum, *N* As mpNum, *Rho2* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function Rho2DistInv returns quantiles and related information for the distribution of the squared population multiple correlation coefficient

**Parameters:**

*Prob*: A real number between 0 and 1.

*p*: An integer greater 2, representing the number of variates

*N*: A real number greater *p*, representing the sample size

*Rho2*: A real number greater or equal 0, representing the squared sample multiple correlation coefficient

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

An approximation to  $Rho_{\alpha, n_1, n_2, \lambda}^2$ , the  $\alpha$ -quantile of the distribution of the squared population multiple correlation coefficient with *p* variates, sample size *N* the distribution of the squared population multiple correlation coefficient and noncentrality parameter  $\rho^2$ , is obtained as

$$F_{\alpha, n_1, n_2, \lambda} \approx c \cdot F_{\alpha, m_1, n_2}, \quad (30.2.16)$$

where  $c = A_1/n_1$ ,  $m_1 = A_1^2/A_2$ ,  $A_1 = (n_1 + n_2)(\gamma - 1)$ ,  $A_2 = (n_1 + n_2)(\gamma^2 - 1)$ ,  $\gamma = 1/(1 - \rho^2)$ , and  $F_{\alpha, m_1, n_2}$  denotes the  $\alpha$ -quantile of a central  $F$ -distribution with  $m_1$  and  $n_2$  degrees of freedom (see section 6.6.3).

### 30.2.4 Properties

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Function **Rho2DistInfo**(*p* As mpNum, *N* As mpNum, *Rho2* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function Rho2DistInfo returns moments and related information for the distribution of the squared population multiple correlation coefficient

**Parameters:**

*p*: An integer greater 2, representing the number of variates

*N*: A real number greater *p*, representing the sample size

*Rho2*: A real number greater or equal 0, representing the squared sample multiple correlation coefficient

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

### 30.2.4.1 Recurrence relations

Lee (1971c) gives the following recurrence relations for the pdf and CDF, valid for all integer values of  $p \geq 6$ :

$$(n-2)\rho^2(1-R^2)f_{R^2}(n,p;\rho^2) = (n-p)R^2(1-\rho^2)f_{R^2}(n-2,p-4;\rho^2) - (p-4)(1-\rho^2)(1-R^2)f_{R^2}(n-2,p-2;\rho^2) \quad (30.2.17)$$

$$\begin{aligned} (n-2)\rho^2(1-R^2)F_{R^2}(n,p;\rho^2) &= (n-p)\rho^2F_{R^2}(n,p-2;\rho^2) \\ &\quad - (p-4)(1-\rho^2)[F_{R^2}(n-2,p-2;\rho^2) - F_{R^2}(n-2,p-4;\rho^2)] \\ &\quad - 2R^2(1-\rho^2)f_{R^2}(n-2,p-4;\rho^2) \end{aligned} \quad (30.2.18)$$

### 30.2.4.2 The moments of $R^2$

Muirhead (1982), p. 178, gives the following formula (see also Johnson *et al.* (1995.), p. 621)

$$\mathbb{E}[(1-R^2)^h] = \frac{\left[\frac{1}{2}(n-m+1)\right]_h}{\left(\frac{1}{2}n\right)_h} (1-\bar{R}^2)^h \times {}_2F_1(h, h, \frac{1}{2}n+h; \bar{R}^2). \quad (30.2.19)$$

where  $(a)_k$  is the Pochammer symbol (see section 9.3) and  ${}_2F_1(\cdot)$  is the Gaussian hypergeometric function (see section 13.4.1).

The lower order moments are given by

$$\mu'_1 = 1 - \frac{(n-m+1)}{n} (1-\bar{R}^2)^h \times {}_2F_1(1, 1, \frac{1}{2}n+1; \bar{R}^2). \quad (30.2.20)$$

$$\mathbb{E}[(1-R^2)^2] = \frac{\left[\frac{1}{2}(n-m+1)\right]_2}{\left(\frac{1}{2}n\right)_2} (1-\bar{R}^2)^2 \times {}_2F_1(2, 2, \frac{1}{2}n+2; \bar{R}^2). \quad (30.2.21)$$

### 30.2.5 Random Numbers

---

Function **Rho2DistRan**(*Size As mpNum*, *p As mpNum*, *N As mpNum*, *Rho2 As mpNum*, *Generator As String*, *Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function Rho2DistRan returns random numbers following the distribution of the squared population multiple correlation coefficient

#### Parameters:

*Size*: A positive integer up to  $10^7$

*p*: An integer greater 2, representing the number of variates

*N*: A real number greater p, representing the sample size

*Rho2*: A real number greater or equal 0, representing the squared sample multiple correlation coefficient

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

A notable result concerning the distribution of  $R^2$  is that  $\tilde{R}^2 = R^2/(1-R^2)$  has the representation

$$\tilde{R}^2 = \frac{(\tilde{\rho}\chi_n + z)^2 + \chi_{p-1}^2}{\chi_{n-p}^2} \quad (30.2.22)$$

where  $\tilde{\rho}^2 = \rho^2/(1-\rho^2)$ ,  $n$  is the sample size less one,  $z$  is a standard normal variate,  $\chi_f$  and  $\chi_f^2$  are chi and chi-square variates on  $f$  degrees of freedom; the variates figuring in this relation are independently distributed and  $\tilde{\rho}$  is taken to be positive [Lee \(1971c\)](#).

### 30.2.6 Confidence Limits for the Noncentrality Parameter

---

Function **Rho2DistNoncentrality**(*alpha* As mpNum, *Rho2* As mpNum, *p* As mpNum, *N* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **Rho2DistNoncentrality** returns confidence limits for the noncentrality parameter  $\rho^2$  and related information for the distribution of the squared population multiple correlation coefficient.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*Rho2*: A real number greater or equal 0, representing the squared sample multiple correlation coefficient

*p*: An integer greater 2, representing the number of variates

*N*: A real number greater *p*, representing the sample size

*Output*: A string describing the output choices

See section [6.1.3.5](#) for the options for *alpha*, *Noncentrality* and *Output*. Algorithms and formulas are given below.

Let  $T$  be a statistic according to the non-central  $t$ -distribution with  $n$  degrees of freedom and a non-centrality parameter  $\delta$ . Then the lower confidence limit  $\hat{\delta}$  of level  $1 - \alpha$  and the two-sided confidence interval  $[\underline{\delta}, \bar{\delta}]$  of the non-centrality parameter  $\delta$  of level  $1 - \alpha$  are given by [Akahira et al. \(1995\)](#):

$$\hat{\delta} = bT - z_\alpha \sqrt{k} + hT^3(z_\alpha^2 - 1)/k, \quad (30.2.23)$$

$$\underline{\delta} = bT - z_{\alpha/2} \sqrt{k} + hT^3(z_{\alpha/2}^2 - 1)/k, \quad (30.2.24)$$

$$\bar{\delta} = bT + z_{\alpha/2} \sqrt{k} - hT^3(z_{\alpha/2}^2 - 1)/k, \quad (30.2.25)$$

where  $k = 1 + (1 - b^2)T^2$ ,  $h$  and  $b$  are defined in equation [\(30.8.19\)](#), and  $z_\alpha$  denotes the  $\alpha$ -quantile of the normal distribution (see section [6.11.3](#)).

### 30.2.7 Sample Size

---

Function **Rho2DistSampleSize**(*alpha* As mpNum, *beta* As mpNum, *p* As mpNum, **Modified-Noncentrality** As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **Rho2DistSampleSize** returns sample size estimates and related information for the distribution of the squared population multiple correlation coefficient

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*beta*: A real number between 0 and 1, specifies the Type II error (or 1 - Power).

*p*: An integer greater 2, representing the number of variates

*ModifiedNoncentrality*: A real number greater 0, representing the modified noncentrality parameter.

*Output*: A string describing the output choices

See section 6.1.3.4 for the options for *alpha*, *beta*, *ModifiedNoncentrality* and *Output*. Algorithms and formulas are given below.

## 30.3 Skew Normal Distribution

### 30.3.1 Definition

In probability theory and statistics, the skew normal distribution is a continuous probability distribution that generalises the normal distribution to allow for non-zero skewness.

See also [http://en.wikipedia.org/wiki/Skew\\_normal\\_distribution](http://en.wikipedia.org/wiki/Skew_normal_distribution).

### 30.3.2 Density and CDF

---

Function **SkewNormalDistBoost**(*x* As mpNum, *a* As mpNum, *b* As mpNum, *c* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **SkewNormalDistBoost** returns pdf, CDF and related information for the skew normal distribution

**Parameters:**

*x*: A real number.

*a*: The location parameter.

*b*: The scale parameter

*c*: The shape parameter

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 30.3.2.1 and 30.3.2.2.

#### 30.3.2.1 Density

The probability density function with location parameter *a*, scale parameter *b*, and shape parameter *c* is

$$f(x; a, b, c) = \frac{2}{b} \phi\left(\frac{x-a}{b}\right) \Phi\left(c\left(\frac{x-a}{b}\right)\right) \quad (30.3.1)$$

#### 30.3.2.2 CDF

$$F(a, b, c) = \Phi\left(\frac{x-a}{b}\right) - 2T\left(\frac{x-a}{b}, c\right) \quad (30.3.2)$$

where  $T_{\text{Owen}}(\cdot, \cdot)$  denotes Owen's *T* function (see section 30.3.6).

### 30.3.3 Quantiles

---

Function **SkewNormalDistInvBoost**(*Prob* As mpNum, *a* As mpNum, *b* As mpNum, *c* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **SkewNormalDistInvBoost** returns quantiles and related information for the the skew normal distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*a*: The location parameter.

*b*: The scale parameter

*c*: The shape parameter

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). The quantile is determined using an iterative algorithm.

### 30.3.4 Properties

Function **SkewNormalDistInfoBoost**(*a* As mpNum, *b* As mpNum, *c* As mpNum, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

The function SkewNormalDistInfoBoost returns moments and related information for the skew normal distribution

**Parameters:**

*a*: The location parameter.

*b*: The scale parameter

*c*: The shape parameter

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 30.3.4.1 Moments

$$\mu_1 = a + bd\sqrt{\frac{2}{\pi}}, \quad \text{where } d = \frac{c}{\sqrt{1+c^2}} \quad (30.3.3)$$

$$\mu_2 = b^2 \left(1 - \frac{2d^2}{\pi}\right) \quad (30.3.4)$$

$$\gamma_1 = \frac{4-\pi}{2} \frac{\left(d\sqrt{2/\pi}\right)^3}{(1-2d^2/\pi)^{3/2}} \quad (30.3.5)$$

$$\gamma_2 = 2(\pi-3) \frac{\left(d\sqrt{2/\pi}\right)^4}{(1-2d^2/\pi)^2} \quad (30.3.6)$$

### 30.3.5 Random Numbers

Function **SkewNormalDistRanBoost**(**Size** As mpNum, *a* As mpNum, *b* As mpNum, *c* As mpNum, **Generator** As String, **Output** As String) As mpNumList

**NOT YET IMPLEMENTED**

The function SkewNormalDistRanBoost returns random numbers following a skew normal distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*a*: The location parameter.

*b*: The scale parameter

*c*: The shape parameter

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*.

### 30.3.6 Owen's T-Function

---

Function **TOwenBoost(*h* As mpNum, *a* As mpNum) As mpNum**

---

The function **TOwenBoost** returns Owen's T-Function

**Parameters:**

*h*: A real number.

*a*: A real number.

Owen's T-Function is defined as ([Owen, 1956](#)):

$$T(h, a) = \frac{1}{2\pi} \int_0^a \frac{\exp\left[-\frac{1}{2}h^2(1+x^2)\right]}{1+x^2} dx \quad (30.3.7)$$

The implementation uses the algorithm described in [Patefield & Tand \(2000\)](#).

## 30.4 Multivariate Normal Distribution

### 30.4.1 Definition

An  $n$ -dimensional random variable  $\mathbf{X}$  with mean vector  $\boldsymbol{\mu}$  and covariance matrix  $\boldsymbol{\Sigma}$  is said to have a nonsingular multivariate normal distribution, in symbols  $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , if  $\boldsymbol{\Sigma}$  is positive definite, and the density function of  $\mathbf{X}$  is of the form given in equation 30.4.1.

See also [Steck & Owen \(1962\)](#) and [Owen & Steck \(1962\)](#)

See [Lai \(2006\)](#)

### 30.4.2 Density of the multivariate normal distribution

The density of the multivariate normal distribution is given by ([Tong, 1990](#)):

$$f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{n/2} |\boldsymbol{\Sigma}|^{1/2}} e^{-Q_n(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})/2}, \quad \mathbf{x} \in \Re^n \quad (30.4.1)$$

where

$$Q_n(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}). \quad (30.4.2)$$

### 30.4.3 Bivariate Normal

The bivariate normal distribution has the following density:

$$g(x, y; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\frac{-(x^2-2\rho xy+y^2)}{2(1-\rho^2)}} \quad (30.4.3)$$

The method developed by [Owen \(1956\)](#) was for a long time the most widely used approach to calculate bivariate normal (BVN) probabilities. Owen showed that

$$\Phi_2(-\infty, b1, b2; \rho) = \frac{\Phi(b1) - \Phi(b2)}{2} - T_{\text{Owen}}(b1, \hat{b}_1) - T_{\text{Owen}}(b2, \hat{b}_2) - c, \quad (30.4.4)$$

where  $\rho$  is the correlation coefficient,  $T_{\text{Owen}}(\cdot, \cdot)$  denotes Owen's  $T$  function (see section 30.3.6), and

$$c = \begin{cases} 0, & \text{if } b_1 b_2 > 0 \text{ or } b_1 b_2 = 0, b_1 + b_2 \geq 0 \\ \frac{1}{2} & \text{otherwise,} \end{cases} \quad (30.4.5)$$

$$\hat{b}_1 = \frac{b_2 - b_1 \rho}{b_1 \sqrt{1 - \rho^2}}, \quad \hat{b}_2 = \frac{b_2 - b_2 \rho}{b_2 \sqrt{1 - \rho^2}}. \quad (30.4.6)$$

[Abramowitz & Stegun. \(1970\)](#) gives the following representation:

$$B(h, k; \rho) = \int_{-\infty}^k \int_{-\infty}^h g(x, y; \rho) dx dy \quad (30.4.7)$$

$$L(h, k; \rho) = \int_k^\infty \int_h^\infty g(x, y; \rho) dx dy \quad (30.4.8)$$

$$L(-h, -k; \rho) = B(h, k; \rho) \quad (30.4.9)$$

Integral representation [Tong \(1990\)](#):

$$B(h, k; \rho) = \int_{-\infty}^{\infty} \prod_{j=1}^2 \left( \frac{d_j \sqrt{|\rho|} z + a_j}{\sqrt{1-\rho}} \right) \phi(z) dz \quad (30.4.10)$$

where  $d_j = -1$  if  $\rho < 0$  and  $j = 2$  and  $d_j = 1$  otherwise.

### 30.4.4 Special structure: zero correlation

This case covers the normal maximum

$$F_{MM}(x, k, \mu) = \prod_{i=1}^k [\Phi(x - \mu_i)] \quad (30.4.11)$$

$$F_M(x, k) = [\Phi(x)]^k \quad (30.4.12)$$

This case covers the normal maximum modulus

$$F_{MM}(x, k, \mu) = \prod_{i=1}^k [\Phi(x - \mu_i) - \Phi(-x - \mu_i)] \quad (30.4.13)$$

$$F_{MM}(x, k, 0) = [\Phi(x) - \Phi(-x)]^k = [2\Phi(x) - 1]^k, \quad x > 0 \quad (30.4.14)$$

See [Stoline & Ury \(1979\)](#)

See [Narula \(1978\)](#)

### 30.4.5 Special structure: Equal-correlated case

With  $\rho_{ij} = \rho$ , we have for a one-sided test [Tong \(1990\)](#):

$$F_n(h; \rho) = \int_{-\infty}^{\infty} \left[ \Phi \left( \frac{h + \sqrt{|\rho|}y}{\sqrt{1-\rho}} \right) \right]^n \phi(y) dy = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(h; \rho) dx_1 \cdots dx_n. \quad (30.4.15)$$

and for a two-sided test:

$$F_n(h; \rho) = \int_{-\infty}^{\infty} \left[ \Phi \left( \frac{h + \sqrt{|\rho|}y}{\sqrt{1-\rho}} \right) - \Phi \left( \frac{-h + \sqrt{|\rho|}y}{\sqrt{1-\rho}} \right) \right]^n \phi(y) dy = \int_{-h}^{\infty} \cdots \int_{-h}^{\infty} g(h; \rho) dx_1 \cdots dx_n. \quad (30.4.16)$$

Note: The distribution of the twosided version can get bi-modal for large arguments, if  $P(x)$  is calculated. This requires modification of the search algorithm.

### 30.4.6 Special structure: Dunnett test

See [Dunnett \(1955\)](#)

See also [Cheng & Meng \(1995\)](#)

For  $\lambda_i = \sqrt{\rho} \geq 0$  for all  $i$ , this reduces to the equicorrelated case.

With  $\rho_{ij} = \lambda_i \lambda_j$ , we have for a one-sided test:

$$F_n(h; \rho_{ij}) = \int_{-\infty}^{\infty} \prod_{i=1}^n \left[ \Phi \left( \frac{(a_i - \mu_i)/\sigma_i + \lambda_i z}{\sqrt{1 - \lambda_i^2}} \right) \right] \phi(y) dz \quad (30.4.17)$$

and for a two-sided test:

$$F_n(h; \rho_{ij}) = \int_{-\infty}^{\infty} \prod_{i=1}^n \left[ \Phi \left( \frac{(a_i - \mu_i)/\sigma_i + \lambda_i z}{\sqrt{1 - \lambda_i^2}} \right) - \Phi \left( \frac{(b_i - \mu_i)/\sigma_i + \lambda_i z}{\sqrt{1 - \lambda_i^2}} \right) \right] \phi(y) dz \quad (30.4.18)$$

Note: The distribution of the twosided version can get bi-modal for large arguments, if  $P(x)$  is calculated. This requires modification of the search algorithm.

### 30.4.7 Special structure: normal range

Let  $X_1, \dots, X_k$  be a random sample of size  $k$  from a  $N(\mu, \sigma^2)$  distribution. Let  $s^2$  be an independent mean square estimate of  $\sigma^2$  with  $n$  degrees of freedom.

Then  $Q_n = \max|X_i - X_j|, 1 < i < j < k$ , has a Normal Range distribution with  $k$  degrees of freedom, and  $Q_t = Q_n/s$  has a Studentized Range distribution with  $k$  and  $n$  degrees of freedom [Hochberg & Tamhane \(1987\)](#).

See also [Stoline \(1978\)](#) and [Harter \(1960\)](#) and [David et al. \(1972\)](#)

#### 30.4.7.1 Density

The density in the central case is given by

$$p(x) = k(k-1) \int_{-\infty}^{\infty} (\Phi(y) - \Phi(y-x))^{k-2} \phi(y) \phi(y-x) dy \quad (30.4.19)$$

#### 30.4.7.2 Normal Range

$$Q(x) = \sum_{i=1}^k \int_{-\infty}^{\infty} \phi(y_i - \mu_i) \left( \prod_{j=1, j \neq i}^k (\Phi(y_j - \mu_j) - \Phi(y_j - \mu_j - x)) \right) dy_i \quad (30.4.20)$$

for  $\mu = 0$  this simplifies to

$$Q(x) = k \int_{-\infty}^{\infty} \phi(y) (\Phi(y) - \Phi(y-x))^{k-1} dy \quad (30.4.21)$$

$$P(x) = 1 - Q(x) = k \int_{-\infty}^{\infty} \phi(y) (L_1^k - [L_1 - L_2]^{k-1}) dy, \quad \text{where} \quad (30.4.22)$$

$$L_1 = \Phi(y), \quad L_2 = \Phi(y-x), \quad L_2 \rightarrow 0 \text{ for } x \rightarrow \infty \quad (30.4.23)$$

$$L_1^k - (L_1 - L_2)^k = L_1^k \left( 1 - \left( 1 - \frac{L_2}{L_1} \right)^k \right) \quad (30.4.24)$$

### 30.4.8 Normal Orthant Probabilities

Let  $X = (X_1, X_2, \dots, X_n)$  be a random vector distributed as  $N(0, R_n)$ , where  $R_n$  is positive definite. The normal orthant probability is defined as

$$P_n = \int_0^{\infty} N(0, R_n) d^n x. \quad (30.4.25)$$

Based on the method of [Sun \(1988a,b\)](#), this probability can be expressed as

$$P_{2k} = \frac{1}{2^{2k}} + \frac{1}{2^{2k-1}\pi} \sum_{i < j=1}^{2k} \arcsin(r_{ij}) + \sum_{j=2}^k \frac{1}{2^{2k-j}\pi^j} \sum_{i_1 < i_2 < \dots < i_{2j}} I_{2j}(R^{(i_1 i_2 \dots i_{2j})}), \quad (30.4.26)$$

$$P_{2k+1} = \frac{1}{2^{2k+1}} + \frac{1}{2^{2k}\pi} \sum_{i < j=1}^{2k+1} \arcsin(r_{ij}) + \sum_{j=2}^k \frac{1}{2^{2k+1-j}\pi^j} \sum_{i_1 < i_2 < \dots < i_{2j}} I_{2j}(R^{(i_1 i_2 \dots i_{2j})}), \quad (30.4.27)$$

where  $R^{(i_1 i_2 \dots i_{2j})}$  denotes the submatrix consisting of the  $(i_1 i_2 \dots i_{2j})^{th}$  rows and columns of  $R_n$ ,

$$I_n(\Lambda_n) = (-2\pi)^{-k} \int_{-\infty}^{\infty} \prod_{i=1}^n \frac{1}{\omega_i} \exp(-\frac{1}{2}\omega^t \Lambda_n \omega) d^n \omega, \quad n = 2k, \quad (30.4.28)$$

and  $\Lambda_n = (\lambda_{ij})$  is a covariance matrix of  $n$  variates.

See [Genz & Bretz \(2002, 2009\)](#) for alternative methods

### 30.4.9 Normal Rank Order Probabilities

Consider the situation where random variables  $X_1, \dots, X_m$  and  $X_1, \dots, Y_n$  are normally distributed with means  $\mu_X$  and  $\mu_Y$ , respectively, and common variance  $\sigma^2$ , all  $m+n$  random variables being mutually independent and  $d = (\mu_Y - \mu_X)/\sigma$ .

Let  $\mathbf{U} = (U_1, \dots, U_{m+n})$ ,  $U_1 < \dots < U_{m+n}$ , denote the order statistics of the random variables  $(X_1, \dots, X_m)$ ,  $(Y_1, \dots, Y_n)$ , and let  $\mathbf{Z} = (Z_1, \dots, Z_{m+n})$  denote a random vector of zeros and ones, where the  $i$ th component  $Z_i$  is 0 (or 1) if  $U_i$  is an  $X$  (or  $Y$ ). Denote by  $\phi(x - \theta)$  the normal density with mean  $\theta$  and variance 1 (see section [6.11.2.1](#)). If  $\mathbf{z} = (z_1, \dots, z_{m+n})$  is a fixed vector of zeros and ones, the probability of the rank order  $z$ ,  $\Pr[\mathbf{Z} = \mathbf{z}]$ , is given by

$$P_{m,n}(\mathbf{z}|d) = m!n! \int_R \cdots \int \prod_{i=1}^{m+n} \phi(t_i - z_id) dt_i, \quad (30.4.29)$$

where the region of integration  $R$  is  $-\infty < t_1 \leq t_2 \leq \dots \leq t_{m+n} < \infty$ . [Milton \(1970\)](#) describes a  $p$ -dimensional midpoint algorithm that economizes the number of arithmetic operations required to evaluate equation (30.4.29), corrects for effects of the edges of the region of integration and involves no high-degree quadrature formulas.

### 30.4.10 Random Numbers

#### 30.4.10.1 Generation of Exchangeable Normal Variates

Consider the situation in which we are interested in generating  $N$  (pseudo) independent  $n$ -dimensional variates  $\mathbf{X}_1, \dots, \mathbf{X}_N$ , such that  $\mathbf{X} = (X_{1t}, \dots, X_{nt})'$  ( $t = 1, \dots, N$ ) has an  $\mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  distribution with a common mean  $\boldsymbol{\mu}$ , a common variance  $\sigma^2$ , and a common correlation coefficient  $\rho \in [0, 1]$ . Then a corresponding algorithm for generating  $\mathbf{X}_1, \dots, \mathbf{X}_N$  is:

Input  $\boldsymbol{\mu}$ ,  $\sigma^2$ ,  $\rho$ ,  $n$ , and  $N$ .

For  $t = 1, \dots, N$  compute

$$X_{it} = \mu + \sigma \left( \sqrt{1 - \rho} Z_{it} + \sqrt{\rho} Z_{0t} \right) \quad (30.4.30)$$

and form  $\mathbf{X} = (X_{1t}, \dots, X_{nt})'$ .

#### 30.4.10.2 Generation of Multivariate Normal Variates with a Special Correlation Structure

In certain statistical applications the covariance matrix  $\boldsymbol{\Sigma} = (\sigma_{ij})$  may be of the form

$$\sigma_{ij} = \begin{cases} \sigma_i & \text{for } i = j \\ \sigma_i \sigma_j \lambda_i \lambda_j & \text{for } i \neq j, \end{cases} \quad (30.4.31)$$

where  $\lambda_i \in [-1, 1](i = 1, \dots, n)$ . In this case, the correlation coefficients are  $\rho_{ij} = \lambda_i \lambda_j$  for all  $i \neq j$ . To generate  $\mathbf{X}_1, \dots, \mathbf{X}_N$  according to an  $\mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  distribution when  $\boldsymbol{\Sigma}$  has such a structure and  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)'$ , we replace equation 30.4.30 by

$$X_{it} = \mu_i + \sigma_i \left( \sqrt{1 - \lambda_i^2} Z_{it} + \lambda_i Z_{0t} \right) \quad (30.4.32)$$

and otherwise follow the algorithm in section 30.4.10.1. Note that here the  $\mu_i$  are not necessarily the same and the correlation coefficients are not necessarily all nonnegative. If  $\mu_i = \mu$ ,  $\sigma_i = \sigma$ , and  $\lambda_i = \sqrt{\rho} \geq 0(i = 1, \dots, n)$ , then this algorithm reduces to the one in section 30.4.10.1 as a special case.

### 30.4.10.3 Generation of Multivariate Normal Variates with an Arbitrary Nonsingular Multivariate Normal Distribution

We are interested in generating  $\mathbf{X}_1, \dots, \mathbf{X}_N$  from an  $\mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  distribution with arbitrary but fixed  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  (which is positive definite).

Let  $\mathbf{T} = (\tau_{ij})$  be the unique lower triangular  $n \times n$  matrix (obtained by Cholesky decomposition) such that  $\mathbf{T}\mathbf{T}' = \boldsymbol{\Sigma}$ . If  $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$ , then  $\mathbf{X} = \mathbf{T}\mathbf{Z} + \boldsymbol{\mu}$  has an  $\mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  distribution.

Consequently, to generate independent random variates  $\mathbf{X}_1, \dots, \mathbf{X}_N$  according to this distribution, the following algorithm may be used:

Compute  $\mathbf{T} = (\tau_{ij})$

Input  $N, n, \boldsymbol{\mu} = (\mu_1, \dots, \mu_n)'$  and  $\mathbf{T} = (\tau_{ij})$ .

Generate  $Z_{1t}, \dots, Z_{nt}$  (which are (pseudo) independent  $\mathcal{N}(0, 1)$  variates) and apply the transformation  $\mathbf{X}_t = \mathbf{T}\mathbf{Z}_t + \boldsymbol{\mu}$ , i.e. compute

$$X_{it} = \sum_{j=1}^i \tau_{ij} Z_j + \mu_i, \quad \text{for } i = 1, \dots, n, \quad (30.4.33)$$

and then form  $\mathbf{X} = (X_{1t}, \dots, X_{nt})'$ .

Repeat this step for  $t = 1, \dots, N$ .

## 30.5 Multivariate t-Distribution

### 30.5.1 Definition

Let  $\mathbf{R} = (\rho_{ij})$  be an  $n \times n$  symmetric matrix such that it is either positive definite or positive semidefinite and  $\rho_{ii} = 1(i = 1, \dots, n)$ . Let  $\mathbf{Z} = (Z_1, \dots, Z_n)'$  have an  $\mathcal{N}_n(\mathbf{0}, \mathbf{R})$  distribution, and let the univariate variable  $S$  be such that  $S$  is independent of  $\mathbf{Z}$ , and  $\nu S^2$  has a  $\chi^2(\nu)$  distribution. Then a natural generalization of the Student's  $t$  variable is

$$\mathbf{t} = (t_1, \dots, t_n)' = \left( \frac{Z_1}{S}, \dots, \frac{Z_n}{S} \right)'. \quad (30.5.1)$$

If  $\mathbf{R}$  is positive definite, then the density of  $\mathbf{t}$  (with correlation matrix  $\mathbf{R}$  and degrees of freedom  $\nu$ ) is given by (Tong, 1990):

$$h(\mathbf{t}; \mathbf{R}, \nu) = \frac{\Gamma((n + \nu)/2)}{(\nu\pi)^{n/2}\Gamma(\nu/2)|\mathbf{R}|^{1/2}} \left( 1 + \frac{1}{\nu} \mathbf{t}' \mathbf{R}^{-1} \mathbf{t} \right)^{-(n+\nu)/2}, \quad \mathbf{t} \in \Re^n. \quad (30.5.2)$$

### 30.5.2 Studentized Maximum and Maximum Modulus

$$\frac{\partial}{\partial c} (F(cx)g(x)) = xg(x)F'(cx) \quad (30.5.3)$$

#### 30.5.2.1 Density and CDF

Let  $X_1, \dots, X_k$  be a random sample of size  $k$  from a  $\mathcal{N}(0, \sigma^2)$  distribution. Let  $s^2$  be an independent mean square estimate of  $\sigma$  with  $n$  degrees of freedom. Then

$$Q = \frac{\max X_j}{s}, \quad j = 1, \dots, k \quad (30.5.4)$$

has a Studentized Maximum distribution with  $k$  and  $n$  degrees of freedom, and

$$Q = \frac{\max |X_j|}{s}, \quad j = 1, \dots, k \quad (30.5.5)$$

has a Studentized Maximum Modulus distribution with  $k$  and  $n$  degrees of freedom.

#### 30.5.2.2 Quantiles

Studentized Maximum Critical Values  $M_{k,\nu,\alpha}$  (One-sided Multivariate  $t$  Critical Values  $t_{k,\nu,\rho=0,\alpha}$ )  
 Studentized Maximum Modulus Critical Values  $|M|_{k,\nu,\alpha}$  (Two-sided Multivariate  $t$  Critical Values  $|t|_{k,\nu,\rho=0,\alpha}$ )

### 30.5.3 Dunnett's t

#### 30.5.3.1 Density and CDF

Let  $X_1, \dots, X_k$  be a random sample of size  $k$  from a  $\mathcal{N}(0, \sigma^2)$  distribution. Let  $s^2$  be an independent mean square estimate of  $\sigma$  with  $n$  degrees of freedom. Then

$$Q = \frac{\max X_1 - X_j}{s}, \quad 1 < j < k \quad (30.5.6)$$

has a onesided Dunnett's  $t$ -distribution with  $k$  and  $n$  degrees of freedom, and

$$Q = \frac{\max_s |X_1 - X_j|}{s}, \quad 1 < j < k \quad (30.5.7)$$

has a twosided Dunnett's  $|t|$ -distribution with  $k$  and  $n$  degrees of freedom.

For Dunnett's test:

$$\lambda_i = \frac{1}{\sqrt{1 + n_0/n_i}} \quad (30.5.8)$$

### 30.5.3.2 Quantiles

One-Sided Multivariate t Critical Values  $t_{k,\nu,\rho=\lambda_i\lambda_j,\alpha}$  Common Correlation  $\rho = 0.5$

Two-Sided Multivariate t Critical Values  $|t|_{k,\nu,\rho=\lambda_i\lambda_j,\alpha}$  Common Correlation  $\rho = 0.5$

### 30.5.4 Studentized Range

Let  $X_1, \dots, X_k$  be a random sample of size  $k$  from a  $\mathcal{N}(0, \sigma^2)$  distribution. Let  $s^2$  be an independent mean square estimate of  $\sigma$  with  $n$  degrees of freedom. Then

$$Q = \frac{\max_s |X_i - X_j|}{s}, \quad 1 < i < j < k \quad (30.5.9)$$

has a Studentized Range distribution with  $k$  and  $n$  degrees of freedom.

#### 30.5.4.1 Density and CDF

$$Q_t(c; n) = \int_0^\infty Q_n(cx)g(x, n)dx, \quad \text{where} \quad (30.5.10)$$

$$g(x; n) = \frac{n^{n/2}}{2^{(n-1)/2}\Gamma(n/2)}x^{n-1}e^{-nx^2/2} \quad \text{is the pdf of } \frac{x}{\sigma} \quad (30.5.11)$$

$$SR(c) = \int_0^\infty Q(cx)g(x)dx, \quad \text{and for } n = \infty, \quad SR(c) = Q(c). \quad (30.5.12)$$

#### 30.5.4.2 Quantiles

Critical Values  $q_{k,\nu,\alpha}$  for the Studentized Range Distribution

See [Hirotsu \(1979\)](#)

### 30.5.5 Special case Owen

Owen considers a special case of the bivariate noncentral t-distribution which is relevant in quality control [Owen \(1965\)](#)

## 30.6 Noncentral Chi-Square Distribution

### 30.6.1 Definition

Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed random variables each following a normal distribution with mean  $\mu_j$  and unit variance. Then  $\chi^2 = \sum_{j=1}^n X_j$  is said to follow a non-central  $\chi^2$ -distribution with  $n$  degrees of freedom and noncentrality parameter  $\lambda = \sum_{j=1}^n (\mu_j - \mu)$ .

### 30.6.2 Density and CDF

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Function **NoncentralCDistEx**(*x As mpNum, n As mpNum, lambda As mpNum, Output As String*) As mpNumList

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**NOT YET IMPLEMENTED**

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The function **NoncentralCDistEx** returns pdf, CDF and related information for the noncentral  $\chi^2$ -distribution

**Parameters:**

*x*: A real number

*n*: A real number greater 0, representing the degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 30.6.2 and 30.6.2.3.

#### 30.6.2.1 Density: Closed form representations

The density of a noncentral chi-square variable with  $n$  degrees of freedom and  $\lambda$  is given by (Wang & Gray, 1993)

$$f_{\chi^2}(n, x; \lambda) = e^{-\lambda/2} f_{\chi^2}(n, x) {}_0F_1\left(-; \frac{n}{2}; \frac{x\lambda}{4}\right) \quad (30.6.1)$$

$$f_{\chi^2}(n, x; \lambda) = \frac{1}{2} e^{-(\lambda+x)/2} \left(\frac{x}{\lambda}\right)^{(n-2)/2} I_{(n-2)/2}(\sqrt{\lambda x}) \quad (30.6.2)$$

where  $f_{\chi^2}(n, \cdot)$  is the pdf of the (central)  $\chi^2$  distribution (see section 6.4.2.1),  ${}_0F_1(\cdot)$  is the confluent hypergeometric limit function (see section 13.1), and  $I_\nu(\cdot)$  is the modified Bessel function of the first kind of order  $\nu$  (see section 4.8.3).

#### 30.6.2.2 Density and CDF: Finite series for odd degrees of freedom

For odd degrees of freedom, the pdf and cdf can be expressed as a finite sum, using the recurrence relations given in section 30.6.4.2, and defining  $h(n, x, \lambda) = e^{(1/2)(x+\lambda)} f_{\chi^2}(n, x; \lambda)$  (András & Baricz, 2008):

$$F_{\chi^2}(1, x; \delta^2) = \Phi(x - \delta) - \Phi(-x - \delta) \quad (30.6.3)$$

$$h(1, x; \lambda) = \frac{\cosh(\sqrt{x\lambda})}{\sqrt{2\pi x}}, \quad h(3, x; \lambda) = \frac{\sinh(\sqrt{x\lambda})}{\sqrt{2\pi\lambda}} \quad (30.6.4)$$

where  $\Phi(\cdot)$  denotes the cdf of the normal distribution (see section 6.11.2.2).

### 30.6.2.3 CDF: General formulas

The cdf of a noncentral chi-square variable with  $n$  degrees of freedom and  $\lambda$  is given by

$$\Pr [\chi^2 \leq x] = F_{\chi^2}(n, x; \lambda) = \int_0^x f_{\chi^2}(n, t; \lambda) dt \quad (30.6.5)$$

### 30.6.2.4 CDF: Infinite series in terms of the central cdf

The cdf of a noncentral chi-square variable with  $n$  degrees of freedom and  $\lambda$  is given by

$$F_{\chi^2}(n, x; \lambda) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!} F_{\chi^2}(n + 2 + j, x) \quad (30.6.6)$$

where  $F_{\chi^2}(n, \cdot)$  is the cdf of the (central)  $\chi^2$  distribution (see section 6.4.2.2).

### 30.6.2.5 CDF: Infinite series in terms of the central pdf

Ding (1992) gives the following representation (this is used by Boost for small lambda):

$$F_{\chi^2}(n, x; \lambda) = 2e^{-\lambda/2} \sum_{i=0}^{\infty} f_{\chi^2}(n + 2 + 2i, x) \left( \sum_{k=0}^i \frac{(\lambda/2)^k}{k!} \right) \quad (30.6.7)$$

where  $f_{\chi^2}(n, \cdot)$  is the pdf of the (central)  $\chi^2$  distribution (see section 6.4.2.1).

### 30.6.2.6 CDF: Integral representation

Chou (1985) gives the following representation for  $n \geq 2$  and  $\lambda \geq 0$ :

$$F_{\chi^2}(n, x; \lambda) = \frac{2^{(1-n)/2} \sqrt{2\pi}}{\Gamma((n-1)/2)} \int_0^x y^{(n-3)/2} \phi(\sqrt{y}) [\Phi(\sqrt{x-y} - \sqrt{\lambda}) - \Phi(-\sqrt{x-y} - \sqrt{\lambda})] dy \quad (30.6.8)$$

where  $\phi(\cdot)$  denotes the pdf of the normal distribution (see section 6.11.2.1) and  $\Phi(\cdot)$  denotes the cdf of the normal distribution (see section 6.11.2.2).

### 30.6.2.7 CDF: 2 moment approximation

Patnaik (1949) gives the following 2-moment approximation based on the central  $\chi^2$ -distribution:

$$F_{\chi^2}(n, x; \lambda) \approx F_{\chi^2}(n_1, x_1; ), \quad \text{where } n_1 = \frac{(n+\lambda)^2}{n+2\lambda}, \quad x_1 = \frac{x(n+\lambda)}{n+2\lambda} \quad (30.6.9)$$

where  $F_{\chi^2}(n, \cdot)$  is the cdf of the (central)  $\chi^2$  distribution (see section 6.4.2.2).

### 30.6.2.8 CDF: 3 moment approximation

Pearson (1959) gives the following 3-moment approximation based on the central  $\chi^2$ -distribution:

$$F_{\chi^2}(n, x; \lambda) \approx F_{\chi^2}(n_1, x_1; ), \quad \text{where } n_1 = \frac{(n+2\lambda)^3}{2(n+3\lambda)^2}, \quad x_1 = \frac{x + \lambda^2/(n+3\lambda)}{2(n+3\lambda)/n + 2\lambda} \quad (30.6.10)$$

where  $F_{\chi^2}(n, \cdot)$  is the cdf of the (central)  $\chi^2$  distribution (see section 6.4.2.2).

### 30.6.2.9 CDF: Saddlepoint approximation

Butler (2007) suggests the following approximation:

$$K(t) = -\frac{n}{2} \log(1 - 2t) + \frac{\lambda t}{1 - 2t}, \quad t \in (-\infty, \frac{1}{2}) \quad (30.6.11)$$

$$K^{(j)}(t) = \frac{2^{j-1}(j-1)!}{(1-2t)^j} \left[ n + \frac{\lambda j}{1-2t} \right] \quad (30.6.12)$$

$$s(x) = -\frac{1}{4x} \left[ n - 2x + \sqrt{n^2 + 4x\lambda} \right], \quad x > 0 \quad (30.6.13)$$

### 30.6.2.10 CDF: Wiener Germ approximation

Penev & Raykov (2000) give the following first and second order Wiener germ approximation:

$$F_{\chi^2}(n, x; \lambda) \approx \Phi \left( \operatorname{sgn}(s) \sqrt{n(s-1)^2(1/(2s) + m - h(1-s)/s) - \ln(A(s)) + 2B(s)/n} \right) \quad (30.6.14)$$

$$\text{where } m = \lambda/n; \quad h(y) = \frac{(1-y)\ln(1-y) + y - \frac{1}{2}y^2}{y^2}; \quad s = \frac{\sqrt{1+4xm/n} - 1}{2m} \quad (30.6.15)$$

$$A(s) = \frac{1}{s} - \frac{2}{s} \cdot \frac{h(1-s)}{1+2ms}; \quad B(s) = \frac{(1+3m)^2}{9(1+2m)^3} \quad (30.6.16)$$

where  $\Phi(\cdot)$  denotes the cdf of the normal distribution (see section 6.11.2.2).

## 30.6.3 Quantiles

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Function **NoncentralCDistInvEx**(*Prob* As mpNum, *n* As mpNum, *lambda* As mpNum, *Output* As String) As mpNumList

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**NOT YET IMPLEMENTED**

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The function NoncentralCDistInvEx returns quantiles and related information for the noncentral  $\chi^2$ -distribution

### Parameters:

*Prob*: A real number between 0 and 1.

*n*: A real number greater 0, representing the degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below

The quantile is approximated as

$$\chi_{n,\lambda,\alpha}^2 \approx (1+b)\chi_{n_1,\lambda,\alpha}^2, \quad \text{where } n_1 = \frac{(n+\lambda)^2}{n+2\lambda}, \quad b = \frac{\lambda}{n+\lambda} \quad (30.6.17)$$

### 30.6.4 Properties

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Function **NoncentralCDistInfoEx**(*n* As mpNum, *lambda* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

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The function NoncentralCDistInfoEx returns moments and related information for the noncentral  $\chi^2$ -distribution

**Parameters:**

*n*: A real number greater 0, representing the degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 30.6.4.1 Moments and Cumulants

The cumulants of noncentral  $\chi^2$  are given by

$$\kappa_r(n, \lambda) = 2^{r-1}(r-1)!(n+r\lambda) \quad (30.6.18)$$

The first 4 cumulants of cube root noncentral  $\chi^2$ ,  $\chi^{2/3}$ , are given by  $\kappa_i^*(n, \lambda) = T_i(1, n, \lambda)$ , with  $r = n + \lambda$ ,  $b = \lambda/r$  (see ([Abdel-Aty, 1954](#))):

$$\begin{aligned} T_1(c, n, \lambda) &= \left(\frac{cr}{n}\right)^{1/3} \left(1 - \frac{2(b+1)}{9r} - \frac{40b^2}{3^4 r^2} + \frac{80(1+3b+33b^2-77b^3)}{3^5 r^3}\right) \\ &+ \left(\frac{cr}{n}\right)^{1/3} \left(\frac{176(1+4b-210b^2+2380b^3-2975b^4)}{3^9 r^4}\right) + o(r^{-4}) \end{aligned} \quad (30.6.19)$$

$$\begin{aligned} T_2(c, n, \lambda) &= \left(\frac{cr}{n}\right)^{2/3} \left(\frac{2(b+1)}{9r} - \frac{16b^2}{3^4 r^2} - \frac{8(13+39b+405b^2-1025b^3)}{3^7 r^3}\right) \\ &+ \left(\frac{cr}{n}\right)^{2/3} \left(\frac{160(1+4b-87b^2+1168b^3-1544b^4)}{3^8 r^4}\right) + o(r^{-4}) \end{aligned} \quad (30.6.20)$$

$$\begin{aligned} T_3(c, n, \lambda) &= \left(\frac{-cr}{n}\right) \left(\frac{8b^2}{3^3 r^2} - \frac{32(1+3b+21b^2-62b^3)}{3^6 r^3}\right) \\ &+ \left(\frac{-cr}{n}\right) \left(\frac{32(8+32b-177b^2+4550b^3-6625b^4)}{3^8 r^4}\right) + o(r^{-4}) \end{aligned} \quad (30.6.21)$$

$$\begin{aligned} T_4(c, n, \lambda) &= -\left(\frac{cr}{n}\right)^{4/3} \left(\frac{16(1+3b+12b^2-44b^3)}{3^6 r^3}\right) \\ &- \left(\frac{cr}{n}\right)^{4/3} \left(\frac{256(1+4b-6b^2+274b^3-458b^4)}{3^8 r^4}\right) + o(r^{-4}) \end{aligned} \quad (30.6.22)$$

### 30.6.4.2 Recurrence Relations

The following recurrence relations hold for the pdf and CDF (Cohen, 1988):

$$f_{\chi^2}(n+4, x; \lambda) = \frac{x \cdot f_{\chi^2}(n, x; \lambda) - n \cdot f_{\chi^2}(n+2, x; \lambda)}{\lambda} \quad (30.6.23)$$

$$F_{\chi^2}(n, x; \lambda) - F_{\chi^2}(n+2, x; \lambda) = 2f_{\chi^2}(n+2, x; \lambda) \quad (30.6.24)$$

$$F_{\chi^2}(n, x; \lambda) - F_{\chi^2}(n-2, x; \lambda) = 2 \frac{\partial}{\partial \lambda} F_{\chi^2}(n-2, x; \lambda) \quad (30.6.25)$$

### 30.6.5 Confidence Limits for the Noncentrality Parameter

Function **NoncentralCDistNoncentralityEx**(*alpha* As mpNum, *lambda* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

**NOT YET IMPLEMENTED**

The function **NoncentralCDistNoncentralityEx** returns confidence limits for the noncentrality parameter lambda and related information for the noncentral  $\chi^2$ -distribution.

#### Parameters:

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*lambda*: A real number greater 0, representing the noncentrality parameter.

*n*: A real number greater 0, representing the degrees of freedom.

*Output*: A string describing the output choices

See section 6.1.3.5 for the options for *alpha*, *Noncentrality* and *Output*. Algorithms and formulas are given below.

(Winterbottom, 1979) gives the following formula to determine the paramter  $\lambda$  of a noncentral  $\chi^2$  distribution with  $n$  degrees of freedom, so that  $F_{\chi^2}(n, x; 0) = 1 - \alpha$  and  $F_{\chi^2}(n, x; \lambda) = 1 - \beta$ . Let  $c$  be the  $1 - \alpha$  percentage point of a  $\chi^2$ -distribution with  $n$  degrees of freedom, let  $x$  be the  $1 - \beta$  percentage point of a  $N(0, 1)$  distribution, and  $T = (c - n)/n$ ,  $Y = 2T + 1$ . Then

$$\begin{aligned} \lambda \approx nT + \sqrt{2nY}x + & \frac{2((3T+2)x^2 + 3T+1))}{3Y} - \frac{(6T+5)x^3 - (36T^2 + 42T + 17)x}{18\sqrt{nY^5/2}} \\ & + \frac{(324T^2 + 594T + 276)x^4}{405nY^4} - \frac{(1080T^3 + 2484T^2 + 976)x^2}{405nY^4} \\ & + \frac{1080T^3 + 1512T^2 + 612T + 148}{405nY^4} - \frac{(10368T^3 + 30780T^2 + 30564T + 10143)x^5}{9720\sqrt{n^3Y^{11}/2}} \\ & + \frac{(25920T^4 + 98928T^3 + 163080T^2 + 137544T + 47188)x^3}{9720\sqrt{n^3Y^{11}/2}} \\ & + \frac{(45360T^4 + 106704T^3 + 80460T^2 + 31092T + 13489)x}{9720\sqrt{n^3Y^{11}/2}} \end{aligned} \quad (30.6.26)$$

### 30.6.6 Sample Size Calculation

---

Function **NoncentralCDistSampleSizeEx**(*alpha* As mpNum, *beta* As mpNum, *n* As mpNum, *ModifiedNoncentrality* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function NoncentralCDistSampleSizeEx returns sample size estimates and related information for the noncentral  $\chi^2$ -distribution.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*beta*: A real number between 0 and 1, specifies the Type II error (or 1 - Power).

*n*: A real number greater 0, representing the denominator degrees of freedom.

*ModifiedNoncentrality*: A real number greater 0, representing the modified noncentrality parameter.

*Output*: A string describing the output choices

See section 6.1.3.4 for the options for *alpha*, *beta*, *ModifiedNoncentrality* and *Output*. Algorithms and formulas are given below.

### 30.6.7 Random Numbers

#### 30.6.7.1 Noncentral Chi-Square

---

Function **NoncentralCDistRanEx**(*Size* As mpNum, *n* As mpNum, *lambda* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function NoncentralCDistRanEx returns random numbers following a noncentral  $\chi^2$ -distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*n*: A real number greater 0, representing the degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below

Random numbers from a non-central chi-square distribution is easily obtained using the definition above by e.g.

1. Put  $\mu = \sqrt{\lambda/n}$
2. Sum  $n$  random numbers from a normal distribution with mean  $\mu$  and variance unity. Note that this is not a unique choice. The only requirement is that  $\lambda = \sum \mu_i^2$ .
3. Return the sum as a random number from a non-central chi-square distribution with  $n$  degrees of freedom and non-central parameter  $\lambda$ .

### 30.6.7.2 Central Wishart distribution

To simulate a standard central Wishart matrix, in symbols  $\mathbf{V} \sim \mathcal{W}_s(k, \mathbf{I}_s, \mathbf{0})$ , we use the method proposed by Odell and Feiveson:

Generate  $s(s - 1)/2$  i.i.d  $\mathcal{N}(0, 1)$  random variables  $N_{ij}$ ,  $1 \leq i \leq j \leq s$ .

Independently generate  $s$  independent random variables  $L_1, L_2, \dots, L_s$ , where  $L_j \sim \chi^2_{k-j+1}$ ,  $j = 1, 2, \dots, s$ . Then  $\mathbf{V} = (V_{ij}) \sim \mathcal{W}_s(k, \mathbf{I}_s, \mathbf{0})$ , with

$$V_{jj} = L_j + \sum_{i=1}^{j-1} N_{ij}^2, \quad j = 1, 2, \dots, s, \quad (30.6.27)$$

$$V_{ij} = N_{ij}\sqrt{L_i} + \sum_{\nu=1}^{j-1} N_{\nu i}N_{\nu j} = V_{ji}, \quad i < j. \quad (30.6.28)$$

### 30.6.7.3 Noncentral Wishart Distribution: Definition

The  $p \times p$  random Matrix is said to have a noncentral Wishart distribution, , in symbols  $\mathbf{A} \sim \mathcal{W}_p(n, \Sigma, \mathbf{M})$ , with noncentrality parameter  $\mathbf{M} : p \times p$  and degrees of freedom  $n$  if there exist independent  $p \times 1$  random vectors  $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$  and constant  $p \times 1$  vectors  $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_n$  such that  $\mathbf{Y}_i \sim \mathcal{N}(0, 1)$ ,  $i = 1, 2, \dots, n$ , and

$$\mathbf{A} = \sum_{i=1}^n \mathbf{Y}_i \mathbf{Y}_i', \quad \mathbf{M} = \Sigma^{\frac{1}{2}} \left[ \sum_{i=1}^p \boldsymbol{\mu}_i \boldsymbol{\mu}_i' \right] \left( \Sigma^{\frac{1}{2}} \right)' \quad (30.6.29)$$

Here,  $\Sigma^{\frac{1}{2}}$  is any square root of  $\Sigma$ .

### 30.6.7.4 Noncentral Wishart distribution: Full Rank

To simulate a non-central Wishart matrix, in symbols  $\mathbf{A} \sim \mathcal{W}_p(n, \Sigma, \mathbf{M})$ , we use the method proposed by Gleser (1976):

$$\mathbf{A} = \Sigma^{\frac{1}{2}} \left[ \sum_{i=1}^p (\mathbf{Z}_i + \mathbf{M}_i)(\mathbf{Z}_i + \mathbf{M}_i)' + \mathbf{V} \right] \left( \Sigma^{\frac{1}{2}} \right)' \quad (30.6.30)$$

where  $\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p$  and  $\mathbf{V}$  are mutually independent,  $\mathbf{M} = (\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_p)$ ,  $\mathbf{Z}_i \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p)$ ,  $i = 1, 2, \dots, p$ , and  $\mathbf{V} \sim \mathcal{W}_p(n-p, \mathbf{I}_p, \mathbf{0})$ .

Generating  $\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p$  requires  $p^2$  i.i.d.  $\mathcal{N}(0, 1)$  variables, while  $p(p-1)/2$  additional i.i.d.  $\mathcal{N}(0, 1)$  variables and  $p$  independent  $\chi^2$  variables are needed to generate  $\mathbf{V}$  by means of the Odell-Feiveson method already described.

### 30.6.7.5 Noncentral Wishart distribution: Less Than Full Rank

As an alternative, the following representation can be used:

$$\mathbf{A} = \Sigma^{\frac{1}{2}} \Gamma \begin{pmatrix} \mathbf{Q} & \mathbf{Q}^{\frac{1}{2}} \mathbf{R} \\ \mathbf{Q}^{\frac{1}{2}} \mathbf{R} & \mathbf{W} + \mathbf{R}' \mathbf{R} \end{pmatrix} \Gamma' \left( \Sigma^{\frac{1}{2}} \right)', \quad \text{where} \quad (30.6.31)$$

$$\mathbf{M} = \Gamma \begin{pmatrix} \mathbf{D}_{\delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \Gamma', \quad (30.6.32)$$

$\mathbf{Q}$ ,  $\mathbf{R}$  and  $\mathbf{W}$  are mutually independent,  $\mathbf{Q} \sim \mathcal{W}_t(n, \mathbf{I}_t, \mathbf{D}_{\delta})$ ,  $\mathbf{W} \sim \mathcal{W}_{p-t}(n-t, \mathbf{I}_{p-t}, \mathbf{0})$ , the scalar elements  $r_{ij}$  of  $\mathbf{R}$  are i.i.d.  $\mathcal{N}(0, 1)$  and  $\mathbf{Q}^{\frac{1}{2}}$  is any square root of  $\mathbf{Q}$

Assuming that  $\Sigma^{\frac{1}{2}}$  and  $\Gamma$  are already calculated, we proceed as follows:

Generate  $\mathbf{W} \sim \mathcal{W}_{p-t}(n - t, \mathbf{I}_{p-t}, \mathbf{0})$  by the Odell-Feiveson method.

Generate  $t(p - t)$  i.i.d.  $\mathcal{N}(0, 1)$  variables to serve as the elements  $r_{ij}$  of  $\mathbf{R}$ .

Generate  $\mathbf{Q} \sim \mathcal{W}_t(n, \mathbf{I}_t, \mathbf{D}_\delta)$  by the standard method (based on equation 30.6.30) outlined above.

Take the square root of  $Q$  using the Cholesky decomposition, yielding a lower triangular matrix.

Form the matrix product  $\mathbf{Q}^{\frac{1}{2}} \mathbf{R}$ , taking advantage of the fact that  $\mathbf{Q}^{\frac{1}{2}}$  is lower triangular. Also, calculate  $\mathbf{R}' \mathbf{R}$ , and then  $\mathbf{W} + \mathbf{R}' \mathbf{R}$ .

Finally, calculate the matrix in equation 30.6.31.

## 30.7 Noncentral Beta-Distribution

### 30.7.1 Definition

If  $X_1$  and  $X_2$  are independent random variables,  $X_1$  following a non-central  $\chi^2$ -distribution with noncentrality parameter  $\lambda$  and  $2a$  degrees of freedom, and  $X_2$  following a  $\chi^2$ -distribution with  $2b$  degrees of freedom, then the distribution of the ratio  $B = \frac{X_1}{X_1+X_2}$  is said to follow a non-central Beta-distribution with  $a$  and  $b$  degrees of freedom.

Note: The univariate version of the noncentral distribution of Wilks Lambda: GLM (see section 30.13.1.1) is equivalent to  $W = 1 - B$

See [Tiwari & Yang \(1997\)](#)

### 30.7.2 Density and CDF

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Function **NoncentralBetaDistBoost**(*x* As mpNum, *m* As mpNum, *n* As mpNum, *lambda* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function NoncentralBetaDistBoost returns pdf, CDF and related information for the central Beta-distribution

#### Parameters:

*x*: A real number

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

#### 30.7.2.1 Density

The density function of the noncentral beta-Distribution is given by ([Wang & Gray, 1993](#)): this needs to be checked, see Paolella 2007:

$$f_{\text{Beta}'}(x; n_1, n_2, \lambda) = e^{-\lambda/2} f_B(x; n_1, n_2) {}_1F_1\left(\frac{1}{2}(m+n), \frac{1}{2}n, \frac{nx\lambda}{2(m+nx)}\right) \quad (30.7.1)$$

where  $f_B(x; m, n)$  is the pdf of the (central) beta-distribution (see section 6.2.2.1) and  ${}_1F_1(\cdot)$  is the confluent hypergeometric function (see section 13.2.1).

#### 30.7.2.2 CDF: Infinite Series

The cdf can be calculated using the following infinite series [Benton & Krishnamoorthy \(2003\)](#):

$$\Pr[F \leq x] = F_{B'}(x; a, b, \lambda) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!} I(a+j, b, x) \quad (30.7.2)$$

### 30.7.3 Quantiles

---

Function **NoncentralBetaDistInvBoost**(*Prob* As mpNum, *m* As mpNum, *n* As mpNum, *lambda* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function `NoncentralBetaDistInvBoost` returns quantiles and related information for the noncentral Beta-distribution

#### Parameters:

*Prob*: A real number between 0 and 1.

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

### 30.7.4 Properties

Function **NoncentralBetaDistInfoBoost**(*m* As *mpNum*, *n* As *mpNum*, *lambda* As *mpNum*, *Output* As String) As *mpNumList*

NOT YET IMPLEMENTED

The function `NoncentralBetaDistInfoBoost` returns moments and related information for the non-central Beta-distribution

#### Parameters:

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 30.7.4.1 Moments of the non-central Beta-distribution

Algebraic moments of the non-central Beta-distribution are given as (see [Walck \(2007\)](#), p. 109)

$$\mathbb{E}(x^k) = e^{-\lambda/2} \sum_{r=0}^{\infty} \frac{(\lambda/2)^r}{r!} \frac{B(p+r+k, q)}{B(p+r, q)} \quad (30.7.3)$$

$$\mathbb{E}(x^k) = \frac{\Gamma(p+k)\Gamma(p+q)}{\Gamma(p)\Gamma(p+q+k)} \times e^{-\lambda/2} {}_2F_1(p+q, p+k; p, p+q+k; \lambda/2) \quad (30.7.4)$$

Lower order moments are given by

$$\mathbb{E}(Y) = \frac{a}{a+b} \times e^{-\lambda/2} {}_2F_1(a+1, a+b; a, a+b+1; \lambda/2) \quad (30.7.5)$$

$$\mathbb{E}(Y^2) = \frac{a(a+1)}{a+b(a+b+1)} \times e^{-\lambda/2} {}_2F_1(a+1, a+b; a, a+b+1; \lambda/2) \quad (30.7.6)$$

Let  $W = 1 - B = |E|/|T + E|$ . The noncentral moments of  $W$  are specified in Theorem 10.5.1 of [Muirhead \(1982\)](#) as

$$E(W^s) = \frac{\Gamma(n/2+s)\Gamma((n+m)/2)}{\Gamma(n/2)\Gamma((n+m)/2+s)} {}_1F_1\left(s; \frac{n+m}{2} + s; -\frac{1}{2}\lambda\right) \quad (30.7.7)$$

where  $\Gamma(\cdot)$  is the gamma function (see section 4.8.6) and  ${}_1F_1(\cdot)$  is the confluent hypergeometric function (see section 13.2.1). From this the moments of  $B$  can be derived in the same way as the moments of  $R^2$  are derived from the moments of  $1 - R^2$  (see section 30.2.4.2).

### 30.7.5 Random Numbers

---

Function **NoncentralBetaDistRanBoost**(*Size As mpNum, m As mpNum, n As mpNum, lambda As mpNum, Generator As String, Output As String*) As mpNumList

---

NOT YET IMPLEMENTED

---

The function NoncentralBetaDistRanBoost returns random numbers following a noncentral Beta-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

Random numbers from a non-central Beta-distribution with integer or half-integer p- and q-values is easily obtained using the definition above i.e. by using a random number from a non-central chi-square distribution and another from a (central) chi-square distribution.

## 30.8 Noncentral Student's t-Distribution

### 30.8.1 Definition

If  $X$  is a random variable following a normal distribution with mean  $\delta$  and variance unity and  $\chi^2$  is a random variable following an independent  $\chi^2$ -distribution with  $n$  degrees of freedom, then the distribution of the ratio  $\frac{X}{\sqrt{\chi^2/n}}$  is called noncentral t-distribution with  $n$  degrees of freedom and noncentrality parameter  $\delta$ .

### 30.8.2 Density and CDF

---

Function **NoncentralTDistBoost**(*x* As mpNum, *n* As mpNum, *delta* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function NoncentralTDistBoost returns pdf, CDF and related information for the noncentral *t*-distribution

**Parameters:**

*x*: A real number

*n*: A real number greater 0, representing the degrees of freedom

*delta*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given in sections 30.8.2 and 30.8.2.3.

#### 30.8.2.1 Density: Closed form representation

The density of a variable following a noncentral t-distribution with  $n$  degrees of freedom and noncentrality parameter  $\delta$  is given by (Wang & Gray, 1993)

$$f_{t'}(n, x, \delta) = K_1 \left[ \frac{\sqrt{2}\delta x \cdot {}_1F_1(m + 1; \frac{3}{2}; y^2)}{a\Gamma(m + \frac{1}{2})} - \frac{{}_1F_1(m + \frac{1}{2}; \frac{1}{2}; y^2)}{\sqrt{a}\Gamma(m + 1)} \right] \quad (30.8.1)$$

$$K_1 = \frac{n^m \Gamma(n + 1) \exp(-\frac{1}{2}\delta^2)}{2^n a^m \Gamma(m)}; \quad m = \frac{n}{2}; \quad a = n + x^2 \quad y^2 = \frac{\delta^2 x^2}{2a}, \quad (30.8.2)$$

where  $\Gamma(\cdot)$  is the Gamma function (see section 4.8.6), and  ${}_1F_1(\cdot)$  is the confluent hypergeometric function (see section 13.2.1). For  $x = 0$  this simplifies to (Witkovsky, 2013):

$$f_{t'}(n, 0, \delta) = \exp(-\frac{1}{2}\delta^2) \frac{\Gamma(m + \frac{1}{2})}{\sqrt{\pi n} \Gamma(m)} \quad (30.8.3)$$

#### 30.8.2.2 Density: Integral representation

The pdf of a variable following a (singly) noncentral t-distribution with  $n$  degrees of freedom and noncentrality parameter  $\delta$  is given by (Witkovsky, 2013)

$$f_{t'}(n, x, \delta) = \int_0^\infty \phi\left(x\sqrt{\frac{q}{n}} - \delta\right) f_{\chi^2}(n, q) \sqrt{\frac{q}{n}} dq. \quad (30.8.4)$$

where  $\phi(\cdot)$  denotes the pdf of the normal distribution (see section 6.11.2.1) and  $f_{\chi^2}(n, \cdot)$  denotes the pdf of the (central)  $\chi^2$  distribution with  $n$  degrees of freedom (see section 6.4.2.1).

### 30.8.2.3 Density: Infinite series

The pdf of a variable following a noncentral t-distribution with  $n$  degrees of freedom and noncentrality parameter  $\delta$  is given by (Bristow *et al.*, 2013)

$$f_{t'}(n, x, \delta) = \frac{nt}{n^2 + 2nt^2 + t^4} + \frac{1}{2} \sum_{i=0}^{\infty} P_i I'_x \left( i + \frac{1}{2}, \frac{n}{2} \right) + \frac{\delta}{\sqrt{2}} Q_i I'_x \left( i + 1, \frac{n}{2} \right), \quad \text{and} \quad (30.8.5)$$

$I'_x(\cdot, \cdot)$  denotes the density of the beta distribution (see section 6.2.2.1), and  $P_i$  and  $Q_i$  are defined in equation 30.8.15.

### 30.8.2.4 CDF: Integral representation

The cdf of a variable following a noncentral t-distribution with  $n$  degrees of freedom and noncentrality parameter  $\delta$  is given by (Witkovsky, 2013)

$$\Pr[X \leq x] = F_{t'}(n, x, \delta) = \int_0^\infty \Phi \left( x \sqrt{\frac{q}{n}} - \delta \right) f_{\chi^2}(n, q) dq. \quad (30.8.6)$$

where  $\Phi(\cdot)$  denotes the cdf of the normal distribution (see section 6.11.2.2) and  $f_{\chi^2}(n, \cdot)$  denotes the pdf of the  $\chi^2$  distribution with  $n$  degrees of freedom (see section 6.4.2.1).

### 30.8.2.5 CDF: Finite series for integer degrees of freedom

Owen (1968) gives the following algorithm:

$$F_{t'}(n, x, \delta) = \Phi(-\delta \sqrt{B}) + 2(M_1 + M_3 + \dots + M_n) \quad \text{for odd degrees of freedom} \quad (30.8.7)$$

$$F_{t'}(n, x, \delta) = \Phi(-\delta) + \sqrt{2\pi}(M_2 + M_4 + \dots + M_n) \quad \text{for even degrees of freedom} \quad (30.8.8)$$

$$A = \frac{t}{\sqrt{n}}, \quad B = \frac{n}{n + t^2} \quad (30.8.9)$$

$$M_1 = T_{\text{Owen}}(\delta \sqrt{B}, A), \quad M_2 = A \sqrt{B} \phi(\delta \sqrt{B}) - \Phi(\delta A \sqrt{B}) \quad (30.8.10)$$

$$M_3 = B(\delta A M_2 + A \phi(\delta)/\sqrt{2\pi}), \quad M_4 = \frac{1}{2} B(\delta A M_3 + M_2) \quad (30.8.11)$$

$$M_k = \frac{k-3}{k-2} B(a_k \delta A M_{k-1} + M_{k-2}), \quad a_k = \frac{1}{(k-4)a_{k-1}} \quad \text{for } k \geq 5, \quad (30.8.12)$$

where  $T_{\text{Owen}}(\cdot, \cdot)$  denotes Owen's T function (see section 30.3.6), and  $\Phi(\cdot)$  and  $\phi(\cdot)$  denote the cdf (see section 6.11.2.2) and pdf (see section 6.11.2.1) of the normal distribution, respectively.

### 30.8.2.6 CDF: Infinite series

The cdf of a variable following a noncentral t-distribution with  $n$  degrees of freedom and noncentrality parameter  $\delta$  is given by (Benton & Krishnamoorthy, 2003; Bristow *et al.*, 2013)

$$F_{t'}(n, x, \delta) = \Phi(-\delta) + \frac{1}{2} \sum_{i=0}^{\infty} P_i I_x\left(i + \frac{1}{2}, \frac{n}{2}\right) + \frac{\delta}{\sqrt{2}} Q_i I_x\left(i + 1, \frac{n}{2}\right), \quad \text{and} \quad (30.8.13)$$

$$1 - F_{t'}(n, x, \delta) = \frac{1}{2} \sum_{i=0}^{\infty} P_i I_y\left(\frac{n}{2}, i + \frac{1}{2}\right) + \frac{\delta}{\sqrt{2}} Q_i I_y\left(\frac{n}{2}, i + 1\right), \quad \text{where} \quad (30.8.14)$$

$$\lambda = \frac{1}{2}\delta^2; \quad P_i = \frac{e^{-\lambda}\lambda^i}{i!}; \quad Q_i = \frac{e^{-\lambda}\lambda^i}{\Gamma(i+3/2)}; \quad x = \frac{t^2}{n+t^2}; \quad y = 1-x, \quad (30.8.15)$$

$I_x(\cdot, \cdot)$  denotes the (normalized) incomplete beta function (see section 6.2.2.2), and  $\Phi(\cdot)$  denotes the cdf of the normal distribution (see section 6.11.2.2).

### 30.8.2.7 CDF: Saddlepoint approximation

Broda & Paolella (2007) give the following saddlepoint approximation:

$$F_{t'}(y_1; n, \mu) \approx \Phi(w) + \phi(w) \left( \frac{1}{w} - \frac{d}{u} \right) \quad (30.8.16)$$

$$f_{t'}(y_1; n, \mu) \approx \phi(w)/u, \quad \text{where} \quad (30.8.17)$$

$$y_2 = [\mu y_1 + \sqrt{4n(y_1^2 + n) + \mu^2 y_1^2}] / (2y_1^2 + 2n),$$

$$t_1 = -\mu + y_1 y_2,$$

$$t_2 = -y_1 t_1 / (2ny_2),$$

$$d = 1/(t_1 y_2),$$

$$u = \sqrt{(\mu y_1 y_2 + 2n)/(2n)}/y_2,$$

$$w = \text{sgn}(y_1 - \mu) \sqrt{-\mu t_1 - 2n \ln(y_2)},$$

and  $\Phi(\cdot)$  and  $\phi(\cdot)$  denote the CDF (see section 6.11.2.2) and pdf (see section 6.11.2.1) of the normal distribution, respectively.

### 30.8.2.8 CDF: Edgeworth expansion

Akahira *et al.* (1995) gives the following approximation:

$$F_{t'}(n, x, \delta) \approx \Phi\left(\frac{\sqrt{k-4ac} - \sqrt{k}}{2a}\right), \quad \text{where} \quad (30.8.18)$$

$$h = \frac{1}{24} \left( \frac{1}{n^2} + \frac{1}{4n^3} \right) \quad b = \sqrt{\frac{2}{n} \frac{\Gamma(\frac{1}{2}n + \frac{1}{2})}{\Gamma(\frac{1}{2}n)}} \quad (30.8.19)$$

$c = bt - a + \delta$ ,  $a = t^3 h/k$ ,  $k = 1 + (1 - b^2)t^2$ , and  $\Phi(\cdot)$  denotes the CDF of the normal distribution (see section 6.11.2.2).

### 30.8.3 Quantiles

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Function **NoncentralTDistInvBoost**(*Prob* As mpNum, *n* As mpNum, *delta* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

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The function NoncentralTDistInvBoost returns quantiles and related information for the noncentral *t*-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*n*: A real number greater 0, representing the degrees of freedom

*delta*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below:

Let  $t_{n,\delta;\alpha}$  denote the  $\alpha$ -quantile of the noncentral *t*-distribution with  $n$  degrees of freedom and noncentrality parameter  $\delta$ . The quantile is calculated starting with one of the approximation below, followed by a Newton procedure using 30.8.2.7 until a relative error of  $10^{-4}$ . The final value is then obtained from this value by a Newton procedure using the "exact" formulas for the CDF.

#### 30.8.3.1 Quantiles: Akahira's approximation

Akahira *et al.* (1995) gives the following approximation:

$$t_{n,\delta;\alpha} \approx \frac{\delta b + u_1 \sqrt{b + c(\delta^2 - u_1^2)}}{b^2 - cu_1^2}, \quad \text{where} \quad (30.8.20)$$

$$u_1 = z_\alpha - \frac{at_1^3}{(1+ct_1^2)^{3/2}}, \quad t_1 = \frac{\delta b + u_0 \sqrt{b + c(\delta^2 - u_0^2)}}{b^2 - cu_0^2}, \quad u_0 = z_\alpha - \frac{at_0^3}{(1+ct_0^2)^{3/2}},$$

$t_0 = \delta + z_\alpha$ ,  $c = 1 - b^2$ ,  $g = c^{-1/2}$ ,  $a = (z_\alpha^2 - 1)h$ ,  $h$  and  $b$  are defined in equation (30.8.19), and  $z_\alpha$  denotes the  $\alpha$ -quantile of the normal distribution (see section 6.11.3). This approximation produces reliable results for  $\delta > 0$ , if either  $-bg < z_\alpha \leq -1$  or  $1 \leq z_\alpha < bg + |z_\alpha^2 - 1|hg$ .

#### 30.8.3.2 Quantiles: van Eeden's approximation

van Eeden (1961) gives the following approximation to the upper percentage point:

$$\begin{aligned} t_{n,\delta;\alpha} \approx z + \frac{z^3 + z}{4n} + \frac{5z^5 + 16z^3 + 3z}{96n^2} + \delta + \frac{\delta(2z^2 + 1) + \delta^2 z}{4n} \\ + \delta \left( \frac{4z^4 + 12z^2 + 1}{32n^2} + \frac{\delta(z^3 + 4z + 4z)}{16n^2} - \frac{\delta^2(z^2 - 1)}{24n^2} - \frac{\delta^3 z}{32n^2} \right) \end{aligned} \quad (30.8.21)$$

where  $z_\alpha$  denotes the  $\alpha$ -quantile of the normal distribution (see section 6.11.3). This approximation produces reliable results for  $\delta > 0$ , if  $z_\alpha \leq 1$ .

### 30.8.4 Properties

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Function **NoncentralTDistInfoBoost**(*n* As mpNum, *delta* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function NoncentralTDistInfoBoost returns moments and related information for the noncentral *t*-distribution

**Parameters:**

*n*: A real number greater 0, representing the degrees of freedom

*delta*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 30.8.4.1 Moments (singly noncentral)

The algebraic moments (defined for  $n > r$ ) are given by

$$\mu'_r = \left(\frac{1}{2}n\right)^{r/2} \frac{\Gamma\left(\frac{1}{2}(n-r)\right)}{\Gamma\left(\frac{1}{2}n\right)} \sum_{i=0}^{[r/2]_G} \binom{r}{2i} \frac{(2i)!}{2^i i!} \delta^{r-2i}. \quad (30.8.22)$$

The first four raw moments are given by

$$\mathbb{E}(t) = \delta \sqrt{\frac{1}{2}n} \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}n\right)} \quad (30.8.23)$$

$$\mathbb{E}(t^2) = (\delta^2 + 1) \frac{n}{n-2} \quad (30.8.24)$$

$$\mathbb{E}(t^3) = \delta(\delta^2 + 3) \sqrt{\frac{1}{8}n^3} \frac{\Gamma\left(\frac{1}{2}(n-3)\right)}{\Gamma\left(\frac{1}{2}n\right)} \quad (30.8.25)$$

$$\mathbb{E}(t^4) = (\delta^4 + 6\delta^2 + 3) \frac{n^2}{(n-2)(n-4)} \quad (30.8.26)$$

#### 30.8.4.2 Recurrence relations

() () The following relation holds (Witkovsky, 2013):

$$f_{t'}(n, x, \delta) = \frac{n}{x} \left[ F_{t'}\left(n+2, x\sqrt{1+2/n}, \delta\right) - F_{t'}(n, x, \delta) \right] \quad (30.8.27)$$

See Wang & Gray (1993) for additional recurrence relations for the density.

### 30.8.5 Random Numbers

---

Function **NoncentralTDistRanBoost**(*Size* As mpNum, *n* As mpNum, *delta* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function `NoncentralTDistRanBoost` returns random numbers following a noncentral  $t$ -distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*n*: A real number greater 0, representing the degrees of freedom

*delta*: A real number greater 0, representing the noncentrality parameter

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below:

Random numbers from a non-central  $t$ -distribution is easily obtained using the definition in terms of the ratio between two independent random numbers from a normal and a central chi-square distribution. This ought to be sufficient for most applications but if needed more efficient techniques may easily be developed e.g. using more general techniques.

### 30.8.6 Confidence Limits for the Noncentrality Parameter

---

Function **NoncentralTDistNoncentrality**(*alpha* As `mpNum`, *delta* As `mpNum`, *theta* As `mpNum`, *n* As `mpNum`, *Output* As `String`) As `mpNumList`

---

**NOT YET IMPLEMENTED**

---

The function `NoncentralTDistNoncentrality` returns confidence limits for the doubly noncentrality parameter *delta* and related information for the noncentral  $t$ -distribution.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*delta*: A real number greater 0, representing the numerator noncentrality parameter

*theta*: A real number greater 0, representing the denominator noncentrality parameter

*n*: A real number greater 0, representing the degrees of freedom.

*Output*: A string describing the output choices

See section 6.1.3.5 for the options for *alpha*, *Noncentrality* and *Output*. Algorithms and formulas are given below.

Let  $T$  be a statistic according to the non-central  $t$ -distribution with  $n$  degrees of freedom and a non-centrality parameter  $\delta$ . Then the lower confidence limit  $\hat{\delta}$  of level  $1 - \alpha$  and the two-sided confidence interval  $[\underline{\delta}, \bar{\delta}]$  of the non-centrality parameter  $\delta$  of level  $1 - \alpha$  are given by Akahira et al. (1995):

$$\hat{\delta} = bT - z_\alpha \sqrt{k} + hT^3(z_\alpha^2 - 1)/k, \quad (30.8.28)$$

$$\underline{\delta} = bT - z_{\alpha/2} \sqrt{k} + hT^3(z_{\alpha/2}^2 - 1)/k, \quad (30.8.29)$$

$$\bar{\delta} = bT + z_{\alpha/2} \sqrt{k} - hT^3(z_{\alpha/2}^2 - 1)/k, \quad (30.8.30)$$

where  $k = 1 + (1 - b^2)T^2$ ,  $h$  and  $b$  are defined in equation (30.8.19), and  $z_\alpha$  denotes the  $\alpha$ -quantile of the normal distribution (see section 6.11.3).

### 30.8.7 Sample Size Function

Function **NoncentralTDistSampleSize**(*alpha* As mpNum, *beta* As mpNum, **ModifiedNoncentrality1** As mpNum, **ModifiedNoncentrality2** As mpNum, **Output** As String) As mpNumList

NOT YET IMPLEMENTED

The function NoncentralTDistSampleSize returns sample size estimates and related information for the doubly noncentral *t*-distribution.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*beta*: A real number between 0 and 1, specifies the Type II error (or 1 - Power).

**ModifiedNoncentrality1**: A real number greater 0, representing the modified numerator noncentrality parameter.

**ModifiedNoncentrality2**: A real number greater 0, representing the modified denominator noncentrality parameter.

**Output**: A string describing the output choices

See section 6.1.3.4 for the options for *alpha*, *beta*, **ModifiedNoncentrality** and **Output**. Algorithms and formulas are given below.

We denote by  $N_{t'}(\alpha, \beta, \tilde{\rho})$  the sample size function of the noncentral *t*-distribution for a given confidence level  $\alpha$ , power  $\beta$  and modified noncentrality parameter  $\tilde{\rho}$ . This function determines the minimal sample size  $N$  for given noncentrality parameter, Type I error  $1 - \alpha$ , and Type II error  $1 - \beta$ , where  $\delta = \sqrt{N}\tilde{\rho}$

## 30.9 Doubly Noncentral Student's t-Distribution

### 30.9.1 Definition

Let  $T = X/\sqrt{Y/n}$ , where  $X$  and  $Y$  are independent,  $X$  has a normal distribution with mean  $\mu$  and unit variance, and  $Y$  has a noncentral  $\chi^2$  distribution with  $n$  degrees of freedom and noncentrality parameter  $\theta$ .

### 30.9.2 Density and CDF

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Function **DoublyNoncentralTDist**(*x As mpNum, n As mpNum, delta As mpNum, theta As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **DoublyNoncentralTDist** returns pdf, CDF and related information for the doubly noncentral  $t$ -distribution

#### Parameters:

*x*: A real number

*n*: A real number greater 0, representing the degrees of freedom

*delta*: A real number greater 0, representing the numerator noncentrality parameter

*theta*: A real number greater 0, representing the denominator noncentrality parameter

*Output*: A string describing the output choices

#### 30.9.2.1 Density (doubly noncentral): Integral representation

The pdf of a variable following a doubly noncentral  $t$ -distribution with  $n$  degrees of freedom and noncentrality parameters  $\delta$  and  $\theta$  is given by ([Witkovsky, 2013](#))

$$f_{t''}(t; n; \delta, \theta) = \int_0^\infty \phi\left(x\sqrt{\frac{q}{n}} - \delta\right) f_{\chi^2}(n, q; \theta) \sqrt{\frac{q}{n}} dq. \quad (30.9.1)$$

where  $\phi(\cdot)$  denotes the pdf of the normal distribution (see section [6.11.2.1](#)) and  $f_{\chi_n^2}(n, \cdot; \theta)$  denotes the pdf of the noncentral  $\chi^2$  distribution with  $n$  degrees of freedom and noncentrality parameter  $\theta$  (see section [30.6.2](#)).

#### 30.9.2.2 Density: Infinite series

[Kocherlakota & Kocherlakota \(1991\)](#) show that

$$f_{t''}(t; n; \mu, \theta) = \sum_{i=0}^{\infty} \omega_{i,\theta} s_{i,n} f_{t'}(s_{i,n} t; n + 2i, \mu), \quad \text{where} \quad (30.9.2)$$

$$\omega_{i,\theta} = \frac{\exp(-\theta/2)(\theta/2)^i}{i!} \quad \text{and} \quad s_{i,n} = \sqrt{\frac{n+2i}{n}} \quad (30.9.3)$$

and  $f_{t'}(\cdot)$  denotes the pdf of the singly noncentral  $t$ -distribution (see section [30.8.2](#)). Note that the terms  $f_{t'}(s_{i,n} t; n + 2i, \mu)$  can be calculated by using the recurrence relation in equation [\(30.8.4.2\)](#) once the first 2 terms have been calculated explicitly using any of the methods given in section [30.8.2](#).

### 30.9.2.3 Density (doubly noncentral): Saddlepoint approximation

Broda & Paoletta (2007) give the following saddlepoint approximation:

$$f_{t''}(t; n; \mu, \theta) \approx \phi(w)/u, \quad \text{where} \quad (30.9.4)$$

where  $w$  and  $u$  are defined in (30.9.9).

### 30.9.2.4 CDF: Integral representation

The cdf of a variable following a doubly noncentral t-distribution with  $n$  degrees of freedom and noncentrality parameters  $\delta$  and  $\theta$  is given by (Witkovsky, 2013)

$$\Pr[X \leq x] = F_{t''}(t; n; \delta, \theta) = \int_0^\infty \Phi\left(x\sqrt{\frac{q}{n}} - \delta\right) f_{\chi^2}(n, q; \theta) dq. \quad (30.9.5)$$

where  $\Phi(\cdot)$  denotes the cdf of the normal distribution (see section 6.11.2.2) and  $f_{\chi_n^2}(n, \cdot; \theta)$  denotes the pdf of the noncentral  $\chi^2$  distribution with  $n$  degrees of freedom and noncentrality parameter  $\theta$  (see section 30.6.2).

### 30.9.2.5 CDF: Infinite series

Kocherlakota & Kocherlakota (1991) show that

$$F_{t''}(t; n; \mu, \theta) = \sum_{i=0}^{\infty} \omega_{i,\theta} s_{i,n} F_{t'}(s_{i,n}t; n+2i, \mu), \quad \text{where} \quad (30.9.6)$$

where  $\omega_{i,\theta}$  and  $s_{i,n}$  are defined in (30.9.3) and  $F_{t'}(\cdot)$  denotes the cdf of the singly noncentral t-distribution (see section 30.8.2.3). Note that the terms  $F_{t'}(s_{i,n}t; n+2i, \mu)$  can be calculated by using the recurrence relation in equation (30.8.4.2) once the first 2 terms have been calculated explicitly using any of the methods given in section 30.8.2.3.

### 30.9.2.6 CDF: Approximation by singly noncentral t

Broda & Paoletta (2007) proposes the following approximation

$$F_{t''}(t; n; \mu, \theta) \approx F_{t'}(t/c; f; \mu), \quad \text{where} \quad (30.9.7)$$

$$f = \frac{7}{2} \left( -1 + \sqrt{15 - 7g^2/h}^{-1} \right), \quad c = \sqrt{h(1 - 2/f)}, \quad g = m_1/\sqrt{\mu^2/2}, h = m_2/(1 + \mu^2), \quad (30.9.8)$$

and  $m_i$  denotes the  $i$ th raw moment of  $t''$ , given in section 30.9.4.1.

### 30.9.2.7 CDF: Saddlepoint approximation CDF

Broda & Paoletta (2007) give the following saddlepoint approximation:

$$F_{t''}(y_1; n; \mu, \theta) \approx \Phi(w) + \phi(w) \left( \frac{1}{w} - \frac{d}{u} \right), \quad \text{where} \quad (30.9.9)$$

$$\begin{aligned}
a &= y_1^4 + 2ny_1^2 + n^2 \\
c_2 &= (-2y_1^3\mu - 2y_1n\mu)/a \\
c_1 &= (y_1^2\mu^2 - ny_1^2 - n^2 - \theta n)/a \\
c_0 &= (y_1n\mu)/a \\
q &= \frac{1}{3}c_1 - \frac{1}{9}c_2^2, \quad r = \frac{1}{6}(c_1c_2 - 3c_0) - \frac{1}{27}c_2^3 \\
y_2 &= \sqrt{-4q} \cos \left( \frac{1}{3} \arccos \left( r/\sqrt{-q^3} \right) \right) - \frac{1}{3}c_2 \\
t_1 &= -\mu + y_1y_2 \\
t_2 &= -y_1t_1/(2ny_2) \\
d &= 1/(t_1y_2) \\
\nu &= 1/(1 - 2t_2) \\
\alpha &= \mu/\sqrt{1 + \theta/n} \\
u &= \sqrt{(y_1^2 + 2nt_2)(2n\nu^2 + 4\theta\nu^3) + 4n^2y_2^2}/(2ny_2^2) \\
w &= \text{sgn}(y_1 - \alpha)\sqrt{-\mu t_1 - n \ln(\nu) - 2\theta\nu t_2}
\end{aligned}$$

, and  $\Phi(\cdot)$  and  $\phi(\cdot)$  denote the cdf (see section 6.11.2.2) and pdf (see section 6.11.2.1) of the normal distribution, respectively.

### 30.9.3 Quantiles

---

Function **DoublyNoncentralTDistInv**(*Prob* As mpNum, *n* As mpNum, *delta* As mpNum, *theta* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function DoublyNoncentralTDistInv returns quantiles and related information for the doubly noncentral *t*-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*n*: A real number greater 0, representing the degrees of freedom

*delta*: A real number greater 0, representing the numerator noncentrality parameter

*theta*: A real number greater 0, representing the denominator noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below: The quantiles are calculated by a Newton iteration, using an first approximation by the singly noncentral distribution as initial estimate in an iteration based on equation 30.9.9, until a relative error of  $10^{-4}$  has been achieved. This new approximation is then used as initial estimate for a Newton iteration based on equation 30.9.6.

### 30.9.4 Properties

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Function **DoublyNoncentralTDistInfo**(*n* As mpNum, *delta* As mpNum, *theta* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function DoublyNoncentralTDistInfo returns moments and related information for the doubly noncentral *t*-distribution

**Parameters:**

*n*: A real number greater 0, representing the degrees of freedom

*delta*: A real number greater 0, representing the numerator noncentrality parameter

*theta*: A real number greater 0, representing the denominator noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 30.9.4.1 Moments

The algebraic moments (defined for  $n > r$ ) are given by

$$\mu'_r = \left(\frac{1}{2}n\right)^{r/2} \frac{\Gamma\left(\frac{1}{2}(n-r)\right)}{\Gamma\left(\frac{1}{2}n\right)} \times \sum_{i=0}^{\lfloor r/2 \rfloor} \binom{r}{2i} \frac{(2i)!}{2^i i!} \delta^{r-2i} \times e^{-\lambda/2} M\left(\frac{1}{2}(n-r), \frac{1}{2}n, \frac{1}{2}\lambda\right). \quad (30.9.10)$$

The first four raw moments are given by

$$\mathbb{E}(t) = \delta \sqrt{\frac{1}{2}n} \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}n\right)} e^{-\lambda/2} M\left(\frac{1}{2}(n-1), \frac{1}{2}n, \frac{1}{2}\lambda\right) \quad (30.9.11)$$

$$\mathbb{E}(t^2) = (\delta^2 + 1) \frac{n}{n-2} e^{-\lambda/2} M\left(\frac{1}{2}(n-2), \frac{1}{2}n, \frac{1}{2}\lambda\right) \quad (30.9.12)$$

$$\mathbb{E}(t^3) = \delta(\delta^2 + 3) \sqrt{\frac{1}{8}n^3} \frac{\Gamma\left(\frac{1}{2}(n-3)\right)}{\Gamma\left(\frac{1}{2}n\right)} e^{-\lambda/2} M\left(\frac{1}{2}(n-3), \frac{1}{2}n, \frac{1}{2}\lambda\right) \quad (30.9.13)$$

$$\mathbb{E}(t^4) = (\delta^4 + 6\delta^2 + 3) \frac{n^2}{(n-2)(n-4)} e^{-\lambda/2} M\left(\frac{1}{2}(n-4), \frac{1}{2}n, \frac{1}{2}\lambda\right) \quad (30.9.14)$$

### 30.9.5 Random Numbers

---

Function **DoublyNoncentralTDistRan**(*Size* As mpNum, *n* As mpNum, *delta* As mpNum, *theta* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function DoublyNoncentralTDistRan returns random numbers following a doubly noncentral *t*-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*n*: A real number greater 0, representing the degrees of freedom

*delta*: A real number greater 0, representing the numerator noncentrality parameter

*theta*: A real number greater 0, representing the denominator noncentrality parameter

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below:

Random numbers from a doubly non-central t-distribution is easily obtained using the definition in terms of the ratio between two independent random numbers from a normal and a non-central chi-square distribution. This ought to be sufficient for most applications but if needed more efficient techniques may easily be developed e.g. using more general techniques.

### 30.9.6 Confidence Limits for the Noncentrality Parameter

---

Function **DoublyNoncentralTDistNoncentrality**(*alpha* As mpNum, *delta* As mpNum, *theta* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

---

NOT YET IMPLEMENTED

---

The function **DoublyNoncentralTDistNoncentrality** returns confidence limits for the doubly noncentrality parameter delta and related information for the noncentral *t*-distribution.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*delta*: A real number greater 0, representing the numerator noncentrality parameter

*theta*: A real number greater 0, representing the denominator noncentrality parameter

*n*: A real number greater 0, representing the degrees of freedom.

*Output*: A string describing the output choices

See section 6.1.3.5 for the options for *alpha*, *Noncentrality* and *Output*. Algorithms and formulas are given below.

Let  $T$  be a statistic according to the non-central *t*-distribution with  $n$  degrees of freedom and a non-centrality parameter  $\delta$ . Then the lower confidence limit  $\hat{\delta}$  of level  $1 - \alpha$  and the two-sided confidence interval  $[\underline{\delta}, \bar{\delta}]$  of the non-centrality parameter  $\delta$  of level  $1 - \alpha$  are given by Akahira *et al.* (1995):

$$\hat{\delta} = bT - z_\alpha \sqrt{k} + hT^3(z_\alpha^2 - 1)/k, \quad (30.9.15)$$

$$\underline{\delta} = bT - z_{\alpha/2} \sqrt{k} + hT^3(z_{\alpha/2}^2 - 1)/k, \quad (30.9.16)$$

$$\bar{\delta} = bT + z_{\alpha/2} \sqrt{k} - hT^3(z_{\alpha/2}^2 - 1)/k, \quad (30.9.17)$$

where  $k = 1 + (1 - b^2)T^2$ ,  $h$  and  $b$  are defined in equation (30.8.19), and  $z_\alpha$  denotes the  $\alpha$ -quantile of the normal distribution (see section 6.11.3).

### 30.9.7 Sample Size Function

---

Function **DoublyNoncentralTDistSampleSize**(*alpha* As mpNum, *beta* As mpNum, **ModifiedNoncentrality1** As mpNum, **ModifiedNoncentrality2** As mpNum, *Output* As String) As mpNumList

---

NOT YET IMPLEMENTED

---

The function `DoublyNoncentralTDistSampleSize` returns sample size estimates and related information for the doubly noncentral  $t$ -distribution.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*beta*: A real number between 0 and 1, specifies the Type II error (or 1 - Power).

*ModifiedNoncentrality1*: A real number greater 0, representing the modified numerator noncentrality parameter.

*ModifiedNoncentrality2*: A real number greater 0, representing the modified denominator noncentrality parameter.

*Output*: A string describing the output choices

See section 6.1.3.4 for the options for *alpha*, *beta*, *ModifiedNoncentrality* and *Output*. Algorithms and formulas are given below.

We denote by  $N_t'(\alpha, \beta, \tilde{\rho})$  the sample size function of the noncentral  $t$ -distribution for a given confidence level  $\alpha$ , power  $\beta$  and modified noncentrality parameter  $\tilde{\rho}$ . This function determines the minimal sample size  $N$  for given noncentrality parameter, Type I error  $1 - \alpha$ , and Type II error  $1 - \beta$ , where  $\delta = \sqrt{N}\tilde{\rho}$

## 30.10 NonCentral F-Distribution

### 30.10.1 Definition

If  $X_1$  and  $X_2$  are independent random variables,  $X_1$  following a non-central  $\chi^2$ -distribution with noncentrality parameter  $\lambda$  and  $m$  degrees of freedom, and  $X_2$  following a  $\chi^2$ -distribution with  $n$  degrees of freedom, then the distribution of the ratio  $F = \frac{X_1/m}{X_2/n}$  is said to follow a non-central F-distribution with noncentrality parameter  $\lambda$  and  $m$  and  $n$  degrees of freedom.

### 30.10.2 Density and CDF

---

Function **NoncentralFDistBoost**(*x* As mpNum, *m* As mpNum, *n* As mpNum, *lambda* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function NoncentralFDistBoost returns pdf, CDF and related information for the noncentral F-distribution

**Parameters:**

*x*: A real number

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

#### 30.10.2.1 Density

#### 30.10.2.2 Density: Closed form representation

The density function of the noncentral F-Distribution is given by ([Wang & Gray, 1993](#))

$$f_{F'}(x; n_1, n_2, \lambda) = e^{-\lambda/2} f_F(x; n_1, n_2) {}_1F_1\left(\frac{1}{2}(m+n), \frac{1}{2}n, \frac{nx\lambda}{2(m+nx)}\right) \quad (30.10.1)$$

where  $f_F(x; m, n)$  is the pdf of the (central) F distribution (see section [6.6.2.1](#)) and  ${}_1F_1(\cdot)$  is the confluent hypergeometric function (see section [13.2.1](#)).

#### 30.10.2.3 CDF: Integral representation

The cdf of a variable following a (singly) noncentral F-distribution with  $n$  and  $m$  degrees of freedom and noncentrality parameter  $\lambda_1$  and is given by ([Chou, 1985](#))

$$\begin{aligned} \Pr[X \leq c] &= F_{F'}(x; m, n; \lambda_1) = \int_0^\infty F_{\chi^2}(n, ncy/m; \lambda_1) f_{\chi^2}(n, y) dy \\ &= \int_0^\infty [1 - F_{\chi^2}(m, mx/(nc))] f_{\chi^2}(n, x; \lambda_1) dx \\ &= 1 - \frac{cn}{m} \int_0^\infty F_{\chi^2}(m, y; \lambda_1) f_{\chi^2}(n, ncy/m) dy \end{aligned} \quad (30.10.2)$$

where  $F_{\chi^2}(n, \cdot; \lambda_1)$  denotes the cdf of the noncentral  $\chi^2$  distribution with  $n$  degrees of freedom and noncentrality parameter  $\lambda_1$  (see section [30.6.2.3](#)) and  $f_{\chi^2}(m, \cdot; \theta)$  denotes the pdf of the noncentral  $\chi^2$  distribution with  $m$  degrees of freedom and noncentrality parameter  $\lambda_2$  (see section [30.6.2](#)).

### 30.10.2.4 CDF: Finite Series for even error degrees of freedom

This is code for noncentral beta and needs to be adapted for noncentral F. The cdf can be calculated using the following finite series, if  $b$  is an integer:

$$I(x; a, b, \lambda) = e^{-\lambda(1-x)} \sum_{n=0}^{b-1} L_n, \quad \text{where} \quad (30.10.3)$$

$$\begin{aligned} L_0 &= 1, \quad L_1 = (1-x)(a + \lambda x), \\ L_n &= \frac{1-x}{n} ((2n-2+a+\lambda x)L_{n-1} - (n+a-2)(1-x)L_{n-2}) \end{aligned}$$

### 30.10.2.5 CDF: Infinite Series

The cdf of a variable following a (singly) noncentral F-distribution with  $n$  and  $m$  degrees of freedom and noncentrality parameter  $\lambda_1$  and is given by

$$\Pr[F \leq x] = F_{F'}(x; m, n, \lambda) = e^{-\lambda} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!} F(m+2j, n, x) \quad (30.10.4)$$

where  $F_F(\cdot)$  denotes the cdf of the central F-distribution (see section 6.6.2.2).

### 30.10.2.6 CDF: 2-moment approximation by central F

[Patnaik \(1949\)](#) suggests the following approximation (see also [Tiku \(1966\)](#)):

$$F_{F'}(x; n_1, n_2, \lambda) \approx F_F(y; m_1, n_2), \quad \text{where} \quad (30.10.5)$$

$$A_1 = (n_1 + \lambda),$$

$$B_1 = (n_1 + 2\lambda),$$

$$m_1 = A_1^2/B_1,$$

$$y = n_1/A_1,$$

and  $F_F(\cdot; m_1, n_2)$  denotes the CDF of a central F distribution with  $m_1$  and  $n_2$  degrees of freedom (see section 6.6.2.2).

### 30.10.2.7 CDF: Saddlepoint approximation

[Butler & Paolella \(2002\)](#) suggest the following second order saddlepoint approximation:

$$F_{F'}(x; n_1, n_2, \lambda) \approx \Phi(w) + \phi(w) \left( \frac{1}{w} - \frac{1}{u} \left( 1 + \frac{\kappa_4}{8} - \frac{5}{24} \kappa_3^2 \right) - \frac{1}{u^3} - \frac{\kappa_3}{2u^2} + \frac{1}{w^3} \right) \quad (30.10.6)$$

$$f_{F'}(x; n_1, n_2, \lambda) \approx \frac{1}{\sqrt{2\pi k_2}} \exp(k - sx) \left( 1 + \frac{\kappa_4}{8} - \frac{5}{24} \kappa_3^2 \right), \quad \text{where} \quad (30.10.7)$$

$$a_1 = 4n_2x(n_1 + n_2)$$

$$a_2 = x^2n_1^3 + 2x^2n_1^2\lambda + 2n_1^2xn_2 + 4x^2n_1n_2\lambda + n_1\lambda^2x^2 + 2n_1\lambda xn_2 + n_2^2n_1 + 4xn_2^2\lambda$$

$$s = [xn_1(n_1 + 2n_2 + \lambda) - n_1n_2 - \sqrt{n_1a_2}]/a_1$$

$$l_1 = n_2/n_1, \quad l_2 = -x$$

$$v_1 = 1/(1 - 2sl_1), v_2 = 1/(1 - 2sl_2)$$

$$g_1 = l_1 v_1, g_2 = l_2 v_2$$

$$k = \frac{1}{2}(n_1 \ln(v_1) + n_2 \ln(v_2)) + s\lambda g_1$$

$$k_2 = 2(g_1^2(n_1 + 2\lambda v_1) + g_2^2 n_2)$$

$$k_3 = 8(g_1^3(n_1 + 3\lambda v_1) + g_2^3 n_2)$$

$$k_4 = 48(g_1^4(n_1 + 4\lambda v_1) + g_2^4 n_2)$$

$$\kappa_3 = k_3/k_2^{3/2}, \kappa_4 = k_4/k_2^2$$

$$w = \text{sgn}(s)\sqrt{2(sx - k)}$$

$$u = s\sqrt{k_2}$$

and  $\Phi(\cdot)$  and  $\phi(\cdot)$  denote the cdf (see section 6.11.2.2) and pdf (see section 6.11.2.1) of the normal distribution, respectively.

### 30.10.3 Quantiles

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Function **NoncentralFDistInvBoost**(*Prob* As mpNum, *m* As mpNum, *n* As mpNum, *lambda* As mpNum, *Output* As String) As mpNumList

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**NOT YET IMPLEMENTED**

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The function NoncentralFDistInvBoost returns quantiles and related information for the non-central *F*-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

#### 30.10.3.1 Quantiles: Central F approximation

An approximation to  $F_{\alpha, n_1, n_2, \lambda}$ , the  $\alpha$ -quantile of a non-central *F*-distribution with  $n_1$  and  $n_2$  degrees of freedom and noncentrality parameter  $\lambda$ , is obtained as

$$F_{\alpha, n_1, n_2, \lambda} \approx c \cdot F_{\alpha, m_1, n_2}, \quad (30.10.8)$$

$$A_1 = (n_1 + \lambda),$$

$$B_1 = (n_1 + 2\lambda),$$

$$m_1 = A_1^2/B_1,$$

$$c = A_1/n_1,$$

and  $F_{\alpha, m_1, n_2}$ , denotes the  $\alpha$ -quantile of a central *F*-distribution with  $m_1$  and  $n_2$  degrees of freedom (see section 6.6.3).

See also Mudholkar *et al.* (1975)

### 30.10.4 Properties

Function **NoncentralFDistInfoBoost**(*m* As mpNum, *n* As mpNum, *lambda* As mpNum, *Output* As String) As mpNumList

**NOT YET IMPLEMENTED**

The function NoncentralFDistInfoBoost returns moments and related information for the noncentral *F*-distribution

**Parameters:**

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 30.10.4.1 Moments

The algebraic moments (defined for  $f_2 > 2r$ ) are given by

$$\mu'_r = \frac{\Gamma(\frac{1}{2}f_1 + r) - \Gamma(\frac{1}{2}f_2 - r)}{\Gamma(\frac{1}{2}f_2)} \sum_{j=0}^r \binom{r}{j} \frac{\frac{1}{2}\lambda f_1)^j}{\Gamma(\frac{1}{2}f_1 + j)}, \quad \text{for } f_2 > 2r. \quad (30.10.9)$$

The first four raw moments (defined for  $n > 2k$ ) are given by

$$\begin{aligned} \mu'_1 &= \frac{n}{m} \frac{m + \lambda}{n - 2} \\ \mu'_2 &= \left(\frac{n}{m}\right)^2 \frac{\lambda^2 + (2\lambda + m)(m + 2)}{(n - 2)(n - 4)} \\ \mu'_3 &= \left(\frac{n}{m}\right)^3 \frac{\lambda^3 + 3(m + 4)\lambda^2 + (3\lambda + m)(m + 4)(m + 2)}{(n - 2)(n - 4)(n - 6)} \\ \mu'_4 &= \left(\frac{n}{m}\right)^4 \frac{\lambda^3 + 4(m + 6)\lambda^3 + 6(m + 6)(m + 4)\lambda^2 + (4\lambda + m)(m + 6)(m + 4)(m + 2)}{(n - 2)(n - 4)(n - 6)(n - 8)} \end{aligned}$$

#### 30.10.4.2 Recurrence relations

Let the density  $g_{m,n}$  be that of  $m/n$  times an  $F_{m,n}$  random variable. Let  $G_{m,n}(y)$  be its distribution function, and let  $g_{m,n}^\lambda$  and  $G_{m,n}^\lambda(y)$  be the density and distribution function of its (singly) noncentral version (the distribution of  $\chi_m^2(\lambda)/\chi_n^2(0)$ ). Then the following recurrence relations hold ([Chattamvelli & Jones, 1995](#))

$$n [G_{m,n}^\lambda(y) - G_{m,n}^\lambda(y)] = -2g_{m,n}^\lambda(y) \quad (30.10.10)$$

$$\lambda(1+y)g_{m+4,n}^\lambda(y) = [\lambda y - m(1+y)]g_{m+2,n}^\lambda(y) + y(m+n)g_{m,n}^\lambda(y). \quad (30.10.11)$$

$$n(1+y)g_{m,n+2}^\lambda(y) = (m+n)g_{m,n}^\lambda(y) + \lambda g_{m+2,n}^\lambda(y). \quad (30.10.12)$$

$$\lambda g_{m+4,n-2}^\lambda(y) + mg_{m+2,n-2}^\lambda(y) = (n-2)y g_{m,n}^\lambda(y). \quad (30.10.13)$$

From equations 30.10.10 to 30.10.13 we obtain

$$\begin{aligned}\lambda(1+y)G_{m+6,n}^\lambda(y) &= [\lambda y - (m+2-\lambda)(1+y)]G_{m+4,n}^\lambda(y) \\ &\quad + [(m+2)(1+y) + y(m+n-\lambda)]G_{m+2,n}^\lambda(y) \\ &\quad - y(m+n)G_{m,n}^\lambda(y)\end{aligned}\tag{30.10.14}$$

$$\begin{aligned}n(1+y)[G_{m,n+2}^\lambda(y) - G_{m+2,n+2}^\lambda(y)] &= (m+n)G_{m,n}^\lambda(y) \\ &\quad + (\lambda - m - n)G_{m+2,n}^\lambda(y) \\ &\quad - \lambda G_{m+4,n}^\lambda(y)\end{aligned}\tag{30.10.15}$$

$$\begin{aligned}(n-2)y[G_{m,n}^\lambda(y) - G_{m+2,n}^\lambda(y)] &= (m+2)G_{m+2,n-2}^\lambda(y) \\ &\quad + (\lambda - m - 2)G_{m+4,n-2}^\lambda(y) \\ &\quad - \lambda G_{m+6,n-2}^\lambda(y)\end{aligned}\tag{30.10.16}$$

See sections 6.6.4.1 for the corresponding expressions for the central F distribution.

#### 30.10.4.3 Relationships to other distributions

$$F_{F'}(x; m, n, \lambda) = F_B\left(\frac{1}{2}n, \frac{1}{2}m, \frac{mx}{mx+n}; \lambda\right)\tag{30.10.17}$$

#### 30.10.5 Random Numbers

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Function **NoncentralFDistRanBoost**(*Size* As mpNum, *m* As mpNum, *n* As mpNum, *lambda* As mpNum, *Generator* As String, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

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The function NoncentralFDistRanBoost returns random numbers following a noncentral *F*-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda*: A real number greater 0, representing the noncentrality parameter

*Generator*: A string describing the random generator

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

Random numbers from a non-central F-distribution is easily obtained using the definition in terms of the ratio between two independent random numbers from central and non-central chi-square distributions. This ought to be sufficient for most applications but if needed more efficient techniques may easily be developed e.g. using more general techniques.

#### 30.10.6 Confidence Limits for the Noncentrality Parameter

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Function **NoncentralFDistNoncentralityEx**(*alpha* As mpNum, *lambda1* As mpNum, *lambda2* As mpNum, *m* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

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The function `NoncentralFDistNoncentralityEx` returns confidence limits for the noncentrality parameter lambda and related information for the noncentral  $F$ -distribution.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).  
*lambda1*: A real number greater 0, representing the numerator noncentrality parameter  
*lambda2*: A real number greater 0, representing the denominator noncentrality parameter  
*m*: A real number greater 0, representing the numerator degrees of freedom.  
*n*: A real number greater 0, representing the denominator degrees of freedom.  
*Output*: A string describing the output choices

See section 6.1.3.5 for the options for *alpha*, *Noncentrality* and *Output*. Algorithms and formulas are given below.

### 30.10.7 Sample Size

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Function **NoncentralFDistSampleSizeEx**(*alpha* As mpNum, *beta* As mpNum, *m* As mpNum, *ModifiedNoncentrality1* As mpNum, *ModifiedNoncentrality1* As mpNum, *Output* As String)  
As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function `NoncentralFDistSampleSizeEx` returns sample size estimates and related information for the noncentral  $F$ -distribution.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).  
*beta*: A real number between 0 and 1, specifies the Type II error (or 1 - Power).  
*m*: A real number greater 0, representing the numerator degrees of freedom.  
*ModifiedNoncentrality1*: A real number greater 0, representing the modified numerator noncentrality parameter.  
*ModifiedNoncentrality1*: A real number greater 0, representing the modified denominator noncentrality parameter.  
*Output*: A string describing the output choices

See section 6.1.3.4 for the options for *alpha*, *beta*, *ModifiedNoncentrality* and *Output*. Algorithms and formulas are given below.

## 30.11 Doubly NonCentral F-Distribution

### 30.11.1 Definition

The doubly noncentral F distribution is determined as

$$F^{(2)} = \frac{U_1/n_1}{U_2/n_2} \quad (30.11.1)$$

where  $U_1$  and  $U_2$  are independent with  $U_i = \chi^2(n_i, \theta_i)$ , where  $n_i$  is the degrees of freedom, and  $\theta_i$  is the noncentrality parameter of the noncentral  $\chi^2$  distribution. We denote the doubly noncentral distribution as  $F^{(2)} = F_{n_1, n_2}(\theta_1, \theta_2)$ . Taking  $\theta_2 = 0$  gives the singly noncentral F and the additional requirement  $\theta_1 = 0$  leads to the central F.

See [Paoletta \(2007\)](#) and [Paoletta \(2006\)](#).

### 30.11.2 Density and CDF

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Function **DoublyNoncentralFDistEx**(*x As mpNum, m As mpNum, n As mpNum, lambda1 As mpNum, lambda2 As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function DoublyNoncentralFDistEx returns pdf, CDF and related information for the noncentral F-distribution

#### Parameters:

*x*: A real number

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda1*: A real number greater 0, representing the numerator noncentrality parameter

*lambda2*: A real number greater 0, representing the denominator noncentrality parameter

*Output*: A string describing the output choices

#### 30.11.2.1 Density: Infinite series

[Butler & Paoletta \(2002\)](#) show that

$$f_{F''}(t; n; \mu, \theta) = \sum_{i=0}^{\infty} \omega_{i,\theta} s_{i,n} f_{F'}(s_{i,n} t; n + 2i, \mu), \quad \text{where} \quad (30.11.2)$$

$$\omega_{i,\theta} = \frac{\exp(-\theta/2)(\theta/2)^i}{i!} \quad \text{and} \quad s_{i,n} = \sqrt{\frac{n+2i}{n}} \quad (30.11.3)$$

and  $f_{F'}(\cdot)$  denotes the pdf of the singly noncentral F-distribution (see section [30.10.2.1](#)). Note that the terms  $f_{F'}(s_{i,n} x; n + 2i, \mu)$  can be calculated by using the recurrence relation in equation [\(30.10.11\)](#) once the first 2 terms have been calculated explicitly using any of the methods given in section [30.10.2.1](#).

### 30.11.2.2 CDF: Integral representation

The cdf of a variable following a doubly noncentral F-distribution with  $n$  and  $m$  degrees of freedom and noncentrality parameters  $\lambda_1$  and  $\lambda_2$  is given by (Chou, 1985)

$$\begin{aligned}\Pr[X \leq c] &= F_{F''}(x; m, n; \lambda_1, \lambda_2) = \int_0^\infty F_{\chi^2}(n, nc/m; \lambda_1) f_{\chi^2}(n, y; \lambda_2) dy \\ &= \int_0^\infty [1 - F_{\chi^2}(m, mx/(nc); \lambda_2)] f_{\chi^2}(n, x; \lambda_1) dx \\ &= 1 - \frac{cn}{m} \int_0^\infty F_{\chi^2}(m, y; \lambda_1) f_{\chi^2}(n, nc/m; \lambda_2) dy\end{aligned}\quad (30.11.4)$$

where  $F_{\chi^2}(n, \cdot; \lambda_1)$  denotes the cdf of the noncentral  $\chi^2$  distribution with  $n$  degrees of freedom and noncentrality parameter  $\lambda_1$  (see section 30.6.2.3) and  $f_{\chi^2}(m, \cdot; \theta)$  denotes the pdf of the noncentral  $\chi^2$  distribution with  $m$  degrees of freedom and noncentrality parameter  $\lambda_2$  (see section 30.6.2).

### 30.11.2.3 CDF: Infinite series

Butler & Paoletta (2002) show that

$$F_{F''}(x; m, n; \lambda_1, \lambda_2) = \sum_{i=0}^{\infty} \omega_{i,\theta} s_{i,n} F_{F'}(s_{i,n} t; m + 2i, n; \lambda_1), \quad \text{where} \quad (30.11.5)$$

where  $\omega_{i,\lambda_1}$  and  $s_{i,n}$  are defined in (30.11.3) and  $F_{F'}(\cdot)$  denotes the cdf of the singly noncentral F-distribution (see section 30.10.2). Note that the terms  $F_{F'}(s_{i,n} t; n + 2i, \mu)$  can be calculated by using the recurrence relation in equation (30.10.14) once the first 3 terms have been calculated explicitly using any of the methods given in section 30.10.2.

### 30.11.2.4 CDF: Central F approximation

Mudholkar *et al.* (1976) suggests an approximation by noncentral  $F$ , which can be converted into an approximation by central  $F$  as follows:

$$\begin{aligned}F_{F'}(x; n_1, n_2, \lambda_1, \lambda_2) &\approx F_F(y; m_1, m_2), \quad \text{where} \\ A_1 &= (n_1 + \lambda_1), A_2 = (n_1 + \lambda_2), \\ B_1 &= (n_1 + 2\lambda_1), B_2 = (n_1 + 2\lambda_2), \\ m_1 &= A_1^2/B_1, m_2 = A_2^2/B_2, \\ y &= x(n_1 A_2)/(n_2 A_1),\end{aligned}\quad (30.11.6)$$

and  $F_F(\cdot; m_1, m_2)$  denotes the CDF of a central  $F$  distribution with  $m_1$  and  $m_2$  degrees of freedom (see section 6.6.2.2).

### 30.11.2.5 CDF: Edgeworth expansion

Mudholkar *et al.* (1976) suggest an Edgeworth-series expansion, which can be converted into an Cornish-Fisher expansion as follows:

$$F_{F'}(x; n_1, n_2, \lambda_1, \lambda_2) \approx \Phi(z), \quad \text{where} \quad (30.11.7)$$

$$u = \frac{-\kappa_1}{\sqrt{\kappa_2}}, \quad z = u + (u^2 - 1)\frac{\gamma_1}{6} + (u^3 - 3u)\frac{\gamma_2}{24} + (4u^3 - 7u)\frac{\gamma_1^2}{36},$$

$$\kappa_i = T_i(1, n_1, \lambda_1) + (-1)^i T_i(x, n_2, \lambda_2), \quad i = 1, 2, 3, 4,$$

$$\gamma_1 = \frac{\kappa_3}{\sqrt{\kappa_2} \kappa_2}, \quad \gamma_2 = \frac{\kappa_4}{\kappa_2^2},$$

$T_i(\cdot)$  is defined in section 30.6.4, and  $\Phi(\cdot)$  denotes the CDF of the normal distribution (see section 6.11.2.2).

### 30.11.2.6 CDF and pdf: Saddlepoint approximation

Butler & Paolella (2002) suggest the following second order saddlepoint approximation::

$$F_{F''}(x; n_1, n_2, \lambda_1, \lambda_2) \approx \Phi(w) + \phi(w) \left( \frac{1}{w} - \frac{1}{u} \left( 1 + \frac{\kappa_4}{8} - \frac{5}{24} \kappa_3^2 \right) - \frac{1}{u^3} - \frac{\kappa_3}{2u^2} + \frac{1}{w^3} \right) \quad (30.11.8)$$

$$f_{F''}(x; n_1, n_2, \lambda_1, \lambda_2) \approx \frac{1}{\sqrt{2\pi k_2}} \exp(k - sx) \left( 1 + \frac{\kappa_4}{8} - \frac{5}{24} \kappa_3^2 \right), \quad \text{where} \quad (30.11.9)$$

$$a = 8x^2 n_2^2 (n_1 + n_2)$$

$$a_0 = (x \lambda_2 n_1^2 - (1-x) n_1^2 n_2 - n_1 n_2 \lambda_1) / a$$

$$a_1 = (2(n_2^2 n_1 + n_1^2 n_2 x^2) - 4x n_1 n_2 (n_1 + n_2 + \lambda_1 + \lambda_2)) / a$$

$$a_2 = (8x(1-x)n_1 n_2^2 + 4x(n_2^3 + \lambda_2 n_2^2 - n_1^2 n_2 x - n_1 n_2 \lambda_1 x)) / (3a)$$

$$p = \sqrt{|3a_2^2 - a_1| / 3}$$

$$q = a_2(2a_2^2 - a_1) + a_0$$

$$s = -2p \cos \left( \frac{1}{3} \arccos(-\frac{1}{2}qp^{-3}) + \frac{1}{3}\pi \right) - a_2$$

$$l_1 = n_2/n_1, \quad l_2 = -x$$

$$v_1 = 1/(1 - 2sl_1), \quad v_2 = 1/(1 - 2sl_2)$$

$$g_1 = l_1 v_1, \quad g_2 = l_2 v_2$$

$$K(s) = \frac{1}{2}(n_1 \ln(v_1) + n_2 \ln(v_2)) + s(\lambda_1 g_1 + \lambda_2 g_2)$$

$$K''(s) = 2(g_1^2(n_1 + 2\lambda_1 v_1) + g_2^2(n_2 + 2\lambda_2 v_2))$$

$$K^{(3)}(s) = 8(g_1^3(n_1 + 3\lambda_1 v_1) + g_2^3(n_2 + 3\lambda_2 v_2))$$

$$K^{(4)}(s) = 48(g_1^4(n_1 + 4\lambda_1 v_1) + g_2^4(n_2 + 4\lambda_2 v_2))$$

$$\kappa_3 = K^{(3)}(s)/K''(s)^{3/2}, \quad \kappa_4 = K^{(4)}(s)/K''(s)^2$$

$$w = \text{sgn}(s) \sqrt{2(sx - K(s))}$$

$$u = s \sqrt{K''(s)}$$

and  $\Phi(\cdot)$  and  $\phi(\cdot)$  denote the cdf (see section 6.11.2.2) and pdf (see section 6.11.2.1) of the normal distribution, respectively.

### 30.11.3 Quantiles

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Function **DoublyNoncentralFDistInvEx**(*Prob* As mpNum, *m* As mpNum, *n* As mpNum, *lambda1* As mpNum, *lambda2* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function DoublyNoncentralFDistInvEx returns quantiles and related information for the the noncentral *F*-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda1*: A real number greater 0, representing the numerator noncentrality parameter

*lambda2*: A real number greater 0, representing the denominator noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

#### 30.11.3.1 Quantiles: Central F approximation

An approximation to  $F_{\alpha, n_1, n_2, \lambda_1, \lambda_2}$ , the  $\alpha$ -quantile of a doubly non-central *F*-distribution with  $n_1$  and  $n_2$  degress of freedom and noncentrality parameters  $\lambda_1$  and  $\lambda_2$ , is obtained as

$$F_{\alpha, n_1, n_2, \lambda_1, \lambda_2} \approx c \cdot F_{\alpha, m_1, m_2}, \quad (30.11.10)$$

$$A_1 = (n_1 + \lambda_1), A_2 = (n_1 + \lambda_2),$$

$$B_1 = (n_1 + 2\lambda_1), B_2 = (n_1 + 2\lambda_2),$$

$$m_1 = A_1^2/B_1, m_2 = A_2^2/B_2,$$

$$c = (n_2 A_1)/(n_1 A_2),$$

and denotes the  $\alpha$ -quantile of a central *F*-distribution with  $m_1$  and  $m_2$  degress of freedom (see section 6.6.3).

### 30.11.4 Properties

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Function **DoublyNoncentralFDistInfoEx**(*m* As mpNum, *n* As mpNum, *lambda1* As mpNum, *lambda2* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function DoublyNoncentralFDistInfoEx returns moments and related information for the non-central *F*-distribution

**Parameters:**

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda1*: A real number greater 0, representing the numerator noncentrality parameter

*lambda2*: A real number greater 0, representing the denominator noncentrality parameter

*Output:* A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

### 30.11.4.1 Moments

The algebraic moments (defined for  $f_2 > 2r$ ) are given by

$$\mu'_r = \frac{\Gamma(\frac{1}{2}f_1 + r) - \Gamma(\frac{1}{2}f_2 - r)}{\Gamma(\frac{1}{2}f_2)} \times \sum_{j=0}^r \binom{r}{j} \frac{\frac{1}{2}\lambda f_1)^j}{\Gamma(\frac{1}{2}f_1 + j)} \times e^{\lambda_2/2} M\left(\frac{1}{2}(n-2r), \frac{1}{2}n, \frac{1}{2}\lambda_2\right) \quad (30.11.11)$$

The first four raw moments are given by

$$\begin{aligned} \mu'_1 &= \frac{n}{m} \frac{m+\lambda}{n-2} \times e^{\lambda_2/2} M\left(\frac{1}{2}(n-2), \frac{1}{2}n, \frac{1}{2}\lambda_2\right) \\ \mu'_2 &= \left(\frac{n}{m}\right)^2 \frac{\lambda^2 + (2\lambda+m)(m+2)}{(n-2)(n-4)} \times e^{\lambda_2/2} M\left(\frac{1}{2}(n-4), \frac{1}{2}n, \frac{1}{2}\lambda_2\right) \\ \mu'_3 &= \left(\frac{n}{m}\right)^3 \frac{\lambda^3 + 3(m+4)\lambda^2 + (3\lambda+m)(m+4)(m+2)}{(n-2)(n-4)(n-6)} \times e^{\lambda_2/2} M\left(\frac{1}{2}(n-6), \frac{1}{2}n, \frac{1}{2}\lambda_2\right) \\ \mu'_4 &= \left(\frac{n}{m}\right)^4 \frac{\lambda^3 + 4(m+6)\lambda^3 + 6(m+6)(m+4)\lambda^2 + (4\lambda+m)(m+6)(m+4)(m+2)}{(n-2)(n-4)(n-6)(n-8)} \\ &\quad \times e^{\lambda_2/2} M\left(\frac{1}{2}(n-8), \frac{1}{2}n, \frac{1}{2}\lambda_2\right) \end{aligned}$$

### 30.11.4.2 Recurrence relations

Let the density  $g_{m,n}$  be that of  $m/n$  times an  $F_{m,n}$  random variable. Let  $G_{m,n}(y)$  be its distribution function, and let  $g_{m,n}^{\lambda_1, \lambda_2}$  and  $G_{m,n}^{\lambda_1, \lambda_2}(y)$  be the density and distribution function of its doubly noncentral version (the distribution of  $\chi_m^2(\lambda_1)/\chi_n^2(\lambda_2)$ ). Then the following recurrence relations hold (Chattamvelli & Jones, 1995)

$$n \left[ G_{m,n}^\lambda(y) - G_{m,n}^\lambda(y) \right] = -2g_{m,n}^\lambda(y) \quad (30.11.12)$$

$$\lambda_1(1+y)g_{m+4,n}^{\lambda_1, \lambda_2}(y) = [\lambda_1y - m(1+y)]g_{m+2,n}^{\lambda_1, \lambda_2}(y) + y(m+n)g_{m,n}^{\lambda_1, \lambda_2}(y) + \lambda_2g_{m,n+2}^{\lambda_1, \lambda_2}(y). \quad (30.11.13)$$

$$\lambda_2(1+y)g_{m,n+4}^{\lambda_1, \lambda_2}(y) = [\lambda_2y - n(1+y)]g_{m+2,n}^{\lambda_1, \lambda_2}(y) + (m+n)g_{m,n}^{\lambda_1, \lambda_2}(y) + \lambda_1g_{m+2,n}^{\lambda_1, \lambda_2}(y). \quad (30.11.14)$$

$$\lambda_1g_{m+4,n-2}^{\lambda_1, \lambda_2}(y) = -mg_{m+2,n}^{\lambda_1, \lambda_2}(y) + nyg_{m,n+2}^{\lambda_1, \lambda_2}(y) + \lambda_2g_{m+2,n}^{\lambda_1, \lambda_2}(y). \quad (30.11.15)$$

From equations 30.11.12 to 30.11.15 we obtain

$$\begin{aligned} \lambda_1(1+y)G_{m+6,n}^{\lambda_1, \lambda_2}(y) &= [\lambda_1y - (m+2-\lambda_1)(1+y)]G_{m+4,n}^{\lambda_1, \lambda_2}(y) \\ &\quad + [(m+2)(1+y) + y(m+n-\lambda_1)]G_{m+2,n}^{\lambda_1, \lambda_2}(y) \\ &\quad - y(m+n)G_{m,n}^{\lambda_1, \lambda_2}(y) + \lambda_2y[G_{m+2,n+2}^{\lambda_1, \lambda_2}(y) - G_{m,n+2}^{\lambda_1, \lambda_2}(y)] \end{aligned} \quad (30.11.16)$$

$$\begin{aligned} [n(1+y) - \lambda_2][G_{m,n+2}^{\lambda_1, \lambda_2}(y) - G_{m+2,n+2}^{\lambda_1, \lambda_2}(y)] &= (m+n)[G_{m+2,n}^{\lambda_1, \lambda_2}(y) - G_{m,n}^{\lambda_1, \lambda_2}(y)] \\ &\quad + \lambda_2(1+y)[G_{m+2,n}^{\lambda_1, \lambda_2}(y) - G_{m+2,n}^{\lambda_1, \lambda_2}(y)] \\ &\quad - \lambda_1[G_{m+2,n}^{\lambda_1, \lambda_2}(y) - G_{m+4,n}^{\lambda_1, \lambda_2}(y)] \end{aligned} \quad (30.11.17)$$

$$(m+2)[G_{m+2,n}^{\lambda_1,\lambda_2}(y) = (\lambda_1 - m - 2)G_{m+4,n}^{\lambda_1,\lambda_2}(y) - \lambda_1 G_{m+6,n}^{\lambda_1,\lambda_2}(y) \\ + y[nG_{m,n}^{\lambda_1,\lambda_2}(y) - nG_{m+2,n}^{\lambda_1,\lambda_2}(y) \\ - \lambda_2 G_{m,n}^{\lambda_1,\lambda_2}(y) - \lambda_2 G_{m+2,n}^{\lambda_1,\lambda_2}(y)]]$$
(30.11.18)

See sections 6.6.4.1 and 30.10.4.2 for the corresponding expressions for the central and singly noncentral F distribution.

### 30.11.5 Random Numbers

---

Function **DoublyNoncentralFDistRanEx**(*Size* As mpNum, *m* As mpNum, *n* As mpNum, *lambda1* As mpNum, *lambda2* As mpNum, *Generator* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function DoublyNoncentralFDistRanEx returns random numbers following a noncentral *F*-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*lambda1*: A real number greater 0, representing the numerator noncentrality parameter

*lambda2*: A real number greater 0, representing the denominator noncentrality parameter

*Generator*: A string describing the random generatorOutput? String? A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

Random numbers from a doubly non-central F-distribution are easily obtained using the definition in terms of the ratio between two independent random numbers from non-central chi-square distributions. This ought to be sufficient for most applications but if needed more efficient techniques may easily be developed e.g. using more general techniques.

### 30.11.6 Confidence Limits for the Noncentrality Parameters

---

Function **DoublyNoncentralFDistNoncentralityEx**(*alpha* As mpNum, *lambda1* As mpNum, *lambda2* As mpNum, *m* As mpNum, *n* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function DoublyNoncentralFDistNoncentralityEx returns confidence limits for the noncentrality parameter lambda and related information for the noncentral *F*-distribution.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*lambda1*: A real number greater 0, representing the numerator noncentrality parameter

*lambda2*: A real number greater 0, representing the denominator noncentrality parameter

*m*: A real number greater 0, representing the numerator degrees of freedom.

*n*: A real number greater 0, representing the denominator degrees of freedom.

*Output*: A string describing the output choices

See section 6.1.3.5 for the options for *alpha*, *Noncentrality* and *Output*. Algorithms and formulas are given below.

### 30.11.7 Sample Size

---

Function **DoublyNoncentralFDistSampleSizeEx**(*alpha* As mpNum, *beta* As mpNum, *m* As mpNum, *ModifiedNoncentrality1* As mpNum, *ModifiedNoncentrality1* As mpNum, *Output* As String) As mpNumList

---

NOT YET IMPLEMENTED

---

The function DoublyNoncentralFDistSampleSizeEx returns sample size estimates and related information for the noncentral *F*-distribution.

**Parameters:**

*alpha*: A real number between 0 and 1, specifies the confidence level (or Type I error).

*beta*: A real number between 0 and 1, specifies the Type II error (or 1 - Power).

*m*: A real number greater 0, representing the numerator degrees of freedom.

*ModifiedNoncentrality1*: A real number greater 0, representing the modified numerator noncentrality parameter.

*ModifiedNoncentrality1*: A real number greater 0, representing the modified denominator noncentrality parameter.

*Output*: A string describing the output choices

See section 6.1.3.4 for the options for *alpha*, *beta*, *ModifiedNoncentrality* and *Output*. Algorithms and formulas are given below.

## 30.12 Noncentral Distribution of Roy's Largest Root

### 30.12.1 Definition

Let  $X, Y$  denote two independent real Gaussian  $p \times n_1$  and  $p \times n_2$  matrices with  $n_1, n_2 \geq p$ , each constituted by zero mean independent, identically distributed columns with common covariance. The Roy's largest root criterion, used in multivariate analysis of variance (MANOVA), is based on the statistic of the largest eigenvalue,  $\Theta_1$ , of  $(A + B)^{-1}B$ , where  $A$  and  $B$  are independent central Wishart matrices. Throughout this section, we define

$$s = p, \quad m = (n_2 - p - 1)/2, \quad n = (n_1 - p - 1)/2 \quad (30.12.1)$$

See Chiani (2012), and Johnstone & Nadler (2013),

### 30.12.2 The exact distribution (null case).

Chiani (2014) proposes the following algorithm:

The CDF of the largest eigenvalue  $\Theta_1$  for a multivariate beta matrix in the null case is:

$$F_{\Theta_1}(\theta_1) = \Pr[\Theta_1 \leq \theta_1] = C \sqrt{|A(\theta_1)|}, \quad \text{where} \quad (30.12.2)$$

$$C = \pi^{s/2} \prod_{i=1}^s \frac{\Gamma\left(\frac{1}{2}(i+2m+2n+s+2)\right)}{\Gamma\left(\frac{i}{2}\right) \Gamma\left(\frac{1}{2}(i+2m+1)\right) \Gamma\left(\frac{1}{2}(i+2n+1)\right)} \quad (30.12.3)$$

When  $s$  is even, we have  $n_{mat} = s$  and the elements of the  $s \times s$  skew-symmetric matrix  $A(\theta_1)$  are:

$$a_{i,j}(\theta_1) = E(\theta_1; m+j, m+i) - E(\theta_1; m+i, m+j) \quad i, j = 1, \dots, s \quad (30.12.4)$$

where

$$E(x; a, b) = \int_0^x t^{a-1} (1-t)^n I(t; b, n+1) dt. \quad (30.12.5)$$

When  $s$  is odd, we have  $n_{mat} = s+1$  and the elements of the  $(s+1) \times (s+1)$  skew-symmetric matrix  $A(\theta_1)$  are as in equation (30.12.4), with the additional elements

$$a_{i,s+1}(\theta_1) = I(\theta_1; m+i, n+1) \quad i = 1, \dots, s \quad (30.12.6)$$

$$a_{s+1,j}(\theta_1) = -a_{j,s+1}(\theta_1) \quad j = 1, \dots, s \quad (30.12.7)$$

$$a_{s+1,s+1}(\theta_1) = 0. \quad (30.12.8)$$

Note that  $a_{i,j}(\theta_1) = -a_{j,i}(\theta_1)$  and  $a_{i,i}(\theta_1) = 0$ . To avoid the numerical integration in equation (30.12.5), we first observe that the incomplete beta functions can be computed iteratively by the relation

$$I(x; a+1, b) = I(x; a, b) - \frac{x^a (1-x)^b}{a+b}. \quad (30.12.9)$$

Then, from equations (30.12.4) and (30.12.5) the following identities can be easily verified:

$$E(x; a, a) = \frac{1}{2} I(x; a, n+1)^2 \quad (30.12.10)$$

$$E(x; a, b+1) = b \frac{E(x; a, b)}{b+n+1} - \frac{I(x; a+b, 2n+2)}{b+n+1} \quad (30.12.11)$$

$$E(x; b, a) = I(x; a, n+1) I(x; b, n+1) - E(x; a, b). \quad (30.12.12)$$

### 30.12.3 Approximation of the CDF(null case)

Chiani (2014) proposes the following algorithm:

$$F_{\Theta_1}(\theta_1) \approx P\left(k, \frac{\log(\theta_1/(1-\theta_1)) - \mu + \sigma\alpha}{\delta}\right) \quad (30.12.13)$$

and for its inverse, useful for evaluating the percentiles,

$$F_{\Theta_1}^{-1}(\theta_1) \approx \frac{\exp[\sigma(\delta P^{-1}(k, y) - \alpha)] + \mu}{1 + \exp[\sigma(\delta P^{-1}(k, y) - \alpha)] + \mu}, \quad (30.12.14)$$

where

$$\mu = 2 \log \tan\left(\frac{\gamma + \phi}{2}\right), \quad \sigma^3 = \frac{16}{(m+n+1)^2} \frac{1}{\sin^2(\gamma + \phi) \sin \gamma \sin \phi} \quad (30.12.15)$$

$$\gamma = \arccos\left(\frac{m+n-2p}{m+n-1}\right), \quad \phi = \arccos\left(\frac{m-n}{m+n-1}\right), \quad (30.12.16)$$

and the constants  $k = 46.446$ ,  $\delta = 0.186054$ ,  $\alpha = 9.84801$  have been chosen to match the moments of the approximation to that of the Tracy-Widom.

## 30.13 Noncentral Distribution of Wilks' Lambda

### 30.13.1 Definitions

Both Wilks  $U$  and Wilks  $\Lambda$  are in use, with  $\Lambda = U^{N/2}$ ,  $U = \Lambda^{2N}$ , and  $z = -2\rho(\Lambda)$ .

Literature: Krishnaiah et al (1971), Walster (1980), Fujikoshi, Pillai (1965), p. 408.

Edgeworth expansion: [Wakaki \(2006\)](#)

#### 30.13.1.1 General Linear Model

Suppose  $E$  is a  $p \times p$  matrix error sums of squares with a central Wishart <sub>$p$</sub> ( $n, \Sigma$ ) distribution. Let  $T = V^T V$  be the treatment sum of squares with a noncentral Wishart <sub>$p$</sub> ( $m, \Omega, \Sigma$ ) distribution with  $m$  degrees of freedom and noncentrality matrix  $\Omega = \Sigma^{-1} M^T M$ . This results when  $V$  is ( $m \times p$ ) with a Normal <sub>$m \times p$</sub> ( $M, \Sigma$ ) distribution, with mean  $e(V) = M$  and columns that are independent with common covariance  $\Sigma$ .

Let  $W_{GLM} = |E|/|T + E|$ .

See [O'Brien & Shieh \(1992\)](#)

See [Lee \(1972a\)](#)

See [Lee \(1971a\)](#)

See [Kulp & Nagarsenker \(1984\)](#)

Noncentral F approximations : [Muller et al. \(1992\)](#) , confidence bounds for power: [Taylor & Muller \(1995\)](#)

#### 30.13.1.2 Block independence

Suppose blocks of variables with dimensions  $p_1$  and  $p_2$  where  $p_1 + p_2 = p$ . Let  $A$  be the matrix of sample covariances with a Wishart <sub>$p$</sub> ( $n, \Sigma$ ) distribution. Specify  $A$  in block form as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (30.13.1)$$

where  $A_{ij}$  is  $p_i \times p_j$ . The likelihood ratio test for block independence rejects for small values of  $W_{CORR} = |A|/(|A_{11}| |A_{22}|)$ .

See [Lee \(1971b\)](#)

See also [Butler & Wood \(2005\)](#)

### 30.13.2 Density and CDF (Overview)

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Function **NoncentralWilksLambdaDist**(*x As mpNum, p As mpNum, m As mpNum, n As mpNum, Omega As mpNum, Output As String*) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **NoncentralWilksLambdaDist** returns pdf, CDF and related information for the non-central WilksLambda-distribution

#### Parameters:

*x*: A real number

*p*: An integer greater 0, representing the number of variates

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Omega*: An array of real numbers representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.1 for the options for *Output*. Algorithms and formulas are given below.

### 30.13.3 Density: Details

#### 30.13.3.1 Exact null distribution

$$\Pr[x \leq x_0] = f_{BetaProd}(x; b_{i=1\dots p}, c_{i=1\dots p}), \quad \text{where} \quad (30.13.2)$$

$$b_i = \frac{1}{2}(n_2 - i + 1); \quad c_i = \frac{1}{2}(n_1 + n_2 - i + 1); \quad i = 1 \dots p. \quad (30.13.3)$$

#### 30.13.3.2 Exact distribution for the noncentral linear case (General Linear Model)

Let  $\delta$  denote the noncentrality parameter in the linear case. Then (Tretter & Walster, 1975), Walster & Tretter (1980)

$$f_{\Lambda,GLM}(p, m, n; x; \delta) = e^{-\delta} \sum_{k=0}^{\infty} \frac{\delta_k}{k!} f_{BetaProd}(x_0; b_{i=1\dots p}, c_{i=1\dots p}), \quad \text{where} \quad (30.13.4)$$

$$b_i = \frac{1}{2}(n_2 - i + 1); \quad i = 1 \dots p. \quad (30.13.5)$$

$$c_1 = \frac{1}{2}(n_1 + n_2 + 2k); \quad c_i = \frac{1}{2}(n_1 + n_2 - i + 1); \quad i = 2 \dots p. \quad (30.13.6)$$

#### 30.13.3.3 Exact distribution for the noncentral linear case (Block Independence)

Let  $\delta$  denote the noncentrality parameter in the linear case. Then

$$f_{\Lambda,CORR}(p, m, n; x; \rho^2) = e^{-\delta} \sum_{k=0}^{\infty} \frac{\delta_k}{k!} f_{BetaProd}(x_0; b_{i=1\dots p}, c_{i=1\dots p}), \quad \text{where} \quad (30.13.7)$$

$$b_i = \frac{1}{2}(n_2 - i + 1); \quad i = 1 \dots p. \quad (30.13.8)$$

$$c_1 = \frac{1}{2}(n_1 + n_2 + 2k); \quad c_i = \frac{1}{2}(n_1 + n_2 - i + 1); \quad i = 2 \dots p. \quad (30.13.9)$$

### 30.13.4 CDF: Details

#### 30.13.4.1 Exact null distribution

$$\Pr[x \leq x_0] = F_{BetaProd}(x; b_{i=1\dots p}, c_{i=1\dots p}), \quad \text{where} \quad (30.13.10)$$

$$b_i = \frac{1}{2}(n_2 - i + 1); \quad c_i = \frac{1}{2}(n_1 + n_2 - i + 1); \quad i = 1 \dots p. \quad (30.13.11)$$

#### 30.13.4.2 Box Expansion (null distribution)

See section 29.6.4.3 for definition of the coefficients of the Box Expansion for Wilks U.

#### 30.13.4.3 Exact distribution for the noncentral linear case (General Linear Model)

Let  $\delta$  denote the noncentrality parameter in the linear case. Then

$$F_{\Lambda,GLM}(p, m, n; x; \delta) = e^{-\delta} \sum_{k=0}^{\infty} \frac{\delta_k}{k!} F_{BetaProd}(x_0; b_{i=1\dots p}, c_{i=1\dots p}), \quad \text{where} \quad (30.13.12)$$

$$b_i = \frac{1}{2}(n_2 - i + 1); \quad i = 1 \dots p. \quad (30.13.13)$$

$$c_1 = \frac{1}{2}(n_1 + n_2 + 2k); \quad c_i = \frac{1}{2}(n_1 + n_2 - i + 1); \quad i = 2 \dots p. \quad (30.13.14)$$

### 30.13.4.4 Exact distribution for the noncentral linear case (Block Independence)

Let  $\delta$  denote the noncentrality parameter in the linear case. Then

$$F_{\Lambda,CORR}(p, m, n; x; \rho^2) = e^{-\delta} \sum_{k=0}^{\infty} \frac{\delta_k}{k!} F_{BetaProd}(x_0; b_{i=1 \dots p}, c_{i=1 \dots p}), \quad \text{where} \quad (30.13.15)$$

$$b_i = \frac{1}{2}(n_2 - i + 1); \quad i = 1 \dots p. \quad (30.13.16)$$

$$c_1 = \frac{1}{2}(n_1 + n_2 + 2k); \quad c_i = \frac{1}{2}(n_1 + n_2 - i + 1); \quad i = 2 \dots p. \quad (30.13.17)$$

### 30.13.4.5 Approximation by non-central Chi-Square (General Linear Model)

Fujikoshi (1973) proposes the following approximation:

$$\begin{aligned} F_{\Lambda,GLM}(p, m, n; x; \Omega) &= F_{\chi^2}(pm, x; \omega_1) + \frac{1}{4n} \sum_{k=1}^3 a_k F_{\chi^2}(pm + 2k, x; \omega_1) \\ &\quad + \frac{1}{96n^2} \sum_{k=1}^6 b_k F_{\chi^2}(pm + 2k, x; \omega_1) \\ &\quad + \frac{1}{96n^3} \sum_{k=1}^9 c_k F_{\chi^2}(pm + 2k, x; \omega_1) + O(n^{-4}) \end{aligned} \quad (30.13.18)$$

$$\Omega = \Lambda^{-1} M M'$$

$$\omega_j = \text{tr } \Omega^j$$

$$m = n + (pq - 1)/2$$

$$x = m \log(x)$$

$$s = (p + q + 1)/4$$

$$r = f(p^2 + q^2 - 5)/48$$

$$a_1 = 2s\omega_1$$

$$a_2 = (2s\omega_1 - \omega_2)$$

$$a_3 = \omega_2$$

$$b_0 = r$$

$$b_1 = 0$$

$$b_2 = r4s^2\omega_1 + 2s^2\omega_1^2 + 2s\omega_2$$

$$b_3 = 4s^2\omega_1 - (1 + 4s^2)\omega_1^2 - (1 + 8s)\omega_2 + 2s\omega_1\omega_2 + (4/3)\omega_3$$

$$b_4 = (1 + 2s^2)\omega_1^2 + (1 + 6s)\omega_2 - 4s\omega_1\omega_2 - 4\omega_3 + \omega_2^2/2$$

$$b_5 = 2s\omega_1\omega_2 + (8/3)\omega_3 - \omega_2^2$$

$$b_6 = \omega_2^2/2$$

$$\begin{aligned}
c_1 &= 2rs\omega_1 & (30.13.19) \\
c_2 &= r(2s\omega_1\omega_2) \\
c_3 &= 2s(r+4s^2)\omega_1 + 2s(1+4s^2)\omega_1^2 + (-r+2s+12s^2)\omega_2 - \frac{4}{3}s^3\omega_1^3 - 4s^2\omega_1\omega_2 - \frac{8}{3}s\omega_3 \\
c_4 &= 2s(r+4s^2)\omega_1 - (1+10s+16s^3)\omega_1^2 - (3+r+10s+36s^2)\omega_2 + 2s(1+2s^2)\omega_1^3 \\
&\quad + 2(2+s+12s^2)\omega_1\omega_2 + 4(1+6s)\omega_3 - 2s^2\omega_1^2\omega_2 - 2s\omega_2^2 - \frac{8}{3}s\omega_1\omega_3 - 2\omega_4 \\
c_5 &= (1+8s+8s^3)\omega_1^2 + (3+r+8s+24s^2)\omega_2 - 4s(1+s^2)\omega_1^3 - 4(3+s+9s^2)\omega_1\omega_2 - 12(1+4s)\omega_3 \\
&\quad + (1+6s^2)\omega_1^2\omega_2 + (1+10s)\omega_2^2 + \frac{32}{3}s\omega_1\omega_3 + 12\omega_4 - \frac{4}{3}\omega_2\omega_3 - s\omega_1\omega_2^2 \\
c_6 &= s(2+\frac{4}{3}s^2)\omega_1^3 + 2(4+s+8s^2)\omega_1\omega_2 + 8(1+\frac{10}{3}s)\omega_3 - 2(1+3s^2)\omega_1^2\omega_2 - 2(1+7s)\omega_2^2 \\
&\quad - \frac{40}{3}s\omega_1\omega_3 - 20\omega_4 + \frac{16}{3}\omega_2\omega_3 + 3s\omega_1\omega_2^2 - \frac{1}{6}\omega_2^3 \\
c_7 &= (1+2s^2)\omega_1^2\omega_2 + (1+6s)\omega_2^2 + \frac{16}{3}s\omega_1\omega_3 + 10\omega_4 - \frac{20}{3}\omega_2\omega_3 - 3s\omega_1\omega_2^2 + \frac{1}{2}\omega_2^3 \\
c_8 &= \frac{8}{3}\omega_2\omega_3 + s\omega_1\omega_2^2 - \frac{1}{2}\omega_2^3 \\
c_9 &= \frac{1}{6}\omega_2^3
\end{aligned}$$

### 30.13.4.6 Approximation by non-central Chi-Square (Block Independence)

Lee (1971b) proposes the following approximation:

$$\begin{aligned}
F_{\Lambda,CORR}(p, m, n; x; P) &= F_{\chi^2}(pm, x; s1) + \frac{1}{4n} \sum_{k=1}^3 a_k F_{\chi^2}(pm + 2k, x; s1) \quad (30.13.20) \\
&\quad + \frac{1}{96n^2} \sum_{k=1}^6 b_k F_{\chi^2}(pm + 2k, x; s1) + O(n^{-3})
\end{aligned}$$

$$\Omega = \Lambda^{-1} M M'$$

$$\omega_j = \text{tr } \Omega^j$$

$$m = n + (pq - 1)/2$$

$$x = m \log(x)$$

$$s = (p + q + 1)/4$$

$$r = f(p^2 + q^2 - 5)/48$$

$$a_0 = -q\omega_1 + \omega_2$$

$$a_1 = (2s + q)\omega_1 - 2\omega_2$$

$$a_2 = -2s\omega_1 + 2\omega_2$$

$$a_3 = -\omega_2$$

$$b_0 = -r - ql\omega_1 + (q + l)\omega_2 + \frac{1}{2}q^2\omega_1^2 - \frac{4}{3}\omega_3 - q\omega_1\omega_2 + \frac{1}{2}\omega_2^2$$

$$b_1 = q^2\omega_1 - 4q\omega_2 - q(q + 2s)\omega_1^2 + 4\omega_3 + (3q + 2s)\omega_1\omega_2 - 2\omega_2^2$$

$$b_2 = r - 2s(q + 2s)\omega_1 + (2p + 6q + 3)\omega_2 + (\frac{1}{2}l^2 + 6qs + 1)\omega_1^2 - 8\omega_3 - (4q + 6s)\omega_1\omega_2 + 4\omega_2^2$$

$$b_3 = 4s^2\omega_1 - (3p + 5q + 5)\omega_2 - (4s^2 + 2qs + 2)\omega_1^2 + \frac{32}{3}\omega_3 + (3q + 8s)\omega_1\omega_2 - 5\omega_2^2$$

$$b_4 = (6s + 1)\omega_2 + (2s^2 + 1)\omega_1^2 - 8\omega_3 - (q + 6s)\omega_1\omega_2 + 4\omega_2^2$$

$$b_5 = \frac{8}{3}\omega_3 + 2s\omega_1\omega_2 - 2\omega_2^2$$

$$b_6 = \frac{1}{2}\omega_2^2$$

### 30.13.4.7 Saddlepoint approximation (Null distribution)

### 30.13.4.8 Saddlepoint approximation (General Linear Model)

Butler & Wood (2005) propose the following approximation

$$F_{\Lambda,GLM}(p, m, n; y; \Omega) \approx 1 - \Phi(r) + \phi(r)(u^{-1} - r^{-1}) \quad (30.13.21)$$

where  $\Phi(\cdot)$  and  $\phi(\cdot)$  denote the cdf (see section 6.11.2.2) and pdf (see section 6.11.2.1) of the normal distribution, respectively,  $r = \text{sgn}(s)\sqrt{2(sy - K(s))}$ ,  $u = s\sqrt{K''(s)}$ , and  $s$  is the solution to the saddlepoint equation

$$K'(s) = y. \quad (30.13.22)$$

$K(s)$  denotes the cumulant generating function and is given by

$$K(s) = \ln \left[ \frac{\Gamma_p(n/2 + s)\Gamma_p((n+m)/2)}{\Gamma_p(n/2)\Gamma_p((n+m)/2+s)} {}_1F_1 \left( s, \frac{1}{2}(n+m) + s, -\frac{1}{2}\Omega \right) \right] \quad (30.13.23)$$

and its first derivative,  $K'(s)$ , is given by

$$K'(s) = \sum_{i=1}^p \left[ \psi \left( \frac{1}{2}n + s - \frac{1}{2}(i-1) \right) - \psi \left( \frac{1}{2}(n+m) + s - \frac{1}{2}(i-1) \right) \right] \quad (30.13.24)$$

$$+ \frac{\partial}{\partial s} \ln \left[ {}_1F_1 \left( s, \frac{1}{2}(n+m) + s, -\frac{1}{2}\Omega \right) \right], \quad (30.13.25)$$

where  $\Gamma_p(\cdot)$  is the multivariate gamma function (see section 28.1.1),  $\psi(\cdot)$  is the digamma function (see section 15.7.2), and  ${}_1F_1(\cdot; \cdot, X)$  is the confluent hypergeometric function of matrix argument (see section 28.1.4). The saddlepoint equation (30.13.22) needs to be evaluated numerically. Also, the computation of  $K''(s)$  is performed using a numerical derivative of  $K'(s)$ .

### 30.13.4.9 Saddlepoint approximation (Block Independence)

Butler & Wood (2005) propose the following approximation

$$F_{\Lambda, \text{CORR}}(p, m, n; y; P) \approx 1 - \Phi(r) + \phi(r)(u^{-1} - r^{-1}) \quad (30.13.26)$$

where  $\Phi(\cdot)$  and  $\phi(\cdot)$  denote the cdf (see section 6.11.2.2) and pdf (see section 6.11.2.1) of the normal distribution, respectively,  $r = \text{sgn}(s)\sqrt{2(sy - K(s))}$ ,  $u = s\sqrt{K''(s)}$ , and  $s$  is the solution to the saddlepoint equation

$$K'(s) = y. \quad (30.13.27)$$

$K(s)$  denotes the cumulant generating function and is given by

$$K(s) = \ln \left[ \frac{\Gamma_{p_1}(n/2)\Gamma_{p_1}((n-p_2)/2+s)}{\Gamma_{p_1}(n/2+s)\Gamma_{p_1}((n-p_2)/2)} \times |I_{p_1} - P^2|^{n/2} {}_2F_1 \left( \frac{n}{2}, \frac{n}{2}; \frac{n}{2} + s; P^2 \right) \right] \quad (30.13.28)$$

and its first derivative,  $K'(s)$ , is given by

$$K'(s) = \sum_{i=1}^p \left[ \psi \left( \frac{1}{2}n + s - \frac{1}{2}(i-1) \right) - \psi \left( \frac{1}{2}(n-p_2) + s - \frac{1}{2}(i-1) \right) \right] \quad (30.13.29)$$

$$+ \frac{\partial}{\partial s} \ln \left[ |I_{p_1} - P^2|^{n/2} {}_2F_1 \left( \frac{n}{2}, \frac{n}{2}; \frac{n}{2} + s; P^2 \right) \right], \quad (30.13.30)$$

where  $\Gamma_p(\cdot)$  is the multivariate gamma function (see section 28.1.1),  $\psi(\cdot)$  is the digamma function (see section 15.7.2), and  ${}_2F_1(\cdot, \cdot, \cdot, X)$  is the Gauss hypergeometric function of matrix argument (see section 28.1.3). The saddlepoint equation (30.13.27) needs to be evaluated numerically. Also, the computation of  $K''(s)$  is performed using a numerical derivative of  $K'(s)$ .

### 30.13.5 Quantiles

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Function **NoncentralWilksLambdaDistInv**(*Prob* As mpNum, *p* As mpNum, *m* As mpNum, *n* As mpNum, *Omega* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function NoncentralWilksLambdaDistInv returns quantiles and related information for the the noncentral WilksLambda-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*p*: An integer greater 0, representing the number of variates

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Omega*: An array of real numbers representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

### 30.13.6 Properties

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Function **NoncentralWilksLambdaDistInfo**(*p* As mpNum, *m* As mpNum, *n* As mpNum, *Omega* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function NoncentralWilksLambdaDistInfo returns moments and related information for the noncentral WilksLambda-distribution

**Parameters:**

*p*: An integer greater 0, representing the number of variates

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Omega*: An array of real numbers representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

#### 30.13.6.1 Moments: Null distribution

#### 30.13.6.2 Moments: General Linear Model

The noncentral moments of  $W_{GLM}$  are specified in Theorem 10.5.1 of Muirhead (1982) as

$$E(W^s) = \frac{\Gamma_p(n/2 + s)\Gamma_p((n+m)/2)}{\Gamma_p(n/2)\Gamma_p((n+m)/2 + s)} {}_1F_1\left(s; \frac{n+m}{2} + s; -\frac{1}{2}\Omega\right) \quad (30.13.31)$$

where  $\Gamma_p(\cdot)$  is the multivariate gamma function (see section 28.1.1) and  ${}_1F_1(\cdot, \cdot, X)$  is the confluent hypergeometric function of matrix argument (see section 28.1.4). See also Butler & Wood (2002).

### 30.13.6.3 Moments: Block Independence

The noncentral moments of  $W_{CORR}$  are specified in Theorem 11.2.6 of Muirhead (1982) as

$$E(W^s) = \frac{\Gamma_{p_1}(n/2)\Gamma_{p_1}((n-p_2)/2+s)}{\Gamma_{p_1}(n/2+s)\Gamma_{p_1}((n-p_2)/2)} \times |I_{p_1} - P^2|^{n/2} {}_2F_1\left(\frac{n}{2}, \frac{n}{2}; \frac{n}{2} + s; P^2\right) \quad (30.13.32)$$

where  $P = \text{diag}\{\rho_1, \dots, \rho_{p_1}\}$  contains the population canonical correlations,  $\Gamma_p(\cdot)$  is the multivariate gamma function (see section 28.1.1) and  ${}_2F_1(\cdot, \cdot, \cdot, X)$  is the Gauss hypergeometric function of matrix argument (see section 28.1.3). See also Butler & Wood (2002).

## 30.13.7 Random Numbers

---

Function **NoncentralWilksLambdaDistRan**(*Size* As mpNum, *p* As mpNum, *m* As mpNum, *n* As mpNum, *Generator* As String, *Omega* As mpNum, *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **NoncentralWilksLambdaDistRan** returns random numbers following a noncentral WilksLambda-distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*p*: An integer greater 0, representing the number of variates

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Omega*: An array of real numbers representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below

### 30.13.7.1 Null distribution

### 30.13.7.2 General Linear Model

### 30.13.7.3 Block Independence

Lee (1971b) gives the following algorithm: The set of transformed canonical corelations  $(\tilde{r}_1^2, \dots, \tilde{r}_p^2)$  is distributed like the roots of the determinantal equation

$$\left| \lambda W - \left[ (\tilde{P}T + Z_1)(\tilde{P}T + Z_1)' + Z_2Z_2' \right] \right| = 0, \quad \text{where} \quad (30.13.33)$$

$$\tilde{r}_i^2 = \frac{r_i^2}{1 - r_i^2}, \quad \tilde{\rho}_i^2 = \frac{\rho_i^2}{1 - \rho_i^2}, \quad \tilde{P} = \text{diag}(\tilde{\rho}_1, \dots, \tilde{\rho}_p), \quad (30.13.34)$$

$$W \sim W_p(I, N - q), \quad T(p \times p) \text{ is such that } TT' \sim W_p(I, N), \quad (30.13.35)$$

and  $Z_1(p \times p)$  and  $Z_2(p \times (q-p))$  are matrices with independent normal variates as their elements. All matrix variates figuring in equation 30.13.33 are independently distributed.

When  $p = 1$ , this reduces to the case of multiple correlation, as given in equation 30.2.22.

## 30.14 Noncentral Distribution of Hotelling's T2

The Lawley-Hotelling generalized  $T_0^2$  and Pillai's  $V$  statistic, defined respectively by

$$T_0^2 = \text{ntr}(AB^{-1}), \quad V = \text{ntr}(A(A+B)^{-1}), \quad (30.14.1)$$

have been suggested as alternatives to Wilk's criterion for testing multivariate linear hypotheses. Here  $A$  and  $B$  are independent  $p \times p$  Wishart matrices on  $q$  and  $n$  degrees of freedom respectively. Davis (1968, 1970b) has found the moments for both criteria and has given recurrence relations for them, together with a formula relating the moments for  $T_0^2$  and  $V$ .

### 30.14.1 Hotelling's T2, Central Moments

The moments of Hotelling's  $T$  exist up to the  $j$ th, where  $j$  is the largest integer such that  $j < \frac{1}{2}(n_2 - m + 1)$ , and can be determined as follows (Davis, 1968):

$$E(T^r) = (-1)^r r!(n_1 + n_2)! \sum_{k=0}^m \frac{l_{kr}}{(m + n_2 - k)!}, \quad \text{where} \quad (30.14.2)$$

$$\mathbf{l}_i = (l_{0i}, \dots, l_{mi})', \quad a_i = \frac{1}{2}(m - i)(n_2 - i), \quad (30.14.3)$$

$$\mathbf{l}_0 = \frac{n_2!}{(n_1 + n_2)!} (0, \dots, 0, 1)', \quad (30.14.4)$$

$$\mathbf{l}_r = \text{diag} \left( \frac{1}{(r - a_0)}, \dots, \frac{1}{(r - a_m)} \right) \sum_{s=0}^{r-1} \mathbf{l}_s, \quad (30.14.5)$$

### 30.14.2 Hotelling's T2, Exact distributions for p=1 and p=2

$$T2(1, n_1, n_2; x) = I \left( \frac{1}{2}n_1, \frac{1}{2}n_2; \frac{x}{1-x} \right) \quad (30.14.6)$$

$$\begin{aligned} T2(2, m, n, x) &= I \left( m - 1, n - 1, \frac{w}{2+w} \right) \\ &\quad - \frac{\sqrt{\pi} \Gamma(\frac{1}{2}(m+n-1))}{\Gamma(\frac{1}{2}m)\Gamma(\frac{1}{2}n)} (1+w)^{(1-n)/2} I \left( \frac{1}{2}(m-1), \frac{1}{2}(n-1); \frac{w^2}{(2+w)^2} \right) \end{aligned} \quad (30.14.7)$$

Lit.: Pillai & Young (1971)

### 30.14.3 Hotelling's T2, Approximation

Let  $m = (n_1 - p - 1)/2$  and  $n = (n_2 - p - 1)/2$ .

$$\mu_1 = p(2m + p + 1)/(2n)$$

$$\mu_2 = \mu_1(2m + 2n + p + 1)(2n + p)/[2n(n - 1)(2n + 1)]$$

$$\mu_3 = 2\mu_2(2m + n + p + 1)(n + p)/[n(n - 2)(n + 1)]$$

$$a = \frac{(2\mu_1 3\mu_2 + 3\mu_1^2 \mu_3 - 6\mu_1 \mu_2^2 - \mu_2 \mu_3)}{(\mu_2 \mu_3 + 4\mu_1 \mu_2^2 - \mu_1^2 \mu_3)}, \quad b = \frac{(a+1)(a+3) - \mu_1^2/\mu_2}{(a+1) - \mu_1^2/\mu_2} \quad (30.14.8)$$

$$K = \frac{\mu_1(b-a-2)}{(a+1)} \quad w = \frac{x}{x+K} \quad (30.14.9)$$

Then  $x = T^2/n_2$  is approximately distributed as  $I_w(a+1, b-a-1)$ .

### 30.14.4 Hotelling's T<sub>2</sub>, Noncentral distribution: GLM

' Fujikoshi (1974), Ann. Inst. Math. Statist., 26, p. 289

$$\begin{aligned} F_{T^2,GLM}(p, m, n; x; \Omega) &= F_{\chi^2}(pm, x; s1) + \frac{1}{4n} \sum_{k=0}^4 a_k F_{\chi^2}(pm + 2k, x; s1) \\ &\quad + \frac{1}{96n^2} \sum_{k=0}^8 b_k F_{\chi^2}(pm + 2k, x; s1) + O(n^{-3}) \end{aligned} \quad (30.14.10)$$

$$\begin{aligned} a_0 &= fg \\ a_1 &= -2g(f - 2\lambda_1) \\ a_2 &= fg - 8g\lambda_1 + 4\lambda_2 \\ a_3 &= 4(g\lambda_1 - 2\lambda_2) \\ a_4 &= 4\lambda_2 \\ b_0 &= fl_0 \\ b_1 &= l_1(f - 2\lambda_1) \\ b_2 &= fl_2 + 2(l_1 - 2l_2)\lambda_1 + 48g^2\lambda_1^2 + 24(f + 4)g\lambda_2 \\ b_3 &= fl_3 + 2(2l_2 - 3l_3)\lambda_1 - 192(g^2 + 1)\lambda_1^2 - 96((f + 8)g + 2)\lambda_2 + 96g\lambda_1\lambda_2 + 128\lambda_3 \\ b_4 &= fl_4 + 2(3l_3 - 4l_4)\lambda_1 + 96(3g^2 + 7)\lambda_1^2 + 48(3(f + 12)g + 14)\lambda_2 - 384g\lambda_1\lambda_2 - 768\lambda_3 + 48\lambda_2^2 \\ b_5 &= 8l_4\lambda_1 - 192(g^2 + 4)\lambda_1^2 - 96((f + 16)g + 8)\lambda_2 + 576g\lambda_1\lambda_2 + 1536\lambda_3 - 192\lambda_2^2 \\ b_6 &= 48(g^2 + 6)\lambda_1^2 + 24((f + 20)g + 12)\lambda_2 - 384g\lambda_1\lambda_2 - 1280\lambda_3 + 288\lambda_2^2 \\ b_7 &= 96g\lambda_1\lambda_2 + 384\lambda_3 - 192\lambda_2^2 \\ b_8 &= 48\lambda_2^2 \end{aligned}$$

### 30.14.5 Hotelling's T<sub>2</sub>, Noncentral distribution: CORR

$$\begin{aligned} F_{T^2,CORR}(p, m, n; x; P) &= F_{\chi^2}(pm, x; s1) + \frac{1}{4n} \sum_{k=0}^4 a_k F_{\chi^2}(pm + 2k, x; s1) \\ &\quad + \frac{1}{96n^2} \sum_{k=0}^8 b_k F_{\chi^2}(pm + 2k, x; s1) + O(n^{-3}) \end{aligned} \quad (30.14.11)$$

$$\begin{aligned} s_1 &= 2\lambda_1 \\ s_2 &= 4\lambda_2 \\ s_3 &= 8\lambda_3 \\ s_1^2 &= s_1 s_1 \\ s_2^2 &= s_2 s_2 \\ p1 &= p + 1 \\ p^3 &= p^2 p \\ p^4 &= p^3 p \\ q^3 &= q^2 q \\ q^4 &= q^3 q \\ h &= qp1 \\ H1 &= 2q + p1 \\ p^2 p &= p^2 + p \\ a_0 &= qp(q - p - 1) - 2qs_1 + s_2 \\ a_1 &= -2q^2 p + 4qs_1 - 2s_2 \\ a_2 &= qp(q + p + 1) - 2(2q + p + 1)s_1 + 2s_2 \\ a_3 &= 2(q + p + 1)s_1 - 2s_2 \end{aligned}$$

$$a_4 = s_2$$

$$b_0 = qp(3qp^3 - 2(3q^2 - 3q + 4)p^2 + 3(q^3 - 2q^2 + 5q - 4)p - 8q^2 + 12q + 4) - 12q^2p(q - p - 1)s_1 - 6q(p^2 - qp + p - 4)s_2 + 12q^2s_1^2 - 16s_3 - 12qs_1s_2 + 3s_2^2$$

$$b_1 = -12q^3p^2(q - p - 1) - 24q^2(p^2 - 2qp + p - 2)s_1 + 12q(p^2 - 2qp + p - 8)s_2 - 48q^2s_1^2 + 48s_3 + 48qs_1s_2 - 12s_2^2$$

$$b_2 = -6q^2p^4 - 12q^2p^3 + 18q^2(q^2 + 1)p^2 + 24q^2(2q + 1)p + 12q(p^3 + 2p^2 - 7(q^2 + 1)p - 16q - 8)s_1 - 6(qp^2 - (7q^2 - q + 8)p - 40q - 12)s_2 + 24(qp + 4q^2 + q + 1)s_1^2 - 12(p + 8q + 1)s_1s_2 - 96s_3 + 24s_2^2$$

$$b_3 = -(12q^3 + 16q)p^3 - (12q^4 + 12q^3 + 96q^2 + 48q)p^2 - (64q^3 + 96q^2 + 64q)p + 12(-qp^3 + (4q^2 - 2q + 4)p^2 + (7q^3 + 4q^2 + 31q + 12)p + 4(7q^2 + 8q + 4))s_1 - 48((q^2 + 3)p + 9q + 5)s_2 - 24(3qp + 5q^2 + 3q + 4)s_1^2 + 176s_3 + 12(3p + 11q + 3)s_1s_2 - 36s_2^2$$

$$b_4 = 3q^2p^4 + (6q^3 + 6q^2 + 24q)p^3 + (3q^4 + 6q^3 + 63q^2 + 60q)p^2 + (24q^3 + 60q^2 + 60q)p - 12(qp^3 + (5q^2 + 2q + 12)p^2 + (4q^3 + 5q^2 + 45q + 32)p + 4(6q^2 + 11q + 9))s_1 + 6(qp^2 + (7q^2 + q + 44)p + 88q + 76)s_2 + 12(p^2 + 2(4q + 1)p + 8q^2 + 8q + 17)s_1^2 - 12(4p + 11q + 4)s_1s_2 - 240s_3 + 42s_2^2$$

$$b_5 = (12qp^3 + 24(q^2 + q + 4)p^2 + 12(q^3 + 2q^2 + 21q + 20)p + 48(2q^2 + 5q + 5))s_1 - 12(qp^2 + (2q^2 + q + 24)p + 32q + 40)s_2 - 24(p^2 + (3q + 2)p + 2q^2 + 3q + 9)s_1^2 + 240s_3 + 48(p + 2q + 1)s_1s_2 - 36s_2^2$$

$$b_6 = (6qp^2 + 6(q^2 + q + 20)p + 120q + 192)s_2 + (12p^2 + 24(q + 1)p + 12(q^2 + 2q + 7))s_1^2 - 12(3p + 4q + 3)s_1s_2 - 160s_3 + 24s_2^2$$

$$b_7 = 48s_3 + 12(q + p + 1)s_1s_2 - 12s_2^2$$

$$b_8 = 3s_2^2$$

End If

## 30.15 Noncentral Distribution of Pillai's V

The Lawley-Hotelling generalized  $T_0^2$  and Pillai's  $V$  statistic, defined respectively by

$$T_0^2 = \text{ntr}(AB^{-1}), \quad V = \text{ntr}(A(A+B)^{-1}), \quad (30.15.1)$$

have been suggested as alternatives to Wilk's criterion for testing multivariate linear hypotheses. Here  $A$  and  $B$  are independent  $p \times p$  Wishart matrices on  $q$  and  $n$  degrees of freedom respectively. [Davis \(1968, 1970b\)](#) has found the moments for both criteria and has given recurrence relations for them, together with a formula relating the moments for  $T_0^2$  and  $V$ .

### 30.15.1 Pillai's V, Central Moments

The moments of Pillai's  $V$  all exist. They can be obtained from equation 30.14.2 by replacing  $n_2$  with  $p - n_1 - n_2 + 1$  and multiplying by  $(-1)^r$ , ( $r = 1, 2, \dots$ ).

Let  $m = (n_1 - p - 1)/2$  and  $n = (n_2 - p - 1)/2$ ,  $s = p$

$$\mu_1 = s(2m + s + 1)/(2(m + n + s + 1)) \quad (30.15.2)$$

$$\mu_2 = \frac{s(2m + s + 1)(2n + s + 1)(2m + 2n + s + 2)}{(4(m + n + s + 1)^2(m + n + s + 2)(2m + 2n + 2s + 1))} \quad (30.15.3)$$

$$m_1 = \frac{\mu_1}{p}, \quad m_2 = \frac{\mu_2}{p^2}, \quad a = \frac{m_1(m_1 - m_1^2 - m_2)}{m_2}, \quad b = \frac{a(1 - m_1)}{m_1}, \quad w = \frac{V}{pn_2} \quad (30.15.4)$$

Then  $w$  is approximately distributed as  $I_w(a, b)$ . Let  $u_\alpha$  be the  $\alpha$  quantile of a beta distribution with  $a$  and  $b$  degrees of freedom. Then  $V_\alpha \approx u_\alpha pn_2$ .

### 30.15.2 Pillai's V, Special cases

$$V(1, n_1, n_2, x) = I(\frac{1}{2}n_1, \frac{1}{2}n_2; x)$$

pdf for  $p = 2$  (Davis 1970):

$$f_{n_1, n_2}(V) = \frac{(\frac{1}{2}V)^{n_1-1}(1 - \frac{1}{2}V)^{n_2-3}}{2B(n_1, n_2 - 1)} {}_2F_1\left(1, \frac{1}{2}(3 - n_2); \frac{1}{2}(n_1 + 1); r^2\right), \quad (0 < V < 1). \quad (30.15.5)$$

$$f_{n_1, n_2}(V) = \frac{(\frac{1}{2}V)^{n_1-3}(1 - \frac{1}{2}V)^{n_2-1}}{2B(n_2, n_1 - 1)} {}_2F_1\left(1, \frac{1}{2}(3 - n_1); \frac{1}{2}(n_2 + 1); r^{-2}\right), \quad (1 < V < 2). \quad (30.15.6)$$

where  $r = V/(2 - V)$ . These functions reduce to polynomials in  $V$  for odd  $n_2 \geq 3$  and odd  $n_1 \geq 3$ , respectively.

### 30.15.3 Pillai's V, Other Properties

$$V(p, n_1, n_2; x) = V(n_1, p, n_2 - p; x)$$

$$V(p, n_1, n_2; x) = 1 - V(p, n_2, n_1; p - x)$$

$$f_{n_1, n_2}(V) = f_{n_2, n_1}(m - V)$$

### 30.15.4 Pillai's V, Noncentral distribution: GLM

' Fujikoshi (1974), Ann. Inst. Math. Statist., 26, p. 289

$$\begin{aligned} F_{V,GLM}(p, m, n; x; \Omega) &= F_{\chi^2}(pm, x; s1) + \frac{1}{4n} \sum_{k=0}^4 a_k F_{\chi^2}(pm + 2k, x; s1) \\ &\quad + \frac{1}{96n^2} \sum_{k=0}^8 b_k F_{\chi^2}(pm + 2k, x; s1) + O(n^{-3}) \end{aligned} \quad (30.15.7)$$

$$l_0 = (3f - 8)g^2 + 4g + 4(f + 2)$$

$$l_1 = -12fg^2$$

$$l_2 = 6(3f + 8)g^2$$

$$l_3 = -4((3f + 16)g^2 + 4g + 4(f + 2))$$

$$l_4 = 3((f + 8)g^2 + 4g + 4(f + 2))$$

$$a_0 = -fg$$

$$a_1 = 2fg$$

$$a_2 = -fg + 4g\lambda_1 + 4\lambda_2$$

$$a_3 = -4g\lambda_1$$

$$a_4 = -4\lambda_2$$

$$b_0 = fl_0$$

$$b_1 = fl_1$$

$$b_2 = fl_2 + 2l_1\lambda_1 - 24fg\lambda_2$$

$$b_3 = fl_3 + 4l_2\lambda_1 + 48(f + 4)g\lambda_2 + 128\lambda_3$$

$$b_4 = fl_4 + 6l_3\lambda_1 + 48(g^2 - 2)\lambda_1^2 - 96(g + 1)\lambda_2 + 96g\lambda_1\lambda_2 + 48\lambda_2^2$$

$$b_5 = 8(l_4\lambda_1 - 12(g^2 + 2)\lambda_1^2 - 6((f + 12)g + 4)\lambda_2 - 12g\lambda_1\lambda_2 - 48\lambda_3)$$

$$b_6 = 8(6(g^2 + 6)\lambda_1^2 + 3((f + 20)g + 12)\lambda_2 - 12g\lambda_1\lambda_2 - 16\lambda_3 - 12\lambda_2^2)$$

$$b_7 = 96(g\lambda_1\lambda_2 + 4\lambda_3)$$

$$b_8 = 48\lambda_2^2$$

### 30.15.5 Pillai's V, Noncentral distribution: CORR

$$\begin{aligned} F_{V,CORR}(p, m, n; x; P) &= F_{\chi^2}(pm, x; s1) + \frac{1}{4n} \sum_{k=0}^4 a_k F_{\chi^2}(pm + 2k, x; s1) \\ &\quad + \frac{1}{96n^2} \sum_{k=0}^8 b_k F_{\chi^2}(pm + 2k, x; s1) + O(n^{-3}) \end{aligned} \quad (30.15.8)$$

$$a_0 = -fg - 4\lambda_2$$

$$a_1 = 2fg$$

$$a_2 = -fg + 4g\lambda_1 + 8\lambda_2$$

$$a_3 = -4g\lambda_1$$

$$a_4 = -4\lambda_2 \quad b_0 = fl_0 + 24fg\lambda_2 - 128\lambda_3 + 48\lambda_2^2$$

$$b_1 = fl_1 - 48fg\lambda_2$$

$$b_2 = fl_2 + 2l_1\lambda_1 + 96\lambda_1^2 - 24(qp^2 + q(q + 1)p - 4)\lambda_2 - 96g\lambda_1\lambda_2 - 192\lambda_2^2$$

$$b_3 = fl_3 + 4l_2\lambda_1 + 96(qp^2 + (q^2 + q + 4)p + 4(q + 1))\lambda_2 + 96g\lambda_1\lambda_2 + 640\lambda_3$$

$$b_4 = fl_4 + 6l_3\lambda_1 + 48(p^2 + 2(q + 1)p + q^2 + 2q - 3)\lambda_1^2 - 24(qp^2 + (q^2 + q + 12)p + 4(3q + 5))\lambda_2 + 192g\lambda_1\lambda_2 + 288\lambda_2^2$$

$$b_5 = 8l_4\lambda_1 - 96(p^2 + 2(q + 1)p + q^2 + 2q + 3)\lambda_1^2 - 48(qp^2 + (q^2 + q + 12)p + 4(3q + 5))\lambda_2 - 192g\lambda_1\lambda_2 - 768\lambda_3$$

$$\begin{aligned}b_6 &= 48(p^2 + 2(q+1)p + q^2 + 2q + 7)\lambda_1^2 + 24(qp^2 + (q^2 + q + 20)p + 4(5q + 8))\lambda_2 - 96g\lambda_1\lambda_2 - 128\lambda_3 - 192\lambda_2^2 \\b_7 &= 96(g\lambda_1\lambda_2 + 4\lambda_3) \\b_8 &= 48\lambda_2^2\end{aligned}$$

## 30.16 Noncentral Distribution of Bartlett's M (2 samples)

### 30.16.1 Definition

Let  $x_1, \dots, x_{N_1}$  be a random sample from a  $N_p(\mu_1, \Sigma_1)$  population and let  $y_1, \dots, y_{N_2}$  be a random sample from a  $N_p(\mu_2, \Sigma_2)$  population. The null hypothesis of equal covariance matrices (ECM) may be written

$$H_{ECM} : \Sigma_1 = \Sigma_2 = \Sigma, \quad (30.16.1)$$

with  $\Sigma, \mu_1, \mu_2$  unrestricted. The Bartlett  $M$  test (or the modified likelihood ratio test) rejects  $H_{ECM}$  for small values of

$$\Lambda_{ECM} = \frac{|A_1^{n_1/n}| |A_2^{n_2/n}|}{|A_1 + A_2|}, \quad \text{where} \quad (30.16.2)$$

$$n_i = N_i - 1, n = n_1 + n_2, \quad A_1 = \sum_{i=1}^{N_1} (x_i - \bar{x})(x_i - \bar{x})^T, \quad A_2 = \sum_{i=1}^{N_2} (y_i - \bar{y})(y_i - \bar{y})^T, \quad (30.16.3)$$

and  $\bar{x}$  and  $\bar{y}$  are the respective sample means. See [Muirhead \(1982\)](#).

The non-central distribution of  $\Lambda_{ECM}$  (i.e. the distribution of  $\Lambda_{ECM}$  when  $\Sigma_1 \neq \Sigma_2$ ) is determined by the quantities  $p, n_1, n_2$ , and  $\Delta = \text{diag}(\delta_1, \dots, \delta_p)$  where  $\delta_1, \dots, \delta_p$  are the eigenvalues of  $\Sigma_1 \Sigma_2^{-1}$ . See Subrahmaniam (1975), Pillai and Nagarsenker (1972) and Manoukian (1986).

### 30.16.2 Density and CDF

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Function **NoncentralBartlettsMDist**(*x As mpNum, p As mpNum, m As mpNum, n As mpNum, Omega As mpNum, Output As String*) As mpNumList

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**NOT YET IMPLEMENTED**

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The function **NoncentralBartlettsMDist** returns pdf, CDF and related information for the noncentral WilksLambda(CORR)-distribution

#### Parameters:

*x*: A real number

*p*: An integer greater 0, representing the number of variates

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Omega*: An array of real numbers representing the noncentrality parameter

*Output*: A string describing the output choices

See section [6.1.3.1](#) for the options for *Output*. Algorithms and formulas are given below.

#### 30.16.2.1 Saddlepoint approximation

[Butler & Wood \(2005\)](#) propose the following approximation

$$F_{\Lambda, ECM}(p, m, n; y; P) \approx 1 - \Phi(r) + \phi(r)(u^{-1} - r^{-1}) \quad (30.16.4)$$

where  $\Phi(\cdot)$  and  $\phi(\cdot)$  denote the cdf (see section 6.11.2.2) and pdf (see section 6.11.2.1) of the normal distribution, respectively,  $r = \text{sgn}(s)\sqrt{2(sy - K(s))}$ ,  $u = s\sqrt{K''(s)}$ , and  $s$  is the solution to the saddlepoint equation

$$K'(s) = y. \quad (30.16.5)$$

$K(s)$  denotes the cumulant generating function and is given by

$$K(s) = \ln \left[ \frac{\Gamma_p(\frac{1}{2}n)\Gamma_p(\frac{1}{2}n_1(1+2s/n))\Gamma_p(\frac{1}{2}n_2(1+2s/n))}{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)\Gamma_p(\frac{1}{2}n(1+2s/n))} \right] \quad (30.16.6)$$

$$+ \ln \left[ |\Delta|^{n_1 s / n_2} {}_2F_1 \left( s, \frac{n_1}{2} \left( 1 + \frac{2s}{n} \right); \frac{n}{2} \left( 1 + \frac{2s}{n} \right); I_p - \Delta \right) \right] \quad (30.16.7)$$

and its first derivative,  $K'(s)$ , is given by

$$K'(s) = \sum_{i=1}^p \left[ \psi \left( \frac{1}{2}n + s - \frac{1}{2}(i-1) \right) - \psi \left( \frac{1}{2}(n+m) + s - \frac{1}{2}(i-1) \right) \right] \quad (30.16.8)$$

$$+ \frac{\partial}{\partial s} \ln \left[ |\Delta|^{n_1 s / n_2} {}_2F_1 \left( s, \frac{n_1}{2} \left( 1 + \frac{2s}{n} \right); \frac{n}{2} \left( 1 + \frac{2s}{n} \right); I_p - \Delta \right) \right] \quad (30.16.9)$$

where  $\Gamma_p(\cdot)$  is the multivariate gamma function (see section 28.1.1),  $\psi(\cdot)$  is the digamma function (see section 15.7.2), and  ${}_2F_1(\cdot, \cdot, \cdot, X)$  is the Gauss hypergeometric function of matrix argument (see section 28.1.3). The saddlepoint equation (30.13.27) needs to be evaluated numerically. Also, the computation of  $K''(s)$  is performed using a numerical derivative of  $K'(s)$ .

### 30.16.3 Quantiles

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Function **NoncentralBartlettsMDistInv**(*Prob* As mpNum, *p* As mpNum, *m* As mpNum, *n* As mpNum, *Omega* As mpNum, *Output* As String) As mpNumList

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NOT YET IMPLEMENTED

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The function NoncentralBartlettsMDistInv returns quantiles and related information for the noncentral WilksLambda(CORR)-distribution

**Parameters:**

*Prob*: A real number between 0 and 1.

*p*: An integer greater than 0, representing the number of variates

*m*: A real number greater than 0, representing the numerator degrees of freedom

*n*: A real number greater than 0, representing the denominator degrees of freedom

*Omega*: An array of real numbers representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.2 for the options for *Prob* and *Output*). Algorithms and formulas are given below.

### 30.16.4 Properties

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Function **NoncentralBartlettsMInfo**(*p* As mpNum, *m* As mpNum, *n* As mpNum, *Omega* As mpNum, *Output* As String) As mpNumList

---

NOT YET IMPLEMENTED

---

The function NoncentralBartlettsMInfo returns moments and related information for the noncentral WilksLambda(CORR)-distribution

**Parameters:**

*p*: An integer greater 0, representing the number of variates

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Omega*: An array of real numbers representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.3 for the options for *Output*. Algorithms and formulas are given below.

### 30.16.4.1 Moments

The noncentral moments of  $W_{CORR}$  are specified in Theorem 11.2.6 of Muirhead (1982) as

$$E(W^s) = \frac{\Gamma_{p_1}(n/2)\Gamma_{p_1}((n-p_2)/2+s)}{\Gamma_{p_1}(n/2+s)\Gamma_{p_1}((n-p_2)/2)} \times |I_{p_1} - P^2|^{n/2} {}_2F_1\left(\frac{n}{2}, \frac{n}{2}; \frac{n}{2} + s; P^2\right) \quad (30.16.10)$$

where  $P = \text{diag}\{\rho_1, \dots, \rho_{p_1}\}$  contains the population canonical correlations,  $\Gamma_p(\cdot)$  is the multivariate gamma function (see section 28.1.1) and  ${}_2F_1(\cdot, \cdot, \cdot, X)$  is the Gauss hypergeometric function of matrix argument (see section 28.1.3). See also Butler & Wood (2002).

### 30.16.5 Random Numbers

---

Function **NoncentralBartlettsMDistRan**(*Size* As mpNum, *p* As mpNum, *m* As mpNum, *n* As mpNum, *Generator* As String, *Omega* As mpNum, *Output* As String) As mpNumList

---

NOT YET IMPLEMENTED

---

The function NoncentralBartlettsMDistRan returns random numbers following a noncentral WilksLambda(CO) distribution

**Parameters:**

*Size*: A positive integer up to  $10^7$

*p*: An integer greater 0, representing the number of variates

*m*: A real number greater 0, representing the numerator degrees of freedom

*n*: A real number greater 0, representing the denominator degrees of freedom

*Generator*: A string describing the random generator

*Omega*: An array of real numbers representing the noncentrality parameter

*Output*: A string describing the output choices

See section 6.1.3.6 for the options for *Size*, *Generator* and *Output*. Algorithms and formulas are given below.

# Chapter 31

## Examples: Discrete Distribution Functions

## 31.1 Noncentral Distribution of Mann-Whitney's U (with Stratification)

### 31.1.1 Definition

See [Wang et al. \(2003\)](#)

See [Mehrotra et al. \(2010\)](#)

See [Divine et al. \(2010\)](#)

See [Zhao \(2006\)](#)

See [Tang \(2011\)](#)

See [Zhao et al. \(2008\)](#)

Although Mann and Whitney developed the MW test under the assumption of continuous responses with the alternative hypothesis being that one distribution is stochastically greater than the other, there are many other ways to formulate the null and alternative hypotheses such that the MW test will give a valid test.

A very general formulation is to assume that:

1. All the observations from both groups are independent of each other,
2. The responses are ordinal (i.e. one can at least say, of any two observations, which is the greater),
3. The distributions of both groups are equal under the null hypothesis, so that the probability of an observation from one population (X) exceeding an observation from the second population (Y) equals the probability of an observation from Y exceeding an observation from X. That is, there is a symmetry between populations with respect to probability of random drawing of a larger observation.
4. Under the alternative hypothesis, the probability of an observation from one population (X) exceeding an observation from the second population (Y) (after exclusion of ties) is not equal to 0.5. The alternative may also be stated in terms of a one-sided test, for example:  $P(X > Y) + 0.5P(X = Y) > 0.5$ .

Let  $x_1, \dots, x_m$  and  $y_1, \dots, y_n$  be two sets of measurements, which we denote by  $X$  and  $Y$ . The test criterion  $U$  of the Mann-Whitney test is then

$$U = \sum_{i=1}^m \sum_{j=1}^n \text{sgn}(x_i - y_j) \quad (31.1.1)$$

### 31.1.2 Central distribution

#### 31.1.2.1 Exact Central distribution

The null distribution of the MW test can be calculated as follows: Let  $p_{n,m}(u)$  denote the probability that  $U = u$  in samples of size  $n$  and  $m$ . Then ([Zimmermann, 1985b](#))

$$(m+n)p_{n,m}(u) = np_{n-1,m}(u-m) + mp_{n,m-1}(u) = np_{n-1,m}(u) + mp_{n,m-1}(u-n) \quad (31.1.2)$$

#### 31.1.2.2 Cumulants

The cumulants of  $U$  are given by ([Robillard, 1972](#)):

$$\begin{aligned} \kappa_{2j} &= \frac{B_{2j}}{2j} \left[ \sum_{s=m+1}^{m+n} s^{2j} - \sum_{s=1}^n s^{2j} \right] \\ &= \frac{B_{2j}}{2j(2j+1)} [B_{2j+1}(n+m+1) + B_{2j+1} - B_{2j+1}(m+1) - B_{2j+1}(n+1)] \end{aligned} \quad (31.1.3)$$

and  $\kappa_{2j+1} = 0$ ,  $j \geq 1$ , and  $B_{2j}$  and  $B_{2j}(x)$  are the Bernoulli numbers and polynomials, respectively, of degree  $2j$  (see section 16.2).

### 31.1.3 Confidence interval for a location parameter

Let  $(D_1, \dots, D_{nm})$  be the ordered statistics of all differences  $x_i - y_j$  from samples of size  $m$  and  $n$ . The median of the  $D_i$  is an estimator of the location parameter  $\theta$  of the shift alternative  $F_1(x) = F_0(x - \theta)$ . A confidence interval for  $\theta$  can be calculated from  $\Pr(D_{r+1} < \theta < D_{nm-r}) \geq 1 - \alpha$ , where  $r$  is the  $\alpha/2$  quantile of the distribution of  $U$ .

### 31.1.4 Noncentral distribution

The calculation of the exact noncentral distribution of  $p = U/(mn)$  requires that either  $F_0$  (the distribution of  $X$ ) and  $F_1$  (the distribution of  $Y$ ) are explicitly known, or that  $F_1$  can be expressed as function of  $F_0$ .

#### 31.1.4.1 Normal alternatives

Consider the situation where random variables  $X_1, \dots, X_m$  and  $Y_1, \dots, Y_n$  are normally distributed with means  $\mu_X$  and  $\mu_Y$ , respectively, and common variance  $\sigma^2$ , all  $m+n$  random variables being mutually independent and  $d = (\mu_Y - \mu_X)/\sigma$ .

Let  $\mathbf{U} = (U_1, \dots, U_{m+n})$ ,  $U_1 < \dots < U_{m+n}$ , denote the order statistics of the random variables  $(X_1, \dots, X_m)$ ,  $(Y_1, \dots, Y_n)$ , and let  $\mathbf{Z} = (Z_1, \dots, Z_{m+n})$  denote a random vector of zeros and ones, where the  $i$ th component  $Z_i$  is 0 (or 1) if  $U_i$  is an  $X$  (or  $Y$ ). Denote by  $\phi(x - \theta)$  the normal density with mean  $\theta$  and variance 1 (see section 6.11.2.1). If  $\mathbf{z} = (z_1, \dots, z_{m+n})$  is a fixed vector of zeros and ones, the probability of the rank order  $z$ ,  $\Pr[\mathbf{Z} = \mathbf{z}]$ , is given by

$$P_{m,n}(\mathbf{z}|d) = m!n! \int_R \cdots \int \prod_{i=1}^{m+n} \phi(t_i - z_i d) dt_i, \quad (31.1.4)$$

where the region of integration  $R$  is  $-\infty < t_1 \leq t_2 \leq \dots \leq t_{m+n} < \infty$ . Milton (1970) describes a  $p$ -dimensional midpoint algorithm that economizes the number of arithmetic operations required to evaluate equation (31.1.4), corrects for effects of the edges of the region of integration and involves no high-degree quadrature formulas.

The distribution of  $U$  is obtained by calculating for all  $\binom{N}{m}$  vectors  $\mathbf{z} = (z_1, \dots, z_{m+n})$  the criterion  $U = t(\mathbf{z}) = \sum_{i=1}^{n+m} z_i r_i$  and summing up for each  $x$  the corresponding probabilities  $P_{m,n}(\mathbf{z})$ :

$$\Pr[U = x] = \sum_{t(z)=x} P_{m,n}(\mathbf{z}). \quad (31.1.5)$$

#### 31.1.4.2 Lehmann alternatives

Lehmann alternatives are of the form  $F_1(x) = [F_0(x)]^k$  or  $F_1(x) = 1 - [1 - F_0(x)]^k$ . The exact distribution under this alternative can be calculated using recurrence relations, similar to the null-distribution (Shorack, 1966): Let  $p_{n,m}(u)$  denote the probability that  $U = u$  in samples of size  $n$  and  $m$ . Then

$$(km + n)p_{n,m}(u) = np_{n-1,m}(u - m) + kmp_{n,m-1}(u) \quad \text{for } F_1 = (F_0)^k \quad (31.1.6)$$

$$(km + n)p_{n,m}(u) = np_{n-1,m}(u) + kmp_{n,m-1}(u - n) \quad \text{for } F_1 = 1 - (1 - F_0)^k \quad (31.1.7)$$

This recursive procedure allows the calculation of the exact noncentral distribution also for larger samples. Under Lehmann alternatives, rank order probabilities can be expressed in closed form for  $F_1 = (F_0)^k$ : let  $S_1, \dots, S_n$  denote the ranks of  $Y$  in the combined sample, e.g.  $\Pr[S_1 = 3, S_2 = 5] = P_{3,2}(0, 0, 1, 0, 1)$ . Then

$$\Pr[S_1 = s_1, \dots, S_n = s_n] = k^n \frac{n!m!}{\Gamma(n+m+1+n(k-1))} \prod_{j=1}^n \frac{\Gamma(s_j + j(k-1))}{\Gamma(s_j + (j-1)(k-1))}. \quad (31.1.8)$$

The first 2 Moments under the alternative  $F_1 = (F_0)^k$  are given by:

$$\mu_1 = k/(k+1) \quad (31.1.9)$$

$$\mu_2 = \frac{k}{mn(k+1)^2} \left[ \frac{m-1}{k+2} + \frac{k(n-1)}{2k+1} + 1 \right]. \quad (31.1.10)$$

### 31.1.4.3 First 4 moments under the general alternative

The first 4 moments under the alternative are given by (Sundrum, 1954):

$$\mu_1 = p \quad (31.1.11)$$

$$\mu_2 = (p^2 + p - q - r)/mn + (q - p^2)/n + (r - p^2)/m + p^2 \quad (31.1.12)$$

$$\begin{aligned} \mu_3 = & 6(p^3 + u - pq - pr)/mn + (2p^3 + s - 3pq)/n^2 + (2p^3 + t - 3pr)/m^2 \\ & + 3(3pq + 2(pr - u - p^3) + q - s - p^2)/mn^2 + 3(3pr + 2(pq - u - p^3) \\ & + r - t - p^2)/m^2n + (4p^3 + 3p^2 + p + 6u + 2(s + t) - 3(q + r)(1 + 2p))/m^2n^2 \end{aligned} \quad (31.1.13)$$

$$\begin{aligned} \mu_4 = & 3(q - p^2)2/n^2 + 6(q - p^2)(r - p^2)/mn + 3(r - p^2)2/m^2 \\ & + (12qp^2 + a - 4sp - 3q^2 - 6p^4)/n^3 + (12rp^2 + b - 4tp - 3r2 - 6p^4)/m^3 \\ & + 6(7(rp^2 - p^4) + 12qp^2 + qp + 2(w - sp) - 3(q^2 + qr) - 8up - p^3)/mn^2 \\ & + (42qp^2 + 72rp^2 + 6rp + 12x - 42p^4 - 18r2 - 18qr - 12tp - 48up - 6p^3)/m^2n \\ & + 6(6(p^4 - rp^2) + 3(q^2 - qp) - 12qp^2 + 4sp + 8up + 2(p^3 + qr - w) + s - a)/mn^3 \\ & + (36p^4 + 18r2 + 12qr - 72rp^2 - 36qp^2 + 24tp - 6b + 48up \\ & - 12x + 12p^3 - 18rp + 6t)/m^3n \\ & + (105p^4 + 42p^3 + 3p^2 + 33q^2 + 33r2 + 54qr - 174qp^2 - 174rp^2 - 42pq \\ & - 42pr + 36sp + 36tp + 192up - 36w - 36x + 6v + 36u)/m^2n^2 \\ & + (132qp^2 + 108rp^2 - 66p^4 - 33q^2 - 36qr - 18r2 - 44sp - 24tp + 11a \\ & - 144up + 36w + 24x - 6v - 36p^3 - 36u - 7p^2 + 54pq + 36pr - 18s + 7q)/m^2n^3 \\ & + (132rp^2 + 108qp^2 - 66p^4 - 33r2 - 36qr - 18q^2 - 44tp - 24sp + 11b \\ & - 144up + 24w + 36x - 6v - 36p^3 - 36u - 7p^2 + 54pr + 36pq - 18t + 7r)/m^3n^2 \\ & + (6(3(q^2 + r2) - 12((q + r)p^2) + 4(p^3 + qr + sp + tp - w - x) - (a + b - v) + 16up \\ & - 6(p^4 + pq + pr + u) + 2(s + t)) - 7(q + r - p^2) + p)/m^3n^3 \end{aligned} \quad (31.1.14)$$

The parameters  $p, q, r, s, t, v, u, a, b, w$  and  $x$  can be calculated from the following rank order probabilities:

$$\begin{aligned}
 p &= P_{1,1}(0, 0) \\
 q &= P_{2,1}(0, 0, 1) \\
 r &= P_{1,2}(0, 1, 1) \\
 s &= P_{3,1}(0, 0, 0, 1) \\
 t &= P_{1,3}(0, 1, 1, 1) \\
 v &= P_{2,2}(0, 0, 1, 1) \\
 u &= v + (1/4)P_{2,2}(0, 1, 0, 1) \\
 a &= P_{4,1}(0, 0, 0, 0, 1) \\
 b &= P_{1,4}(0, 1, 1, 1, 1) \\
 w &= 2a + (2/3)P_{3,2}(0, 0, 1, 0, 1) + (1/6)P_{3,2}(0, 1, 0, 0, 1) \\
 x &= 2b + (2/3)P_{2,3}(0, 1, 0, 1, 1) + (1/6)P_{2,3}(0, 1, 1, 0, 1)
 \end{aligned} \tag{31.1.15}$$

### 31.1.4.4 Estimation of rank order probabilities from the sample

The rank order probabilities which are required for the calculation of the first 4 moments can be estimated from the sample as follows: Let  $U_i$  be the number of  $Y$ 's in the sample greater than  $X_{(i)}$ , where  $X_{(i)}$  is the  $i$ th ordered values of the  $X$ 's amongst themselves. Then

$$P_{1,1}(0, 1) = \frac{1}{mn} \sum_{i=1}^m U_i \tag{31.1.16}$$

$$P_{2,1}(0, 0, 1) = \frac{2}{mn(m-1)} \sum_{i=1}^m (i-1)U_i \tag{31.1.17}$$

$$P_{3,1}(0, 0, 0, 1) = \frac{3}{mn(m-1)(m-2)} \sum_{i=1}^m (i-1)(i-2)U_i \tag{31.1.18}$$

$$P_{4,1}(0, 0, 0, 0, 1) = \frac{4}{mn(m-1)(m-2)(m-3)} \sum_{i=1}^m (i-1)(i-2)(i-3)U_i \tag{31.1.19}$$

$$P_{1,2}(0, 1, 1) = \frac{1}{mn(n-1)} \sum_{i=1}^m U_i(U_i - 1) \tag{31.1.20}$$

$$P_{1,3}(0, 1, 1, 1) = \frac{1}{mn(n-1)(n-2)} \sum_{i=1}^m U_i(U_i - 1)(U_i - 2) \tag{31.1.21}$$

$$P_{1,4}(0, 1, 1, 1, 1) = \frac{1}{mn(n-1)(n-2)(n-3)} \sum_{i=1}^m U_i(U_i - 1)(U_i - 2)(U_i - 3) \tag{31.1.22}$$

$$P_{2,2}(0, 0, 1, 1) = \frac{2}{mn(m-1)(n-1)} \sum_{i=1}^m (i-1)U_i(U_i - 1) \tag{31.1.23}$$

$$P_{3,2}(0, 0, 0, 1, 1) = \frac{3}{mn(m-1)(m-2)(n-1)} \sum_{i=1}^m (i-1)(i-2)U_i(U_i - 1) \tag{31.1.24}$$

$$P_{2,3}(0, 0, 1, 1, 1) = \frac{2}{mn(m-1)(n-1)(n-2)} \sum_{i=1}^m (i-1)U_i(U_i - 1)(U_i - 2) \tag{31.1.25}$$

$$P_{2,2}(0, 1, 0, 1) = \frac{4}{mn(m-1)(n-1)} \sum_{i=1}^m \sum_{j=i+1}^m (U_i - U_j) U_j \quad (31.1.26)$$

$$P_{3,2}(0, 0, 1, 0, 1) = \frac{12}{mn(m-1)(m-2)(n-1)} \sum_{i=1}^m \sum_{j=i+1}^m (i-1)(U_i - U_j) U_j \quad (31.1.27)$$

$$P_{3,2}(0, 1, 0, 0, 1) = \frac{12}{mn(m-1)(m-2)(n-1)} \sum_{i=1}^m \sum_{j=i+1}^m (i-j-1)(U_i - U_j) U_j \quad (31.1.28)$$

$$P_{2,3}(0, 1, 0, 1, 1) = \frac{6}{mn(m-1)(n-1)(n-2)} \sum_{i=1}^m \sum_{j=i+1}^m (U_i - U_j) U_j (U_j - 1) \quad (31.1.29)$$

$$P_{2,3}(0, 1, 0, 1, 1) = \frac{6}{mn(m-1)(n-1)(n-2)} \sum_{i=1}^m \sum_{j=i+1}^m (U_i - U_j) U_j (U_j - 1) \quad (31.1.30)$$

$$P_{2,3}(0, 1, 1, 0, 1) = \frac{6}{mn(m-1)(n-1)(n-2)} \sum_{i=1}^m \sum_{j=i+1}^m (U_i - U_j) (U_i - U_j - 1) U_j \quad (31.1.31)$$

### 31.1.4.5 Confidence interval for $\Pr(X > Y)$

If  $F_0$  and  $F_1$  are known, or if  $F_1$  is a function of  $F_0$ , then the (noncentral) distribution of  $U$  depends only on  $m, n$  and  $p$ , and we can write the pdf as  $f(i|m, n; p)$ . The positive solutions of the two equations

$$\sum_{i=0}^x f(i|m, n; p_U) = \alpha/2, \quad \text{and} \quad \sum_{i=x}^m f(i|m, n; p_L) = \alpha/2 \quad (31.1.32)$$

then form a confidence interval of  $p$  in the sense that

$$\Pr[p_L \leq p \leq p_U] \geq 1 - \alpha. \quad (31.1.33)$$

### 31.1.5 Power for Ordered Categorical Data

See [Kolassa \(1995\)](#)

## 31.2 Noncentral Distribution of Wilcoxon's Signed Rank Test (with Stratification)

See [Good \(2005\)](#)

See [Wang et al. \(2003\)](#)

See [Govindarajulu \(2007\)](#)

See [Shieh et al. \(2007\)](#)

See [Agresti \(2010\)](#)

### 31.2.1 Definition

We consider  $N$  continuously distributed random variables  $D_i, i = 1 \dots N$ , with common pdf  $h_0$ . In a sample  $(d_1, \dots, d_N)$  of size  $N$  let  $r_i$  be the rank of  $d_i$  in the ordered sample.

#### 31.2.1.1 The $T_N$ and $W_N$ test criteria

The test criterion of Wilcoxon's Signed Rank is  $T_N = \sum_{i=1}^N S(d_i)r_i$ , where  $S(d_i) = 1$  for  $x > 0$  and  $S(d_i) = 0$  for  $x < 0$ .  $T_N$  can assume values between 0 and  $\frac{1}{2}N(N + 1)$  in steps of 1.

Equivalent to  $T_N$  is the criterion  $W_N = \sum_{i=1}^N \text{sgn}(d_i)r_i$ .  $W_N$  can assume values between  $-\frac{1}{2}N(N + 1)$  and  $\frac{1}{2}N(N + 1)$  in steps of 2.

We have  $W_N = 2T_N - \frac{1}{2}N(N + 1)$  and  $T_N = (W_N + \frac{1}{2}N(N + 1))$ .

#### 31.2.1.2 The $\tilde{T}_N$ and $\tilde{W}_N$ test criteria

$T$  can also be written as  $T = T_1 + \tilde{T}_N$ ,  $T_1 = \sum_{i=1}^N S(d_i)$ ,  $\tilde{T}_N = \sum_{i=1}^{N-1} \sum_{j=i+1}^N S(d_i + d_j)$ .  $T_1$  is the test criterion of the Sign test (see section ...).

$\tilde{T}_N$  is the test criterion of the modified Signed Rank test.

$\tilde{T}_N$  can also be written as  $\tilde{T}_N = \sum_{i=1}^N s(d_i)(r_i - 1)$ .  $\tilde{T}_N$  can assume values between 0 and  $\frac{1}{2}N(N - 1)$  in steps of 1.

Equivalent to  $\tilde{T}_N$  is the criterion  $\tilde{W}_N = \sum_{i=1}^N \text{sgn}(d_i)(r_i - 1)$ .

$\tilde{W}_N$  can assume values between  $-\frac{1}{2}N(N - 1)$  and  $\frac{1}{2}N(N - 1)$  in steps of 2.

We have  $\tilde{W}_N = 2\tilde{T}_N - \frac{1}{2}N(N - 1)$  and  $\tilde{T}_N = \frac{1}{2}(\tilde{W}_N + \frac{1}{2}N(N - 1))$ .

#### 31.2.1.3 Parameters, expected values and hypotheses

We define  $p_1 = \Pr[D_1 > 0]$ ,  $p_2 = \Pr[D_1 + D_2 > 0]$ ,  $p_{XY} = \Pr[X > Y] - \Pr[X < Y]$ .

Here  $X$  and  $Y$  are random variables: the  $X$  are distributed like positive  $D$  with pdf  $g_0$ , and the  $Y$  like the absolute values of negative  $D$  with pdf  $g_1$  (see below).  $p_{XY}$  can assume values between  $-1$  and  $1$ . The following relationships hold ([Noether, 1967](#); [Noether & Dueker, 1990](#)):

$$p_1 = \frac{1}{2p_{XY}} \left( p_{XY} + 1 - \sqrt{(p_{XY} + 1)^2 - 4p_{XY}p_2} \right) \quad p_2 = (p_1 - p_1^2)p_{XY} + p_1, \quad p_{XY} = \frac{p_2 - p_1}{p_1 - p_1^2}. \quad (31.2.1)$$

Consistent estimators for  $p_1$  and  $p_2$  are  $\hat{p}_1 = T_1/N$  and  $\hat{p}_2 = \tilde{T}_N/(N(N - 1))$ .

The expected values of  $T_N$  and  $W_N$  are weighted sums of the parameters  $p_1$  and  $p_2$ :

$$e(T_N) = Np_1 + \frac{1}{2}N(N - 1)p_2,$$

$$e(W_N) = 2E(T_N) - \frac{1}{2}N(N + 1),$$

whereas the expected values of  $\tilde{T}_N$  and  $\tilde{W}_N$  only depend on  $p_2$ :

$$e(\tilde{T}_N) = \frac{1}{2}N(N - 1)p_2.$$

$$e(\tilde{W}_N) = 2E(\tilde{T}_N) - \frac{1}{2}N(N-1).$$

Under the null hypothesis we have:  $p_{XY} = 0; p_1 = p_2 = \frac{1}{2}; e(T_N) = \frac{1}{4}N(N+1)p_2; e(W_N) = 0$ . In the special case of a shift-hypothesis, assuming a normal distribution, i.e.  $h_0 = \phi(x - \delta)$ , we have

$$p_1 = \Phi(\delta), \quad p_2 = \Phi(\delta\sqrt{2}), \quad \text{and } p_{XY} = \frac{\Phi(\delta\sqrt{2}) - \Phi(\delta)}{\Phi(\delta)\Phi(-\delta)}. \quad (31.2.2)$$

The two-sided test is consistent against alternatives with  $|p_2 - \frac{1}{2}| \neq 0$ . In particular, the test is consistent against alternatives with a median of zero (i.e.  $p_1 = \frac{1}{2}$ ) if  $p_{XY} \neq 0$ , which implies a certain amount of asymmetry. If the distribution of the  $D$  is symmetric around  $c \neq 0$ , then  $|p_2 - \frac{1}{2}|c \neq 0$ , and a right-sided test is consistent against alternatives with  $c > 0$ , a left-sided against  $c < 0$ , and a two-sided against  $c \neq 0$  (see 31.2.2).

### 31.2.2 Central Density

Let  $p_N(w)$  denote the probability  $\Pr[W_N = w]$  in a sample of size  $N$ . Then the following recurrence relation holds (Zimmermann, 1985a) :

$$p_N(w) = \frac{1}{2} (p_{N-1}(w) + p_{N-1}(w-N)). \quad (31.2.3)$$

#### 31.2.2.1 Cumulants of $W_N$

The cumulants of  $W$  are given by (Fellingham & Stoker, 1964) :

$$\kappa_{2j}(W_N) = \frac{2^{2j}(2^{2j}-1)B_{2j}}{2j} \sum_{i=1}^N r_i^{2j} = \frac{2^{2j}(2^{2j}-1)B_{2j}}{2j} \frac{B_{2j+1}(N+1) - B_{2j+1}}{2j+1}, \quad \text{and} \quad (31.2.4)$$

$$\kappa_{2j+1}(W_N) = 0, \quad \text{for } j \geq 0. \quad (31.2.5)$$

In particular,  $\kappa_1(W_N) = N(N+1)/4$ , and  $\kappa_2(W_N) = N(N+1)(2N+1)/24$ , and  $B_{2j}$  and  $B_{2j}(x)$  are the Bernoulli numbers and polynomials, respectively, of degree  $2j$  (see section 16.2).

#### 31.2.2.2 Cumulants of $\tilde{W}_N$

The cumulants of  $\tilde{W}_N$  are given by

$$\kappa_{2j}(\tilde{W}_N) = \frac{2^{2j}(2^{2j}-1)B_{2j}}{2j} \sum_{i=1}^N (r_i - 1)^{2j} = \frac{2^{2j}(2^{2j}-1)B_{2j}}{2j} \frac{B_{2j+1}(N) - B_{2j+1}}{2j+1}, \quad \text{and} \quad (31.2.6)$$

$$\kappa_{2j+1}(\tilde{W}_N) = 0, \quad \text{for } j \geq 0. \quad (31.2.7)$$

In particular,  $\kappa_1(\tilde{W}_N) = N(N-1)/4$ , and  $\kappa_2(\tilde{W}_N) = N(N-1)(2N-1)/24$ , and  $B_{2j}$  and  $B_{2j}(x)$  are the Bernoulli numbers and polynomials, respectively, of degree  $2j$  (see section 16.2).

### 31.2.3 Confidence interval for the median

We assume that the differences  $D_i$  are distributed symmetrically around then median  $c$ . If all  $\frac{1}{2}N(N+1)$  values  $\frac{1}{2}(D_i + D_j)$  are ordered in the sample, and we denoted the ordered statistics as  $(D_1, \dots, D_{N(N+1)/2})$ , then a confidence interval for  $c$  is obtained as

$$\Pr[D_{r+1} < c < D_{N(N+1)/2} - r] \geq 1 - \alpha, \quad (31.2.8)$$

where  $r$  denotes the  $\alpha/2$  quantile of the distribution of  $W_N$ .

### 31.2.4 Noncentral distribution

We consider a sample of size of the random variable  $D$ , which can assume positive or negative values. This sample can be partitioned in  $N + 1$  different ways in sub-samples of size  $m$  (number of positive values) and  $N - m$  (number of negative values), where  $m = 0, \dots, N$ .

#### 31.2.4.1 Method 1

Let  $Z_i = \frac{1}{2}(\text{sgn}(X_i) + 1)$ . For each partition with  $m$  positive values there exist  $\binom{N}{m}$  different permutations  $Z_{N-m,m} = (Z_1, \dots, Z_N)$ , which represent a random vector of  $N - m$  zeros and  $m$  ones. We define

$$f_0(x) = h_0(x) \text{ for } x > 0, \text{ and } 0 \text{ otherwise.}$$

$$f_1(x) = h_0(-x) \text{ for } x < 0, \text{ and } 0 \text{ otherwise.}$$

Then for each fixed vector  $\mathbf{z} = (z_1, \dots, z_N)$  of zeros and ones, the (unconditional) probability of the rank order  $z$ ,  $\Pr[\mathbf{Z} = \mathbf{z}]$ , is given by

$$P_{N-m,m}(\mathbf{z}) = N! \int_R \cdots \int \prod_{i=1}^N f_{z_i}(t_i) dt_i, \quad (31.2.9)$$

where the region of integration  $R$  is  $-\infty < t_1 \leq t_2 \leq \cdots \leq t_N < \infty$  ([Milton, 1970](#)). See also [Klotz \(1963\)](#). The distribution of  $T_N$  is obtained by calculating for all  $\sum_{m=0}^N \binom{N}{m} = 2^N$  vectors  $\mathbf{z} = (z_1, \dots, z_N)$  the criterion  $T_N = t(\mathbf{z}) = \sum_{i=1}^N z_i r_i$  and summing up for each  $T$  the corresponding probabilities  $P_{N-m,m}(\mathbf{z})$ :

$$\Pr[T_N = x] = \sum_{t(z)=x} P_{N-m,m}(\mathbf{z}). \quad (31.2.10)$$

See [Arnold \(1965\)](#) and [Klotz \(1963\)](#).

#### 31.2.4.2 Method 2

The probability that in a sample of size  $N$  exactly  $m$  values are positive is

$$\Pr[M = m] = \binom{N}{m} p_1^m (1 - p_1)^{N-m}. \quad (31.2.11)$$

We define

$$g_0 = \frac{1}{p_1} f_0 \text{ and } g_1 = \frac{1}{1 - p_1} f_1, \text{ so that } \int_0^\infty g_0(x) dx = 1 \text{ and } \int_0^\infty g_1(x) dx = 1. \quad (31.2.12)$$

For a given number of  $m$  positive values the conditional probability  $\Pr[T_{N-m,m} = x]$  is the same as  $\Pr[U_{N-m,m} = x]$  of the Mann-Whitney U-statistic with sample sizes  $m$  and  $N - m$ , when using the distributions  $g_0$  and  $g_1$ . Since  $\Pr[M = m]$  and  $\Pr[T_{N-m,m} = x]$  are independent, the nonconditional distribution of  $T$  is then obtained as

$$\Pr[T_N = x] = \sum_{m=0}^N \Pr[M = m] \times \Pr[U_{N-m,m} = x]. \quad (31.2.13)$$

### 31.2.5 Moments of the noncentral distribution

For the raw moments of the noncentral distribution we have

$$\mu_r(T_N) = \sum_{m=0}^N \Pr[M = m] \times \mu_r(U_{N-m,m}). \quad (31.2.14)$$

This assumes using the distributions  $g_0$  and  $g_1$  for the calculation of the rank order probabilities. The positive values will be assigned to  $X$  and the negative to  $Y$  if the rank order probabilities are estimated from the sample.

### 31.2.6 Sample Size

See [Noether \(1987\)](#)

## 31.3 Noncentral Distribution of Kendall's Tau

### 31.3.1 Definition

See [Wang et al. \(2003\)](#)

Let  $(X_1, Y_1), \dots, (X_N, Y_N)$  be independent random variables, the  $X_i$  with a continuous distribution  $F_0$ , the  $Y_i$  with df  $G_0$ . Let  $R_i$  and  $S_i$  be the ranks of  $X_i$  and  $Y_i$ , respectively. Consider the Kendall rank correlation coefficient

$$\tau = \frac{1}{N(N-1)} \sum_{i=1}^N \sum_{j=1}^N \text{sgn}(R_i - R_j) \text{sgn}(S_i - S_j) \quad (31.3.1)$$

and its transformation  $T_N$  defined by

$$T_N = \frac{1}{4}(\tau + 1)N(N-1) \quad (31.3.2)$$

Then  $e(T_N) = N(N-1)/4$ ,  $\text{Var}(T_N) = N(N-1)(2N+5)/72$ , and the distribution of  $T_N$  is asymptotically normal.  $T_N$  can assume values between 0 and  $N(N-1)/2$ .

### 31.3.2 Central distribution

#### 31.3.2.1 Exact distribution

Let  $p_N(t) = \Pr[T_N = t]$ . Then the following recurrence relation holds:

$$p_N(t) = p_N(t-1) + [p_{N-1}(t) - p_{N-1}(t-N)]/N, \quad (31.3.3)$$

where  $p_N(t) = 0$  for  $t < 0$  or  $t > N(N-1)/2$ , and  $p_N(0) = 1/N!$ .

#### 31.3.2.2 Cumulants

The cumulants of  $T_N$  are given by

$$\kappa_{2j}(T_N) = \frac{B_{2j}}{2j} \sum_{s=1}^N s^{2j} = \frac{B_{2j}}{2j} \left[ \frac{B_{2j+1}(N+1) - B_{2j+1}}{2j+1} - N \right], \quad \text{and} \quad (31.3.4)$$

$$\kappa_{2j+1}(T_N) = 0, \quad \text{for } j \geq 1. \quad (31.3.5)$$

In particular,  $\kappa_1(T_N) = N(N-1)/4$ , and  $\kappa_2(T_N) = N(N-1)(2N+5)/72$ , and  $B_{2j}$  and  $B_{2j}(x)$  are the Bernoulli numbers and polynomials, respectively, of degree  $2j$  (see section [16.2](#)).

### 31.3.3 Noncentral distribution

#### 31.3.3.1 General case

Consider  $k$  pairs of random variables  $(x_1, y_1), \dots, (x_k, y_k)$ . Let  $r_1, \dots, r_k$  be a permutation of  $1, \dots, k$  and let  $s_1, \dots, s_k$  be its reciprocal. We define as in [Snow \(1962\)](#)

$P(r_1, \dots, r_k) = \Pr[\text{if } x\text{'s are ranked in order, ranks of corresponding } y\text{'s are } r_1, \dots, r_k]$ .

$P(r_1, \dots, r_k) = \Pr[\text{if } y\text{'s are ranked in order, ranks of corresponding } x\text{'s are } s_1, \dots, s_k]$ .

Let  $H(t) = 1$  if  $t > 0$  and  $H(t) = 0$  otherwise. Then

$$P(r_1, \dots, r_k) = k! P \left[ \prod_{i=2}^k H(x_i - x_{i-1}) H(y_{s_i} - y_{s_{i-1}}) \right] = k! P \left[ \prod_{i=2}^k H(y_i - y_{i-1}) H(x_{r_i} - x_{r_{i-1}}) \right] \quad (31.3.6)$$

### 31.3.3.2 Bivariate normal distribution

When  $(x_i, y_i)$ ,  $i = 1, 2, \dots, k$ , are normal variables with correlation  $\rho$ , then

$$P(r_1, \dots, r_k) = k!P(U_1 > 0, U_2 > 0, \dots, U_{2k-2} > 0), \quad (31.3.7)$$

where the  $U$ 's are normal variables defined below with zero mean and correlation matrix  $R$ :

$$R = \begin{pmatrix} A & \rho B \\ \rho B' & A \end{pmatrix} \quad (31.3.8)$$

$$A_{k-1,k-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \end{pmatrix} \quad (31.3.9)$$

$$B = (b_{ij}), \quad b_{i,j} = \frac{1}{2} (\delta_{r_{i+1}, j+1} - \delta_{r_{i+1}, j} - \delta_{r_i, j+1} - \delta_{r_i, j}) \quad (31.3.10)$$

### 31.3.3.3 First 4 moments in the general case

The first 4 moments of  $T_N$  are given as [Sundrum \(1953\)](#):

$$\mu_1 = p \quad (31.3.11)$$

$$\mu_2 = 2[p(1-p) + 2(k-p^2)(n-2)]/[n(n-1)] \quad (31.3.12)$$

$$\begin{aligned} \mu_3 = 4[(t - 18kp + 10p^3)n^2 + (6k + 2u - 6p^2 - 5t + 72kp - 34p^3)n \\ + (p + 9p^2 + 30p^3 - 12k - 4u + 6t - 72kp)]/[n(n-1)]^2 \end{aligned} \quad (31.3.13)$$

$$\begin{aligned} \mu_4 = 8[6(k-p^2)2n^4 + (6kp + 2y - 6p^3 - 16tp - 84k2 + 270kp^2 - 108p4)n^3 \\ + (1.5p^2 + 6t + 2b - 126kp - 24up - 18y + 75p^3 + 120tp + 426k2 - 1446kp^2 + 505.5p4)n^2 \\ + (14k + 12u - 15.5p^2 - 30t - 10b + 444kp + 96up + 52y - 213p^3 - 296tp - 924k2 \\ + 2988kp^2 - 943.5p4)n \\ + (p - 28k - 24u + 21p^2 + 36t + 12b - 432kp - 96up - 48y + 180p^3 + 240tp + 720k2 \\ - 2160kp^2 + 630p4)]/[n(n-1)]^3 \end{aligned} \quad (31.3.14)$$

The parameters  $p, k, u, t, b, y$  can be calculated from the following rank order probabilities, writing  $P_{(1423)}$  for the probability that the ranks of the  $Y$  occur in this order when the  $X$  are sorted in ascending order:

$$p = P_{(12)} \quad (31.3.15)$$

$$u = P_{(123)}$$

$$k = P_{(123)} + \frac{1}{4}P_{(132)}$$

$$t = 8(P_{(1234)} + P_{(1243)}) + 6P_{(1342)} + 4P_{(1324)} + 2P_{(2143)} + P_{(2413)} + P_{(1432)}$$

$$b = 15P_{(1234)} + 10P_{(1243)} + 5P_{(1324)} + 4P_{(1342)} + P_{(2143)}$$

$$\begin{aligned} y = (160P_{(12453)} + 150(P_{(12435)} + P_{(12354)}) + 125P_{(12345)} + 96P_{(21453)} + 90P_{(13254)} + 84P_{(13524)} \\ + 80P_{(13425)} + 64P_{(13452)} + 45P_{(21354)} + 40P_{(12543)} + 32(P_{(23514)} + P_{(14352)} + P_{(13542)}) \\ + 24P_{(21543)} + 22P_{(24153)} + 20P_{(14325)} + 18P_{(14523)} + 16P_{(25143)} + 12(P_{(24513)} + P_{(14532)}) \\ + 6(P_{(25314)} + P_{(15342)}) + 4P_{(25413)} + 2(P_{(15432)} + P_{(35142)}))/5 \end{aligned}$$

When  $(x_i, y_i)$ ,  $i = 1, 2, \dots, k$ , are normal variables with correlation  $\rho$ , then closed form expressions exist for  $p, k$  and  $u$  (writing  $a = \arcsin(\rho) / \pi$  and  $b = 2 \arcsin\left(\frac{1}{2}\rho\right) / \pi$ ):

$$\begin{aligned} p &= \frac{1}{2} + a \\ k &= \frac{1}{4}\left(\frac{10}{9} + 4a + 4a^2 - b^2\right) \\ u &= \frac{3}{8}\left(\frac{4}{9} + 4a - 2b + 4a^2 - b^2\right) \end{aligned} \tag{31.3.16}$$

The parameters  $t, b$  and  $y$  have to be evaluated by numerical intergration of the rank order probabilities listed in (31.3.15), as described in section 31.3.3.2.

## 31.4 Noncentral Distribution of Jonckheere-Terpsta's S

### 31.4.1 Definition

The Jonckheere-Terpstra test is a test for an ordered alternative hypothesis within an independent samples (between-participants) design. It is similar to the Kruskal-Wallis test in that the null hypothesis is that several independent samples are from the same population. However, with the Kruskal-Wallis test there is no a priori ordering of the populations from which the samples are drawn. When there is an a priori ordering, the Jonckheere test has more statistical power than the Kruskal-Wallis test.

The null and alternative hypotheses can be conveniently expressed in terms of population medians for  $k$  populations (where  $k > 2$ ). Letting  $\theta_i$  be the population median for the  $i$ th population, the null hypothesis is:

The alternative hypothesis is that the population medians have an a priori ordering e.g.: with at least one strict inequality.

### 31.4.2 Central distribution of Jonckheere-Terpsta's S

See [Murakami & Kamakura \(2009\)](#) for saddlepoint approximation.

See also [Neuhäuser & Hothorn \(1998\)](#)

Consider two vectors of scores  $X = (x_1, \dots, x_N)$  and  $Y = (y_1, \dots, y_N)$ . Let  $J_N$  denote Kendall's  $T_N$  statistic (see ...) with ties in  $X$ , let  $k$  denote the number of blocks with ties in  $X$  ( $k \leq N$ ), let  $n_i$  denote the size of the  $i$ th block.

#### 31.4.2.1 Exact distribution

Let  $p(n_1, \dots, n_k; t) = \Pr[J_N = t]$ . If  $J_N$  is based on  $k$  independent samples of sizes  $n_1, \dots, n_k$ , then (Skillings 1980):

$$p(n_1, \dots, n_k; t) = \sum_x p(n_1, \dots, n_k; x) \times p(n_1, \dots, n_k; t - x) \quad (31.4.1)$$

where the sum is over all  $x$  with positive  $p(\cdot)$ .

#### 31.4.2.2 Cumulants

The cumulants of  $J_N$  are given by (Robillard, 1972):

$$\begin{aligned} \kappa_{2j} &= \frac{B_{2j}}{2j} \left[ \sum_{s=1}^N s^{2j} - \sum_{i=1}^k \sum_{s=1}^{n_i} s^{2j} \right] \\ &= \frac{B_{2j}}{2j(2j+1)} \left[ B_{2j+1}(N+1) + (k-1)B_{2j+1} - \sum_{i=1}^k B_{2j+1}(n_i+1) \right] \end{aligned} \quad (31.4.2)$$

and  $\kappa_{2j+1} = 0$ ,  $j \geq 1$ , and  $B_{2j}$  and  $B_{2j}(x)$  are the Bernoulli numbers and polynomials, respectively, of degree  $2j$  (see section 16.2).

See also [Skillings \(1980\)](#)

## 31.5 Noncentral Distribution of Spearman's Rho

### 31.5.1 Definition

Spearman's  $\rho$  is calculated like Pearson's correlation coefficient, using the rank-transform on the  $x_i$  and  $y_i$ . The exact test for  $H_0$  is the permutation test. Techniques for obtaining the exact distribution and approximations are given below

### 31.5.2 Central Cumulants

The first 8 cumulants are given by ([David et al., 1951](#))

$$\kappa_2 = \frac{1}{n-1} \quad (31.5.1)$$

$$\kappa_4 = \frac{-6(19n^2 + 5n - 36)}{25n(n+1)(n-1)^3} \quad (31.5.2)$$

$$\kappa_6 = \frac{48(583n^6 + 723n^5 - 2603n^4 - 2637n^3 + 4054n^2 + 2760n - 1800)}{(245n^3(n-1)^5n1^3)} \quad (31.5.3)$$

$$\begin{aligned} \kappa_8 &= \frac{144(41939n^{10} - 83709n^9 + 304254n^8 + 578442n^7 - 1012323n^6 - 1690125n^5)}{875n^5(n+1)^5(n-1)^7} \\ &\quad + \frac{144(1800776n^4 + 2358048n^3 - 1616688n^2 - 1080567n + 846720)}{875n^5(n+1)^5(n-1)^7} \end{aligned} \quad (31.5.4)$$

Note: See `PermCumulants.SpearmanCum` for further details.

### 31.5.3 Noncentral Moments

The expected value for  $\rho_s$  in samples from a bivariate normal population with parameter  $rho$  is

$$E(r_s) = \frac{6}{\pi(n+1)}(\arcsin(\rho) + (n-2)\arcsin(\rho/2)) \quad (31.5.5)$$

See [Xu et al. \(2013\)](#) for exact 2nd noncentral moment, and also [van de Wiel & Di Buccianico \(2001\)](#), and [Iman et al. \(1975\)](#), and [Koning & Does \(1988\)](#), and [Lee & Lin \(1992\)](#)

### 31.5.4 Recursive algorithm

Let  $X = (x_1, \dots, x_N)$  and  $Y = (y_1, \dots, y_N)$  denote 2 fixed ordered sets of scores.

Let  $Z = (z_1, \dots, z_n)$  denote any ordered subset of  $Y$  of size  $n$ . Let  $S = x_1z_1 + \dots + x_nz_n$ .

Let  $W_n(S; (x_1, z_1), \dots, (x_n, z_n))$  denote the number of ways of forming  $S$  when based on the first  $n$  elements of  $X$  and  $Z$ . Then the following recurrence relation holds:

$$W_n(S; (x_1, z_1), \dots, (x_n, z_n)) = \sum_{i=1}^n W_{n-1}(S - x_nz_i; (x_1, z_1), \dots, (x_{i-1}, z_{i-1}), (x_i, z_{i+1}), \dots, (x_{n-1}, z_n)) \quad (31.5.6)$$

Algorithm 2: Complete enumeration of all permutations

See [van de Wiel & Di Buccianico \(2001\)](#), and [Iman et al. \(1975\)](#), and [Koning & Does \(1988\)](#), and [Lee & Lin \(1992\)](#)

## 31.6 Noncentral Distribution of Page's L

### 31.6.1 Definition

Spearman's  $L$  is calculated like Pearson's correlation coefficient, using the rank-transform on the  $x_i$  and  $y_i$ . The exact test for  $H_0$  is the permutation test. Techniques for obtaining the exact distribution and approximations are given below

### 31.6.2 Central Cumulants

The first 8 cumulants are given by ([David et al., 1951](#))

$$\kappa_2 = \frac{1}{n-1} \quad (31.6.1)$$

$$\kappa_4 = \frac{-6(19n^2 + 5n - 36)}{25n(n+1)(n-1)^3} \quad (31.6.2)$$

$$\kappa_6 = \frac{48(583n^6 + 723n^5 - 2603n^4 - 2637n^3 + 4054n^2 + 2760n - 1800)}{(245n^3(n-1)^5n1^3)} \quad (31.6.3)$$

$$\begin{aligned} \kappa_8 &= \frac{144(41939n^{10} - 83709n^9 + 304254n^8 + 578442n^7 - 1012323n^6 - 1690125n^5)}{875n^5(n+1)^5(n-1)^7} \\ &\quad + \frac{144(1800776n^4 + 2358048n^3 - 1616688n^2 - 1080567n + 846720)}{875n^5(n+1)^5(n-1)^7} \end{aligned} \quad (31.6.4)$$

Note: See `PermCumulants.SpearmanCum` for further details.

### 31.6.3 Noncentral Moments

The expected value for  $L_s$  in samples from a bivariate normal population with parameter  $L$  is

$$E(r_s) = \frac{6}{\pi(n+1)}(\arcsin(L) + (n-2)\arcsin(L/2)) \quad (31.6.5)$$

See [Xu et al. \(2013\)](#) for exact 2nd noncentral moment, and also [van de Wiel & Di Buccianico \(2001\)](#), and [Iman et al. \(1975\)](#), and [Koning & Does \(1988\)](#), and [Lee & Lin \(1992\)](#)

### 31.6.4 Recursive algorithm

Let  $X = (x_1, \dots, x_N)$  and  $Y = (y_1, \dots, y_N)$  denote 2 fixed ordered sets of scores.

Let  $Z = (z_1, \dots, z_n)$  denote any ordered subset of  $Y$  of size  $n$ . Let  $S = x_1z_1 + \dots + x_nz_n$ .

Let  $W_n(S; (x_1, z_1), \dots, (x_n, z_n))$  denote the number of ways of forming  $S$  when based on the first  $n$  elements of  $X$  and  $Z$ . Then the following recurrence relation holds:

$$W_n(S; (x_1, z_1), \dots, (x_n, z_n)) = \sum_{i=1}^n W_{n-1}(S - x_nz_i; (x_1, z_1), \dots, (x_{i-1}, z_{i-1}), (x_i, z_{i+1}), \dots, (x_{n-1}, z_n)) \quad (31.6.6)$$

Algorithm 2: Complete enumeration of all permutations

See [van de Wiel & Di Buccianico \(2001\)](#), and [Iman et al. \(1975\)](#), and [Koning & Does \(1988\)](#), and [Lee & Lin \(1992\)](#)

## 31.7 Noncentral Distribution of Kruskal-Wallis' H

### 31.7.1 Definition

The exact or approximate distributions of may commonly used rank-based procedures in a one-way or two-way layout can be obtained using algorithms which first calculate the density of the total scores and then in a second step the statistic of interest (an analogue to parametric ANOVA, Many-one comparisons, all-pairs comparisons, etc). In this section the general steps are covered, and relationships to some specific procedures are given.

### 31.7.2 Density of total scores of a one-way layout

#### 31.7.2.1 Null distribution: Recursive generation of the pdf of all total scores

Under the null-hypothesis, all permutations are equally likely to occur, and recursive generation of the whole pdf is possible for small sample sizes. Let  $S_N$  denote the  $N^{th}$  element in an ordered vector of scores, let

$$P_{n_1, \dots, n_k} = (T_1, \dots, T_i, \dots, T_k)W(n_1, \dots, n_i, \dots, n_k) \quad (31.7.1)$$

denote the number of ways of forming the score totals  $T_1, \dots, T_i, \dots, T_k$  with samples of sizes  $n_1, \dots, n_i, \dots, n_k$ , and let  $N = n_1 + n_2 + \dots + n_k$ . Then the following recursion relation holds:

$$P_{n_1, \dots, n_k} == \sum_{i=1}^k (T_1, \dots, T_i - S_N, \dots, T_k)W(n_1, \dots, n_i - 1, \dots, n_k) \quad (31.7.2)$$

#### 31.7.2.2 Noncentral distribution: General case

Under a non-null hypothesis, the assumption that all permutations are equally likely to occur usually does not hold, and the recursive scheme above cannot be applied. In this case, for each vector  $T$  of  $k$  total scores a complete enumeration of all permutations forming this vector is necessary (see [Chase \(1970\)](#) and [Stockmal \(1962\)](#)). For each such permutation, the probability to occur needs to be calculated, based on the specification of the non-null hypothesis, and the sum of all these probabilities is the probability of the is the probability of  $T$ .

#### 31.7.2.3 Noncentral distribution: Alternatives based on the Normal distribution

We consider  $k$  continuous random variables  $X_i$  with density functions  $f_i$  and sample sizes  $n_i$ ,  $i = 1 \dots k$ , and  $N = n_1 + \dots + n_k$ .

Let  $\mathbf{U} = (U_1, \dots, U_N)$ ,  $U_1 < \dots < U_N$ , denote the order statistics of the random variables  $(X_{1,1}, \dots, X_{1,n_1}, \dots, X_{k,1}, \dots, X_{k,n_k})$ , and let  $\mathbf{Z} = (Z_1, \dots, Z_N)$  denote a random vector of integers  $1, 2, \dots, k$ , where the  $i$ th component  $\mathbf{Z}_i$  is  $i$  if  $U_i$  is an  $X_i$ . If  $\mathbf{z} = (z_1, \dots, z_N)$  is a fixed vector of integers  $1, 2, \dots, k$  (with each  $i$  occurring  $n_i$  times), the probability of the rank order  $z$ ,  $\Pr[\mathbf{Z} = \mathbf{z}]$ , is given by

$$P_{n_1, \dots, n_k}(\mathbf{z}|d) = n_1! \dots n_k! \int_R \dots \int \prod_{i=1}^N f_i(t_i) dt_i, \quad (31.7.3)$$

where the region of integration  $R$  is  $-\infty < t_1 \leq t_2 \leq \dots \leq t_N < \infty$  ([Milton, 1970](#)).

### 31.7.2.4 Noncentral distribution: Lehmann Alternatives

Lehmann alternatives are of the form  $F_1(x) = [F_0(x)]^k$  or  $F_1(x) = 1 - [1 - F_0(x)]^k$ . The exact distribution under this alternative can be calculated using recurrence relations, similar to the null-distribution ([Shorack, 1966](#)): Let  $p_{n,m}(u)$  denote the probability that  $U = u$  in samples of size  $n$  and  $m$ . Then

$$(km + n)p_{n,m}(u) = np_{n-1,m}(u - m) + kmp_{n,m-1}(u) \quad \text{for } F_1 = (F_0)^k \quad (31.7.4)$$

$$(km + n)p_{n,m}(u) = np_{n-1,m}(u) + kmp_{n,m-1}(u - n) \quad \text{for } F_1 = 1 - (1 - F_0)^k \quad (31.7.5)$$

This recursive procedure allows the calculation of the exact noncentral distribution also for larger samples.

### 31.7.3 Moments and cumulants of linear rank statistics for the one-way layout

For both central and noncentral distributions, the moments, and from the moments the cumulants (see section [29.1](#)) of linear rank statistics can be determined numerically from the pdf. Linear rank statistics for which this can be easily done are orthogonal polynomials, the Jonckhere-Terpstra statistic and Halperin type Mann-Whitney statistics.

### 31.7.4 Special cases for the one-way layout (H-type)

#### 31.7.4.1 Kruskal-Wallis Test

See [Spurrier \(2003\)](#)

See [Robinson \(1980\)](#)

See [Di Bucchianico & van de Wiel \(2005\)](#)

See [van de Wiel \(2004\)](#)

See [Fan \*et al.\* \(2011\)](#)

See [Fan & Zhang \(2012\)](#)

#### 31.7.4.2 Koziol Test

See also [Koziol \(1982\)](#)

## 31.8 Noncentral Distributions of Friedman's S

### 31.8.1 Density of total scores in a two-way layout

Under the assumption of independent blocks, the following procedures are valid both for the central and noncentral distributions described above.

#### 31.8.1.1 Linear Rank Statistics

For the linear rank statistics described above, the pdf of the statistics can be first calculated per block. Then the full distribution can be calculated as

$$p(n_1, \dots, n_k; t) = \sum_x p(n_1, \dots, n_k; x) \times p(n_1, \dots, n_k; t - x) \quad (31.8.1)$$

#### 31.8.1.2 Friedman Type

Consider a two-way layout of  $n$  rows ("blocks") and  $k$  columns ("treatments"), where the observations in different blocks are assumed to be independent. Let  $W(n; T_1, \dots, T_k)$  denote the number of ways of forming the ordered vector of column rank totals  $(T_1, \dots, T_k)$  when based on  $(n-1)$  blocks, and let  $(X_{1(i)}, \dots, X_{k(i)})$  denote the  $i^{th}$  permutation of the rank totals in the  $n^{th}$  block. Then the following recurrence relation holds:

$$W(n; T_1, \dots, T_k) = \sum_{i=1}^k W(n-1; T_1 - X_{1(i)}, \dots, T_k - X_{k(i)}) \quad (31.8.2)$$

For literature, see Odeh(1977b), Patil (1975)

#### 31.8.1.3 General case

Consider a two-way layout of  $k$  columns ("treatments"), and  $l$  independent blocks with sample sizes  $n_{ij}$  for the  $i^{th}$  treatment in the  $j^{th}$  block,  $i = 1 \dots k$  and  $j = 1 \dots l$ . Let  $Q(l; T_1, \dots, T_k)$  denote the probability of the column-total vector  $(T_1, \dots, T_k)$ , based on  $l$  blocks, and let  $p_j(x_1, \dots, x_k)$  denote the probability of the column-total vector  $(x_1, \dots, x_k)$  in the  $j^{th}$  block. Then the following recurrence relationship holds:

$$Q(l+1; T_1, \dots, T_k) = \sum_{x_1, \dots, x_k} Q(l; T_1 - x_1, \dots, T_k - x_k) \times p_{n+1}(x_1, \dots, x_k) \quad (31.8.3)$$

where the sum is over all  $x$  with positive  $p(\cdot)$ .

### 31.8.2 Moments and cumulants of linear rank statistics for the two-way layout

For both central and noncentral distributions, the cumulants of linear rank statistics which have been determined per block as described in section 31.7.3 can be used to calculate the cumulants for the two-way layout, since the blocks are assumed to be independent. Let  $\kappa_{r,j}$  denote the  $r^{th}$  cumulant of the linear rank statistic in the  $i^{th}$  block. Then the  $r^{th}$  cumulant of the combined statistic is given by

$$\kappa_r = \sum_{i=1}^l \kappa_{r,i} \quad (31.8.4)$$

### 31.8.3 Special cases for the two-way layout

#### 31.8.3.1 Friedman Test

See [Skillings & Mack \(1981\)](#)

See [Röhmel \(1997\)](#)

See [Gansky \*et al.\* \(2001\)](#)

#### 31.8.3.2 Cochran Test

#### 31.8.3.3 Quade Test

See [Quade \(1979\)](#)

#### 31.8.3.4 Mack-Skillings Test

See also [Mack & Skillings \(1980\)](#)

See also [Skillings & Mack \(1981\)](#)

#### 31.8.3.5 Koziol Test

See also [Koziol \(1982\)](#)

# Chapter 32

## Examples: Statistical Procedures

### 32.1 Introduction to Inferential Statistics Functions

Reference [Bortz et al. \(1990\)](#)

Reference [Conover & Iman \(1981\)](#)

#### 32.1.1 The Linear Model

This section describes the following computational and conceptual relationships (see "a tale of two regressions")

- Simple Linear regression between two random variables and between one random variable and one design (dummy) variable (Classical t-test)
- Multiple Linear regression between one predicted random variable and multiple predictor variables and between one predicted random variable and multiple design (dummy) variables (Classical ANOVA)
- Canonical correlation analysis between two sets of random variables and between one set of random variables and one set of design (dummy) variables (Classical MANOVA)

#### 32.1.2 Transformations

This section describes the following computational and conceptual relationships

- The t-test for 2 independent groups, the Mann-Whitney Test, and the  $2 \times 2$  contingency table
- The classical ANOVA for  $k$  independent groups, the Kruskal-Wallis Test, and the  $r \times 2$  contingency table
- The classical MANOVA for  $k$  independent groups, the Multivariate Kruskal-Wallis Test, and the  $r \times c$  contingency table

#### 32.1.3 Randomisation, Block-Randomisation, and Permutation Tests

This section reviews Randomisation and Block-Randomisation and discusses the implications for permutation tests and normal-theory based procedures like classical ANOVA.

### 32.1.4 Commonly Used Function Types

#### 32.1.4.1 Functions returning p-Values and Confidence Intervals

These functions have the form `Test?(Proc; Alpha; [Parameters;], OutputString)`. Here “?” is a placeholder for the type data to be analyzed, e.g. “2i” for 2 independent samples, “Proc” specifies the procedures which should be performed, e.g. “TTest” for Student’s t-test, “Alpha” is the type I error used for the test, it is also used to specify the confidence level (1-Alpha) which will be used for the construction of confidence intervals, “[Parameters;]” denote any parameters (like sample sizes, means, standard deviations) which the tests require, and “OutputString” specifies the computed results which will be returned. This can be any of the following:

- **Input:** a list of the input used for the test.
- **Description:** a list of all test-specific descriptive results, like degrees of freedom, t-values etc. If only some of these are desired, then they need to be listed individually, as stated in the description of each test.
- **TCrit1:** the critical value for a one-sided test
- **TCrit2:** the critical value for a two-sided test
- **PValueH01[method]:** the p-value for  $H_{01}$  (depends on the test, e.g.  $\mu_1 \geq \mu_2$ )
- **PValueH02[method]:** the p-value for  $H_{02}$  (depends on the test, e.g.  $\mu_1 \leq \mu_2$ )
- **PValueH03[method]:** the p-value for  $H_{03}$  (depends on the test, e.g.  $\mu_1 = \mu_2$ )
- **LowerCL1[item],[method]:** lower confidence limit, 1-sided
- **UpperCL1[item],[method]:** upper confidence limit, 1-sided
- **LowerCL2[item],[method]:** lower confidence limit, 2-sided
- **UpperCL2[item],[method]:** upper confidence limit, 2-sided
- **LengthCI2[item],[method]:** length of the 2-sided confidence interval

Note: the optional parameter **[method]** allows you to specify the method of calculation for p-values and confidence intervals (e.g. an exact method or an approximation). The optional parameter **[item]** allows you to choose for what the confidence interval will be calculated (e.g. difference of the means or effect size). The available choices depend on the test. If these optional parameters are omitted, the defaults will be used.

As an example, for a test for 2 independent samples, “2i” is used to specify the type of data to be analyzed, and there are 3 test parameters, the sample size **N**, the means **Mean**, and the standard deviations **StDev**.

**Test2i(Proc As String;  $\alpha$  As mpNum, N As mpNum[], Mean As mpNum[], StDev As mpNum[], OutputString As String) As mpNumList,**

and an actual call to the function, requesting Student’s t-test with description, the critical value for a two-sided test, the p-value for  $H_{03}$  (in the case of TTest this is  $\mu_1 \neq \mu_2$ ), for 2 independent

samples of size 10 and standard deviation 1 each, with means 2.3 and 4.5, and a type I error  $\alpha = 0.05$  would be

---

```
Result = Test2i("TTest", 0.05, 10, {2.3, 4.5}, 1, "Description + TCrit2 + PValueH03")
mp.Print Result
```

---

which produces the output

```
df: 18
Difference of Means: -2.2
t-value: -4.9193496
TCrit2: 2.100922
PValueH03: 0.0001106
```

### 32.1.4.2 Functions returning Power Estimates

These functions have the form `TestPower?(Proc; Alpha; [Parameters;], OutputString)`. Here “?” is a placeholder for the type data to be analyzed, e.g. “2i” for 2 independent samples, “Proc” specifies the procedures which should be performed, e.g. “TTest” for Student’s t-test, “Alpha” is the type I error used for the test, it is also used to specify the confidence level (1-Alpha) which will be used for the construction of confidence intervals, “[Parameters;]” denote any parameters (like sample sizes, means, standard deviations) which the tests require, and “OutputString” specifies the computed results which will be returned. This can be any of the following:

- **Input:** a list of the input used for the test.
- **Description:** a list of all test-specific descriptive results, like degrees of freedom, t-values etc. If only some of these are desired, then they need to be listed individually, as stated in the description of each test.
- **TCrit1:** the critical value for a one-sided test
- **TCrit2:** the critical value for a two-sided test
- **PowerHA1[method]:** the power to reject  $H_{01}$  (depends on the test, e.g.  $\mu_1 \geq \mu_2$ ) and to accept  $H_{A1}$
- **PowerHA2[method]:** the power to reject  $H_{02}$  (depends on the test, e.g.  $\mu_1 \leq \mu_2$ ) and to accept  $H_{A2}$
- **PowerHA3[method]:** the power to reject  $H_{03}$  (depends on the test, e.g.  $\mu_1 = \mu_2$ ) and to accept  $H_{A3}$

Note: the optional parameter **[method]** allows you to specify the method of calculation for p-values and confidence intervals (e.g. an exact method or an approximation). The available choices depend on the test. If these optional parameters are omitted, the defaults will be used.

As an example, for a test for 2 independent samples, “2i” is used to specify the type of data to be analyzed, and there are 3 test parameters, the sample size **N**, the means **Mean**, and the standard deviations **StDev**.

**TestPower2i(Proc As String;  $\alpha$  As mpNum, **N** As mpNum[], **Mean** As mpNum[], **StDev** As mpNum[], OutputString As String) As mpNumList,**

and an actual call to the function, requesting Student’s t-test with description, the critical value for a two-sided test, the power for  $H_{A3}$  (in the case of TTest this is  $\mu_1 \neq \mu_2$ ), for 2 independent samples of size 10 and standard deviation 1 each, with means 2.3 and 4.5, and a type I error  $\alpha = 0.05$  would be

---

```
Result = TestPower2i("TTest", 0.05, 10, {2.3, 4.5}, 1, "Description + TCrit2 +
    PowerHA3")
mp.Print Result
```

---

which produces the output

```
df: 18
Difference of Means: -2.2
t-value: -4.9193496
TCrit2: 2.100922
PValueH03: 0.996354
```

### 32.1.4.3 Functions returning Sample Size Estimates

These functions have the form `TestSampleSize?(Proc; Alpha; Beta; [Parameters;], OutputString)`. Here

”?” is a placeholder for the name of the distribution,

”Proc” specifies the procedures which should be performed, e.g. ”TTest” for Student’s t-test,

”Alpha” specifies the confidence level (or Type I error),

”Beta” specifies the Type I error (or  $1 - \text{Power}$ ),

”[Parameters;]” denote any additional parameters of the distribution (if any) which are not a function of the sample size, and

”OutputString” specifies the computed results which will be returned. This can be any of the following:

- **ExactN**: returns an ”exact”, i.e. typically non-integer sample size estimate
- **UpperN**: upper integer sample size estimate
- **LowerN**: lower integer sample size estimate
- **UpperNPower**: actual power when using UpperN
- **LowerNPower**: actual power when using LowerN

As an example, for a test for 2 independent samples, ”2i” is used to specify the type of data to be analyzed, and there are 2 test parameters, the means **Mean**, and the standard deviations **StDev**.

`TestSampleSize2i(Proc As String,  $\alpha$  As mpNum,  $\beta$  As mpNum, Mean As mpNum[], StDev As mpNum[], OutputString As String) As mpNumList`

and an actual call to the function, requesting an upper sample size estimate (and actual power) for  $\alpha = 0.95$ ,  $\beta = 0.1$ , and standard deviations  $\sigma_1 = \sigma_2 = 1$ , means  $\mu_1 = 2.3$  and  $\mu_2 = 4.5$ , would be

---

```
Result = TestSampleSize2i("TTest", 0.05, 0.1, {2.3, 4.5}, 1, "UpperN + UpperNPower")
mp.Print Result
```

---

which produces the output

```
UpperN: 2 x 6 = 12
UpperNPower: 0.9285919
```

## 32.2 Tests for the mean from 1 sample (Student's t-test)

### 32.2.1 Overview

Let  $(X_1, X_2, \dots, X_N)$  denote a random sample of size  $N$  from a normal distribution with mean  $\mu$  and variance  $\sigma^2$ , and let

$$\bar{x}_1 = \frac{1}{N} \sum_{i=1}^N X_i \quad \text{and} \quad s^2 = \frac{1}{N-1} \sum_{i=1}^N (X_i - \bar{x}_1)^2 \quad (32.2.1)$$

be the usual sample estimates of the unknown population mean  $\mu$  and unknown population variance  $\sigma^2$ . Then Student's t-test can be used to test hypotheses concerning  $\mu$  with regard to a reference value  $\mu_0$ .

### 32.2.2 Tests and Confidence Intervals

---

Function **StudentTTest1**(*Type1Error* As *mpNum*, *N* As *mpNum[]*, *Mean* As *mpNum[]*, *StDev* As *mpNum[]*, *Output* As *String*) As *mpNumList*

---

**NOT YET IMPLEMENTED**

---

The function **StudentTTest1** returns p-values, confidence intervals and related information for Student's t-test

#### Parameters:

*Type1Error*: A real number greater than 0 and less than 1.

*N*: An array of integers greater than 1, representing the sample sizes

*Mean*: An array of reals, representing the means

*StDev*: An array of positive reals, representing the standard deviations

*Output*: A string describing the output choices

See section 32.1.4.1 for the options for *Type1Error* and *Output*). Algorithms and formulas are given in sections 32.2.2.1 and 32.2.2.3.

#### 32.2.2.1 Tests: algorithms and formulas

Let  $F_t(\cdot, \nu)$  denote the CDF (see section 6.13.2.2) and let  $t_{\nu, \alpha}$  denote the  $\alpha$ -quantile (see section 6.13.3) of the  $t$ -distribution with  $\nu$  degrees of freedom. Define

$$t = \frac{\bar{x}_1 - \mu_0}{s}, \quad s = \sqrt{s_1^2/N}, \quad \nu = N - 1. \quad (32.2.2)$$

Then  $p$ -values and rejection criteria for  $H_0$  can be calculated as summarized in Table 32.1.

Test problem	<i>p</i> -value	Reject $H_0$
$H_{01} : \mu \leq \mu_0$ vs $H_{A1} : \mu > \mu_0$	$F_t(-t, \nu)$	$t > t_{\nu; 1-\alpha}$
$H_{02} : \mu \geq \mu_0$ vs $H_{A2} : \mu < \mu_0$	$F_t(t, \nu)$	$t > t_{\nu; \alpha}$
$H_{03} : \mu = \mu_0$ vs $H_{A3} : \mu \neq \mu_0$	$F_t(t, \nu) - F_t(-t, \nu)$	$t > t_{\nu; 1-\alpha/2}$ or $t > t_{\nu; \alpha/2}$

**Table 32.1:** Student's t-test, 1 sample, tests for the mean

The test can also be expressed in terms of a correlation coefficient  $r$  between the combined  $X$  and an indicator variable, where  $t$  and  $r$  are related by

$$r = \frac{t}{\sqrt{t^2 + \nu}}, \quad t = \nu \frac{r}{1 - r^2}. \quad (32.2.3)$$

### 32.2.2.2 Tests: examples

An actual call to the function, requesting Student's t-test with description, the critical value for a two-sided test, the p-value for  $H_{03}$  (in the case of `TTest` this is  $\mu_1 \neq \mu_2$ ), for 2 independent samples of size 10 and standard deviation 1 each, with means 2.3 and 4.5, and a type I error  $\alpha = 0.05$  would be

---

```
Result = Test2i("TTest", 0.05, 10, {2.3, 4.5}, 1, "Description + TCrit2 + PValueH03")
mp>Show Result
```

---

which produces the output

```
df: 18
Difference of Means: -2.2
t-value: -4.9193496
TCrit2: 2.100922
PValueH03: 0.0001106
```

### 32.2.2.3 Confidence Intervals: algorithms and formulas

Let  $A_1 = t_{\nu,\alpha} \cdot s$  and  $A_2 = t_{\nu,\alpha/2} \cdot s$ , where  $s$  and  $\nu$  are defined in (32.2.2), and  $t_{\nu,\alpha}$  denotes the  $\alpha$ -quantile of the (central)  $t$ -distribution with  $\nu$  degrees of freedom (see section 6.13.3).

Then confidence intervals for  $\mu_1 - \mu_0$  can be calculated as summarized in Table 32.2.

Type	Confidence Interval (Difference of Means)
Left-sided	$-\infty \leq \mu_1 - \mu_0 \leq (\bar{x}_1 - \mu_0) + A_1$
Right-sided	$(\bar{x}_1 - \mu_0) - A_1 \leq \mu_1 - \mu_0 \leq +\infty$
Two-sided	$(\bar{x}_1 - \mu_0) - A_2 \leq \mu_1 - \mu_0 \leq (\bar{x}_1 - \mu_0) + A_2$

**Table 32.2:** Student's t-test, 1 sample, confidence intervals for the mean

### 32.2.2.4 Confidence Intervals: examples

An actual call to the function, requesting Student's t-test with description, the critical value for a two-sided test, the p-value for  $H_{03}$  (in the case of `TTest` this is  $\mu_1 \neq \mu_2$ ), for 2 independent samples of size 10 and standard deviation 1 each, with means 2.3 and 4.5, and a type I error  $\alpha = 0.05$  would be

---

```
Result = Test2i("TTest", 0.05, 10, {2.3, 4.5}, 1, "Description + TCrit2 + PValueH03")
mp>Show Result
```

---

which produces the output

```
df: 18
Difference of Means: -2.2
```

t-value: -4.9193496  
TCrit2: 2.100922  
PValueH03: 0.0001106

### 32.2.3 Power

---

Function **StudentTTestPower1**(*Type1Error* As *mpNum*, *N* As *mpNum[]*, *Mean* As *mpNum[]*, *StDev* As *mpNum[]*, *Output* As *String*) As *mpNumList*

---

**NOT YET IMPLEMENTED**

---

The function **StudentTTestPower1** returns power estimations and related information for Student's t-test

**Parameters:**

*Type1Error*: An real number greater then 0 and less than 1.

*N*: An array of integers greater than 1, representing the samples sizes

*Mean*: An array of reals, representing the means

*StDev*: An array of positive reals, representing the standard deviations

*Output*: A string describing the output choices

See section 32.1.4.2 for the options for *Type1Error* and *Output*). Algorithms and formulas are given in section 32.2.3.1.

#### 32.2.3.1 Power calculations: algorithms and formulas

Let  $\sigma_1^2 = \sigma^2$  and  $\nu = N - 1$ . Define

$$\tilde{\rho} = \frac{\mu_1 - \mu_0}{\sigma} \text{ and } \delta = \sqrt{\nu} \tilde{\rho}. \quad (32.2.4)$$

Let  $F_{t'}(\cdot, \nu, \delta)$  denote the CDF of the (singly) noncentral  $t$ -distribution with  $\nu$  degrees of freedom and noncentrality parameter  $\delta$  (see section and let  $t_{\nu, \alpha}$  denote the  $\alpha$ -quantile of the central  $t$ -distribution with  $\nu$  degrees of freedom (see section 6.13.3)).

Then the power for accepting  $H_A$  at the confidence level  $\alpha$  can be calculated as summarized in Table 32.3.

Test	Null Hypothesis	Alternative	Power
1 sided	$H_{01} : \mu \leq \mu_0$	$H_{A1} : \mu > \mu_0$	$F_{t'}(-t_{\nu;1-\alpha}, \nu, \delta)$
1 sided	$H_{02} : \mu \geq \mu_0$	$H_{A2} : \mu < \mu_0$	$F_{t'}(t_{\nu;1-\alpha}, \nu, \delta)$
2 sided	$H_{03} : \mu = \mu_0$	$H_{A1} : \mu > \mu_0$	$F_{t'}(-t_{\nu;1-\alpha/2}, \nu, \delta)$
2 sided	$H_{03} : \mu = \mu_0$	$H_{A2} : \mu < \mu_0$	$F_{t'}(t_{\nu;1-\alpha/2}, \nu, \delta)$
2 sided	$H_{03} : \mu = \mu_0$	$H_{A3} : \mu \neq \mu_0$	$F_{t'}(t_{\nu;1-\alpha/2}, \nu, \delta) - F_t(-t_{\nu;1-\alpha/2}, \nu, \delta)$

**Table 32.3:** Student's t-test, 1 sample, power calculations

#### 32.2.3.2 Power calculations: examples

An actual call to the function, requesting Student's t-test with description, the critical value for a two-sided test, the power for  $H_{A3}$  (in the case of TTest this is  $\mu_1 \neq \mu_2$ ), for 2 independent samples of size 10 and standard deviation 1 each, with means 2.3 and 4.5, and a type I error  $\alpha = 0.05$  would be

---

```
Result = TestPower2i("TTest", 0.05, 10, {2.3, 4.5}, 1, "Description + TCrit2 +
PowerHA3")
```

---

```
mp.Print Result
```

---

which produces the output

```
df: 18
Difference of Means: -2.2
t-value: -4.9193496
TCrit2: 2.100922
PValueH03: 0.996354
```

### 32.2.4 Sample Size Calculation

---

Function **StudentTTestSampleSize1**( *Type1Error* As mpNum, *Type2Error* As mpNum, *Mean* As mpNum[], *StDev* As mpNum[], *Output* As String) As mpNumList

---

**NOT YET IMPLEMENTED**

---

The function **StudentTTestSampleSize1** returns sample size estimations and related information for Student's t-test

**Parameters:**

*Type1Error*: An real number greater then 0 and less than 1.

*Type2Error*: An real number greater then 0 and less than 1.

*Mean*: An array of reals, representing the means

*StDev*: An array of positive reals, representing the standard deviations

*Output*: A string describing the output choices

See section 32.1.4.2 for the options for *Type1Error*, *Type2Error* and *Output*. Algorithms and formulas are given in section 32.2.4.1.

#### 32.2.4.1 Sample size calculations: algorithms and formulas

Let  $N_{t'}(\alpha, \beta, \tilde{\rho})$  denote the sample size function of the (singly) noncentral  $t$ -distribution (see section 30.8.7 ) for a given confidence level  $\alpha$ , power  $\beta$  and noncentrality parameter  $\tilde{\rho}$  (as defined in equation 32.2.4).

The required total sample size  $N$  can be calculated as summarized in Table 32.4.

Test	Null Hypothesis	Alternative	Minimal sample size
1 sided	$H_{01} : \mu \leq \mu_0$	$H_{A1} : \mu > \mu_0$	$N_{t'}(\alpha, \beta, \tilde{\rho})$
1 sided	$H_{02} : \mu \geq \mu_0$	$H_{A2} : \mu < \mu_0$	$N_{t'}(\alpha, \beta, \tilde{\rho})$
2 sided	$H_{03} : \mu = \mu_0$	$H_{A1} : \mu > \mu_0$	$N_{t'}(\alpha, \beta, \tilde{\rho})$
2 sided	$H_{03} : \mu = \mu_0$	$H_{A2} : \mu < \mu_0$	$N_{t'}(\alpha, \beta, \tilde{\rho})$
2 sided	$H_{03} : \mu = \mu_0$	$H_{A3} : \mu \neq \mu_0$	$N2_{t'}(\alpha, \beta, \tilde{\rho})$

**Table 32.4:** Student's t-test, 1 sample, sample size calculations

Note that the returned value of  $N$  will in general not be an integer, and rounding up may be required.

### 32.2.4.2 Sample size calculations: examples

An actual call to the function, requesting an upper sample size estimate (and actual power) for  $\alpha = 0.95$ ,  $\beta = 0.1$ , and standard deviations  $\sigma_1 = \sigma_2 = 1$ , means  $\mu_1 = 2.3$  and  $\mu_2 = 4.5$ , would be

---

```
Result = TestSampleSize2i("TTest", 0.05, 0.1, {2.3, 4.5}, 1, "UpperN + UpperNPower")
mp.Print Result
```

---

which produces the output

```
UpperN: 2 x 6 = 12
UpperNPower: 0.9285919
```

### 32.2.5 Confidence Interval (Effect size)

Let  $T$  be a statistic according to the non-central  $t$ -distribution with  $n$  degrees of freedom and a non-centrality parameter  $\delta$ . Then the lower confidence limit  $\hat{\delta}$  of level  $1 - \alpha$  and the two-sided confidence interval  $[\underline{\delta}, \bar{\delta}]$  of the non-centrality parameter  $\delta$  of level  $1 - \alpha$  are given by [Akahira et al. \(1995\)](#):

$$\hat{\delta} = bT - z_\alpha \sqrt{k} + hT^3(z_\alpha^2 - 1)/k, \quad (32.2.5)$$

$$\underline{\delta} = bT - z_{\alpha/2} \sqrt{k} + hT^3(z_{\alpha/2}^2 - 1)/k, \quad (32.2.6)$$

$$\bar{\delta} = bT + z_{\alpha/2} \sqrt{k} - hT^3(z_{\alpha/2}^2 - 1)/k, \quad (32.2.7)$$

where  $k = 1 + (1 - b^2)T^2$ ,  $h$  and  $b$  are defined in equation (30.8.19), and  $z_\alpha$  denotes the  $\alpha$ -quantile of the normal distribution (see section 6.11.3).

Let  $A_1 = t_{\nu, \alpha} \cdot s$  and  $A_2 = t_{\nu, \alpha/2} \cdot s$ , where  $s$  and  $\nu$  are defined in (32.3.3), and  $t_{\nu, \alpha}$  denotes the  $\alpha$ -quantile of the (central)  $t$ -distribution with  $\nu$  degrees of freedom (see section 6.13.3). Then confidence intervals for the difference of 2 means can be calculated (needs to be worked out).

### 32.2.6 Tolerance Intervals

Let  $X$  be a normally distributed variable following a  $N(\mu, \sigma^2)$  distribution, with  $\mu, \sigma^2$  unknown, and let  $X_1, X_2, \dots, X_n$  be a random sample. A one-sided tolerance interval covers with confidence  $1 - \alpha$  at least the fraction  $p$  of values of the distribution  $N(\mu, \sigma^2)$ .

Right-hand tolerance interval  $(-\infty, \bar{X} + kS) : P[P(X < \bar{X} + kS) \geq p] = 1 - \alpha]$ ,

Leftt-hand tolerance interval  $(\bar{X} - kS, \infty) : P[P(X > \bar{X} - kS) \geq p] = 1 - \alpha]$ ,

where  $\bar{X}$  and  $S$  denote the estimates of mean and standard deviation from the sample, and

$$k = -\frac{t_\alpha(n-1, -u_p \sqrt{n})}{\sqrt{n}} = \frac{t_{1-\alpha}(n-1, u_p \sqrt{n})}{\sqrt{n}} \quad (32.2.8)$$

$$k = -\frac{t_{n-1, -\delta; \alpha}}{\sqrt{n}} = \frac{t_{n-1, \delta; 1-\alpha}}{\sqrt{n}} \quad (32.2.9)$$

where  $t_{n, \delta; \alpha}$  denotes the  $\alpha$ -quantile of the noncentral  $t$ -distribution with  $n$  degrees of freedom and noncentrality parameter  $\delta$  (see section ,  $\delta = z_p \sqrt{n}$ , and  $z_p$  denotes the  $p$ -quantile of the normal distribution (see section 6.11.3)).

See Janiga, 2006.

See [Odeh & Owen \(1980\)](#)

### 32.2.7 Prediction Intervals

The following formulas are from "06. Statistical Details for the Distribution Platform", which refer to Hahn and Meeker (1991), pages 61-64. But see also 04. Odeh, 1989, for a different definition.

For  $m$  future observations:

$$[\underline{y}_m, \bar{y}_m] = \bar{X} \pm t_{n-1;1-\alpha/(2m)} \times \sqrt{1 + \frac{1}{n}} \times s, \text{ for } m \geq 1. \quad (32.2.10)$$

For the mean of  $m$  future observations:

$$[Y_l, Y_u] = \bar{X} \pm t_{n-1;1-\alpha/2} \times \sqrt{\frac{1}{m} + \frac{1}{n}} \times s, \text{ for } m \geq 1. \quad (32.2.11)$$

where  $m$  is the number of future observations, and  $n$  is the number of points in the current analysis sample. The one-sided intervals are formed by using  $1 - \alpha$  in the quantile functions.

## 32.3 Tests for means from 2 independent samples (Student's t-test)

### 32.3.1 Overview

Let  $(X_1, X_2, \dots, X_N)$  denote a random sample of size  $N$  from a normal distribution with mean  $\mu$  and variance  $\sigma^2$ , and let

$$\bar{x}_1 = \frac{1}{N} \sum_{i=1}^N X_i \quad \text{and} \quad s^2 = \frac{1}{N-1} \sum_{i=1}^N (X_i - \bar{x}_1)^2 \quad (32.3.1)$$

be the usual sample estimates of the unknown population mean  $\mu$  and unknown population variance  $\sigma^2$ . Then Student's t-test can be used to test hypotheses concerning  $\mu$  with regard to a reference value  $\mu_0$ .

### 32.3.2 Tests and Confidence Intervals

---

Function **StudentTTest2i**(*Type1Error* As *mpNum*, *N* As *mpNum[]*, *Mean* As *mpNum[]*, *StDev* As *mpNum[]*, *Output* As *String*) As *mpNumList*

---

**NOT YET IMPLEMENTED**

---

The function **StudentTTest2i** returns p-values, confidence intervals and related information for Student's t-test

#### Parameters:

*Type1Error*: A real number greater than 0 and less than 1.

*N*: An array of integers greater than 1, representing the samples sizes

*Mean*: An array of reals, representing the means

*StDev*: An array of positive reals, representing the standard deviations

*Output*: A string describing the output choices

See section 32.1.4.1 for the options for *Type1Error* and *Output*). Algorithms and formulas are given in sections 32.3.2.1 and 32.3.2.3.

#### 32.3.2.1 Tests: algorithms and formulas

Let  $F_t(\cdot, \nu)$  denote the CDF (see section 6.13.2.2) and let  $t_{\nu, \alpha}$  denote the  $\alpha$ -quantile (see section 6.13.3) of the *t*-distribution with  $\nu$  degrees of freedom. Define

$$t = \frac{(\bar{x}_1 - \bar{x}_2)}{s}, \quad \text{where} \quad (32.3.2)$$

$$s = \sqrt{MSE \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}, \quad MSE = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{\nu}, \quad \nu = n_1 + n_2 - 2. \quad (32.3.3)$$

Then *p*-values and rejection criteria for  $H_0$  can be calculated as summarized in Table 32.5.

The test can also be expressed in terms of a correlation coefficient  $r$  between the combined  $X$  and an indicator variable, where  $t$  and  $r$  are related by

$$r = \frac{t}{\sqrt{t^2 + \nu}}, \quad t = \nu \frac{r}{1 - r^2}. \quad (32.3.4)$$

Test problem	$p$ -value	Reject $H_0$
$H_{01} : \mu_1 \leq \mu_2$ vs $H_{A1} : \mu_1 > \mu_2$	$F_t(-t, \nu)$	$t > t_{\nu;1-\alpha}$
$H_{02} : \mu_1 \geq \mu_2$ vs $H_{A2} : \mu_1 < \mu_2$	$F_t(t, \nu)$	$t > t_{\nu;\alpha}$
$H_{03} : \mu_1 = \mu_2$ vs $H_{A3} : \mu_1 \neq \mu_2$	$F_t(t, \nu) - F_t(-t, \nu)$	$t > t_{\nu;1-\alpha/2}$ or $t > t_{\nu;\alpha/2}$

**Table 32.5:** Student's t-test, 2 independent samples, tests for the mean

### 32.3.2.2 Tests: examples

An actual call to the function, requesting Student's t-test with description, the critical value for a two-sided test, the p-value for  $H_{03}$  (in the case of TTest this is  $\mu_1 \neq \mu_2$ ), for 2 independent samples of size 10 and standard deviation 1 each, with means 2.3 and 4.5, and a type I error  $\alpha = 0.05$  would be

```
Result = Test2i("TTest", 0.05, 10, {2.3, 4.5}, 1, "Description + TCrit2 + PValueH03")
mp>Show Result
```

which produces the output

```
df: 18
Difference of Means: -2.2
t-value: -4.9193496
TCrit2: 2.100922
PValueH03: 0.0001106
```

### 32.3.2.3 Confidence Intervals: algorithms and formulas

Let  $A_1 = t_{\nu,\alpha} \cdot s$  and  $A_2 = t_{\nu,\alpha/2} \cdot s$ , where  $s$  and  $\nu$  are defined in (32.3.3), and  $t_{\nu,\alpha}$  denotes the  $\alpha$ -quantile of the (central)  $t$ -distribution with  $\nu$  degrees of freedom (see section 6.13.3). Then confidence intervals for  $\mu_1 - \mu_2$  can be calculated as summarized in Table 32.6.

Type	Confidence Interval (Difference of Means)
Left-sided	$-\infty \leq \mu_1 - \mu_2 \leq (\bar{x}_1 - \bar{x}_2) + A_1$
Right-sided	$(\bar{x}_1 - \bar{x}_2) - A_1 \leq \mu_1 - \mu_2 \leq +\infty$
Two-sided	$(\bar{x}_1 - \bar{x}_2) - A_2 \leq \mu_1 - \mu_2 \leq (\bar{x}_1 - \bar{x}_2) + A_2$

**Table 32.6:** Student's t-test, 2 independent samples, confidence intervals for the difference of the means

### 32.3.2.4 Confidence Intervals: examples

An actual call to the function, requesting Student's t-test with description, the critical value for a two-sided test, the p-value for  $H_{03}$  (in the case of TTest this is  $\mu_1 \neq \mu_2$ ), for 2 independent samples of size 10 and standard deviation 1 each, with means 2.3 and 4.5, and a type I error  $\alpha = 0.05$  would be

```
Result = Test2i("TTest", 0.05, 10, {2.3, 4.5}, 1, "Description + TCrit2 + PValueH03")
mp>Show Result
```

which produces the output

### 32.3. TESTS FOR MEANS FROM 2 INDEPENDENT SAMPLES (STUDENT'S T-TEST)731

```
df: 18
Difference of Means: -2.2
t-value: -4.9193496
TCrit2: 2.100922
PValueH03: 0.0001106
```

### 32.3.3 Power

---

Function **StudentTTestPower2i**(*Type1Error* As *mpNum*, **N** As *mpNum*[], **Mean** As *mpNum*[], **StDev** As *mpNum*[], **Output** As String) As *mpNumList*

---

**NOT YET IMPLEMENTED**

---

The function **StudentTTestPower2i** returns power estimations and related information for Student's t-test

**Parameters:**

*Type1Error*: An real number greater then 0 and less than 1.

*N*: An array of integers greater than 1, representing the samples sizes

*Mean*: An array of reals, representing the means

*StDev*: An array of positive reals, representing the standard deviations

*Output*: A string describing the output choices

See section 32.1.4.2 for the options for *Type1Error* and *Output*). Algorithms and formulas are given in section 32.2.3.1.

#### 32.3.3.1 Power calculations: algorithms and formulas

Let  $\sigma_1^2 = \sigma_2^2 = \sigma^2$ ,  $N = n_1 + n_2$ ,  $q_1 = n_1/N$ ,  $q_2 = n_2/N$ , and  $\nu = N - 2$ . Define

$$\tilde{\rho} = \frac{\mu_1 - \mu_2}{\sigma} \sqrt{\frac{q_1 + q_2}{q_1 \cdot q_2}}, \text{ and } \delta = \sqrt{N}\tilde{\rho}. \quad (32.3.5)$$

Let  $F_{t'}(\cdot, \nu, \delta)$  denote the CDF of the (singly) noncentral *t*-distribution with  $\nu$  degrees of freedom and noncentrality parameter  $\delta$  (see section and let  $t_{\nu, \alpha}$  denote the  $\alpha$ -quantile of the central *t*-distribution with  $\nu$  degrees of freedom (see section 6.13.3)).

Then the power for accepting  $H_A$  at the confidence level  $\alpha$  can be calculated as summarized in Table 32.7.

Test	Null Hypothesis	Alternative	Power
1 sided	$H_{01} : \mu_1 \leq \mu_2$	$H_{A1} : \mu_1 > \mu_2$	$F_{t'}(-t_{\nu;1-\alpha}, \nu, \delta)$
1 sided	$H_{02} : \mu_1 \geq \mu_2$	$H_{A2} : \mu_1 < \mu_2$	$F_{t'}(t_{\nu;1-\alpha}, \nu, \delta)$
2 sided	$H_{03} : \mu_1 = \mu_2$	$H_{A1} : \mu_1 > \mu_2$	$F_{t'}\left(-t_{\nu;1-\alpha/2}, \nu, \delta\right)$
2 sided	$H_{03} : \mu_1 = \mu_2$	$H_{A2} : \mu_1 < \mu_2$	$F_{t'}\left(t_{\nu;1-\alpha/2}, \nu, \delta\right)$
2 sided	$H_{03} : \mu_1 = \mu_2$	$H_{A3} : \mu_1 \neq \mu_2$	$F_{t'}\left(t_{\nu;1-\alpha/2}, \nu, \delta\right) - F_{t'}\left(-t_{\nu;1-\alpha/2}, \nu, \delta\right)$

**Table 32.7:** Student's t-test, 2 independent samples, power calculations

#### 32.3.3.2 Power calculations: examples

An actual call to the function, requesting Student's t-test with description, the critical value for a two-sided test, the power for  $H_{A3}$  (in the case of TTest this is  $\mu_1 \neq \mu_2$ ), for 2 independent samples of size 10 and standard deviation 1 each, with means 2.3 and 4.5, and a type I error  $\alpha = 0.05$  would be

---

```
Result = TestPower2i("TTest", 0.05, 10, {2.3, 4.5}, 1, "Description + TCrit2 +
PowerHA3")
```

---

```
mp.Print Result
```

which produces the output

```
df: 18
Difference of Means: -2.2
t-value: -4.9193496
TCrit2: 2.100922
PValueH03: 0.996354
```

### 32.3.4 Sample Size Calculation

---

Function **StudentTTestSampleSize2i**(*Type1Error* As *mpNum*, *Type2Error* As *mpNum*, *Mean* As *mpNum*[], *StDev* As *mpNum*[], *Output* As String) As *mpNumList*

---

**NOT YET IMPLEMENTED**

---

The function **StudentTTestSampleSize2i** returns sample size estimations and related information for Student's t-test

**Parameters:**

*Type1Error*: An real number greater then 0 and less than 1.

*Type2Error*: An real number greater then 0 and less than 1.

*Mean*: An array of reals, representing the means

*StDev*: An array of positive reals, representing the standard deviations

*Output*: A string describing the output choices

See section 32.1.4.2 for the options for *Type1Error*, *Type2Error* and *Output*. Algorithms and formulas are given below.

#### 32.3.4.1 Sample size calculations: algorithms and formulas

Let  $N_t'(\alpha, \beta, \tilde{\rho})$  denote the sample size function of the (singly) noncentral  $t$ -distribution (see section 30.8.7 ) for a given confidence level  $\alpha$ , power  $\beta$  and noncentrality parameter  $\tilde{\rho}$  (as defined in equation 32.3.5).

The required total sample size  $N$  can be calculated as summarized in Table 32.8.

Test	Null Hypothesis	Alternative	Minimal sample size
1 sided	$H_{01} : \mu_1 \leq \mu_2$	$H_{A1} : \mu_1 > \mu_2$	$N_t'(\alpha, \beta, \tilde{\rho})$
1 sided	$H_{02} : \mu_1 \geq \mu_2$	$H_{A2} : \mu_1 < \mu_2$	$N_t'(\alpha, \beta, \tilde{\rho})$
2 sided	$H_{03} : \mu_1 = \mu_2$	$H_{A1} : \mu_1 > \mu_2$	$N_t'(\alpha, \beta, \tilde{\rho})$
2 sided	$H_{03} : \mu_1 = \mu_2$	$H_{A2} : \mu_1 < \mu_2$	$N_t'(\alpha, \beta, \tilde{\rho})$
2 sided	$H_{03} : \mu_1 = \mu_2$	$H_{A3} : \mu_1 \neq \mu_2$	$N_{2t'}(\alpha, \beta, \tilde{\rho})$

**Table 32.8:** Student's t-test, 2 independent samples, sample size calculations

Note that the returned value of  $N$  will in general not be an integer, and rounding up may be required to ensure that all  $n_i$  are integers.

#### 32.3.4.2 Sample size calculations: examples

An actual call to the function, requesting an upper sample size estimate (and actual power) for  $\alpha = 0.95$ ,  $\beta = 0.1$  , and standard deviations  $\sigma_1 = \sigma_2 = 1$  , means  $\mu_1 = 2.3$  and  $\mu_2 = 4.5$ , would be

---

```
Result = TestSampleSize2i("TTest", 0.05, 0.1, {2.3, 4.5}, 1, "UpperN + UpperNPower")
mp.Print Result
```

---

which produces the output

```
UpperN: 2 x 6 = 12
UpperNPower: 0.9285919
```

### 32.3.5 Confidence Interval (Effect size)

Let  $T$  be a statistic according to the non-central  $t$ -distribution with  $n$  degrees of freedom and a non-centrality parameter  $\delta$ . Then the lower confidence limit  $\hat{\delta}$  of level  $1 - \alpha$  and the two-sided confidence interval  $[\underline{\delta}, \bar{\delta}]$  of the non-centrality parameter  $\delta$  of level  $1 - \alpha$  are given by [Akahira et al. \(1995\)](#):

$$\hat{\delta} = bT - z_\alpha \sqrt{k} + hT^3(z_\alpha^2 - 1)/k, \quad (32.3.6)$$

$$\underline{\delta} = bT - z_{\alpha/2} \sqrt{k} + hT^3(z_{\alpha/2}^2 - 1)/k, \quad (32.3.7)$$

$$\bar{\delta} = bT + z_{\alpha/2} \sqrt{k} - hT^3(z_{\alpha/2}^2 - 1)/k, \quad (32.3.8)$$

where  $k = 1 + (1 - b^2)T^2$ ,  $h$  and  $b$  are defined in equation (30.8.19), and  $z_\alpha$  denotes the  $\alpha$ -quantile of the normal distribution (see section 6.11.3).

Let  $A_1 = t_{\nu,\alpha} \cdot s$  and  $A_2 = t_{\nu,\alpha/2} \cdot s$ , where  $s$  and  $\nu$  are defined in (32.3.3), and  $t_{\nu,\alpha}$  denotes the  $\alpha$ -quantile of the (central)  $t$ -distribution with  $\nu$  degrees of freedom (see section 6.13.3). Then confidence intervals for the difference of 2 means can be calculated (needs to be worked out).

# **Part VI**

## **Appendices**

# Appendix A

## Python

### A.1 Overview

Python is a widely used general-purpose, high-level programming language. Its design philosophy emphasizes code readability, and its syntax allows programmers to express concepts in fewer lines of code than would be possible in languages such as C. The language provides constructs intended to enable clear programs on both a small and large scale.

Python supports multiple programming paradigms, including object-oriented, imperative and functional programming or procedural styles. It features a dynamic type system and automatic memory management and has a large and comprehensive standard library.

Like other dynamic languages, Python is often used as a scripting language, but is also used in a wide range of non-scripting contexts. Using third-party tools, Python code can be packaged into standalone executable programs. Python interpreters are available for many operating systems.

#### A.1.1 Downloading and installing mpMath

mpmath is a free (BSD licensed) Python library for real and complex floating-point arithmetic with arbitrary precision. It has been developed by Fredrik Johansson since 2007, with help from many contributors. mpmath runs on Python 2.5 or higher (including Python 3.x), both on 32 and 64 bit, with no other required dependencies.

mpmath can be downloaded from

<http://mpmath.org/>

The latest version is 0.19, released 2014-06-10. Download: mpmath-0.19.tar.gz., extract it, open a Windows command prompt in the extracted directory, and run

```
python setup.py install
```

The license is the New BSD License can be found in appendix G.2.2)

The contributors are listed in section F.1.1

#### A.1.2 Running tests

It is recommended that you run mpmath's full set of unit tests to make sure everything works. The tests are located in the tests subdirectory of the main mpmath directory. They can be run in the interactive interpreter using the runtests() function:

---

```
import mpmath
mpmath.runtests()
```

---

## A.1.3 Precision management

### A.1.3.1 autoprec()

`mpmath.autoprec(ctx, f, maxprec=None, catch=(), verbose=False)`

Return a wrapped copy of `f` that repeatedly evaluates `f` with increasing precision until the result converges to the full precision used at the point of the call.

This heuristically protects against rounding errors, at the cost of roughly a 2x slowdown compared to manually setting the optimal precision. This method can, however, easily be fooled if the results from `f` depend ‘discontinuously’ on the precision, for instance if catastrophic cancellation can occur. Therefore, `autoprec()` should be used judiciously.

### Examples

Many functions are sensitive to perturbations of the input arguments. If the arguments are decimal numbers, they may have to be converted to binary at a much higher precision. If the amount of required extra precision is unknown, `autoprec()` is convenient:

---

```
>>> from mpmath import *
>>> mp.dps = 15
>>> mp.pretty = True
>>> besselj(5, 125 * 10**28) # Exact input
-8.03284785591801e-17
>>> besselj(5, '1.25e30') # Bad
7.12954868316652e-16
>>> autoprec(besselj)(5, '1.25e30') # Good
-8.03284785591801e-17
```

---

The following fails to converge because  $\sin(\pi) = 0$  whereas all finite-precision approximations of  $\pi$  give nonzero values:

---

```
>>> autoprec(sin)(pi)
Traceback (most recent call last):
...
NoConvergence: autoprec: prec increased to 2910 without convergence
```

---

As the following example shows, `autoprec()` can protect against cancellation, but is fooled by too severe cancellation:

---

```
>>> x = 1e-10
>>> exp(x)-1; expm1(x); autoprec(lambda t: exp(t)-1)(x)
1.00000008274037e-10
1.00000000005e-10
1.00000000005e-10
>>> x = 1e-50
>>> exp(x)-1; expm1(x); autoprec(lambda t: exp(t)-1)(x)
0.0
```

---

---

```
1.0e-50
0.0
```

---

With `catch`, an exception or list of exceptions to intercept may be specified. The raised exception is interpreted as signaling insufficient precision. This permits, for example, evaluating a function where a too low precision results in a division by zero:

---

```
>>> f = lambda x: 1/(exp(x)-1)
>>> f(1e-30)
Traceback (most recent call last):
...
ZeroDivisionError
>>> autoprec(f, catch=ZeroDivisionError)(1e-30)
1.0e+30
```

---

### A.1.3.2 workprec()

`mpmath.workprec(ctx, n, normalize_output=False)`  
The block

---

```
with workprec(n):
<code>
```

---

sets the precision to n bits, executes `|code|`, and then restores the precision.

`workprec(n)(f)` returns a decorated version of the function `f` that sets the precision to n bits before execution, and restores the precision afterwards. With `normalize_output=True`, it rounds the return value to the parent precision.

### A.1.4 workdps()

`mpmath.workdps(ctx, n, normalize_output=False)`

This function is analogous to `workprec` (see documentation) but changes the decimal precision instead of the number of bits.

#### A.1.4.1 extraprec()

`mpmath.extraprec(ctx, n, normalize_output=False)`

The block

---

```
with extraprec(n):
<code>
```

---

increases the precision n bits, executes `|code|`, and then restores the precision.

`extraprec(n)(f)` returns a decorated version of the function `f` that increases the working precision by n bits before execution, and restores the parent precision afterwards. With `normalize_output=True`, it rounds the return value to the parent precision.

### A.1.4.2 extradps()

`mpmath.extradps(ctx, n, normalize_output=False)`

This function is analogous to `extraprec` (see documentation) but changes the decimal precision instead of the number of bits.

## A.1.5 Performance and debugging

### A.1.5.1 memoize()

`mpmath.memoize(ctx, f)`

Return a wrapped copy of `f` that caches computed values, i.e. a memoized copy of `f`. Values are only reused if the cached precision is equal to or higher than the working precision:

---

```
>>> from mpmath import *
>>> mp.dps = 15; mp.pretty = True
>>> f = memoize(maxcalls(sin, 1))
>>> f(2)
0.909297426825682
>>> f(2)
0.909297426825682
>>> mp.dps = 25
>>> f(2)
Traceback (most recent call last):
...
NoConvergence: maxcalls: function evaluated 1 times
```

---

### A.1.5.2 maxcalls()

`mpmath.maxcalls(ctx, f, N)`

Return a wrapped copy of `f` that raises `NoConvergence` when `f` has been called more than `N` times:

---

```
>>> from mpmath import *
>>> mp.dps = 15
>>> f = maxcalls(sin, 10)
>>> print(sum(f(n) for n in range(10)))
1.95520948210738
>>> f(10)
Traceback (most recent call last):
...
NoConvergence: maxcalls: function evaluated 10 times
```

---

### A.1.5.3 monitor()

`mpmath.monitor(f, input='print', output='print')`

Returns a wrapped copy of `f` that monitors evaluation by calling `input` with every input (`args`, `kwargs`) passed to `f` and `output` with every value returned from `f`. The default action (specify using the special string value '`'print'`') is to print inputs and outputs to `stdout`, along with the total evaluation count:

---

```
>>> from mpmath import *
>>> mp.dps = 5; mp.pretty = False
>>> diff(monitor(exp), 1) # diff will eval f(x-h) and f(x+h)
in 0 (mpf('0.9999999906867742538452148'),) {}
out 0 mpf('2.7182818259274480055282064')
in 1 (mpf('1.0000000009313225746154785'),) {}
out 1 mpf('2.7182818309906424675501024')
mpf('2.7182808')
```

---

To disable either the input or the output handler, you may pass None as argument.

Custom input and output handlers may be used e.g. to store results for later analysis:

---

```
>>> mp.dps = 15
>>> input = []
>>> output = []
>>> findroot(monitor(sin, input.append, output.append), 3.0)
mpf('3.1415926535897932')
>>> len(input) # Count number of evaluations
9
>>> print(input[3]); print(output[3])
((mpf('3.1415076583334066'),), {})
8.49952562843408e-5
>>> print(input[4]); print(output[4])
((mpf('3.1415928201669122'),), {})
-1.66577118985331e-7
```

---

#### A.1.5.4 timing()

`mpmath.timing(f, *args, **kwargs)`

Returns time elapsed for evaluating `f()`. Optionally arguments may be passed to time the execution of `f(*args, **kwargs)`.

If the first call is very quick, `f` is called repeatedly and the best time is returned.

## A.2 CPython

CPython, the reference implementation of Python, is free and open source software and has a community-based development model, as do nearly all of its alternative implementations. CPython is managed by the non-profit Python Software Foundation.

For further information on Python, see [Wikipedia: Python](#) (the text above has been copied from this reference), or the [Python Homepage](#). Support for COM is included in the distribution of the [ActivePython Community Edition](#).

Python can use GMP und MPFR thanks to [GMPY2](#), with documentation [here](#).

IPython is an integrations platform for various scientific libraries (NumPy, SciPy, matplotlib, pandas etc.) <http://ipython.org/>. Popular distributions are the Community Edition of Anaconda: <http://docs.continuum.io/anaconda/index.html>,

Book recommendation: [McKinney \(2012\)](#).

Example for using the library

---

```
#Enable COM support
from win32com.client import Dispatch

#Load the mpNumerics library
mp = Dispatch("mpNumerics.mp_Lib")

#Set Floating point type to MPFR with 60 decimal digits precision
mp.FloatingPointType = 3
mp.Prec10 = 60

#Assign values to x1 and x2
x1 = mp.Real(4.5)
x2 = mp.Real(1.21)

#Calculate x3 = x1 / x2
x3 = x1.Div(x2)

#print the value of x3
print (x3.Str())
```

---

Example for using Excel

---

```
#Enable COM support
from win32com.client import Dispatch

#Load the Excel library
xl = Dispatch("Excel.Application")
xl.Visible = 1
xl.Workbooks.Add()
xl.Cells(1,1).Value = "Hello442"
print("From Python")
```

---

To compile the mpmath library libraries, Python 2.7 is required.

### A.2.1 Downloading and installing CPython 2.7

ActivePython is ActiveState's complete and ready-to-install distribution of Python. It provides a one-step installation of all essential Python modules, as well as extensive documentation. The Windows distribution ships with PyWin32 – a suite of Windows tools developed by Mark Hammond, including bindings to the Win32 API and Windows COM. ActivePython can be downloaded from

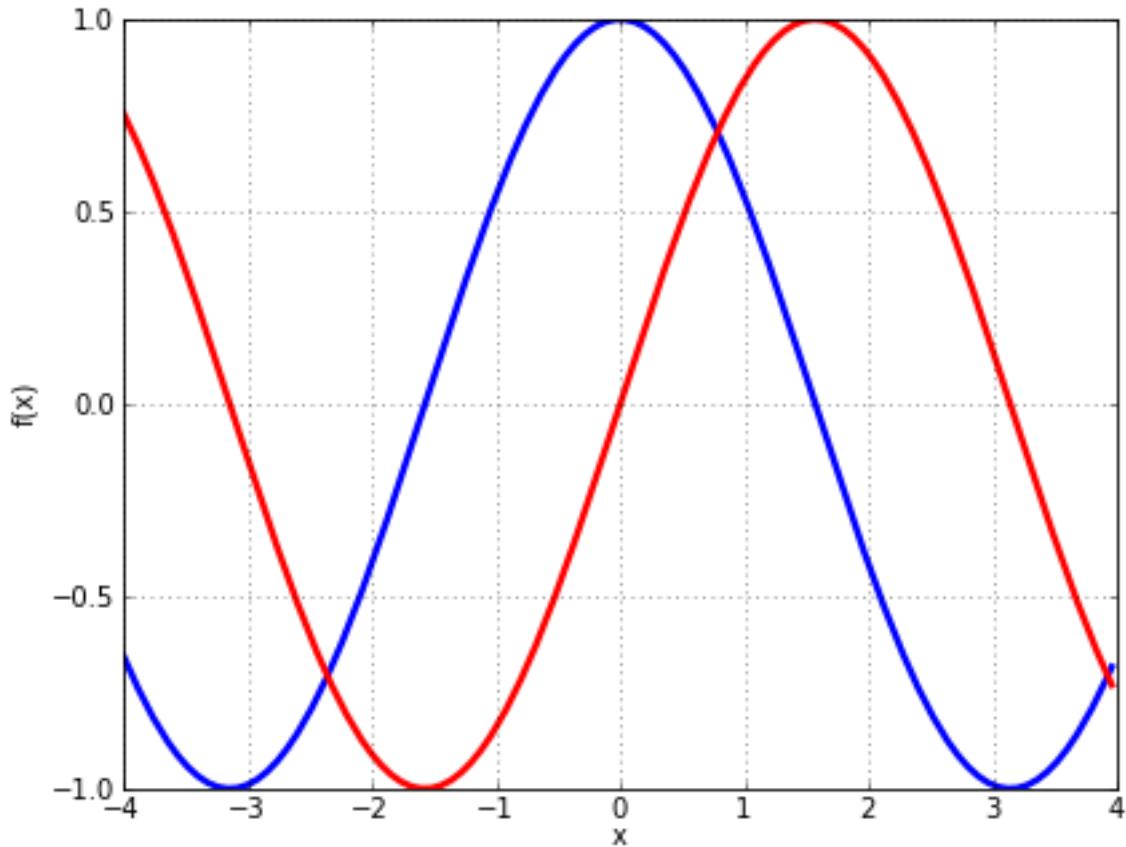
<http://www.activestate.com/activepython/downloads>.

The latest release version of the 2.7x series is 2.7.6.9. You need to download 2 separate files to support compilation of both 32 bit and 64 bit dlls.

## A.2.2 Plotting

If matplotlib is available, the functions plot and cplot in mpmath can be used to plot functions respectively as x-y graphs and in the complex plane. Also, splot can be used to produce 3D surface plots.

### A.2.2.1 Function curve plots



**Figure A.1:** MpMath 2Dplot

Output of `plot([cos, sin], [-4, 4])`

```
mpmath.plot(ctx, f, xlim=[-5, 5], ylim=None, points=200, file=None, dpi=None, singularities=[], axes=None)
```

Shows a simple 2D plot of a function  $f(x)$  or list of functions  $[f_0(x), f_1(x), \dots, f_n(x)]$  over a given interval specified by `xlim`. Some examples:

---

```
plot(lambda x: exp(x)*li(x), [1, 4])
plot([cos, sin], [-4, 4])
plot([fresnels, fresnelc], [-4, 4])
plot([sqrt, cbrt], [-4, 4])
plot(lambda t: zeta(0.5+t*j), [-20, 20])
plot(floor, ceil, abs, sign, [-5, 5])
```

---

Points where the function raises a numerical exception or returns an infinite value are removed from the graph. Singularities can also be excluded explicitly as follows (useful for removing erroneous vertical lines):

---

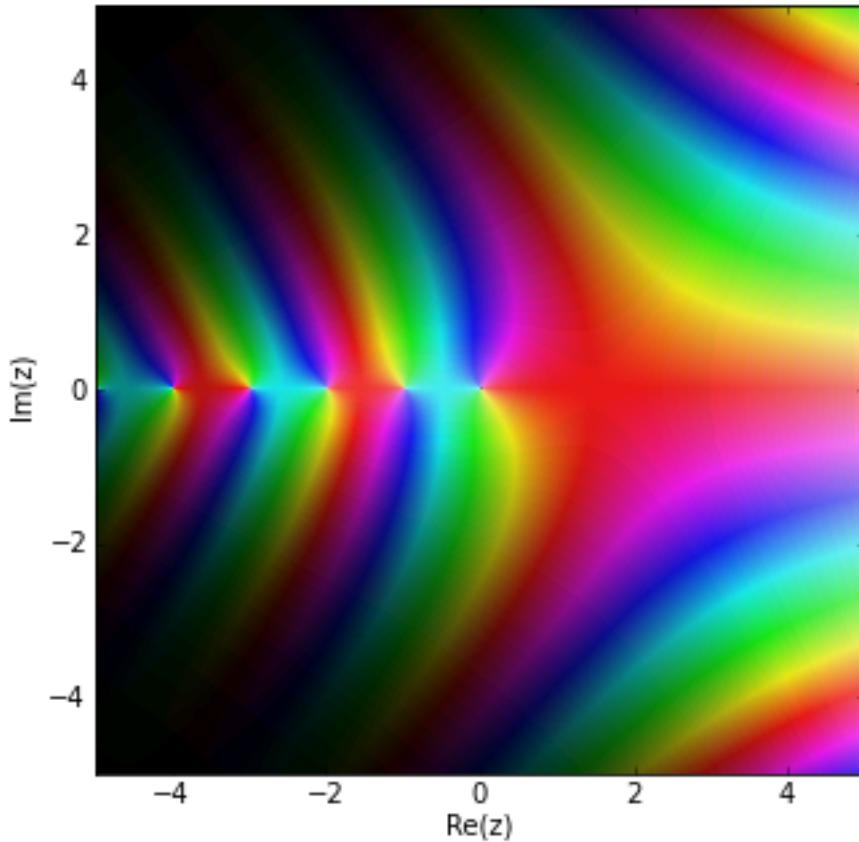
```
plot(cot, ylim=[-5, 5]) # bad
plot(cot, ylim=[-5, 5], singularities=[-pi, 0, pi]) # good
```

---

For parts where the function assumes complex values, the real part is plotted with dashes and the imaginary part is plotted with dots.

Note: This function requires matplotlib (pylab).

### A.2.2.2 Complex function plots



**Figure A.2:** MpMath cplot

Output of `fp.cplot(fp.gamma, points=100000)`

```
mpmath.cplot(ctx, f, re=[-5, 5], im=[-5, 5], points=2000, color=None, verbose=False, file=None, dpi=None, axes=None)
```

Plots the given complex-valued function `f` over a rectangular part of the complex plane specified by the pairs of intervals `re` and `im`. For example:

---

```
cplot(lambda z: z, [-2, 2], [-10, 10])
cplot(exp)
cplot(zeta, [0, 1], [0, 50])
```

---

By default, the complex argument (phase) is shown as color (hue) and the magnitude is show as brightness. You can also supply a custom color function (color). This function should take a complex number as input and return an RGB 3-tuple containing floats in the range 0.0-1.0.

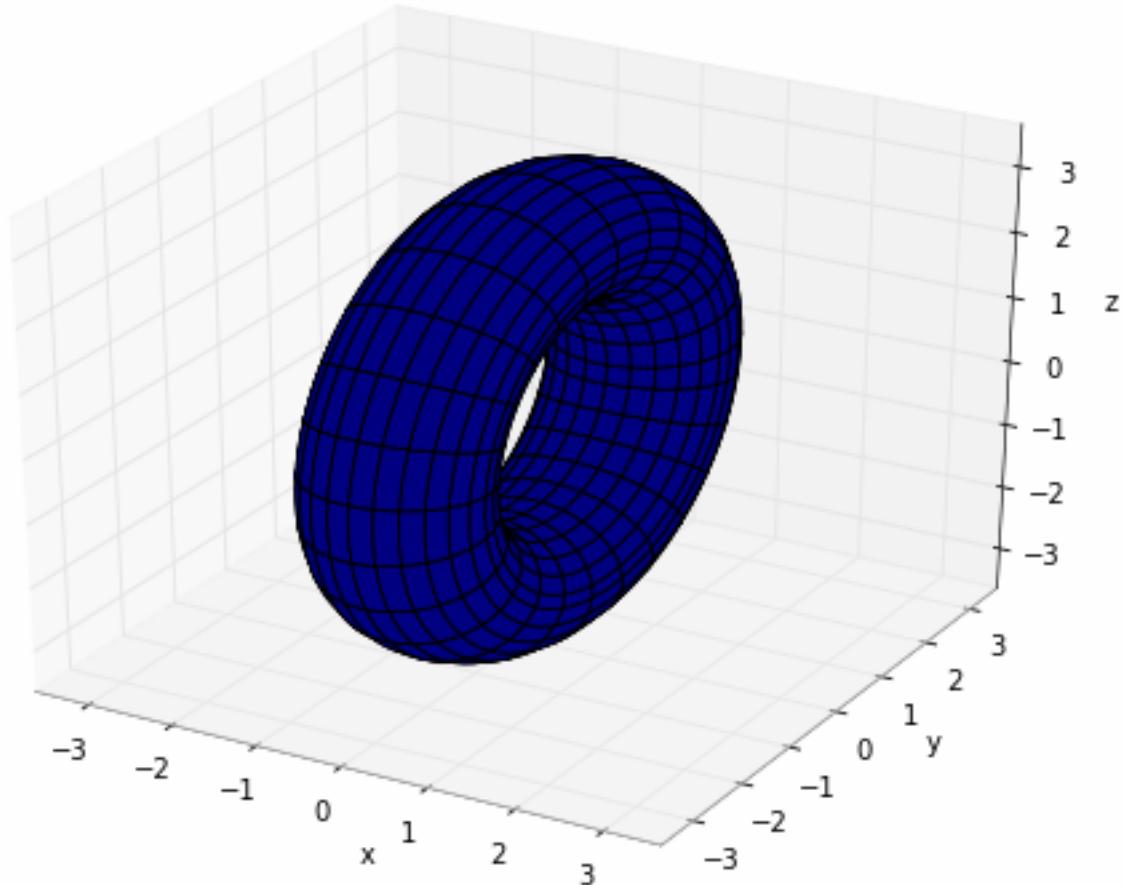
To obtain a sharp image, the number of points may need to be increased to 100,000 or thereabout. Since evaluating the function that many times is likely to be slow, the 'verbose' option is useful to display progress.

Note: This function requires matplotlib (pylab).

### A.2.2.3 3D surface plots

Output of splot for the donut example.

```
mpmath.splot(ctx, f, u=[-5, 5], v=[-5, 5], points=100, keep_aspect=True, wireframe=False, file=None, dpi=None, axes=None)
```



**Figure A.3:** MpMath Surface plot

Plots the surface defined by  $f$ .

If  $f$  returns a single component, then this plots the surface defined by  $z = f(x, y)$  over the rectangular domain with  $x = u$  and  $y = v$ .

If  $f$  returns three components, then this plots the parametric surface  $x, y, z = f(u, v)$  over the pairs of intervals  $u$  and  $v$ .

For example, to plot a simple function:

---

```
>>> from mpmath import *
>>> f = lambda x, y: sin(x+y)*cos(y)
>>> splot(f, [-pi,pi], [-pi,pi])
```

---

Plotting a donut:

---

```
>>> r, R = 1, 2.5
>>> f = lambda u, v: [r*cos(u), (R+r*sin(u))*cos(v), (R+r*sin(u))*sin(v)]
```

---

```
>>> splot(f, [0, 2*pi], [0, 2*pi])
```

Note: This function requires matplotlib (pylab) 0.98.5.3 or higher.

### A.2.3 Using the C-API

This document describes how to write modules in C or C++ to extend the Python interpreter with new modules. Those modules can not only define new functions but also new object types and their methods. The document also describes how to embed the Python interpreter in another application, for use as an extension language. Finally, it shows how to compile and link extension modules so that they can be loaded dynamically (at run time) into the interpreter, if the underlying operating system supports this feature.

This document assumes basic knowledge about Python. For an informal introduction to the language, see The Python Tutorial. The Python Language Reference gives a more formal definition of the language. The Python Standard Library documents the existing object types, functions and modules (both built-in and written in Python) that give the language its wide application range.

For a detailed description of the whole Python/C API, see the separate Python/C API Reference Manual.

#### . Extending Python with C or C++

It is quite easy to add new built-in modules to Python, if you know how to program in C. Such extension modules can do two things that cannot be done directly in Python: they can implement new built-in object types, and they can call C library functions and system calls.

To support extensions, the Python API (Application Programmers Interface) defines a set of functions, macros and variables that provide access to most aspects of the Python run-time system. The Python API is incorporated in a C source file by including the header "Python.h". The compilation of an extension module depends on its intended use as well as on your system setup; details are given in later chapters.

Note: The C extension interface is specific to CPython, and extension modules do not work on other Python implementations. In many cases, it is possible to avoid writing C extensions and preserve portability to other implementations. For example, if your use case is calling C library functions or system calls, you should consider using the ctypes module or the cffi library rather than writing custom C code. These modules let you write Python code to interface with C code and are more portable between implementations of Python than writing and compiling a C extension module.

#### A.2.3.1 Downloading and installing Gmpy2

GMPY and GMPY2 are C-coded Python extension modules that support fast multiple-precision arithmetic. GMPY only supports the GMP library and provides fast multiple-precision integer and rational arithmetic. The limited mpf type from GMP is also supported. GMPY2 supports the GMP library for integer and rational arithmetic but GMPY2 adds support for multiple-precision real and complex arithmetic as provided by the MPFR and MPC libraries.

GMPY2 can be downloaded from

<https://code.google.com/p/gmpy/>.

The file which needs to download is specific for the Python version. For Python 2.7x 32 bit, the file gmpy2-2.0.3.win32-py2.7.exe needs to be downloaded. After download, start the executable file and follow the instructions.

The license can be found in appendix [G.1.3](#))

The contributors are listed in section [F.1.1](#)

## A.2.4 Interfaces to the C family of languages

### A.2.4.1 Windows, GNU/Linux, Mac OSX: GNU Compiler Collection

The GNU Compiler Collection (GCC) is a compiler system produced by the GNU Project supporting various programming languages. GCC is a key component of the GNU toolchain. The Free Software Foundation (FSF) distributes GCC under the GNU General Public License (GNU GPL). GCC has played an important role in the growth of free software, as both a tool and an example.

Originally named the GNU C Compiler, because it only handled the C programming language, GCC 1.0 was released in 1987 and the compiler was extended to compile C++ in December of that year.<sup>[1]</sup> Front ends were later developed for Objective-C, Objective-C++, Fortran, Java, Ada, and Go among others.<sup>[3]</sup>

As well as being the official compiler of the unfinished GNU operating system, GCC has been adopted as the standard compiler by most other modern Unix-like computer operating systems, including Linux and the BSD family. A port to RISC OS has also been developed extensively in recent years. There is also an old (3.0) port of GCC to Plan9, running under its ANSI/POSIX Environment (APE).<sup>[4]</sup> GCC is also available for Microsoft Windows operating systems and for the ARM processor used by many portable devices.

For further information on the GNU Compiler Collection, see [Wikipedia: GCC](#) (the text above has been copied from this reference), or the [GCC Homepage](#).

### A.2.4.2 Windows: MSVC

Microsoft Visual C++ (often abbreviated as MSVC or VC++) is a commercial (free version available), integrated development environment (IDE) product from Microsoft for the C, C++, and C++/CLI programming languages. It features tools for developing and debugging C++ code, especially code written for the Microsoft Windows API, the DirectX API, and the Microsoft .NET Framework.

Although the product originated as an IDE for the C programming language, the compiler's support for that language conforms only to the original edition of the C standard, dating from 1989. The later revisions of the standard, C99 and C11, are not supported.<sup>[41]</sup> According to Herb Sutter, the C compiler is only included for "historical reasons" and is not planned to be further developed. Users are advised to either use only the subset of the C language that is also valid C++, and then use the C++ compiler to compile their code, or to just use a different compiler such as Intel C++ Compiler or the GNU Compiler Collection instead.<sup>[42]</sup>

For further information on Microsoft Visual C++, see [Wikipedia: MSVC](#) (the text above has been copied from this reference), or the [MSVC Homepage](#).

### A.2.4.3 Windows, GNU/Linux, Mac OSX: C

In computing, C is a general-purpose programming language initially developed by Dennis Ritchie between 1969 and 1973 at AT&T Bell Labs.[5][6] Like most imperative languages in the ALGOL tradition, C has facilities for structured programming and allows lexical variable scope and recursion, while a static type system prevents many unintended operations. Its design provides constructs that map efficiently to typical machine instructions, and therefore it has found lasting use in applications that had formerly been coded in assembly language, most notably system software like the Unix computer operating system.[7]

C is one of the most widely used programming languages of all time,[8][9] and C compilers are available for the majority of available computer architectures and operating systems.

Many later languages have borrowed directly or indirectly from C, including D, Go, Rust, Java, JavaScript, Limbo, LPC, C#, Objective-C, Perl, PHP, Python, Verilog (hardware description language),[4] and Unix's C shell. These languages have drawn many of their control structures and other basic features from C. Most of them (with Python being the most dramatic exception) are also very syntactically similar to C in general, and they tend to combine the recognizable expression and statement syntax of C with underlying type systems, data models, and semantics that can be radically different. C++ and Objective-C started as compilers that generated C code; C++ is currently nearly a superset of C,[10] while Objective-C is a strict superset of C.[11]

Before there was an official standard for C, many users and implementors relied on an informal specification contained in a book by Dennis Ritchie and Brian Kernighan; that version is generally referred to as "K&R" C. In 1989 the American National Standards Institute published a standard for C (generally called "ANSI C" or "C89"). The next year, the same specification was approved by the International Organization for Standardization as an international standard (generally called "C90"). ISO later released an extension to the internationalization support of the standard in 1995, and a revised standard (known as "C99") in 1999. The current version of the standard (now known as "C11") was approved in December 2011.[12]

For further information on C++, see [Wikipedia: C](#) (the text above has been copied from this reference), or the [Wikipedia: GCC](#) (the text above has been copied from this reference), or the [GCC Homepage](#).

Example in C

---

```
#include <iostream>
#include "mpreal.h"

using mpfr::mpreal;
using std::cout;
using std::endl;

// double - version
double schwefel(double x)
{
    return 418.9829 - x * sin(sqrt(abs(x)));
}

//MPFR C - version
void mpfr_schwefel(mpfr_t y, mpfr_t x)
```

```
{  
mpfr_t t;  
mpfr_init(t);  
mpfr_abs(t,x,GMP_RNDN);  
mpfr_sqrt(t,t,GMP_RNDN);  
mpfr_sin(t,t,GMP_RNDN);  
mpfr_mul(t,t,x,GMP_RNDN);  
mpfr_set_str(y,"418.9829",10,GMP_RNDN);  
mpfr_sub(y,y,t,GMP_RNDN);  
mpfr_clear(t);  
}  
  
// MPFR C++ - version  
mpreal mpfr_schwefel(mpreal& x)  
{  
    return "418.9829" - x*sin(sqrt(abs(x)));  
}  
  
  
int main(int argc, char* argv[])  
{  
    const int digits = 50;  
    mpreal::set_default_prec(mpfr::digits2bits(digits));  
    const mpreal pi      = mpfr::const_pi();  
    mpreal x      = "-343.5";  
    mpreal SResult = mpfr_schwefel(x);  
    cout.precision(digits); // Show all the digits  
    cout << "pi      = " << pi      << endl;  
    cout << "SResult = " << SResult << endl;  
    return 0;  
}
```

---

#### A.2.4.4 Windows, GNU/Linux, Mac OSX: C++

C++ (pronounced "see plus plus") is a statically typed, free-form, multi-paradigm, compiled, general-purpose programming language. It is regarded as an intermediate-level language, as it comprises both high-level and low-level language features.[3] Developed by Bjarne Stroustrup starting in 1979 at Bell Labs, C++ was originally named C with Classes, adding object oriented features, such as classes, and other enhancements to the C programming language. The language was renamed C++ in 1983,[4] as a pun involving the increment operator.

C++ is one of the most popular programming languages[5][6] and is implemented on a wide variety of hardware and operating system platforms. As an efficient compiler to native code, its application domains include systems software, application software, device drivers, embedded software, high-performance server and client applications, and entertainment software such as video games.[7] Several groups provide both free and proprietary C++ compiler software, including the GNU Project, LLVM, Microsoft, Intel and Embarcadero Technologies. C++ has greatly influenced many other popular programming languages, most notably C# and Java.

C++ is also used for hardware design, where the design is initially described in C++, then analyzed, architecturally constrained, and scheduled to create a register-transfer level hardware description language via high-level synthesis.[8]

The language began as enhancements to C, first adding classes, then virtual functions, operator overloading, multiple inheritance, templates and exception handling, among other features. After years of development, the C++ programming language standard was ratified in 1998 as ISO/IEC 14882:1998. The standard was amended by the 2003 technical corrigendum, ISO/IEC 14882:2003. The current standard extending C++ with new features was ratified and published by ISO in September 2011 as ISO/IEC 14882:2011 (informally known as C++11).[9]

For further information on C++, see [Wikipedia: C++](#) (the text above has been copied from this reference), or the [GCC Homepage](#).

Example in C++

---

```
#include <iostream>
#include "mpreal.h"

int main(int argc, char* argv[])
{
using mpfr::mpreal;
using std::cout;
using std::endl;

// Required precision of computations in decimal digits
// Play with it to check different precisions
const int digits = 50;

// Setup default precision for all subsequent computations
// MPFR accepts precision in bits - so we do the conversion
mpreal::set_default_prec(mpfr::digits2bits(digits));

// Compute all the vital characteristics of mpreal (in current precision)
// Analogous to lamch from LAPACK
```

```
const mpreal one      = 1.0;
const mpreal zero     = 0.0;
const mpreal eps       = std::numeric_limits<mpreal>::epsilon();
const int   base       = std::numeric_limits<mpreal>::radix;
const mpreal prec      = eps * base;
const int   bindigits  = std::numeric_limits<mpreal>::digits(); // eqv. to
                     mpfr::mpreal::get_default_prec();
const mpreal rnd        = std::numeric_limits<mpreal>::round_error();
const mpreal maxval    = std::numeric_limits<mpreal>::max();
const mpreal minval    = std::numeric_limits<mpreal>::min();
const mpreal small      = one / maxval;
const mpreal sfmin     = (small > minval) ? small * (one + eps) : minval;
const mpreal round      = std::numeric_limits<mpreal>::round_style();
const int   min_exp     = std::numeric_limits<mpreal>::min_exponent;
const mpreal underflow  = std::numeric_limits<mpreal>::min();
const int   max_exp      = std::numeric_limits<mpreal>::max_exponent;
const mpreal overflow   = std::numeric_limits<mpreal>::max();

// Additionally compute pi with required accuracy - just for fun :)
const mpreal pi        = mpfr::const_pi();

cout.precision(digits); // Show all the digits
cout << "pi      = " << pi      << endl;
cout << "eps     = " << eps     << endl;
cout << "base    = " << base    << endl;
cout << "prec    = " << prec    << endl;
cout << "b.digits = " << bindigits << endl;
cout << "rnd     = " << rnd     << endl;
cout << "maxval  = " << maxval  << endl;
cout << "minval  = " << minval  << endl;
cout << "small   = " << small   << endl;
cout << "sfmin   = " << sfmin   << endl;
cout << "1/sfmin = " << 1 / sfmin << endl;
cout << "round   = " << round   << endl;
cout << "max_exp = " << max_exp << endl;
cout << "min_exp = " << min_exp << endl;
cout << "underflow = " << underflow << endl;
cout << "overflow = " << overflow << endl;

return 0;
}
```

#### A.2.4.5 Mac OSX: Objective C

Objective-C is a general-purpose, object-oriented programming language that adds Smalltalk-style messaging to the C programming language. It is the main programming language used by Apple for the OS X and iOS operating systems, and their respective application programming interfaces (APIs), Cocoa and Cocoa Touch.

The programming language Objective-C was originally developed in the early 1980s. It was selected as the main language used by NeXT for its NeXTSTEP operating system, from which OS X and iOS are derived.[2] Generic Objective-C programs that do not use the Cocoa or Cocoa Touch libraries, or using parts that may be ported or reimplemented for other systems can also be compiled for any system supported by GCC or Clang.

Objective-C source code program files usually have .m filename extensions, while Objective-C header files have .h extensions, the same as for C header files. Objective-C++ files are denoted with a .mm file extension.

For further information on C++, see [Wikipedia: Objective C](#) (the text above has been copied from this reference), or the [GCC Homepage](#).

Example in Objective C

---

```
# import "Forwarder.h"
# import "Recipient.h"

int main(void) {
    Forwarder *forwarder = [Forwarder new];
    Recipient *recipient = [Recipient new];

    [forwarder setRecipient:recipient]; //Set the recipient.
    /*
     * Observe forwarder does not respond to a hello message! It will
     * be forwarded. All unrecognized methods will be forwarded to
     * the recipient
     * (if the recipient responds to them, as written in the Forwarder)
    */
    [forwarder hello];

    [recipient release];
    [forwarder release];

    return 0;
}
```

---

#### A.2.4.6 Mac OSX: Objective C++

Objective-C++ is a language variant accepted by the front-end to the GNU Compiler Collection and Clang, which can compile source files that use a combination of C++ and Objective-C syntax. Objective-C++ adds to C++ the extensions that Objective-C adds to C. As nothing is done to unify the semantics behind the various language features, certain restrictions apply:

A C++ class cannot derive from an Objective-C class and vice versa.

C++ namespaces cannot be declared inside an Objective-C declaration.

Objective-C declarations may appear only in global scope, not inside a C++ namespace

Objective-C classes cannot have instance variables of C++ classes that do not have a default constructor or that have one or more virtual methods,[citation needed] but pointers to C++ objects can be used as instance variables without restriction (allocate them with new in the -init method).

C++ "by value" semantics cannot be applied to Objective-C objects, which are only accessible through pointers.

An Objective-C declaration cannot be within a C++ template declaration and vice versa. However, Objective-C types, (e.g., Classname \*) can be used as C++ template parameters.

Objective-C and C++ exception handling is distinct; the handlers of each cannot handle exceptions of the other type. This is mitigated in recent runtimes as Objective-C exceptions are either replaced by C++ exceptions completely (Apple runtime), or partly when Objective-C++ library is linked (GNUstep libobjc2).

Care must be taken since the destructor calling conventions of Objective-C and C++'s exception run-time models do not match (i.e., a C++ destructor will not be called when an Objective-C exception exits the C++ object's scope). The new 64-bit runtime resolves this by introducing interoperability with C++ exceptions in this sense.[15]

Objective-C blocks and C++11 lambdas are distinct entities, however a block is transparently generated on Mac OS X when passing a lambda where a block is expected.[16]

Objective-C++ is Objective-C (probably with COCOA Framework) with the ability to link with C++ code (probable classes).

Yes, you can use this language in XCODE to develop for Mac OS X, iPhone/iPodTouch, iPad. It works very well.

You don't have to do anything weird in your project to use Objective-C++. Just name your Objective-C files with the extension .mm (instead of .m) and you are good to go.

It is my favorite architecture: develop base class library of my game/application in C++ so I can reuse it in other platforms (Windows, Linux) and use COCOA just for the iPhone/iPad UI specific stuff.

For further information on Objective C++, see [Wikipedia: C++](#) (the text above has been copied from this reference), or the [GCC Homepage](#).

Example in C++

---

```
#include <iostream>
#include "mpreal.h"

int main(int argc, char* argv[])
{
    using mpfr::mpreal;
    using std::cout;
```

```

using std::endl;

// Required precision of computations in decimal digits
// Play with it to check different precisions
const int digits = 50;

// Setup default precision for all subsequent computations
// MPFR accepts precision in bits - so we do the conversion
mpreal::set_default_prec(mpfr::digits2bits(digits));

// Compute all the vital characteristics of mpreal (in current precision)
// Analogous to lamch from LAPACK
const mpreal one      = 1.0;
const mpreal zero     = 0.0;
const mpreal eps       = std::numeric_limits<mpreal>::epsilon();
const int   base       = std::numeric_limits<mpreal>::radix;
const mpreal prec      = eps * base;
const int   bindigits  = std::numeric_limits<mpreal>::digits(); // eqv. to
                     // mpfr::mpreal::get_default_prec();
const mpreal rnd       = std::numeric_limits<mpreal>::round_error();
const mpreal maxval    = std::numeric_limits<mpreal>::max();
const mpreal minval    = std::numeric_limits<mpreal>::min();
const mpreal small     = one / maxval;
const mpreal sfmin     = (small > minval) ? small * (one + eps) : minval;

// Additionally compute pi with required accuracy - just for fun :)
const mpreal pi        = mpfr::const_pi();

cout.precision(digits); // Show all the digits
cout << "pi      = " << pi      << endl;
cout << "eps     = " << eps     << endl;
cout << "base    = " << base    << endl;
cout << "prec    = " << prec    << endl;
cout << "b.digits = " << bindigits << endl;
cout << "rnd     = " << rnd     << endl;
cout << "maxval  = " << maxval  << endl;
cout << "minval  = " << minval  << endl;
cout << "small   = " << small   << endl;
cout << "sfmin   = " << sfmin   << endl;
cout << "1/sfmin = " << 1 / sfmin << endl;

return 0;
}

```

---

## A.3 Cython: C extensions for the Python language

[Cython] is a programming language that makes writing C extensions for the Python language as easy as Python itself. It aims to become a superset of the [Python] language which gives it high-level, object-oriented, functional, and dynamic programming. Its main feature on top of these is support for optional static type declarations as part of the language. The source code gets translated into optimized C/C++ code and compiled as Python extension modules. This allows for both very fast program execution and tight integration with external C libraries, while keeping up the high programmer productivity for which the Python language is well known.

The primary Python execution environment is commonly referred to as CPython, as it is written in C. Other major implementations use Java (Jython [Jython]), C# (IronPython [IronPython]) and Python itself (PyPy [PyPy]). Written in C, CPython has been conducive to wrapping many external libraries that interface through the C language. It has, however, remained non trivial to write the necessary glue code in C, especially for programmers who are more fluent in a high-level language like Python than in a close-to-the-metal language like C.

Originally based on the well-known Pyrex [Pyrex], the Cython project has approached this problem by means of a source code compiler that translates Python code to equivalent C code. This code is executed within the CPython runtime environment, but at the speed of compiled C and with the ability to call directly into C libraries. At the same time, it keeps the original interface of the Python source code, which makes it directly usable from Python code. These two-fold characteristics enable Cython's two major use cases: extending the CPython interpreter with fast binary modules, and interfacing Python code with external C libraries.

While Cython can compile (most) regular Python code, the generated C code usually gains major (and sometime impressive) speed improvements from optional static type declarations for both Python and C types. These allow Cython to assign C semantics to parts of the code, and to translate them into very efficient C code. Type declarations can therefore be used for two purposes: for moving code sections from dynamic Python semantics into static-and-fast C semantics, but also for directly manipulating types defined in external libraries. Cython thus merges the two worlds into a very broadly applicable programming language.

## A.4 PyPy: a fast alternative Python interpreter

PyPy is a fast, compliant alternative implementation of the Python language (2.7.8 and 3.2.5). Thanks to its Just-in-Time compiler, Python programs often run faster on PyPy than on CPython. PyPy's Python Interpreter is written in RPython and implements the full Python language. This interpreter very closely emulates the behavior of CPython. It contains the following key components: a bytecode compiler responsible for producing Python code objects from the source code of a user application; a bytecode evaluator responsible for interpreting Python code objects; a standard object space, responsible for creating and manipulating the Python objects seen by the application.

The bytecode compiler is the preprocessing phase that produces a compact bytecode format via a chain of flexible passes (tokenizer, lexer, parser, abstract syntax tree builder, bytecode generator). The bytecode evaluator interprets this bytecode. It does most of its work by delegating all actual manipulations of user objects to the object space. The latter can be thought of as the library of built-in types. It defines the implementation of the user objects, like integers and lists, as well as the operations between them, like addition or truth-value-testing.

This division between bytecode evaluator and object space gives a lot of flexibility. One can plug in different object spaces to get different or enriched behaviours of the Python objects.

### A.4.1 Installation

#### A.4.1.1 Windows

#### A.4.1.2 Mac OSX

#### A.4.1.3 GNU/Linux

## A.5 Jython: Python for the Java Virtual Machine

This list of JVM Languages comprises notable computer programming languages that are used to produce software that runs on the Java Virtual Machine (JVM). Some of these languages are interpreted by a Java program, and some are compiled to Java bytecode and JITcompiled during execution as regular Java programs to improve performance. The JVM was initially designed to support only the Java programming language. However, as time passed, ever more languages were adapted or designed to run on the Java platform.

Apart from the Java language itself, the most common or well-known JVM languages are:

- Clojure, a Lisp dialect
- Groovy, a programming and scripting language
- Scala, an object-oriented and functional programming language[1]
- JRuby, an implementation of Ruby
- Jython, an implementation of Python

Jython is an implementation of the Python language for the Java platform. Throughout this book, you will be learning how to use the Python language, and along the way we will show you where the Jython implementation differs from CPython, which is the canonical implementation of Python written in the C language. It is important to note that the Python language syntax remains consistent throughout the different implementations. At the time of this writing, there are three mainstream implementations of Python. These implementations are: CPython, Jython for the Java platform, and IronPython for the .NET platform. At the time of this writing, CPython is the most prevalent of the implementations. Therefore if you see the word Python somewhere, it could well be referring to that implementation.

This book will reference the Python language in sections regarding the language syntax or functionality that is inherent to the language itself. However, the book will reference the name Jython when discussing functionality and techniques that are specific to the Java platform implementation. No doubt about it, this book will go in-depth to cover the key features of Jython and you'll learn concepts that only adhere to the Jython implementation. Along the way, you will learn how to program in Python and advanced techniques.

Developers from all languages and backgrounds will benefit from this book. Whether you are interested in learning Python for the first time or discovering Jython techniques and advanced concepts, this book is a good fit. Java developers and those who are new to the Python language will find specific interest in reading through Part I of this book as it will teach the Python language from the basics to more advanced concepts. Seasoned Python developers will probably find more interest in Part II and Part III as they focus more on the Jython implementation specifics. Often in this reference, you will see Java code compared with Python code.

### A.5.1 Installing Java

A.5.1.1 Windows

A.5.1.2 Mac OSX

A.5.1.3 GNU/Linux

### A.5.2 Installing Eclipse with support for JPython

A.5.2.1 Windows

A.5.2.2 Mac OSX

A.5.2.3 GNU/Linux

## A.6 IronPython: Python for .NET Framework and Mono

IronPython is an implementation of the Python programming language targeting the .NET Framework and Mono. Jim Hugunin created the project and actively contributed to it up until Version 1.0 which was released on September 5, 2006.[2] Thereafter, it was maintained by a small team at Microsoft until the 2.7 Beta 1 release; Microsoft abandoned IronPython (and its sister project IronRuby) in late 2010, after which Hugunin left to work at Google.[3] IronPython 2.0 was released on December 10, 2008.[4] The project is currently maintained by a group of volunteers at Microsoft's CodePlex open-source repository. It is free and open-source software, and can be implemented with Python Tools for Visual Studio, which is a free and open-source extension for free, isolated, and commercial versions of Microsoft's Visual Studio IDE.[5] [6]

IronPython is written entirely in C#, although some of its code is automatically generated by a code generator written in Python.

IronPython is implemented on top of the Dynamic Language Runtime (DLR), a library running on top of the Common Language Infrastructure that provides dynamic typing and dynamic method dispatch, among other things, for dynamic languages.[7] The DLR is part of the .NET Framework 4.0 and is also a part of trunk builds of Mono. The DLR can also be used as a library on older CLI implementations.

For further information, see [Wikipedia: IronPython](#) (the text above has been copied from this reference).

The original distribution of SharpDevelop includes IronPython (it is not included in the mpFormulaPy IDE). Ironpython can be downloaded from the [IronPython Homepage](#). Visual Studio integration is available through [Python Tools for Visual Studio](#).

---

```
#Load CLR
import clr

#Load the mpNumerics library
clr.AddReference('MatrixClass2')
from MatrixClass2 import mp, mpNum

#Set Floating point type to MPFR with 60 decimal digits precision
mp.FloatingPointType = 3;
mp.Prec10 = 60;

#Assign values to x1 and x2
x1 = mp.CNum("32.47");
x2 = mp.CNum("12.41");

#Calculate x3 = x1 / x2
x3 = x1 / x2;

#print the value of x3
print "Result: ", x3.Str();
```

---

## A.7 Brython: a Python to JavaScript compiler

Brython is a Python to JavaScript compiler, but it does the whole job in the browser and so makes Python appear to be a client scripting language.

Brythonic is an ancient Celtic language, but in this case Brython stands for Browser Python, or I suppose it could be reference to the Life of Brian. Wherever the name derives, Brython is another of the growing examples of treating JavaScript as an assembly language. In this case, though, the approach is slightly different. The idea is that you can write Python in the browser as if it was a language with built-in browser support.

For example:

---

```
<script type="text/python">
def echo():
    alert("Hello %s !" %doc["zone"].value) </script>
```

---

You can see that this looks like a native script. You can also see that it is Python with a few extensions to enable it to work in the browser environment - doc represents the DOM and you can use it to select an element using indexing doc[”zone”] is the element with id ”zone”.

## A.8 QPython: Python for Android

QPython is a script engine that runs Python on android devices. It lets your android device run Python scripts and projects. It contains the Python interpreter, console, editor, and the SL4A Library for Android. It's Python on Android! It offers the development kit which lets you easily develop Python projects and scripts on your Android device.

[[ Main Features ]]

- \* Supports Python programming on Android including web apps, games, and SL4A programming etc
- \* Run Python scripts / projects on Android devices
- \* Can execute Python code & files from QRCode
- \* QEdit lets you create/edit Python scripts / projects easily
- \* Includes many useful python libraries
- \* Support pip

[[ Programming Features ]]

- \* Supports Web App programming, which let you develop mobile apps with web development framework, this speeds up your mobile development greatly
- \* Supports native UI programming, which let you develop games more easily by using scripts
- \* Supports SL4A programming to access Android's features: network, Bluetooth, GPS, and more

## A.9 Pythonista: Python for iOS

Pythonista is an integrated development environment for writing Python scripts on iOS. You can create interactive experiments and prototypes using multi-touch, animations, and sound â€” or just use the interactive prompt as a powerful calculator.

With extensive support for URL schemes, access to the iOS clipboard and photo library, and all the powerful libraries that come with Python, it is also possible to use Pythonista as a flexible automation tool for text or image processing.

Pythonista is also an excellent companion for learning the Python language â€” You can get started easily with lots of ready-to-run examples, the interactive prompt helps you experiment, and you can read the entire documentation right within the app.

The code editor has everything you'd expect: Syntax highlighting, code completion and an extended keyboard specifically designed for writing Python. You can even extend the editor with your own scripts.

The Standard Library has tons of modules for doing math, processing text, working with data from the web, and much more. Additionally, you have access to the iOS clipboard, an in-app web browser and the powerful Python Imaging Library â€” including support for accessing photos in your camera roll.

Pythonista includes the NumPy and matplotlib packages for scientific computation and data visualization.

With the integrated UI Editor, you can create user interfaces for your scripts without writing any code.

# Appendix B

## LibreOffice Calc and Microsoft Excel

### B.1 LibreOffice Calc

The mpFormulaPy library provides multiprecision arithmetic versions of functions which are equivalent to the numerically orientated functions of LibreOffice Calc.

The following references are useful for automating LibreOffice:

[www.pitonyak.org/oo.php](http://www.pitonyak.org/oo.php).

[http://www.pitonyak.org/OOME\\_3.0.pdf](http://www.pitonyak.org/OOME_3.0.pdf).

[http://www.wollmux.net/wiki/images/f/f9/Makro\\_Kochbuch.pdf](http://www.wollmux.net/wiki/images/f/f9/Makro_Kochbuch.pdf).

[https://wiki.documentfoundation.org/images/6/63/Makroprogrammierung\\_V41.pdf](https://wiki.documentfoundation.org/images/6/63/Makroprogrammierung_V41.pdf).

#### B.1.1 Windows

Installation on Windows, LibreOffice Portable:

The mpmath directory needs to be copied into the LibreOffice Python site-packages directory:

\LibreOfficePortable\App\libreoffice\program\python-core-3.3.3\lib\site-packages

If you want gmpy2 support, copy the files

gmpy2.pyd  
gmpy2-2.0.5-py3.3.egg-info

into the same directory.

The sheetFunction.py file needs to be copied into the LibreOffice Python Scripts directory:

\LibreOfficePortable\App\libreoffice\share\Scripts\python

#### B.1.2 Mac OSX

Installation on the Macintosh:

In Finder, navigate to:

Macintosh HD/Applications/LibreOffice

Right-Click on LibreOffice, and choose 'Show Package Contents'. This will open the LibreOffice 'Contents' subdirectory. The mpmath directory needs to be copied into the Python site-packages directory:

Contents/Frameworks/LibreOfficePython.framework/Versions/3.3/lib/python3.3/site-packages

The `sheetFunction.py` file needs to be copied into the LibreOffice Python Scripts directory:

Contents/Resources/Scripts/Python

### B.1.3 GNU/Linux

Installation on Ubuntu 14.04

To install support for python scripting from within LibreOffice:

```
sudo apt-get install libreoffice-script-provider-python
```

To install mpmath for Python 3.x:

```
sudo apt-get install python3	mpmath
```

if nautilus is not already working, do the following:

```
sudo mkdir -p /root/.config/nautilus  
gksu nautilus
```

if gksu was not already installed, you need to install it:

```
sudo apt-get install gksu
```

The `sheetFunction.py` file needs to be copied into the LibreOffice Python Scripts directory:

Computer/usr/lib/libreoffice/share/Scripts/python

## B.2 Microsoft Excel

The mpFormulaPy library provides multiprecision arithmetic versions of functions which are equivalent to the numerically orientated functions of MS Excel.

### B.2.1 Windows

The functions are enabled by selecting and activating the mpFormula32.xll or mpFormula64.xll Addin.

### B.2.2 Mac OSX

The connection to MS Excel on OSX still needs to be worked out.

## B.3 Spreadsheet functions implemented in mpFormulaPy

The mpFormulaPy library provides multiprecision arithmetic versions of functions which are equivalent to the numerically orientated functions of MS Excel and OpenOffice Calc.

### B.3.1 Spreadsheet Mathematical Functions

The mathematical functions listed below are supported by MS Excel 2013, Apache Open Office 4.0, and LibreOffice 4.1. Earlier versions of these programs support a subset of these functions. All of these functions have multiprecision arithmetic equivalents in mpFormulaPy.

**Table B.1:** Spreadsheet Mathematical Functions

Function	Description	Section
ABS	Returns the absolute value of a number	<a href="#">3.6.7 (page 50)</a>
ACOS	Returns the arccosine of a number	<a href="#">4.6.2 (page 140)</a>
ACOSH	Returns the inverse hyperbolic cosine of a number	<a href="#">4.7.2 (page 148)</a>
ACOT	Returns the inverse cotangent of a number	<a href="#">4.6.5 (page 145)</a>
ACOTH	Returns the inverse hyperbolic cotangent of a number	<a href="#">4.7.4 (page 150)</a>
ASIN	Returns the arcsine of a number	<a href="#">4.6.1 (page 138)</a>
ARABIC	Returns a Roman number to Arabic, as a number	<a href="#">3.3.1 (page 34)</a>
ASINH	Returns the inverse hyperbolic sine of a number	<a href="#">4.7.1 (page 147)</a>
ATAN	Returns the arctangent of a number	<a href="#">4.6.3 (page 142)</a>
ATAN2	Returns the arctangent from x- and y-coordinates	<a href="#">4.6.4 (page 144)</a>
ATANH	Returns the inverse hyperbolic tangent of a number	<a href="#">4.7.3 (page 149)</a>
BASE	Converts a number into a text representation with the given radix (base)	<a href="#">3.3.6 (page 37)</a>
CEILING	Rounds a number to the nearest integer or to the nearest multiple of significance	<a href="#">3.5.2 (page 40)</a>
CEILING.MATH	Rounds a number to the nearest integer or to the nearest multiple of significance	<a href="#">3.5.2 (page 41)</a>
COMBIN	Returns the number of combinations for a given number of objects	<a href="#">4.9.3 (page 156)</a>
COMBINA	Returns the number of combinations with repetitions for a given number of objects	<a href="#">4.9.3 (page 156)</a>
COS	Returns the cosine of a number	<a href="#">4.4.5 (page 114)</a>
COSH	Returns the hyperbolic cosine of a number	<a href="#">4.5.2 (page 128)</a>
COT	Returns the cotangent of a number	<a href="#">4.4.9 (page 122)</a>
COTH	Returns the hyperbolic cotangent of a number	<a href="#">4.5.6 (page 136)</a>
CSC	Returns the cosecant of a number	<a href="#">4.4.8 (page 120)</a>
CSCH	Returns the hyperbolic cosecant of a number	<a href="#">4.5.5 (page 134)</a>
DECIMAL	Converts a text representation of a number in a given base into a decimal number	<a href="#">3.3.6 (page 38)</a>
DEGREES	Converts radians to degrees	<a href="#">4.4.2 (page 110)</a>
EVEN	Rounds a number up to the nearest even integer	<a href="#">3.5.5 (page 43)</a>
EXP	Returns e raised to the power of a given number	<a href="#">4.2.1 (page 90)</a>

*Continued on next page*

Table B.1 – *Continued from previous page*

Function	Description	Section
FACT	Returns the factorial of a number	4.9.1 (page 156)
FACTDOUBLE	Returns the double factorial of a number	4.9.2 (page 156)
FLOOR	Rounds a number down, toward zero	3.5.3 (page 41)
FLOOR.MATH	Rounds a number down, to the nearest integer or to the nearest multiple of significance	3.5.3 (page 42)
GCD	Returns the greatest common divisor	4.9.6 (page 157)
GCD_ADD	*Returns the greatest common divisor	4.9.6 (page 157)
INT	Rounds a number down to the nearest integer	3.5.7 (page 43)
LCM	Returns the least common multiple	4.9.7 (page 158)
LCM_ADD	*Returns the least common multiple	4.9.7 (page 158)
LN	Returns the natural logarithm of a number	4.2.4 (page 95)
LOG	Returns the logarithm of a number to a specified base	4.2.4.1 (page 96)
LOG10	Returns the base-10 logarithm of a number	4.2.5 (page 97)
MDETERM	Returns the matrix determinant of an array	5.1.1 (page 159)
MINVERSE	Returns the matrix inverse of an array	5.1.2 (page 159)
MMULT	Returns the matrix product of two arrays	3.12.7 (page 85)
MUNIT	Returns the unit matrix	3.12.1 (page 80)
MOD	Returns the remainder from division	3.7.10 (page 66)
MROUND	Returns a number rounded to the desired multiple	3.5.11 (page 45)
MULTINOMIAL	Returns the multinomial of a set of numbers	4.9.4 (page 157)
ODD	Rounds a number up to the nearest odd integer	3.5.6 (page 43)
PI	Returns the value of pi	4.1.1 (page 87)
POWER	Returns the result of a number raised to a power	4.3.2 (page 100)
PRODUCT	Multiplies its arguments	3.7.6 (page 61)
QUOTIENT	Returns the integer portion of a division	3.5.12 (page 46)
RADIANS	Converts degrees to radians	4.4.2 (page 110)
RAND	Returns a random number between 0 and 1	3.11.1 (page 77)
RANDBETWEEN	Returns a random number between the numbers you specify	3.11.1 (page 77)
ROMAN	Converts an arabic numeral to roman, as text	3.3.1 (page 34)
ROUND	Rounds a number to a specified number of digits	3.5.1 (page 40)
ROUNDDOWN	Rounds a number down, toward zero	3.5.9 (page 45)
ROUNDUP	Rounds a number up, away from zero	3.5.10 (page 45)
SERIESSUM	Returns the sum of a power series based on the formula	3.7.2 (page 57)
SIGN	Returns the sign of a number	3.6.9 (page 52)
SEC	Returns the secant of the given angle	4.4.7 (page 118)
SECH	Returns the hyperbolic secant of the given angle	4.5.4 (page 132)
SIN	Returns the sine of the given angle	4.4.4 (page 112)
SINH	Returns the hyperbolic sine of a number	4.5.1 (page 126)
SQRT	Returns a positive square root	4.3.3 (page 103)
SQRTPI	Returns the square root of (number * pi)	4.4.3 (page 111)
SUBTOTAL	Returns a subtotal in a list or database	7.5.1 (page 281)
SUM	Adds its arguments	7.3.1 (page 267)

*Continued on next page*

Table B.1 – *Continued from previous page*

Function	Description	Section
SUMIF	Adds the cells specified by a given criteria	<a href="#">7.3.1 (page 267)</a>
SUMIFS	!Adds the cells in a range that meet multiple criteria	<a href="#">7.3.1 (page 267)</a>
SUMPRODUCT	Returns the sum of the products of corresponding array components	<a href="#">3.7.2 (page 57)</a>
SUMSQ	Returns the sum of the squares of the arguments	<a href="#">3.7.2 (page 57)</a>
SUMX2MY2	Returns the sum of the difference of squares of corresponding values in two arrays	<a href="#">3.7.2 (page 56)</a>
SUMX2PY2	Returns the sum of the sum of squares of corresponding values in two arrays	<a href="#">3.7.2 (page 56)</a>
SUMXMY2	Returns the sum of squares of differences of corresponding values in two arrays	<a href="#">3.7.2 (page 56)</a>
TAN	Returns the tangent of a number	<a href="#">4.4.6 (page 116)</a>
TANH	Returns the hyperbolic tangent of a number	<a href="#">4.5.3 (page 130)</a>
TRUNC	Truncates a number to an integer	<a href="#">3.5.4 (page 43)</a>

### B.3.2 Spreadsheet Engineering Functions

The Engineering functions listed below are supported by MS Excel 2013, Apache Open Office 4.0, and LibreOffice 4.1. Earlier versions of these programs support a subset of these functions. All of these functions have multiprecision arithmetic equivalents in mpFormulaPy. The implementation is based on Apache Open Office. Functions marked with an !! are only available in MS Excel.

**Table B.2:** Spreadsheet Engineering Functions

Function	Description	Section
BESSELI	Returns the modified Bessel function $I_n(x)$	4.8.3 (page 152)
BESSELJ	Returns the Bessel function $J_n(x)$	4.8.1 (page 152)
BESSELK	Returns the modified Bessel function $K_n(x)$	4.8.4 (page 153)
BESSELY	Returns the Bessel function $Y_n(x)$	4.8.2 (page 152)
BIN2DEC	Converts a binary number to decimal	3.3.2 (page 34)
BIN2HEX	Converts a binary number to hexadecimal	3.3.2 (page 34)
BIN2OCT	Converts a binary number to octal	3.3.2 (page 34)
BITAND	Returns a value shifted left by shift_amounts bits	3.8.1 (page 67)
BITLSHIFT	Returns a value shifted left by shift_amounts bits	3.7.7 (page 63)
BITOR	Returns a value shifted left by shift_amounts bits	3.8.2 (page 67)
BITRSHIFT	Returns a value shifted right by shift_amounts bits	3.7.9 (page 65)
BITXOR	Returns a value shifted left by shift_amounts bits	3.8.3 (page 67)
COMPLEX	Converts real and imaginary coefficients into a complex number	3.6.3 (page 48)
CONVERT	Converts a number from one measurement system to another	7.2.1 (page 263)
DEC2BIN	Converts a decimal number to binary	3.3.3 (page 35)
DEC2HEX	Converts a decimal number to hexadecimal	3.3.3 (page 35)
DEC2OCT	Converts a decimal number to octal	3.3.3 (page 35)
DELTA	Tests whether two values are equal	3.9.7 (page 70)
ERF	Returns the error function	4.8.4 (page 153)
ERFC	Returns the complementary error function	4.8.5 (page 153)
GAMMA	Returns the Gamma function value	4.8.6 (page 154)
GESTEP	!!Tests whether a number is greater than a threshold value	3.9.7 (page 70)
HEX2BIN	Converts a hexadecimal number to binary	3.3.4 (page 36)
HEX2DEC	Converts a hexadecimal number to decimal	3.3.4 (page 36)
HEX2OCT	Converts a hexadecimal number to octal	3.3.4 (page 36)
IMABS	Returns the absolute value (modulus) of a complex number	3.6.7 (page 50)
IMAGINARY	Returns the imaginary coefficient of a complex number	3.6.6 (page 49)
IMARGUMENT	Returns the argument theta, an angle expressed in radians	3.6.8 (page 51)
IMCONJUGATE	Returns the complex conjugate of a complex number	3.6.10 (page 52)
IMCOS	Returns the cosine of a complex number	4.4.5 (page 114)
IMCOSH	Returns the hyperbolic cosine of a complex number	4.5.2 (page 128)
IMCOT	Returns the cotangent of a complex number	4.4.9 (page 122)
IMCSC	Returns the hyperbolic cosine of a complex number	4.4.8 (page 120)

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Table B.2 – *Continued from previous page*

<b>Function</b>	<b>Description</b>	<b>Section</b>
IMCSCH	Returns the cotangent of a complex number	4.5.5 (page 134)
IMDIV	Returns the quotient of two complex numbers	3.7.8 (page 64)
IMEXP	Returns the exponential of a complex number	4.2.1 (page 90)
IMLN	Returns the natural logarithm of a complex number	4.2.4 (page 95)
IMLOG10	Returns the base-10 logarithm of a complex number	4.2.5 (page 97)
IMLOG2	Returns the base-2 logarithm of a complex number	4.2.6 (page 98)
IMPOWER	Returns a complex number raised to an integer power	4.3.2 (page 100)
IMPRODUCT	Returns the product of complex numbers	3.7.6 (page 62)
IMREAL	Returns the real coefficient of a complex number	3.6.5 (page 49)
IMSEC	Returns the secant of a complex number	4.4.7 (page 118)
IMSECH	Returns the hyperbolic secant of a complex number	4.5.4 (page 132)
IMSIN	Returns the sine of a complex number	4.4.4 (page 112)
IMSINH	Returns the hyperbolic sine of a complex number	4.5.1 (page 126)
IMSQRT	Returns the square root of a complex number	4.3.3 (page 103)
IMSUB	Returns the difference between two complex numbers	3.7.3 (page 58)
IMSUM	Returns the sum of complex numbers	3.7.2 (page 55)
IMTAN	Returns the tangent of a complex number	4.4.6 (page 116)
OCT2BIN	Converts an octal number to binary	3.3.5 (page 37)
OCT2DEC	Converts an octal number to decimal	3.3.5 (page 37)
OCT2HEX	Converts an octal number to hexadecimal	3.3.5 (page 37)

### B.3.3 Spreadsheet Statistical Functions

The Statistical functions listed below are supported by MS Excel 2013, Apache Open Office 4.0, and LibreOffice 4.1. Earlier versions of these programs support a subset of these functions. Functions marked with an !! are only available in MS Excel. Functions marked with an asterix are only available in Open Office. All of these functions have multiprecision arithmetic equivalents in mpFormulaPy.

**Table B.3:** Spreadsheet Statistical Functions

Function	Description	Section
AVEDEV	Returns the average of the absolute deviations of data points from their mean	<a href="#">7.3.7 (page 272)</a>
AVERAGE	Returns the average of its arguments	<a href="#">7.3.2 (page 268)</a>
AVERAGEA	Returns the average of its arguments, including numbers, text, and logical values	<a href="#">7.3.2 (page 268)</a>
AVERAGEIF	!!Returns the average (arithmetic mean) of all the cells in a range that meet a given criteria	<a href="#">7.3.2 (page 268)</a>
AVERAGEIFS	!!Returns the average (arithmetic mean) of all cells that meet multiple criteria.	<a href="#">7.3.2 (page 268)</a>
BETADIST	Returns the beta cumulative distribution function	<a href="#">6.2.2 (page 177)</a>
BETAINV	Returns the inverse of the cumulative distribution function for a specified beta distribution	<a href="#">6.2.3 (page 178)</a>
BINOMDIST	Returns the individual term binomial distribution probability	<a href="#">6.3.1 (page 182)</a>
CHIDIST	Returns the one-tailed probability of the chi-squared distribution	<a href="#">6.4.2 (page 186)</a>
CHIINV	Returns the inverse of the one-tailed probability of the chi-squared distribution	<a href="#">6.4.3 (page 188)</a>
CHITEST	Returns the test for independence	<a href="#">7.6.8 (page 286)</a>
CONFIDENCE	Returns the confidence interval for a population mean	<a href="#">7.6.1 (page 283)</a>
CORREL	Returns the correlation coefficient between two data sets	<a href="#">7.7.2 (page 288)</a>
COUNT	Counts how many numbers are in the list of arguments	<a href="#">7.1.1 (page 260)</a>
COUNTA	Counts how many values are in the list of arguments	<a href="#">7.1.1 (page 260)</a>
COUNTBLANK	Counts the number of blank cells within a range	<a href="#">7.1.1 (page 260)</a>
COUNTIF	Counts the number of cells within a range that meet the given criteria	<a href="#">7.1.1 (page 261)</a>
COUNTIFS	!!Counts the number of cells within a range that meet multiple criteria	<a href="#">7.1.1 (page 261)</a>
COVAR	Returns covariance, the average of the products of paired deviations	<a href="#">7.7.1 (page 288)</a>
CRITBINOM	Returns the smallest value for which the cumulative binomial distribution is less than or equal to a criterion value	<a href="#">6.3.2 (page 183)</a>

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Table B.3 – *Continued from previous page*

Function	Description	Section
DEVSQ	Returns the sum of squares of deviations	7.3.8 (page 272)
EXPONDIST	Returns the exponential distribution	6.5.1 (page 192)
FDIST	Returns the F probability distribution	6.6.2 (page 195)
FINV	Returns the inverse of the F probability distribution	6.6.3 (page 197)
FISHER	Returns the Fisher transformation	7.7.3 (page 289)
FISHERINV	Returns the inverse of the Fisher transformation	7.7.3 (page 290)
FORECAST	Returns a value along a linear trend	7.8.3 (page 291)
FREQUENCY	Returns a frequency distribution as a vertical array	7.1.2 (page 261)
FTEST	Returns the result of an F-test	7.6.5 (page 285)
GAMMADIST	Returns the gamma distribution	6.7.1 (page 201)
GAMMAINV	Returns the inverse of the gamma cumulative distribution	6.7.2 (page 202)
GAMMALN	Returns the natural logarithm of the gamma	4.8.6 (page 154)
GEOMEAN	Returns the geometric mean	7.3.3 (page 269)
GROWTH	Returns values along an exponential trend	5.2.2 (page 161)
HARMEAN	Returns the harmonic mean	7.3.4 (page 269)
HYPGEOMDIST	Returns the hypergeometric distribution	6.8.2 (page 205)
INTERCEPT	Returns the intercept of the linear regression line	7.8.1 (page 291)
KURT	Returns the kurtosis of a data set	7.3.10 (page 274)
LARGE	Returns the k-th largest value in a data set	7.4.5 (page 276)
LINEST	Returns the parameters of a linear trend	5.1.3 (page 160)
LOGEST	Returns the parameters of an exponential trend	5.2.1 (page 161)
LOGINV	Returns the inverse of the lognormal distribution	6.9.3 (page 210)
LOGNORMDIST	Returns the cumulative lognormal distribution	6.9.2 (page 208)
MAX	Returns the maximum value in a list of arguments	7.4.2 (page 275)
MAXA	Returns the maximum value in a list of arguments, including numbers, text, and logical values	7.4.2 (page 275)
MEDIAN	Returns the median of the given numbers	7.4.3 (page 275)
MIN	Returns the minimum value in a list of arguments	7.4.1 (page 275)
MINA	Returns the smallest value in a list of arguments, including numbers, text, and logical values	7.4.1 (page 275)
MODE	Returns the most common value in a data set	7.4.4 (page 276)
NEGBINOMDIST	Returns the negative binomial distribution	6.10.1 (page 212)
NORMDIST	Returns the normal cumulative distribution	6.11.2 (page 215)
NORMINV	Returns the inverse of the normal cumulative distribution	6.11.3 (page 217)
NORMSDIST	Returns the standard normal cumulative distribution	6.11.2 (page 216)
NORMSINV	Returns the inverse of the standard normal cumulative distribution	6.11.3 (page 218)
PEARSON	Returns the Pearson product moment correlation coefficient	7.7.2 (page 289)
PERCENTILE	Returns the k-th percentile of values in a range	7.4.7 (page 277)
PERCENTRANK	Returns the percentage rank of a value in a	7.4.8 (page 278)

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Table B.3 – *Continued from previous page*

Function	Description	Section
PERMUT	Returns the number of permutations for a given number of objects	4.9.5 (page 157)
POISSON	Returns the Poisson distribution	6.12.2 (page 220)
PROB	Returns the probability that values in a range are between two limits	7.4.11 (page 280)
QUARTILE	Returns the quartile of a data set	7.4.9 (page 278)
RANK	Returns the rank of a number in a list of numbers	7.4.10 (page 279)
RSQ	Returns the square of the Pearson product moment correlation coefficient	7.7.2 (page 289)
SKEW	Returns the skewness of a distribution	7.3.9 (page 273)
SLOPE	Returns the slope of the linear regression line	7.8.2 (page 291)
SMALL	Returns the k-th smallest value in a data set	7.4.6 (page 277)
STANDARDIZE	Returns a normalized value	7.2.2 (page 265)
STDEV	Estimates standard deviation based on a sample	7.3.6 (page 271)
STDEVA	Estimates standard deviation based on a sample, including numbers, text, and logical values	7.3.6 (page 271)
STDEVP	Calculates standard deviation based on the entire population	7.3.6 (page 271)
STDEVPA	Calculates standard deviation based on the entire population, including numbers, text, and logical values	7.3.6 (page 272)
STEYX	Returns the standard error of the predicted y-value for each x in the regression	7.8.4 (page 292)
TDIST	Returns the Student's t-distribution	6.13.2 (page 223)
TINV	Returns the inverse of the Student's t-distribution	6.13.3 (page 225)
TREND	Returns values along a linear trend	5.1.4 (page 160)
TRIMMEAN	Returns the mean of the interior of a data set	7.2.3 (page 265)
TTEST	Returns the probability associated with a Student's t-test	7.6.4 (page 284)
VAR	Estimates variance based on a sample	7.3.5 (page 269)
VARA	Estimates variance based on a sample, including numbers, text, and logical values	7.3.5 (page 270)
VARP	Calculates variance based on the entire population	7.3.5 (page 270)
VARPA	Calculates variance based on the entire population, including numbers, text, and logical values	7.3.5 (page 270)
WEIBULL	Returns the Weibull distribution	6.14.1 (page 228)
ZTEST	Returns the one-tailed probability-value of a z-test	7.6.3 (page 284)
PERMUTATIONA	Returns the number of permutations for a given number of objects (with repetitions) that can be selected from the total objects	4.9.5 (page 157)
PHI	Returns the value of the density function for a standard normal distribution	6.11.2 (page 216)
GAUSS	Returns 0.5 less than the standard normal cumulative distribution	6.11.2 (page 216)

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Table B.3 – *Continued from previous page*

Function	Description	Section
CHISQDIST	*Calc	<a href="#">6.4.2 (page 186)</a>
CHISQINV	*Calc	<a href="#">6.4.3 (page 189)</a>

### B.3.4 Spreadsheet Revised Statistical Functions

The Statistical functions listed below are supported by MS Excel 2010, 2013, but not by earlier versions. They are not supported by Apache Open Office or LibreOffice. All of these functions have multiprecision arithmetic equivalents in mpFormulaPy.

**Table B.4:** Spreadsheet Revised Statistical Functions

Function	Description	Section
AGGREGATE	Returns an aggregate in a list or database	7.5.2 (page 281)
BETA.DIST	Returns the cumulative beta probability density function.	6.2.2 (page 177)
BETA.INV	Returns the inverse of the cumulative beta probability density function.	6.2.3 (page 179)
BINOM.DIST	Returns the individual term binomial distribution probability.	6.3.1 (page 182)
BINOM.DIST.RANGE	Returns the probability of a trial result using a binomial distribution.	6.3.1 (page 183)
BINOM.INV	Returns the smallest number which is the cumulative binomial distribution that is greater than a criterion value.	6.3.2 (page 184)
CEILING.PRECISE	Rounds a number the nearest integer or to the nearest multiple of significance. Regardless of the sign of the number, the number is rounded up.	3.5.2 (page 40)
CHISQ.DIST	Returns the cumulative beta probability density function	6.4.2 (page 186)
CHISQ.DIST.RT	Returns the one tailed probability of the chi-squared distribution.	6.4.2 (page 187)
CHISQ.INV	Returns the inverse of the chi-squared distribution.	6.4.3 (page 188)
CHISQ.INV.RT	Returns the inverse of the one tailed probability of the chi-squared distribution.	6.4.3 (page 189)
CHISQ.TEST	Returns the value from the chi-squared test	7.6.8 (page 286)
CONFIDENCE.NORM	Returns the confidence interval for a population mean.	7.6.1 (page 283)
CONFIDENCE.T	Returns the confidence interval for a population using a Student's t distribution	7.6.2 (page 283)
COVARIANCE.P	Returns the covariance of two lists of numbers.	7.7.1 (page 288)
COVARIANCE.S	Returns the sample covariance, the average of the products deviations for each data point pair in two data sets	7.7.1 (page 288)
ERF.PRECISE	Returns the error function	4.8.4 (page 153)
ERFC.PRECISE	Returns the complementary ERF function integrated between x and infinity	4.8.5 (page 153)
EXPON.DIST	Returns the exponential distribution.	6.5.1 (page 192)
F.DIST	Returns the F probability distribution	6.6.2 (page 195)

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Table B.4 – *Continued from previous page*

Function	Description	Section
F.DIST.RT	Returns the F probability distribution.	6.6.2 (page 195)
F.INV	Returns the inverse of the F probability distribution.	6.6.3 (page 197)
F.INV.RT	Returns the inverse of the F probability distribution.	6.6.3 (page 197)
F.TEST	Returns the result of an F-test.	7.6.5 (page 285)
FLOOR.PRECISE	Rounds a number the nearest integer or to the nearest multiple of significance. Regardless of the sign of the number, the number is rounded up.	3.5.3 (page 41)
GAMMA.DIST	Returns the gamma distribution.	6.7.1 (page 201)
GAMMA.INV	Returns the inverse of the gamma distribution.	6.7.2 (page 202)
GAMMALN.PRECISE	Returns the natural logarithm of the gamma function, $\Gamma(x)$	4.8.6 (page 154)
HYPGEOM.DIST	Returns the hyper geometric distribution for a finite population.	6.8.2 (page 206)
LOGNORM.DIST	Returns the cumulative lognormal distribution of x, where $\ln(x)$ is normally distributed with parameters mean and stdev.	6.9.2 (page 209)
LOGNORM.INV	Returns the inverse of the lognormal cumulative distribution function	6.9.3 (page 210)
MODE.MULT	Returns a vertical array of the most frequently occurring, or repetitive values in an array or range of data	7.4.4 (page 276)
MODE.SNGL	Returns the number that occurs most frequently in a range.	7.4.4 (page 276)
NEGBINOM.DIST	Returns the negative binomial distribution.	6.10.1 (page 212)
NETWORKDAYS.INTL	Returns the number of whole workdays between two dates using parameters to indicate which and how many days are weekend days	8.3.2 (page 307)
NORM.DIST	Returns the normal cumulative distribution.	6.11.2 (page 215)
NORM.INV	Returns the inverse of the normal cumulative distribution.	6.11.3 (page 218)
NORM.S.DIST	Returns the standard normal cumulative distribution.	6.11.2 (page 216)
NORM.S.INV	Returns the inverse of the standard normal cumulative distribution.	6.11.3 (page 218)
PERCENTILE.EXC	Returns the Kth percentile of values in a range, where k is in the range 0..1, exclusive	7.4.7 (page 277)
PERCENTILE.INC	The Kth percentile of values in an array of numbers.	7.4.7 (page 277)
PERCENTRANK.EXC	Returns the rank of a value in a data set as a	7.4.8 (page 278)

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Table B.4 – *Continued from previous page*

Function	Description	Section
PERCENTRANK.INC	percentage (0..1, exclusive) of the data set Returns the rank of a value in a data set as a percentage of the data set.	7.4.8 (page 278)
POISSON.DIST	Returns the Poisson distribution.	6.12.2 (page 220)
QUARTILE.EXC	Returns the quartile of the data set, based on percentile values from 0..1, exclusive	7.4.9 (page 279)
QUARTILE.INC	Returns the quartile of a data set.	7.4.9 (page 279)
RANK.AVG	Returns the rank of a number in a list of numbers.	7.4.10 (page 280)
RANK.EQ	Returns the rank of a number in a list of numbers.	7.4.10 (page 280)
SKEW.P	Returns the skewness of a distribution based on the entire population.	7.3.9 (page 273)
STDEV.P	Returns the standard deviation based on the entire population.	7.3.6 (page 272)
STDEV.S	Returns the standard deviation based on a sample.	7.3.6 (page 271)
T.DIST	Returns the Percentage Points (probability) for the Student t-distribution	6.13.2 (page 223)
T.DIST.2T	Returns the two-tailed Student's t-distribution	6.13.2 (page 224)
T.DIST.RT	Returns the right-tailed Student's t-distribution	6.13.2 (page 223)
T.INV	Returns the T-value of the Student's t-distribution	6.13.3 (page 225)
T.INV.2T	Returns the t-value of the Student's t-distribution	6.13.3 (page 225)
T.TEST	Returns the probability associated with the F-Test.	7.6.4 (page 285)
VAR.P	Calculates variance based on the population.	7.3.5 (page 270)
VAR.S	Returns the variance based on a sample.	7.3.5 (page 269)
WEIBULL.DIST	Returns the Weibull distribution.	6.14.1 (page 228)
WORKDAY.INTL	Returns the serial number of the date before or after a specified number of workdays	8.2.11 (page 304)
Z.TEST	Returns the two-tailed P-value of a z-test	7.6.3 (page 284)

### B.3.5 Spreadsheet Database Functions

The Database functions listed below are supported by MS Excel 2013, Apache Open Office 4.0, and LibreOffice 4.1. Earlier versions of these programs support a subset of these functions. All of these functions have multiprecision arithmetic equivalents in mpFormulaPy. The implementation is based on Apache Open Office.

**Table B.5:** Spreadsheet Database Functions

Function	Description	Section
DAVERAGE	Returns the average of selected database entries	<a href="#">7.9.5 (page 294)</a>
DCOUNT	Counts the cells that contain numbers in a database	<a href="#">7.9.3 (page 293)</a>
DCOUNTA	Counts nonblank cells in a database	<a href="#">7.9.3 (page 294)</a>
DGET	Extracts from a database a single record that matches the specified criteria	<a href="#">7.9.1 (page 293)</a>
DMAX	Returns the maximum value from selected database entries	<a href="#">7.9.9 (page 296)</a>
DMIN	Returns the minimum value from selected database entries	<a href="#">7.9.8 (page 296)</a>
DPRODUCT	Multiplies the values in a particular field of records that match the criteria in a database	<a href="#">7.9.2 (page 293)</a>
DSTDEV	Estimates the standard deviation based on a sample of selected database entries	<a href="#">7.9.7 (page 295)</a>
DSTDEVP	Calculates the standard deviation based on the entire population of selected database entries	<a href="#">7.9.7 (page 295)</a>
DSUM	Adds the numbers in the field column of records in the database that match the criteria	<a href="#">7.9.4 (page 294)</a>
DVAR	Estimates variance based on a sample from selected database entries	<a href="#">7.9.6 (page 295)</a>
DVARP	Calculates variance based on the entire population	<a href="#">7.9.6 (page 295)</a>

### B.3.6 Spreadsheet Financial Functions

The financial functions listed below are supported by MS Excel 2013, Apache Open Office 4.0, and LibreOffice 4.1. Earlier versions of these programs support a subset of these functions. All of these functions have multiprecision arithmetic equivalents in mpFormulaPy. The implementation is based on Apache Open Office. Functions marked with an asterix are only available in Open Office.

**Table B.6:** Spreadsheet Financial Functions

Function	Description	Section
ACCRINT	Returns the accrued interest for a security that pays periodic interest	<a href="#">8.5.1 (page 313)</a>
ACCRINTM	Returns the accrued interest for a security that pays interest at maturity	<a href="#">8.5.2 (page 314)</a>
AMORDEGRC	Returns the depreciation for each accounting period by using a depreciation coefficient	<a href="#">8.7.7 (page 327)</a>
AMORLINC	Returns the depreciation for each accounting period	<a href="#">8.7.6 (page 326)</a>
COUPDAYBS	Returns the number of days from the beginning of the coupon period to the settlement date	<a href="#">8.4.1 (page 309)</a>
COUPDAYS	Returns the number of days in the coupon period that contains the settlement date	<a href="#">8.4.2 (page 310)</a>
COUPDAYSNC	Returns the number of days from the settlement date to the next coupon date	<a href="#">8.4.3 (page 310)</a>
COUPNCD	Returns the next coupon date after the settlement date	<a href="#">8.4.4 (page 310)</a>
COUPNUM	Returns the number of coupons payable between the settlement date and maturity date	<a href="#">8.4.5 (page 311)</a>
COUPPCD	Returns the previous coupon date before the settlement date	<a href="#">8.4.6 (page 311)</a>
CUMIPMT	Returns the cumulative interest paid between two periods	<a href="#">8.8.9 (page 331)</a>
CUMIPMT_ADD	*Returns the cumulative interest paid between two periods	<a href="#">8.8.9 (page 331)</a>
CUMPRINC	Returns the cumulative principal paid on a loan between two periods	<a href="#">8.8.10 (page 332)</a>
CUMPRINC_ADD	*Returns the cumulative principal paid on a loan between two periods	<a href="#">8.8.10 (page 332)</a>
DB	Returns the depreciation of an asset for a specified period by using the fixed-declining balance method	<a href="#">8.7.4 (page 325)</a>
ddb	Returns the depreciation of an asset for a specified period by using the double-declining balance method or some other method that you specify	<a href="#">8.7.1 (page 324)</a>
DISC	Returns the discount rate for a security	<a href="#">8.5.3 (page 314)</a>
DOLLARDE	Converts a dollar price, expressed as a fraction, into a dollar price, expressed as a decimal number	<a href="#">8.10.1 (page 338)</a>
DOLLARFR	Converts a dollar price, expressed as a decimal number, into a dollar price, expressed as a fraction	<a href="#">8.10.2 (page 338)</a>

*Continued on next page*

Table B.6 – *Continued from previous page*

Function	Description	Section
DURATION	Returns the annual duration of a security with periodic interest payments	8.4.7 (page 311)
DURATION_ADD	*Returns the annual duration of a security with periodic interest payments	8.4.7 (page 311)
EFFECT	Returns the effective annual interest rate	8.8.11 (page 332)
EFFECT_ADD	*Returns the effective annual interest rate	8.8.11 (page 332)
FV	Returns the future value of an investment	8.8.1 (page 328)
FVSCHEDULE	Returns the future value of an initial principal after applying a series of compound interest rates	8.8.13 (page 333)
INTRATE	Returns the interest rate for a fully invested given period	8.5.4 (page 315)
IPMT	Returns the interest payment for an investment for a security	8.8.7 (page 331)
IRR	Returns the internal rate of return for a series of cash flows	8.9.1 (page 334)
ISPMT	Calculates the interest paid during a specific period of an investment	8.8.14 (page 333)
MDURATION	Returns the Macauley modified duration for a security with an assumed par value of \$100	8.4.8 (page 312)
MIRR	Returns the internal rate of return where positive and negative cash flows are financed at different rates	8.9.3 (page 335)
NOMINAL	Returns the annual nominal interest rate	8.8.12 (page 332)
NOMINAL_ADD	*Returns the annual nominal interest rate	8.8.12 (page 332)
NPER	Returns the number of periods for an investment	8.8.4 (page 329)
NPV	Returns the net present value of an investment based on a series of periodic cash flows and a discount rate	8.9.4 (page 335)
ODDFPRICE	Returns the price per \$100 face value of a security with an odd first period	8.5.5 (page 315)
ODDFYIELD	Returns the yield of a security with an odd first period cash flows	8.5.6 (page 316)
ODDLPRICE	Returns the price per \$100 face value of a security with an odd last period	8.5.7 (page 317)
ODDLYIELD	Returns the yield of a security with an odd last period	8.5.8 (page 317)
PDURATION	Returns the number of periods required by an investment to reach a specified value	8.8.5 (page 330)
PMT	Returns the periodic payment for an annuity	8.8.3 (page 329)
PPMT	Returns the payment on the principal for an investment for a given period	8.8.8 (page 331)
PRICE	Returns the price per \$100 face value of a security that pays periodic interest	8.5.9 (page 318)
PRICEDISC	Returns the price per \$100 face value of a discounted security	8.5.10 (page 319)

*Continued on next page*

Table B.6 – *Continued from previous page*

Function	Description	Section
PRICEMAT	Returns the price per \$100 face value of a security that pays interest at maturity	8.5.11 (page 319)
PV	Returns the present value of an investment	8.8.2 (page 328)
RATE	Returns the interest rate per period of an annuity	8.8.6 (page 330)
RECEIVED	Returns the amount received at maturity for a fully invested security	8.5.12 (page 320)
RRI	Returns an equivalent interest rate for the growth of an investment	8.9.2 (page 334)
SLN	Returns the straight-line depreciation of an asset for one period	8.7.2 (page 324)
SYD	Returns the sum-of-years' digits depreciation of an asset for a specified period	8.7.3 (page 325)
TBILLEQ	Returns the bond-equivalent yield for a Treasury bill	8.6.1 (page 322)
TBILLPRICE	Returns the price per \$100 face value for a Treasury bill	8.6.2 (page 322)
TBILLYIELD	Returns the yield for a Treasury bill	8.6.3 (page 322)
VDB	Returns the depreciation of an asset for a specified or partial period by using a declining balance method	8.7.5 (page 326)
XIRR	Returns the internal rate of return for a schedule of cash flows that is not necessarily periodic	8.9.5 (page 336)
XNPV	Returns the net present value for a schedule of cash flows that is not necessarily periodic	8.9.6 (page 336)
YIELD	Returns the yield on a security that pays periodic interest	8.5.13 (page 320)
YIELDDISC	Returns the annual yield for a discounted security; for example, a Treasury bill	8.5.14 (page 321)
YIELDMAT	Returns the annual yield of a security that pays	8.5.15 (page 321)

### B.3.7 Spreadsheet Date and Time Functions

The Date and Time functions listed below are supported by MS Excel 2013, Apache Open Office 4.0, and LibreOffice 4.1. All of these functions have equivalents in mpFormulaPy, but there are of course no multiprecision versions.

**Table B.7:** Spreadsheet Date and Time Functions

Function	Description	Section
DATE	Returns the serial number of a particular date	<a href="#">8.2.1 (page 301)</a>
DATEVALUE	Converts a date in the form of text to a serial number	<a href="#">8.2.3 (page 301)</a>
DAY	Converts a serial number to a day of the month	<a href="#">8.1.4 (page 298)</a>
DAYS	Returns the number of days between two dates	<a href="#">8.1.5 (page 298)</a>
DAYS360	Calculates the number of days between two dates based on a 360-day year	<a href="#">8.3 (page 306)</a>
EDATE	Returns the serial number of the date that is the indicated number of months before or after the start date	<a href="#">8.2.4 (page 302)</a>
EOMONTH	Returns the serial number of the last day of the month before or after a specified number of months	<a href="#">8.2.5 (page 302)</a>
EASTERSUNDAY	*Returns the serial number of Easter Sunday	<a href="#">8.2.2 (page 301)</a>
HOUR	Converts a serial number to an hour	<a href="#">8.1.3 (page 298)</a>
ISOWEEKNUM	Returns the number of the ISO week number of the year for a given date	<a href="#">8.1.9 (page 300)</a>
MINUTE	Converts a serial number to a minute	<a href="#">8.1.2 (page 297)</a>
MONTH	Converts a serial number to a month	<a href="#">8.1.6 (page 298)</a>
NETWORKDAYS	Returns the number of whole workdays between two dates	<a href="#">8.3.1 (page 306)</a>
NOW	Returns the serial number of the current date and time	<a href="#">8.2.6 (page 303)</a>
SECOND	Converts a serial number to a second	<a href="#">8.1.1 (page 297)</a>
TIME	Returns the serial number of a particular time	<a href="#">8.2.7 (page 303)</a>
TIMEVALUE	Converts a time in the form of text to a serial number	<a href="#">8.2.8 (page 303)</a>
TODAY	Returns the serial number of today's date	<a href="#">8.2.9 (page 303)</a>
WEEKDAY	Converts a serial number to a day of the week	<a href="#">8.1.8 (page 299)</a>
WEEKNUM	Converts a serial number to a number representing where the week falls numerically with a year	<a href="#">8.1.9 (page 299)</a>
WEEKNUM_ADD	*Converts a serial number to a number representing where the week falls numerically with a year	<a href="#">8.1.9 (page 299)</a>
WORKDAY	Returns the serial number of the date before or after a specified number of workdays	<a href="#">8.2.10 (page 304)</a>
YEAR	Converts a serial number to a year	<a href="#">8.1.7 (page 299)</a>
YEARFRAC	Returns the year fraction representing the number of whole days between start_date and end_date	<a href="#">8.3.3 (page 308)</a>

## B.4 Spreadsheet functions not implemented in mpFormu-laPy

### B.4.1 Spreadsheet Text Functions

The Text functions listed below are supported by MS Excel 2013, Apache Open Office 4.0, and LibreOffice 4.1. Earlier versions of these programs support a subset of these functions. All of these functions can be used in mpFormulaPy functions, but do not have an own implementation.

**Table B.8:** Spreadsheet Text Functions

Function	Description
ASC	Changes full-width (double-byte) English letters or katakana within a character string to half-width (single-byte) characters
BAHTTEXT	Converts a number to text, using the Â§ (baht) currency format
CHAR	Returns the character specified by the code number
CLEAN	Removes all nonprintable characters from text
CODE	Returns a numeric code for the first character in a text string
CONCATENATE	Joins several text items into one text item
DOLLAR	Converts a number to text, using the \$ (dollar) currency format
EXACT	Checks to see if two text values are identical
FIND	Finds one text value within another (case-sensitive)
FINDB	Finds one text value within another (case-sensitive)
FIXED	Formats a number as text with a fixed number of decimals
JIS	Changes half-width (single-byte) English letters or katakana within a character string to full-width (double-byte) characters
LEFT	Returns the leftmost characters from a text value
LEFTB	Returns the leftmost characters from a text value
LEN	Returns the number of characters in a text string
LENB	Returns the number of characters in a text string
LOWER	Converts text to lowercase
MID	Returns a specific number of characters from a text string starting at the position you specify
MIDB	Returns a specific number of characters from a text string starting at the position you specify
NUMBERVALUE	Converts a text argument to a number in a locale-independent manner
PHONETIC	Extracts the phonetic (furigana) characters from a text string
PROPER	Capitalizes the first letter in each word of a text value
REPLACE	Replaces characters within text

*Continued on next page*

Table B.8 – *Continued from previous page*

<b>Function</b>	<b>Description</b>
REPLACEB	Replaces characters within text
REPT	Repeats text a given number of times
RIGHT	Returns the rightmost characters from a text value
RIGHTB	Returns the rightmost characters from a text value
SEARCH	Finds one text value within another (not case-sensitive)
SEARCHB	Finds one text value within another (not case-sensitive)
SUBSTITUTE	Substitutes new text for old text in a text string
T	Converts its arguments to text
TEXT	Formats a number and converts it to text
TRIM	Removes spaces from text
UPPER	Converts text to uppercase
VALUE	Converts a text argument to a number
UNICHAR	Returns the Unicode character that is referenced by the given numeric value
UNICODE	Returns the number (code point) that corresponds to the first character of the text

## B.4.2 Spreadsheet Information Functions

The Information functions listed below are supported by MS Excel 2013, Apache Open Office 4.0, and LibreOffice 4.1. Earlier versions of these programs support a subset of these functions. All of these functions have multiprecision arithmetic equivalents in mpFormulaPy. The implementation is based on Apache Open Office. Functions marked with an !! are only available in MS Excel. Functions marked with an asterix are only available in Open Office.

**Table B.9:** Spreadsheet Information Functions

Function	Description
CELL	Returns information about the formatting, location or contents of a cell
ERROR.TYPE	Returns a number corresponding to an error type
INFO	Returns information about the current operating environment
ISBLANK	Returns TRUE if the value is blank
ISERR	Returns TRUE if the value is any error value except #N/A
ISERROR	Returns TRUE if the value is any error value
ISEVEN	Returns TRUE if the number is even
ISEVEN_ADD	*Returns TRUE if the number is even
ISLOGICAL	Returns TRUE if the value is a logical value
ISNA	Returns TRUE if the value is the #N/A error value
ISNONTEXT	Returns TRUE if the value is not text
ISNUMBER	Returns TRUE if the value is a number
ISODD	Returns TRUE if the number is odd
ISODD_ADD	*Returns TRUE if the number is odd
ISREF	Returns TRUE if the value is a reference
ISTEXT	Returns TRUE if the value is text
N	Returns a value converted to a number
NA	Returns the error value #N/A
TYPE	Returns a number indicating the data type of a value
FORMULA	*Calc
FORMULATEXT	!!Returns the formula at the given reference as text
ISFORMULA	Returns TRUE if there is a reference to a cell that contains a formula
CURRENT	*Calc

### B.4.3 Spreadsheet Logical Functions

The Information functions listed below are supported by MS Excel 2013, Apache Open Office 4.0, and LibreOffice 4.1. Earlier versions of these programs support a subset of these functions. All of these functions can be used in mpFormulaPy functions, but do not have an own implementation. Functions marked with an !! are only available in MS Excel.

**Table B.10:** Spreadsheet Logical Functions

Function	Description
AND	Returns TRUE if all of its arguments are TRUE
FALSE	Returns the logical value FALSE
IF	Specifies a logical test to perform
IFNA	Specifies a logical test to perform
IFERROR	!!Returns a value you specify if a formula evaluates to an error; otherwise, returns the result of the formula
NOT	Reverses the logic of its argument
OR	Returns TRUE if any argument is TRUE
TRUE	Returns the logical value TRUE
XOR	Returns a logical exclusive OR of all arguments

#### B.4.4 Spreadsheet Lookup Functions

The Lookup functions listed below are supported by MS Excel 2013, Apache Open Office 4.0, and LibreOffice 4.1. Earlier versions of these programs support a subset of these functions. All of these functions can be used in mpFormulaPy functions, but do not have an own implementation. Functions marked with an asterix are only available in Open Office.

**Table B.11:** Spreadsheet Lookup Functions

Function	Description
ADDRESS	Returns a reference as text to a single cell in a worksheet
AREAS	Returns the number of areas in a reference
CHOOSE	Chooses a value from a list of values
COLUMN	Returns the column number of a reference
COLUMNS	Returns the number of columns in a reference
HLOOKUP	Looks in the top row of an array and returns the value of the indicated cell
HYPERLINK	Creates a shortcut or jump that opens a document stored on a network server, an intranet, or the Internet
INDEX	Uses an index to choose a value from a reference or array
INDIRECT	Returns a reference indicated by a text value
LOOKUP	Looks up values in a vector or array
MATCH	Looks up values in a reference or array
OFFSET	Returns a reference offset from a given reference
ROW	Returns the row number of a reference
ROWS	Returns the number of rows in a reference
RTD	!!Retrieves real-time data from a program that supports COM automation
TRANSPOSE	Returns the transpose of an array
VLOOKUP	Looks in the first column of an array and moves across
DDE	*Calc
STYLE	*Calc
SHEET	Returns the sheet number of a referenced sheet
SHEETS	Returns the number of sheets in a reference

### B.4.5 Spreadsheet Automation, Cube and Web Functions

The functions listed below are supported by MS Excel. They are not supported by Apache Open Office or LibreOffice. All of these functions can be used in mpFormulaPy functions, but do not have an own implementation.

**Table B.12:** Spreadsheet Automation Functions

Function	Description
CALL	Calls a procedure in a dynamic link library or code resource
EUROCONVERT	Converts a number to euros, converts a number from euros to a euro member currency, or converts a number from one euro member currency to another by using theeuro as an intermediary (triangulation)
GETPIVOTDATA	Returns data stored in a PivotTable report
REGISTER.ID	Returns the register ID of the specified dynamic link library (DLL) or code resource that has been previously registered
SQL.REQUEST	Connects with an external data source and runs a queryfrom a worksheet, then returns the result as an array

**Table B.13:** Spreadsheet Cube Functions

Function	Description
CUBEKPIMEMBER	Returns a key performance indicator (KPI) name, property, and measure, and displays the name and property in the cell. A KPI is a quantifiable measurement, such as monthly gross profit or quarterly employee turnover, used to monitor an organization's performance.
CUBEMEMBER	Returns a member or tuple in a cube hierarchy. Use to validate that the member or tuple exists in the cube.
CUBEMEMBERPROPERTY	Returns the value of a member property in the cube. Use to validate that a member name exists within the cube and to return the specified property for this member.
CUBERANKEDMEMBER	Returns the nth, or ranked, member in a set. Use to return one or more elements in a set, such as the top sales performer or top 10 students.
CUBESET	Defines a calculated set of members or tuples by

*Continued on next page*

Table B.13 – *Continued from previous page*

<b>Function</b>	<b>Description</b>
CUBESETCOUNT	sending a set expression to the cube on the server, which creates the set, and then returns that set to Microsoft Office Excel.
CUBEVALUE	Returns the number of items in a set. Returns an aggregated value from a cube.

**Table B.14:** Spreadsheet Web Functions

<b>Function</b>	<b>Description</b>
ENCODEURL	Returns a URL-encoded string
FILTERXML	Returns specific data from the XML content by using the specified XPath
WEBSERVICE	Returns data from a web service

# Appendix C

## Languages with CLR Support

### C.1 Visual Basic .NET

Visual Basic .NET (VB.NET) is an object-oriented computer programming language that can be viewed as an evolution of the classic Visual Basic (VB), implemented on the .NET Framework. Microsoft currently supplies two main editions of IDEs for developing in Visual Basic: Microsoft Visual Studio 2012, which is commercial software and Visual Basic Express Edition 2012, which is free of charge. The command-line compiler, VBC.EXE, is installed as part of the freeware .NET Framework SDK. Mono also includes a command-line VB.NET compiler. The most recent version is VB 2012, which was released on August 15, 2012.

#### C.1.1 Visual Basic 2005

Visual Basic 2005 was the name used to refer to Visual Basic .NET, as Microsoft decided to drop the .NET portion of the title.

For this release, Microsoft added many features, including:

Edit and Continue

Design-time expression evaluation.

The My pseudo-namespace (overview, details), which provides easy access to certain areas of the .NET Framework that otherwise require significant code to access dynamically generated classes (notably My.Forms)

The Using keyword, simplifying the use of objects that require the Dispose pattern to free resources

Just My Code, which when debugging hides (steps over) boilerplate code written by the Visual Studio .NET IDE and system library code

Data Source binding, easing database client/server development

Generics

Partial classes, a method of defining some parts of a class in one file and then adding more definitions later; particularly useful for integrating user code with auto-generated code

Operator overloading and nullable Types

Support for unsigned integer data types commonly used in other languages

#### C.1.2 Visual Basic 2008

Visual Basic 2008 was released together with the Microsoft .NET Framework 3.5 on 19 November 2007.

For this release, Microsoft added many features, including:

A true conditional operator, "If(condition as boolean, truepart, falsepart)", to replace the "IIf" function. Anonymous types

Support for LINQ

Lambda expressions

XML Literals

Type Inference

Extension methods

### C.1.3 Visual Basic 2010

In April 2010, Microsoft released Visual Basic 2010. Microsoft had planned to use the Dynamic Language Runtime (DLR) for that release[8] but shifted to a co-evolution strategy between Visual Basic and sister language C# to bring both languages into closer parity with one another. Visual Basic's innate ability to interact dynamically with CLR and COM objects has been enhanced to work with dynamic languages built on the DLR such as IronPython and IronRuby.[9] The Visual Basic compiler was improved to infer line continuation in a set of common contexts, in many cases removing the need for the "\_" line continuation character. Also, existing support of inline Functions was complemented with support for inline Subs as well as multi-line versions of both Sub and Function lambdas.[10]

### C.1.4 Visual Basic 2012

The latest version of Visual Basic .NET, which uses .NET framework 4.5.

Async Feature,

Iterators,

Call Hierarchy,

Caller Information and

Global Keyword in Namespace Statements

are some of the major features introduced in this version of VB.

### C.1.5 Relation to older versions of Visual Basic

Whether Visual Basic .NET should be considered as just another version of Visual Basic or a completely different language is a topic of debate. This is not obvious, as once the methods that have been moved around and that can be automatically converted are accounted for, the basic syntax of the language has not seen many "breaking" changes, just additions to support new features like structured exception handling and short-circuited expressions. Two important data type changes occurred with the move to VB.NET. Compared to VB6, the Integer data type has been doubled in length from 16 bits to 32 bits, and the Long data type has been doubled in length from 32 bits to 64 bits. This is true for all versions of VB.NET. A 16-bit integer in all versions of VB.NET is now known as a Short. Similarly, the Windows Forms GUI editor is very similar in style and function to the Visual Basic form editor.

The version numbers used for the new Visual Basic (7, 7.1, 8, 9, ...) clearly imply that it is viewed by Microsoft as still essentially the same product as the old Visual Basic.

The things that have changed significantly are the semantics from those of an object-based programming language running on a deterministic, reference-counted engine based on COM to a fully object-oriented language backed by the .NET Framework, which consists of a combination of the Common Language Runtime (a virtual machine using generational garbage collection and a

just-in-time compilation engine) and a far larger class library. The increased breadth of the latter is also a problem that VB developers have to deal with when coming to the language, although this is somewhat addressed by the My feature in Visual Studio 2005.

The changes have altered many underlying assumptions about the "right" thing to do with respect to performance and maintainability. Some functions and libraries no longer exist; others are available, but not as efficient as the "native" .NET alternatives. Even if they compile, most converted VB6 applications will require some level of refactoring to take full advantage of the new language. Documentation is available to cover changes in the syntax, debugging applications, deployment and terminology.[11]

For further information on Visual Basic .NET, see [Wikipedia: Visual Basic .NET](#) (the text above has been copied from this reference).

Example for using the library

---

```
Imports System
Imports System.Console
Imports Microsoft.VisualBasic
Imports Microsoft.VisualBasic.Strings
Imports MatrixClass2

Module Module1
Sub Main()
mp.Prec10() = 100 : mp.FloatingPointType() = 3
Dim Y1, Y2, Y3, Y4 As New mpNum
Y1 = mp.Sqrt(2)
Writeline("#Sqrt(12): ")
Writeline("@" & Y1)
Y2 = Sqrt(2)
Writeline("#Sqrt(12): ")
Writeline("@" & Y2)
Y3 = Y1 - Y2
Y4 = Y3 + CNum("1.4")
Writeline("#Diff:")
Writeline("@" & Y4)
End Sub
End Module
```

---

Example for using Excel

---

```
Imports System
Imports System.Console
Imports Microsoft.VisualBasic
Imports Microsoft.VisualBasic.Strings

Module Module1
Sub DemoExcel()
Dim objExcel As Object
objExcel = CreateObject("Excel.Application")
'objExcel.Workbooks.Open("C:\Extra\mpNumerics\Output\mpTemp00.html")
objExcel.Visible = True
```

```
objExcel.Workbooks.Add
objExcel.Cells(1, 1).Value = "Test value"
objExcel = Nothing
End Sub

Sub Main()
Call DemoExcel()
End Sub
End Module
```

---

Example for using Forms

---

```
Imports System.Windows.Forms

Partial Class MyForm : Inherits Form
    'Component's Declaration
    Friend WithEvents lblFirstName As Label = New Label
    Friend WithEvents lblLastName As Label = New Label
    Friend WithEvents txtFirstName As TextBox = New TextBox
    Friend WithEvents txtLastName As TextBox = New TextBox
    Friend WithEvents btnShow As Button = New Button

    Private Sub InitializeComponent()
        Me.Text = "My Second Example Form"

        'lblFirstName Setting
        lblFirstName.Text = "First Name : "
        'Set the label into AutoSize
        lblFirstName.AutoSize = True
        'Set the location/position of the lblFirstName Object relative to the form
        'System.Drawing.Point(x, y)
        lblFirstName.Location = New System.Drawing.Point(10, 10)

        'lblLastName Setting
        lblLastName.Text = "Last Name : "
        lblLastName.AutoSize = True
        lblLastName.Location = New System.Drawing.Point(10, 60)

        'txtFirstName Setting
        txtFirstName.MaxLength = 50
        txtFirstName.Size = New System.Drawing.Size(150, 40)
        txtFirstName.Location = New System.Drawing.Point(100, 10)

        'txtLastName Setting
        txtLastName.MaxLength = 50
        txtLastName.Size = New System.Drawing.Size(150, 40)
        txtLastName.Location = New System.Drawing.Point(100, 60)

        'btnShow Setting
        btnShow.Text = "&Show"
        btnShow.Size = New System.Drawing.Size(50, 30)
        btnShow.Location = New System.Drawing.Point(10, 100)

        'Adding the control/component into the Form
        Me.Controls.Add(lblLastName)
        Me.Controls.Add(lblFirstName)
        Me.Controls.Add(txtLastName)
        Me.Controls.Add(txtFirstName)
        Me.Controls.Add(btnShow)
        Me.Size = New System.Drawing.Size(txtLastName.Right + 20, btnShow.Top + 70)
        Me.StartPosition = FormStartPosition.CenterScreen
```

```
End Sub

Private Sub btnShow_Clicked(ByVal sender As System.Object, ByVal e As
    System.EventArgs) Handles btnShow.Click
    MessageBox.Show("Welcome " & txtFirstName.Text & " " & txtLastName.Text, "Welcome")
End Sub

Public Sub New()
    InitializeComponent()
End Sub

End Class

Module Module1

    Function Main(ByVal cmdArgs() As String) As Integer
        Application.EnableVisualStyles()
        Dim theForm As New MyForm
        theForm.ShowDialog()
        Return 0
    End Function

End Module
```

---

Example for using .NET Charts

---

```
Imports System.Windows.Forms
Imports System.Windows.Forms.DataVisualization.Charting

Module Module1

Function Main(ByVal cmdArgs() As String) As Integer
Dim Chart1 As System.Windows.Forms.DataVisualization.Charting.Chart
Chart1 = New Chart()
Dim chartArea1 As New ChartArea()
Chart1.ChartAreas.Add("Default")
Chart1.Series.Add("Default")

' Populate series data
Dim yValues As Double() = {65.62, 75.54, 60.45, 34.73, 85.42}
Dim xValues As String() = {"France", "Canada", "Germany", "USA", "Italy"}
Chart1.Series("Default").Points.DataBindXY(xValues, yValues)

' Set Doughnut chart type
Chart1.Series("Default").ChartType = SeriesChartType.Doughnut

' Set labels style
Chart1.Series("Default")("PieLabelStyle") = "Outside"

' Set Doughnut radius percentage
Chart1.Series("Default")("DoughnutRadius") = "60"

' Explode data point with label "Italy"
Chart1.Series("Default").Points(4)("Exploded") = "true"

' Enable 3D
Chart1.ChartAreas("Default").Area3DStyle.Enable3D = false

' Set drawing style
chart1.Series("Default")("PieDrawingStyle") = "SoftEdge"

' Set Chart control size
Chart1.Size = New System.Drawing.Size(360, 260)

Dim FileName As String
FileName = cmdArgs(0)
'FileName = "I:\mpNew\mpNumerics\VBNET.emf"
'Chart1.SaveImage(FileName, ChartImageFormat.EmfDual)
Chart1.Serializer.Save(FileName)
Return 0
End Function

End Module
```

---

Example for using the speech synthesizer

---

```
Imports System.Windows.Forms
Imports System.Speech.Synthesis

Module Module1

Function Main(ByVal cmdArgs() As String) As Integer
Dim speaker as New SpeechSynthesizer()
speaker.Rate = 1
speaker.Volume = 100
speaker.Speak("Hello world")
speaker.SetOutputToWaveFile("c:\soundfile.wav")
speaker.Speak("Hellow world.")
speaker.SetOutputToDefaultAudioDevice()
'Must remember to reset out device or the next call to speak
'will try to write to a file
End Function

End Module
```

---

Example for using Matlab as a COM Server from Visual Basic

This example calls a user-defined MATLAB function named solve\_bvp from a Microsoft Visual Basic client application through a COM interface. It also plots a graph in a new MATLAB window and performs a simple computation:

```
Dim MatLab As Object
Dim Result As String
Dim MReal(1, 3) As Double
Dim MIImag(1, 3) As Double

MatLab = CreateObject("Matlab.Application")

'Calling MATLAB function from VB
'Assuming solve_bvp exists at specified location
Result = MatLab.Execute("cd d:\matlab\work\bvp")
Result = MatLab.Execute("solve_bvp")

'Executing other MATLAB commands
Result = MatLab.Execute("surf(peaks)")
Result = MatLab.Execute("a = [1 2 3 4; 5 6 7 8]")
Result = MatLab.Execute("b = a + a ")
'Bring matrix b into VB program
MatLab.GetFullMatrix("b", "base", MReal, MIImag)
```

---

The following examples require NetOffice to be installed.

Example for calling Excel using NetOffice

---

```

Imports NetOffice
' Imports Office = NetOffice.OfficeApi
Imports Excel = NetOffice.ExcelApi
' Imports NetOffice.ExcelApi.Enums

Module Program

Private Sub GetActiveExcel()
Dim xlProxy As Object =
    System.Runtime.InteropServices.Marshal.GetActiveObject("Excel.Application")
Dim xlApp As Excel._Application = New Excel._Application(Nothing, xlProxy)
Dim workBook As Excel.Workbook = xlApp.ActiveWorkbook
Dim workSheet As Excel.Worksheet = xlApp.ActiveSheet
Dim wbName As String = workBook.Name
System.Console.WriteLine(wbName)
' VERY IMPORTANT! OTHERWISE LATER CALLS WILL FAIL!
xlApp.Dispose()
End Sub

Sub Main()
GetActiveExcel()
End Sub

End Module

```

---

Example for calling Word using NetOffice

---

```

Imports NetOffice
Imports Office = NetOffice.OfficeApi
Imports Word = NetOffice.WordApi
Imports NetOffice.WordApi.Enums

Module Program

Private Sub GetActiveWord()
Dim wdProxy As Object =
    System.Runtime.InteropServices.Marshal.GetActiveObject("Word.Application")
Dim wdApp As Word._Application = New Word._Application(Nothing, wdProxy)
' VERY IMPORTANT! OTHERWISE LATER CALLS WILL FAIL!
wdApp.Dispose()
End Sub

Sub Main()
GetActiveWord()
End Sub

End Module

```

---

Example for calling PowerPoint using NetOffice

---

```
Imports NetOffice
Imports Office = NetOffice.OfficeApi
Imports PowerPoint = NetOffice.PowerPointApi
Imports NetOffice.PowerPointApi.Enums

Module Program

Private Sub GetActivePowerpoint()
Dim ppProxy As Object =
    System.Runtime.InteropServices.Marshal.GetActiveObject("Powerpoint.Application")
Dim ppApp As Powerpoint._Application = New Powerpoint._Application(Nothing, ppProxy)

' VERY IMPORTANT! OTHERWISE LATER CALLS WILL FAIL!
ppApp.Dispose()
End Sub

Sub Main()
GetActivePowerpoint()
End Sub

End Module
```

---

## C.2 C# 4.0

C# C#[note 1] (pronounced see sharp) is a multi-paradigm programming language encompassing strong typing, imperative, declarative, functional, procedural, generic, object-oriented (class-based), and component-oriented programming disciplines. It was developed by Microsoft within its .NET initiative and later approved as a standard by Ecma (ECMA-334) and ISO (ISO/IEC 23270:2006). C# is one of the programming languages designed for the Common Language Infrastructure.

C# is intended to be a simple, modern, general-purpose, object-oriented programming language.[6] Its development team is led by Anders Hejlsberg. The most recent version is C# 5.0, which was released on August 15, 2012.

C# has the following syntax:

Semicolons are used to denote the end of a statement. Curly braces are used to group statements. Statements are commonly grouped into methods (functions), methods into classes, and classes into namespaces. Variables are assigned using an equals sign, but compared using two consecutive equals signs. Square brackets are used with arrays, both to declare them and to get a value at a given index in one of them

By design, C# is the programming language that most directly reflects the underlying Common Language Infrastructure (CLI).[30] Most of its intrinsic types correspond to value-types implemented by the CLI framework. However, the language specification does not state the code generation requirements of the compiler: that is, it does not state that a C# compiler must target a Common Language Runtime, or generate Common Intermediate Language (CIL), or generate any other specific format. Theoretically, a C# compiler could generate machine code like traditional compilers of C++ or Fortran. Some notable features of C# that distinguish it from C and C++ (and Java, where noted) are:

C# supports strongly typed implicit variable declarations with the keyword var, and implicitly typed arrays with the keyword new[] followed by a collection initializer. Meta programming via C# attributes is part of the language. Many of these attributes duplicate the functionality of GCC's and VisualC++'s platform-dependent preprocessor directives.

Like C++, and unlike Java, C# programmers must use the keyword virtual to allow methods to be overridden by subclasses. Extension methods in C# allow programmers to use static methods as if they were methods from a class's method table, allowing programmers to add methods to an object that they feel should exist on that object and its derivatives.

The type dynamic allows for run-time method binding, allowing for JavaScript like method calls and run-time object composition. C# has strongly typed and verbose function pointer support via the keyword delegate.

Like the Qt framework's pseudo-C++ signal and slot, C# has semantics specifically surrounding publish-subscribe style events, though C# uses delegates to do so. C# offers Java-like synchronized method calls, via the attribute [MethodImpl(MethodImplOptions.Synchronized)], and has support for mutually-exclusive locks via the keyword lock. The C# language does not allow for global variables or functions. All methods and members must be declared within classes. Static members of public classes can substitute for global variables and functions.

Local variables cannot shadow variables of the enclosing block, unlike C and C++.

A C# namespace provides the same level of code isolation as a Java package or a C++ namespace,

with very similar rules and features to a package. C# supports a strict Boolean data type, bool. Statements that take conditions, such as while and if, require an expression of a type that implements the true operator, such as the boolean type. While C++ also has a boolean type, it can be freely converted to and from integers, and expressions such as if(a) require only that a is convertible to bool, allowing a to be an int, or a pointer. C# disallows this "integer meaning true or false" approach, on the grounds that forcing programmers to use expressions that return exactly bool can prevent certain types of programming mistakes common in C or C++ such as if (a = b) (use of assignment = instead of equality ==).

In C#, memory address pointers can only be used within blocks specifically marked as unsafe, and programs with unsafe code need appropriate permissions to run. Most object access is done through safe object references, which always either point to a "live" object or have the well-defined null value; it is impossible to obtain a reference to a "dead" object (one that has been garbage collected), or to a random block of memory. An unsafe pointer can point to an instance of a value-type, array, string, or a block of memory allocated on a stack. Code that is not marked as unsafe can still store and manipulate pointers through the System.IntPtr type, but it cannot dereference them. Managed memory cannot be explicitly freed; instead, it is automatically garbage collected. Garbage collection addresses the problem of memory leaks by freeing the programmer of responsibility for releasing memory that is no longer needed.

In addition to the try...catch construct to handle exceptions, C# has a try...finally construct to guarantee execution of the code in the finally block, whether an exception occurs or not.

Multiple inheritance is not supported, although a class can implement any number of interfaces. This was a design decision by the language's lead architect to avoid complication and simplify architectural requirements throughout CLI. When implementing multiple interfaces that contain a method with the same signature, C# allows the programmer to implement each method depending on which interface that method is being called through, or, like Java, allows the programmer to implement the method once and have that be the single invocation on a call through any of the class's interfaces.

C#, unlike Java, supports operator overloading. Only the most commonly overloaded operators in C++ may be overloaded in C#. C# is more type safe than C++. The only implicit conversions by default are those that are considered safe, such as widening of integers. This is enforced at compile-time, during JIT, and, in some cases, at runtime. No implicit conversions occur between booleans and integers, nor between enumeration members and integers (except for literal 0, which can be implicitly converted to any enumerated type). Any user-defined conversion must be explicitly marked as explicit or implicit, unlike C++ copy constructors and conversion operators, which are both implicit by default.

C# has explicit support for covariance and contravariance in generic types, unlike C++ which has some degree of support for contravariance simply through the semantics of return types on virtual methods.

Enumeration members are placed in their own scope. C# provides properties as syntactic sugar for a common pattern in which a pair of methods, accessor (getter) and mutator (setter) encapsulate operations on a single attribute of a class. No redundant method signatures for the getter/setter implementations need be written, and the property may be accessed using attribute syntax rather than more verbose method calls.

Checked exceptions are not present in C# (in contrast to Java). This has been a conscious

decision based on the issues of scalability and versionability. For further information on C#, see [Wikipedia: C#](#) (the text above has been copied from this reference)

Example for using the library

---

```
using System;
using System.Collections.Generic;
using System.Text;
using MatrixClass2;

namespace ConsoleSimple
{
    class Program
    {
        static void Main(string[] args)
        {
            mp.Prec10 = 339;
            mp.FloatingPointType = 3;
            double x1 = 15.0;
            mpNum Y1 = "5.12";
            mpNum Y2 = Y1 * x1;
            mpNum Y3 = mp.Sqrt(Y1);
            Console.WriteLine(" x1: " + x1 + "; Y1: ");
            Console.WriteLine("@" + Y1.Str());
            Console.WriteLine(" Y2: " + Y2.Str() + "; Y3: " );
            Console.WriteLine("@" + Y3.Str());
        }
    }
}
```

---

Example for using Excel

---

```
using System;

namespace DemoExcel
{
    class Program
    {
        static void Main(string[] args)
        {
            dynamic xlApp = Activator.CreateInstance(Type.GetTypeFromProgID("Excel.Application"));
            xlApp.Visible = true;
            xlApp.Workbooks.Add();
            xlApp.Cells(1, 1).Value = "Test value";
        }
    }
}
```

---

## C.3 JScript 10.0

JScript .NET is a .NET programming language developed by Microsoft.

The primary differences between JScript and JScript .NET can be summarized as follows:

Firstly, JScript is a scripting language, and as such programs (or more suggestively, scripts) can be executed without the need to compile the code first. This is not the case with the JScript .NET command-line compiler, since this next-generation version relies on the .NET Common Language Runtime (CLR) for execution, which requires that the code be compiled to Common Intermediate Language (CIL), formerly called Microsoft Intermediate Language (MSIL), code before it can be run. Nevertheless, JScript .NET still provides full support for interpreting code at runtime (e.g., via the Function constructor or the eval function) and indeed the interpreter can be exposed by custom applications hosting the JScript .NET engine via the VSA[jargon] interfaces.

Secondly, JScript has a strong foundation in Microsoft's ActiveX/COM technologies, and relies primarily on ActiveX components to provide much of its functionality (including database access via ADO, file handling, etc.), whereas JScript .NET uses the .NET Framework to provide equivalent functionality. For backwards-compatibility (or for where no .NET equivalent library exists), JScript .NET still provides full access to ActiveX objects via .NET / COM interop using both the ActiveXObject constructor and the standard methods of the .NET Type class.

Although the .NET Framework and .NET languages such as C# and Visual Basic .NET have seen widespread adoption, JScript .NET has never received much attention, by the media or by developers. It is not supported in Microsoft's premier development tool, Visual Studio .NET. However, ASP.NET supports JScript .NET.

For further details, see [Wikipedia: JScript.NET](#) (the text above has been copied from this reference).

Example for using the library:

---

```
//Load the mpNumerics library
import MatrixClass2;

//Set Floating point type to MPFR with 60 decimal digits precision
mp.FloatingPointType = 3;
mp.Prec10 = 60;

//Assign values to x1 and x2
var x1 = mp.CNum("32.47");
var x2 = mp.CNum("12.41");

//Calculate x3 = x1 / x2
var x3 = x1 / x2;

//Print the value of x3
print("Result: ", x3.Str());
```

---

Example for using Excel:

---

```
// Declare the variables
var Excel, Book;

// Create the Excel application object.
Excel = new ActiveXObject("Excel.Application");
```

```
// Make Excel visible.  
Excel.Visible = true;  
  
// Create a new work book.  
Book = Excel.Workbooks.Add()  
  
// Place some text in the first cell of the sheet.  
Book.ActiveSheet.Cells(1,1).Value = "This is column A, row 1";  
  
// Save the sheet.  
Book.SaveAs("C:\\TEST.XLS");  
  
// Close Excel with the Quit method on the Application object.  
Excel.Application.Quit();
```

---

## C.4 C++ 10.0, Visual Studio

C++/CLI (Common Language Infrastructure) is a language specification created by Microsoft and intended to supersede Managed Extensions for C++. It is a complete revision that aims to simplify the older Managed C++ syntax, which is now deprecated.[1] C++/CLI was standardized by Ecma as ECMA-372. It is currently available in Visual Studio 2005, 2008, 2010 and 2012, including the Express editions.

Syntax changes[edit]C++/CLI should be thought of as a language of its own (with a new set of keywords, for example), instead of the C++ superset-oriented Managed C++. Because of this, there are some major syntactic changes, especially related to the elimination of ambiguous identifiers and the addition of .NET-specific features.

Many conflicting syntaxes, such as the multiple versions of operator new() in MC++ have been split: in C++/CLI, .NET reference types are created with the new keyword gcnew. Also, C++/CLI has introduced the concept of generics (conceptually similar to standard C++ templates, but quite different in their implementation).

In C++/CLI the only type of pointer is the normal C++ pointer, and the .NET reference types are accessed through a handle, with the new syntax ClassName^ instead of ClassName. This new construct is especially helpful when managed and standard C++ code is mixed; it clarifies which objects are under .NET automatic garbage collection and which objects the programmer must remember to explicitly destroy.

Operator overloading works analogously to standard C++. Every \* becomes a ; every & becomes an %, but the rest of the syntax is unchanged, except for an important addition: Operator overloading is possible not only for classes themselves, but also for references to those classes. This feature is necessary to give a ref class the semantics for operator overloading expected from .NET ref classes. In reverse, this also means that for .Net framework ref classes, reference operator overloading often is implicitly implemented in C++/CLI.

For further information, see [Wikipedia: C++/CLI](#) (the text above has been copied from this reference).

Example for using the library

---

```
// compile with: /clr
using namespace System;
using namespace MatrixClass2;

int main()
{
    mp^ MP = gcnew mp;
    MP->Prec10 = 30;
    MP->FloatingPointType = 3;
    mpNum^ x1 = gcnew mpNum;
    x1 = "3.4";
    mpNum^ x2 = gcnew mpNum;
    x2 = "13.4";
    mpNum^ x3 = gcnew mpNum;
    x3 = x1 / x2;
    String^ Result = x1->Str();
    Console::WriteLine("Result: {0} ", Result);
    return 0;
}
```

---

Example for mixing managed and unmanaged code

```
// compile with: /clr
using namespace System;
using namespace MatrixClass2;

int main()
{
// pragma_directives_managed_unmanaged.cpp
// compile with: /clr
#include <stdio.h>
#include <iostream>

// func1 is managed
void func1() {
System::Console::WriteLine("In managed C++ function (func1).");
}

// #pragma unmanaged
// push managed state on to stack and set unmanaged state
#pragma managed(push, off)

// func2 is unmanaged
void func2() {
printf("In unmanaged C function (func2).\n");
}

// func3 is unmanaged
void func3() {
std::cout << "In unmanaged C++ function (func3)." << std::endl;
}

// #pragma managed
#pragma managed(pop)

// main is managed
int main() {
func1();
func2();
func3();
}
```

## C.5 F# 3.0

F# (pronounced F Sharp) is a strongly typed, multi-paradigm programming language encompassing functional, imperative and object-oriented programming techniques. F# is most often used as a cross-platform CLI language, but can also be used to generate JavaScript[3] and GPU[4] code.

F# is developed by the F# Software Foundation,[5] Microsoft and open contributors. An open source, cross-platform compiler for F# is available from the F# Software Foundation.[6] F# is also a fully supported language in Visual Studio.[7] Other tools supporting F# development[clarification needed] include Mono, MonoDevelop, SharpDevelop and WebSharper

F# originated as a variant of ML and has been influenced by OCaml, C#, Python, Haskell,[2] Scala and Erlang.

For further information, see the [F# Homepage](#) or [Wikipedia: F#](#) (the text above has been copied from this reference).

Example for using the library

---

```

open System.Windows.Forms
open MatrixClass2

// Create a window and set a few properties
let form = new Form(Visible=true, TopMost=true, Text="Welcome to F#")

// mp.FloatingPointType = 3 // Does not work, need mp.SetFloatingPointType(3)

let x1 = mp.CNum("32.47")
let x2 = mp.CNum("32.47")
let x3 = x1 + x2
let s = x3.Str()

let label = new Label(Text = s)

// Add the label to the form
form.Controls.Add(label)

// Finally, run the form
[<System.STAThread>]
Application.Run(form)

```

---

Example for using functions

---

```

/// Iteration using a 'for' loop
let printList lst =
  for x in lst do
    printfn "%d" x

/// Iteration using a higher-order function
let printList2 lst =
  List.iter (printfn "%d") lst

/// Iteration using a recursive function and pattern matching

```

```
let rec printList3 lst =
  match lst with
  | [] -> ()
  | h :: t ->
    printfn "%d" h
    printList3 t
```

## C.6 MatLab (.NET interface)

MATLAB (matrix laboratory) is a numerical computing environment and fourth-generation programming language. Developed by MathWorks, MATLAB allows matrix manipulations, plotting of functions and data, implementation of algorithms, creation of user interfaces, and interfacing with programs written in other languages, including C, C++, Java, and Fortran.

Although MATLAB is intended primarily for numerical computing, an optional toolbox uses the MuPAD symbolic engine, allowing access to symbolic computing capabilities. An additional package, Simulink, adds graphical multi-domain simulation and Model-Based Design for dynamic and embedded systems.

The MATLAB application is built around the MATLAB language, and most use of MATLAB involves typing MATLAB code into the Command Window (as an interactive mathematical shell), or executing text files containing MATLAB codes, including scripts and/or functions.[6]

Variables are defined using the assignment operator, `=`. MATLAB is a weakly typed programming language because types are implicitly converted.[7] It is a dynamically typed language because variables can be assigned without declaring their type, except if they are to be treated as symbolic objects,[8] and that their type can change. Values can come from constants, from computation involving values of other variables, or from the output of a function.

As suggested by its name (a contraction of "Matrix Laboratory"), MATLAB can create and manipulate arrays of 1 (vectors), 2 (matrices), or more dimensions. In the MATLAB vernacular, a vector refers to a one dimensional ( $1 \times n$  or  $n \times 1$ ) matrix, commonly referred to as an array in other programming languages. A matrix generally refers to a 2-dimensional array, i.e. an  $m \times n$  array where  $m$  and  $n$  are greater than 1. Arrays with more than two dimensions are referred to as multidimensional arrays. Arrays are a fundamental type and many standard functions natively support array operations allowing work on arrays without explicit loops.

MATLAB can call functions and subroutines written in the C programming language or Fortran. A wrapper function is created allowing MATLAB data types to be passed and returned. The dynamically loadable object files created by compiling such functions are termed "MEX-files" (for MATLAB executable).[13][14]

Libraries written in Java, ActiveX or .NET can be directly called from MATLAB and many MATLAB libraries (for example XML or SQL support) are implemented as wrappers around Java or ActiveX libraries. Calling MATLAB from Java is more complicated, but can be done with a MATLAB extension,[15] which is sold separately by MathWorks, or using an undocumented mechanism called JMI (Java-to-MATLAB Interface),[16] which should not be confused with the unrelated Java Metadata Interface that is also called JMI.

As alternatives to the MuPAD based Symbolic Math Toolbox available from MathWorks, MATLAB can be connected to Maple or Mathematica.[17]

Libraries also exist to import and export MathML. MATLAB has a COM interface which is described in section [D.10](#).

For further information on MatLab, see [Wikipedia: MatLab](#) (the text above has been copied from this reference), or the [MatLab Homepage](#). See also [MatLab External Interfaces](#).

Example for using the library

---

```
x = 4.3;
fprintf('Start: x is equal to %6.2f.\n',x);

%Load mpFormulaPy .NET assembly
NET.addAssembly('MatrixClass2');

%Instantiate local reference to mpFormulaPy
mp = MatrixClass2.mp;

%Set Floating point type to MPFR with 60 decimal digits precision
NET.setStaticProperty('MatrixClass2.mp.FloatingPointType', 3);
NET.setStaticProperty('MatrixClass2.mp.Prec10', 50);
Myprec = mp.Prec10;

fprintf('End: Myprec is equal to %6.2f.\n',Myprec);

%Assign values to x1 and x2
x1 = mp.CNum(4.5);
x2 = mp.CNum('1.1');
x3 = x1 / x2;
x4 = MatrixClass2.mpNum;

x4 = mp.CNum(4.5356346345);
s = x3.Str();
s2 = char(s);
fprintf('s is equal to %s.\n',s2);
fprintf('End: x is equal to %6.2f.\n',x);
quit;
```

---

## C.7 Java (via jni4net)

Java is a general-purpose, concurrent, class-based, object-oriented computer programming language that is specifically designed to have as few implementation dependencies as possible. It is intended to let application developers "write once, run anywhere" (WORA), meaning that code that runs on one platform does not need to be recompiled to run on another. Java applications are typically compiled to bytecode (class file) that can run on any Java virtual machine (JVM) regardless of computer architecture. Java is, as of 2012, one of the most popular programming languages in use, particularly for client-server web applications, with a reported 10 million users.[10][11] Java was originally developed by James Gosling at Sun Microsystems (which has since merged into Oracle Corporation) and released in 1995 as a core component of Sun Microsystems' Java platform. The language derives much of its syntax from C and C++, but it has fewer low-level facilities than either of them.

The original and reference implementation Java compilers, virtual machines, and class libraries were developed by Sun from 1991 and first released in 1995. As of May 2007, in compliance with the specifications of the Java Community Process, Sun relicensed most of its Java technologies under the GNU General Public License. Others have also developed alternative implementations of these Sun technologies, such as the GNU Compiler for Java and GNU Classpath.

The syntax of Java is largely derived from C++. Unlike C++, which combines the syntax for structured, generic, and object-oriented programming, Java was built almost exclusively as an object-oriented language. All code is written inside a class, and everything is an object, with the exception of the primitive data types (e.g. integers, floating-point numbers, boolean values, and characters), which are not classes for performance reasons. Unlike C++, Java does not support operator overloading or multiple inheritance for classes.

Oracle Corporation is the current owner of the official implementation of the Java SE platform, following their acquisition of Sun Microsystems on January 27, 2010. This implementation is based on the original implementation of Java by Sun. The Oracle implementation is available for Mac OS X, Windows and Solaris. Because Java lacks any formal standardization recognized by Ecma International, ISO/IEC, ANSI, or other third-party standards organization, the Oracle implementation is the de facto standard.

The Oracle implementation is packaged into two different distributions: The Java Runtime Environment (JRE) which contains the parts of the Java SE platform required to run Java programs and is intended for end-users, and the Java Development Kit (JDK), which is intended for software developers and includes development tools such as the Java compiler, Javadoc, Jar, and a debugger. OpenJDK is another notable Java SE implementation that is licensed under the GPL. The implementation started when Sun began releasing the Java source code under the GPL. As of Java SE 7, OpenJDK is the official Java reference implementation.

Programs written in Java have a reputation for being slower and requiring more memory than those written in C++. However, Java programs' execution speed improved significantly with the introduction of Just-in-time compilation in 1997/1998 for Java 1.1, the addition of language features supporting better code analysis (such as inner classes, the StringBuffer class, optional assertions, etc.), and optimizations in the Java virtual machine itself, such as HotSpot becoming the default for Sun's JVM in 2000. As of December 2012, microbenchmarks show Java 7 is approximately 44% slower than C++.

For further information on Java, see [Wikipedia: Java](#) (the text above has been copied from this

reference), or the [Java Homepage](#).

The Java SDK can be downloaded from [Java SDK Download Homepage](#). Currently versions 1.7 and 1.8 are supported.

Java programs can access the functions provided by the mpFormulaPy Library using the jni4net bridge between Java and .NET (see the [jni4net Homepage](#) for more information).

For this reason, a Java program using the mpFormulaPy Library needs to include a number of import statements for the mpLib and the mpNum types. It is also necessary to initialize the bridge between Java and .NET and to load and register the relevant assembly, as shown in the program below.

The supporting .dll and .jar files are located in

.. \mpFormula40\Toolbox\mpFormula4java\Win32\work for 32 bit, and  
.. \mpFormula40\Toolbox\mpFormula4java\Win64\work for 64 bit.

A copy of the example program below can be found in

.. \mpFormula40\Toolbox\mpFormula4java\Win32 for 32 bit and  
.. \mpFormula40\Toolbox\mpFormula4java\Win64 for 64 bit.

The batch files in these directories assume that the Java SDK is installed and that the directory containing `javac.exe` and `java.exe` is in the path. It is also assumed that on Windows 64 bit only the binaries of the 64 bit (but not of the 32 bit) version the Java SDK are in the path. To compile and/or run a 32 bit Java program on Windows 64 bit, use the batch files `CompileAndRun32bitOnW64.cmd` and `RunOnly32bitOnW64.cmd`, modifying the absolute paths in these batch files as needed.

Example for using the library

---

```

import java.io.IOException;

//Import Java2Net Bridge and mpFormulaPy classes
import net.sf.jni4net.Bridge;
import mpformula4java.mpLib;
import mpformula4java.mpLibT;
import mpformula4java.mpNum;
import mpformula4java.mpNumT;

public class MyCalcUsageInJava {
    public static void main(String arsg[]) throws IOException {

        //Initialize Java2Net Bridge and load relevant assemblies
        System.out.printf("Opening Java2Net Bridge ... \n");
        Bridge.init();
        Bridge.LoadAndRegisterAssemblyFrom(new java.io.File("mpFormula4java.j4n.dll"));

        //Initialize the mpNumerics library
        mpLib mp = new mpLibT();

        //Set Floating point type to MPFR with 60 decimal digits precision
        mp.SetFloatingPointType(3);
        mp.SetPrec10(50);

        //Assign values to x1 and x2
    }
}
```

```

mpNum x1 = mp.Num("12.0");
System.out.printf("Value of x1 is : " + x1.Str() + "\n");

mpNum x2 = mp.Sqrt(x1);
System.out.printf("Value of x2 is : " + x2.Str() + "\n");

mpNum x3 = x1.Plus(x2);
System.out.printf("Value of x1 + x2 is : " + x3.Str() + "\n");

x3 = x1.Minus(x2);
System.out.printf("Value of x1 - x2 is : " + x3.Str() + "\n");

x3 = x1.Times(x2);
System.out.printf("Value of x1 * x2 is : " + x3.Str() + "\n");

//Calculate x3 = x1 / x2
x3 = x1.Div(x2);

//Print the value of x3
System.out.printf("Value of x1 / x2 is : " + x3.Str() + "\n");

System.out.printf("Closing Java2Net Bridge ... \n");

}
}

```

---

### C.7.0.1 Downloading and installing the Java SDK

The Java SDK can be downloaded from the [Java SDK Download Homepage](#). Currently releases 1.7 and 1.8 are available for download. The prebuild binaries of mpFormulaPy use the 1.7 release. To support both 32 bit and 64 bit builds, you need to install the 32 and 64 bit editions of the SDK separately.

The batch files used for building Java support assume that the Java SDK is installed and that the directory containing `javac.exe` and `java.exe` is in the path. It is also assumed that you build on Windows 64 bit and that only the binaries of the 64 bit (but not of the 32 bit) version the Java SDK are in the path.

### C.7.0.2 Downloading and installing jni4net

jni4net: [jni4net](#).

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The contributors are listed in section [F.2.5](#)

The required binary files are contained in the file `jni4net-0.8.6.0-bin.zip`, which can be downloaded from

<http://sourceforge.net/projects/jni4net/files/>.

Unzip this file and copy the content of the resulting folder to `\mpFormula40\Java2Net`.

### C.7.0.3 Applying jni4net to build the Java interface

The source code which is specific to providing Java support for mpFormulaPy is automatically generated when building the documentation and is contained in the file  
 \Toolbox\mpFormula4java\Source\CSharp\Calc.cs.

In the folder \Toolbox\mpFormula4java\Source\CSharp, open the solution mpFormula4java.sln, and compile the assembly mpFormula4java.dll for 32 and 64 bit (Release).

From \Toolbox\mpFormula4java\Source, copy the subfolder \MakeJavaBridge into the folder \mpFormula40\Java2Net.

Within the folder \mpFormula40\Java2Net\MakeJavaBridge run the command file Make.cmd. This will do the following:

- generateProxies will copy all dependencies from jni4net lib to work directory
- generateProxies will run the proxygen tool to wrap mpFormula4java.dll
- proxygen: generate java proxies
- proxygen: generate C# proxies
- proxygen: generate build.cmd
- generateProxies will run work\build2.cmd to compile the output generated by proxygen.
- build2.cmd : run javac to compile java classes
- build2.cmd : run jar to package classes into .JAR file
- build2.cmd : run csc to compile C# classes and produce .DLL

Once this has been completed without error messages, run the command file Make Install.cmd.

This will install the appropriate .cmd, .jar and .dll files into  
 \Toolbox\mpFormula4java\Win32 for 32 bit and  
 \Toolbox\mpFormula4java\Win64, for 64 bit.

### C.7.0.4 Testing the Java-.NET Bridge

A copy of the example program below can be found in

..\\mpFormula40\\Toolbox\\mpFormula4java\\Win32 for 32 bit and  
 ..\\mpFormula40\\Toolbox\\mpFormula4java\\Win64 for 64 bit.

The batch files in these directories assume that the Java SDK is installed and that the directory containing javac.exe and java.exe is in the path. It is also assumed that on Windows 64 bit only the binaries of the 64 bit (but not of the 32 bit) version the Java SDK are in the path. To compile and/or run a 32 bit Java program on Windows 64 bit, use the batch files CompileAndRun32bitOnW64.cmd and RunOnly32bitOnW64.cmd, modifying the absolute paths in these batch files as needed.

To confirm that the installation was successful, run the command files CompileAndRun32bit.cmd and CompileAndRun64bit.cmd in their respective folders.

# Appendix D

## Component Object Model (COM) Interface

Component Object Model (COM) is a binary-interface standard for software components introduced by Microsoft in 1993. It is used to enable interprocess communication and dynamic object creation in a large range of programming languages. COM is the basis for several other Microsoft technologies and frameworks, including OLE, OLE Automation, ActiveX, COM+, DCOM, the Windows shell, DirectX, and Windows Runtime

Overview[edit source — edit]The essence of COM is a language-neutral way of implementing objects that can be used in environments different from the one in which they were created, even across machine boundaries. For well-authored components, COM allows reuse of objects with no knowledge of their internal implementation, as it forces component implementers to provide well-defined interfaces that are separated from the implementation. The different allocation semantics of languages are accommodated by making objects responsible for their own creation and destruction through reference-counting. Casting between different interfaces of an object is achieved through the `QueryInterface` method. The preferred method of inheritance within COM is the creation of sub-objects to which method calls are delegated.

COM is an interface technology defined and implemented as standard only on Microsoft Windows and Apple's Core Foundation 1.3 and later plug-in API,[1] that in any case implement only a subset of the whole COM interface.[2] For some applications, COM has been replaced at least to some extent by the Microsoft .NET framework, and support for Web Services through the Windows Communication Foundation (WCF). However, COM objects can be used with all .NET languages through .NET COM Interop. Networked DCOM uses binary proprietary formats, while WCF encourages the use of XML-based SOAP messaging. COM is very similar to other component software interface technologies, such as CORBA and Java Beans, although each has its own strengths and weaknesses.

Unlike C++, COM provides a stable ABI that does not change between compiler releases.[3] This makes COM interfaces attractive for object-oriented C++ libraries that are to be used by clients compiled using different compiler versions.

For further information, see [Wikipedia: Component Object Model](#) (the text above has been copied from this reference).

Missing:

A description of how to build the relevant projects.

## D.1 VBScript (Windows Script Host)

VBScript (Visual Basic Scripting Edition) is an Active Scripting language developed by Microsoft that is modeled on Visual Basic. It is designed as a â€œlightweightâ€ language with a fast interpreter for use in a wide variety of Microsoft environments. VBScript uses the Component Object Model to access elements of the environment within which it is running; for example, the FileSystemObject (FSO) is used to create, read, update and delete files.

VBScript has been installed by default in every desktop release of Microsoft Windows since Windows 98;<sup>[1]</sup> in Windows Server since Windows NT 4.0 Option Pack;<sup>[2]</sup> and optionally with Windows CE (depending on the device it is installed on).

A VBScript script must be executed within a host environment, of which there are several provided with Microsoft Windows, including: Windows Script Host (WSH), Internet Explorer (IE), and Internet Information Services (IIS).<sup>[3]</sup> Additionally, the VBScript hosting environment is embeddable in other programs, through technologies such as the Microsoft Script Control (msscript.ocx).

VBScript can also be used to create applications that run directly on a workstation running Microsoft Windows. The simplest example is a script that makes use of the Windows Script Host (WSH) environment. Such a script is usually in a stand-alone file with the file extension .vbs. The script can be invoked in two ways. Wscript.exe is used to display output and receive input through a GUI, such as dialog and input boxes. Cscript.exe is used in a command line environment.

VBScript can be included in two other types of scripting files: Windows Script Files, and HTML Applications.

A Windows Script File (WSF) is styled after XML. A WSF file can include multiple VBS files. As a result WSF files provide a means for code reuse: one can write a library of classes or functions in one or more .vbs files, and include those files in one or more WSF files to use and reuse that functionality in a modular way. The files have extension .wsf and can be executed using wscript.exe or cscript.exe, just like a .vbe file.

An HTML Application (HTA) is styled after HTML. The HTML in the file is used to generate the user interface, and a scripting language such as VBScript is used for the program logic. The files have extension .hta and can be executed using mshta.exe.

VBScript (and JScript) can also be used in a Windows Script Component - an ActiveX-enabled script class that can be invoked by other COM-enabled applications.<sup>[13]</sup> These files have extension .wsc.

For further information on VBScript, see [Wikipedia: VBScript](#) (the text above has been copied from this reference).

Example for using the library

---

```
Option Explicit
```

```
Sub Print(s)
WScript.Echo s
End Sub
```

```

Sub DemoBetadist()
Dim mp, x, df1, df2, px1
Set mp = CreateObject("mpNumerics.mp_Lib")
With mp
'FloatingPointType: Single = 1, Double = 2, Multi = 3, Interval = 4, Decimal = 5,
    Rational = 6
.FloatingPointType() = 3
.Prec10() = 36
Set x = .Real(0.1)
Set df1 = .Real(13.0)
Set df2 = .Real(23.0)
Set px1 = .Real(0.0005)
' pdf = 1, LeftTail = 2, RightTail = 3, LeftQuantile = 4, RightQuantile = 5
Print ("#TableStart#")
Print ("Item" & "|" & "Value")
Print ("Density:" & "|" & .BetaDist(1, x, df1, df2).Str())
Print ("LeftTail:" & "|" & .BetaDist(2, x, df1, df2).Str())
Print ("RightTail:" & "|" & .BetaDist(3, x, df1, df2).Str())
Print ("LeftQuantile:" & "|" & .BetaDist(4, px1, df1, df2).Str())
Print ("RightQuantile:" & "|" & .BetaDist(5, px1, df1, df2).Str())
Print("#TableEnd#")
Print("")
End With
Set mp = Nothing
End Sub

Call DemoBetadist()

```

---

Example for using Excel

---

```

Option Explicit

Sub Print(s)
WScript.Echo s
End Sub

Sub DemoExcel()
Dim objExcel 'As Excel.Application
Set objExcel = CreateObject("Excel.Application")
'objExcel.Workbooks.Open("C:\Extra\mpNumerics\Output\mpTemp00.html")
objExcel.Visible = True
objExcel.Workbooks.Add
objExcel.Cells(1, 1).Value = "Test value"
Set objExcel = Nothing
End Sub

Call DemoExcel()

```

---

## D.2 JScript (Windows Script Host)

JScript is Microsoft's dialect of the ECMAScript standard[1] that is used in Microsoft's Internet Explorer.

JScript is implemented as a Windows Script engine.[2] This means that it can be "plugged in" to any application that supports Windows Script,[3] such as Internet Explorer, Active Server Pages, and Windows Script Host. It also means that any application supporting Windows Script can use multiple languages (JScript, VBScript, Perl, and others).

JScript was first supported in the Internet Explorer 3.0 browser released in August 1996. Its most recent version is JScript 9.0, included in Internet Explorer 9.

JScript supports conditional compilation, which allows a programmer to selectively execute code within block comments. This is an extension to the ECMAScript standard that is not supported in other JavaScript implementations.

The original JScript is an Active Scripting engine. Like other Active Scripting languages, it is built on the COM/OLE Automation platform and provides scripting capabilities to host applications.

This is the version used when hosting JScript inside a Web page displayed by Internet Explorer, in an HTML application, in classic ASP, in Windows Script Host scripts and several other Automation environments.

JScript is sometimes referred to as "classic JScript" or "Active Scripting JScript" to differentiate it from newer .NET-based versions.

Some versions of JScript are available for multiple versions of Internet Explorer and Windows. For example, JScript 5.7 was introduced with Internet Explorer 7.0 and is also installed for Internet Explorer 6.0 with Windows XP Service Pack 3, while JScript 5.8 was introduced with Internet Explorer 8.0 and is also installed with Internet Explorer 6.0 on Windows Mobile 6.5.

Microsoft's implementation of ECMAScript 5th Edition in Windows 8 Consumer Preview is called JavaScript and the corresponding Visual Studio 11 Express Beta includes a completely new, full-featured JavaScript editor with IntelliSense enhancements for HTML5 and ECMAScript 5 syntax, annotations for multiple overloads, simplified DOM configuration, brace matching, collapsible outlining and "go to definition".[6]

For further information on JScript, see [Wikipedia: JScript](#) (the text above has been copied from this reference). For further information on JavaScript, see [Wikipedia: JavaScript](#).

Example for using the library

---

```

var stdin = WScript.StdIn;
var stdout = WScript.StdOut;

var mp = new ActiveXObject ("mpNumerics.mp_Lib");
mp.Prec10 = 60;
mp.FloatingPointType = 3;
var x = mp.Real(2);
var y = mp.Sqrt(x);
var s = y.Str();

stdout.WriteLine("Sqrt(2):");

```

```
stdout.WriteLine(s);
```

---

Example for using Excel

---

```
var xls = new ActiveXObject ("Excel.Application");
xls.visible = true;
var newBook = xls.Workbooks.Add;
newBook.Worksheets.Add;
newBook.Worksheets(1).Activate;
newBook.Worksheets(1).Cells(1,1).value="First Column, First Cell";
newBook.Worksheets(1).Cells(2,1).value="First Column, Second Cell";
newBook.Worksheets(1).Cells(1,2).value="Second Column, First Cell";
newBook.Worksheets(1).Cells(2,2).value="Second Column, Second Cell";
newBook.Worksheets(1).Name="WorkSheet from Javascript";
// newBook.Worksheets(1).SaveAs("C:\\temp\\TEST2.XLS");
```

---

## D.3 Visual Basic for Applications, Visual Basic 6.0

Visual Basic for Applications (VBA) is an implementation of Microsoft's event-driven programming language Visual Basic 6 and its associated integrated development environment (IDE).

Visual Basic for Applications enables building user defined functions, automating processes and accessing Windows API and other low-level functionality through dynamic-link libraries (DLLs). It supersedes and expands on the abilities of earlier application-specific macro programming languages such as Word's WordBasic. It can be used to control many aspects of the host application, including manipulating user interface features, such as menus and toolbars, and working with custom user forms or dialog boxes.

As its name suggests, VBA is closely related to Visual Basic and uses the Visual Basic Runtime Library, but it can normally only run code within a host application rather than as a standalone program. It can, however, be used to control one application from another via OLE Automation. For example, it is used to automatically create a Word report from Excel data, which are automatically collected by Excel from polled observation sensors. VBA has the ability to use (but not create) (ActiveX/COM) DLLs, and later versions add support for class modules.

VBA is built into most Microsoft Office applications, including Office for Mac OS X (apart from version 2008) and other Microsoft applications such as Microsoft MapPoint and Microsoft Visio. For further information, see [Wikipedia: VBA](#) (the text above has been copied from this reference).

Visual Basic is a third-generation event-driven programming language and integrated development environment (IDE) from Microsoft for its COM programming model first released in 1991. Microsoft intends Visual Basic to be relatively easy to learn and use.[1][2] Visual Basic was derived from BASIC and enables the rapid application development (RAD) of graphical user interface (GUI) applications, access to databases using Data Access Objects, Remote Data Objects, or ActiveX Data Objects, and creation of ActiveX controls and objects. The scripting language VBScript is a subset of Visual Basic.

A programmer can create an application using the components provided by the Visual Basic program itself. Programs written in Visual Basic can also use the Windows API, but doing so requires external function declarations. Though the program has received criticism for its perceived faults, version 3 of Visual Basic was a runaway commercial success, and many companies offered third party controls greatly extending its functionality.

The final release was version 6 in 1998. Microsoft's extended support ended in March 2008 and the designated successor was Visual Basic .NET (now known simply as Visual Basic).

For further information, see [Wikipedia: VB6](#) (the text above has been copied from this reference).

```
' Imports System
' Imports System.Console
' Imports Microsoft.VisualBasic.Strings
' Imports mpNumericsLib
```

```
'Module Module1
```

```
Sub Print(s As String)
    WriteLine(s)
End Sub
```

```
Sub DemoBetadist()
Dim mp As New mp_Lib
Dim x, df1, df2, px1 As New mp_Real
With mp
'FloatingPointType: Single = 1, Double = 2, Multi = 3, Interval = 4, Rational = 5
.FloatingPointType() = 3
.Prec10() = 40
x = .Real(0.1)
df1 = .Real(13.0)
df2 = .Real(23.0)
px1 = .Real(0.0005)
' pdf = 1, LeftTail = 2, RightTail = 3, LeftQuantile = 4, RightQuantile = 5
Print ("Density:" & .BetaDist(1, x, df1, df2).Str())
Print ("LeftTail:" & .BetaDist(2, x, df1, df2).Str())
Print ("RightTail:" & .BetaDist(3, x, df1, df2).Str())
Print ("LeftQuantile:" & .BetaDist(4, px1, df1, df2).Str())
Print ("RightQuantile:" & .BetaDist(5, px1, df1, df2).Str())
End With
mp = Nothing
End Sub

Sub Main()
Call DemoBetadist()
End Sub

'End Module
```

---

## D.4 OpenOffice Basic

OpenOffice Basic (formerly known as StarOffice Basic or StarBasic or OOoBasic) is a dialect of the programming language BASIC that is included with the OpenOffice, StarOffice and LibreOffice office suites.

Although OpenOffice Basic itself is similar to other dialects of Basic, such as Microsoft's VBA, the application programming interface (API) is very different, as the example below of a macro illustrates. While there is a much easier way to obtain the "paragraph count" document property, the example shows the fundamental methods for accessing each paragraph in a text document, sequentially.

For further information, see [Wikipedia: OpenOffice Basic](#) (the text above has been copied from this reference).

For help regarding the language, see the [OpenOffice.org BASIC Programming Guide](#). Information on the OpenOffice API is available from [OpenOffice API](#).

---

```
Sub ParaCount
    '
    ' Count number of paragraphs in a text document
    '

    Dim Doc As Object, Enum As Object, TextEl As Object, Count As Long
    Doc = ThisComponent
    ' Is this a text document?
    If Not Doc.SupportsService("com.sun.star.text.TextDocument") Then
        MsgBox "This macro must be run from a text document", 64, "Error"
        Exit Sub
    End If
    Count = 0
    ' Examine each component - paragraph or table?
    Enum = Doc.Text.CreateEnumeration
    While Enum.HasMoreElements
        TextEl = Enum.NextElement
        ' Is the component a paragraph?
        If TextEl.SupportsService("com.sun.star.text.Paragraph") Then
            Count = Count + 1
        End If
    Wend
    'Display result
    MsgBox Count, 0, "Paragraph Count"
End Sub
```

---

Example for using the library

---

```
Sub DemoBetadist()
    Dim mp, x, df1, df2, px1
    Set mp = CreateObject("mpNumerics.mp_Lib")
    With mp
        'FloatingPointType: Single = 1, Double = 2, Multi = 3, Interval = 4, Decimal = 5,
        Rational = 6
        .FloatingPointType() = 3
        .Prec10() = 36
    End With
End Sub
```

```
Set x = .Real(0.1)
Set df1 = .Real(13.0)
Set df2 = .Real(23.0)
Set px1 = .Real(0.0005)
' pdf = 1, LeftTail = 2, RightTail = 3, LeftQuantile = 4, RightQuantile = 5
Print ("#TableStart#")
Print ("Item" & "|" & "Value")
Print ("Density:" & "|" & .BetaDist(1, x, df1, df2).Str())
Print ("LeftTail:" & "|" & .BetaDist(2, x, df1, df2).Str())
Print ("RightTail:" & "|" & .BetaDist(3, x, df1, df2).Str())
Print ("LeftQuantile:" & "|" & .BetaDist(4, px1, df1, df2).Str())
Print ("RightQuantile:" & "|" & .BetaDist(5, px1, df1, df2).Str())
Print("#TableEnd#")
Print("")
End With
Set mp = Nothing
End Sub
```

---

## D.5 Lua

Lua is a lightweight multi-paradigm programming language designed as a scripting language with "extensible semantics" as a primary goal. Lua is cross-platform since it is written in ISO C. Lua has a relatively simple C API, thus "Lua is especially useful for providing end users with an easy way to program the behavior of a software product without getting too far into its innards."

Lua is commonly described as a multi-paradigm language, providing a small set of general features that can be extended to fit different problem types, rather than providing a more complex and rigid specification to match a single paradigm. Lua, for instance, does not contain explicit support for inheritance, but allows it to be implemented relatively easily with metatables. Similarly, Lua allows programmers to implement namespaces, classes, and other related features using its single table implementation; first-class functions allow the employment of many powerful techniques from functional programming; and full lexical scoping allows fine-grained information hiding to enforce the principle of least privilege.

In general, Lua strives to provide flexible meta-features that can be extended as needed, rather than supply a feature-set specific to one programming paradigm. As a result, the base language is light—the full reference interpreter is only about 180 kB compiled[1]—and easily adaptable to a broad range of applications.

Lua is a dynamically typed language intended for use as an extension or scripting language, and is compact enough to fit on a variety of host platforms. It supports only a small number of atomic data structures such as boolean values, numbers (double-precision floating point by default), and strings. Typical data structures such as arrays, sets, lists, and records can be represented using Lua's single native data structure, the table, which is essentially a heterogeneous associative array.

Lua implements a small set of advanced features such as first-class functions, garbage collection, closures, proper tail calls, coercion (automatic conversion between string and number values at run time), coroutines (cooperative multitasking) and dynamic module loading.

By including only a minimum set of data types, Lua attempts to strike a balance between power and size.

For further information on Lua, see [Wikipedia: Lua](#) (the text above has been copied from this reference), or the [Lua for Windows Homepage](#).

Example for using the library

---

```
--Enable COM support
require("luacom")

--Load the mpNumerics library
mp = luacom.CreateObject("mpNumerics.mp_Lib")

--Set Floating point type to MPFR with 60 decimal digits precision
mp.FloatingPointType = 3
mp.Prec10 = 60

--Assign values to x1 and x2
x1 = mp:Real(4.5)
x2 = mp:Real("1.1")
```

```
--Calculate x3 = x1 / x2
x3 = x1:Div(x2)

--Print the value of x3
print ("Result: ", x3:Str())
```

---

Example for using Excel

---

```
require('luacom')
excel = luacom.CreateObject("Excel.Application")
excel.Visible = true
wb = excel.Workbooks:Add()
ws = wb.Worksheets(1)

for i=1, 20 do
    ws.Cells(i,1).Value2 = i
end

-- excel.DisplayAlerts = false
-- excel:Quit()
-- excel = nil
```

---

## D.6 Ruby

Ruby is a dynamic, reflective, general-purpose object-oriented programming language that combines syntax inspired by Perl with Smalltalk-like features. It was also influenced by Eiffel and Lisp.[8] Ruby was first designed and developed in the mid-1990s by Yukihiro "Matz" Matsumoto in Japan.

Ruby supports multiple programming paradigms, including functional, object oriented and imperative. It also has a dynamic type system and automatic memory management; it is therefore similar in varying respects to Smalltalk, Python, Perl, Lisp, Dylan, and CLU.

The syntax of Ruby is broadly similar to that of Perl and Python. Class and method definitions are signaled by keywords. In contrast to Perl, variables are not obligatorily prefixed with a sigil. When used, the sigil changes the semantics of scope of the variable. One difference from C and Perl is that keywords are typically used to define logical code blocks, without braces (i.e., pair of `begin` and `end`). For practical purposes there is no distinction between expressions and statements.[39] Line breaks are significant and taken as the end of a statement; a semicolon may be equivalently used. Unlike Python, indentation is not significant.

One of the differences of Ruby compared to Python and Perl is that Ruby keeps all of its instance variables completely private to the class and only exposes them through accessor methods (`attr_writer`, `attr_reader`, etc.). Unlike the "getter" and "setter" methods of other languages like C++ or Java, accessor methods in Ruby can be created with a single line of code via metaprogramming; however, accessor methods can also be created in the traditional fashion of C++ and Java. As invocation of these methods does not require the use of parentheses, it is trivial to change an instance variable into a full function, without modifying a single line of code or having to do any refactoring achieving similar functionality to C# and VB.NET property members.

For further information on Ruby, see [Wikipedia: Ruby](#) (the text above has been copied from this reference), or the [Ruby Homepage](#). An easy-to-install package for Windows can be found at [RubyForge](#).

Example for using the library

---

```
#Enable COM support
require 'win32ole'

#Load the mpNumerics library
mp = WIN32OLE.new("mpNumerics.mp_Lib")

#Set Floating point type to MPFR with 60 decimal digits precision
mp.FloatingPointType = 3
mp.Prec10 = 60

#Assign values to x1 and x2
x1 = mp.Real(4.5)
x2 = mp.Real("1.1")

#Calculate x3 = x1 / x2
x3 = x1.Div(x2)

#print the value of x3
```

```
puts x3.Str

obj = WIN32OLE_VARIANT.new([[1.345345,2,3],[4,5,6]])
p obj[0,0]
p obj[1,0]
obj[0,0] = 7
p obj.value
```

---

Example for using Excel

```
require 'win32ole'
xl = WIN32OLE.new("Excel.Application")

puts "Excel failed to start" unless xl

xl.Visible = true

workbook = xl.Workbooks.Add
sheet = workbook.Worksheets(1)

#create some fake data
data_a = []
(1..10).each{|i| data_a.push i }

data_b = []
(1..10).each{|i| data_b.push((rand * 100).to_i) }

#fill the worksheet with the fake data
#showing 3 ways to populate cells with values
(1..10).each do |i|
sheet.Range("A#{i}").Select
xl.ActiveCell.Formula = data_a[i-1]

sheet.Range("B#{i}").Formula = data_b[i-1]

cell = sheet.Range("C#{i}")
cell.Formula = "=A#{i} - B#{i}"

end

#chart type constants (via http://support.microsoft.com/kb/147803)
xlArea = 1
xlBar = 2

xlColumn = 3
xlLine = 4
xlPie = 5
xlRadar = -4151
```

```
xlXYScatter = -4169
xlCombination = -4111
xl3DArea = -4098

xl3DBar = -4099
xl3DColumn = -4100
xl3DLine = -4101

xl3DPie = -4102
xl3DSurface = -4103
xlDoughnut = -4120

#creating a chart
chart_object = sheet.ChartObjects.Add(10, 80, 500, 250)

chart = chart_object.Chart
chart_range = sheet.Range("A1", "B10")

chart.SetSourceData(chart_range, nil)
chart.ChartType = xlXYScatter

#get the value from a cell

val = sheet.Range("C1").Value
puts val

#saving as pre-2007 format
excel97_2003_format = -4143

pwd = Dir.getcwd.gsub('/', '\\') << '\\'

#otherwise, it sticks it in default save directory- C:\Users\Sam\Documents on my
#system
#workbook.SaveAs("#{pwd}whatever.xls", excel97_2003_format)

#xl.Quit
```

---

## D.7 PHP CLI

PHP is a server-side scripting language designed for web development but also used as a general-purpose programming language. PHP is now installed on more than 244 million websites and 2.1 million web servers.[2] Originally created by Rasmus Lerdorf in 1995, the reference implementation of PHP is now produced by The PHP Group.[3] While PHP originally stood for Personal Home Page,[4] it now stands for PHP: Hypertext Preprocessor, a recursive acronym.[5]

PHP code is interpreted by a web server with a PHP processor module which generates the resulting web page: PHP commands can be embedded directly into an HTML source document rather than calling an external file to process data. It has also evolved to include a command-line interface capability and can be used in standalone graphical applications.[6]

PHP is free software released under the PHP License, which is incompatible with the GNU General Public License (GPL) due to restrictions on the usage of the term PHP.[7] PHP can be deployed on most web servers and also as a standalone shell on almost every operating system and platform, free of charge.[8]

The PHP interpreter only executes PHP code within its delimiters. Anything outside its delimiters is not processed by PHP (although non-PHP text is still subject to control structures described in PHP code). The most common delimiters are `?php` to open and `?<` to close PHP sections. `script language="php"<` and `!</script>` delimiters are also available, as are the shortened forms `?<` or `?=<` (which is used to echo back a string or variable) and `?<`, as well as ASP-style short forms `?%<` or `?%=<` and `%<`. While short delimiters are used, they make script files less portable as support for them can be disabled in the PHP configuration, and so they are discouraged.[37] The purpose of all these delimiters is to separate PHP code from non-PHP code, including HTML.[38]

The first form of delimiters, `?php` and `?<`, in XHTML and other XML documents, creates correctly formed XML 'processing instructions'.[39] This means that the resulting mixture of PHP code and other markup in the server-side file is itself well-formed XML.

Variables are prefixed with a dollar symbol, and a type does not need to be specified in advance. Unlike function and class names, variable names are case sensitive. Both double-quoted ("") and heredoc strings provide the ability to interpolate a variable's value into the string.[40] PHP treats newlines as whitespace in the manner of a free-form language (except when inside string quotes), and statements are terminated by a semicolon.[41] PHP has three types of comment syntax: `/* */` marks block and inline comments; `//` as well as `#` are used for one-line comments.[42] The echo statement is one of several facilities PHP provides to output text, e.g., to a web browser.

In terms of keywords and language syntax, PHP is similar to most high level languages that follow the C style syntax. if conditions, for and while loops, and function returns are similar in syntax to languages such as C, C++, C#, Java and Perl.

PHP CLI is a short for PHP Command Line Interface. As the name implies, this is a way of using PHP in the system command line. Or by other words it is a way of running PHP Scripts that aren't on a web server (such as Apache web server or Microsoft IIS). People usually treat PHP as web development, server side tool. However, PHP CLI applies all advantages of PHP to shell scripting allowing to create either service side supporting scripts or system application even with GUI.

For further information on PHP, see [Wikipedia: PHP](#) (the text above has been copied from this reference), or the [PHP for Windows Homepage](#), or the [PHP CLI Homepage](#).

Example for using the library

---

```
#PHP Command line example
<?php
#Load the mpNumerics library
$mp = new COM("mpNumerics.mp_Lib") or die("Cannot open library");

#Set Floating point type to MPFR with 60 decimal digits precision
$mp->FloatingPointType = 3;
$mp->Prec10 = 60;

#Assign values to x1 and x2
$x1 = $mp->Real(4.5);
$x2 = $mp->Real('1.1');

#Calculate x3 = x1 / x2
$x3 = $x1->Div($x2);

echo "Hello world of PHP CLI! \n";

#print the value of x3
echo $x3->Str();
?>
```

---

Example for using Excel

---

```
#PHP.ini has to be stored in the PDP application directory (derived from the sample
inis)
#Need to enable the win extension_dir directive
#Need to add:

#[COM_DOT_NET]
#extension=php_com_dotnet.dll

#as explained in
#http://www.php.net/manual/en/com.installation.php

<?php
$xlApp = new COM("Excel.Application");
$xlApp->Workbooks->Add();
$xlApp->Range("A1:C6")->Select();
$xlApp->ActiveCell->Formula = "Hello World!";
$xlApp->Visible = 1;
?>
```

---

## D.8 Perl

Perl is a family of high-level, general-purpose, interpreted, dynamic programming languages. The languages in this family include Perl 5 and Perl 6.[4]

Though Perl is not officially an acronym,[5] there are various backronyms in use, such as: Practical Extraction and Reporting Language.[6] Perl was originally developed by Larry Wall in 1987 as a general-purpose Unix scripting language to make report processing easier.[7] Since then, it has undergone many changes and revisions. The latest major stable revision of Perl 5 is 5.18, released in May 2013. Perl 6, which began as a redesign of Perl 5 in 2000, eventually evolved into a separate language. Both languages continue to be developed independently by different development teams and liberally borrow ideas from one another.

The Perl languages borrow features from other programming languages including C, shell scripting (sh), AWK, and sed.[8] They provide powerful text processing facilities without the arbitrary data-length limits of many contemporary Unix tools,[9] facilitating easy manipulation of text files. Perl 5 gained widespread popularity in the late 1990s as a CGI scripting language, in part due to its parsing abilities.[10]

In addition to CGI, Perl 5 is used for graphics programming, system administration, network programming, finance, bioinformatics, and other applications. It's nicknamed "the Swiss Army chainsaw of scripting languages" because of its flexibility and power,[11] and possibly also because of its perceived "ugliness".[12] In 1998, it was also referred to as the "duct tape that holds the Internet together", in reference to its ubiquity and perceived inelegance.[13]

The overall structure of Perl derives broadly from C. Perl is procedural in nature, with variables, expressions, assignment statements, brace-delimited blocks, control structures, and subroutines.

Perl also takes features from shell programming. All variables are marked with leading sigils, which unambiguously identify the data type (for example, scalar, array, hash) of the variable in context. Importantly, sigils allow variables to be interpolated directly into strings. Perl has many built-in functions that provide tools often used in shell programming (although many of these tools are implemented by programs external to the shell) such as sorting, and calling on operating system facilities.

Perl takes lists from Lisp, hashes ("associative arrays") from AWK, and regular expressions from sed. These simplify and facilitate many parsing, text-handling, and data-management tasks. Also shared with Lisp are the implicit return of the last value in a block, and the fact that all statements have a value, and thus are also expressions and can be used in larger expressions themselves.

Perl 5 added features that support complex data structures, first-class functions (that is, closures as values), and an object-oriented programming model. These include references, packages, class-based method dispatch, and lexically scoped variables, along with compiler directives (for example, the strict pragma). A major additional feature introduced with Perl 5 was the ability to package code as reusable modules.

All versions of Perl do automatic data-typing and automatic memory management. The interpreter knows the type and storage requirements of every data object in the program; it allocates and frees storage for them as necessary using reference counting (so it cannot deallocate circular data structures without manual intervention). Legal type conversions – for example, conversions from number to string – are done automatically at run time; illegal type conversions are fatal errors.

ActivePerl is a closed source distribution from ActiveState that has regular releases that track the core Perl releases.[65] The distribution also includes the Perl package manager (PPM),[66] a popular tool for installing, removing, upgrading, and managing the use of common Perl modules.

Strawberry Perl is an open source distribution for Windows. It has had regular, quarterly releases since January 2008, including new modules as feedback and requests come in. Strawberry Perl aims to be able to install modules like standard Perl distributions on other platforms, including compiling XS modules.

For further information on Perl, see [Wikipedia: Perl](#) (the text above has been copied from this reference), or the [Perl Homepage](#).

ActivePerl is available from [ActivePerl Homepage](#). This distribution includes support for COM. See [here](#) for an example.

Example for using the library

---

```
#Enable COM support
use Win32::OLE;

#Load the mpNumerics library
$mp = Win32::OLE->new('mpNumerics.mp_Lib');

#Set Floating point type to MPFR with 60 decimal digits precision
$mp->{FloatingPointType} = 3;
$mp->{Prec10} = 60;

#Assign values to x1 and x2
$x1 = $mp->Real(4.5);
$x2 = $mp->Real('1.1');

#Calculate x3 = x1 / x2
$x3 = $x1->Div($x2);

#print the value of x3
print $x3->Str();

# Wait for user input...
# print "Press <return> to continue...";
# $x = <STDIN>;
```

---

Example for using the Excel

---

```
use Win32::OLE;

# Start Excel and make it visible
$xlApp = Win32::OLE->new('Excel.Application');
$xlApp->{Visible} = 1;

# Create a new workbook
$xlBook = $xlApp->Workbooks->Add;

# Our data that we will add to the workbook...
```

```
$mydata = [[{"Item": "Category", "Price"},  
["Nails", "Hardware", "5.25"],  
["Shirt", "Clothing", "23.00"],  
["Hammer", "Hardware", "16.25"],  
["Sandwich", "Food", "5.00"],  
["Pants", "Clothing", "31.00"],  
["Drinks", "Food", "2.25"]];  
  
# Write all the data at once...  
$rng = $xlBook->ActiveSheet->Range("A1:C7");  
$rng->{Value} = $mydata;  
  
# Create a PivotTable for the data...  
$tbl = $xlBook->ActiveSheet->PivotTableWizard(1, $rng, "", "MyPivotTable");  
  
# Set pivot fields...  
$tbl->AddFields("Category", "Item");  
$tbl->PivotFields("Price")->{Orientation} = 4; # 4=xlDataField  
  
# Create a chart too...  
$chart = $xlBook->Charts->Add;  
$chart->SetSourceData($rng, 2);  
$chart->{ChartType} = 70; # 3D-pie chart  
$chart->Location(2, "Sheet4");  
  
# Wait for user input...  
# print "Press <return> to continue...";  
# $x = <STDIN>;  
  
# Clean up  
# $xlBook->{Saved} = 1;  
# $xlApp->Quit;  
# $xlBook = 0;  
# $xlApp = 0;  
# $xlApp = 0;  
print "All done.";
```

---

## D.9 R (Statistical System)

R is a free software programming language and a software environment for statistical computing and graphics. The R language is widely used among statisticians and data miners for developing statistical software[2][3] and data analysis.[3] Polls and surveys of data miners are showing R's popularity has increased substantially in recent years.[4][5][6]

R is an implementation of the S programming language combined with lexical scoping semantics inspired by Scheme. S was created by John Chambers while at Bell Labs. R was created by Ross Ihaka and Robert Gentleman[7] at the University of Auckland, New Zealand, and is currently developed by the R Development Core Team, of which Chambers is a member. R is named partly after the first names of the first two R authors and partly as a play on the name of S.[8]

R is a GNU project.[9][10] The source code for the R software environment is written primarily in C, Fortran, and R.[11] R is freely available under the GNU General Public License, and pre-compiled binary versions are provided for various operating systems. R uses a command line interface; however, several graphical user interfaces are available for use with R.

R provides a wide variety of statistical and graphical techniques, including linear and nonlinear modeling, classical statistical tests, time-series analysis, classification, clustering, and others. R is easily extensible through functions and extensions, and the R community is noted for its active contributions in terms of packages. There are some important differences, but much code written for S runs unaltered. Many of R's standard functions are written in R itself, which makes it easy for users to follow the algorithmic choices made. For computationally intensive tasks, C, C++, and Fortran code can be linked and called at run time. Advanced users can write C, C++[12] or Java[13] code to manipulate R objects directly.

R is highly extensible through the use of user-submitted packages for specific functions or specific areas of study. Due to its S heritage, R has stronger object-oriented programming facilities than most statistical computing languages. Extending R is also eased by its lexical scoping rules.[14]

Another strength of R is static graphics, which can produce publication-quality graphs, including mathematical symbols. Dynamic and interactive graphics are available through additional packages.[15]

R has its own LaTeX-like documentation format, which is used to supply comprehensive documentation, both on-line in a number of formats and in hard copy.

R is an interpreted language; users typically access it through a command-line interpreter. If a user types "2+2" at the R command prompt and presses enter, the computer replies with "4", as shown below:

```
i 2+2
[1] 4
```

Like many other languages, R supports matrix arithmetic. R's data structures include scalars, vectors, matrices, data frames (similar to tables in a relational database) and lists.[16] R's extensible object-system includes objects for (among others): regression models, time-series and geo-spatial coordinates.

R supports procedural programming with functions and, for some functions, object-oriented programming with generic functions. A generic function acts differently depending on the type of arguments passed to it. In other words, the generic function dispatches the function (method)

specific to that type of object. For example, R has a generic print() function that can print almost every type of object in R with a simple "print(objectname)" syntax.

Although mostly used by statisticians and other practitioners requiring an environment for statistical computation and software development, R can also operate as a general matrix calculation toolbox - with performance benchmarks comparable to GNU Octave or MATLAB.[17]

For further information on R, see [Wikipedia: R](#) (the text above has been copied from this reference), or the [R Homepage](#).

COM support can be obtained by installing the [R RDCOMClient](#)

Installation of the binary should be as straightforward as any other R package for Windows, e.g. use the command

```
install.packages("RDCOMClient", repos = "http://www.omegahat.org/R")
```

or use the Packages menu and make certain to include the Omegahat repository in the list of repositories to search.

There exists also a commercial [R for Excel distribution](#).

A popular GUI for R is [Rstudio](#).

Within RStudio:

Tools - Install Packages.

Type RD in the dialogue box.

RDCOMClient should appear in the drop down box.

Select RDCOMClient and click on Install.

Needs to be done separately for 32 bit and 64 bit.

R contains packages which provide interfaces to

GMP (<http://mulcyber.toulouse.inra.fr/projects/gmp>) and

MPFR (<http://rmpfr.r-forge.r-project.org/>).

Book recommendation: [Adler \(2012\)](#).

Book recommendation: [Verzani \(2011\)](#).

Book recommendation: [Chang \(2012\)](#).

Example for using the library

---

```
#Enable COM support
require("RDCOMClient")

#Load the mpNumerics library
mp = COMCreate("mpNumerics.mp_Lib")

#Set Floating point type to MPFR with 60 decimal digits precision
mp[["Prec10"]] = 160
mp[["FloatingPointType"]] = 3

#Assign values to x1 and x2
x1 = mp$Real("4.5")
x2 = mp$Real("1.1")

#Perform arithmetic operations
```

```
x3 = x1$Plus(x2)
x4 = x1$Div(x2)
```

```
#Display output
mp[["Prec10"]]
x3$Str()
x4$Str()
```

---

Example for using Excel

---

```
#Load Library
require("rcom")

#Create instance of Excel
xlApp = comCreateObject("Excel.Application")

#Add 1 workbook and make it visible
wb = xlApp[["Workbooks"]]$Add()
xlApp[["Visible"]] = TRUE

#Display the name of the 1st worksheet
ws = wb[["Worksheets", 1]]
wsname = ws[["Name"]]
wsname

#Assign values to a range
mrange = ws[["Range", "A1:B10"]]
mrange[["Value"]] = 10.3

#Display the values of the range
d = mrange[["Value"]]
d
$
```

---

Within RStudio, on the menu bar, click Tools -> Install packages.  
In the textbox "Packages" enter "RMpfr" and click on "Install".  
This should install the RMpfr and GMP packages.

---

```
#Load Library
require("Rmpfr")

n1.25 <- mpfr(5, precBits = 256)/4
n1.25

n1.25 ^ c(1:7, 20, 30)

exp(n1.25)

getGroupMembers("Math")

showMethods(classes = "mpfr")
```

```
showMethods(classes = "mpfrArray")
```

---

## D.10 MatLab (COM interface)

MATLAB has already been introduced in section C.6 Apart from the the .NET interface described in this section, MATLAB has also a COM interface. Its use is illustrated by the examples below.

Example for using the library via COM

---

```
%Open a COM server on Matlab

mp = actxserver('mpNumerics.mp_Lib');
mp.Prec10 = 60;
mp.FloatingPointType = 3;

x = mp.Real(2);
y = mp.Sqrt(x);
s = y.Str();

s2 = char(s);
fprintf('s is equal to %s.\n',s2);

quit;
```

---

Example for using Excel

---

```
%Open a COM server on Matlab

x = 4.3;
fprintf('x is equal to %6.2f.\n',x);

Excel = actxserver('Excel.Application');

Excel.Workbooks.Add();
Excel.Visible = true;
quit;
```

---

# Appendix E

## Building the toolbox

Building the toolbox and the library from scratch is a much more involved process than just using them.

Conceptually, it could be described as a top-down process which starts at the level of the modification of the source files for the documentation, the following (automated) generation of various \*.xml, \*.cs, \*.h files and their processing with appropriate tools, which create the .NET, COM, native DLL and spreadsheet interfaces, ultimately leading to the connecting point of the mpNumC.h header file.

It could also be described as a bottom-up process, starting with the compilation of the \*.c, \*.h and \*.asm of the GMP, MPFR and FLINT library. followed by the compilation of the Eigen and Boost template libraries with the various supported data types, again leading to the connecting point of the mpNumC.h header file.

In practice, it is easiest to start any rebuilding of the toolbox or the library with an already working installation, with the following steps in mind:

- When changing a function, or introducing a new one, always start at the documentation, and provide all information which is required for automated generation of dependent files.
- Compile the documentation in latex, and process the output with makemenu etc.
- Run the routines which are necessary to update the .NET, COM, native DLL and spreadsheet interfaces.
- Decide whether you need to update the mpNumC.h header file.

Alternatively, you could start with a breaking change in one of the underlying libraries (e.g. GMP), recompile them first, then recompile all of the dependent libraries.

## E.1 Building the documentation and standard interfaces

### E.1.1 Documentation

The following software was used to build the documentation:

miktex 2.9.4813 : [miktex](#).

texniccenter 2.02: [texniccenter](#).

After installing these programs the following additional steps are needed:

Build → Select Output Profile: Latex ⇒ PDF

Build → Define Output Profile: Latex ⇒ PDF.

In this dialogue box, select the tab "(La)Tex". Find the item "Path to (La)Tex compiler". Depending on your system, it will have an entry like

```
C:\Program Files\MiKTeX 2.9\miktex\bin\pdflatex.exe
```

Change this to

```
C:\Program Files\MiKTeX 2.9\miktex\bin\pdflatex.exe -enable-write18
```

Still in the same In this dialogue box, select the tab "Postprocessor"

In the Listbox "Processors", add an item and name it "Nomenclature"

In the Textfield "Executable:", enter the full path to "miktex-makeindex.exe" ,like

```
C:\Program Files\MiKTeX 2.9\miktex\bin\x64\miktex-makeindex.exe
```

In the Textfield "Arguments:", enter the following:

```
-s nomencl.list "%tm.nlo" -o "%tm.nls"
```

### E.1.2 C interface

The source code for C interface is automatically generated when building the documentation.

### E.1.3 C++ interface

The source code for C++ interface is automatically generated when building the documentation.

### E.1.4 COM interface

The source code for COM interface is automatically generated when building the documentation.

### E.1.5 .NET interface

The source code for .NET interface is automatically generated when building the documentation.

## E.2 Building the specific interfaces

### E.2.1 Excel support via Excel-DNA

ExcelDNA: [ExcelDNA](#).

Copyright (c) 2005-2009 Govert van Drimmelen

The license can be found in appendix [G.2.4](#)

The contributors are listed in section [F.2.4](#)

### E.2.2 OpenOffice.org/Apache OpenOffice/LibreOffice Calc support

Support for Calc is provided by an Add-in, which connects Calc with mpFormulaPy using the built-in Basic scripting language.

There is also an external program, which is used to install the locally available functions to the global library.

## E.3 Building the Toolbox GUI

The GUI is written in CSharp, using the following libraries:

### E.3.1 NetOffice

NetOffice: [NetOffice](#).

Copyright (c) 2011 Sebastian Lange.

The license is the MIT License (see appendix [G.2.3](#))

The contributors are listed in section [F.2.3](#)

## E.4 Other Software

### E.4.1 Downloads

The following references include suggested downloads:

Microsoft .NET Framework 4 (Standalone Installer): [Microsoft .NET Framework 4](#).

Adobe Reader: [Adobe Reader](#).

gnuplot: <http://www.gnuplot.info/> .

R: <http://www.r-project.org/> .

LibreOffice: [LibreOffice](#).

LibreOffice API: [LibreOffice API](#).

OpenOffice Basic Guide: [OpenOffice Basic Guide](#).

Microsoft Office Compatibility Pack : [Microsoft Office Compatibility Pack](#) .

Microsoft Access Database Engine 2010 Redistributable: [Microsoft Access Database Engine](#).

Connection Strings: [Connection Strings](#).

Chart Controls 3.5: [Chart Controls 3.5](#).

Samples Environment for Microsoft Chart Controls 3.5 [Samples 3.5](#).

Samples Environment for Microsoft Chart Controls 4.0 [Samples 4.0](#).

High performance WPF 3D Chart: [High performance WPF 3D Chart](#).

Interactive 3D bar chart custom control in WPF: [Interactive 3D bar chart custom control in WPF](#).

WPF-Print-Engine-Part-I: [WPF-Print-Engine-Part-I](#).

Printing-in-wpf: [printing-in-wpf](#).

EMF-Printer-Driver: [EMF-Printer-Driver](#).

3-D Graphics Overview: [3-D Graphics Overview](#).

csharphelper: [csharphelper](#).

vbhelper: [vbhelper](#).

SVG Adobe: [SVG Adobe](#).

### E.4.2 Other References

The following references also influenced the design:

Apophenia: <http://apophenia.info/> .

Burkhard: <http://people.sc.fsu.edu/~jburkardt/> .

Gladman: <http://www.gladman.me.uk/> .

Arndt: <http://www.jjj.de/fxt/fxtbook> .

Vogt: <http://www.axelvogt.de/axalom/index.html> .

L'Ecuyer: <http://www.iro.umontreal.ca/~lecuyer/> .

CVM: <http://www.cvmlib.com/> .

Alglib: <http://www.alglib.net/> .

GSL: <https://www.gnu.org/software/gsl/> .

FFTW: <http://www.fftw.org/> .

Function Parser: <http://warp.povusers.org/FunctionParser/fparser.html> .

Xnumbers v.6.0 for Excel 2010: <http://www.thetropicalevents.com/Xnumbers60.htm> .

NIST: <http://dlmf.nist.gov/> .

Mathjax: <http://docs.mathjax.org/en/latest/start.htmltex-and-latex-input> .

Vbsedit: <http://www.vbsedit.com/>.

STATA: [STATA](#).

gpower3: [gpower3](#).

TIOBE: TIOBE Programming Community Index.

PYPL: PYPL PopularitY of Programming Language index.

ilnumerics: ilnumerics.

pdfBooks: <http://www.pdfbook.co.ke/>.

### E.4.3 Previous version

SQLite: <http://www.sqlite.org/about.html>.

System.Data.SQLite: System.Data.SQLite.

SQLite2009 Pro Enterprise Manager: SQLite2009 Pro Enterprise Manager.

Process Caller: Process Caller.

Interprocess-communication: Interprocess-communication.

MDI Tabcontrol: MDI Tabcontrol.

ScintillaNET: ScintillaNET.

Fast-Colored-TextBox: Fast-Colored-TextBox.

Spreadsheet-Control: Spreadsheet-Control.

Spreadsheet-Control: Spreadsheet-Control.

DataGridView Printer: DataGridView Printer.

HTML Editor: HTML Editor.

HTML Editor: HTML Editor.

Formula Engine: Formula Engine.

Grammatica: Grammatica.

Excel xll add-in library: Excel xll add-in library.

jni4net: jni4net.

jedit: <http://www.jedit.org/>.

Setting up jedit for compiling: <http://courses.cs.washington.edu/courses/cse413/02au/jEdit.html>.

FSharp MSDN: FSharp MSDN.

FSharp Wiki: FSharp Wiki.

FSharp Home: FSharp Home.

FSharp .NET: FSharp .NET.

Tsunami: Tsunami.

Gnuplot CSharp: Gnuplot CSharp.

oxyplot: oxyplot.

Propertygrid: Propertygrid.

MSDN MS Office: MSDN MS Office.

Open XML SDK 2.5 for Microsoft Office: Open XML SDK 2.5 for Microsoft Office.

famfamfam-silk-icons 16x16: famfamfam-silk-icons.

famfamfam-silk-icons 32x32: famfamfam-silk-icons.

silk-companion-1-icons: silk-companion-1-icons.

Fugue Icons: Fugue Icons.

Wix Toolset: Wix Toolset.

Wix Edit: Wix Edit.

Speech: Speech Synthesizer.

A-Calculation-Engine-for-NET: A-Calculation-Engine-for-NET.

## E.5 To Do

### E.5.1 New GUI

- To be developed with SharpDevelop
- all mpFormulaPy panels to be used as standalone (with messaging, from mp.), as in-process COM server, as .NET dll.
- mpFunction panel for selection of functions and macros (procedures)
- mpChart panel for selection of XLM charts
- mpOptions panel for selection of options
- mpFormulaPy pane: Display of images (EMF, JPG)
- mpFormulaPy pane: Display of HTML output, incl PDF, navigation pane, export to Word
- mpFormulaPy pane: Display of tabular output
- mpFormulaPy pane: Navigator for the output of all zzFormulas, including ranges, matrices and charts, and inspector of full precision in decimal and binary
- optional: Support for simple scripts (VB.NET, cSharp, JScript2010, ReoScript, VBScript, JScript).
- optional: Transfer of simple VB.Net and CSharp scripts to SharpDevelop or Visual Studio.
- optional: Transfer of charts to SharpDevelop for editing.

### E.5.2 GUI

- Work out and implement starting Java with jni4net.
- Add item for F# in project explorer
- Redesign the project explorer so that it becomes usable if only certain types of files are present
- Redesign the GUI as a DLL that can be run from other applications.
- Figure out necessary changes to GUI to support ultra high resolutions (DPI aware)
- Add "Program finished" and timing after returning from "Program started. Please stand by.."
- Avoid opening of output after call to forms or just opening word.
- Display an error message when an executable like Matlab cannot be found BEFORE trying to run the program
- Implement property pages for code modules, including .exe, linkage, post-build actions and post-run actions, save and load as xml
- Implement full options module including saving and loading as xml.

- Implement live connection to workbooks as new category in project tree. Implement `mp.GetActiveWorkbook()`.
- Implement starting VBA files as Add-ins (needs to activate "Trust access to the VBA object model")

### E.5.3 C++ Library

- Fix Win64 Eigen Decimal output crashes
- Fix Win64 Eigen mpfrb crashes
- Implement Random module

### E.5.4 Manual

- Make a release version of the manual, removing all currently unsupported features, with automatic generation of interface files
- Add a roadmap appendix to the manual
- Re-introduce description of all Toolbox components in introduction (like NetOffice etc)
- Describe how to make the build for the `mpFormulaPy.msi` file
- Describe how to make the build for the GUI, including the changes to ScintillaNET etc.
- Describe how to make the build for the xll function, the main dialogue, and the ribbon and task pane project for Excel using Excel DNA
- Descriptions of nonparametric procedures, as currently implemented in Stats32
- Complete chapter on discrete distributions
- Transfer remaining text from Word
- Figure out recursion relation for noncentral t, and document them

### E.5.5 New Programming

- Figure out recursion relation for noncentral t, and implement it in doubly noncentral t
- Implement doubly noncentral F with recursion relations
- Wilks noncentral: Implement algorithm by Butler and multivariate hypergeometric functions
- Implement 2nd noncentral moment of Spearman rho
- Implement  $R^2$  as in Benton/Boost
- Implement integral versions of noncentral Chi2, t, F
- Implement hypergeometric functions from AMath

# Appendix F

## Acknowledgements

### F.1 Contributors to libraries used in the numerical routines

#### F.1.1 Contributors to mpMath

The following text has been copied from the mpMath manual (0.19):

xxxx

### F.2 Contributors to libraries used in the GUI

#### F.2.1 Contributors to Sharp Develop

The site: <https://github.com/icsharpcode/SharpDevelop/wiki/Contributors> states the following:  
"Non-Developers: Christoph Wille (PM), Bernhard Spuida (Kalfaktor).

Developers: Daniel Grunwald (Technical Lead), Matt Ward, David Srbecký (Debugger), Siegfried Pammer, Martin Koníček, Peter Forstmeier (SharpDevelop Reports), Tomáš Linhart, Kumar Devvrat, Eusebiu Marcu.

Past developers:

This list is by no means exhaustive: Mike Krüger (Project Founder), Alexandre Semenov, Andrea Paatz, Christian Hornung, David Alpert, Denis ERCHOFF, Dickon Field, Georg Brandl, Ifko Kovalčák, Itai Bar-Haim, Ivan Shumilin, John Reilly, John Simons, Justin Dearing, Markus Palme, Mathias Simmack, Matt Everson, Nathan Allan, Nikola Kavaldjiev, Philipp Maihart, Poul Staunsgaard, Robert Pickering, Robert Zaunere, Roman Taranchenko, Russell Wilkins, Scott Ferrett, Sergej Andrejev, Shinsaku Nakagawa, Tomasz Tretkowski, Troy Simpson."

#### F.2.2 Contributors to Unmanaged Exports

The main author of Unmanaged Exports is Robert Giesecke.

#### F.2.3 Contributors to NetOffice

The main author of NetOffice is Sebastian Lange.

#### **F.2.4 Contributors to Excel-DNA**

The main author of Excel-DNA is Govert van Drimmelen.

#### **F.2.5 Contributors to jni4net**

The main author of jni4net is Pavel Šavara.

#### **F.2.6 System.Data.SQLite**

The main author of System.Data.SQLite is R. Hipp.

# Appendix G

## Licenses

### G.1 GNU Licenses

#### G.1.1 GNU General Public License, Version 2

GNU LIBRARY GENERAL PUBLIC LICENSE  
Version 2, June 1991

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Finally, any free program is threatened constantly by software patents. We wish to avoid the danger that companies distributing free software will individually obtain patent licenses, thus in effect transforming

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Most GNU software, including some libraries, is covered by the ordinary GNU General Public License, which was designed for utility programs. This license, the GNU Library General Public License, applies to certain designated libraries. This license is quite different from the ordinary one; be sure to read it in full, and don't assume that anything in it is the same as in the ordinary license.

The reason we have a separate public license for some libraries is that they blur the distinction we usually make between modifying or adding to a program and simply using it. Linking a program with a library, without changing the library, is in some sense simply using the library, and is analogous to running a utility program or application program. However, in a textual and legal sense, the linked executable is a combined work, a derivative of the original library, and the ordinary General Public License treats it as such.

Because of this blurred distinction, using the ordinary General Public License for libraries did not effectively promote software sharing, because most developers did not use the libraries. We concluded that weaker conditions might promote sharing better.

However, unrestricted linking of non-free programs would deprive the users of those programs of all benefit from the free status of the libraries themselves. This Library General Public License is intended to permit developers of non-free programs to use free libraries, while preserving your freedom as a user of such programs to change the free libraries that are incorporated in them. (We have not seen how to achieve this as regards changes in header files, but we have achieved it as regards changes in the actual functions of the Library.) The hope is that this will lead to faster development of free libraries.

The precise terms and conditions for copying, distribution and modification follow. Pay close attention to the difference between a "work based on the library" and a "work that uses the library". The former contains code derived from the library, while the latter only works together with the library.

Note that it is possible for a library to be covered by the ordinary General Public License rather than by this special one.

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The "Library", below, refers to any such software library or work which has been distributed under these terms. A "work based on the Library" means either the Library or any derivative work under copyright law: that is to say, a work containing the Library or a portion of it, either verbatim or with modifications and/or translated straightforwardly into another language. (Hereinafter, translation is included without limitation in the term "modification".)

"Source code" for a work means the preferred form of the work for making modifications to it. For a library, complete source code means all the source code for all modules it contains, plus any associated interface definition files, plus the scripts used to control compilation and installation of the library.

Activities other than copying, distribution and modification are not covered by this License; they are outside its scope. The act of running a program using the Library is not restricted, and output from such a program is covered only if its contents constitute a work based on the Library (independent of the use of the Library in a tool for writing it). Whether that is true depends on what the Library does and what the program that uses the Library does.

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These requirements apply to the modified work as a whole. If identifiable sections of that work are not derived from the Library, and can be reasonably considered independent and separate works in themselves, then this License, and its terms, do not apply to those sections when you distribute them as separate works. But when you distribute the same sections as part of a whole which is a work based on the Library, the distribution of the whole must be on the terms of this License, whose permissions for other licensees extend to the entire whole, and thus to each and every part regardless of who wrote it.

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In addition, mere aggregation of another work not based on the Library with the Library (or with a work based on the Library) on a volume of a storage or distribution medium does not bring the other work under the scope of this License.

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This option is useful when you wish to copy part of the code of the Library into a program that is not a library.

4. You may copy and distribute the Library (or a portion or derivative of it, under Section 2) in object code or executable form under the terms of Sections 1 and 2 above provided that you accompany it with the complete corresponding machine-readable source code, which must be distributed under the terms of Sections 1 and 2 above on a medium customarily used for software interchange.

If distribution of object code is made by offering access to copy from a designated place, then offering equivalent access to copy the source code from the same place satisfies the requirement to distribute the source code, even though third parties are not compelled to copy the source along with the object code.

5. A program that contains no derivative of any portion of the Library, but is designed to work with the Library by being compiled or linked with it, is called a "work that uses the Library". Such a work, in isolation, is not a derivative work of the Library, and therefore falls outside the scope of this License. However, linking a "work that uses the Library" with the Library creates an executable that is a derivative of the Library (because it contains portions of the Library), rather than a "work that uses the library".

The executable is therefore covered by this License. Section 6 states terms for distribution of such executables.

When a "work that uses the Library" uses material from a header file that is part of the Library, the object code for the work may be a derivative work of the Library even though the source code is not. Whether this is true is especially significant if the work can be linked without the Library, or if the work is itself a library. The threshold for this to be true is not precisely defined by law.

If such an object file uses only numerical parameters, data structure layouts and accessors, and small macros and small inline functions (ten lines or less in length), then the use of the object file is unrestricted, regardless of whether it is legally a derivative work. (Executables containing this object code plus portions of the Library will still fall under Section 6.)

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6. As an exception to the Sections above, you may also compile or link a "work that uses the Library" with the Library to produce a work containing portions of the Library, and distribute that work under terms of your choice, provided that the terms permit modification of the work for the customer's own use and reverse engineering for debugging such modifications.

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4. You may copy and distribute the Library (or a portion or derivative of it, under Section 2) in object code or executable form under the terms of Sections 1 and 2 above provided that you accompany it with the complete corresponding machine-readable source code, which must be distributed under the terms of Sections 1 and 2 above on a medium customarily used for software interchange.

If distribution of object code is made by offering access to copy from a designated place, then offering equivalent access to copy the source code from the same place satisfies the requirement to distribute the source code, even though third parties are not compelled to copy the source along with the object code.

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If such an object file uses only numerical parameters, data structure layouts and accessors, and small macros and small inline functions (ten lines or less in length), then the use of the object file is unrestricted, regardless of whether it is legally a derivative work. (Executables containing this object code plus portions of the Library will still fall under Section 6.)

Otherwise, if the work is a derivative of the Library, you may distribute the object code for the work under the terms of Section 6. Any executables containing that work also fall under Section 6, whether or not they are linked directly with the Library itself.

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# Appendix H

## Mathematical Notation

**Table H.1:** Notation related to Sets

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$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integer numbers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers

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# **Part VII**

## **Back Matter**

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# Nomenclature

$\chi_{\nu,\alpha}^2$	$\alpha$ quantile of the central $\chi^2$ -distribution with $\nu$ degrees of freedom (page 188)
$\Gamma(x)$	Gamma Function (page 154)
$\Gamma_p(x)$	Multivariate Gamma Function (page 592)
$\mathbb{C}$	Set of complex numbers (page 887)
$\mathbb{N}$	Set of natural numbers (page 887)
$\mathbb{Q}$	Set of rational numbers (page 887)
$\mathbb{R}$	Set of real numbers (page 887)
$\mathbb{Z}$	Set of integer numbers (page 887)
$\Phi(x)$	CDF of the standardized normal distribution (page 217)
$\phi(x)$	pdf of the standardized normal distribution (page 216)
$\Phi^{-1}(\alpha)$	Inverse CDF of the standardized normal distribution (page 217)
$F_F(m, n, x)$	CDF of the central $F$ -distribution (page 196)
$f_F(m, n, x)$	pdf of the central $F$ -distribution (page 196)
$F_N(x; \mu, \sigma^2)$	CDF of the normal distribution with mean $\mu$ and variance $\sigma^2$ (page 217)
$F_N(x; \mu, \sigma^2)$	pdf of the normal distribution with mean $\mu$ and variance $\sigma^2$ (page 216)
$F_N^{-1}(\alpha; \mu, \sigma^2)$	Inverse CDF of the normal distribution with mean $\mu$ and variance $\sigma^2$ (page 217)
$f_R(r, N; \rho)$	pdf of the Distribution of the Sample Correlation Coefficient (page 616)
$F_t(n, x)$	CDF of the central $t$ -distribution (page 224)
$f_t(n, x)$	pdf of the central $t$ -distribution (page 224)
$F_{\chi^2}(n, x)$	CDF of the central chi-square distribution (page 187)
$f_{\chi^2}(n, x)$	pdf of the central chi-square distribution (page 187)
$F_{\chi^2}(n, x; \lambda)$	CDF of the noncentral chi-square distribution (page 642)
$f_{\chi^2}(n, x; \lambda)$	pdf of the noncentral chi-square distribution (page 641)
$F_{\nu_1, \nu_2, \alpha}$	$\alpha$ quantile of the central $F$ -distribution with $\nu_1$ and $\nu_2$ degrees of freedom (page 196)
$F_{\text{Beta}'}(x; a, b, \lambda)$	CDF of the (singly) noncentral Beta-distribution (page 649)
$f_{\text{Beta}'}(x; a, b, \lambda)$	pdf of the (singly) noncentral Beta-distribution (page 649)
$F_{\text{Beta}}(a, b, x)$	CDF of the central Beta-distribution (page 178)
$f_{\text{Beta}}(a, b, x)$	pdf of the central Beta-distribution (page 178)
$F_{\text{Bin}}(n, k; p)$	CDF of the binomial distribution (page 183)
$f_{\text{Bin}}(n, k; p)$	pmf of the binomial distribution (page 183)
$F_{\text{NegBin}}(n, k; p)$	CDF of the negative binomial distribution (page 213)
$f_{\text{NegBin}}(n, k; p)$	pmf of the negative binomial distribution (page 213)
$f_{F''}(x; m, n)$	pdf of the doubly noncentral $F$ -distribution (page 671)
$f_{F'}(x; m, n)$	CDF of the (singly) noncentral $F$ -distribution (page 665)
$f_{F'}(x; m, n)$	pdf of the (singly) noncentral $F$ -distribution (page 665)
$F_{R^2}(x; p, n, \rho^2)$	CDF of the Square of the Multiple Sample Correlation Coefficient (page 625)
$f_{R^2}(x; p, n, \rho^2)$	pdf of the Square of the Multiple Sample Correlation Coefficient (page 625)

$F_{t''}(t; n; \delta, \theta)$	CDF of the doubly noncentral t-square distribution (page 660)
$f_{t''}(t; n; \delta, \theta)$	pdf of the doubly noncentral t-square distribution (page 659)
$F_{t'}(n, x, \delta)$	CDF of the (singly) noncentral t-distribution (page 653)
$f_{t'}(n, x, \delta)$	pdf of the (singly) noncentral t-distribution (page 652)
$I_\nu(z)$	Modified Bessel function of the first kind of real order $\nu$ (page 152)
$J_\nu(z)$	Bessel function of the first kind of real order $\nu$ (page 152)
$K_\nu(z)$	Modified Bessel function of the second kind of real order $\nu$ (page 153)
$N_{Rho}(\alpha, \beta, \tilde{\rho})$	Sample size function of the noncentral $t$ -distribution for a given confidence level $\alpha$ , power $\beta$ and modified noncentrality parameter $\tilde{\rho}$ (page 623)
$N_{t''}(\alpha, \beta, \tilde{\rho})$	Sample size function of the doubly noncentral $t$ -distribution for a given confidence level $\alpha$ , power $\beta$ and modified noncentrality parameter $\tilde{\rho}$ (page 658)
$N_{t''}(\alpha, \beta, \tilde{\rho})$	Sample size function of the doubly noncentral $t$ -distribution for a given confidence level $\alpha$ , power $\beta$ and modified noncentrality parameter $\tilde{\rho}$ (page 663)
$t_{\nu, \alpha}$	$\alpha$ quantile of the central $t$ -distribution with $\nu$ degrees of freedom (page 225)
$T_{Owen}(a, b)$	Owen's T-Function (page 633)
$t_{n, \delta; \alpha}$	$\alpha$ quantile of the noncentral $t$ -distribution with $\nu$ degrees of freedom and noncentrality parameter $\delta$ (page 655)
$Y_\nu(z)$	Bessel function of the second kind of real order $\nu$ (page 152)
$z_\alpha$	$\alpha$ quantile of the standardized normal distribution (page 217)
${}_0\tilde{F}_1(b; x)$	Regularized Confluent Hypergeometric Limit Function (page 422)
${}_0F_1(a; \Omega)$	Confluent Hypergeometric Limit Function for Matrix Argument (page 596)
${}_1\tilde{F}_1(a, b; z)$	Kummer's Regularized Confluent Hypergeometric Function (page 424)
${}_1F_1(a, b; \Omega)$	Kummer's Confluent Hypergeometric Function for Matrix Argument (page 595)
${}_1F_1(a, b; z)$	Kummer's Confluent Hypergeometric Function (page 423)
${}_2\tilde{F}_1(a, b; c; z)$	Gauss Regularized Hypergeometric Function (page 431)
${}_2F_1(a, b; c; \mathbf{T})$	Gauss Hypergeometric Function of Matrix Argument (page 594)
${}_2F_1(a, b; c; x)$	Gauss Hypergeometric Function (page 429)
CDF	cumulative distribution function (page 170)
pdf	probability density function (page 170)
pmf	probability mass function (page 170)

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