Problem Set 10

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1)

$$U_0 = \frac{3}{4}E \tag{1}$$

$$\psi_{\text{inc}} = e^{ikx} \tag{2}$$

1.a)

$$k \equiv \sqrt{\frac{2mE}{\hbar^2}} \tag{3}$$

We know that $E > U_0$, so we can set up the relevant equations. We will call region 1 the region to the left of the step ($x < 0$) and region 2 to the right ($x > 0$).

$$\psi_1(x) = e^{ikx} + Be^{-ikx} \tag{4}$$

$$k' \equiv \sqrt{\frac{2m(E - U_0)}{\hbar^2}} \tag{5}$$

$$\psi_2(x) = Ce^{ik'x} \tag{6}$$

We can impose the required continuity conditions at $x = 0$.

$$\begin{aligned} \frac{B}{1} &= \frac{k - k'}{k + k'} \\ B &= \frac{\sqrt{\frac{2mE}{\hbar^2}} - \sqrt{\frac{2m(E-U_0)}{\hbar^2}}}{\sqrt{\frac{2mE}{\hbar^2}} + \sqrt{\frac{2m(E-U_0)}{\hbar^2}}} \\ &= \frac{\sqrt{E} - \sqrt{E - U_0}}{\sqrt{E} + \sqrt{E - U_0}} \\ &= \frac{\sqrt{E} - \sqrt{\frac{E}{4}}}{\sqrt{E} + \sqrt{\frac{E}{4}}} \\ &= \frac{1}{3} \end{aligned} \tag{7}$$

We know that $C = 1 + B$:

$$\begin{aligned} C &= 1 + B \\ C &= 1 + \frac{1}{3} \\ C &= \frac{4}{3} \end{aligned} \tag{8}$$

Therefore the wave function in the 2 regions is:

$$\begin{aligned} \psi_1(x) &= e^{ikx} + \frac{1}{3}e^{-ikx} \\ \psi_2(x) &= \frac{4}{3}e^{ik'x} \end{aligned} \tag{9}$$

1.b)

$$\begin{aligned} R &= \frac{B \cdot B}{A \cdot A} \\ &= \frac{1}{9} \end{aligned} \tag{10}$$

Equation (6-7):

$$\begin{aligned} R &= \frac{\left(\sqrt{E} - \sqrt{E - U_0}\right)^2}{\left(\sqrt{E} + \sqrt{E - U_0}\right)^2} \\ &= \frac{\left(\sqrt{E} - \sqrt{\frac{E}{4}}\right)^2}{\left(\sqrt{E} + \sqrt{\frac{E}{4}}\right)^2} \\ &= \frac{E + \frac{E}{4} - E}{E + \frac{E}{4} + E} \\ &= \frac{\frac{E}{4}}{9\frac{E}{4}} \\ &= \frac{1}{9} \end{aligned} \tag{11}$$

2)

$$B = -\frac{K + ik}{K - ik}A \tag{12}$$

2.a)

We can rewrite this in terms of the $\delta(E)$, which is the phase shift of the energy.

$$\begin{aligned} B &= -\frac{e^{i\delta(E)}}{e^{-i\delta(E)}}A \\ &= -e^{2i\delta(E)}A \end{aligned} \tag{13}$$

For $x < 0$

$$\begin{aligned} \psi(x) &= Ae^{ikx} - Ae^{i\delta(E)}e^{-ikx} \\ &= Ae^{i\delta(E)}\left(e^{ikx}e^{-i\delta(E)} - e^{ikx}e^{i\delta(E)}\right) \\ &= 2iAe^{i\delta(E)}\sin(kx - \delta(E)) \end{aligned} \tag{14}$$

Now that we know the wave function, we can take the absolute valued squared to find the probability density.

$$
\begin{aligned}
|\psi(x)|^2 &= \psi^*(x)\psi(x) \\
&= \left(-2iAe^{-i\delta(E)}\sin(kx-\delta(E))\right)\left(2iAe^{i\delta(E)}\sin(kx-\delta(E)\right) \\
&= 4|A|^2\sin^2(kx-\delta(E))
\end{aligned}
\tag{15}
$$

2.b)

As $k \to 0$, the energy must also be approaching 0. Therefore, C (amplitude inside of the step) must also approach 0 as the probability of finding the particle in the step will be VERY small (ie: if the energy is zero, then the amount of tunneling is zero). We also know that $\delta(E)$ must approach 0 because $\arctan(0) = 0$.

$$
\begin{aligned}
C &= \lim_{k\to 0} A - \frac{K+ik}{K-ik}A \\
&= A - A \\
&= 0
\end{aligned}
\tag{16}
$$

As $K \to 0$, $\delta(E) = \arctan(\infty)$ will approach $\frac{\pi}{2}$. Therefore, we get the maximum phase shift when $K \to 0$.

$$
\begin{aligned}
C &= \lim_{K\to 0} A - \frac{K+ik}{K-ik}A \\
&= 2A
\end{aligned}
\tag{17}
$$

When $K \to 0, U_0 \to E$ so we are approaching the case when $E > U_0$. So we penetrate deeper into the forbidden region (it is at its highest when $U_0 \to E$).

3)

$$
E = U_0 + \frac{\pi^2\hbar^2}{2mL^2}
\tag{18}
$$

$$
k \equiv \sqrt{\frac{2mE}{\hbar^2}}
\tag{19}
$$

$$
\begin{aligned}
k' &\equiv \sqrt{\frac{2m(E-U_0)}{\hbar^2}} \\
&= \sqrt{\frac{2m\frac{\pi^2\hbar^2}{2mL^2}}{\hbar^2}} \\
&= \frac{\pi}{L}
\end{aligned}
\tag{20}
$$

We have the following continuity requirements:

$$
\begin{aligned}
A + B &= C + D \\
Fe^{ikL} &= Ce^{ikL} + De^{-ik'L} \\
k(A - B) &= k'(C - D) \\
Fke^{ikL} &= Cke^{ikL} - Dk'e^{-ik'L}
\end{aligned}
\tag{21}
$$

We know that $k' \cdot L = \pi$.

$$
\begin{aligned}
Fe^{ikL} &= Ce^{i\pi} + De^{-i\pi} \\
Fe^{ikL} &= -C - D
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
Fke^{ikL} &= k'\left(Ce^{ik'L} - De^{ik'L}\right) \\
&= k'(D - C)
\end{aligned}
\tag{23}
$$

We can divide Equation 22 and Equation 23:

$$
\frac{k'}{k} = \frac{C + D}{C - D} \tag{24}
$$

From Equation 21, we can find the ratio of $\frac{k'}{k}$:

$$
\begin{aligned}
\frac{k'}{k} &= \frac{A - B}{C - D} \\
\implies \text{first cont eq.} \quad &= \frac{C + D}{C - D}
\end{aligned}
\tag{25}
$$

Therefore, we know that $A - B = C + D$, but we know that $C + D = A + B$:

$$
A - B = C + D = A + B \tag{26}
$$

The only way for Equation 26 to be true is if $B = 0$, so we have shown that B must equal 0 in this case.

4)

$$
U_0 = -3E \tag{27}
$$

4.a)

Classically, the particle would increase in kinetic energy (momentum) because the potential energy decreases and $E = T + U$ at all positions and times.

The kinetic energy would increase from E to $4E$ "instantly."

4.b)

Region 1 is the region to the left of the step $(x < 0)$ and region 2 is the region to the right of the step $(x > 0)$.

$$
\psi_1(x) = e^{ikx} + Be^{-ikx} \tag{28}
$$

$$
\psi_2(x) = Ce^{ik'x} \tag{29}
$$

$$k \equiv \sqrt{\frac{2mE}{\hbar^2}} \tag{30}$$

$$k' \equiv \sqrt{\frac{8mE}{\hbar^2}} \tag{31}$$

Continuity:

$$1 + B = C \tag{32}$$

$$\begin{aligned} k - Bk &= Ck' \\ k(1 - B) &= Ck' \\ \frac{k}{k'} &= \frac{C}{1 - B} \\ \frac{1}{2} &= \frac{C}{1 - B} \\ 2C &= 1 - B \\ C &= \frac{1 - B}{2} \end{aligned} \tag{33}$$

$$\begin{aligned} 1 + B &= \frac{1}{2} - \frac{B}{2} \\ \frac{3B}{2} &= -\frac{1}{2} \\ B &= -\frac{1}{3} \end{aligned} \tag{34}$$

$$C = \frac{2}{3} \tag{35}$$

$$\psi_1(x) = e^{ikx} - \frac{1}{3}e^{-ikx} \tag{36}$$

$$\psi_2(x) = \frac{2}{3}e^{ik'x} \tag{37}$$

4.c)

We want to find the reflextion coefficient.

$$\begin{aligned} R &= \frac{B^* B}{A^* A} \\ &= \frac{1}{9} \\ &= 0.\overline{1} \end{aligned} \tag{38}$$

This is the probability that an incoming particle is reflected.