Homework 5

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1) 2.6

1.a)

$$E' = \{\text{limit points of E}\} \quad (1)$$

Show that E' is closed.

Closed means that every limit point of E' is a point of E'.

We can show that $(E')^c$ is open.

Proof. For all $p \in (E')^c$, there exists an $r > 0$ such that for all $q \in B_r(p), q \notin E$ (with $q \neq p$).

Assume that $q \in E'$, with $d(p, q) = \varepsilon$. Then $B_{\frac{r-\varepsilon}{2}}(q) \subset B_r(p)$, so we have found a ball that avoids E. Hence q is not a limit point of E, so $q \notin E'$.

Hence every point of $(E')^c$ is an interior point, so $(E')^c$ is an open set. □

1.b)

Show that E and $\overline{E}$ have the same limit points. $\overline{E} = E \cup E'$.

Proof. $\left(\Longrightarrow, E' \subset \left(\overline{E}\right)'\right)$ We know that $E \subset \overline{E}$. Because of this, any limit point of E is a limit point of $\overline{E}$. Hence $E' \subset \left(\overline{E}\right)'$.

$\left(\Longleftarrow, \left(\overline{E}\right)' \subset E'\right)$

We know that for all $p \in \left(\overline{E}\right)'$, for all $r > 0$ there exists $q \in B_r$ with $q \in \overline{E}$ (with $p \neq q$). Therefore $q \in E$ or $q \in E'$.

Assume for all $x \in B_r(p), x \notin E$. Let $d(p, q) = \varepsilon$. Consider $B_{\frac{r-\varepsilon}{2}}(q) \subset B_r(p)$. So for all $a \in B_{\frac{r-\varepsilon}{2}}(q), a \notin E$. So $q \notin E'$ ↯. We assumed for all $x \in B_r(p), x \notin E$, so our assumption must be wrong. So then $q \notin E$. So $q \in E'$, which implies that for any neighborhood around any point in $\left(\overline{E}\right)'$, there is another point that is inside of E. Hence $\left(\overline{E}\right)'$ is a subset of E'. □

1.c)

Do E and E' always have the same limit points?

In $(\mathbb{R}, d)$.

Let $E = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Then $E' = \{0\}$ and $E'' = \emptyset$. So not true.

2) 2.7

$A_1, A_2, ..., A_n$ are subset of a metric space.

2.a)

If $B_n = \bigcup_{i=1}^n A_i$, show that $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$.

Proof. $(\Longleftarrow)$ Suppose $y \in \bigcup_{i=1}^n \overline{A_i}$. Then $y \in \overline{A_i}$ for some $1 \leq i \leq n$. So $y \in A_i \cup A_{i'}'$ for some i.

- If $y \in A_i$, then $y \in B_n$ by the given, so $y \in \overline{B_n}$.
- If $y \in A_i'$, then y is a limit point of A_i. $A_i \subset B_n$, which implies that y is a limit point of $B_n \implies y \in \overline{B_n}$.

($\implies$) Suppose $y \in \overline{B_n} = B_n \cup B_n'$, then $y \in B_n$ or $y \in B_n'$.

- If $y \in B_n$ then $y \in A_i$ for some i. This implies that $y \in \bigcup_{i=1}^{n} A_i \implies y \in \bigcup_{i=1}^{n} \overline{A_i}$.
- If $y \in B_n'$ then y is a limit point of B_n. Because y is a limit point of B_n, for all $r > 0$ $N_r(y)$ contains a point $x \in B_n$ with $x \neq y$. Because $x \in B_n$, there exists an i such that $x \in A_i$. Note that is only works for finite i as if we have an infinite union, then we might not be able to pick out a specific A_i that x is a part of (see (c)). Because we can find this x for any neighborhood of $y \in A_i$, $y \in A_i'$. Hence $y \in \overline{A_i}$, which means that y is in the RHS of the problem.

$\square$

2.b)

If $B = \bigcup_{i=1}^{\infty} A_i$, show $\overline{B} \supset \bigcup_{i=1}^{\infty} \overline{A_i}$.

Proof. Suppose $x \in \bigcup_{i=1}^{\infty}$. Then $x \in \overline{A_i} = A_i + A_i'$ for some i.

- If $x \in A_i$, then $x \in B$ by given. So $x \in \overline{B}$ as $B \subset \overline{B}$.
- If $x \in A_i'$, then x is a limit point of A_i. $A_i \subset B \implies x$ is a limit point of $B \implies x \in \overline{B}$.

$\square$

2.c)

Example:

$$\begin{aligned} A_i &= \left\{\frac{1}{i}\right\} \\ B &= \bigcup_{i=1}^{\infty} A_i = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} \end{aligned} \tag{2}$$

So $0 \in \overline{B}$.

$\overline{A_i} = A_i$ as they consist of a single point. So $0 \notin \bigcup_{i=1}^{\infty} \overline{A_i}$ as 0 is not in any A_i, but $0 \in \overline{B}$.

3) 2.9

$$E^\circ = \{\text{interior points of } E\} \tag{3}$$

3.a)

Show E° is always open.

Proof. For all $x \in E^\circ$, there exists $r > 0$ such that $N_r(x) \subset E$. This is because for all $N_r(x)$, we can pick a $q \in N_r(x) \implies \exists N_{r_1}(q) \subset N_r(x) \subset E$. This is because there exists a neighborhood around q if we make r_1 smaller than $\frac{r-\varepsilon}{2}$, where $\varepsilon = d(p,q)$. Hence any point in $N_r(x) \in E^\circ$, which implies that $N_r(x) \subset E^\circ$.

Because we can always find a neighborhood around any $x \in E^\circ$ that is a subset of E°, we have shown that E° is an open set. $\square$

3.b)

Show that E open $\iff E^\circ = E$

Proof. ($\Longrightarrow$) Assume that E is open. So for all $x \in E$, x is an interior point. So $E = \{\text{interior point of } E\} = E^\circ$.

($\Longleftarrow$) Assume $E^\circ = E$. Then $E = \{\text{interior points of } E\}$. We know that E° is open, which implies that E must be open. □

3.c)

Show that if $G \subset E$ and G is open, then $G \subset E^\circ$.

Proof. Because G is open, $G = G^\circ$. For all $g \in G$, there exists $r > 0$ such that $B_r(g) \subset G \subset E$. Hence every g is an interior point of E. Hence $G \subset E^\circ$. □

3.d)

Show that $(E^\circ)^c = \overline{E^c}$.

Proof. ($\Longrightarrow$) Suppose $y \in (E^\circ)^c$. So y is not an interior point of E. So irregardless of how small we make our neighborhood of y, it will always contain a least one $x \in E^c$. So we have 2 options for y:

- if $y \in E^c$, then we are done (as $E^c \subset \overline{E^c}$)
- if $y \notin E^c$, then y must be a limit point of E^c as all neighborhoods of y must contains at least one point of E^c, or else y would be an interior point of E. Hence $y \in (E^c)' \subset \overline{E^c}$.

($\Longleftarrow$) Suppose $y \in \overline{E^c}$. Then $y \in E^c$ or $y \in (E^c)'$.

- If $y \in E^c$, then y cannot be an interior point of E so $y \in (E^\circ)^c$.
- If $y \in (E^c)'$, then y is a limit point of E^c. This means that all neighborhoods of y contain another (distinct) point $q \in E^c$. Because all neighborhoods always contains another point in E^c, y cannot be an interior point of E. Hence $y \in (E^\circ)^c$.

□

3.e)

Do E and $\overline{E}$ have the same interiors?

Let us be in $\mathbb{R}$.

Let $E = \{x \mid x \in \mathbb{Q}, 0 < x < 1\}$.

$E^\circ = \emptyset$ as any neighborhood $r > 0$ contains points of $\mathbb{R}$ not in $\mathbb{Q}$.

$\overline{E} = (0, 1)$ as $\mathbb{Q}$ dense in $\mathbb{R}$.

$\left(\overline{E}\right)^\circ = (0, 1) \neq \emptyset$. Hence E and $\overline{E}$ do not necessarily have the same interiors.

3.f)

Do E and E° have the same closures?

Let us be in $\mathbb{R}$.

Let $E = \{x \mid x \in \mathbb{Q}, 0 < x < 1\}$. Then $\overline{E} = [0, 1]$.

But $E^\circ = \emptyset$ as we are in $\mathbb{R}$ and any neighborhood of any point in E cannot be a subset of E as it will contain another point in $\mathbb{R}$ that is not in $\mathbb{Q}$ (it will contain an *irrational* number). Hence $\overline{E^\circ} = \emptyset$.

So no as $\overline{E} \neq \overline{E^\circ}$.

4) 2.10

X is an infinite set. $p, q \in X$

$$d(p,q) = \begin{cases} 1 \text{ if } p \neq q \\ 0 \text{ if } p = q \end{cases} \tag{4}$$

4.a)

Show this is a metric.

(a) $d(p,q) = 0 \Longrightarrow p = q$. $d(p,q) = 1 > 0$ otherwise.

(b) $d(p,q) = d(q,p)$ by definition of metric.

(c) $r, p, q \in X$

We need $d(r,q) \leq d(r,p) + d(p,q)$ to hold for all p, q, r.

Case 1: $r = p = q$. Then all terms are 0 so it holds.

Case 2: $r \neq p, r = q \Longrightarrow 0 \leq 1 + 1$.

Case 3: $r = p, r \neq q \Longrightarrow 1 \leq 0 + 1$.

4.b)

Let $E \subset X$.

For all $x \in E$, $B_{r<1}(x) = \{x\} \subset E$ as $x \in E$. So all nonempty subsets of X are open.

4.c)

We know that all nonempty subsets of X are open, so if $E \subset X$, then E^c is open as E^c is a nonempty subset of X. So $(E^c)^c = E$ is closed. If $E = \emptyset$, then E is closed. So all subsets of X are closed.

4.d)

All finite subsets of X are compact as we just need to select all points in the subset as the finite subcover.

Let $A \subset X$, with A infinite.

Suppose $0 < r < 1$, then $\bigcup_{p \in A} B_r(p)$ is a cover of A. But it cannot be a finite subcover as $B_r(p)$ only contains one point of A. If there exists a finite subcover, then A must be finite $\lightning$. So there does not exist a finite subcover of A is A is infinite.

5) 2.14

Construct open cover of $(0,1)$ which has no finite subcover.

Let $G_i = \left(\frac{1}{i}, 1\right)$. This covers $(0,1)$ if we have union of G_i for $2 \leq i < \infty$. The sets are nested, meaning that the outermost set will approach containing (with $i \to \infty$) $(0,1)$.

If we have a finite subcover, then we just take the max index of our subcover. Call the max index m. Then we have missed all points between $\left(0, \frac{1}{m}\right)$ so any finite subcover will not cover $(0,1)$.

6) 2.22

$$\mathbb{Q}^k = \{(a_1, a_2, ..., a_k) \mid a_1, a_2, ..., a_k \in \mathbb{Q}\} \tag{5}$$

We know that $\mathbb{Q}^k \subset \mathbb{R}^k$ and that $\mathbb{Q}^k$ is countable.

$\mathbb{Q}^k$ is countable because it is the cartesian product of countable sets, so it must be countable.

Now we must show that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$.

Let $x \in \mathbb{R}^k, x \notin \mathbb{Q}^k$ (denote the components of x by $x_1, x_2, \ldots$).

We know that $\mathbb{Q}$ is dense in $\mathbb{R}$, so $\exists y_i \in \mathbb{Q}$ such that $|x_i - y_i| < \frac{r}{\sqrt{k}}$ for any $r > 0$.

$$(x_i - y_i)^2 < \frac{r^2}{k} \tag{6}$$

$$|x - y| = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2} < \sqrt{\sum_{i=1}^{k} \frac{r^2}{k}} = r \tag{7}$$

For all $r > 0$. Hence $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$.

Because $\mathbb{Q}^k$ is countable and is dense in $\mathbb{R}^k$, we have shown that $\mathbb{R}^k$ is separable.