Homework 6

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1) 4.3.7

1.a)

$$\begin{aligned} E(Y|X=1) &= 0.66 \cdot 1 + 0.34 \cdot 2 \\ &= 1.34 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(Y|X=1) &= (1-1.34)^2 \cdot 0.66 + (2-1.34)^2 \cdot 0.34 \\ &= 0.2244 \end{aligned}$$

1.b)

		y = 1	y = 2
x	1	0.132	0.068
	2	0.24	0.06
	3	0.33	0.17

1.c)

$$\begin{aligned} P(Y=1) &= 0.132 + 0.24 + 0.33 \\ &= 0.702 \end{aligned}$$

1.d)

$$P(X=1|Y=1) = \frac{0.132}{0.702} = 0.188$$

2) 4.3.11

$$f(x,y) = \begin{cases} x+y \text{ for } 0<x<1 \text{ and } 0<y<1 \\ 0 \text{ otherwise} \end{cases}$$

2.a)

$$\begin{aligned} f_X(x) &= \int_0^1 (x+y)\,\mathrm{d}y \\ &= \frac{y^2}{2} + xy \bigg|_{y=0}^{1} \\ &= \frac{1}{2} + x \end{aligned}$$

$$
\begin{aligned}
f_{Y|X=x}(x) &= \frac{f(x,y)}{f_X(x)} \\
&= \frac{x+y}{x+\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
P(0.3 < Y < 0.5|X = x) &= \int_{0.3}^{0.5} \frac{x+y}{x+\frac{1}{2}}\,\mathrm{d}y \\
&= \frac{1}{x+\frac{1}{2}} \int_{0.3}^{0.5} (x+y)\,\mathrm{d}y \\
&= \frac{1}{x+\frac{1}{2}} \left[xy + \frac{y^2}{2} \Big|_{y=0.3}^{0.5} \right] \\
&= \frac{1}{x+\frac{1}{2}} \left[0.5x + \frac{1}{8} - 0.3x - 0.045 \right] \\
&= \frac{1}{x+\frac{1}{2}} \left[0.2x + \frac{2}{25} \right] \\
&= \frac{0.2x + \frac{2}{25}}{x+\frac{1}{2}}
\end{aligned}
$$

2.b)

We can integrate over all of x because we have already found the conditional probability for Y between $0.3 < Y < 0.5$.

$$
\begin{aligned}
P(0.3 < Y < 0.5) &= \int_0^1 \frac{0.2x + \frac{2}{25}}{x+\frac{1}{2}} \left(x + \frac{1}{2} \right) \mathrm{d}x \\
&= \int_0^1 \left(0.2x + \frac{2}{25} \right) \mathrm{d}x \\
&= \frac{x^2}{10} + \frac{2}{25}x \Big|_{x=0}^{1} \\
&= 0.18
\end{aligned}
$$

3) 4.3.15

$$
f(x,y) = \frac{e^{-\frac{x}{y}} e^{-y}}{y} \text{ for } x > 0 \text{ and } y > 0
$$

$$
f_{X|Y=y}(x) = \frac{1}{y} e^{-\frac{x}{y}} \text{ for } x > 0
$$

This equation is not separable, so therefore X and Y are not independent.

4) 4.4.4

4.a)

Let T = total waiting time.

$$T = X_1 + X_2 + X_3 + X_4 + X_5 + Y_1 + Y_2 + Y_3$$

4.b)

$$\begin{aligned} E(T) &= E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) + E(Y_1) + E(Y_2) + E(Y_3) \\ &= 5 \cdot 3 + 3 \cdot 6 \\ &= 33 \end{aligned}$$

Because the events are independent (or at least we are modeling them as such), we can add the variances.

$$\begin{aligned} \mathrm{Var}(T) &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3) + \mathrm{Var}(X_4) + \mathrm{Var}(X_5) + \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + \mathrm{Var}(Y_3) \\ &= 5 \cdot 2 + 3 \cdot 4 \\ &= 22 \end{aligned}$$

5) 4.4.8

$$f(x, y) = \begin{cases} 24xy \text{ for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \text{ and } x + y \leq 1 \\ 0 \text{ otherwise} \end{cases}$$

$$\begin{aligned} E(X) &= \int_0^1 \int_0^{1-x} 24x^2 y \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^1 12x^2 (1-x)^2 \, \mathrm{d}x \\ &= \int_0^1 (12x^2 - 24x^3 + 12x^4) \, \mathrm{d}x \\ &= 4x^3 - 6x^4 + \frac{12}{5} x^5 \Big|_{x=0}^1 \\ &= \frac{2}{5} \end{aligned}$$

$E(X) = E(Y)$ because the function is symmetric.

$$\begin{aligned} E(XY) &= \int_0^1 \int_0^{1-x} xy 24xy \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^1 \int_0^{1-x} 24x^2 y^2 \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^1 8x^2 (1-x)^3 \, \mathrm{d}x \\ &= \frac{2}{15} \end{aligned}$$

$$
\begin{aligned}
\operatorname{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\
&= \frac{2}{15} - \frac{4}{25} \\
&= -\frac{2}{75}
\end{aligned}
$$