Homework 3

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1)

$$\psi(x) = N e^{-\frac{\alpha}{2}\ |x|} e^{ik_0 x}$$

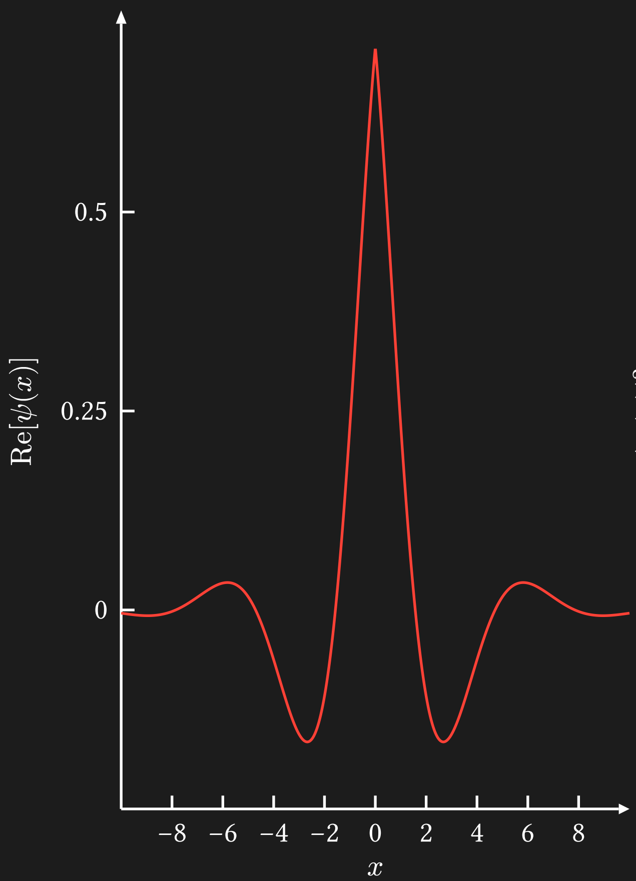
1.a)

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi(x)|^2 \ dx \\
&= \int_{-\infty}^{\infty} \psi^*(x)\psi(x) dx \\
&= N^2 \int_{-\infty}^{\infty} e^{-\alpha\ |x|} dx \\
&= 2N^2 \int_0^{\infty} e^{-\alpha x} dx \\
1 &= 2\frac{N^2}{\alpha} \\
\implies N &= \sqrt{\frac{\alpha}{2}}
\end{aligned}$$

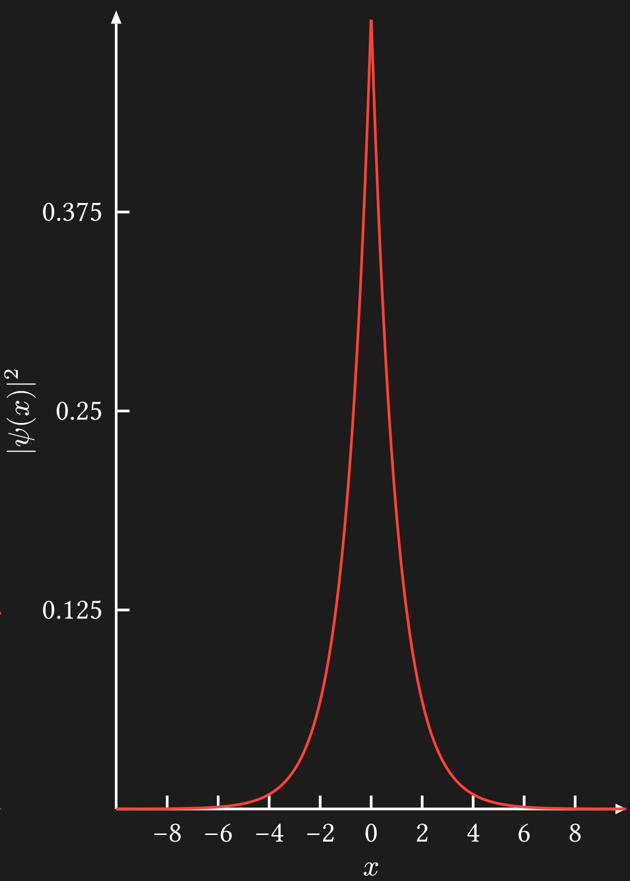
1.b)

$$
\begin{aligned}
A(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x)e^{-ikx}dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} Ne^{-\frac{\alpha}{2}\,|x|}e^{ik_0x}e^{-ikx}dx \\
&= \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}\,|x|}e^{ix(k_0-k)}dx \\
&= \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}\,|x|\,+ix(k_0-k)}dx \\
&= \frac{N}{2\pi}\left[\int_{0}^{\infty} e^{x\left(-\frac{\alpha}{2}+i(k_0-k)\right)}dx + \int_{-\infty}^{0} e^{x\left(\frac{\alpha}{2}+i(k_0-k)\right)}dx\right] \\
&= \frac{N}{2\pi}\left[\left.\frac{e^{x\left(-\frac{\alpha}{2}+i(k_0-k)\right)}}{-\frac{\alpha}{2}+i(k_0-k)}\right|_{x=0}^{\infty} + \left.\frac{e^{x\left(\frac{\alpha}{2}+i(k_0-k)\right)}}{\frac{\alpha}{2}+i(k_0-k)}\right|_{x=-\infty}^{0}\right] \\
&= \frac{N}{2\pi}\left[\frac{\cancel{e^{-\infty\frac{\alpha}{2}}e^{i\infty(k_0-k)}} - e^{0}}{-\frac{\alpha}{2}+i(k_0-k)} + \frac{e^{0} - \cancel{e^{-\infty\frac{\alpha}{2}}e^{\infty(k_0-k)}}}{\frac{\alpha}{2}+i(k_0-k)}\right] \\
&= \frac{N}{2\pi}\left[\frac{-1}{-\frac{\alpha}{2}+i(k_0-k)} + \frac{1}{\frac{\alpha}{2}+i(k_0-k)}\right] \\
&= \frac{N}{2\pi}\left[\frac{-1}{-\frac{\alpha}{2}+i(k_0-k)}\cdot\frac{\frac{\alpha}{2}+i(k_0-k)}{\frac{\alpha}{2}+i(k_0-k)} + \frac{1}{\frac{\alpha}{2}+i(k_0-k)}\cdot\frac{-\frac{\alpha}{2}+i(k_0-k)}{-\frac{\alpha}{2}+i(k_0-k)}\right] \\
&= \frac{N}{2\pi}\left[\frac{-2\left(\frac{\alpha}{2}\right)}{\frac{\alpha^2}{4}-(k_0-k)^2}\right] \\
&= \frac{N}{2\pi}\frac{-\alpha}{-\frac{\alpha^2}{4}-(k_0-k)^2} \\
&= \frac{N}{2\pi}\frac{4\alpha}{\alpha^2+4(k_0-k)^2}
\end{aligned}
$$

1.c)

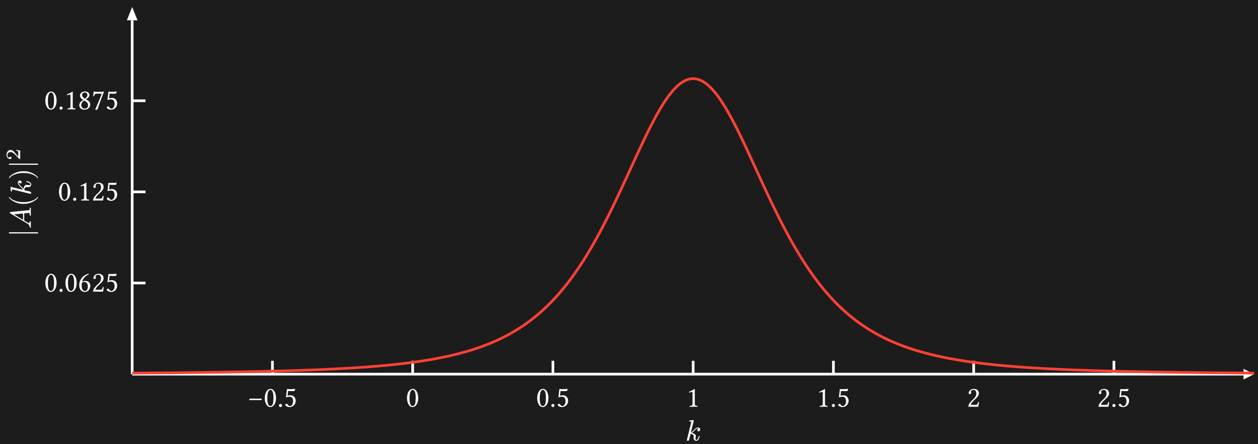


Graph 1: $\mathrm{Re}[\psi(x)]$ for $\alpha = 1$



Graph 2: $|\psi(x)|^2$ for $\alpha = 1$

$$
\begin{aligned}
|A(k)|^2 &= \left| \frac{N}{2\pi} \frac{4\alpha}{\alpha^2 + 4(k_0 - k)^2} \right|^2 \\
&= \frac{N^2 16\alpha^2}{4\pi^2 \left(\alpha^2 + 4(k_0 - k)^2\right)^2}
\end{aligned}
$$



Graph 3: $|A(k)|^2$ for $\alpha = 1, k_0 = 1$

1.d)

We can plug $p = \hbar(0.5k_0)$ into $|A(k)|^2$ to find the probability of finding the particle with momentum p (using the relation $p = \hbar k$).

$$\begin{aligned} \hbar(0.5k_0) &= \hbar k \\ 0.5k_0 &= k \\ \Longrightarrow |A(0.5k_0)|^2 &= \frac{N^2 16\alpha^2}{4\pi^2\left(\alpha^2 + 4(k_0 - 0.5k_0)^2\right)^2} \end{aligned}$$

We can repeat the same process for $p = \hbar(1.1k_0)$.

$$\begin{aligned} \hbar(1.1k_0) &= \hbar k \\ 1.1k_0 &= k \\ \Longrightarrow |A(1.1k_0)|^2 &= \frac{N^2 16\alpha^2}{4\pi^2\left(\alpha^2 + 4(k_0 - 1.1k_0)^2\right)^2} \end{aligned}$$

By inspection, we can see that when $k = 1.1k_0$, the denominator will be smaller than when $k = 0.5k_0$. Therefore, the probability of finding the particle with momentum $p = \hbar(1.1k_0)$ will be higher than the probability of finding the particle with momentum $p = \hbar(0.5k_0)$ (unless $k_0 = 0$, when they are equal).

2)

$$\psi(x) = \begin{cases} 0 & \text{for } x < -3a \\ C & \text{for } -3a \leq x \leq a \\ 0 & \text{for } x > a \end{cases}$$

2.a)

$$\begin{aligned} 1 &= C^2 \int_{-3a}^{a} dx \\ &= 4aC^2 \\ \Longrightarrow C &= \frac{1}{2\sqrt{a}} \end{aligned}$$

2.b)

$$\begin{aligned} P(0 \leq X \leq a) &= \int_0^a \frac{1}{16a^2} dx \\ &= \frac{1}{16a} \end{aligned}$$

2.c)

In order to find the plausible values for momenta, we need to take the Fourier transform of $\psi(x)$.

$$
\begin{aligned}
A(k) &= \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx \\
&= C \int_{-3a}^{a} e^{-ikx} dx \\
&= C \left[\frac{ie^{-ikx}}{k} \bigg|_{x=-3a}^{a} \right] \\
&= \frac{C}{k} \left(ie^{-ika} - ie^{3ika} \right)
\end{aligned}
$$

We can set $|A(k)|^2 = 0$ to find impossible values for the momentum.

$$
\begin{aligned}
0 &= |A(k)|^2 \\
0 &= \left| \frac{C}{k} \left(ie^{-ika} - ie^{3ika} \right) \right|^2 \\
0 &= \frac{C^2}{k^2} \left| ie^{-ika} - ie^{3ika} \right|^2 \\
0 &= \left(\frac{C^2}{k^2} \right) (\sin(3ka) + \sin(ka) + i(\cos(ka) - \cos(3ka)))(\sin(3ka) + \sin(ka) - i(\cos(ka) - \cos(3ka))) \\
0 &= \left(\frac{C^2}{k^2} \right) (\sin^2(3ka) + \cos^2(3ka) + \sin^2(ka) + \cos^2(ka) - 2\cos(3ka)\cos(ka) + 2\sin(3ka)\sin(ka)) \\
0 &= \left(\frac{C^2}{k^2} \right) (2 - 2\cos(3ka)\cos(ka) + 2\sin(3ka)\sin(ka)) \\
1 &= \cos(3ka)\cos(ka) - \sin(3ka)\sin(ka) \\
1 &= \cos(4ka) \\
\implies 4ka &= 2n\pi \\
k &= \frac{n\pi}{2a} \quad n \in \mathbb{Z}
\end{aligned}
$$

We know that $p = \hbar k$, so we can write the values for which $k = 0$ in terms of p.

$$
\begin{aligned}
k &= \frac{n\pi}{2a} \\
\frac{p}{\hbar} &= \frac{n\pi}{2a} \\
p &= \frac{n\pi\hbar}{2a} \quad n \in \mathbb{Z}
\end{aligned}
$$

3)

$$
A(k) = \frac{C\alpha}{\sqrt{\pi}} e^{-\alpha^2 k^2}
$$

$$
\begin{aligned}
\psi(x) &= \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk \\
&= \frac{C\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 k^2} e^{ikx}\, dk \\
\text{gaussian integral : } a = \alpha^2, b = ix \implies \quad &= \frac{C\cancel{\alpha}}{\cancel{\sqrt{\pi}}} \frac{\cancel{\sqrt{\pi}}}{\cancel{\alpha}} e^{-\frac{x^2}{4\alpha^2}} \\
&= Ce^{-\frac{x^2}{4\alpha^2}}
\end{aligned}
$$

By inspection of $\psi(x)$, we can see that α corresponds to the uncertainty/standard deviation (ε) of the Gaussian. Therefore $\Delta x = \alpha$.

We can do a similar analysis for $A(k)$ to find the uncertainty in k.

$$
\begin{aligned}
\alpha^2 &= \frac{1}{4\varepsilon^2} \\
\implies \varepsilon = \Delta k &= \frac{1}{2\alpha}
\end{aligned}
$$

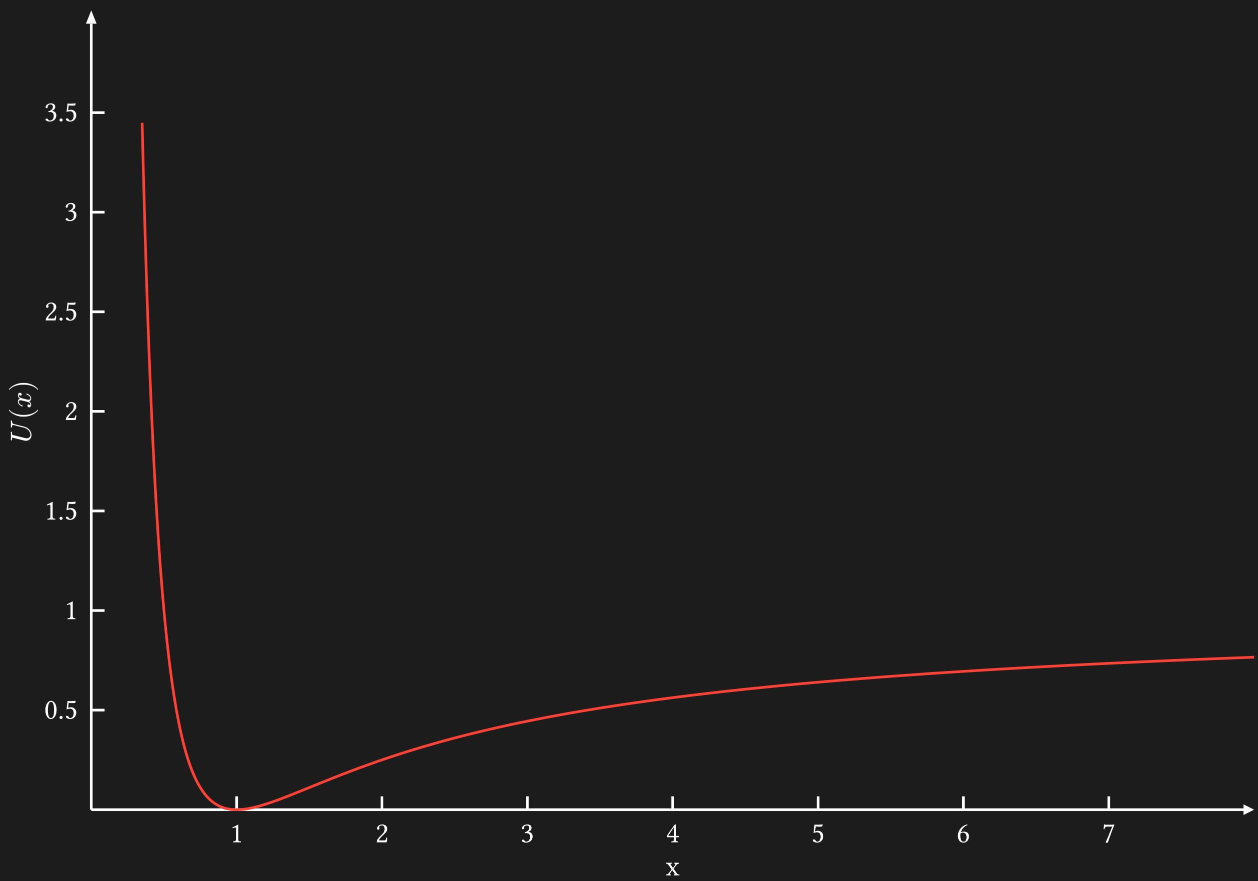
$$
\begin{aligned}
\Delta x \Delta k &\leq \frac{1}{2} \\
\frac{\cancel{\alpha}}{2\cancel{\alpha}} &\leq \frac{1}{2} \\
\frac{1}{2} &= \frac{1}{2}
\end{aligned}
$$

Because this is a Gaussian, $\Delta x \Delta k$ attains the minimum value of $\frac{1}{2}$.

4)

$$
U(x) = \frac{1}{x^2} - \frac{2}{x} + 1
$$

4.a)



Graph 4: Potential Energy vs. Position of $U(x) = \frac{1}{x^2} - \frac{2}{x} + 1$

$$\lim_{x \to 0} U(x) = \infty$$

$$\lim_{x \to \infty} U(x) = \frac{1}{\infty} - \frac{2}{\infty} + 1 = 1$$

4.b)

If the particle has 0.5 Joules of energy, it will have 2 turning points and is bound. We can find those turning points by setting $U(x) = 0.5$ and assuming that $x \neq 0$.

$$U(x) = 0.5$$

$$\frac{1}{x^2} - \frac{2}{x} + 1 = 0.5$$

$$\frac{1}{x^2} - \frac{2}{x} = -0.5$$

$$\frac{x - 2x^2}{x^3} = -0.5$$

$$x - 2x^2 = -0.5x^3$$

$$0.5x^2 - 2x + 1 = 0$$

$$\implies x = 2 \pm \sqrt{2}$$

4.c)

When a particle has 2 Joules of energy, it is not bound as there is only one time when $U(x) = 2$ for $x > 0$. The limit of $U(x)$ as $x \to \infty$ is 1, so the particle will never get "stopped" by the potential energy (or turned around).

$$\begin{aligned} f(x) &= x^2 \\ f(1) &= 1^2 \\ &= 1 \end{aligned}$$