Baby Rudin 3rd Edition
Chapter 2: Basic Topology

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Highlights:

- §Finite, Countable, and Uncountable Sets
  - **2.5 Example** - Show that $\mathbb{Z}$ is countable with explicit formula.
  - **2.8 Theorem** - Show that every infinite subset of a countable set is countable.
  - **2.12 Theorem** - Show that the union of countable sets is countable.
  - **2.13 Theorem and Corollary** - Show that $\mathbb{Q}$ is countable.
  - **2.14 Theorem** - Show that the set of all *infinite* binary sequences is uncountable.
- §Metric Spaces
  - **2.19 Theorem** - Show that every neighborhood is an open set (Hint: show that any point of this neighborhood is an interior point).
  - **2.20 Theorem** - Show that every neighborhood of a limit point $p \in E$ contains infinitely many points of E (Hint: use contrapositive).
  - **2.23 Theorem** - Show that a set E is open if and only if its complement is closed.
  - **2.30 Theorem** - Show that a subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.
- §Compact Sets
  - something
- §Connected Sets
  - something
- §Perfect Sets
  - something

Exercises

2.1. Prove that the empty set is a subset of every set.

Proof. From equations (14):

If $A \subset B$, then

$$A \cup B = B,\ A \cap B = A$$

For the empty set this gives

$\varnothing \cup B = B$, $\varnothing \cap B = \varnothing$ $\square$

2.2. A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N$$

Proof. Let E be the set of all polynomials in (1) with the coefficients that are given in (2). That is, E_1 is the set of polynomials of (1) with $N = 1$ in (2) and E_2 is the set of polynomials of (1) with $N = 2$ in (2), ...

Then the set of algebraic numbers $\mathbb{A}$ is

$$\bigcup_{n=1}^{\infty} \{z \in \mathbb{C} \mid p(z) = 0 \text{ for } p(z) \in E_n\}$$

which is a union of countably many finite sets and is therefore countable. □

2.3. Prove that there exist real numbers which are not algebraic.

Proof. The set of all algebraic numbers was shown to be countable in Exercise 2.2 above and **Theorem 2.14** shows there exists infinite sets that are uncountable so therefore there must exist real numbers that are not algebraic. □

2.4. Is the set of all irrational real numbers countable?

Proof. We know that $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$. Furthermore, we know that $\mathbb{Q}$ is countable from the *Corollary* of **Theorem 2.13** and **Theorem 2.14** implies that $\mathbb{R}$ is uncountable. Therefore, $\mathbb{I}$ must be uncountable.

□

2.5. Construct a bounded set of real numbers with exactly three limit points.

Let us restate the definition of a limit point: **Definition 2.18 (a)** A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.

Construction. If we have the set $\{1/n \mid n \in \mathbb{N}\} \cup \{1 + 1/n \mid n \in \mathbb{N}\} \cup \{2 + 1/n \mid n \in \mathbb{N}\}$, then we would have three limit points at 0, 1, and 2. The elements a of this set are also bounded $|a| \leq 2$. □

2.6. Let $E^{'}$ be the set of all limit points of a set E. Prove that $E^{'}$ is closed. Prove that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E} = E \cup E^{'}$.) Do E and $E^{'}$ always have the same limit points?

First, we will prove that $E^{'}$ is closed. To do this we need to show that every limit point of $E^{'}$ is an element of $E^{'}$. Since we also need to prove that E and $\overline{E}$ have the same limit points, we will show that every limit point of $\overline{E} = E \cup E^{'}$ is an element of $E^{'}$, which will prove that $E^{'}$ is closed. Following this, we will show the converse of this initial conditional statement to prove that E and $\overline{E}$ have the same limit points, i.e. that $E^{'} = \overline{E}^{'}$.

Proof.

$\Longrightarrow$: $\overline{E}^{'} \subset E^{'}$

If p is a limit point of $\overline{E}^{'}$, then since it is a limit point, every neighborhood $N_r(p)$ with $r > 0$, contains $q \neq p$

such that $q \in \overline{E}$. In other words, $0 < d(p,q) < r$. If $q \in E$, we are done as this shows that $p \in E^{'}$. If $q \notin E$, then $q \in E^{'}$ and we know it is a limit point of E and therefore every neighborhood $N_s(q)$ with $s > 0$, where $s = min(d(p,q), r - d(p,q))$, contains $t \neq q$ such that $t \in E$. In other words, $0 < d(q,t) < s$. From the triangle inequality we know $d(p,t) \leq d(p,q) + d(q,t) < d(p,q) + r - d(p,q) = r$, showing in any case that $E^{'}$ is closed $\square$.

$\Longleftarrow$: $E^{'} \subset \overline{E}^{'}$

Since we just showed that $\overline{E}^{'} \subset E^{'}$, we will now show its converse.

If $p \in E^{'}$, then every $N_r(p)$ with $r > 0$ contains a $q \neq p$ such that $q \in E \implies q \in \overline{E}$. Therefore $p \in \overline{E}^{'}$ and $E^{'} \subset \overline{E}^{'} \implies E^{'} = \overline{E}^{'}$. $\square$

E and $E^{'}$ do not always have the same limit points. For example if $E = \{1/n \mid n \in \mathbb{N}\}$, then $E^{'} = \{0\}$ but $(E^{'})^{'} = \varnothing$. Note that a finite set has no limit points.

2.7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^{n} \overline{A_i}$, for $n = 1, 2, 3, \ldots$.

(b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\overline{B} \supset \bigcup_{i=1}^{\infty} \overline{A_i}$.

Proof (a). Since $B_n = \bigcup_{i=1}^{n} A_i$, we need to show that $\overline{B_n} = \overline{\bigcup_{i=1}^{n} = A_i} = \bigcup_{i=1}^{n} \overline{A_i}$, for $n = 1, 2, 3, \ldots$, using induction. First, to construct the base case we must show that the closure of the union of two sets is equal to the union of the closure of two sets: $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

If $x \in \overline{A} \cup \overline{B}$, then $x \in A \cup A' \cup B \cup B' = A \cup B \cup A' \cup B' = A \cup B \cup (A \cup B)' = \overline{A \cup B}$

If $x \in \overline{A \cup B}$, then $x \in A \cup B \cup (A \cup B)' = A \cup B \cup A' \cup B' = A \cup A' \cup B \cup B' = \overline{A} \cup \overline{B}$

Therefore, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Base case n=2: $\overline{B_2} = \overline{\bigcup_{i=1}^{2} A_i} = \overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$.

Inductive case (assume $\overline{B_n} \bigcup_{i=1}^{n} \overline{A_i}$ holds for $n \geq 0$):

$$\begin{aligned}
\overline{\bigcup_{i=1}^{n+1} A_i} &= \overline{A_{n+1}} \cup \overline{\bigcup_{i=1}^{n} A_i} \quad \text{[base case]} \\
&= \overline{A_{n+1}} \cup \bigcup_{i=1}^{n} \overline{A_i} \quad \text{[induction hypothesis]} \\
&= \bigcup_{i=1}^{n+1} \overline{A_i}
\end{aligned}$$

$\square$

Proof (b). If $x \in \bigcup_{i=1}^{\infty} \overline{A_i}$, then $x \in \overline{A_i}$ for some i. Therefore, $x \in \overline{B} = \overline{A_1} \cup \overline{A_2} \cup \overline{A_3} \cup \ldots$ showing that $\overline{B} \supset \bigcup_{i=1}^{\infty} \overline{A_i}$. $\square$

2.8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Explanation. Yes, every point of E is a limit point because $E \subset \mathbb{R}^2$ so if $x \in E$, where E is an open set we have that x is an interior point of E. Therefore, there is a neighborhood N of x such that $N \subset E$. Since we are dealing with the real numbers we can find another point of E in N which shows that x is a limit point. There are closed sets for which this statement is not true. For example, finite sets are closed and do not contain any limit points.

2.9. Let E^o denote the set of all interior points of a set E. [See **Definition 2.18(e)**; E^o is called the *interior* of E.]

(a) Prove that E^o is always open.

(b) Prove that E is open if and only if $E^o = E$.

(c) If $G \subset E$ and G is open, prove that $G \subset E^o$.

(d) Prove that the complement of E^o is the closure of the complement of E.

(e) Do E and $\overline{E}$ always have the same interiors?

(f) Do E and E^o always have the same closures?

Proof (a). If $x \in E^o$, then x is an interior point of E and $\exists$ a neighborhood $N \subset E$. However, for any $q \in N$, q is also an interior point of E and so $q \in E^o$. Therefore, $N \subset E^o$ and E^o is always open. ☐

Proof (b). If E is open, then all its points are interior points. Therefore, $E^o = E$ [**Definition 2.18(f)**]. If $E^o = E$, then all points in E are interior points and E is open. ☐

Proof (c). If $G \subset E$ and G is open, then $x \in G \implies x \in N \subset G \subset E \implies x \in N \subset E$ and $x \in E^o$. Therefore, $G \subset E^o$. ☐

Proof (d). If $x \in (E^o)^c$, then $x \in E^c \cup E \setminus E^o$. If $x \in E^c$, then $x \in \overline{E^c}$ (Remember that $\overline{E} = E \cup E'$). If $x \in E \setminus E^o$, then any neighborhood of x will contain points that are not in E. Therefore, $x \in (E^c)'$ so that $x \in \overline{E^c}$. This shows that $(E^o)^c \subset \overline{E^c}$.

If $x \in \overline{E^c}$, then $x \in E^c \cup (E^c)'$. If $x \in E^c$, then $x \in (E^o)^c$. If $x \in (E^c)'$, then any neighborhood of x will contain points that are not in E so that $x \in (E^o)^c$. This shows that $\overline{E^c} \subset (E^o)^c$.

Therefore, $(E^o)^c = \overline{E^c}$. ☐

Explanation (e). No they do not. For example, let $E = (0,1) \cup (1,2) \cup (2,3)$. In this case $E = E^o$ but the closure of E is $\overline{E} = [0,3]$, which has an interior of $\overline{E}^o = (0,3)$.

Explanation (f). No they do not. For example, let $E = (0,1) \cup \{2,3\}$, which has $E^o = (0,1)$. Then $\overline{E} = [0,1] \cup \{2,3\}$ and $\overline{E^o} = [0,1]$.

2.10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Proof. We will show that this meets parts *(a)-(c)* of **Definition 2.15** for a metric.

If $p \neq q$, then $d(p,q) = 1 > 0$. If $p = q$, then $d(p,q) = d(p,p) = 0$. [Part *(a)*]

For any $p \neq q, d(p,q) = d(q,p) = 1$. [Part *(b)*]

For any $r \in X, d(p,q) \leq d(p,r) + d(r,q)$, we notice that r cannot be equal to both p and q, unless of course we have the case where $p = q = r$

If $p = q = r$, then $0 \leq 0 + 0 = 0$.

If $p = q \neq r$, then $0 \leq 1 + 1 = 2$.

If $p \neq q = r$, then $1 \leq 1 + 0 = 1$. If $p = r \neq q$, then $1 \leq 0 + 1 = 1$.

If $p \neq q$ and $p \neq r$ and $q \neq r$, then $1 \leq 1 + 1 = 2$. [Part *(c)*]

Therefore, we have shown that this is a metric. This metric is known as the *Discrete Metric*.

Each point is an open set in this metric space with the unit ball of radius $1/2$, $B_{1/2}(x) \subset \{x\}$. Every subset of this metric space, being the union of open sets, is open. But if its complement is also open then that means it is also closed.

Only finite subsets of this metric space are compact since any infinite subset of this metric space has an open covering from the union of its one point subsets and this cannot be reduced to a finite subcovering.

2.11. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \\ d_3(x,y) &= |x^2 - y^2|, \\ d_4(x,y) &= |x - 2y|, \\ d_5(x,y) &= \tfrac{|x-y|}{1+|x-y|}, \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Proof.

$d_1(x,y) = (x-y)^2$: No. Let $x = 5, y = 1, r = 3$, then the triangle inequality for this becomes $(x-y)^2 \leq (x-r)^2 + (r-y)^2 \implies (5-1)^2 \leq (5-3)^2 + (3-1)^2 \implies 16 \leq 4 + 4$, which is obviously false.

$d_2(x,y) = \sqrt{|x-y|}$: Yes. If we square both sides we see that the right hand side is larger than right hand side of **Theorem 1.37(f)**.

$d_3(x,y) = |x^2 - y^2|$: No. Any $x = -y$ gives $d_3(x,y) = 0 \not> 0$.

$d_4(x,y) = |x - 2y|$: No. $x = 1, y = 1/2$ gives $d_4(1, 1/2) = 0 \not> 0$. Also, $d_4(x,y) \neq d_4(y,x)$.

$d_5(x,y) = \frac{|x-y|}{1+|x-y|}$: Yes. Let $|x-y|, |x-r|, |r-y|$ be represented by u, v, w, respectively. Then we have, $\frac{u}{1+u} \leq \frac{v}{1+v} + \frac{w}{1+w}$, which if we simplify becomes $u + uv + uw + uvw \leq v + w + uv + uw + 2vw + 2uvw$, which we can see is true.

2.12. Let $K \subset \mathbb{R}^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$ Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Proof. [Had to look this one up. Pretty impressed with the analytical skills that produced this proof.]

Suppose $K \subset U_\alpha$, where U_α is open. Then 0 must be in some subset U_{α_0}. Since U_{α_0} is open, $\exists \delta > 0$ such that $(-\delta, \delta) \subset U_{\alpha_0}$. $1/n \in U_{\alpha_0}$ if $n > 1/\delta$. Let N be the largest integer in $1/\delta$, and let $\alpha_j, j = 1, \ldots, N$, be such that $1/j \in U_{\alpha_j}$. Then $K \subset \bigcup_{j=0}^{N} U_{\alpha_j}$. □

2.13. Construct a compact set of real numbers whose limit points form a countable set.

Construction. Any finite set is compact as you can take the finite union of the open covers of the isolated points. Finite sets don't have any limit points, which is obviously countable.

2.14. Give an example of an open cover of the segment $(0,1)$ which has no finite sub-cover.

Example. Let $U_n = (0, 1 - 1/n), n \in \mathbb{N}$. This covers $(0,1)$ but does not have a finite sub-cover.

2.15. Show that **Theorem 2.36** and its Corollary become false (in $\mathbb{R}^1$, for example) if the word "compact" is replaced by "closed" or by "bounded".

Example. Lets look for an example that can show that this is false. To refresh, **Theorem 2.36** states: If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha\}$ is nonempty.

The sets $A_n = (0,1)$ are bounded but not closed. Finite collections of these sets have nonempty intersections but in the limit, the intersection is empty.

The sets $B_n = [n, \infty)$ are closed but not bounded. Finite collections of these sets have nonempty intersections but in the limit, the intersection is empty.

These sets also are an example to show that the Corollary is false because these sets are nested.

that if every intersection of a finite subcollection of closed (or bounded) sets is nonempty, then the intersection of the entire collection is nonempty. Let $A_n = (0, 1/n)$ and $B_n = [n, \infty)$, where we can see that A_n is bounded but not closed and B_n is closed but not bounded that both have nonempty finite intersections but in in the limit are empty.

2.16. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p,q) = |p-q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

Notes: To prove that E is closed we will show that $p \in E^c$ is an interior point of E^c, thereby showing that E is closed. We will need to show that this is true for $p^2 \leq 2$ as well as for $p^2 \geq 3$, as this is on both sides of the interval $2 < p^2 < 3$.

Proof. If $p \in E^c$, either $p^2 \leq 2$ or $p^2 \geq 3$. First, suppose $p^2 \leq 2$. Since E^c is $\mathbb{Q}/E$, this becomes $p^2 < 2$ as there is no rational number whose square is 2. If $p = 0$, let $\delta = 1$; otherwise let $\delta = min(\sqrt{(\frac{2-p^2}{3}}), \frac{2-p^2}{3|p|})$. Then if $y \in (p-\delta, p+\delta)$, we have $y^2 < 2$.

2.17. Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

Proof.

2.18. Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?

Proof.

2.19.

(a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

(b) Prove the same for disjoint open sets.

(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

Proof (a).

Proof (b).

Proof (c).

2.20. Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)

Proof.

2.21. Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}^1$. Put $A_o = \mathbf{p}^{-1}(A)$, $B_o = \mathbf{p}^{-1}(B)$. [Thus $t \in A_o$ if and only if $\mathbf{p}(t) \in A$.]

(a) Prove that A_o and B_o are separated subsets of $\mathbb{R}^1$.

(b) Prove that there exists $t_o \in (0,1)$ such that $\mathbf{p}(t_o) \notin A \cup B$.

(c) Prove that every convex subset of $\mathbb{R}^k$ is connected.

Proof (a).

Proof (b).

Proof (c).

2.22. A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable. *Hint:* Consider the set of points which have only rational coordinates.

Proof.

2.23. A collection V_α of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \supset X$ such that $x \in G$, we have $x \in V_\alpha \supset G$ for some α. In other words, every open set in X is the union of a subcollection of V_α.

Prove that every separable metric space has a *countable* base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.

Proof.

2.24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n (n = 1, 2, 3, \ldots)$, and consider the centers of the corresponding neighborhoods.

Proof.

2.25. Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint:* For every positive integer n, there are finitely many neighborhoods of radius $1/n$ whose union covers

K.

Proof.

2.26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint:* By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a *countable* subcover G_n, $n = 1, 2, 3, \dots$ If no finite subcollection of G_n covers X, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n, but $\bigcap F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E, and obtain a contradiction.

Proof.

2.27. Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.

Suppose $E \subset \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. *Hint:* Let V_n be a countable base of $\mathbb{R}^k$, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Proof.

2.28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary:* Every countable closed set in $\mathbb{R}^k$ has isolated points.) *Hint:* Use Exercise 27.

Proof.

2.29. Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments. *Hint:* Use Exercise 22.

Proof.

2.30. Imitate the proof of **Theorem 2.43** to obtain the following result:

If $\mathbb{R}^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.

Equivalent statement: If G_n is a dense open subset of $\mathbb{R}^k$, for $n = 1, 2, 3, \dots$, then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in $\mathbb{R}^k$).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

Proof.