Geometry: Euclid and Beyond
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Chapter 1 - Euclid's Geometry

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1 A First Look at Euclid's Elements

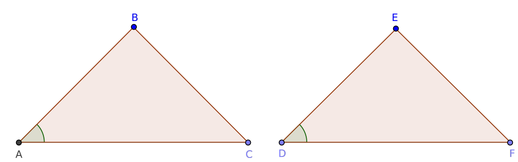
1.2 Read Euclid's *Elements*, Book I, Propositions 1-34. Be prepared to explain the statements and present proofs of (I.4), (I.5), (I.8), (I.15), (I.26), (I.27), (I.29), (I.30), and (I.32).

Note: Even though these propositions are proven in Euclid's Elements, below are my proofs when working out the exercises.

(I.4) - *If two triangles have the two sides equal to two sides respectively, and have the angles contained by the qual straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angels will be equal to the remaining angles respectively, namely those which the equal sides subtend.*

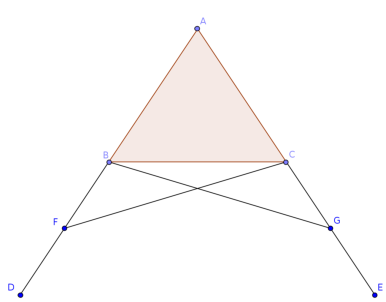
Proof. This is the notorious Side-Angle-Side (SAS) proposition from Euclid. Probably one of the most used propositions in the first several books of the *Elements*. In fact, this proposition of *superposition* is so important that Hilbert's axioms for geometry actually take (SAS) as an axiom in itself. Suppose we have triangles $\triangle ABC$ and $\triangle DEF$ where $\overline{AC} = \overline{DF}, \overline{AB} = \overline{DE}$ and $\angle BAC = \angle EDF$.

Place point A from triangle $\triangle ABC$ on point D of triangle $\triangle DEF$. Then, since $\overline{AC} = \overline{DF}$, line them up so $\overline{AC}$ coincides with $\overline{DF}$. Since $\angle BAC = \angle EDF$, we will then have that $\overline{AB}$ will line up and coincide with $\overline{DE}$ so that all points, sides, and angles of the two triangles will coincide with one another. Therefore, $\triangle ABC = \triangle DEF$. ■



(I.5) - *In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.*

Proof.



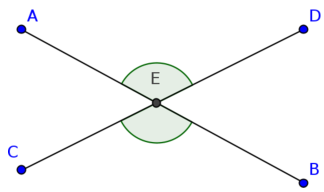
Let $\triangle ABC$ be isosceles. Produce $\overline{AD}$ from $\overline{AB}$ and $\overline{AE}$ from $\overline{AC}$ [post. 2]. Pick F on $\overline{BD}$ at random and make $\overline{AG}$ equal to $\overline{AF}$ [I.3]. Join $\overline{FC}$ and $\overline{BG}$ [post. 1]. Since $\overline{AB} = \overline{AC}, \overline{AF} = \overline{AG}$ and $\angle BAC$ is common; we have $\triangle AFC = \triangle AGB$ [I.4]. By subtracting $\triangle ABC$ from both, we also see that $\triangle BFC = \triangle CGB$ and therefore the angles under the base are equal, $\angle BCG = \angle CBF$. We also see that $\angle GBC = \angle FCB$ so if we add back $\triangle ABC$ to $\triangle BFC, \triangle CGB$ we have that $\angle FCA = \angle GBA$. But we see that $\angle FCA = \angle FCB + \angle BCA$ and $\angle GBA = \angle GBC + \angle CBA$ and we already know that $\angle FCB = \angle GBC$, therefore $\angle ABC = \angle CBA$. ■

(I.8) - *If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.*

Proof. This is the Side-Side-Side (SSS) proposition. Euclid proves this proposition using [I.7] which is a construction proposition showing that it is impossible to construct another triangle using equal sides from the same base. [I.7] is proved impossible by contradiction because of the isosceles triangles it makes between it and the original. We won't go through the proof of [I.8] as the majority of it relies on [I.7] after having the bases of the triangles match up and coincide with one another. ■

(I.15) - *If two straight lines cut one another, they make the vertical angles equal to one another.*

Proof.

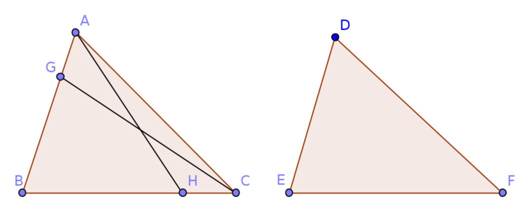


Let $\overline{AB}, \overline{CD}$ intersect at point E. $\angle AED + \angle DEB$ is equal to two right angles [I.13]. Similarly, $\angle CEB + \angle DEB$ is equal to two right angles [I.13]. Subtract $\angle DEB$ from both and we have $\angle AED = \angle CEB$ ■

(I.26) - *If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle.*

Proof. This is the Angle-Side-Angle (ASA) and Side-Angle-Angle (SAA) or Angle-Angle-Side (AAS) proposition.

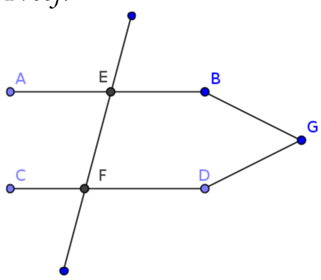
(ASA): For $\triangle ABC$ and $\triangle DEF$ let $\angle ABC = \angle DEF$, $\overline{BC} = \overline{EF}$, and $\angle BCA = \angle EFD$.
Now suppose $\overline{AB} \neq \overline{DE}$ and that $\overline{AB} > \overline{DE}$. Let $\overline{GB}$ be equal to $\overline{DE}$ [I.3] and join C to G [post. 1]. Then $\triangle GBC = \triangle DEF$ [I.4] but this is impossible because $\angle BCA = \angle EFD$ by hypothesis and we have seen that $\angle BCG = \angle EFD$ and $\angle BCG \neq \angle BCA$ as it is smaller. Thus, $\overline{AB} = \overline{DE}$. Now, since $\overline{AB} = \overline{DE}$ and $\overline{BC} = \overline{EF}$ and $\angle ABC = \angle DEF$, we have $\triangle ABC = \triangle DEF$.



(SAA): For $\triangle ABC$ and $\triangle DEF$ let $\overline{AB} = \overline{DE}$, $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$.
Now suppose $\overline{BC} > \overline{EF}$. Make $\overline{BH}$ the same length as $\overline{EF}$ [I.3] and join A and H [post. 1]. Then $\triangle ABH = \triangle DEF$ [I.4] so that $\angle BHA = \angle EFD$ but $\angle BCA = \angle EFD$ and therefore $\angle BHA = \angle BCA$, which is impossible (greater to the lesser) as $\angle BHA$ is the exterior angle from $\triangle AHC$ and therefore must be greater than $\angle BCA$ [I.16]. Thus, $\overline{BC} = \overline{EF}$ and $\triangle ABC = \triangle DEF$ [I.4]. ■

(I.27) - *If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.*

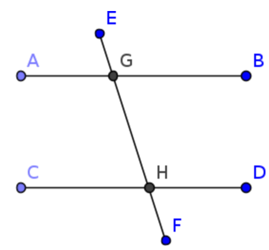
Proof.



Let $\overline{AB}$ and $\overline{CD}$ be two straight lines with the straight line $\overline{EF}$ falling on them. If $\angle AEF = \angle EFD$, then $\angle AEF + \angle FEB$ is equal to two right angles [I.13]. Therefore, $\angle EFD + \angle FEB$ is also equal to two right angles. Since these two angles are not less than two right angles the lines if produced further will not meet [post. 5]. A similar argument can be shown for the side that A and C lie on. Therefore, the two straight lines are parallel (Note: this is a different method of proof than Euclid uses). ■

(I.29) - *A straight line falling on parallel straight lines makes the alternates angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side euql to two right angles.*

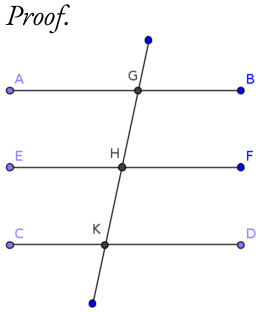
Proof.
Let $\overline{AB}$ and $\overline{CD}$ be parallel lines with $\overline{EF}$ falling on them.



Suppose $\angle AGH$ is not equal to $\angle GHD$. Then one of them must be greater. Let $\angle AGH$ be greater. Add $\angle BGH$ to both $\angle AGH$ and $\angle GHD$. Since $\angle AGH + \angle HGB$ is equal to two right angles [I.13] $\angle GHD + \angle BGH$ is less than two right angles so if B and D are extended indefinitely they will meet which is a contradiction. Thus, $\angle AGH = \angle GHD$. $\angle EGB = \angle AGH$ [I.15] and thus $\angle EGB = \angle GHD$. Additionally, since $\angle AGH + \angle BGH$ is equal to two right angles we have that $\angle BGH + \angle GHD$ is equal to two right angles. ■

(I.30) - *Staight lines parallel to the same straight lines are also parallel to one another.*

Proof.



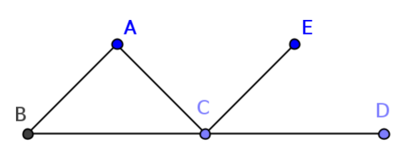
Suppose that $\overline{AB} \parallel \overline{EF}$ and $\overline{CD} \parallel \overline{EF}$.

$\angle AGH = \angle GHF$ [I.29] and $\angle GHF = \angle HKD$ [I.29], thus $\angle AGH = \angle HKD$ and by [I.27] $\overline{AB} \parallel \overline{CD}$. ■

(I.32) - *In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the trangle are equal to two right angles.*

Proof. Angles of a triangle add up to two right angles.

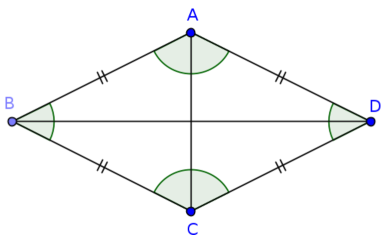
From $\triangle ABC$ let $\overline{BD}$ be produced from point C [post. 2]. Make $\overline{CE}$ parallel to $\overline{AB}$ [I.31]. $\angle ECD = \angle ABC$ and $\angle ACE = \angle BAC$ [I.29]. Therefore, $\angle ACD = \angle ECD + \angle ACE = \angle ABC + \angle BAC$. $\angle ACD + \angle ACB$ is equal to two right angles [I.13], therefore $\angle ABC + \angle BAC + \angle ACB$ is equal to two right angles. ■



For the following Exercises 1.4-1.10, present proofs in the style of Euclid, using any results you like from Book I,1-34 (excluding the theory of area, which starts with (1.35)). Be sure to refer to Euclid's definitions, postulates, common notions, and propositions by number whenever you use one.

1.4 A *rhombus* is a figure with four equal sides. Show that the diagonals of a rhombus meet at right angles, and that the four small triangles thus formed are congruent to each other.

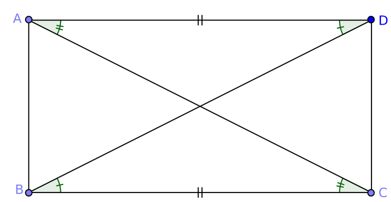
Proof. Technically a rhombus is a parallelogram with equal sides and where one set of equal interior and opposite angles are acute and the other pair obtuse. Let $ABCD$ be the given rhombus with $\angle BAD = \angle BCD$ and $\angle ABC = \angle ADC$.



Join the diagonals of the rhombus, $\overline{AC}$ and $\overline{BD}$ [post. 1]. These diagonals bisect the area of the parallelogram [I.34]. $\triangle ABC = \triangle ADC$ and $\triangle ABD = \triangle CBD$ [I.5], therefore the two base angles $\angle BAC, \angle BCA, \angle DAC, \angle DCA$ are all equal and with a similar argument so are $\angle CBD, \angle CDB, \angle ABD, \angle ADB$. Therefore the smaller triangles are all congruent by Angle-Side-Angle [I.26] and the angles adjacent to each other on the straight line diagonals are right as they are equal to each other [definition 10]. ■

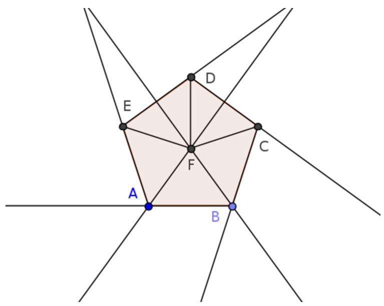
1.5 A *rectangle* is a four-sided figure with four right angles. Show that the two diagonals of a rectangle are congruent and bisect each other.

Proof.
Let $ABCD$ be a rectangle. Join $\overline{AC}$ and $\overline{BD}$ [post. 1]. $\triangle ABC = \triangle ADC$ [I.34], thus $\overline{AC} = \overline{BD}$. To show that $\overline{AC}$ and $\overline{BD}$ are bisected where they cross we can show that the two segments are equal. Since $\overline{AD} \parallel \overline{BC}$ and $\overline{AC}$ and $\overline{BD}$ fall on these parallel lines we have congruent angles $\angle CBD, \angle BDA$ and $\angle DAC, \angle ACB$. Therefore, say the bisection of the diagonals is a point O, then we have $\triangle OBC = \triangle OAD$ [I.26] showing us that the lengths of the diagonals have been bisected. ■



1.6 The exterior angles of a pentagon, with sides extended, add up to four right angles.

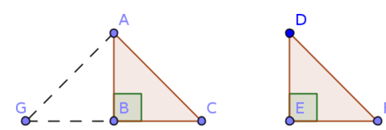
Proof.



Let $ABCDE$ be a regular pentagon (equilateral and equiangular). Bisect the interior angles of the pentagon at points A and B and let them meet at F. Then $\angle FAB = \angle FBA$ so $\overline{FA} = \overline{FB}$ [I.6] and therefore $\triangle FAB$ is isosceles. Join $\overline{EF}, \overline{CF}, \overline{DF}$ [post. 1]. These lines also bisect their respective angles because the original angles we bisected were arbitrary and therefore all of the inner triangles are congruent and isosceles. Thus the exterior angle at point A with $\angle EAF$ and $\angle FAB$ are equal to two right angles [I.13]. But since the inner triangles are isosceles these angles are equal and therefore the angle at point A is equal to the angle at the vertex of the isosceles triangles. Therefore, summing up the five exterior angles we see that they are are equal to the sum of of the five vertex angles for the inner isosceles triangles which are equal to four right angles (360 degrees). ■

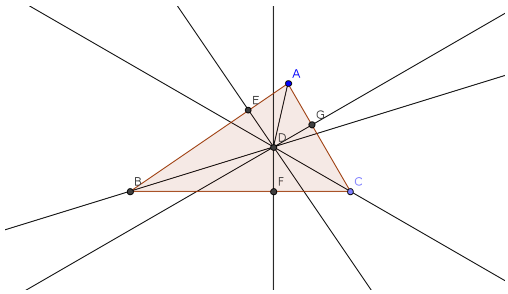
1.7 If two right triangles have one side and the hypotenuse respectively congruent, then the triangles are congruent. (We call this the right-angle-side-side theorem (RASS). Note in general that "ASS" is false: If two triangles have an angle and two sides equal, they need not be congruent.)

Proof.
Let $\triangle ABC$ and $\triangle DEF$ be two right triangles having $\overline{AC} = \overline{DF}$ and $\overline{AB} = \overline{DE}$. On $\triangle ABC$ from $\overline{BC}$ make $\overline{BG}$ equal to $\overline{EF}$ [I.3]. Join $\overline{AG}$ [post. 1]. Now we have that $\triangle AGB = \triangle DEF$ since $\overline{GB} = \overline{EF}$, $\overline{AB} = \overline{DE}$ and $\angle ABG, \angle DEF$ are both right [I.4]. $\triangle AGC$ is isosceles because $\overline{AC} = \overline{DF}$, and thus $\overline{AG} = \overline{DF}$ so $\overline{AG} = \overline{AC}$. Therefore, $\angle AGB = \angle DFE = \angle ACB$. Since $\angle DFE = \angle ACB$ and $\angle ABC = \angle DEF$, as both are right, and $\overline{AC} = \overline{DF}$ we see that $\triangle ABC = \triangle DEF$ [I.26]. ■



1.8 Show that the three angle bisectors of a triangle meet in a point. Be careful how you make your construction, and in what order you do the steps of your proof.

Proof.

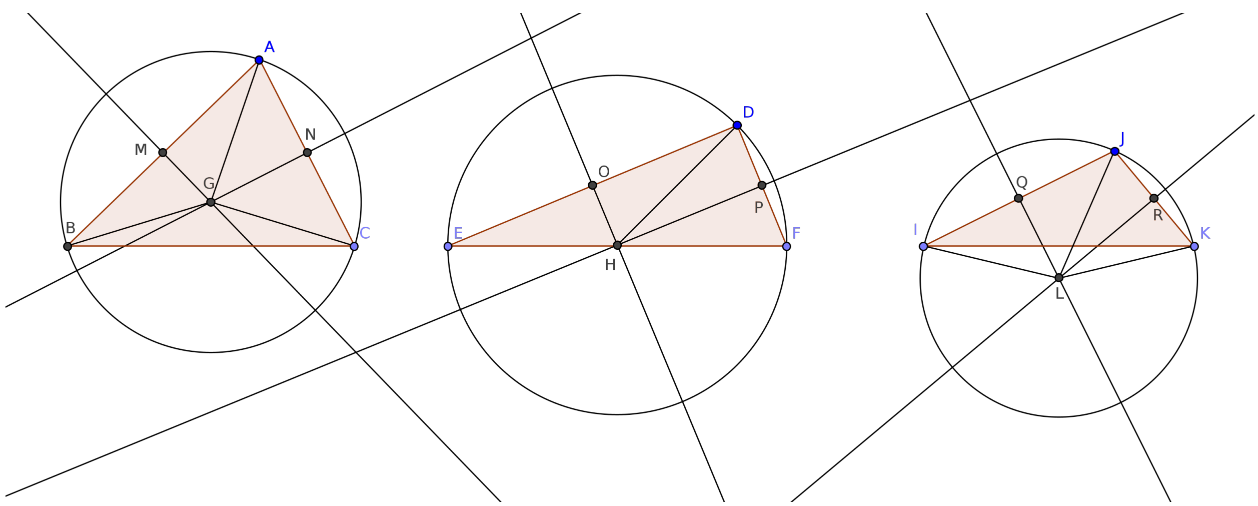


Let $\triangle ABC$ be the given triangle. Let $\angle ABC, \angle ACB$ be bisected by the straight lines $\overline{BD}, \overline{CD}$ [I.9] and let these meet one another at the point D. From D let $\overline{DE}, \overline{DF}, \overline{DG}$ be drawn perpendicular to the straight lines $\overline{AB}, \overline{BC}, \overline{CA}$ [I.11]. Now, since $\angle ABD$ is equal to $\angle CBD$, and the right angle $\angle BED$ is also equal to the other right angle $\angle BFD$, we have $\triangle EBD = \triangle FBD$ because $\overline{BD}$ is also common [I.26]. Therefore, $\overline{DE} = \overline{DF}$ and similarly we can show that $\overline{DG} = \overline{DF}$ for the other angle bisector. Thus, the three straight lines $\overline{DE}, \overline{DF}, \overline{DG}$ are equal to one another. Join $\overline{AD}$ [post. 1]. Furthermore, since $\overline{DE} = \overline{DG}$, $\angle DEA = \angle DGA$ (both right), and $\overline{DA}$ is common, we have that $\triangle DEA = \triangle DGA$ [RASS - problem 1.7 above]. Therefore, $\angle DAE = \angle DAG$ so that $\angle BAC$ has been bisected by $\overline{AD}$. Therefore, we have shown that the three angle bisectors of a triangle meet in a point. ■

1.9 The three perpendicular bisectors of the sides of a triangle meet in a single point. Be sure to give a reason why they should meet at all.

Proof. First, the perpendicular bisectors of a triangle will meet because the only way that two perpendicular lines won't intersect if produced indefinitely is if they are produced from a straight line or from line that has an angle greater than two right angles. For a triangle these two cases are impossible, therefore they must meet.

Now we can show that they also will meet in a single point. There are three cases for where the perpendicular bisectors can meet: inside the triangle, on one of the sides of the triangle, or outside the triangle. We will look at each case.



(Meets inside the triangle):

For $\triangle ABC$, let $\overline{AB}$ and $\overline{AC}$ be bisected at the points M, N respectively [I.10] and from M, N let $\overline{MG}$ and $\overline{NG}$ be drawn at right angles to $\overline{AB}, \overline{AC}$ [I.11]. Let them meet at the point G.

Join $\overline{GA}, \overline{GB}, \overline{GC}$ [post. 1]. Then, since $\overline{AM} = \overline{BM}$, $\overline{MG}$ is common and $\angle AMG = \angle BMG$, we have $\overline{AG} = \overline{BG}$. Doing this with the other perpendicular bisector shows us that $\overline{CG}$ is also equal to $\overline{AG}$ and $\overline{BG}$. Therefore, $\triangle GBC$ is isosceles. Bisect $\angle BGC$ with, say, point X (not shown on diagram) on $\overline{BC}$ so that $\triangle BGX = \triangle CGX$ [I.4]. Thus, $\overline{BX} = \overline{CX}$ and therefore $\overline{BC}$ has been bisected. Additionally, since $\angle GXC = \angle GXB$ and they both lie on the same straight line and adjacent to one another, they are both right [def. 10].

(Meets on one of the sides of the triangle):

For $\triangle DEF$, let $\overline{DE}$ and $\overline{DF}$ be bisected at the points O, P respectively [I.10] and from O, P let $\overline{OH}$ and $\overline{PH}$ be drawn at right angles to $\overline{DE}, \overline{DF}$ [I.11]. Let them meet at the point H.

Join $\overline{HD}$ [post. 1]. Then, since $\overline{DO} = \overline{EO}$, $\overline{OH}$ is common and $\angle DOH = \angle EOH$, we have $\overline{DH} = \overline{HE}$. Doing this with the other perpendicular bisector shows us that $\overline{FH}$ is also equal to $\overline{EH}$ and therefore $\overline{EF}$ has been bisected. No, perpendicular bisector is needed for point H as it already coincides with the other perpendicular bisectors from the other sides of the triangle.

(Meets outside the triangle):

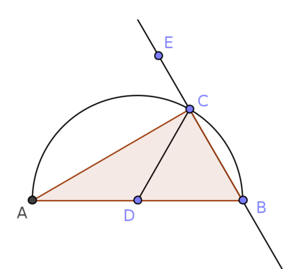
For $\triangle DEF$, let $\overline{JI}$ and $\overline{JK}$ be bisected at the points Q, R respectively [I.10] and from Q, R let $\overline{QL}$ and $\overline{RL}$ be drawn at right angles to $\overline{JI}, \overline{JK}$ [I.11]. Let them meet at the point L.

Join $\overline{LJ}, \overline{LI}, \overline{LK}$ [post. 1]. Then, since $\overline{JQ} = \overline{IQ}$, $\overline{QL}$ is common and $\angle JQL = \angle IQL$, we have $\overline{JL} = \overline{IL}$. Doing this with the other perpendicular bisector shows us that $\overline{KL}$ is also equal to $\overline{JL}$ and $\overline{IL}$. Therefore, $\triangle LIK$ is isosceles. Bisect $\angle KLI$ with, say, point Y (not shown on diagram) on $\overline{KI}$ so that $\triangle KLY = \triangle ILY$ [I.4]. Thus, $\overline{KY} = \overline{IY}$ and therefore $\overline{KI}$ has been bisected. Additionally, since $\angle LYI = \angle LYK$ and they both lie on the same straight line and adjacent to one another, they are both right [def. 10].

Therefore, the three perpendicular bisectors of the sides of a triangle meet in a single point. ■

1.10 Still using only results from Book I, show that if $\overline{AB}$ is the diameter of a circle, and C lies on the circle, then the angle $\angle ACB$ is a right angle.

Proof.
Bisect $\overline{AB}$ to find center, label it D [I.10]. Join $\overline{CD}$ [post. 1] and extend $\overline{BC}$ to E [post. 2]. $\overline{AD} = \overline{DC}$, thus $\triangle ADC$ is isosceles and $\angle CAD = \angle ACD$ [I.5]. $\overline{CD} = \overline{DB}$, so $\triangle DCB$ is also isosceles and $\angle DCB = \angle DBC$ [I.5]. Thus, $\angle ACD + \angle DCB = \angle DAC + \angle DBC$. But $\angle DAC + \angle DBC = \angle ACE$ as the exterior angle is equal to the two opposite and interior angles [I.32]. Thus, if two adjacent angles are equal to each other on a straight line they both are right angles [def. 10]. Therefore, $\angle ACB$ is a right angle. ■

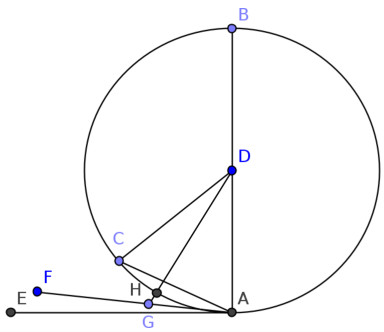


1.11 Read the *Elements*, Book III, Propositions 1-34. Be prepared to present statements and proofs of (III.16), (III.18),

(III.20), (III.21), (III.22), (III.31), and (III.32). For the following exercises, present proofs in the style of Euclid, using any results you like from (I.1)-(I.34) and (III.1)-(III.34) (still excluding the theory of area).

(III.16) - *The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.*

Proof. A straight line drawn at right angles to the diameter of a circle from its end does not fall within the circle.

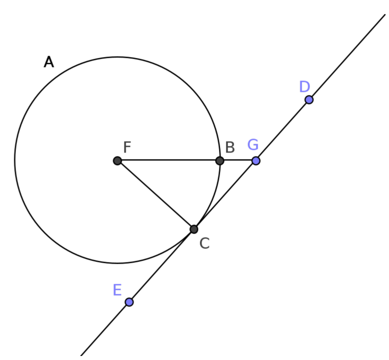


Describe circle ABC [post. 3] with center D and diameter $\overline{BA}$. Make line $\overline{AE}$ perpendicular to $\overline{BA}$ [I.11]. Suppose that $\overline{AE}$ does not fall outside the circle but falls within it as, say $\overline{CA}$ and let $\overline{DC}$ be joined [post. 1]. $\triangle DCA$ is isosceles because $\overline{DC} = \overline{DA}$, therefore $\angle DAC = \angle DCA$ [I.5]. But $\angle DAC$ is right but this is impossible as a triangle cannot have two right angles. Therefore, the straight line drawn from the point A at right angles to $\overline{BA}$ will not fall within the circle. Note: There are other portions of the proof but we will not go over these. ∎

(III.18) - *If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.*

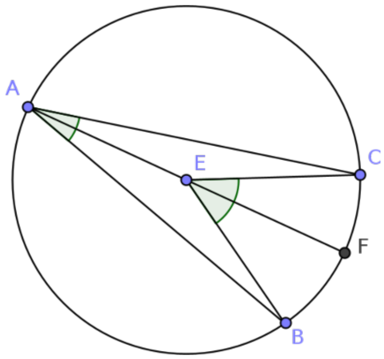
Proof. A straight line from the center of a circle to the point of contact of a tangent line is perpendicular with the given tangent line.

Suppose $\overline{FC}$ is not perpendicular to $\overline{DE}$. Instead, suppose that $\overline{FG}$ is perpendicular to $\overline{DE}$. Then, $\angle FGC$ is right and $\overline{FC} > \overline{FG}$ [I.19] but $\overline{FC} = \overline{FB}$ and $\overline{FB} > \overline{FG}$ which is impossible (the lesser to the greater). Thus, $\overline{FC}$ is perpendicular to $\overline{DE}$ as any other line would yield the same contradiction as the previous attempt. ∎



(III.20) - *In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.*

Proof.

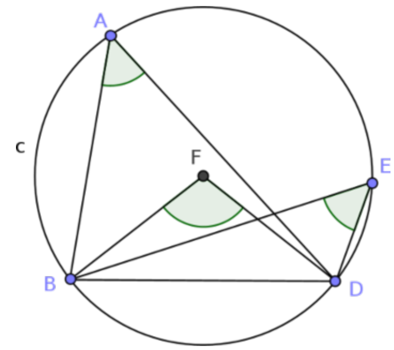


Describe circle ABC [post. 3] with $\overline{AF}$ as diameter and E as the center. Join $\overline{EC}, \overline{EB}, \overline{AC}, \overline{AB}$ [post. 1]. Since $\overline{EC} = \overline{EA} \implies \angle EAC = \angle ECA$ [I.5]. Then, $\angle CEF = \angle EAC + \angle ECA$ [I.32] and therefore $\angle CEF$ is twice $\angle CAE$. We can show the same for $\angle FEB$ being twice $\angle BAE$. Thus, $\angle CEB$ is twice $\angle CAB$. ■

(III.21) - *In a circle the angles in the same segment are equal to on another.*

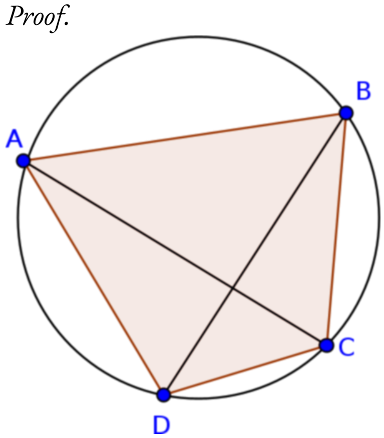
Proof.

Describe circle $ABCDE$ [post. 3] with center F where $\angle BAD$ and $\angle BED$ are in the same segment. Join $\overline{BF}, \overline{FD}$ [post. 1] so that $\angle BFD$ is in the same segment as $\angle BAD$ and $\angle BED$. Since $\angle BFD$ is twice $\angle BAD$ and also twice $\angle BED$ [III.20] we have that $\angle BAD = \angle BED$. ■



(III.22) - *The opposite angles in cyclic quadrilaterals are equal to two right angles.*

Proof.

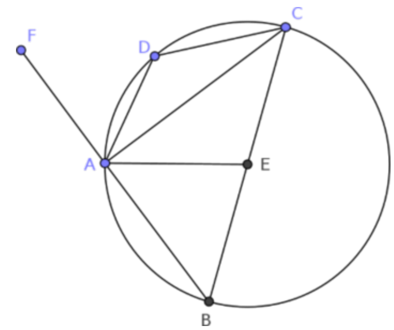


Describe circle $ABCD$ [post. 3] and join $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ [post. 1] to make the cyclic quadrilateral $ABCD$. Join the diagonals $\overline{AC}, \overline{BD}$ [post. 1]. The interior angles of $\triangle ABC$, $\angle BAC + \angle ACB + \angle CBA$, which equal two right angles [I.32]. $\angle BAC = \angle BDC, \angle ACB = \angle ADB$ [III.21]. Then, since $\angle ADC = \angle ADB + \angle BDC$, we have that $\angle ADC = \angle ACB + \angle BAC$. Now add $\angle CBA$ to both sides so that $\angle ADC + \angle CBA = \angle ACB + \angle BAC + \angle CBA$, which we saw is equal to two right angles as these are the interior angles of $\triangle ABC$. We can show the same thing for $\angle DAB + \angle DCB$. Thus, we have shown that the opposite angles in a cyclic quadrilateral equal two right angles. ■

(III.31) - *In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; and futher the angle of the greater segment is greater than a right angle, and the angle of the less segment less than a right angle.*

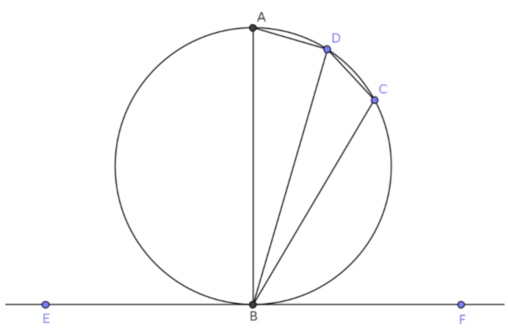
Proof.

We have already shown that the $\angle BAC$ is right in the semicircle (see exercise 1.10 above). Since, $\angle ABC, \angle BAC$ are less than two right angles [I.17] and $\angle BAC$ was shown to be right, then $\angle ABC$ is less than a right angle and this is the angle in the segment greater than the semicircle (segment ABC). Since cyclic quadrilaterals have opposite angles that sum to two right angles [III.22], we can see that $\angle ADC$ in the cyclic quadrilateral $ABCD$ must be less than a right angle since $\angle ABC$ was shown to be greater than a right angle and this is the angle in the segment lesser than the semicircle (segment ADC). ■



(III.32) - *If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equl to the angles in the alternate segments of the circle.*

Proof.

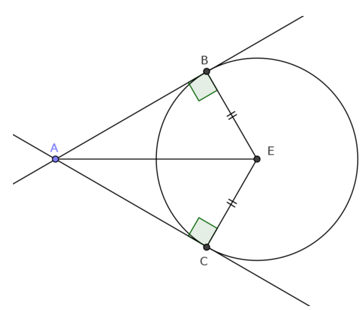


Draw $\overline{EF}$ tangent to the circle $ABCD$ at point B. From B draw a straight line $\overline{BD}$ cutting the circle. Also, draw $\overline{BA}$ from B at right angle to $\overline{EF}$ [I.11] which will be the diameter [III.19]. Let point C be taken at random on the circumference BD and let $\overline{AD}, \overline{DC}, \overline{DC}, \overline{CB}$ be joined [post. 1]. Since $\overline{BA}$ is the diameter of the circle $\angle ADB$ is right [III.31] and therefore the remaining angles in $\triangle BAD$ are equal to one right angle [I.32]. $\angle ABF$ is also right and therefore is equal to $\angle BAD + \angle ABD$. Subtracting $\angle ABD$ from both sides we see that $\angle DBF = \angle BAD$, which are in the alternate segment of the circle. Since $ABCD$ is a cyclic quadrilateral the opposite angles equal two right angles [III.22]. Thus, $\angle DBF + \angle DBE = \angle BAD + \angle BCD$ and $\angle BAD$ was shown to be equal to $\angle DBF$. Therefore, $\angle DBE$ is equal to $\angle BCD$ in the alternate segment DCB of the circle. ■

1.12 Let $\overline{AB}$ and $\overline{AG}$ be two tangent lines from a point A outside a given circle. Show that $\overline{AB} \cong \overline{AG}$.

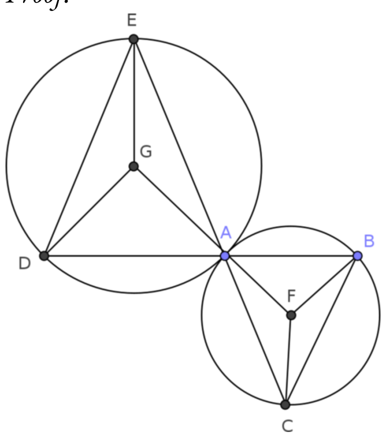
Proof.

Find the center of the circle and label it E [III.1]. Join $\overline{BE}, \overline{EC}$ [post. 1]. $\angle ABE = \angle ACE$ as both are right [III.18]. Join $\overline{AE}$ [post. 1]. Therefore, $\overline{AC} = \overline{AB}$ (RASS, see exercise 1.7 above). ■



1.13 Let two circles be tangent at a point A. Draw two lines through A meeting the circles at further points B, G, D, E. Show that $\overline{BC}$ is parallel to $\overline{DE}$.

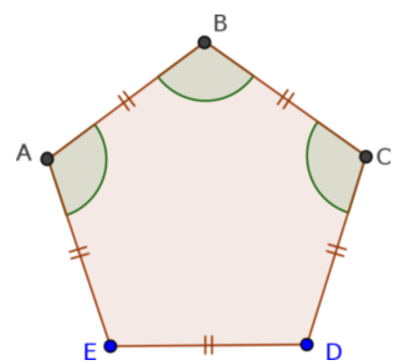
Proof.



Since $\overline{DB}$ and $\overline{EC}$ are straight lines we can show that $\overline{BC} \parallel \overline{DE}$ if either of the alternate angles are equal; $\angle DEA = \angle ACB$ or $\angle CBA = \angle ADE$ [I.27]. Find centers of circles DEA, ABC [III.1] and label them G and F, respectively. Join $\overline{FG}$ [post. 1] and note that this goes through point A [III.12]. Join $\overline{FB}$ and $\overline{DG}$ [post. 1]. $\triangle DGA$ and $\triangle BFA$ are isosceles [I.5] since $\overline{DG} = \overline{AG}$ and $\overline{AF} = \overline{BF}$. Additionally, $\angle DAG = \angle FAB$ [I.15] and since $\triangle DGA$ and $\triangle BFA$ are isosceles, their base angles are equal [I.5]. Therefore, $\angle BFA = \angle DGA$ [I.32] which means that $\angle DEA = \angle ACB$ [III.20] since they are both half of $\angle DGA$ and $\angle BFA$, respectively. Therefore, $\overline{BC}$ is parallel to $\overline{DE}$ [I.27]. ■

1.14 Given a pentagon $ABCDE$. Assume that all five sides are equal, and that the angles at A, B, C are equal. Prove that in fact all five angles are equal (so it is a *regular* pentagon).

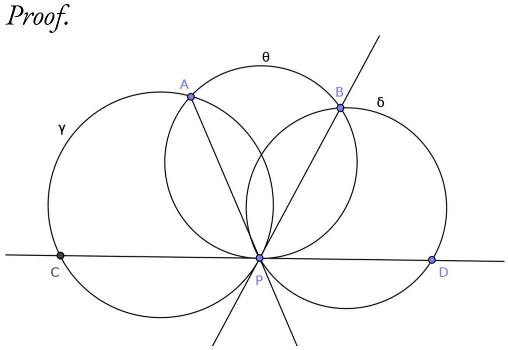
Proof.



Join $\overline{AC}, \overline{BD}, \overline{CE}$ [post. 1]. $\triangle ABC, \triangle BCD, \triangle CDE, \triangle DEA, \triangle EAB$ are isosceles [I.5]. We can see that $\angle AEB = \angle ABE$, which is half of $\angle ABC$. The same thing can be shown for the rest of the isosceles triangles. Thus, all of the isosceles triangles are equal and therefore all of the interior angles are equal. Therefore, the pentagon is both equilateral and equiangular and thus it is *regular*. ■

1.15 Let two circles γ and δ meet at a point P. Let the tangent to γ at P meet δ again at B, and let the tangent to δ at P meet γ again at A. Let θ be the circle through A, B, P. Let the tangent to θ at P meet γ and δ ant C, D. Prove that $\overline{PC} \cong \overline{PD}$.

Proof.



In progress – this one is hard! ■

2 Ruler and Compass Constructions

Haven't done these yet but may do so at some point with Geogebra

3 Euclid's Axiomatic Method

3.1 Explain what is wrong with the "proof" in (Example 3.1).

If the point E lies outside of the triangle as in the second diagram of Example 3.1, then the step in the proof **Drop perpendiculars $\overline{EF}$ and $\overline{EG}$ to the sides of the triangle** is not possible as the perpendiculars will be *outside* of the triangle. Additionally, looking at the second part of this example we can clearly see that even though the proof says that $\overline{BF} = \overline{CG}$, we can see that this is obviously false as $\overline{BF} > \overline{CG}$.

3.2 Read Euclid (I.35)-(I.48), Book II, and (III.35)-(III.37). Be prepared to present proofs of (I.35), (I.41), (I.43), (I.47), (II.6), (II.11), and (III.36).

NOTE: EVEN THOUGH THESE PROPOSITIONS ARE PROVEN IN EUCLID'S ELEMENTS, BELOW ARE MY PROOFS WHEN WORKING OUT THE EXERCISES.

(I.35) - *Parallelgrams which are on the same base and in the same parallels are equal to one another.*

Proof. Let $ABCD$ and $EBCF$ be parallelograms on the same base $\overline{BC}$ and in the same parallels $\overline{AF}, \overline{BC}$. We will show that $ABCD = EBCF$ (sides and angles equal).

We know that $\overline{AD} = \overline{EF} = \overline{BC}$ as well as $\overline{AB} = \overline{DC}$ and $\overline{BE} = \overline{CF}$ [I.34]. Furthermore, we see that $\overline{AE} = \overline{DF}$, since $\overline{DE}$ is common. $\angle ECD = \angle EAB$ [I.29] and $\triangle AEB = \triangle FDC$ [I.4]. Subtracting $\triangle DEG$ and then adding $\triangle BGC$ from both $\triangle AEB$ and $\triangle FDC$, we see that $ABCD = EBCF$.

TODO - diagram ∎

(I.41) - *If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle.*

Proof. Parallelogram $ABCD$ has the same base, $\overline{BC}$, as $\triangle EBC$ and both are in the same parallels $\overline{AE}, \overline{BC}$. Join $\overline{AC}$. Then $\triangle ABC$ is equal to $\triangle EBC$ [I.37] since they have same base and are in the same parallels. The diagonal $\overline{AC}$ bisects $ABCD$ [I.34] so $ABCD$ is twice the triangle BCE.

TODO - diagram ∎

(I.43) - *In any parallelogram the complements of the parallelograms about the diameter are equal to on another.*

Proof. Let $ABCD$ be a parallelogram, and $\overline{AC}$ its diameter and about $\overline{AC}$ let $\overline{EH}, \overline{FG}$ be parallelograms, and $\overline{BK}, \overline{KD}$ the so-called complements. Show that $\overline{BK} = \overline{KD}$.

$\triangle ABC = \triangle ACD, \triangle KGC = \triangle KFC$ and $\triangle AEK \triangle AKH$ [I.34]. Subtract these respective triangles from $\triangle ABC, \triangle ACD$. All that is left is $\overline{BK}$ and $\overline{KD}$, which are equal to each other.

TODO - diagram ■

(I.47) - *In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.*

Proof. The famous proof of Pythagorean Theorem. TODO ■

(II.6) - *If a straight line be bisected and a straight line be andded to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.*

Proof. ■

(II.11) - *To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.*

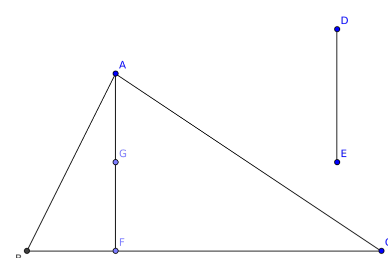
Proof. ■

(III.36) - *If a point be taken outside a cirlce and from it there fall on the circle two staight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straigtht line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.*

Proof. ■

3.3 Given a triangle $\triangle ABC$ and given a segment $\overline{DE}$, construct a rectangle with content equal to the triangle $\triangle ABC$, and with one side equal to $\overline{DE}$.

The general formula for the area of a triangle is $\frac{1}{2}b \cdot h$, where b is the base of the triangle and h is the height of the triangle.. For the triangle $\triangle ABC$ in the figure, the base is $\overline{BC}$ and the height is $\overline{AF}$ (G at the midpoint of $\overline{AF}$) so that the area is $\overline{BC} \cdot \overline{GF}$. This equation is actually an equation of a rectangle (in the sense of content). Follow rest of I.44 after the parallelogram BEFG was created (our rectangle takes the place of the parallelogram BEFG).



3.4 Given a rectangle, construct a square with the same content.

Proof. TODO - diagram

Let AB, BC be the given rectangle. If AB, BC is already square, we are done. If not, $\overline{AB}$ or $\overline{BC}$ is greater. Let $\overline{AB}$ be greater (see figure) and let it be produced to F [post. 2]. Let $\overline{BF}$ be made equal to $\overline{BC}$ [I.2] and let $\overline{AF}$ be bisected at G [I.10]. With center G and one of the lines $\overline{AG}, \overline{GF}$ let the semi-circle AHF be described [post. 3]. Let $\overline{BH}$ be produced from $\overline{BC}$ to H [post. 2] and $\overline{GH}$ joined [post. 1]. Since $\overline{AF}$ is cut equally at G and unequally at B the rectangle AB, BF together with the square on $\overline{BG}$ is equal to the square on $\overline{GF}$ [II.5]. But $\overline{GF} = \overline{GH}$, therefore the rectangle AB, BF together with the square on $\overline{GB}$ is equal to the square on $\overline{GH}$ [?]. However, the squares on $\overline{HB}, \overline{GB}$ are equal to the square on $\overline{GH}$ [I.47]; therefore the rectangle $\overline{AB}, \overline{BF}$ which remains is equal to the square on $\overline{BH}$. But since $\overline{BF} = \overline{BC}$, the rectangle $\overline{AB}, \overline{BF}$ is equal to the rectangle $\overline{AB}, \overline{BC}$. Therefore, the given rectangle is equal to the square on $\overline{BH}$. ■

3.5 Given a line l and given two points A, B not on l, construct a circle passing through A, B and tangent to l.

3.6 Given two lines l, m and a point P not on either line, construct a circle passing through P and tangent to both l and m.

3.7 Given a triangle $\triangle ABC$, let $\overline{DE}$ be a line parallel to the base $\overline{BC}$, let F be the midpoint of $\overline{DE}$, and let $\overline{AF}$ meet $\overline{BC}$ in G. Prove that G is the midpoint of $\overline{BC}$.

Proof. $\triangle ABC = \triangle FCB$ and $\triangle ABF = \triangle FCA$ [I.34]. Produce $\overline{BF} \parallel \overline{AC}$ and $\overline{CF} \parallel \overline{AB}$ [I.31, I.3]. $\overline{BC}$ falls on parallels $\overline{AC}$ and $\overline{BF}$, thus $\angle FBC = \angle BCA$ and $\angle ABC = \angle BCF$. Since $\angle FBC = \angle BCA$ and $\angle BGF = \angle AGC$ [I.15] and $\overline{AG} = \overline{GF}$ then $\triangle AGC = \triangle BGF$ [I.26] so that $\angle BFG = \angle GAC$. Therefore, $\overline{BG} = \overline{GC}$.

TODO - diagram ■

3.8 Let Γ be a circle with center O. Let $\overline{AB}$ and $\overline{AC}$ be tangents to Γ from a point A outside the circle. Let $\overline{BC}$ meet $\overline{OA}$ at D. Prove that $\overline{OA} \cdot \overline{OD} = \overline{OB}^2$ (meaning the rectangle on $\overline{OA}$ and $\overline{OD}$ has equal content to the square on $\overline{OB}$).

Proof. Join $\overline{OB}, \overline{OC}$ [post. 1]

Since $\overline{AO}$ is cut at random by $\overline{BC}$, the square on $\overline{AO}$ and the square on $\overline{DO}$ are equal to twice the rectangle contained by $\overline{AO}, \overline{DO}$ ($2\overline{AO} \cdot \overline{DO}$) and the square on $\overline{AD}$. Thus, $\overline{AO}^2 + \overline{DO}^2 = 2\overline{AO} \cdot \overline{DO} + \overline{AD}^2$ [II.7]

It is easy to show that the triangles $\triangle ABD, \triangle ODB$ are also right. (need to show this)

Thus, $\overline{DO}^2 = \overline{OB}^2 - \overline{BD}^2, \overline{AO}^2 = \overline{AB}^2 + \overline{OB}^2, \overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2$.

Then, $\overline{AO}^2 + \overline{DO}^2 = 2\overline{AO} \cdot \overline{DO} + \overline{AD}^2 = \overline{AB}^2 + \overline{OB}^2 + \overline{OB}^2 - \overline{BD}^2 = \overline{AD}^2 + \overline{BD}^2 + 2\overline{OB}^2 - \overline{BD}^2 = \overline{AD}^2 + 2\overline{OB}^2$

Therefore, $\overline{AD}^2 + 2\overline{OB}^2 = 2\overline{AO} \cdot \overline{DO} + \overline{AD}^2 \implies \overline{OB}^2 = 2\overline{AO} \cdot \overline{DO}$.

TODO - diagram ■

3.9 Let $\triangle ABC$ be a right triangle, and let $\overline{AD}$ be the altitude from the right angle A to the hypotenuse $\overline{BC}$. Prove that $\overline{AD}^2 = \overline{BD} \cdot \overline{DC}$ (in the sense of content).

Proof. $\overline{BD}^2 = \overline{AB}^2 + \overline{AC}^2, \overline{AB}^2 = \overline{BD}^2 + \overline{AD}^2, \overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2$ [I.47].

$(\overline{BD} + \overline{DC})^2 = \overline{AB}^2 + \overline{AC}^2$

$\overline{BD}^2 + 2\overline{BD} \cdot \overline{DC} + \overline{DC}^2 = \overline{BD}^2 + \overline{AD}^2 + \overline{AD}^2 + \overline{DC}^2$

$2\overline{BD} \cdot \overline{DC} = 2\overline{AD}^2 \implies \overline{AD}^2 = \overline{BD} \cdot \overline{DC}$

TODO - diagram ■

3.10 Problem: Given a triangle $\triangle ABC$, and given a point D on $\overline{BC}$, draw a line through D that will divide the triangle into two pieces of equal content.

Solution (Peletier): Let E be the midpoint of $\overline{BC}$. Draw $\overline{AD}$; draw $\overline{EF}$ parallel to $\overline{AD}$. Then $\overline{DF}$ divides the triangle in half.

Prove that the content of the quadrilateral $ABDF$ is equal to the content of the triangle $\triangle DFC$.

3.11 (Campanus). Use the theory of content to show that the line $\overline{DE}$ joining the midpoint of two sides of a triangle is parallel to the third side.

4 Construction of the Regular Pentagon

4.1 Read Euclid, Book IV.

This one is self explanatory.

4.2 Explain why the construction of (Problem 4.3) gives a regular pentagon.

4.3 Given a circle, but not given its center, construct an inscribed equilateral triangle in as few steps as possible (par=7).

Proof. Say we are given circle ABC.

1. Let $\overline{GH}$ be drawn touching the circle ABC at A [III.16].
2. On the straight line $\overline{AH}$ and at the point A on it, let the angle $\angle HAC$ be constructed equal to a third of two right angles ($\pi/3$ radians, i.e. 60 degrees).
3. Draw line $\overline{BC}$ parallel to $\overline{GH}$ [I.31].
4. Join $\overline{AB}$ to form the triangle $\triangle ABC$.

The equilateral $\triangle ABC$ has been inscribed in circle ABC.

TODO - diagram ■

4.4 Construct a square in as few steps as possible (par=9).

Proof.

1. Draw a straight line $\overline{AB}$.

2. Draw $\overline{AC}$ perpendicular to $\overline{AB}$ from the point A [I.11].

3. Let $\overline{AD}$ be made equal to $\overline{AB}$ [I.3].

4. From D draw $\overline{DE}$ parallel to $\overline{AB}$ [I.31].

5. From B draw $\overline{BE}$ parallel to $\overline{AD}$.

$ABED$ is a square.

TODO - diagram ■

4.5 Given a line segment $\overline{AB}$, construct a regular pentagon having $\overline{AB}$ as a side (par=11).

4.6 Given a circle Γ and given its center O, construct inside Γ three equal circles, each one tangent to Γ and to the other two (par=13).

Proof. ■

4.7 Let $\triangle ABC$ be an equilateral triangle inscribed in a circle. Let D, E be the midpoints of two sides, and extend $\overline{DE}$ to meet the circle at F. Prove that E divides the segment $\overline{DF}$ in extreme and mean ratio, i.e. the rectangle $\overline{EF} \cdot \overline{DF}$ equals the square $\overline{DE}^2$.

4.8 Take a long thin piece of paper. Tie a simple overhand knot in the paper, and fold the knot flat. Explain why the flat knot makes a regular pentagon.

5 Some Newer Results

5.1 (2SAS) Suppose we are given two triangles $\triangle ABC$ and $\triangle A'B'C'$. Assume that $\overline{AB} \cong 2\overline{A'B'}$ and $\overline{AC} \cong 2\overline{A'C'}$, and the angles at A and A' are equal. Prove that $\triangle ABC \cong 2\triangle A'B'C'$.

Proof. Let D, E, F be the midpoints of the sides of the triangle $\triangle ABC$ [I.10] and join $\overline{DE}, \overline{DF}, \overline{EF}$ [post. 1].

$\overline{AD} \cong \frac{1}{2}\overline{AB} \cong \overline{A'B'}$ and $\overline{AE} \cong \frac{1}{2}\overline{AC} \cong \overline{A'C'}$ and $\angle B'A'C' \cong \angle DAE$.

Thus, $\triangle ADE \cong \triangle A'B'C'$ (SAS) [I.4].

With the same reasoning we see that $\triangle DBF \cong \triangle DEF \cong \triangle EFC$. Therefore, $\overline{BF} \cong \frac{1}{2}\overline{BC} \cong \overline{B'C'}$. Thus, all sides of the triangle $\triangle ABC$ are doubles of the sides of the triangle $\triangle A'B'C'$ and similarly all angles of the triangle $\triangle ABC$ are equal to the angles of the triangle $\triangle A'B'C'$. Therefore, $\triangle ABC \cong 2\triangle A'B'C'$.

TODO - recheck proof and diagram ■

5.2 (2SSS) Suppose we are given two triangles $\triangle ABC$ and $\triangle A'B'C'$ and assume that $\overline{AB} \cong 2\overline{A'B'}, \overline{AC} \cong 2\overline{A'C'}$, and $\overline{BC} \cong 2\overline{B'C'}$. Prove that $\triangle ABC \cong 2\triangle A'B'C'$.

Proof. Similar to Exercise 5.1, take $\triangle ABC$ and find the midpoints of its sides to get $\triangle DEF$

Since $\overline{DE} \cong \frac{1}{2}\overline{BC} \cong \overline{B'C'}$, *overlineAD* $\cong \frac{1}{2}\overline{AB} \cong \overline{A'B'}$ and *overlineAE* $\cong \frac{1}{2}\overline{AC} \cong \overline{A'C'}$, then $\triangle ADE \cong \triangle A'B'C'$ [I.8] and therefore all the angles are equal.

Thus, all sides are double of $\triangle A'B'C'$ and all angles are equal. Therefore, $\triangle ABC \cong 2\triangle A'B'C'$.

TODO - diagram ■

5.3 Let *l, m, n* be three parallel lines. Suppose they cut off equal segments $\overline{AB} \cong \overline{BC}$ on a transversal line. Show that the segments $\overline{DE}, \overline{EF}$ cut off by any other transversal line are equal.

Proof. From points A and B draw perpendicular lines to m and n, respectively [I.11].

$\angle l'AB \cong \angle ABm \cong \angle m'BC \cong \angle BCn$ [I.29, I.15], therefore the triangles created are congruent (AAS) [I.26]. Thus the lengths between the parallel lines are equal. The same argument can be used for the line $\overline{DE}$ falling on the parallels. This will in turn create congruent triangles (AAS) [I.26] so that $\overline{DE} \cong \overline{EF}$.

TODO - diagram ■

5.4 Given three line segments, make a ruler and compass construction of a triangle whose medians are congruent to the three given segments. What condition on the segments is necessary for this to be possible?

5.5 Let *ABCD* be a quadrilateral. Show that the figure formed by joining the midpoints of the four sides is a parallelogram.

5.6 In any triangle, show that the center X of the nine-point circle lies on the Euler line (Proposition 5.7), and is the midpoint of the segment $\overline{OH}$ joining the circumcenter O to the orthocenter H.

5.7 Use cyclic quadrilaterals to give another proof of Proposition 5.6, as follows. Let $\triangle ABC$ be the given triangle. Let the altitudes $\overline{BL}$ and $\overline{CK}$ meet at H. Let $\overline{AH}$ meet the opposite side at M. Then show that $\overline{AM} \perp \overline{BC}$. (This proof is probably the one known to Archimedes.)

Proof. *BKLC* is a cyclic quadrilateral because K and E are both right angles subtending $\overline{BC}$ ($\angle BLC$ and $\angle BKC$). Thus, the vertices of *BKLC* lie on a circle.

We see that $\angle KBL \cong \angle ABL \cong \angle AML$ and $\angle AMK \cong \angle LCK \cong \angle KBL \cong \angle AML$. Thus, $\overline{HM}$ bisects $\angle KML$.

AKMC is a cyclic quadrilateral because $\angle AMK \cong \angle ACK$, thus $\angle AMC \cong \angle AKC$. But $\angle AKC$ is right, therefore $\angle AMC$ is right. Thus, $\overline{HM}$ has made a right angle with $\overline{BC}$.

TODO - diagram ■

5.8 Show that the opposite angles α, γ of a quadrilateral *ABCD* add to two right angles if and only if *A, B, C, D* lie on a circle.

Proof. TODO - If the opposite angles α, γ of a quadrilateral $ABCD$ add to two right angles, then...

Conversely, suppose that A, B, C, D lie on a circle. Join $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ and also the diagonals $\overline{AC}, \overline{BD}$. Since A, B, C, D lie on a circle, Proposition 5.8 shows that angles subtending the same circumference are equal. Thus, $\angle BAC \cong \angle BDC, \angle ABD \cong \angle ACD, \angle DBC \cong \angle DAC, \angle BCA \cong \angle ADB$.

The angles of triangle $\triangle ABC$ sum to two right angles: $\angle BAC + \angle ABD + \angle DBC + \angle ACB$ [I.32].

Thus, $\angle B = \angle ABD + \angle DBC$ and $\angle D = \angle ADB + \angle BDC$, which when summed together is $\angle B + \angle D = \angle ABD + \angle DBC + \angle ADB + \angle BDC \implies \angle ABD + \angle DBC + \angle ACB + \angle BAC$ and this was shown to be equal to two right angles. Therefore, if A, B, C, D lie on a circle then the opposite angles α, γ of a quadrilateral $ABCD$ add to two right angles.

TODO - put in terms of alpha and gamma to clean up the algebra of all the angles and diagram ■

5.9 Let $\overline{AB}$ be the diameter of a circle Γ. Show that a triangle $\triangle ABC$ has a right angle at C if and only if C lies on the circle Γ.

Proof. If C lies on the circle, we have already seen that any triangle with base as the diameter, has a right angle at the other point, in this case C, not on the diameter [III.31] (Thale's Theorem).

If $\angle ACB$ is right, we must now show that C lies on Γ.

1. Find the center of the circle by bisecting the diameter, label this O.
2. Draw a line through O until it intersects the other side of the circle at D.
3. Join $\overline{AD}$ and $\overline{DB}$, to complete the rectangle.

We know that the angle at D is right, since we are sure it lies on the circle D [III.31]... to be continued.

TODO - finish and diagram ■

5.10 Let B, C and D, E lie on two rays emanating from a point A. Show that B, C, D, E lie on a circle if and only if $\overline{AB} \cdot \overline{AC} = \overline{AD} \cdot \overline{AE}$ (in the sense of content).

5.11 In the construction to Problem 5.11, prove that $\overline{DE}$ is parallel to $\overline{AB}$.

5.12 In the construction to Problem 5.11, show that the circle through A, B, C is tangent to Γ. Thus this construction solves the problem, "given a circle Γ and given two points A, B, to find a circle passing through A, B, and tangent to Γ." This is a special case of the problem of Apollonius (Section 38).

5.13 (The Simson line). Let $\triangle ABC$ be any triangle. Let P be a point on the circumscribed circle ABC. Let D, E, F be the feet of the perpendiculars from P to the sides of the triangle (extended as necessary). Then D, E, F lie on a line.

5.14 (The Miquel point). Let $\triangle ABC$ be a triangle. Let D, E, F be points on the sides of the triangle. Show that the circles through $\overline{ADE}$, $\overline{BDF}$, and $\overline{CEF}$ all meet in a common point G.

5.15 (Pappus's theorem). Let A, B, C be points on a line l, and let A', B', C' be points on a line m. Assume that $\overline{AC'} \parallel \overline{A'C}$ and $\overline{B'C} \parallel \overline{BC'}$. Show that $\overline{AB'} \parallel \overline{A'B}$.

5.16 Construct three circles of different radii, each one tangent to the other two, with noncollinear centers, in as few steps as possible (par=7).

5.17 Let A, B, C, D be four points on a circle Γ. Let four more circles pass through $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$, respectively, meeting in further points A', B', C', D'. Show that $A'B'C'D'$ is a cyclic quadrilateral.

5.18 (Painting the plane). If the plane as been colored so that each point has one of three colors (red, yellow, blue), prove that for any interval $\overline{AB}$ there exist two points C, D of the same color, with $\overline{AB} \cong \overline{CD}$.

5.19 Given an angle with vertex O and a point P inside the angle, drop perpendiculars $\overline{PA}, \overline{PB}$ to the two sides of the angle, draw $\overline{AB}$, and drop perpendiculars $\overline{OC}, \overline{PD}$ to the line $\overline{AB}$. Then show that $\overline{AC} = \overline{BD}$.

5.20 Given any triangle $\triangle ABC$, let D, E, F be the feet of the altitudes. Show that the six projections G, H, I, J, K, L of D, E, F onto the other sides of the triangle lie on a circle.

5.21 (Wentworth). Let $\triangle ABC$ be a triangle. Construct with ruler and compass a line parallel to $\overline{BC}$, meeting $\overline{AD}$ in D and $\overline{AC}$ in E, such that $\overline{DE} = \overline{DB} + \overline{EC}$.