Peano Arithmetic

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1 The Peano Axioms

This is a formalization of first-order Peano Arithmetic. It is based on a definition of so-called statements via the notion of sets. This approach allows us to formulate the axiom of induction and applications of it in various proofs in a very natural way. Be aware that this is quite experimental and the main purpose of this formalization is to investigate how axiom schemas and the like can be implemented in Naproche-SAD without using such an "abuse of notation" via sets.
This formalization was successfully checked with E[1] 2.5.

> [read FLib/Statements/Library/statements.ftl]
> [prover eprover-2.5]

1.1 The language of Peano Arithmetic

In order to formulate Peano's axioms of arithmetic we introduce the notion of natural numbers. We extend the signature of the structure of natural numbers by two constant symbols 0 and 1 and a binary function symbol $+$. In the following sections this signature will be extended by further symbols.

> **Signature 1.1 (0101)** *A natural number is a notion. Let k, l, m, n denote natural numbers.*
>
> **Signature 1.2 (0102)** *0 is a natural number.*
>
> **Signature 1.3 (0103)** *1 is a natural number.*
>
> **Signature 1.4 (0104)** *$n + m$ is a natural number. Let the sum of n and m stand for $n + m$.*

1.2 The axioms

Now we can state the axioms of Peano Arithmetic.

> **Axiom 1.5 (0105)** *$n + 1$ is a natural number.*
>
> **Axiom 1.6 (0106)** *If $n + 1 = m + 1$ then $n = m$.*
>
> **Axiom 1.7 (0107)** *For no n we have $n + 1 = 0$.*
>
> **Axiom 1.8 (0108)** *Let P be a statement such that $P(0)$ and for all n if $P(n)$ then $P(n + 1)$. Then we have $P(n)$ for all n.*

[1] https://wwwlehre.dhbw-stuttgart.de/~sschulz/E/E.html

1.3 Immediate consequences

As two direct consequences of the above axioms we will prove that every non-zero number is a successor of some number and that no number is equal to its successor.

Proposition 1.9 (0109) *n = 0 or n = m + 1 for some natural number m.*
Proof. [prove off] Define P = natural number x — x = 0 or x = y + 1 for some natural number y. [/prove]
Then P(0) and for all natural numbers x if P(x) then P(x + 1). Hence we have P(x) for every natural number x. qed.

Proposition 1.10 (0110) *n /neq n + 1.*
Proof. [prove off] Define P = natural number x — x /neq x + 1. [/prove]
Then P(0).
For all natural numbers x if P(x) then P(x + 1). proof. Let x be a natural number. Assume P(x). Then x /neq x + 1. If x + 1 = (x + 1) + 1 then x = x + 1. Hence we have P(x + 1). end.
Therefore P holds for every natural number. qed.

1.4 Additional constants

To complete this section let us introduce some new constant symbols to represent the first few numbers.

Definition 1.11 (0111) *2 = 1 + 1.*

Definition 1.12 (0112) *3 = 2 + 1.*

Definition 1.13 (0113) *4 = 3 + 1.*

Definition 1.14 (0114) *5 = 4 + 1.*

Definition 1.15 (0115) *6 = 5 + 1.*

Definition 1.16 (0116) *7 = 6 + 1.*

Definition 1.17 (0117) *8 = 7 + 1.*

Definition 1.18 (0118) *9 = 8 + 1.*