Problem Set 4

Mark Schulist

1) 10.10

$$V(x) = \frac{1}{2} m\omega^2 x^2 \tag{1}$$

$$H' = \gamma x^3 \tag{2}$$

1.a)

First order energy corrections.

$$E_n^{(1)} = \langle n|H'|n\rangle = 0 \tag{3}$$

Because we can never get back to the n state with 3 ladder operators.

1.b)

Second order energy corrections.

$$E_n^{(2)} = \sum_{m \neq n} \frac{\left(|\langle n^{(0)}|H'|m^{(0)}\rangle|\right)^2}{E_n^{(0)} - E_m^{(0)}} \tag{4}$$

We know that $H' = \gamma x^3$, so we can use ladder operators to solve.

$$\begin{aligned} x^3 &= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(a^\dagger + a\right)^3 \\ &= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \Bigg[\underbrace{a^\dagger a^\dagger a^\dagger}_{+3, \sqrt{(n+3)(n+2)(n+1)}} + \underbrace{aaa}_{-3, \sqrt{n(n-1)(n-2)}} + \underbrace{a^\dagger a^\dagger a}_{+1, n\sqrt{n+1}} + \underbrace{a a^\dagger a^\dagger}_{+1, \sqrt{n+1}(n+2)} \\ &\quad + \underbrace{a^\dagger a a^\dagger}_{+1, (n+1)^{\frac{3}{2}}} + \underbrace{a^\dagger a a}_{-1, n\sqrt{n-1}} + \underbrace{a a^\dagger a}_{-1, n^{\frac{3}{2}}} + \underbrace{a a a^\dagger}_{-1, \sqrt{n}(n+1)} \Bigg] \end{aligned} \tag{5}$$

As we can see, we can only get to $n \pm \{1, 3\}$ from where we started. Hence we only need to consider states where m is 1 or 3 away from n as otherwise the inner product will return 0.

$$E_0^2 = \frac{|\langle 0|H'|3\rangle|^2}{E_0 - E_3} + \frac{|\langle 0|H'|1\rangle|^2}{E_0 - E_1} \tag{6}$$

$$E_1^2 = \frac{|\langle 1|H'|4\rangle|^2}{E_1 - E_4} + \frac{|\langle 1|H'|0\rangle|^2}{E_1 - E_0} + \frac{|\langle 1|H'|2\rangle|^2}{E_1 - E_2} \tag{7}$$

$$E_2^2 = \frac{|\langle 2|H'|1\rangle|^2}{E_2 - E_1} + \frac{|\langle 2|H'|3\rangle|^2}{E_2 - E_3} + \frac{|\langle 2|H'|5\rangle|^2}{E_2 - E_5} \tag{8}$$

Now we can compute these inner products.

$$\begin{aligned} E_n &= \hbar\omega\left(\frac{1}{2}+n\right) \\ E_n - E_m &= \hbar\omega(n-m) \end{aligned} \tag{9}$$

$$\frac{|\langle 0|H'|3\rangle|^2}{E_0 - E_3} = \left(\frac{\hbar}{2m\omega}\right)^3 \gamma^2 \frac{6}{-3\hbar\omega} \tag{10}$$

$$\frac{|\langle 0|H'|1\rangle|^2}{E_0 - E_1} = \left(\frac{\hbar}{2m\omega}\right)^3 \gamma^2 \frac{9}{-\hbar\omega} \tag{11}$$

$$\frac{|\langle 1|H'|4\rangle|^2}{E_1 - E_4} = \left(\frac{\hbar}{2m\omega}\right)^3 \gamma^2 \frac{24}{-3\hbar\omega} \tag{12}$$

$$\frac{|\langle 1|H'|0\rangle|^2}{E_1 - E_0} = \left(\frac{\hbar}{2m\omega}\right)^3 \gamma^2 \frac{9}{\hbar\omega} \tag{13}$$

$$\frac{|\langle 1|H'|2\rangle|^2}{E_1 - E_2} = \left(\frac{\hbar}{2m\omega}\right)^3 \gamma^2 \frac{72}{-\hbar\omega} \tag{14}$$

$$\frac{|\langle 2|H'|1\rangle|^2}{E_2 - E_1} = \left(\frac{\hbar}{2m\omega}\right)^3 \gamma^2 \frac{72}{\hbar\omega} \tag{15}$$

$$\frac{|\langle 2|H'|3\rangle|^2}{E_2 - E_3} = \left(\frac{\hbar}{2m\omega}\right)^3 \gamma^2 \frac{243}{-\hbar\omega} \tag{16}$$

$$\frac{|\langle 2|H'|5\rangle|^2}{E_2 - E_5} = \left(\frac{\hbar}{2m\omega}\right)^3 \gamma^2 \frac{60}{-3\hbar\omega} \tag{17}$$

Plugging in we get:

$$E_0^2 = \left(\frac{\hbar}{2m\omega}\right)^3 \frac{\gamma^2}{\hbar\omega}(-2-9) = \left(\frac{\hbar}{2m\omega}\right)^3 \frac{\gamma^2}{\hbar\omega}(-11) \tag{18}$$

$$E_1^2 = \left(\frac{\hbar}{2m\omega}\right)^3 \frac{\gamma^2}{\hbar\omega}\left(9 - \frac{24}{3} - 72\right) = \left(\frac{\hbar}{2m\omega}\right)^3 \frac{\gamma^2}{\hbar\omega}(-71) \tag{19}$$

$$E_2^2 = \left(\frac{\hbar}{2m\omega}\right)^3 \frac{\gamma^2}{\hbar\omega}\left(72 - 243 - \frac{60}{3}\right) = \left(\frac{\hbar}{2m\omega}\right)^3 \frac{\gamma^2}{\hbar\omega}(-191) \tag{20}$$

1.c)

$$|n^1\rangle = \sum_{m\neq n} \frac{\langle m|H'|n\rangle}{E_n - E_m}|m^0\rangle \tag{21}$$

$$
\begin{aligned}
|0^1\rangle &= \frac{\langle 1|H'|0\rangle}{E_0 - E_1}|1\rangle + \frac{\langle 3|H'|0\rangle}{E_0 - E_3}|3\rangle \\
&= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \frac{\gamma^2}{\hbar\omega}\left[-3|1\rangle - \frac{\sqrt{6}}{3}|3\rangle\right]
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
|1^1\rangle &= \frac{\langle 0|H'|1\rangle}{E_1 - E_0} + \frac{\langle 2|H'|1\rangle}{E_1 - E_2} + \frac{\langle 4|H'|1\rangle}{E_1 - E_4} \\
&= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \frac{\gamma^2}{\hbar\omega}\left[3|0\rangle - 6\sqrt{2}|2\rangle - \frac{\sqrt{24}}{3}|4\rangle\right]
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
|2^1\rangle &= \frac{\langle 1|H'|2\rangle}{E_2 - E_1} + \frac{\langle 3|H'|2\rangle}{E_2 - E_3} + \frac{\langle 5|H'|2\rangle}{E_2 - E_5} \\
&= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \frac{\gamma^2}{\hbar\omega}\left[6\sqrt{2}|1\rangle - 9\sqrt{3}|3\rangle - \frac{\sqrt{60}}{3}|5\rangle\right]
\end{aligned} \tag{24}
$$

2) 10.24

$$
H = V_0 \begin{pmatrix} 3 & \varepsilon & 0 & 0 \\ \varepsilon & 3 & 2\varepsilon & 0 \\ 0 & 2\varepsilon & 5 & \varepsilon \\ 0 & 0 & \varepsilon & 7 \end{pmatrix} \tag{25}
$$

2.a)

When $\varepsilon = 0$, then the eigenvalues are 3, 3, 5, 7 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

We find these by just reading them off of the diagonal.

2.b)

We know that $|1\rangle, |2\rangle$ are degenerate and $|3\rangle, |4\rangle$ are not. Let's find the non-degenerate energy corrections first.

$$
H' = V_0 \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 2\varepsilon & 0 \\ 0 & 2\varepsilon & 0 & \varepsilon \\ 0 & 0 & \varepsilon & 0 \end{pmatrix} \tag{26}
$$

We can see that $\langle 3|H'|3\rangle = \langle 4|H'|4\rangle = 0$ so we need to go to the second order to find a nonzero term.

$$
\begin{aligned}
E_3^2 &= \frac{|\langle 2|H'|3\rangle|^2}{E_3 - E_2} + \frac{|\langle 4|H'|3\rangle|^2}{E_3 - E_4} \\
&= \frac{4V_0^2\varepsilon^2}{2V_0} + \frac{V_0^2\varepsilon^2}{2V_0} \\
&= \frac{3}{2}\varepsilon^2 V_0
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
E_4^2 &= \frac{|\langle 3|H'|4\rangle|^2}{E_4 - E_3} \\
&= \frac{V_0^2\varepsilon^2}{2V_0} \\
&= \frac{1}{2}V_0\varepsilon^2
\end{aligned}
\tag{28}
$$

Now we can handle the degenerate cases by solving for the eigenvalues of the subspace containing the degenerate states.

$$
H_d' = V_0 \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}
\tag{29}
$$

Now we diagonalize:

$$
\begin{aligned}
\begin{vmatrix} -\lambda & V_0\varepsilon \\ V_0\varepsilon & -\lambda \end{vmatrix} &= \lambda^2 - (V_0\varepsilon)^2 = 0 \\
\implies \lambda &= \pm\varepsilon V_0
\end{aligned}
\tag{30}
$$

Hence the corrected energies for all 4 states are:

$$
\begin{aligned}
E_4 &= V_0\left(5 + \frac{3}{2}\varepsilon^2\right) \\
E_3 &= V_0\left(7 + \frac{1}{2}\varepsilon^2\right) \\
E_2 &= V_0(3 + \varepsilon) \\
E_1 &= V_0(3 - \varepsilon)
\end{aligned}
\tag{31}
$$