# Homework 9

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## 1) 6.3.4

### 1.a)

$$f(x) = \frac{x}{\theta^2} e^{\frac{x^2}{2\theta^2}}, \quad x \geq 0,$$

$$\mu = \theta \sqrt{\frac{\pi}{2}}$$

$$\sigma^2 = \theta^2 \frac{4 - \pi}{2}$$

Methods of moments. We have a single parameter $(\theta)$ so we only need to find the first moment.

$$\widehat{\mu_1} = \frac{1}{n} \sum_{i=1}^{n} x_i = \overline{x}$$

$$\mu_1 = \mu = \theta \sqrt{\frac{\pi}{2}}$$

We can set them equal and solve for $\theta$:

$$\overline{x} = \theta \sqrt{\frac{\pi}{2}}$$

$$\overline{x} \sqrt{\frac{2}{\pi}} = \hat{\theta}$$

## 2) 6.3.6

### 2.a)

$$X_i \overset{\text{iid}}{\sim} \text{Poisson}(\lambda)$$

$$p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Maximum likelihood: we will find the log-likelihood.

$$\begin{aligned} \ln(p_X(x)) &= \ln\left(\frac{\lambda^x e^{-\lambda}}{x!}\right) \\ &= \ln(\lambda^x) + \ln\left(e^{-\lambda}\right) - \ln(x!) \end{aligned}$$

$$
\begin{aligned}
\mathcal{L}(\theta) &= n\ln\left(e^{-\theta}\right) + \sum_{i=1}^{n}\left[\ln\left(\theta^{x_i}\right) - \ln(x_i!)\right] \\
&= -n\theta + \sum_{i=1}^{n}\left[x_i\ln(\theta) - \ln(x_i!)\right] \\
&= -n\theta + \ln(\theta)\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\ln(x_i!)
\end{aligned}
$$

We can set the derivative of the log-likelihood equal to zero to find the maximum.

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{L}(\theta)}{\mathrm{d}\theta} &= -n + \frac{1}{\theta}\sum_{i=1}^{n}x_i \\
\implies \quad 0 &= -n + \frac{1}{\theta}\sum_{i=1}^{n}x_i \\
n\theta &= \sum_{i=1}^{n}x_i \\
\hat{\theta} &= \frac{1}{n}\sum_{i=1}^{n}x_i \\
&= \overline{x}
\end{aligned}
$$

### 2.b)

$$
\begin{aligned}
\overline{x} &= \frac{12 + 2\cdot 11 + 3\cdot 14 + 4\cdot 9}{50} \\
&= 2.24
\end{aligned}
$$

Therefore, $\hat{\lambda} = 2.24$.

### 2.c)

Model-based variance:

$$
\begin{aligned}
\mathrm{var}(X) &= \hat{\lambda} \\
&= 2.24
\end{aligned}
$$

Sample variance:

$$
\begin{aligned}
S_x &= \frac{1}{n-1}\sum_{i=1}^{n}x_i \\
&= 1.533
\end{aligned}
$$

If the data truly assumes a Poisson distribution, then we should use the model-based variance as it more accurately represents the population.

## 3) 6.3.7

### 3.a)

$$
X \sim \mathrm{nbinom}(r = 5, p = ?\,)
$$

$$P(X = x) = \binom{x+4}{x}(1-p)^x p^5$$

$$\mathcal{L}(p) = 5\ln(p) + \ln\binom{X+4}{X} + \ln(1-p)X$$

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}p} = \frac{5}{p} - \frac{X}{1-p}$$

$$\implies \quad 0 = \frac{5}{p} - \frac{X}{1-p}$$

$$\hat{p} = \frac{5}{5+X}$$

**3.b)**

**3.c)**

$$\hat{p} = \frac{5}{5+47}$$

$$= 0.096$$