Outline

Problem

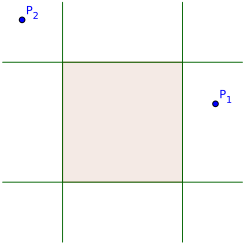
Find the answers for each of the three separate questions. Note that the result of a solved question, appeared earlier in the order of appearance, can be used in answer to the following one.

1. *(5 points) Given a square $\mathcal{S}$ and a point P outside of the square. Prove, by contradiction, that one side of $\mathcal{S}$ can be extended into a line that divides the plane into two half-planes, one contains the square $\mathcal{S}$ and the other contains the point P.*
2. *(5 points) A square $\mathcal{S}$ is inside a circle $\mathcal{C}$ such that $\mathcal{S}$ does not contains the centre of $\mathcal{C}$. Prove that there exists a diameter of $\mathcal{C}$, parallel to one side of $\mathcal{S}$, divides $\mathcal{C}$ into two half-circles, and one of them contains $\mathcal{S}$.*
3. *(15 points) A square $\mathcal{S}$ is inside a circle $\mathcal{C}$ radius 1, such that $\mathcal{S}$ does not contains the centre of $\mathcal{C}$. Prove that the side of S cannot be longer than $\sqrt{\frac{4}{5}}$.*

Problem

1. *(5 points) Given a square $\mathcal{S}$ and a point P outside of the square. Prove, by contradiction, that one side of $\mathcal{S}$ can be extended into a line that divides the plane into two half-planes, one contains the square $\mathcal{S}$ and the other contains the point P.*

Lets extend every side of the square $\mathcal{S}$ into a line. It is easy to see that the lines divide the plane into 8 parts, none of them contains $\mathcal{S}$.

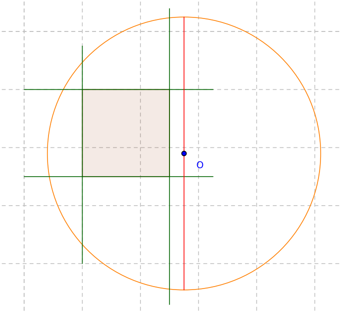


Assume the opposite, then point P cannot be in any of these parts, because any point in these parts shall be separated from $\mathcal{S}$ by a line extended from a side of $\mathcal{S}$. Thus, P shall be inside of $\mathcal{S}$, which is a contradiction to the given condition.

Problem

2. *(5 points) A square $\mathcal{S}$ is inside a circle $\mathcal{C}$ such that $\mathcal{S}$ does not contains the centre of $\mathcal{C}$. Prove that there exists a diameter of $\mathcal{C}$, parallel to one side of $\mathcal{S}$, divides $\mathcal{C}$ into two half-circles, and one of them contains $\mathcal{S}$.*

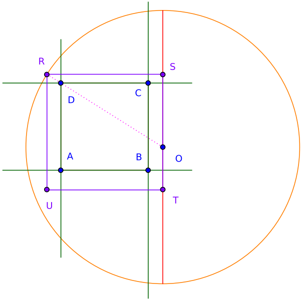
By the previous question, the centre O of $\mathcal{C}$ is outside of $\mathcal{S}$, so there is a line extended from a side of $\mathcal{S}$ and divides $\mathcal{C}$ into two parts, one contain $\mathcal{S}$ and the other contains O. A diameter through O parallel with this line divides $\mathcal{C}$ into two half-circles, one contains $\mathcal{S}$ and the other contains O.



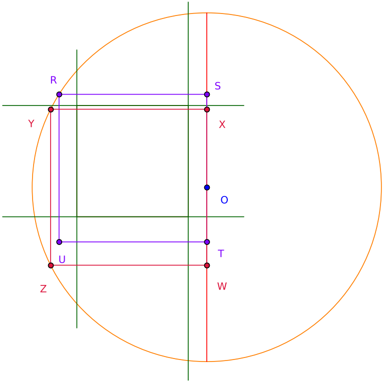
Problem

3. *(15 points) A square $\mathcal{S}$ is inside a circle $\mathcal{C}$ radius 1, such that $\mathcal{S}$ does not contains the centre of $\mathcal{C}$. Prove that the side of S cannot be longer than $\sqrt{\frac{4}{5}}$.*

Now, let A, B, C, and D be $\mathcal{S}$'s vertices, WLOG line through BC iseparates $\mathcal{S}$ from O. Both A and D are inside the circle. R is the intersection of line through OD with the circle. *Enlarge* the square $ABCD$ into the square $RSTU$, in other words, construct the square $RSTU$, where vertex S is the foot of the perpendicular line from R to the diameter, note that $BC \leq ST$.



If vertex T is not on the perimeter, the square $RSTU$ can be translated (move) along the diameter and *enlarge* into the square $WXYZ$, such that O becomes the midpoint of ST.



$$1 = OY^2 = XY^2 + XO^2 = 5XO^2 \Rightarrow XO = \frac{1}{\sqrt{5}} \Rightarrow XW = 2XO = \boxed{\sqrt{\frac{4}{5}}.}$$

- Perfect solutions: 0.
- Near-perfect solutions: 2 (23/25 pts) - Ha-Anh Le (S20) and Minh Nguyen (S11).
- Good solutions: 6 (20/25 pts) - Anthony Pham (S3), Benny Le (S8), Karl Le (S37), Laetitia Baud (S58), Quan Le Anh (S60), and Vu-Lam Le nguyen (S28).
- Better-than-average solutions: 0.

Let (a_1, a_2, a_3, a_4) be a permutation of $(1, 2, 3, 4)$,

- The absolute value $|a_1 - a_2|$ is the positive difference between a_1 and a_2.
  For example for $a_1 = 2, a_2 = 4, a_3 = 3, a_4 = 1$, $|a_1 - a_2| = |2 - 4| = |-2| = 2$.
- The tables below list all triples $(a_1, a_2, |a_1 - a_2|)$:

a_1	a_2	$\lvert a_1 - a_2\rvert$
1	2	1
1	3	2
1	4	3

a_1	a_2	$\lvert a_1 - a_2\rvert$
2	1	1
2	3	1
2	4	2

a_1	a_2	$\lvert a_1 - a_2\rvert$
3	1	2
3	2	1
3	4	1

a_1	a_2	$\lvert a_1 - a_2\rvert$
4	1	3
4	2	2
4	3	1

Problem

Find the answers for each of the three separate questions. Note that the result of a solved question, appeared earlier in the order of appearance, can be used in answer to the following one.

1. *(2 points) For a pair* (a_1, a_2)*, how many permutations* (a_1, a_2, a_3, a_4) *of* $(1, 2, 3, 4)$ *are there?*
2. *(3 points) For all permutations* (a_1, a_2, a_3, a_4) *of* $(1, 2, 3, 4)$*, find the average value of*
$$|a_1 - a_2|.$$
3. *(5 points) For all permutations* (a_1, a_2, a_3, a_4) *of* $(1, 2, 3, 4)$*, find the average value of the sum*
$$|a_1 - a_2| + |a_3 - a_4|.$$
4. *(15 points) For all permutations* $(a_1, a_2, \ldots, a_{2022})$ *of* $(1, 2, \ldots, 2022)$*, find the average value of the sum*
$$|a_1 - a_2| + |a_3 - a_4| + \ldots + |a_{2021} - a_{2022}|.$$

Problem

1. *(2 points) For a pair* (a_1, a_2), *how many permutations* (a_1, a_2, a_3, a_4) *of* $(1, 2, 3, 4)$ *are there?*

a_1	a_2	$\lvert a_1 - a_2 \rvert$
1	2	1
1	3	2
1	4	3

a_1	a_2	$\lvert a_1 - a_2 \rvert$
2	1	1
2	3	1
2	4	2

a_1	a_2	$\lvert a_1 - a_2 \rvert$
3	1	2
3	2	1
3	4	1

a_1	a_2	$\lvert a_1 - a_2 \rvert$
4	1	3
4	2	2
4	3	1

For the first question, note that for each pair (a_1, a_2) listed in the table, there are actually two different permutations (a_1, a_2, a_3, a_4) with the same pair (a_1, a_2), for example

$$(1, 2, 3, 4), \ (1, 2, 4, 3),$$

since there are $2! = \boxed{2}$ ways to permute the remaining numbers a_3, a_4.

Problem

2. *(3 points) For all permutations* (a_1, a_2, a_3, a_4) *of* $(1, 2, 3, 4)$, *find the average value of*

$$|a_1 - a_2|.$$

a_1	a_2	$\|a_1 - a_2\|$
1	2	1
1	3	2
1	4	3

a_1	a_2	$\|a_1 - a_2\|$
2	1	1
2	3	1
2	4	2

a_1	a_2	$\|a_1 - a_2\|$
3	1	2
3	2	1
3	4	1

a_1	a_2	$\|a_1 - a_2\|$
4	1	3
4	2	2
4	3	1

For the second question, the sum of all the coloured values $|a_1 - a_2|$ listed in the tables is

$$\underbrace{1 + 1 + \ldots + 1}_{6} + \underbrace{2 + \ldots + 2}_{4} + \underbrace{3 + 3}_{2} = 1 \cdot 6 + 2 \cdot 4 + 3 \cdot 2 = 20.$$

By the first question, the sum of all values of $|a_1 - a_2|$ is twice of that, thus it is 40. The number of all permutations (a_1, a_2, a_3, a_4) is $4!$, thus the average value of $|a_1 - a_2|$ is $\frac{40}{4!} = \boxed{\frac{5}{3}}.$

Problem

3. *(5 points) For all permutations* (a_1, a_2, a_3, a_4) *of* $(1, 2, 3, 4)$, *find the average value of the sum*

$$|a_1 - a_2| + |a_3 - a_4|.$$

For the third question, it is easy to see that the pair (a_3, a_4) repeats all possible values of (a_1, a_2), thus the average value of $|a_3 - a_4|$ is the same as the average value of $|a_1 - a_2|$, therefore the average value of the sum $|a_1 - a_2| + |a_3 - a_4|$ is twice the average value of $|a_1 - a_2|$, or $\boxed{\frac{10}{3}.}$

Problem

3. *(15 points) For all permutations* $(a_1, a_2, \ldots, a_{2022})$ *of* $(1, 2, \ldots, 2022)$, *find the average value of the sum*

$$|a_1 - a_2| + |a_3 - a_4| + \ldots + |a_{2021} - a_{2022}|.$$

- The fourth question is just a special case of the generalization, where $n = 1000$. Following the reasoning of the simple case with $n = 2$, we just need to find the average value of $|a_1 - a_2|$, since it is the same as the average value of $|a_3 - a_4|, \ldots, |a_{2n-1} - a_{2n}|$.
- Now, consider $a_1 = k$, where $1 \le k \le 2n$. Basically it the same as if we examine the k^{th} table in the simple case, the sum of all possible values of $|a_1 - a_2|$ in this case is,

$$\begin{aligned} &|k-1| + |k-2| + \ldots + |k-(k-1)| + |k-(k+1)| + \ldots + |k-2n| \\ &= (k-1) + (k-2) + \ldots + 1 + 1 + 2 + \ldots + (2n-k) \\ &= \frac{(k-1)k}{2} + \frac{(2n-k)(2n-k+1)}{2} = k^2 - (2n+1)k + n(2n+1) \end{aligned}$$

- There are $2n-1$ pairs of values (a_1, a_2) ($2n-1$ lines in the k^{th} table), thus the average value of $|a_1 - a_2|$,

$$\frac{k^2 - (2n+1)k + n(2n+1)}{2n-1}.$$

Problem

3. *(15 points) For all permutations* $(a_1, a_2, \ldots, a_{2022})$ *of* $(1, 2, \ldots, 2022)$, *find the average value of the sum*

$$|a_1 - a_2| + |a_3 - a_4| + \ldots + |a_{2021} - a_{2022}|.$$

- For all possible values of a_1 where $a_1 \in \{1, 2, \ldots, 2n\}$, the average value of $|a_1 - a_2|$ is

$$\frac{1}{2n}\sum_{k=1}^{2n} \frac{k^2 - (2n+1)k + n(2n+1)}{2n-1}$$

$$= \frac{1}{2n(2n-1)}\left(\sum_{k=1}^{2n} k^2 - (2n+1)\sum_{k=1}^{2n} k + (2n)(n)(2n+1)\right)$$

$$= \frac{1}{2n(2n-1)}\left(\frac{2n(2n+1)(4n+1)}{6} - (2n+1)\frac{2n(2n+1)}{2} + (2n)(n)(2n+1)\right) = \frac{2n+1}{3}$$

- The average of the sum $|a_1 - a_2| + |a_3 - a_4| + \ldots + |a_{2n-1} - a_{2n}|$ is $n|a_1 - a_2|$, so

$$\sum_{k=1}^{n} |a_{2k-1} - a_{2k}| = \frac{n(2n+1)}{3}, \text{ for } n = 1011, \ \frac{1011 \cdot 2023}{3} = \boxed{681751.}$$

- Perfect solutions: 2 (25/25 pts) - Benny Le (S8) and Minh Nguyen (S11).
- Near-perfect solutions: 0.
- Good solutions: 0.
- Better-than-average solutions: 2 (15/25 pts) - Anthony Pham (S3) and Chi Ton Nguyen (S55).

Theorem (Wilson's Theorem)

If integer $p > 1$, then $(p-1)! + 1$ is divisible by p if and only if p is prime.

Proof.

($\Rightarrow$) Let that p be a prime number, we prove that $(p-1)! + 1 \equiv 0 \pmod{p}$.

- First, we prove that if $a \in \{1, 2, \dots p-1\}$ then there exist one and only one $b \in \{1, 2, \dots p-1\}$ so $ab \equiv 1 \pmod{p}$.
  - Assume that there exist $b_1 \neq b_2$ and $b_1, b_2 \in \{1, 2, \dots, p-1\}$, such that $ab_1 \equiv ab_2 \equiv 1 \pmod{p}$.
  - None of the products $1a, 2a, \dots, (p-1)a$ should have all residues 0 modulo p, in other words, none of them is divisible by p.
  - Furthermore, for any $b_1 \neq b_2 \in \{1, 2, \dots, p-1\}$, $ab_1 - ab_2 = a(b_1 - b_2)$, and since since $a \not\equiv 0 \pmod{p}$, $b_1 - b_2 \neq 0$, and $-p < b_1 - b_2 < p$, so $1a, 2a, \dots, (p-1)a$ should have different residues modulo p, in other words, different remainders when divided by p.
- Now, since a pair (a, a) has residue 1 modulo p is equivalent to $p \mid a^2 - 1 = (a-1)(a+1)$, or $a = 1$, or $a = p - 1$; so the numbers $2, 3, \dots p-2$ are groupped into $\frac{p-3}{2}$ pairs of distinct numbers such that the product of them has residue 1 modulo p, in other words, has a remainder 1 when divided by p. Therefore

$$(p-1)! + 1 \equiv 1(2 \cdot 3 \cdots (p-2))(p-1) + 1 \equiv 1(p-1) + 1 \equiv 0 \pmod{p}.$$

Theorem (Wilson's Theorem)

If integer $p > 1$, then $(p-1)! + 1$ is divisible by p if and only if p is prime.

Proof.

$(\Leftarrow)$ Let $(p-1)! + 1 \equiv 0 \pmod{p}$, we prove that p is a prime.

- Lets ssume that p is composite, then p has a prime factor q such that $1 < q < p$, thus $q \mid (p-1)!$. Since $q \mid p \mid (p-1)! + 1$. Hence, $q \mid ((p-1)! + 1) - (p-1)! = 1$. Impossible. Thus, p is a prime.

$\square$

Problem

Find the answers for each of the three separate questions. Note that the result of a solved question, appeared earlier in the order of appearance, can be used in answer to the following one.

1. *(5 points)* $n+1$ *is a composite number, find the greatest common divisor of* $n!+1$ *and* $(n+1)!$.
2. *(10 points)* $n+1$ *is a prime number, find the greatest common divisor of* $n!+1$ *and* $(n+1)!$.
3. *(10 points)* p *is an odd prime number. Note that*

$$1 \equiv -(p-1) \pmod{p}, \; 3 \equiv -(p-3) \pmod{p}, \; \ldots$$

By Wilson's Theorem,

$$(1 \cdot 3 \cdots (p-2))\,(2 \cdot 4 \cdots (p-1)) \equiv -1 \pmod{p}.$$

Prove that

$$1 \cdot 3 \cdots (p-2) \equiv (-1)^{\frac{p+1}{2}} \pmod{p} \text{ and } 2 \cdot 4 \cdots (p-1) \equiv (-1)^{\frac{p+1}{2}} \pmod{p}$$

Problem

1. *(5 points) $n+1$ is a composite number, find the greatest common divisor of $n!+1$ and $(n+1)!$.*

For the first question, if $n+1$ is a composite number, then for any prime number $p < n+1$ or $p \leq n$, so $p \mid n!$, thus $p \nmid n!+1$. However $p \mid (n+1)!$, thus $\gcd(n!+1, (n+1)!) = \boxed{1.}$

Problem

2. *(10 points) $n+1$ is a prime number, find the greatest common divisor of $n!+1$ and $(n+1)!$.*

- For the second question, any prime factor of both numbers $n!+1, (n+1)!$, shall be a prime factor of $(n+1)!$, meaning $n+1$ is a prime number or there exists $q \leq n$ prime number. As in the previous question, for any $q \leq n$ prime number, $q \mid n!$, so $q \not| \; n!+1$, so q cannot be a common factor of both numbers $n!+1, (n+1)!$.
- Now, $n+1$ is a prime, by Wilson's Theorem $n!+1 \equiv 0 \pmod{n+1}$. It is obvious that $n+1 \mid (n+1)!$, so $n+1$ is a common factor of both $n!+1, (n+1)!$. However $n+1 \not| \; n!$, so $(n+1)^2 \not| \; (n+1)!$. Thus, the greatest commmon divisor of $n!+1$ and $(n+1)!$ is $\boxed{\text{n+1.}}$

Problem

3. *(10 points) p is an odd prime number. Prove that*

$$1 \cdot 3 \cdots (p-2) \equiv (-1)^{\frac{p+1}{2}} \pmod{p} \text{ and } 2 \cdot 4 \cdots (p-1) \equiv (-1)^{\frac{p+1}{2}} \pmod{p}$$

- For the third question, let $A = 1 \cdot 3 \cdots (p-2)$, $B = 2 \cdot 4 \cdots (p-1)$, by rearranging the congruence equality in the Wilson's Theorem,

$$AB = (1 \cdot 3 \cdots (p-2))\,(2 \cdot 4 \cdots (p-1)) \equiv -1 \pmod{p}$$

- On the other hand,

$$A = 1 \cdot 3 \cdots (p-2) \equiv (-(p-1))(-(p-3)) \cdots (-(p-(p-2))) = (-1)^{\frac{p-1}{2}}(p-1)(p-3)\cdots$$
$$\Rightarrow A \equiv (-1)^{\frac{p-1}{2}} B \pmod{p} \Rightarrow (AB)A \equiv (-1)(-1)^{\frac{p-1}{2}} B \pmod{p}.$$

- Since $p \not| B$, thus $\boxed{A^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}.}$ Similarly $\boxed{B^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}.}$

- Perfect solutions: 2 (25/25 pts) - Benny Le (S8) and Minh Nguyen (S11).
- Near-perfect solutions: 0.
- Good solutions: 0.
- Better-than-average solutions: 1 (15/25 pts) - Albert Dinh-Le (S24).

Definition (Polynomial)

For n positive integer, $P(x)$ is a $n-$degree polynomial if there exist real number $a_n \neq 0, a_{n-1}, \ldots, a_1, a_0$ such that

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

The $P(x) = c$, where c is a real number is called zero-degree, or constant polynomial.

Definition (Quadratic Polynomial)

A second degree polynomial $P(x)$ is called a *quadratic.* In other words, if there exist $a \neq 0, b$, and c real number such that

$$P(x) = ax^2 + bx + c.$$

Definition (Definition of Root)

For n positive integer, $P(x)$ is a $n-$degree polynomial. A real number r is called a **root** of $P(x)$ if $P(r) = 0$.

Fact (Factorization by roots)

n is a positive integer, $P(x)$ is a $n-$degree polynomial. If real number $r_1, r_2, \ldots, r_m$ are roots of $P(x)$ then there exist a $(n-m)-$degree polynomial $Q(x)$ such that

$$P(x) = (x - r_1)(x - r_2)\ldots(x - r_m)Q(x).$$

If $n = m$ then $Q(x)$ is a constant polynomial.

Fact (Existence of unique coefficients)

$P(x)$ and $Q(x)$ are both $n-$degree polynomials,

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \text{ where } a_n \neq 0.$$
$$Q(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0, \text{ where } n_n \neq 0.$$

If $P(x) = Q(x)$, for all real values of x, then $a_n = b_n,\ a_{n-1} = b_{n-1},\ \ldots,\ a_1 = b_1,\ a_0 = b_0$.

Problem

1. *(5 points)* $P(x) = x^2 + bx + 2$ *is a second degree polynomial (with the coefficient of* x^2 *is* 1*). The number* 1 *is a root of* $P(x)$*. Find the other root of* $P(x)$*.*
2. *(10 points) Find a second degree polynomial* $P(x)$ *such that*
$$(x+1)P(x) = (x-2)P(x+1).$$
3. *(10 points) For an arbitrary* n *positive integer, find the polynomial* $P(x)$ *such that*
$$(x+1)P(x) = (x-n)P(x+1).$$

Problem

1. *(5 points) $P(x) = x^2 + bx + 2$ is a second degree polynomial (with the coefficient of x^2 is 1). The number 1 is a root of $P(x)$. Find the other root of $P(x)$.*

For the first question, since 1 is a root or $P(x)$, by Factorization by roots, there exists $Q(x) = b_1 x + b_0$,

$$x^2 + bx + 2 = (x-1)(b_1 x + b_0) \Rightarrow x^2 + bx + 2 = b_1 x^2 + (b_0 - b_1)x - b_0.$$

By Existence of unique coefficients,

$$\begin{cases} 1 = b_1 \\ b = b_0 - b_1 \\ 2 = -b_0 \end{cases} \Rightarrow b_0 = -2, b_1 = 1, b = -3 \Rightarrow P(x) = x^2 - 3x + 2 = (x-1)(x-2).$$

Therefore the second root of $P(x)$ is $\boxed{2}$.

Problem

2. *(10 points) Find a second degree polynomial $P(x)$ such that*

$$(x+1)P(x) = (x-2)P(x+1).$$

- By substitutions,

$$(x+1)P(x) = (x-1)P(x+1) \Rightarrow \begin{cases} x=-1 & \Rightarrow (0)P(-1) = (-3)P(0) \Rightarrow P(0) = 0 \\ x=0 & \Rightarrow (+1)P(0) = (-2)P(1) \Rightarrow P(1) = 0 \\ x=+1 & \Rightarrow (+2)P(1) = (-1)P(2) \Rightarrow P(2) = 0 \end{cases}$$

- Thus, by Definition of Root $0, 1$ and 2 are roots of $P(x)$. $P(x)$ is a Quadratic Polynomial, there exist a, b, and c real numbers ($a \neq 0$), such that $P(x) = ax^2 + bx + c \quad (*)$
- On the other hand by Factorization by roots, there exist a constant polynomial $Q(x) = d$, where $d \neq 0$ real number, such that

$$P(x) = (x-0)(x-1)(x-2)Q(x) = dx^3 - 3dx^2 + 2dx \quad (**)$$

- (*) and (**) imply a contradiction: $P(x)$ cannot be both second- and third-degree polynomial. Hence, $\boxed{\text{there is no such } P(x).}$

Problem

3. *(10 points) For an arbitrary n positive integer, find the polynomial $P(x)$ such that*

$$(x+1)P(x) = (x-n)P(x+1).$$

- So $0, 1, \ldots, n$ are roots of $P(x)$, thus by Factorization by roots, there exist polynomial $Q(x)$, such that

$$P(x) = x(x-1)\ldots(x-n)Q(x).$$

- Substituting this into the given equation $(x+1)P(x) = (x-n)P(x+1)$,

$$(x+1)x(x-1)\ldots(x-n)Q(x) = (x-n)(x+1)(x)\ldots(x-n+1)Q(x+1) \Rightarrow Q(x) = Q(x+1).$$

- Thus $Q(x) = Q(x+1)$ for all real value of x. This can only be possible if $Q(x)$ is a constant polynomial. This can only be possible if and only if $Q(x) = c$ is a constant polynomial, where c is a real number.
- Therefore $\boxed{P(x) = cx(x-1)\ldots(x-n),}$ where c is an arbitrary real number.

- Perfect solutions: 1 (25/25 pts) - Benny Le (S8).
- Near-perfect solutions: 2 (22/25 pts) - Chi Ton Nguyen (S55) and Minh Nguyen (S11).
- Good solutions: 0.
- Better-than-average solutions: 1 (15/25 pts) - Laetitia Baud (S58).