Homework 3

Mark Schulist

1) 1.12

We want to show that

$$|r_1 + r_2 + \ldots + r_n| \leq |r_1| + |r_2| + \ldots + |r_n| \quad (1)$$

Proof. We can use induction on n to show this is true regardless of the number of terms in our sum.

When $n = 1$, this is trivially true as $|r_1| \leq |r_1|$.

Suppose Equation 1 holds when $n = k$. If $n = k + 1$ then:

$$|r_1 + r_2 + \ldots + r_k + r_{k+1}| \leq |r_1 + r_2 + \ldots + r_k| + |r_{k+1}| \quad (2)$$

by the triangle inequality.

We can break up the RHS further using the inductive hypothesis:

$$|r_1 + r_2 + \ldots + r_k| + |r_{k+1}| \leq |r_1| + |r_2| + \ldots + |r_k| + |r_{k+1}| \quad (3)$$

This above equation is essentially the inductive hypothesis plus a constant positive term, so the above inequality holds for any value of n. □

2) 1.13

For $x, y \in \mathbb{R}$, show that $\|x| - |y\| \leq |x - y|$.

Proof. We can start with the LHS and square it. We know that abs always returns a positive value, so we do not need to worry about square rooting and squaring negative values.

$$\|x| - |y\|^2 = x^2 + y^2 - 2|x\|y| \quad (4)$$

And now do the same to the RHS.

$$|x - y|^2 = x^2 + y^2 - 2xy \quad (5)$$

We can observe that $2|x\|y| \geq 2xy$. Hence $-2|x\|y| \leq -2xy$. We can add $x^2 + y^2$ to both sides:

$$x^2 + y^2 - 2|x\|y| \leq x^2 + y^2 - 2xy \quad (6)$$

From Equation 4 and Equation 5 we can see

$$\begin{aligned} &\|x| - |y\|^2 \leq |x - y|^2 \\ \implies &\|x| - |y\| \leq |x - y| \end{aligned} \quad (7)$$

We can take this last square root because both elements inside of it have absolute value signs, meaning they must be positive. □

3) 1.15

When does equality hold for the Cauchy-Schwartz inequality?

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \cdot \sum_{i=1}^{n} |b_i|^2 \tag{8}$$

If we treat $\vec{a}$ and $\vec{b}$ as vectors in $\mathbb{R}^n$, the equality holds if $\vec{a} = \lambda\vec{b}$ for some $\lambda \in \mathbb{R}$. In other words, if they are scalar multiples of each other then equality holds in Equation 8.

Looking at each component, this means that $a_i = \lambda b_i$ for all $i \in \{1, ..., n\}$.

The LHS of Equation 8 becomes:

$$\left|\sum_{i=0}^{n} \lambda a_i^2\right|^2 = \lambda^2 \left(\sum_{i=0}^{n} a_i^2\right)^2 \tag{9}$$

And the RHS of Equation 8:

$$\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} \lambda^2 a_i^2 = \lambda^2 \left(\sum_{i=1}^{n} a_i^2\right)^2 \tag{10}$$

Hence we can see that Equation 9 and Equation 10 are equal, so equality holds when $a_i = \lambda b_i$.

If all of the $b_i = 0$ or all of the $a_i = 0$, then we trivially get equality as $0 = 0$.

4) 1.16

$$k \geq 3 \quad x, y \in \mathbb{R}^k \quad |x - y| = d > 0 \quad r > 0 \tag{11}$$

Intuitively, we can think of this problem as 2 (hyper) spheres of radius r that may or may not intersect (as determined by the constraints of r). If the 2 spheres intersect at more than one (infinite) points, then there are ∞ solutions to z. If they intersect at a single point (the midpoint of the spheres), then there is a single solution for z. If they do not intersect at all, then there are no solutions to z.

4.a)

We can think about the possible values for z with a *cartoon picture* for the case when $2r > d \Longrightarrow r > \frac{d}{2}$.

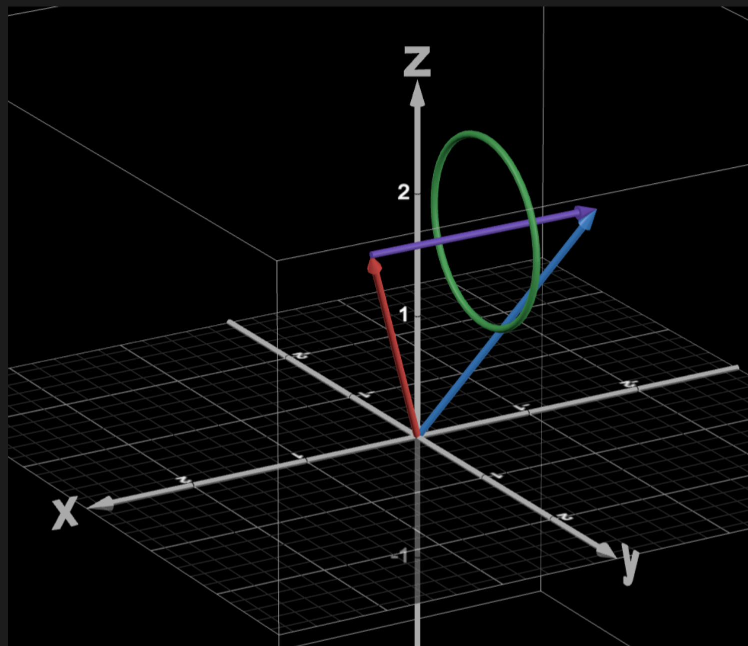

Figure 1: Cartoon picture of all of the possible z vectors with a fixed r in $\mathbb{R}^3$

Because $r > \frac{d}{2}$, we know that all of the possible vectors z will not lie on the purple vector in Figure 1. We can generalize to higher dimensions below.

We know that in $\mathbb{R}^k$, $k \geq 3$ there are an infinite number of vectors orthogonal to any other vector. Let us write this orthogonal vector as v. Then:

$$v \cdot (x - y) = 0 \tag{12}$$

has infinite solutions to v.

Let us define $z = m + \delta v$ where m is the midpoint between x and y $\left(\frac{x+y}{2}\right)$ and $\delta = \sqrt{r^2 - \frac{d^2}{4}}$. δ is found using the Pythagorean Theorem, where r is the distance from z to x, y and $d = |x - y|$.

Now we can define $z = m + \delta v$. We know that z is equidistant from x and y by definition of m and v. m is the midpoint and v is orthogonal to v, so adding them will retain the equal distance between z and x, y.

Examining δ we can see that when $r > \frac{d}{2}$, we get a positive value for δ meaning that there are infinite solutions to z. This is because there are infinite solutions to our vector v, and our definition of z will have infinite solutions if the coefficient in front of v (δ in this case) is nonzero (which it is in this case, as we showed above).

Hence, z has infinite solutions when $r > \frac{d}{2}$.

4.b)

When $r = \frac{d}{2}$, then the only possible z vector is the one that lies on the purple vector in Figure 1, half way between the tip of the red and blue vectors (because of the constraint that $|z - x| = |z - y|$). Any other point would either not be equidistant from x and y or have a $r > \frac{d}{2}$.

Generalizing to higher dimensions, we know that $|x - y| = d$ so there must exist a unique midpoint z such that $|x - z| = \frac{d}{2} = r$ and $|x - y| = \frac{d}{2} = r$. The triangle inequality only holds when the triange made by x, y, z as vertices is degenerate. So z must live on the line segment connecting x and y and must be equidistant from x and y, hence z is unique (as the midpoint between x and y).

4.c)

When $r < \frac{d}{2}$, it is clear that no z can satisfy the constraints of the problem. The distance between x and y is d, so if $2r < d$ then z would have nowhere to lie.

We know that the x, y, z form a triangle in $\mathbb{R}^n$, so hence the lenghts of the 3 sides must satisfy the triangle inequality.

$$\begin{aligned} d = |x - y| \leq |x - z| + |y - z| < \frac{d}{2} + \frac{d}{2} \\ \implies d < d \end{aligned} \tag{13}$$

Which is a contradiction, so the three sides cannot form a triangle and hence there is no possible value for z that works with $r < \frac{d}{2}$.

5) 1.18

$$k \geq 2 \quad x \in \mathbb{R}^k \tag{14}$$

We need to show that there exists $y \in \mathbb{R}^k$ with $y \neq 0$ such that $x \cdot y = 0$.

Proof. We need to prove that we can always find an orthogonal vector to x in $\mathbb{R}^k$, $k \geq 2$.

If $x = 0$, then $y \cdot x = 0$ for any $y \in \mathbb{R}^k$.

If $x \neq 0$ but at least a single component of x is zero (say, the jth component), then we set all components in y to be zero except for the jth component which we set to any nonzero value.

If $x \neq 0$ and all components of x are nonzero, then we cook up y according to the following rules:

$$y = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \\ \dots \\ 0 \end{pmatrix} \tag{15}$$

We know that $x \cdot y = 0$ in this case because we are essentially doing the following computation:

$$-x_1 \cdot x_2 + x_2 \cdot x_1 = 0 \tag{16}$$

And we know that all of the $x_i \neq 0$ because we have already handled that case above (this is, the case where one of the $x_i = 0$ is handled above). □

This is not true when $k = 1$ because in $\mathbb{R}^1$, there is no nonzero “vector” that, when multiplied by any other vector, gives us zero. For example, $1 \cdot x \neq 0$ if $x \neq 0$.

6) 1.20

Property 3 of cut: If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

We will remove this requirement of a cut and see what it does to
(1) LUB property (this should still hold)
(2) Satisfies (A1-4) of addition axioms, but fails (A5).

We will refer to this new set without property 3 of cuts as R.

(1)

Let A be a nonempty subset of R bounded above by $\beta \in R$.

$$\gamma = \bigcup_{\alpha \in A} \alpha \tag{17}$$

Claim: $\gamma \in R = \sup A$.

We first need to show that γ is a cut.

1. A is nonempty, so there exists $\alpha_0 \in A$. α_0 is a cut so $\alpha_0 \neq \emptyset$. So there exists $p \in \mathbb{Q}$ such that $p \in \alpha_0$. Then $p \in \gamma$. So $\gamma \neq \emptyset$. β is a cut, so $\beta \neq \mathbb{Q}$. So there exists $q \in \mathbb{Q}$ such that $q \notin \beta$. Since β is an upper bound of A, then for all $\alpha \in A$, $\alpha \leq \beta \iff \alpha \subset \beta$. Hence $q \notin \alpha \forall \alpha \in A \implies q \notin \gamma$. Hence $\gamma \neq \mathbb{Q}$.
2. Choose $p \in \gamma$ then $p \in \alpha$ for some $\alpha \in A$. Since α is a cut, then any $q < p$ is in α. Then all $q \in \mathbb{Q}$ such that $q < p$ are elements of γ.

Now we can show that $\gamma = \sup A$.

Claim: $\gamma \leq \alpha \forall \alpha \in A$.

$\gamma \subset \alpha$ is true by the definition of γ as a union of all $\alpha \in A$.

Claim: If $s < \gamma$ then δ is not an upper bound of A.

Since $\delta < \gamma$, then there exists $s \in \gamma$ such that $s \notin \delta$. But since $\gamma = \bigcup_{\alpha \in A} \alpha$, then $s \in \alpha_0$ for some $\alpha_0 \in A$. Hence $\alpha_0 > \delta$ then δ is not an upper bound of A. Hence γ is the least upper bound of A.

(2)

(A1) $x, y \in R \implies x + y \in R$

We need to show that is $\alpha, \beta \in R$, then $\alpha + \beta \in R$. Being in R means that $\alpha + \beta$ is a cut.

1. It is clear that $\alpha + \beta \subset Q, \alpha + \beta \neq \emptyset$. Take $r' \notin \alpha, s' \notin \beta$. Then $r' + s' > r + s$ for all $r \in \alpha, s \in \beta$. Hence $r' + s' \notin \alpha + \beta$ so $\alpha + \beta \neq \mathbb{Q}$ (which is a requirement of property 1 of cut).
2. Pick $p \in \alpha + \beta \iff p = r + s$ for some $r \in \alpha, s \in \beta$. If $q < p$ then $q < r + s \implies q - s < r \implies q - s \in \alpha$. But $q = \underbrace{(q - s)}_{\in \alpha} + \underbrace{s}_{\in \beta} \implies q \in \alpha + \beta$.

(A2) $x, y \in R \implies x + y = y + x$

WTS: $\alpha + \beta = \beta + \alpha$

Since any element in $\alpha + \beta$ is of the form $r + s$ (due to the definition of addition of cuts), $\underbrace{r}_{\beta} + \underbrace{s}_{\in \alpha} = s + r$.

(A3) $x, y, z \in R \implies (x + y) + z = x + (y + z)$

WTS: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

Using the same logic as above, we know that $\alpha + \beta$ can be written in the form $r + s$, so $(\alpha + \beta) + \gamma$ can be written as $(s + r) + p = s + (r + p)$, where $s \in \alpha, r \in \beta, p \in \gamma$. We know addition on $\mathbb{Q}$ is associative, so we can clearly see that addition on R is also associative.

(A4) There exists a $0 \in R$ such that $0 + x = x \forall x \in R$.

We can define the zero element as $\{p \mid p \in \mathbb{Q}, p \leq 0\}$. This is different from the "normal" definition of the zero element as now we need to include $0 \in \mathbb{Q}$ in the set (where "the set" refers to $0 \in R$).

We can see that $0 + \alpha$ where $0, \alpha \in R$ is equal to zero for any $\alpha \in R$.

To show the first inclusion: If $r \in \alpha$ and $s \in 0$, then $r + s \leq r$. Hence $\alpha + 0 \subset \alpha$.

Now the second inclusion: Pick $r, p \in \alpha$ with $r > p$. Then $p - r \in 0$ and $p = r + (p - r) \in \alpha + 0$. Hence $\alpha \subset \alpha + 0$.

Hence $\alpha + 0 = \alpha$.

(A5) If $x \neq 0, \exists! - x \in R$ such that $x + (-x) = 0$

We want to show that this is FALSE without the third property of cuts.

Because we do not have the third property of cuts, the following set is in R.

$$\alpha = \{p \mid p \in \mathbb{Q}, p \leq 2\} \in R \tag{18}$$

The definition of $-\alpha$ yields:

$$-\alpha = \{p \mid \exists r > 0, -p - r \notin \alpha\} \tag{19}$$

We can also think about $-\alpha$ as $-\alpha = \{t \mid t \in \mathbb{Q}, t < -2\}$. We can see that $\alpha + (-\alpha)$ does not include $0 \in \mathbb{Q}$, but $0 \in \mathbb{Q}$ is an element of the cut corresponding to zero. Hence, this property does not hold.

If we were to change the definition of $-\alpha$ to:

$$-\alpha = \{p \mid \exists r \geq 0, -p - r \notin \alpha\} \tag{20}$$

we would still run into the same problem with $\alpha - \alpha \neq 0$. Because now we are allowed to both have set with and without maximum elements but we still need to have a single zero element, we conclude it is impossible to satisfy (A5).