Sparse Computing, Databases, and Semirings

Birds of a feather flock together

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Outline

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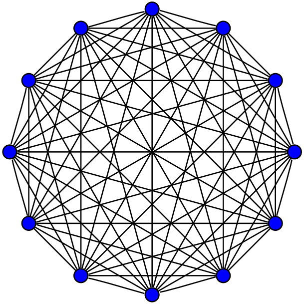
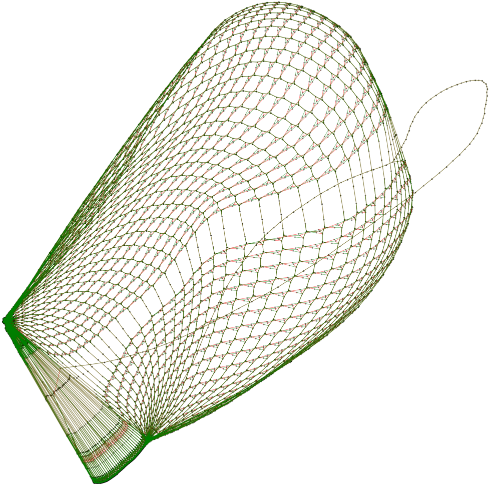
Examples of sparsity



Complete graph on 12 vertices, not sparse!

https://commons.wikimedia.org/wiki/File:11-simplex_graph.svg



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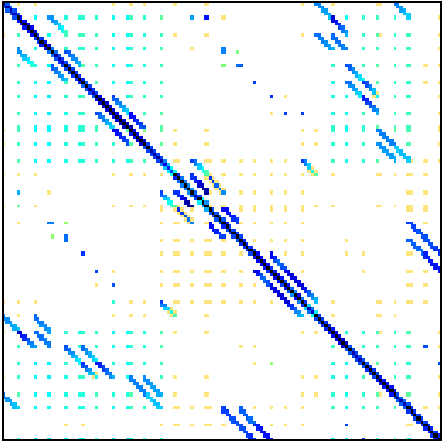

`https://sparse.tamu.edu/Norris/torso1`

Examples of sparsity

name: char[32]	**salary**: uint32
bob	80000
alice	85000

	0	1	...	80000	...	85000	...	4294967296
a	0	0	0	0	0	0	0	0
⋮								
alice	0	0	0	0	0	1	0	0
⋮								
bob	0	0	0	1	0	0	0	0
⋮								
z...z	0	0	0	0	0	0	0	0

Navigating sparsity: graph traversal

Graph: $R(\mathsf{src}, \mathsf{dst})$ or adjacency matrix A. Find all pairs of nodes reachable via paths of length 3.

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$$\text{relabel: } R_1(\text{src}, X_1), R_2(X_1, X_2), R_3(X_2, \text{dst}),$$

$$Q = \prod_{\text{src,dst}} R_1 \bowtie R_2 \bowtie R_3.$$

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Using matrix A:

$$(A^2)_{ij} = \sum_k A_{ik} A_{kj} = \#\text{ of ways } i \text{ can reach } j \text{ in 2 hops}$$

$$Q_{i\ell} = (A^3)_{ij} = \sum_j (A^2)_{ij} A_{j\ell} = \sum_{k,j} A_{ik} A_{kj} A_{j\ell}.$$

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Find min cost of all paths of length (exactly) 3 between pairs of nodes.

Edge weights: $R(\mathsf{src}, \mathsf{dst}, W)$, or $A_{ij} \in \mathbb{R}$.

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With A,

$$Q_{ij} = \min_j \left(\min_k A_{ik} + A_{kj} \right) + A_{j\ell} = \min_{j,k} A_{ik} + A_{kj} + A_{j\ell}.$$

$+$ *distributes* over min.

Suspicious...

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One algebra to rule them all

Definition

A *semiring* is a tuple $(S, +, \cdot, 0, 1)$ where

- 0 is the identity for $+$,
- 1 is the identity for $\cdot$,
- 0 is an annihilator for $\cdot$,
- $+$ and $\cdot$ are associative, and $+$ is commutative,
- $a \cdot (b + c) = a \cdot b + a \cdot c$,
- $(b + c) \cdot a = b \cdot a + c \cdot a$.

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Examples

- $(\mathbb{B}, \vee, \wedge, \bot, \top)$,
- $(\mathbb{N}, +, \cdot, 0, 1)$,
- $(\mathbb{R}, \min, +, \infty, 0)$.

One algebra to rule them all

Definition

A *schema* Γ is a product of sets $A_1 \times \cdots \times A_n$. Each A_i is an *attribute*.

Definition

An S-*relation* is a function $R : \Gamma \to S$ with *finite support*, where S is a semiring. The support of R is the set $\{\mathbf{a} \in \Gamma \mid R(\mathbf{a}) \neq 0\}$.

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Examples

- $\mathbb{B}$-relations: database tables under set semantics,
- $\mathbb{N}$-relations: ... bag semantics,
- Trop$^+$-relations: shortest paths,
- a tensor $T \in \mathbb{R}^{d_1 \times \cdots \times d_n} \cong T : [d_1] \times \cdots \times [d_n] \to \mathbb{R}$.

S-relations were introduced by [GKT07] for provenance tracking.

S-relational algebra

Equijoin. $\Gamma(R_1) = \mathbf{A} \times \mathbf{A}_1$, $\Gamma(R_2) = \mathbf{A} \times \mathbf{A}_2$.

$$[\![R_1 \bowtie R_2]\!](\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2) = R_1(\mathbf{a}, \mathbf{a}_1) \otimes R_2(\mathbf{a}, \mathbf{a}_2).$$

Union. Just entry-wise addition.

Projection. $\Gamma(R) = \mathbf{A} \times \mathbf{A}'$.

$$[\![\prod_{\mathbf{A}'} R]\!](\mathbf{a}') = \bigoplus_{\mathbf{a} \in \mathbf{A}} R(\mathbf{a}, \mathbf{a}').$$

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$$[\![\prod_{\mathbf{A}'} R]\!](\mathbf{a}') = \bigoplus_{\mathbf{a} \in \mathbf{A}} R(\mathbf{a}, \mathbf{a}').$$

This is well-defined because R has finite support!

Brief note on datalog over semirings

What about datalog over semirings? Relevant for iterative computations: shortest paths, BFS, betweenness centrality, PageRank, eigensolvers, ...

We need to figure out:

- Some notion of (partial) order?
- Monotonicity with respect to this order?
- Infinite Herbrand universe, how to prove termination?
- Generalization of subtraction? if we want semi-naïve.

Brief note on datalog over semirings

See the datalog$^\circ$ paper by Mahmoud and others [AKNP$^+$22].

Basic takeaways:

- Separate partial order from semiring natural order: $a \sqsubseteq b \not\equiv \exists c.\ a \oplus c = b$,
- Be careful about $\perp$ in poset vs 0 in semiring,
- Semiring needs certain algebraic properties (stability) for convergence,
- Semi-naïve needs $+$ to be idempotent, poset must form complete distributive lattice.

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Approaches to optimization

Traditionally, (DB) queries and tensor programs are optimized differently.

Query optimization. High-level, fixed set of operators e.g. $\sigma, \bowtie, \prod$. E-graphs good! Opt's include: physical operator choice (nested loop, hash, merge); join reordering; aggregate push-down. Cost estimation is sophisticated. See [Gra95].

Tensor optimization. Lower-level, nested loops over variables. Variables are harder to handle, not impossible but different techniques like polyhedral analysis show up.

Factorization - thinking in terms of variables

x	y	z
1	2	3
1	3	4
1	3	5
2	1	1

Table: Listing representation of $R(x, y, z)$.

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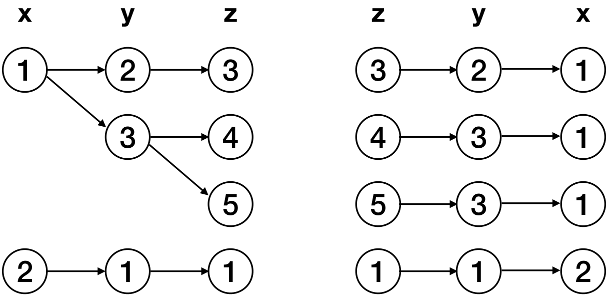

Factorization - thinking in terms of variables

As sets,

$$\begin{aligned} R = &\{1\} \times (\{2\} \times \ \{3\} \\ &\quad \cup \ \{3\} \times (\{4\} \\ &\qquad \cup \ \{5\})) \\ \cup\, &\{2\} \times \ \{1\} \times \ \{1\}. \end{aligned}$$

Distributivity again! $\times$ distributes over $\cup$.

Factorization

Factorization is not new in sparse computing.

- adjacency list
- compressed sparse row/column (CSR/CSC) format
- ELL format (row $\rightarrow$ fixed $\#$ of nonzero cols) - bounded degree graph

Factorizing saves time, not just space

Assume $M = A = B$, but this query is valid in general.

$$C = \sum_{i,j,k} M(i,j) \cdot A(j,k) \cdot B(i,k).$$

Typical DB: *binary join plan*, e.g.

$$C = \text{COUNT}\ (M \bowtie A) \bowtie B.$$

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All binary plans take worst-case $\Omega(N^2)$ time, where N is the $\#$ of edges.

Exercise: prove that $C = O(N^{1.5})$.

Factorizing saves time, not just space

What if we tried to write dense tensor algebra code?

```
C = 0
for i in range(I):
    for j in range(J):
        for k in range(K):
            C += M[i,j] * A[j,k] * B[i,k]
```

Factorizing saves time, not just space

Now let's try to make it sparse.

```
C = 0
for i in M.i ∩ B.i:
    for j in M.j ∩ A.j:
        for k in B.k ∩ A.k:
            C += M[i,j] * A[j,k] * B[i,k]
```

Intuition: the loop body expression is only nonzero when all inputs have values for the given i, j, k.

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Important: if $i_0 \notin M.i \cap B.i$, then don't iterate over any tuples (i_0, j, k)! $M.j$ and $B.k$ depend on the current value of i.

Factorizing saves time, not just space

```
# M: i -> j -> value
# A: j -> k -> value
# B: i -> k -> value
C = 0
for i in M:
    M_j = M[i]; B_k = B[i]?
    for j in M_j:
        A_k = A[j]?
        for k in B_k:
            A_k[k]?
            C += M[i][j] * A[j][k] * B[i][k]
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```

If we make sure to iterate over the smaller set in every intersection, **we obtain the worst-case optimal runtime** $O(N^{1.5})$. This is the Generic Join algorithm [NRR13].

Factorizing saves time, not just space

Lots of exciting research in last decade from DB theory folks:

- Factorized Database (FDB) research: Dan Olteanu's group [OZ12]
- Functional Aggregate Queries (FAQ): RelationalAI folks [KNR23]

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Where do we go next?

In no particular order,

- Bring high-level query planning and low-level loop optimization closer together. Build a unified optimizer. See Free Join [WWS23], SQDLite [SSS23] for initial steps.
- Figure out how to efficiently **parallelize** factorized joins. Build/exploit factorized statistics? Fine-grained parallelism? Need to handle skew.
- Support more "exotic" tensor programs. Allow affine arithmetic on variables. How does this affect optimality guarantees? How do we optimize this?

Thank you!

Questions?

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