Planetary Motion

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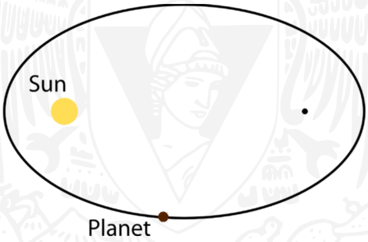
Contenido

The equations of motion

One of the main contributions of Newton's Laws was the verification of Kepler's Laws of Planetary Motion.



In this project we pretend to solve numerically this equations of motion with Newton's theory and analyze the results

The equations of motion

Suppose we have a planet of mass m that orbits around another planet of mass $M \gg m$.

Using polar coordinates, with r the radial distance between the center of both planets and θ the angle between r and the axis of symmetry, we obtain using Newton's Second Law and the Universal Gravitation Law that:

The equations of motion

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \quad \text{y} \quad m(\ddot{r} - r\dot{\theta}^2) = -\frac{GMm}{r^2} \tag{1}$$

The first of this equations is just the conservation of angular momentum and is equivalent to:

$$h = r^2\dot{\theta} = \text{constant} \tag{2}$$

The equations of motion

Finally, substituting (2) in (1), we obtain the equations of planetary motion:

$$\ddot{r} = \frac{h^2}{r^3} - \frac{GM}{r^2} \quad \text{y} \quad \dot{\theta} = \frac{h}{r^2} \tag{3}$$

that, without dimensions, are simply:

Equations of planetary motion

$$\ddot{r} = \frac{1}{r^3} - \frac{1}{r^2} \quad \text{y} \quad \dot{\theta} = \frac{1}{r^2} \tag{4}$$

If we define $r_p = \dot{r}$, the equations become a system of 3 ODE:

$$\dot{r_p} = \frac{1}{r^3} - \frac{1}{r^2} \quad , \quad \dot{r} = r_p \quad \text{y} \quad \dot{\theta} = \frac{1}{r^2} \tag{5}$$

Taylor's Method

We can solve the system of equations in (5) using the second order Taylor Method. This method basically consists in solving an ODE in the following way:

$$\frac{dy}{dt} = f(t, y) \quad \text{with:} \quad y(a) = y_{ini} \tag{6}$$

in an interval $[a, b]$ with constant spacing between the points given by $h = (b - a)/n$, where n is the number of points, approximating the values of the function with a Taylor Series expansion:

Taylor's Method

$$y(t + h) \approx y(t) + y'(t) \cdot h + y''(t) \cdot \frac{h^2}{2} \tag{7}$$

Taylor's Method

For r and r_p:

$$\dot{r_p} = \frac{1}{r^3} - \frac{1}{r^2} \quad \text{y} \quad \dot{r} = r_p \quad \text{with} \quad r_p(0) = 0.2 \quad \text{and} \quad r(0) = 1 \tag{8}$$

with $h = 0.001$ in the interval $[0, 20]$, we approximate with the Taylor Series expansion:

$$r_p(t+h) \approx r_p(t) + \dot{r_p}(t) \cdot h + \ddot{r_p}(t) \cdot \frac{h^2}{2}$$

$$r(t+h) \approx r(t) + \dot{r}(t) \cdot h + \ddot{r}(t) \cdot \frac{h^2}{2}$$

where the second derivatives are given by the ODE and are:

$$\ddot{r_p} = \frac{-3\dot{r}}{r^4} + \frac{2\dot{r}}{r^3} \quad \text{y} \quad \ddot{r} = \dot{r_p} = \frac{1}{r^3} - \frac{1}{r^2}$$

Taylor's Method

Also, for θ:

$$\dot{\theta} = \frac{1}{r^2} \quad \text{with} \quad \theta(0) = 0 \tag{9}$$

with $h = 0.001$ in the interval $[0, 20]$, we approximate with the Taylor Series expansion:

$$\theta(t+h) \approx \theta(t) + \dot{\theta}(t) \cdot h + \ddot{\theta}(t) \cdot \frac{h^2}{2}$$

where the second derivative is given by the ODE and is:

$$\ddot{\theta} = -\frac{2\dot{r}}{r^3} \tag{10}$$

The next step is to implement this on Fortran.

Plot for $\theta(t)$

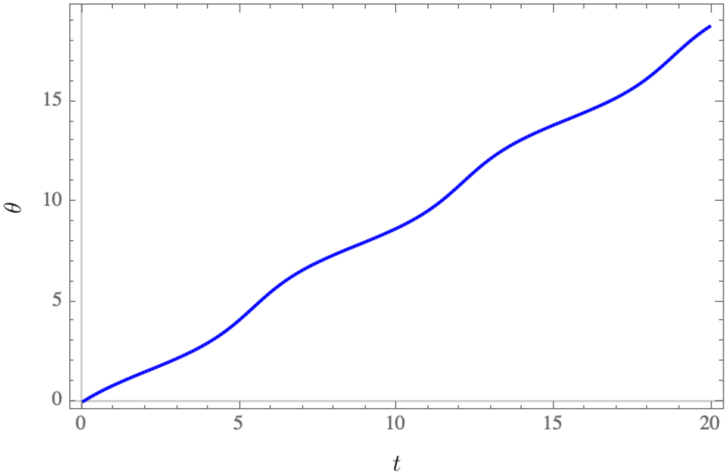


Figure: Plot of θ vs t.

In polar coordinates...

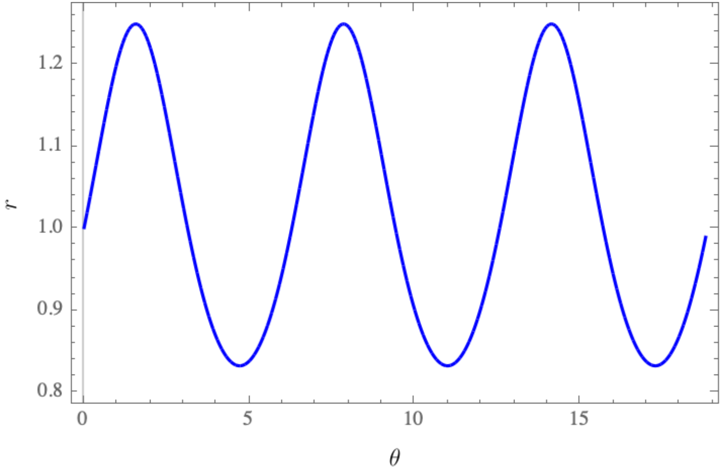


Figure: Plot of r vs θ.

Finally, in cartesian coordinates...

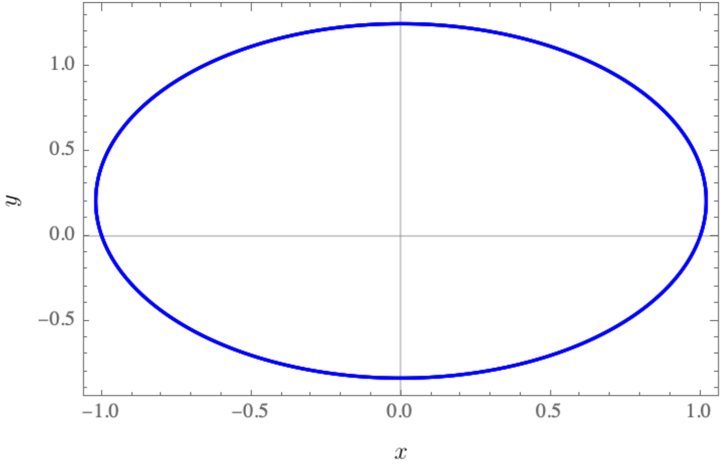


Figure: Plot of y vs x.

Conclusion

From the last plot, we can conclude that when we make the transformation from polar to cartesian coordinates:

$$x = r\cos(\theta) \quad \text{y} \quad y = r\sin(\theta) \tag{11}$$

we see that the trajectory of the planet of mass m is an **elliptic** trajectory. This is a numeric proof of Kepler's 1st Law, who in 1609 wrote that:

Kepler's 1st Law

Every planet moves around the Sun describing an elliptic orbit where the Sun is at one of the foci of the ellipse.

¡Gracias!

谢谢!

ありがとう!

Thanks!

Grazie!

Merci!

Mulțmesc!

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