Homework 3

Mark Schulist

1)

$$\psi(x) = N e^{-\frac{\alpha}{2}\,|x|} e^{i k_0 x}$$

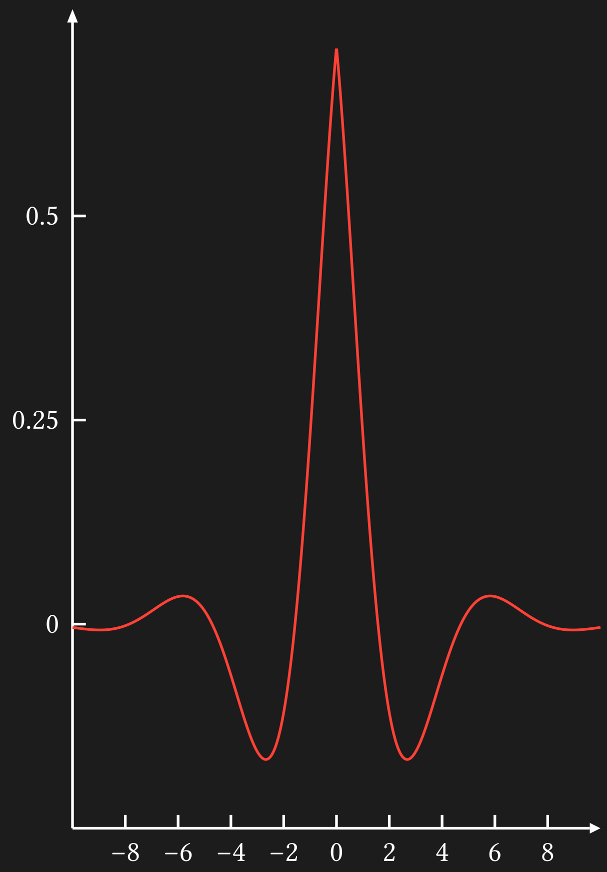
1.a)

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx \\
&= \int_{-\infty}^{\infty} \psi^*(x)\psi(x) dx \\
&= N^2 \int_{-\infty}^{\infty} e^{-\alpha\,|x|} dx \\
&= 2N^2 \int_{0}^{\infty} e^{-\alpha x} dx \\
1 &= 2\frac{N^2}{\alpha} \\
\implies N &= \sqrt{\frac{\alpha}{2}}
\end{aligned}$$

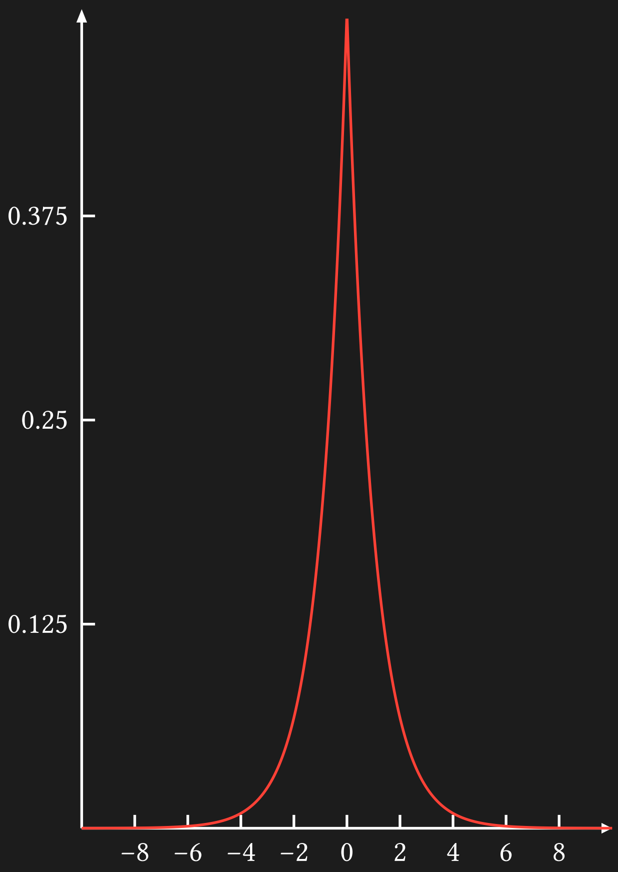
1.b)

$$\begin{aligned}
A(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} N e^{-\frac{\alpha}{2}\,|x|} e^{i k_0 x} e^{-ikx} dx \\
&= \frac{N}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}\,|x|} e^{ix(k_0 - k)} dx \\
&= \frac{N}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}\,|x| \,+ix(k_0 - k)} dx \\
&= \frac{N}{2\pi} \frac{4\alpha}{4k_0^2 - 8kk_0 + 4k^2 + \alpha^2}
\end{aligned}$$

1.c)



Graph 1: $\mathrm{Re}[\psi(x)]$ for $\alpha = 1$



Graph 2: $|\psi(x)|^2$ for $\alpha = 1$

2)

$$\psi(x) = \begin{cases} 0 & \text{for } x < -3a \\ C & \text{for } -3a \leq x \leq a \\ 0 & \text{for } x > a \end{cases}$$

2.a)

$$\begin{aligned} 1 &= C \int_{-3a}^{a} dx \\ &= 4aC \\ \implies C &= \frac{1}{4a} \end{aligned}$$

2.b)

$$\begin{aligned} P(0 \leq X \leq a) &= \int_{0}^{a} \frac{1}{16a^2} dx \\ &= \frac{1}{16a} \end{aligned}$$

2.c)

3)

$$A(k) = \frac{C\alpha}{\sqrt{\pi}} e^{-\alpha^2 k^2}$$

$$\begin{aligned}
\psi(x) &= \int_{-\infty}^{\infty} A(k) e^{ikx} dk \\
&= \frac{C\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 k^2} e^{ikx} dk \\
&= \frac{C\cancel{\alpha}}{\cancel{\sqrt{\pi}}} \frac{\cancel{\sqrt{\pi}}}{\cancel{\alpha}} e^{-\frac{x^2}{4\alpha^2}} \\
&= C e^{-\frac{x^2}{4\alpha^2}}
\end{aligned}$$

By inspection, we can see that α corresponds to the uncertainty/standard deviation (ε) of the Gaussian. Therefore $\Delta x = \alpha$.

We can do a similar analysis for $A(k)$ to find the uncertainty in k.

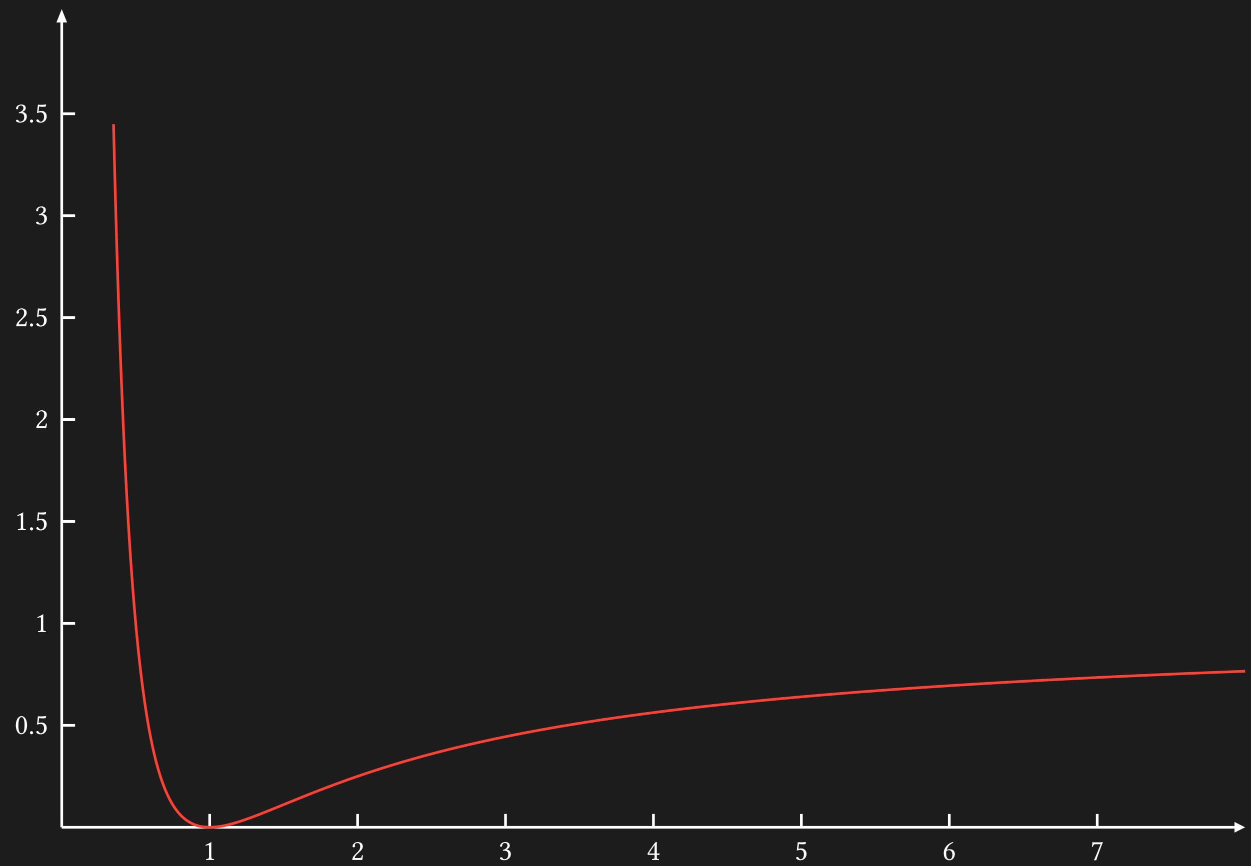
$$\begin{aligned}
\alpha^2 &= \frac{1}{4\varepsilon^2} \\
\implies \varepsilon = \Delta k &= \frac{1}{2\alpha}
\end{aligned}$$

$$\begin{aligned}
\Delta x \Delta k &\leq \frac{1}{2} \\
\frac{\cancel{\alpha}}{2\cancel{\alpha}} &\leq \frac{1}{2} \\
\frac{1}{2} &\leq \frac{1}{2}
\end{aligned}$$

4)

$$U(x) = \frac{1}{x^2} - \frac{2}{x} + 1$$

4.a)



Graph 3: Potential Energy vs. Position of $U(x) = \frac{1}{x^2} - \frac{2}{x} + 1$

$$\lim_{x \to 0} U(x) = \infty$$

$$\lim_{x \to \infty} U(x) = \frac{1}{\infty} - \frac{2}{\infty} + 1 = 1$$

4.b)

If the particle has 0.5 Joules of energy, it will have 2 turning points and is bound. We can find those turning points by setting $U(x) = 0.5$ and assuming that $x \neq 0$.

$$U(x) = 0.5$$

$$\frac{1}{x^2} - \frac{2}{x} + 1 = 0.5$$

$$\frac{1}{x^2} - \frac{2}{x} = -0.5$$

$$\frac{x - 2x^2}{x^3} = -0.5$$

$$x - 2x^2 = -0.5x^3$$

$$0.5x^2 - 2x + 1 = 0$$

$$\implies x = 2 \pm \sqrt{2}$$

4.c)

When a particle has 2 Joules of energy, it is not bound as there is only one time when $U(x) = 2$ for $x > 0$. The limit of $U(x)$ as $x \to \infty$ is 1, so the particle will never get "stopped" by the potential energy (or turned around).

$$\begin{aligned} f(x) &= x^2 \\ f(1) &= 1^2 \\ &= 1 \end{aligned}$$