Homework 4

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1) 2.2

Show that the set of algebraic numbers is countable.

An algebraic number is a $z \in \mathbb{C}$ that, with the some set of coefficients, satisfies the following equation:

$$a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n z^0 = 0 \quad (1)$$

We know that for each $N, n \in \mathbb{N}$, $n + |a_0| + |a_1| + \ldots + |a_n| = N$ has a finite number of solutions.

Let C_{nN} be the set of complex numbers satisfying Equation 1 with $n + |a_0| + |a_1| + \ldots + |a_n| = N$, which we know has a finite number of solutions (where solution means set of coefficients that satisfies the equation) for a fixed n, N.

Hence $C_n = \bigcup_{i=1}^{\infty} C_{ni}$ is at most countable as we are taking an infinite union of finite sets. And finally $A = \bigcup_{i=1}^{\infty} C_i$ is at most countable as we are taking the union of many at most countable sets.

We know that the integers are a subset of A (all integers are algebraic), and the integers are an infinite set. Hence A must be a countable set.

We know that A contains all possible algebraic numbers due to the following: each algebraic number can be represented by *some* polynomial of degree n (using Equation 1), and we have *every possible polynomial for each degree* due to our contruction. We have every degree as we are taking the union over all n. For each n, we have all possible set of coefficients as, by construction, we are taking all possible $n + |a_0| + |a_1| + \ldots + |a_n| = N$ for each degree. Hence A contains all algebraic numbers.

2) 2.3

Show that there exists real numbers which are not algebraic.

Proof.

From the above problem, we know that the set of algebraic numbers if countable. Call it A.

Suppose $\mathbb{R} \subset A$, meaning that all real numbers are algebraic. Since $\mathbb{R}$ is uncountable, A must also be $\lightning$.

So A is not a subset of $\mathbb{R}$, hence there exist real numbers which are not algebraic. $\square$

3) 2.4

Is the set of irrational numbers countable?

We know that the irrational numbers (call the set A) satisfies $A = \mathbb{R} \setminus \mathbb{Q}$, which implies that $A \cup \mathbb{Q} = \mathbb{R}$. We know that $\mathbb{Q}$ is countable, so if we assume that A is countable then we get that the union of 2 countable sets is uncountable $\lightning$. Hence A must be uncountable.

4) 2.5

Construct a bounded set with 3 limit points.

$$
\begin{aligned}
A &= \left\{\frac{1}{n+1} \mid n \in \mathbb{N}\right\} \\
B &= \left\{1 + \frac{1}{n+1} \mid n \in \mathbb{N}\right\} \\
C &= \left\{2 + \frac{1}{n+1} \mid n \in \mathbb{N}\right\}
\end{aligned}
\tag{2}
$$

We can show that set A has a limit point as 0. Pick an arbitrary open ball $\frac{1}{r}, r > 0$ centered at 0. Any point $\frac{1}{n}, n > r$ is in this ball. Hence there are are infinitely many points from A in any ball about 0, therefore 0 must be a limit point. We can apply this same logic to show that B and C also have respective limit points of 1, 2.

If we pick any other point in A, I claim it cannot be a limit point of A. Suppose we pick $p < 0$. Then we can find a neighborhood around p that does not contain any elements of A by setting the radius to any value less than $-p$.

Suppose we pick a point $0 < p = \frac{1}{k} \leq \frac{1}{2}$. If $p \in A$ then we just make the ball $B_r(p)$ have radius $r < \frac{1}{k(k+1)}$, which guarantees that no other point of A will be in this open interval. If $p \notin A$, then we just find the distance to the nearest point in A and set the radius to be smaller than that distance.

If we pick a point $p > \frac{1}{2}$, then we just set the radius of the neighborhood to be less than $p - \frac{1}{2}$ which guarantees that no elements of A are in the neighborhood.

With these 3 cases, we have shown that A only has a single limit point.

A is also bounded below by 0 and above by $\frac{1}{2}$. $\frac{1}{n}$ cannot be negative, so 0 is the lower bound. The smallest n is 1, so $\frac{1}{1+1} = \frac{1}{2}$ which is the upper bound.

Each of these sets has 3 limit points (0, 1, 2 in particular), so $A \cup B \cup C$ must also have 3 limit points as the sets are disjoint and their individual limit points are distinct.

5) 2.8

Is every limit point of every open set $E \subset \mathbb{R}^2$ a limit point of E?

E is an open set if for all $x \in E$, there exists $r > 0$ such that $B_r(x) \subset E$.

x is a limit point of E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.

Because E is open, we can always find a neighborhood $B_r(x)$ around any $x \in E$ such that $B_r(x) \subset E$. Because we can always find this neighborhood, we can find a point $q \in B_r(x)$ such that $q \neq p$ and $q \in E$. If we make the neighborhood larger (larger r), the point q will still be in $B_r(x)$ (meaning that q is in all neighborhoods of larger radius than r, satisfying part of the possible neighborhoods). If we make the neighborhood smaller, the same q is no longer guaranteed to still be in the neighborhood, but $B_r(x)$ will contain other points in E (as $B_r(x) \subset E$ for all $r > 0$) so we can still find another point $q \in B_r(x), q \neq p$ such that $q \in E$.

For closed sets, no! Take $X = \{(x, y) \mid x^2 + y^2 \leq 1, x, y \in \mathbb{R}\} \cup \{(10, 10)\}$.

This set is closed, but the point $(10, 10) \in X$ is not a limit point of X. Hence not every point of X is a limit point of X if X is closed.

6) 2.11

6.a)

$$d_1(x, y) = (x - y)^2 \tag{3}$$

(a) $(x - y)^2 > 0$ and $0 = (x - y)^2 \implies 0 = x - y \implies x = y$, so satisfied.

(b) $(x - y)^2 = (y - x)^2$ so is satisfied.

(c) We want to see if $(x - z)^2 \leq (x - y)^2 + (y - z)^2$. If we set $x = 0, y = 0.5, z = 1$ we get:

$$\begin{aligned} (-1)^2 &\leq 0.5^2 + 0.5^2 \\ 1 &\nleq 0.25 + 0.25 \end{aligned} \tag{4}$$

So metric (1) is not a valid metric.

6.b)

$$d_2(x, y) = \sqrt{|x - y|} \tag{5}$$

(a) $\sqrt{|x - y|} > 0$ unless $x = y$, as $\sqrt{|x - y|} = 0 \implies |x - y| = 0 \implies x = y$. So (a) is satisfied.

(b) $\sqrt{|x - y|} = \sqrt{|y - x|}$. We take multiply by −1 in the abs and nothing changes.

(c) We want to show that $\sqrt{|x - z|} \leq \sqrt{|x - y|} + \sqrt{|y - z|}$. We can show this by squaring both sides.

$$\underbrace{|x - z| \leq |x - y| + |y - z|}_{\text{triangle ineq, thm 1.37, prop f}} + \underbrace{2\sqrt{|x - y|\ |y - z|}}_{>0} \tag{6}$$

We know that the first part is true by the "regular" triangle inequality, and if we add a positive term to the greater then or equal to side, the inequality still holds. Hence this metric satisfies all 3 properties of being a valid metric.

6.c)

$$d_3(x, y) = |x^2 - y^2| \tag{7}$$

(a) $\left|(-1)^2 - (1)^2\right| = 0$, but $x, y \neq 0$. So this is not a valid metric!

6.d)

(b) $|x^2 - 2y| \neq |y^2 - 2x|\ \forall x, y \in \mathbb{R}$ so does not satisfy symmetric property (as $x^2 \neq 2x$, they are not the same operation being applied to both x and y).

6.e)

$$d_5(x, y) = \frac{|x - y|}{1 + |x - y|} \tag{8}$$

(a) $\frac{|x-y|}{1+|x-y|} > 0$ if $x, y \neq 0$ as we have positive divided by a positive. If $0 = \frac{|x-y|}{1+|x-y|} \implies |x - y| = 0 \implies x = y$.

(b) $\frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|}$ as same operations are being applied to x and y (just labeling).

(c) This is a a bit longer so will continue onto another line.

We want to show that $\frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$.

First, we will prove a Lemma denoted as *Andy's Lemma* (named after the fellow classmate I worked on the homework with).

Lemma (Andy's Lemma)
Given $c \leq a, c \neq b, x \in \mathbb{R}, b + a > 0$ with $\frac{x+a}{b+a} < 1$, the following is true:

$$\frac{x+a}{b+a} \geq \frac{x+c}{b+c} \tag{9}$$

Proof. Let $n = \frac{x+a}{b+a}$. We know that $n < 1$ by the given.

$$\begin{aligned} n &= \frac{x+a}{b+a} \\ nb + na &= x + a \\ nb - x &= a(1-n) \\ n < 1 \implies \quad na + nb &\geq c(1-n) \\ nb - x &\geq c - cn \\ nb + cn &\geq c + x \\ n(b+c) &\geq c + x \\ n &\geq \frac{c+x}{b+c} \\ \implies \quad \frac{x+a}{b+a} &\geq \frac{c+x}{c+b} \end{aligned} \tag{10}$$

□

Now we can apply the lemma to the problem. We will set $a = \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$.

$$\begin{aligned} a &= \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \\ a &= \frac{|x-y| + |y-z| + 2|x-y||y-z|}{1 + |x-y||y-z| + |x-y| + |y-z|} \\ \text{get rid of 2} \implies \quad a &\geq \frac{|x-y| + |y-z| + |x-y||y-z|}{1 + |x-y||y-z| + |x-y| + |y-z|} \end{aligned} \tag{11}$$

We know that $a < 1$ (due to the +1 in the denominator) so we can apply the lemma. Using the normal triangle inequality (thm 1.37, prop f), we know that $|x-z| \leq |x-y| + |y-z|$. Hence we can replace $|x-y| + |y-z|$ with $|x-z|$ in both the numerator and denominator and keep the inequality valid.

$$a \geq \frac{|x-z| + |x-y||y-z|}{1 + |x-z| + |x-y||y-z|} \tag{12}$$

Now we again apply the lemma and replace $|x-y||y-z|$ with 0. This fits all the requirements of the lemma as all we need is for $0 \leq |x-y||y-z|$ which is true as abs must be non-negative.

$$a \geq \frac{|x-z|}{|x-z|+1} \tag{13}$$

Now we plug back into the definition of a.

$$\begin{aligned} \frac{|x-z|}{|x-z|+1} &\leq a = \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \\ \frac{|x-z|}{|x-z|+1} &\leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \end{aligned} \tag{14}$$

And hence this is a valid metric!