Homework 5

Mark Schulist

1)

$$\text{If } A \subseteq B \text{ and there is an injection } f : B \to A\text{, then } |A| = |B|$$

Proof: If there exists an injection $f : B \hookrightarrow A$, then we know that $|A| \geq |B|$. Similarly, we know that if $A \subseteq B$, then $|B| \geq |A|$. This is because if $A \subseteq B$, then there exists an injection $g : A \hookrightarrow B$. By the Cantor-Schroeder-Bernstein theorem, if there is an injection in both directions between A and B, then there exists a bijection between A and B. Because there is a bijection between A and B, $|A| = |B|$. □

2)

$$A = \{1, 2, 3, 4, 5, 6\}$$

2.a)

$$\begin{aligned} <= \{&(1,2), (1,3), (1,4), (1,5), (1,6) \\ &(2,3), (2,4), (2,5), (2,6) \\ &(3,4), (3,5), (3,6) \\ &(4,5), (4,6) \\ &(5,6)\} \subset A \times A \end{aligned}$$

2.b)

$$\begin{aligned} \leq= \{&(1,1), (1,2), (1,3), (1,4), (1,5), (1,6) \\ &(2,2), (2,3), (2,4), (2,5), (2,6) \\ &(3,3), (3,4), (3,5), (3,6) \\ &(4,4), (4,5), (4,6) \\ &(5,5), (5,6) \\ &(6,6)\} \subset A \times A \end{aligned}$$

2.c)

$$\begin{aligned} | = \{&(1,1), (2,2), (3,3), (4,4), (5,5), (6,6) \\ &(1,2), (1,3), (1,4), (1,5), (1,6) \\ &(2,4), (2,6) \\ &(3,6)\} \subset A \times A \end{aligned}$$

3)

3.a)

$$R = \mathbb{R} \times \mathbb{R} \setminus \{(x, x) \mid x \in \mathbb{R}\}$$

This is the relation $\neq$ for two real numbers. If we graph this set in the Cartesian plane, we get all points *except* for points on the diagonal, which are the points where $x = y$.

3.b)

$$S = \{(x, y) \mid x, y \in \mathbb{R}, x > y\}$$

This is the relation "greater than" for two real numbers. I changed the notation of S to show the $>$ more clearly.

4)

4.a) All Relations

A relation is a subset of $A \times A$, and the set of all subsets of $A \times A$ is $\mathcal{P}(A \times A)$. This has a cardinality of $2^{|A \times A|} = 2^{|A|^2}$.

4.b) Reflexive Relations

A relation R is reflexive if $(a, a) \in R$ for all $a \in A$.

We need to make sure that the middle diagonal is in the set, and then turn on and off all other elements in the relation. There are $(|A|^2) - |A|$ total elements that can be turned on and off, so we just raise 2 to that value.

$$2^{|A|^2 - |A|}$$

4.c) Symmetric Relations

A relation R is symmetric if $(a, b) \in R$ implies $(b, a) \in R$ for all $a, b \in A$.

Here, we just need to find the number of ways that we can turn on and off half of the elements including the diagonal. Essentially, we just need to add back the other half of the diagonal $\left(\frac{|A|}{2}\right)$ to $\frac{|A|^2}{2}$. So we get $\frac{|A|}{2} + \frac{|A|^2}{2}$ possible items to turn on and off, which corresponds to the following number of symmetric relations.

$$2^{\frac{|A|^2 + |A|}{2}}$$

4.d) Transitive Relations

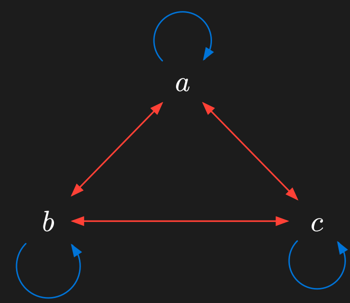
A relation R is transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

It has been shown that there is no polynomial function that generates the number of transitive relations, and nobody has found a general function for the number of transitive relations. So our hope of finding one is...minimal...

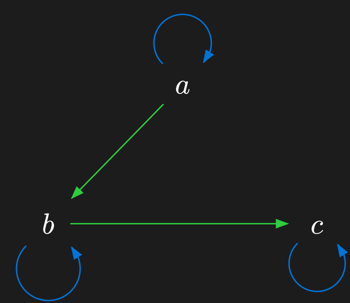
5)

$$A = \{a, b, c\}$$

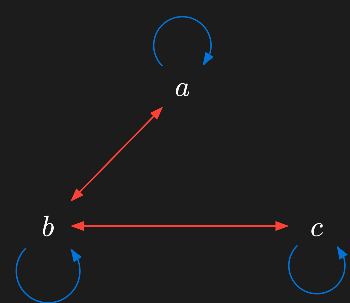
1. Reflexive, symmetric, and transitive relation on A:



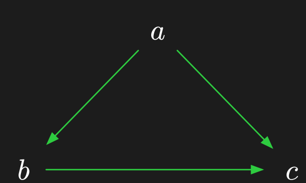
2. Reflexive, non-symmetric, non-transitive relation on A:



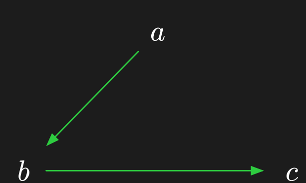
3. Reflexive, symmetric, non-transitive relation on A:



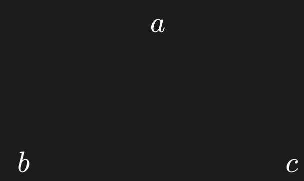
4. Non-reflexive, non-symmetric, transitive relation on A:



5. Non-reflexive, non-symmetric, non-transitive relation on A:

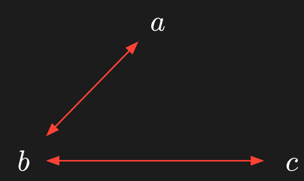


6. Non-reflexive, symmetric, transitive relation on A:



This is where $R = \emptyset$.

7. Non-reflexive, symmetric, non-transitive relation on A:



8. Reflexive, non-symmetric, transitive relation on A: