Problem Set 1

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1) 1.2

$$
\begin{aligned}
|\psi_1\rangle &= \frac{1}{\sqrt{3}}|+\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|-\rangle \\
|\psi_2\rangle &= \frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}|-\rangle \\
|\psi_3\rangle &= \frac{1}{\sqrt{2}}|+\rangle + e^{i\pi/4}\frac{1}{\sqrt{2}}|-\rangle
\end{aligned} \tag{1}
$$

1.a)

1.a.a) First State

$$
|\phi_1\rangle = a|+\rangle + b|-\rangle \tag{2}
$$

$$
\begin{aligned}
0 &= \langle\phi_1|\psi_1\rangle \\
&= a^*\frac{1}{\sqrt{3}} + b^* i\frac{\sqrt{2}}{\sqrt{3}} \\
\implies |a|^2 &= 2|b|^2
\end{aligned} \tag{3}
$$

We can use the normality condition:

$$
\begin{aligned}
1 &= |a|^2 + |b|^2 \\
1 &= 2|b|^2 + |b|^2 \\
1 &= 3|b|^2 \\
\implies b &= \frac{1}{\sqrt{3}} = b^*
\end{aligned} \tag{4}
$$

Then we can plug our result of b^* into the orthonormality condition.

$$
\begin{aligned}
a^* &= -ib^*\sqrt{2} \\
a^* &= -i\frac{\sqrt{2}}{\sqrt{3}} \\
\implies a &= i\frac{\sqrt{2}}{\sqrt{3}}
\end{aligned} \tag{5}
$$

Therefore:

$$
|\phi_1\rangle = i\frac{\sqrt{2}}{\sqrt{3}}|+\rangle + \frac{1}{\sqrt{3}}|-\rangle \tag{6}
$$

1.a.b) Second State

$$|\phi_2\rangle = a|+\rangle + b|-\rangle \tag{7}$$

$$\begin{aligned} 0 &= \langle\phi_2|\psi_2\rangle \\ 0 &= \frac{a^*}{\sqrt{5}} - b^*\frac{2}{\sqrt{5}} \\ a^* &= 2b^* \end{aligned} \tag{8}$$

Normality:

$$\begin{aligned} |a|^2 + |b|^2 &= 1 \\ 4|b|^2 + |b|^2 &= 1 \\ b &= \pm\frac{1}{\sqrt{5}} \end{aligned} \tag{9}$$

Therefore, $a = \pm\frac{2}{\sqrt{5}}$

There are two possible solutions, we will just use one here.

$$|\phi_2\rangle = \frac{2}{\sqrt{5}}|+\rangle + \frac{1}{\sqrt{5}}|-\rangle \tag{10}$$

1.a.c) Third State

$$|\phi_3\rangle = a|+\rangle + b|-\rangle \tag{11}$$

$$\begin{aligned} 0 &= \langle\phi_3|\psi_3\rangle \\ 0 &= a^*\frac{1}{\sqrt{2}} + b^* e^{i\pi/4}\frac{1}{\sqrt{2}} \\ a^* &= -b^* e^{i\pi/4} \\ b &= -a e^{i\pi/4} \end{aligned} \tag{12}$$

Normality:

$$|a|^2 + |b|^2 = 1 \tag{13}$$

$$\begin{aligned} |b|^2 &= \left(-a^* e^{-i\pi/4}\right)\left(-a e^{i\pi/4}\right) \\ |b|^2 &= |a|^2 \end{aligned} \tag{14}$$

Plugging into normality condition:

$$\begin{aligned} |a|^2 + |a|^2 &= 1 \\ 2|a|^2 &= 1 \\ |a|^2 &= \frac{1}{2} \\ a &= \pm\frac{1}{\sqrt{2}} \end{aligned} \tag{15}$$

$$b = \mp e^{i\pi/4} \frac{1}{\sqrt{2}} \tag{16}$$

Therefore one option is:

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}|+\rangle - e^{i\pi/4}\frac{1}{\sqrt{2}}|-\rangle \tag{17}$$

1.b)

$$\langle\psi_1|\psi_2\rangle = 1 \tag{18}$$

$$\begin{aligned} \langle\psi_1|\psi_2\rangle &= \frac{1}{\sqrt{15}} + \left(\frac{2\sqrt{2}}{\sqrt{15}}i\right) \\ &= \frac{1}{\sqrt{15}}\big(1 + i2\sqrt{2}\big) \end{aligned} \tag{19}$$

$$\begin{aligned} \langle\psi_1|\psi_2\rangle &= \frac{1}{\sqrt{6}} - i\frac{\sqrt{2}}{\sqrt{3}}e^{i\pi/4}\frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}} - ie^{i\pi/4}\right) \\ &= \frac{1}{\sqrt{6}}(2 - i) \end{aligned} \tag{20}$$

$$\begin{aligned} \langle\psi_2|\psi_1\rangle &= \frac{1}{\sqrt{15}} - \frac{2}{\sqrt{5}}i\frac{\sqrt{2}}{\sqrt{3}} \\ &= \frac{1}{\sqrt{15}}\big(1 - i2\sqrt{2}\big) \end{aligned} \tag{21}$$

$$\langle\psi_2|\psi_2\rangle = 1 \tag{22}$$

$$\begin{aligned} \langle\psi_2|\psi_3\rangle &= \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}e^{i\pi/4}\frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{10}}\big(1 - 2e^{i\pi/4}\big) \\ &= \frac{1}{\sqrt{10}}\big(1 - \sqrt{2} - i\sqrt{2}\big) \end{aligned} \tag{23}$$

$$\begin{aligned} \langle\psi_3|\psi_1\rangle &= \frac{1}{\sqrt{6}} + i\frac{\sqrt{2}}{\sqrt{3}}e^{-i\pi/4}\frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}} + ie^{-i\pi/4}\right) \\ &= \frac{1}{\sqrt{6}}(2 + i) \end{aligned} \tag{24}$$

$$\begin{aligned}\langle\psi_3|\psi_2\rangle &= \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}\frac{1}{\sqrt{2}}e^{-i\pi/4} \\ &= \frac{1}{\sqrt{10}}\left(1+\sqrt{2}i-\sqrt{2}\right)\end{aligned} \tag{25}$$

$$\langle\psi_3|\psi_3\rangle = 1 \tag{26}$$

2) 1.4

Show $|b|^2 = |c|^2 = |d|^2 = \frac{1}{2}$

$$\begin{aligned}|+\rangle_x &= a|+\rangle + b|-\rangle \\ |-\rangle_- &= c|+\rangle + d|-\rangle\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{1}{2} &= |_x\langle-|+\rangle|^2 \\ &= |c^*\langle+|+\rangle + d^*\langle-|+\rangle|^2 \\ &= |c^*|^2 = |c|^2\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{1}{2} &= |_x\langle+|+\rangle|^2 \\ &= |a^*\langle+|-\rangle + b^*\langle-|-\rangle|^2 \\ &= |a^*|^2 = |a|^2\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{1}{2} &= |_x\langle-|-\rangle|^2 \\ &= |c^*\langle+|-\rangle + d^*\langle-|-\rangle|^2 \\ &= |d^*|^2 = |d|^2\end{aligned} \tag{30}$$

3) 9.21

$$V_M(R) = -D_e + \frac{1}{2}\mu\omega^2(R-R_0)^2 \tag{31}$$

$$V_{\mathrm{HO}}(R) = D_e\left(e^{-2\alpha(R-R_0)} - 2e^{-\alpha(R-R_0)}\right) \tag{32}$$

$$\alpha = \omega\sqrt{\frac{\mu}{2D_e}} \tag{33}$$

We can do a Taylor expansion around the point R_0.

$$V_M(R_0) = -D_e \tag{34}$$

$$\begin{aligned}V_M'(R) &= D_e\left(-2\alpha e^{-2\alpha(R-R_0)} + 2\alpha e^{-\alpha(R-R_0)}\right) \\ V_M'(R_0) &= 0\end{aligned} \tag{35}$$

$$\begin{aligned} V_M''(R) &= D_e\left[4\alpha^2 e^{-2\alpha(R-R_0)} - 2\alpha^2 e^{-\alpha(R-R_0)}\right] \\ V_M''(R_0) &= D_e(2\alpha^2) = \omega^2\mu \end{aligned} \quad (36)$$

Plugging into the defintion of Taylor expansion:

$$-D_e + \frac{1}{2}\mu\omega^2(R - R_0)^2 \quad (37)$$

Which is exactly what V_{HO} is!

We can find the cubic term by finding the third degree Taylor expansion.

$$\begin{aligned} V_M'''(R) &= D_e\left(-8\alpha^3 e^{-2\alpha(R-R_0)} + 2\alpha^3 e^{-\alpha(R-R_0)}\right) \\ V_M'''(R_0) &= D_e\left(-8\alpha^3 + 2\alpha^3\right) \\ &= -D_e 6\alpha^3 \\ &= -\frac{3}{\sqrt{2}}\frac{1}{\sqrt{D_e}}\omega^3\mu^{3/2} \end{aligned} \quad (38)$$

Therefore the cubic correction term is:

$$-D_e\frac{6}{6}\alpha^3(R - R_0)^3 \quad (39)$$