Numerical Analysis Assignment 2

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Course: AMATH 740

Problem 4. Conditioning of linear systems.

Solution. **(a)**

We begin by showing $\kappa_2(A)$ is always equal to 1 if A is an orthogonal matrix.

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 \tag{4.1}$$

We can expand these norms out as follows where we use $A^{-1} = A^\intercal$ because for an othognal matrix $AA^\intercal = A^\intercal A = \mathbf{1}$.

$$\begin{aligned} \|A\|_2^2 &= \lambda_{\max}(A^\intercal A) \\ &= \lambda_{\max}(\mathbf{1}) \\ &= 1 \end{aligned} \qquad \begin{aligned} \|A^{-1}\|_2^2 &= \lambda_{\max}\left(\left(A^{-1}\right)^\intercal A^{-1}\right) \\ &= \lambda_{\max}(AA^\intercal) \\ &= \lambda_{\max}(\mathbf{1}) = 1 \end{aligned}$$

Plugging these two equations into eq. (4.1) we see $\kappa_2(A) = 1$ for all orthogonal A.

(b)

Here we show the matrix $A = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$ is not orthogonal.

$$AA^\intercal = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\mathbf{1} \neq \mathbf{1}$$

We can see here that $A^{-1} = \frac{1}{2}A^\intercal$ which we will use below in our calculation of $\kappa_2(A)$.

$$\begin{aligned} \kappa_2(A) &= \|A\|_2 \|A^{-1}\|_2 = \sqrt{\lambda_{\max}(A^\intercal A)}\sqrt{\lambda_{\max}\left((A^{-1})^\intercal A^{-1}\right)} \\ &= \sqrt{\lambda_{\max}(2 \cdot \mathbf{1})}\sqrt{\tfrac{1}{4}\lambda_{\max}(AA^\intercal)} \\ &= \sqrt{2}\sqrt{\tfrac{1}{2}} = 1 \end{aligned}$$

Because $\kappa_2(A) = 1$, we can conclude the matrix is well-conditioned.

When computing the solution to the perturbed problem, the change in solution from the unperturbed problem is very small. This is because the matrix is well conditioned (κ_2 is small)

(c)

First, is the matrix $B = \left[\begin{smallmatrix} 1 & -1+\delta \\ 1 & -1 \end{smallmatrix}\right]$ orthogonal?

$$BB^{\mathsf{T}} = \begin{bmatrix} 1 & -1+\delta \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1+\delta & -1 \end{bmatrix} = \begin{bmatrix} 1+(\delta-1)^2 & 2-\delta \\ 2-\delta & 2 \end{bmatrix} \neq \mathbf{1}$$

We conclude the matrix is not orthogonal. Via python we can calculate the matrix condition number to be $\kappa_2(B) = 39,999,947,698.45045$, and hence we conclude the matrix is not well conditioned.

When computing the solution to the perturbed problem, the change in solution from the unperturbed problem is massive (on the order 10^6). This is because the matrix is very ill conditioned (κ_2 is *very* large).