Homework 6

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1)

$R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(2,3),(3,2),(4,5),(5,4),(4,6),(6,4),(5,6),(6,5)\}$

$$[1] = \{1\} \subset R$$
$$[2] = \{2,3\} \subset R$$
$$[4] = \{4,5,6\} \subset R$$

2)

$$A = \{1,2,3,4,5,6\}$$

Proof: We can first prove reflexivity. We know that $\forall a \in A, a = a \cdot 1$. Therefore, we have shown that $|$ is reflexive.

To prove transitivity, we need to show that if $a|b$ and $b|c$, then $a|c$.

$$b = a \cdot n_1 \quad c = b \cdot n_2$$
$$c = a \cdot (n_1 \cdot n_2)$$
$$\Longrightarrow a|c$$

We have shown that divides is transitive. □

3)

$$xRy \Leftrightarrow x^2 + y^2 \text{ is even}$$

3.a)

Proof: We can first prove that R is reflexive. We know that $x^2 + x^2 = 2x^2$ is even because any number multiplied by 2 is even. Therefore, we have shown that R is reflexive.

To prove that R is symmetric, we need to show that if xRy, then yRx.

$$\begin{aligned} x^2 + y^2 &\text{ is even} \\ \implies y^2 + x^2 &\text{ is even} \end{aligned}$$

Addition is commutative, so we have shown that R is symmetric.

To prove that R is transitive, we need to show that if xRy and yRz, then xRz.

$$\begin{aligned} x^2 + y^2 \text{ is even and } y^2 + z^2 \text{ is even} & \\ x^2 + y^2 = 2n_1 \quad y^2 + z^2 = 2n_2 & \text{ for some } n_1, n_2 \in \mathbb{Z} \\ x^2 + 2y^2 + z^2 = 2n_1 + 2n_2 & \\ x^2 + z^2 = 2(n_1 + n_2 - y^2) & \\ \implies xRz & \end{aligned}$$

$\square$

3.b)

This relation just compares the parity of $x, y \in \mathbb{Z}$. If x, y are both even or both odd, then xRy. Otherwise, $x \neq Ry$.

Therefore, the two equivalence classes are:

$$\begin{aligned} [0] &= \{..., -4, -2, 0, 2, 4, ...\} \\ [1] &= \{..., -5, -3, -1, 1, 3, 5, ...\} \end{aligned}$$

4)

4.a)

Proof: We can construct a relation that is symmetric and transitive but not reflexive.

a

b $\qquad$ c

In this case, $R = \emptyset$. This relation is symmetric because there are no tuples to check for symmetry. It is also transitive because there are no tuples to check for transitivity. However, it is not reflexive because not every element is related to itself. $\square$

4.b)

Proof: Suppose R, S are equivalence relations on A. We need to show that $R \cup S$ is also an equivalence relation on A.

We can show that $R \cup S$ is reflexive. We know that R, S are reflexive, so $\forall a \in A, aRa \land aSa$. Therefore, $\forall a \in A, a(R \cup S)a$. R and S both contain the diagonal elements, so their union will also contain the diagonal elements.

We can show that $R \cup S$ is symmetric. We know that R, S are symmetric, so $\forall a, b \in A, aRb \implies bRa \land aSb \implies bSa$. Therefore, $\forall a, b \in A, a(R \cup S)b \implies b(R \cup S)a$. R and S both contain the symmetric elements, so their union will also contain the symmetric elements.

We can show that $R \cup S$ is transitive. We know that for each set individually, $aRb \land bRc \implies aRc \land aSb \land bSc \implies aSc$. Therefore, $\forall a, b, c \in A, a(R \cup S)b \land b(R \cup S)c \implies a(R \cup S)c$. R and S both contain the transitive elements, so their union will also contain the transitive elements. $\square$