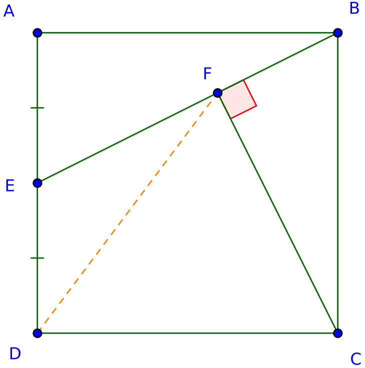
- Be open minded.
- Ask anything.
- Enjoy.

Outline

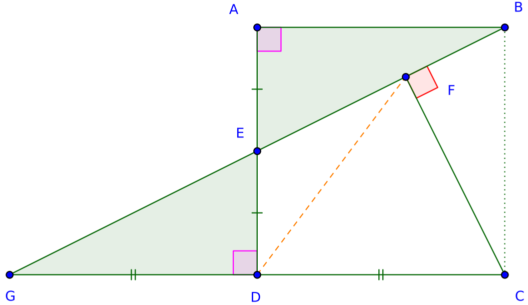
- Congruent triangles.
- Right triangles. Midpoint of the hypotenuse.
- Median segments in a triangle.
- Isosceles triangles.
- Angles, arcs, and chords in a circle.
- Triangle trigonometry. Law of cosines.
- Cyclic quadrilateral. Ptolemy theorem.

Example

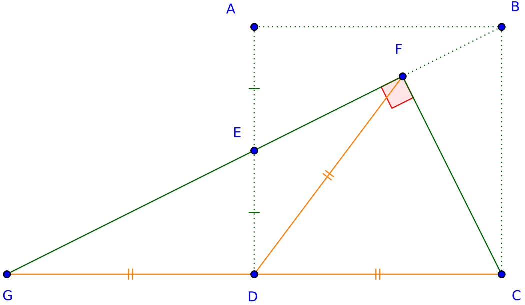
In the square $ABCD$, E is the midpoint of side AD, F is the foot of the altitude from C to BE. Prove that $DF = DA$.



Extending FE to intersect CD at G.



By angle-side-angle (ASA) $\triangle EAB \cong \triangle EDG \Rightarrow DG = AB = DC$.

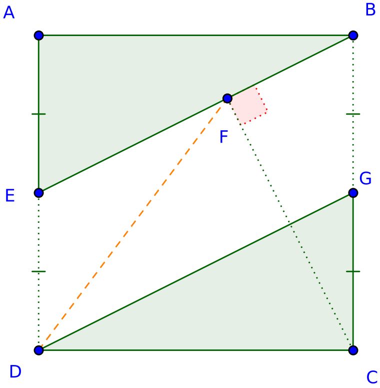


$$DG = DC,\ \angle GFC = 90^\circ \Rightarrow DF = DC.$$

Theorem

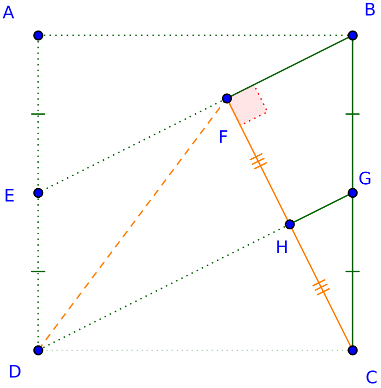
The midpoint of the hypotenuse of a right triangle is at equidistance from all vertices.

G is midpoint of BC.

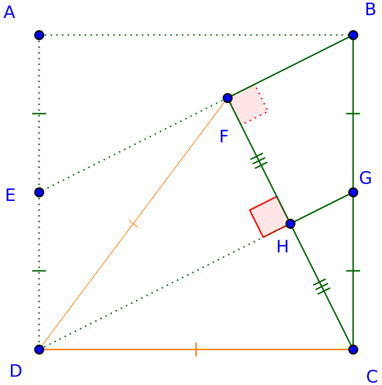


By side-side-side (SSS) $\triangle EAB \cong \triangle GCD$, so $\angle ABE = \angle CDG \Rightarrow EB \parallel DG$.

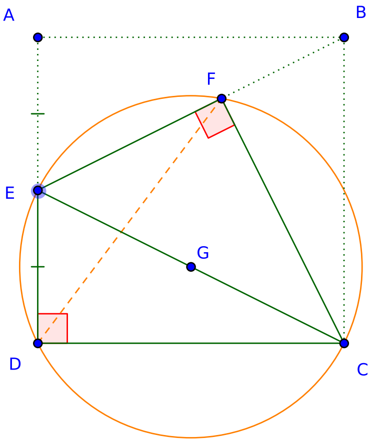
DG intersects FC at H.

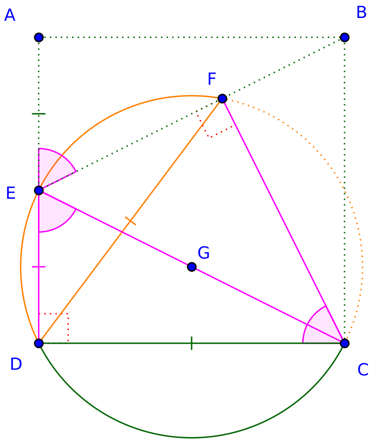


In $\triangle FBC$, $GH \parallel BF$, G is midpoint of BC, so H is midpoint of FC.



In $\triangle DFC$, $DH \perp FC$, H is midpoint of FC,
thus by side-angle-side (SAS) $\triangle DHF \cong \triangle DHC$, hence $DF = DC$.

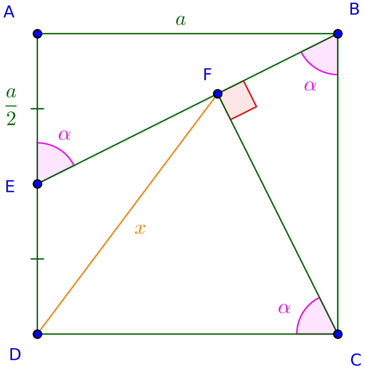




Both $\triangle EDC$ and $\triangle EFC$ are right at D and F.
C, D, E, and F are on the circle centred at G.

$\angle DEC = \angle AEB = 180^\circ - \angle DEF = \angle DCF$.
$\angle DEC = \angle DFC$ (subtends arc DC).
$\angle DCF = \angle DFC$, $\triangle DCF$ is isosceles, $DC = DF$.

Let $\alpha = \angle DCF = \angle FBC = \angle AEB$. $EB = \sqrt{AB^2 + AE^2} = \frac{a\sqrt{5}}{2}$.
$\cos\alpha = \frac{AE}{EB} = \frac{1}{\sqrt{5}}$, $\frac{FC}{BC} = \sin\alpha = \frac{AB}{EB} = \frac{2}{\sqrt{5}}$, so $FC = BC\sin\alpha = \frac{2a}{\sqrt{5}}$.

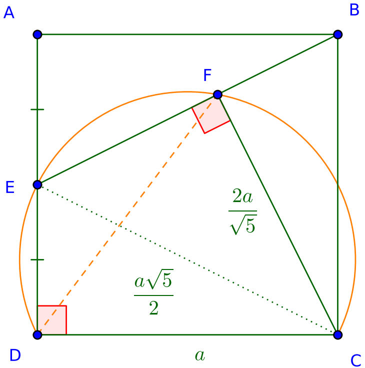


$$DF = \sqrt{DC^2 + FC^2 - 2(DC)(FC)\cos\alpha} = \sqrt{a^2 + \left(\frac{2a}{\sqrt{5}}\right)^2 - 2a\frac{2a}{\sqrt{5}}\frac{1}{\sqrt{5}}} = a.$$

Solution Five

Ptolemy theorem

Let $DC = a \Rightarrow EB = EC = \frac{a\sqrt{5}}{2}$. $\frac{FC}{BC} = \frac{AB}{EB} \Rightarrow FC = \frac{2a}{\sqrt{5}}$. $\frac{FB}{BC} = \frac{AE}{EB} \Rightarrow FB = \frac{a}{\sqrt{5}}$.



By Ptolemy theorem for the cyclic quadrilateral $CDEF$,

$$DF \cdot EC = ED \cdot FC + EF \cdot DC = \frac{a}{2} \cdot \frac{2a}{\sqrt{5}} + \left(\frac{a\sqrt{5}}{2} - \frac{a}{\sqrt{5}}\right) \cdot a = \frac{a^2\sqrt{5}}{2} \Rightarrow DF = a.$$

- Divisibility rules.
- Sum of the digits.
- Divisibility rules for perfect squares, cubes, and powers
- Variables and Diophantine equations.
- Algebaric identities.
- Modular arithmetic.

Example

What is the greatest multiple of 8 whose digits are all different?

The *divisibility rule for* 8 states that the last three digits of a multiple of 8 must be divisible by 8. To create the largest 8−digit number, the last three digits must be $0, 1$, and 2. Thus, the largest 3−digit multiple of 8 with those digits is 120. Thus the desired number is $\boxed{9876543120.}$

Example

What is the least multiple of 36 that contains only digits 4 and 5.

Divisibility rule for 9 state that the sum of digits of the number must be $9, 18, \ldots$. Because 45 or 54 are not divisible by 4, so $9 = 4 + 5$ is not a possible sum. For 18, the 2−digit multiple of 4 can be made from two pairs of 4 and 5 is 44. Thus the number is $\boxed{5544}$.

Example

Find a 7−digit number containing only digits 2 or digits 3 such that there are more of digits 2 than of digits 3 and the number is divisible by both 3 and 4.

By divisibility rule for 4, its last two digits must be 32. The sum of 5 remaining digits must have a remainder 1. Since there must be at least 3 digits of 2, hence all 5 remaining digits must be 2. The number is $\boxed{2222232.}$

Example

Is there a 5−digit perfect square whose sum of digits is 29?

A perfect square is divisible by 3 or has a remainder of 1 when divided by 3 (why?). Since the remainder of a number when divided by 3 is the same as the remainder of its sum of digits when divided by 3, and 29 has a remainder of 2 when divided by 3 so there is no such number.

Example

Find the perfect cube **n** such that all digits of n are 9 except the unit digit, which is 5.

There is no such perfect cube since a perfect cube has a remainder $0, 1$, or 8 when divided by 9.

Example

Find $\mathbf{n} > 3$ such that the $(n+1)$-digit binary number $\overline{10\ldots01_2}$ is a perfect power of 3.

Let $\overline{10\ldots01_2} = 2^n + 1 = 3^m$.
If m is odd, then $2^n = 3^m - 1 = (3-1)(3^{m-1} + \ldots + 1)$, where $3^{m-1} + \ldots + 1$ is a sum of odd number of odd terms. Contradiction.
If m is even, then let $m = 2k$, thus since 3^k is odd, $3^k = 2a + 1$, so $3^{2k} = (2a+1)^2$, therefore $2^n = 3^{2k} - 1 = 4a(a+1)$. This means a or $a+1$ odd. So $a = 1$, $2^n = 8 \Rightarrow n = 3$. Contradiction.

Example

Digits **a**, **b**, and **c** are used to form 3−digit numbers $\overline{abc}, \overline{bca}$, and $\overline{cab}$. The sum of these numbers is 1332, find $a+b+c$.

$\overline{abc} = 100a + 10b + c$, similarly with others. Their sum is $111(a+b+c) = 1332$, $a+b+c = 12$.

Example

Find all 4−digit number **n** whose sum of digits is $2010 - n$.

Let $n = \overline{abcd}$. Then $1001a + 101b + 11c + 2d = 2010$. If $a = 1$, then $b = 9$, so $11c + 2d = 100$, so $c = 8, d = 2$. If $a = 2$, then $b = c = 0, d = 4$. The solutions are $\boxed{1982,\ 2004}$.

Example

Find a potitive integer **a** such that $(1 + 2 + \ldots + a) - 1000a$ is a 3−digit number.

Let the $3 - digit\ k = (1 + 2 + \ldots + a) - 1000a$. Then $k = \frac{a(a+1)}{2} - 1000a = a\left(\frac{a+1}{2} - 1000\right)$. If $a < 1999$, then $k < 0$. If $a \geq 2000$, then $k \geq 2000 \cdot \frac{1}{2} = 1000$. Thus $a = \boxed{1999.}$

Example

What is the last digit of $\left(\ldots\left((7)^7\right)^7\ldots\right)^7$? There are 1001 digits 7.

By testing $7 \equiv 7 \pmod{10}, 7^7 = (7)(7^2)^3 \equiv -7 \pmod{10}, (7^7)^7 \equiv (-7)^7 \equiv 7 \pmod{10}, \ldots$ By Induction Principle, it can be proved that the last digit of the generic expression is 7 if it has an odd amount of 7, otherwise it is 3. The given one has an odd number of 7, so its last digit is $\boxed{7.}$

Example

In how many zeros can the number $1^n + 2^n + 3^n + 4^n$ end for **n** positive integer?

For $n = 1$, and 2, the sum ends in one and two zeros. Now, for all $n \geq 3$, $2^n, 4^n$ are divisible by 8, and $1^n + 3^n$ congruent to 2 or 4 modulo 8. Thus, the sum cannot end in three or more zeros.

Example

Find the last five digit of 5^{1981}.

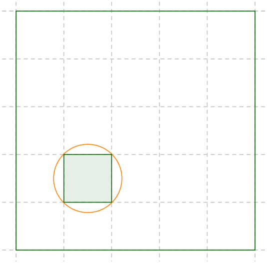
Let $n = 5^{1981} - 5^5 = 5^5(5^{1976} - 1)$, since $1976 = 8 \cdot 247$, so $5^8 - 1$ is a divisor of $5^{1976} - 1$. $5^8 - 1 = (5-1)(5+1)(5^2+1)(5^4+1)$ is divisible by 32, so n is divisible by $5^5 \cdot 2^5 = 100000$. Note $5^5 = 3125$. Therefore $5^{1981} = 100000 \cdot a + 3125$. The last five digits are $\boxed{03125.}$

- The Piegeonhole Principle.
- The Extremal Principle.

Example

There are 101 points in a unit square. Prove that five of them can be covered by a circle radius $\frac{1}{7}$.

Lets divide the unit square into $5 \times 5 = 25$ squares as shown below.



According to the Piegonhole Principle, there exists a square where at least $\left\lceil \frac{101}{25} \right\rceil = 5$ points reside. The circle centered at the center of the square radius $\frac{1}{7}$ cover the whole square since,

$$\text{half of the diagonal of the square, } \frac{1}{2} \cdot \frac{\sqrt{2}}{5} < \boxed{\frac{1}{7}.}$$

Example

For **n** ≥ 1, on a $2n \times 2n$ board, $3n$ squares are marked. Prove that n rows and n columns can be selected so that they contain all marked squares.

In the example below, where $n = 4$, in a 8×8 board we can choose rows $2, 5, 7$ and 8; then columns d, f, g and h to cover all marked squares.

	a	b	c	d	e	f	g	h
8	x							
7		x						x
6							x	
5			x		x			
4						x		
3				x				
2			x	x		x		
1								x

	a	b	c	d	e	f	g	h
8	x							
7		x						x
6							x	
5			x		x			
4						x		
3				x				
2			x	x		x		
1								x

- Lets choose n rows such that **each of them cover as many squares as possible.** We prove that these rows cover at least $2n$ marked squares. *In the example we choose rows* $2, 5, 7$, *and* 8. *They covered* 8 *marked squares.*
- Assume that the number of marked squares covered by these square is less than $2n$, then there are more than $3n - 2n = n$ squares *not covered* by them, therefore the *non-selected* n rows cover at least $n + 1$ squares.
- According to the Piegonhole Principle, there is one *non-selected* row that contains at least two marked squares. But by choice as above, **each of the selected rows should have at least as many markered squares as a non-selected rows,** thus the selected ones should cover at least $2n$ squares.
- Then the number of marked squares covered by all square is at least $2n + (n + 1) = 3n + 1$. This exceeds the number of marked squares, which is $3n$, thus it is impossible.

The Piegonhole Principle

Greedy algorithm

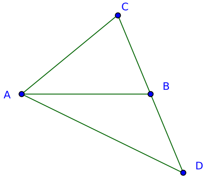
	a	b	c	d	e	f	g	h
8	x							
7		x						x
6							x	
5			x		x			
4						x		
3				x				
2			x	x		x		
1								x

- Hence, at least $2n$ marked squares are covered by $\boxed{n \text{ selected rows.}}$ For at most n remaining uncovered squares, it is easy to choose $\boxed{n \text{ columns.}}$
- A **greedy algorithm** is any algorithm that follows the problem-solving heuristic of making the locally optimal choice at each stage. A greedy strategy might not produce an optimal solution, but a greedy heuristic can yield locally optimal solutions that is close to a globally optimal solution in a reasonable amount of time.
- For example, **the travelling salesman problem** *"Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?"* is of high computational complexity.
- A solution uses the greedy algorithm with the following heuristic *"At each step of the journey, visit the nearest unvisited city."* This heuristic does not intend to find the best solution, but it terminates in a reasonable number of steps.

Example

Ω is a set of points on the plane. Every point in Ω is a midpoint of two points in Ω. Show that Ω is infinite set.

- Suppose that Ω is a finite set. According to the Extremal Principle, **there exists two points** $A, B \in \Omega$, **such that the distance** AB **is maximal.**
- Now, since $B \in \Omega$, there exist two points $C, D \in \Omega$ so that B is the midpoint of CD.



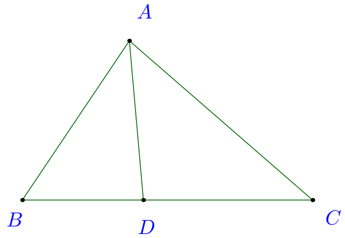
- Since one of the angles $\angle ABC$, $\angle ABD$, says $\angle ABD$ is at least 90°, thus in $\triangle ABD$, $AD > AB$. This contradicts the assumption that A, B are the two points in Ω, such that the distance AB is maximal.

Thus, there are no such two points A, B, so $\boxed{\Omega \text{ is infinite set.}}$

Example

There are n ($n \geq 4$) red and blue points on a plane with the following interesting property: every line segment that joins two points of the same colour contains a point of another colour. Prove that all the points lie on a single straight line.

If the points were not a single straight line, different triangles can be formed with the points as vertices. By the Extremal Principle, **let $\triangle ABC$ be the triangle with smallest area.**



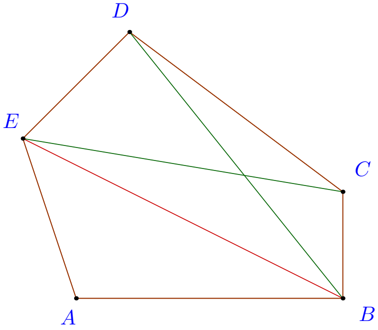
At least two of the vertices of this triangle have the same colour, let them be B, C. Between them there exists a point D of different colour, see above. Both $\triangle ABD$, $\triangle BCD$ have smaller area than $\triangle ABC$. This is contradiction!

Therefore, $\boxed{\text{Hence, all the points must lie on a single straight line.}}$

Example

For a convex pentagon, three diagonals can be chosen to form a triangle.

By the Extremal Principle, **let BE be the longest diagonal of the pentagon**, F be the intersection of BD and CE.



$BD + CE = (BF + FD) + (CF + FE) = (BF + FE) + (CF + FD) > BE + CD > BE$.

Since BE is the largest among BD, CE, BE, these $\boxed{\text{three diagonals form a triangle.}}$

There is no homework for this session. If you have enjoy the discussion, come next time. There is no need for preparation of anything for the next session.