Problem Set 7

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1)

$$\psi(x) = \begin{cases} 2\sqrt{a^3} x e^{-ax} \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}$$

1.a)

$$\begin{aligned} \int_0^\infty |\psi(x)|^2 \, \mathrm{d}x &= \int_0^\infty \left|2\sqrt{a^3} x e^{-ax}\right|^2 \mathrm{d}x \\ &= \int_0^\infty 4a^3 x^2 e^{-2ax} \, \mathrm{d}x \\ &= 4a^3 \int_0^\infty x^2 e^{-2ax} \, \mathrm{d}x \\ &= \frac{1}{4a^3} \end{aligned}$$

1.b)

We can set $\frac{\mathrm{d}}{\mathrm{d}x} |\psi(x)|^2 = 0$ to find the maximum of the function. We know that as $x \to \infty$, $e^{-2ax} \to 0$, so we do not need to worry about maximums at the bounds.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} |\psi(x)|^2 &= \frac{\mathrm{d}}{\mathrm{d}x} 4a^3 x^2 e^{-2ax} \\ &= e^{-2ax}(2x - 2ax^2) \end{aligned}$$

$$\begin{aligned} 0 &= e^{-2ax}(2x - 2ax^2) \\ 0 &= 2x - 2ax^2 \\ 2x &= 2ax^2 \\ x \neq 0 \Longrightarrow x &= \frac{1}{a} \end{aligned}$$

Therefore, the maximum of the function is at $x = \frac{1}{a}$.

1.c)

$$\begin{aligned} \int_0^{\frac{1}{a}} |\psi(x)|^2 \, \mathrm{d}x &= \int_0^{\frac{1}{a}} \left|2\sqrt{a^3} x e^{-ax}\right|^2 \mathrm{d}x \\ &= \int_0^{\frac{1}{a}} 4a^3 x^2 e^{-2ax} \, \mathrm{d}x \\ &= 1 - \frac{5}{e^2} \end{aligned}$$

2)

We will use the symmetric well as the solution is much nicer. We know that $n = 4$, so we can use the "odd" solution that uses sin.

$$\psi(x) = \begin{cases} Ae^{Kx} \text{ if } x < -a \\ A\frac{e^{-Ka}}{\sin(ka)} \sin(kx) \text{ if } -a \leq x \leq a \\ Ae^{-Kx} \text{ if } x > a \end{cases}$$

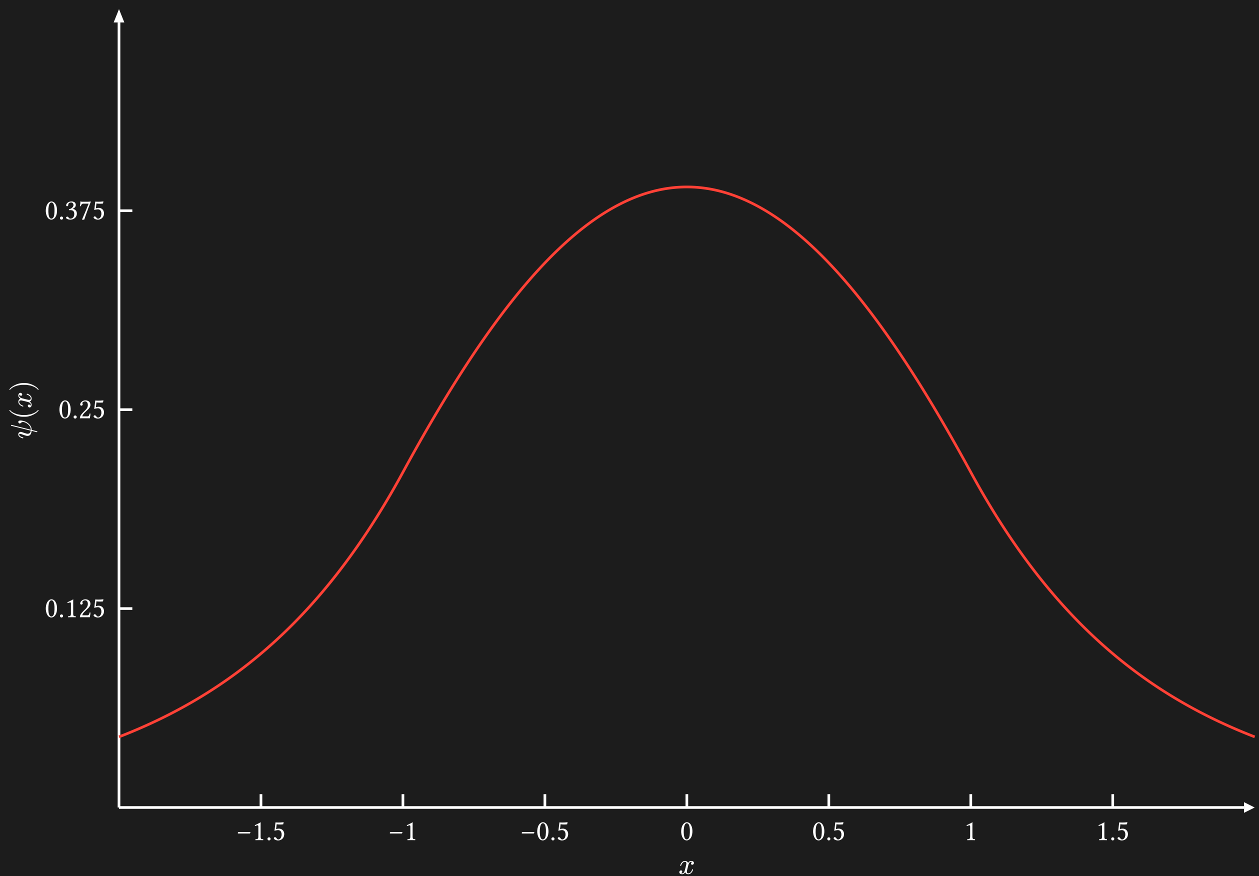
To determine the normalization constant, we can numerically integrate the function and set it equal to 1. We will set $a = 1$.

Running the following code will give us the normalization constant.

```
import numpy as np
import scipy.integrate as integrate

def psi(x):
    a = 1
    k = 1
    K = k * np.tan(k * a)

    if x < -a:
        return np.exp(K * x)
    elif -a <= x <= a:
        return np.exp(-K * a) / np.cos(k * a) * np.cos(k * x)
    else:
        return np.exp(-K * x)

A = 1 / integrate.quad(psi, -100, 100)
```

After running the code, we find that $A \approx 0.926788$.

We can plot the wave function.



Graph 1: $\psi(x)$ for $a = 1, k = 1$

FIX, NEED TO HAVE $n = 4$, this shows $n = 1$.

3)

$$U(x) = \begin{cases} 0 & \text{if } |x| > a \\ -U_0 & \text{if } |x| < a \end{cases}$$

3.a)

$$E_1 = -\frac{1}{2}U_0$$

We know that the first excited state will use an odd function as there will be one node in $\psi(x)$.

Let's define the following quantities:

$$\eta = -\varepsilon \cot(\varepsilon)$$

4)

$$\psi(x) = Axe^{-\frac{x^2}{L^2}}$$

We can plug this function into the Schroedinger equation to find $U(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi(x) = Ae^{-\frac{x^2}{L^2}}\left(\frac{-2x^2}{L^2}\right) + Ae^{-\frac{x^2}{L^2}}$$

$$= \frac{\psi(x)}{x}\left(1 - \frac{2x^2}{L^2}\right)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\psi(x)}{x}\left(1 - \frac{2x^2}{L^2}\right)$$

$$= \frac{\psi(x)}{x}\left(1 - \frac{4x}{L^2}\right) + \left(1 - \frac{2x^2}{L^2}\right)\left(\frac{-2\psi(x)}{L^2}\right)$$

$$= Ae^{-\frac{x^2}{L^2}}\left(-\frac{6}{L^2} + \frac{4x^2}{L^4}\right)$$

We can (finally) plug this into the SE.

$$-\frac{2m}{\hbar^2}U(x)\psi(x) = \psi(x)\left(-\frac{6}{L^2} + \frac{4x^2}{L^4}\right)$$

$$U(x) = \frac{\hbar^2}{2mL^2}(4x^2 - 6)$$

This is a parabola that opens upwards.

$$U(0) = \frac{-6\hbar^2}{2mL^2}$$

At $x = 0$, the potential energy is $\frac{-3\hbar^2}{mL^2}$.