Homework 6

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1)

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 3), (3, 2), (4, 5), (5, 4), (4, 6), (6, 4), (5, 6), (6, 5)\}$

$$[1] = \{1\}$$
$$[2] = \{2, 3\}$$
$$[4] = \{4, 5, 6\}$$

2)

$$A = \{1, 2, 3, 4, 5, 6\}$$

Proof: We can first prove reflexivity. We know that $\forall a \in A, a = a \cdot 1$. Therefore, we have shown that $|$ is reflexive.

To prove transitivity, we need to show that if $a|b$ and $b|c$, then $a|c$.

$$b = a \cdot n_1 \quad c = b \cdot n_2$$
$$c = a \cdot (n_1 \cdot n_2)$$
$$\Longrightarrow a|c$$

We have shown that divides is transitive. □

3)

$$xRy \Leftrightarrow x^2 + y^2 \text{ is even}$$

3.a)

Proof: We can first prove that R is reflexive. We know that $x^2 + x^2 = 2x^2$ is even because any number multiplied by 2 is even. Therefore, we have shown that R is reflexive.

To prove that R is symmetric, we need to show that if xRy, then yRx.

$$\begin{aligned} x^2 + y^2 &\text{ is even} \\ \implies y^2 + x^2 &\text{ is even} \end{aligned}$$

Addition is commutative, so we have shown that R is symmetric.

To prove that R is transitive, we need to show that if xRy and yRz, then xRz.

$$\begin{aligned} x^2 + y^2 \text{ is even and } y^2 + z^2 \text{ is even} \\ x^2 + y^2 = 2n_1 \quad y^2 + z^2 = 2n_2 \text{ for some } n_1, n_2 \in \mathbb{Z} \\ x^2 + 2y^2 + z^2 = 2n_1 + 2n_2 \\ x^2 + z^2 = 2(n_1 + n_2 - y^2) \\ \implies xRz \end{aligned}$$

□

3.b)

This relation just compares the parity of $x, y \in \mathbb{Z}$. If x, y are both even or both odd, then xRy. Otherwise, $x \neq Ry$.

Therefore, the two equivalence classes are:

$$\begin{aligned} [0] &= \{..., -4, -2, 0, 2, 4, ...\} \\ [1] &= \{..., -5, -3, -1, 1, 3, 5, ...\} \end{aligned}$$

4)

4.a)

Proof: We can construct a relation that is symmetric and transitive but not reflexive.

a

b c

Figure 1:

In this case, $R = \emptyset$. This relation is symmetric because there are no tuples to check for symmetry. It is also transitive because there are no tuples to check for transitivity. However, it is not reflexive because not every element is related to itself. □

4.b)

Proof: We can show that this is false by creating two relations R, S that are individually equivalence relations, but their union is not.

We will make sets on $A = \{a, b, c\}$.

$$R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$$
$$S = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$$

These relations are both equivalence relations.

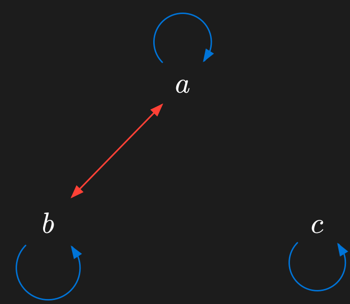

Figure 2: Relation R

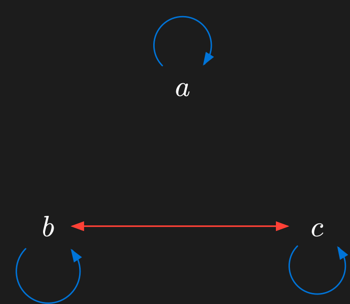

Figure 3: Relation S

When we take the union of $R \cup S$ we get the following relation:

$$R \cup S = \{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}$$

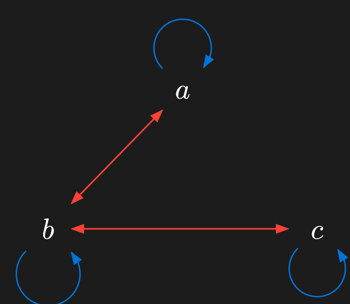

Figure 4: Relation $R \cup S$

Clearly, this relation is not an equivalence relation as aRb and bRc but $a \not R c$, which means that $R \cup S$ is not transitive. $\square$