Problem Set 8

Mark Schulist

1)

$$
\begin{aligned}
k &= 480 \ \frac{\mathrm{N}}{\mathrm{m}} \\
m &= 1.00784 \ \mathrm{m} \\
&= 1.67 \times 10^{-27} \ \mathrm{kg}
\end{aligned}
$$

We can find the angular frequency given the values of m and k above.

$$\omega = \sqrt{\frac{k}{m}}$$

$$
\begin{aligned}
\Delta E &= \left(\frac{3}{2}\right)\hbar\omega - \left(0 + \frac{1}{2}\right)\hbar\omega \\
&= \hbar\omega
\end{aligned}
$$

$$E_{\text{photon}} = h\frac{c}{\lambda}$$

We can set the energies equal to each other and solve for λ.

$$
\begin{aligned}
\hbar\omega &= h\frac{c}{\lambda} \\
\lambda &= \frac{hc}{\hbar\omega} \\
\lambda &= \frac{2\pi c}{\omega}
\end{aligned}
$$

Substituting in values:

$$
\begin{aligned}
\lambda &= \frac{2\pi c}{\sqrt{\frac{k}{m}}} \\
&= 3516 \ \mathrm{nm}
\end{aligned}
$$

2)

The momentum operator in coordinate space is:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

We know that in the box, the wave function is:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

We can find the uncertainty by finding $\langle p \rangle$ and $\langle p^2 \rangle$.

$$
\begin{aligned}
\langle p \rangle &= -\frac{2i\hbar}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \left[\frac{\mathrm{d}}{\mathrm{d}x} \sin\left(\frac{n\pi x}{L}\right)\right] \mathrm{d}x \\
&= -\frac{2i\hbar}{L} \cdot \frac{n\pi x}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle p^2 \rangle &= \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \hat{p}\hat{p} \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\
&= \frac{2}{L} \cdot \frac{\pi n}{L} \cdot (-i\hbar) \int_0^L \sin\left(\frac{n\pi x}{L}\right) \hat{p} \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\
&= \frac{2}{L} \cdot \frac{\pi n}{L} \cdot (-i\hbar) \cdot \frac{\pi n}{L} \cdot (-i\hbar) \int_0^L \sin\left(\frac{n\pi x}{L}\right) \left(-\sin\left(\frac{n\pi x}{L}\right)\right) \mathrm{d}x \\
&= \frac{-2\hbar^2\pi^2 n^2}{L^3} \cdot \frac{L(-2n\pi + \sin(2n\pi))}{4n\pi} \\
\Longrightarrow n \in \mathbb{N} \quad &= \frac{-2\hbar^2\pi^2 n^2}{L^3} \cdot \frac{-2\pi n L}{4\pi n} \\
&= \frac{\hbar^2\pi^2 n^2}{L^2}
\end{aligned}
$$

$$
\begin{aligned}
\Delta p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{\langle p^2 \rangle} \\
&= \sqrt{\frac{\hbar^2\pi^2 n^2}{L^2}} \\
&= \frac{\hbar\pi n}{L}
\end{aligned}
$$

3)

We know that the wave function for the harmonic oscillator in the ground state takes the form:

$$
\begin{aligned}
\psi(x) &= C_0 e^{-\alpha x^2} \\
\alpha &\equiv \frac{m\omega}{2\hbar}
\end{aligned}
$$

We can solve for the expectation value of the position. We know that the position operator is just the position: $\hat{x} = x$.

$$
\begin{aligned}
C_0 \int_{-\infty}^{\infty} x e^{-2\alpha x^2} \,\mathrm{d}x &= \frac{C_0}{4\alpha} e^{-2\alpha x^2} \Big|_{x=-\infty}^{\infty} \\
&= 0
\end{aligned}
$$

So $\langle x \rangle = 0$, which makes sense because in the ground state it is symmetric about $x = 0$.

4)

$$\hat{L} = -i\hbar \frac{\partial}{\partial \phi}$$

Well-defined means that the function must obey the following equation:

$$\hat{L} f(\phi) = \lambda f(\phi)$$

We can plug in $\hat{L}$ and solve for a possible function:

$$-i\hbar \frac{\partial}{\partial \phi} f(\phi) = \lambda f(\phi)$$
$$\frac{\partial}{\partial \phi} f(\phi) = \frac{i\lambda}{\hbar} f(\phi)$$

We can see that $f(\phi)$ must take the form of an exponential. We can define the following quantity:

$$a \equiv \frac{i\lambda}{\hbar}$$

$$\frac{\partial}{\partial \phi} f(\phi) = a f(\phi)$$

We can see the following:

$$f(\phi) = e^{a\phi}$$
$$f(\phi) = e^{\frac{i\lambda}{\hbar}\phi}$$

5)

$$\psi(x) = \begin{cases} 2\sqrt{a^3} x e^{-ax} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

5.a)

$$E = 0$$

$$-\frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \psi(x) + U(x)\psi(x) = E\psi(x)$$

$$U(x) 2\sqrt{a^3} x e^{-ax} = \frac{\hbar^2}{2m} 2a\sqrt{a^3} e^{-ax}(ax - 2)$$
$$U(x) = \frac{\hbar}{2m} \frac{a}{x}(ax - 2)$$
$$= \frac{\hbar^2 a^2}{2m} - \frac{\hbar^2 a}{mx}$$

5.b)

$$U(x) = \infty \quad x < 0$$

This must be true because it is impossible for the particle to be there as the wave function is zero for $x < 0$.

5.c)

We want to find when $U(x) = 0$ which is when we get to the forbidden region.

$$0 = \frac{\hbar^2 a^2}{2m} - \frac{\hbar^2 a}{mx}$$
$$x = \frac{2}{a}$$

We can now integrate $|\psi(x)|^2$ from $\frac{2}{a}$ to ∞.

$$\begin{aligned} 4a^3 \int_{\frac{2}{a}}^{\infty} x^2 e^{-2ax} \, \mathrm{d}x &= \frac{4a^3}{4a^3} \left[-e^{-2ax} (1 + 2ax + 2a^2x^2) \Big|_{x=\frac{2}{a}}^{\infty} \right] \\ &= e^{-4}(1 + 4 + 8) \\ &= 13e^{-4} \\ &\approx 0.238 \end{aligned}$$

5.d)

$$\begin{aligned} \langle x \rangle &= 4a^3 \int_0^{\infty} x x^2 e^{-2ax} \, \mathrm{d}x \\ &= \frac{3}{2a} \end{aligned}$$

$$\begin{aligned} \langle x^2 \rangle &= 4a^3 \int_0^{\infty} x^2 x^2 e^{-2ax} \, \mathrm{d}x \\ &= \frac{3}{a^2} \end{aligned}$$

$$\begin{aligned} \Delta x &= \sqrt{\frac{3}{a^2} - \frac{9}{4a^2}} \\ &= \sqrt{\frac{3}{4a^2}} \\ &= \frac{\sqrt{3}}{2a} \end{aligned}$$

5.e)

$$\hat{p} = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x}$$

$$
\begin{aligned}
\langle p \rangle &= -i\hbar \int_0^\infty 2\sqrt{a^3} x e^{-ax} \frac{\mathrm{d}}{\mathrm{d}x} \left[2\sqrt{a^3} x e^{-ax}\right] \mathrm{d}x \\
&= \int_0^\infty 2\sqrt{a^3} x e^{-ax} \left(2\sqrt{a^3} e^{-ax} - 2\sqrt{a^5} x e^{-ax}\right) \mathrm{d}x \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle p^2 \rangle &= -\hbar^2 \int_0^\infty \left[\left(2\sqrt{a^3} x e^{-ax}\right)\left(-4\sqrt{a^5} e^{-ax} + 2\sqrt{a^7} x e^{-ax}\right)\right] \mathrm{d}x \\
&= \hbar^2 a^2
\end{aligned}
$$

$$
\begin{aligned}
\Delta p &= \sqrt{\hbar^2 a^2} \\
&= \hbar a
\end{aligned}
$$

5.f)

$$
\Delta p \Delta x \geq \frac{\hbar}{2}
$$

$$
\hbar \cancel{a} \cdot \frac{\sqrt{3}}{2\cancel{a}} = \hbar \frac{\sqrt{3}}{2} \geq \frac{\hbar}{2}
$$

So yes it is satisfied!