Problem Set 6

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1)

$$n = 3 \quad L = 10 \text{ nm}$$

We know that when $x < 0$ and $x > L$, $\Psi(x,t) = 0$ due to the infinite potential well.

We will focus on the middle region where $0 < x < L$.

We know that the energy of the particle is given by:

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2 m_e L^2}$$

$$E_3 = \frac{3^2 \pi^2 \hbar^2}{2 m_e L^2}$$

$$= 5.422 \times 10^{-29} \text{ Joules}$$

The total wave function is given by:

$$\Psi(x,t) = \psi(x)\phi(t)$$

From class, we know that $\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ as well as $\phi(t) = e^{-\frac{iEt}{\hbar}}$. We can combine these to get our total wave function:

$$\Psi(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{iEt}{\hbar}}$$

We can plug in the energy and other constants:

$$\Psi(x,t) = \sqrt{\frac{2}{10^{-8}}} \sin\left(\frac{3\pi x}{10^{-8}}\right) e^{-i(5.14 \times 10^{13})t}$$

$$= 4472 \sin(6.28 \times 10^{7} x) e^{-i(5.14 \times 10^{13})t}$$

2)

$$n = 4 \quad L = 5 \text{ nm}$$

We can find the energy of the electron in both the ground state $(n = 1)$, and its current excited state $(n = 4)$.

$$E_1 = \frac{1^2 \pi^2 \hbar^2}{2 m_e L^2}$$

$$E_4 = \frac{4^2 \pi^2 \hbar^2}{2 m_e L^2}$$

$$
\begin{aligned}
E_{\text{photon}} &= E_4 - E_1 \\
&= \frac{\pi^2 \hbar^2}{2 m_e L^2}(16 - 1)
\end{aligned}
$$

We can find the photon wavelength using the energy of the photon:

$$
\begin{aligned}
E_{\text{photon}} &= \frac{hc}{\lambda} \\
\lambda &= \frac{hc}{E_{\text{photon}}} \\
&= \frac{hc}{15\pi^2\hbar^2} 2 m_e L^2 \\
&= 5.5 \times 10^{-6} \quad \text{m}
\end{aligned}
$$

3)

$$
\begin{aligned}
P\left(\frac{L}{4} < X < \frac{3L}{4}\right) &= \int_{\frac{L}{4}}^{3\frac{L}{4}} |\psi(x)|^2 \, \mathrm{d}x \\
&= \int_{\frac{L}{4}}^{\frac{3L}{4}} \left(\frac{2}{L}\right) \sin^2\left(\frac{\pi x}{L}\right) \mathrm{d}x \\
&= \left(\frac{2}{L}\right) \int_{\frac{L}{4}}^{\frac{3L}{4}} \sin^2\left(\frac{\pi x}{L}\right) \mathrm{d}x \\
&= \left(\frac{1}{L}\right) \int_{\frac{L}{4}}^{\frac{3L}{4}} \left(1 - \cos\left(\frac{2\pi x}{L}\right)\right) \mathrm{d}x \\
&= \frac{1}{L}\left[x - \frac{\sin\left(\frac{2\pi x}{L}\right)}{2\pi}\right]\Bigg|_{x=\frac{L}{4}}^{\frac{3L}{4}} \\
&= \frac{1}{L}\left[\frac{3L}{4} - \frac{L}{4} - \frac{L\sin\left(\frac{3}{2}\pi\right) + L\sin\left(\frac{\pi}{2}\right)}{2\pi}\right] \\
&= \frac{1}{L}\left[\frac{L}{2} + \frac{L}{\pi}\right] \\
&= \frac{1}{2} + \frac{1}{\pi}
\end{aligned}
$$

4)

$$
U(x) = \begin{cases} 0 \text{ if } |x| < \frac{a}{2} \\ \infty \text{ if } |x| > \frac{a}{2} \end{cases}
$$

We know that the following SE must be satisfied:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = -k^2\psi(x)$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$

The general solution to this differential equation is:

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

We can write the two boundary conditions for $x = \pm\frac{L}{2}$:

$$0 = \psi\left(x = -\frac{L}{2}\right) = A\sin\left(-k\frac{L}{2}\right) + B\cos\left(-k\frac{L}{2}\right)$$
$$0 = \psi\left(x = \frac{L}{2}\right) = A\sin\left(k\frac{L}{2}\right) + B\cos\left(k\frac{L}{2}\right)$$

Cosine is an even function, which means that $\cos(x) = \cos(-x)$.

We know that when $n = 1$, there should be zero nodes in $\psi(x)$. Additionally, this function should be symmetric around $x = 0$ due to that fact that is has 0 nodes. In fact, when n takes an odd value, there will be an even number of nodes, which implies that $\psi(x)$ must be symmetric. Therefore, when n is odd, $\psi(x)$ must be entirely composed of cos.

When $n = 2$, there should be one node. Because there is one node, $\psi(x)$ cannot be symmetric. Following the same reasoning as above (although reversed), when n is even, $\psi(x)$ must be entirely composed of sin.

We can first solve for when n is odd. Let's first find what k is.

$$\psi_{n=\text{ odd}}\left(\frac{L}{2}\right) = A\cos\left(k\frac{L}{2}\right) = 0$$
$$\Longrightarrow k = \frac{n\pi}{L}$$

This is because the argument to cos becomes $\frac{n\pi}{2}$, and $\cos\left(\frac{n\pi}{2}\right) = 0\ \forall n \in [1]$, where [1] denotes the set of all even numbers.

We can now find A by normalizing.

$$
\begin{aligned}
1 &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \left|A\cos\left(\frac{n\pi x}{L}\right)\right|^2 \mathrm{d}x \\
&= \int_{-\frac{L}{2}}^{\frac{L}{2}} A^2\cos^2\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\
&= \frac{A^2}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \left[1+\cos\left(\frac{2n\pi x}{L}\right)\right] \mathrm{d}x \\
&= \frac{A^2}{2}\left[x+\frac{L\sin\left(\frac{2n\pi x}{L}\right)}{2n\pi x}\right]\Bigg|_{x=-\frac{L}{2}}^{\frac{L}{2}} \\
&= \frac{A^2}{2}[L+0] \\
\implies A &= \sqrt{\frac{2}{L}}
\end{aligned}
$$

Finally, we can use the values of A and k in our equation for $\psi_{n=\text{ odd}}(x)$:

$$\psi_{n=\text{ odd}}(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x}{L}\right)$$

For when n is even, the function will look identical except the cos will be replaced with sin. In the integral when we found A, the cos term cancelled out. When n is even, $\cos(n\pi x) - \cos(-n\pi x)$ will be 0, similar to how the sines cancelled out in the odd example.

Therefore, the equation for the $\psi(x)$ when n is even is:

$$\psi_{x=\text{ even}}(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

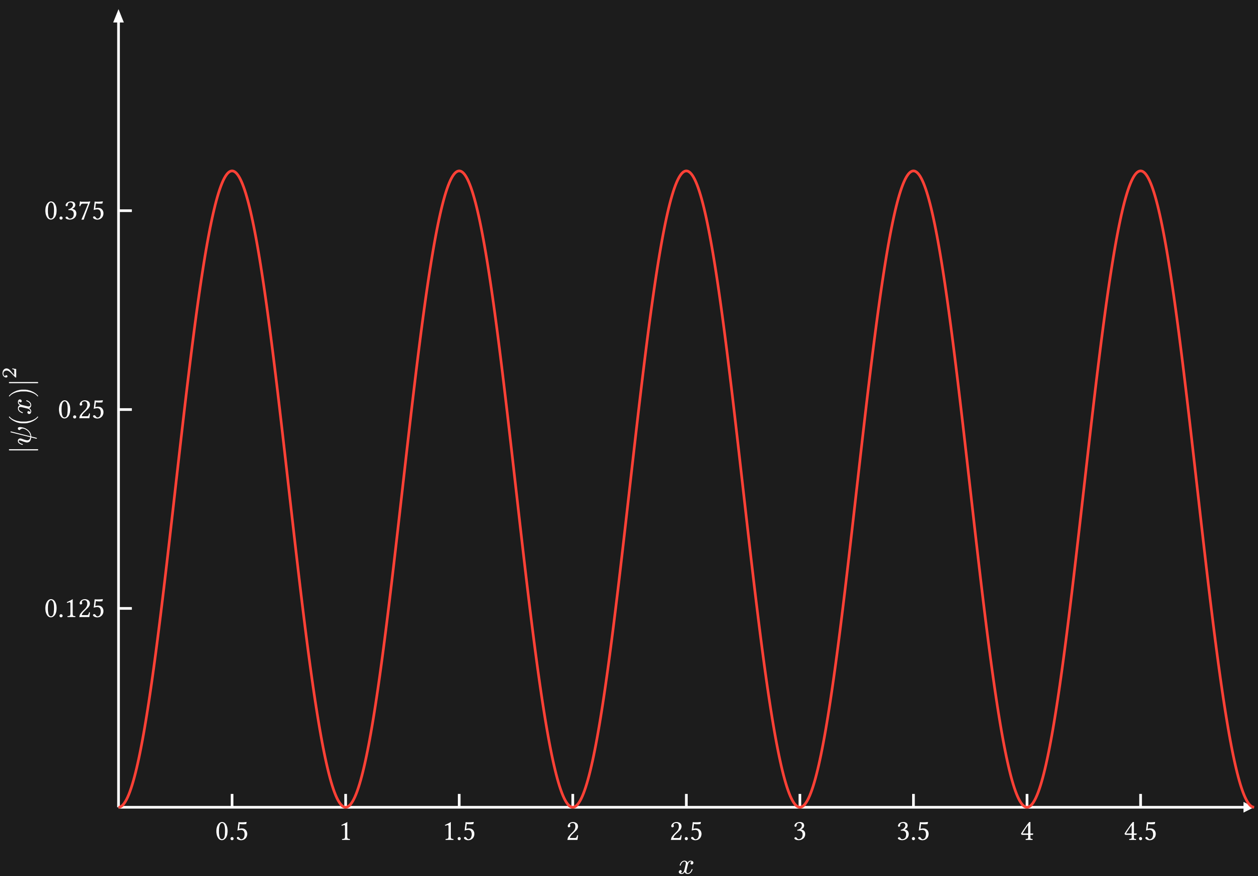
5)

$$n = 5$$

To find where the particle is most likely to be, we can solve for when $|\psi(x)|^2$ is largest.

$$
\begin{aligned}
|\psi(x)|^2 &= \left|\sqrt{\frac{2}{L}}\sin\left(\frac{5\pi x}{L}\right)\right|^2 \\
&= \frac{2}{L}\sin^2\left(\frac{5\pi x}{L}\right)
\end{aligned}
$$

We can plot this function to better understand what the probability density looks like for $0 < x < L$.



Graph 1: $|\psi(x)|^2$ for $L = 5$

We can see that the probability is at a maximum when $x = n + \frac{1}{2}$ for $n \in \mathbb{Z}$ in the case that $L = 5$.

More generally, we want $\frac{5\pi x}{L} = \pi n + \frac{\pi}{2}$ for $n \in \mathbb{Z}$. This simplifies to $x = \frac{\pi n L}{5\pi} + \frac{\pi L}{10\pi} = \frac{2nL+L}{10}$. This formula gives us the position of the maximum probability density for any n.

6)

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

We need to show that $\int \psi_n(x)\psi_m^*(x) = 0$ if $n \neq m$.

First we can find the complex conjugate of $\psi_n(x)$:

$$\psi_n^*(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

This is the same as $\psi_n(x)$ because there is no imaginary component.

$$
\begin{aligned}
\int_0^L \psi_n(x)\psi_m^*(x)\,\mathrm{d}x &= \int_0^L \left(\frac{2}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x \\
&= \frac{1}{L}\int_0^L \left[\cos\left((n-m)\frac{\pi x}{L}\right) - \cos\left((n+m)\frac{\pi x}{L}\right)\right]\mathrm{d}x \\
&= \frac{1}{L}\left[\frac{L}{(n-m)\pi}\sin\left((n-m)\frac{\pi x}{L}\right) - \frac{L}{(n+m)\pi}\sin\left((n+m)\frac{\pi x}{L}\right)\right]\Bigg|_{x=0}^{L} \\
&= \frac{1}{(n-m)\pi}\sin((n-m)\pi) - \frac{1}{(n+m)\pi}\sin((n+m)\pi)
\end{aligned}
$$

This function is 0 when $n \neq m$. This is because $\sin((n-m)\pi) = 0$ and $\sin((n+m)\pi) = 0$ when $n \neq m$.