Homework 1

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1) 1.1

$$r \in \mathbb{Q} \quad x \notin \mathbb{Q} \tag{1}$$

Show $r + x$ and $r \cdot x$ are not rational.

Proof. Suppose $r + x \in \mathbb{Q}$, where $r = \frac{p}{q}$

$$\begin{aligned} \frac{p}{q} + x &= \frac{a}{b} \\ \implies \quad x &= \frac{a}{b} + \frac{p}{q} \\ \implies \quad x &= \frac{aq - pb}{bp} \quad ↯ \end{aligned} \tag{2}$$

This contradicts assumption that x is not rational, hence $r + x \notin \mathbb{Q}$.

Suppose $r \cdot x \in \mathbb{Q}$, with $r = \frac{p}{q}$.

$$\begin{aligned} \frac{p}{q}x &= \frac{a}{b} \\ \implies \quad x &= \frac{aq}{bp} \quad ↯ \end{aligned} \tag{3}$$

Contradicts assumption that x is not rational, hence $r \cdot x \notin \mathbb{Q}$. $\square$

2) 1.2

Show $\nexists r \in \mathbb{Q}$ such that $r^2 = 12$.

Proof. Let $r = \frac{p}{q}$.

$$\begin{aligned} \frac{p^2}{q^2} &= 12 \\ p^2 &= 12q^2 \end{aligned} \tag{4}$$

We can see that 3 must divide both the LHS and RHS, so $3|p^2 \implies 3|p$.

Therefore, we can write $p = 3a$.

$$\begin{aligned} 9a^2 &= 12q^2 \\ \implies \quad 3a^2 &= 4q^2 \end{aligned} \tag{5}$$

Now we can see that $3|3a^2$ so 3 must divide q. But then if 3 divides q and p, the quotient cannot be in lowest terms. ↯

Hence, there does not exists a $r \in \mathbb{Q}$ with $r^2 = 12$. $\square$

3) 1.4

E is nonempty subset of ordered set.

α is lower bound of E

β is upper bound of E

Show $\alpha \leq \beta$

Proof.

$$\begin{gathered} \alpha \leq p \forall p \in E \\ \beta \geq p \forall p \in E \end{gathered} \tag{6}$$

Therefore if we combine the inequalities:

$$\begin{gathered} \alpha \leq p \forall p \in E \leq \beta \\ \implies a \leq \beta \end{gathered} \tag{7}$$

□

4) 1.5

A nonempty subset of $\mathbb{R}$, bounded below.

$-A = \{-x \mid x \in A\}$

Show that $\inf A = -\sup(-A)$.

Proof. Suppose $y \leq x \forall x \in A$, and y is a lower bound of A.

Then $-y \geq -x \forall -x \in (-A)$

So $-A$ is bounded above by $-y$ which is $-\inf(A)$

So:

$$\begin{aligned} -\inf(A) &= \sup(-A) \\ \inf(A) &= -\sup(-A) \end{aligned} \tag{8}$$

□