Homework 10

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1) 6.2.6

1.a)

$$\hat{p}_1 = \frac{X}{m}$$
$$\hat{p}_2 = \frac{Y}{n}$$

We need to show that the expected value of the estimator $(\hat{p}_1 - \hat{p}_2)$ is the estimator itself $(p_1 - p_2)$.

$$\begin{aligned} E(\hat{p}_1 - \hat{p}_2) &= E(\hat{p}_1) - E(\hat{p}_2) \\ &= \frac{E(X)}{m} - \frac{E(Y)}{n} \\ &= \frac{p_1 m}{m} - \frac{p_2 n}{n} \\ &= p_1 - p_2 \end{aligned}$$

We can see that this is an unbiased estimator.

1.b)

To find the standard error, we need to find the error of the estimator.

$$\operatorname{var}(\hat{p}_1 - \hat{p}_2) = \frac{p_1(1 - p_1)}{m} + \frac{p_2(1 - p_2)}{n}$$
$$\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{p_1(1 - p_1)}{m} + \frac{p_2(1 - p_2)}{n}}$$

We know that the standard error is just the square root of the variance of the estimator, but we don't actually know the values of the values of p_1 and p_2. We can estimate them with $\hat{p}_1$ and $\hat{p}_2$.

$$S_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{m} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n}}$$

1.c)

$$\begin{aligned} \hat{p}_1 - \hat{p}_2 &= \frac{70}{100} - \frac{160}{200} \\ &= -0.1 \end{aligned}$$

$$\begin{aligned} S_{\hat{p}_1 - \hat{p}_2} &= \sqrt{\frac{0.7 \cdot 0.3}{100} + \frac{0.8 \cdot 0.2}{200}} \\ &= 0.054 \end{aligned}$$

2) Additional Problem

$$f(x \mid \theta) = \theta x^{\theta-1}, 0 < x < 1, \theta > 0$$

2.a)

$$\begin{aligned}\mathcal{L}(\theta) &= (\theta-1)\sum \ln x_i + n \ln \theta \\ &= \theta \sum \ln x_i - \sum \ln x_i + n \ln \theta\end{aligned}$$

$$\begin{aligned}\frac{\partial d\mathcal{L}}{\partial \theta} &= \sum \ln x_i + \frac{n}{\theta} \\ \implies \quad 0 &= \sum \ln x_i + \frac{n}{\theta} \\ -\frac{n}{\theta} &= \sum \ln x_i \\ \hat{\theta} &= -\frac{n}{\sum \ln x_i}\end{aligned}$$

2.b)

$$I(\theta) = -E\left[\frac{\partial^2}{\partial \theta^2} \ln f(X;\theta)\right]$$

Find the second derivative of $\mathcal{L}$.

$$\begin{aligned}\mathcal{L}(\theta) &= (\theta-1)X + \ln \theta \\ \frac{\partial \mathcal{L}(\theta)}{\partial \theta} &= X + \frac{1}{\theta} \\ \frac{\partial^2 \mathcal{L}}{\partial \theta^2} &= -\frac{1}{\theta^2}\end{aligned}$$

Find the negative expected value of $\frac{\partial^2 \mathcal{L}}{\partial \theta^2}$

$$E\left(\frac{1}{\theta^2}\right) = \frac{1}{\theta^2}$$

2.c)

$$\begin{aligned}\hat{\theta} &\dot{\sim} \mathcal{N}\left(\theta, \frac{1}{nI(\theta)}\right) \\ &\dot{\sim} \mathcal{N}\left(3, \frac{1}{100\frac{1}{9}}\right) \\ &\dot{\sim} \mathcal{N}(3, 0.09)\end{aligned}$$

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pnorm(3.1, 3, 0.3) - pnorm(2.9, 3, 0.3)
[1] 0.2611173
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