

Problem Set 10

Mark Schulist

1)

$$U_0 = \frac{3}{4}E \quad (1)$$

$$\psi_{\text{inc}} = e^{ikx} \quad (2)$$

1.a)

$$k \equiv \sqrt{\frac{2mE}{\hbar^2}} \quad (3)$$

We know that $E > U_0$, so we can set up the relevant equations. We will call region 1 the region to the left of the step ($x < 0$) and region 2 to the right ($x > 0$).

$$\psi_1(x) = e^{ikx} + Be^{-ikx} \quad (4)$$

$$k' \equiv \sqrt{\frac{2m(E - U_0)}{\hbar^2}} \quad (5)$$

$$\psi_2(x) = Ce^{ik'x} \quad (6)$$

We can impose the required continuity conditions at $x = 0$.

$$\begin{aligned} \frac{B}{1} &= \frac{k - k'}{k + k'} \\ B &= \frac{\sqrt{\frac{2mE}{\hbar^2}} - \sqrt{\frac{2m(E - U_0)}{\hbar^2}}}{\sqrt{\frac{2mE}{\hbar^2}} + \sqrt{\frac{2m(E - U_0)}{\hbar^2}}} \\ &= \frac{\sqrt{E} - \sqrt{E - U_0}}{\sqrt{E} + \sqrt{E - U_0}} \\ &= \frac{\sqrt{E} - \sqrt{\frac{E}{4}}}{\sqrt{E} + \sqrt{\frac{E}{4}}} \\ &= \frac{1}{3} \end{aligned} \quad (7)$$

We know that $C = 1 + B$:

$$\begin{aligned} C &= 1 + B \\ C &= 1 + \frac{1}{3} \\ C &= \frac{4}{3} \end{aligned} \quad (8)$$

Therefore the wave function in the 2 regions is:

$$\begin{aligned}\psi_1(x) &= e^{ikx} + \frac{1}{3}e^{-ikx} \\ \psi_2(x) &= \frac{4}{3}e^{ik'x}\end{aligned}\tag{9}$$

1.b)

$$\begin{aligned}R &= \frac{B \cdot B}{A \cdot A} \\ &= \frac{1}{9}\end{aligned}\tag{10}$$

Equation (6-7):

$$\begin{aligned}R &= \frac{(\sqrt{E} - \sqrt{E - U_0})^2}{(\sqrt{E} + \sqrt{E - U_0})^2} \\ &= \frac{(\sqrt{E} - \sqrt{\frac{E}{4}})^2}{(\sqrt{E} + \sqrt{\frac{E}{4}})^2} \\ &= \frac{E + \frac{E}{4} - E}{E + \frac{E}{4} + E} \\ &= \frac{\frac{E}{4}}{9\frac{E}{4}} \\ &= \frac{1}{9}\end{aligned}\tag{11}$$

2)

$$B = -\frac{K + ik}{K - ik}A\tag{12}$$

2.a)

We can rewrite this in terms of the $\delta(E)$, which is the phase shift of the energy.

$$\begin{aligned}B &= -\frac{e^{i\delta(E)}}{e^{-i\delta(E)}}A \\ &= -e^{2i\delta(E)}A\end{aligned}\tag{13}$$

For $x < 0$

$$\begin{aligned}\psi(x) &= Ae^{ikx} - Ae^{i\delta(E)}e^{-ikx} \\ &= Ae^{i\delta(E)}(e^{ikx}e^{-i\delta(E)} - e^{-ikx}e^{i\delta(E)}) \\ &= 2iAe^{i\delta(E)}\sin(kx - \delta(E))\end{aligned}\tag{14}$$

Now that we know the wave function, we can take the absolute valued squared to find the probability density.

$$\begin{aligned}
|\psi(x)|^2 &= \psi^*(x)\psi(x) \\
&= (-2iAe^{-i\delta(E)} \sin(kx - \delta(E))) (2iAe^{i\delta(E)} \sin(kx - \delta(E))) \\
&= 4|A|^2 \sin^2(kx - \delta(E))
\end{aligned} \tag{15}$$

2.b)

As $k \rightarrow 0$, the energy must also be approaching 0. Therefore, C (amplitude inside of the step) must also approach 0 as the probability of finding the particle in the step will be VERY small. We also know that $\delta(E)$ must approach 0 because $\arctan(0) = 0$.

$$\begin{aligned}
C &= \lim_{k \rightarrow 0} A - \frac{K + ik}{K - ik} A \\
&= A - A \\
&= 0
\end{aligned} \tag{16}$$

As $K \rightarrow 0$, $\delta(E) = \arctan(\infty)$ will approach $\frac{\pi}{2}$. Therefore, we get the maximum phase shift when $K \rightarrow 0$.

$$\begin{aligned}
C &= \lim_{K \rightarrow 0} A - \frac{K + ik}{K - ik} A \\
&= 2A
\end{aligned} \tag{17}$$

So $B = A$ TODO: figure out the meaning of this.

3)

$$E = U_0 + \frac{\pi^2 \hbar^2}{2mL^2} \tag{18}$$

$$k \equiv \sqrt{\frac{2mE}{\hbar^2}} \tag{19}$$

$$\begin{aligned}
k' &\equiv \sqrt{\frac{2m(E - U_0)}{\hbar^2}} \\
&= \sqrt{\frac{2m \frac{\pi^2 \hbar^2}{2mL^2}}{\hbar^2}} \\
&= \frac{\pi}{L}
\end{aligned} \tag{20}$$

We have the following continuity requirements:

$$\begin{aligned}
A + B &= C + D \\
Fe^{ikL} &= Ce^{ikL} + De^{-ik'L} \\
k(A - B) &= k'(C - D) \\
Fke^{ikL} &= Cke^{ikL} - Dk'e^{-ik'L}
\end{aligned} \tag{21}$$

We know that $k' \cdot L = \pi$.

$$\begin{aligned} Fe^{ikL} &= Ce^{i\pi} + De^{-i\pi} \\ Fe^{ikL} &= -C - D \end{aligned} \quad (22)$$

$$\begin{aligned} Fke^{ikL} &= k'(Ce^{ik'L} - De^{ik'L}) \\ &= k'(D - C) \end{aligned} \quad (23)$$

We can divide Equation 22 and Equation 23:

$$\frac{k'}{k} = \frac{C + D}{C - D} \quad (24)$$

From Equation 21, we can find the ratio of $\frac{k'}{k}$:

$$\begin{aligned} \frac{k'}{k} &= \frac{A - B}{C - D} \\ \Rightarrow \text{first cont eq.} &= \frac{C + D}{C - D} \end{aligned} \quad (25)$$

Therefore, we know that $A - B = C + D$, but we know that $C + D = A + B$:

$$A - B = C + D = A + B \quad (26)$$

The only way for Equation 26 to be true is if $B = 0$, so we have shown that B must equal 0 in this case.

4)

$$U_0 = -3E \quad (27)$$

4.a)

Classically, the particle would increase in kinetic energy (momentum) because the potential energy decreases and $E = T + U$ at all positions and times.

4.b)

Region 1 is the region to the left of the step ($x < 0$) and region 2 is the region to the right of the step ($x > 0$).

$$\psi_1(x) = e^{ikx} + Be^{ikx} \quad (28)$$

$$\psi_2(x) = Ce^{ik'x} \quad (29)$$

$$k \equiv \sqrt{\frac{2mE}{\hbar^2}} \quad (30)$$

$$k' \equiv \sqrt{\frac{8mE}{\hbar^2}} \quad (31)$$

Continuity:

$$1 + B = C \quad (32)$$

$$\begin{aligned}
k + Bk &= Ck' \\
k(1 + B) &= Ck' \\
\frac{k}{k'} &= \frac{C}{1 + B} \\
\frac{1}{2} &= \frac{C}{1 + B} \\
2C &= 1 + B \\
C &= \frac{1 + B}{2}
\end{aligned} \tag{33}$$

$$\begin{aligned}
1 + B &= \frac{1}{2} + \frac{B}{2} \\
\frac{B}{2} &= -\frac{1}{2}
\end{aligned} \tag{34}$$

$$\begin{aligned}
B &= -1 \\
C &= 0
\end{aligned} \tag{35}$$