MCC House Championship

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August 25, 2022

Contents

Part I

Winter-Spring 2022

Chapter 1

MIC R5

1.1 Topics

Algebra

1. Equations. Radicals. Substitutions.
2. Sums and Products.
3. Sequences.

Combinatorics

1. Counting. Correction over Counting.
2. Sets.
3. Permuations. Combinations.

Geometry

1. Right triangles.
2. Squares.
3. Trapezoids.
4. Circles.
5. Computational Geometry.

Number Theory

1. Divisibility.
2. Modular Arithmetic.
3. Number Bases.

1.2 Rules

- The total time to complete the test is 90 minutes.
- The test consists of 10 multiple-choice and 4 show-you-work problems. To answer each of 10 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 6 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.

1.3 Problems

Problem 1.3.1 (MIC-2022-SM1-R5-P1). (*6 points*)

What is the value of $5^4 - 4^3 \times 3^2 - 2^1 - 1^0$?

(A) 24 (B) 45 (C) 46 (D) 64 (E) 549

Problem 1.3.2 (MIC-2022-SM1-R5-P2). (*6 points*)

How many two-digit positive integers contain at least one digit equal to 6?

(A) 9 (B) 18 (C) 19 (D) 20 (E) 30

Problem 1.3.3 (MIC-2022-SM1-R5-P3). (*6 points*)

Points E and F lie inside rectangle $ABCD$ with $AE = DE = BF = CF = EF$. If $AB = 22$ and $BC = 16$, find the area of the quadrilateral $AEFB$.

(A) 32 (B) $64\sqrt{2}$ (C) $64\sqrt{3}$ (D) 128 (E) 160

Problem 1.3.4 (MIC-2022-SM1-R5-P4). (*6 points*)

The **??** below shows a large square divided into nine congruent smaller squares. There are circles inscribed in five of the smaller squares. The total area covered by all the five circles is 100π. Find the area of the large square.

(A) 144 (B) 288 (C) 560 (D) 684 (E) 720

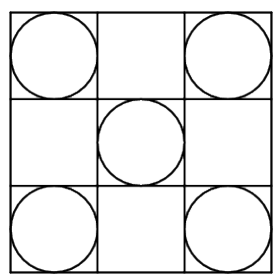

Figure 1.1: **??**

Problem 1.3.5 (MIC-2022-SM1-R5-P5). (*6 points*)

Define $f(x) = 2x - 3$ and suppose that $g(x + 2) = f(f(x - 1) \cdot f(x + 1) + f(x))$. Find $g(6)$.

(A) 49 (B) 79 (C) 129 (D) 239 (E) 259

Problem 1.3.6 (MIC-2022-SM1-R5-P6). (*6 points*)

Pentagon $ABCDE$ consists of a square $ACDE$ and an equilateral triangle ABC that share the side $\overline{AC}$. A circle centered at C has area 36. The intersection of the circle and the pentagon has half the area of the pentagon. Find the area of the pentagon.

(A) 20 (B) 24 (C) 25 (D) 28 (E) 30

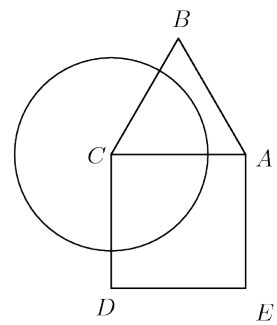

Figure 1.2: **??**

Problem 1.3.7 (MIC-2022-SM1-R5-P7). (*6 points*)

The following sequence lists all the positive rational numbers that do not exceed $\frac{1}{2}$ by first listing the fraction with denominator 2, followed by the one with denominator 3, followed by the two fractions with denominator 4 in increasing order, and so forth so that the sequence is

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{4}, \frac{1}{5}, \frac{2}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots .$$

Let m and n be relatively prime positive integers so that the 2022th fraction in the list is equal to $\frac{m}{n}$. Find $m+n$.

(A) 19 (B) 22 (C) 35 (D) 61 (E) 81

Problem 1.3.8 (MIC-2022-SM1-R5-P8). (*6 points*)

A teacher suggests four possible books for students to read. Each of five students selects one of the four books. How many ways can these selections be made if each of the books is read by at least one student?

(A) 168 (B) 180 (C) 240 (D) 360 (E) 480

Problem 1.3.9 (MIC-2022-SM1-R5-P9). (*6 points*)

Find the remainder when $2^{5^7} + 5^{7^2} + 7^{2^5}$ is divided by 11.

(A) 2 (B) 3 (C) 6 (D) 7 (E) 9

Problem 1.3.10 (MIC-2022-SM1-R5-P10). (*6 points*)

Let m and n be positive integers such that $x = m+\sqrt{n}$ is a solution to the equation $x^2-10x+1 = \sqrt{x}(x+1)$. Find the difference $n-m$.

(A) 5 (B) 17 (C) 41 (D) 48 (E) 55

Problem 1.3.11 (MIC-2022-SM1-R5-P11). (*10 points*)

Helena drives 5 km to the northeast, then 15 km to the southeast, then 25 km to the southwest, then 35 km to the northwest, and finally 20 km to the northeast.

How many miles is Helena from where she started and in which direction?

Problem 1.3.12 (MIC-2022-SM1-R5-P12). (*10 points*)

At the first show, all the seats in the theater were taken, and 65 percent of the audience was children. At the second show, again, all the seats were taken, but this time only 50 percent of the audience was children. Of all the people who attended either of the shows, 57 percent were children although there were 12 adults and 28 children who attended both shows.

(*5 points*) What is the total number of children who attended either of the shows?

(*5 points*) What is the number of adults who attended the first show?

Problem 1.3.13 (MIC-2022-SM1-R5-P13). (*10 points*)

A circle in the first quadrant with center on the curve $y = 2x^2 - 27$ is tangent to the y-axis and the line $4x = 3y$. The radius of the circle is $\frac{m}{n}$ where M and n are relatively prime positive integers.

Find the product mn.

Problem 1.3.14 (MIC-2022-SM1-R5-P14). (*10 points*)

Let $S = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + 49 \cdot 50 \cdot 51$.

(*5 points*) How many distinct prime numbers are divisors of S?

(*5 points*) What is the sum of distinct prime numbers that are divisors of S?

1.4 Grading

Answers for multiple-choice problems.

Problem 1: C

Problem 2: B

Problem 3: D

Problem 4: E

Problem 5: A

Problem 6: E

Problem 7: B

Problem 8: C

Problem 9: A

Problem 10: C

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 11: 2 points if figure out that travelling in opposite directions cancelling each other.

Problem 12: Separately grading for each part,

(a) 2 points if can express correct the number of children attended either of the shows with variables.

(b) 2 points if can express correct the number of children attended either of the shows with variables.

Problem 13: 2 points if can establish the equation for distance from the center to the line $4x = 3y$.

Problem 14: 2 points if can use the Hockey Stick Idenity or the summation identity. 2 points if can manually figure out the value of the sum.

1.5 Solutions

Solution. **??** $5^4 - 4^3 \times 3^2 - 2^1 - 1^0 = 625 - 64 \times 9 - 2 - 1 = 46$. The answer is $\boxed{C.}$ □

Solution. [First solution] **??** There are 9 two-digit numbers that have 6 as unit digit $16, 26, \ldots, 96$. There are 10 two-digit numbers that have 6 as tens digit $60, 61, \ldots, 69$. The number 66 counted twice, so the number of them is $9 + 10 - 1 = 18$. The answer is $\boxed{B.}$ □

Solution. [Second solution] **??** Let count the two-digit numbers that have no 6 as a digit. There are 8 ways to chose the tens digit $1, 2, 3, 4, 5, 7, 8, 9$. There are 9 ways to chose the unit digit $0, 1, 2, 3, 4, 5, 7, 8, 9$. Thus the number of two-digit numbers that have at least one digit to be 6 is $90 - 8 \cdot 9 = 18$. The answer is $\boxed{B.}$ □

Solution. **??** Let $AE = DE = EF = FB = FC = x$ as shown in **??**. By Pythagorean theorem,

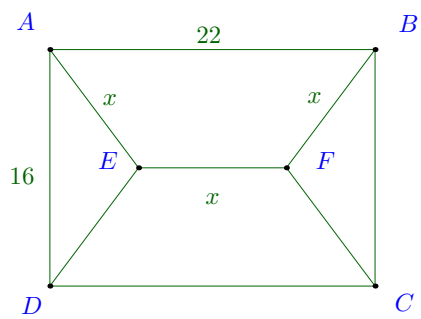

Figure 1.3: **??**

$$22 = 2\sqrt{x^2 - \left(\frac{16}{2}\right)^2} + x \Rightarrow 22 - x = 2\sqrt{x^2 - 64} \Rightarrow x^2 - 44x + 484 = 4x^2 - 128$$
$$\Rightarrow 3x^2 + 44x - 740 = 0 \Rightarrow (x - 10)(3x + 74) = 0 \Rightarrow x = 10$$

Thus, $[AEFB] = \frac{1}{2}(EF + AB) \cdot \frac{1}{2}AD = \frac{32 \cdot 16}{4} = 128$ The answer is $\boxed{D.}$ □

Solution. **??** Each small circle has area $\dfrac{100\pi}{5} = 20\pi$, so their radii are $2\sqrt{5}$. The side length of a small square is $2 \cdot 2\sqrt{5} = 4\sqrt{5}$, so it's area is $(4\sqrt{5})^2 = 80$. Thus, the area of the large square is $9 \cdot 80 = 720$. The answer is $\boxed{E.}$ □

Solution. **??**

$$g(6) = g(4 + 2) = f(f(4 - 1) \cdot f(4 + 1) + f(4)) = f(f(3) \cdot f(5) + f(4))$$
$$f(3) = 2 \cdot 3 - 3 = 3, f(4) = 2 \cdot 4 - 3 = 5, f(5) = 2 \cdot 5 - 3 = 7$$
$$\Rightarrow g(6) = f(3 \cdot 7 + 5) = f(26) = 2 \cdot 26 - 3 = 49$$

The answer is $\boxed{A.}$ □

Solution. **??** The area of the intersection is $\left(\frac{1}{6} + \frac{1}{4}\right) 36 = 15$. Thus, the area of the pentagon is $2 \cdot 15 = 30$. The answer is $\boxed{E.}$ □

Solution. **??** First, note that the number of fractions with denominator n is $\lfloor \frac{n}{2} \rfloor$,

$$\underbrace{\frac{1}{2}}_{1}, \underbrace{\frac{1}{3}}_{1}, \underbrace{\frac{1}{4}, \frac{2}{4}}_{2}, \underbrace{\frac{1}{5}, \frac{2}{5}}_{2}, \underbrace{\frac{1}{6}, \frac{2}{6}, \frac{3}{6}}_{3}, \underbrace{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}}_{3}, \ldots.$$

To find the denominator of the 2022th term, we need to find such n that:

$$1+1+2+2+\ldots+n+n < 2020 \leq 1+1+2+2+\ldots+(n+1)+(n+1) \Rightarrow n(n+1) < 2020 < (n+1)(n+2)$$

It is easy to see that $n = 44$. The denominator of the 2022th fraction is 90 (the denominators start with 2) and its numerator is $2022 - 1980 = 42$. Thus, the fraction is $\frac{42}{90} = \frac{7}{15} = \frac{m}{n}$. Therefore $m + n = 7 + 15 = 22$. The answer is $\boxed{B.}$ □

Solution. **??** Let each book be A, B, C, D. Since the question asks for the number of ways can these selections be made if each of the books is read by at least one student, we fix A, B, C, D in five slots, such that there are one remaining empty slot,

$$A,\ B,\ C,\ D,\ \{\}$$

There are four ways to select letter for the slot, and each configuration can be ordered in $\frac{5!}{2!}$ ways (because of the duplicated letter), thus the total number of ways for this case is:

$$4 \cdot \frac{5!}{2!} = 240$$

The answer is $\boxed{C.}$ □

Solution. **??** By Fermat's Little Theorem

$$\begin{aligned}
&2^{10} \equiv 1 \pmod{11} \Rightarrow 2^{5^7} \equiv 2^5 \equiv 10 \pmod{11} \\
&5^{10} \equiv 1 \pmod{11} \Rightarrow 5^{7^2} \equiv 5^9 = 625 \cdot 625 \cdot 5 \equiv 9 \cdot 9 \cdot 5 \equiv 9 \pmod{11} \\
&7^{10} \equiv 1 \pmod{11} \Rightarrow 7^{2^5} \equiv 7^2 \equiv 5 \pmod{11}
\end{aligned}$$

Thus, $2^{5^7} + 5^{7^2} + 7^{2^5} \equiv 10 + 9 + 5 \equiv 2 \pmod{11}$. The answer is $\boxed{A.}$ □

Solution. **??** Dividing both sides by x,

$$x - 10 + \frac{1}{x} = \sqrt{x} + \frac{1}{\sqrt{x}}$$

Let $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, then $x + \frac{1}{x} = y^2 - 2$,

$$y^2 - y - 12 = 0 \Rightarrow y = 4 \text{ (since } y > 0) \Rightarrow \sqrt{x} + \frac{1}{\sqrt{x}} = 4 \Rightarrow x - 4\sqrt{x} + 1 = 0 \Rightarrow x = 7 + 4\sqrt{3}$$

Thus, $m = 7, n = 16 \cdot 3 = 48 \Rightarrow n - m = 41$. The answer is $\boxed{C.}$ □

Solution. **??** Helena travels $5+20 = 25$ km to the northeast and 25 km to the southwest. These two cancel each other. She travels 15 km to the southeast, and 35 km to the northwest. So she is $\boxed{35 - 15 = 20 \text{ km northwest}}$ of her original position. □

Solution. **??** Let n be the total number of seats in the theater. The total number of children in the first show is $0.65n$ and in the second show $0.5n$. Since 28 of them attended both shows, so the total number of children attended either of the shows is $0.65n + 0.5n - 28$. This is 57% of the total number of people attened either of the shows, which is $2s - 12 - 28 = 2s - 40$, thus

$$0.65n + 0.5n - 28 = 0.57(2s - 40) \Rightarrow s = 520$$

Therefore the total number of children attended either of the shows is $\boxed{0.57(2s - 40) = 570,}$ and the number of adults attended the first show is $\boxed{0.35s = 182.}$ □

Solution. **??** Let the center of the circle be $(a, 2a^2 - 27)$. Then its radius is a, since it is tangent to the y-axis. Also the distance from the center to the line $4x = 3y$ is a. Thus we have

$$\begin{aligned}
&\frac{|4(a) - 3(2a^2 - 27)|}{\sqrt{3^2 + 4^2}} = \frac{|-6a^2 + 4a + 81|}{5} = a \\
&\Rightarrow -6a^2 + 4a + 81 = -5a \text{ (since the circle is in the first quadrant)} \\
&\Rightarrow 3(2a^2 - 3a - 27) = 3(2a - 9)(a + 3) = 0 \Rightarrow a = \frac{9}{2}
\end{aligned}$$

Thus, $m = 9, n = 2$ and $\boxed{mn = 9 \cdot 2 = 18.}$ □

Solution. [First solution] **??** Note that

$$S = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + 49 \cdot 50 \cdot 51 = 6\left(\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \cdots + \binom{51}{3}\right)$$

According to the The Hockey Stick Identity,

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \cdots + \binom{51}{3} = \binom{51+1}{3+1} = \binom{52}{4}$$

Therefore $S = 6\binom{52}{4} = 1624350 = 2 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$.

Thus S has $\boxed{\text{6 distinct prime divisors}}$ and $\boxed{\text{their sum is } 2 + 3 + 5 + 7 + 13 + 17 = 47.}$ □

Solution. [Second solution] **??** Note that

$$\sum_{k=1}^{n} k \cdot (k+1) \cdot (k+2) = \frac{n \cdot (n+1) \cdot (n+2) \cdot (n+3)}{4}.$$

Thus,

$$S = \frac{49 \cdot 50 \cdot 51 \cdot 52}{4} = 1624350 = 2 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$$

Thus S has $\boxed{\text{6 distinct prime divisors}}$ and $\boxed{\text{their sum is } 2 + 3 + 5 + 7 + 13 + 17 = 47.}$ □

Chapter 2

HC R6

2.1 Topics

Chess

1. Moves: capture, check, double check, promotion, stalemate, checkmate
2. Analysis: whose turn is it.
3. Retrograde: last move, last two moves, last multiple of moves.
4. Proof games: White and Black, instead of making the best moves, they co-operate to make the game legally into a set position.
5. Symmetric position with odd number of moves.
6. Switchback: a piece moved out and came back into that same square.

Logic

1. Casework: what if A is true, what if A is false.
2. Process of elimination: If A is not true, B is not true, then C should be true.
3. Reverse argument: if there is at least ..., then there is atmost ...
4. Implication from truth: if A told the truth and A said X, then X is true.
5. Conflict of truth: if B said A lied, then both cannot be truth tellers.

Algebra

1. Equations. Radicals.
2. Sum and Products.
3. Sequences.
4. Inequalities. Estimations.

Combinatorics

1. Permutations.
2. Combinations.
3. Stars and bars.
4. Counting in two ways.
5. Algorithms and Processes. Pigeonhole Principle.

Geometry

1. Right triangles.
2. Rectangles.
3. Areas.

Number Theory

1. Divisibility.
2. Factorials. Perfect Powers.
3. Diophantine equations.

2.2 Problems

Problem 2.2.1 (HC-2022-SM1-R6-P1)**.** (*Beginner Level*)

In the **??** below, each side made *exactly* three (03) moves.

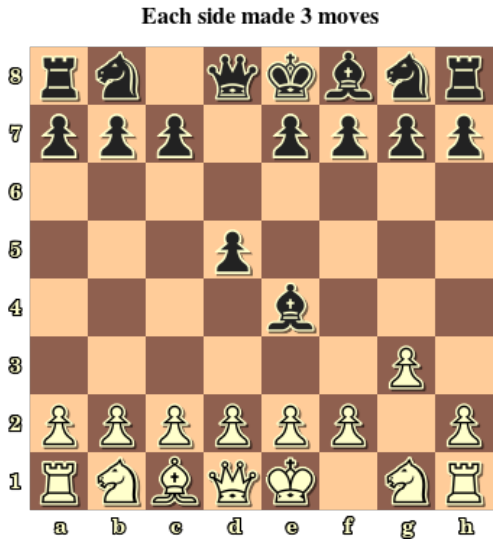

Figure 2.1: **??**

What were the moves?

How to provide your answer:

- If you think the moves were 1.e4 e5 2.d4 d5 3.exd5 exd4, submit 1.e4 e5 2.d4 d5 3.exd5 exd4
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 2.2.2 (HC-2022-SM1-R6-P2)**.** (*Beginner Level*)

In Seville, a barber shaved all of and only those inhabitants who did not shave themselves. Sam met four local people Alain, Bernard, Columbus, and Daniel. Here were what they said:

- Alain: I am the only barber in this town.
- Bernard: There is only one barber in this town.
- Columbus: One of them is lying. I shaved both of them.
- Daniel: Alain is lying.

If exactly two of them telling the truth, then who were shaved?

How to provide your answer:

- If you think Alain and Bernard were not shaved, Columbus and Daniel were shaved, then submit NA, NB, SC, SD.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 2.2.3 (HC-2022-SM1-R6-P3)**.** (*Beginner Level*)

The product of four factors $1! \cdot 2! \cdot 3! \cdot 4! = 288$ can be turned into a perfect square by removing the $2!$ factor,

$$1! \cdot 3! \cdot 4! = 144 = 12^2$$

Remove one factor from the product $P = 1! \cdot 2! \cdot 3! \cdot 4! \cdot 5! \cdot 6! \cdot 7! \cdot 8!$ to turn it into a perfect square.

How to provide your answer:

- If you think the product P can be turned into a perfect square by removing $1!$, submit $1!$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 2.2.4 (HC-2022-SM1-R6-P4)**.** (*Beginner Level*)

The recent Math Individual Contest has three (03) problems, each of which was given a score from $\{0, 1, 2, \ldots, 7\}$. It is know that for any two participants, there is at most one problem in which they had received the same score. *For example, there are no two participants whose ordered scores for the three problems are* $7, 1, 2$ *and* $7, 1, 5$*, but there might be two participants whose ordered scores are* $7, 1, 2$ *and* $7, 2, 1$.

What would be the maximum number of participants?

How to provide your answer:

- If you think that the maximum number of participants is 12, submit 12.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly smaller* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger* than your answer.

Problem 2.2.5 (HC-2022-SM1-R6-P5)**.** (*Beginner Level*)

A rectangle has perimeter 10 and diagonal $\sqrt{15}$. What is its area?

How to provide your answer:

- If you think that the area is 10, submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 2.2.6 (HC-2022-SM1-R6-P6)**.** (*Intermediate Level*)

In the **??** below, each side made *exactly* four (04) moves.

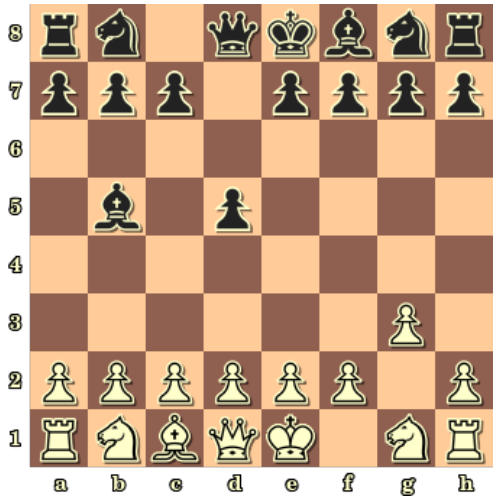

Figure 2.2: **??**

What were the moves?

How to provide your answer:

- If you think the moves were 1.e4 e5 2.d4 d5 3.exd5 exd4 4. f4 f5, submit 1.e4 e5 2.d4 d5 3.exd5 exd4 4. f4 f5
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 2.2.7 (HC-2022-SM1-R6-P7)**.** (*Intermediate Level*)

In Milan, the barbers' club obeys the following conditions:

- Condition 1: If any member has shaved any member, then all members have shaved him, through not necessarily all at the same time. No member has shaved himself.
- Condition 2: Four of the members are named Khoa, Minh, Quan, and Tung. Khoa has shaved Tung.
- Condition 3: After being shaved by any member of the club, a member must pay 1 Euro into the club's treasury.

Was Quan shaved by Minh? And if the club has 100 members, then what is the minimum amount paid by the members into the treasury?

How to provide your answer:

- If you think Quan was not shaved by Minh, and the minimum amount paid into the treasury is at least 1 Euro, then submit $N, 1$.
- If you think Quan was shaved by Minh, and the minimum amount paid into the treasury is at least 2 Euro, then submit $Y, 2$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly larger* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly smaller* than your answer.

Problem 2.2.8 (HC-2022-SM1-R6-P8)**.** (*Intermediate Level*)

Find the *positive* real number x, if

$$\sqrt{2x + 3\sqrt{2x + 3\sqrt{2x + \ldots + 3\sqrt{5x}}}} = x$$

How to provide your answer:

- If you think that the value of x is 3.14, submit 3.14
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 2.2.9 (HC-2022-SM1-R6-P9)**.** (*Intermediate Level*)

What is the maximum number of ways are there to insert + signs between the digits of

$$\underbrace{111111111111111}_{\text{fifteen 1's}}$$

so that the result will be a multiple of 30?

How to provide your answer:

- If you think the maximum number of ways is 100, then submit 100.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly smaller* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger* than your answer.

Problem 2.2.10 (HC-2022-SM1-R6-P10)**.** (*Intermediate Level*)

The Challenge The Bezout's Lemma states that for any positive integer pair (m, n), where $m < n$ and $\gcd(m, n) = 1$, there exist many pairs of integers (x, y) such that $mx+ny = 1$. *For example, for* $(m, n) = (3, 5)$

$$3 \cdot 2 + 5 \cdot (-1) = 3 \cdot (-3) + 5 \cdot 2 = 3 \cdot 7 + 5 \cdot (-4) = \ldots = 1$$

Among all possible (x, y) pairs , you have to find the pair (x, y) where $|x| + |y|$ is minimal. *The* $(2, -1)$ *pair in the above example, is the minimal pair, since*

$$|2| + |-1| = 3 < |-3| + |2| = 5 < |7| + |-4| = 11 < \ldots$$

Input The input (hc-2022-1-6-10-input.txt) will consist of a set of lines, each line contains two positive integers $((m, n))$, where $m < n$ and $\gcd(m, n) = 1$, each of m and n is less than $1,000,000,001$, and separating by a space.

Output The output (hc-2022-1-6-10-output.txt) will consist of a set of lines, each line contains two positive integers $((x, y))$ separating by a space, which is the corresponding *minimal* pair for the given two positive integers $((m, n))$.

Sample Input

```
3 5
5 7
7 10
```

Output

```
2 -1
3 -2
3 -2
```

How to your troop can compete in this Coding Challenge?

- **Preparation:** first you have to study the problem specified and should fully understand the problem. If you have any questions, please ask. You have *one week* before the contest to make preparation based on the description of the challenge. You have **to code a problem solver for each problem**. The code have to be tested thoroughly to make sure that it works correctly.
- **Solving the Challenge:** your troop *receives the Input* file in the designated shared folder. Depending on the challeng, an Input file might contain data for one or more cases (of the same challenge). Your problem solver has to solve each and every case, then output the results as required into an Output file. Your troop have 30 **minutes** to work on the challenge and *submits the Output* file in the designated folder.
- **Verification:** The jury verify your submissions. Any submissions that contain errors will be discarded. No resubmission is allowed. The **number of correctly solved** and **successfully submitted** cases is recorded as your troop's *score* for the challenge.

How your troop's performance is graded:

- If your troop's *score* is higher or equal to the *score* of you opponent for the challenge then your troop scores 1 hit, otherwise 0 hit.
- If your troop cannot submit the output for the challenge, then your troop scores 0 hit, regardless of the score of you opponent.
- It is allowed that your troop create/modify the code during the contest, or attempt to solve the challenge without coding.

Problem 2.2.11 (HC-2022-SM1-R6-P11)**.** (*Advanced Level*)

Consider the **??** below. First, *remove* one chess piece and then *check mate* Black in two (2) moves.

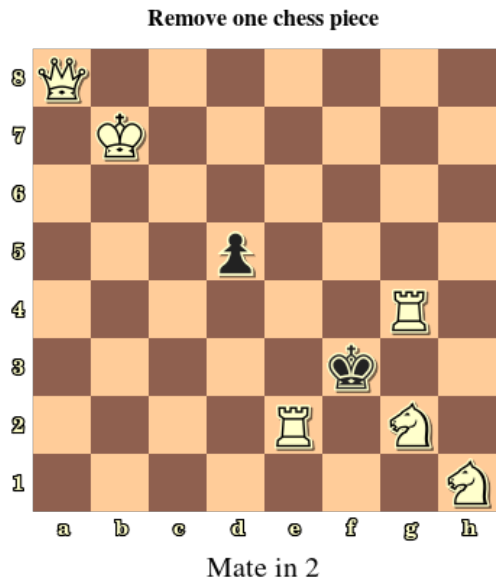

Figure 2.3: **??**

Which piece to be removed? What were the moves to mate?

How to provide your answer:

- If you think that the White rook at g4 has to be removed and the two moves to mate are 1. Re4 Kf4 2. Re3, then submit Rg4, 1. Re4 Kf4 2. Re3
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 2.2.12 (HC-2022-SM1-R6-P12)**.** (*Advanced Level*)

The population of Barcelona is approximately 5,624,000. The famous Barber's Association of Barcelona allows only local barbers to join. It has the following rules:

- Rule 1: Every member of the association has shaved at least one member.
- Rule 2: No member has ever shaved himself.
- Rule 3: No member has ever been shaved by more than one member.
- Rule 4: There is only one member who has never been shaved at all.

At most how many members can the association have?

How to provide your answer:

- If you think the association has at most 10 members, submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly smaller* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger* than your answer.

Problem 2.2.13 (HC-2022-SM1-R6-P13)**.** (*Advanced Level*)

A product of two different odd positive integers $(2a+1)(2b+1)$ (a and b are positive integers, $a \neq b$) can be written as a difference of two *non-zero* perfect squares as follow,

$$(2a+1)(2b+1) = (a+b+1)^2 - (a-b)^2$$

For example $3 \cdot 5 = 15 = 16 - 1 = 4^2 - 1^2,\ 5 \cdot 11 = 55 = 64 - 9 = 8^2 - 3^2$. However, the above formula cannot be used to express any *odd perfect square* $(2a+1)^2$ as a difference of two *non-zero* perfect squares, since the term $a - b$ is 0 when $a = b$.

Find a way to express any *odd perfect square* $(2a+1)^2$ as a difference of two *non-zero* perfect squares.

How to provide your answer:

- If you think $(2a)^2$ and a^2 are the two non-zero perfect squares whose difference is $(2a+1)^2$, submit $(2a, a)$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 2.2.14 (HC-2022-SM1-R6-P14)**.** (*Advanced Level*)

In the Fall House Championship each pair of houses play 2 games. The winner of a match gets 2 points, the loser gets 0, and there are no draws. A single house wins the championship with 26 points and exactly two houses share the last position with 20 points.

How many houses participated in the championship?

How to provide your answer:

- If you think there were 5 houses, then submit 5.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 2.2.15 (HC-2022-SM1-R6-P15)**.** (*Intermediate Level*)

The Challenge You are given a list of lowcase words, each does not exceed 16 letters. Your task is to **all pairs of words** in the list such that they **differ by a single letter.** *For example,* booster *and* rooster *differ by their first letters, but* coasted *and* coasts *do not.*

Input The input (hc-2022-1-6-15-input.txt) will consist of a set of lines, each line contains one lowcase word that does not exceed 16 letters.

Output The output (hc-2022-1-6-15-output.txt) will consist of a set of lines, each line contains a pairs of words that differ by a single letter. The first word in the pair must preceed the second word by the order of appearance in the input.

Sample Input

```
boosts
booster
rooster
roaster
coasted
roasted
coastal
postal
posted
```

Output

```
booster rooster
rooster roaster
roaster roasted
coasted roasted
```

How to your troop can compete in this Coding Challenge?

- **Preparation:** first you have to study the problem specified and should fully understand the problem. If you have any questions, please ask. You have *one week* before the contest to make preparation based on the description of the challenge. You have **to code a problem solver for each problem**. The code have to be tested thoroughly to make sure that it works correctly.
- **Solving the Challenge:** your troop *receives the Input* file in the designated shared folder. Depending on the challeng, an Input file might contain data for one or more cases (of the same challenge). Your problem solver has to solve each and every case, then output the results as required into an Output file. Your troop have 30 **minutes** to work on the challenge and *submits the Output* file in the designated folder.
- **Verification:** The jury verify your submissions. Any submissions that contain errors will be discarded. No resubmission is allowed. The **number of correctly solved** and **successfully submitted** cases is recorded as your troop's *score* for the challenge.

How your troop's performance is graded:

- If your troop's *score* is higher or equal to the *score* of you opponent for the challenge then your troop scores 1 hit, otherwise 0 hit.
- If your troop cannot submit the output for the challenge, then your troop scores 0 hit, regardless of the score of you opponent.
- It is allowed that your troop create/modify the code during the contest, or attempt to solve the challenge without coding.

2.3 Coding Challenge Inputs

?? Input as below (see hc-2022-1-6-10-input.txt)

```
13 31
121 457
997 823772
3421 43772
123456788 1000000003
```

?? Input as below (see hc-2022-1-6-15-input.txt)

```
warm
ward
card
cord
cold
table
cable
carle
carls
carps
corps
coops
crops
crows
crown
stone
shone
shine
chine
chins
coins
corns
cores
cones
coney
money
```

2.4 Answers

Problem 1: 1. g3 d5 2. Bg2 Bf5 3. Be4 Bxe4

Problem 2: NA, SB, SC, SD

Problem 3: 4!

Problem 4: 64

Problem 5: 5

Problem 6: 1. g3 d6! 2. Bg2 Bd7 3. Bc6 d5 4. Bb5 Bxb5

Problem 7: Y, 4950

Problem 8: 5

Problem 9: 2002

Problem 10: Output as below

```
12 -5
34 -9
273489 -331
28661 -2240
569306932 -70284805
```

Problem 11: Several solutions

- Ng2, 1. Rgg2! Kf4 2. Qf8++
- Ng2, 1. Reg2! Qe3 2. Qa3++
- d5, 1.. Kxe2! 2. Qd8 Kf1/Kf3 3. Qd1++
- d5, 1.. Kxg4! 2. Qa5 Kf3/Kh3 3. Qh5++

Problem 12: 0

Problem 13: $(2a^2 + 2a + 1,\ 2a^2 + 2a)$

Problem 14: 12

Problem 15: Output as below

```
warm ward
ward card
card cord
cord cold
table cable
cable carle
carle carls
carls carps
carps corps
corps coops
corps corns
corps cores
coops crops
crops crows
crows crown
stone shone
shone shine
shine chine
chine chins
chins coins
coins corns
corns cores
cores cones
cones coney
coney money
```

2.5 Solutions

Solution. **??**

It's pretty straight forward.

Figure 2.4: 1. g3 d5

Figure 2.5: 2. Bg2 Bf5

Both bishops took a *pause*: $\boxed{\text{1. g3 d5 2. Bg2 Bf5 3. Be4 Bxe4.}}$ □

Solution. **??**

If Alain told the truth, then Bernard did too. In this case, Columbus's statement was false, so he was not a barber. Daniel lied too. Thus, Alain shaved Bernard, Columbus, and Daniel. Since there was no other barber in town, so Alain was not shaved.

If Alain lied, then Bernard, Columbus, and Daniel all told the truth. This is impossible.

The answer is $\boxed{NA, SB, SC, SD.}$ □

Solution. **??**

Note that $2 = 1 \cdot 2,\ 4 = 2 \cdot 2,\ 6 = 3 \cdot 2,\ 8 = 4 \cdot 2$, so

$$2! \cdot 4! \cdot 6! \cdot 8! = (1! \cdot 1 \cdot 2)(3! \cdot 2 \cdot 2)(5! \cdot 3 \cdot 2)(7! \cdot 4 \cdot 2) = (1! \cdot 3! \cdot 5! \cdot 7!) \cdot 2^4 \cdot \underbrace{1 \cdot 2 \cdot 3 \cdot 4}_{4!}$$

$$P = 1! \cdot 2! \cdot \cdots \cdot 8! = (1! \cdot 3! \cdot 5! \cdot 7!) \cdot (2! \cdot 4! \cdot 6! \cdot 8!) = (1! \cdot 3! \cdot 5! \cdot 7! \cdot 2^2)^2 \cdot 4!$$

Thus, by removing 4!, P can be turned into a perfect square. The answer is $\boxed{4!.}$ □

Solution. **??**

Assume there exists 65 participants. By the Pigeonhole Principle, at least $\lceil \frac{65}{8} \rceil = 9$ participants got the same score on one problem. Then, again by the Pigeonhole Principle, from these 9 participants, at least $\lceil \frac{9}{8} \rceil = 2$ participants got the same points on another problem. This is the contradicition, so the number of contestants cannot be more than 64.

The maximum 64 is attainable. Consider the lattice points $(x, y, x - y)$ such that $x, y, z \in [0, 7]$, where values are calculated mod 8. □

Solution. **??**

Let w and ℓ be the width and length of the rectangle, by Pythagorean theorem,

$$\begin{cases} \sqrt{w^2 + \ell^2} = \sqrt{15} \\ 2(w + \ell) = 10 \end{cases} \Rightarrow \begin{cases} w^2 + \ell^2 = 15 \\ w + \ell = 5 \end{cases} \Rightarrow (w + \ell)^2 - (w^2 + \ell^2) = 10 \Rightarrow 2w\ell = 10 \Rightarrow w\ell = 5$$

Thus, the area is $\boxed{5.}$ □

Solution. **??**

The key move is the first move of Black: 2.. d6.

Figure 2.6: 1. g3 d6!

Figure 2.7: 2. Bg2 Bd7

Then, the other nice move is the third move of White: 3. Bc6

Figure 2.8: 3. Bc6 d5

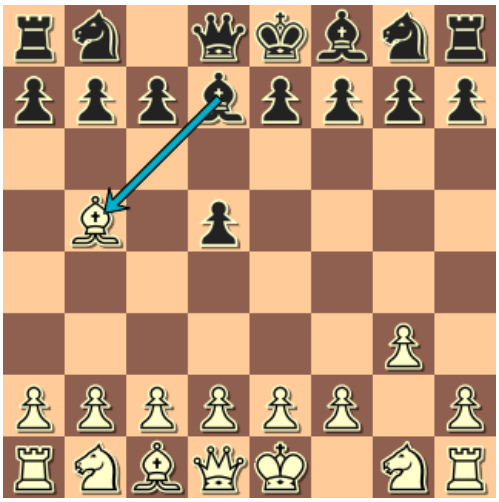

Figure 2.9: 4. Bb5 Bxb5

Thus, the answer is $\boxed{\text{1. g3 d6! 2. Bg2 Bd7 3. Bc6 d5 4. Bb5 Bxb5.}}$ □

Solution. **??** According to *Condition 2* Khoa has shaved Tung, so by *Condition 1*, every member of the clubs shaved Khoa. Thus, by *Condition 1 and 2*, both Minh and Quan shaved Khoa. Therefore, by *Condition 1*, Minh and Quan were also shaved by others. This implies that Minh shaved Quan. By the same reasoning, everyone was shaved by any other member. So, by *Condition 3*, the amount was paid is at least $\frac{1}{2}(100 \cdot 99) = 4950$ Euro.

The answer is $\boxed{\text{Y, 4950.}}$ □

Solution. **??** Note that

$$\sqrt{2x + 3\sqrt{2x + 3\sqrt{2x... + 3\sqrt{5x}}}} = x$$

$$\Rightarrow \sqrt{2x + 3\underbrace{\sqrt{2x + 3\sqrt{2x... + 3\sqrt{5x}}}}_{x}} = x$$

$$\Rightarrow \sqrt{2x + 3x} = x \Rightarrow 5x = x^2 \Rightarrow x = 5$$

The answer is $\boxed{5.}$ □

Solution. **??**

First, since there are 15 digits of 1, so it does not matter how many + signs we insert into the sequence, the result always is divisible by 3. Therefore, we just need to consider how to make the result divisible by 10. Since there are 15 digits, each ends with 1, so we will need to create 10 numbers. Thus, the number of + signs is 9. They are to be inserted into $15 - 1 = 14$ positions.

Thus, the number of possible ways is $\boxed{\binom{14}{9} = 2002.}$ □

Solution. **??** Following is an example solution in Python.

```
def bezout(m, n):
    d, r = m, n # d becomes gcd(m, n), r is remainder
    x, y = 1, 0 # the coefficient of m, n
    u, v = 0, 1 # helpers
    while r:
        q = d // r
        d, r = r, d % r
        x, u = u, x - q * u
        y, v = v, y - q * v
    return x % (n // d), y % (-m // d) # modulus in this way so that y is positive and z is negative

if __name__ == "__main__":
    with open('hc-2022-1-6-10-input.txt', 'rt') as i_file:
        lines = i_file.readlines()

        with open('hc-2022-1-6-10-output.txt', 'wt') as o_file:
            for line in lines:
                m, n = [int(e) for e in line.strip().split()]
                x, y = bezout(m, n)
                o_file.write(f"{x} {y}\n")
```

□

Solution. **??**

There are two possible solutions. The first is to remove the white knight at g2 and then using the

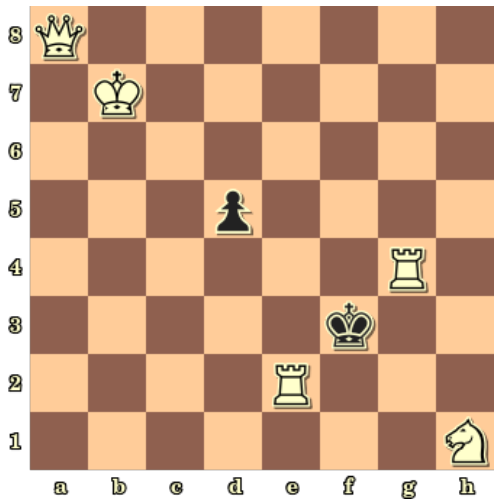

Figure 2.10: Remove Ng2

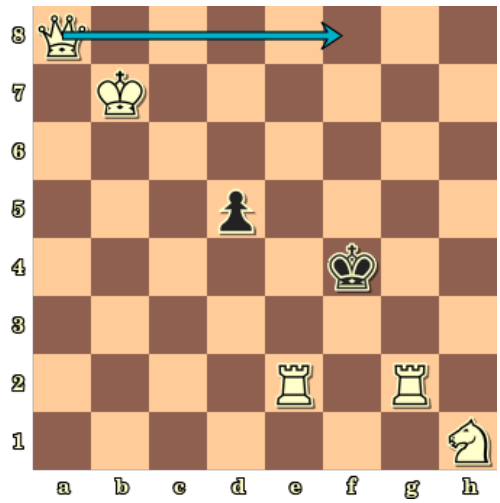

Figure 2.11: 1. Rgg2! Kf4 2. Qf8++

combination of the white queen and two rooks to mate: 1. Rgg2! Kf4 2. Qf8++. A symmetrical solution is 1. Reg2! Qe3 2. Qa3++. Thus, the first set of solutions is to remove the white knight at g2: Ng2, 1. Rgg2! Kf4 2. Qf8++ or Ng2, 1. Reg2! Qe3 2. Qa3++.

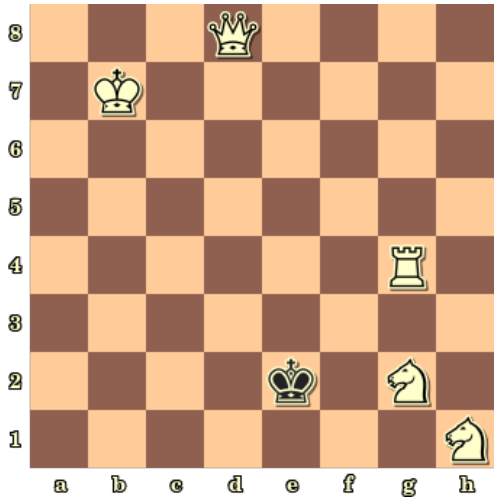

Figure 2.12: 1.. Kxe2! 2. Qd8

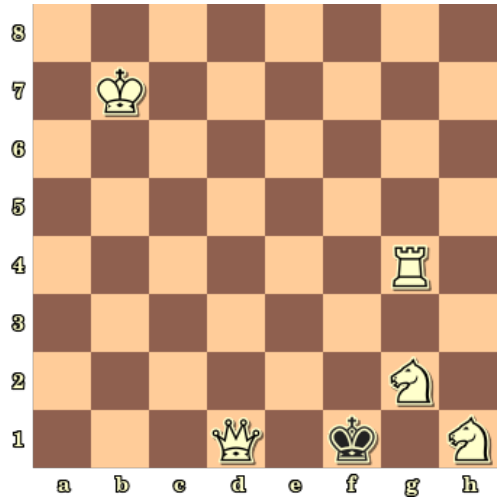

Figure 2.13: 2.. Kf1/Kf3 3. Qd1++

The other approach is to remove the black pawn at d5. There are two ways to check mate. The first one is 1.. Kxe2! 2. Qd8 Kf1/Kf3 3. Qd1++. and a symmetrical solution is 1.. Kxg4! 2. Qa5 Kf3/Kh3 3. Qh5++. Thus, the second set of solution is to remove the black pawn at d5: d5, 1.. Kxe2! 2. Qd8 Kf1/Kf3 3. Qd1++ or d5, 1.. Kxg4! 2. Qa5 Kf3/Kh3 3. Qh5++. □

Solution. **??**

Assume that the association has at least one member. Then according to *Rule 4*, there is a member B_1 who has never been shaved at all. Since, according to *Rule 1*, B_1 has shaved at least one *other* member who according to *Rule 2* is not B_1.

Let him be B_2. Since B_2 shaved some one who is not B_1 (never been shaved at all), so it is another person, say B_3.

Since B_3 shaved someone, who was neither B_1 (never been shaved at all), nor B_2 (because he was shaved by B_1), so that would be another person, say B_4.

Now, B_4 shaved someone, who was neither B_1 (never been shaved at all), nor B_2 (because he was shaved by B_1), nor B_3 (because he was shaved by B_2), so there would be another person, say B_5.

By following this logic, there exists an *infinite sequence* of members, who shaved each other. The number of inhabitants of Barcelona is *finite*, so it is a contradiction.

Therefore the association has $\boxed{\text{0 members.}}$ □

Solution. **??** It is simple to verify that,

$$(2a+1)(2a+1) = 4a^2+4a+1 = (2a^2+2a+1)^2 - (2a^2+2a)^2$$

We can also solve the Diophantine equation for positive integers x and y,

$$x^2 - y^2 = (2a+1)^2 = (4a^2+4a+1)\cdot 1 \Rightarrow \begin{cases} x+y = 4a^2+4a+1 \\ x-y = 1 \end{cases} \quad \Rightarrow (x,y) = (2a^2+2a+1, 2a^2+2a)$$

Thus, the answer is $\boxed{(2a^2+2a+1,\ 2a^2+2a).}$ □

Solution. **??**

Let assume that there were n houses. Since each pair of houses play 2 games, the number of games is $\frac{1}{2}n(n-1)$, and the total number of points for a game is 2. So the total number of points from all the games is,

$$2 \cdot \frac{1}{2}n(n-1)\cdot 2 = 2n(n-1).$$

Since for a game, a house can earn 0 or 2 points, so the score of each house is an even number. The highest score is 26 and the lowest is 20, so the other scores can only be 22 or 24.

Let assume that there are x houses who scored 22 and y houses who scores 24, then

$$\begin{cases} x+y = n-3 \\ 2\cdot 20 + 22x + 24y + 26 = 2n(n-1) \end{cases} \quad \Rightarrow x = -n^2+13n-3,\ y = n(n-12)$$

Since $y \geq 0$, so $n \geq 12$, and since $x = -n(n-13)-3 \geq 0$, so $n < 13$, therefore $n = 12,\ (x,y) = (9,0)$.

Thus, the number of houses is $9+2+1 = 12.$. The answer is $\boxed{12.}$ □

Solution. **??** Following is an example solution in Python.

```
def find_pairs(words):
    pairs = []
    count = len(words)

    for f in range(0, count-1):
        for s in range(f+1, count):
            if len(words[f]) == len(words[s]) and \
                sum([0 if words[f][i] == words[s][i] else 1 for i in range(0,
                    len(words[f]))]) == 1:
                pairs.append([words[f], words[s]])

    return pairs

if __name__ == "__main__":
    with open('hc-2022-1-6-15-input.txt', 'rt') as i_file:
        words = [w.strip() for w in i_file.readlines()]

        pairs = find_pairs(words)

        with open('hc-2022-1-6-15-output.txt', 'wt') as o_file:
            for first, second in pairs:
                o_file.write(f"{first} {second}\n")
```

□

Chapter 3

MIC R6

3.1 Topics

Algebra

1. Ratios. Proportions.
2. Exponents. Logarithms.
3. Sums and Products. Identities. Factoring.
4. Inequalities. Estimations.

Combinatorics

1. Counting. Complementary Counting. Direct Counting. Casework.
2. Permutations.
3. Sets. Principle of Inclusion-Exclusion.
4. Combinatorial Geometry.

Geometry

1. Right triangles.
2. Squares. Rectangles.
3. Area.
4. Cirle.
5. Cubes. Tetrahedron. Volume.

Number Theory

1. Divisibility. GCD. LCM.
2. Diophantine Equations.

3.2 Rules

- The total time to complete the test is 90 minutes.
- The test consists of 10 multiple-choice and 4 show-you-work problems. To answer each of 10 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 6 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.

3.3 Problems

Problem 3.3.1 (MIC-2022-SM1-R6-P1). (*6 points*)

The following diagram shows an eight-sided polygon $ABCDEFGH$ with side lengths $8, 15, 8, 8, 8, 6, 8$, and 29 as shown. All of its angles are right angles. Turn this eight-sided polygon into a six-sided polygon by connecting B to D with an edge and E to G with an edge to form polygon $ABDEGH$. See **??**.

Find the area of $ABDEGH$.

(A) 80 (B) 232 (C) 256 (D) 296 (E) 380

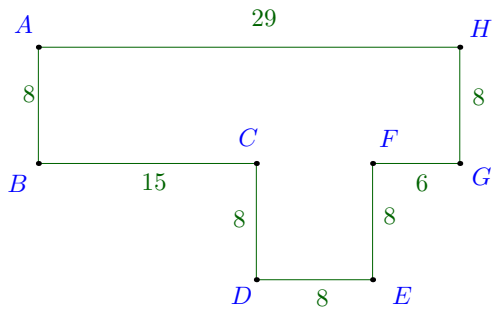

Figure 3.1: **??**

Problem 3.3.2 (MIC-2022-SM1-R6-P2). (*6 points*)

Pete's research shows that the number of nuts collected by the squirrels in any park is proportional to the square of the number of squirrels in that park. If Pete notes that five squirrels in a park collect 60 nuts, how many nuts are collected by 20 squirrels in a park?

(A) 360 (B) 720 (C) 960 (D) 1500 (E) 1800

Problem 3.3.3 (MIC-2022-SM1-R6-P3). (*6 points*)

How many six-digit positive integers do not either start or end with 6?

(A) $180,000$ (B) $710,000$ (C) $720,000$ (D) $730,000$ (E) $7,200,00$

Problem 3.3.4 (MIC-2022-SM1-R6-P4). (*6 points*)

The greatest common divisor of n and 480 is 12. The least common multiple of n and 480 is 3360. Find the sum of the digits of n.

(A) 8 (B) 9 (C) 10 (D) 12 (E) 14

Problem 3.3.5 (MIC-2022-SM1-R6-P5). (*6 points*)

There are relatively prime positive integers m and n so that

$$\frac{m}{n} = \log_4\left(32^{\log_9 27}\right).$$

Find the product mn.

(A) 60 (B) 30 (C) 15 (D) 10 (E) 5

Problem 3.3.6 (MIC-2022-SM1-R6-P6). (*6 points*)

In the following **??** two sides of a square are tangent to a circle with diameter 4. One corner of the square lies on the circle. There are positive integers m and n so that the area of the square is $m + \sqrt{n}$. Find $m + n$.

(A) 36 (B) 38 (C) 252 (D) 510 (E) 536

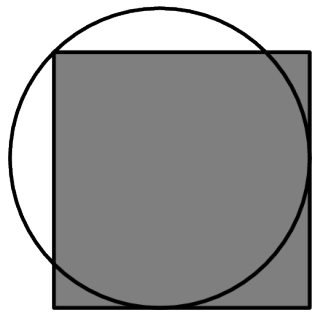

Figure 3.2: **??**

Problem 3.3.7 (MIC-2022-SM1-R6-P7). (*6 points*)

How many triangles appear in the **??** below?

(A) 48 (B) 56 (C) 72 (D) 76 (E) 84

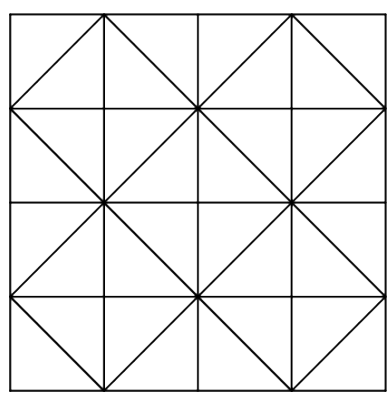

Figure 3.3: **??**

Problem 3.3.8 (MIC-2022-SM1-R6-P8). (*6 points*)

Let a, b, and c be positive real numbers such that $a^2+b^2+c^2 = 2163$ and $(a+b)^2+(b+c)^2+(c+a)^2 = 7788$. Find $a + b + c$.

(A) 32 (B) 47 (C) 56 (D) 75 (E) 87

Problem 3.3.9 (MIC-2022-SM1-R6-P9). (*6 points*)

n is the largest three-digit number such that it is equal to the sum of its digits plus twice the product of its digits. Find the sum of the digits of n.

(A) 12 (B) 15 (C) 19 (D) 21 (E) 24

Problem 3.3.10 (MIC-2022-SM1-R6-P10). (*6 points*)

In how many ways can you write 12 as an ordered sum of *three, four, or five* positive integers where the smallest of those integers is equal to 2? *For example,* $2+5+5$, $2+2+3+5$, *and* $2+2+2+2+4$ *are three such ways, but* $10+2$ *is not.* Furthermore, $2+5+5$ and $5+2+5$ are considered the same.

(A) 9 (B) 10 (C) 11 (D) 12 (E) 13

Problem 3.3.11 (MIC-2022-SM1-R6-P11). (*10 points*)

Asheville, Bakersfield, Charter, and Darlington are four small towns along a straight road in that order. The distance from Bakersfield to Charter is one-third the distance from Asheville to Charter and one-quarter the distance from Bakersfield to Darlington.

If it is 15 miles from Bakersfield to Charter, how many miles is it from Asheville to Darlington?

Problem 3.3.12 (MIC-2022-SM1-R6-P12). (*10 points*)

Find integer n such that both $n-2174$ and $n+2174$ are perfect cubes. *A positive integer a is a perfect cube if there exists a positive integer b such that $a=b^3$.*

Problem 3.3.13 (MIC-2022-SM1-R6-P13). (*10 points*)

Six children stand in a line outside their classroom, when they enter the classroom, they sit in a circle in random order. Let name the children standing in the line as $ABCDEF$.

(*5 points*) List all possible arrangements when the children sit in a circle, in clockwise direction, starting with the child A.

(*5 points*) There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that *no two children who stood next to each other* in the line end up *sitting next to each other* in the circle.

Find the difference $n-m$.

Problem 3.3.14 (MIC-2022-SM1-R6-P14). (*10 points*)

Let A, B, C, D, E, F, G, H be the eight vertices of a $6\times 6\times 6$ cube as shown. The two figures $ACFH$ and $BDEG$ are congruent regular tetrahedra.

(*5 points*) Find the volume of $ACFH$.

(*5 points*) Find the volume of the intersection of these two tetrahedra.

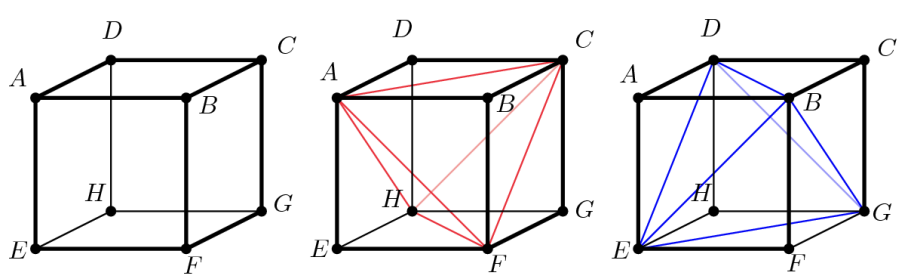

Figure 3.4: **??**

3.4 Grading

Answers for multiple-choice problems.

Problem 1: E

Problem 2: C

Problem 3: C

Problem 4: D

Problem 5: A

Problem 6: B

Problem 7: E

Problem 8: D

Problem 9: E

Problem 10: B

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 11: 2 points if figure out that $BC = 15$ or $AC = 45$.

Problem 12: 2 points if can establish an equation of $b^3 - a^3 = 4348$. 2 additional points if can figure out 1087 is a prime number or $b - a = 4$.

Problem 13: Separately grading for each part,

(a) 2 points if can list at least 5 cases correctly.

(b) 1 points if can find the total arrangements is $5! = 120$ (not $6! = 720$ becaus they sit around a table)

Problem 14: Separately grading for each part,

(a) 2 points if can state the tetrahedron's volume with a correct formula

(b) 2 points if can dissect the calculation of the intersection based on smaller tetrahedra.

3.5 Solutions

Solution. **??**

Let I and J be points as shown in **??**.

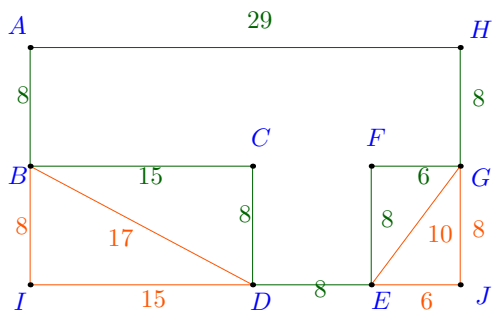

Figure 3.5: I and J

$$[ABDEGH] = [AIJH] - [BID] - [EJG] = 16 \cdot 29 - \frac{1}{2}15 \cdot 8 - \frac{1}{2}6 \cdot 8 = 380$$

The answer is $\boxed{E.}$ □

Solution. **??** Five squirrels collect 60 nuts, so 20 squirrels collect $\left(\frac{20}{5}\right)^2 \cdot 60 = \boxed{960}$ nuts The answer is $\boxed{C.}$ □

Solution. [First solution] **??**

The number of six-digit numbers is $9 \cdot 10^5$. The number of six-digit numbers that start with 6 is 10^5, the number of six-digit numbers that end with 6 is $9 \cdot 10^3 \cdot 9$, and the number of six-digit numbers that start and end with 6 is 10^4. Thus, by Principle of Inclusion-Exclusion, the number of six-digit numbers that do not either start with 6 is

$$9 \cdot 10^5 - (10^5 + 9 \cdot 10^3 \cdot 9 - 10^4) = 720000$$

The answer is $\boxed{C.}$ □

Solution. [Second solution] **??**

By direct counting, the number of six-digit numbers that do not either start with 7 is

$$8 \cdot 10^4 \cdot 9 = 720000$$

The answer is $\boxed{C.}$ □

Solution. **??**

$$\gcd(n, 480) \cdot \operatorname{lcm}(n, 480) = n \cdot 480 \Rightarrow n = \frac{\gcd(n, 480) \cdot \operatorname{lcm}(n, 480)}{480} = \frac{12 \cdot 3360}{480} = 84$$

The answer is $\boxed{D.}$ □

Solution. **??**

$$\log_9 27 = \frac{3}{2} \Rightarrow (32)^{\frac{3}{2}} = (2)^{\frac{15}{2}} = (4)^{\frac{15}{4}} \Rightarrow \log_4 (32)^{\frac{3}{2}} = \log_4 (4)^{\frac{15}{4}} = \frac{15}{4}.$$

Thus, $mn = 15 \cdot 4 = 60$. The answer is $\boxed{A.}$ □

Solution. **??**

Let the radius of the circle be r and the side of the square be s. Then the diagonal of the square is $r + r\sqrt{2}$ (by drawing the center and Pythagorean theorem),

$$2 + 2\sqrt{2} = s\sqrt{2} \Rightarrow s^2 = \frac{1}{2}4(1+\sqrt{2})^2 = 2(3+2\sqrt{2}) = 6 + 4\sqrt{2} = 6 + \sqrt{32}$$

Then $m + n = 6 + 32 = 38$. The answer is $\boxed{B.}$ □

Solution. **??** We do case work.

Case 1: we count the $1 - 1 - \sqrt{2}$ triangles. There are 2 in each of the 16 squares, for a total of 32.

Case 2: we count the $2 - 2 - 2\sqrt{2}$ triangles next. We see that there are $4 \cdot 4 = 16$ of them, four in each of 4 squares.

Case 3: we count the $\sqrt{2} - \sqrt{2} - 2$ triangles next. There are four in each $\sqrt{2}$ square, for a total of $5 \cdot 4 = 20$. But then there are also four on the sides, so we get 24.

Case 4: we count the $2\sqrt{2} - 2\sqrt{2} - 4$ triangles next. There are 4 of them.

Case 5: we count the $3 - 3 - 3\sqrt{2}$ triangles next. There are 8 of them.

Thus, there are in total $32 + 16 + 24 + 4 + 8 = 84$. The answer is $\boxed{E.}$ □

Solution. **??**

$$\begin{aligned} &a^2 + b^2 + c^2 = 2163, (a+b)^2 + (b+c)^2 + (c+a)^2 = 7788 \\ \Rightarrow\ &(a+b+c)^2 = (a^2+b^2+c^2) + 2(ab+bc+ca) = 7788 - 2163 = 5625 = 75^2 \end{aligned}$$

Thus, $a + b + c = 75$. The answer is $\boxed{D.}$ □

Solution. **??** Let $n = \overline{abc} = 100a + 10b + c$,

$$100a + 10b + c = (a+b+c) + 2abc \Rightarrow 2abc = 99a + 9b$$

In order to it be largest, $a = 9$

$$2bc = 99 + b \Rightarrow b(2c-1) = 9 \cdot 11 \Rightarrow b = 9, c = 6$$

Thus, the number is 996 and its sum of digits is $9 + 9 + 6 = 24$. The answer is $\boxed{E.}$ □

Solution. **??** Following are the number of ways 12 can be expressed as a sum of positive integers, each is at least 2. Thus, it is easy to see that there are $\boxed{11}$ ways 12 can be expressed as a sum of positive integers, and

2+2+2+2+2+2	
2+2+2+2+4	1
2+2+2+3+3	2
2+2+2+6	3
2+2+3+5	4
2+2+4+4	5
2+3+3+4	6
3+3+3+3	
2+2+8	7
2+3+7	8
2+4+6	9
2+5+5	10
3+3+6	
3+4+5	
4+4+4	
2+10	
3+9	
4+8	
5+7	
6+6	
12	

at least one of them is 2, The answer is $\boxed{C.}$ □

Solution. **??** Let A, B, C, D denote the cities. We get $BC = 15$, $AC = 3 \cdot 15 = 45$, and $BD = 4 \cdot 15 = 60$. Therefore $AB = AC - BC = 30$, $CD = BD - BC = 45$.

Thus $\boxed{AB + BC + CD = 15 + 30 + 45 = 90.}$ □

Solution. **??** Let a and b be positive integers such that,

$$\begin{cases} a^3 = n - 2174 \\ b^3 = n + 2174 \end{cases} \Rightarrow b^3 - a^3 = 4348 \Rightarrow (b-a)(b^2 + ba + a^2) = 2^2 \cdot 1087$$

Now, note that 1087 is a prime number, thus the only possible factoring is

$$\begin{cases} b - a = 4 \\ b^2 + ba + a^2 = 1087 \end{cases} \Rightarrow b^2 + ba + a^2 = (b-a)^2 + 3ab = 1087 \Rightarrow ab = 357 \Rightarrow a(a+4) = 357 = 17 \cdot 21$$

Thus $a = 17, b = 21$, and $\boxed{n = 7087.}$ □

Solution. **??** By rearranging them according to the requirements, we have only 10 possible arrangements

$$ACEBDF$$
$$ACEBFD$$
$$ACFDBE$$
$$ADBECF$$
$$ADBFCE$$
$$ADFBEC$$
$$AEBDFC$$
$$AECFBD$$
$$AFCEBD$$
$$AFDBEC$$

Thus, $\frac{m}{n} = \frac{10}{120} = \frac{1}{12} \Rightarrow m = 1, n = 12$ and $\boxed{n - m = 11.}$ □

Solution. **??** First, let d be the length of the cube side. It is easy to calculate the volume of a *corner* tetrahedron

$$\frac{1}{3} \cdot \frac{d \cdot d}{2} \cdot d = \frac{d^3}{6}$$

Therefore

$$V_{ACFH} = d^3 - 4 \cdot \frac{d^3}{6} = \frac{d^3}{3} = \frac{6^3}{3} = 72$$

Thus the volume of $ACFH$ is $\boxed{72.}$

For the second question, the two tetrahedra's intersection is show as below

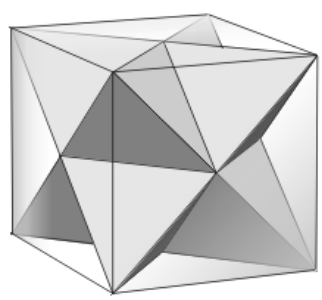

Figure 3.6: Intersection of the tetrahedra

Notice that the intersection of the tetrahedra is one of the bigger tetrahedra with the four smaller tetrahedra on its points cut off. Using the tetrahedron volume formula, and using $d\sqrt{2}$ and $d\frac{\sqrt{2}}{2}$ as the side lengths, thus the volume of the intersection as $\boxed{\frac{d^3}{3} - 4 \times \frac{d^3}{24} = \frac{d^3}{6} = \frac{6^3}{6} = 36.}$ □

Chapter 4

PC R1

4.1 Topics

Algebra

1. Fractions. Ratios. Percent.
2. Sums and Products.
3. Sequences. Periodic Sequences.
4. Equations. System of Equations.
5. Estimation. Inequalities.

Combinatorics

1. Counting. Corrections over Counting.
2. Complementary Counting.
3. Permutations.
4. Combinations.
5. Probability.

Geometry

1. Triangles. Right triangles. Isosceles Triangles.
2. Angle Chasing.
3. Area. Perimeter.
4. Similar Triangles. Similarly Ratio.

Number Theory

1. Divisibility.
2. Diophantine Equations.

Coding

1. String operations
2. Permutations.
3. Lists. Sets.
4. Math functions.
5. Number bases.
6. Comparing multiple conditions at once.

4.2 Rules

- The *total time* to complete the test is 60 minutes. The problems that have no answers or incorrect answers will be discussed after the test.
- **Middle school teams** are given **60 minutes** to work on **20 problems**, and **high school teams** are given **90 minutes** to work on **30 problems.**
- The teams may fill in answers to the contest problems during the prescribed time for their teams (60 minutes for middle school teams, 90 minutes for high school teams) following the start of their contest clock. At any time during the competition, contestants can submit answers to the problems. Answers may be submitted multiple times by the same team, but for each problem **only the last answer** received before the contest ends will be accepted and graded. Partial submissions are accumulated. Consequently, if a team submits an answer to problem 1 and later wises to submit an answer to problem 2, there is no need to resubmit the answer to problem 1. All members are allowed to work on the problems as a team. No help may be provided by persons not on their team. Team members must not reveal or discuss contest problems with anyone other than their immediate team members until the test ends.
- All problems are designed so that they can be worked without the aid of calculating devices. On the other hand, participants are *permitted to use calculators or computers*, for example, to write a computer program in order solve or better understand a problem. Team members may also *refer to books, computer programs, or other reference materials* available to them *locally* as long as the students seek this material without aid from persons not on their team. Any team with a member attempted to do *violate this rule* of the test *will be disqualified.*
- Team members *shall not* use computers or any other communication devices to *seek help* with problem solutions *from people or Internet resources* outside of their own team members.
- Grading:
  1. For a **correct answer** of *each problem*, the team is awarded with 1 point. For an *unanswered* problem or for an *incorrect answer*, the team is awarded with 0 point.
  2. The score of the team is the *sum of awarded points* for all the problems.

4.3 Problems

Problem 4.3.1 (PCT-2022-SM1-R1-P1)**.** In the **??** below $ABCD$ is a square and both $\triangle CFD$ and $\triangle CBE$ are equilateral. Find the degree measure of $\angle BEF$.

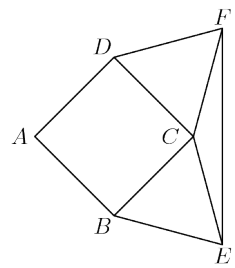

Figure 4.1: **??**

Problem 4.3.2 (PCT-2022-SM1-R1-P2)**.** $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{m}{n}$, where m and n are relatively prime positive integers. Find the difference $n-m$.

Problem 4.3.3 (PCT-2022-SM1-R1-P3)**.** The cross in the **??** below is made up of five congruent squares. The perimeter of the cross is 84. Find its area.

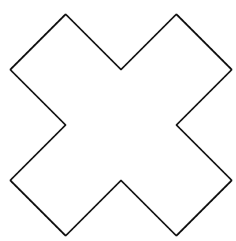

Figure 4.2: **??**

Problem 4.3.4 (PCT-2022-SM1-R1-P4)**.** One-third of the students who attend Grant School are taking Algebra. One-quarter of the students taking Algebra are also on the track team. There are 15 students on the track team who take Algebra. Find the number of students who attend Grant School and are not taking Algebra.

Problem 4.3.5 (PCT-2022-SM1-R1-P5)**.** The **??** below shows a 9×7 arrangement of 3×3 squares. Alternate squares of the grid are split into two triangles with one of the triangles shaded. Find the area of the shaded region.

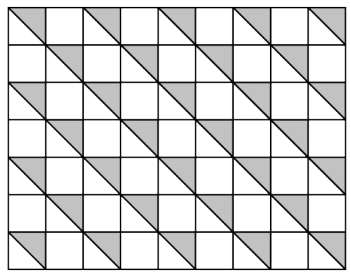

Figure 4.3: **??**

Problem 4.3.6 (PCT-2022-SM1-R1-P6)**.** The twenty-first century began on January 1, 2001 and runs through December 31, 2100. Note that February 1, 2020 fell on Saturdays and there were five Saturdays in February 2022. In how many years of the twenty-first century does February have five Saturdays or five Sundays?

Problem 4.3.7 (PCT-2022-SM1-R1-P7)**.** Inside the 3×4 rectangle $ABCD$ in the **??** below, point E is chosen a distance $AE = 1.5$ and $DE = 3.5$. The line segments from E to the four vertices of the rectangle are drawn. Find the area of the shaded region.

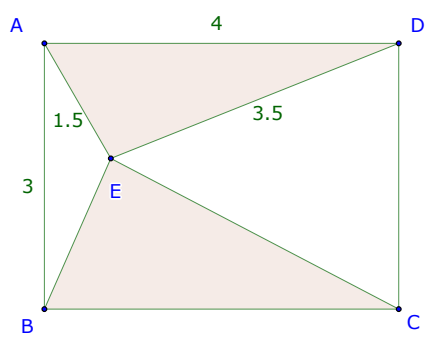

Figure 4.4: **??**

Problem 4.3.8 (PCT-2022-SM1-R1-P8)**.** Johan is swimming laps in the pool. At 12:17 he realized that he had just finished one-third of his workout. By 12:24 he had completed seven more laps, and he realized that he had just finished six-elevenths of his workout. After 12:24 how many more minutes must Johan swim to complete his workout?

Problem 4.3.9 (PCT-2022-SM1-R1-P9)**.** The **??** below shows a shaded region bounded by a semicircular arc of a large circle and two smaller semicircular arcs. The smallest semicircle has radius 8, and the shaded region has area 160π. Find the diameter of the large circle.

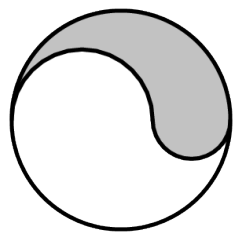

Figure 4.5: **??**

Problem 4.3.10 (PCT-2022-SM1-R1-P10)**.** Real numbers x and y satisfy the two equations,

$$\begin{cases} \frac{1}{x} + \frac{1}{y} & = 5 \\ \frac{2}{x} + \frac{3}{y} & = 9 \end{cases}$$

Evaluate $\dfrac{7-4y}{x}$.

Problem 4.3.11 (PCT-2022-SM1-R1-P11)**.** Shenelle has some square tiles. Some of the tiles have side length 5 cm while the others have side length 3 cm. The total area that can be covered by the tiles is exactly 2022cm^2. Find the least number of tiles that Shenelle can have.

Problem 4.3.12 (PCT-2022-SM1-R1-P12)**.** The first number in the following sequence is 1. It is followed by two 1's and two 2's. This is followed by three 1's, three 2's, and three 3's. The sequence continues in this fashion.

$$1, 1, 1, 2, 2, 1, 1, 1, 2, 2, 2, 3, 3, 3, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, \ldots.$$

Find the 2022th term of this sequence.

Problem 4.3.13 (PCT-2022-SM1-R1-P13)**.** A jar was filled with jelly beans so that 50% of the beans were red, 34% of the beans were green, and 16% of the beans were blue. Alan then removed the same number of red jelly beans and green jelly beans from the jar so that now 20% of the jelly beans in the jar are blue. What percent of the jelly beans in the jar are now red?

Problem 4.3.14 (PCT-2022-SM1-R1-P14)**.** Steve needed to address a letter to 2743 Becker Road. He remembered the digits of the address, but he forgot the correct order of the digits, so he wrote them down in random order. The probability that Steve got exactly two of the four digits in their correct positions is $\dfrac{m}{n}$ where m and n are relatively prime positive integers. Find the difference $n - m$.

Problem 4.3.15 (PCT-2022-SM1-R1-P15)**.** A large rectangle is tiled by some 1×1 tiles. In the center there is a small rectangle tiled by some white tiles. The small rectangle is surrounded by a red border which is five tiles wide. That red border is surrounded by a white border which is five tiles wide. Finally, the white border is surrounded by a red border which is five tiles wide. The resulting pattern is pictured below. The area of the large rectangle is 4819.

How many red tiles are used to tile the large rectangle?

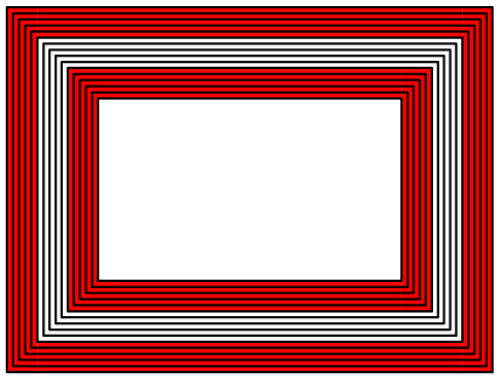

Figure 4.6: **??**

Problem 4.3.16 (PCT-2022-SM1-R1-P16)**.** The Bell Zoo has the same number of rhinoceroses as the Carlton Zoo has lions. The Bell Zoo has three more elephants than the Carlton Zoo has lions. The Bell Zoo has the same number of elephants as the Carlton Zoo has rhinoceroses. The Carlton Zoo has two more elephants than rhinoceroses. The Carlton Zoo has twice as many monkeys as it has rhinoceroses, elephants, and lions combined, and it has two more penguins than monkeys. The Bell Zoo has two-thirds as many monkeys as the Carlton Zoo has penguins. The Bell Zoo has two more penguins than monkeys but only half as many lions as penguins. The total of the numbers of rhinoceroses, elephants, lions, monkeys, and penguins in the Bell Zoo is 60.

Find the total of the numbers of rhinoceroses, elephants, lions, monkeys, and penguins in the Carlton Zoo.

Problem 4.3.17 (PCT-2022-SM1-R1-P17)**.** Right triangle ABC has a right angle at C. Point D on side $\overline{AB}$ is the base of the altitude of $\triangle ABC$ from C. Point E on side $\overline{BC}$ is the base of the altitude of $\triangle CBD$ from D. Given that $\triangle ACD$ has area 48 and $\triangle CDE$ has area 36, find the area of $\triangle DBE$.

Problem 4.3.18 (PCT-2022-SM1-R1-P18)**.** Find the number of subsets of $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ where the elements in the subset add to 47.

Problem 4.3.19 (PCT-2022-SM1-R1-P19)**.** Let n be a positive integer such that $\lfloor \sqrt{n} \rfloor - 3$ divides $n - 4$ and $\lfloor \sqrt{n} \rfloor + 3$ divides $n + 4$. Find the greatest such n less than 1000. *Note: $\lfloor x \rfloor$ refers to the greatest integer less than or equal to x.*

Problem 4.3.20 (PCT-2022-SM1-R1-P20)**.** Rectangle $ABCD$ measures 51 by 28. Eighteen points (including A and C) are marked on the diagonal AC dividing the diagonal into 17 congruent pieces. Twenty-two points (including A and B) are marked on the side AB dividing the side into 21 congruent pieces. Seventeen non-overlapping triangles are constructed as shown. Each triangle has two vertices that are two of these adjacent marked points on the side of the rectangle, and one vertex that is one of the marked points along the diagonal of the rectangle. Only the left 17 of the 21 congruent pieces along the side of the rectangle are used as bases of these triangles.

Find the sum of the areas of these 17 triangles.

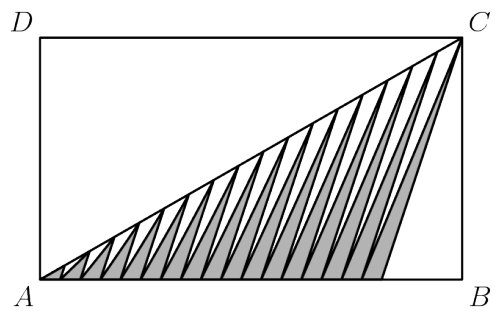

Figure 4.7: **??**

4.4 Grading

Answers for the problems.

Problem 1: 75
Problem 2: 23
Problem 3: 245
Problem 4: 120
Problem 5: 144
Problem 6: 7
Problem 7: 6
Problem 8: 15
Problem 9: 40
Problem 10: 66
Problem 11: 34
Problem 12: 14
Problem 13: 50
Problem 14: 3
Problem 15: 2200
Problem 16: 72
Problem 17: 108
Problem 18: 21
Problem 19: 56
Problem 20: 306

4.5 Solutions

Solution. **??**

Since $ABCD$ is a square and both $\triangle CFD$ and $\triangle CBE$ are equilateral, $\angle DCB = 90^\circ$ and $\angle FCD$ and $\angle BCE = 60^\circ$. Thus $\angle ECF = 360^\circ - 60^\circ - 60^\circ - 90^\circ = 150^\circ$. Furthermore $\triangle ECF$ is an isosceles triangle, so

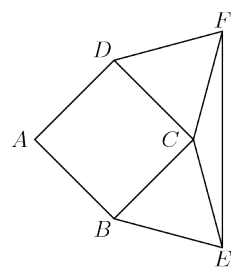

Figure 4.8: **??**

$\angle CEF = \dfrac{180^\circ - 150^\circ}{2} = 15^\circ$. Because $\angle BEF = \angle BEC + \angle CEF$, therefore $\boxed{\angle BEF = 60^\circ + 15^\circ = 75^\circ.}$ □

Solution. **??**

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{m}{n} \Rightarrow \frac{23}{15} = \frac{11}{12} + \frac{m}{n} \Rightarrow \frac{92}{60} - \frac{55}{60} = \frac{m}{n} \Rightarrow \frac{m}{n} = \frac{37}{60}$$

Therefore $\boxed{n - m = 60 - 37 = 23.}$ □

Solution. **??**

Each side is $\dfrac{84}{12} = 7$ The area of one square is $7^2 = 49$. Therefore the total area is $\boxed{49 \cdot 5 = 245.}$ □

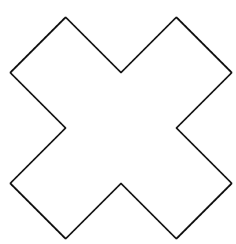

Figure 4.9: **??**

Solution. **??**

15 students on the track team who take Algebra. Thus, $4 \cdot 15 = 60$ students are taking Algebra. The number of students in the school is $3 \cdot 60 = 180$. Therefore the number of students in the school who did not take Algebra is $\boxed{180 - 60 = 120.}$ □

Solution. **??**

Diagonally from bottom-left to top-right, there are $1+3+5+7+7+5+3+1=32$ shaded triangles. Together they form $\frac{32}{2}=16$ squares. Therefore the total area is $\boxed{16\cdot 3\cdot 3=144.}$ □

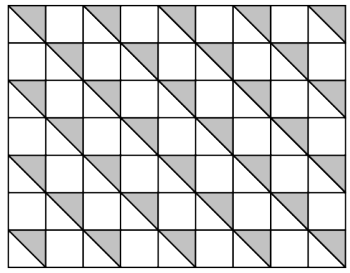

Figure 4.10: **??**

Solution. **??**

If February has 5 Saturdays, then there are 29 days from the first to the last Saturdays, including all Saturdays, shown as below,

$$\text{Sa}\ \underbrace{\text{------}}_{\text{6 days}}\ \text{Sa}\ \underbrace{\text{------}}_{\text{6 days}}\ \text{Sa}\ \underbrace{\text{------}}_{\text{6 days}}\ \text{Sa}\ \underbrace{\text{------}}_{\text{6 days}}\ \text{Sa}$$

Thus, a year has five Saturdays in February is a leap year. In every non-leap year of $365=7\times 52+1$ days, the day-of-the-week for February 1 is shifted by *one*: from Monday to Tuesday, from Tuesday to Wednesday, and so on. In a leap year, because February has an extra day, so the day-of-the-week for February 1 is shifted by *two*: from Monday to Wednesday, from Tuesday to Thursday, and so on. Therefore, the day-of-the-week for February 1 in a leap year is Saturday, then it is shifted by *five* weekdays in the next leap year,

$$\underbrace{\text{Sa}}_{2020}\ \text{M Tu W}\ \underbrace{\text{Th}}_{2024}\ \text{Sa Su M}\ \underbrace{\text{Tu}}_{2028}$$

Because a week has 7 days and the shift is 5, so after 7 leap years, or 28 years, the day-of-the-week for February 1 will be the same again.

$$\underbrace{\text{Sa Th Tu Su F W M}}_{\text{period 7}}\ \text{Sa}$$

Thus, the following leap years have the day-of-the-week for February 1 as Saturday,

$$2020,\ 2048,\ 2076$$

By the periodic sequence Sa Th Tu Su F W M, since the day-of-the-week for February 1 in 2020 is a Saturday, the first leap year after 2000 has five Sundays is 2004. The following leap years have the day-of-the-week for February 1 as Saturday,

$$2004,\ 2032,\ 2060,\ 2088$$

Therefore the number of such years are $\boxed{3+4=7.}$ □

Solution. **??**

The base of each triangle is the length of a side of the rectangle. The combined height of both triangles is height of the rectangles. Therefore their combined area is $\boxed{\frac{3\cdot 4}{2} = 6.}$ □

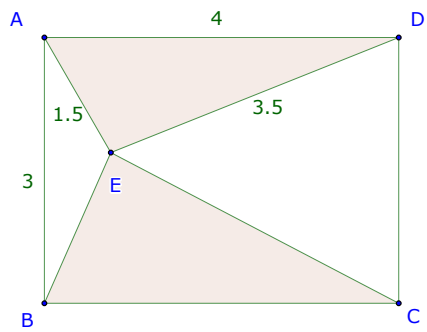

Figure 4.11: **??**

Solution. **??**

At $12:24$, Johan did $\frac{6}{11} - \frac{1}{3} = \frac{7}{33}$ of his workout. This was done by 7 laps, thus his workout consists of $7 \div \frac{7}{33} = 33$ laps. He has $1 - \frac{6}{11} = \frac{5}{11}$ of his workout to be done, or $\frac{5}{11} \cdot 33 = 15$ more laps. In 7 minutes, he did 7 laps, thus he needed $\boxed{15}$ more minutes to finish the workout. □

Solution. **??**

Let R be the radius of the big circle. Since 8 is the radius of the small semicircle, $R - 8$ is the radius of the larger semicircle. The shaded area is equal to the area of the top half of the circle, plus the small semicircle, and minus the large semicircle.

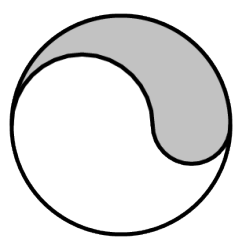

Figure 4.12: **??**

$$\frac{1}{2}\pi R^2 + \frac{1}{2}(64\pi) - \frac{1}{2}\pi(R-8)^2 = 160\pi \Rightarrow 8R - 160 = 0 \Rightarrow R = \frac{160}{8} = 20$$

Therefore the diameter is $\boxed{20 \cdot 2 = 40.}$ □

Solution. **??**

$$\begin{cases} \frac{1}{x}+\frac{1}{y} &= 5 \\ \frac{2}{x}+\frac{3}{y} &= 9 \end{cases} \Rightarrow 3(\frac{1}{x}+\frac{1}{y})-\frac{2}{x}+\frac{3}{y}=3\cdot 5-9 \Rightarrow \frac{1}{x}=6 \Rightarrow x=\frac{1}{6}, y=-1$$

Therefore $\boxed{\frac{7-4y}{x}=11\cdot 6=66.}$ □

Solution. **??**

Let m and n be the number of 5×5 and 3×3 tiles, then $25m+9n=2022$, so $3 \mid m$. Let $m=3m_1$, so

$$25m_1+3n=674 \Rightarrow 674-25m_1=3n \Rightarrow m_1 \leq 26$$

In order to minimize the number of tiles, Shenelle needs as many 5×5 tiles as possible. Easy to see that if $m_1=26$, then $n=8$ and $(m,n)=(26,8)$ is the solution of $25m+9n=2022$, where $\boxed{m+n=34}$ is minimal. □

Solution. **??**

It is easy to ses that the sequence consists of blocks, the nth block has n numbers 1, n numbers 2, ..., n numbers n, all together n^2 numbers.

$$\underbrace{1}, \underbrace{1,1,2,2}, \underbrace{1,1,1,2,2,2,3,3,3}, \underbrace{1,1,1,1,2,2,2,2,3,3,3,3,4,4,4,4}, \ldots.$$

Thus, the number of terms from the first to the nth blocks is:

$$1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

It is also easy to test that

$$\frac{17(17+1)(2\cdot 17+1)}{6}=1785<2022<=2109\frac{18(18+1)(2\cdot 18+1)}{6}$$

Since $2109=2022+87=2022+15+18*4$, the 2022th number must be $\boxed{14.}$ □

Solution. **??**

We use a simple technique here for easy calculating with percentages. Let there be 100 total jelly beans in the jar, then 50 jelly beans are red, 34 jelly beans are green, and 16 jelly beans are blue. Since 16 of the jelly beans in the jar becomes 20% after Alan removes some green and red jelly beans. Thus the number of jelly beans after removal is $16\cdot 5=80$. It means that 20 of the original beans were removed, and $\frac{1}{2}20=10$ are the red beans.

Therefore, $\boxed{\frac{50-10}{80}=\frac{40}{80}=50\%}$ of the jelly beans in the jar are red. □

Solution. **??**

There are $\binom{4}{2}=6$ ways to choose the two digits that he got correct. Once those are determined, then we simply swap the remaining two digits to make them incorrect, and this is the only way for the other two to be incorrect. There are a total of $4!=24$ orderings. The probability is $\frac{6}{24}=\frac{1}{4}$, and $\boxed{n-m=3.}$ □

Solution. **??** Let $m > n$ be the length and width of the most inner border. There are two red sections. The area of the first red section is $(m+10)\cdot(n+10) - m\cdot n = 10m+10n+100$. The area of the second red section is $(m+30)\cdot(n+30) - (m+20)\cdot(n+20) = 10m+10n+500$. Thus, the total area of the two red sections is,

$$10m + 10n + 100 + 10m + 10n + 500 = 20(m+n) + 600.$$

Now, the area of the large rectangle is $(m+30)\cdot(n+30) = 4819 = 79\cdot 61 \Rightarrow m = 49, n = 31$. Thus the

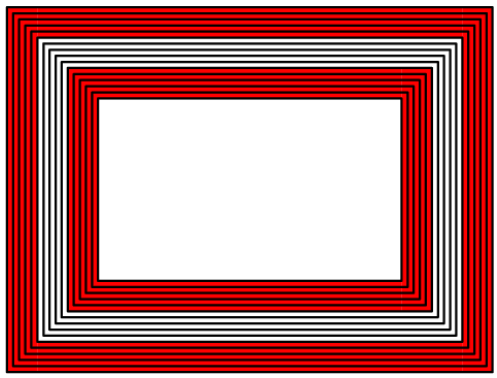

Figure 4.13: **??**

number of red tiles used is $\boxed{20(m+n) + 600 = 20(49+31) + 600 = 2200.}$ □

Solution. **??** Let the numbers of *rhinoceroses in the Bell* Zoo be r.

- The Bell Zoo has the same number of rhinoceroses as the Carlton Zoo has lions, so the number of *lions in the Carlton* Zoo is r.
- The Bell Zoo has three more elephants than the Carlton Zoo has lions, so the number of *elephants in the Bell* Zoo is $r+3$.
- The Bell Zoo has the same number of elephants as the Carlton Zoo has rhinoceroses, so the number of *rhinoceroses in the Carlton* Zoo is $r+3$.
- The Carlton Zoo has two more elephants than rhinoceroses, so the number of *elephants in the Carlton* Zoo is $r+5$
- The Carlton Zoo has twice as many monkeys as it has rhinoceroses, elephants, and lions combined, so the number of *monkeys in the Carlton* Zoo is $2\left[(r+3)+(r+5)+r\right] = 6r+16$.
- The Carlton Zoo has two more penguins than monkeys, so the number of *penguins in the Carlton* Zoo is $6r+18$.
- The Bell Zoo has two-thirds as many monkeys as the Carlton Zoo has penguins, so the number of *monkeys in the Bell* Zoo is $\frac{2}{3}(6r+18) = 4r+12$.
- The Bell Zoo has two more penguins than monkeys, so the number of *penguins in the Bell* Zoo is $4r+14$.
- The Bell Zoo has only half as many lions as penguins, so the number of *lions in the Bell* Zoo is $\frac{1}{2}(4r+14) = 2r+7$.

The total of the numbers of rhinoceroses, elephants, lions, monkeys, and penguins in the Bell Zoo is 60,

$$r + (r+3) + (2r+7) + (4r+12) + (4r+14) = 60 \Rightarrow 12r + 36 = 60 \Rightarrow r = 2$$

Thus, the total of the numbers of animals in the Carlton Zoo is

$$(r+3) + (r+5) + r + (6r+16) + (6r+18) = 15r + 42 = 72.$$

□

Solution. **??** First, since $\triangle ACD \sim \triangle CDE$, see **??** below, thus

$$\frac{[ACD]}{[CDE]} = \left(\frac{CD}{DE}\right)^2$$

Similarly $\triangle CDB \sim \triangle DEB$, so

$$\frac{[CDB]}{[DEB]} = \left(\frac{CD}{DE}\right)^2 \Rightarrow \frac{[CDE]+[DEB]}{[DEB]} = \frac{[ACD]}{[CDE]} \Rightarrow \frac{[CDE]+[DEB]}{[DEB]} = \frac{48}{36} = \frac{4}{3}.$$

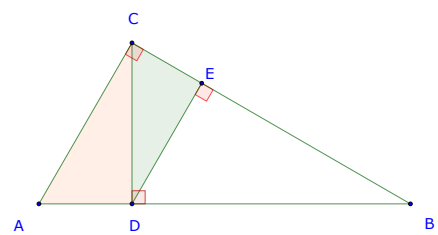

Figure 4.14: **??**

Thus, $\boxed{[DEB] = 3[CDE] = 3 \cdot 36 = 108.}$ □

Solution. **??**

If S be a subset whose elements sum up to 47, $S \in \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$, then S can be represented by a base-2 number for example $\overline{1001110001}_2$, because

$$1 \cdot 1 + 0 \cdot 3 + 0 \cdot 5 + 1 \cdot 7 + 1 \cdot 9 + 1 \cdot 11 + 0 \cdot 13 + 0 \cdot 15 + 0 \cdot 17 + 1 \cdot 19 = 1 + 7 + 9 + 11 + 19 = 47$$

```
m = 10                                  # Initially there are 10 odd numbers
numbers = [2*i+1 for i in range(0, m)]  # the odd numbers 1, 3, 5, ..., 19 in a list
c = 0                                   # counter of the solutions

for n in range(0, 2**m):                # ranging from 0 to 2^m, look at all numbers
    b = bin(n)                          # convert n from decimal into binary format
    x = bin(n)[2:]                      # chop the unsuable '0b' prefix
    f = x.zfill(m)                      # padding the string with 0 digits
    digits = [int(i) for i in f]        # convert each character into a 0 or 1
    S, s = set(), 0                     # S is the subset, s is the sum

    for i, j in zip(digits, numbers):   # go through both arrays
        s += i * j
        if i:                           # if the position is not 0
            S.add(j)

    if s == 47:                         # verify the sum
        c += 1                          # if found, increase the number of solutions
        print(c, S)                     # print the solution with counter
```

Below are $\boxed{21}$ solutions,

```
1 {19, 13, 15}
2 {19, 17, 11}
3 {5, 7, 9, 11, 15}
4 {3, 7, 9, 13, 15}
5 {3, 7, 9, 11, 17}
6 {3, 5, 11, 13, 15}
7 {3, 5, 9, 13, 17}
8 {3, 5, 9, 11, 19}
9 {3, 5, 7, 15, 17}
10 {3, 5, 7, 13, 19}
11 {1, 7, 11, 13, 15}
12 {1, 7, 9, 13, 17}
13 {1, 7, 9, 11, 19}
14 {1, 5, 11, 13, 17}
15 {1, 5, 9, 15, 17}
16 {1, 5, 9, 13, 19}
17 {1, 5, 7, 15, 19}
18 {1, 3, 11, 15, 17}
19 {1, 3, 11, 13, 19}
20 {1, 3, 9, 15, 19}
21 {1, 3, 7, 17, 19}
```

□

Solution. **??**

```
from math import floor, sqrt                    # floor and sqrt functions

n=999                                           # upper limit of n
while n > 4:                                    # until n is at least 4
    if (n-4) % (floor(sqrt(n)) - 3) == 0:       # check the first condition
        if (n+4) % (floor(sqrt(n)) + 3) == 0:   # check the second condition
            print(n)                            # if found, print
            break                               # quit the loop
    n -= 1                                      # not found, decrease n by 1
```

Below is the $\boxed{56}$ solution,

```
56
```

□

Solution. **??**

Let ℓ and h be the length and height of the rectangle.

The base of each of the 17 triangles is $\frac{AB}{21} = \frac{\ell}{21}$. From similar triangles, the heights of the triangles (from left to right) are

$$\frac{h}{17}, 2 \cdot \frac{h}{17}, 3 \cdot \frac{h}{17}, \ldots 17 \cdot \frac{h}{17}$$

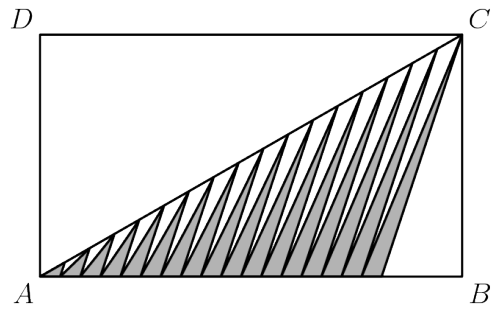

Figure 4.15: **??**

Therefore, the sum of the areas of the 17 triangles is,

$$\left(\frac{1}{2} \cdot \frac{\ell}{21} \cdot \frac{h}{17}\right) + \left(\frac{1}{2} \cdot \frac{\ell}{21} \cdot 2 \cdot \frac{h}{17}\right) + \ldots + \left(\frac{1}{2} \cdot \frac{\ell}{21} \cdot 17 \cdot \frac{h}{17}\right)$$
$$= \frac{1}{2} \cdot \frac{\ell}{21} \cdot \frac{h}{17} (1 + 2 + 3 + \cdots + 17) = \frac{1}{2} \cdot \frac{\ell}{21} \cdot \frac{h}{17} \cdot 17 \cdot 9 = \frac{3\ell h}{2 \cdot 7}$$

With $\ell = 51$ and $h = 28$, the sum is $\boxed{\frac{3\ell h}{2 \cdot 7} = \frac{3 \cdot 51 \cdot 28}{2 \cdot 7} = 306.}$ □

Chapter 5

PC R2

5.1 Topics

Algebra

1. Fraction. Average. Mean.
2. Equations. Substitution.
3. Average.
4. Distance-Time-Speed. Average Speed.
5. Sums and Products. Factorials. Telescoping sums and products.
6. Identities. Factoring sum of powers.
7. Quadratics. Roots.
8. Polynomials. Sum of roots. Vieta's theorem.
9. Logarithm. Rules of logarithm.
10. Determinant. Rules of determinants.
11. Cauchy Schwarz Inequality. Equality condition.
12. Complex number. Real and imaginary parts. Pure imaginary number. Absolute value.
13. Functions. Number of mappings between elements of domain and range.

Combinatorics

1. Counting. Worse case, best case.
2. Grids. Lattice points. Polygon with latice points as vertices. Pick's theorem.
3. Probability. Expected value.
4. Coloring. Cube face colouring. Rotation.
5. Casework. Permutations,. Combinations.
6. Correction over Counting.

Geometry

1. Area.
2. Quadrilateral. Perpendicular diagonals.
3. Angle Chasing. Sum of angles in a triangle. Angles subtending same arc in a circle.
4. Circle.

5. Triangles. Right triangles.
6. Parallelogram.
7. Rhombus.
8. Computational Geometry. Intersection points. Distance. Area.
9. Locust.
10. Trigonometry.
11. Three dimensional shapes: cones, pyramid. Volume, based, height.

Number Theory

1. Divisibility. Rules for divisibility by 2,3, 5, 9, and 11. Parity.
2. Number bases.
3. Factoring.
4. Perfect powers.
5. Diophantine equations.

Coding

1. Iterate over a range of numbers.
2. Check if a number is a prime.
3. Sum of digits of a positive integers.
4. Binary numbers. Represent a subset of a set as a binary number.
5. Conversions of decimal to binary numbers.
6. Set comparison.

5.2 Rules

- The *total time* to complete the test is 90 minutes. The problems that have no answers or incorrect answers will be discussed after the test.
- **High school teams** are given **90 minutes** to work on **30 problems.**
- The teams may fill in answers to the contest problems during the prescribed time for their teams (60 minutes for middle school teams, 90 minutes for high school teams) following the start of their contest clock. At any time during the competition, contestants can submit answers to the problems. Answers may be submitted multiple times by the same team, but for each problem **only the last answer** received before the contest ends will be accepted and graded. Partial submissions are accumulated. Consequently, if a team submits an answer to problem 1 and later wises to submit an answer to problem 2, there is no need to resubmit the answer to problem 1. All members are allowed to work on the problems as a team. No help may be provided by persons not on their team. Team members must not reveal or discuss contest problems with anyone other than their immediate team members until the test ends.
- All problems are designed so that they can be worked without the aid of calculating devices. On the other hand, participants are *permitted to use calculators or computers*, for example, to write a computer program in order solve or better understand a problem. Team members may also *refer to books, computer programs, or other reference materials* available to them *locally* as long as the students seek this material without aid from persons not on their team. Any team with a member attempted to do *violate this rule* of the test *will be disqualified.*
- Team members *shall not* use computers or any other communication devices to *seek help* with problem solutions *from people or Internet resources* outside of their own team members.
- Grading:
  1. For a **correct answer** of *each problem*, the team is awarded with 1 point. For an *unanswered* problem or for an *incorrect answer*, the team is awarded with 0 point.
  2. The score of the team is the *sum of awarded points* for all the problems.

5.3 Problems

Problem 5.3.1 (PCT-2022-SM1-R2-P1)**.** The five numbers 17, 98, 39, 50, and n have a mean equal to n. Find n.

Problem 5.3.2 (PCT-2022-SM1-R2-P2)**.** How many sets of two positive prime numbers $\{p, q\}$ have the property that $p + q = 120$?

Problem 5.3.3 (PCT-2022-SM1-R2-P3)**.** The Fahrenheit temperature (F) is related to the Celsius temperature (C) by $F = \frac{9}{5} \cdot C + 32$. What is the temperature in Fahrenheit degrees that is one-ninth as large if measured in Celsius degrees?

Problem 5.3.4 (PCT-2022-SM1-R2-P4)**.** Seven boxes are numbered 1, 2, 3, 4, 5, 6, and 7. Suppose that there are N balls distributed among these seven boxes. Find the least N for which it is guaranteed that for at least one k, box number k contains at least k^2 balls.

Problem 5.3.5 (PCT-2022-SM1-R2-P5)**.** The two diagonals of a quadrilateral have lengths 12 and 15, and the two diagonals are perpendicular to each other. Find the area of the quadrilateral.

Problem 5.3.6 (PCT-2022-SM1-R2-P6)**.** a and b are digits so that the 15−digit number $7a7ba7ab7ba7b77$ is divisible by 99. Find $7a + 5b$.

Problem 5.3.7 (PCT-2022-SM1-R2-P7)**.** Talya went for a 6 kilometer run. She ran 2 kilometers at 12 kilometers per hour followed by 2 kilometers at 10 kilometers per hour followed by 2 kilometers at 8 kilometers per hour. Talya's average speed for the 6 kilometer run was $\frac{m}{n}$ kilometers per hour, where m and n are relatively prime positive integers. Find the difference $m - n$.

Problem 5.3.8 (PCT-2022-SM1-R2-P8)**.** In the **??** below $\angle LAM = \angle LBM = \angle LCM = \angle LDM$, and $\angle AEB = \angle BFC = \angle CGD = 36$ degrees. Given that $\angle KLM = \angle KML$, find the degree measure of $\angle AEF$.

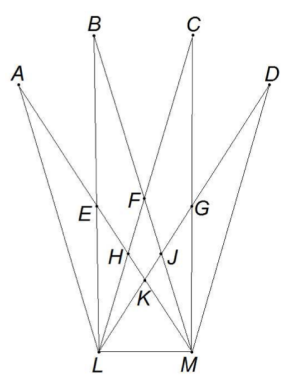

Figure 5.1: **??**

Problem 5.3.9 (PCT-2022-SM1-R2-P9)**.** Find the sum of all positive integers n with the property that the digits of n add up to $2022 - 2 \cdot n$.

Problem 5.3.10 (PCT-2022-SM1-R2-P10)**.** Find the sum of all the real values of x satisfying

$$(x + \frac{1}{x} - 18)^2 = x + \frac{1}{x} + 18.$$

Problem 5.3.11 (PCT-2022-SM1-R2-P11)**.** Suppose that the vertices of a polygon all lie on a rectangular lattice of points where adjacent points on the lattice are a distance 1 apart. Then the area of the polygon can be found using Pick's Formula:

$$I + \frac{B}{2} - 1,$$

where I is the number of lattice points inside the polygon, and B is the number of lattice points on the boundary of the polygon.

Pat applied Pick's Formula to find the area of a polygon but mistakenly interchanged the values of I and B. As a result, Pat's calculation of the area was too small by 38. Using the correct values for I and B, the ratio $n = \frac{I}{B}$ is an integer. Find the greatest possible value of n.

Problem 5.3.12 (PCT-2022-SM1-R2-P12)**.** The product $20! \cdot 21! \cdot 22! \cdots 28!$ can be expressed as $m \cdot n^3$, where m and n are positive integers, and m is not divisible by the cube of any prime. Find the sum of all prime factors of n that are not prime factors of m.

Problem 5.3.13 (PCT-2022-SM1-R2-P13)**.** The **??** below shows a parallelogram $ABCD$ with $AB = 33$ and $AD = 55$. Diagonal BD is perpendicular to side AB. Points E and F bisect sides AD and BC, respectively. Points G and H are the intersections of BD with AF and CE, respectively. Find the area of quadrilateral $EGFH$.

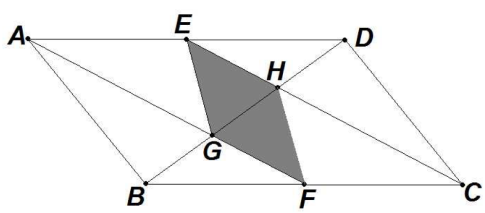

Figure 5.2: **??**

Problem 5.3.14 (PCT-2022-SM1-R2-P14)**.** The expression below

$$\frac{\log_{10} 20^2 \cdot \log_{20} 30^2 \cdots \log_{9990} 10000^2}{\log_{10} 11^2 \cdot \log_{11} 12^2 \cdot \log_{12} 13^2 \cdots \log_{999} 1000^2}.$$

is evaluated as $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 5.3.15 (PCT-2022-SM1-R2-P15)**.** On the inside of a square with side length 60, construct four congruent isosceles triangles each with base 60 and height 50, and each having one side coinciding with a different side of the square. Find the area of the octagonal region common to the interiors of all four triangles.

Problem 5.3.16 (PCT-2022-SM1-R2-P16)**.**

$$\left(1 + \frac{1}{1+2^1}\right)\left(1 + \frac{1}{1+2^2}\right)\left(1 + \frac{1}{1+2^3}\right)\cdots\left(1 + \frac{1}{1+2^{2022}}\right) = \frac{m}{n},$$

where m and n are relatively prime positive integers. Find the value of $\frac{m}{n-1}$.

Problem 5.3.17 (PCT-2022-SM1-R2-P17)**.** How many subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ have the property that no two of its elements differ by more than 4? For example, count the sets $\{3\}$, $\{8, 11, 12\}$, and $\{5, 6, 7, 8, 9\}$ but not the set $\{1, 3, 5, 7\}$.

Problem 5.3.18 (PCT-2022-SM1-R2-P18)**.** Define the determinant $D_1 = |1|$, the determinant $D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$, and the determinant $D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{vmatrix}$. In general, for positive integer n, let the determinant D_n have 1s in every position of its first row and first column, 3s in the remaining positions of the second row and second column, 5s in the remaining positions of the third row and third column, and so forth.

Find the least n so that $D_n \geq 1000$.

The value of 2×2 or 3×3 determinants can be computed as follow,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_2, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Higher-order determinants can also be computed recursively from lower-order determinants as shown above how a 3rd order determinant is based on 2nd determinants.

Let A be an $n \times n$ matrix, and $det(A)$ be its determinant, then

1. *If we multiply a row (column) of A by a number, the determinant of A will be multiplied by the same number.*
2. *If the ith row (column) in A is a sum of the ith row (column) of a matrix B and the ith row (column) of a matrix C and all other rows in B and C are equal to the corresponding rows in A (that is B and C differ from A by one row only), then $det(A) = det(B) + det(C)$.*
3. *If some column of A can be expressed as a linear combination of the other columns, then $det(A) = 0$.*
4. *If we add a row (column) of A multiplied by a scalar k (a real number) to another row (column) of A, then the determinant will not change.*
5. *If we swap two rows (columns) in A, the determinant will change its sign.*
6. *The determinant of the transpose of A (the transpose of a matrix is an operator which flips a matrix over its diagonal) equals the determinant of A, $det(A) = det(A_T)$.*
7. *If A is a triangular matrix, i.e. all elements above (or below) a main diagonal are 0, then its determinant equals the product of the diagonal entries.*

Problem 5.3.19 (PCT-2022-SM1-R2-P19)**.** The **??** below shows a 36×36 square $ABCD$. Points E and F lie on sides AD and CD, respectively, so that $DE = DF = 12$. Set X consists of the shaded triangle ABC with its interior, while set Y consists of the shaded triangle DEF with its interior. Set Z consists of all the points that are midpoints of segments connecting a point in set X with a point in set Y. That is,

$$Z = \{z | z \text{ is the midpoint of } xy \text{ for } x \in X \text{ and } y \in Y\}.$$

Find the area of the set Z.

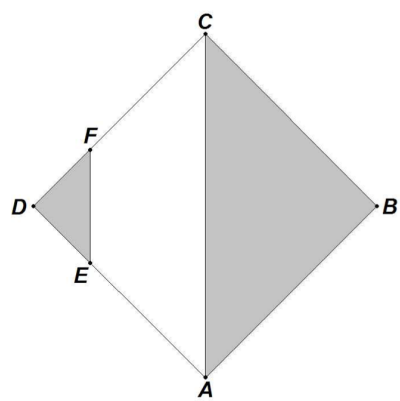

Figure 5.3: **??**

Problem 5.3.20 (PCT-2022-SM1-R2-P20)**.** For integers a, b, c, and d the polynomial $p(x) = ax^3+bx^2+cx+d$ satisfies $p(5) + p(25) = 1906$. Find the minimum possible value for $|p(15)|$.

Problem 5.3.21 (PCT-2022-SM1-R2-P21)**.** Find the remainder when $8^{2022} + 6^{2022}$ is divided by 100.

Problem 5.3.22 (PCT-2022-SM1-R2-P22)**.** Let x be a real number between 0 and $\dfrac{\pi}{2}$ for which the function $3\sin^2 x + 8\sin x \cos x + 9\cos^2 x$ obtains its maximum value, M. Find the value of $2M + 15\cos^2 x$.

Problem 5.3.23 (PCT-2022-SM1-R2-P23)**.** Larry and Diane start 100 miles apart along a straight road. Starting at the same time, Larry and Diane drive their cars toward each other. Diane drives at a constant rate of 30 miles per hour. To make it interesting, at the beginning of each 10 mile stretch, if the two drivers have not met, Larry flips a fair coin. If the coin comes up heads, Larry drives the next 10 miles at 20 miles per hour. If the coin comes up tails, Larry drives the next 10 miles at 60 miles per hour. Larry and Diane stop driving when they meet. The expected number of times that Larry flips the coin is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the difference $m - n$.

Problem 5.3.24 (PCT-2022-SM1-R2-P24)**.** The complex number w has positive imaginary part and satisfies $|w| = 3$. The triangle in the complex plane with vertices at w, w^2, and w^3 has a right angle at w. Find the real part of w^3.

Let $A(a)$, $B(b)$, and $C(c)$ be the three complex numbers in the complex plane. segments AB and BC are perpendicular if and only if

$$\frac{a-b}{b-c} \in i\mathbb{R}(\textit{an imaginary number with no real part.})$$

Problem 5.3.25 (PCT-2022-SM1-R2-P25)**.** You have a collection of small wooden blocks that are rectangular solids measuring $3 \times 4 \times 6$. Each of the six faces of each block is to be painted a solid color, and you have three colors of paint to use. Find the number of distinguishable ways you could paint the blocks. (*Two blocks are distinguishable if you cannot rotate one block so that it looks identical to the other block.*)

Problem 5.3.26 (PCT-2022-SM1-R2-P26)**.** Seven people of seven different ages are attending a meeting. The seven people leave the meeting one at a time in random order. Given that the youngest person leaves the meeting sometime before the oldest person leaves the meeting, the probability that the third, fourth, and fifth people to leave the meeting do so in order of their ages (youngest to oldest) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the difference $n - m$.

Problem 5.3.27 (PCT-2022-SM1-R2-P27)**.** A container is shaped like a right circular cone open at the top surmounted by a frustum which is open at the top and bottom as shown in the **??** below. The lower cone has a base with radius 2 centimeters and height 6 centimeters while the frustum has bases with radii 2 and 8 centimeters and height 6 centimeters. If there is a rainfall measuring 2 centimeter of rain, the rain falling into the container will fill the container to a height of $m + 3\sqrt{n}$ cm, where m and n are positive integers. Find the difference $m - n$.

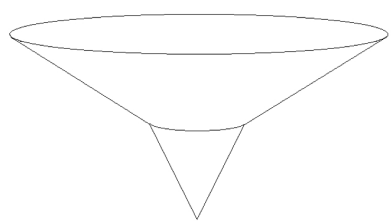

Figure 5.4: **??**

Problem 5.3.28 (PCT-2022-SM1-R2-P28)**.** Let $A = 1, 2, 3, 4, 5$ and $B = 0, 1, 2$. Find the number of pairs of functions $\{(f, g)\}$ where both f and g map the set A into the set B and there are exactly two elements $x \in A$ where $f(x) = g(x)$. For example, the function f that maps

$$1 \to 0,\ 2 \to 1,\ 3 \to 0,\ 4 \to 2,\ 5 \to 1$$

and the constant function g which maps each element of A to 0 form such a pair of functions.

Problem 5.3.29 (PCT-2022-SM1-R2-P29)**.** Ten spherical balls are stacked in a pyramid. The bottom level of the stack has six balls each with radius 6 arranged in a triangular formation with adjacent balls tangent to each other. The middle level of the stack has three balls each with radius 5 arranged in a triangular formation each tangent to three balls in the bottom level. The top level of the stack has one ball with radius 6 tangent to the three balls in the middle level. The **??** shows the stack of ten balls with the balls in the middle shaded. The height of this stack of balls is m $+\sqrt{n}$, where m and n are positive integers. Find the difference $m - n$.

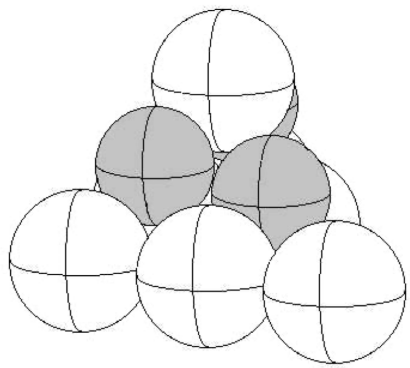

Figure 5.5: **??**

Problem 5.3.30 (PCT-2022-SM1-R2-P30)**.** Cindy and Neil wanted to paint the side of a staircase in the six-square pattern shown below in the **??** so that each of the six squares is painted a solid color, and no two squares that share an edge are the same color. Cindy draws all n patterns that can be colored using the four colors red, white, blue, and green. Neil looked at these patterns and claimed that k of the patterns Cindy drew were incorrect because two adjacent squares were colored with the same color. This is because Neil is color-blind and cannot distinguish red from green. Find $n + k$.

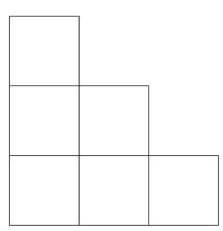

Figure 5.6: **??**

5.4 Grading

Answers for the problems.

Problem 1: 51
Problem 2: 12
Problem 3: 40
Problem 4: 134
Problem 5: 90
Problem 6: 50
Problem 7: 323
Problem 8: 108
Problem 9: 3015
Problem 10: 37
Problem 11: 20
Problem 12: 81
Problem 13: 242
Problem 14: 2051
Problem 15: 360
Problem 16: 2
Problem 17: 144
Problem 18: 11
Problem 19: 432
Problem 20: 47
Problem 21: 0
Problem 22: 34
Problem 23: 285
Problem 24: 23
Problem 25: 243
Problem 26: 17
Problem 27: 364
Problem 28: 9720
Problem 29: 280
Problem 30: 1166

5.5 Solutions

Solution. **??**

It is obvious that the mean of $17, 98, 39$, and 50 is n. Thus $\boxed{n = \frac{17+98+39+50}{4} = 51.}$ □

Solution. **??**

The simplest way is to solve by a simple program

```
from math import floor, sqrt

def is_prime(n):
    for i in range(2, floor(sqrt(n))+1):
        if n % i == 0:
            return False
    return True

count = 0
for i in range(1, 30):  # for p < 60
    p = 2 * i + 1
    q = 120 - p
    if p > q:
        break
    if is_prime(p) and is_prime(q):
        count += 1
        print(count, p, q)
```

There are $\boxed{12}$ such pairs

```
1 7 113
2 11 109
3 13 107
4 17 103
5 19 101
6 23 97
7 31 89
8 37 83
9 41 79
10 47 73
11 53 67
12 59 61
```

□

Solution. **??**

Since $F = \frac{9}{5} \cdot C + 32$, we need to find F such that,

$$F = 9C \Rightarrow F = 9\frac{5}{9}(F - 32) \Rightarrow F = 5(F - 32) \Rightarrow F = 40$$

Therefore the temperature is $\boxed{40}$ in Fahrenheit. □

Solution. **??**

It is easy to see that the larger number of balls such that box 1 has less than 1^2 balls, box 2 has less than 2^2 balls, ..., box 7 has less than 7^2 balls is

$$(1^2 - 1) + (2^2 - 1) + (3^2 - 1) + (4^2 - 1) + (5^2 - 1) + (6^2 - 1) + (7^2 - 1) = 133.$$

Therefore the least number of balls needed is $\boxed{133 + 1 = 134.}$ □

Solution. **??**

Let $ABCD$ be a quadrilateral whose diagonals are perpendicular to each other. The diagonals divide the

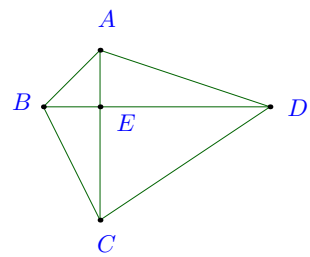

Figure 5.7: **??**

quadrilateral into four right triangles, thus

$$[ABCD] = [ABE]+[BCE]+[CDE]+[DAE] = \frac{1}{2}(AE\cdot EB+BE\cdot CE+CE\cdot DE+DE\cdot EA) = \frac{1}{2}(AC\cdot BD)$$

Thus, the area of $ABCD$ is $\boxed{\frac{12\cdot 15}{2} = 90.}$ □

Solution. **??**

Let $n = 7a7ba7ab7ba7b77$. Since $99 \mid n$ so,

$$9 \mid 7+a+7+b+a+7+a+b+7+b+a+7+b+7+7 = 49+4(a+b) \Rightarrow 9 \mid a+b+1$$
$$11 \mid (7+7+a+a+7+a+b+7)-(a+b+7+b+b+7+7) = 7+2(a-b)$$

Since $-11 = 7+2(0-9) \leq 7+2(a-b) \leq 7+2(9-0) = 25$, so the odd number $7+2(a-b)$ is divisible by 11 only if it is 11 or -11. Thus $a-b \in \{-9, 2\}$.

Now, $9 \mid a+b+1$, so $a+b \in \{8, 17\}$. If $a+b = 17$, then $(a,b) \in \{(8,9),(9,8)\}$, thus $a-b \in \{\pm 1\}$, which is not possible.

Thus, $a+b = 8$, and because $a-b$ has the same parity as $a+b$, so $a-b = 2$. Therefore $a = 5, b = 3$, and $\boxed{7a+5b = 35+15 = 50.}$

□

Solution. **??**

Talya's average speed is

$$\frac{6}{\frac{2}{12}+\frac{2}{10}+\frac{2}{8}} = \frac{360}{37} = \frac{m}{n}$$

Thus, $\boxed{m-n = 360-37 = 323.}$ □

Solution. **??**

$\angle LAM = \angle LBM = \angle LCM = \angle LDM$ hence $A,\ B,\ C,\ D,\ M$ and N lie on the same circle c_1. Similarly $\angle AEB = \angle BFC = \angle CGD$, so $E,\ F,\ G,\ M$ and N line on the circle c_2. Since $\angle AEB = \angle BFC = \angle CGD$, so the arcs $\widehat{AB},\ \widehat{BC},\ \widehat{CA}$ are the same. Let $x = \angle LAM = \angle LBM = \angle LCM = \angle LDM$, $y = \angle ALB = \angle BLC = \angle CLD$, and $z = \angle KLM = \angle KML$, see **??**. Furthermore in $\triangle ALM$,

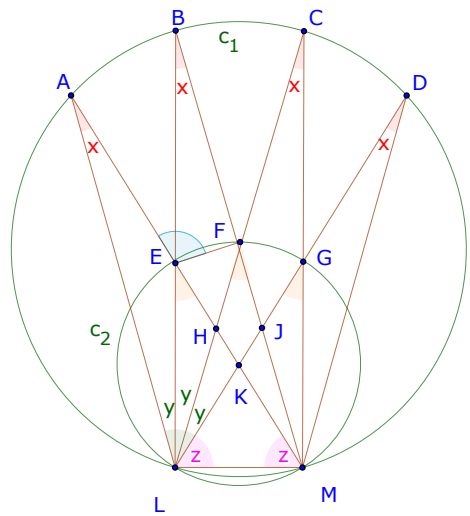

Figure 5.8: x, y, z angles

$$
\begin{aligned}
& x + 3y + 2z = 180^\circ,\ \angle AEB = x + y \Rightarrow x + y = 36^\circ \\
\Rightarrow\ & \angle AEF = \angle AEB + \angle BEF = 36^\circ + (180^\circ - \angle LEF) = 36^\circ + (180^\circ - (180^\circ - \angle LMF)) \\
\Rightarrow\ & \angle AEF = 36^\circ + \angle LMF = 36^\circ + z + y
\end{aligned}
$$

Since $y + z = \dfrac{(x + 3y + 2z) - (x + y)}{2}$, therefore $\boxed{\angle AEF = 36^\circ + \dfrac{180^\circ - 36^\circ}{2} = 108^\circ.}$ □

Solution. **??** The simplest way is to solve by a simple program

```
def sum_of_digits(n):
    return sum([int(d) for d in str(n)])

count, limit = 0, 2022
for n in range(1, limit):
    if 2*n + sum_of_digits(n) == limit:
        count += 1
        print(count, n, sum_of_digits(n), limit-2*n)
```

The output is shown as below.

```
1 998 26 26
2 1007 8 8
3 1010 2 2
```

The sum of all these numbers is $\boxed{3015.}$ □

Solution. **??**

Let $y = x + \frac{1}{x}$, then $(x + \frac{1}{x} - 18)^2 = x + \frac{1}{x} + 18$ can be written as,

$$(y-18)^2 = y + 18 \Rightarrow y^2 - 37y + 306 = 0 \Rightarrow \Delta = 37^2 - 4 \cdot 306 = 145 \Rightarrow y = \frac{37 \pm \sqrt{145}}{2}$$

For each value of y, we obtain some values of x by solving the equation $x + \frac{1}{x} = y$, or $x^2 - yx + 1 = 0$. By Viete's theorem, the sum of the roots for $x^2 - yx + 1 = 0$ is y. Therefore the sum of the values of all x solutions to the given equation is the sum of all y values, which is

$$\frac{37 - \sqrt{145}}{2} + \frac{37 + \sqrt{145}}{2} = 37.$$

The answer is $\boxed{37.}$ □

Solution. **??**

As stated by the problem,

$$I + \frac{1}{2}B - 1 = (B + \frac{1}{2}I - 1) + 38 \Rightarrow \frac{1}{2}I = \frac{1}{2}B + 38 \Rightarrow I = B + 76 \Rightarrow \frac{I}{B} = 1 + \frac{76}{B}$$

Since B is a factor of 76, and B is the number of lattice points on the borders (inlcuding the vertices), so it cannot be 2, the least value of B is 4. Thus the largest value for $\frac{I}{B}$ is $\boxed{1 + \frac{76}{4} = 20.}$ □

Solution. **??**

Note that for three consecutive factorials, such as 20!, 21!, and 22!, $20! \cdot 21! \cdot 22! = (20!)^3 \cdot (21)^2 \cdot 22$. Thus,

$$\begin{aligned} &20! \cdot 21! \cdots 28! = (20! \cdot 23! \cdot 26!)^3 \cdot ((21)^2 \cdot 22) \cdot ((24)^2 \cdot 25) \cdot ((27)^2 \cdot 28) \\ &= (20! \cdot 23! \cdot 26!)^3 \cdot 3^2 \cdot 7^2 \cdot 2 \cdot 11 \cdot 2^6 \cdot 3^2 \cdot 5^2 \cdot 3^6 \cdot 2^2 \cdot 7 \\ &= (20! \cdot 23! \cdot 26!)^3 \cdot 2^9 \cdot 3^{10} \cdot 5^2 \cdot 7^3 \cdot 11 = (20! \cdot 23! \cdot 26! \cdot 2^3 \cdot 3^3 \cdot 7)^3 \cdot 3 \cdot 5^2 \cdot 11 \\ &\Rightarrow m = 3 \cdot 5^2 \cdot 11,\ n = 20! \cdot 23! \cdot 26! \cdot 2^3 \cdot 3^3 \cdot 7 \end{aligned}$$

Thus, the prime factors of n that are not prime factors of m are $2, 7, 13, 17, 19, 23$, and their sum is $\boxed{2 + 7 + 13 + 17 + 19 + 23 = 81.}$ □

Solution. **??**

It is easy to show that $BG = GH = HD$. Since ABD is a right triangle, from Pythagorean theorem we find that $BD = \sqrt{55^3 - 33^2} = 44$, thus $GH = \frac{44}{3}$. $EGFH$ is parallelogram. $EF \parallel AB$, so $EF \perp BD$, or

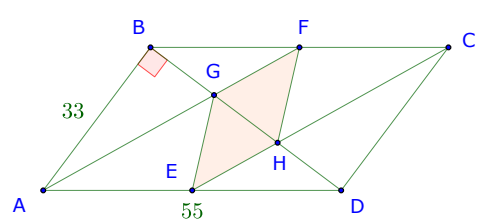

Figure 5.9: $EGFH$ is a rhombus

$EF \perp GH$ which means that $EGFH$ is a rhombus, therefore $\boxed{[EFGH] = \frac{1}{2} \cdot EF \cdot GH = 242.}$ □

Solution. **??**

First, since $\log_a(b^n) = n\log_a b$,

$$\log_{10} 20^2 \cdot \log_{20} 30^2 \cdots \log_{9990} 10000^2 = 2^{999} \log_{10} 20 \cdot \log_{20} 30 \cdots \log_{9990} 10000$$

Furtheremore, $\log_a b \cdot \log_b c = \log_a c$,

$$\log_{10} 20 \cdot \log_{20} 30 \cdots \log_{9990} 10000 = \log_{10} 10000 = \log_{10} 10^4 = 4 = 2^2.$$

Therefore the numerator of the given fraction is 2^{1001}.

Similarly, the denominator of the given fraction is

$$\log_{10} 11^2 \cdot \log_{11} 12^2 \cdots \log_{999} 1000^2 = 2^{990} \log_{10} 11 \cdot \log_{11} 12 \cdots \log_{999} 1000 = 2^{990} \log_{10} 1000 = 2^{990} \cdot 3$$

Thus the value of the fraction is $\frac{2^{1001}}{2^{990} \cdot 3} = \frac{2^{11}}{3}$. Therefore $m = 2^1 1 = 2048, n = 3$ and $\boxed{m + n = 2051.}$ □

Solution. **??**

We use computational geometry approach for this problem.

Let the center of the square O be the origin. Let E, F, G, H be the vertices of the four congruent isosceles triangles inside $ABCD$. It is easy to see that $A(-30, -30), B(30, -30), C(30, 30)$, and $D(-30, 30)$. Furthermore $E(0, 20), F(-20, 0), G(0, -20)$, and $H(20, 0)$. Now let $IJKLMNQP$ be the common octogon, see **??**.

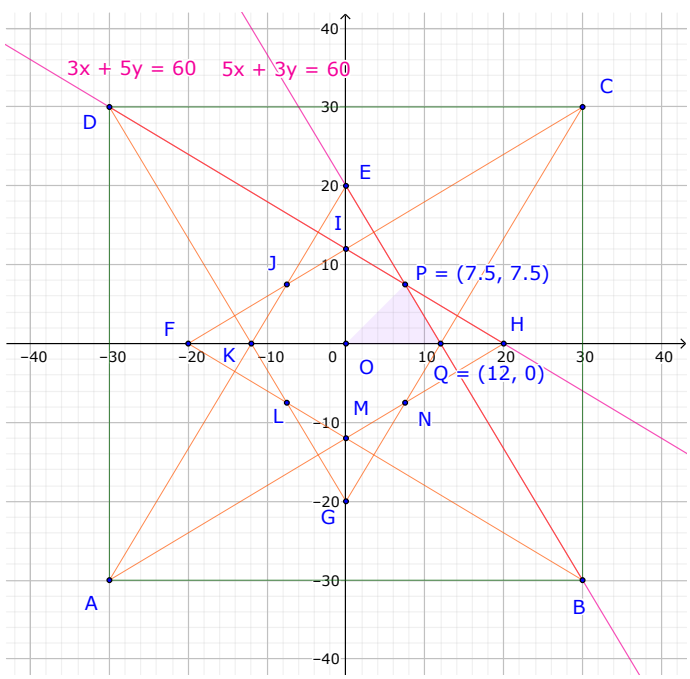

Figure 5.10: $IJKLMNQP$ octogon

The octogon is symmetric. It is composed by eight identical shapes of $\triangle OPQ$. Using the coordinates of A, B, C, D and E, F, G, H, we determined the lines through DH and BE as $3x + 5y = 60$ and $5x + 3y = 60$, therefore they have $Q(12, 0)$ and $I(0, 12)$ as $x-$ and $y-$intercept, and $P(7.5, 7.5)$ as the intersection.

It is now simple to see that $[OPQ] = \frac{1}{2} \cdot 7.5 \cdot 12 = 45$. Thus, the area of the of the octogon is $\boxed{8 \cdot 45 = 360.}$ □

Solution. **??**

Note that, for $k \geq 1$,

$$1 + \frac{1}{1+2^k} = \frac{2+2^k}{1+2^k} = 2\frac{1+2^{k-1}}{1+2^k}$$

Therefore the given product can be written as,

$$\begin{aligned}&\left(1+\frac{1}{1+2^1}\right)\left(1+\frac{1}{1+2^2}\right)\left(1+\frac{1}{1+2^3}\right)\cdots\left(1+\frac{1}{1+2^{10}}\right)\\ &= 2\frac{1+2^0}{1+2^1}\cdot 2\frac{1+2^1}{1+2^2}\cdots 2\frac{1+2^{2021}}{1+2^{2022}} = 2^{2022}\frac{1+2^0}{1+2^{2022}} = \frac{2^{2023}}{1+2^{2022}}\end{aligned}$$

Therefore $\boxed{\frac{m}{n-1} = \frac{2^{2023}}{2^{2022}} = 2.}$ □

Solution. **??**

We can take a coding approach here. First, any subset the given set $\{1,2,3,4,5,6,7,8,9,10,11,12\}$, for example $\{8,11,12\}$ can be consider as a 12−length *binary string* of 0 and 1, with 0 indicates the absence and 1 indicates the presence of the numbers at that position,

$$\{8,11,12\} \Leftrightarrow \underline{0000000}\overline{10011}_2$$

Similarly $\{2,5,7\} \Leftrightarrow \underline{0}\overline{10010100000}_2$. Thus, there is a bijection between the subsets of the given set and the numbers between 0 and $2^{12}-1$.

We just need to go through these numbers, tranform each of them into a subset and verify if the subset satisfies the required condition. For example, the number 19 is the binary $\overline{10011}_2$ which corresponds to $\{8,11,12\}$, and this is a set where the difference between any two elements does not exceed 4.

```
m = 12
counter = 0
for n in range(0, 2**m):                # ranging from 0 to 2^m, look at all numbers
    b = bin(n)                          # convert n from decimal into binary format
    x = bin(n)[2:]                      # chop the unsuable '0b' prefix
    f = x.zfill(m)                      # padding the string with 0 digits

    subset = set()                      # create a new subset
    for i in range(0, m):               # for each position of the binary string
        if f[i] == '1':                 # if it is 1
            subset.add(i+1)             # then add the number i+1 to the subset

    is_good_set = True
    subset = sorted(subset)             # sort the subset
    length = len(subset)
    for i in range(0, length-1):        # go through the subset elements
        for j in range(i+1, length):    # check the following ones
            if subset[j]-subset[i] > 4: # verify the condition
                is_good_set = False     # if the condition is violated
                break
        if not is_good_set:
            break

    if is_good_set:                     # is it a good set?
        counter += 1
        print(counter, sorted(subset))  # print it
```

The result is $\boxed{144}$ such subset. □

Solution. **??**

Consider the property 4, consider D_n,

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 3 & \cdots & 3 \\ 1 & 3 & 5 & \cdots & 5 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 3 & 5 & \cdots & 2n+1 \end{vmatrix}$$

First lets substract the first row from all rows below it,

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 2 & 4 & \cdots & 4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 2 & 4 & \cdots & 2n \end{vmatrix}$$

Then lets substract the second row from all rows below it,

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 2 & \cdots & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 2 & \cdots & 2n-2 \end{vmatrix}$$

By continuing doing so, we arrive at an upper triangular matrix with all 1s in the first row, and all 2s on and above the diagonals in all other rows,

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 2 & \cdots & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 \end{vmatrix}$$

Thus, by the property 7, $D_n = 2^{n-1}$, and the first n such that $D_n > 1000$ is $\boxed{n = 11.}$ □

Solution. **??**

Let denote the diagram as shown in **??**.

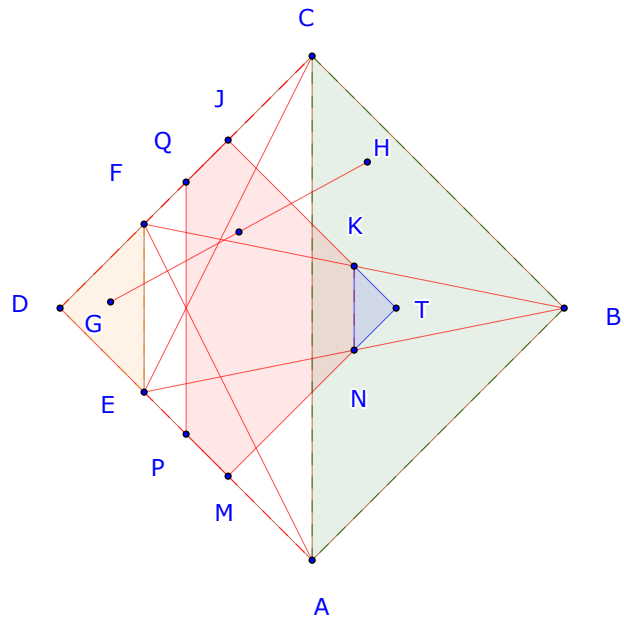

Figure 5.11: $[JKNMPQ]$

The midpoints of segments connecting the vertices of set X in green and set Y in orange form a red hexagon. The red midpoint of a pair green H, orange G is shown inside the hexagon as an example. J is midpoint of FC, so F and J trisects DC, $DF = FJ = JC$. M is midpoint of EA, so E and M trisects DA, $DE = EM = MA$. In addition K and N are midpoints of FB and EB, respectively.

$$[JKNMPQ] = [DJTM] - [DQP] - [KTN] = [DJTM] - [DQP] - \frac{[DEF]}{4} = 24 \cdot 24 - \frac{18 \cdot 18}{2} + \frac{\frac{1}{2} \cdot 12 \cdot 12}{4} = 432$$

Therefore $\boxed{[JKNMPQ] = 432.}$ □

Solution. **??**

Let $n = p(15)$, since $p(5) + p(25) = 1906$, so

$$\begin{aligned}
&n = p(15) = a \cdot 15^3 + b \cdot 15^2 + c \cdot 15 + d = 3375a + 225b + 15c + d \\
&1906 = p(5) + p(25) = (a \cdot 5^3 + b \cdot 5^2 + c \cdot 5 + d) + (a \cdot 25^3 + b \cdot 25^2 + c \cdot 25 + d) \\
\Rightarrow\ &1906 = 15750a + 650b + 30c + 2d = (9000a + 200b) + 2(3375a + 225b + 15c + d) = (9000a + 200b) + 2n \\
\Rightarrow\ &1906 - 2n = 200(45a + b) \Rightarrow 200 \mid 1906 - 2n
\end{aligned}$$

The smallest value of $|n|$ such that $200 \mid 1906 - 2n$ is $n = -47$, or $\boxed{|n| = 47,}$ it happens when $a = 1$, $b = -35$, $c = -296$, $d = -13$. □

Solution. **??**

Note that for any positive odd integer n,

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \ldots + b^{n-1}) \Rightarrow a + b \mid a^n + b^n$$

Let $S = 8^{2022} + 6^{2022}$,

$$S = 8^{2022} + 6^{2022} = (8^2)^{1011} + (6^2)^{1011} \Rightarrow 8^2 + 6^2 \mid S \Rightarrow 100 \mid S$$

Therefore the remainder is $\boxed{0.}$ □

Solution. **??**

Note that, $\sin^2 x + \cos^2 x = 1,\ \sin 2x = 2\sin x \cos x,\ \cos 2x = 2\cos^2 x - 1$, so

$$3\sin^2 x + 8\sin x\cos x + 9\cos^2 x = 3(\sin^2 x + \cos^2 x) + 4(2\sin x \cos x) + 6\cos^2 x = 6 + 4\sin 2x + 3\cos 2x$$

Now, by Cauchy-Schwarz inequality,

$$4\sin 2x + 3\cos 2x \leq \sqrt{(4^2+3^2)(\sin^2 2x + \cos^2 2x)} = 5$$

Thus $3\sin^2 x + 8\sin x\cos x + 9\cos^2 x \leq 11$. This happens when

$$\frac{\sin 2x}{4} = \frac{\cos 2x}{3} \Rightarrow \left(\frac{\cos 2x}{3}\right)^2 = \frac{(\sin 2x)^2 + (\cos 2x)^2}{4^2+3^2} = \frac{1}{25}$$
$$\Rightarrow \cos 2x = \frac{3}{5} \Rightarrow \cos^2 x = \frac{1}{2}\left(1 + \frac{3}{5}\right) = \frac{4}{5}$$

Therefore $\boxed{2M + 15\cos^2 x = 2 \cdot 11 + 15 \cdot \frac{4}{5} = 34.}$ □

Solution. **??**

For each head, Larry gained 10 miles in 30 minutes. For each tail, he gained 10 miles in 10 minutes. For every 10 miles, Diane needs 20 minutes.

When Larry flips only heads, then he is slowest. Since they goes in opposite direction at speed of $20 + 30 = 50$, so they will meet after $\frac{100}{50} = 2$ hours. Larry completes 40 miles, he must flip 4 times. The probability to have the number of flips to be 4 is

$$\left(\frac{1}{2}\right)^4 = \frac{1}{16}.$$

Now, if they meet after he flips 5 times, then he goes between 40 and less then 50 miles, and Diane goes between 50 and between 60 miles, or between 100 minutes and 120 minutes. Larry can only flip three heads and two tails, because

$$3 \cdot 30 + 2 \cdot 10 = 110.$$

The probability to have the number of turns to be 5 is,

$$\binom{3}{5}\left(\frac{1}{2}\right)^5 = \frac{5}{8}$$

Now, if they meet after he flips 6 times, then he goes between 50 and 60 miles, and Diane goes between 40 and 50 miles, or between 80 minutes, and 100 minutes. Larry can flip one heads and fives tails, because

$1 \cdot 30 + 5 \cdot 10 = 80$, or Larry can flip two heads and four tails, because $2 \cdot 30 + 4 \cdot 10 = 100$. The probability to have the number of turns to be 6 is,

$$\binom{1}{6}\left(\frac{1}{2}\right)^6 + \binom{2}{6}\left(\frac{1}{2}\right)^6 = \frac{21}{64}$$

Finally, if they meet after he flips 7 times, then he goes between 60 and 70 miles, and Diane goes between 30 and 40 miles, Larry must flip six tails first, and then anything afterwards, since $6 \cdot 10 = 60$. The probability to have the number of flips to be 7 is,

$$\left(\frac{1}{2}\right)^6 = \frac{1}{64}.$$

Therefore the expected number of turn is,

$$4 \cdot \frac{1}{16} + 5 \cdot \frac{5}{8} + 6 \cdot \frac{21}{64} + 7 \cdot \frac{1}{64} = \frac{349}{64}$$

Thus $m = 349, n = 64$ and $\boxed{m - n = 285.}$ □

Solution. **??**

Since

$$\frac{w^3 - w^2}{w^2 - w} = w + 1 \in i\mathbb{R} \Rightarrow w + 1 = a \cdot i,\ a \in \mathbb{R} \Rightarrow w = -1 + ai$$

Because $|w| = 3$, therefore $\sqrt{1 + a^2} = 9 \Rightarrow a = 2\sqrt{2}$. Therefore $w^3 = (-1 + 2i\sqrt{2})^3 = 23 - 10i\sqrt{2}$, hence its real part is $\boxed{23.}$ □

Solution. **??**

Each face has 3 ways to colour. So in total 3^6 ways. However each pair of opposite faces are identical, thus the number of distinct ways to colour them is $\boxed{\frac{3^6}{3} = 3^5 = 243.}$ □

Solution. **??** We count the number of orders satisfying both conditions (*good* orders). and then put that over the number of orders where the youngest person leaves before the oldest person. By symmetry, the total number is $\frac{7!}{2} = 2520$.

Let call the youngest person 1 and the oldest 7. We consider the four following cases: (i) 1 and 7 both are not part of the third, fourth, and fifth people to leave; (ii) 1 is part of but 7 is not; (iii) 7 is part of but 1 is not, (iv) both 1 and 7 are part of.

For the first case, we have $\binom{4}{2} = 6$ ways to chose where 1, 7 are. Then, by symmetry, $\frac{1}{3!} = \frac{1}{6}$ of the permutations in this case have the third, fourth, and fifth people leaving in order of ages. This has $\frac{6 \cdot 5!}{6} = 120$ *good* orders.

For the second case, we have 2 ways to chose where 7 is. 1 is forced to be the third person because no-one else can be younger. By symmetry, $\frac{1}{2!} = \frac{1}{2}$ of the permutations have the fourth and fifth people leaving in order of ages. This has $\frac{2 \cdot 5!}{2} = 120$ *good* orders.

For the third case, we have 2 ways to chose where 1 is. 7 is forced to be the fifth person because no-one else can be older. By symmetry, $\frac{1}{2!} = \frac{1}{2}$ of the permutations have the third and fourth people leaving in order of ages. This has $\frac{2 \cdot 5!}{2} = 120$ *good* orders.

For the fourth case, 1 is forced to be the third and 7 is forced to be the fifth person. Note that among all 5! permutations now it does not matter who is fourth as they have to be younger than 7 and older than 1. So there are 120 good orders.

Our probability is $\frac{480}{2520} = \frac{4}{21}$, thus $\boxed{n - m = 21 - 4 = 17.}$ □

Solution. **??**

A vertical cut in **??** shows a part as if the frustum is a cone.

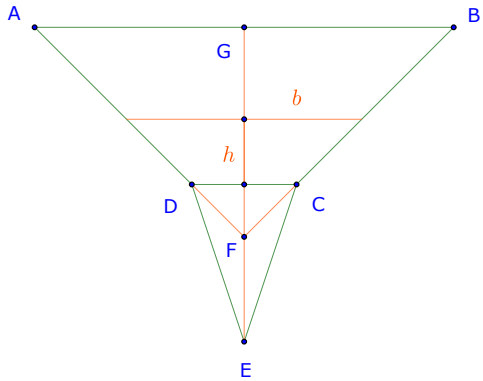

Figure 5.12: A vertical cut

Since base $DC = 4$, $AB = 16$, easy to see that the total height $FG = 8$.

The total volume of 2cm rain into the base radius 8cm is $\pi \cdot 8^2 \cdot 2 = 128\pi$. The amouunt of rain going into the frustum is

$$128\pi - \frac{1}{3}\pi \cdot 2^2 \cdot 6 = 120\pi$$

If this fills upto the height h, measuring from F upward, then since $b = h$, the filled volume is,

$$120\pi = \frac{1}{3}\pi \cdot h^2 \cdot h - \frac{1}{3}\pi \cdot 2^2 \cdot 2 \Rightarrow h^3 = 368$$

Therefore, the total height is $4 + \sqrt[3]{368}$, thus $m = 4, n = 368$, and $\boxed{n - m = 364.}$ □

Solution. **??**

Since A has 5 elements, B has 3 elements, so there are 3^5 ways to construct an f function. Now, a g function satisfying the conditions can be constructed by choosing two elements of A, and map them to the same elements of B as f does. There are $\binom{5}{2}$ ways.

The three remaining three elements of A can be mapped by g into two different elements of B, therefore 2^3 ways to map them.

Finally, note the roles of f and g are interchangable, so the total number of ways is $\boxed{\frac{3^5\binom{5}{2}\cdot 2^3}{2} = 9720.}$ □

Solution. **??**

The height of the stack can be divided into:

1. the distance DG from the ground to the center of one of the large sphere, and
2. the distance GE from the level of the center of one of the large sphere to the level of the center of one of the shaded sphere, and
3. the distance FH from the level of the center of one of the shaded sphere to the level of the center top large sphere, and
4. the distance HJ from the level of the center of the top large sphere to its highest point, see **??**.

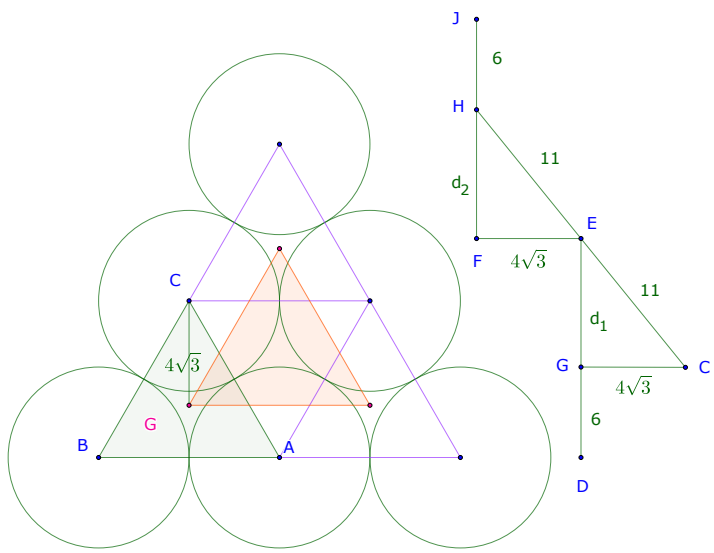

Figure 5.13: A vertical cut

Now, the distance from the ground to the center of one of the large sphere is 6, shown as the length of the segment DG. The distance from from the level of the centres A, B, or C of one of the large sphere to the level of the center of one of the shaded sphere is basically the distance GE. This GE height is equal to the distance from the center G of the equilateral triangle ABC formed by three centres A, B, and C of the three tangent large spheres to the centre of the shaded sphere E. $\triangle CGE$ is a right triangle with CE is the sum of the two radii of a large and a shaded spheres, and CG is two-third of the altitude of the $\triangle ABC$, thus

$$GE = \sqrt{11^2 - \left(\frac{12\sqrt{3}}{2} \cdot \frac{2}{3}\right)^2} = \sqrt{73}$$

Furtheremore, three centres of the three shaded spheres form an equilateral triangle congruent to $\triangle ABC$. By similar reason, the distance FH from the level of the center of one of the shaded sphere to the level of the top large sphere is also $\sqrt{73}$.

Finally, the distance HJ from the level of the center of the top large sphere to its highest point is 6.

Thus, the total height is $2 \cdot 6 + 2 \cdot \sqrt{73} = 12 + \sqrt{292}$. Therefore $m = 12, n = 292$, and $\boxed{n - m = 280.}$ $\square$

Solution. **??**

Let number the squares as show in the **??**.

1		
2	3	
4	5	6

Figure 5.14: Numbered squares

To count Cindy's total n, we colour the squares from $1-6$, each time verify if a pair of squares that share a side are coloured differently. Once such a colouring is found, we then verify if pair of squares that share a side are coloured by red and green. The sum of those colouring is Ryan's.

```
colours = ['r', 'w', 'b', 'g']
c_count, r_count = 0, 0

for s1 in colours:
    for s2 in colours:
        if s2 == s1:
            continue

        for s3 in colours:
            if s3 == s2:
                continue

            for s4 in colours:
                if s4 == s2:
                    continue

                for s5 in colours:
                    if s5 == s4 or s5 == s3:
                        continue

                    for s6 in colours:
                        if s6 == s5:
                            continue

                        c_count += 1
                        sets = [{s1, s2}, {s2, s3}, {s2, s4}, {s3, s5}, {s4, s5}, {s5, s6}]
                        if any([s == {'r', 'g'} for s in sets]):
                            r_count += 1

print(c_count + r_count)
```

The answer is $\boxed{1166.}$ □

Chapter 6

ICLT February

6.1 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.2 Problems for Level 1

Problem 6.2.1 (ICLT-2022-SM1-2-L1-P1). (*4 points*)

I am now four years younger than twice what my age was five years ago. What is my age now?

(A) 10 (B) 11 (C) 12 (D) 13 (E) 14

Problem 6.2.2 (ICLT-2022-SM1-2-L1-P2). (*4 points*)

Given that $WV \parallel YZ$ and $WZ \parallel VY$ in **??**, find x.

(A) 20 (B) 22.5 (C) 25 (D) 27.5 (E) 30

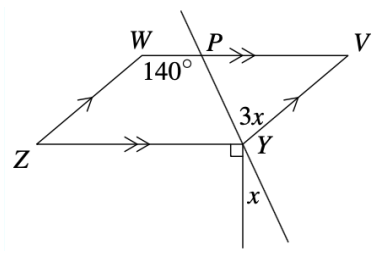

Figure 6.1: **??**

Problem 6.2.3 (ICLT-2022-SM1-2-L1-P3). (*4 points*)

Jon teaches a fourth grade class at an elementary school where class sizes are always at least 32 students and at most 40. One day Jon decides that he wants to arrange the students in their desks in a rectangular grid with no gaps. Unfortunately for Jon he discovers that doing so could only result in one straight line of desks. How many students does Jon have in his class?

(A) 31 (B) 33 (C) 35 (D) 37 (E) 39

Problem 6.2.4 (ICLT-2022-SM1-2-L1-P4). (*6 points*)

How many distinct words can you make by rearrange the letters of OTTAWA?

(A) 60 (B) 120 (C) 180 (D) 360 (E) 720

Problem 6.2.5 (ICLT-2022-SM1-2-L1-P5). (*6 points*)

A two-digit prime number $\overline{ab}$ is called *nice* if exactly one of its digits (a or b) is a prime number. *For example 59 is such a number because 5 is a prime number but 9 is not. 23 is not such a number because both 2 and 3 are prime numbers.* How many *nice* numbers are there?

(A) 13 (B) 14 (C) 15 (D) 16 (E) 17

Problem 6.2.6 (ICLT-2022-SM1-2-L1-P6). (*6 points*)

Our math club has 18 members and 3 officers: President, Vice President, and Treasurer. Ali has a huge crush on Brenda, and won't be an officer unless she is one too. Brenda is unaware of Ali's affections, and doesn't care if he is an officer or not; she's perfectly happy to be an officer even if Ali isn't one. In how many ways can the club choose its officers?

(A) 3212 (B) 4176 (C) 5050 (D) 5922 (E) 7680

Problem 6.2.7 (ICLT-2022-SM1-2-L1-P7)**.** (*10 points*)

In **??**, $m \parallel n$, $AB \perp m$, $\angle ADC = \angle BCE = 3x$, $\angle CEB = 50°$, and $\angle BCD = x$.

1. (*5 points*) Find x.
2. (*5 points*) Find $\angle ACD$.

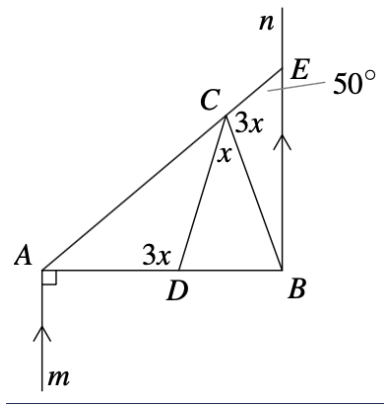

Figure 6.2: **??**

Problem 6.2.8 (ICLT-2022-SM1-2-L1-P8)**.** (*10 points*)

Write each of the following expressions as a single fraction:

1. (*5 points*) $\dfrac{4}{3y} - \dfrac{5y-3}{5y^2}$.
2. (*5 points*) $\dfrac{3}{s} - \dfrac{7s-2}{s+2}$.

Problem 6.2.9 (ICLT-2022-SM1-2-L1-P9)**.** (*10 points*)

(p, q, r) is a triple of prime numbers.

1. (*5 points*) How many such triples are there if $p < q < r$ and $p + q + r = 21$.
2. (*5 points*) How many such triples are there if $p \leq q \leq r$ $p + q + r = 21$.

6.3 Grading for Level 1

Answers for multiple-choice problems.

Problem 1:	E	14
Problem 2:	C	25
Problem 3:	D	37
Problem 4:	C	180
Problem 5:	A	13
Problem 6:	B	4176

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 2 point if can find $\angle ACD = 140^\circ - 3x$

(b) 2 point if can reuse $\angle ACD = 140^\circ - 3x$

Problem 8: Separately for each part,

(a) 2 points if can correctly bring both fractions to the same denominator.

(b) 2 points if can correctly bring both fractions to the same denominator.

Problem 9: Separately for each part,

(a) 2 points if can find 3 as the smallest prime number.

(b) 2 points if can find that there are two even primes.

6.4 Solutions for Level 1

Solution. **??** Let x be my age now. Five years ago, my age was $x-5$. Twice my age six years ago is $2(x-5)$. I am now four years younger than this quantity, so my age now must be $2(x-5)-4$. Since my age now is also x, we have an equation:

$$x = 2(x-5) - 4 \Rightarrow x = 2x - 14 \Rightarrow x = 14$$

Thus, my age now is $\boxed{14.}$ The answer is $\boxed{E.}$ □

Solution. **??** Since $WZ \parallel VY$ We find $\angle PVY = 180° - \angle ZWP = 180° - 140° = 40°$. Similarly, $\angle WZY = 40°$, and $\angle ZYV = 180° - \angle WZY = 140°$.

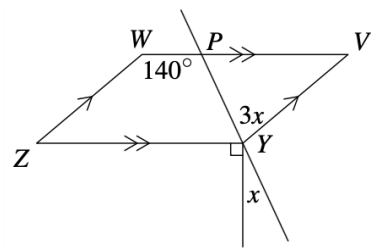

Figure 6.3: $\angle PVY = 40°$

Since $\angle PYV = 3x$, we have

$$\angle PYZ = \angle VYZ - \angle PYV = 140° - 3x.$$

Since this angle, the 90° angle, and the angle with measure x together give us straight line through PY, we have

$$\angle PYZ + 90° + x = 180° \Rightarrow (140° - 3x) + 90° + x = 180° \Rightarrow x = 25°$$

Therefore $\boxed{x = 37.}$ The answer is $\boxed{C.}$ □

Solution. **??** Let m be the number of rows in the grid of students and let n be the number of columns. The total number of students is the product mn. If the only way to express mn as a product of positive integers is for one of the integers to be 1, then 1 and mn are the only divisors of mn, so mn is prime.

The number of students in Jon's class is $\boxed{37,}$ the only prime between 32 and 40. The answer is $\boxed{D.}$ □

Solution. **??** Let pretend that the letters are all different, and we have $OT_1T_2A_1WA_2$. We have 6! permutations (since all the letters are different). Because the pair (A_1, A_2) can have 2! different arrangements but in fact result in the same

$$\begin{cases} __A_1_A_2_ \Rightarrow __A_A_ \\ __A_2_A_1_ \Rightarrow __A_A_ \end{cases}$$

Thus, we need to correct the counting by divide the total by the number of ways to permute (A_1, A_2). Similarly, the total has to be divided by the number of ways to permute (T_1, T_2).

Therefore the number of ways is $\boxed{\dfrac{6!}{2! \cdot 2!} = 180.}$ The answer is $\boxed{C.}$ □

Solution. **??** The two-digit prime numbers are

$$11, \underline{13}, \underline{17}, 19, 23, \underline{29}, \underline{31}, 37, 41, \underline{43}, \underline{47}, 53, \underline{59}, 61, \underline{67}, \underline{71}, 73, \underline{79}, \underline{83}, \underline{89}, \underline{97}$$

Note that 1 is not a prime number. By elimination, there is $\boxed{13}$ such numbers. The answer is $\boxed{A.}$ □

Solution. **??** We use complementary counting approach: first we count all the permutations (without restrictions), and then subtract the ones that are not allowed.

Since there are 4 different officers, so there are $18 \cdot 17 \cdot 16$ ways to choose the officers without restrictions. The cases that we have to exclude are those where Ali is an officer but Brenda is not. So let's count the number of these cases by constructive counting. There are 3 choices for the office that Ali will hold. There are $18 - 2 = 16$ choices for the first remaining office (since Ali is already chosen and we're not allowed to choose Brenda). There are 15 choices for the last remaining office.

In total there are $\boxed{18 \cdot 17 \cdot 16 - 3 \cdot 16 \cdot 15 = 4176}$. The answer is $\boxed{B.}$ □

Solution. **??**

First, $\angle ABE = 90^\circ$ because $m \parallel n$ and $m \perp AB$. Then, from $\triangle ABE$ we have

$$\angle EAB + \angle ABE + \angle AEB = 180^\circ \Rightarrow \angle EAB = 180^\circ - \angle ABE - \angle AEB = 40^\circ.$$

We can then use $\triangle ACD$

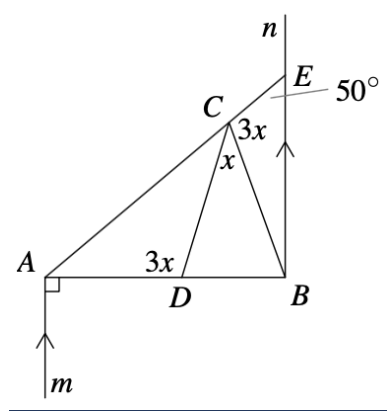

Figure 6.4: $\angle ACD = 140^\circ - 3x$

$$\angle ACD = 180^\circ - \angle DAC - \angle ADC = 180^\circ - 40^\circ - 3x = 140^\circ - 3x.$$

We have three angles with vertices at C that together make a straight line, so we have

$$\angle ACD + \angle DCB + \angle BCE = 180^\circ \Rightarrow 140^\circ - 3x + x + 3x = 180^\circ \Rightarrow x = 40^\circ.$$

Thus, $\boxed{x = 40^\circ.}$ It is easy to see that $\boxed{\angle ACD = 140^\circ - 3x = 20^\circ.}$ □

Solution. **??**

$$\frac{4}{3y} - \frac{5y-3}{5y^2} = \frac{4(5y) - (5y-3)3}{15y^2} = \frac{5y+9}{15y^2}$$

$$\frac{3}{s} - \frac{7s-2}{s+2} = \frac{3(s+2) - (7s-2)s}{s(s+2)} = \frac{-7s^2 + 5s + 6}{s(s+2)}$$

□

Solution. **??** Because $p + q + r = 21$ and 21 is an odd number, so all p, q, r should be odd or exactly two of them are odd.

For the first question, because $p < q < r$, thus all p, q, r are odd primes. If all p, q, r are larger then 3, then $p + q + r \geq 5 + 7 + 11 = 23 > 21$, therefore $p = 3$ and $q + r = 21 - 3 = 18$. There are only two possible solutions $5 + 13 = 7 + 11 = 18$. So there are $\boxed{2}$ triples $\boxed{(3, 5, 13), (3, 7, 11).}$

For the second question, we need to investigate the case when two of them are even. This means that $p = q = 2$. In this case we have an additional triple $(2, 2, 17)$. Thus, $\boxed{(2, 2, 17), (3, 5, 13), (3, 7, 11)}$ are the solutions. $\square$

6.5 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.6 Problems for Level 2

Problem 6.6.1 (ICLT-2022-SM1-2-L2-P1). (*6 points*)

Our math club has 18 members and 3 officers: President, Vice President, and Treasurer. Ali has a huge crush on Brenda, and won't be an officer unless she is one too. Brenda is unaware of Ali's affections, and doesn't care if he is an officer or not; she's perfectly happy to be an officer even if Ali isn't one. In how many ways can the club choose its officers?

(A) 3212 (B) 4176 (C) 5050 (D) 5922 (E) 7680

Problem 6.6.2 (ICLT-2022-SM1-2-L2-P2). (*6 points*)

In **??**, $m \parallel n$, $AB \perp m$, $\angle ADC = \angle BCE = 3x$, $\angle CEB = 50°$, and $\angle BCD = x$. Find $\angle ACD$.

(A) 15 (B) 20 (C) 25 (D) 30 (E) 35

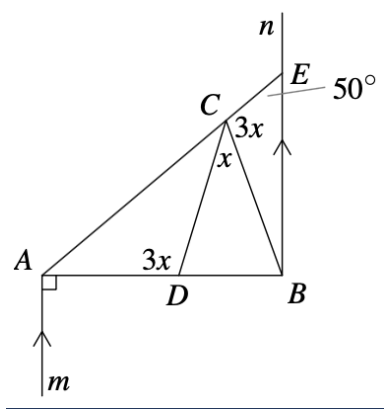

Figure 6.5: **??**

Problem 6.6.3 (ICLT-2022-SM1-2-L2-P3). (*6 points*)

(p, q, r) is a triple of prime numbers. How many such triples are there if $p \leq q \leq r$ $p + q + r = 21$.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 6.6.4 (ICLT-2022-SM1-2-L2-P4). (*6 points*)

Evaluate

$$3(2y^2 + 4x - 2) - 6(x^2 - 2y - 7)$$

when $x = -y$.

(A) $2y + 1$ (B) 12 (C) $2y^2$ (D) 24 (E) 36

Problem 6.6.5 (ICLT-2022-SM1-2-L2-P5). (*6 points*)

My debate club has 10 students: 5 girls and 5 boys. For the upcoming competition I must choose 5 students to form a team. One of the five chosen students is the team captain. Among the remaining four chosen students, two must be girls and two must be boys. How many different team can I choose?

(A) 200 (B) 300 (C) 600 (D) 1000 (E) 5400

Problem 6.6.6 (ICLT-2022-SM1-2-L2-P6). (*6 points*)

Two different rectangles have the same width. All four sides of both rectangles have integer lengths. The areas of the rectangles are 1818 and 1746. Find the largest possible value of the common width of the rectangles.

(A) 18 (B) 12 (C) 6 (D) 3 (E) 2

Problem 6.6.7 (ICLT-2022-SM1-2-L2-P7)**.** (*10 points*)

In **??**, $PQ = PR$ and $\angle PQY = \angle PRX$.

1. (*5 points*) Prove that $QY = RX$.
2. (*5 points*) Prove that $XN = YN$.

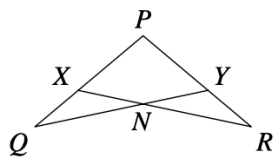

Figure 6.6: **??**

Problem 6.6.8 (ICLT-2022-SM1-2-L2-P8)**.** (*10 points*)

Tim has four balls of different colours: blue, green, red, and white.

1. (*5 points*) He has two wooden boxes, both look exactly the same. In how many ways can he put the four balls into the two boxes?
2. (*5 points*) If he paints the boxes so that they look completely different, then in how many ways can he put the four balls into these two boxes?

Problem 6.6.9 (ICLT-2022-SM1-2-L2-P9)**.** (*10 points*)

x and y are positive real numbers such that,

$$\begin{cases} \sqrt{x} + 2\sqrt{y} = \frac{5}{6} \\ 3\sqrt{x} - \sqrt{y} = \frac{1}{6} \end{cases}$$

1. (*5 points*) What is the value of $\sqrt{x}$?
2. (*5 points*) Solve the equation.

6.7 Grading for Level 2

Answers for multiple-choice problems.

Problem 1:	B	4176
Problem 2:	B	20
Problem 3:	D	3
Problem 4:	E	36
Problem 5:	C	600
Problem 6:	A	18

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 2 point if can find $\triangle YQP \cong \triangle XRP$.

(b) 2 point if can find $XQ = PQ - PX = PR - PY = YR$.

Problem 8: Separately for each part,

(a) 2 points if can casework the number of balls into a box.

(b) 2 points if can reuse the first case.

Problem 9: Separately for each part,

(a) 2 points if can use elimination to find the value of x.

(b) 2 points if can use substitution.

6.8 Solutions for Level 2

Solution. **??** We use complementary counting approach: first we count all the permutations (without restrictions), and then subtract the ones that are not allowed.

Since there are 4 different officers, so there are $18 \cdot 17 \cdot 16$ ways to choose the officers without restrictions. The cases that we have to exclude are those where Ali is an officer but Brenda is not. So let's count the number of these cases by constructive counting. There are 3 choices for the office that Ali will hold. There are $18 - 2 = 16$ choices for the first remaining office (since Ali is already chosen and we're not allowed to choose Brenda). There are 15 choices for the last remaining office.

In total there are $\boxed{18 \cdot 17 \cdot 16 - 3 \cdot 16 \cdot 15 = 4176}$. The answer is $\boxed{B.}$ □

Solution. **??**

First, $\angle ABE = 90^\circ$ because $m \parallel n$ and $m \perp AB$. Then, from $\triangle ABE$ we have

$$\angle EAB + \angle ABE + \angle AEB = 180^\circ \Rightarrow \angle EAB = 180^\circ - \angle ABE - \angle AEB = 40^\circ.$$

We can then use $\triangle ACD$

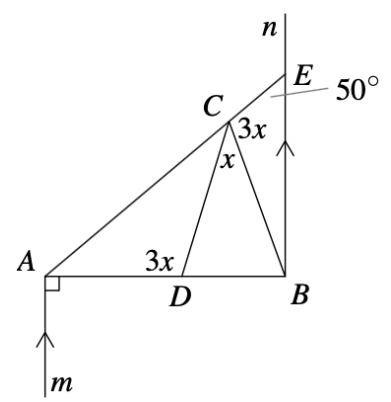

Figure 6.7: $\angle ACD = 140^\circ - 3x$

$$\angle ACD = 180^\circ - \angle DAC - \angle ADC = 180^\circ - 40^\circ - 3x = 140^\circ - 3x.$$

We have three angles with vertices at C that together make a straight line, so we have

$$\angle ACD + \angle DCB + \angle BCE = 180^\circ \Rightarrow 140^\circ - 3x + x + 3x = 180^\circ \Rightarrow x = 40^\circ.$$

Thus, $x = 40^\circ$. It is easy to see that $\boxed{\angle ACD = 140^\circ - 3x = 20^\circ.}$ The answer is $\boxed{B.}$ □

Solution. **??** Because $p + q + r = 21$ and 21 is an odd number, so all p, q, r should be odd or exactly two of them are odd.

First, because $p < q < r$, thus all p, q, r are odd primes. If all p, q, r are larger then 3, then $p + q + r \geq 5 + 7 + 11 = 23 > 21$, therefore $p = 3$ and $q + r = 21 - 3 = 18$. There are only two possible solutions $5 + 13 = 7 + 11 = 18$. So there are two triples $(3, 5, 13), (3, 7, 11)$.

Second, we need to investigate the case when two of them are even. This means that $p = q = 2$. In this case we have an additional triple $(2, 2, 17)$.

Thus, there are $\boxed{3}$ solutions $\boxed{(2, 2, 17), (3, 5, 13), (3, 7, 11).}$ The answer is $\boxed{D.}$ □

Solution. **??**

When $x = -y$, the given expression becomes

$$3(2y^2 + 4x - 2) - 6(x^2 - 2y - 7) = 3(2y^2 - 4y - 2) - 6(y^2 - 2y - 7) = 36$$

Thus, it is $\boxed{36.}$ The answer is $\boxed{E.}$ □

Solution. **??**

First, for a boy captain, there is $\binom{5}{1} = 5$ ways to choose. Then, for 2 boy members, there are $\binom{4}{2}$ ways to choose them. For the 2 girl members, there are $\binom{5}{2}$ ways to choose them. So there are $5 \cdot \binom{4}{2} \cdot \binom{5}{2} = 300$ ways to choose.

It is the same numbers of ways if the team has a girl as a captain.

Thus, the total number of ways is $\boxed{2 \cdot 300 = 600.}$ The answer is $\boxed{C.}$ □

Solution. **??**

Let ℓ_1 be the length of the larger rectangle and ℓ_2 be the length of the smaller rectangle and let w be the width of both rectangles, then

$$\begin{cases} \ell_1 \cdot w = 1818 \\ \ell_2 \cdot w = 1746 \end{cases}$$

Our goal is to find the largest possible value of w. We see that ℓ_1 and n are divisors of 1818 while ℓ_2 and n are divisors of 1746. This means that w is common divisor of 1818 and 1746. Since we are trying to find the largest possible value of w, our goal is actually to find the greatest common divisor of 1818 and 1746.

It is easy to compute that

$$\gcd(1818, 1746) = \gcd(2 \cdot 3^2 \cdot 101, 2 \cdot 3^2 \cdot 97) = 2 \cdot 3^2 = 18.$$

Therefore the largest value of w is $\boxed{18.}$ The answer is $\boxed{A.}$ □

Solution. **??**

Since $PQ = PR$ and $\angle PQY = \angle PRX$. Triangles $\triangle YQP$ and $\triangle XRP$ also have angle $\angle QPR$ in common, so by ASA, they are congruent. Since QY and RX are corresponding parts of triangles $\triangle YQP$ and $\triangle XRP$, we have $QY = RX$.

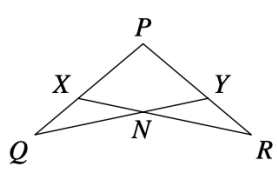

Figure 6.8: $QY = RX$

Since $\triangle YQP \cong \triangle XRP$, we have $PX = PY$ because they are corresponding sides of the two triangles. We also have $PQ = PR$, and we can combine these two side length equalities to show $XQ = YR$:

$$XQ = PQ - PX = PR - PY = YR.$$

Our congruent triangles also give us $\angle Q = \angle R$. Also, $\angle XNQ = \angle YNR$ since they are vertical angles. $XQ = YR$, $\angle Q = \angle R$, and $\angle XNQ = \angle YNR$ give us $\triangle XNQ \cong \triangle YNR$ by AAS Congruence, so $\boxed{XN = YN.}$ □

Solution. **??**

If the two boxes are the same, then we just need to find the number of ways to choose from the 4 balls to put some into one box. He can chose all 4 balls in 1 way, or 3 balls in $\binom{4}{3} = 4$ ways, or 2 balls in $\binom{4}{2} = 6$ ways, or 1 ball in $\binom{4}{1} = 4$ ways, or no ball in 1 way. Note that the two boxes are the same, choosing 3 balls into one box is the same as choosing 1 ball into the other box.

Thus, the number of ways should be $\boxed{\frac{1}{2}(1+4+6+4+1) = 8.}$

For the second question, if the two box are different, then we don't have to divide the choices, thus the total number of ways is $\boxed{16.}$ □

Solution. **??**

$$\begin{cases} \sqrt{x} + 2\sqrt{y} = \frac{5}{6} \\ 3\sqrt{x} - \sqrt{y} = \frac{1}{6} \end{cases} \Rightarrow (\sqrt{x} + 2\sqrt{y}) + 2(3\sqrt{x} - \sqrt{y}) = \frac{5}{6} + 2\frac{1}{6}$$
$$\Rightarrow 7\sqrt{x} = \frac{7}{6}$$

Thus $\boxed{\sqrt{x} = \frac{1}{6}.}$ Therefore $x = \frac{1}{36}$. By substitution $\sqrt{y} = 3\sqrt{x} - \frac{1}{6} = \frac{1}{3} \Rightarrow y = \frac{1}{9}$. So $\boxed{(x, y) = (\frac{1}{36}, \frac{1}{9}).}$ □

6.9 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.10 Problems for Level 3

Problem 6.10.1 (ICLT-2022-SM1-2-L3-P1). (*6 points*)

My debate club has 10 students: 5 girls and 5 boys. For the upcoming competition I must choose 5 students to form a team. One of the five chosen students is the team captain. Among the remaining four chosen students, two must be girls and two must be boys. How many different team can I choose?

(A) 200 (B) 300 (C) 600 (D) 1000 (E) 5400

Problem 6.10.2 (ICLT-2022-SM1-2-L3-P2). (*6 points*)

Two different rectangles have the same width. All four sides of both rectangles have integer lengths. The areas of the rectangles are 1818 and 1746. Find the largest possible value of the common width of the rectangles.

(A) 18 (B) 12 (C) 6 (D) 3 (E) 2

Problem 6.10.3 (ICLT-2022-SM1-2-L3-P3). (*6 points*)

x and y are positive real numbers such that,

$$\begin{cases} \sqrt{x} + 2\sqrt{y} = \frac{5}{6} \\ 3\sqrt{x} - \sqrt{y} = \frac{1}{6} \end{cases}$$

Find the ratio $\frac{y}{x}$.

(A) 3 (B) 4 (C) 6 (D) 8 (E) 9

Problem 6.10.4 (ICLT-2022-SM1-2-L3-P4). (*6 points*)

In **??**, $ABCD$ is a square. $AB = 4$. M is the midpoint of AD and $BM \perp MN$. Find BN.

(A) $2+\sqrt{2}$ (B) 4 (C) $3\sqrt{2}$ (D) 5 (E) $4\sqrt{2}$

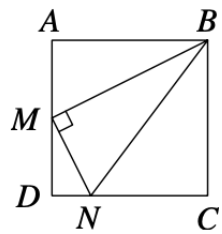

Figure 6.9: **??**

Problem 6.10.5 (ICLT-2022-SM1-2-L3-P5). (*6 points*)

The **??** shows a grid. How many 9-step paths are there going from G to E through F?

(A) 12 (B) 24 (C) 30 (D) 36 (E) 40

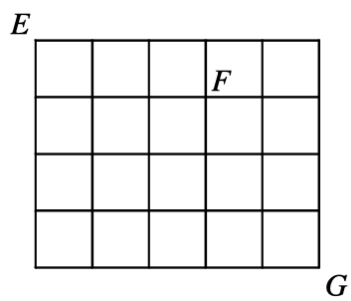

Figure 6.10: **??**

Problem 6.10.6 (ICLT-2022-SM1-2-L3-P6). (*6 points*)

Let $n = 2^a \cdot 3^b$, where a and b are positive integers. If n has exactly 22 divisors, including 1 and itself, how many divisors does n^2 have?

(A) 22 (B) 44 (C) 63 (D) 66 (E) 84

Problem 6.10.7 (ICLT-2022-SM1-2-L3-P7)**.** (*10 points*)

There are 805 animals in the store. They are dogs, cats, and mice. The ratio of dogs to cats to mice is $2:3:30$.

1. (*5 points*) How many cats are in the store?
2. (*5 points*) If 23 cats are added to the store, what is then the ratio of cats to dogs?

Problem 6.10.8 (ICLT-2022-SM1-2-L3-P8)**.** (*10 points*)

Given the side lengths shown in **??**,

1. (*5 points*) Prove that $AE \parallel BC$ and $AB \parallel DE$.
2. (*5 points*) What is the ratio of areas $\frac{[\triangle ABC]}{[\triangle ADE]}$?

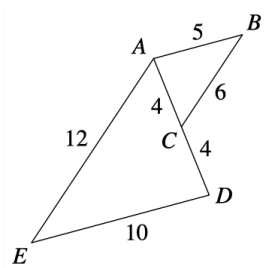

Figure 6.11: **??**

Problem 6.10.9 (ICLT-2022-SM1-2-L3-P9)**.** (*10 points*)

Let $n = (30)^2 = 900$.

1. (*5 points*) Find the number of distinct positive divisors of n excluding 1 and n.
2. (*5 points*) Find the number of distinct positive divisors of n^2 excluding 1 and n^2.

6.11 Grading for Level 3

Answers for multiple-choice problems.

Problem 1:	C	600
Problem 2:	A	18
Problem 3:	B	4
Problem 4:	D	5
Problem 5:	E	40
Problem 6:	C	63

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 2 point if find 35 as the number of animals in a group, or a common denominator.

(b) 2 point if find the total number of dogs.

Problem 8: Separately for each part,

(a) 2 points if can use SSS for similarity.

(b) 2 points if can deduce the ratio of the heights.

Problem 9: Separately for each part,

(a) 1 points if correctly factor n

(b) 1 points if correctly factor n^2

6.12 Solutions for Level 3

Solution. **??**

First, for a boy captain, there is $\binom{5}{1} = 5$ ways to choose. Then, for 2 boy members, there are $\binom{4}{2}$ ways to choose them. For the 2 girl members, there are $\binom{5}{2}$ ways to choose them. So there are $5 \cdot \binom{4}{2} \cdot \binom{5}{2} = 300$ ways to choose.

It is the same numbers of ways if the team has a girl as a captain.

Thus, the total number of ways is $\boxed{2 \cdot 300 = 600.}$ The answer is $\boxed{C.}$ □

Solution. **??**

Let ℓ_1 be the length of the larger rectangle and ℓ_2 be the length of the smaller rectangle and let w be the width of both rectangles, then

$$\begin{cases} \ell_1 \cdot w = 1818 \\ \ell_2 \cdot w = 1746 \end{cases}$$

Our goal is to find the largest possible value of w. We see that ℓ_1 and n are divisors of 1818 while ℓ_2 and n are divisors of 1746. This means that w is common divisor of 1818 and 1746. Since we are trying to find the largest possible value of w, our goal is actually to find the greatest common divisor of 1818 and 1746.

It is easy to compute that

$$\gcd(1818, 1746) = \gcd(2 \cdot 3^2 \cdot 101, 2 \cdot 3^2 \cdot 97) = 2 \cdot 3^2 = 18.$$

Therefore the largest value of w is $\boxed{18.}$ The answer is $\boxed{A.}$ □

Solution. **??**

$$\begin{cases} \sqrt{x} + 2\sqrt{y} = \frac{5}{6} \\ 3\sqrt{x} - \sqrt{y} = \frac{1}{6} \end{cases} \Rightarrow (\sqrt{x} + 2\sqrt{y}) + 2(3\sqrt{x} - \sqrt{y}) = \frac{5}{6} + 2\frac{1}{6}$$
$$\Rightarrow 7\sqrt{x} = \frac{7}{6}$$

Thus $\boxed{\sqrt{x} = \frac{1}{6}.}$ Therefore $x = \frac{1}{36}$. By substitution $\sqrt{y} = 3\sqrt{x} - \frac{1}{6} = \frac{1}{3} \Rightarrow y = \frac{1}{9}$. So $\boxed{\frac{y}{x} = \frac{\frac{1}{9}}{\frac{1}{36}} = 4.}$ The answer is $\boxed{B.}$ □

Solution. **??**

First,

$$\angle NMD = 180^\circ - \angle NMB - \angle BMA = 180^\circ - 90^\circ - \angle BMA = 90^\circ - \angle BMA = \angle ABM.$$

and $\angle MDN = \angle A$, we have $\triangle MDN \sim \triangle BAM$.

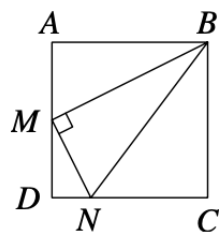

Figure 6.12: $\triangle MDN \sim \triangle BAM$

Thus

$$\frac{MN}{BM} = \frac{MD}{AB} = \frac{1}{2} = \frac{AM}{AB} \Rightarrow \triangle BAM \sim \triangle BMN \Rightarrow \frac{BN}{BM} = \frac{BM}{BA}$$

Therefore $BN = \frac{BM^2}{BA} = \frac{BA^2+AM^2}{BA} = \frac{4^2+2^2}{4} = 5$. The answer is $\boxed{D.}$ □

Solution. **??**

From G to F, it is 2 steps to the left and 3 steps up, for a total of $\binom{5}{2} = 10$ different paths. From F to E,

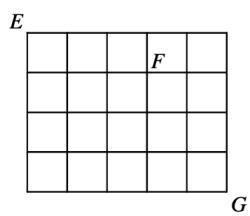

Figure 6.13: $\binom{5}{2} = 10$ path from G to F

it is 3 steps to the left and 1 step up, for a total of $\binom{4}{1} = 4$ different paths.

So there are $\boxed{4 \cdot 10 = 40}$ paths from G to F. The answer is $\boxed{E.}$ □

Solution. **??**

Since n is in the format of $2^a 3^b$, the number of divisors of n is

$$22 = (a+1)(b+1)$$

Thus the pair (a, b) can only be $(1, 10)$ or $(10, 1)$, since n^2 is

$$n^2 = \left(2^a 3^b\right)^2 = (2^{2a} 3^{2b})$$

Thus, the number of divisors of n^2 is $\boxed{(2a+1)(2b+1) = 3 \cdot 21 = 63.}$ The answer is $\boxed{C.}$ □

Solution. **??**

For the first question, for every 2 dogs, there are 3 cats and 30 mice. Therefore, we group the animals such that each group contains 2 dogs, 3 cats, and 30 mice. Each such group has $2+3+30=35$ animals, and there are 805 total animals, so there must be 23 groups. There are 3 cats in each of these groups, so there are $\boxed{23cdot3=69}$ cats.

The number of dogs is $23 \cdot 2 = 46$. If we increase the number of cats by 23 to a total of 92, then the ratio of cats to dogs now is $\boxed{92:46=2:1.}$ □

Solution. **??**

Since

$$\frac{AB}{DE}=\frac{AC}{AD}=\frac{BC}{AE}=\frac{1}{2},$$

so $\triangle ABC \sim \triangle DEA$ by SSS Similarity. Therefore, $\angle BAC = \angle EDA$, so $AB \parallel DE$. Also, $\angle DAE = \angle ACB$,

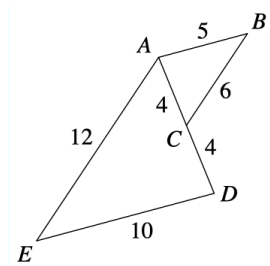

Figure 6.14: $\triangle ABC \sim \triangle DEA$

so $AE \parallel BC$.

It is easy to see that that ratio of heights between the two triangles $\triangle ABC$ and $\triangle DEA$ is the same as their side ratio. Therefore the ratio of their areas is the square of their ratio, thus $\boxed{\frac{[\triangle ABC]}{[\triangle ADE]}=\left(\frac{1}{2}\right)^2=\frac{1}{4}.}$ □

Solution. **??**

Note that $n=30^2=(2\cdot3\cdot5)^2=2^2\cdot3^2\cdot5^2$. Thus, the number of divisors of n, including 1 and itself, is,

$$(2+1)(2+1)(2+1)=27.$$

So there are $\boxed{27-2=25}$ divisors of n not including 1 and n.

Similarly, $n^2=30^4=2^4\cdot3^4\cdot5^4$ has $\boxed{(4+1)(4+1)(4+1)-2=123}$ divisors, not including 1 and n^2. □

6.13 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.14 Problems for Level 4

Problem 6.14.1 (ICLT-2022-SM1-2-L4-P1). (*6 points*)

In **??**, $ABCD$ is a square. $AB = 4$. M is the midpoint of AD and $BM \perp MN$. Find BN.

(A) $2+\sqrt{2}$ (B) 4 (C) $3\sqrt{2}$ (D) 5 (E) $4\sqrt{2}$

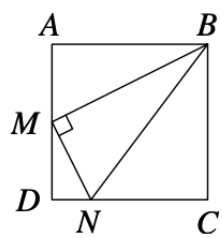

Figure 6.15: **??**

Problem 6.14.2 (ICLT-2022-SM1-2-L4-P2). (*6 points*)

Let $n = 2^a \cdot 3^b$, where a and b are positive integers. If n has exactly 22 divisors, including 1 and itself, how many divisors does n^2 have?

(A) 22 (B) 44 (C) 63 (D) 66 (E) 84

Problem 6.14.3 (ICLT-2022-SM1-2-L4-P3). (*6 points*)

There are 805 animals in the store. They are dogs, cats, and mice. The ratio of dogs to cats to mice is $2:3:30$. How many cats are in the store?

(A) 23 (B) 46 (C) 69 (D) 92 (E) 115

Problem 6.14.4 (ICLT-2022-SM1-2-L4-P4). (*6 points*)

Suppose 2 distinct integers are chosen from between 0 and 9, inclusive. What is the probability that they are both odd?

(A) $\frac{2}{9}$ (B) $\frac{5}{18}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) $\frac{5}{9}$

Problem 6.14.5 (ICLT-2022-SM1-2-L4-P5). (*6 points*)

Find the standard form of the equation of the line through $(6,1)$ that is perpendicular to the line $2x = -3y + 7$.

(A) $3x-2y=10$ (B) $3x-2y=20$ (C) $2x-3y=10$ (D) $2x-3y=20$ (E) $3x-4y=20$

Problem 6.14.6 (ICLT-2022-SM1-2-L4-P6). (*6 points*)

In triangle $\triangle PQR$, $PQ = 3$, $\angle Q = 90^\circ$, and $QR = 4$. Find the inradius of $\triangle PQR$.

(A) 2 (B) $2-\sqrt{2}$ (C) $\frac{3}{2}$ (D) $\frac{1+\sqrt{2}}{2}$ (E) 1

Problem 6.14.7 (ICLT-2022-SM1-2-L4-P7)**.** (*10 points*)

The Pirates play 4 games with the Mathematicians. The Pirates are 70% likely to win any given game.

1. (*5 points*) What is the probability that The Pirates win the first two games?
2. (*5 points*) What is the probability that the Mathematicians win exactly one game?

Problem 6.14.8 (ICLT-2022-SM1-2-L4-P8)**.** (*10 points*)

In $\triangle XYZ$, $XZ = 8$. Point D is on side YZ such that $\angle YXD = \angle ZXD$, $YD = 3$, and $DZ = 4$.

1. (*5 points*) Find XY.
2. (*5 points*) Find the aread of $\triangle XYZ$.

Problem 6.14.9 (ICLT-2022-SM1-2-L4-P9)**.** (*10 points*)

Find all two-digit number $\overline{ab}$, if

1. (*5 points*) $a^2 + b^2 = 4(a + b) - 8$.
2. (*5 points*) $a^2 + b^2 = 4(a + b) - 3$.

6.15 Grading for Level 4

Answers for multiple-choice problems.

Problem 1:	D	5
Problem 2:	C	63
Problem 3:	C	69
Problem 4:	A	$\frac{2}{9}$
Problem 5:	B	$3x - 2y = 20$
Problem 6:	E	1

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 1 point if can find the probability is a product of probabilities to win both the first and the second games,

(b) 1 point if can casework which game the Mathematicians wins.

Problem 8: Separately for each part,

(a) 2 points if use the Angle Bisector Theorem.

(b) 2 points if use the Heron's Formula

Problem 9: Separately for each part,

(a) 2 points if can reduce to sum of squares

(b) 2 points if can argue $5 = 1^2 + 2^2$ is the only possible sum of perfect squares.

6.16 Solutions for Level 4

Solution. **??**

First,

$$\angle NMD = 180^\circ - \angle NMB - \angle BMA = 180^\circ - 90^\circ - \angle BMA = 90^\circ - \angle BMA = \angle ABM.$$

and $\angle MDN = \angle A$, we have $\triangle MDN \sim \triangle BAM$.

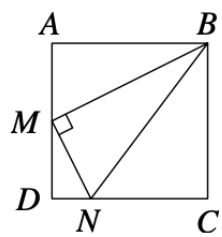

Figure 6.16: $\triangle MDN \sim \triangle BAM$

Thus

$$\frac{MN}{BM} = \frac{MD}{AB} = \frac{1}{2} = \frac{AM}{AB} \Rightarrow \triangle BAM \sim \triangle BMN \Rightarrow \frac{BN}{BM} = \frac{BM}{BA}$$

Therefore $BN = \frac{BM^2}{BA} = \frac{BA^2+AM^2}{BA} = \frac{4^2+2^2}{4} = 5$. The answer is $\boxed{D.}$ □

Solution. **??**

Since n is in the format of 2^a3^b, the number of divisors of n is

$$22 = (a+1)(b+1)$$

Thus the pair (a, b) can only be $(1, 10)$ or $(10, 1)$, since n^2 is

$$n^2 = \left(2^a3^b\right)^2 = (2^{2a}3^{2b})$$

Thus, the number of divisors of n^2 is $\boxed{(2a+1)(2b+1) = 3 \cdot 21 = 63.}$ The answer is $\boxed{C.}$ □

Solution. **??**

For the first question, for every 2 dogs, there are 3 cats and 30 mice. Therefore, we group the animals such that each group contains 2 dogs, 3 cats, and 30 mice. Each such group has $2 + 3 + 30 = 35$ animals, and there are 805 total animals, so there must be 23 groups. There are 3 cats in each of these groups, so there are $\boxed{23cdot3 = 69}$ cats. The answer is $\boxed{C.}$ □

Solution. **??**

Let's consider an outcome to be a choice of 2 numbers without regard to order. There are $\binom{10}{2} = 45$ ways to choose 2 numbers from our set of 10 numbers if we don't care about order. There are $\binom{5}{2} = 10$ ways to choose 2 numbers from the 5 odd numbers without regard to order. Therefore, the probability is $\boxed{\frac{10}{45} = \frac{2}{9}.}$ The answer is $\boxed{A.}$ □

Solution. **??**

We find the slope of the given line by writing it in slope-intercept form. Subtracting 7 from both sides and dividing both sides by -3 gives

$$y = -\frac{2}{3}x + \frac{7}{3}.$$

So, the slope of the given line is $-2/3$. Therefore, the slope of a line perpendicular to the given line is $-1/(-2/3) = 3/2$. We find a point-slope equation of the line through $(6, 1)$ with slope $3/2$, and then convert it to standard form:

$$y - 1 = \frac{3}{2}(x - 6) \Rightarrow 2(y - 1) = 3(x - 6) \Rightarrow \ 2y - 2 = 3x - 18 \Rightarrow \ 3x - 2y = 20$$

Thus, the standard form of the line is $\boxed{3x - 2y = 20.}$ The answer is $\boxed{B.}$ □

Solution. **??**

Since $\triangle PQR$ is a right triangle, we have $[PQR] = (PQ)(QR)/2 = 6$. $PR = \sqrt{PQ^2 + QR^2} = 5$. The semiperimeter of $\triangle PQR$ is $(3 + 4 + 5)/2 = 6$. Thus, the inradius is $\boxed{6/6 = 1.}$ The answer is $\boxed{E.}$ □

Solution. **??**

The Pirates wins the first two games if they wins both the first and the second games,

$$P(\text{Pirates win the 1st game}) \times P(\text{Pirates win the 2nd game}) = 0.7^2 = 0.49$$

Thus the probability that the Pirates wins the first two games is $\boxed{0.49.}$

The Mathematicians win exactly one game if the Pirates wins all three other games, for example for the Mathematicians to win the first game, then lose all other, the probability is

$$P(\text{Pirates lose the 1st game}) \times P(\text{Pirates win a game})^3 = 0.3 \cdot 0.7^3 = 0.1029$$

There are 4 such cases, so the probability is $\boxed{4 \cdot 0.1029 = 0.4116.}$ □

Solution. **??**

Since $\angle YXD = \angle ZXD$, we know that XD bisects $\angle YXZ$. From the Angle Bisector Theorem, we have $\frac{XY}{YD} = \frac{XZ}{DZ} = 2$. Therefore, $\boxed{XY = 2YD = 6.}$

The semi-perimeter of the $\triangle XYZ$ with sides $XY = 6, YZ = 7, ZX = 8$ is $\frac{21}{2}$. Thus, its area is $\sqrt{\frac{21}{2}\left(\frac{21}{2} - 8\right)\left(\frac{21}{2} - 7\right)\left(\frac{21}{2} - 6\right)}$, or $\boxed{\frac{21\sqrt{15}}{4}.}$ □

Solution. **??**

$$a^2 + b^2 = 4(a+b) - 8 \Rightarrow (a^2 - 4a + 4) + (b^2 - 4b + 4) = 0 \Rightarrow (a-2)^2 + (b-2)^2 = 0$$

The left side is a sum of two perfect square. It can be 0 if and only if both perfect square is 0. Hence $\boxed{(a,b) = (2,2).}$

$$a^2 + b^2 = 4(a+b) - 3 \Rightarrow (a^2 - 4a + 4) + (b^2 - 4b + 4) = 5 \Rightarrow (a-2)^2 + (b-2)^2 = 5$$

The left side is a sum of two perfect square. It can be 5 if and only if one of them is 1 and the other is 4.

Case 1: $|a-2| = 1, |b-2| = 2$.

$$\begin{cases} a-2 = 1, b-2 = 2 \\ a-2 = 1, b-2 = -2 \\ a-2 = -1, b-2 = 2 \\ a-2 = -1, b-2 = -2 \end{cases} \Rightarrow \overline{ab} \in \{34, 30, 14, 10\}$$

Case 2: $|a-2| = 4, |b-2| = 1$.

$$\begin{cases} a-2 = 2, b-2 = 1 \\ a-2 = 2, b-2 = -1 \end{cases} \Rightarrow \overline{ab} \in \{43, 41\}$$

Therefore the possible solutions are $\boxed{\{10, 14, 30, 34, 41, 43\}.}$ □

6.17 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
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     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.18 Problems for Level 5

Problem 6.18.1 (ICLT-2022-SM1-2-L5-P1). (*6 points*)

Suppose 2 distinct integers are chosen from between 0 and 9, inclusive. What is the probability that they are both odd?

$(A)\ \frac{2}{9}$ $(B)\ \frac{5}{18}$ $(C)\ \frac{1}{3}$ $(D)\ \frac{1}{2}$ $(E)\ \frac{5}{9}$

Problem 6.18.2 (ICLT-2022-SM1-2-L5-P2). (*6 points*)

Find the standard form of the equation of the line through $(6, 1)$ that is perpendicular to the line $2x = -3y + 7$.

$(A)\ 3x-2y = 10$ $(B)\ 3x-2y = 16$ $(C)\ 2x-3y = 10$ $(D)\ 2x-3y = 20$ $(E)\ 3x-4y = 20$

Problem 6.18.3 (ICLT-2022-SM1-2-L5-P3). (*6 points*)

In $\triangle XYZ$, $XZ = 8$. Point D is on side YZ such that $\angle YXD = \angle ZXD$, $YD = 3$, and $DZ = 4$. Find the aread of $\triangle XYZ$.

$(A)\ \frac{63}{4}$ $(B)\ \frac{21\sqrt{5}}{4}$ $(C)\ 21$ $(D)\ \frac{21\sqrt{15}}{4}$ $(E)\ 24$

Problem 6.18.4 (ICLT-2022-SM1-2-L5-P4). (*6 points*)

The sum of digits of a number written in base 2 is calculated in the same way as in decimal number base. *For example the sum of digits of the number* $\overline{101}_2$ *in base 2 is* $1 + 0 + 1 = 2$.

Let $S(n)$ be sum of digit of the number n in base 2. Similarly $S(2n)$, $S(4n)$, $S(8n)$ are the sum of digits of the number $2n$, $4n$, $8n$ in base 2.

Find $S(n)$ if $S(n) + S(2n) + S(4n) + S(8n) = 32$.

$(A)\ 6$ $(B)\ 7$ $(C)\ 8$ $(D)\ 9$ $(E)\ 10$

Problem 6.18.5 (ICLT-2022-SM1-2-L5-P5). (*6 points*)

$\sqrt{29 + 12\sqrt{5}}$ can be expressed in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by any perfect square larger than 1. Find $a + b + c$.

$(A)\ 6$ $(B)\ 7$ $(C)\ 8$ $(D)\ 9$ $(E)\ 10$

Problem 6.18.6 (ICLT-2022-SM1-2-L5-P6). (*6 points*)

Each side of an equilateral triangle is 6 centimeters long. An altitude of this triangle is used as the side of a square. What is the number of square centimeters in the area of the square?

$(A)\ 15$ $(B)\ 18$ $(C)\ 20$ $(D)\ 24$ $(E)\ 27$

Problem 6.18.7 (ICLT-2022-SM1-2-L5-P7). (*10 points*)

The set S_1 of seven parallel lines $\ell_1, \ell_2, \ldots, \ell_7$in a plane intersect another set S_2 of of n parallel $k_1, k_2, \ldots, k_n$ lines that go in another direction. The lines form a total of 210 parallelograms, many of which overlap each other.

1. (*5 points*) In how many ways can you choose two lines from set S_1 and two lines from set S_2 so that they can form a pararellogram?
2. (*5 points*) Find n.

Problem 6.18.8 (ICLT-2022-SM1-2-L5-P8). (*10 points*)

In base$-b$, where b is a positive integer larger than 6, the number n is expressed as

$$n = \overline{1452}_b = 1 \cdot b^3 + 4 \cdot b^2 + 5 \cdot b + 2$$

1. (*5 points*) Is there a base b where n is a prime number?
2. (*5 points*) For which b, n has at least two distinct factor larger then 1 and less than n?

Problem 6.18.9 (ICLT-2022-SM1-2-L5-P9). (*10 points*)

Let $S(n) = 1 + i + i^2 + \ldots + i^n$, where $i = \sqrt{-1}$ the imaginary number.

1. (*5 points*) Find $S(3)$.
2. (*5 points*) Find $S(2022)$.

6.19 Grading for Level 5

Answers for multiple-choice problems.

Problem 1:	A	$\frac{2}{9}$
Problem 2:	B	$3x - 2y = 16$
Problem 3:	D	$\frac{21\sqrt{15}}{4}$
Problem 4:	C	8
Problem 5:	B	10
Problem 6:	E	27

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 2 point if can determine how to chose a set of 2×2 lines to form a pararellogram.

(b) 2 point if can express the equation for n

Problem 8: Separately for each part,

(a) 2 points if can recognize or factor the polynomial $b^3 + 4b^2 + 5b + 2$.

(b) 2 points if can completelt factor the polynomial $b^3 + 4b^2 + 5b + 2$.

Problem 9: Separately for each part,

(a) 2 points if can apply $i^2 = -1$

(b) 2 points if can recognize a block of four consecutive power of i will have a sum of 0.

6.20 Solutions for Level 5

Solution. **??**

Let's consider an outcome to be a choice of 2 numbers without regard to order. There are $\binom{10}{2} = 45$ ways to choose 2 numbers from our set of 10 numbers if we don't care about order. There are $\binom{5}{2} = 10$ ways to choose 2 numbers from the 5 odd numbers without regard to order. Therefore, the probability is $\boxed{\frac{10}{45} = \frac{2}{9}.}$ The answer is $\boxed{A.}$ □

Solution. **??**

We find the slope of the given line by writing it in slope-intercept form. Subtracting 7 from both sides and dividing both sides by -3 gives

$$y = -\frac{2}{3}x + \frac{7}{3}.$$

So, the slope of the given line is $-2/3$. Therefore, the slope of a line perpendicular to the given line is $-1/(-2/3) = 3/2$. We find a point-slope equation of the line through $(6, 1)$ with slope $3/2$, and then convert it to standard form:

$$y - 1 = \frac{3}{2}(x - 6) \Rightarrow 2(y - 1) = 3(x - 6) \Rightarrow \ 2y - 2 = 3x - 18 \Rightarrow \ 3x - 2y = 16$$

Thus, the standard form of the line is $\boxed{3x - 2y = 16.}$ The answer is $\boxed{B.}$ □

Solution. **??**

Since $\angle YXD = \angle ZXD$, we know that XD bisects $\angle YXZ$. From the Angle Bisector Theorem, we have $\frac{XY}{YD} = \frac{XZ}{DZ} = 2$. Therefore, $\boxed{XY = 2YD = 6.}$

The semi-perimeter of the $\triangle XYZ$ with sides $XY = 6, YZ = 7, ZX = 8$ is $\frac{21}{2}$. Thus, its area is $\sqrt{\frac{21}{2}\left(\frac{21}{2} - 8\right)\left(\frac{21}{2} - 7\right)\left(\frac{21}{2} - 6\right)}$, or $\boxed{\frac{21\sqrt{15}}{4}.}$ The answer is $\boxed{D.}$ □

Solution. **??**

Note that the number $2n$ in base 2 has the same sequence of digits as the number n with an addition of 0 at the end, for exemplae

$$\overline{1}_2 \cdot 2 = \overline{10}_2, \ \overline{1001}_2 \cdot 2 = \overline{10010}_2$$

Therefore $S(n) = S(2n) = S(4n) = S(8n)$, thus $\boxed{S(n) = \frac{32}{4} = 8.}$ The answer is $\boxed{C.}$ □

Solution. **??**

Let

$$\sqrt{29 + 12\sqrt{5}} = a + b\sqrt{c} \Rightarrow 29 + 12\sqrt{5} = (a^2 + b^2c) + 2ab\sqrt{c} \Rightarrow c = 5 \Rightarrow \begin{cases} a^2 + 5b^2 = 29 \\ ab = 6 \end{cases}$$

From $ab = 6$ and a, b are positive integers, by testing $(a, b) = (3, 2)$. Thus $\boxed{a + b + c = 3 + 2 + 5 = 10.}$ The answer is $\boxed{B.}$ □

Solution. **??**

Let ABC be the triangle and AD an altitude. Then triangle $\triangle ABD$ is a 30-60-90 triangle with $AD = (\sqrt{3}/2)(AB) = 3\sqrt{3}$. The area of the square is then $\boxed{(3\sqrt{3})^2 = 27.}$ The answer is $\boxed{E.}$ □

Solution. **??**

Since set S_1 has 7 lines, so there are $\binom{7}{2}$ ways to choose two lines from it. Similarly there are $\binom{n}{2}$ ways to choose another two lines from it. The intersection of these four lines forms a parallelo gram. Thus, the number of ways to form a pararellogram is $\boxed{\binom{7}{2}\binom{n}{2}.}$

Now, since

$$\binom{7}{2}\binom{n}{2} = 210 \Rightarrow \frac{21n(n-1)}{2} = 210 \Rightarrow n(n-1) = 20 \Rightarrow n = 5$$

So there are $\boxed{n = 5}$ lines in set S_2. □

Solution. **??**

Note that,

$$n = \overline{1452}_b = 1 \cdot b^3 + 4 \cdot b^2 + 5 \cdot b + 2 = (b^3 + b^2) + (3b^2 + 3b) + (2b + 2) = (b+1)(b^2 + 3b + 2) = (b+1)^2(b+2)$$

Thus, $\boxed{n \text{ is always a composite number}}$, regardless of what value b might have. Furthermore $\boxed{b + 1\ neb + 2}$ are the two distinct factors of n, which are larger then 1 and less than n. □

Solution. **??**

Note that,

$$S(3) = 1 + i + i^2 + i^3 = 1 + i + (-1) + (-i) = 0$$

Therefore $\boxed{S(3) = 0.}$

Furthermore, for any $k > 1$, $i^{4k-4} + i^{4k-3} + i^{4k-2} + i^{4k-1} = i^{4k-4}S(3) = 0$. Therefore for any $n = 4k - 1$,

$$S(4k-1) = \underbrace{1 + i + i^2 + i^3} + \ldots + \underbrace{i^{4k-4} + i^{4k-3} + i^{4k-2} + i^{4k-1}} = 0$$

Thus, $S(4k+2) = S(4k-1) + i^{4k} + i^{4k+1} + i^{4k+2} = 1 + i + (-1) = i$. Therefore $\boxed{S(2022) = i.}$ □

6.21 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.22 Problems for Level 6

Problem 6.22.1 (ICLT-2022-SM1-2-L6-P1). (*6 points*)

The sum of digits of a number written in base 2 is calculated in the same way as in decimal number base. *For example the sum of digits of the number* $\overline{101}_2$ *in base 2 is* $1 + 0 + 1 = 2$.

Let $S(n)$ be sum of digit of the number n in base 2. Similarly $S(2n)$, $S(4n)$, $S(8n)$ are the sum of digits of the number $2n$, $4n$, $8n$ in base 2.

Find $S(n)$ if $S(n) + S(2n) + S(4n) + S(8n) = 32$.

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Problem 6.22.2 (ICLT-2022-SM1-2-L6-P2). (*6 points*)

$\sqrt{29 + 12\sqrt{5}}$ can be expressed in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by any perfect square larger than 1. Find $a + b + c$.

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Problem 6.22.3 (ICLT-2022-SM1-2-L6-P3). (*6 points*)

The set S_1 of seven parallel lines $\ell_1, \ell_2, \ldots, \ell_7$in a plane intersect another set S_2 of of n parallel $k_1, k_2, \ldots, k_n$ lines that go in another direction. The lines form a total of 210 parallelograms, many of which overlap each other. Find n.

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Problem 6.22.4 (ICLT-2022-SM1-2-L6-P4). (*6 points*)

$A_1A_2 \ldots A_9$ is a regular nonagon (9−gon). Find $\angle A_1A_4A_5$.

(A) 154 (B) 140 (C) 132 (D) 100 (E) 95

Problem 6.22.5 (ICLT-2022-SM1-2-L6-P5). (*6 points*)

Find the positive constant c that makes the quadratic $25x^2 + cx + 49$ the square of a binomial.

(A) 70 (B) 75 (C) 80 (D) 85 (E) 90

Problem 6.22.6 (ICLT-2022-SM1-2-L6-P6). (*6 points*)

Find the unit digit of 3^{2022}.

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

Problem 6.22.7 (ICLT-2022-SM1-2-L6-P7). (*10 points*)

A real number x is selected randomly such that $0 \leq x \leq 3$.

1. (*5 points*) What is the probability that $|x - \frac{1}{3}| \leq \frac{1}{2}$?
2. (*5 points*) What is the probability that $(x-1)(2-x) \geq 0$.

Problem 6.22.8 (ICLT-2022-SM1-2-L6-P8). (*10 points*)

$A_1A_2 \ldots A_8$ is a regular octagon ($8-$gon), where $A_1A_2 = 1$. Find the area of $A_1A_2A_4A_7$.

Problem 6.22.9 (ICLT-2022-SM1-2-L6-P9). (*10 points*)

P is a parabola with axis of symmetry parallel to the x-axis such that its graph passes through the points $(3, 0)$, $(-2, 1)$, and $(12, -1)$.

1. (*5 points*) What is the equation of P?
2. (*5 points*) At what points does P intersect the $y-$axis?

6.23 Grading for Level 6

Answers for multiple-choice problems.

Problem 1:	C	8
Problem 2:	B	10
Problem 3:	E	27
Problem 4:	D	100
Problem 5:	A	70
Problem 6:	E	9

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 2 point if can determine that $0 \leq x \leq \frac{5}{6}$.

(b) 2 point if can determine that $1 \leq x \leq 2$.

Problem 8: Separately for each part,

(a) 2 points if can find that $\triangle A_3A_4A_5 \cong \triangle A_4A_5A_6$.

(b) 2 points if can find that $A_1A_3A_4A_6$ is an isosceles trapezoid.

Problem 9: Separately for each part,

(a) 2 points if can determine the generic equation $x = ay^2 + by + c$.

(b) 2 points if find that its $y-$ intercepts are the intersection points.

6.24 Solutions for Level 6

Solution. **??**

Note that the number $2n$ in base 2 has the same sequence of digits as the number n with an addition of 0 at the end, for exemplae

$$\overline{1}_2 \cdot 2 = \overline{10}_2,\ \overline{1001}_2 \cdot 2 = \overline{10010}_2$$

Therefore $S(n) = S(2n) = S(4n) = S(8n)$, thus $\boxed{S(n) = \frac{32}{4} = 8.}$ The answer is $\boxed{C.}$ □

Solution. **??**

Let

$$\sqrt{29 + 12\sqrt{5}} = a + b\sqrt{c} \Rightarrow 29 + 12\sqrt{5} = (a^2 + b^2c) + 2ab\sqrt{c} \Rightarrow c = 5 \Rightarrow \begin{cases} a^2 + 5b^2 = 29 \\ ab = 6 \end{cases}$$

From $ab = 6$ and a, b are positive integers, by testing $(a, b) = (3, 2)$. Thus $\boxed{a + b + c = 3 + 2 + 5 = 10.}$ The answer is $\boxed{B.}$ □

Solution. **??**

Since set S_1 has 7 lines, so there are $\binom{7}{2}$ ways to choose two lines from it. Similarly there are $\binom{n}{2}$ ways to choose another two lines from it. The intersection of these four lines forms a parallelo gram. Thus, the number of ways to form a pararellogram is $\boxed{\binom{7}{2}\binom{n}{2}.}$

Now, since

$$\binom{7}{2}\binom{n}{2} = 210 \Rightarrow \frac{21n(n-1)}{2} = 210 \Rightarrow n(n-1) = 20 \Rightarrow n = 5$$

So there are $\boxed{n = 5}$ lines in set S_2. The answer is $\boxed{A.}$ □

Solution. **??**

Since the polygon is a regular 9−gon, each exterior angle is $360/9 = 40$ degrees. Therefore, each interior angle is $180° - 40° = 140°$, or for example $\angle A_1A_2A_3 = 140°$, see **??**. Since $\triangle A_1A_2A_3$ is isosceles,

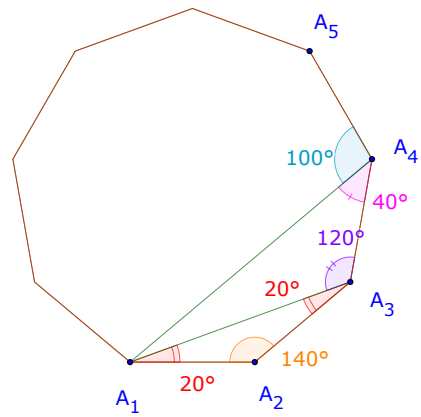


Figure 6.17: $\angle A_1A_2A_3 = 140°$

$\angle A_2A_1A_3 = \angle A_2A_3A_1$. The sum of the angles of $\triangle A_1A_2A_3$ is $180°$, thus

$$\angle A_2A_3A_1 = \frac{1}{2}(180° - 140°) = 20°.$$

Thus, $\angle A_1A_3A_4 = \angle A_2A_3A_4 - \angle A_2A_3A_1 = 120°$. Since $A_1A_2A_3A_4$ is an isosceles trapezoid, where $A_2A_3 \parallel A_1A_4$, so

$$\angle A_3A_4A_1 = 180° - \angle A_2A_3A_4 = 40° \Rightarrow \angle A_1A_4A_5 = \angle A_3A_4A_5 - \angle A_3A_4A_1 = 100°.$$

Therefore $\boxed{\angle A_1A_4A_5 = 100°.}$ The answer is $\boxed{D.}$ □

Solution. **??**

Suppose the quadratic is the square of the binomial $ax + b$. We have

$$25x^2 + cx + 49 = (ax + b)^2 = (ax)^2 + 2(ax)(b) + b^2 = a^2x^2 + 2abx + b^2. \Rightarrow \begin{cases} a^2 = 25 \\ 2ab = c \\ b^2 = 49 \end{cases}$$

It is easy to see that $a = \pm 5, b = \pm 7$, and $\boxed{c = 2ab = 70.}$ The answer is $\boxed{A.}$ □

Solution. [First solution] **??**

Note that

$$3^1 = 3, 3 \cdot 3 = 9, 9 \cdot 3 = 27, 7 \cdot 3 = 21, 1 \cdot 3 = 3$$

Thus the unit digits of $3^1, 3^2, \ldots$ is the periodic sequence

$$\underbrace{3, 9, 7, 1}_{\text{period 4}}, 3, 9, 7, 1, 3 \ldots$$

Since 2022 has a remainder 2 when divided by 4, thus it has the same unit digit as 3^2, which is $\boxed{9}$. The answer is $\boxed{E.}$ □

Solution. [Second solution] **??**

$$3^4 = 81 \equiv 1 \pmod{10} \Rightarrow 3^{2022} = 3^2 \cdot (3^4)^{505} \equiv 3^2 \equiv 1 \pmod{10}$$

Thus 2022 the unit digit of $\boxed{9}$. The answer is $\boxed{E.}$ □

Solution. **??**

For the first question

$$|x - \frac{1}{3}| \leq \frac{1}{2} \Rightarrow -\frac{1}{2} \leq x - \frac{1}{3} \leq \frac{1}{2} \Rightarrow \frac{1}{3} - \frac{1}{2} \leq x \leq \frac{1}{3} + \frac{1}{2} \Rightarrow -\frac{1}{6} \leq x \leq \frac{5}{6}$$

Since $0 \leq x \leq 3$, thus $0 \leq x \leq \frac{5}{6}$, the probability is $\boxed{\frac{\frac{5}{6}}{3} = \frac{5}{18}.}$

For the second question

$$(x-1)(2-x) \geq \Rightarrow \begin{cases} x-1 \geq 0, 2-x \geq 0 \text{ or} \\ x-1 \leq 0, 2-x \leq 0 \end{cases} \Rightarrow \begin{cases} 1 \leq x \leq 2 \text{ or} \\ x \leq 1, 2 \leq x \end{cases}$$

The second case is impossible, so $1 \leq x \leq 2$, the probability is $\boxed{\frac{1}{3}.}$ □

Solution. **??**

For the first question, it is easy to see that $\triangle A_3A_4A_5 \cong \triangle A_4A_5A_6$, thus $[A_1A_3A_5A_6] = [A_1A_3A_4A_6]$, see **??**.

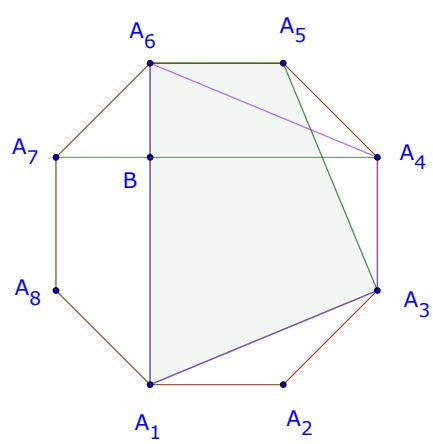


Figure 6.18: $\angle A_1A_2A_3 = 140°$

Let $A_1A_2 = s$. Now, $\triangle A_6BA_7$ is a $45-45-90$ right triangle, so $A_7B = \frac{s}{\sqrt{2}} = \frac{s\sqrt{2}}{2}$. Thus, $A_4B = A_5A_6 + A_7B = s + \frac{s\sqrt{2}}{2} = \frac{(2+\sqrt{2})s}{2}$. Since $A_1A_3A_4A_6$ is an isosceles trapezoid,

$$[A_1A_3A_4A_6] = \frac{1}{2}(A_3A_4 + A_1A_6) \cdot A_4B = \frac{1}{2}(s + s + 2\frac{s\sqrt{2}}{2})\frac{(2+\sqrt{2})s}{2} = \frac{(3+2\sqrt{2})s^2}{2}$$

With $s = 1$, $\boxed{[A_1A_3A_4A_6] = \frac{3+2\sqrt{2}}{2}.}$ □

Solution. **??**

Because the axis of symmetry is parallel to the x-axis, it opens either leftward or rightward. Therefore, its general form is $x = ay^2 + by + c$. Substituting the three given points in this equation gives us the system

$$\begin{cases} 3 = c \\ -2 = a - b + c \\ 12 = a + b + c \end{cases} \Rightarrow \begin{cases} -5 = a - b \\ 9 = a + b \end{cases} \Rightarrow (a, b, c) = (2, -7, 3)$$

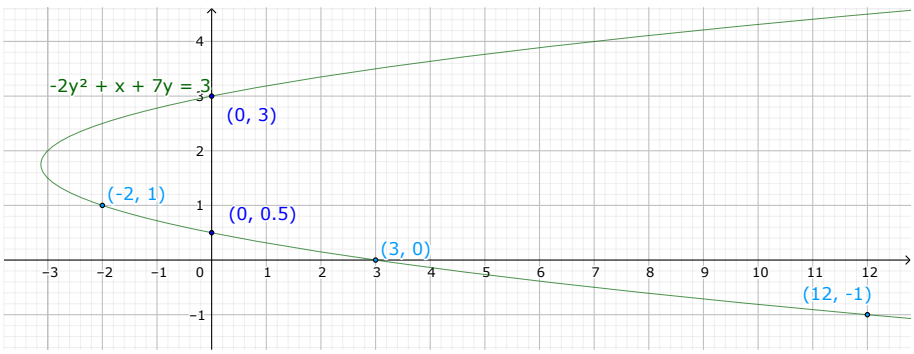


Figure 6.19: **??**

Thus the parabola's equation is $x = 2y^2 - 7y + 3$.

For the second question, its $y-$ intercepts are points with $x-$coordinates $0 : 2y^2 - 7y + 3 = 0 \Rightarrow y_{1,2} = \frac{1}{2},\ 3$ Therefore the intercepts are $\boxed{\left(0, \frac{1}{2}\right), (0, 3).}$ □

6.25 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.26 Problems for Level 7

Problem 6.26.1 (ICLT-2022-SM1-2-L7-P1). (*6 points*)

$A_1A_2\ldots A_9$ is a regular nonagon (9−gon). Find $\angle A_1A_4A_5$.

(A) 154 (B) 140 (C) 132 (D) 100 (E) 95

Problem 6.26.2 (ICLT-2022-SM1-2-L7-P2). (*6 points*)

Find the positive constant c that makes the quadratic $36x^2 + cx + 49$ the square of a binomial.

(A) 70 (B) 75 (C) 80 (D) 85 (E) 90

Problem 6.26.3 (ICLT-2022-SM1-2-L7-P3). (*6 points*)

A real number x is selected randomly such that $0 \le x \le 3$. What is the probability that $(x-\frac{1}{2})(2-x) \ge 0$.

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{5}$ (E) $\frac{3}{5}$

Problem 6.26.4 (ICLT-2022-SM1-2-L7-P4). (*6 points*)

How many of the positive divisors of 6^{2022} have a units digit of 9?

(A) 2022 (B) 2020 (C) 1024 (D) 505 (E) 506

Problem 6.26.5 (ICLT-2022-SM1-2-L7-P5). (*6 points*)

In the figure, PC is tangent to the circle and PD bisects $\angle CPE$. If $\widehat{CD} = 70°$, $\widehat{DE} = 50°$, and $\angle DQE = 40°$, then determine the measure of the arc from A to D that does not include point C.

(A) 200 (B) 250 (C) 125 (D) 120 (E) 275

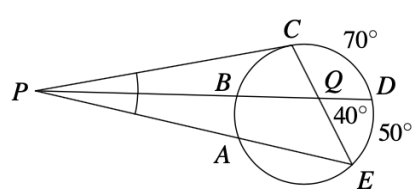


Figure 6.20: **??**

Problem 6.26.6 (ICLT-2022-SM1-2-L7-P6). (*6 points*)

Find the 2022st digit past the decimal point in the expansion of $\frac{1}{39}$.

(A) 1 (B) 2 (C) 4 (D) 5 (E) 6

Problem 6.26.7 (ICLT-2022-SM1-2-L7-P7). (*10 points*)

In a bag, there are 6 banknotes: $\$1, \$2, \$5, \$10, \$20$, and $\$50$.

1. (*5 points*) The probability for randomly taking any one out of the bag is the same. What is the expected value when randomly taking out one of the banknotes?
2. (*5 points*) If the probability to take the notes $\$1, \$2, \$5, \$10, \$20$, and $\$50$, is $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, and $\frac{1}{32}$, then what is the expected value when randomly taking out one of the banknotes?

Problem 6.26.8 (ICLT-2022-SM1-2-L7-P8). (*10 points*)

Chord UV intersect chord ST at point M. $ST = 12$ and $UV = 13$.

1. (*5 points*) If $SM = MT$, find all possible values of UM.
2. (*5 points*) If the ratio $SM : MT$ is $2 : 1$, find all possible values of UM.

Problem 6.26.9 (ICLT-2022-SM1-2-L7-P9). (*10 points*)

For $n \geq m \geq r$ positive integers, prove that:

1. (*5 points*)
$$\binom{2n}{n} + \binom{2n}{n-1} = \frac{1}{2}\binom{2n+2}{n+1}$$
2. (*5 points*)
$$\binom{n}{m} \cdot \binom{m}{r} = \binom{n}{r} \cdot \binom{n-r}{m-r}$$

6.27 Grading for Level 7

Answers for multiple-choice problems.

Problem 1:	D	100
Problem 2:	A	70
Problem 3:	C	$\frac{1}{2}$
Problem 4:	E	506
Problem 5:	B	250
Problem 6:	A	1

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 1 points if can correctly divise the expected value formula.

(b) 1 points if can correctly divise the expected value formula.

Problem 8: Separately for each part,

(a) 1 points if can apply the power of the point formula.

(b) 1 points if can apply the power of the point formula.

Problem 9: Separately for each part,

(a) 1 points if can express binomial coefficients as fraction of factorials.

(b) 1 points if can express binomial coefficients as fraction of factorials.

6.28 Solutions for Level 7

Solution. **??**

Since the polygon is a regular 9−gon, each exterior angle is $360/9 = 40$ degrees. Therefore, each interior angle is $180° - 40° = 140°$, or for example $\angle A_1A_2A_3 = 140°$, see **??**. Since $\triangle A_1A_2A_3$ is isosceles,

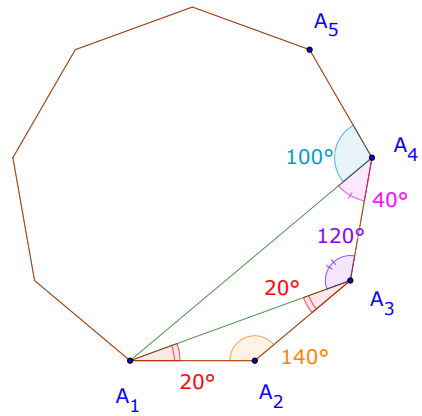


Figure 6.21: $\angle A_1A_2A_3 = 140°$

$\angle A_2A_1A_3 = \angle A_2A_3A_1$. The sum of the angles of $\triangle A_1A_2A_3$ is $180°$, thus

$$\angle A_2A_3A_1 = \frac{1}{2}(180° - 140°) = 20°.$$

Thus, $\angle A_1A_3A_4 = \angle A_2A_3A_4 - \angle A_2A_3A_1 = 120°$. Since $A_1A_2A_3A_4$ is an isosceles trapezoid, where $A_2A_3 \parallel A_1A_4$, so

$$\angle A_3A_4A_1 = 180° - \angle A_2A_3A_4 = 40° \Rightarrow \angle A_1A_4A_5 = \angle A_3A_4A_5 - \angle A_3A_4A_1 = 100°.$$

Therefore $\boxed{\angle A_1A_4A_5 = 100°.}$ The answer is $\boxed{D.}$ □

Solution. **??**

Suppose the quadratic is the square of the binomial $ax + b$. We have

$$36x^2 + cx + 49 = (ax + b)^2 = (ax)^2 + 2(ax)(b) + b^2 = a^2x^2 + 2abx + b^2. \Rightarrow \begin{cases} a^2 = 25 \\ 2ab = c \\ b^2 = 49 \end{cases}$$

It is easy to see that $a = \pm 5, b = \pm 7$, and $\boxed{c = 2ab = 70.}$ The answer is $\boxed{E.}$ □

Solution. **??**

For the second question

$$(x - \frac{1}{2})(2 - x) \geq \Rightarrow \begin{cases} x - \frac{1}{2} \geq 0, 2 - x \geq 0 \text{ or} \\ x - \frac{1}{2} \leq 0, 2 - x \leq 0 \end{cases} \Rightarrow \begin{cases} \frac{1}{2} \leq x \leq 2 \text{ or} \\ x \leq \frac{1}{2}, 2 \leq x \end{cases}$$

The second case is impossible, so $\frac{1}{2} \leq x \leq 2$, the probability is $\boxed{\frac{2-\frac{1}{2}}{3-0} = \frac{1}{2}.}$ The answer is $\boxed{C.}$ □

Solution. **??**

First, if number has a unit digit 9, then it is an odd number. Thus, only divisors of 6^{2022} in the format of 3^n, $0 \leq n \leq 2022$, can have an odd unit digit. Furthermore,

$$3^1 = 3, 3 \cdot 3 = 9, 9 \cdot 3 = 27, 7 \cdot 3 = 21, 1 \cdot 3 = 3$$

Thus the unit digits of $3^1, 3^2, \ldots$ is the periodic sequence

$$\underbrace{3, 9, 7, 1}_{\text{period 4}}, 3, 9, 7, 1, 3 \ldots$$

Since $2022 = 4 \times 505 + 2$. There are $\boxed{505 + 1 = 506}$ such numbers. The answer is $\boxed{E.}$ □

Solution. **??**

First, we note that since $\angle DQE$ is the average of $\widehat{DE}$ and $\widehat{CB}$, we have $\widehat{CB} = 30°$.

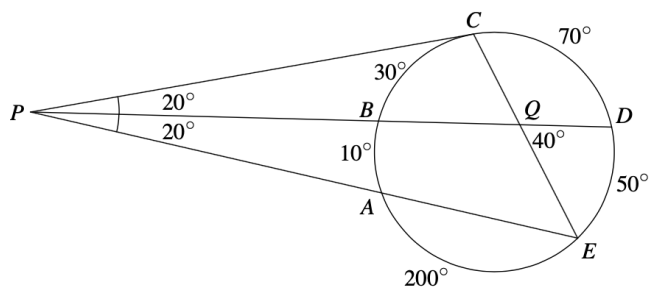


Figure 6.22: $\widehat{AD}$

Now, $\angle CPD = (\widehat{CD} - \widehat{CB})/2 = 20°$. Since DP bisects $\angle CPE$, $\angle DPE = \angle CPD = 20°$. Because $\angle DPE = (\widehat{DE} - \widehat{AB})/2$, $\widehat{AB} = 10°$.

By subtracting from 360°:

$$\widehat{AE} = 360° - 50° - 70° - 30° - 10° = 200°$$

Thus $\boxed{\widehat{AD} = 250°.}$ The answer is $\boxed{B.}$ □

Solution. **??**

By long division $\frac{1}{39} = 0.\underbrace{025641}_{\text{period 6}}$. Since 2022 is divisible by 6, so the $2022^{\text{th}}-$digit is $\boxed{1.}$

The answer is $\boxed{A.}$ □

Solution. **??**

If the probability to take any note is the same, then the expected value is

$$\frac{1}{6}(1) + \frac{1}{6}(2) + \frac{1}{6}(5) + \frac{1}{6}(10) + \frac{1}{6}(20) + \frac{1}{6}(50) = \frac{44}{3}.$$

For the second question, the expected value is

$$\frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(5) + \frac{1}{16}(10) + \frac{1}{32}(20) + \frac{1}{32}(50) = \frac{71}{16}.$$

□

Solution. **??**

For the first question, since M is the midpoint of ST, we have $SM = MT = 6$. The power of point M gives $(MS)(MT) = (MU)(MV)$. Let $MU = x$, so that $MV = 13 - x$. Therefore, we have $36 = x(13 - x)$, so $x^2 - 13x + 36 = 0$, from which we have $(x - 9)(x - 4) = 0$. So, the two possible values of UM are $\boxed{4 \text{ and } 9.}$

For the first question, if $SM : MT = 2 : 1$, then $SM = 8, MT = 4$. Then for $MU = x$,

$$(MS)(MT) = (MU)(MV) \Rightarrow 32 = x(13 - x) \Rightarrow x = \frac{1}{2}(13 + \sqrt{41})$$

□

Solution. **??**

For the first question,

$$\binom{2n}{n} + \binom{2n}{n-1} = \frac{(2n)!}{n! \cdot n!} + \frac{(2n)!}{(n-1)! \cdot (n+1)!} = \frac{(2n)!}{(n-1)! \cdot n!}\left(\frac{1}{n} + \frac{1}{n+1}\right)$$
$$= \frac{(2n)!}{(n-1)! \cdot n!}\frac{2n+1}{n \cdot (n+1)} = \frac{(2n+1)!}{(n)! \cdot (n+1)!}$$

For the second question,

$$\binom{n}{m} \cdot \binom{m}{r} = \frac{n!}{m!(n-m)!}\frac{m!}{r!(m-r)!} = \frac{n!}{(n-m)!(m-r)!r!}$$
$$\binom{n}{r} \cdot \binom{n-r}{m-r} = \frac{n!}{r!(n-r)!}\frac{(n-r)!}{(n-m)!(m-r)!} = \frac{n!}{(n-m)!(m-r)!r!}$$

Thus, they are the same. □

Chapter 7

PC R3

7.1 Topics

Algebra

1. Fraction. Percent.
2. Sums and Products.
3. Equations. System of Equations.
4. Sum of Squares.

Combinatorics

1. Permutations
2. Combinations.
3. Multiple independent events.
4. Inclusion-Exclusion Principle. Sets.
5. Casework.
6. Grids. Tiling.
7. Probability. Multiple repeatitions of events.

Geometry

1. Squares.
2. Right Triangles.
3. Perimeter.
4. Circles.
5. Locust.
6. Polygons. Regular Polygons.
7. Area.
8. Volume of a rectangular prism. Volume of a pyramid.

Number Theory

1. Perfect Powers.
2. Divisibility. GCD, LCM. Linear combinations. Divisors.
3. Modular Arithmetic.

4. Diophantine Equations.

Coding

1. Divisors of a positive integers. Count the number of divisors.
2. Check if a number is a prime.
3. Sum of digits of a positive integers.
4. Binary numbers. Represent a subset of a set as a binary number.
5. Conversions of decimal to binary numbers.
6. Set comparison.

7.2 Rules

- **Middle school teams** are given **60 minutes** to work on **20 problems**, and **high school teams** are given **90 minutes** to work on **30 problems.**
- The teams may fill in answers to the contest problems during the prescribed time for their teams (60 minutes for middle school teams, 90 minutes for high school teams) following the start of their contest clock. At any time during the competition, contestants can submit answers to the problems. Answers may be submitted multiple times by the same team, but for each problem **only the last answer** received before the contest ends will be accepted and graded. Partial submissions are accumulated. Consequently, if a team submits an answer to problem 1 and later wises to submit an answer to problem 2, there is no need to resubmit the answer to problem 1. All members are allowed to work on the problems as a team. No help may be provided by persons not on their team. Team members must not reveal or discuss contest problems with anyone other than their immediate team members until the test ends.
- All problems are designed so that they can be worked without the aid of calculating devices. On the other hand, participants are *permitted to use calculators or computers*, for example, to write a computer program in order solve or better understand a problem. Team members may also *refer to books, computer programs, or other reference materials* available to them *locally* as long as the students seek this material without aid from persons not on their team. Any team with a member attempted to do *violate this rule* of the test *will be disqualified.*
- Team members *shall not* use computers or any other communication devices to *seek help* with problem solutions *from people or Internet resources* outside of their own team members.
- Grading:
  1. For a **correct answer** of *each problem*, the team is awarded with 1 point. For an *unanswered* problem or for an *incorrect answer*, the team is awarded with 0 point.
  2. The score of the team is the *sum of awarded points* for all the problems.

  The problems that have no answers or incorrect answers will be discussed after the test.

7.3 Problems

Problem 7.3.1 (PCT-2022-SM1-R3-P1)**.** Mike has 14 books, Sean has 9 books, and little Sherry has only 7 books. Find the percentage of these books that Sean has.

Problem 7.3.2 (PCT-2022-SM1-R3-P2)**.** The **??** below was formed by taking four squares, each with side length 6, and putting one on each side of a square with side length 21. Find the perimeter of the **??** below.

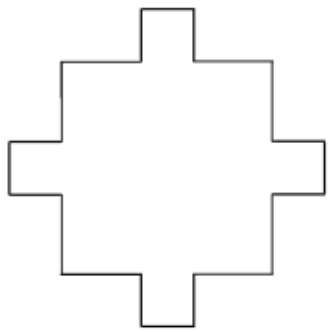


Figure 7.1: **??**

Problem 7.3.3 (PCT-2022-SM1-R3-P3)**.** The sum of the numbers $3a - 4$, $3b - 4$, and $3c - 4$ is 2022. Find the sum of the numbers $4a - 3$, $4b - 3$, and $4c - 3$.

Problem 7.3.4 (PCT-2022-SM1-R3-P4)**.** The following diagram shows a square where each side has four dots that divide the side into three equal segments. The shaded region has area 119. Find the area of the original square.

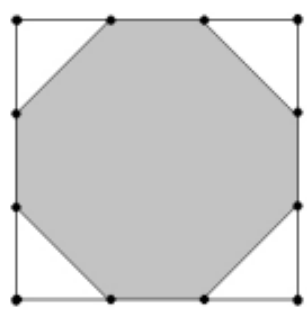


Figure 7.2: **??**

Problem 7.3.5 (PCT-2022-SM1-R3-P5)**.** A 2 meter long bookshelf is filled end-to-end with 34 books. Some of the books are 5 centimeters thick while all the others are 7 centimeters thick. Find the number of books on the shelf that are 5 centimeters thick.

Problem 7.3.6 (PCT-2022-SM1-R3-P6)**.** Find the number of three-digit *odd* positive integers where the digits are three different prime numbers. For example, count 235 but not 553 or 532.

Problem 7.3.7 (PCT-2022-SM1-R3-P7)**.** Melanie has $4\frac{2}{5}$ cups of flour. The recipe for one batch of cookies calls for $1\frac{1}{2}$ cups of flour. Melanie plans to make $2\frac{1}{2}$ batches of cookies. When she is done, she will have $\frac{m}{n}$ cups of flour remaining, where m and n are relatively prime positive integers. Find the difference $n - m$.

Problem 7.3.8 (PCT-2022-SM1-R3-P8)**.** The **??** below has a 1×1 square, a 2×2 square, a 3×3 square, a 4×4 square, and a 5×5 square. Each of the larger squares shares a corner with the 1×1 square. Find the area of the region covered by the 1×1, 2×2, 3×3, and 4×4 squares, but not covered by the 5×5 square.

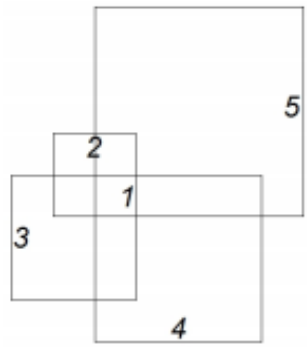


Figure 7.3: **??**

Problem 7.3.9 (PCT-2022-SM1-R3-P9)**.** Find the value of x such that $3^{x+2} - 3^{x-1} = 2106$.

Problem 7.3.10 (PCT-2022-SM1-R3-P10)**.** Mildred the cow is tied with a rope to the side of a square shed with side length 10 meters. The rope is attached to the shed at a point two meters from one corner of the shed. The rope is 14 meters long. The area of grass growing around the shed that Mildred can reach is given by $n\pi$ square meters, where n is a positive integer. Find n.

Problem 7.3.11 (PCT-2022-SM1-R3-P11)**.** One evening a theater sold 400 tickets for a concert. Each ticket sold for \$40, and all tickets were purchased using \$5, \$10, and \$20 bills. At the end of the evening the theater had received twice as many \$10 bills as \$20 bills, and 20 more \$5 bills than \$10 bills. How many bills did the theater receive altogether?

Problem 7.3.12 (PCT-2022-SM1-R3-P12)**.** Find the number of squares such that the sides of the square are segments in the following **??** and where the dot is inside the square.

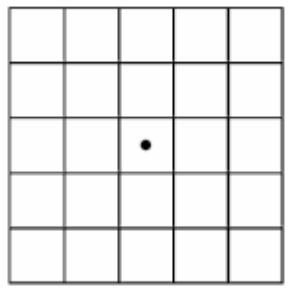


Figure 7.4: **??**

Problem 7.3.13 (PCT-2022-SM1-R3-P13)**.** One afternoon Elizabeth noticed that twice as many cars on the expressway carried only a driver as compared to the number of cars that carried a driver and one passenger. She also noted that twice as many cars carried a driver and one passenger as those that carried a driver and two passengers. 10% of the cars carried a driver and three passengers, and no car carried more than four people. Any car containing at least three people was allowed to use the fast lane. Elizabeth calculated that $\frac{m}{n}$ of the people in cars on the expressway were allowed to ride in the fast lane, where m and n are relatively prime positive integers. Find $n - m$.

Problem 7.3.14 (PCT-2022-SM1-R3-P14)**.** Find the number of positive integers n such that a regular polygon with n sides has internal angles with measures equal to an integer number of degrees.

Problem 7.3.15 (PCT-2022-SM1-R3-P15)**.** The real numbers x, y, and z satisfy the system of equations

$$\begin{cases} x^2 + 27 = -8y + 10z \\ y^2 + 196 = 18z + 13x \\ z^2 + 119 = -3x + 30y \end{cases}$$

Find $5x + 3y + z$.

Problem 7.3.16 (PCT-2022-SM1-R3-P16)**.** The **??** below shows a barn in the shape of two congruent pentagonal prisms that intersect at right angles and have a common center. The ends of the prisms are made of a 12 foot by 7 foot rectangle surmounted by an isosceles triangle with sides 10 feet, 10 feet, and 12 feet. Each prism is 40 feet long. Find the volume of the barn in cubic feet.

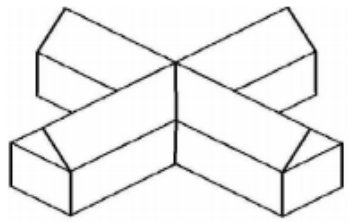


Figure 7.5: **??**

Problem 7.3.17 (PCT-2022-SM1-R3-P17)**.** Suzie flips a fair coin 6 times. The probability that Suzie flips 3 heads in a row but not 4 heads in a row is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the difference $n - m$.

Problem 7.3.18 (PCT-2022-SM1-R3-P18)**.** Find the least positive integer N that is 50 times the number of positive integer divisors that N has.

Problem 7.3.19 (PCT-2022-SM1-R3-P19)**.** Find the positive integer n such that the least common multiple of n and $n - 30$ is $n + 4900$.

Problem 7.3.20 (PCT-2022-SM1-R3-P20)**.** The 24 unshaded squares in the **??** below showing a 5×5 grid can be tiled with twelve 1×2 tiles. One such tiling is shown. Find the number of ways the grid can be tiled.

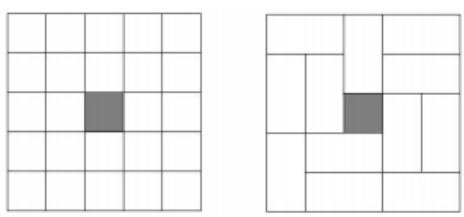


Figure 7.6: **??**

7.4 Grading

Answers for the problems.

Problem 1: 30
Problem 2: 132
Problem 3: 2703
Problem 4: 153
Problem 5: 19
Problem 6: 18
Problem 7: 7
Problem 8: 19
Problem 9: 5
Problem 10: 144
Problem 11: 1610
Problem 12: 19
Problem 13: 45
Problem 14: 22
Problem 15: 72
Problem 16: 9168
Problem 17: 13
Problem 18: 1800
Problem 19: 175
Problem 20: 196

7.5 Solutions

Solution. **??** The percentage of books that Sean has is $\frac{9}{14+9+7} = 0.3$, or $\boxed{30\%.}$ □

Solution. **??** The perimeter of the **??** is the perimeter of the large 21×21 square plus *twice* the perimeter of the 6×6 square. Thus our answer is $4 \cdot 21 + 2 \cdot 4 \cdot 6 = \boxed{132.}$ □

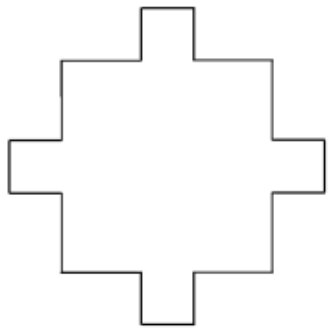


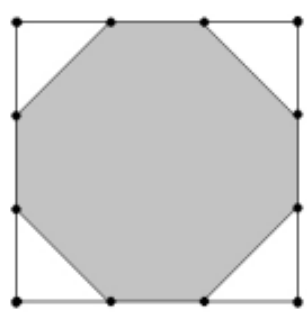
Figure 7.7: **??**

Solution. **??**

$$(3a-4)+(3b-4)+(3c-4) = 2022 \Rightarrow 3(a+b+c)-12 = 2022 \Rightarrow a+b+c = 678$$
$$(4a-3)+(4b-3)+(4c-3) = 4(a+b+c)-3\cdot 3 = 4\cdot 678 - 9 = \boxed{2703.}$$

□

Solution. **??** A pair of two corner triangles make up $\frac{1}{9}$ of the whole square. The area of the shaded region is



$1 - 2 \cdot \frac{1}{9} = \frac{7}{9}$ of the whole square. The area of the whole square is $119 \cdot \frac{9}{7} = \boxed{153.}$ □

Solution. **??**

Let m be the number of 5 cm wide books and let n be the number of 7 cm wide books

$$\left.\begin{aligned} m+n &= 34 \\ 5m+7n &= 200 \end{aligned}\right\} \Rightarrow 7(m+n)-(5m+7n) = 7\cdot 34 - 200 \Rightarrow 2m = 38 \Rightarrow m = 18 = 9$$

Therefore there are $m = \boxed{19}$ books that are 5−cm wide. □

Solution. **??** First, we count all three-digit numbers whose digits are single-digit prime numbers $2, 3, 5, 7$, each can be used once in a three-digit number,

- there are $\binom{4}{3} = 4$ ways to choose three primes from $2, 3, 5, 7$,
- there are $3! = 6$ ways to permute these three selected primes,

thus the total number of three-digit numbers with prime digits is $4 \cdot 6 = 24$.

Now, in the similar way the number of those three-digit numbers with 2 as a unit digit is the number of ways to choose two single-digit primes other than 2 from $2, 3, 5, 7,$, which is $\binom{3}{2} \cdot 2! = 6$.

Therefore the number of three-digit odd positive integers where the digits are three different prime numbers is $24 - 6 = \boxed{18.}$ □

Solution. **??** The amount of flour she will use is $\frac{3}{2} \cdot \frac{5}{2} = \frac{15}{4}$, there remains $\frac{22}{5} - \frac{15}{4} = \frac{13}{20}$. Thus, $n - m = 20 - 13 = \boxed{7.}$ □

Solution. **??** Let $S_1, S_2, \ldots, S_5$ denote the area of the squares 1×1, 2×2, $\ldots$, and 5×5. Let A_1 be the area

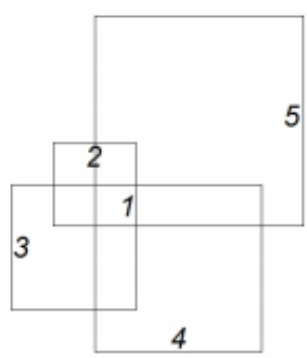


Figure 7.8: S_1, S_2, S_3, S_4, S_5

covered by *at least one* of the S_1, S_2, S_3, S_4 squares, which is $1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4 = 30$. Let A_2 covered by *at least two* of the squares, or by pairs of $(S_1, S_2), (S_1, S_3), \ldots, (S_3, S_4)$,

$$|S_1 \cap S_2| = 1, |S_1 \cap S_3| = 1, |S_1 \cap S_4| = 1, |S_2 \cap S_3| = 2, |S_2 \cap S_4| = 1, |S_3 \cap S_4| = 3$$
$$\Rightarrow A_2 = 1 + 1 + 1 + 2 + 1 + 3 = 9$$

Let A_3 covered by *at least three* of the squares, or by $(S_1, S_2, S_3), (S_1, S_2, S_4), \ldots, (S_2, S_3, S_4)$,

$$|S_1 \cap S_2 \cap S_3| = 1, |S_1 \cap S_2 \cap S_4| = 1, |S_1 \cap S_3 \cap S_4| = 1, |S_2 \cap S_3 \cap S_4| = 1$$
$$\Rightarrow A_3 = 1 + 1 + 1 + 1 = 4$$

Finally, the area A_4 covered by *all* of the S_1, S_2, S_3, S_4 squares is 1. Therefore, by the *Inclusion-Exclusion Principle,* the area covered by all of S_1, S_2, S_3, S_4 squares is

$$A_1 - A_2 + A_3 - A_4 = 30 - 9 + 4 - 1 = 24.$$

It is easy to see S_5 covers an area of 5 of this region. Thus the area of the region covered by S_1, S_2, S_3, S_4 squares, but not covered by the S_5 square is $24 - 5 = \boxed{19.}$ □

Solution. **??** $3^{x+2} - 3^{x-1} = 2106 \Rightarrow 3^{x-1}(3^3 - 1) = 81 \cdot 26 \Rightarrow 3^{x-1} = 81 \Rightarrow x = \boxed{5.}$ □

Proof. **??** The total area is the sum of the areas of

- Half of the blue circle centred at E with radius 14,
- A quarter of the orange circle centred at A with radius 6,
- A quarter of the green circle centred at D with radius 12,
- A quarter of the red circle centred at C with radius 2.

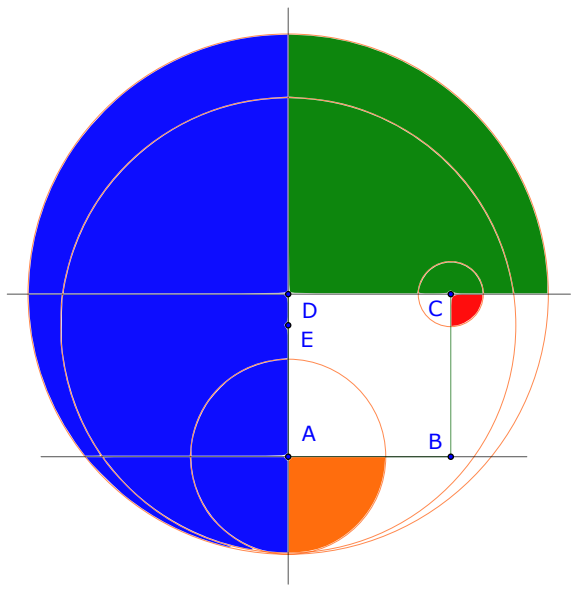


Figure 7.9: **??**

All together $\left(\frac{14^2}{2} + \frac{6^2}{4} + \frac{12^2}{4} + \frac{2^2}{4}\right)\pi = \boxed{144\pi}$. □

Proof. **??** Since there are 400 tickets and each were sold for \$40 thus they made a total of $400 \cdot 40 = \$16000$. Now, let n be the number of 20 dollar bills, then the number of 10 dollar bills is $2n$ and the number of 5 dollar bills is $2n + 20$,

$$20 \cdot n + 10 \cdot 2n + (2n + 20) \cdot 5 = 16000 \Rightarrow 50n = 15900 \Rightarrow n = 318$$

Therefore the number of bills is $318 + 2 \cdot 318 + 2 \cdot 318 + 20 = \boxed{1610.}$ □

Proof. **??** We proceed with casework, see **??**.

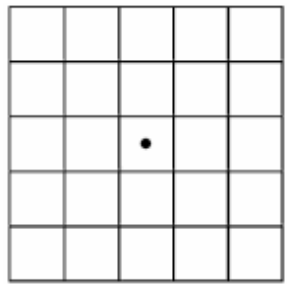


Figure 7.10: **??**

Case 1: 1×1 square. By inspection there is only 1 square.

Case 2: 2×2 square. By inspection there are only 4 of 2×2 squares.

Case 3: 3×3 square. There are 9 of 3×3 squares.

Case 4: 4×4 square. There are 4 of 4×4 squares.

Case 5: 5×5 square. There is only 1 5×5 square.

Therefore the number of squares is $1 + 4 + 9 + 4 + 1 = \boxed{19.}$ □

Proof. **??**

Let x be the fraction of cars that have only a driver. Then there are $2x$ cars that have a driver and a passenger, and $4x$ cars that a driver and 2 passengers. Since $\frac{1}{10}$ of the cars have a driver and 3 passengers, this means that

$$x + 2x + 4x = 1 - \frac{1}{10} = \frac{9}{10} \Rightarrow 7x = \frac{9}{10} \Rightarrow x = \frac{9}{70}$$

Thus there are $\frac{9}{70}$ cars with only a driver, $\frac{9}{35}$ with a driver and a passenger, $\frac{18}{35}$ with a driver and 2 passenger, and $\frac{1}{10}$ of the cars have a driver and 3 passengers.

Let say we have 70 cars, then 9 cars with only a driver, 18 with a driver and a passenger, 36 with a driver and 2 passenger, and 7 have a driver and 3 passengers. Thus the number of people in cars with at least 3 people in it is, $3 \cdot 36 + 4 \cdot 7 = 136$. The total number of people in cars is, $1 \cdot 9 + 2 \cdot 18 + 3 \cdot 36 + 4 \cdot 7 = 181$. Their fraction is $\frac{m}{n} = \frac{136}{181}$.

Thus $n - m = \boxed{45.}$

Note that, we assume the number of the cars as 70 *to make the calculation easier. However, it does not change the ratio of the number of people to the number of cars.* □

Proof. **??** The measure of an internal angle is $180 - \dfrac{360}{n}$, so $\dfrac{360}{n}$ is an integer. Thus, n is a divisor of 360. The number of factors of $360 = 2^3 \cdot 3^2 \cdot 5$ is $(3+1)(2+1)(1+1) = 24$. Note that $n \neq 1, 2$, so the answer is $24 - 2 = \boxed{22.}$ □

Proof. **??**

$$\left.\begin{aligned} x^2 + 27 &= -8y + 10z \\ y^2 + 196 &= 18z + 13x \\ z^2 + 119 &= -3x + 30y \end{aligned}\right\} \Rightarrow (x-5)^2 + (y-11)^2 + (z-14)^2 = 0 \Rightarrow (x, y, z) = (5, 11, 14)$$

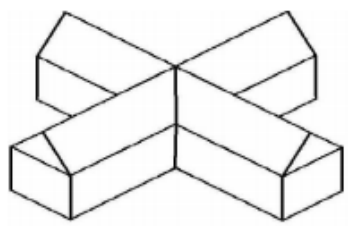
Thus $5x + 3y + z = 25 + 33 + 14 = \boxed{72}$. □

Proof. **??** The volume is the sum of the volume of the 2 pentagonal prisms minus the common center area.

Lets split the pentagonal prism into 2 parts: the top part which is where the isoceles triangle is and the bottom part which is the rectangle underneath the isoceles triangle.

- The top part is a triangular prism with a triangular base $10, 10, 12$, height 40, so its volume is 1920.
- The bottom part is a box with side lengths $7, 12, 40$, thus its volume is $7 \cdot 12 \cdot 40 = 3360$.

Consequently, the volume of a pentagonal prism is $1920 + 3360 = 5280$.



Now all we need to find the volume of the common center. We can split the common center place into 2 parts as well. The top part which is a pyramid and the bottom which is a box.

- The top part volume is a pyramid with base side length 12, height 8, so its volume is $\dfrac{12 \cdot 12 \cdot 8}{3} = 384$.
- The bottom part is a box which has side lengths $12, 12, 7$ thus its volume is $12 \cdot 12 \cdot 7 = 1008$

Therefore, the total volume of the common center is $1008 + 384 = 1392$.

Thus, the answer is $2 \cdot 5280 - 1392 = \boxed{9168}$. □

Proof. **??**

Three consecutive head, but not four:

$$\text{THHHTX, XTHHHT, HHHTXX, XXTHHH}$$

where H denote head, T denote tail and X denote either head or tail. There are 2 string of the format THHHTX, because X can be fill in 2 ways, 2 string of the format XTHHHT, because X can be fill in 2 ways, $2 \cdot 2$ strings of HHHTXX, and $2 \cdot 2$ strings of XXTHHH. Thus total number of favourable cases are $2 + 2 + 4 + 4 = 12$. Total possible outcome is $2^6 = 64$. Hence probability is $\frac{12}{64} = \frac{3}{16}$. Hence $m = 3, n = 16$ and $n - m = \boxed{13}$. □

Proof. **??**

```python
def count_divisors(n):
    counter = 0                     # set counter as 0
    for i in range(1, n+1):         # iterates from 1 to n
        if n % i == 0:              # if it is a divisor of n
            counter += 1            # add it to the counter
    return counter                  # return counter

n = 1
while True:
    if n == 50 * count_divisors(n): # if n is 50 times number of its divisor
        print(n)
        break
    n += 1
```

Thus, $\boxed{1800}$ is the answer. □

Proof. **??**

$n \mid n + 4900$, thus $n \mid 4900$. Furthermore $n - 30 \mid n + 4900 = (n - 30) + 4930$, so $n - 30 \mid 4930$.

Since $4930 = 2 \cdot 5 \cdot 17 \cdot 29$, the divisors of 4930 that are larger then 30 and less than 4900 are,

$$34, 58, 85, 145, 170, 290, 493, 986, 2465.$$

Among these only $n - 30 = 145$ is such a number that $n = 175$ is also divisor of 4900.

Thus, $\boxed{175}$ is the answer. □

Proof. **??**

Lets investigate how the *two left-most columns* of the 5×5 square can be *seprated from the square* into a 5×2 rectangle. There are 4 different ways as shown in the **??** below,

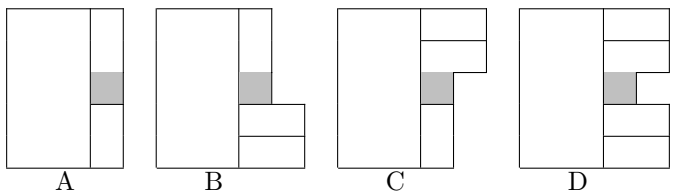


Figure 7.11: A, B, C, and D

Case A: There are 8 ways to tile first two columns of A, see **??**. It means that there are also the last two

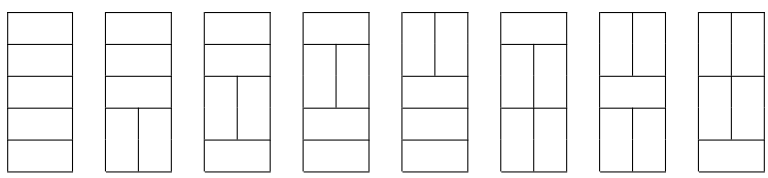


Figure 7.12: Tile two columns

columns of the 5×5 square. So in total, there are $8 \times 8 = 64$ ways in this case.

Case B, C: We count the number of ways for case B. It is easy to see that there is 3 ways to tile the 3×2 block of B in **??**. The number of ways for both B and C is $2 \cdot 8 \cdot 3 = 48$.

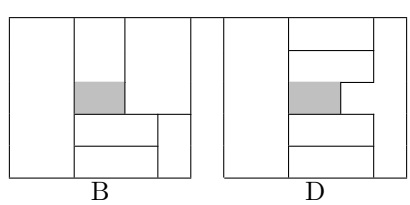
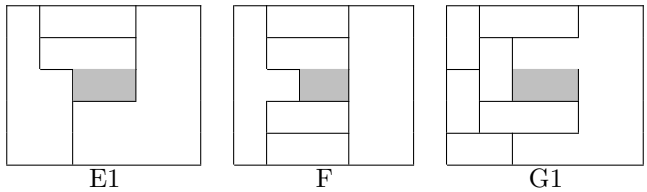


Figure 7.13: B, C, and D

Case D: It is only one way to tile the right most block of D, see **??**. There are $8 \cdot 1 = 8$ ways.

Now, let take a look when the two left-most columns cannot be separated into a 5×2 rectangle. There are five cases, see **??**. Note that case $E2$ is the reflection of $E1$ over the middle row, and case $G2$ is the reflection of $G1$ over the middle row.



E1 F G1

Figure 7.14: $E1, F$, and $G1$

Case E1, E2: AS seen in above, there are 3 ways to tile the left block and $8 + 3 = 11$ ways for the right block. So the number of ways for both $E1$ and $E2$ is $2 \cdot 3 \cdot 11 = 66$.

Case F: It is easy to see that the number of ways for F is 8.

Case G1, G2: It is easy to see that the number of ways for $G1$ and $G2$ is 2.

Thus the total number of ways is $\boxed{64 + 48 + 8 + 66 + 8 + 2 = 196.}$ □

Chapter 8

PC R4

8.1 Topics

Algebra

1. Sums and Products. Telescopic sums and products. Primary sequence $a_1, a_2, \ldots, a_n$, secondary sequence $b_1 = a_1, b_2 = a_1 a_2, \ldots b_n = a_1 a_2 \cdots a_n$.
2. Difference of reciprocals (see below.)
3. Sequences. Arithmetic sequences.
4. Equations.
5. Polynomials. Vieta's theorem. Coefficients (see below).
6. Estimation. Inequalities.
7. Radical $10 \pm 6\sqrt{3} = (1 \pm \sqrt{3})^3, 52 \pm 30\sqrt{3} = (5 \pm 3\sqrt{3})^2$.
8. Rationalize equations with radicals (bring to power).
9. Ratio. Speed-Distance-Time.
10. Sum of squares. Inequalities.

Combinatorics

1. Counting. Casework.
2. Counting in two ways.
3. Principle of Inclusion-Exclusion.
4. Probability.
5. Expected value (see below).
6. Geometric Probability.
7. Permutations.
8. Combinations.
9. Bijections.

Geometry

1. Right triangles. Pythagorean Theorem.
2. Equilateral triangles. Altitude.
3. Similar triangles.

4. Squares.
5. Hexagon.
6. Area. Triangles with same base.
7. Computational geometry. Equation of the plane $ax + by + cz + d = 0$. (see below). Distance from a point to a plane with equation (see below).
8. Tangent circles.
9. Cone base. Radius of base. Height.
10. Hypercubes. Number of $m-$dimensional hypercubes (see below).
11. Menelaus Theorem.

Number Theory

1. Divisibility. Prime Factorization.
2. Perfect powers.
3. Parity.
4. Multiples. Divisors. GCD. LCM.
5. Number of trailing 0 digits.
6. Diophantine equations.
7. No perfect square between two consecutive perfect squares. No perfect cube between two consecutive perfect cubes.

Coding

1. Permutations of strings.
2. Create a set of unique elements from a list.
3. Sort lists, sort sets.
4. Extract digits from a positive integer.
5. Finding square root.
6. Determine (using pow(), round()) if a number is a perfect square, if it is a perfect cube.
7. Choose all $n-$element subsets from a set.

Definition (Expected value)**.** $x_1, x_2, \ldots x_n$ are the outcomes of an event E. Value x_1 occurs with probability p_1, value x_2 occurs with probability p_2, and so on, where $p_1 + p_2 + \cdots + p_n = 1$. The expected value of the outcome is defined as *the sum of the probabilities of the outcomes times the value of the outcomes*:

$$\mu(E) = p_1x_1 + p_2x_2 + \cdots + p_nx_n.$$

Fact. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, where $a_n \neq 0$, then the leading term of $P(x+1) + P(x-1)$ is $2a_nx^n$.

Fact. The *co-efficients* $a, b, c,$ *and* d of the equation of the plane $ax + by + cz + d = 0$ through three points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), are *the solutions of the system of equations*,

$$\begin{cases} ax_1 + by_1 + cz_1 + d = 0 \\ ax_2 + by_2 + cz_2 + d = 0 \\ ax_3 + by_3 + cz_3 + d = 0 \end{cases}$$

Fact. The distance from point (x_0, y_0, z_0) to the plane with the equation $ax + by + cz + d = 0$ is,

$$\frac{|ax_0 + by_0 + cz_0|}{\sqrt{a^2 + b^2 + c^2}}$$

Theorem (Number of $m-$dimensional hypercubes)

An $n-$dimensional hypercube is more commonly referred to as an $n-$cube or sometimes as an $n-$dimensional cube. The number of the $m-$dimensional hypercubes of a $n-$cube is

$$E(n,m) = 2^{n-m}\binom{n}{m}$$

Proof. **??** For each of the 2^n vertices of the hypercube, there are $\binom{n}{m}$ ways to choose a collection of m edges incident to that vertex. Each of these collections defines one of the $m-$dimensional faces incident to the considered vertex. Doing this for all the vertices of the hypercube, each of the $m-$dimensional faces of the hypercube is counted 2^m times since it has that many vertices, and we need to divide $2^n\binom{n}{m}$ by this number. Thus, $E(n,m) = 2^{n-m}\binom{n}{m}$. ■

Theorem (Menelaus Theorem)

Let ABC be a triangle, and let D, F be points on lines BC, AB, respectively. E is on the extension of CA. Points D, E, F are collinear *if and only if*:

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

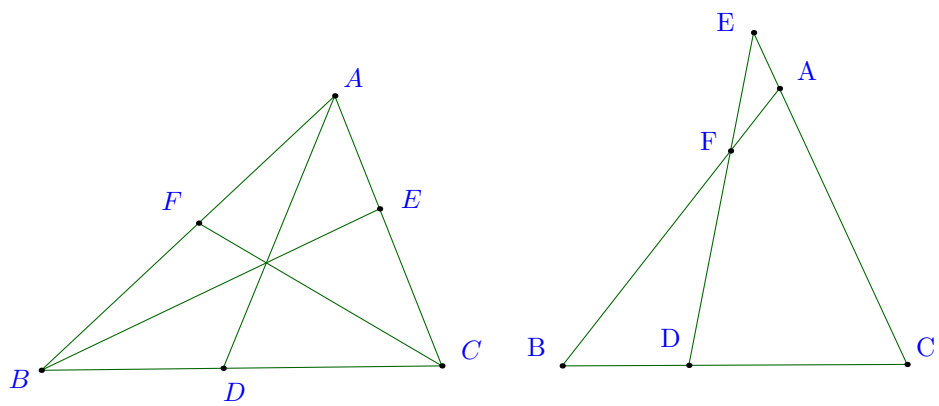


Figure 8.1: Ceva (left) and Menelaus (right) theorems

Fact.

$$\frac{1}{(k+m)(k+n)} = \frac{1}{n-m}\left(\frac{1}{k+m} - \frac{1}{k+n}\right).$$

8.2 Rules

- The *total time* to complete the test is 90 minutes. The problems that have no answers or incorrect answers will be discussed after the test.
- **High school teams** are given **90 minutes** to work on **30 problems.**
- The teams may fill in answers to the contest problems during the prescribed time for their teams (60 minutes for middle school teams, 90 minutes for high school teams) following the start of their contest clock. At any time during the competition, contestants can submit answers to the problems. Answers may be submitted multiple times by the same team, but for each problem **only the last answer** received before the contest ends will be accepted and graded. Partial submissions are accumulated. Consequently, if a team submits an answer to problem 1 and later wises to submit an answer to problem 2, there is no need to resubmit the answer to problem 1. All members are allowed to work on the problems as a team. No help may be provided by persons not on their team. Team members must not reveal or discuss contest problems with anyone other than their immediate team members until the test ends.
- All problems are designed so that they can be worked without the aid of calculating devices. On the other hand, participants are *permitted to use calculators or computers*, for example, to write a computer program in order solve or better understand a problem. Team members may also *refer to books, computer programs, or other reference materials* available to them *locally* as long as the students seek this material without aid from persons not on their team. Any team with a member attempted to do *violate this rule* of the test *will be disqualified.*
- Team members *shall not* use computers or any other communication devices to *seek help* with problem solutions *from people or Internet resources* outside of their own team members.
- Grading:
  1. For a **correct answer** of *each problem*, the team is awarded with 1 point. For an *unanswered* problem or for an *incorrect answer*, the team is awarded with 0 point.
  2. The score of the team is the *sum of awarded points* for all the problems.

8.3 Problems

Problem 8.3.1 (PCT-2022-SM1-R4-P1)**.** Paul starts at 1 and counts by twenty: $1, 21, 41, 61, \ldots$. At the same time and at the same speed, Penny counts backwards from 2022 by twenty three: $2022, 1999, 1976, 1953, \ldots$. Find the one number that both Paul and Penny count at the same time.

Problem 8.3.2 (PCT-2022-SM1-R4-P2)**.** The **??** below shows a large square divided into 9 congruent smaller squares. A shaded square bounded by some of the diagonals of those smaller squares has area 18. Find the area of the large square.

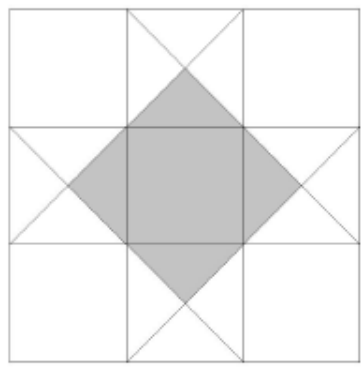


Figure 8.2: **??**

Problem 8.3.3 (PCT-2022-SM1-R4-P3)**.** When Phil and Shelley stand on a scale together, the scale reads 97 kilograms. When Shelley and Ryan stand on the same scale together, the scale reads 113 kilograms. When Phil and Ryan stand on the same scale together, the scale reads 106 kilograms. Find the number of pounds Shelley weighs.

Problem 8.3.4 (PCT-2022-SM1-R4-P4)**.** Find the least positive integer m such that $\operatorname{lcm}(21, m) = \operatorname{lcm}(45, m)$. Here $\operatorname{lcm}(a, b)$ is the least common multiple of a and b.

Problem 8.3.5 (PCT-2022-SM1-R4-P5)**.** A store had 366 chocolate bars. Mini bought some of the bars, and Maxi bought 41 more of the bars than Mini bought. After that, the store still had three times as many chocolate bars as Mini bought. Find the number of chocolate bars that Maxi bought.

Problem 8.3.6 (PCT-2022-SM1-R4-P6)**.** For some constant k the polynomial $p(x) = 2x^2 + kx + 117$ has the property that $p(1) = p(10)$. Evaluate $p(20)$.

Problem 8.3.7 (PCT-2022-SM1-R4-P7)**.** Consider an alphabetized list of all the arrangements of the letters in the word OTTAWA. Then AAOTTW would be in position 1 in the list, AAOTWT would be in position 2 in the list, and so forth. Find the position that OTTAWA would be in the list.

Problem 8.3.8 (PCT-2022-SM1-R4-P8)**.** Find the number of trailing zeros at the end of the base-10 representation of the integer $375^{25^2} \cdot 456^{52^5}$.

Problem 8.3.9 (PCT-2022-SM1-R4-P9)**.** The **??** below shows $\triangle ABC$ with point D on side $\overline{BC}$. Three lines parallel to side $\overline{BC}$ divide segment $\overline{AD}$ into four equal segments. In the triangle, the ratio of the area of the shaded region to the area of the unshaded region is $\frac{47}{34}$ and $\frac{BD}{CD} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

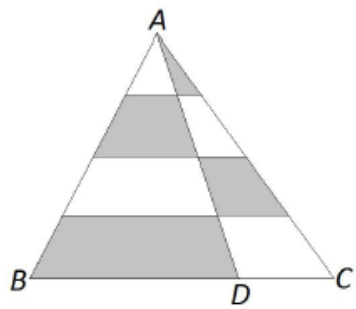


Figure 8.3: **??**

Problem 8.3.10 (PCT-2022-SM1-R4-P10)**.** Find the number of positive integers less than or equal to 2022 that have at least one pair of adjacent digits that are both even. For example, count the numbers 24, 1862, and 2012, but not 4, 58, or 1276.

Problem 8.3.11 (PCT-2022-SM1-R4-P11)**.** Dave has a pile of fair standard six-sided dice. In round one, Dave selects 6 of the dice and rolls them. He calculates the sum of the numbers face up on those dice to get r_1. In round two, Dave selects $r_1 - 1$ dice and rolls them. He calculates the sum of the numbers face up on those dice to get r_2. In round three, Dave selects $r_2 - 2$ dice and rolls them. He calculates the sum of the numbers face up on those dice to get r_3. Find the expected value of r_3.

Problem 8.3.12 (PCT-2022-SM1-R4-P12)**.** Let P be a polynomial satisfying $P(x + 1) + P(x - 1) = x^3$ for all real numbers x. Find the value of $P(10)$.

Problem 8.3.13 (PCT-2022-SM1-R4-P13)**.** Let $ABCDE$ be a pentagon with area 2017 such that four of its sides AB, BC, CD, and EA have integer length. Suppose that $\angle A = \angle B = \angle C = 90^o$, $AB = BC$, and $CD = EA$. The maximum possible perimeter of $ABCDE$ is $a + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a + b + c$.

Problem 8.3.14 (PCT-2022-SM1-R4-P14)**.** Find the product of all integers n for which $n - 3$ and $n^2 + 4$ are both perfect cubes.

Problem 8.3.15 (PCT-2022-SM1-R4-P15)**.** For real numbers a, b, and c the polynomial $p(x) = 3x^7 - 291x^6 + ax^5 + bx^4 + cx^2 + 154x - 2$ has 7 real roots whose sum is 97. Find the sum of the reciprocals of those 7 roots.

Problem 8.3.16 (PCT-2022-SM1-R4-P16)**.** Let $a_1 = 1 + \sqrt{2}$ and for each $n \geq 1$ dene $a_{n+1} = 2 - \frac{1}{a_n}$. Find the greatest integer less than or equal to the product $a_1 a_2 a_3 \cdots a_{150}$.

Problem 8.3.17 (PCT-2022-SM1-R4-P17)**.** The expression $\left(1 + \sqrt[6]{26 + 15\sqrt{3}} - \sqrt[6]{26 - 15\sqrt{3}}\right)^6 = m + n\sqrt{p}$, where m, n, and p are positive integers, and p is not divisible by the square of any prime. Find the value of the expression $m - n + p$.

Problem 8.3.18 (PCT-2022-SM1-R4-P18)**.** In the 3-dimensional coordinate space nd the distance from the point $(21, 21, 21)$ to the plane that passes through the points $(336, 36, 36)$, $(36, 636, 36)$, and $(36, 36, 336)$.

Problem 8.3.19 (PCT-2022-SM1-R4-P19)**.** Find the greatest integer $n < 2000$ for which $4n^3 - 3n$ is the product of two consecutive odd integers.

Problem 8.3.20 (PCT-2022-SM1-R4-P20)**.** Let a be a solution to the equation $\sqrt{x^2+3} = \sqrt[3]{x^3+33}$. Evaluate the ratio of $\frac{1062}{a^2}$ to $3a^2 - 22a + 9$.

Problem 8.3.21 (PCT-2022-SM1-R4-P21)**.** The diagram below shows a large circle. Six congruent medium-sized circles are each internally tangent to the large circle and tangent to two neighboring medium-sized circles. Three congruent small circles are mutually tangent to one another and are each tangent to two medium-sized circles as shown. The ratio of the area of the large circle to the area of one of the small circles can be written as $m + \sqrt{n}$, where m and n are positive integers. Find the difference $n - m$.

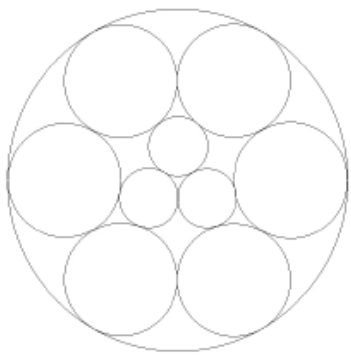


Figure 8.4: **??**

Problem 8.3.22 (PCT-2022-SM1-R4-P22)**.** Find the number of functions f that map the set $\{1, 2, 3, 4, 5\}$ into itself such that the range of the function $f(x)$ is the same as the range of the function $f(f(x))$.

Problem 8.3.23 (PCT-2022-SM1-R4-P23)**.** The familiar 3-dimensional cube has 6 2-dimensional faces, 12 1-dimensional edges, and 8 0-dimensional vertices. Find the number of 12-dimensional sub-subfaces in a 15-dimensional cube.

Problem 8.3.24 (PCT-2022-SM1-R4-P24)**.** Eight red boxes and eight blue boxes are randomly placed in four stacks of four boxes each. The probability that exactly one of the stacks consists of two red boxes and two blue boxes is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $n - m$.

Problem 8.3.25 (PCT-2022-SM1-R4-P25)**.** Leaving his house at noon, Jim walks at a constant rate of 4 miles per hour along a 4 mile straight route arriving to the beach at 1 PM. At a randomly chosen time between noon and 1 PM, Sally chooses a random location along Jim's route and begins running at a constant rate of 7 miles per hour along Jim's route in the same direction that Jim is walking until she arrives to the beach. The probability that Sally runs past Jim while he is walking is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

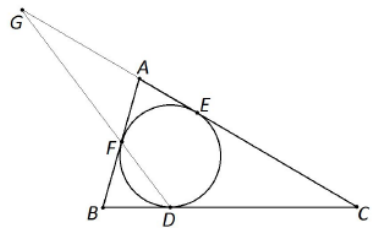


Figure 8.5: **??**

Problem 8.3.26 (PCT-2022-SM1-R4-P26)**.** The incircle of $\triangle ABC$ is tangent to sides $\overline{BC}, \overline{AC}$, and $\overline{AB}$ at D, E, and F, respectively. Point G is the intersection of lines AC and DF as shown. The sides of $\triangle ABC$ have lengths $AB = 73, BC = 123$, and $AC = 120$. Find the length AG.

Problem 8.3.27 (PCT-2022-SM1-R4-P27)**.** Find the minimum value of $4(x^2 + y^2 + z^2 + w^2) + (xy - 7)^2 + (yz - 7)^2 + (zw - 7)^2 + (wx - 7)^2$ as x, y, z, and w range over all real numbers.

Problem 8.3.28 (PCT-2022-SM1-R4-P28)**.** Let $T_k = \frac{k(k+1)}{2}$ be the k-th triangular number. The infinite series

$$\sum_{k=4}^{\infty} \frac{1}{(T_{k-1} - 1)(Tk - 1)(T_{k+1} - 1)}$$

has the value $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the difference $n - m$.

Problem 8.3.29 (PCT-2022-SM1-R4-P29)**.** Find the number of three-element subsets of $\{1, 2, 3, ..., 13\}$ that contain at least one element that is a multiple of 2, at least one element that is a multiple of 3, and at least one element that is a multiple of 7 such as $\{2, 3, 7\}$ or $\{6, 10, 14\}$.

Problem 8.3.30 (PCT-2022-SM1-R4-P30)**.** A container is shaped like a right circular cone with base diameter 18 and height 12. The vertex of the container is pointing down, and the container is open at the top. Four spheres, each with radius 3, are placed inside the container as shown. The first sphere sits at the bottom and is tangent to the cone along a circle. The second, third, and fourth spheres are placed so they are each tangent to the cone and tangent to the first sphere, and the second and fourth spheres are each tangent to the third sphere. The volume of the tetrahedron whose vertices are at the centers of the spheres is $m\sqrt{n}$. where m and n are positive integers and n is not divisible by any perfect square larger than 1. Find $m + n$. *Note that the second and fourth spheres are not necessarily tangent to each other.*

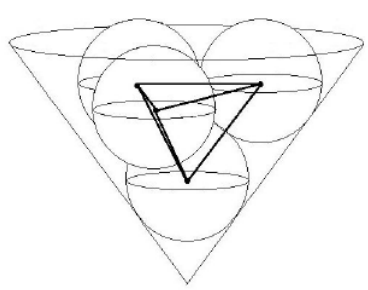


Figure 8.6: **??**

8.4 Grading

Answers for the problems.

Problem 1: 941
Problem 2: 81
Problem 3: 52
Problem 4: 315
Problem 5: 106
Problem 6: 477
Problem 7: 80
Problem 8: 1875
Problem 9: 162
Problem 10: 743
Problem 11: 238
Problem 12: 485
Problem 13: 172
Problem 14: 22
Problem 15: 77
Problem 16: 213
Problem 17: 31
Problem 18: 225
Problem 19: 1935
Problem 20: 3
Problem 21: 273
Problem 22: 1305
Problem 23: 3640
Problem 24: 587
Problem 25: 17
Problem 26: 84
Problem 27: 96
Problem 28: 449
Problem 29: 29
Problem 30: 19

8.5 Solutions

Solution. **??** At the beginning, Paul was at 1 and Penny was at 2022, the difference between the two numbers is $2022 - 1 = 2021$. after every step, Paul's number is incresed by 20, Penny's number is decreased by 23, thus the difference between the two numbers is decreased by $20 + 23 = 43$. Since $2021 = 34 \cdot 47$, after 47 step the difference between their numbers is 0.

They both arrive at the number $\boxed{941 = 1 + 47 \cdot 20 = 2022 - 47 \cdot 23.}$ □

Solution. **??** It is easy to see that the four shaded rectangles make up the small shaded square at the centre. Therefore the area of a small square if $\frac{18}{2} = 9$. The large square consists of 9 small squares, thus its area is

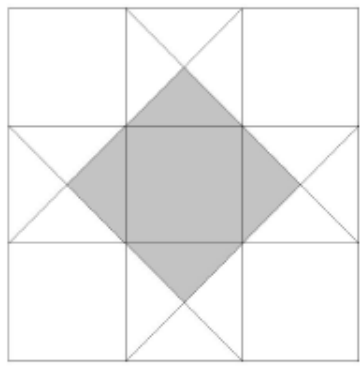


Figure 8.7: **??**

$\boxed{9 \cdot 9 = 81.}$ □

Solution. **??** Since the weight of Phil and Shelley plus the weight of Shelley and Ryan substracts the weight of Phil and Ryan is twice the weight of Shelley, so the weight of Shelley is $\boxed{\frac{1}{2}(97 + 113 - 106) = 52.}$ □

Solution. **??** It is easy to see that the value of both least common multiples it is a multiple of m.

On the other hand, the least common multiple $\text{lcm}(21, m)$ is a multiple of 3 and 7. This number is also the least common multiple of $\text{lcm}(45, m)$, thus, it is also multiple of 5 and 9. Therefore, it is a multiple of $\text{lcm}(5, 7, 9) = 315$.

The least value for m, such that its multiple is a multiple of m, is $\boxed{m = 315.}$ □

Solution. **??** According to the problem, the total number of chocolates the store had, before Mini bought some cholocate, was four times the number of cholocates that Mini bought plus the number of cholocates that Maxi bought. In other words, it is five times the number of cholocates that Mini bought plus 41. Thus, the number of chocolates that Mini bought is $\frac{366-41}{5} = 65$.

Therefore the number of chocolates that Maxi bought $\boxed{65 + 41 = 106.}$ □

Solution. **??** Since $p(1) = p(10)$ for $p(x) = 2x^2 + kx + 117$,

$$2 \times (1)^2 + k \times 1 + 117 = 2 \times (10)^2 + k \times 10 + 117 \Rightarrow -9k - 198 = 0 \Rightarrow k = -22$$

Therefore $\boxed{p(20) = 2 \times (20)^2 + (-22) \times 20 + 117 = 477.}$ □

Solution. **??** The Python function *permutations* from the standard *itertools* helps to generate list of all permutation of a string 'OTTAWA', which is considered as a list, into a list of list,

starts with: ('A', 'A', 'O', 'T', 'T', 'W')
then continues with: ('A', 'A', 'O', 'T', 'W', 'T')
...
and ends with: ('W', 'T', 'T', 'O', 'A', 'A')

```
from itertools import permutations

counter = 1
all_permutations = permutations('OTTAWA')            # permute all letters
distinct_permutations = set(all_permutations)        # remove duplicates
sorted_permutations = sorted(distinct_permutations) # sort them

for permutation in sorted_permutations:              # iterate through the list
    text = ''.join(permutation)                      # convert to a string
    if text == 'OTTAWA':
        print(counter, text)
    counter += 1
```

Its output is $80OTTAWA$. Thus, $\boxed{80}$ is the answer. □

Solution. **??** The number of trailing 0s depends on the least number of the exponents of the powers of 2 and 5. Since,

$$525^{25^2} \cdot 456^{52^5} = (3 \cdot 5^3)^{25^2} \cdot (2^3 \cdot 3 \cdot 19)^{52^5}$$

Thus the exponent of the power of 5 is $3 \cdot 25^2$ and the exponent of the power of 2 is $3 \cdot 52^5$. It is easy to see that $2 \cdot 25^2 < 2 \cdot 52^5$. Therefore the number of traling zeros is $\boxed{1875.}$ □

Solution. **??** First, drawing parallel lines with the sides of $\triangle ABC$, denote I, J, and M as shown in **??**. It is easy to see that the shaded area inside $\triangle ADC$ is $6[AJM]$. and the shaded area inside $\triangle ADB$ is $10[AIM]$. Thus, the area of the shaded region is $10[AIM]+6[AJM]$. Similarly, the unshaded area is $6[AIM]+10[AJM]$. Now, let $r = \frac{BD}{DC} = \frac{IM}{MJ}$,

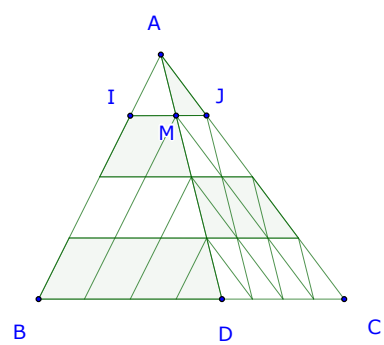


Figure 8.8: **??**

$$\frac{[AIM]}{[AJM]} = \left(\frac{IM}{MJ}\right) = r, \Rightarrow \frac{10[AIM]+6[AJM]}{6[AIM]+10[AJM]} = \frac{10\frac{[AIM]}{[AJM]}+6}{6\frac{[AIM]}{[AJM]}+10} = \frac{10r+6}{6r+10} \Rightarrow \frac{10r+6}{6r+10} = \frac{47}{34} \Rightarrow r = \frac{133}{29}$$

Therefore $m = 133, n = 29$, and $\boxed{m+n=162.}$ □

Solution. [Solution 1] **??** Let consider a positive integer n that is less than or equal to 2022 and n has no pair of adjacent digits that are both even.

Case 1: There are 9 1−digit such possitive integers.

Case 2: There are 5 choices for an odd ten digit and 10 choices for a unit digit with different parity, or 4 choices for an even ten digit and 5 choices for an odd unit digit, so there are $5 \cdot 10 + 4 \cdot 5 = 70$.

Case 3: There are 5 choices for an odd ten digit, 9 choices for a hundred digit, and 10 choices for a unit digit; or 5 choices for an even ten digit and 5×5 choices for both odd hundred and unit digits; so there are $5 \cdot 9 \cdot 10 + 5 \cdot 5 \cdot 5 = 575$.

Case 4: Let 1 be the thousands digit. There are 5 choices for an odd ten digit, 10 choices for a hundred digit, and 10 choices for a unit digit; or 5 choices for an even ten digit and 5×5 choices for both odd hundred and unit digits; so there are $5 \cdot 10 \cdot 10 + 5 \cdot 5 \cdot 5 = 625$.

Case 5: From 2000 to 2022 there is no such number.

The total of them is $9 + 70 + 575 + 625 = 1279$.

Thus, the number of positive integer less than or equal to 2022 and have at least one pair of adjacent digits that are both even is $\boxed{2022 - 1279 = 743.}$ □

Solution. [Solution 2] **??**

```
counter = 0

for n in range(1, 2023):
    s = str(n)
    found = False
    for i in range(0, len(s)):
        if i + 1 <= len(s) - 1:
            if int(s[i]) % 2 == 0 and int(s[i+1]) % 2 == 0:
                found = True
                counter += 1
                break

print(counter)
```

Its output is $\boxed{743.}$ □

Solution. **??** The expected value while rolling one fair dice inside

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} = \frac{7}{2}.$$

Thus, the expected value $r_1 = 6 \times \frac{7}{2} = 21$. Similarly $r_2 = (21 - 1)\frac{7}{2} = 70$, and $\boxed{r_3 = (70 - 2)\frac{7}{2} = 238.}$ □

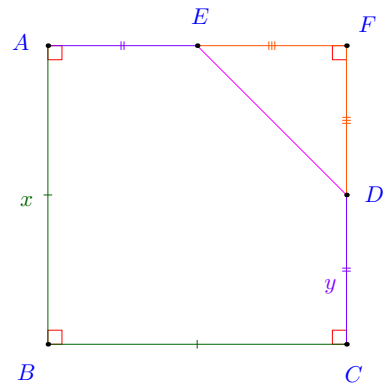
Solution. **??** Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then the leading term of $P(x+1) + P(x-1)$ is $2a_n x^n = x^3$, thus $a_n = \frac{1}{2}, n = 3$. Now let $P(x) = \frac{1}{2}x^3 + a_2 x^2 + a_1 x + a_0$, then

$$P(x+1) + P(x-1) = \frac{1}{2}(x+1)^3 + a_2(x+1)^2 + a_1(x+1) + a_0 + \frac{1}{2}(x-1)^3 + a_2(x-1)^2 + a_1(x-1) + a_0$$

$$= x^3 + 2a_2x^2 + (2a_1 + 3)x + 2(a_2 + a_0) \Rightarrow a_2 = 0, a_1 = -\frac{3}{2}, a_0 = 0 \Rightarrow P(x) = \frac{x^3}{2} - \frac{3x}{2}$$

Therefore $\boxed{P(10) = \frac{10^3}{2} - \frac{3 \cdot 10}{2} = 485.}$ □

Solution. **??** It is easy to see that the pentagon is part of a square as shown below. Let $AB = BC = x, CD =$



$EA = y$,

$$x^2 - \frac{1}{2}(x-y)^2 = 2018 \Rightarrow \begin{cases} 2018 \geq x^2 - \frac{1}{2}(x-1)^2 = \frac{1}{2}(x+1)^2 - 1 \Rightarrow x \leq \left\lceil 2\sqrt{2019} \right\rceil = 90 \\ 2018 \leq x^2 - \frac{1}{2}1^2 \Rightarrow x \geq \left\lceil \sqrt{2019} \right\rceil = 45 \end{cases}$$

On the other hand $x^2 - \frac{1}{2}(x-y)^2 = 2018 \Rightarrow y = x - \sqrt{2(x^2 - 2018)}$, where $P_[ABCDE] = 2x + 2y + (x-y)\sqrt{2}$.

```
from math import floor, sqrt

max_p = [0, 0, 0]
for x in range(45, 91):
    sx = sqrt(2 * (x * x - 2018))
    if sx == floor(sx):
        y = int(x - sx)
        p = 2*x + 2*y + (x-y) * sqrt(2)
        if max_p[0] < p:
            max_p = [p, x, y]

print(max_p)
```

Its output is $175.79898987322332, 46, 32$. Thus, $x = 46, y = 32$, and $P_[ABCDE] = 2 \cdot 46 + 2 \cdot 32 + 14\sqrt{2} = 156 + 14\sqrt{2}$. The answer is $\boxed{156 + 14 + 2 = 172.}$ □

Solution. **??** Let $n-3=a^3,\ n^2+4=b^3$, then $(ab)^3=a^3b^3=(n^2+4)(n-3)=n^3-3n^2+4n-12$. Since

$$\begin{aligned}n^3-(ab)^3&=3n^2-4n+12=n(3n-4)+12\\(ab)^3-(n-1)^3&=n-11\end{aligned}$$

Case 1: $n>11$, then $n^3>(ab)^3>(n-1)^3$, this means there is no solution.

Case 2: $0\le n\le 11$. We can investigate whether there is any solution by

```
def cubic_root(x):
    if x < 0:
        x = abs(x)
        r = x**(1/3)*(-1)
    else:
        r = x**(1/3)
    return r

for n in range(0, 12):
    a = int(cubic_root(n-3))
    b = int(cubic_root(n*n+4))
    if a**3 == n - 3 and b**3 == n*n + 4:
        print(n, a, b)
```

There is two solution $n=2, n-3=(-1)^3, n^2+4=8=2^3$, and $n=11, n-3=2^3, n^2+4=125=5^3$.

Case 3: $n<0$. Let substitute n by $-n$, where $n\ge 1$, then both $-n-3$ and n^2+4 are perfect cubes, thus $n+3$ and n^2+4 are also perfect cubes. Similarly to the first case

$$\begin{aligned}&(n^2+4)(n+3)-(n+1)^3=n+11>0\\&((n+2)^3-(n^2+4)(n+3)=n(3n+8)-4>0\end{aligned}$$

Therefore the product of all such n is $\boxed{2\cdot 11=22.}$ □

Solution. **??** By Vieta's theorem, if $p(x)=3x^7-291x^6+ax^5+bx^4+cx^2+154x-2$ has 7 real roots $x_1,x_2,\ldots,x_7$, then

$$\begin{cases}x_1+x_2+\ldots+x_7=(-1)^2\frac{-291}{3}=97\\x_1x_2\cdots x_6+x_1x_2\cdots x_5x_7+\ldots+x_2x_3\cdots x_7=(-1)^6\frac{154}{3}=\frac{134}{3}\\x_1x_2\cdots x_6x_7=(-1)^7\frac{-2}{3}=\frac{2}{3}\end{cases}$$
$$\Rightarrow\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_7}=\frac{x_1x_2\cdots x_6+x_1x_2\cdots x_5x_7+\ldots+x_2x_3\cdots x_7}{x_1x_2\cdots x_6x_7}=\frac{154}{3}\cdot\frac{3}{2}=77$$

Therefore the sum of the reciprocals of the roots is $\boxed{77.}$ □

Solution. **??** Let $b_n=a_1a_2\cdot a_n$, then

$$b_n=a_1a_2\cdots a_{n-1}a_n=a_1a_2\cdots a_{n-1}\left(2-\frac{1}{a_{n-1}}\right)=2a_1a_2\cdots a_{n-1}-a_1a_2\cdots a_{n-2}=2b_{n-1}-b_{n-1}$$
$$b_n-b_{n-1}=b_{n-1}-b_{n-2}=\ldots=b_2-b_1=a_1a_2-a_1=a_1(a_2-1)=a_1\left(1-\frac{1}{a_1}\right)=a_1-1=\sqrt{2}$$
$$b_n=b_{n-1}+\sqrt{2}=b_{n-2}+2\sqrt{2}=\ldots=b_2+(n-2)\sqrt{2}=1+2\sqrt{2}+(n-2)\sqrt{2}=1+n\sqrt{2}$$

Therefore $b_{150}=1+150\sqrt{2}$, and $\boxed{\lfloor 1+150\sqrt{2}\rfloor=213.}$ □

Solution. **??** Note that,

$$(\sqrt{3}+1)^3 = (\sqrt{3})^3 + 3(\sqrt{3})^2 + 3(\sqrt{3}) + 1 = 3\sqrt{3} + 9 + 3\sqrt{3} + 1 = 2(5+3\sqrt{3}) \Rightarrow 5+3\sqrt{3} = \frac{(\sqrt{3}+1)^3}{2}$$

$$(5+3\sqrt{3})^2 = 25 + 27 + 30\sqrt{3} = 2(26+15\sqrt{3}) \Rightarrow 26+15\sqrt{3} = \frac{(5+3\sqrt{3})^2}{2} = \frac{\left(\frac{(\sqrt{3}+1)^3}{2}\right)^2}{2} = \frac{(\sqrt{3}+1)^6}{2^3}$$

$$\Rightarrow \sqrt[6]{26+15\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{2}}, \sqrt[6]{26-15\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{2}} \Rightarrow 1 + \sqrt[6]{26+15\sqrt{3}} - \sqrt[6]{26-15\sqrt{3}} = 1+\sqrt{2}$$

$$\Rightarrow (1 + \sqrt[6]{26+15\sqrt{3}} + \sqrt[6]{26-15\sqrt{3}})^6 = (1+\sqrt{2})^6 = 99 + 70\sqrt{2}.$$

Therefore $m = 99, n = 70, n = 2$, and $\boxed{m - n + p = 31.}$ □

Solution. **??** The *co-efficients* $a, b, c,$ *and* d of the equation of the plane $ax + by + cz + d = 0$ through three points $(636, 36, 36)$, $(36, 336, 36)$, and $(36, 36, 336)$, are *the solutions of the system of equations*,

$$\begin{cases} 336a + 36b + 36c + d = 0 \\ 36a + 636b + 36c + d = 0 \\ 36a + 36b + 336c + d = 0 \end{cases}$$

One of the solution is $(a, b, c, d) = (2, 1, 2, -780)$. The distance from point $(21, 21, 21)$ to the plane with the equation $2x + y + 2z - 780 = 0$ is,

$$\frac{|1 \cdot 21 + 2 \cdot 21 + 2 \cdot 22 - 780|}{\sqrt{2^2 + 1^2 + 2^2}} = \boxed{225.}$$

□

Solution. **??** Note that if there exists k such that $(2k-1)(2k+1) = 4n^3 - 3n$, then $4k^2 = 4n^3 - 3n + 1$. Thus, the problem reduce to find if $4n^3 - 3n + 1$ is an even perfect square.

```
from math import sqrt

for n in range(1999, 0, -1):
    m = 4 * pow(n, 3) - 3 * n + 1
    p = round(sqrt(m))
    if p * p == m and p % 2 == 0:
        print(n)
        break
```

Its output is $\boxed{1935.}$ □

Solution. **??** Note that

$$\sqrt{x^2+3} = \sqrt[3]{x^3+33} \Rightarrow (x^2+3)^3 = (x^3+33)^2 \Rightarrow 9x^4 - 66x^3 + 27x^2 - 1062 = 0 \Rightarrow 3(3x^2 - 22x + 9) = \frac{1062}{x^2}$$

Therefore, for any a root of the equation $\boxed{\frac{1062}{a^2} : 3a^2 - 22a + 9 = 3.}$ □

Solution. **??** Let O be the centre of the large circle; $A_1, A_2, \ldots, A_6$ be the centres of the medium-size circles; and O_1, O_2, and O_3 be the centres of the small circles, see **??**. Let R be the radius of the medium-sized triangle and r be the radius of the small circle. First, it is easy to see that each $\triangle OA_iA_{i+1}, i = 1, 2, \ldots, 6$

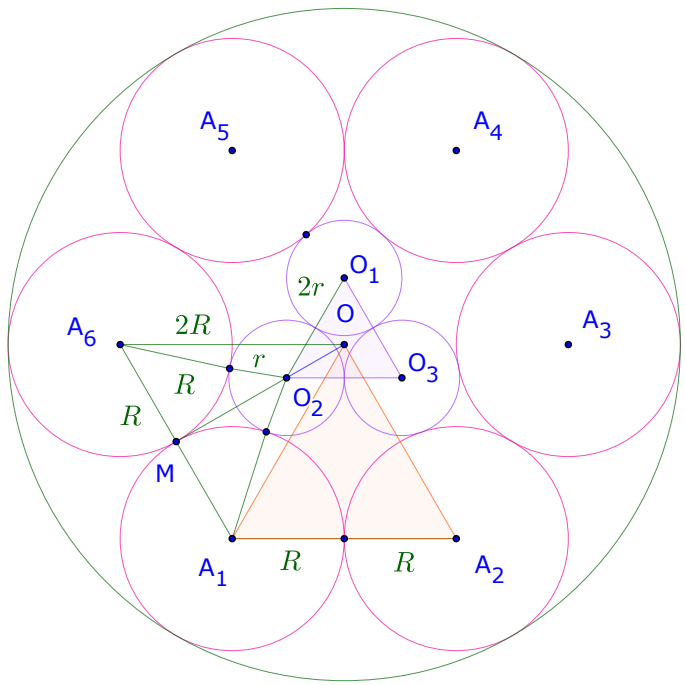


Figure 8.9: **??**

is isosceles triangle and the sum $\sum \angle OA_iA_{i+1}$ is $360°$, so each angle is $60°$, therefore all $\triangle OA_iA_{i+1}$ are equilateral. Thus $OA_i = 2R, i = 1, 2, \ldots, 6$.

Now, consider the triangle OA_6A_1 with O_2 on the altitude from O to the midpoint M of A_6A_1. The distance OO_2 is $\frac{2}{3}$ of the altitude of $\triangle O_1O_2O_2$ with side length $2r$. The distance O_2M is a leg of the right triangle $\triangle O_2A_6M$, so

$$OM = OO_2 + O_2M \Rightarrow \frac{(2R)\sqrt{3}}{2} = \frac{2}{3}\frac{(2r)\sqrt{3}}{2} + \sqrt{(R+r)^2 - R^2}$$

$$3R - 2r = \sqrt{3}\sqrt{2Rr + r^2} \Rightarrow 9R^2 - 12Rr + 4r^2 = 3(2Rr + r^2) \Rightarrow 9R^2 - 18Rr + r^2 \Rightarrow R = \frac{(3 + 2\sqrt{2})r}{3}$$

Therefore the ratio of the area of the large circle to the area of one of the small circles is $\frac{\pi(3R)^2}{\pi r^2} = \left(\frac{3R}{r}\right)^2 = (3 + 2\sqrt{2})^2 = 17 + \sqrt{288}$.

Thus, $m = 17, n = 288$, and $\boxed{n - m = 273.}$ □

Solution. **??** Let R be the range of f, $R \subseteq \{1,2,3,4,5\}$. Note that if the range of $f(f(x))$ is same as the range of f, then the range of $f(f(x))$ is R, and for every element $e \in R$, $f(e) \in R$, in other words, f maps R onto itself.

Thus, if k is the elements of $|R| = k$, then $1 \leq k \leq 5$. The number of f mappings is the number of ways to choose k elements from n elements (which elements belong to R) multiply with the number of ways to permute k elements (which elements of R map to other elements of R) multiply with the number of ways to map the remaining $5-k$ elements onto some of k elements, for $k \in \{1,2,3,4,5\}$, see **??** for an example.

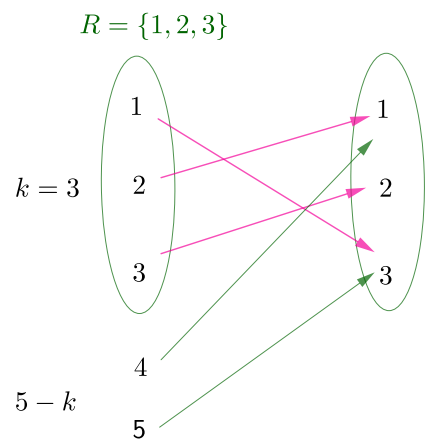


Figure 8.10: $k = 3,\ R = \{1,2,3\}$

Therefore the total number of such mappings is,

$$S = \sum_{k=1}^{5} \binom{5}{k} \cdot k! \cdot k^{5-k} = \boxed{1305.}$$

□

Solution. **??** Applying directly the theorem, $\boxed{E(15,12) = 2^{15-12}\binom{15}{12} = 3640.}$ □

Solution. **??** Instead of considering it as 4 separate stacks, think of it as one big stack separated into 4 pieces. The only ways to satisfy the conditions are,

$$\text{RRRR} \mid \text{RRBB} \mid \text{RBBB} \mid \text{RBBB}, \text{ or}$$
$$\text{BBBB} \mid \text{RRBB} \mid \text{RRRB} \mid \text{RRRB}.$$

The total number of ways to chose 8 Rs and 8 B to create one of the patterns is $\binom{16}{8} = \frac{16!}{8!*8!} = 12870$. Now the number of ways to arrange the first pattern

$$\text{RRRR} \mid \text{RRBB} \mid \text{RBBB} \mid \text{RBBB}$$

is $\frac{4!}{2!} = 12$ ways for the four 4−blocks placement, then $\binom{4}{2} = 6$ ways to arrange $RRBB$, then $\binom{4}{3} = 4$ ways to arrange $RBBB$, and $\binom{4}{3} = 4$ ways to arrange the second $RBBB$. Similarly for the second pattern, thus the total number of permutations is $12 \cdot 6 \cdot 4 \cdot 4 \cdot 2 = 2304$.

The requested probability is $\frac{2304}{12870} = \frac{128}{715}$, thus $m = 128, n = 715$, and $\boxed{n - m = 587.}$ □

Solution. **??**

Let the time when Sally starts, counting from noon, is x, where $0 \le x \le 1$. Let the distance from Jim's house to the place where Sally starts is y, where $0 \le y \le 4$. Let the time when Sally catches up with Jim is t, then the distance Jim walks is $4t$, and the distance Sally runs is $y + 7(t - x)$, thus

$$4t = y + 7(t - x) \Rightarrow 7x - y = 3t.$$

Because Jim walks past the place where Sally starts so $x \le t$. Furthermore Sally must catch up with Jim before he walks 1 hour, so $t \le 1$. Therefore $3x \le 7x - y \le 3 \Rightarrow 7x - 3 \le y \le 4x$. Thus the possible pair of (time, distance) values must satisfy, or the area of $\triangle ABO$, as shown in **??**.

$$7x - 3 \le y \le 4x; 0 \le x \le 1; 0 \le y \le 4$$

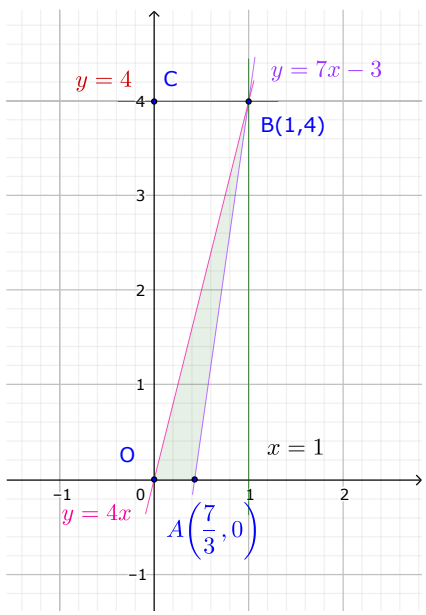


Figure 8.11: $[ABO]$ area

Therefore $[ABO] = \frac{1}{2} \cdot \frac{3}{7} \cdot 4 = \frac{6}{7}$, and the probability $\frac{[ABO]}{4} = \frac{6}{28} = \frac{3}{14}$. Thus $\boxed{m + n = 17.}$ □

Solution. **??** By Menelaus theorem and since $BD = BF, DC = CE, AF = AE$, let $GA = x$, see **??**.

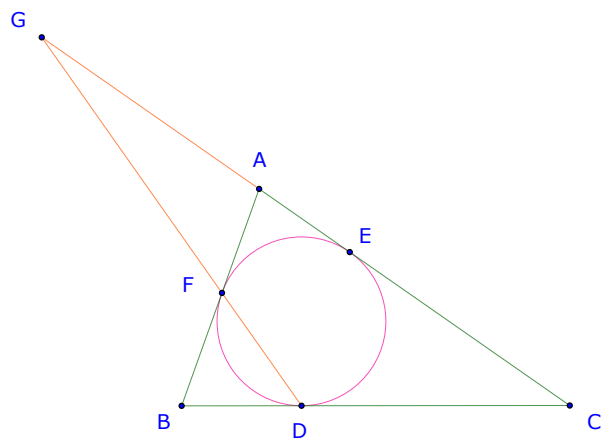


Figure 8.12: **??**

$$\frac{BD}{DC} \cdot \frac{CG}{GA} \cdot \frac{AF}{FB} = 1 \Rightarrow \frac{CG}{GA} \cdot \frac{AE}{EC} = 1 \Rightarrow \frac{AC + x}{x} \cdot \frac{AE}{EC} = 1$$

Now, $AE = \frac{AB+AC-BC}{2} = \frac{73+120-123}{2} = 35$, and $EC = AC - AE = 120 - 35 = 85$, thus

$$\frac{120 + x}{x} \cdot \frac{35}{85} = 1 \Rightarrow x = \boxed{84}$$

□

Solution. **??**

$$\begin{aligned}
&2(x^2 + y^2) + (xy - 7)^2 = 2(x^2 + y^2 - 2xy) + (x^2y^2 - 10xy + 25) + 24 = 2(x - y)^2 + (xy - 5)^2 + 24 \\
&\Rightarrow 4(x^2 + y^2 + z^2 + w^2) + (xy - 7)^2 + (yz - 7)^2 + (zw - 7)^2 + (wx - 7)^2 \\
&= 2\left[(x - y)^2 + (y - z)^2 + (z - w)^2 + (w - x)^2\right] + (xy - 5)^2 + (yz - 5)^2 + (zw - 5)^2 + (wx - 5)^2 + 4 \cdot 24 \\
&\geq 4 \cdot 24 = \boxed{96}
\end{aligned}$$

□

Solution. **??** Note that

$$T_k - 1 = \frac{k(k+1) - 2}{2} = \frac{(k-1)(k+2)}{2},\ T_{k-1} - 1 = \frac{(k-2)(k+1)}{2},\ T_{k+1} - 1 = \frac{(k)(k+3)}{2}$$

$$\frac{1}{(T_{k-1} - 1)(T_k - 1)(T_{k+1} - 1)} = \frac{8}{(k-2)(k-1)(k)(k+1)(k+2)(k+3)}.$$

Now, by follow that,

$$\frac{1}{(k+m)(k+n)} = \frac{1}{n-m}\left(\frac{1}{k+m} - \frac{1}{k+n}\right)$$

multiple times for pairs of $k-2, k-1, k, k+1, k+2$, and $k+3$, we obtain

$$\frac{8}{(k-2)(k-1)(k)(k+1)(k+2)(k+3)} = \frac{A}{k-2} + \frac{B}{k-1} + \frac{C}{k} + \frac{D}{k+1} + \frac{E}{k+2} + \frac{F}{k+3},$$

where $A = \frac{1}{15},\ B = -\frac{1}{3},\ C = \frac{2}{3},\ D = -\frac{2}{3},\ E = \frac{1}{3},\ F = -\frac{1}{15}$.

Therefore the given sum is

$$\sum_{k=4}^{\infty} \frac{1}{15}\left(\frac{1}{k-2}-\frac{1}{k+3}\right)-\frac{1}{3}\left(\frac{1}{k-1}-\frac{1}{k+2}\right)+\frac{2}{3}\left(\frac{1}{k}-\frac{1}{k+1}\right).$$

These are telescopic sums, and by further simplifying we get the limit,

$$\frac{1}{15}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)-\frac{1}{3}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)+\frac{2}{3}\left(\frac{1}{4}\right)=\frac{1}{15}\cdot\frac{29}{20}-\frac{1}{3}\cdot\frac{47}{60}+\frac{2}{3}\cdot\frac{1}{4}=\frac{1}{450}$$

Thus, the answer is $\boxed{450-1=449.}$ □

Solution. **??**

```
def has_a_multiple_of(numbers, factor):
    for n in numbers:
        if n % factor == 0:
            return True
    return False

counter = 1
for a in range(1, 12):
    for b in range(a+1, 13):
        for c in range(b+1, 14):
            if has_a_multiple_of([a,b,c], 2) and \
                has_a_multiple_of([a,b,c], 3) and \
                    has_a_multiple_of([a,b,c], 7):
                    print(counter, a,b,c)
                    counter += 1
```

Its output shows $\boxed{29}$ subsets. □

Solution. **??** First, let O be the centre of the base and V ve the top vertex of the cone. Let O_1, O_2, O_3, and O_4 be the centres of the sphere. Let α be the plane through O_2, O_3, and O_4. Let I be the intersection of VO_1 with the plane α. Let AB be the diameter of the cone such that VB is parallel to O_2O_3. See **??**.

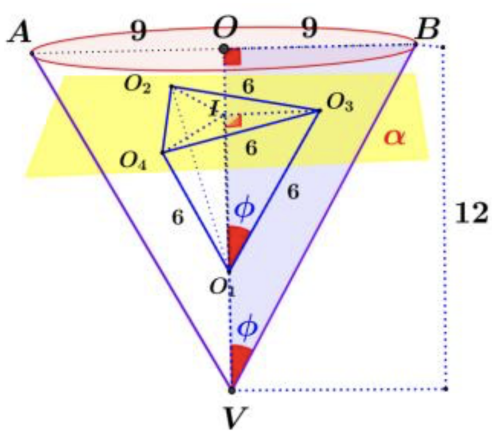


Figure 8.13: **??**

Note that $\triangle O_2O_3O_4$ is an isosceles triangle $O_2O_3 = O_2O_4 = 6$. The distance from O_1 to any of O_2, O_3, and O_4 is 6. Note that $\triangle OVB \sim \triangle IO_1O_3$, so

$$\phi = \angle(O_1O_3, O_1I) = \angle(VB, VO_2) \Rightarrow \frac{IO_3}{IO_1} = \frac{3}{4}$$

Now, since each of $\triangle IO_1O_2$, $\triangle IO_1O_3$, and $\triangle IO_1O_4$ is right triangle and similar to $\triangle OVB$, thus let $O_1I = 4k$, then $IO_2 = IO_3 = IO_4 = 3k$. Easy to see that $5k = 6$, so $k = \frac{6}{5}$. See **??**.

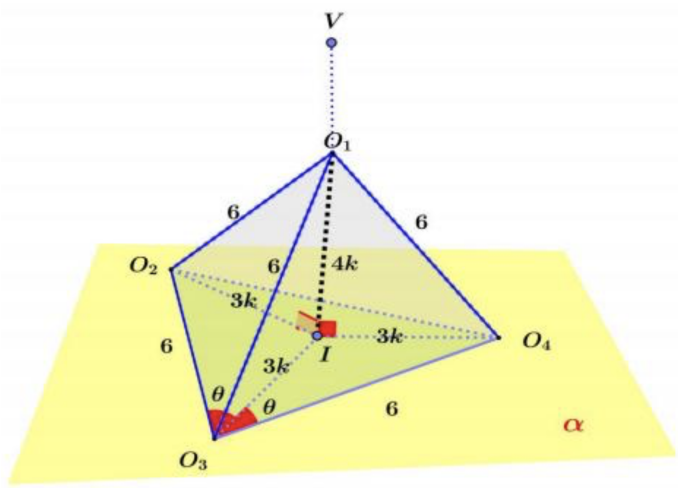


Figure 8.14: **??**

Now, let $\angle O_2O_3O_4 = 2\theta$, then by the Law of Cosines in $\triangle O_2IO_3$,

$$(3k)^2 + 6^2 - 2 \cdot 6 \cdot 3k \cdot \cos\theta = (3k)^2 \Rightarrow \cos\theta = \frac{1}{k} = 56$$

$$[O_2O_3O_4] = \frac{1}{2} \cdot O_2O_3 \cdot O_3O_4 \cdot \sin 2\theta = \frac{6^2}{2} \cdot 2 \cdot \sin\theta \cdot \cos\theta = 36 \cdot \frac{\sqrt{11}}{6} \cdot \frac{5}{6} = 5\sqrt{11}$$

$$K = \frac{1}{3}O_2O_3O_4]O_1I = \frac{1}{3} \cdot 5\sqrt{11} \cdot \frac{24}{5} = 8\sqrt{11}.$$

Therefore $m = 8, n = 11$, and $\boxed{m + n = 19.}$ □

Chapter 9

HC R7

9.1 Topics

Chess

1. Moves: capture, check, double check, promotion, stalemate, checkmate
2. Analysis: whose turn is it.
3. Retrograde: last move, last two moves, last multiple of moves.
4. Proof games: White and Black, instead of making the best moves, they co-operate to make the game legally into a set position.
5. Symmetric position with odd number of moves.
6. Switchback: a piece moved out and came back into that same square.

Logic

1. Casework: what if A is true, what if A is false.
2. Process of elimination: If A is not true, B is not true, then C should be true.
3. Reverse argument: if there is at least ..., then there is atmost ...
4. Implication from truth: if A told the truth and A said X, then X is true.
5. Conflict of truth: if B said A lied, then both cannot be truth tellers.

Algebra

1. Ratio.
2. Sums and Products. Telescopic Products.

Combinatorics

1. Counting in two ways.
2. Probability. Favourables outcomes.

Geometry

1. Paralellograms.
2. Rhombus.
3. Area.
4. Symmetry. Axis of symmetry.
5. Right triangles. $45 - 45 - 90$ triangles.

Number Theory

1. Divisibility.

9.2 Problems

Problem 9.2.1 (HC-2022-SM1-R7-P1)**.** (*Beginner Level*)

After the session, the teacher gave each student in the math club 7 candies, and she still has 24 candies. In order to give each student 2 more candies, she needs to have an extra 32 candies.

How many students are there? And how many candies does the teacher have?

How to provide your answer:

- If you think there are 5 students and the teacher has 10 candies, submit $5, 10$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 9.2.2 (HC-2022-SM1-R7-P2)**.** (*Beginner Level*)

Hassan, the horse-trader, challenged three buyers to guess correctly the colour of his best horse. The horse's colour was either black, brown, or gray.

- Helena: The horse is not black.
- Gabriella: The horse is either gray or brown.
- Isabella: The horse is brown.

Hassan said that at least one of the buyers guessed right and at least one guessed wrong.

Who were correct and who were not?

How to provide your answer:

- If you think Helena was correct, Gabriella was incorrect, and Isabella was incorrect too, submit HC, GI, II.
- If you think Helena was incorrect, Gabriella was correct, and Isabella was incorrect, submit HI, GC, II.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 9.2.3 (HC-2022-SM1-R7-P3)**.** (*Beginner Level*)

In the **??**, each side made exactly four *legal* moves.

Figure 9.1: **??**

What were the moves?

How to provide your answer:

- If you think the moves were 1. e4 e5 2. d4 d5 3. c4 c5 4. f4 f5, then submit 1. e4 e5 2. d4 d5 3. c4 c5 4. f4 f5.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 9.2.4 (HC-2022-SM1-R7-P4)**.** (*Beginner Level*)

The **??** shows a parallelogram and the point P of intersection of its diagonals is marked.

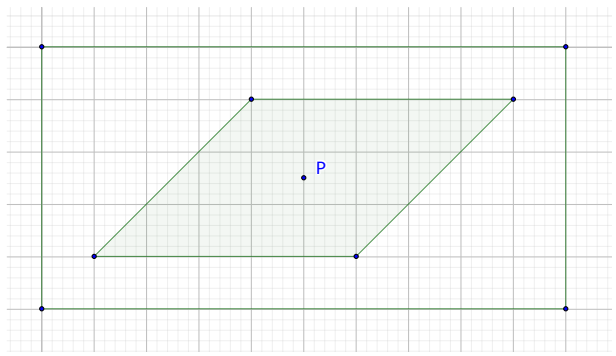


Figure 9.2: **??**

Draw a straight line through P so that it breaks the parallelogram into two parts, from which you can form a *rhombus.*

How to provide your answer:

- If you think that it is possible, submit a drawing (that scales)
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 9.2.5 (HC-2022-SM1-R7-P5). (*Beginner Level*)

Some positive integers can be created by using the digits $1, 2, 3$, and 4, with four arithmetic operations $+,\ -, \div, \times$. Each digit is used once and each operation can be used none, once, or several times. *For example, the numbers* $1, 2, 3$ *and* 4 *can be created as below,*

$$(2+3)-(4\times 1)=\boxed{1}, (4-(3+1))+2=\boxed{2}, (4+2\times 1)-3=\boxed{3}, (14-2)\div 3=\boxed{4}.$$

In how many ways can you express the number 21 with the digits $1, 2, 3$, and 4, and four arithmetic operations $+,\ -, \div, \times$? *Note that* $1+2$ *and* $2+1$ *are considered as the same.*

How to provide your answer:

- If you think that the number 21 can be expressed, show how do you do it. The more ways you get, the higher your score for this problem.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that shows *strictly smaller* number of ways to express the number 21 then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has *strictly larger* number of ways to express the number 21 than your answer.

Problem 9.2.6 (HC-2022-SM1-R7-P6)**.** (*Intermediate Level*)

Mai and Dao went on a picnic in a natural conservation. Around noon, they met a kid who got lost. They decided to stop for lunch. Mai had *three* loaves of bread, Dao had *two loaves*, and the five loaves were divided equally among the three of them. Shortly after lunch, they found the parents of the kid. The parents were very happy, so they gave the girls $10 as to express their gratitude.

How can Mai and Dao divide the $10 *fairly*?

How to provide your answer:

- If you think that Mai should get $5, and Dao should receives $5, then submit 5, 5.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 9.2.7 (HC-2022-SM1-R7-P7)**.** (*Intermediate Level*)

Phong and Quynh, each randomly choose a number from $1, 2, \ldots, 1000$. What is the probability that Phong's number is larger than Quynh's number?

How to provide your answer:

- If you think the probability is $\frac{1}{2}$, then submit $\frac{1}{2}$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly larger* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly smaller* than your answer.

Problem 9.2.8 (HC-2022-SM1-R7-P8)**.** (*Intermediate Level*)

In the **??**, it's White turn and it can checkmate Black in two moves.

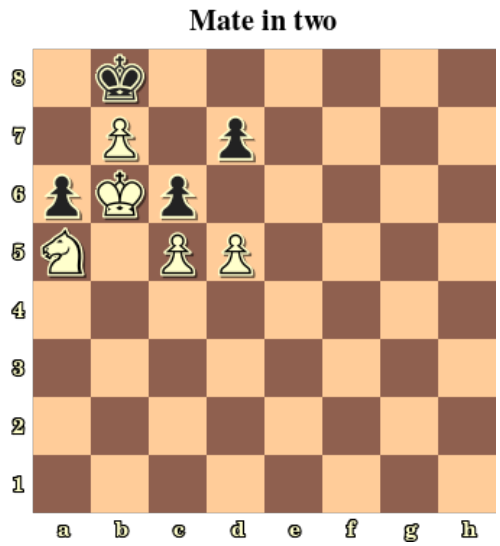


Figure 9.3: **??**

What are the moves?

How to provide your answer:

- If you think that the moves are 1. Nb3 d6. 2. Kxa6; submit 1. Nb3 d6. 2. Kxa6.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 9.2.9 (HC-2022-SM1-R7-P9)**.** (*Intermediate Level*)

The axis of symmetry is a straight line that divides a shape into two identical parts, The shape on the left of the **??** has a single axis of symmetries, while the shape on the right has two (different) axes of symmetries.

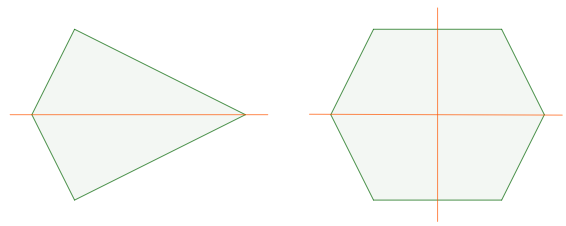


Figure 9.4: Shape with axes of symmetries

Now, assemble the four pieces in the **??** below into a shape that has at least two axes of symmetries.

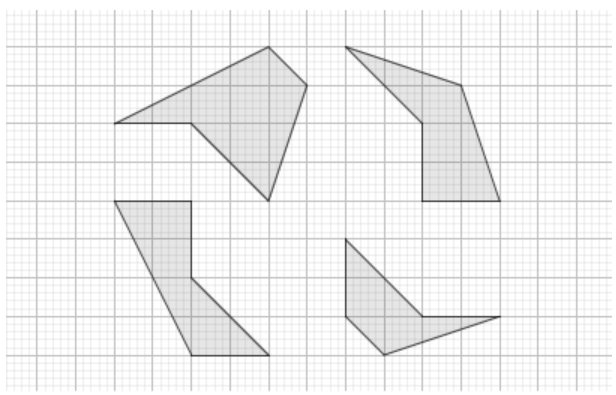


Figure 9.5: **??**

How to provide your answer:

- If you think the maximum number of ways is 100, then submit 100.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 9.2.10 (HC-2022-SM1-R7-P10)**.** (*Intermediate Level*)

The Challenge Given two strings a and b, each contains low-case letters $a, b, \ldots, z$, the string x is called *ordered common permutation* of a and b, if there is a permutation of the letters of a and a permutation of the letters of b such that each contains the string x. *For example,*

$$(a, b) = (\text{nice}, \text{pita}) \Rightarrow x = \text{i}$$
$$(a, b) = (\text{kind}, \text{rum}) \Rightarrow x =$$
$$(a, b) = (\text{women}, \text{walking}) \Rightarrow x = \text{nw}$$
$$(a, b) = (\text{beauty}, \text{pretty}) \Rightarrow x = \text{ety}$$

It is important to note that the letters in x are *sorted alphabetically, so* $x = \text{nw}$ *is an ordered common permutation, while* $x = \text{wn}$ *is not.* Furthermore if a and b have no *common permutation,* then x is an empty string.

Input (*hc-2022-1-7-10-inputput.txt*) The input file contains several cases, each case is a line. Each line contains two string of lowercase characters, separating by a space character, with first string of denoting a and the second one denoting b. Each string consists of at most 1000 characters.

Output (*hc-2022-1-7-10-output.txt*) The output file shall consist of a set of lines, each line contains a single string denoting x, which is the *common permutation* of the two string a and b in the corresponding line of the input.

Sample Input (*hc-2022-1-7-10-sample-input.txt*)

```
nice pita
kind rum
women walking
beauty pretty
```

Sample Output (*hc-2022-1-7-10-sample-output.txt*)

```
i

nw
ety
```

How to your troop can compete in this Coding Challenge?

- **Preparation:** first you have to study the problem specified and should fully understand the problem. If you have any questions, please ask. You have *one week* before the contest to make preparation based on the description of the challenge. You have **to code a problem solver for each problem**. The code have to be tested thoroughly to make sure that it works correctly.
- **Solving the Challenge:** your troop *receives the Input* file in the designated shared folder. Depending on the challeng, an Input file might contain data for one or more cases (of the same challenge). Your problem solver has to solve each and every case, then output the results as required into an Output file. Your troop have 30 **minutes** to work on the challenge and *submits the Output* file in the designated folder.
- **Verification:** The jury verify your submissions. Any submissions that contain errors will be discarded. No resubmission is allowed. The **number of correctly solved** and **successfully submitted** cases is recorded as your troop's *score* for the challenge.

How your troop's performance is graded:

- If your troop's *score* is higher or equal to the *score* of you opponent for the challenge then your troop scores 1 hit, otherwise 0 hit.
- If your troop cannot submit the output for the challenge, then your troop scores 0 hit, regardless of the score of you opponent.
- It is allowed that your troop create/modify the code during the contest, or attempt to solve the challenge without coding.

Problem 9.2.11 (HC-2022-SM1-R7-P11)**.** (*Advanced Level*)

The Magic Tree in the Forest of Fangorn was planted by Fingolfin. On the first day it increased its height by $\frac{1}{2}$, on the second day by a $\frac{1}{3}$, on the third day by a $\frac{1}{4}$, and so on. In how many days did it take to grow to five hundred times its original height?

How to provide your answer:

- If you think that it takes 100 days, submit 100.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 9.2.12 (HC-2022-SM1-R7-P12)**.** (*Advanced Level*)

Prince Billy the Bookworm is visiting the court of Uttaku. The High King has two daughters. The elder one is beautiful, but not too smart. The younger one is very smart, but not as beautiful. The King tells Billy that he seems to be a very clever and personable man, and the King likes hime, so he allows Billy to make a statement, such that

> **If the statement is true, then I will give you my younger daugther in marriage, but if the statement is false, then you might not marry her.**

However, Billy had his heart set on the *elder* daughter, not the younger one! What statement would *force* the King to let him marry the elder daughter?

How to provide your answer:

- If you think you have such statement, submit that.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 9.2.13 (HC-2022-SM1-R7-P13)**.** (*Advanced Level*)

Show that, in the **??**, either White or Black, can mate in one.

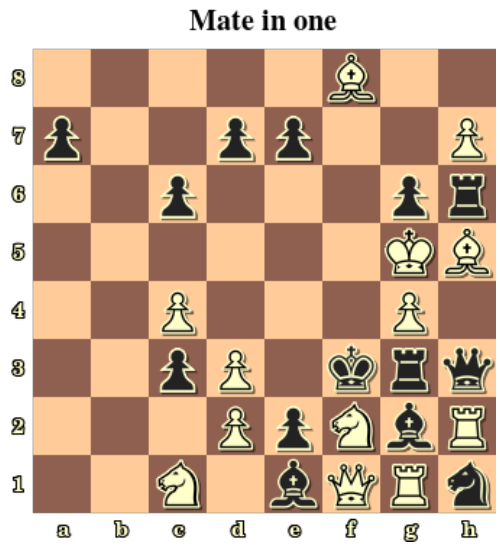


Figure 9.6: **??**

What are the moves?

How to provide your answer:

- If you think that if White moves, it can mate in one with Rc1; and if Black moves, it can mate in one with Nc2, then submit Rc1 Nc2.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 9.2.14 (HC-2022-SM1-R7-P14)**.** (*Advanced Level*)

There are three rectangles in the **??**. The lengths of some segments are shown.

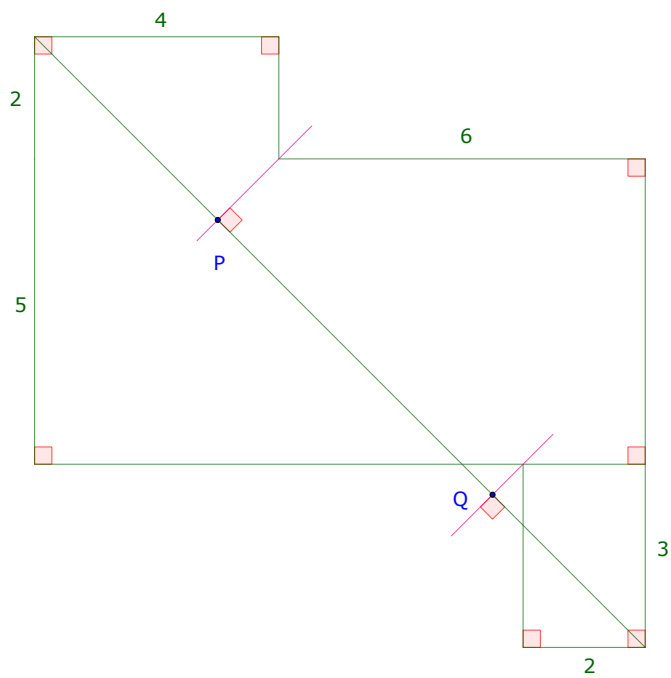


Figure 9.7: **??**

Find the length of the segment PQ.

How to provide your answer:

- If you think that $PQ = 5$, then submit 5.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 9.2.15 (HC-2022-SM1-R7-P15)**.** (*Advanced Level*)

The Challenge The *reverse and add* function starts with a number, reverses its digits, and adds the reverse to the original. If the sum is not a palindrome (meaning it does not give the same number read from left to right and right to left), we repeat this procedure until it does. *For example, if we start with 195 as the initial number, we get 9,339 as the resulting palindrome after the fourth addition:*

$$\begin{aligned} 195 + 591 &= 786 \\ 786 + 687 &= 1473 \\ 1473 + 3741 &= 5214 \\ 5214 + 4125 &= 9339 \end{aligned}$$

This method leads to palindromes in a few steps for almost all of the integers. But there are interesting exceptions. 196 is the first number for which no palindrome has been found. It has never been proven, however, that no such palindrome exists.

Your task is to write a program that takes a given number and gives the resulting palindrome (if one exists) and the number of iterations/additions it took to find it. You may assume that all the numbers used as test data will terminate in an answer with less than $1,000$ iterations (additions), and yield a palindrome that is not greater than $4,294,967,295$.

Input The input (hc-2022-1-7-15-input.txt) will consist of a set of lines, each line contains one integer between 1 and 1000.

Output The output (hc-2022-1-7-15-output.txt) will consist of a set of lines, each line contains the minimum number of iterations to find the palindrome and the resulting palindrom itself, separating by a space character.

Sample Input

```
195
265
750
```

Output

```
4 9339
5 45254
3 6666
```

How to your troop can compete in this Coding Challenge?

- **Preparation:** first you have to study the problem specified and should fully understand the problem. If you have any questions, please ask. You have *one week* before the contest to make preparation based on the description of the challenge. You have **to code a problem solver for each problem**. The code have to be tested thoroughly to make sure that it works correctly.
- **Solving the Challenge:** your troop *receives the Input* file in the designated shared folder. Depending on the challeng, an Input file might contain data for one or more cases (of the same challenge). Your problem solver has to solve each and every case, then output the results as required into an Output file. Your troop have 30 **minutes** to work on the challenge and *submits the Output* file in the designated folder.
- **Verification:** The jury verify your submissions. Any submissions that contain errors will be discarded. No resubmission is allowed. The **number of correctly solved** and **successfully submitted** cases is recorded as your troop's *score* for the challenge.

How your troop's performance is graded:

- If your troop's *score* is higher or equal to the *score* of you opponent for the challenge then your troop scores 1 hit, otherwise 0 hit.

- If your troop cannot submit the output for the challenge, then your troop scores 0 hit, regardless of the score of you opponent.
- It is allowed that your troop create/modify the code during the contest, or attempt to solve the challenge without coding.

9.3 Coding Challenge Inputs

?? Input as below (see hc-2022-1-7-10-input.txt)

```
repeating string
repetition repeat
grandfather grandmother
desalination vocalization
honorificabilitudinitatibus thyroparathyroidectomized
floccinaucinihilipilification pseudopseudohypoparathyroidism
supercalifragilisticexpialidocious hippopotomonstrosesquippedaliophobia
pneumonoultramicroscopicsilicovolcanoconiosis pneumonoultramicroscopicsilicovolcanoconiosis
```

?? Input as below (see hc-2022-1-7-15-input.txt)

```
4
12
89
109
149
177
187
376
494
495
512
768
861
977
999
```

9.4 Answers

Problem 1: 28, 220

Problem 2: HC, GC, II

Problem 3: 1. d4 c6 2. Bd2 Qa5 3. Bb4 Qa4 4. Bc5 e6

Problem 4: See the solution.

Problem 5: There are at least 10 ways.

$$\begin{aligned}(43-1)\div 2&=21\\ 42\div(3-1)&=21\\ 24-3\times 1&=21\\ 21\times(4-3)&=21\\ (24-3)\times 1&=21\\ (4\times(2+3))+1&=21\\ 13+2\times 4&=21\\ (4+3)\times(2+1)&=21\\ (4+2+1)\times 3&=21\\ (4\times 2-1)\times 3&=21\end{aligned}$$

Problem 6: 8, 2

Problem 7: $\frac{999}{2000}$

Problem 8: 1. dxc6 dxc6 2. Nxd6++ or 1. dxc6 ... 2. c7++.

Problem 9: See the solution.

Problem 10: Output as below

```
ginrt
eeprt
adeghnrrt
aaiilnot
aacdhiioortt
aahiiooptu
aadeeiiiiloopprssstu
aaccccceiiiiiilllmmnnnnoooooooooopprrsssstuuv
```

Problem 11: 998.

Problem 12: You will not allow me to marry either of your daughter.

Problem 13: Qxe2++ Bxd2++

Problem 14: $\frac{9\sqrt{2}}{2}$

Problem 15: Output as below

```
0 4
1 33
24 8813200023188
2 1111
2 1991
15 8836886388
23 8813200023188
3 15851
0 494
5 79497
1 727
2 6996
3 13431
6 475574
0 999
```

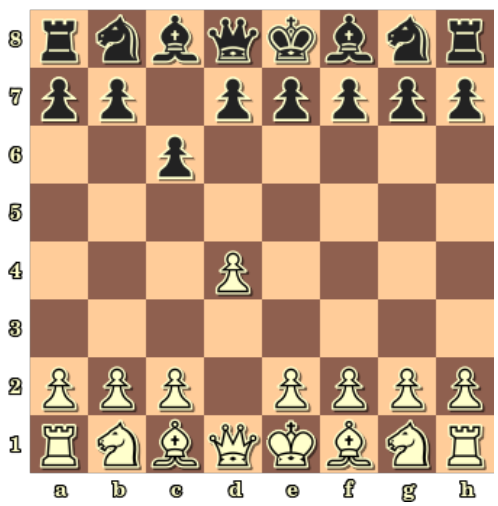
9.5 Solutions

Solution. **??** The additional candies, 2 for each student, come from the 24 candies the teacher has plus the 32 extra candies that the teacher needs. Thus, there are $\boxed{\frac{24+32}{2} = 28}$ students and $\boxed{28 \cdot 7 + 24 = 220}$ candies. □

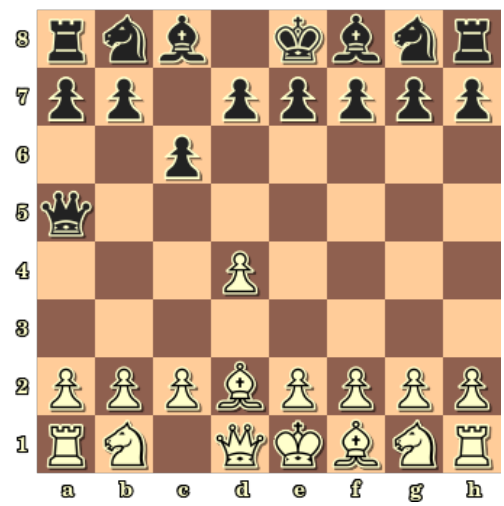
Solution. **??** If the horse were black. Then Helena, Gabriella, and Isabella were wrong. This contradicts the given condition, since there is no one who guessed right. If the horse were brown. Then Helena, Gabriella, and Isabella were right. This again contradicts the given condition, since there is no one who guessed wrong. Therefore the horse is gray. In this case Helena and Gabriella were right, and Isabella were wrong. This satisfies the given condition. The answer is $\boxed{HC,\ GC,\ II.}$ □

Solution. **??** Thus, the answer is $\boxed{\text{1. d4 c6. 2. Bd2 Qa5 3. Bb4 Qa4 4. Bc5 e6}}$ □

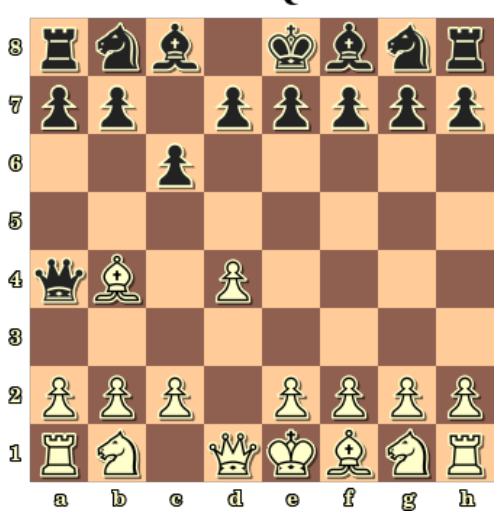
1. d4 c5



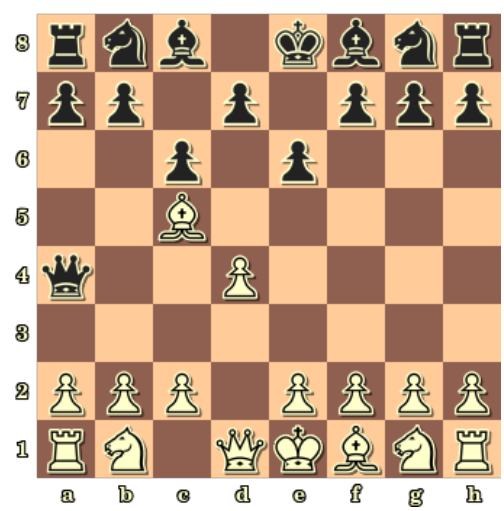
1. Bd2 Qa5



3. Bb4 Qa4



4. Bc5 e6



Solution. **??** There are at least two possible solutions, one is shown as below in **??**. Note that the length of the segment between the two intersection points is the length of the horiontal side.

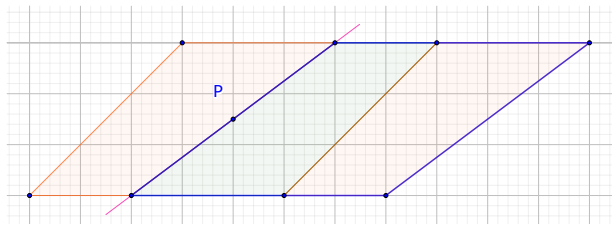


Figure 9.8: Line intersects horizontal sides

The second solution shows in the **??**. Note that the length of the segment between the two intersection points is the length of the vertical side. □

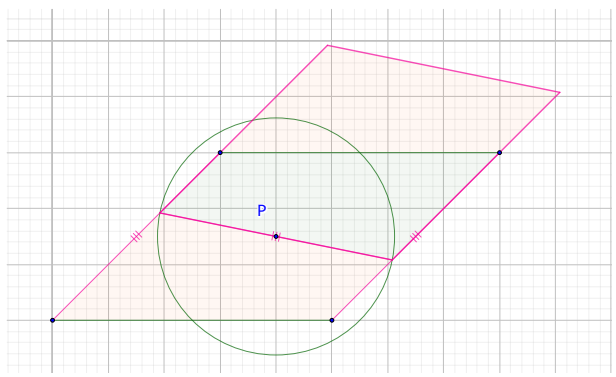


Figure 9.9: Line intersects vertical sides

Solution. **??** There are at least 10 ways.

$$
\begin{aligned}
(43-1)\div 2 &= 21\\
42\div(3-1) &= 21\\
24-3\times 1 &= 21\\
21\times(4-3) &= 21\\
(24-3)\times 1 &= 21\\
(4\times(2+3))+1 &= 21\\
13+2\times 4 &= 21\\
(4+3)\times(2+1) &= 21\\
(4+2+1)\times 3 &= 21\\
(4\times 2-1)\times 3 &= 21
\end{aligned}
$$

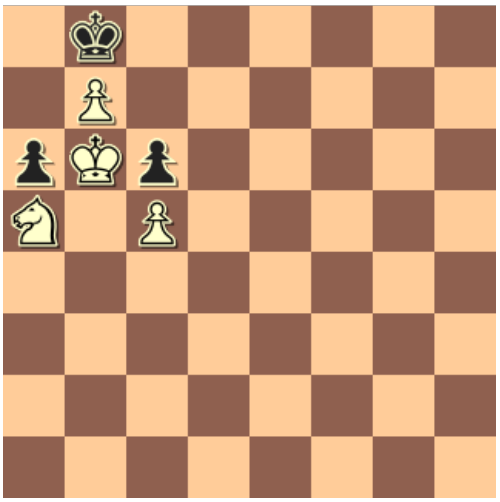
□

Solution. **??** Each of them got $\frac{5}{3}$ loaves. Thus Mai gave the kid $3-\frac{5}{3}=\frac{4}{3}$ loaves, and Dao gave the kid $2-\frac{5}{3}=\frac{1}{3}$ of a loaf. Therefore, Mai should get $\frac{4}{5}\cdot 10=\boxed{\$8,}$ and Dao should have $\frac{1}{5}\cdot 10=\boxed{\$2.}$ □

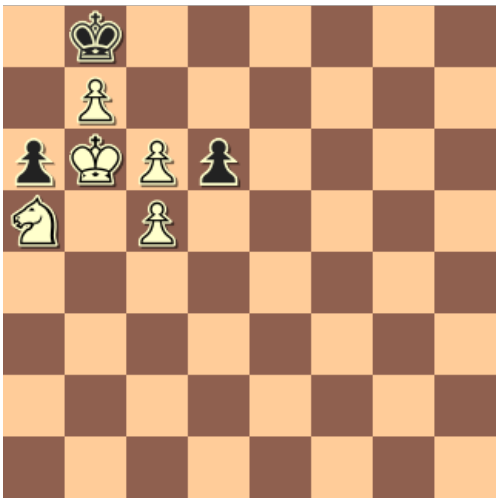
Solution. **??** First, it is easy to see that there are $1000 \cdot 1000$ pairs of numbers from $1, 2, \ldots, 1000$. Among these there are exactly 1000 pairs where Phong's and Quynh's numbers are the same. So there are $1000 \cdot 1000 - 1000 = 999 \cdot 1000$ pairs whose numbers are not equal. Easy to see that exactly half of those are pairs whose the first number is larger then the second, (and the other half of those are pairs whose the first number is smaller then the second.) Thus, the probability is $\dfrac{\frac{1}{2} \cdot 999 \cdot 1000}{1000 \cdot 1000} = \boxed{\dfrac{999}{2000}}$. □

Solution. **??** White shall start with 1. dxc6. If Black takes this pawn, then Nxd6++. If Black moves the pawn

1. dx6 ex6 2. Nxd6++



1. dx6 ... 2. c7++



on the *d* column (other Black pieces cannot move), then c7++. Thus, the answer is 1. dxc6 dxc6 2. Nxd6++ or 1. dxc6 ... 2. c
□

Solution. **??** □

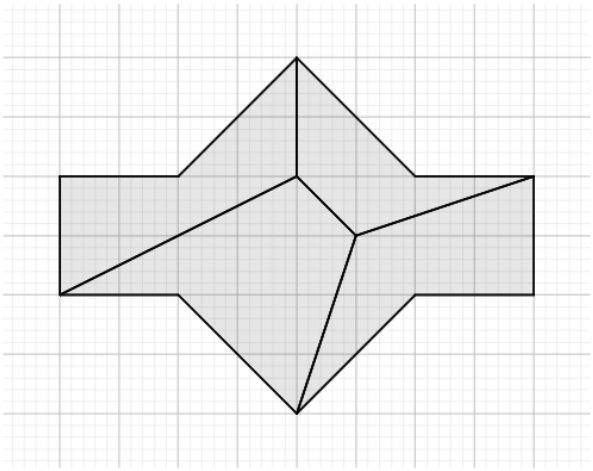


Figure 9.10: Assembled shape

Solution. **??** Following is an example solution in Python.

```
def find_common_permutations(ab_pairs):
    x_list = []
    for a, b in ab_pairs:
        x = [
            l * min(a.count(l), b.count(l))
            for l in set(a).intersection(set(b))
        ]
        x_list.append(x)
    return x_list

if __name__ == "__main__":
    with open('hc-2022-1-7-10-input.txt', 'rt') as i_file:
        ab_pairs = [w.strip().split() for w in i_file.readlines()]
        x_list = find_common_permutations(ab_pairs)
        with open('hc-2022-1-7-10-output.txt', 'wt') as o_file:
            for x in x_list:
                o_file.write(f"{''.join(sorted(x))}\n")
```

□

Solution. **??** Note that after the 1st days, the tree's height is $\left(1+\frac{1}{2}\right)=\frac{3}{2}$ of the original. After the 2nd days, the tree's height is $\frac{3}{2}\left(1+\frac{1}{3}\right)=\frac{4}{2}$ of the original. After the 3nd days, the tree's height is $\frac{4}{2}\left(1+\frac{1}{4}\right)=\frac{5}{2}$ of the original. Thus, after the n^{nd} days, the tree's height is $\frac{n+2}{2}$ of the original. If $\frac{n+2}{2}=500$, then $\boxed{n=1000-2=998 \text{ days}}$. □

Solution. **??** The correct statement should be false, otherwise the King will make Billy to marry the younger daughter. First, since Billy wants to marry the elder daughter, so the statement should includes the fact that the King does *not allow him to marry the elder.* Second, the fact that the King make him marry the younger daughter shall falsify the statement, then he *shall not marry the younger.* Thus the statement should consists both propositions.

Therefore, he should say $\boxed{\text{You will not allow me to marry either of your daughter.}}$

Why does it work? If the statement is true, then the King should make him marry the younger daughter, but that shall falsify the statement! If the statement is false, then the King should let him marry one of the daugthers. He could not marry the younger one, because that happens only if the statement were true. Thus, he is marrying the elder daughter. □

Solution. **??** The solutions are fairly simple as shown above $\boxed{\text{Qxe2++ Bxd2++}}$ □

White moves

Black moves

Solution. **??** First, let denote A and H as in **??**. It is easy to see that AH is the hypotenus of a $45-45-90$ right triangle with side length 10. Therefore $AH = 10\sqrt{2}$. Furthermore,

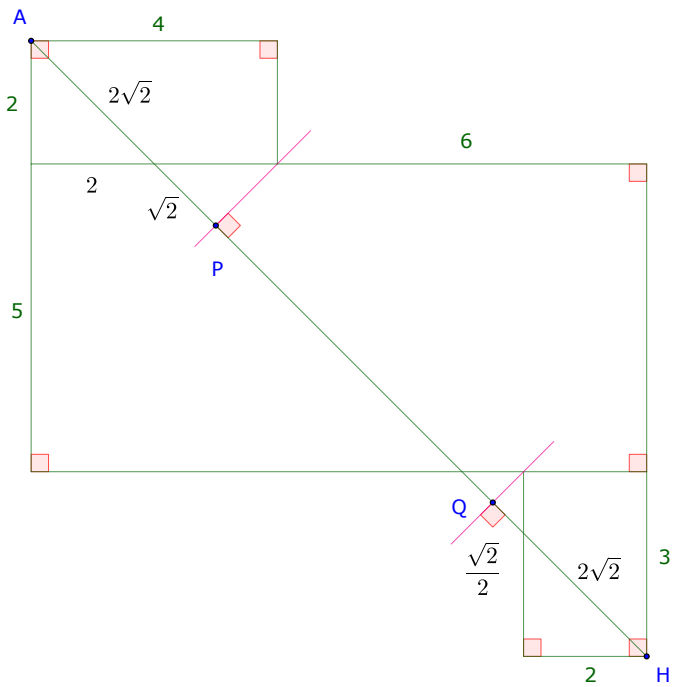


Figure 9.11: $AH = 10\sqrt{2}$

$$AP = 2\sqrt{2} + \frac{2}{\sqrt{2}} = 3\sqrt{2}, QH = 2\sqrt{2} + \frac{\sqrt{2}}{2} = \frac{5\sqrt{2}}{2} \Rightarrow PQ = 10\sqrt{2} - \frac{11\sqrt{2}}{2} = \boxed{\frac{9\sqrt{2}}{2}}$$

□

Solution. **??** Following is an example solution in Python.

```
def is_palindrome(n):
    s = str(n)
    l = len(s)
    return all(s[i] == s[l-1-i] for i in range(0, l // 2))

if __name__ == "__main__":
    with open('hc-2022-1-7-15-input.txt', 'rt') as i_file:
        n_list = [int(w.strip()) for w in i_file.readlines()]

        with open('hc-2022-1-7-15-output.txt', 'wt') as o_file:
            for n in n_list:
                counter = 0
                while True:
                    if not is_palindrome(n):
                        counter += 1
                        n = n + int(str(n)[::-1])
                    else:
                        o_file.write(f"{counter} {n}\n")
                        break
```

□

Chapter 10

PC R5

10.1 Topics

Algebra

1. Ratio.
2. Percent.
3. Fractions.
4. Sums and Products.
5. Equations. Substitution. Elimination.
6. Inequalities. Estimations. Testing.
7. Factoring $n(n+1)(n+2)(n+3)+1$.
8. Second degree identities such as $(a+b)^2 = a^2 + 2ab + b^2, a^2 - b^2 = (a-b)(a+b)$.
9. Sum of squares is non-negative. Equality case when sum of squares is zero.
10. Inequation with absolute value.

Combinatorics

1. Counting.
2. Complementary Counting.
3. Correction over Counting.
4. Permutation.
5. Combination.
6. Inclusion-Exclusion Principle.
7. Sets.
8. Probability. Complementaty Probability. Probability of Independently Repeated Events.

Geometry

1. Triangles. Similar Triangles
2. Square.
3. Circle.
4. Trapezoid.
5. Area. Ratio of areas of similar triangles.

6. Perimeter.
7. Cone. Sphere. Volume.

Number Theory

1. Divisibility. Leap years.
2. Prime factorizations.
3. Divisors. Number of divisors. Largest even/odd divisors.
4. Modular Arithmetic.
5. Perfect power. Sequence of remainders of powers of consecutive exponents, such as $7^1, 7^2, 7^3, \ldots$

Coding

1. Divisibility. Remainder.
2. Powers.
3. Square. Perfect Squares.

10.2 Rules

- **Middle school teams** are given **60 minutes** to work on **20 problems**, and **high school teams** are given **90 minutes** to work on **30 problems.**
- The teams may fill in answers to the contest problems during the prescribed time for their teams (60 minutes for middle school teams, 90 minutes for high school teams) following the start of their contest clock. At any time during the competition, contestants can submit answers to the problems. Answers may be submitted multiple times by the same team, but for each problem **only the last answer** received before the contest ends will be accepted and graded. Partial submissions are accumulated. Consequently, if a team submits an answer to problem 1 and later wises to submit an answer to problem 2, there is no need to resubmit the answer to problem 1. All members are allowed to work on the problems as a team. No help may be provided by persons not on their team. Team members must not reveal or discuss contest problems with anyone other than their immediate team members until the test ends.
- All problems are designed so that they can be worked without the aid of calculating devices. On the other hand, participants are *permitted to use calculators or computers*, for example, to write a computer program in order solve or better understand a problem. Team members may also *refer to books, computer programs, or other reference materials* available to them *locally* as long as the students seek this material without aid from persons not on their team. Any team with a member attempted to do *violate this rule* of the test *will be disqualified.*
- Team members *shall not* use computers or any other communication devices to *seek help* with problem solutions *from people or Internet resources* outside of their own team members.
- Grading:
  1. For a **correct answer** of *each problem*, the team is awarded with 1 point. For an *unanswered* problem or for an *incorrect answer*, the team is awarded with 0 point.
  2. The score of the team is the *sum of awarded points* for all the problems.

  The problems that have no answers or incorrect answers will be discussed after the test.

10.3 Problems

Problem 10.3.1 (PCT-2022-SM1-R5-P1)**.** Caden, Zoe, Noah, and Sophia shared a pizza. Caden ate 16 percent of the pizza. Zoe ate 50 percent more of the pizza than Caden ate. Noah ate 50 percent more of the pizza than Zoe ate, and Sophia ate the rest of the pizza.

Find the percentage of the pizza that Sophia ate.

Problem 10.3.2 (PCT-2022-SM1-R5-P2)**.** The **??** below was made by gluing together 5 non-overlapping congruent squares. The figure has area 80.

Find the perimeter of the figure.

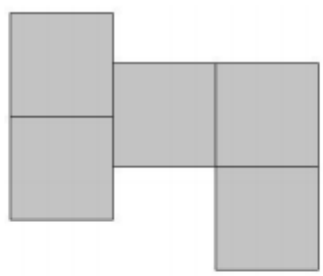


Figure 10.1: **??**

Problem 10.3.3 (PCT-2022-SM1-R5-P3)**.** The Festival of Eating a Lot happens every 39 years. If tit happens in 2022, what is the next leap year when it happens again?

Problem 10.3.4 (PCT-2022-SM1-R5-P4)**.** The following diagram includes six circles with radius 8, one circle with radius 16, and one circle with radius 32. The area of the shaded region is $k\pi$. Find k.

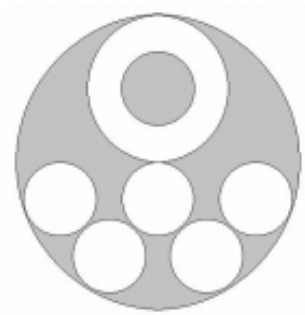


Figure 10.2: **??**

Problem 10.3.5 (PCT-2022-SM1-R5-P5)**.** Find the greatest odd divisor of 480^3.

Problem 10.3.6 (PCT-2022-SM1-R5-P6)**.** On a typical morning Aiden gets out of bed, goes through his morning preparation, rides the bus, and walks from the bus stop to work arriving at work 120 minutes after getting out of bed. One morning Aiden got out of bed late, so he rushed through his morning preparation getting onto the bus in half the usual time, the bus ride took 25 percent longer than usual, and he ran from the bus stop to work in half the usual time it takes him to walk arriving at work 96 minutes after he got out of bed. The next morning Aiden got out of bed extra early, leisurely went through his morning preparation taking 50 percent longer than usual to get onto the bus, his bus ride took 50 percent less time than usual, and he walked slowly from the bus stop to work taking 50 percent longer than usual.

How many minutes after Aiden got out of bed did he arrive at work that day?

Problem 10.3.7 (PCT-2022-SM1-R5-P7)**.** Find the number of positive integers less than 100 that are divisors of 600.

Problem 10.3.8 (PCT-2022-SM1-R5-P8)**.** The positive integer m is a multiple of 111, and the positive integer n is a multiple of 31. Their sum is 2022.

Find $n - m$.

Problem 10.3.9 (PCT-2022-SM1-R5-P9)**.** In $\triangle ADE$ points B and C are on side AD and points F and G are on side AE so that $BG \parallel CF \parallel DE$, as shown in **??**. The area of $\triangle ABG$ is 48, the area of trapezoid $CFED$ is 192, and $AB = CD$.

Find the area of trapezoid $BGFC$.

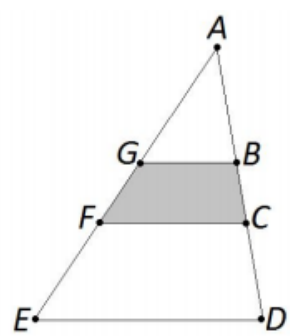


Figure 10.3: **??**

Problem 10.3.10 (PCT-2022-SM1-R5-P10)**.** Find the number of rearrangements of the letters in the word **SASKATOON** that begin and end with the same letter such as SAKATOONS.

Problem 10.3.11 (PCT-2022-SM1-R5-P11)**.** Find the sum of all prime divisors of $29! + 33!$.

Problem 10.3.12 (PCT-2022-SM1-R5-P12)**.** Let x, y, and z be real numbers such that

$$\begin{cases} 12x - 9y^2 = 7 \\ 6y - 9z^2 = -2 \\ 12z - 9x^2 = 4 \end{cases}$$

Find $3x^2 + 6y^2 + 9z^2$.

Problem 10.3.13 (PCT-2022-SM1-R5-P13)**.** Find the number of positive integer divisors of 45^{18} that are either perfect squares or perfect cubes.

Problem 10.3.14 (PCT-2022-SM1-R5-P14)**.** Let a and b be positive integers such that $a + ab = 1443$ and $ab + b = 1444$.

Find $a + 10b$.

Problem 10.3.15 (PCT-2022-SM1-R5-P15)**.** Find the remainder when 7^{7^7+7} is divided by 1000.

Problem 10.3.16 (PCT-2022-SM1-R5-P16)**.** The set of positive real numbers x that satisfy $2|x^2 - 9| \leq 9|x|$ is an interval $[m, M]$.

Find $10m + M$.

Problem 10.3.17 (PCT-2022-SM1-R5-P17)**.** Let a_0, a_1, ..., a_6 be real numbers such that $a_0 + a_1 + ... + a_6 = 1$ and

$$\begin{cases} a_0 + a_2 + a_3 + a_4 + a_5 + a_6 = \frac{1}{2} \\ a_0 + a_1 + a_3 + a_4 + a_5 + a_6 = \frac{2}{3} \\ a_0 + a_1 + a_2 + a_4 + a_5 + a_6 = \frac{7}{8} \\ a_0 + a_1 + a_2 + a_3 + a_5 + a_6 = \frac{29}{30} \\ a_0 + a_1 + a_2 + a_3 + a_4 + a_6 = \frac{143}{144} \\ a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = \frac{839}{840} \end{cases}$$

The value of a_0 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the difference $n - m$.

Problem 10.3.18 (PCT-2022-SM1-R5-P18)**.** Omar has four fair standard six-sided dice. Omar invented a game where he rolls his four dice over and over again until the number 1 does not appear on the top face of any of the dice. Omar wins the game if on that roll the top faces of his dice show at least one 2 and at least one 5. The probability that Omar wins the game is $\frac{m}{n}$, where m and n are relatively prime positive integers.

Find $m + n$.

Problem 10.3.19 (PCT-2022-SM1-R5-P19)**.** Find the sum of all values of the product ab, where (a, b) is an ordered pair of positive integers and $a^2 + \sqrt{2017 - b^2}$ is a perfect square.

Problem 10.3.20 (PCT-2022-SM1-R5-P20)**.** A right circular cone has a height equal to three times its base radius and has volume 1. The cone is inscribed inside a sphere as shown in **??**. The volume of the sphere is $\frac{m}{n}$, where m and n are relatively prime positive integers.

Find $m + n$.

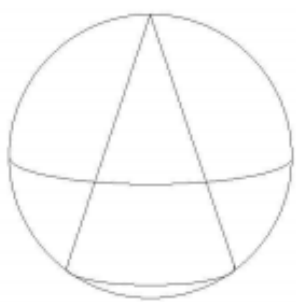


Figure 10.4: **??**

10.4 Grading

Answers for the problems.

Problem 1: 24
Problem 2: 48
Problem 3: 2256
Problem 4: 512
Problem 5: 3375
Problem 6: 78
Problem 7: 18
Problem 8: 24
Problem 9: 60
Problem 10: 3780
Problem 11: 1120
Problem 12: 6
Problem 13: 263
Problem 14: 417
Problem 15: 249
Problem 16: 33
Problem 17: 5039
Problem 18: 431
Problem 19: 266
Problem 20: 581

10.5 Solutions

Solution. **??** Since Caden ate 16 percent of the pizza, Zoe ate 50 percent more of the pizza than Caden ate, which means that Zoe ate $16 + 16 \cdot 0.5 = 24$ percent. Noah ate 50 percent more of the pizza than Zoe ate, means that Noah ate $24 + 24 \cdot 0.5 = 36$.

Sophia ate the rest of the pizza, which means she ate $100 - 36 - 24 - 16 = \boxed{24}$ percent. □

Solution. **??** First, the length of a side is $\sqrt{\frac{80}{5}} = 4$, Second, it is easy to see that there are $2 \cdot 4 = 8$ sides of

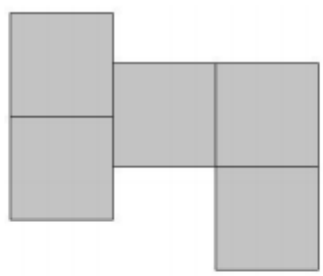


Figure 10.5: 4 sides not on perimeter

the squares are not on the perimeter. Thus, the length of the perimeter is $(5 \cdot 4 - 8) \cdot 4 = \boxed{48}$. □

Solution. [Solution 1] **??** A leap years is divisible by 4 but not 100, or it is divisible by 400. So, n is an integer such that $2022 + 39n \equiv 0 \pmod 4$. The least two values of n satisfy that

$$2022 + 39n \equiv 0 \pmod 4 \Rightarrow 22 - n \equiv 0 \pmod 4, \text{ are 2 and 6}$$

In those cases, $2022 + 39n$ can be 2100 and 2256. For both values of n, $2022 + 39n$ is not divisible by 400. Only $\boxed{2256}$ is not divisible by 100, thus it is a leap year. □

Solution. [Solution 2] **??**

```
n = 1
while True:
    y = 2022 + 39 * n
    if (y % 4 == 0 and y % 100 != 0) or (y % 400 == 0):
        print(y, n)
        break
    n += 1
```

The output is $\boxed{2256}$ 6. □

Solution. **??** The area of the whole is $32^2\pi$, the area of each small circle is $8^2\pi$, and the area of the ring is $16^2\pi - 8^2\pi$, so the shaded area would be equivelent to

$$32^2\pi - 5 \cdot (8^2\pi) - (16^2\pi - 8^2\pi) = (1024 - 4 \cdot 64 - 256)\pi = 512\pi \Rightarrow k = \boxed{512}$$

□

Solution. **??** Note that $480^3 = (48 \cdot 10)^3 = (32 \cdot 3 \cdot 5)^3 = 2^{15}15^3$. The largest odd divisior of 480^3 is $15^3 = \boxed{3375}$. □

Solution. **??** Let a be how long it takes him for his morning preparation. Let b be how long it takes him to ride the bus. Let c be how long it takes him to walk from the bus stop to work.

$$\begin{cases} a+b+c & = 120 \\ 0.5a+1.25b+0.5c & = 96 \end{cases}$$

Doubling both sides of the second equation and subtracting the first one from it, we get

$$(a+2.5b+c)-(a+b+c) = 192-120 \Rightarrow 1.5b = 72 \Rightarrow b = 48 \Rightarrow 0.5a+1.25\cdot 48+0.5c = 96 \Rightarrow 0.5a+0.5b = 36$$

Thus, $1.5a+0.5b+1.5c = 1.5\cdot 36+0.5\cdot 48 = \boxed{78}$. □

Solution. **??** Note that $600 = 6\cdot 100 = 2^3\cdot 3\cdot 5^2$. Thus the number of divisors of 600 is $(3+1)(1+1)(2+1) = 24$. Among these $600, 300, 200, 150, 120$, and 100 are the ones larger or equal to 100. Thus, the number of divisors less than 100 is $24-6 = \boxed{18}$. □

Solution. [Solution 2] **??**

```
count = 0
for n in range(1, 100):
    if 600 % n == 0:
        count += 1

print(count)
```

The output is $\boxed{18}$. □

Solution. **??** Let a and b be positive integers such that $m = 111a$, $n = 31b$,

$$111a+31b = 2022 \Rightarrow 2022-111a = 31b \Rightarrow 3(674-37a) = 31b \Rightarrow 3 \mid b$$

Let $b = 3c$, then $674 = 37a+31c$. Note that $674 \div 37 \approx 18.21$, thus first by testing if $a = 18$, then by reducing it by 1 each time,

$$\begin{aligned} 674 &= 37\cdot 18+8 \\ 674 &= 37\cdot 17+31\cdot 1+6\cdot 1+8 \\ 674 &= 37\cdot 16+31\cdot 2+6\cdot 2+8 \\ &\cdots \\ 674 &= 37\cdot 9+31\cdot 9+\underbrace{6\cdot 9+8}_{62} \end{aligned}$$

The remainder sequence $8, 14, 20, 26, 32, 38, 44, 50, 56, \underbrace{62}$ contains 62, which is divisible by 31. Therefore $a = 18-9 = 9, c = 11$, and $m = 111\cdot 9, n = 31\cdot 33$, thus $n-m = \boxed{24}$. □

Solution. [Solution 2] **??**

```
a = 1
while True:
    m = 111 * a
    if (2022 - m) % 31 == 0:
        print(m, 2022 - m, 2022 - 2*m)
    a += 1
    if m >= 2022:
        break
```

The output is 966 2013 $\boxed{24}$. □

Solution. **??** Let $BG = a$, $CF = b$, both heights of the triangle ABG (from A) and the trapezoid $CDEF$ be h. Let the height of the trapezoid $BCFG$ be x, see **??**. It is easy to see that $[CDEF] = [ABG] + bh$, thus

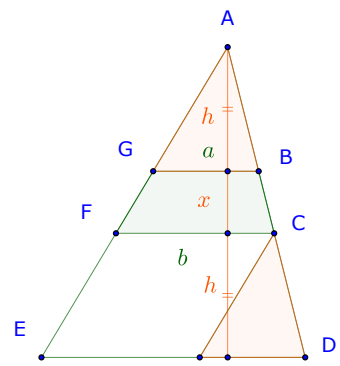


Figure 10.6: $BG = a, CF = b$

$bh = 192 - 48 = 144$. In addition $ah = 2 \cdot 48 = 96$, so $\frac{a}{b} = \frac{2}{3}$, Now,

$$\frac{h}{h+x} = \frac{a}{b} \Rightarrow \frac{h}{h+x} = \frac{2}{3} \Rightarrow x = \frac{h}{2}$$

Therefore $[BCFG] = \frac{(a+b)x}{2} = \frac{(a+b)h}{4} = \frac{192+48}{4} = \boxed{60}$. □

Proof. **??** First, the number of ways to choose a pair of same letters is 3: two letters S, A, or O. Once that chosen, there are $9 - 2 = 7$ remaining letters, with two pairs of same letters. There are $\frac{7!}{2!2!} = 1260$ ways to permutate them. Thus the number of ways is $3 \cdot 1260 = \boxed{3780}$. □

Proof. **??** First,

$$29! + 33! = 29!(1 + 30 \cdot 31 \cdot 32 \cdot 33)$$

Note that, let $n = 30$, then

$$n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n)(n^2 + 3n + 2) + 1 = 1 + (n^2 + 3n)^2 + 2(n^2 + 3n) + 1 = (n^2 + 3n + 1)^2$$

Therefore $29! + 33! = 29! \cdot (30^2 + 3 \cdot 30 + 1)^2 = 29! \cdot 991^2$. The sum of the prime factors of $29! + 33!$ is

$$2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 + 991 = \boxed{1120}$$

□

Proof. **??**

$$\begin{cases} 12x - 9y^2 = 7 \\ 6y - 9z^2 = -2 \\ 12z - 9x^2 = 4 \end{cases}$$

By adding the equations together $12x - 9x^2 + 6y - 9y^2 + 12z - 9z^2 = 9$. Note that by the identity $(a+b)^2 = a^2 + 2ab + b^2$,

$$\begin{aligned} 9x^2 - 12x &= (3x)^2 - 2(3x)2 + 2^2 - 2^2 = (3x-2)^2 - 4 \\ 9y^2 - 6y &= (3y)^2 - 2(3y)1 + 1^2 - 1^2 = (3y-1)^2 - 1 \\ 9z^2 - 12z &= (3z)^2 - 2(3z)2 + 2^2 - 2^2 = (3z-2)^2 - 4 \\ \Rightarrow 0 &= 12x - 9x^2 + 6y - 9y^2 + 12z - 9z^2 - 9 = -(3x-2)^2 - (3y-1)^2 - (3z-2)^2 \end{aligned}$$

This can be true if and only if $x = z = \frac{2}{3}, y = \frac{1}{3}$. Thus, $3x^2 + 6y^2 + 9z^2 = (3+9)\left(\frac{2}{3}\right)^2 + 6\left(\frac{1}{3}\right)^2 = \boxed{6}$. □

Proof. **??** Let $n = 45^{18} = (3^2)^{18}5^{18} = 3^{36}5^{18}$. If d divisor of n is a perfect square, then its prime factorizations contains an even power of 2 and an even power of 5,

$$d = 2^{2a}5^{2b}, \text{ where } 0 \le a \le 18, 0 \le b \le 9,$$

so there are $(19+1)(9+1) = 200$ such divisors.

Similarly if d is a perfect cube, then

$$d = 2^{3a}5^{3b}, \text{ where } 0 \le a \le 12, 0 \le b \le 6,$$

so there are $(12+1)(6+1) = 91$ such divisors.

Now if d can be both perfect square and perfect cube, then

$$d = 2^{6a}5^{6b}, \text{ where } 0 \le a \le 6, 0 \le b \le 3,$$

so there are $(6+1)(3+1) = 28$ such divisors.

In total, there are $200 + 91 - 28 = 263$ such divisors. □

Proof. **??**

$$\begin{cases} a + ab = 1443 \\ ab + b = 1444 \end{cases} \Rightarrow b = a + 1 \Rightarrow 1444 = b(a+1) = b^2 \Rightarrow b = 38 \Rightarrow a = 37$$

Thus, $a + 10b = 37 + 380 = \boxed{417}$. □

First solution. **??** Note that

$$\begin{aligned} &7^4 = 2401 \equiv 401 \pmod{1000} \Rightarrow 7^8 \equiv 401^2 \equiv 801 \pmod{1000} \Rightarrow 7^{16} \equiv 801^2 \equiv 601 \pmod{1000} \\ &\Rightarrow 7^{32} \equiv 601^2 \equiv 201 \pmod{1000} \Rightarrow 7^{64} \equiv 201^2 \equiv 401 \pmod{1000} \Rightarrow 7^{64} \equiv 7^4 \Rightarrow 7^{60} \equiv 1 \pmod{1000} \end{aligned}$$

Since $7^7 = 7^4 \cdot 7^3 = 2401 \cdot 343 \equiv 1 \cdot 43 \pmod{60}$, thus

$$7^{7^7+7} \equiv 7^{50} = 7^{32} \cdot 7^{16} \cdot 7^2 \equiv 201 \cdot 601 \cdot 49 \equiv \boxed{249} \pmod{1000}.$$

□

Proof. **??** *Case 1:* $x > 3$. Then

$$2|x^2-9| = 2(x^2-9), 9|x| = 9x \Rightarrow 2(x^2-9) \le 9x \Rightarrow \left(x+\frac{3}{2}\right)(x-6) \le 0 \Rightarrow -\frac{3}{2} \le x \le 6 \Rightarrow 3 < x \le 6$$

Case 2: $0 \le x \le 3$. Then

$$2|x^2-9| = 2(9-x^2), 9|x| = 9x \Rightarrow 2(9-x^2) \le 9x \Rightarrow \left(x-\frac{3}{2}\right)(x+6) \ge 0 \Rightarrow x \le -6 \text{ or } x \ge \frac{3}{2} \Rightarrow \frac{3}{2} \le x \le 3$$

Thus $[m, M] = \left[\frac{3}{2}, 6\right]$, therefore, $2m + 5M = 2 \cdot \frac{3}{2} + 5 \cdot 6 = \boxed{33}$. □

Proof. **??** Note that by summing up all equations, we obtain $6a_0$ and $5a_i$, $i = 1, 2, \ldots, 6$

$$\begin{cases} a_0+a_2+a_3+a_4+a_5+a_6 = \frac{1}{2} \\ a_0+a_1+a_3+a_4+a_5+a_6 = \frac{2}{3} \\ a_0+a_1+a_2+a_4+a_5+a_6 = \frac{7}{8} \\ a_0+a_1+a_2+a_3+a_5+a_6 = \frac{29}{30} \\ a_0+a_1+a_2+a_3+a_4+a_6 = \frac{143}{144} \\ a_0+a_1+a_2+a_3+a_4+a_5 = \frac{839}{840} \end{cases} \Rightarrow 6a_0 + 5(a_1 + \ldots + a_6) = \frac{25201}{5040}$$

Since $a_0 + a_1 + ... + a_6 = 1$, so $a_0 = \frac{25201}{5040} - 5 = \frac{1}{5040}$. Therefore $n - m = \boxed{5039}$. □

Proof. **??** The probability that he throws no 1 in a round is $\left(\frac{5^4}{6^4}\right)$. Thus, the probability that he throws at least one 1 is $p = 1 - \left(\frac{5^4}{6^4}\right)$.

The probability of the nth−round for Omar to wins the game when

1. When he throws at least one 1 in any of the previous $n-1$ round or p^{n-1}.
2. His dice show no 1 and at least one 2 and at least one 5.

In order to have no 1, the dice can only have $2, 3, \ldots, 6$.

Case 1: $2, 5$, and 2 different values from $\{3, 4, 6\}$. Then there are $\binom{3}{2} = 3$ ways to chose a pair of distinct values from $\{3, 4, 6\}$. Together with $2, 5$, there are 4! ways of permutations. In total $3 \cdot 24 = 72$ ways.

Case 1: $2, 5$, and 1 value from $\{3, 4, 6\}$. Then there are $\binom{3}{1} = 3$ ways to chose a value from $\{3, 4, 6\}$. $\binom{4}{2} = 6$ ways to chose positions for $2, 5$, and 2 ways to permutate them. In total $3 \cdot 6 \cdot 2 = 36$ ways.

Case 2: $2, 2, 5$, plus one value a from $\{3, 4, 6\}$ 3 ways to chose a from $\{3, 4, 6\}$, $\binom{4}{2} = 6$ ways to chose positions for $a, 5$, and 2 ways to permutate them. The duplicate $2, 2$ go into remaining positions. In total $3 \cdot 6 \cdot 2 = 36$ ways.

Case 3: $2, 5, 5$, plus one value a from $\{3, 4, 6\}$ 3 ways to chose a from $\{3, 4, 6\}$, $\binom{4}{2} = 6$ ways to chose positions for $a, 2$, and 2 ways to permutate them. The duplicate $5, 5$ go into remaining positions. In total $3 \cdot 6 \cdot 2 = 36$ ways.

Case 3: Only values from 2 and 5. Easy to see that there are $2^4 - 2 = 14$ such cases.

Thus the total number of ways is $72 + 36 + 36 + 36 + 14 = 194$, the probability for Omar to win at round n^{th} is,

$$\frac{194}{1296} p^{n-1}$$

Thus the probability for him to win at one of the rounds is

$$\frac{194}{1296} \cdot (1 + p + p^2 + \ldots) = \frac{194}{1296} \cdot \frac{1}{1-p} = \frac{194}{1296} \cdot \frac{6^4}{5^4} = \frac{194}{625}$$

Thus $n - m = 625 - 194 = \boxed{431}$. □

Counting the winning cases. **??** For counting the winning cases, we can use the below code.

```
s = []
count = 0
for a in range(1, 7):
    s.append(a)
    for b in range(1, 7):
        s.append(b)
        for c in range(1, 7):
            s.append(c)
            for d in range(1, 7):
                s.append(d)
                if 1 not in s and 2 in s and 5 in s:
                    count += 1
                    print(''.join('%s' % e for e in s))
                s.pop()
            s.pop()
        s.pop()
    s.pop()
print(count)
```

There are $\boxed{194}$ such cases. □

Proof. **??** If the sum $a^2 + \sqrt{2017 - b^2}$ is integral and a is integral, that means that $\sqrt{2017 - b^2}$ is integral. With testing, it is easy to find that $b = 44$ and $b = 9$ are the only solutions, meaning that $\sqrt{2017 - (44)^2} = 9$ or $\sqrt{2017 - 9^2} = 44$.

Case 1: $a^2 + 9 = c^2$.

$$(c - a)(c + a) = 9 \Rightarrow c = 5, a = 4$$

Case 2:$a^2 + 44 = c^2$

$$(c - a)(c + a) = 44 \Rightarrow c = 12, a = 10$$

Thus, there are two solutions for $(a, b) : (10, 9)$ and $(4, 44)$, The product ab are 90 and 176, for a total of $\boxed{266}$. □

Find b so $\sqrt{2017 - b^2} \in \mathbb{Z}+$. **??** To find $\sqrt{2017 - b^2}$ we can apply the following code,

```
from math import sqrt

for b in range(1, round(sqrt(2017))):
    c = round(sqrt(2017 - b**2))
    if c*c == 2017 - b**2:
        print(b)
```

There are two values $\boxed{b = 44 \text{ and } b = 9}$ □

Proof. **??** First, we determine the dimensions of the cone. We have that $h = 3r$ and

$$V = \frac{h \cdot r^2\pi}{3} \implies \frac{3r \cdot r^2\pi}{3} = 1 \implies r^3 = \frac{1}{\pi}.$$

Consider the cross-section of the cone that goes through both the center of the sphere, the apex of the cone, and is perpendicular to the base of the cone. Note that this also goes through the center of the cone's base. Let A and B be the vertexes formed from the cone's base in the cross-section, C be the apex of the cone, and O be the center of the sphere. Also let the sphere's radius be x. Then, note that $x = OA = OB = OC$ as they're all on the sphere, see **??**.

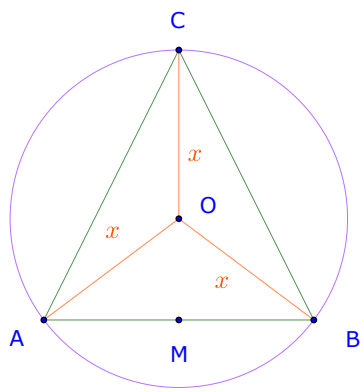


Figure 10.7: $x = OA = OB = OC$

This makes O the circumcenter of $\triangle ABC$. Additionally, $BC = AC = \sqrt{9r^2 + r^2} = \sqrt{10r^2} = r\sqrt{10}$ as they're the slant heights of the cone.

Also, letting the center of the base be M, we have that M is the midpoint of A and B since A and B are diametrically opposed. We also have $AB = 2r$ as they're diametrically opposed. Then, $MC = 3x$ is just the height of the cone. So, $[ABC] = \frac{3r \cdot 2r}{2} = 3r^2$.

By the formula for circumradius,

$$\frac{20r^3}{4x} = 3r^2 \implies \frac{20r}{3} = 4x \implies x = \frac{5r}{3}.$$

So, the volume of the sphere is

$$\frac{4x^3}{3} = \frac{4 \cdot \frac{5r}{3}^3 \cdot \pi}{3} = \frac{4 \cdot 125r^3 \cdot \pi}{81} = \frac{500}{81}$$

Therefore $n - m = 500 - 81 = \boxed{419}$. □

Chapter 11

PC R6

11.1 Topics

Algebra

1. $13^2 + 43^3 = 2018$. $35^2 + 28^2 + 3^2 = 2018$. $35 \cdot 28 + 35 \cdot 3 - 28 \cdot 3 = 1001$.
2. Fractions.
3. Ratio.
4. Average (mean) of a sequence of numbers.
5. Factoring.
6. Sums and Products.
7. Equation. System of Equations.
8. Inequality. Estimation.
9. Identities
$$\begin{aligned}
&ab - an - bm + mn = (a-m)(b-n)\\
&a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3\\
&a^3 + b^3 = (a+b)(a^2 - ab + b^2)\\
&8abc + 4ab + 4bc + 4ca + 2a + 2b + 2c + 1 = (2a+1)(2b+1)(2c+1),\\
&8abc - 4ab - 4ac + 2a - 4bc + 2b + 2c - 1 = (2a-1)(2b-1)(2c-1)\\
&x^3 + y^3 + z^3 - 3xzy = \frac{1}{2}(x+y+z)\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right]\\
&a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})
\end{aligned}$$
10. System of linear inequations. Maximizing linear combinations.
11. Polynomials. Cubic polynomials. Vieta's Theorem.
12. Sum of geometric sequence $1, r, r^2, \ldots$, where $|r| < 1 : 1 + r + r^2 + \ldots = \frac{1}{1-r}$.

Combinatorics

1. Counting, Correction over Counting. Complementary Counting.
2. Permutations.
3. Combinations.
4. Grids.

Geometry

1. Triangle. Sum of angles in a triangle.
2. Right triangles. Pythagorean Theorem. $30-60-90$ right triangles.
3. Angle Chasing.
4. Squares. Rectangles.
5. Cirles. Diameters. Angles subtends same arcs.
6. Trapezoids.
7. Area.
8. Trigonometry Identities
$$\begin{aligned}
&-1 \le \sin\alpha,\ \cos\alpha \le 1\\
&(\sin x)^2 + (\cos x)^2 = 1,\\
&\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},\\
&\sin\left(x + \frac{\pi}{2}\right) = \cos(x),\\
&\sin\alpha\cos\beta - \sin\beta\cos\alpha = \sin(\alpha-\beta).
\end{aligned}$$
9. Trigonometry Identities II.
$$\begin{aligned}
&\cos\beta = \frac{r}{\sqrt{r^2+s^2}}, \sin\beta = \frac{s}{\sqrt{r^2+s^2}}\\
&\Rightarrow \sin\alpha\cdot\frac{r}{\sqrt{r^2+s^2}} + \cos\alpha\frac{s}{\sqrt{r^2+s^2}} = \sin(\alpha+\beta)
\end{aligned}$$
10. Trigonometry Identities III.
$$\begin{aligned}
&\sin(x+a) = 1, \Rightarrow x = \frac{\pi}{2} - a.\\
&\cos(x+a) = 1, \Rightarrow x = -a.
\end{aligned}$$

Number Theory

1. Divisibility.
2. Perfect powers. Remainders of a sequence of powers $11^1, 11^2, \ldots, 11^n, \ldots$
3. Modular Arithmetic.
4. Number bases. Binary representation of a number. Counting digits of a binary number.
5. Fermat's little Theorem.

Coding

1. Division. Remainder. Remainder of a power when divided by a number.
2. Convert decimal into binary number.
3. Counting the number of digits 1 of a binary number.
4. Iterate all possible cases when throwing randomly four fair dice $(1-6)$.

Fact. *Let $A(a)$, $B(b)$, and $C(c)$ be the three complex numbers in the complex plane. A, B, and C are collinear if and only if*
$$a - c = \lambda(b-c),\ \textit{where } \lambda \in \mathbb{R}.$$

Fact. Any cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ $(a \neq 0)$ has at least one real root.

Theorem 11.1.1 (Angle Bisector Theorem)
For $\triangle ABC$, D on BC, AD is the internal angle bisector of $\angle BAC$, then

$$\frac{AB}{BD} = \frac{AC}{CD}.$$

Fact (Length of a Median in a Triangle)**.** In $\triangle ABC$, M is the midpoint of BC, then

$$AM = \frac{1}{2}\sqrt{2c^2 + 2b^2 - a^2}.$$

Theorem (Heron's formula)
In $\triangle ABC$, $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

Fact. If a sphere rests on one lateral edge of three hexagonal base pyramids, then the distance the center of the sphere to each of those edges is the same, thus the length of the projection of the vector $< x, y, z >$ onto each edge must be the same.

If the vectors pointing from the origin to the three apices of the pyramids, all have the same length. then the dot products of $< x, y, z >$ with each of these vectors must give the same value.

11.2 Rules

- The *total time* to complete the test is 90 minutes. The problems that have no answers or incorrect answers will be discussed after the test.
- **High school teams** are given **90 minutes** to work on **30 problems.**
- The teams may fill in answers to the contest problems during the prescribed time for their teams (60 minutes for middle school teams, 90 minutes for high school teams) following the start of their contest clock. At any time during the competition, contestants can submit answers to the problems. Answers may be submitted multiple times by the same team, but for each problem **only the last answer** received before the contest ends will be accepted and graded. Partial submissions are accumulated. Consequently, if a team submits an answer to problem 1 and later wises to submit an answer to problem 2, there is no need to resubmit the answer to problem 1. All members are allowed to work on the problems as a team. No help may be provided by persons not on their team. Team members must not reveal or discuss contest problems with anyone other than their immediate team members until the test ends.
- All problems are designed so that they can be worked without the aid of calculating devices. On the other hand, participants are *permitted to use calculators or computers*, for example, to write a computer program in order solve or better understand a problem. Team members may also *refer to books, computer programs, or other reference materials* available to them *locally* as long as the students seek this material without aid from persons not on their team. Any team with a member attempted to do *violate this rule* of the test *will be disqualified.*
- Team members *shall not* use computers or any other communication devices to *seek help* with problem solutions *from people or Internet resources* outside of their own team members.
- Grading:
  1. For a **correct answer** of *each problem*, the team is awarded with 1 point. For an *unanswered* problem or for an *incorrect answer*, the team is awarded with 0 point.
  2. The score of the team is the *sum of awarded points* for all the problems.

11.3 Problems

Problem 11.3.1 (PCT-2022-SM1-R6-P1)**.** Find the positive integer n such that

$$\frac{1}{2} \cdot \frac{3}{4} + \frac{5}{6} \cdot \frac{7}{8} + \frac{9}{10} \cdot \frac{11}{12} = \frac{n}{480}.$$

Problem 11.3.2 (PCT-2022-SM1-R6-P2)**.** In the **??**, a triangle with side lengths 18, 20, and 23 has a circle with radius 8 centered at each vertex.

Find n so that the total area inside the three circles but outside of the triangle is $n\pi$.

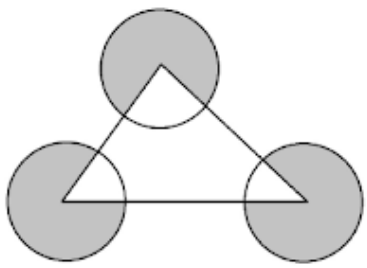


Figure 11.1: **??**

Problem 11.3.3 (PCT-2022-SM1-R6-P3)**.** Find x so that the arithmetic mean of $x, 3x, 1000$, and 3000 is 2022.

Problem 11.3.4 (PCT-2022-SM1-R6-P4)**.** The following diagram shows a grid of 49 cells.

Find the number of rectangles pictured in the diagram that contain at least three cells of the grid.

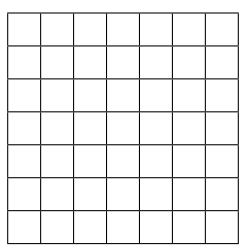


Figure 11.2: **??**

Problem 11.3.5 (PCT-2022-SM1-R6-P5)**.** One afternoon at the park there were twice as many dogs as there were people, and there were twice as many people as there were snakes. The sum of the number of eyes plus the number of legs on all of these dogs, people, and snakes was 544.

Find the number of dogs that were at the park.

Problem 11.3.6 (PCT-2022-SM1-R6-P6)**.** Triangle ABC has $AB = AC$. Point D is on side $\overline{BC}$ so that $AD = CD$ and $\angle BAD = 36°$.

Find the degree measure of $\angle ABC$.

Problem 11.3.7 (PCT-2022-SM1-R6-P7)**.** In 10 years the product of Melanie's age and Phil's age will be 400 more than it is now.

Find what the sum of Melanie's age and Phil's age will be 7 years from now.

Problem 11.3.8 (PCT-2022-SM1-R6-P8)**.** Let a and b be positive integers such that $2a - 9b + 18ab = 2024$.

Find $b - a$.

Problem 11.3.9 (PCT-2022-SM1-R6-P9)**.** Trapezoid has side lengths $10, 10, 10$, and 22. Each side of the trapezoid is the diameter of a semicircle with the two semicircles on the two parallel sides of the trapezoid facing outside the trapezoid and the other two semicircles facing inside the trapezoid as shown in **??**. The region bounded by these four semicircles has area $m + n\pi$, where m and n are positive integers.

Find the difference $m - n$.

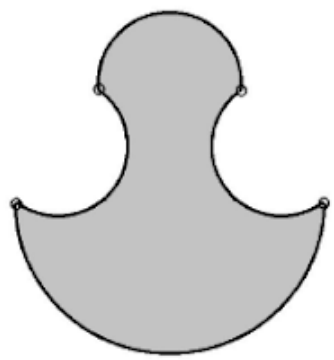


Figure 11.3: **??**

Problem 11.3.10 (PCT-2022-SM1-R6-P10)**.** Find the remainder when 11^{2022} is divided by 100.

Problem 11.3.11 (PCT-2022-SM1-R6-P11)**.** Find the number of positive integers $k \leq 13000$ for which there exist integers m and n so that $k = 2^m + 2^n$. For example, $64 = 2^5 + 2^5$, $65 = 2^0 + 2^6$, and $66 = 2^1 + 2^6$.

Problem 11.3.12 (PCT-2022-SM1-R6-P12)**.** A jeweler can get an alloy that is 40% gold for 200 dollars per ounce, an alloy that is 60% gold for 300 dollar per ounce, and an alloy that is 90% gold for 400 dollars per ounce. The jeweler will purchase some of these gold alloy products, melt them down, and combine them to get an alloy that is 50% gold.

Let $\frac{m}{n}$ be the ratio of the 40% gold to the 90% gold in the new 50% gold alloy, where m and n are relatively prime positive integers.

Find $m + n$.

Problem 11.3.13 (PCT-2022-SM1-R6-P13)**.** Five lighthouses are located, in order, at points A, B, C, D, and E along the shore of a circular lake with a diameter of 10 miles. Segments AD and BE are diameters of the circle. At night, when sitting at A, the lights from B, C, D, and E appear to be equally spaced along the horizon. The perimeter in miles of pentagon $ABCDE$ can be written $m + \sqrt{n}$, where m and n are positive integers.

Find $m + n$.

Problem 11.3.14 (PCT-2022-SM1-R6-P14)**.** A complex number z whose real and imaginary parts are *integers* satisfies

$$(Re(z))^4 + \left(Re(z^2)\right)^2 + |z|^4 = (2018)(81),$$

where $Re(w)$ and $Im(w)$ are the real and imaginary parts of w, respectively.

Find $|z|^2$.

Problem 11.3.15 (PCT-2022-SM1-R6-P15)**.** Let a and b be real numbers such that

$$\frac{1}{a^2} + \frac{3}{b^2} = 2018a \text{ and } \frac{3}{a^2} + \frac{1}{b^2} = 290b.$$

Then $\frac{ab}{b-a} = \frac{m}{n}$, where m and n are relatively prime positive integers.

Find the difference $n - m$.

Problem 11.3.16 (PCT-2022-SM1-R6-P16)**.** If you roll four standard, fair six-sided dice, the top faces of the dice can show just one value (for example, 3333), two values (for example, 2666), three values (for example, 5215), or four values (for example, 4236). The mean number of values that show is $\frac{m}{n}$, where m and n are relatively prime positive integers.

Find the difference $m - n$.

Problem 11.3.17 (PCT-2022-SM1-R6-P17)**.** One afternoon a bakery finds that it has 300 cups of flour and 300 cups of sugar on hand. Annie and Sam decide to use this to make and sell some batches of cookies and some cakes. Each batch of cookies will require 1 cup of flour and 3 cups of sugar. Each cake will require 2 cups of flour and 1 cup of sugar. Annie thinks that each batch of cookies should sell for 2 dollars and each cake for 1 dollar, but Sam thinks that each batch of cookies should sell for 1 dollar and each cake should sell for 3 dollars.

Find the sum of the maximum dollars of income they can receive if they use Sam's selling plan and the maximum dollars of income they can receive if they use Annie's selling plan.

Problem 11.3.18 (PCT-2022-SM1-R6-P18)**.** Find the positive integer k such that the roots of

$$x^3 - 15x^2 + kx - 845$$

are three distinct collinear points in the complex plane.

Problem 11.3.19 (PCT-2022-SM1-R6-P19)**.** Suppose that a and b are positive real numbers such that

$$3\log_{11}\left(\frac{1331 - a - b}{3ab}\right) = 3 - 2\log_{11}(ab).$$

Find $101 - \sqrt[3]{a} - \sqrt[3]{b}$.

Problem 11.3.20 (PCT-2022-SM1-R6-P20)**.** Aileen plays badminton where she and her opponent stand on opposite sides of a net and attempt to bat a birdie back and forth over the net. A player wins a point if their opponent fails to bat the birdie over the net. When Aileen is the server (the first player to try to hit the birdie over the net), she wins a point with probability $\frac{9}{10}$. Each time Aileen successfully bats the birdie over the net, her opponent, independent of all previous hits, returns the birdie with probability $\frac{3}{4}$. Each time Aileen bats the birdie, independent of all previous hits, she returns the birdie with probability $\frac{m}{n}$, where m and n are relatively prime positive integers.

Find $n - m$.

Problem 11.3.21 (PCT-2022-SM1-R6-P21)**.** Let x be in the interval $\left(0, \frac{\pi}{2}\right)$ such that $\sin x - \cos x = \frac{1}{2}$. Then $\sin^3 x + \cos^3 x = \frac{m\sqrt{p}}{n}$, where m, n, and p are relatively prime positive integers, and p is not divisible by the square of any prime.

Find the product mnp.

Problem 11.3.22 (PCT-2022-SM1-R6-P22)**.** Positive integers a and b satisfy

$$\begin{cases} a^3 + 32b + 2c = 2022 \\ b^3 + 32a + 2c = 1119. \end{cases}$$

Find $a^2 + b^2 + c^2$.

Problem 11.3.23 (PCT-2022-SM1-R6-P23)**.** Let a, b, and c be integers simultaneously satisfying the equations

$$\begin{cases} 4abc + a + b + c = 2018 \\ \quad ab + bc + ca = -507 \end{cases}$$

Find $2|a| + 3|b| + 5|c|$.

Problem 11.3.24 (PCT-2022-SM1-R6-P24)**.** Five girls and five boys randomly sit in ten seats that are equally spaced around a circle. The probability that there is at least one diameter of the circle with two girls sitting on opposite ends of the diameter is $\frac{m}{n}$, where m and n are relatively prime positive integers.

Find the difference $n - m$.

Problem 11.3.25 (PCT-2022-SM1-R6-P25)**.** If a and b are in the interval $\left(0, \frac{\pi}{2}\right)$ such that

$$13(\sin a + \sin b) + 43(\cos a + \cos b) = 2\sqrt{2018},$$

then $\tan a + \tan b = \frac{m}{n}$, where m and n are relatively prime positive integers.

Find the difference $n - m$.

Problem 11.3.26 (PCT-2022-SM1-R6-P26)**.** Let a, b, and c be real numbers.

$$\begin{cases} u = a^2 + b^2 + c^2 \\ v = 2ab + 2bc + 2ca \end{cases}$$

Suppose $2018u = 1001v + 256$. Find the maximum possible value of $35a - 28b - 3c$.

Problem 11.3.27 (PCT-2022-SM1-R6-P27)**.** Suppose $p < q < r < s$ are prime numbers such that

$$pqrs + 1 = 4^{p+q}.$$

Find $r + s$.

Problem 11.3.28 (PCT-2022-SM1-R6-P28)**.** In $\triangle ABC$ points D, E, and F lie on side $\overline{BC}$ such that $\overline{AD}$ is an angle bisector of $\angle BAC$, $\overline{AE}$ is a median, and $\overline{AF}$ is an altitude. Given that $AB = 154$ and $AC = 128$, and $9 \times DE = EF$, find the side length BC.

Problem 11.3.29 (PCT-2022-SM1-R6-P29)**.** Find the three-digit positive integer n for which

$$\binom{n}{3}\binom{n}{4}\binom{n}{5}\binom{n}{6}$$

is a perfect square.

Problem 11.3.30 (PCT-2022-SM1-R6-P30)**.** One right pyramid has a base that is a regular hexagon with side length 1, and the height of the pyramid is 8. Two other right pyramids have bases that are regular hexagons with side length 4, and the heights of those pyramids are both 7. The three pyramids sit on a plane so that their bases are adjacent to each other and meet at a single common vertex. A sphere with radius 2 rests above the plane supported by these three pyramids. The distance that the center of the sphere is from the plane can be written as $\frac{p\sqrt{q}}{r}$, where p, q, and r are relatively prime positive integers, and q is not divisible by the square of any prime.

Find $p + q + r$.

11.4 Grading

Answers for the problems.

Problem 1: 926
Problem 2: 160
Problem 3: 1022
Problem 4: 651
Problem 5: 64
Problem 6: 48
Problem 7: 44
Problem 8: 36
Problem 9: 80
Problem 10: 21
Problem 11: 105
Problem 12: 5
Problem 13: 55
Problem 14: 369
Problem 15: 11
Problem 16: 455
Problem 17: 690
Problem 18: 219
Problem 19: 90
Problem 20: 1
Problem 21: 560
Problem 22: 234
Problem 23: 185
Problem 24: 8
Problem 25: 17
Problem 26: 16
Problem 27: 282
Problem 28: 94
Problem 29: 489
Problem 30: 82

11.5 Solutions

Solution. **??** Note that

$$\frac{1}{2}\cdot\frac{3}{4}+\frac{5}{6}\cdot\frac{7}{8}+\frac{9}{10}\cdot\frac{11}{12}=\frac{463}{240}=\frac{926}{480}\Rightarrow n=\boxed{926}$$

□

Solution. **??** Since the sum of the internal angles of a triangle is $180°$, so the total area of the circles overlapped by the triangle makes half of a circle. Therefore, the shaded area is

$$3\cdot\pi\cdot r^2-\frac{1}{2}\pi\cdot r^2=\pi\frac{5r^2}{2}.$$

With $r=8$, $\pi\frac{5r^2}{2}=\pi\cdot 5\cdot 32=\boxed{160}\pi$. □

Solution. **??** If 2022 is the arithmetic mean of $x, 3x, 1000$, and 3000, then

$$2022=\frac{x+3x+1000+3000}{4}=x+1000\Rightarrow x=\boxed{1022}$$

□

Solution. **??** In order to choose a rectangle in the grid, we need to choose two rows and two columns. For example in **??**, rows $2,4$ and columns $2,5$ are choosen. A rectangle with less than 3 cells are the ones choosen

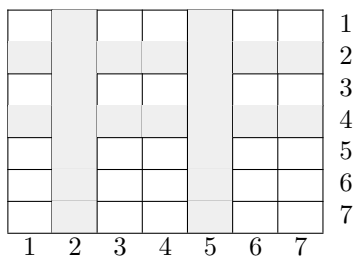


Figure 11.4: Rows $2,4$ and columns $2,5$

by:

1. the two rows are the same and the two columns are the same, or in $7\cdot 7=49$ ways,
2. the two rows differ by 1 and the two columns are the same, or in $6\cdot 7=42$ ways,
3. the two rows are the same and the two columns are the same, or in $7\cdot 6=42$ ways.

In total $49+42+42=133$ ways.

On the other hand, the total number of ways to choose any two rows is $\binom{7}{2}$ (for two different ones) plu 7 (for two same ones), or 28; and any two columns is also 21, therefore, the number of rectangles is $28^2=784$.

Thus the number of rectangles with at least three cells is $784-133=\boxed{651}$. □

Solution. **??** Let d, p, and s be the number of dog, people, and snakes in the park. It is known that $d=2p$, and $p=2s$, thus $d=2p=4s$. Furthermore,

$$2(d+p+s)+(4d+2p)=544\Rightarrow 2(4s+2s+s)+(16s+4s)=544\Rightarrow 34s=544\Rightarrow s=16\Rightarrow d=4s=\boxed{64}$$

□

Solution. **??** Since $AB = AC$ and $AD = DC$, so both $\triangle ABC$ and $\triangle ADC$ are isosceles. Let $x = \angle ABC = \angle ACB = \angle DAC$, see **??**.

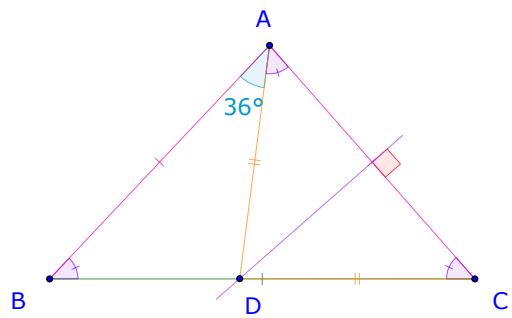


Figure 11.5: **??**

Now the sum of angles in $\triangle ABC$ is $180°$, thus

$$x + 36° + 2x = 180° \Rightarrow x = \boxed{48°.}$$

□

Solution. **??** Let m and p be the ages of Melanie and Phil, then

$$(m+10)(p+10) = mp + 400 \Rightarrow mp + 10(m+p) + 100 = mp + 400 \Rightarrow m + p = 30$$

Therefore in 7 years from now, the sum of their ages is $30 + 2 \cdot 7 = \boxed{44}$. □

Solution. **??** The key to the solution of this problem is factoring $18ab + 2a - 9b$,

$$18ab + 2a - 9b = \underbrace{(2a)(9b) + (2a)(1)}_{2a(9b+1)} + \underbrace{(9b)(-1) + (-1)}_{(-1)(9b+1)} - 1 \Rightarrow (2a-1)(9b+1) = 2023$$

Since $1, 7, 17, 119, 289, 2023$ are 6 divisors of 2023, and only 289 has a remainder of 1 when divided by 9,

$$\begin{cases} 2a - 1 = 7 \\ 9b + 1 = 289 \end{cases} \Rightarrow a = 4, b = 32 \Rightarrow a + b = \boxed{36}$$

□

Solution. **??** The area of the shaded region is equal to

1. the area of the trapezoid,
2. minus two half circles, radius 5, on the left and right sides,
3. plus one half circle, radius 5, on top,
4. and plus one half circle, radius 11, at the bottom.

The height of the trapezoid is equal to the leg of a right triangle with hypotenuse 10, another leg of length $\frac{22-10}{2} = 6$, which is $\sqrt{10^2 - 6^2} = 8$.

Therefore the area of the shaded region is

$$\frac{8 \cdot (10+22)}{2} - 2\frac{\pi \cdot 5^2}{2} + \frac{\pi \cdot 5^2}{2} + \frac{\pi \cdot 11^2}{2} = 128 + 48\pi \Rightarrow m = 128, n = 48 \Rightarrow m - n = \boxed{80}$$

□

Solution. [First solution] **??** Let look at the remainder of $11^1, 11^2, 11^3, \ldots$ when divided by 100.

$$\begin{aligned} 11^1 &\equiv 11 \pmod{100} \\ 11^2 \equiv 11 \cdot 11 = 121 &\equiv 21 \pmod{100} \\ 11^3 \equiv 21 \cdot 11 = 231 &\equiv 31 \pmod{100} \\ &\cdots \\ 11^{10} \equiv 91 \cdot 11 = 1001 &\equiv 01 \pmod{100} \end{aligned}$$

Therefore $11^{2022} \equiv 11^2 \equiv \boxed{21} \pmod{100}$. □

Solution. [Second solution] **??**

$$\begin{aligned} &11^{20} = (11^2)^{10} \equiv (21)^{10} \equiv (41)^5 \equiv 41 \cdot (81)^2 \equiv 41 \cdot 61 \equiv 1 \pmod{100} \\ &\Rightarrow 11^{2022} = 11^{20 \cdot 101} \cdot 11^2 \equiv \boxed{21} \pmod{100} \end{aligned}$$

□

Solution. [Third solution] **??**

```
n = 1
for m in range(1, 2023):
    n = n*11 % 100
print(n)
```

The output is $\boxed{21}$. □

Solution. [First solution] **??** If $k = 2^m + 2^n$, where m, n are non-negative integers then

1. if $m = n$, $k = 2^{m+1} = \overline{1\underbrace{00\ldots0}_{m+1}}_2$, where $m + 1 \geq 0$.
2. if $m > n$, $k = 2^m + 2^n = \overline{1\underbrace{00\ldots0}_{m-n-1}1\underbrace{00\ldots0}_{n}}_2$

This would mean that k is a number within 1 and 13000 having at most two digits of 1 in its binary representation. There are 14 numbers having exactly one digit of 1 in its binary representation, with the power at least 1.

$$1 = \overline{1}_2,\ 2 = \overline{10}_2, \ldots,\ 8192 = \overline{1\underbrace{00\ldots0}_{13}}_2.$$

The largest number less than or equal to 13000 and having two digits of 1 in its binary representation is

$$12288 = 2^{13} + 2^{12} = \overline{11\underbrace{00\ldots0}_{12}}_2.$$

Thus, in 14 positions there are $\binom{14}{2}$ ways to chose 2 positions for the digit 1, resulting in 91 ways, or 91 such numbers. Therefore in total, there are $14 + 91 = \boxed{105}$ such numbers. □

Solution. [Second solution] **??** Similar to the solution above, k is a number within 1 and 13000 having at most two digits of 1 in its binary representation. The code below convert a decimal number between 1 and 13000 and verify the number of digits 1.

```
counter = 0

for n in range(1, 13001):
    b = bin(n)                      # convert n from decimal into binary format
    x = b[2:]                       # chop the unsuable '0b' prefix
    sod = sum(int(i) for i in x)    # sum of digits
    if sod in [1, 2]:
        counter += 1
        print(counter, n)

print(counter)
```

The output is $\boxed{105}$. □

Solution. [Third solution] **??** The code below counts directly sum of powers and compare with the upper limit. The exponents runs from -1 since $2^{(}-1)+2^{(}-1)=1$.

```
from math import floor, log2

m = floor(log2(13000)+1)
s = set()

for i in range(-1, m):
    for j in range(-1, m):
        k = 2**i + 2**j
        if k == int(k) and k < 13001:
            s.add(k)

print(len(s))
```

The output is $\boxed{105}$. □

Solution. **??** Assume that the jeweler need x fraction of an alloy of the 40% gold, y fraction of an alloy of the 60% gold, and z fraction of an alloy of the 90% gold, to make an alloy of 50% gold. Thus where $x \geq 0, y \geq 0, z \geq 0$, $x+y+z=1$, and $0.4x+0.6y+0.9z=0.5$.

We need to find x, y, z such that $200x+300y+400z$ is minimal. The key to the solution is to

1. reduce the number of variables in the expression $200x+300y+400z$ using the fact that $x+y+z=1$ and $0.4x+0.6y+0.9z=0.5$,
2. then estimate the minimal value based on the fact that $x \geq 0, y \geq 0, z \geq 0$.

$$\begin{cases} 0.4x+0.6y+0.9z=0.5 \Rightarrow 0.4x+0.6y+0.9-0.9x-0.9y=0.5 \Rightarrow 0.4=0.5x+0.3y \Rightarrow 5x+3y=4 \\ 200x+300y+400z=200x+300y+400(1-x-y)=400-200x-100y \end{cases}$$

$$\Rightarrow 400-200x-100y=400-40(50x+30y)+20y=240+2y$$

Since $y \geq 0$, so $240+2y \geq 240$. The minimal value of $200x+300y+400z$ is reached if

$$y=0,\ 5x+3y=4 \Rightarrow x=\frac{4}{5}, z=1-x-y=\frac{1}{5} \Rightarrow \frac{x}{z}=\frac{1}{4} \Rightarrow m+n=\boxed{5}.$$

□

Solution. **??** Since, when sitting at A, the lights from B, C, D, and E appear to be equally spaced along the horizon, so $\angle BAC = \angle CAD = \angle DAE$, see **??**.

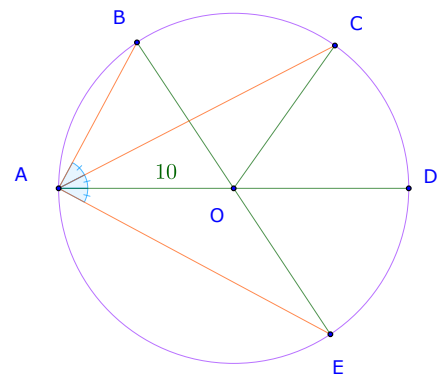


Figure 11.6: $\angle BAC = \angle CAD = \angle DAE$

Therefore $\angle BAC = \angle CAD = \angle DAE = \frac{\angle BAE}{3} = 30°$. Thus, $\triangle ABD, \triangle ACD, \triangle ADE$ are $30 - 60 - 90$ triangles, thus

$$AB = CD = DE = \frac{AD}{2}, BC = \frac{AD}{2}, \text{ and } AE = AD\sqrt{3}2$$

The perimeter of $ABCDE$ is $20 + 5\sqrt{3} = 20 + \sqrt{75}$, thus $m = 20, n = 75$, and $n - m = \boxed{55}$. □

Solution. **??** Let $z = a + bi$, where $a, b \in \mathbb{R}, i = \sqrt{-1}$. Then $a = Re(z), b = Im(z)$, and

$$\begin{cases} z^2 = a^2 - b^2 - 2abi \Rightarrow Re(z^2) = a^2 - b^2 \\ |z| = \sqrt{a^2 + b^2} \Rightarrow |z|^4 = (a^2 + b^2)^2 \end{cases}$$
$$\Rightarrow (Re(z))^4 + \left(Re(z^2)\right)^2 + |z|^4 = a^4 + (a^2 - b^2)^2 + (a^2 + b^2)^2 = 3a^4 + 2b^4$$
$$\Rightarrow 3a^4 + 2b^4 = (2018)(81) \Rightarrow 3 \mid b \Rightarrow 3 \mid a$$
$$3a_1^4 + 2b_1^4 = 2018, \text{ where } a = 3a_1, b = 3b_1$$

It is easy to test that the only possible solutions are $(a_1, b_1) = (\pm 4, \pm 5)$. Thus $|z|^2 = a^2 + b^2 = (3a_1)^2 + (3b_1)^2 = 144 + 225 = \boxed{369}$. □

Solution. **??**

$$\begin{cases} \frac{1}{a^2} + \frac{3}{b^2} = 2018a \\ \frac{3}{a^2} + \frac{1}{b^2} = 290b \end{cases} \Rightarrow \begin{cases} \frac{1}{a^3} + \frac{1}{a}\frac{3}{b^2} = 2018 \\ \frac{1}{b^3} + \frac{3}{a^2}\frac{1}{b} = 290 \end{cases} \Rightarrow \frac{1}{a^3} - 3\frac{1}{a^2}\frac{1}{b} + 3\frac{1}{a}\frac{1}{b^2} - \frac{1}{b^3} = 1728$$
$$\Rightarrow \left(\frac{1}{a} - \frac{1}{b}\right)^3 = 12^3 \Rightarrow \frac{b-a}{ab} = 12 \Rightarrow \frac{ab}{b-a} = \frac{1}{12} \Rightarrow m = 1, n = 12 \Rightarrow n - m = \boxed{11}$$

□

Solution. [First solution] **??** The code below counts all number of values for every case when throwing 4 fair dice.

```
sum_of_values = 0
s = []
for a in range(1, 7):
    s.append(a)
    for b in range(1, 7):
        s.append(b)
        for c in range(1, 7):
            s.append(c)
            for d in range(1, 7):
                s.append(d)
                sum_of_values += len(set(s))
                s.pop()
            s.pop()
        s.pop()
    s.pop()
print(sum_of_values)
```

The output is 4026, thus the mean value is $\frac{4026}{6^4} = \frac{671}{216}$, so $m = 671, n = 126$, and $m - n = \boxed{455}$. □

Solution. [Second solution] **??**

Case 1: One value. There is 6 ways to choose a value from $1, 2, \ldots, 6$,

$$a, a, a, a.$$

For each value there is a single ways for all four dice to have the same value. Thus the sum of all values is $6 \cdot 1 = 6$.

Case 2: Two values, two pairs. There is $\binom{6}{2} = 15$ ways to choose two pairs of different values from $1, 2, \ldots, 6$,

$$a, a, b, b.$$

For each pair there is a $\binom{4}{2} = 6$ to choose two dice to have the first value, 1 way for the remaining value. Thus the sum of all values is $15 \cdot 6 \cdot 2 = 180$.

Case 3: Two values, one single, one triple. There is $\binom{6}{2} = 15$ ways to choose two pairs of different values from $1, 2, \ldots, 6$. Two ways to split the two values to from a single and a triple,

$$a, a, a, b.$$

For each pair there is a $\binom{4}{3} = 4$ to choose three dice to have the first value, 1 way for the remaining value. Thus the sum of all values is $15 \cdot 4 \cdot 2 \cdot 2 = 240$.

Case 4: Three values. There is $\binom{6}{3} = 20$ ways to choose a triple of different values from $1, 2, \ldots, 6$. There are 3 ways to chose a value for the remaining dice, thus total 60 triples,

$$a, a, b, c.$$

There are $\binom{4}{2} = 6$ ways to choose two dice to have the two different values, 1 way for the remaining same values. There are 2 ways to permute the two different values. Thus the sum of all values is $20 \cdot 3 \cdot 6 \cdot 2 \cdot 3 = 2160$.

Case 5: Four values. There is $\binom{6}{4} = 15$ ways to choose a quadruple of different values from $1, 2, \ldots, 6$.

$$a, b, c, d.$$

For each quadruple there are 4! ways to permute the dice. Thus the sum of all values is $15 \cdot 4! \cdot 4 = 1440$.

Thus, the mean of the values is $\frac{6+180+240+2160+1440}{6^4} = \frac{671}{216}$, so $m = 671, n = 126$, and $m - n = \boxed{455}$. □

Solution. **??** Let $x \geq 0$ and $y \geq 0$ be the number of cookie and cake batches, respectively. The amount of flour and sugar cups required is $x + 2y$ and $3x + y$, respectively. Each of these quantities is less than or equal to 300, therefore the following system of inequations:

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x + 2y \leq 300 \\ 3x + y \leq 300 \end{cases}$$

The four conditions in the **??** establish the shaded region $ABCD$, where $A(0, 150)$, $B(60, 120)$, $C(100, 0)$, and $D(0, 0)$.

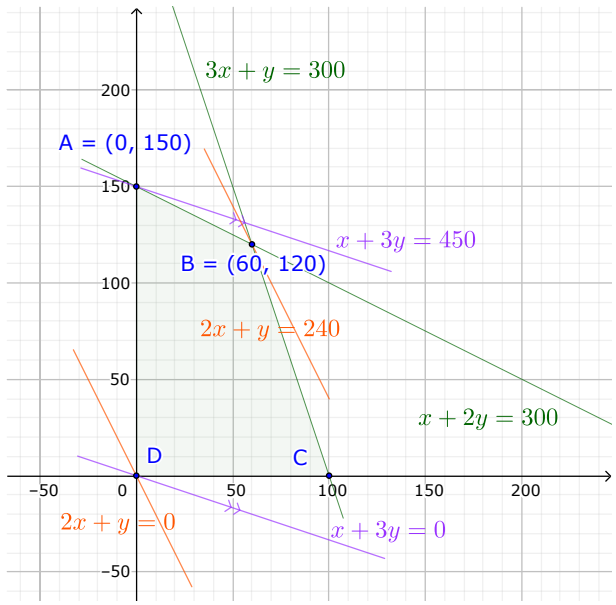


Figure 11.7: $\max\{2x+y\} = 240,\ \max\{x+3y\} = 450$

We need to maximize the amount of dollars Annie and Sam can receive, which are $2x + y$ and $x + 3y$, respectively, where (x, y) are within $ABCD$.

First, it is easy to draw lines $2x + y = 0$ and $3x + y = 0$ through $D(0, 0)$. $2x + y = a$ and $x + 3y = b$ values can be maximized if we can draw lines parallel with $2x + y = 0$ and $3x + y = 0$ that intersect $ABCD$ while maximizes a and b. Second, lines through the vertices of $ABCD$ and parallel to $2x + y = 0$ and $3x + y = 0$ shall maximize values of a and b.

Thus, line $2x + y = 240$ through $B(60, 120)$ maximizes a, and line $x + 3y = 450$ through $A(0, 150)$ maximizes b. Therefore $a + b = 450 + 240 = \boxed{690.}$ □

Solution. **??** Let $P(x) = x^3 - 15x^2 + kx - 1105$. It is know that $P(x)$ has at least one real root and let that be $r \in \mathbb{R}$. Let $a_1 + ib_1$ and $a_2 + ib_2$ be the other two roots of $P(x)$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$, and $i = \sqrt{-1}$.

By Vieta's Theorem,

$$\begin{cases} r + (a_1 + ib_1) + (a_2 + ib2) = 15 \\ r(a_1 + ib_1) + r(a_2 + ib2) + (a_1 + ib_1)(a_2 + ib2) = k \\ r(a_1 + ib_1)(a_2 + ib2) = 845 \end{cases}$$

Since $15 = r + (a_1 + ib_1) + (a_2 + ib2) = (r + a_1 + a_2) + i(b_1 + b_2)$, the left side 15 is a real number, thus the imaginary part of the right side $i(b_1 + b_2)$ should be 0, so $b_1 + b_2 = 0$, or $b_2 = -b_1$.

Furthermore $\frac{1105}{r} = (a_1 + ib_1)(a_2 + ib2) = (a_1a_2 - b_1b_2) + i(a_2b_1 + a_1b_2)$. Similarly $a_2b_1 + a_1b_2 = 0$, or $(a_2 - a_1)b_1 = 0$, thus $a_1 = a_2$, or $b_1 = 0$.

Case 1: $a_1 = a_2$ and $b_2 = -b_1$. Let $a = a_1 = a_2$, $b = b_1 = -b_2$, then the three roots are r, $a + ib$, and $a - ib$. Since they are collinear, so there exists $\lambda \in \mathbb{R}$,

$$r - (a + ib) = \lambda((a + ib) - (a - ib)) \Rightarrow r - a - ib = 2\lambda ib \Rightarrow r - a = (1 + 2\lambda)ib$$
$$\Rightarrow \begin{cases} r - a = 0,\ b = 0,\ \text{or} \\ r - a = 0,\ b \neq 0,\ 1 + 2\lambda = 0 \end{cases}$$

Case 1: if $r = a$ and $b = 0$, then alls three roots are r. Thus $r = 5$, but $r^3 \neq 1105$. So no solution for this case.

Case 1: if $r = a$ and $b \neq 0$, $\lambda = -\frac{1}{2}$. Then alls three roots are $r, r+ib$, and $r-ib$. $r+(r+ib)+(r-ib) = 15$, so $r = 5$. $r(r^2 + b^2) = 845$, so $b^2 = 144$, or $b = 12$ (the case $b = -12$ leads to the permutation of the two roots $r + ib$, and $r - ib$.) Therefore,

$$k = r(r + ib) + r(r - ib) + (r + ib)(r - ib) = 2r^2 + r^2 + b^2 = 3r^2 + b^2 = 75 + 144 = \boxed{219}$$

It is easy to verify that $x^3 - 15x^2 + 219x - 845 = (x - 5)(x - (5 + 12i))(x - (5 - 12i))$. □

Solution. **??**

$$3\log_{11}\left(\frac{1331 - a - b}{3ab}\right) + 2\log_{11}(ab) = \log_{11}\left(\frac{1331 - a - b}{3ab}\right)^3 \cdot (ab)^2 = \log_{11}\left(\frac{(1331 - a - b)^3}{3^3ab}\right)$$
$$\Rightarrow \frac{(1331 - a - b)^3}{3^3ab} = 11^3 \Rightarrow 1331 - a - b = 11 \cdot 3 \cdot \sqrt[3]{ab}$$

Thus the given equation becomes

$$1331 - a - b - 3 \cdot 11 \cdot \sqrt[3]{ab} = 0 \Rightarrow 11^3 + (-\sqrt[3]{a})^3 + (-\sqrt[3]{b})^3 - 3 \cdot 11(-\sqrt[3]{a})(-\sqrt[3]{b}) = 0$$

By with $x = 11, y = -\sqrt[3]{a}, z = -\sqrt[3]{b}$, applying the identity

$$x^3 + y^3 + z^3 - 3xzy = \frac{1}{2}(x + y + z)\left[(x - y)^2 + (y - z)^2 + (z - x)^2\right]$$
$$x^3 + y^3 + z^3 - 3xzy = 0 \Rightarrow (x + y + z)\left[(x - y)^2 + (y - z)^2 + (z - x)^2\right] = 0$$

Since $a, b > 0$, so $x - y = 11 + \sqrt[3]{a} > 0$, $x - z = 11 + \sqrt[3]{b} > 0$, so $(x - y)^2 + (y - z)^2 + (z - x)^2 > 0$, thus $x + y + z = 0$, or $11 - \sqrt[3]{a} - \sqrt[3]{b} = 0$. Therefore $101 - \sqrt[3]{a} - \sqrt[3]{b} = 101 - 11 = \boxed{90}$. □

Solution. **??** Assume that Aileen wins in $2k$ ($k \geq 1$) bats: she bats k times including the serve, her opponents bats k times, and her opponent fails to bat the birdie over the net in the kth attempt. Thus the probability for the $2k$ bats is,

$$\left(\frac{m}{n}\right)^k \cdot \left(\frac{3}{4}\right)^{k-1} \frac{1}{4} = \frac{m}{4n}\left(\frac{3m}{4n}\right)^{k-1}$$

Therefore, the total probability is

$$\frac{9}{10} = \frac{m}{4n}\sum_{k=1}^{+\infty}\left[\left(\frac{3m}{4n}\right)^{k-1}\right] \Rightarrow \frac{9}{10} = \frac{m}{4n}\frac{1}{1-\frac{3m}{4n}} = \frac{m}{4n-3m}$$
$$\Rightarrow 9(4n-3m) = 10m \Rightarrow 37m = 36n \Rightarrow \frac{m}{n} = \frac{36}{37} \Rightarrow n - m = \boxed{1}$$

□

Solution. **??**

$$\begin{aligned}
\sin x - \cos x =& \sqrt{2}\left(\sin x \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \cos x\right) \\
=& \sqrt{2}\left(\sin x \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos x\right) \quad (\text{because } \sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}) \\
=& \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) \quad (\text{because } \sin\alpha\cos\beta - \sin\beta\cos\alpha = \sin(\alpha-\beta)) \\
\sin x - \cos x =& \frac{1}{2} \Rightarrow \sin\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{4} \Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \sqrt{1 - \left(\frac{\sqrt{2}}{4}\right)^2} = \sqrt{\frac{7}{8}}
\end{aligned}$$

In addition, and note that $(\sin x)^2 + (\cos x)^2 = 1$,

$$(\sin x - \cos x)^2 = \left(\frac{1}{2}\right)^2 \Rightarrow (\sin x)^2 - 2\sin x\cos x + (\cos x)^2 = \frac{1}{4} \Rightarrow \sin x \cos x = \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}$$

Similar as above, and note that $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$,

$$\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{7}}{2}$$

Therefore,

$$(\sin x)^3 + (\cos x)^3 = (\sin x + \cos x)\left[(\sin x)^2 - \sin x\cos x + (\cos x)^2\right] = \frac{\sqrt{7}}{2} \cdot \left(1 - \frac{3}{8}\right) = \frac{5\sqrt{7}}{16}$$

Thus $m = 5, p = 7$, and $n = 16$, so $m \cdot n \cdot p = \boxed{560}$. □

Solution. **??**

$$\begin{cases} a^3 + 32b + 2c = 2022 \\ b^3 + 32a + 2c = 1119. \end{cases} \Rightarrow a^3 - b^3 - 32(a-b) = 903 \Rightarrow (a-b)((a-b)^2 + 3ab - 32) = 3 \cdot 7 \cdot 43$$

Note that any perfect square has remainder 0 or 1 when divided by 3, thus $(a-b)^2 + 3ab - 32$ is not divisible by 3. Therefore $a - b$ is divisible by 3. If $a - b \geq 3 \cdot 7 = 21$, then $(a-b)^2 + 32ab - 32 \geq 441 - 32 > 43$.

Thus the only possible factoring is $a - b = 3, (a-b)^2 + 3ab - 32 = 7 \cdot 43$. Then $ab = \frac{7 \cdot 43 + 32 - 9}{3} = 108 = 9 \cdot 12$. Thus $(a.b) = (12, 9)$.

Therefore $c = 3$. Thus $a^2 + b^2 + c^2 = 144 + 81 + 9 = \boxed{234.}$ □

Solution. **??** Since $2(4abc + a + b + c) - 4(ab + bc + ca) - 1 = (2a - 1)(2b - 1)(2c - 1)$,

$$(2a - 1)(2b - 1)(2c - 1) = 2 \cdot 2018 + 4 \cdot 507 - 1 = 6063 = 3 \cdot 43 \cdot 47$$

Note that not all of a, b, or c can be positive since $ab + bc + ca < 0$. Because $(2a - 1)(2b - 1)(2c - 1) > 0$, so two of $2a - 1, 2b - 1, 2c - 1$ have to be negative and one of them is positive.

If $2a - 1 = 3, 2b - 1 = -43,\ 2c - 1 = -47 \Rightarrow a = 2, b = -21, c = -23$, then $ab + bc + ca = 395$.

If $2a - 1 = -3, 2b - 1 = 43,\ 2c - 1 = -47 \Rightarrow a = -1, b = 22, c = -23$, then $ab + bc + ca = -505$.

If $2a - 1 = -3, 2b - 1 = -43,\ 2c - 1 = 47 \Rightarrow a = -1, b = -21, c = 24$, then $ab + bc + ca = -507$.

Thus $2|a| + 3|b| + 5|c| = 2|1| + 3|21| + 5|24| = \boxed{185.}$ □

Solution. **??** Let the girls be indistinguishable and the boys be indistinguishable. There are 2^5 ways to place the people on the circle such that no girl is sitting across another girl (by dividing the seats into 5 pairs of opposite seats, in each pair a girl can sit in one of the seats). There are $\binom{10}{5} = 252$ ways to randomly place the people (choose any 5 seats for 5 girls.) Thus, the probability to place the girls, such that one or more girls are sitting on opposite ends, is

$$1 - \frac{2^5}{252} = 1 - \frac{8}{63} = \frac{55}{63} \Rightarrow m = 55, n = 63, n - m = 63 - 55 = \boxed{8}.$$

□

Solution. **??** Note that $13^2 + 43^2 = 2018$, let $\sin\alpha = \frac{43}{\sqrt{2018}}$, then $\cos\alpha = \frac{13}{\sqrt{2018}}$,

$$13(\sin a + \sin b) + 43(\cos a + \cos b) = 2\sqrt{2018} \Rightarrow (\sin a\cos\alpha + \sin\alpha\cos a) + (\sin b\cos\alpha + \sin\alpha\cos b) = 2$$
$$\Rightarrow \sin(a + \alpha) + \sin(b + \alpha) = 2 \Rightarrow \sin(a + \alpha) = \sin(b + \alpha) = 1 \Rightarrow a = b = \frac{\pi}{2} - \alpha$$
$$\Rightarrow \sin a = \sin b = \cos\alpha,\ \cos a = \cos b = \sin\alpha \Rightarrow \tan a + \tan b = 2\cot\alpha = 2\frac{13}{43}$$
$$\Rightarrow m = 26, n = 43 \Rightarrow n - m = \boxed{17}$$

□

Solution. **??** Note that,

$$\begin{aligned}&(35a - 28b - 3c)^2 + (35b - 28c - 3a)^2 + (35c - 28a - 3b)^2\\ &= 2018a^2 - 2002ab - 2002ac + 2018b^2 - 2002bc + 2018c^2\\ &= 2018u - 1001v = 256 \Rightarrow 35a - 28b - 3c \le \sqrt{1024} = \boxed{16}\end{aligned}$$

□

Solution. **??** Note that p, q, r, s are all odd, otherwise $pqrs + 1$ is an odd number. Since $4^{p+q} - 1 = (4 - 1)(4^{p+q-1} + \ldots + 1)$,, so $3 \mid 4^{p+q} - 1$, thus $p = 3$.

Now, $4^{p+q} - 1 = 3(4^{q+2} + 4^{q+1} + \ldots + 4 + 1)$, therefore

$$q \mid 4^{q+2} + 4^{q+1} + \ldots + 4 + 1.$$

According to Fermat's little Theorem,

$$4^{q-1} \equiv 1 \pmod q \Rightarrow q \mid \frac{4^{q-1} - 1}{3} = 4^{q-2} + 4^{q-1} + \ldots + 4 + 1$$
$$\Rightarrow q \mid 4^{q+2} + 4^{q+1} + 4^q + 4^{q-1} = 4^{q-1}(4^3 + 4^2 + 4^1 + 1) = 85 \cdot 4^{q-1}$$

Since $85 = 5 * 17$, so $q = 5$, or $q = 17$. If $q = 17$, then $p + q = 20$, and $4^{20} - 1 = (4^5)^4 - 1$ having more than 4 distinct divisors larger than 1. If $q = 5$, then $4^{p+q} - 1 = 4^8 - 1 = 3 \cdot 5 \cdot 17 \cdot 257$. Thus $r = 17, s = 257$. Thus $q = 5$, and $p + q + r + s = 3 + 5 + 17 + 257 = \boxed{282}$. □

Solution. **??** Let $BC = a, CA = b$, and $AB = c$ be the lengths of the sides of the $\triangle ABC$, see **??**. First, the square of the length of the median $AE^2 = \dfrac{2b^2 + 2c^2 - a^2}{4}$. The length of the altitude AF is determined based

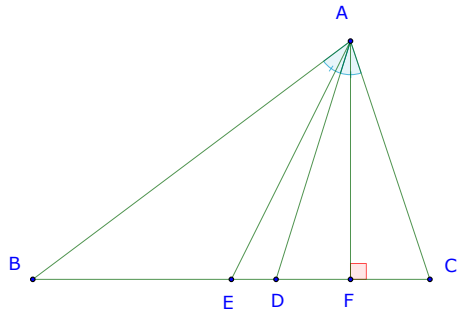


Figure 11.8: **??**

on the Heron's Formula,

$$AF^2 = \left(\frac{2[ABC]}{a}\right)^2 = \frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{4a^2} = \frac{((b+c)^2 - a^2)(a^2 - (b-c)^2)}{4a^2}$$
$$= \frac{((b^2+c^2-a^2)+2bc)(2bc-(b^2+c^2-a^2))}{4a^2} = \frac{4b^2c^2 - (b^2+c^2-a^2)^2}{4a^2}$$
$$EF^2 = AE^2 - AF^2 = \frac{2b^2+2c^2-a^2}{4} - \frac{(4b^2c^2-(b^2+c^2-a^2)^2}{4a^2} = \frac{(c^2-b^2)^2}{4a^2} \Rightarrow EF = \frac{c^2-b^2}{2a}$$

Now by Angle Bisector Theorem

$$\frac{DB}{DC} = \frac{AB}{AC} = \frac{c}{b} \Rightarrow \frac{DE + \frac{a}{2}}{DE - \frac{a}{2}} = \frac{c}{b} \Rightarrow \frac{\frac{a}{2} + DE}{c} = \frac{\frac{a}{2} - DE}{b} = \frac{2DE}{c-b} = \frac{a}{c+b} \Rightarrow DE = \frac{a(c-b)}{2(c+b)}$$

Since

$$EF = 9DE \Rightarrow \frac{c^2-b^2}{2a} = 9\frac{a(c-b)}{2(c+b)} \Rightarrow a^2 = \frac{(c+b)^2}{9} \Rightarrow a = \boxed{94}.$$

□

Solution. **??** Expanding the given expression, and removing all occurrences of perfect squares, the term $\frac{(n-3)(n-5)}{2\cdot 3}$ has to be a perfect square. Let

$$\frac{(n-3)(n-5)}{6} = m^2 \Rightarrow (n-4)^2 - 1 = 6m^2 \Rightarrow (n-4)^2 = 6m^2 + 1$$

The remaining part of the solution is based on a simple code to test the minimal value of n.

```
from math import sqrt

for n in range(100, 1000):
    k = (n-4)*(n-4) - 1
    if k % 6 == 0:
        m = round(sqrt(k/6))
        if m * m == k/6:
            print(n)
            break
```

Thus $n = \boxed{489}$. □

Solution. **??** Place the three pyramids in 3−dimensional coordinate space so that they sit on the $xy-$plane with their common vertex at the origin, and the center of the base of the pyramid with base side length 1 at the point $(1, 0, 0)$. The apex of that pyramid is then at the point $(1, 0, 8)$. The other two pyramids then have centers at $(-2, 2\sqrt{3}, 0)$ and $(-2, -2\sqrt{3}, 0)$ and apices at $(-2, 2\sqrt{3}, 7)$ and $(-2, -2\sqrt{3}, 7)$, respectively.

Let the center of the sphere be at (x, y, z). The sphere rests on one lateral edge of each pyramid. Because the distance the center of the sphere to each of those edges is the same, the length of the projection of the vector $< x, y, z >$ onto each edge must be the same. Note that vectors $< 1, 0, 8 >$, $< -2, 2\sqrt{3}, 7 >$, and $< -2, -2\sqrt{3}, 7 >$, which point from the origin to the three apices of the pyramids, all have the same length. This implies that the dot products of $< x, y, z >$ with each of these vectors must give the same value, and thus,

$$x + 0y + 8z = -2x + 2\sqrt{3}y + 7z = -2x - 2\sqrt{3}y + 7z \Rightarrow y = 0, 3x = -z$$

Hence, for some positive real number t, the vector $< -t, 0, 3t >$ points from the origin to the center of the sphere, where t is determined so that the component of $< -t, 0, 3t >$ perpendicular to $< 1, 0, 8 >$ is 2, the radius of the sphere. The square of the length of that component is given by

$$| < -t, 0, 3t > |^2 - \frac{(< -t, 0, 3t > \cdot < 1, 0, 8 >)^2}{| < 1, 0, 8 > |^2} = \frac{121t^2}{65}$$

Because this should be the square of the radius of the sphere, it equals 4 implying that $t = \frac{2\sqrt{65}}{11}$, and the center of the sphere is at $\frac{2\sqrt{65}}{11} < -1, 0, 3 >$.

The required height is $\frac{6\sqrt{65}}{11}$, and the requested sum is $6 + 65 + 11 = \boxed{82.}$ □

Chapter 12

HC R8

12.1 Topics

Chess

1. Moves: capture, check, double check, promotion, stalemate, checkmate
2. Analysis: whose turn is it.
3. Retrograde: last move, last two moves, last multiple of moves.
4. Proof games: White and Black, instead of making the best moves, they co-operate to make the game legally into a set position.
5. Symmetric position with odd number of moves.
6. Switchback: a piece moved out and came back into that same square.

Logic

1. Casework: what if A is true, what if A is false.
2. Process of elimination: If A is not true, B is not true, then C should be true.
3. Reverse argument: if there is at least ..., then there is atmost ...
4. Implication from truth: if A told the truth and A said X, then X is true.
5. Conflict of truth: if B said A lied, then both cannot be truth tellers.

Algebra

1. Fraction. Comparison. Fraction with smallest denominator.
2. Speed-Distance-Time.
3. Identities $(x+1)^3, (x-1)^3, (x+1)^4, (x-1)^4$.
4. Sums and Products.

Combinatorics

1. Casework.
2. Permutations.
3. Combinations.
4. Grid. Rotations.
5. Counting the same quantity in two (or multiple) ways.

12.2 Problems

Problem 12.2.1 (HC-2022-SM1-R8-P1)**.** (*Beginner Level*)

The mighty cat visited a large store. She was very good at catching mice.

- On the first day, she caught one third of the mice.
- On the next day, she caught one third of the remaining mice.
- On the third day, she caught one third of the remaining mice.
- On the fourth day, she caught eight mice.

How many mice were there at the beginning of the first day?

How to provide your answer:

- If you think there are 10 mice, submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.2 (HC-2022-SM1-R8-P2)**.** (*Beginner Level*)

On the Island of Knights and Liars, there are only two types of people: the Knights, who always tell the truth, and the liars, who always lie. My Linh visited the island and met three persons, Anna, Bebe, and Cicci.

- She asked Anna: *'Are you a Knight or a Liar?'*. Anna angrily replied: *'I refuse to tell you'* and then she left.
- Bebe told My Linher that *'Anna is a Liar.'*.
- Cicci told her that *'Don't listent to Bebe. He is a Liar.'*.

Who was which?

How to provide your answer:

- If you think Anna is a Liar, Bebe is the Knight, and Cicci is a Liar, submit AL, BK, CL.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.3 (HC-2022-SM1-R8-P3)**.** (*Beginner Level*)

In the **??**, White just made a legal move.

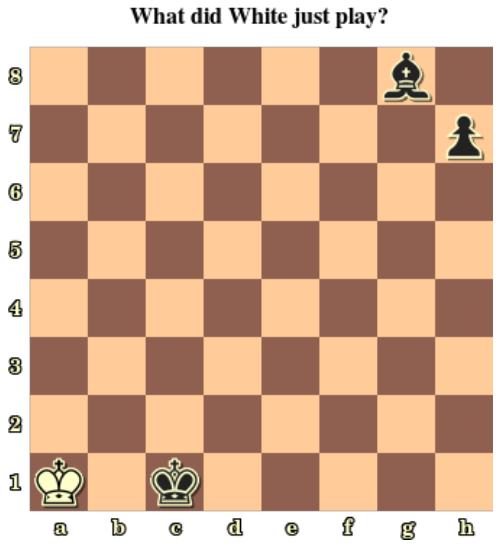


Figure 12.1: **??**

What was the last move of Black and the last move of White?

How to provide your answer:

- If you think the last moves of Black was Kd1c1, and the last move of White was Kb1a1, submit Kc1 Ka1.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.4 (HC-2022-SM1-R8-P4)**.** (*Beginner Level*)

Find the fraction $\frac{m}{n}$, where m and n are positive integers and n has the *smallest possible value* such that,

$$\frac{11}{13} < \frac{m}{n} < \frac{17}{19}$$

How to provide your answer:

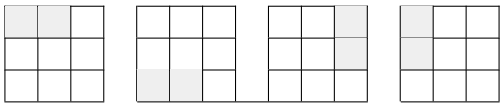
- If you think that it $\frac{10}{9}$, submit $\frac{10}{9}$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly larger* than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly smaller* than your answer.

Problem 12.2.5 (HC-2022-SM1-R8-P5)**.** (*Beginner Level*)

How many different patterns can be made by shading exactly two of the nine squares? Patterns that can be matched by flips and/or turns are not considered different. *For example, the patterns shown below are not considered different.*



How to provide your answer:

- If you think that the answer is 21, then submit 21.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.6 (HC-2022-SM1-R8-P6)**.** (*Intermediate Level*)

Hong Anh started up a mountain trail at eight o'clock in the morning. She climbed up at a constant rate of one and half kilometres per hour. When she reached the top, she camped there for twelve hours. Then she climbed down at a rate of four and half kilometres per hour, reaching the place where she started at noon the next day.

How long was the trail from the bottom to the top of the mountain?

How to provide your answer:

- If you think that the length of the trail is 10 kilometres, submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.7 (HC-2022-SM1-R8-P7)**.** (*Intermediate Level*)

On the Island of Knights and Liars, there are only two types of people: the Knights, who always tell the truth, and the liars, who always lie. The King of Canada sent a spy named Minh was sent to this island. The police somehow knew it and they caught three people, Ajja, Belle, and Coco. It is knew that one of them was a Knight, one of them was a Liar, and the third was the spy, but it is not known that who was which.

Here were what they said,

- Ajja: My name is Minh.
- Belle: That is true.
- Coco: I am Minh.

Can you help Inspector Double Smarty to find out who was which?

How to provide your answer:

- If you think Ajja is the Liar, Belle is the spy, and Coco is the Knight, then submit AL, BS, CK.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.8 (HC-2022-SM1-R8-P8)**.** (*Intermediate Level*)

In the **??**, the pieces are all White. Some pieces are wrongly coloured.

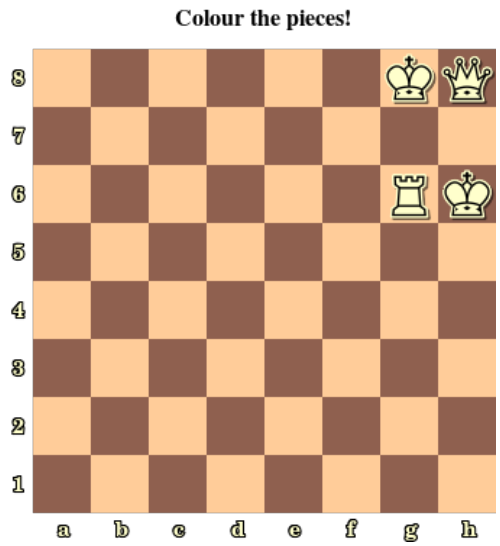


Figure 12.2: **??**

Colour them to Black and determine what was the last move?

How to provide your answer:

- If you think that the pieces on g6, g8 should be Black and the last move was White with Qh7h8+, then submit g6, g8, W, Qh7h8+.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.9 (HC-2022-SM1-R8-P9)**.** (*Intermediate Level*)

Find the largest integer not exeeding the below expression,

$$\frac{2023^3}{2021 \cdot 2022} - \frac{2021^3}{2022 \cdot 2023}$$

How to provide your answer:

- If you think the largest of such number is 100, then submit 100.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly smaller* than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger* than your answer.

Problem 12.2.10 (HC-2022-SM1-R8-P10)**.** (*Intermediate Level*)

The Challenge Given a decimal integer such as 10, the task is to convert it into the **Roman Numerals.** *For example,*

$$4 \Rightarrow IV$$
$$9 \Rightarrow IX$$
$$39 \Rightarrow XXXIX$$
$$160 \Rightarrow CLX$$

Input (*hc-2022-1-8-10-inputput.txt*) The input file contains several cases, each case is a line. Each line contains a positive integer less than or equal to 3999.

Output (*hc-2022-1-8-10-output.txt*) The output file shall consist of a number of lines, each line contains a single string, which is the roman numeral of the integer in the corresponding line of the input.

Sample Input (*hc-2022-1-8-10-sample-input.txt*)

```
5
9
39
101
160
```

Sample Output (*hc-2022-1-8-10-sample-output.txt*)

```
V
IX
XXXIX
CI
CLX
```

How to your troop can compete in this Coding Challenge?

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How your troop's performance is graded:

- If your troop's *score* is higher or equal to the *score* of you opponent for the challenge then your troop scores 1 hit, otherwise 0 hit.
- If your troop cannot submit the output for the challenge, then your troop scores 0 hit, regardless of the score of you opponent.
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Problem 12.2.11 (HC-2022-SM1-R8-P11)**.** (*Advanced Level*)

King Anthony went for hunting trip with his knights. At night, they stayed in one of the king's hunting lodges in the forest. In this house, there are nine rooms. The king slept in the central room. The knights stayed in the eight surrounding rooms such that in each direction north, south, east, west the total number of knights in the *three rooms* facing that direction should always be *nine. For example, the* **??** *below shows a possible case with each room has 3 knights. The letter K indicates the King staying in the central room.*

What would be the smallest number of knights were with the king? What would be the largest?

How to provide your answer:

- If you think that the smallest and largest number of knights would be are 10 and 20, submit 10, 20.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

3	3	3
3	K	3
3	3	3

Figure 12.3: **??**

Problem 12.2.12 (HC-2022-SM1-R8-P12)**.** (*Advanced Level*)

On the Island of Knights and Liars, there are only two types of people: the Knights, who always tell the truth, and the liars, who always lie. On the day of the Festival of Life, some gods and demons visited the island. It is known that the gods always tell the truth and the demons always lie.

Melaine was also visiting the island on that day. She met someone who made a statement so that she knew for sure that he was a god.

What was the statement?

How to provide your answer:

- If you think you have such statement, submit that.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- If your troop's *score* is higher or equal to the *score* of you opponent for the challenge then your troop scores 1 hit, otherwise 0 hit.
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Problem 12.2.13 (HC-2022-SM1-R8-P13)**.** (*Advanced Level*)

It is not know whose turn is it in the **??**. However it is a situation when White *check-mates Black in one*.

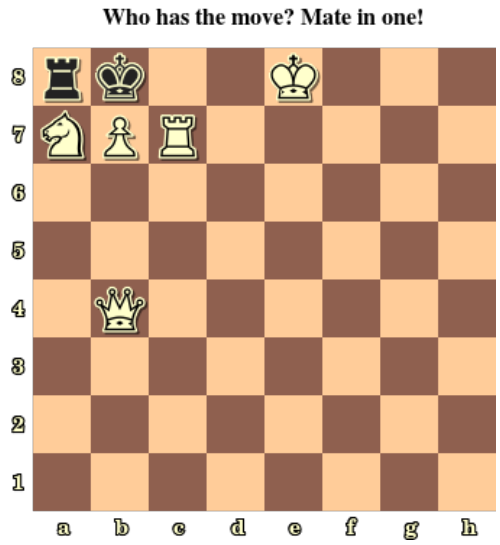

Figure 12.4: **??**

Whose turn is it? In how many ways does *mate in one* happen? Show all solutions.

How to provide your answer:

- If you think that if White moves, and Rc8 is the move that checkmate Black, submit W, Rc8++.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that has *strictly smaller* number of solutions than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has *strictly larger* number of solutions than your answer.

Problem 12.2.14 (HC-2022-SM1-R8-P14)**.** (*Advanced Level*)

a, b, and c are non-zero real numbers such that

$$\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}=\frac{50}{7}, \text{ and } a+b+c=15$$

What is the value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

How to provide your answer:

- If you think that the value of the sum is 5, then submit 5.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.15 (HC-2022-SM1-R8-P15)**.** (*Advanced Level*)

The Challenge In the *Blackjack 21* each player receives a number of cards. The objective is by summing up the cards to get the highest possible less than or equal to 21. Each number card is worth as the number on the card. A royal Jack, Queen, or King is worth 10. An ace is worth 1 or 11. *For example, below are five blackjack hands (set of cards) with their scores sorted.*

$$\begin{aligned} 2+3+4+5+A &= 15 \\ 5+7+4 &= 16 \\ A+6 &= 11+6 = 17 \\ A+J &= 11+10 = 21 \\ J+Q+2 &= 10+10+2 = 22 \end{aligned}$$

Note that any score larger then 21 (called *busted*) is worth less than any score larger than 1 but less than or equal to 21. Note that it is possible to use the A in different ways to get the highest possible score.

Input The input (hc-2022-1-8-15-input.txt) will consist of a set of lines, each line contains a player identifier and a list of cards. The player identifier and the list of cards are separated by a space character. The cards in the list are separated by a comma. Each card is a number from $2, 3, \ldots, 10$, or a letter from J, Q, K, A.

Output The output (hc-2022-1-8-15-output.txt) will consist a set of lines, each contains a player identifier followed by the highest possible score of that player separated by a space. *Note that the order of players with same scores is* **important.** *Also note that players with the busted scores are sorted the end of the list, obviously the higher busted score, the lower it ranks. Players with the same score should be sorted by their identifiers*

Sample Input

```
P1 5,7,4
P2 2,3,4,5,A
P3 J,Q,2
P4 A,J
P5 A,6
```

Output

```
P4 21
P5 17
P1 16
P2 15
P3 22
```

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12.3 Coding Challenge Inputs

?? Input as below (see hc-2022-1-8-10-input.txt)

```
1
4
5
9
12
49
51
89
99
101
345
499
521
1078
1111
1999
2432
2000
2999
3999
```

?? Input as below (see hc-2022-1-8-15-input.txt)

```
P1 5,7,4,10
P2 2,3,4,5,A
P3 J,Q
P4 A,J,Q
P5 K,Q,6
P6 K,Q,A
P7 10,10,A
P8 2,3,4,5,6,7
P9 2,3,4,5,6
P10 2,3,4,5,5
```

12.4 Answers

Problem 1: 27

Problem 2: AK, BL, CK

Problem 3: Nb3a1+ Ka2×a1

Problem 4: $\frac{6}{7}$

Problem 5: 8

Problem 6: 18

Problem 7: AS, BK, CL

Problem 8: g8, W, g7×h8Q+.

Problem 9: 8

Problem 10: Output as below

```
I
IV
V
IX
XII
XLIX
LI
LXXXIX
XCIX
CI
CCCXLV
CDXCIX
DXXI
MLXXVIII
MCXI
MCMXCIX
MMCDXXXII
MM
MMCMXCIX
MMMCMXCIX
```

Problem 11: 18, 36

Problem 12: I am not a Knight.

Problem 13: B, ..Kxa7 b7b8=N++; or B, ..Kxc7 b7xa8=N++; or B, ..Rxa7 Rc8++.

Problem 14: $\frac{71}{105}$

Problem 15: Output as below

```
P4 21
P6 21
P7 21
P3 20
P9 20
P10 19
P2 15
P1 26
P5 26
P8 27
```

12.5 Solutions

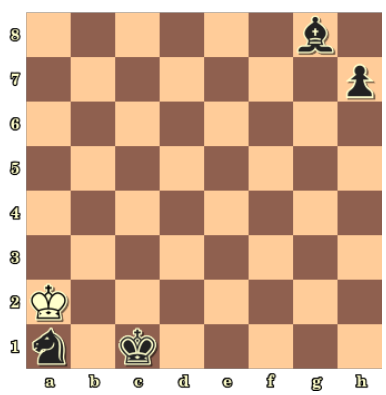
Solution. **??** Note that the cat caught $\frac{1}{3}$ of the mice on the first day, so there were $\frac{2}{3}$ of the mice at the beginning of the second day, thus she caught $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$ of the mice at the end of the second day. Thus there were $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$ of the mice at the beginning of the third day. By similar reasoning, the fraction of the mice remaining after the third day is $\frac{4}{9} \cdot \frac{2}{3} = \frac{8}{27}$.

This fraction represents the number of mice caught on the fourth day, which is 8. Therefore the total number of mice at the beginning of the first day is $\boxed{27.}$ □

Solution. **??** The key is the *action* of Anna, she left without saying anything. In fact, she *did refuse* to tell anything about herself. Thus, she is a Knight. Therefore Bebe is a Liar and Cicci is another Knight. The answer is $\boxed{\text{AK, BL, CK.}}$ □

Solution. **??** The answer is $\boxed{\text{Nb3a1+ Ka2×a1.}}$ □

-1.. Na1+ 1.Kxa1



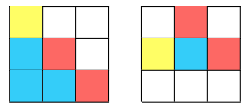
Solution. **??** Note that,

$$\begin{aligned}
&\frac{2}{3} < \frac{11}{13} < \frac{17}{19} < \frac{3}{3}\\
&\frac{3}{4} < \frac{11}{13} < \frac{17}{19} < \frac{4}{4}\\
&\frac{4}{5} < \frac{11}{13} < \frac{17}{19} < \frac{5}{5}\\
&\frac{5}{6} < \frac{11}{13} < \frac{17}{19} < \frac{6}{6}\\
&\Rightarrow m \geq 7
\end{aligned}$$

On the other hand, $\frac{11}{13} < \frac{6}{7} < \frac{17}{19}$, therefore the fraction of $\frac{m}{n}$ is $\boxed{\frac{6}{7}.}$ □

Solution. **??**

Case 1: At least one square is a square at a *vertex* of the grid. See the left grid on the figure below. Without loss of generality (WLOG), since the grid can be rotated, let one of the squares is the upper-left corner one, coloured yellow. Now consider the diagonal through that square. The second square can be one of the two squares, coloured red, on the diagonal or one of the three squares, coloured teal, outside of the diagonal (these squares can be reflectionally symmetric about this diagonal, so we only consider only the



squares on one of the sides of the diagonal.) Thus, in this case, there are $2 + 3 = 5$ distinct choices for the second square.

Case 2: At least one square is on an *edge*, coloured yellow, but no square is on a vertex. See the right grid on the figure below. For the second square, there is a single choice for non-vertex square, coloured teal, on the diagonal and two choices for non-vertex square, coloured red, outside of the diagonal. In other words, there are clearly two edge-edge and one edge-center combinations, so this case has 3 choices.

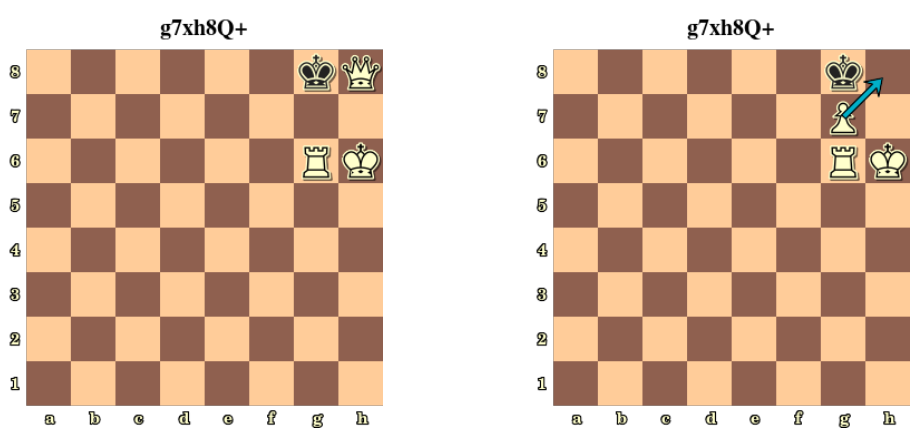
In total, there are $5 + 3 = \boxed{8}$ distinct patterns. □

Solution. **??** Hong Anh spent in total 28 hours from eight o'clock in the morning to noon the next day. Since she spent 12 hours at the top, so the amount of time she spent on the trail is $28 - 12 = 16$. Note that she climbed down 4.5 km/h, or three times the speed of 1.5 km/h when she climbed up, thus, she spent thrice the amount of time on the way up than on the way down. Therefore, she spent a quarter of the time to climb down, or $16 \cdot \frac{1}{4} = 4$ hours on the way down.

Thus, the distance is $4 \times 4.5 = \boxed{18}$ km. □

Solution. **??** Since Ajja claimed that Ajja is the spy, Ajja cannot be the Knight, so Ajja is the spy or Ajja is the Liar. It is the same situation with Coco. Therefore one of them, Ajja or Coco, is the spy and the other one is the Liar. Therefore Belle is the Knight. Since Belle said that Ajja made a true statement, so Ajja is the spy. Thus, Coco is the Liar. The answer is $\boxed{AS,\ BK,\ CL.}$ □

Solution. **??** Because both Kings can never be in check simultaneously, one of them is in double check by Rg6



and Qh8. The only possible explanation to this double check is that the last move was g7xh8=Q+. Thus we conclude that Kh6, Qh8 and Rg6 are White; Kg8 is Black. Last move is not completely determined because both g7xBh8=Q+ and g7xNh8=Q+ are possible. Thus, the answer is g8, W, g7×h8Q+. □

Solution. **??** Let $x = 2022$, then the given expression is

$$\begin{aligned}
&\frac{(x+1)^3}{(x-1)(x)} - \frac{(x-1)^3}{(x)(x+1)} = \frac{(x+1)^4}{(x-1)(x)(x+1)} - \frac{(x-1)^4}{(x-1)(x)(x+1)} \\
&= \frac{(x^4+4x^3+6x^2+4x+1)-(x^4-4x^3+6x^2-4x+1)}{(x-1)(x)(x+1)} \\
&= \frac{8x^3+8x}{(x-1)(x)(x+1)} = \frac{8x^2+8}{(x-1)(x+1)} = \frac{8x^2+8}{x^2-1} = 8 + \frac{16}{x^2-1}
\end{aligned}$$

Thus, for $x = 2022$, then $\left\lfloor 8 + \frac{16}{2022^2-1} \right\rfloor = \boxed{8}$. □

Solution. **??** Following is an example solution in Python.

```python
num = [1, 4, 5, 9, 10, 40, 50, 90, 100, 400, 500, 900, 1000]
sym = ["I", "IV", "V", "IX", "X", "XL", "L", "XC", "C", "CD", "D", "CM", "M"]

def decimal_to_roman(decimal):
    roman = []
    i = 12

    while decimal:
        div = decimal // num[i]
        decimal %= num[i]

        while div:
            roman.append(sym[i])
            div -= 1
        i -= 1

    return roman

if __name__ == "__main__":
    with open('hc-2022-1-8-10-input.txt', 'rt') as i_file:
        decimals = [int(w.strip()) for w in i_file.readlines()]
        with open('hc-2022-1-8-10-output.txt', 'wt') as o_file:
            for decimal in decimals:
                roman = decimal_to_roman(decimal)
                o_file.write(f"{''.join(roman)}\n")
```

□

Solution. **??** Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3$, and b_4 be the numbers of knights in each room as shown in the **??**. Let $S = a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4$, then

$$36 = (a_1 + b_1 + a_2) + (a_2 + b_2 + a_3) + (a_3 + b_4 + a_4) + (a_4 + b_4 + a_1) \Rightarrow \begin{cases} S = 36 - (a_1 + a_2 + a_3 + a_4) \\ S = \frac{1}{2}(36 + (b_1 + b_2 + b_3 + b_4)) \end{cases}$$

a_1	b_1	a_2
b_4	K	b_2
a_4	b_3	a_3

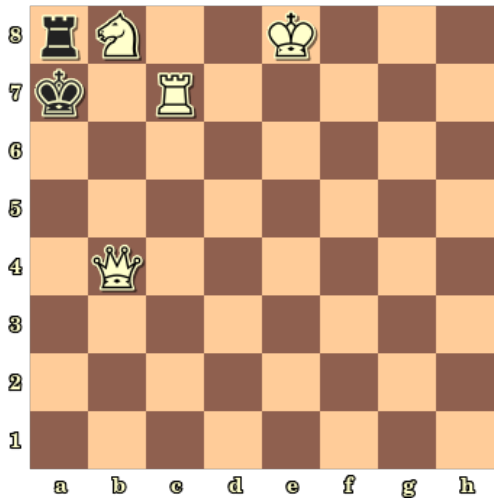
Figure 12.5: With variables

Thus, the least value of S is $\boxed{18,}$ if $b_1 = b_2 = b_3 = b_4 = 0$, and $a_1 = a_3 = 5, a_2 = a_4 = 4$. The most value of S is $\boxed{36,}$ if $a_1 = a_2 = a_3 = a_4 = 0$, and $b_1 = b_2 = b_3 = b_4 = 9$. □

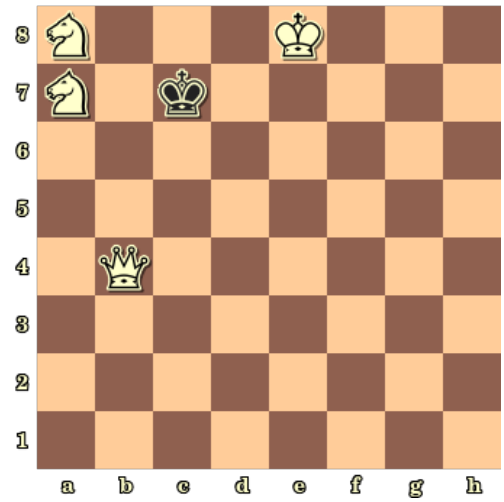
Solution. **??** The statement is *I am not a Knight.* Neither a Liar, a Knight, nor a demon could make such statement. Thus the one who made the statement should be a god. □

Solution. **??** Note that Black could not move last. Thus Black should move next. There are three possible

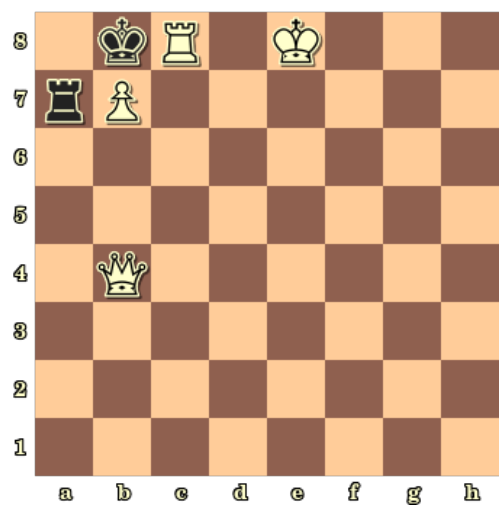
.. Kxa7 b8=N++



.. Kxc7 bxa8=N++



.. Rxa7 Rc8++



move sequences: $\boxed{\text{B, ..Kxa7 b7b8=N++; B, ..Kxc7 b7xa8=N++; or B, ..Rxa7 Rc8++.}}$ □

Solution. **??**

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}+3$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\left(\frac{50}{7}+3\right)\frac{1}{15}=\boxed{\frac{71}{105}}$$

□

Solution. **??** Following is an example solution in Python.

```python
from collections import defaultdict
from itertools import product

ROYALS = ['J', 'Q', 'K']
ACE = 'A'

def score(hand):
    aces = 0
    score = 0

    for card in hand:
        if card in ROYALS:
            score += 10
        elif card not in ACE:
            score += int(card)
        else:
            aces += 1

    if aces == 0:
        return score

    combinations = [score + sum(ace_values) for ace_values in product((1,11),
        repeat=aces)]
    delta = [abs(21 - s) for s in combinations if s <= 21]
    return combinations[delta.index(min(delta))]


if __name__ == "__main__":
    good_score_dict = defaultdict(list)
    bust_score_dict = defaultdict(list)
    with open('hc-2022-1-8-15-input.txt', 'rt') as i_file:
        players = [w.strip().split() for w in i_file.readlines()]
        for player_id, hand in players:
            hand = hand.split(',')
            s = score(hand)
            if s <= 21:
                good_score_dict[s].append(player_id)
            else:
                bust_score_dict[s].append(player_id)

    with open('hc-2022-1-8-15-output.txt', 'wt') as o_file:
        for s in sorted(good_score_dict, reverse=True):
            for player in good_score_dict[s]:
                o_file.write(f"{player} {s}\n")
        for s in sorted(bust_score_dict):
            for player in bust_score_dict[s]:
                o_file.write(f"{player} {s}\n")
```

□

Chapter 13

MIC R7

13.1 Topics

Algebra

1. Ratio. Average.
2. Rate of change.
3. Compare perfect squares with consecutive pairs of perfect squares.
4. Inequalities. Cauchy-Schwarz inequality.
5. Sums and Products.

Combinatorics

1. Counting. Casework. Weekdays.
2. Probability. Favourable outcomes.
3. Games. Invariance principle. Winning positions.

Geometry

1. Right triangles. Midpoint of the hypotenus.
2. Similar triangles.
3. Paralellograms.
4. Kite.
5. Angle chasing. Angles in Circles. Angles in cyclic quadrilaterals.
6. Circles. Right triangles in circle.
7. Heron's Formula.
8. Area of triangle by trigonometry.

Number Theory

1. Divisibility.
2. Prime Numbers.
3. Remainders of a perfect square when divided by 3, by 4, by 5.
4. Modular Arithmetic.
5. Number bases. Digits.
6. Diophantine Equations.

13.2 Rules

- The total time to complete the test is 90 minutes.
- The test consists of 10 multiple-choice and 4 show-you-work problems. To answer each of 10 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 6 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.

13.3 Problems

Problem 13.3.1 (MIC-2022-SM1-R7-P1). (*6 points*)

This year, Lan's birthday is on a Thursday. What day of the week will it be 59 days after her birthday?

(A) *Monday* (B) *Wednesday* (C) *Thursday* (D) *Friday* (E) *Sunday*

Problem 13.3.2 (MIC-2022-SM1-R7-P2). (*6 points*)

For every 3° rise in temperature, the volume of a certain gas expands by 4 cubic centimeters. If the volume of the gas is 24 cubic centimeters when the temperature is 32°, what was the volume of the gas in cubic centimeters when the temperature was 17°?

(A) 4 (B) 8 (C) 12 (D) 15 (E) 18

Problem 13.3.3 (MIC-2022-SM1-R7-P3). (*6 points*)

If two dice are tossed, what is the probability that the product of the numbers showing on the tops of the dice is at least 12 and at most 16?

(A) $\frac{1}{36}$ (B) $\frac{1}{24}$ (C) $\frac{1}{12}$ (D) $\frac{7}{36}$ (E) $\frac{2}{9}$

Problem 13.3.4 (MIC-2022-SM1-R7-P4). (*6 points*)

The letters A, B, C and D represent digits in the addition and substraction shown in the **??**.

What is the value of D?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

$$\begin{array}{rcc} & A & B \\ + & C & A \\ \hline & D & A \end{array} \qquad \begin{array}{rcc} & A & B \\ - & C & A \\ \hline & & A \end{array}$$

Figure 13.1: **??**

Problem 13.3.5 (MIC-2022-SM1-R7-P5). (*6 points*)

Let n be a positive integer greater than 6. It is known that both $n-1$ and $n+1$ are prime numbers.

What is the largest positive integer that always divides $n^2(n^2+16)$?

(A) 288 (B) 196 (C) 172 (D) 168 (E) 144

Problem 13.3.6 (MIC-2022-SM1-R7-P6). (*6 points*)

In the Decimal country, gold coins have value of $1, 10$, or 100. In how many ways an invoice of 222 can be paid? *For example, the invoice of* 222 *can be paid by* 2 *pieces of* $100-$*coin,* 2 *pieces of* $10-$*coins, and* 2 *pieces of* $1-$*coin.*

(A) 36 (B) 39 (C) 42 (D) 45 (E) 48

Problem 13.3.7 (MIC-2022-SM1-R7-P7). (*6 points*)

One number is removed from the set of integers from 1 to n. The average of the remaining numbers is $9\frac{13}{17}$. Which integer was removed?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Problem 13.3.8 (MIC-2022-SM1-R7-P8). (*6 points*)

ABC is a triangle, right-angled at C. The internal bisectors of $\angle BAC$ and $\angle ABC$ meet BC and CA at P and Q, respectively. M and N are the feet of the perpendiculars from P and Q to AB. Find $\angle MCB$, if $\angle NCA = 18°$.

(A) 24 (B) 27 (C) 30 (D) 35 (E) 37

Problem 13.3.9 (MIC-2022-SM1-R7-P9). (*6 points*)

How many pairs of (m, n) integers are there, such that

$$n^2 - 6n = m^2 + m - 10$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 13.3.10 (MIC-2022-SM1-R7-P10). (*6 points*)

n is a positive integer such that its first digit is 1 and if this digit is transferred to the end of it, then the number is tripled.

Find the sum of the digits of the smallest n.

(A) 22 (B) 24 (C) 26 (D) 28 (E) 30

Problem 13.3.11 (MIC-2022-SM1-R7-P11). (*10 points*)

In the **??** below, P is the midpoint of side AB in rectangle $ABCD$. Q is the foot of the altitude from C to DP.

Prove that $\angle BQC = \angle BCQ$.

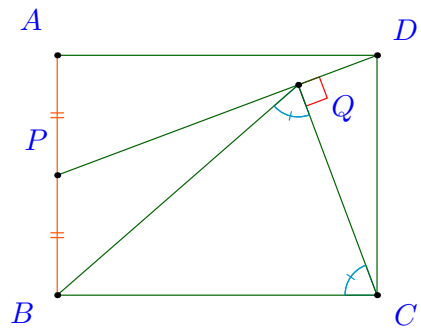


Figure 13.2: **??**

Problem 13.3.12 (MIC-2022-SM1-R7-P12). (*10 points*)

(*5 points*) In triangle ABC, $AB = c, BC = a, CA = b$. Prove that

$$[ABC] \leq \frac{1}{4}(a^2 + b^2)$$

where $[ABC]$ denotes the area of $\triangle ABC$.

(*5 points*) Now a, b, c are arbitrary real numbers. Prove that

$$(a + b - c)(b + c - a)(c + a - b)(a + b + c) \leq (a^2 + b^2)^2$$

Problem 13.3.13 (MIC-2022-SM1-R7-P13). (*10 points*)

(*2 points*) Prove that any *odd* positive integers larger or equal to 1, can be expressed as the different of two perfect squares. *For example* $1^2 - 0^2 = 1$, $5 = 3^4 - 2^2$, *and* $21 = 11^2 - 10^2$.

(*3 points*) Prove that any *multiple* of 4 that is larger or equal to 4, can be expressed as the different of two perfect squares. *For example* $2^2 - 0^2 = 4$, $8 = 3^2 - 1^2$, *and* $16 = 5^2 - 3^2$.

(*5 points*) The numbers from 1 to 100 are written on a board. Van and Vu take turn to play a two-player game. Van goes first. In his turn the player can choose a number on the board and earse it, such that the sum of all earsed numbers (including the one the player just earse) can be wrriten as *the difference of two perfect squares.* Who will have the strategy to win the game and why?

Problem 13.3.14 (MIC-2022-SM1-R7-P14). (*10 points*)

It is well known that the sum of the first n cube is always a perfect square,

$$1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Is there an m positive integer such that the sum below is a perfect square?

$$(m+1)^3 + (m+2)^3 + \ldots + (2m)^3$$

13.4 Grading

Answers for multiple-choice problems.

Problem 1: E Sunday

Problem 2: A 4

Problem 3: D $\frac{7}{36}$

Problem 4: E 9

Problem 5: E 144

Problem 6: B 39

Problem 7: A 5

Problem 8: B 27

Problem 9: E 4

Problem 10: C 26

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 11: There are several approaches to the solution,

(a) 2 points if extend PD to intersect BC at E, and prove that $\triangle BEC$ is right.

(b) 2 points if use R midpoint of DC and can prove that $QC \perp BR$.

(c) 2 points if use R midpoint of DC, S as intersection of BR and QC and can prove that $\triangle CSR \sim \triangle CQD$.

Problem 12: Separately grading for each part,

(a) 2 points if can use $\frac{1}{2}ab\sin C$ formula for triangle area. 2 points for inequality $\frac{1}{2}ab \leq \frac{1}{4}(a^2+b^2)$.

(b) 2 points for Heron's formula.

Problem 13: Separately grading for each part,

(a) 1 point if can investigate $(k+1)^2 - k^2$.

(b) 1 point if can investigate $(k+2)^2 - k^2$.

(c) 2 point if can investigate the $4k+2$ numbers.

Problem 14: 1 point if can apply the formula to find the value of the expression. 1 point if can simplify the expression. 1 point if can casework for m even and odd.

13.5 Solutions

Solution. **??** Since $59 = 7 \cdot 8 + 3$, so 59 days after a Thursday is a $\boxed{\text{Sunday.}}$ The answer is $\boxed{E.}$ □

Solution. **??** It is easy to see that by descreasing every $3°$, the volume of the gas reduced by 4 cubic centimeters. Thus By decreasing $32° - 17° = 15° = 5 \cdot 3°$, the volume of the gas reduced by $5 \cdot 4 = 20$ cubic centimeters. Thus the volume of the gas at $15°$ is $24 - 20 = \boxed{4}$ cubic centimeters. The answer is $\boxed{A.}$ □

Solution. [First solution] **??** If the product of the numbers on the two dice at least 12 and at most 16, then the two numbers can be

1. 12, there are two pairs $\{2, 6\}$ and $\{3, 4\}$, each pair has two permutations.
2. 15, there is one pair $\{3, 5\}$, the pair has two permutations.
3. 16, there is one pair $\{4, 4\}$, the pair has a single permutation.

The probability is $\frac{2\cdot 2+1\cdot 2+1\cdot 1}{36} = \boxed{\frac{7}{36}}$. The answer is $\boxed{D.}$ □

Solution. **??** Note that $B + A$ has a unit digit of A, thus $B = 0$. Then $0 - A$ has the unit digit A, so $A = 5$. Now, $A - 1 - C = 0$, so $C = 4$. $D = C + 5$ since there is no carry-over. so $D = \boxed{9.}$ The answer is $\boxed{E.}$ □

Solution. **??** Since $n > 6$, $n - 1$ and $n + 1$ are prime numbers, thus n has to be even and n is divisible 3. Let $n = 6m$, then

$$P = n^2(n^2 + 16) = 36m^2(36m^2 + 16) = 144m^2(9m^2 + 4).$$

Hence, $144 \mid P$. On the other hand, with

$$\begin{cases} n = 6, n - 1 = 5, n + 1 = 7, n^2(n^2 + 16) = 13 \cdot 144 \\ n = 12, n - 1 = 11, n + 1 = 13, n^2(n^2 + 16) = 160 \cdot 144 \end{cases} \Rightarrow \gcd(6^2(6^2 + 16), 12^2(12^2 + 16)) = 144$$

Thus $\boxed{144}$ is the largest integer that divides $n^2(n^2 + 16)$, for any n such that $n - 1$ and $n + 1$ are prime numbers.

The answer is $\boxed{E.}$ □

Solution. **??** The value of the invoice is 222, thus it is easy to see that 2 pieces of $1-$coins have to be used. Now, we need to determine in how many ways the remaining 220 can be paid.

Case 1: If we use no $100-$coin, then there are 23 ways to use only $10-$coins and $1-$coins:

$$(22 \times 10, 0 \times 1),\ (21 \times 10, 10 \times 1), \ldots,,\ (0 \times 10, 220 \times 1)$$

Case 2: If we use 1 piece of $100-$coin, then to pay the remaining $220 - 100 = 120$ there are 13 ways to use only $10-$coins and $1-$coins:

$$(12 \times 10, 0 \times 1),\ (11 \times 10, 10 \times 1), \ldots,\ (0 \times 10, 120 \times 1)$$

Case 3: If we use 2 piece of $100-$coin, then to pay the remaining $220 - 200 = 20$ there are 3 ways to use only $10-$coins and $1-$coins:

$$(2 \times 10, 0 \times 1),\ (1 \times 10, 10 \times 1),\ (0 \times 10, 20 \times 1)$$

Then $23 + 13 + 3 = \boxed{39}$ The answer is $\boxed{B.}$ □

Solution. **??** Let m be the positive integer, $1 \le m \le n$, that was removed, then

$$\frac{(1+2+\ldots+n)-m}{n-1} = 9\frac{13}{17} \Rightarrow m = \frac{n(n+1)}{2} - \frac{166(n-1)}{17} \Rightarrow 17 \mid n-1 \quad (1)$$

On the other hand,

$$1 \le m \le n \Rightarrow \frac{n(n+1)}{2} - \frac{166(n-1)}{17} \le n \Rightarrow \frac{1}{34}(n-1)(17n-332) \le 0 \Rightarrow n \le \frac{332}{17} \approx 19.53 \quad (2)$$

Thus, from (1) and (2), $n = 18$. Therefore $m = \frac{18 \cdot 19}{2} - \frac{166 \cdot 17}{17} = \boxed{5.}$

The answer is $\boxed{A.}$ □

Solution. **??** It is easy to see that in the quadrilateral $BCQN$, $\angle BCQ = \angle BNQ = 90°$. Therefore it is a cyclic quadrilateral. See **??**.

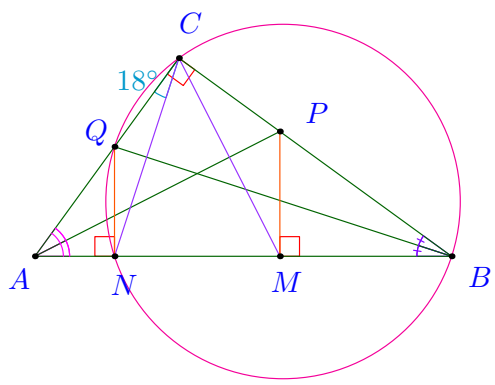


Figure 13.3: **??**

Thus $\angle QBN = \angle QCN = 18°$, so $\angle ABC = 2\angle QBN = 36°$. Hence $\angle BAC = 90° - 36° = 54°$. Therefore since $ACPM$ is a cyclic quadrilateral, $\angle MCB = \angle MAP = \frac{1}{2}\angle BAC = \boxed{27°.}$

The answer is $\boxed{B.}$ □

Solution. **??**

$$n^2 - 6n = m^2 + m - 10 \Rightarrow n^2 - 6n + 9 = m^2 + m - 1 \Rightarrow (n-3)^2 = m^2 + m - 1$$

It is easy to see that $m \ne 0$, so we investigate two cases.

Case 1: if $m > 0$, then $m \ge 1$, and $(n-3)^2$ is between two consecutive perfect squares. Since

$$m^2 \le m^2 + m - 1 < m^2 + 2m + 1 = (m+1)^2 \Rightarrow m^2 + m - 1 = m^2 \Rightarrow m = 1.$$

$(n-3)^2 = m^2 = 1$, thus $n = 2$ or $n = 4$, and $(m, n) \in \{(1, 2), (1, 4)\}$.

Case 2: if $m < 0$, then let $p = -m \ge 1$, and $m^2 + m - 1 = p^2 - p - 1$. Similarly $(n-3)^2$ is between two consecutive perfect squares,

$$(p-1)^2 = p^2 - 2p + 1 \ge p^2 - p - 1 < p^2 \Rightarrow p^2 - 2p + 1 \ge p^2 - p - 1 \Rightarrow p = 2.$$

$m = -p = -2$, $(n-3)^2 = (p-1)^2 = 1$, so $n = 2$, or $n = 4$, and $(m, n) \in \{(-2, 2), (-2, 4)\}$.

There are in total $\boxed{4}$ solutions. The answer is $\boxed{E.}$ □

Solution. **??** Let $n = \overline{1m}$, where $m = \underbrace{\cdots}_{k \text{ digits}}$ a $k-$ digit number, then

$$\overline{\underbrace{m}_{k \text{ digits}} 1} = 3(\overline{1 \underbrace{m}_{k \text{ digits}}}) \Rightarrow 10m + 1 = 3(10^k + m) \Rightarrow 7m = 3 \cdot 10^k - 1 \Rightarrow 3 \cdot 10^k - 1 \equiv 0 \pmod 7.$$

Now, since $3 \cdot 10^k - 1 \equiv 3 \cdot 3^k - 1 \equiv 3^{k+1} - 1 \pmod 7$, thus $3^{k+1} - 1 \equiv 0 \pmod 7$.) By Fermat's little Theorem $3^6 - 1 \equiv 0 \pmod 7$, and for $n = 1, 2, \ldots, 5$, $3^n - 1$ is not divisible by 7. Thus, the smallest such k is 5, and therefore $m = \frac{3 \cdot 10^5 - 1}{7} = 42857$.

The sum of the digits of m is $4 + 2 + 8 + 5 + 7 = \boxed{26.}$ The answers is $\boxed{C.}$ □

Solution. **??** Extend DP and CB to intersect each other at E. Since $BP = \frac{1}{2}CD$, so BP is the midsegment of $\triangle ECD$, thus $EB = BC$. See **??**.

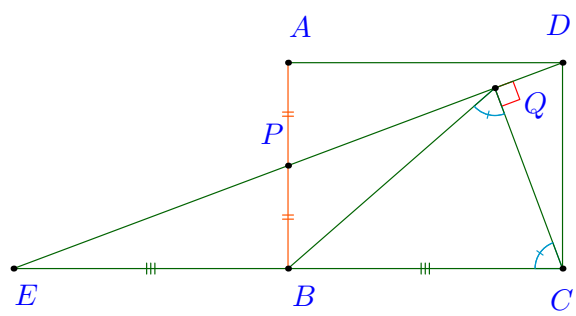


Figure 13.4: $EB = BC$

Now, $\triangle ECQ$ is a right triangle at Q, and B is the midpoint of the hypotenuse EC, therefore $EB = BQ = BC$, or $\triangle BQC$ is isosceles at B.

Thus $\boxed{\angle BQC = \angle BCQ.}$ □

Solution. **??** Let a, b, c are arbitrary real numbers.

First, if they are not sides of a triangle, then one of $a+b-c$, $b+c-a$, and $c+a-b$ is a negative. WLOG, let $a + b < c$, then $a + b - c < 0$, $b + c - a > 0$, and $c + a - b > 0$, thus

$$(a + b - c)(b + c - a)(c + a - b)(a + b + c) < 0 \leq (a^2 + b^2)^2$$

Now, let there be a triangle ABC, $AB = c, BC = a, CA = b$, by Heron's Formula,

$$[ABC]^2 = \frac{(a + b - c)(b + c - a)(c + a - b)(a + b + c)}{16}.$$

On the other hand,

$$[ABC] = \frac{1}{2}ab \sin C \leq \frac{1}{2}ab \leq \frac{1}{4}(a^2 + b^2)$$

Thus $\boxed{(a + b - c)(b + c - a)(c + a - b)(a + b + c) \leq (a^2 + b^2)^2.}$ □

Solution. **??** For the first question, for any odd number $2k+1$, $(k \geq 0)$,

$$2k+1 = (k+1)^2 - k^2.$$

For the second question, for any multiple of 4, that is larger than 4, or $4(k+1)$, $(k \geq 0)$,

$$4(k+1) = (k+2)^2 - (k)^2.$$

Thus, any number in format of $4k, 4k+1$, or $4k+3$, can be written as the difference of any two perfect squares. Now, since any perfect square si divisible by 4, or has a reaminder 1 when divided by 4, thus $4k+2$ cannot be written as the difference of any two perfect squares.

For the last question, let $S_0 = \{4, 8, \ldots, 100\}$ be the set of multiples of 4, $S_1 = \{1, 5, \ldots, 97\}$, $S_2 = \{2, 6, \ldots, 98\}$, and $S_3 = \{3, 7, \ldots, 99\}$. Van will win this game by choosing an element of S_0 at the beginning. From the results of the first and second question, Vu should choose a number such that the sum of Van and Vu numbers must be in the format of $4k, 4k+1$, or $4k+3$. Therefore Vu has to choose a number from one the sets S_0, S_1, or S_3.

1. If he chooses from S_0, then Van chooses the next one from S_0.
2. If he chooses from S_1, then Van chooses the next one from S_3.
3. If he chooses from S_3, then Van chooses the next one from S_1.

From the second turns, after a number of pairs of turns of (Van, Vu), the number of elements of S_0 is even, the number of elements of S_1 and S_3 are the same. At a certain moment, there is no element in any of these three sets, then Vu is forced to take an element from S_2 and he loses at that moment. Van will win. □

Solution. **??** Note that, a perfect square has a remainder of 0 or 1 when divided by 4. Now,

$$(m+1)^3 + (m+2)^3 + \ldots + (2m)^3 = \left(1^3 + 2^3 + \ldots + (2m)^3\right) - \left(1^3 + 2^3 + \ldots + (m)^3\right)$$
$$= \left(\frac{2m(2m+1)}{2}\right)^2 - \left(\frac{m(m+1)}{2}\right)^2 = \frac{m^2(15m^2+14m+3)}{4}$$

Case 1: if m is even. Then $\frac{m^2}{4}$ is a perfect square and $15m^2+14m+3$ is a perfect square too. But if m is even, then $15m^2+14m+3$ has a remainder of 3 when divided by 4, so it cannot be a perfect square.

Case 2: if m is an odd number. Then let $m = 2k+1$

$$\frac{15m^2+14m+3}{4} = \frac{15(2k+1)^2+14(2k+1)+3}{4} = 15k^2+22k+8$$

having remainder of 2 when divided by 4 if k is even, and having remainder of 3 when divided by 4 if k is odd. This is impossible for $15k^2+22k+8$ to be a perfect square.

Thus the given sum cannot be a perfect square for any m. □

Chapter 14

PC 2022

14.1 Problems

Problem 14.1.1 (PCT-2022-SM1-HS-P1)**.** Find the maximum possible value obtainable by inserting a single set of parentheses into the expression $1 + 2 \times 3 + 4 \times 5 + 6$.

Solution. **??**

$$1 + 2 \times (3 + 4) \times 5 + 6 = 1 + 2 \times 7 \times 5 + 6 = \boxed{77.}$$

□

Problem 14.1.2 (PCT-2022-SM1-HS-P2)**.** Call a date mm/dd/yy *multiplicative* if its month number times its day number is a two-digit integer equal to its year expressed as a two-digit year. For example, 01/21/21, 03/07/21, and 07/03/21 are multiplicative. Find the number of dates between January 1, 2022 and December 31, 2030 that are multiplicative.

Solution. **??**

```
from datetime import date, timedelta

now = date(2022, 1, 1)
delta = timedelta(days=1)
count = 0
done = False

while not done:
    if now.day * now.month == (now.year - 2000):
        print(f"{now.day:2} {now.month:2} {now.year - 2000:2}")
        count += 1
    now += delta
    done = (now.year == 2030) and (now.month  == 12) and (now.day == 31)

print(f"Total: {count}")
```

The number of multiplicative dates is $\boxed{29.}$ □

Problem 14.1.3 (PCT-2022-SM1-HS-P3)**.** An isosceles triangle has a base with length 12 and the altitude to the base has length 18. Find the area of the region of points inside the triangle that are a distance of at most 3 from that altitude.

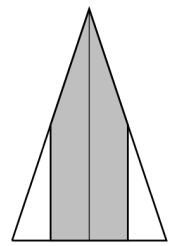


Figure 14.1: **??**

Solution. **??** The two unshaded triangles are two congruent right triangles with legs 9 and 3. Thus, the unshaded area is $9 \cdot 3 = 27$. Thus the area of the shaded region is $\frac{12 \cdot 18}{2} - 27 = \boxed{81.}$ □

Problem 14.1.4 (PCT-2022-SM1-HS-P4)**.** Of 450 students assembled for a concert, 40 percent were boys. After a bus containing an equal number of boys and girls brought more students to the concert, 41 percent of the students at the concert were boys. Find the number of students on the bus.

Solution. **??** 40% of 450 is $450 \cdot 0.4 = 180$. Let $2x$ be the number of students on the bus.

$$180 + x = (450 + 2x) \cdot 0.41 \Rightarrow 180 + x = 184.5 + 0.82x \Rightarrow 0.18x = 4.5 \Rightarrow x = 25 \Rightarrow 2x = \boxed{50.}$$

□

Problem 14.1.5 (PCT-2022-SM1-HS-P5)**.** Below is a diagram showing a 6×8 rectangle divided into four 6×2 rectangles and one diagonal line. Find the total perimeter of the four shaded trapezoids.

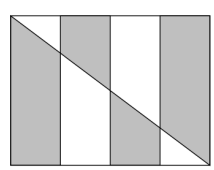


Figure 14.2: **??**

Solution. **??** It is easy to see that the left-most and right-most trapezoids are congruent. The two middle trapezoids are congruent, too. Since the perimeter of the left-most shaded trapezoid is Thus the total perimeter is

$$2\left(8 + \frac{10}{4} + 6 + 2\right) + 2\left(2 + \frac{10}{4} + 4 + 2\right) = 2(10 + 5 + 10 + 4) = \boxed{58.}$$

□

Problem 14.1.6 (PCT-2022-SM1-HS-P6)**.** Let $a_1 = 2021$, and for $n \geq 1$, let $a_{n+1} = \sqrt{4 + a_n}$. Then a_5 can be written as

$$\sqrt{\frac{m+\sqrt{n}}{2}} + \sqrt{\frac{m-\sqrt{n}}{2}}$$

Solution. **??** First,

$$a_2 = \sqrt{4+2021} = \sqrt{2025} = 45, a_3 = \sqrt{4+45} = \sqrt{49} = 7, a_4 = \sqrt{4+7} = \sqrt{11}$$

Now, $a_5 = \sqrt{4+\sqrt{11}}$, let

$$\sqrt{\frac{m+\sqrt{n}}{2}} + \sqrt{\frac{m-\sqrt{n}}{2}} = \sqrt{4+\sqrt{11}} \Rightarrow \left(\frac{m+\sqrt{n}}{2}\right) + \left(\frac{m-\sqrt{n}}{2}\right) + 2\left(\sqrt{\frac{m^2-n}{4}}\right) = 4+\sqrt{11}$$

$$\Rightarrow m + \sqrt{m^2-n} = 4+\sqrt{11} \Rightarrow \begin{cases} m & = 4 \\ m^2 - n & = 11 \end{cases} \Rightarrow m = 4, n = 5$$

Therefore $10m + n = 10 \cdot 4 + 5 = \boxed{45.}$ □

Problem 14.1.7 (PCT-2022-SM1-HS-P7)**.** In a room there are 144 people. They are joined by n other people who are each carrying k coins. When these coins are shared among all $n + 144$ people, each person has 2 of these coins. Find the minimum possible value of $2n + k$.

Solution. **??** First

$$nk = (144+n) \cdot 2 \Rightarrow k = \frac{2(n+144)}{n} = 2 + \frac{288}{n} \Rightarrow 2n + k = 2 + 2n + \frac{288}{n} = 2 + 2\left(n + \frac{144}{n}\right)$$

By AM-GM inequality $a + b \geq 2\sqrt{ab}$, thus

$$n + \frac{144}{n} \geq 2 \cdot \sqrt{n \cdot \frac{144}{n}} = 24 \Rightarrow 2n + k \geq 2 + 2 \cdot 24 = \boxed{50.}$$

□

Problem 14.1.8 (PCT-2022-SM1-HS-P8)**.** The product

$$\left(\frac{1+1}{1^2+1} + \frac{1}{4}\right)\left(\frac{2+1}{2^2+1} + \frac{1}{4}\right)\left(\frac{3+1}{3^2+1} + \frac{1}{4}\right)\cdots\left(\frac{2022+1}{2022^2+1} + \frac{1}{4}\right)$$

can be written as $\frac{q}{2^r \cdot s}$, where r is a positive integer, and q and r are relatively prime odd positive integers. Find s.

Solution. **??** Note that

$$\frac{k+1}{k^2+1} + \frac{1}{4} = \frac{4k+4+k^2+1}{4(k^2+1)} = \frac{1}{4} \cdot \frac{(k+2)^2+1}{k^2+1}$$

Therefore the given product is,

$$\frac{1}{4^{2022}} \cdot \frac{(3^2+1)(4^2+1)\ldots(2024^2+1)}{(1^2+1)(2^2+1)\ldots(2022^2+1)} = \frac{1}{2^{4044}} \cdot \frac{(2023^2+1)(2024^2+1)}{(1^2+1)(2^2+1)}$$

Since $1^2 + 1 = 2 \mid 2023^2 + 1$, and $2^2 + 1 = 5 \mid 2023^2 + 1$, thus the given product can be written as $\frac{q}{2^r \cdot s}$, where

$$q = \frac{(2023^2+1)(2024^2+1)}{(1^2+1)(2^2+1)}, r = 4044, \text{ and } s = \boxed{1.}$$

□

Problem 14.1.9 (PCT-2022-SM1-HS-P9)**.** For positive integer n let $z_n = \sqrt{\frac{3}{n}} + i$, where $i = \sqrt{-1}$. Find $|z_1 \cdot z_2 \cdots z_{47}|$.

Solution. **??** Note that for any two complex number $x = a + bi, y = c + di$, then

$$|xy| = |(ac - bd) + (ad + bc)i| = \sqrt{(ac - bd)^2 + (ad + bc)^2} = \sqrt{a^2 + b^2}\sqrt{c^2 + d^2} = |x| \cdot |y|$$

Therefore, $|z_1 \cdot z_2 \cdots z_{47}| = |z_1| \cdot |z_2| \cdots |z_{47}|$, thus for $n \geq 1$,

$$|z_n| = \sqrt{\frac{3}{n} + 1} = \sqrt{\frac{n+3}{n}} \Rightarrow |z_1 \cdot z_2 \cdots z_{47}| = \sqrt{\frac{4 \cdot 5 \cdots 50}{1 \cdot 2 \cdots 47}} = \sqrt{\frac{48 \cdot 49 \cdot 50}{1 \cdot 2 \cdot 3}} = \boxed{140}$$

□

Problem 14.1.10 (PCT-2022-SM1-HS-P10)**.** Let a be positive real number such that $4a^2 + \frac{1}{a^2} = 117$. Find

$$8a^3 + \frac{1}{a^3}$$

Solution. **??** Note that

$$4a^2 + \frac{1}{a^2} = (2a)^2 + \frac{1}{a^2} = \left(2a + \frac{1}{a}\right)^2 - 2 \cdot 2a \cdot \frac{1}{a} \Rightarrow 2a + \frac{1}{a} = \sqrt{117 + 4} = 11$$
$$8a^3 + \frac{1}{a^3} = (2a)^3 + \frac{1}{a^3} = \left(2a + \frac{1}{a}\right)\left((2a)^2 - 2a \cdot \frac{1}{a} + \frac{1}{a^2}\right) = 11(117 - 2) = \boxed{1265.}$$

□

Problem 14.1.11 (PCT-2022-SM1-HS-P11)**.** In quadrilateral $ABCD$, let $AB = 7$, $BC = 11$, $CD = 3$, $DA = 9$, $\angle BAD = \angle BCD = 90°$, and diagonals AC and BD intersect at E. The ratio $\frac{BE}{DE} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. [Solution 1] **??** It is easy to see that in any cyclic quadrilateral,

$$\frac{BE}{DE} = \frac{AB}{AD} \cdot \frac{CB}{CD}.$$

This can be proved by the Law of Sines as below,

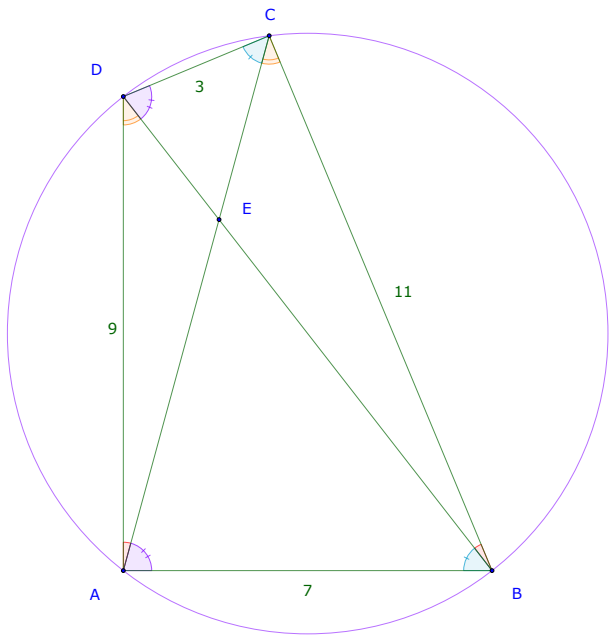


Figure 14.3: **??**

$$\frac{BE}{AE} = \frac{\sin\angle BAE}{\sin\angle ABE}, \frac{AE}{DE} = \frac{\sin\angle ADE}{\sin\angle DAE}, \frac{AB}{AD} = \frac{\sin\angle ADE}{\sin\angle ABE}, \frac{CB}{CD} = \frac{\sin\angle CDE}{\sin\angle CBE} = \frac{\sin\angle BAE}{\sin\angle DAE}$$
$$\Rightarrow \frac{BE}{DE} = \frac{BE}{AE} \cdot \frac{AE}{DE} = \frac{AB}{AD} \cdot \frac{CB}{CD} = \frac{7}{9} \cdot \frac{11}{3} = \frac{77}{27} \Rightarrow 77 + 27 = \boxed{104.}$$

□

Solution. [Solution 2] **??** Let $A(0,0)$, $B(7,0)$, and $D(9,0)$ be the points A, B, and D. C is the intersection of the circle radius 11 centred at B and radius 3 centred at D.

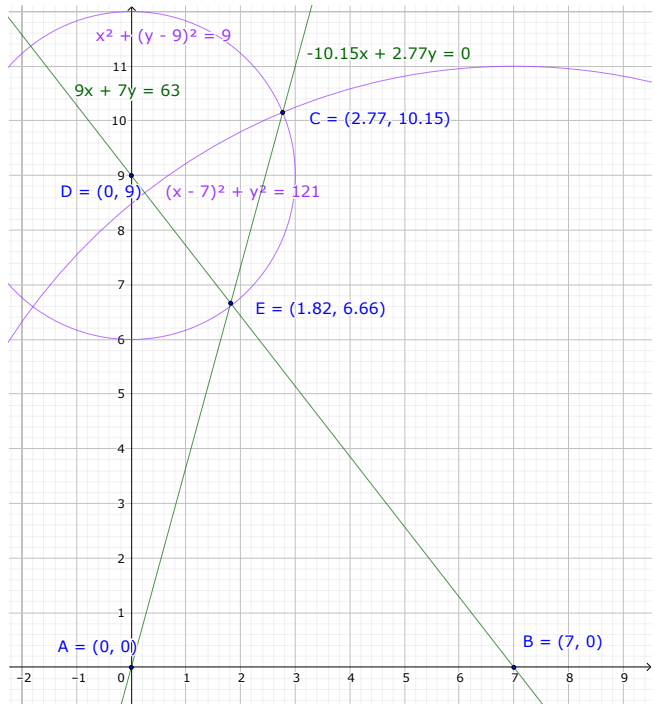


Figure 14.4: **??**

$$\begin{cases}(x-7)^2+y^2=11^2\\ x^2+(y-9)^2=3^2\end{cases} \Rightarrow \begin{cases}x^2-14x+y^2=72\\ x^2+y^2-18y=-72\end{cases} \Rightarrow -14x+18y=144 \Rightarrow 9y=7x+72 \Rightarrow y=\frac{7}{9}x+8$$

$$(x-7)^2+\left(\frac{7}{9}x+8\right)^2=11^2 \Rightarrow \frac{130}{81}x^2-\frac{14}{9}x-8=0 \Rightarrow 65x^2-63x-324=0 \Rightarrow (5x+9)(13x-36)=0$$

Thus, $x=\frac{36}{13}, y=\frac{132}{13}$, $C\left(\frac{36}{13},\frac{132}{13}\right)$ and E is the intersection of line $AC:\ y=\frac{11}{3}x$ and $BD:\ 9x+7y=63$,

$$\begin{cases}y=\frac{11}{3}x\\ 9x+7y=63\end{cases} \Rightarrow 9x+\frac{77}{3}x=63 \Rightarrow x=\frac{189}{104} \Rightarrow y=\frac{693}{104}$$

Therefore $BE=\sqrt{(7-\frac{189}{104})^2+(0-\frac{693}{104})^2}=\frac{77\sqrt{130}}{104}$, $DE=\sqrt{(0-\frac{189}{104})^2+(9-\frac{693}{104})^2}=\frac{27\sqrt{130}}{104}$, so $\frac{BE}{DE}=\frac{77}{27} \Rightarrow m+n=77+27=\boxed{104.}$ □

Problem 14.1.12 (PCT-2022-SM1-HS-P12)**.** Let a and b be positive real numbers satisfying

$$\frac{a}{b}\left(\frac{a}{b}+2\right)+\frac{b}{a}\left(\frac{b}{a}+2\right)=2022$$

Find the positive integer n such that

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}=\sqrt{n}$$

Solution. **??** Let $x=\sqrt{\frac{a}{b}}$, then $\sqrt{\frac{b}{a}}=\frac{1}{x}$, let $y=x+\frac{1}{x}$, then

$$\begin{aligned}
&\frac{a}{b}+\frac{b}{a}=y^2-2, \left(\frac{a}{b}\right)^2+\left(\frac{b}{a}\right)^2=(y^2-2)^2-2\\
\Rightarrow\ &\frac{a}{b}\left(\frac{a}{b}+2\right)+\frac{b}{a}\left(\frac{b}{a}+2\right)=(y^2-2)^2+2(y^2-2)-2\\
\Rightarrow\ &((y^2-2)+1)^2=2025\Rightarrow(y^2-2)+1=45\Rightarrow y^2=46\Rightarrow y=\sqrt{46}\Rightarrow n=\boxed{46}
\end{aligned}$$

□

Problem 14.1.13 (PCT-2022-SM1-HS-P13)**.** Find the number of positive divisors of 20^{22} that are perfect squares or perfect cubes.

Solution. [Solution 1] **??** First let $n=20^{22}=\left(2^2\cdot 5\right)^{22}=2^{44}5^{22}$. If d divisor of n is a perfect square, then its prime factorizations contains an even power of 2 and an even power of 5,

$$d=2^{2a}5^{2b},\ \text{where } 0\le 2a\le 44, 0\le 2b\le 22\Rightarrow 0\le a\le 22, 0\le b\le 11,$$

so there are $(22+1)(11+1)=23\cdot 12=276$ such divisors.

Similarly if d is a perfect cube, then

$$d=2^{3a}5^{3b},\ \text{where } 0\le 3a\le 44, 0\le 3b\le 22\Rightarrow 0\le a\le 14, 0\le b\le 7,$$

so there are $(14+1)(7+1)=120$ such divisors.

Now if d can be both perfect square and perfect cube, then

$$d=2^{6a}5^{6b},\ \text{where } 0\le 6a\le 44, 0\le 6b\le 22\Rightarrow 0\le a\le 7, 0\le b\le 3,$$

so there are $(7+1)(3+1)=32$ such divisors.

In total, there are $276+120-32=\boxed{364}$ such divisors. □

Solution. [Solution 2] **??**

```
count = 0

for a in range(0, 45):
    for b in range(0, 23):
        if ((a % 2 == 0) and (b % 2 == 0)) or ((a % 3 == 0) and (b % 3 == 0)):
            count +=1

print(count
```

The number of such powers is $\boxed{364.}$ □

Problem 14.1.14 (PCT-2022-SM1-HS-P14)**.** Of the integers a, b, and c that satisfy $0 < c < b < a$ and $a^3 - b^3 - c^3 - abc + 1 = 2022$, let c' be the least value of c appearing in any solution, let a' be the least value of a appearing in any solution with $c = c'$, and let b' be the value of b in the solution where $c = c'$ and $a = a'$. Find $a' + b' + c'$.

Solution. **??**

```
found = False
a = 13

while not found:
    for b in range(a-1, 0, -1):
        for c in range(b-1, 0, -1):
            if a*a*a - b*b*b - c*c*c - a*b*c + 1 == 2022:
                print(a+b+c, a,b,c)
                found = True
                break
    a += 1
```

The sum of $a + b + c$ is $\boxed{19.}$ □

Problem 14.1.15 (PCT-2022-SM1-HS-P15)**.** Let a be real number such that

$$5\sin\left(\frac{a}{2}\right)^4 + 12\cos a = 5\cos\left(\frac{a}{2}\right)^4 + 12\sin a$$

There are relatively prime positive integers m and n such that $\tan a = \frac{m}{n}$. Find $10m + n$.

Solution. **??** First, note that $\sin a^2 + \cos a^2 = 1, \cos a^2 - \sin a^2 = \cos 2a$, thus

$$5\cos\left(\frac{a}{2}\right)^4 - 5\sin\left(\frac{a}{2}\right)^4 = 5(\cos\frac{a}{2}^2 - \sin\frac{a}{2}^2)(\cos\frac{a}{2}^2 + \sin\frac{a}{2}^2) = 5\cos a$$

Therefore

$$12\cos a = 5\cos a + 12\sin a \Rightarrow 12\sin a = 7\cos a \Rightarrow \tan a = \frac{7}{12} \Rightarrow 10m + n = 10 \cdot 7 + 12 = \boxed{82.}$$

□

Problem 14.1.16 (PCT-2022-SM1-HS-P16)**.** The sum of the solutions to the equation

$$x^{\log_2 x} = \frac{64}{x}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. **??** First, let $y = \log_2 x$, then $x = 2^y$, thus

$$(2^y)^y = \frac{64}{2^y} \Rightarrow 2^{y^2+y} = 2^6 \Rightarrow y^2 + y - 6 = 0 \Rightarrow y \in \{2, -3\} \Rightarrow x \in \{4, \frac{1}{8}\} \Rightarrow 4 + \frac{1}{8} = \frac{33}{8} \Rightarrow m + n = \boxed{41.}$$

□

Problem 14.1.17 (PCT-2022-SM1-HS-P17)**.** Find the least positive integer with the property that if its digits are reversed and then 450 is added to this reversal, the sum is the original number. For example, 621 is not the answer because it is not true that $621 = 126 + 450$.

Solution. **??**

```
found = False
n = 451

while not found:
    s = str(n)
    r = s[::-1]
    m = int(r)
    if m + 450 == n:
        print(n)
        found = True
        break
    n +=1
```

The least such number is $\boxed{1501.}$ □

Problem 14.1.18 (PCT-2022-SM1-HS-P18)**.** In $\triangle ABC$, let D be on BC such that $AD \perp BC$. Suppose also that $\tan B = 4 \sin C$, $AB^2 + CD^2 = 17$, and $AC^2 + BC^2 = 21$. Find the measure of $\angle C$ in degrees between $0°$ and $180°$.

Solution. **??** Since $\tan B = \frac{AD}{BD}, \sin C = \frac{AD}{AC}$, so $\tan B = 4 \sin C \Rightarrow AC = 4BD$. Furthermore

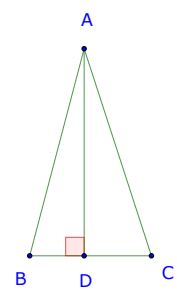


Figure 14.5: **??**

$$AB^2 + CD^2 = AD^2 + BD^2 + CD^2 = BD^2 + (AD^2 + CD^2) = BD^2 + AC^2 \Rightarrow 17BD^2 = 17 \Rightarrow BD = 1$$
$$\Rightarrow AC = 4, BC^2 = 5, \Rightarrow BC = \sqrt{5} \Rightarrow CD = \sqrt{5} - 1$$
$$\Rightarrow \cos C = \frac{CD}{AC} = \frac{\sqrt{5} - 1}{4} \Rightarrow \angle C = \boxed{72°}$$

□

Problem 14.1.19 (PCT-2022-SM1-HS-P19)**.** Let x be a real number such that $(\sqrt{6})^x - 3^x = 2^{x-2}$. Evaluate $\frac{4^{x+1}}{9^{x-1}}$.

Solution. **??** Dividing both sides by 3^x,

$$(\sqrt{6})^x - 3^x = 2^{x-2} \Rightarrow \left(\sqrt{\frac{2}{3}}\right)^x - 1 = \frac{1}{4}\left(\frac{2}{3}\right)^x$$

Now let $y = \left(\sqrt{\frac{2}{3}}\right)^x$, then

$$y - 1 = \frac{y^2}{4} \Rightarrow y^2 - 4y + 4 = 0 \Rightarrow y = 2 \Rightarrow \frac{4^x}{9^x} = \left(\frac{2}{3}\right)^{2x} = \left(\sqrt{\frac{2}{3}}\right)^{4x} = 16 \Rightarrow \frac{4^{x+1}}{9^{x-1}} = 36 \cdot \frac{4^x}{9^x} = \boxed{576.}$$

□

Problem 14.1.20 (PCT-2022-SM1-HS-P20)**.** Let $ABCD$ be a convex quadrilateral inscribed in a circle with $AC = 7$, $AB = 3$, $CD = 5$, and $AD - BC = 3$. Then $BD = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. **??** Let $x = BC$, then $AD = x + 3$. Since $ABCD$ is a cyclic quadrilateral, $\angle ADC = 180° - \angle ABC$, or $\cos\angle ADC = -\cos\angle ABC$. By the Law of Cosines for $\triangle ADC$ and $\triangle ABC$,

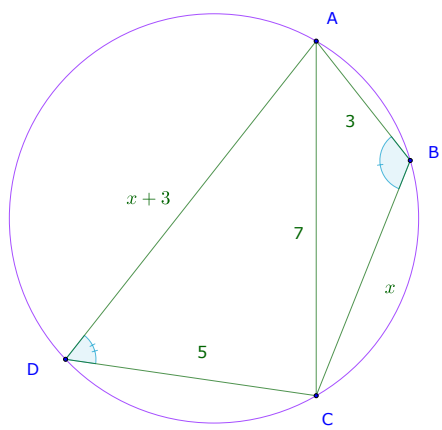


Figure 14.6: **??**

$$\frac{(x+3)^2 + 5^2 - 7^2}{2 \cdot 5 \cdot (x+3)} = -\frac{x^2 + 3^2 - 7^2}{2 \cdot 3 \cdot x}$$
$$\Rightarrow \frac{(x-5)(8x^2 + 73x + 120)}{30x(x+3)} = 0 \Rightarrow x = 5$$

Now, $AD = BC + 3 = 8$, by the Ptolemy's Theorem:

$$BD = \frac{AB \cdot CD + AD \cdot BC}{AC} = \frac{3 \cdot 5 + 8 \cdot 5}{7} = \frac{55}{7} \Rightarrow m + n = \boxed{62.}$$

□

Problem 14.1.21 (PCT-2022-SM1-HS-P21)**.** Find the number of sequences of 10 letters where all the letters are either A or B, the first letter is A, the last letter is B, and the sequence contains no three consecutive letters reading ABA. For example, count $AAABBABBAB$ and $ABBBBBBBAB$ but not $AABBAABABB$ or $AAAABBBBBA$.

Solution. **??** In the following program, we go through the list of number from 0 to $2^{10} - 1$, each number represent a possible sequence of 10 letters where all the letters are either A or B. The letter A is represented by 0, the letter B is represented by 1.

We first take a decimal number, convert it into binary format. Then we need to remove the prefix $0b$ that indicate the string represents a binary number. The binary string is then prefixed with 0 to ensure it contains 10 digits. Once done, we need to investigate whether the required conditions are satisfied (i) the number starts with 0, (ii) ends with 1, and does not include a substring 010.

```
count  = 0

for n in range(0, 2**10):
    b = bin(n)      # convert n from decimal into binary format
    x = b[2:]       # chop the unsuable '0b' prefix
    s = x.zfill(10) # padding the string with 0 digits

    if s.startswith('0') and s.endswith('1') and '010' not in s:
        count += 1

print(count)
```

The number of such sequence is $\boxed{86.}$ □

Problem 14.1.22 (PCT-2022-SM1-HS-P22)**.** Circle ω_1 has radius 7 and center C_1. Circle ω_2 has radius 23 and center C_2 with $C_1C_2 = 34$. Let a common internal tangent of ω_1 and ω_2 pass through A_1 on ω_1 and A_2 on ω_2, and let a common external tangent of ω_1 and ω_2 pass through B_1 on ω_1 and B_2 on ω_2 such that A_1 and B_1 lie on the same side of the line C_1C_2. Let P be the intersection of lines A_1A_2 and B_1B_2. Find the area of quadrilateral $PC_1A_2C_2$.

Solution. **??** Let E be the intersection of A_1A_2 and C_1C_2. Let F be the foot of the altitude from C_1 to C_2B_2.

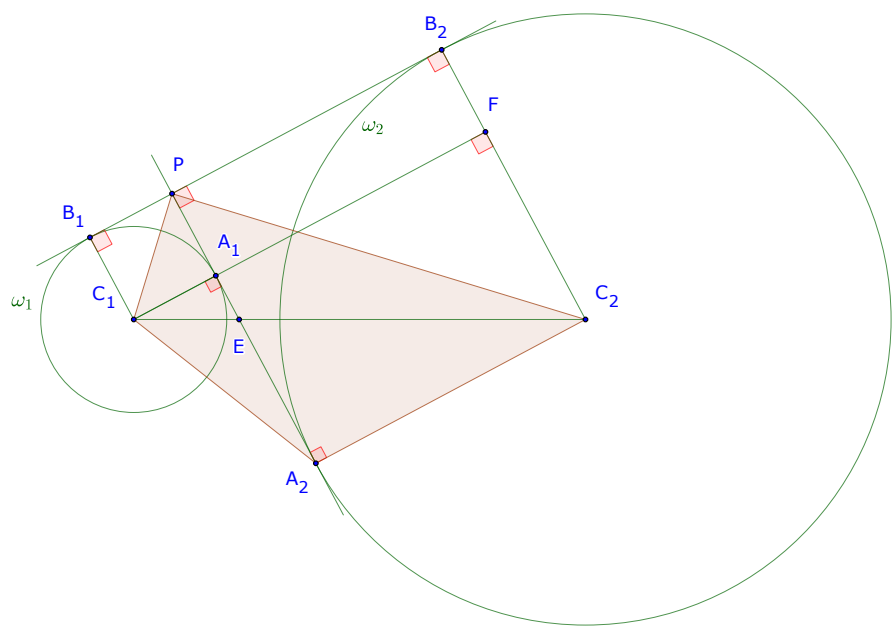


Figure 14.7: **??**

$$\frac{C_1F}{C_1C_2} = \frac{\sqrt{34^2 - (23-7)^2}}{34} = \frac{15}{17}, \ \frac{C_1A_1}{C_1E} = \frac{C_2A_2}{C_2E} = \frac{C_1A_1 + C_2A_2}{34} = \frac{15}{17}, \text{ so } C_1, A_1, F \text{ are collinear}$$

$$\Rightarrow [PC_1A_2C_2] = [B_1C_1A_2C_2B_2] - [B_1C_1P] - [B_2C_2P] = [B_1C_1FB_2] + [C_1A_2C_2F] - [B_1C_1P] - [B_2C_2P]$$

$$= B_1C_1 \cdot B_1B_2 + \frac{(C_1F + A_2C_2) \cdot C_2F}{2} - \frac{B_1C_1 \cdot B_1P}{2} - \frac{B_2C_2 \cdot PB_2}{2}$$

Thus, $[PC_1A_2C_2] = 7 \cdot 30 + \frac{53 \cdot 16}{2} - \frac{7 \cdot 7}{2} - \frac{23 \cdot 23}{2} = \boxed{345}$. □

Problem 14.1.23 (PCT-2022-SM1-HS-P23)**.** There are prime numbers a, b, and c such that the system of equations

$$\begin{cases} a \cdot x - 3 \cdot y + 6 \cdot z = 8 \\ b \cdot x + 3\frac{1}{2} \cdot y + 2\frac{1}{3} \cdot z = -28 \\ c \cdot x - 5\frac{1}{2} \cdot y + 18\frac{1}{3} \cdot z = 0 \end{cases}$$

has infinitely many solutions for (x, y, z). Find the product $a \cdot b \cdot c$.

Solution. **??** In order to have infinitely many solutions, the determinant must be zero,

$$\det A = \begin{vmatrix} a & -3 & 6 \\ b & \frac{7}{2} & \frac{7}{3} \\ c & -\frac{11}{2} & \frac{55}{3} \end{vmatrix}$$

Note that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Thus

$$\det A = a(\frac{7}{2} \cdot \frac{55}{3} - \frac{7}{3} \cdot (-\frac{11}{2})) - (-3)(b \cdot \frac{55}{3} - \frac{7}{3} \cdot c) + 6(b \cdot (-\frac{11}{2}) - c \cdot \frac{7}{2}) = 77a + 22b - 28c$$

Since $77a + 22b - 28c = 0 \Rightarrow 2 \mid a,\ 7 \mid b,\ 11 \mid c$, thus $a = 2, b = 7, c = 11 \Rightarrow abc = 2 \cdot 7 \cdot 11 = \boxed{154}$. □

Problem 14.1.24 (PCT-2022-SM1-HS-P24)**.** Find the number of permutations of the letters $AAABBBCCC$ where no letter appears in a position that originally contained that letter. For example, count the permutations $BBBCCCAAA$ and $CBCAACBBA$ but not the permutation $CABCACBAB$.

Solution. **??**

```
from itertools import permutations

count = 0
for p in permutations('AAABBBCCC'):
    s = ''.join(p)
    if 'A' not in s[0:3] and 'B' not in s[3:6] and 'C' not in s[6:9]:
        count += 1

print(count)
```

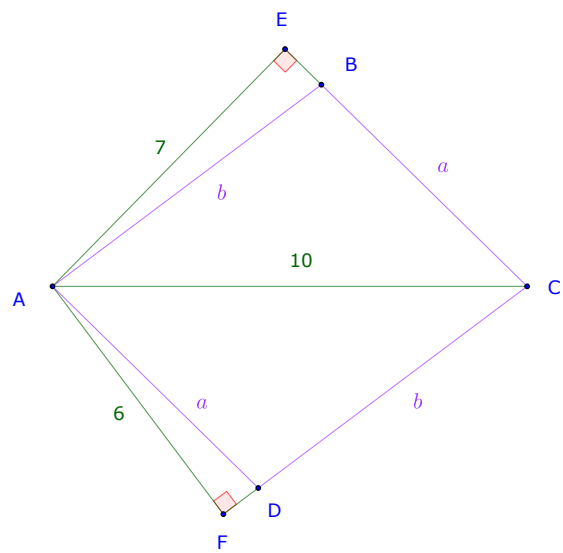
The number of such permutations is $\boxed{12096.}$

```
from itertools import permutations

count = 0
l = 3
o = l * 'A' + l * 'B' + l * 'C'
p_all = set()
for p in permutations(o):
    s = ''.join(p)
    if 'A' not in s[0:3] and 'B' not in s[3:6] and 'C' not in s[6:9]:
        if s not in p_all:
            p_all.add(s)
            count += 1

print(count)
```

□

Problem 14.1.25 (PCT-2022-SM1-HS-P25)**.** Let $ABCD$ be a parallelogram with diagonal $AC = 10$ such that the distance from A to line CD is 6 and the distance from A to line BC is 7. There are two non-congruent configurations of $ABCD$ that satisfy these conditions. The sum of the areas of these two parallelograms is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. **??** Let $BC = ADa, AB = CD = b$ as shown in the diagram below. First $EB = EC - a =$



$\sqrt{10^2 - 7^2} - a = \sqrt{51} - a$, and $FD = FC - DC = \sqrt{10^2 - 6^2} - b = 8 - b$. Furtheremore $\triangle ADF \sim \triangle ABE$, so

$$\begin{cases} \frac{AB}{AE} = \frac{AD}{AF} \Rightarrow \frac{b}{7} = \frac{a}{6} \\ \frac{EB}{AE} = \frac{FD}{AF} \Rightarrow \frac{\sqrt{51}-a}{7} = \frac{8-b}{6} \end{cases} \Rightarrow a = \frac{12}{13}(28 - 3\sqrt{51}),\ b = \frac{14}{13}(28 - 3\sqrt{51})$$

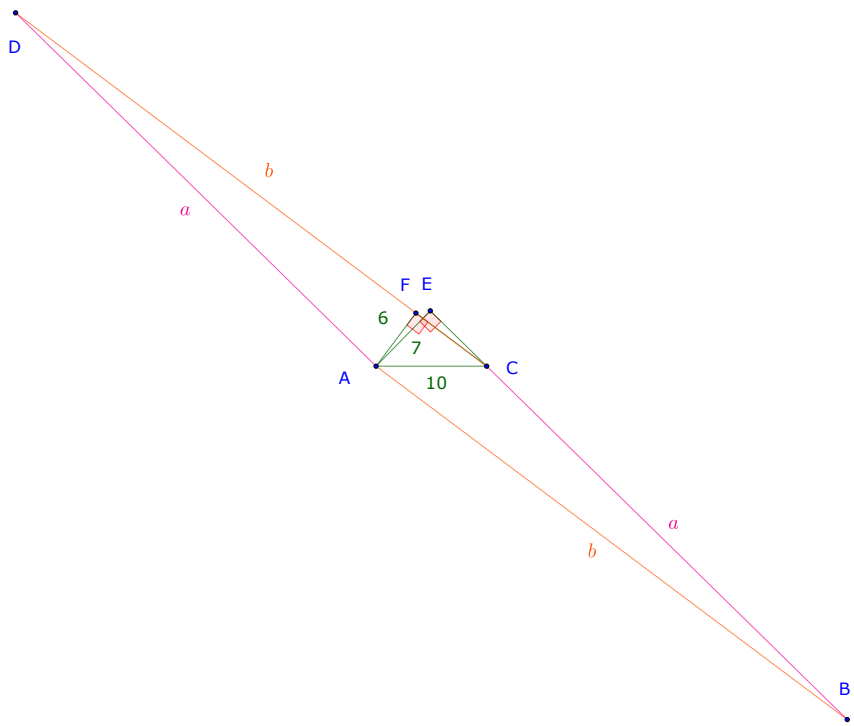
Note: using the below for quick solution with *wolfram*

```
solve \frac{b}{7} = \frac{a}{6}, \frac{\sqrt{51} - a}{7} = \frac{8 - b}{6} for a, b
```

Thus,

$$[ABCD] = \frac{1}{2}(7a + 6b) = \frac{1}{2} \cdot 7 * (\frac{12}{13}(28 - 3\sqrt{51})) + 6(\frac{14}{13}(28 - 3\sqrt{51})) = \frac{84}{13}(28 - 3\sqrt{51})$$

For the second case, First $EB = EC + a = \sqrt{10^2 - 7^2} + a = \sqrt{51} + a$, and $FD = DC - FC =$



$b - \sqrt{10^2 - 6^2} = b - 8$. Furtheremore $\triangle ADF \sim \triangle ABE$, so

$$\begin{cases} \frac{AB}{AE} = \frac{AD}{AF} \Rightarrow \frac{b}{7} = \frac{a}{6} \\ \frac{EB}{AE} = \frac{FD}{AF} \Rightarrow \frac{\sqrt{51}+a}{7} = \frac{b-8}{6} \end{cases} \Rightarrow a = \frac{12}{13}(28 + 3\sqrt{51}),\ b = \frac{14}{13}(28 + 3\sqrt{51})$$

Note: using the below for quick solution with *wolfram*

```
solve \frac{b}{7} = \frac{a}{6}, \frac{\sqrt{51} + a}{7} = \frac{b-8}{6} for a, b
```

Thus,

$$[ABCD] = \frac{1}{2}(7a + 6b) = \frac{1}{2} \cdot 7 * (\frac{12}{13}(28 + 3\sqrt{51})) + 6(\frac{14}{13}(28 + 3\sqrt{51})) = \frac{84}{13}(28 + 3\sqrt{51})$$

Therefore the sum of the two areas is $\frac{84}{13}(28-3\sqrt{51})+\frac{84}{13}(28+3\sqrt{51}) = \frac{4704}{13}$, $m+n = 4704+13 = \boxed{4717}$. □

Problem 14.1.26 (PCT-2022-SM1-HS-P26)**.** Antonio plays a game where he continually flips a fair coin to see the sequence of heads (H) and tails (T) that he flips. Antonio wins the game if he sees on four consecutive flips the sequence $TTHT$ before he sees the sequence $HTTH$. The probability that Antonio wins the game is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. **??** The probability that $TTHT$ preceeds $HTTH$ is $\frac{5}{12}$.

Thus, $m = 7, n = 12, m + n = 5 + 12 = \boxed{17}$. □

Problem 14.1.27 (PCT-2022-SM1-HS-P27)**.** For integer $k \geq 1$, let $a_k = \frac{k}{4k^4+1}$. Find the least integer n such that $a_1 + a_2 + a_3 + \ldots + a_n > \frac{505.45}{2022}$.

Solution. **??** Note that

$$\begin{aligned}
&4k^4 + 1 = 4k^4 + 4k^2 + 1 - 4k^2 = (2k^2+1)^2 - (2k)^2 = (2k^2+2k+1)(2k^2-2k+1) \\
&\frac{1}{2k^2-2k+1} - \frac{1}{2k^2+2k+1} = \frac{4k}{(2k^2+2k+1)(2k^2-2k+1)} \\
&\frac{1}{2k^2+2k+1} = \frac{1}{(k+1)^2+k^2}, \ \frac{1}{2k^2-2k+1} = \frac{1}{k^2+(k-1)^2} \\
\Rightarrow\ & a_k = \frac{1}{4}\left(\frac{1}{k^2+(k-1)^2} - \frac{1}{(k+1)^2+k^2}\right) \\
\Rightarrow\ & a_1 + a_2 + a_3 + \ldots + a_n = \frac{1}{4}\left(\frac{1}{1^2+0^2} - \frac{1}{(n+1)^2+n^2}\right) = \frac{1}{4}\left(1 - \frac{1}{(n+1)^2+n^2}\right)
\end{aligned}$$

Since $a_1 + a_2 + a_3 + \ldots + a_n > \frac{505.45}{2022}$,

$$1 - \frac{1}{(n+1)^2+n^2} > \frac{4 \cdot 505.45}{2022} \Rightarrow (n+1)^2 + n^2 > 10110 \Rightarrow n \geq \boxed{71}$$

Note that

$$\sum_{k=1}^{n} \frac{k}{4k^4+1} = \frac{1}{4}\left(1 - \frac{1}{(n+1)^2+n^2}\right) \quad \text{and} \quad 1 - \frac{1}{72^2+71^2} > \frac{2021.8}{2022} > 1 - \frac{1}{71^2+70^2}$$

□

Problem 14.1.28 (PCT-2022-SM1-HS-P28)**.** Six gamers play a round-robin tournament where each gamer plays one game against each of the other five gamers. In each game there is one winner and one loser where each player is equally likely to win that game, and the result of each game is independent of the results of the other games. The probability that the tournament will end with exactly one gamer scoring more wins than any other player is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. **??** Let represent the scores by a table as below. 1 is at row i and column j means player p_i wins against player p_j, 0 means the opposite. Since only the upper-half of the table is filled (to prevent dealing with too many numbers), the scores of the players need to be calculated as follow. The column $'1'$ contains how many wins for each player in the rows, against the players listed in the top row. The row $'0'$ means how many wins for each player in the columns, against the players listed in the left-most column. The scores for player $p_1, p_2, \ldots, p_6$ is listed in the w column, each is the sum of the corresponding wins in the column $'1'$ and row $'0'$.

	p1	p2	p3	p4	p5	p6	'1'	w					
p1		0	1	0	1	1	**3**	3	0	1	2	3	4
p2			1	1	1	1	**4**	5		5	6	7	8
p3				1	0	0	**1**	1			9	10	11
p4					1	0	**1**	2				12	13
p5						0	**0**	1					14
p6							**0**	3					
'0'		**1**	**0**	**1**	**1**	**3**							

Figure 14.8: **??**

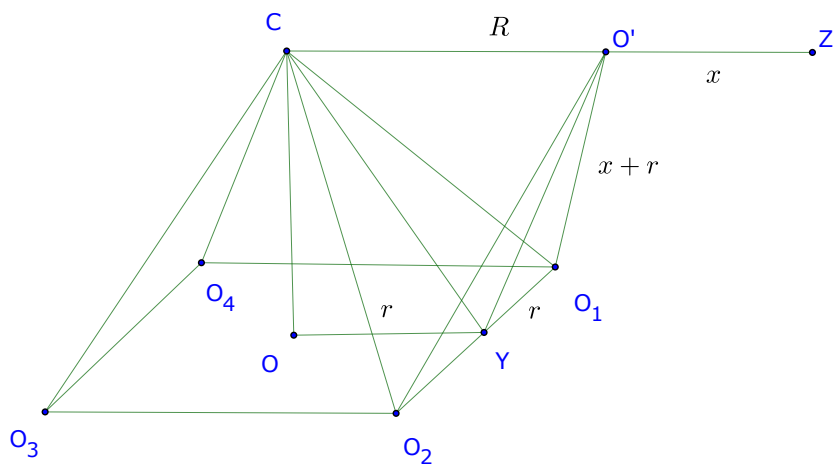
A $15-digit$ binary number is formed by putting together rows $1, 2, \ldots, 5$, for example 010111111100100. The correct cases are counted as in below. We need to find among the five counts $p1, p2, \ldots, p6$, there is only one number strictly larger then all other numbers.

```
count  = 0

for n in range(0, 2**15):
    b = bin(n)      # convert n from decimal into binary format
    x = b[2:]       # chop the unsuable '0b' prefix
    s = x.zfill(15) # padding the string with 0 digits
    l = [int(i) for i in s]

    p1 = l[0] + l[1] + l[2] + l[3] + l[4]
    p2 = (1-l[0]) + l[5] + l[6] + l[7] + l[8]
    p3 = (1-l[1]) + (1-l[5]) + l[9] + l[10] + l[11]
    p4 = (1-l[2]) + (1-l[6]) + (1-l[9]) + l[12] + l[13]
    p5 = (1-l[3]) + (1-l[7]) + (1-l[10]) + (1-l[12]) + l[14]
    p6 = (1-l[4]) + (1-l[8]) + (1-l[11]) + (1-l[13]) + (1-l[14])

    # print([p1, p2, p3, p4, p5, p6])
    m, c = 0, 0
    for p in [p1, p2, p3, p4, p5, p6]:
        if m < p:
            m = p
            c = 1
        elif m == p:
            c += 1
    if c == 1:
        count += 1

print(count)
```

The number of such cases is 20544, therefore the probability is $\frac{20544}{2^{15}} = \frac{321}{512}$, thus $m+n = 321+512 = \boxed{833}$. $\square$

Problem 14.1.29 (PCT-2022-SM1-HS-P29)**.** Sphere S with radius 100 has diameter AB and center C. Four small spheres all with radius 17 have centers that lie in a plane perpendicular to AB such that each of the four spheres is internally tangent to S and externally tangent to two of the other small spheres. Find the radius of the smallest sphere that is both externally tangent to two of the four spheres with radius 17 and internally tangent to S at a point in the plane perpendicular to AB at C.

Solution. **??** Let $O_1, \ldots, O_4$ be the centres of the small spheres. Let O' be the centre of the sixth plane. The plane containing them intersects AB at O. Let Y be the tangent point of (O_1) and (O_2). Let Z be the point on the plane perpendicular to AB at C where (O') tangent with S. Let x be the radius of (O'). $\triangle CO_1O_2$ is



isosceles, $CO_1 = R - r, O_1Y = r$, thus $CY = \sqrt{(R-r)^2 - r^2} = \sqrt{R^2 - 2Rr}$.

$\triangle COY$ is right at O, $OY = r$, so $CO = \sqrt{CY^2 - OY^2} = \sqrt{R^2 - 2Rr - r^2}$.

$\angle CYO = \angle YCZ \Rightarrow \cos\angle YCZ = \cos\angle CYO = \frac{r}{\sqrt{R^2-2Rr}}$.

In $\triangle O_1O_2O'$, $O'Y = \sqrt{(x+r)^2 - r^2} = \sqrt{x^2 + 2xr}$.

By the Law of Cosines for $\triangle O'YC$,

$$\begin{aligned}
&O'Y^2 = O'C^2 + CY^2 - 2 \cdot O'C \cdot CY \cdot \cos\angle YCO' \\
&\Rightarrow x^2 + 2xr = (R-x)^2 + (R^2 - 2Rr) - 2(R-x)\sqrt{R^2 - 2Rr}\frac{r}{\sqrt{R^2 - 2Rr}} \\
&\Rightarrow x^2 + 2xr = R^2 - 2Rx + x^2 + R^2 - 2Rr - 2Rr + 2xr \Rightarrow 2Rx = 2R^2 - 4Rr \\
&\Rightarrow x = R - 2r = 100 - 34 = \boxed{66}
\end{aligned}$$

□

Problem 14.1.30 (PCT-2022-SM1-HS-P30)**.** There is a positive integer s such that there are s solutions to the equation

$$64\sin^2(2x) + \tan^2(x) + \cot^2(x) = 46$$

in the interval $\left[0, \frac{\pi}{2}\right]$ all of the form $\frac{m_k}{n_k}\pi$, where m_k and n_k are relatively prime positive integers, for $k = 1, 2, 3, \ldots, s$. Find $(m_1 + n_1) + (m_2 + n_2) + (m_3 + n_3) + \ldots + (m_s + n_s)$.

Solution. **??**

$$\tan^2(x) + \cot^2(x) + 2 = (\tan x + \cot x)^2 = \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)^2 = \left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right)^2 = \left(\frac{1}{\frac{\sin 2x}{2}}\right)^2 = \frac{4}{\sin^2 2x}$$

Now, let $y = \sin^2 2x$, then

$$64y + \frac{4}{y} - 2 = 46 \Rightarrow y = \frac{3 \pm \sqrt{5}}{8} = \left(\frac{\sqrt{5} \pm 1}{4}\right)^2 \Rightarrow \sin 2x = \frac{\sqrt{5} \pm 1}{4} \Rightarrow 2x \in \left\{\frac{\pi}{10}, \frac{3\pi}{10}\right\} \Rightarrow x \in \left\{\frac{\pi}{20}, \frac{3\pi}{20}\right\}$$

Thus, the desired sum is $1 + 20 + 3 + 20 = \boxed{44.}$ $\square$

Chapter 15

ICLT April

15.1 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

15.2 Problems for Level 1

Problem 15.2.1 (ICLT-2022-SM1-4-L1-P1). (*4 points*)

Factor $2a^3 - 4a^2 - 2a$ as completely as you can.

(A) $2(a^3-2a^2-a)$ (B) $2a(a^2-2a-1)$ (C) $2a(a-1)^2$ (D) $2a(a^2-1)$ (E) $2a(a+1)^2$

Problem 15.2.2 (ICLT-2022-SM1-4-L1-P2). (*4 points*)

What is the maximum number of possible points of intersection of a circle and a square?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 15.2.3 (ICLT-2022-SM1-4-L1-P3). (*4 points*)

There are 27 cats at the pound. 14 of them are short-haired. 11 of them are kittens. 6 of them are long-haired adult cats (not kittens). How many of them are short-haired kittens?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 15.2.4 (ICLT-2022-SM1-4-L1-P4). (*6 points*)

Take 30 toothpicks and make a single pile out of them. Because you can make 1 pile of 24 toothpicks, 1 is a divisor of 24. From the 30 toothpicks you can also make 2 piles with an equal number of toothpicks in each pile. This means that 2 is also a divisor of 24. How many different integers that can be the number of equal piles of toothpicks out of the original 24?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 15.2.5 (ICLT-2022-SM1-4-L1-P5). (*6 points*)

Cindy was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3, giving an answer of 40. What would her answer have been had she worked the problem correctly?

(A) 13 (B) 14 (C) 15 (D) 16 (E) 17

Problem 15.2.6 (ICLT-2022-SM1-4-L1-P6). (*6 points*)

Given $AB \parallel CD$ in the diagram, find $\angle AHF$.

(A) 58° (B) 132° (C) 68° (D) 122° (E) 112°

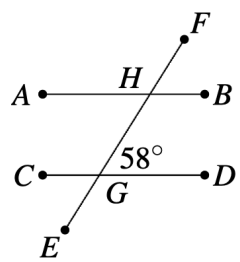


Figure 15.1: **??**

Problem 15.2.7 (ICLT-2022-SM1-4-L1-P7). (*10 points*)

Let S be a cube.

1. (*5 points*) How many planes which pass through at least three vertices of S such that those vertices also on the same face of S?
2. (*5 points*) Compute the number of planes which pass through at least three vertices of S.

Problem 15.2.8 (ICLT-2022-SM1-4-L1-P8). (*10 points*)

Each week, between 40 and 70 students show up for an archery class run by Betty, Wilma. Usually the students break up into groups of equal size for target practice. However, this week, Betty noticed that she could not break the students up into multiple groups of equal size. Wilma noticed that if Betty and Wilma joined the students in practicing, they still could not break the archers up into groups of equal size.

1. (*5 points*) What is the minimal number of students showed up to the archery class this week to make this situation possible?
2. (*5 points*) There are several cases with different numbers of students to make this situation possible. Find all of them.

Problem 15.2.9 (ICLT-2022-SM1-4-L1-P9). (*10 points*)

Eight points are marked on the circumference of a circle.

1. (*5 points*) How many different chords can be drawn by connecting two of these eight points?
2. (*5 points*) How many different chords can be drawn by connecting two of these eight points such that no two chords can share any common point, including their vertices?

15.3 Grading for Level 1

Answers for multiple-choice problems.

Problem 1:	C	$2a(a^2 - 2a - 1)$
Problem 2:	E	8
Problem 3:	A	4
Problem 4:	E	8
Problem 5:	B	14
Problem 6:	D	122

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 2 if find 6 faces of the cube.

(b) 2 if can find planes that can contain parallel opposite edges of the cube (not on the same face).

Problem 8: Separately for each part,

(a) 2 points if can figure out that the solution is the smaller prime in a prime pair with difference of 2.

(b) 2 points if can identify all prime pairs with difference of 2 (such as $(41, 43)$) with in 40 and 70.

Problem 9: Separately for each part,

(a) 2 points if can find count the number of other points (7) can be connected from a point.

(b) 2 points if can find that none of the point can be a vertex of two different chords.

15.4 Solutions for Level 1

Solution. **??**

$$2a^3 - 4a^2 + 2a = (2a)a^2 - (2a)2a - (2a)1 = 2a(a^2 - 2a - 1)$$

The answer is $\boxed{B.}$ □

Solution. **??** It is easy to see that maximum number of intersections of a circle and a segment is two. Since the square has four sides, so the maximum number of intersection points of a circle and a square is $2 \cdot 4 = \boxed{8.}$

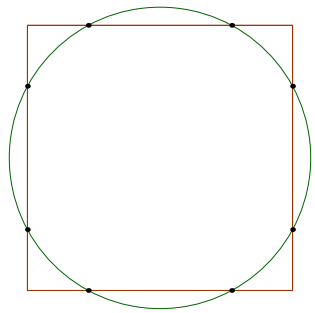


Figure 15.2: 8 intersection points

The answer is $\boxed{E.}$ □

Solution. **??** We draw a Venn diagram. Let x be the number of short hair kittens. The number of short-hair

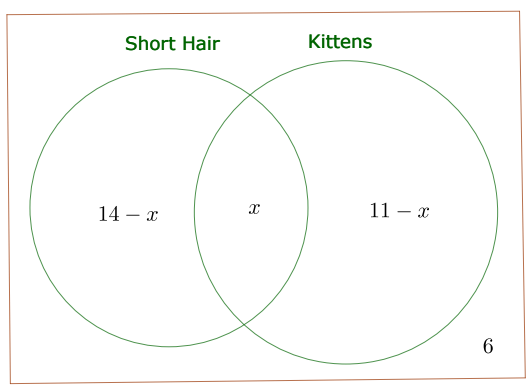


Figure 15.3: x short hair kittens

cats that are not kittens is $14 - x$. The number of kittens cats that are not short hair is $11 - x$. Therefore,

$$(14 - x) + x + (11 - x) + 6 = 27 \Rightarrow x = 4$$

The answer is $\boxed{A.}$ □

Solution. **??** We begin by translating this problem into the language of mathematics. The goal is to find possible numbers of groups containing equal numbers of toothpicks where the total in all the groups is 24. In other words, we are looking for the positive divisors of 24.

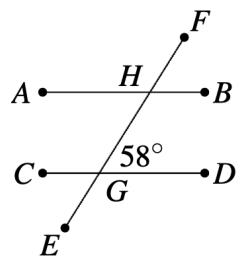
Now that we have turned this word problem into a math problem, we hunt for the positive divisors of 24. We can test 24 for divisibility by all the positive integers from 1 to 24. The positive divisors of 24 are $1, 2, 3, 4, 6, 8, 12$, and 24.

The answer is $\boxed{E.}$ □

Solution. **??** Let Cindy's number be c. She subtracted 9 then divided the result by 3, so she computed $\frac{c-9}{3}$. She got an answer of 40 when she did this, so $\frac{c-9}{3} = 40$. We multiply both sides by 3 to get $c - 9 = 120$. We then add 9 to both sides to find $c = 129$. Now, we compute the answer Cindy should have found. She was supposed to subtract 3 from this number, which would have given her $129 - 3 = 126$. Then, she was supposed to divide by 9, which would give her $\frac{126}{9} = \boxed{14}$.

The answer is $\boxed{B.}$ □

Solution. **??** BHG and $\angle HGD$ are same-side interior angles, so $\angle BHG + \angle HGD = 180°$ and $\angle BHG =$



$180° - 58° = 122°$. $\angle AHF$ is vertical to $\angle BHG$, so $\angle AHF = \boxed{122°}$.

The answer is $\boxed{D.}$ □

Solution. **??** There are three types of planes that can go through three points in the cube. The plane can contain an entire face of the cube, or it can contain parallel opposite edges of the cube (not on the same face), or it can fail to contain an edge of the cube.

Case 1: The plane contains a face. There are 6 faces, and therefore $\boxed{6}$ planes that pass through them. This is the answer for the first question.

Case 2: The plane contains parallel opposite edges. There are 12 edges, and so $\frac{12}{2} = 6$ pairs of parallel opposite edges.

Case 3: The plane does not contain an edge. For example, in the diagram below, the plane passes through vertices A, B, and C. Note that A, B and C all share an edge with vertex D. Every plane which does not contain an edge of the cube must pass through 3 vertices which all share an edge with a unique vertex as in this example, and each vertex uniquely determines such a plane. Since the cube has 8 vertices, there are 8 planes that do not contain an edge of the cube.

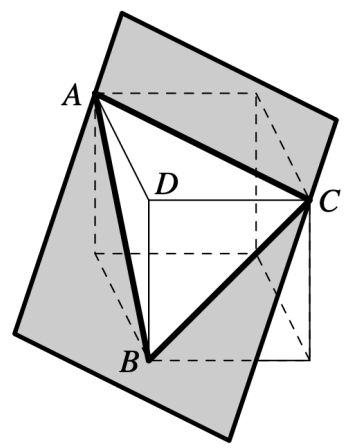


Figure 15.4: Plane through A, B, and C

For the second question, the sum of the three cases is $6 + 6 + 8 = \boxed{20}$ planes. □

Solution. **??** For the first question, let n be the number of students at class. Betty noticed that n has no divisors between 1 and itself, so n is prime. Wilma noticed that $n + 2$ is also prime. This means we are looking for the smaller of two primes that differ by 2 that are between 40 and 70. The primes in that range are 41, 43, 47, 53, 59, 61, and 67. The smallest of such prime is $\boxed{41.}$

For the second question, there are $\boxed{2}$ such pairs $(41, 43)$, and $(59, 61)$. □

Solution. **??** For the first question, we can choose two out of eight points (without regard to order) in $8 \times 72 = 28$ ways, so there are $\boxed{28}$ chords.

For the second question, it is easy to see that the chords cannot share any vertices so, each of these eight points is connected with only one other vertex by a single chord. There are at most $\boxed{4}$ such chords.

We can easily show that it is possible by separating the eight points $A_1, A_2, \ldots, A_8$, ordered clockwise, into four pairs $(A_1, A_2), \ldots, (A_3, A_4)$, then going clockwise and connecting the pairs. □

15.5 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
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     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
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     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

15.6 Problems for Level 3

Problem 15.6.1 (ICLT-2022-SM1-4-L3-P1). (*4 points*)

What is the ratio of x to y if

$$\frac{10x-3y}{15x-2y}=\frac{3}{5}.$$

$(A)\ \frac{9}{2}$ $(B)\ \frac{9}{5}$ $(C)\ \frac{9}{7}$ $(D)\ \frac{9}{11}$ $(E)\ \frac{9}{13}$

Problem 15.6.2 (ICLT-2022-SM1-4-L3-P2). (*4 points*)

Find the product $BC \cdot DC$ given $AD = 3, BD = 4$, and $AB = 5$.

$(A)\ \frac{8}{3}$ $(B)\ \frac{16}{3}$ $(C)\ \frac{20}{3}$ $(D)\ \frac{220}{9}$ $(E)\ \frac{320}{9}$

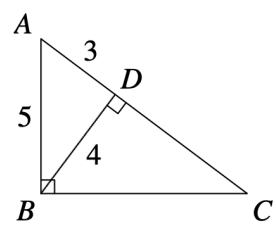


Figure 15.5: **??**

Problem 15.6.3 (ICLT-2022-SM1-4-L3-P3). (*4 points*)

How many of the positive divisors of 120 are even?

$(A)\ 4$ $(B)\ 6$ $(C)\ 8$ $(D)\ 12$ $(E)\ 16$

Problem 15.6.4 (ICLT-2022-SM1-4-L3-P4). (*6 points*)

I have a bag with coins and beads. Each object in the bag is either silver or gold. Twenty percent of the objects in the bag are beads and half of the coins in the bag are gold. What percent of the objects are silver coins?

$(A)\ 30$ $(B)\ 35$ $(C)\ 40$ $(D)\ 45$ $(E)\ 50$

Problem 15.6.5 (ICLT-2022-SM1-4-L3-P5). (*6 points*)

Five chickens eat 10 bags of food in 20 days. How long does it take 22 chickens to eat 55 bags of food?

$(A)\ 55$ $(B)\ 55\frac{5}{9}$ $(C)\ 56$ $(D)\ 56\frac{5}{9}$ $(E)\ 57$

Problem 15.6.6 (ICLT-2022-SM1-4-L3-P6). (*6 points*)

Given $AC = 4, CD = 5$, and $AB = 6$ as in the diagram. Find BD if the perimeter of $\triangle BCD$ is 20.

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

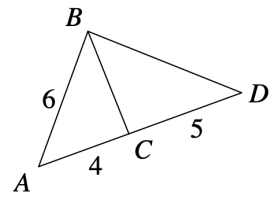


Figure 15.6: **??**

Problem 15.6.7 (ICLT-2022-SM1-4-L3-P7). (*10 points*)

1. (*5 points*) How many numbers between $1, 2, \ldots, 30$ are divisible by 3.
2. (*5 points*) Find the highest power of 3 that divides 30!.

Problem 15.6.8 (ICLT-2022-SM1-4-L3-P8). (*10 points*)

64 unit cubes ($1 \times 1 \times 1$ cubes) are glued together to form a $4 \times 4 \times 4$ large cube.

1. (*5 points*) At most how many of faces of the large cube are visible from a single point in space?
2. (*5 points*) At most how many of these unit cubes are visible from a single point in space?

Problem 15.6.9 (ICLT-2022-SM1-4-L3-P9). (*10 points*)

In the diagram at right, the angle equalities and side lengths of $\triangle PQR$ and $\triangle ABC$ are given.

1. (*5 points*) What is the value of $y + 2x$?
2. (*5 points*) What is the ratio of the areas $\frac{[PQR]}{[ABC]}$?

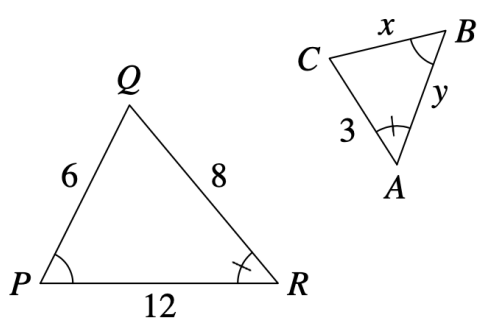


Figure 15.7: **??**

15.7 Grading for Level 3

Answers for multiple-choice problems.

Problem 1:	B	$\frac{9}{5}$
Problem 2:	E	$\frac{320}{9}$
Problem 3:	D	12
Problem 4:	C	40
Problem 5:	A	55
Problem 6:	D	9

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 2 if can find all multiples of 3 between 1 and 30.

(b) 2 if can find the multiples of $3^2, 3^3$ between 1 and 30.

Problem 8: Separately for each part,

(a) 2 points if can figure out that 4 faces of the large cube can be seen from a single point.

(b) 2 points if can figure out that there is a $3 \times 3 \times 3$ cube hidden from view.

Problem 9: Separately for each part,

(a) 2 points if can find $\triangle ABC \sim \triangle RPQ$.

(b) 2 points if can show ratio of areas is square of ratio of similarity.

15.8 Solutions for Level 3

Solution. **??**

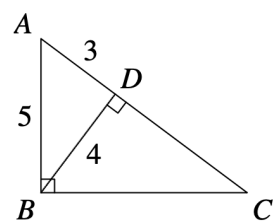
$$\frac{10x-3y}{15x-2y}=\frac{3}{5}\Rightarrow 5(10x-3y)=3(15x-2y)\Rightarrow 50x-15y=45x-6y\Rightarrow 5x=9y\Rightarrow \frac{x}{y}=\boxed{\frac{9}{5}}$$

The answer is $\boxed{B.}$ □

Solution. **??** Since $\angle BAD=\angle CAB$ and $\angle BDA=\angle CBA$, we have $\triangle BAD\sim\triangle CAB$ by AA Similarity. Therefore, we have

$$\frac{BC}{BD}=\frac{AB}{AD}=\frac{5}{3}\Rightarrow BC=\left(\frac{5}{3}\right)(BD)=\frac{20}{3}.$$

We can use this same similarity to find AC, and then subtract AD to get CD We could also note that



$\angle BCD=\angle BCA$ and $\angle BDC=\angle CBA$, so $\triangle BCD\sim\triangle ACB$ by AA Similarity. Therefore,

$$\frac{CD}{BD}=\frac{BC}{AB}=\frac{\frac{20}{3}}{5}=\frac{4}{3}\Rightarrow CD=\left(\frac{4}{3}\right)(BD)=\frac{16}{3}.$$

Thus, $BD\cdot DC=\frac{20}{3}\cdot\frac{16}{3}=\boxed{\frac{320}{9}.}$

The answer is $\boxed{E.}$ □

Solution. [First solution] **??** It's usually helpful in a divisor problem to start with prime factorization:

$$120=2^3\cdot 3^1\cdot 5^1.$$

We know that each positive divisor of 120 has a prime factorization in the form $2^a\cdot 3^b\cdot 5^c$ where a is a whole number no greater than 3, where b is a whole number no greater than 1, and where c is a whole number no greater than 1. We want to count only the even divisors, which are the ones with prime factorizations where a is at least 1. This leaves us with 3 choices for a (1, 2, or 3), and 2 choices for each of b and c (0 or 1). Multiplying the numbers of choices for each exponent in the prime factorization of an even divisor of 120, we find that the total number of even divisors of 120 is $3\cdot 2\cdot 2=\boxed{12}$.

The answer is $\boxed{D.}$ □

Solution. [Second solution] **??** If a divisor of 120 is an even number, let say $2n$, then n is a divisor of 60, and since,

$$60=2^2\cdot 3^1\cdot 5^1.$$

The total number of divisors of 60 is $3\cdot 2\cdot 2=\boxed{12}$.

The answer is $\boxed{D.}$ □

Solution. **??** Since 20% of the objects in the bag are beads, $100\% - 20\% = 80\%$ of the objects in the bag are coins. Half of the coins in the bag are gold, so half of the coins must be silver. Thus $\frac{80\%}{2} = \boxed{40\%}$ of the objects are silver coins.

The answer is $\boxed{C.}$ □

Solution. **??** Doubling the number of chickens doubles the bags of food eaten, so the number of chickens and bags of food are *directly proportional.* If we want twice as many bags of food eaten but we have the same number of chickens, we must double the number of days, so the bags of food and the amount of time are *directly proportional.*

Putting these together, we see that the bags of food is *jointly proportional* to the number of chickens and the amount of time. So, we see that the expression

$$\frac{(\text{Time in days})(\text{Number of chickens})}{(\text{Bags of scratch})}$$

is constant. Using the given information, we find:

$$\frac{(\text{Time in days})(\text{Number of chickens})}{(\text{Bags of scratch})} = \frac{(20)(5)}{10} = 10.$$

So, if we have 22 chickens and we want to find out how long it takes them to eat 55 bags of scratch, we must have

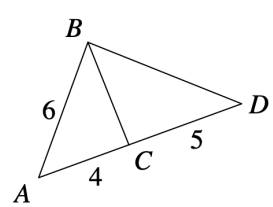
$$\frac{(\text{Time in days})(22)}{55} = 10 \Rightarrow \frac{(\text{Time in days})(2)}{5} = 10$$

Multiplying both sides by $\frac{5}{2}$, we find that the required time is

$$10\left(\frac{5}{2}\right) = \boxed{55}\text{days.}$$

The answer is $\boxed{A.}$ □

Solution. **??**



$$\frac{AC}{AB} = \frac{4}{6} = \frac{2}{3} \quad \text{and} \quad \frac{AB}{AD} = \frac{6}{9} = \frac{2}{3} \Rightarrow \triangle ACB \sim \triangle ABD \text{ by SAS}$$

Since the sides of $\triangle ACB$ are $\frac{2}{3}$ the corresponding sides of $\triangle ABD$, we have $BC = \frac{2}{3}(BD)$. Now we can use that perimeter information. Since

$$BC + CD + DB = 20 \Rightarrow \frac{2}{3}(BD) + 5 + BD = 20 \Rightarrow \frac{5}{3}(BD) = 15 \Rightarrow BD = \boxed{9.}$$

The answer is $\boxed{D.}$ □

Solution. **??** For the first question, note that there are $\frac{30}{3} = \boxed{10}$ multiples of 3 between 1 and 30,

$$1 \cdot 3 = 3, 2 \cdot 3 = 6, \ldots, 10 \cdot 3 = 30.$$

For the second question, each multiple of 3 has a factor of 3 in its prime factorization. Furthermore, there are $\left\lfloor \frac{30}{3^2} \right\rfloor = 3$ multiples of 3^2 between 1 and 30,

$$1 \cdot 3^2 = 9, 2 \cdot 3^2 = 18, \ldots, 3 \cdot 3^2 = 27.$$

and each of these number has an addtional factor of 3 in its prime factorization. Finally there is $\left\lfloor \frac{30}{3^3} \right\rfloor = 1$ multiples of 3^3 between 1 and 30, which is $1 \cdot 3^3 = 27$, and this number has an addtional factor of 3 in its prime factorization.

Thus, the total number of factors of 3 in 30! is $10 + 3 + 1 = \boxed{14}$. □

Solution. [First solution] **??** For the first question, wWhen you look from a point so that you can see at most $\boxed{3}$ faces.

For the second question, to maximize the number of unit cubes visible, you need to see 3 faces. Therefore a $3 \times 3 \times 3$ cube is hidden from view, thus the number of small visible cubes is $4^3 - 3^3 = \boxed{37}$. □

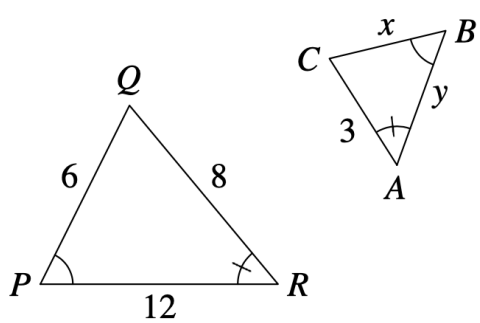
Solution. [Second solution] **??** For the first question, when you look from a point so that you can see at most $\boxed{3}$ faces.

For the second question, to maximize the number of unit cubes visible, you need to see 3 faces. If we add the numbers of cubes on the faces together, then every two faces share four cubes, and three faces share on cube, thus

$$3 \cdot 4 \cdot 4 - 3 \cdot 4 + 1 = \boxed{37}.$$

□

Solution. **??** For the first question, Since $\angle A = \angle R$ and $\angle B = \angle P$, we have $\triangle ABC \sim \triangle RPQ$.



Therefore, we have $\frac{AB}{RP} = \frac{BC}{PQ} = \frac{AC}{RQ}$. Substitution gives $\frac{y}{12} = \frac{x}{6} = \frac{3}{8}$, from which we find $y = (12)(\frac{3}{8}) = \frac{9}{2}$ and $x = (6)(\frac{3}{8}) = \frac{9}{4}$. Thus $y + 2x = \boxed{9}$.

For the second question, $\frac{[PQR]}{[ABC]} = \left(\frac{RQ}{AC}\right)^2 = \left(\frac{8}{3}\right)^2 = \boxed{\frac{64}{9}.}$ □

15.9 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

15.10 Problems for Level 4

Problem 15.10.1 (ICLT-2022-SM1-4-L4-P1). (*4 points*)

Five chickens eat 10 bags of food in 20 days. How long does it take 22 chickens to eat 55 bags of food?

(A) 55 (B) $55\frac{5}{9}$ (C) 56 (D) $56\frac{5}{9}$ (E) 57

Problem 15.10.2 (ICLT-2022-SM1-4-L4-P2). (*4 points*)

Given $AC = 4, CD = 5$, and $AB = 6$ as in the diagram. Find BD if the perimeter of $\triangle BCD$ is 20.

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

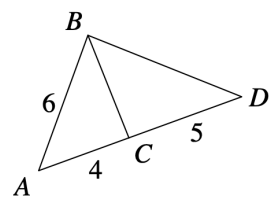


Figure 15.8: **??**

Problem 15.10.3 (ICLT-2022-SM1-4-L4-P3). (*4 points*)

Find the highest power of 3 that divides 30!.

(A) 10 (B) 12 (C) 14 (D) 16 (E) 18

Problem 15.10.4 (ICLT-2022-SM1-4-L4-P4). (*6 points*)

64 unit cubes ($1 \times 1 \times 1$ cubes) are glued together to form a $4 \times 4 \times 4$ large cube. At most how many of these unit cubes are visible from a single point in space?

(A) 27 (B) 37 (C) 40 (D) 45 (E) 48

Problem 15.10.5 (ICLT-2022-SM1-4-L4-P5). (*6 points*)

Five points A, B, C, D, E are given with coordinates. There is a straight line passes through exactly four points. Find the point that the straight line does not pass through.

(A) $A(1,3)$ (B) $B(4,1)$ (C) $C(7,-1)$ (D) $D(-2,5)$ (E) $E(-1,4)$

Problem 15.10.6 (ICLT-2022-SM1-4-L4-P6). (*6 points*)

Find the area of $\triangle CEF$.

(A) 24 (B) 48 (C) 64 (D) 96 (E) 192

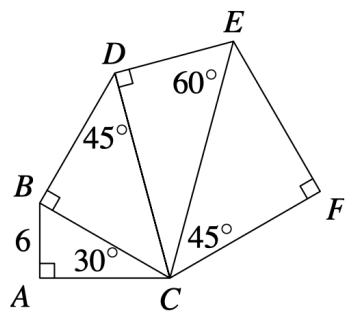


Figure 15.9: **??**

Problem 15.10.7 (ICLT-2022-SM1-4-L4-P7). (*10 points*)

1. (*5 points*) Factor the expression $a^4 - b^2$.
2. (*5 points*) Find all (a, b) positive integer pair such that $a^4 - b^2 = 45$.

Problem 15.10.8 (ICLT-2022-SM1-4-L4-P8). (*10 points*)

1. (*5 points*) We have a cube with 4 blue faces and 2 red faces. What is the probability that when it is rolled, a red face will be facing up?
2. (*5 points*) We have a cube with 3 blue faces, 2 red faces, and 1 white face. What is the probability that when it is rolled, a red face will be facing up?

Problem 15.10.9 (ICLT-2022-SM1-4-L4-P9). (*10 points*)

Three cards are chosen at random from a standard 52-card deck.

1. (*5 points*) What is the probability that they are all the same color?
2. (*5 points*) What is the probability that they are not all the same color?

15.11 Grading for Level 4

Answers for multiple-choice problems.

Problem 1:	A	55
Problem 2:	D	9
Problem 3:	C	14
Problem 4:	B	37
Problem 5:	E	$E(-1, 4)$
Problem 6:	D	96

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 2 if can use $a^2 - b^2 = (a - b)(a + b)$.

(b) 2 if can casework for three different factorings.

Problem 8: Separately for each part,

(a) 2 points if can establish the number of favourable outcomes and total number of outcomes.

(b) 2 points if can establish the number of favourable outcomes and total number of outcomes, even if there are more then two types of outcomes.

Problem 9: Separately for each part,

(a) 2 points if can find the number of favorable outcomes $2 \times \binom{26}{3}$.

(b) 2 points if can use complementary probability (result from the first question).

15.12 Solutions for Level 4

Solution. **??**

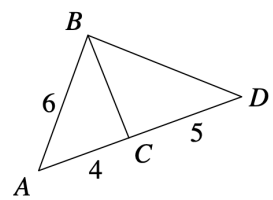
$$\frac{10x-3y}{15x-2y}=\frac{3}{5}\Rightarrow 5(10x-3y)=3(15x-2y)\Rightarrow 50x-15y=45x-6y\Rightarrow 5x=9y\Rightarrow \frac{x}{y}=\boxed{\frac{9}{5}}$$

The answer is $\boxed{B.}$ □

Solution. **??** Since $\angle BAD=\angle CAB$ and $\angle BDA=\angle CBA$, we have $\triangle BAD\sim\triangle CAB$ by AA Similarity. Therefore, we have

$$\frac{BC}{BD}=\frac{AB}{AD}=\frac{5}{3}\Rightarrow BC=\left(\frac{5}{3}\right)(BD)=\frac{20}{3}.$$

We can use this same similarity to find AC, and then subtract AD to get CD We could also note that



$\angle BCD=\angle BCA$ and $\angle BDC=\angle CBA$, so $\triangle BCD\sim\triangle ACB$ by AA Similarity. Therefore,

$$\frac{CD}{BD}=\frac{BC}{AB}=\frac{\frac{20}{3}}{5}=\frac{4}{3}\Rightarrow CD=\left(\frac{4}{3}\right)(BD)=\frac{16}{3}.$$

Thus, $BD\cdot DC=\frac{20}{3}\cdot\frac{16}{3}=\boxed{\frac{320}{9}.}$

The answer is $\boxed{E.}$ □

Solution. [First solution] **??** It's usually helpful in a divisor problem to start with prime factorization:

$$120=2^3\cdot 3^1\cdot 5^1.$$

We know that each positive divisor of 120 has a prime factorization in the form $2^a\cdot 3^b\cdot 5^c$ where a is a whole number no greater than 3, where b is a whole number no greater than 1, and where c is a whole number no greater than 1. We want to count only the even divisors, which are the ones with prime factorizations where a is at least 1. This leaves us with 3 choices for a (1, 2, or 3), and 2 choices for each of b and c (0 or 1). Multiplying the numbers of choices for each exponent in the prime factorization of an even divisor of 120, we find that the total number of even divisors of 120 is $3\cdot 2\cdot 2=\boxed{12}$.

The answer is $\boxed{D.}$ □

Solution. [Second solution] **??** If a divisor of 120 is an even number, let say $2n$, then n is a divisor of 60, and since,

$$60=2^2\cdot 3^1\cdot 5^1.$$

The total number of divisors of 60 is $3\cdot 2\cdot 2=\boxed{12}$.

The answer is $\boxed{D.}$ □

Solution. **??** Since 20% of the objects in the bag are beads, $100\% - 20\% = 80\%$ of the objects in the bag are coins. Half of the coins in the bag are gold, so half of the coins must be silver. Thus $\frac{80\%}{2} = \boxed{40\%}$ of the objects are silver coins.

The answer is $\boxed{C.}$ □

Solution. **??** We can check if A, B, and C are on the same line by comparing the slope of the line through A and B to the slope of the line through A and C. If these slopes are the same, then we must go in the same direction to get from A to B as we do to get from A to C. In other words, the three points are on the same line.

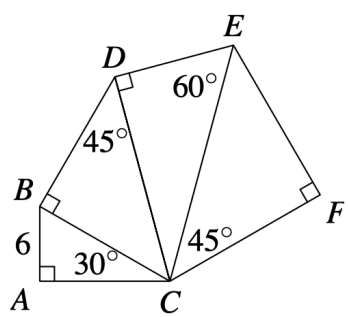
Note that, the slope between (x_1, y_1) and (x_2, y_2) is $\frac{y_2-y_1}{x_2-x_1}$. The slopes of AB, AC, AD, and AE are,

$$\text{Slope of } AB = \frac{4-1}{1-3} = -\frac{3}{2}, \quad \text{Slope of } AC = \frac{7-1}{(-1)-3} = -\frac{3}{2},$$
$$\text{Slope of } AD = \frac{(-2)-1}{5-3} = -\frac{3}{2}, \quad \text{Slope of } AE = \frac{(-1)-1}{4-3} = -2$$

Now, it is sufficient to see that E does not line on the straight line formed by A, B, C, and D.

The answer is $\boxed{E.}$ □

Solution. **??** Because $\triangle ABC$ is a 30-60-90 triangle and AB is opposite the 30° angle, we have $AC = AB\sqrt{3} = 6\sqrt{3}$ and $BC = 2(AB) = 12$. Because $\triangle BDC$ is a 45-45-90 triangle, we have $BD = BC = 12$ and $CD = BC\sqrt{2} = 12\sqrt{2}$. Because $\triangle EDC$ is a 30-60-90 triangle and $\overline{CD}$ is the longer leg, we have



$CD = DE\sqrt{3}$. Since $CD = 12\sqrt{2}$, we have

$$DE = \frac{CD}{\sqrt{3}} = \frac{12\sqrt{2}}{\sqrt{3}} = \frac{12\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{6}}{3} = 4\sqrt{6}.$$

We also have $EC = 2ED = 8\sqrt{6}$. Finally, $\triangle ECF$ is a 45-45-90 triangle with hypotenuse $EC = 8\sqrt{6}$, so $EC = EF = 8\sqrt{3}$. Therefore, $[CEF] = \frac{1}{2} \cdot 8\sqrt{3} \cdot 8\sqrt{3} = \boxed{96}$.

The answer is $\boxed{D.}$ □

Solution. **??** For the first question,

$$a^4 - b^2 = (a^2)^2 - b^2 = \boxed{(a^2 - b)(a^2 + b).}$$

For the second question,

$$a^4 - b^2 = 45 \Rightarrow (a^2 - b)(a^2 + b) = 45 \Rightarrow \begin{cases} a^2 - b = 1,\ a^2 + b = 45 \\ a^2 - b = 3,\ a^2 + b = 15 \\ a^2 - b = 5,\ a^2 + b = 9 \end{cases}$$

It is easy to see that for the first case, $2a^2 = 46$, or $a^2 = 23$, and for the third case, $2a^2 = 14$, or $a^2 = 7$, which is impossible. The second case implies that $2a^2 = 18$, or $\boxed{a = 3, b = 6.}$ □

Solution. [First solution] **??** For the first question, the probability a red face will be facing up is $\frac{2}{6} = \boxed{\frac{1}{3}}$.

For the second question, the probability a red face will be facing up is also $\frac{2}{6} = \boxed{\frac{1}{3}}$. □

Solution. [Second solution] **??** For the first question, wWhen you look from a point so that you can see at most $\boxed{3}$ faces.

For the second question, to maximize the number of unit cubes visible, you need to see 3 faces. If we add the numbers of cubes on the faces together, then every two faces share four cubes, and three faces share on cube, thus

$$3 \cdot 4 \cdot 4 - 3 \cdot 4 + 1 = \boxed{37}.$$

□

Solution. **??** For the first question, there are 26 cards of each color, so 3 of them can be selected in $\binom{26}{3}$ ways, and of course there are 2 colors. So the answer is

$$2 \cdot \frac{\binom{26}{3}}{\binom{52}{3}} = \boxed{\frac{4}{17}.}$$

For the second question, we know the probability they are all same color, so we just need to subtract that from 1. Thus, the probability is $1\frac{4}{17} = \boxed{\frac{13}{17}.}$ □

15.13 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

15.14 Problems for Level 5

Problem 15.14.1 (ICLT-2022-SM1-4-L5-P1). (*4 points*)

Five points A, B, C, D, E are given with coordinates. There is a straight line passes through exactly four points. Find the point that the straight line does not pass through.

(A) $A(1,3)$ (B) $B(4,1)$ (C) $C(7,-1)$ (D) $D(-2,5)$ (E) $E(-1,4)$

Problem 15.14.2 (ICLT-2022-SM1-4-L5-P2). (*4 points*)

Find the area of $\triangle CEF$.

(A) 24 (B) 48 (C) 64 (D) 96 (E) 192

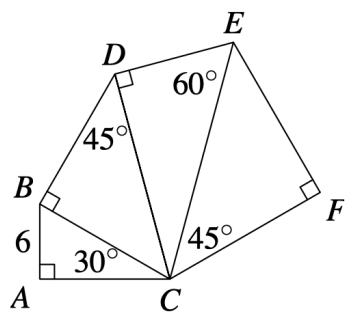


Figure 15.10: **??**

Problem 15.14.3 (ICLT-2022-SM1-4-L5-P3). (*4 points*)

Find the sum of $10a + b$ where (a, b) is a positive integer pair such that $a^4 - b^2 = 45$.

(A) 9 (B) 27 (C) 36 (D) 45 (E) 63

Problem 15.14.4 (ICLT-2022-SM1-4-L5-P4). (*6 points*)

We have a cube with 3 blue faces, 2 red faces, and 1 white face. What is the probability that when it is rolled, a red face will be facing up?

(A) $\frac{1}{3}$ (B) $\frac{2}{5}$ (C) $\frac{1}{6}$ (D) $\frac{3}{5}$ (E) $\frac{1}{2}$

Problem 15.14.5 (ICLT-2022-SM1-4-L5-P5). (*6 points*)

Compute

$$\frac{1}{\sqrt{9}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{7}} + \ldots + \frac{1}{\sqrt{2}+\sqrt{1}}$$

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $1+\sqrt{2}$ (E) 3

Problem 15.14.6 (ICLT-2022-SM1-4-L5-P6). (*6 points*)

In quadrilateral $ABCD$, $\angle A = \angle B = 128°$, and $\angle D$ is $10°$ less than 5 times $\angle C$. Find $10\angle C - 2\angle D$.

(A) 18 (B) 20 (C) 22 (D) 24 (E) 26

Problem 15.14.7 (ICLT-2022-SM1-4-L5-P7). (*10 points*)

Determine which numbers below are multiples of 12, and which one are not,

(A) 727_8 (B) 202120_3 (C) 12340_5 (D) 6543_7 (E) $CB10_{13}$

Problem 15.14.8 (ICLT-2022-SM1-4-L5-P8). (*10 points*)

1. (*5 points*) How many ways to choose two distinct digits from $1, 2, \ldots, 8$ to form a two-digit number that is a multiple of 4?
2. (*5 points*) The numbers 1 through 8 are arranged to form an eight-digit number. What is the probability that the number is a multiple of 4?

Problem 15.14.9 (ICLT-2022-SM1-4-L5-P9). (*10 points*)

S is the square with vertices $(0, 0), (3, 0), (3, 3)$, and $(0, 3)$. P is a point chosen at random inside S (P can also be on any side of S.)

1. (*5 points*) Determine the region where P is closer to $(0, 0)$ than to $(4, 2)$?
2. (*5 points*) What is the probability that P is closer to $(0, 0)$ than to $(4, 2)$?

15.15 Grading for Level 5

Answers for multiple-choice problems.

Problem 1:	E	$E(-1,4)$
Problem 2:	D	96
Problem 3:	C	36
Problem 4:	A	$\frac{1}{3}$
Problem 5:	C	2
Problem 6:	B	20

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: 2 points if can convert non-decimal numbers into decimals.

Problem 8: Separately for each part,

(a) 2 points if can identify the two-digits numbers that divisible by 4 (might include 0 and 9 as digits).

(b) 2 points if can find the number of permutation of 6 (or 8) distinct digits as 6! (or 8!).

Problem 9: Separately for each part,

(a) 2 points if can establish the equation of the perpendicular bisector of $(0,0)$ and $(4,2)$.

(b) 2 points if can apply ratio of areas for geometric probability.

15.16 Solutions for Level 5

Solution. **??** We can check if A, B, and C are on the same line by comparing the slope of the line through A and B to the slope of the line through A and C. If these slopes are the same, then we must go in the same direction to get from A to B as we do to get from A to C. In other words, the three points are on the same line.

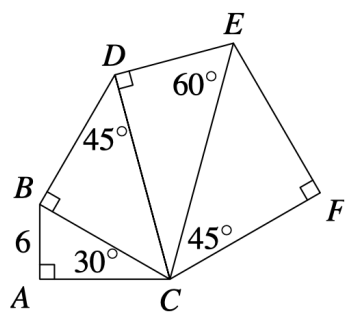
Note that, the slope between (x_1, y_1) and (x_2, y_2) is $\frac{y_2-y_1}{x_2-x_1}$. The slopes of AB, AC, AD, and AE are,

$$\text{Slope of } AB = \frac{4-1}{1-3} = -\frac{3}{2}, \quad \text{Slope of } AC = \frac{7-1}{(-1)-3} = -\frac{3}{2},$$
$$\text{Slope of } AD = \frac{(-2)-1}{5-3} = -\frac{3}{2}, \quad \text{Slope of } AE = \frac{(-1)-1}{4-3} = -2$$

Now, it is sufficient to see that E does not line on the straight line formed by A, B, C, and D.

The answer is $\boxed{E.}$ □

Solution. **??** Because $\triangle ABC$ is a 30-60-90 triangle and AB is opposite the $30°$ angle, we have $AC = AB\sqrt{3} = 6\sqrt{3}$ and $BC = 2(AB) = 12$. Because $\triangle BDC$ is a 45-45-90 triangle, we have $BD = BC = 12$ and $CD = BC\sqrt{2} = 12\sqrt{2}$. Because $\triangle EDC$ is a 30-60-90 triangle and $\overline{CD}$ is the longer leg, we have



$CD = DE\sqrt{3}$. Since $CD = 12\sqrt{2}$, we have

$$DE = \frac{CD}{\sqrt{3}} = \frac{12\sqrt{2}}{\sqrt{3}} = \frac{12\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{6}}{3} = 4\sqrt{6}.$$

We also have $EC = 2ED = 8\sqrt{6}$. Finally, $\triangle ECF$ is a 45-45-90 triangle with hypotenuse $EC = 8\sqrt{6}$, so $EC = EF = 8\sqrt{3}$. Therefore, $[CEF] = \frac{1}{2} \cdot 8\sqrt{3} \cdot 8\sqrt{3} = \boxed{96}$.

The answer is $\boxed{D.}$ □

Solution. **??**

$$a^4 - b^2 = (a^2)^2 - b^2 = (a^2 - b)(a^2 + b) = 45 \Rightarrow \begin{cases} a^2 - b = 1, \ a^2 + b = 45 \\ a^2 - b = 3, \ a^2 + b = 15 \\ a^2 - b = 5, \ a^2 + b = 9 \end{cases}$$

It is easy to see that for the first case, $2a^2 = 46$, or $a^2 = 23$, and for the third case, $2a^2 = 14$, or $a^2 = 7$, which is impossible. The second case implies that $2a^2 = 18$, or $a = 3, b = 6$, thus $10a + b = \boxed{36}$.

The answer is $\boxed{C.}$ □

Solution. **??** The probability a red face will be facing up is also $\frac{2}{6} = \boxed{\frac{1}{3}}$.

The answer is $\boxed{A.}$ □

Solution. **??** Note that

$$\frac{1}{\sqrt{(n+1)}+\sqrt{n}} = \frac{\sqrt{(n+1)}-\sqrt{n}}{(\sqrt{(n+1)}+\sqrt{n})(\sqrt{(n+1)}-\sqrt{n})} = \frac{\sqrt{(n+1)}-\sqrt{n}}{(\sqrt{(n+1)})^2-(\sqrt{n})^2} = \sqrt{(n+1)}-\sqrt{n}$$

Therefore,

$$\frac{1}{\sqrt{9}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{7}} + \ldots + \frac{1}{\sqrt{2}+\sqrt{1}} = \sqrt{9}-\sqrt{8}+\sqrt{8}-\sqrt{7}+\ldots+\sqrt{2}-\sqrt{1} = \boxed{2}$$

The answer is $\boxed{C.}$ □

Solution. **??** Let $\angle C = x$, so $\angle D = 5x - 10°$. Since

$$\angle A + \angle B + \angle C + \angle D = 360° \Rightarrow 128° + 128° + x + (5x - 10°) = 360° \Rightarrow x = 19°$$

. Therefore, $\angle D = 5x - 10° = 85°$. Thus, $10\angle C - 2\angle D = 190° - 170° = \boxed{20°}$.

The answer is $\boxed{B.}$ □

Solution. **??**

$$727_8 = 7 \cdot 8^2 + 2 \cdot 8 + 7 = 471 \not\vdots\, 12$$

$$202120_3 = 2 \cdot 3^5 + 2 \cdot 3^3 + 1 \cdot 3^2 + 2 \cdot 3 = 555 \not\vdots\, 12$$

$$12340_5 = 1 \cdot 5^4 + 2 \cdot 5^3 + 3 \cdot 5^2 + 4 \cdot 5 = 970 \not\vdots\, 12$$

$$6543_7 = 6 \cdot 7^3 + 5 \cdot 7^2 + 4 \cdot 7 + 3 = 2334 \not\vdots\, 12$$

$$CB10_{13} = 12 \cdot 13^3 + 11 \cdot 13^2 + 10 \cdot 1 = 28236 \vdots\, 12$$

Thus, the numbers $727_8, 202120_3, 12340_5, 6543_7$ are not divisible by 12, and the number $CB10_{13}$ is divisible by 12. □

Solution. **??** For the first question, there are $\boxed{14}$ two-digit numbers with two distinct digits from $1, 2, \ldots, 8$ are multiple of 4,

$$12, 16, 24, 28, 32, 36, 48, 52, 56, 64, 68, 72, 76, 84$$

For the second question, if a number is a multiple of 4, then the number formed by its last two digits must be divisible by 4. For each eight-digit number with the last two digits chose, there are 6! ways to permute the remaining 6 digits. Therefore the probability to form an eight-digit number, that is also multiple of 4, is,

$$\frac{14 \cdot 6!}{8!} = \boxed{\frac{1}{4}.}$$

□

Solution. **??** For the first question, since P is closer to $(0,0)$ than to $(4,2)$, its location must be on the left side of the perpendicular bisector ℓ of $(0,0)$ and $(4,2)$. Since the line through $(0,0)$ and $(4,2)$ is $y = \frac{1}{2}x$, thus the slope of ℓ is $\frac{-1}{\frac{1}{2}} = -2$.

The midpoint of $(0,0)$ and $(4,2)$ is $(2,1)$, thus the equation of ℓ is

$$y - 1 = (-2)(x-2) \Rightarrow 2x + y = 5.$$

ℓ intersect the square at $(1,3)$ (where $y = 3$, so $x = 1$), and $(2.5, 0)$ (where $x = 0$, so $y = 2.5$), thus the region where P is closer to $(0,0)$ than to $(4,2)$ is $\boxed{\text{the shaded trapezoid}}$ shown as below.

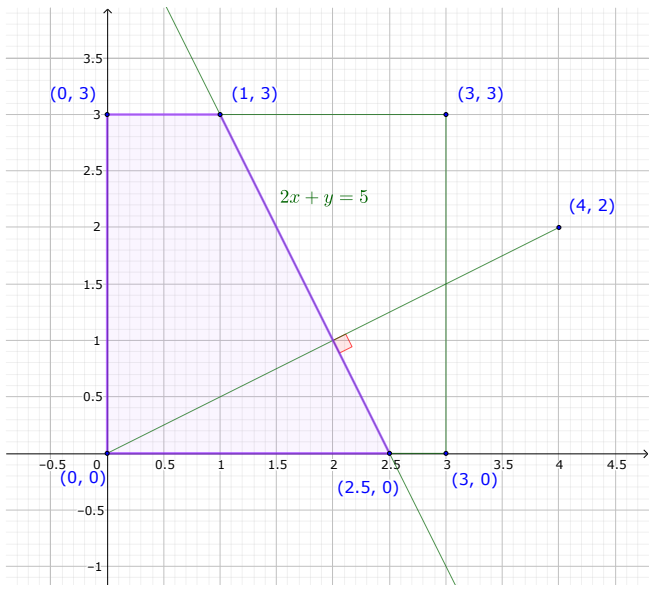


Figure 15.11: **??**

For the second question, the area of the trapezoid is $\frac{(1+2.5)(3)}{2} = 5.25$, so the probability is $\frac{5.25}{9} = \boxed{\frac{7}{12}.}$ $\square$

15.17 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

15.18 Problems for Level 7

Problem 15.18.1 (ICLT-2022-SM1-4-L7-P1). (*4 points*)

Let $x \star y = \sqrt{x^2 + y^2}$. Find the product of all values of t such that $(2t+1) \star (2t-1) = 15$.

$(A)\ -\frac{223}{8}$ $(B)\ \frac{223}{8}$ $(C)\ -\frac{225}{8}$ $(D)\ -\frac{225}{8}$ $(E)\ \frac{112}{4}$

Problem 15.18.2 (ICLT-2022-SM1-4-L7-P2). (*4 points*)

K is a point outside of the circle (T) (centred at T), KP is tangent to (T) at point P. Line through K intersects circle (T) at point Y such that Y lies between K and T. It is known that $YK = 6$ and $YT = 2$. Find the length of KP.

$(A)\ 2$ $(B)\ 2\sqrt{11}$ $(C)\ 2\sqrt{13}$ $(D)\ 2\sqrt{15}$ $(E)\ 3$

Problem 15.18.3 (ICLT-2022-SM1-4-L7-P3). (*4 points*)

Jenny has 8 stamp books that each contain 43 pages. Each page in her books contains 6 stamps. Jenny decides to reorganize her stamp books such that each page contains 10 stamps. This will give her more space to collect stamps without having to buy new books. Under her new system, Jenny fills up 4 complete books, still with 43 pages per book. Her fifth book now contains 34 pages filled with 10 stamps per page and 1 last page with the remaining stamps. How many stamps are on that last page?

$(A)\ 2$ $(B)\ 4$ $(C)\ 6$ $(D)\ 8$ $(E)\ 0$

Problem 15.18.4 (ICLT-2022-SM1-4-L7-P4). (*6 points*)

A penny, nickel, dime, quarter, and a dollar are simultaneously flipped. What is the expected value of the amount of the coins which come up heads?

(A) 20.5¢ (B) 40¢ (C) 55.5¢ (D) 70.5¢ (E) 71¢

Problem 15.18.5 (ICLT-2022-SM1-4-L7-P5). (*6 points*)

The graph of $y = f(x)$ has a y-intercept at $(0, a)$, where $a \neq 0$. Furtheremore $f(3) \neq a$. How many of the following graphs of functions must have the same $y-$ intercept as the graph of $y = f(x)$?

$$y = f(\frac{x}{3}),\ y = 3f(x),\ y = f(x+3),\ y = f(x) - 3,\ y = f(x) + 3.$$

$(A)\ 0$ $(B)\ 1$ $(C)\ 2$ $(D)\ 3$ $(E)\ 4$

Problem 15.18.6 (ICLT-2022-SM1-4-L7-P6). (*6 points*)

Find the ratio $\frac{AB}{AF}$ in the figure below.

$(A)\ \frac{1}{2}$ $(B)\ \frac{\sqrt{2}}{2}$ $(C)\ \frac{\sqrt{3}}{2}$ $(D)\ \frac{\sqrt{6}}{2}$ $(E)\ \frac{\sqrt{6}}{3}$

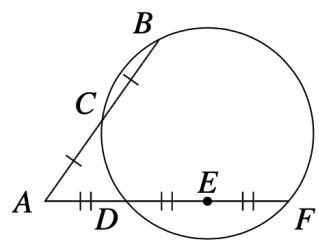


Figure 15.12: **??**

Problem 15.18.7 (ICLT-2022-SM1-4-L7-P7). (*10 points*)

1. (*5 points*) How many numbers are there between 1 and 100 that is a multiple of 2 and 5?
2. (*5 points*) How many numbers between 1 and 100 (including both 1 and 100) have reciprocals that are repeating decimals?

Problem 15.18.8 (ICLT-2022-SM1-4-L7-P8). (*10 points*)

1. (*5 points*) Prove that, for any positive integer n,

$$\binom{2n}{2} - 2\binom{n}{2} = n^2.$$

2. (*5 points*) Prove that, for any positive integers n and r such that $n \geq r+1$,

$$\binom{n}{r} = \frac{n}{n-r}\binom{n-1}{r}.$$

Problem 15.18.9 (ICLT-2022-SM1-4-L7-P9). (*10 points*)

The sides of $\triangle ABC$ are $AB = 13, BC = 5, CA = 12$. Let I be the centre of the incircle of $\triangle ABC$ and r be the inradius.

1. (*5 points*) Let X and Y be the tangent points of the incircle with CA and CB. Prove that $CXIY$ is a square.
2. (*5 points*) Find the length of the inradius.

15.19 Grading for Level 7

Answers for multiple-choice problems.

Problem 1:	A	$-\frac{223}{8}$
Problem 2:	D	$2\sqrt{15}$
Problem 3:	B	4
Problem 4:	D	70.5¢
Problem 5:	C	2
Problem 6:	E	$\frac{\sqrt{6}}{3}$

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately for each part,

(a) 2 points if can identify the multiples of 10 that have no other prime factors than 2 and 5.

(b) 2 points if can find all powers of 2 less than 100.

Problem 8: Separately for each part,

(a) 2 points if can convert binomial terms into algebra fractions, such as $\binom{n}{2} = \frac{n(n-1)}{2}$.

(b) 2 points if can convert binomial terms into algebra fractions, such as $\binom{n-1}{r} = \frac{(n-1)!}{(n-1-r)!r!}$.

Problem 9: Separately for each part,

(a) 2 points if can determine that $\triangle ABC$ is a right triangle.

(b) 2 points if can apply the tangent property to find that $r = \frac{BC+CA-AB}{2}$.

15.20 Solutions for Level 7

Solution. **??**

$$(2t+1)\star(2t-1)=\sqrt{(2t+1)^2+(2t-1)^2}=\sqrt{4t^2+4t+1+4t^2-4t+1}=\sqrt{8t^2+2}$$
$$\sqrt{8t^2+2}=15\Rightarrow t^2=\frac{223}{8}$$

Now, there are two values of t $\pm\sqrt{\frac{223}{8}}$ and their product is $\boxed{-\frac{223}{8}.}$

The answer is $\boxed{A.}$ □

Solution. **??**

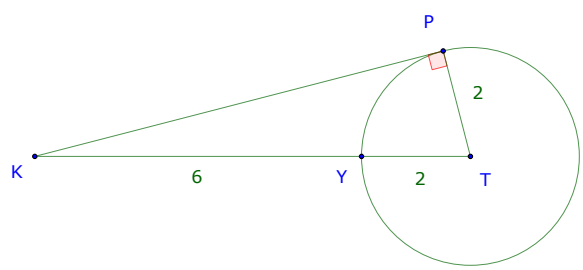


Figure 15.13: **??**

$$KP=\sqrt{KT^2-PT^2}=\sqrt{(6+2)^2-2^2}=\boxed{2\sqrt{15}}$$

The answer is $\boxed{D.}$ □

Solution. **??** Since Jenny's new system of arranging stamps includes 10 on each page, the number of the last page will be the units digit of her total number of stamps (in base 10). That units digit is the same as the units digit of $8\cdot3\cdot6=144$, which is $\boxed{4}$.

The answer is $\boxed{B.}$ □

Solution. **??** There is a $\frac{1}{2}$ probability that each coin comes up heads, so the expected value of the coins that come up heads is

$$\frac{1}{2}(1¢+5¢+10¢+25¢+100¢)=\boxed{70.5¢}.$$

The answer is $\boxed{D.}$ □

Solution. **??** The graph of $\boxed{y = f(\frac{x}{3})}$ is a horizontal scaling of the graph of $y = f(x)$ by a factor of 3. So, if the point $(0, a)$ is on the graph of $y = f(x)$, it is also on the graph of $y = f(\frac{x}{3})$. So, the graphs of $y = f(x)$ and $y = f(\frac{x}{3})$ must have the same y-intercept.

The graph of $\boxed{y = f(3x)}$ is a horizontal scaling of the graph of $y = f(x)$ by a factor of $\frac{1}{3}$. So, if the point $(0, a)$ is on the graph of $y = f(x)$, it is also on the graph of $y = f(3x)$. So, the graphs of $y = f(x)$ and $y = f(3x)$ must have the same y-intercept.

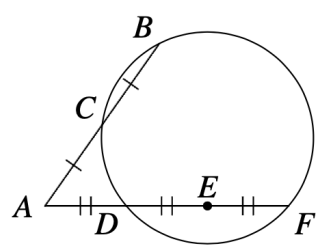
The graph of $y = 3f(x)$ is a vertical scaling of $y = f(x)$ by a factor of 3. So, the y-intercept of $y = 3f(x)$ is thrice as far from the x-axis as the y-intercept of $y = f(x)$. Therefore, these graphs only have the same y-intercept if the graph of $y = f(x)$ passes through the origin, which is not possible since $a \neq 0$.

The graph of $y = f(x + 3)$ is a 3-unit leftward shift of the graph of $y = f(x)$. So, the y-intercept of $y = f(x + 3)$ is the value of $f(3)$, which is not a. Therefore, the graphs of $y = f(x)$ and $y = f(x + 3)$ do not have the same y-intercept.

The graph of $y = f(x) + 3$ is a 3-unit vertical shift upward of the graph of $y = f(x)$. So, the y-intercept of $y = f(x) + 3$ is 3 units above the y-intercept of $y = f(x)$. Therefore, they cannot be the same point.

The answer is $\boxed{C.}$ □

Solution. **??** The power of point A gives $(AC)(AB) = (AD)(AF)$. From the diagram, we have $AC = \frac{AB}{2}$



and $AD = \frac{AD}{3}$, so we have $\frac{1}{2}AB^2 = \frac{1}{3}AF^2$. Therefore, $\frac{AB^2}{/}AF^2 = \frac{2}{3}$. Taking the positive square root of both sides gives $\frac{AB}{AF} = \sqrt{\frac{2}{3}} = \boxed{\frac{\sqrt{6}}{3}.}$

The answer is $\boxed{E.}$ □

Solution. **??** For the first question, there are 6 multiples of powers of 2 and powers of 5 less than or equal 100,

$$10, 20, 40, 50, 80, 100.$$

For the second question, we don't need to convert a single fraction. Beside the number 1, from the first question, there are 6 multiples of powers of 2 and powers of 5 less than or equal 100. Now, we count the ones with prime factorizations include only primes 2 or 5. There are 6 powers of 2 less than or equal 100 are

$$2, 4, 8, 16, 32, 64.$$

There are 2 powers of 5 less than or equal 100 are

$$5, 25.$$

Therefore, in total there are $1 + 6 + 2 + 6 = 15$ such numbers. Thus, there are $100 - 15 = \boxed{85}$ numbers that have reciprocals that are repeating decimals. □

Solution. **??** For the first question,

$$\binom{2n}{2} - 2\binom{n}{2} = \frac{2n(2n-1)}{2} - 2 \cdot \frac{n(n-1)}{2} = n(2n-1) - n(n-1) = n^2.$$

For the second question,

$$\frac{n}{n-r}\binom{n-1}{r} = \frac{n}{n-r}\frac{(n-1)!}{(n-1-r)!r!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}.$$

□

Solution. **??** For the first question, $AB^2 = BC^2 + CA^2$, so $\triangle ABC$ is right at C. It is easy to see that $CXIY$

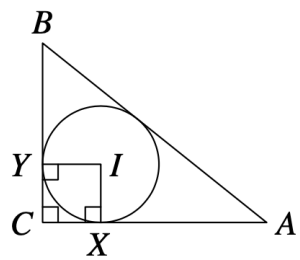


Figure 15.14: **??**

is a rectangle because it has three right angles at C, X, and Y. Furthermore $IX = IY$, so $CXIY$ is a $\boxed{\text{square}}$.

Now, by the tangent property,

$$IX = CX = \frac{1}{2}(BC + CA - AB) = \frac{5 + 12 - 13}{2} = \boxed{2}.$$

□

15.21 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

15.22 Problems for Level 8

Problem 15.22.1 (ICLT-2022-SM1-4-L8-P1). (*4 points*)

The graph of $y = f(x)$ has a y-intercept at $(0, a)$, where $a \neq 0$. Furtheremore $f(3) \neq a$. How many of the following graphs of functions must have the same $y-$ intercept as the graph of $y = f(x)$?

$$y = f(\frac{x}{3}),\ y = 3f(x),\ y = f(x+3),\ y = f(x) - 3,\ y = f(x) + 3.$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 15.22.2 (ICLT-2022-SM1-4-L8-P2). (*4 points*)

Find the ratio $\frac{AB}{AF}$ in the figure below.

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{6}}{2}$ (E) $\frac{\sqrt{6}}{3}$

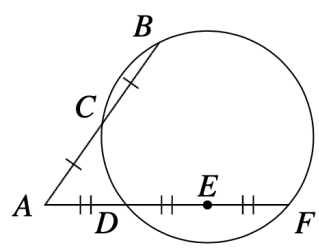


Figure 15.15: **??**

Problem 15.22.3 (ICLT-2022-SM1-4-L8-P3). (*4 points*)

How many numbers between 1 and 100 (including both 1 and 100) have reciprocals that are repeating decimals?

(A) 94 (B) 92 (C) 90 (D) 86 (E) 85

Problem 15.22.4 (ICLT-2022-SM1-4-L8-P4). (*6 points*)

What is the value of

$$\binom{2n}{2} - 2\binom{n}{2}?$$

(A) $2n$ (B) $n(n-1)$ (C) n^2 (D) $n(n+1)$ (E) $2n^2$

Problem 15.22.5 (ICLT-2022-SM1-4-L8-P5). (*6 points*)

Suppose a and b are constants such that

$$(x^3 + bx^2 - 7x + 9)(x^2 + ax + 5) = x^5 + 18x^4 + 70x^3 + 27x^2 + 19x + 45$$

Find the value of $5b - 7a$.

(A) 20 (B) 18 (C) 16 (D) 14 (E) 12

Problem 15.22.6 (ICLT-2022-SM1-4-L8-P6). (*6 points*)

For how many values of n, where $40 \leq n \leq 80$ is $n \equiv -n \pmod{13}$?

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

Problem 15.22.7 (ICLT-2022-SM1-4-L8-P7). (*10 points*)

On the center of each face of a large cube with side length 3 centimeters, we glue a small cube of side length 1 centimeter (so that an entire face of each small cube is glued to the large cube). We then paint the entire resulting figure red.

1. (*5 points*) What is the total surface area of the large that is not painted?
2. (*5 points*) How many square centimeters of red paint will we use?

Problem 15.22.8 (ICLT-2022-SM1-4-L8-P8). (*10 points*)

1. (*5 points*) Compute
$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3}.$$
2. (*5 points*) Compute
$$\binom{23}{22} + \binom{24}{22} + \ldots + \binom{44}{22} - \binom{45}{23}.$$

Problem 15.22.9 (ICLT-2022-SM1-4-L8-P9). (*10 points*)

Cone $\mathcal{A}$ has the radius r, height h. Cone $\mathcal{B}$ has the radius $3r$, height $\frac{h}{9}$. What is the ratio of the volume of $\mathcal{A}$ to the volume of $\mathcal{B}$?

15.23 Grading for Level 8

Answers for multiple-choice problems.

Problem 1:	C	2
Problem 2:	E	$\frac{\sqrt{6}}{3}$
Problem 3:	E	85
Problem 4:	C	n^2
Problem 5:	B	18
Problem 6:	E	3

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately for each part,

(a) 2 points if can identify the area covered by the small cube on each face of the large cube.

(b) 2 points if can use complementary counting (subtracting the non-painted area.)

Problem 8: Separately for each part,

(a) 2 points if can use the Hockey Stick Identity.

(b) 2 points if can use the Hockey Stick Identity.

Problem 9: 2 points if can determine the formula for the volume of a cone.

15.24 Solutions for Level 8

Solution. **??** The graph of $\boxed{y = f(\frac{x}{3})}$ is a horizontal scaling of the graph of $y = f(x)$ by a factor of 3. So, if the point $(0, a)$ is on the graph of $y = f(x)$, it is also on the graph of $y = f(\frac{x}{3})$. So, the graphs of $y = f(x)$ and $y = f(\frac{x}{3})$ must have the same y-intercept.

The graph of $\boxed{y = f(3x)}$ is a horizontal scaling of the graph of $y = f(x)$ by a factor of $\frac{1}{3}$. So, if the point $(0, a)$ is on the graph of $y = f(x)$, it is also on the graph of $y = f(3x)$. So, the graphs of $y = f(x)$ and $y = f(3x)$ must have the same y-intercept.

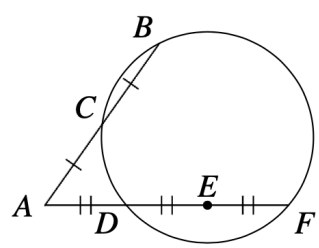
The graph of $y = 3f(x)$ is a vertical scaling of $y = f(x)$ by a factor of 3. So, the y-intercept of $y = 3f(x)$ is thrice as far from the x-axis as the y-intercept of $y = f(x)$. Therefore, these graphs only have the same y-intercept if the graph of $y = f(x)$ passes through the origin, which is not possible since $a \neq 0$.

The graph of $y = f(x + 3)$ is a 3-unit leftward shift of the graph of $y = f(x)$. So, the y-intercept of $y = f(x + 3)$ is the value of $f(3)$, which is not a. Therefore, the graphs of $y = f(x)$ and $y = f(x + 3)$ do not have the same y-intercept.

The graph of $y = f(x) + 3$ is a 3-unit vertical shift upward of the graph of $y = f(x)$. So, the y-intercept of $y = f(x) + 3$ is 3 units above the y-intercept of $y = f(x)$. Therefore, they cannot be the same point.

The answer is $\boxed{C.}$ □

Solution. **??** The power of point A gives $(AC)(AB) = (AD)(AF)$. From the diagram, we have $AC = \frac{AB}{2}$



and $AD = \frac{AD}{3}$, so we have $\frac{1}{2}AB^2 = \frac{1}{3}AF^2$. Therefore, $\frac{AB^2}{/}AF^2 = \frac{2}{3}$. Taking the positive square root of both sides gives $\frac{AB}{AF} = \sqrt{\frac{2}{3}} = \boxed{\frac{\sqrt{6}}{3}.}$

The answer is $\boxed{E.}$ □

Solution. **??** We don't need to convert a single fraction. Beside the number 1, we count the ones with prime factorizations include only primes 2 or 5. There are 6 powers of 2 less than or equal 100 are

$$2, 4, 8, 16, 32, 64.$$

There are 2 powers of 5 less than or equal 100 are

$$5, 25.$$

Finally, there are 6 multiples of powers of 2 and powers of 5 less than or equal 100,

$$10, 20, 40, 50, 80, 100.$$

Therefore, in total there are $1+6+2+6=15$ such numbers. Thus, there are $100-15=\boxed{85}$ numbers that have reciprocals that are repeating decimals.

The answer is $\boxed{E.}$ □

Solution. **??**

$$\binom{2n}{2}-2\binom{n}{2}=\frac{2n(2n-1)}{2}-2\cdot\frac{n(n-1)}{2}=n(2n-1)-n(n-1)=\boxed{n^2.}$$

The answer is $\boxed{C.}$ □

Solution. **??** Note that,

$$(x^3+bx^2-7x+9)(x^2+ax+5)=x^5+(a+b)x^4+(ab-2)x^3+(-7a+5b+9)x^2+(9a-35)x+45$$
$$x^5+(a+b)x^4+(ab-2)x^3+(-7a+5b+9)x^2+(9a-35)x+45=x^5+18x^4+70x^3+27x^2+19x+45$$

Therefore, by comparing the coefficients of the powers of x,

$$\begin{cases} a+b=18 \\ ab-2=70 \\ -7a+5b+9=27 \\ 9a-35=19 \end{cases} \Rightarrow a=6, b=12 \Rightarrow 5b-7a=\boxed{18.}$$

The answer is $\boxed{B.}$

Note, there is a shortcut when comparing the coefficient of x^2 : $-7a+5b+9=27 \Rightarrow 5b-7a=18$. However without solving the system of equations, there is no guarantee that a and b exist (!). □

Solution. **??**

$$n\equiv -n \pmod{13} \Rightarrow 2n\equiv 0 \pmod{13},$$

since $\gcd(2,13)=1$, so $nn\equiv 0 \pmod{13}$.

Between 40 and 80, there are $\boxed{3}$ such values: $52, 65$, and 78.

The answer is $\boxed{E.}$ □

Solution. **??** For the first question, it is easy to see that for each face of the large cube, there is an area equivalent to the area of a face of the small cube, $1\cdot 1=1$ square centimeter, which is not painted. Thus the total surface area of the large cube not painted is $6\cdot 1=\boxed{6}$ square centimeters.

For the second question, note that the total surface painted is equal to the total surface of the large cube and the surfaces of the small cubes subtract twice the total surface area of the large cube that not painted. Thus, it is

$$6\cdot 3\cdot 3+6\cdot 1\cdot 1-2\cdot 6=\boxed{48.}$$

□

Solution. **??** The Hockey Stick Identity states that, for any n, r positive integers,

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

For the first question, by the Hockey Stick Identity,

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} = \binom{6+1}{3+1} = \binom{7}{4} = \boxed{35.}$$

For the second question, by the Hockey Stick Identity,

$$\binom{22}{22} + \binom{23}{22} + \binom{24}{22} + \ldots + \binom{44}{22} = \binom{44+1}{22+1} = \binom{45}{23}$$
$$\Rightarrow \binom{23}{22} + \binom{24}{22} + \ldots + \binom{44}{22} - \binom{45}{23} = \boxed{-1}.$$

□

Solution. **??**

The volume of cone $\mathcal{A}$ is $\frac{1}{3}\pi r^2 h$. The volume of cone $\mathcal{B}$ is $\frac{1}{3}\pi(3r)^2\left(\frac{h}{9}\right) = \frac{1}{3}\pi r^2 h$.

Thus, their volume ratio is $\boxed{1}$. □

Chapter 16

HC R9

16.1 Topics

Chess

1. Moves: capture, check, double check, promotion, stalemate, checkmate
2. Analysis: whose turn is it.
3. Retrograde: last move, last two moves, last multiple of moves.
4. Proof games: White and Black, instead of making the best moves, they co-operate to make the game legally into a set position.
5. Symmetric position with odd number of moves.
6. Switchback: a piece moved out and came back into that same square.

Logic

1. Casework: what if A is true, what if A is false.
2. Process of elimination: If A is not true, B is not true, then C should be true.
3. Reverse argument: if there is at least ..., then there is atmost ...
4. Implication from truth: if A told the truth and A said X, then X is true.
5. Conflict of truth: if B said A lied, then both cannot be truth tellers.

Algebra

1. Estimations.
2. Inequalities. Upper limit. Lower limit.
3. Second- and Third degree identities.
4. Sums and Products.

Combinatorics

1. Counting in two ways.
2. Permutations.
3. Combinations.
4. Invariant.

Geometry

1. Right triangles. Pythagorean theorem.

2. Reflection. Symmetry.
3. Circle, square, triangle areas.
4. Area ratio of triangles, same base or same height.

Number Theory

1. Prime numbers.
2. Unit digits.
3. Number bases.

Coding

1. Sum of multiple variables equal to a constant.
2. Nested loops.
3. Lists and Sets.

16.2 Problems

Problem 16.2.1 (HC-2022-SM1-R9-P1)**.** (*Beginner Level*)

In the Forest of Forgetfullness, there is a Lion and a Unicorn. The Lion lies on Mondays, Tuesdays, and Wednesdays, and tells the truth on the other days of the week. The Unicorn, on the other hand, lies on Thursdays, Fridays, and Saturdays, but tells the truth on the other days of the week.

One day, when visiting the Forest, Julie met both the Lion and the Unicorn. One of them said,

He lied yesterday and I will lie tomorrow.

Who said the statement above? What days of the week could that day be?

How to provide your answer:

- If you think it was the Lion and that day can be Wednesday or Thursday, submit L, Wed, Thu.
- If you think it was the Unicorn and that day can be Friday, or Lion and Saturday submit U, Fri and L, Sat.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that has *strictly less* solutions than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly more* solutions than your answer.

Problem 16.2.2 (HC-2022-SM1-R9-P2)**.** (*Beginner Level*)

Is it possible to choose four positive integers such that the sum of any three of them is a prime number? *For example* $3, 11, 17, 19$ *cannot be such four numbers because* $3 + 11 + 17 = 31$ *is a prime number, but* $3 + 11 + 19 = 33$ *is not a prime number.*

How to provide your answer:

- If you think that $3, 11, 17, 19$ are such four numbers, submit $3, 11, 17, 19$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 16.2.3 (HC-2022-SM1-R9-P3)**.** (*Beginner Level*)

Place the numbers $1, 2, \ldots, 6$ at the edges of a tetrahedron, shown below, in such a way that the sums of the numbers on the egdes connected to each vertex are the same,

$$a + b + c = a + d + f = b + d + e = c + e + f.$$

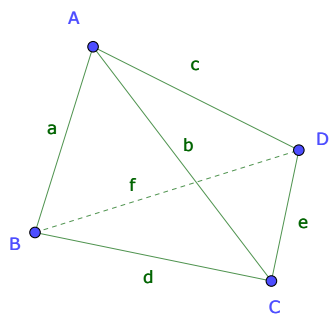


Figure 16.1: **??**

How to provide your answer:

- If you have think that $a = 1, b = 2, c = 3, d = 4, e = 5, f = 6$ is a solution, submit $1, 2, 3, 4, 5, 6$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 16.2.4 (HC-2022-SM1-R9-P4)**.** (*Beginner Level*)

A single tea bag can be used to make 2 or 3 cups of tea. Minh and Linh each brought one box of tea bags to the club. Minh made 57 cups and Linh made 83 cups of tea by using all the tea bags from each own box.

What is the number of tea bags in a box?

How to provide your answer:

- If you think that it is 10, submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 16.2.5 (HC-2022-SM1-R9-P5)**.** (*Beginner Level*)

A rectangular piece of paper of width 8 inches is folded over so that its corner coincides with point C on the opposite side. If $BC = 5$ inches, find the length in inches of fold ℓ.

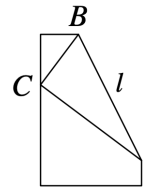


Figure 16.2: **??**

How to provide your answer:

- If you think the length of ℓ is 10, then submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 16.2.6 (HC-2022-SM1-R9-P6)**.** (*Intermediate Level*)

x and y are real numbers such that $x \neq y$ and $x \neq -y$,

$$\begin{cases} x^3 = 13x + 3y \\ y^3 = 3x + 13y \end{cases}$$

Find the value of $\left(x^2 - y^2\right)^2$.

How to provide your answer:

- If you think that the length of the trail is 10 kilometres, submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 16.2.7 (HC-2022-SM1-R9-P7)**.** (*Intermediate Level*)

Given the square in the figure with side length 6 and four semicircles which have the sides of the square as their diameters as shown, find the area of the shaded figure in the diagram.

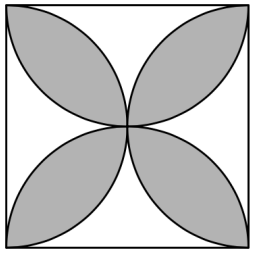


Figure 16.3: **??**

How to provide your answer:

- If you think that the area is 10, then submit 10. You can use the symbol π, where appropriate.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 16.2.8 (HC-2022-SM1-R9-P8)**.** (*Intermediate Level*)

A is the product of two identical base-2 numbers,

$$A = \underbrace{11\ldots11_2}_{2022 \text{ digits}} \times \underbrace{11\ldots11_2}_{2022 \text{ digits}}$$

How many digits of 1 does A have?

How to provide your answer:

- If you think that A has 10 digits of 1, submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 16.2.9 (HC-2022-SM1-R9-P9)**.** (*Intermediate Level*)

Four students A, B, C, and D went scouting. They came to an small old bridge over a fast river. The bridge was so narrow and shaky that it could hold *no more than two persons at a time.* It was night. Any party who tried to cross the bridge must have a flashlight too see. Unfortunately they have *only one flashlight.* The flashlight must be walked back and forth (it cannot be thrown.) Each student walks at different speed: A, B, C, and D can cross the bridge in $2, 3, 6$, and 8 minutes, respectively. However, when walking together, a pair must walk together at the rate of the slower person's pace.

In at least how many minutes could all of them cross the bridge?

How to provide your answer:

- If you think that all of them could cross the bridge in 5 minutes, then submit 5.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly larger* than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly smaller* than your answer.

Problem 16.2.10 (HC-2022-SM1-R9-P10)**.** (*Intermediate Level*)

The Coding Challenge

How many positive integer solutions are there to

$$\begin{cases} a+b+c+d=100 \\ a+b \geq 15 \\ c+d \leq 25 \end{cases}$$

How to provide your answer:

- If your code shows the answer is 100 minutes, then submit 100. You have to also submit the code for verification. If your code does not produce the result you submit, then the submission will not be qualified.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 16.2.11 (HC-2022-SM1-R9-P11)**.** (*Advanced Level*)

On the Island of Knights and Liars, there are exactly three types of people: the Knights, who always tell the truth, the Liars, who always lie, and the normals, who sometimes lie and sometimes tell the truth.

Tung visited the island and met three beautiful girls. It is known that *one of them is a Knight*, *one of them is a normal*, and *one of them is a Liar.* It is scary to know that the normal one is a *werewolf*, who used to devour people at full moon.

It is also known that *the Liar is younger than the normal*, and *the normal is younger than the Knight.*

Tung wants to marry for the Knight or the Liar and to avoid the normal one. He can **only ask a single question** from *only one of the girls* who he can freely choose among the three, and the question can only be answered with *Yes* or *No.*

What would be his question?

How to provide your answer:

- If you think that you have the question, submit the question with a short analysis why it works.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 16.2.12 (HC-2022-SM1-R9-P12)**.** (*Advanced Level*)

The plus $+$ and $-$ signs can be put between the sequence of numbers

$$1 \quad 2 \quad 3 \quad 4 \quad 5$$

to make different *non-negative* values, for example

$$1-2+3-4+5=3,\ 1+2+3+4-5=5,\ 1-2+3+4-5=1$$

What is the minimal value can be made by putting the plus $+$ and $-$ signs between the numbers,

$$1 \quad 2 \quad 3 \quad \ldots \quad 2024 \quad 2025$$

How to provide your answer:

- If you think the minimal number you can made it 4004, submit an expression that can compute to= 4004.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that has *strictly larger sum* than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has *strictly smaller sum* than your answer.

Problem 16.2.13 (HC-2022-SM1-R9-P13)**.** (*Advanced Level*)

In the figure, $\triangle ABC$ is divided into smaller triangles with the areas of some triangles are shown.

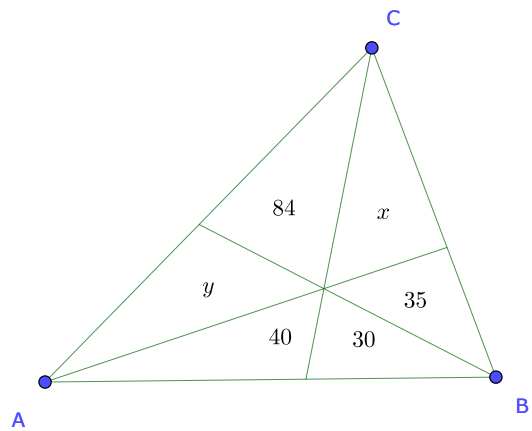


Figure 16.4: **??**

Find the area of $\triangle ABC$.

How to provide your answer:

- If you think that the area is 10, then submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 16.2.14 (HC-2022-SM1-R9-P14)**.** (*Advanced Level*)

Initially you are given the sequence $1, 2, \ldots, 2022$. In a turn, you earse any two number m and n and replace them with $mn + m + n$. It is easy to see that after 2021 turns you will be left with a single number.

What can be the maximum value of that number?

How to provide your answer:

- If you think that the maximum value of that number is 10, then submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly smaller* than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger* than your answer.

Problem 16.2.15 (HC-2022-SM1-R9-P15)**.** (*Advanced Level*)

The Coding Challenge

Nine square tiles are placed in a row, and each can be painted with one of the four colors red (R), yellow (Y), blue (B), and white (W). Find the number of ways this can be done so that each block of five adjacent tiles contains at least one tile of each color. That is, count the patterns RWBWYRRBW and WWBYRWYBW but not RWBYYBWWR because the five adjacent tiles colored BYYBW does not include the color red.

How to provide your answer:

- If your code shows the answer is 100 minutes, then submit 100. You have to also submit the code for verification. If your code does not produce the result you submit, then the submission will not be qualified.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

16.3 Answers

Problem 1: $\boxed{L, Sun}$ or $\boxed{U, Wed}$.

Problem 2: $1, 3, 7, 9$

Problem 3: 0

Problem 4: 28

Problem 5: $5\sqrt{5}$

Problem 6: 133

Problem 7: $18\pi - 36$

Problem 8: 2022 (digits of 1)

Problem 9: 19 (minutes)

Problem 10: 24500

Problem 11: Is B older then C?

Problem 12: 1

Problem 13: 315

Problem 14: $2023! - 1$

Problem 15: 3768

16.4 Solutions

Solution. **??** Let casework. The Lion lies on Mondays, Tuesdays, and Wednesdays, and tells the truth on the other days of the week. The Unicorn, on the other hand, lies on Thursdays, Fridays, and Saturdays, but tells the truth on the other days of the week.

Case 1: if it was the Lion's statement.

Case 1a: If the Lion lied, then since it said that it lied on the next day, that means it told the truth on the next day and lied on the current day. Therefore it was Wednesday. Furthermore it lied that the Unicorn told the truth on the previous day, which was Tuesday, which contracdicts the given condition that the Unicorn lied only on Thursdays, Fridays, and Saturdays.

Case 1b: If the Lion told the truth, then since it said that it lied on the next day, that means it told the lied on the next day and told the truth on the current day. Therefore it was Sunday. Furthermore it told the truth that the Unicorn lied on the previous day, which was Saturday, which was correct.

Case 2: if it was the Unicorn's statement.

Case 2a: If the Unicorn lied, then since it said that it lied on the next day, that means it told the truth on the next day and lied on the current day. Therefore it was Saturday. Furthermore it lied that the Lion told the truth on the previous day, which was Friday, which contracdicts the given condition that the Lion lied only on Mondays, Tuesdays, and Wednesdays.

Case 2b: If the Unicorn told the truth, then since it said that it lied on the next day, that means it told the lied on the next day and told the truth on the current day. Therefore it was Wednesday. Furthermore it told the truth that the Lion lied on the previous day, which was Tuesday, which was correct.

Therefore there are two solutions $\boxed{L, Sun}$ or $\boxed{U, Wed}$. □

Solution. **??** $\boxed{1, 3, 7, 9}$ is a solution,

$$1 + 3 + 7 = 11,\ 1 + 3 + 9 = 13,\ 1 + 7 + 9 = 17,\ 3 + 7 + 9 = 19,$$

are all prime number. □

Solution. **??** Since all the sums have the same value. Let S be that value. By adding all the sums at the vertices A, B, C, and D,

$$4S = 2(a + b + c + d + e + f) = 2(1 + 2 + \ldots + 6) = 2 \cdot \frac{6 \cdot 7}{2} = 42.$$

This is impossible because 4 is not a divisor of 42. There is $\boxed{\text{no solution.}}$ □

Solution. **??** Let x be the number of tea bags in a box. Since Minh brewed 57 cups using each bag at least twice, we know that $2x \leq 57$, so $x \leq 28$. On the other hand, Linh used each bag at most 3 times, so $3x \geq 83$ and $x \geq 28$. Thus, there are $\boxed{28}$ bags in a box. □

Solution. **??** We draw the piece of paper before it is folded and after, where F is where the corner was before

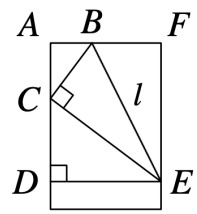


Figure 16.5: $FE = x$

the fold and C where the corner is after. Thus $BF = CB = 5$ and $FE = CE$. Since $AF = 8$, we have $AB = 8 - 5 = 3$.

From right triangle ABC we find $AC = 4$. Let $FE = x$. Thus $CD = x - 4$ and $CE = x$. Since $ED = AF = 8$, we have $(x-4)^2 + 8^2 = x^2$ from $\triangle CDE$. Hence $x = 10$ and from right $\triangle BCE$ we find $\ell = \sqrt{25 + 100} = \boxed{5\sqrt{5}}$.. □

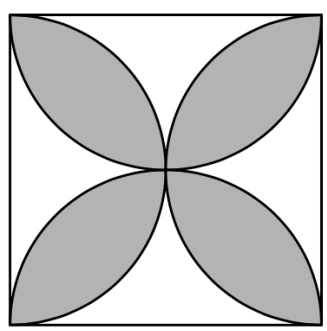
Solution. **??** Note that $x^3 - y^3 = (x-y)(x^2+xy+y^2)$, $x^3 + y^3 = (x+y)(x^2-xy+y^2)$, and $x \neq \pm y$, so,

$$\begin{cases} x^3 = 13x + 3y \\ y^3 = 3x + 13y \end{cases} \Rightarrow \begin{cases} x^3 - y^3 = 10(x-y) \Rightarrow x^2 + xy + y^2 = 10 \\ x^3 + y^3 = 16(x+y) \Rightarrow x^2 - xy + y^2 = 16 \end{cases} \Rightarrow \begin{cases} x^2 + y^2 = 13 \\ xy = -3 \end{cases}$$

$$\Rightarrow \begin{cases} (x+y)^2 = 13 - 2 \cdot 3 = 19 \\ (x-y)^2 = 13 + 2 \cdot 3 = 7 \end{cases} \Rightarrow \left(x^2 - y^2\right)^2 = (x+y)^2(x-y)^2 = 19 \cdot 7 = \boxed{133}$$

□

Solution. **??** The simple figures we have here are 4 semicircles and a square. The desired area is the region where semicircles overlap. Hence, we note that by adding together the areas of the four semicircles, we exceed



the area of the square by the total area of the desired region. (Make sure you see this; it is because each *leaf* is in two of the semicircles.) We are *overcounting* the area covered by the semicircles by twice counting the amount of area in the shaded regions. Hence, the area of the desired region is the total area of the four semicircles minus the area of the square, or

$$4 \cdot \frac{3^2\pi}{2} - 6^2 = \boxed{18\pi - 36}.$$

□

Solution. **??** Note that

$$\underbrace{11\ldots1_2}_{n \text{ digits}} = 2^n - 1 \Rightarrow \underbrace{11\ldots1_2}_{n \text{ digits}} \cdot \underbrace{11\ldots1_2}_{n \text{ digits}} = (2^n-1)^2 = 2^{2n} - 2\cdot 2^n + 1$$
$$= \underbrace{10\ldots0_2}_{2n \text{ digits}} - \underbrace{10\ldots0_2}_{n+1 \text{ digits}} + 1$$
$$= \underbrace{11\ldots1}_{n-1}\underbrace{00\ldots0}_{n-1}1_2$$

Thus, A has $(n-1)+1 = n$ digits of 1. With $n = 2022$, the answer is $\boxed{2022.}$ □

Solution. **??** Let t_1, t_2, t_3, and t_4 be the time needed for each student to cross. The most obvious solution is to let the fastest person (A) accompany each other person over the bridge and return alone with the flashlight. We write this schedule as

$$+\{A,B\} - A + \{A,C\} - A + \{A,D\},$$

denoting forward and backward movement by $+$ and $-$, respectively.

The total duration of this solution is

$$t_2 + t_1 + t_3 + t_1 + t_4 = 2t_1 + t_2 + t_3 + t_4 = 2\cdot 2 + 3 + 6 + 8 = 21 \text{ minutes.}$$

The interesting twist of the puzzle is that the obvious solution is not optimal. A second thought reveals that it might pay off to let the two slow persons (C and D) cross the bridge together, to avoid having both terms t_3 and t_4 in the total time. However, starting with

$$+\{C,D\} - C + \ldots \qquad \text{or} \qquad +\{C,D\} - D + \ldots,$$

incurs the penalty of having C or D cross at least three times in total(!).

The correct solution in this case is to let persons 3 and 4 cross in the middle,

$$+\{A,B\} - A + \{C,D\} - B + \{A,B\},$$

with a total time of,

$$t_2 + t_1 + t_4 + t_2 + t_2 = t_1 + 3t_2 + t_4 = 2 + 3\cdot 3 + 8 = \boxed{19} \text{ minutes.}$$

□

Solution. **??** Following is an example solution in Python.

```
count = 0
for a in range(1, 98):
    for b in range(1, 98):
        for c in range(1, 98):
            for d in range(1, 98):
                if a+b+c+d == 100 and a+b >= 15 and c+d <= 25:
                    count += 1
print(count)
```

□

Solution. **??** First, let sort the girls by age, then,

$$\text{Knight} > \text{normal} > \text{Liar}.$$

Now, Tung can ask any of the girls, let say A, by pointing to the two girls, indicating them B and C,

$$\boxed{\text{Is B older then C?}}$$

Case 1: if A is a Knight. Then the answer is *Yes* if B is a normal and C is a Liar. The answer is *No* if B is a Liar and C is a normal.

Case 2: if A is a Liar. Then the answer is *Yes* if B is a normal and C is a Knight. The answer is *No* if B is a Knight and C is a normal.

From the analysis of the answers in the first two cases, Tung can safely marry C if the answer is *Yes*, and marry B if the answer is *No.*

Case 3: if A is a normal. It does not matter what she said. Tung can marry any one. □

Solution. **??** First, note that since we have 2025 numbers, among them 1013 odd numbers

$$1, 3, 5, \ldots, 2025,$$

thus the sum $1 \pm 2 \pm 3 \ldots \pm 2025$ is an odd number. Thus the minimal non-negative possible value is 1.

Note that for four consecutive integers $n - (n+1) - (n+2) + (n+3) = 0$, thus

$$1 + (2 - 3 - 4 + 5) + (6 - 7 - 8 + 9) + \ldots + (2022 - 2023 - 2024 + 2025) = \boxed{1}$$

□

Solution. **??** By the fact that the area ratio of two triangles with the same height is the ratio of their bases,

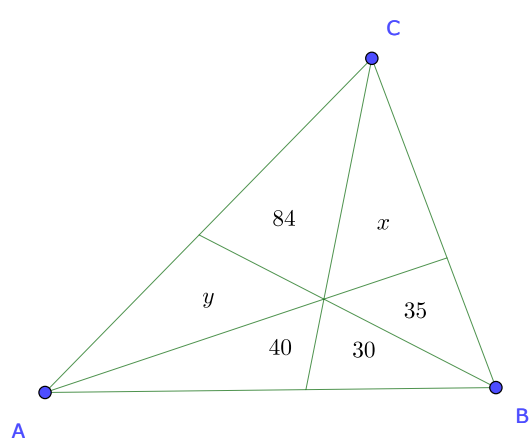


Figure 16.6: x, y denotes unknown areas

$$\frac{40}{30} = \frac{40 + y + 84}{30 + 35 + x}, \ \frac{35}{x} = \frac{35 + 30 + 40}{x + 84 + y}, \ \frac{84}{y} = \frac{84 + x + 35}{y + 40 + 30}$$
$$\Rightarrow x = 70, \ y = 56.$$

Thus $[ABC] = 30 + 35 + 70 + 84 + 56 + 40 = \boxed{315.}$ □

Solution. **??** Let assume that at a certain step, the sequence contains

$$a_1, a_2, \ldots, a_k.$$

Let P be the expression $P = (a_1 + 1)(a_2 + 1) \ldots (a_k + 1)$.

Since $mn + m + n + 1 = (m + 1)(n + 1)$, so m, n are replaced with $(m + 1)(n + 1) - 1$, so WLOG, by combining a_{k-1}, a_k, the sequence becomes

$$a_1, a_2, \ldots, a_{k-2}, (a_{k-1} + 1)(a_k + 1) - 1.$$

However,

$$P = (a_1 + 1)(a_2 + 1) \ldots (a_{k-1} + 1)\,[(a_{k-1} + 1)(a_k + 1) - 1 + 1] = (a_1 + 1)(a_2 + 1) \ldots (a_k + 1),$$

retains its value. Thus it is an invariant. Thus after $k - 1$ steps, we shall receive the number $(a_1 + 1)(a_2 + 1) \ldots (a_k + 1) - 1$.

Therefore the final number is $(1 + 1) \cdot (2 + 1) \cdot (2022 + 1) - 1 = \boxed{2023! - 1.}$ □

Solution. **??** Following is an example solution in Python.

```
count = 0

for a0 in range(0, 4):
    for a1 in range(0, 4):
        for a2 in range(0, 4):
            for a3 in range(0, 4):
                for a4 in range(0, 4):
                    for a5 in range(0, 4):
                        for a6 in range(0, 4):
                            for a7 in range(0, 4):
                                for a8 in range(0, 4):
                                    c0 = len({a0,a1,a2,a3,a4}) == 4
                                    c1 = len({a1,a2,a3,a4,a5}) == 4
                                    c2 = len({a2,a3,a4,a5,a6}) == 4
                                    c3 = len({a3,a4,a5,a6,a7}) == 4
                                    c4 = len({a4,a5,a6,a7,a8}) == 4
                                    if c0 and c1 and c2 and c3 and c4:
                                        count += 1

print(count)
```

□

Chapter 17

SM 67

17.1 Seminar Grade 6-7

Problem 17.1.1 (SM-2022-SM1-67-P1)**.** How many odd numbers with middle digit 5 and no digit repeated are there between 20000 and 69999?

Solution. **??**

We can put the 5 in the middle first, and turn our attention to the last digit, which must be $1, 3, 7$, or 9. If this last digit is 3, it restricts the first digit, which can be one of $2, 3, 4$, or 6, while the first digit is unrestricted if the last digit is not 3. We thus consider two cases.

Case 1: The last digit is 3. Here there are only the possibilities $2, 4$, and 6 for the first digit. The remaining two digits can be chosen from the 7 choices we haven't used in $7 \cdot 6$ ways. The total for this case is thus $1 \cdot 1 \cdot 3 \cdot 7 \cdot 6 = 126$.

Case 2: The last digit is not 3. For the last digit we have the choices $1, 7$, and 9, and for the first digit we have the four choices $2, 3, 4$, and 6. Again, the remaining two digits may be picked in $7 \cdot 6$ ways, for a total of $1 \cdot 3 \cdot 4 \cdot 7 \cdot 6 = 504$.

The total number of numbers is the sum of the totals found by the two cases: $126 + 504 = \boxed{\mathbf{630}}$. □

Remark. ??

Only **Khoa Vu** attempted to solve this problem by combinatorics (counting). His solution is good and almost perfect if he would have done casework as shown above. Using a single case to count as he did would not prevent the first and last digit to be the same digit of 3.

Ryan Nguyen noted that he solved the problem by coding and got the correct answer. However no code was submmitted, thus it cannot be verified.

Problem 17.1.2 (SM-2022-SM1-67-P2)**.** What is $\frac{\sqrt{2}}{5-\sqrt{2}-\sqrt{3}}$ when its denominator is rationalized?

Solution. **??**

First, lets prove a generic result,

Claim — The product of $a\sqrt{b}+c\sqrt{d}$ and $a\sqrt{b}-c\sqrt{d}$ is always rational if a, b, c, and d are rational.

Proof.

$$(a\sqrt{b}+c\sqrt{d})(a\sqrt{b}-c\sqrt{d}) = (a\sqrt{b})^2-(c\sqrt{d})^2 = a^2b-c^2d.$$

It is easy to see that the expression a^2b-c^2d is always rational if a, b, c, and d are rational. ■

Now, to rationalize the denominator $5-\sqrt{2}-\sqrt{3} = (5-\sqrt{2})-\sqrt{3}$, we first multiple it with $(5-\sqrt{2})+\sqrt{3}$,

$$((5-\sqrt{2})-\sqrt{3})((5-\sqrt{2})+\sqrt{3}) = (5-\sqrt{2})^2-3 = 25-10\sqrt{2}+2-3 = 24-10\sqrt{2} = 2(12-5\sqrt{2}).$$

The numerator becomes $\sqrt{2}((5-\sqrt{2})+\sqrt{3}) = 5\sqrt{2}-2+\sqrt{6}$, and the fraction is

$$\frac{5\sqrt{2}-2+\sqrt{6}}{2(12-5\sqrt{2})}.$$

To rationalize $12-5\sqrt{2}$, we multiple it with $12+5\sqrt{2}$, the fraction becomes,

$$\frac{(5\sqrt{2}-2+\sqrt{6})(12+5\sqrt{2})}{2(12-5\sqrt{2})(12+5\sqrt{2})} = \frac{26+50\sqrt{2}+10\sqrt{3}+12\sqrt{6}}{2(12^2-25\cdot 2)} = \boxed{\frac{\mathbf{13+25\sqrt{2}+5\sqrt{3}+6\sqrt{6}}}{\mathbf{94}}}.$$

□

Remark. ??

Only **Ryan Nguyen** submitted a solution and he has provided a perfect one for this problem.

Problem 17.1.3 (SM-2022-SM1-67-P3)**.** Given $9^6 = 531,441$, how would you represent $531,440$ in base 9?

Solution. **??**

Since $9^6 = 531,441$, thus in base-9 the number $531,441$ is represented by

$$\overline{1000000}_9.$$

Now, note that $9 = \overline{10}_9,\ 8 = \overline{8}_9$, so $531,440 = \boxed{\mathbf{\overline{888888}_9}.}$ □

Remark. ??

Only **Ryan Nguyen** submitted a solution and he has provided a perfect one for this problem.

Problem 17.1.4 (SM-2022-SM1-67-P4)**.** Simplify the following expression as much as possible:

$$\left(\frac{a^3-1}{a^2-1}\right)\left(\frac{a^2+2a+1}{a^3+1}\right)\left(\frac{a^2-a+1}{a+1}\right).$$

Solution. **??**

First, some expressions can be factored using well-known second- and third-degree identities

$$\begin{aligned} a^2+2a+1 &= (a+1)^2 \\ a^2-1 &= (a-1)(a+1) \\ a^3-1 &= (a-1)(a^2+a+1) \\ a^3+1 &= (a+1)(a^2-a+1) \end{aligned}$$

Thus, the given fraction can be rewritten as,

$$\frac{(a-1)(a^2+a+1)}{(a-1)(a+1)}\frac{(a+1)^2}{(a+1)(a^2-a+1)}\frac{a^2-a+1}{a+1} = \frac{\cancel{(a-1)}(a^2+a+1)}{\cancel{(a-1)}(a+1)}\frac{\cancel{(a+1)^2}}{\cancel{(a+1)(a^2-a+1)}}\frac{\cancel{a^2-a+1}}{\cancel{a+1}}$$

Thus, the remaining fraction is $\boxed{\mathbf{\frac{a^2+a+1}{a+1}}}$. □

Remark. ??

Only **Ryan Nguyen** submitted a solution and he has provided a perfect one for this problem.

Problem 17.1.5 (SM-2022-SM1-67-P5)**.** The two circles in the figure are tangent at G. Prove that $\angle AEB = \angle CFD$.

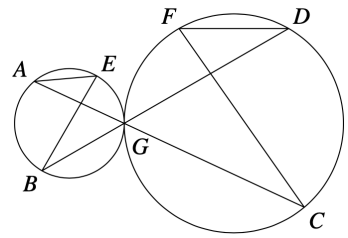


Figure 17.1: **??**

Solution. **??**

First, note that $\angle AEB = \angle AGB$, since both subend the same small arc $\widehat{AB}$ (which does contain neither E nor G.) Similarly $\angle CFD = \angle CGD$ because they subend the same small arc $\widehat{CD}$. Finally $\angle AGB = \angle CGD$ because they are vertically opposite angles, thus $\boxed{\angle \mathbf{AEB} = \angle \mathbf{CFD}}$. □

Remark. ??

Only **Khoa Vu** submitted an almost perfect solution. Instead of applying *alternate segment theorem*, which refers to a different situation, his solution would be perfect if that would have used the property that *two angles subtending the same arc have the same measure.*

Problem 17.1.6 (SM-2022-SM1-67-P6)**.** Show that if h_a, h_b, and h_c are the altitudes of a triangle, then

$$\frac{1}{h_a} < \frac{1}{h_b} + \frac{1}{h_c}.$$

Solution. **??**

The *triangle inequality* states that

Claim — If a, b, and c are the three sides of a triangle then $a + b < c$.

Now, let h_a, h_b, and h_c be the altitudes to the sides a, b, and c, respectively, and let S be the area of the triangle, then

$$2S = ah_a = bh_b = ch_c \Rightarrow \frac{1}{h_a} = \frac{a}{2S},\ \frac{1}{h_b} = \frac{b}{2S},\ \frac{1}{h_c} = \frac{c}{2S}$$

Therefore, the desired inequality can be rewritten as *equivalent* inequalities, as shown below,

$$\frac{1}{h_a} < \frac{1}{h_b} + \frac{1}{h_c} \Leftrightarrow \frac{a}{2S} < \frac{b}{2S} + \frac{c}{2S} \Leftrightarrow a < b + c.$$

Thus $\boxed{\textbf{the given inequality is equivalent to the triangle inequality}}$, which is true. □

Remark. ??

Only **Khoa Vu** submitted a solution and he has provided a perfect one for this problem.

Problem 17.1.7 (SM-2022-SM1-67-P7)**.** Let $ABCD$ be a parallelogram of area 10 with $AB = 3$ and $BC = 5$. Locate E, F, and G on segments AB, BC, and AD, respectively, with $AE = BF = AG = 2$. Let the line through G parallel to EF intersect CD at H. Find the area of the quadrilateral $EFHG$.

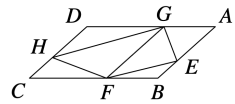


Figure 17.2: **??**

Solution. **??**

A generic result can be proven that,

Claim — In the parallelogram $CDGF$, H is on CD, then $[FGH] = \frac{1}{2}[CDGF]$.

Proof. Since H is on CD, so the altitudes from C, D, and H to FG are the same. Therefore $[FGH] = [FGC]$ because both triangles share the same base FG and have the same height. It is also easy to see that $\triangle CFG \cong \triangle GDC$ because they have three corresponding pairs of equal sides. Hence, $[CFG] = [GDC]$. Thus, $[FGH] = [FGC] = \frac{1}{2}[CDGF]$. ■

The proof is simple based on the claim above. Because $AG = BF$, thus $AGFB$ and $CFGD$ are parallelograms,

$$[GEFH] = [GEF] + [GHF] = \frac{1}{2}[AGFB] + \frac{1}{2}[CFGD] = \frac{1}{2}[ABCD].$$

Therefore $[GEFH] = \frac{10}{2} = \boxed{\mathbf{5.}}$ □

Remark. ??

Only **Khoa Vu** and **Ryan Nguyen** submitted perfect solutions for this problem.

Problem 17.1.8 (SM-2022-SM1-67-P8)**.** The two regular hexagons shown share the side EF. Given that the perimeter of quadrilateral $ABCD$ is $44 + 22\sqrt{3}$, find EF.

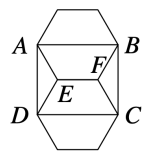


Figure 17.3: **??**

Solution. **??**

Let EF intersects AD and BC at X and Y, respectively. $BF = FC$, so $\triangle BFC$ is isosceles.

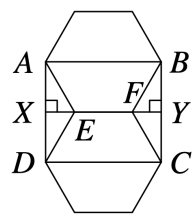


Figure 17.4: EF intersects AD, BC at X, Y

$$\angle BFC = 360^\circ - \angle BFE - \angle CFE = 120^\circ \Rightarrow \angle FBC = \angle FCB = 30^\circ$$
$$\angle ABF = \angle CDE = 60^\circ \Rightarrow \angle ABC = \angle DCB = 90^\circ.$$

Similarly $\angle ADC = \angle DAB = 90^\circ$, thus $ABCD$ is a rectangle.

Now, let $EF = x$. Since the longest diagonal of a regular hexagon is twice the length of a side, we have $AB = CD = 2x$. It is easy to see that $\triangle EXA = \triangle EXD$ by *angle-side-angle.* Therefore

$$\angle XEA = \angle XED = 60^\circ \Rightarrow EX = \frac{1}{2}EA = \frac{x}{2} \Rightarrow AX = \frac{x\sqrt{3}}{2} \Rightarrow AD = BC = x\sqrt{3}.$$

Finally, the perimeter of $ABCD$ can be computed based on x as follow,

$$2(AB + BC) = 2(2x + x\sqrt{3}) = 2x(2 + \sqrt{3}) \Rightarrow 44 + 22\sqrt{3} = 2x(2 + \sqrt{3}) \Rightarrow x = \boxed{\mathbf{11.}}$$

□

Remark. ??

Only **Ryan Nguyen** submitted a solution for this problem. A small calculation mistake regrettably ruined a nearly perfect solution.

Problem 17.1.9 (SM-2022-SM1-67-P9)**.** Less than 50 people are at a party. Each person shakes everyone else's hand. If there is an odd number of total handshakes at the party, what is the largest number of people that could be at the party?

Solution. **??**

Let the number of people be n, so that the total number of handshakes is $\binom{n}{2} = \frac{n(n-1)}{2}$. For this to be odd, there must be only one factor of 2 in the product $n(n-1)$; otherwise the 2 in the denominator would cancel one 2, but others would remain to make the number even.

Since either n or $n-1$ must be even, whichever is even must be divisible by 2 and not 4. Thus for $(n, n-1)$ the pairs $(49, 48)$ and $(48, 47)$ are ruled out, since 4 divides 48. The next smaller pair, $(47, 46)$, works fine, since 2 only divides 46 once. Thus $n = \boxed{\mathbf{47}.}$ □

> **Remark. ?? Khoa Vu**, **Minh Nhat Henry Trinh** and **Ryan Nguyen** submitted perfect solutions for this problem.

Problem 17.1.10 (SM-2022-SM1-67-P10)**.** How many lines are determined by 12 points in a plane, no three of which are collinear? (That is, how many lines are formed if all possible lines through two of the points are drawn?)

Solution. **??**

Since no three of the points lie on a single line, each pair of two points determines a distinct line. The total number of lines is thus equal to the number of pairs of points, or $\binom{12}{2} = \frac{12 \cdot 11}{2} = \boxed{\mathbf{66}.}$ □

Remark. ??

Only **Ryan Nguyen** submitted a solution for this problem. A small calculation mistake regrettably ruined a nearly perfect solution.

Problem 17.1.11 (SM-2022-SM1-67-P11)**.** Find the probability of rolling a 12 with two six-sided dice.

Solution. **??**

We could do this as follows: there is only 1 way to get a 12 $(6-6)$, while there are 36 total possibilities (anything-anything). Thus the probability is $\boxed{\mathbf{\frac{1}{36}}}$.

However, we can also use the multiplication rule for probability as follows. The only possible roll which yields a sum of 12 is $6-6$. The probability of a 6 on the first die is obviously $\frac{1}{6}$, the probability of 6 on the second die is $\frac{1}{6}$. Since the two dice are uncorrelated, the probability of both occurring is the product, or $(\frac{1}{6})(\frac{1}{6}) = \frac{1}{36}$. □

Remark. ??

Khoa Vu, **Minh Nhat Henry Trinh** and **Ryan Nguyen** submitted perfect solutions for this problem.

Problem 17.1.12 (SM-2022-SM1-67-P12)**.** Ignatius plays a game in which he chooses at random from a penny, a nickel, a dime, and a quarter. How much can he pay to play and still break even?

Solution. **??**

The amount Ignatius can pay to play a game and still hope to break even is exactly the expected value. In this game, Ignatius chooses each coin with probability $\frac{1}{4}$, so his expected value is

$$\left(\frac{1}{4}\right)(1) + \left(\frac{1}{4}\right)(5) + \left(\frac{1}{4}\right)(10) + \left(\frac{1}{4}\right)(25) = \boxed{\mathbf{10.25}.}$$

□

Remark. ??

Only **Ryan Nguyen** submitted a solution for this problem. The solution should be worded with more clarity.

Problem 17.1.13 (SM-2022-SM1-67-P13)**.** How many subsets of

$$\{\, n \mid n \text{ is a multiple of 3 less than } 100 \,\}$$

are also subsets of

$$\{\, n \mid n \text{ is a multiple of 4 less than } 100 \,\}?$$

Solution. **??**

The first set is given by $\{3, 6, 9, \ldots, 99\}$. The second set is $\{4, 8, \ldots, 96\}$.

A set is a subset of both sets if and only if all its elements are divisible by both 3 and 4; that is, are divisible by 12. Thus every desired subset must also be a subset of $\{12, 24, 36, 48, 60, 72, 84, 96\}$, a set with 8 elements. *Every subset of this latter set is one of the desired subsets and every desired subset is also a subset of this set.*

Hence the number we wish is just the number of subsets of this 8−element set, or $2^8 = \boxed{\mathbf{256}.}$ □

Remark. ??

Only **Ryan Nguyen** submitted a solution for this problem. The solution was not complete. It provided the number of elements (8) in the set of $\{12, 24, 36, 48, 60, 72, 84, 96\}$. However, the problem asked for the number of subsets that contain *any number* of these 8 elements, which is the number of subsets of the set $\{12, 24, 36, 48, 60, 72, 84, 96\}$.

Problem 17.1.14 (SM-2022-SM1-67-P14)**.** (*Advanced Level*)

Show that for all positive integers n,

$$7 + 6 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \cdots + 6 \cdot 7^n = 7^{n+1}.$$

Solution. **??**

We use *The Induction Principle* to prove the desired equality.

For *the base case* when $n = 1 : 7 + 6 \cdot 7 = 7(1 + 6) = 7^2$, thus the given hypothesis is correct.

Now, for *the inductive step*, lets assume that the hypothesis is correct for $n \geq 1$, or $7 + 6 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \cdots + 6 \cdot 7^n = 7^{n+1}$. We have to prove that it is also correct for $n + 1$, or

$$7 + 6 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \cdots + 6 \cdot 7^n + 6 \cdot 7^{n+1} = 7^{n+2}.$$

By the assumption of the inductive step,

$$7 + 6 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \cdots + 6 \cdot 7^n + 6 \cdot 7^{n+1} = 7^{n+1} + 6 \cdot 7^{n+1} = 7^{n+1}(1 + 6) = 7^{n+2}.$$

Thus, the hypothesis is also correct for $n + 1$. Hence, $\boxed{\textbf{it is correct for all } \mathbf{n \geq 1}.}$ □

Remark. ??

Only **Ryan Nguyen** submitted a solution and he has provided a perfect one for this problem.

Problem 17.1.15 (SM-2022-SM1-67-P15)**.** (*Advanced Level*)

Each of 6 points in space is connected to the other 5 points by line segments. Each segment thus formed is colored green or purple. Show that it is impossible to color all the segments without forming a triangle in which all three segments are the same color.

Definition. A *monochromatic triangle* is triangle in which the sides are all the same color.

Solution. **??**

Lets consider connecting one point, say P_1, to the other 5 points. Since we are coloring these 5 segments with 2 colors, by *The Pigeonhole Principle* at least three of them, say P_2, P_3, and P_4, must be the same color, say C_1.

Now, if we connect any two of these three points, say P_2 and P_3, with that color C_1, we form a *monochromatic triangle* $\triangle P_1P_2P_3$. Thus we must connect these three points with the other color, say C_2. However, in doing this we form a monochromatic triangle $\triangle P_2P_3P_4$ with these three points as vertices.

Hence, there is $\boxed{\textbf{no way to connect}}$ the 6 points as suggested without forming a monochromatic triangle. □

Remark. ??

No student submitted solution for this problem.

Chapter 18

MIC R8

18.1 Rules

- The total time to complete the test is 90 minutes.
- The test consists of 10 multiple-choice and 4 show-you-work problems. To answer each of 10 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 6 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.

18.2 Problems

Problem 18.2.1 (MIC-2022-SM1-R8-P1). (*6 points*)

When evaluated, the sum of the digits of the integer equal to $10^{2022} - 2022$ is

(A) 18176 (B) 18167 (C) 18153 (D) 18185 (E) 18193

Problem 18.2.2 (MIC-2022-SM1-R8-P2). (*6 points*)

In the diagram below, there are three spinners. They are used to define (from left to right) the hundreds, tens, and ones digits of a three-digit number. *For example, the diagram below defines the number* 380.

How many such three-digit numbers formed this way are divisible by 18?

(A) 6 (B) 9 (C) 11 (D) 12 (E) 16

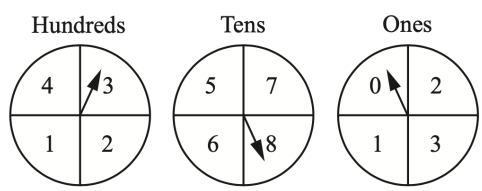


Figure 18.1: **??**

Problem 18.2.3 (MIC-2022-SM1-R8-P3). (*6 points*)

In the square $PQRS$ shown in the diagram below, $PQ = 30$, Segments ST, SU, SV, and SW divide the square into 5 regions of *equal* area.

$$[SPT] = [STU] = [SUQV] = [SVW] = [SWR].$$

The ratio $\left(\frac{SU}{ST}\right)^2$ can be written as $\frac{m}{n}$, where m and n are positive relatively prime integers. Find $m + n$.

(A) 48 (B) 70 (C) 82 (D) 144 (E) 287

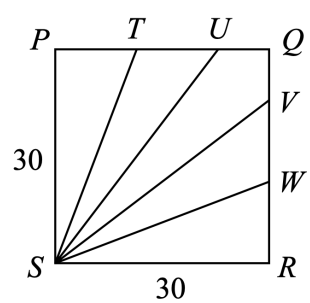


Figure 18.2: **??**

Problem 18.2.4 (MIC-2022-SM1-R8-P4). (*6 points*)

For a regular dice, the number of dots on opposite faces add up to 7. Four regular dice are stacked as shown in the diagram below.

What is the sum of the number of dots hidden *between* the dice?

(A) 16 (B) 17 (C) 19 (D) 23 (E) 25

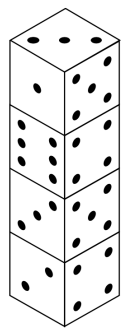


Figure 18.3: **??**

Problem 18.2.5 (MIC-2022-SM1-R8-P5). (*6 points*)

Integers a, b, c, d, and e satisfying the following conditions:

1. $2 \leq a < b < c < d < e < 100$,
2. $\gcd(a, e) = 1$,
3. a, b, c, d, and e form a geometric sequence.

Find the value of c.

(A) 32 (B) 36 (C) 40 (D) 42 (E) 64

Problem 18.2.6 (MIC-2022-SM1-R8-P6). (*6 points*)

In the room, there are twenty people. m of them are men and w of them are women. Each pair of men shakes hands, and each pair of women shakes hands, but there are no handshakes between a man and a woman. The total number of handshakes is 94.

Determine the value of $|m - w|$. $|x|$ *is the absolute value of* x, $|3 - 2| = |1| = |2 - 3| = |-1| = 1$.

(A) 10 (B) 8 (C) 6 (D) 4 (E) 2

Problem 18.2.7 (MIC-2022-SM1-R8-P7). (*6 points*)

Regular 10-sided polygon $ABCDEFGHIJ$ has area 2022 square units. Determine the area (in square units) of the rectangle $CDHI$.

(A) 804.4 (B) 806 (C) 806.8 (D) 808 (E) 808.8

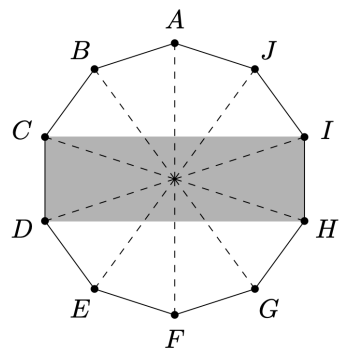


Figure 18.4: **??**

Problem 18.2.8 (MIC-2022-SM1-R8-P8). (*6 points*)

In the 4×4 grid below each cell at row i and column j contains the value equal to $i \cdot j$.

We define a *path* in the grid is a sequence of squares, such that consecutive squares share an edge and no square occurs twice in the sequence. Furthermore, the *score* of a path is the sum of the point values of all squares in the path.

Determine the highest possible score of a path that begins with the bottom left corner of the grid (where the number 1 stands) and ends with the top right corner (where the number 16 stands.)

(A) 90 (B) 84 (C) 96 (D) 98 (E) 100

4	8	12	16
3	6	9	12
2	4	6	8
1	2	3	4

Problem 18.2.9 (MIC-2022-SM1-R8-P9). (*6 points*)

Seven points are chosen on a circle, and chords are drawn connecting every pair of points. *No three chords intersect in a single point inside the circle.*

How many triangles are there with all three vertices *strictly* in the interior of the circle (no vertex can be on the perimeter of the circle)?

(A) 84 (B) 56 (C) 28 (D) 14 (E) 7

Problem 18.2.10 (MIC-2022-SM1-R8-P10). (*6 points*)

Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that

$$\begin{cases} P(1) = P(3) = P(5) = P(7) = a \\ P(2) = P(4) = P(6) = P(8) = -a \end{cases}$$

What is the smallest possible value of a?

(A) 8! (B) 7! (C) 945 (D) 315 (E) 105

Problem 18.2.11 (MIC-2022-SM1-R8-P11). (*10 points*)

For a school performance, exactly 120 tickets were sold. The tickets cost 12 dollars for section A, 10 for section B, and 6 for section C. The number of tickets sold for section A was equal to the number of tickets sold for section C. The total revenue from the sales was 1150. Find the number of tickets sold for each section.

Problem 18.2.12 (MIC-2022-SM1-R8-P12). (*10 points*)

Let $ABCD$ be a unit square. Points X and Y are on sides BC and CD, respectively. Furtheremore the following triangles have the same area,

$$[ABX] = [XCY] = [YDA]$$

Find the ratio $\left(\frac{[CXY]}{[AXY]}\right)^2$.

Problem 18.2.13 (MIC-2022-SM1-R8-P13). (*10 points*)

Find all pairs (x, y) of real numbers which satisfy

$$\begin{cases} x^3 - y^3 = 4(x - y) \\ x^3 + y^3 = 2(x + y) \end{cases}$$

Problem 18.2.14 (MIC-2022-SM1-R8-P14). (*10 points*)

For all real numbers x, y define

$$x \star y = \frac{x + y}{1 + xy}$$

Evaluate the expression

$$(\cdots(((2 \star 3) \star 4) \star 5) \star \cdots) \star 2022.$$

18.3 Grading

Answers for multiple-choice problems.

Problem 1: E

Problem 2: A

Problem 3: B

Problem 4: C

Problem 5: B

Problem 6: D

Problem 7: E

Problem 8: D

Problem 9: E

Problem 10: D

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 11: 2 points if can establish a correct equations with variable(s).

Problem 12: 2 points if can compute side length BX, CX, CY, or DY.

Problem 13: 2 points for each case $x = y$ or $x = -y$.

Problem 14: 2 points if can discover telescopic product (not necessarily able to solve it)

18.4 Solutions

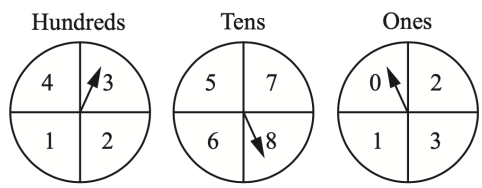
Solution. **??**

$$10^{2022} = 1 + \underbrace{99\ldots9}_{2022\ 9s} \Rightarrow 10^{2022} - 2022 = \underbrace{99\ldots9}_{2022\ 9s} - 2021 = \underbrace{99\ldots9}_{2018\ 9s}7978$$

Therefore the sum of the digits of the given number is equal to $2019 \cdot 9 + 8 + 2 \cdot 7 = \boxed{18193}$.

The answer is $\boxed{E.}$ □

Solution. **??** A number is divisible by 18 if it is divisible by 2 and 9. Since it is divisible by 2, the unit digit can only be 0 or 2.



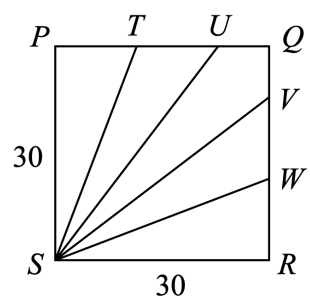
Case 1: If the unit digit is 0, then the sum of the hundreds and tens digits must be 9. There are four such cases $1 + 8 = 2 + 7 = 3 + 6 = 4 + 5 = 9$.

Case 2: If the unit digit is 2, then the sum of the hundreds and tens digits must be 7 or 16. However since none of the number in the left-most spinner is larger than 5, thus 16 is not a possible sum. For the sum equal to 7, there are two cases $1 + 6 = 2 + 5 = 7$.

Thus, in total there are $4 + 2 = \boxed{6}$ cases.

The answer is $\boxed{A.}$ □

Solution. **??**

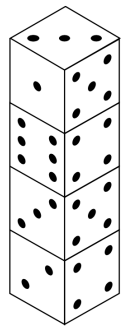


$$[PQRS] = 30^2 = 900 \Rightarrow [SPT] = [STU] = [SVW] = [SWR] = [SUQV] = \frac{900}{5} = 180$$
$$[SPT] = \frac{PS \cdot PT}{2} = 15(PT) \Rightarrow PT = 12,\ [STU] = \frac{PS \cdot TU}{2} = 15(TU) \Rightarrow TU = 12 \Rightarrow PU = 24$$
$$ST = \sqrt{PS^2 + PT^2} = \sqrt{30^2 + 12^2} = \sqrt{1044}$$
$$SU = \sqrt{PS^2 + PU^2} = \sqrt{30^2 + 24^2} = \sqrt{1476}$$

Thus, $\left(\frac{SU}{ST}\right)^2 = \frac{1476}{1044} = \frac{41}{29}$. Therefore, $m = 41, n = 29$, so $m + n = \boxed{70}$.

The answer is $\boxed{B.}$ □

Solution. **??** On the bottom die, the two visible faces are showing 2 dots and 4 dots. Since the number of dots on opposite faces of this die add to 7, then there are 5 dots on the face opposite the face having 2 dots, and 3 dots on the face opposite the face having 4 dots. Therefore, the top face of this bottom die (which is a face that is hidden between the dice) has either 1 dot on it or it has 6 dots on it.



On the second die from the bottom, the sum of the number of dots on the top and bottom faces (the faces hidden between the dice) is 7 since the number of dots on opposite faces add to 7. (We do not need to know which faces these are, though we could determine that they must have 1 and 6 dots on them.) Similarly, on the third die from the bottom, the sum of the number of dots on the top and bottom faces (the faces hidden between the dice) is also 7.

Finally, the top face of the top die shows 3 dots, and so the bottom face of this die (which is a face that is hidden between the dice) contains 4 dots. Therefore, the sum of the number of dots hidden between the dice is either $4+7+7+1=19$ or $4+7+7+6=24$. Of these two possible answers, $\boxed{19}$ is the only answer which appears among the five given answers.

The answer is $\boxed{C.}$ □

Solution. **??** Let r be the common ratio of the geometric sequence a, b, c, d, e. Since $a < b < c < d < e$, so $r > 1$, and $(a, b, c, d, e) = (a, ar, ar^2, ar^3, ar^4)$.

Because $\gcd(a, e) = 1$, so r is not an integer, let $r = \frac{x}{y}$, where $x > y > 1$. Thus $b = \frac{ax}{y}$, $c = \frac{ax^2}{y^2}$, $d = \frac{ax^3}{y^3}$, $e = \frac{ax^4}{y^4}$).

Now, because e is an integer, so $y^4 \mid a$. Thus, there exists k integer such that $a = ky^4$. Then $b = kxy^3, c = kx^2y^2, d = kx^3y, e = kx^4$. Hence $\gcd(a, e) = \gcd(ky^4, kx^4) = k$, so $k = 1$. Hence $a = y^4$ and $e = x^4$.

Now $2 < a = y^4 < b < c < d < e = x^4 < 100$. This means that $y^4 < x^4$ are the two 4th powers of positive integers between 2 and 100. Thus $y^4 = 2^4$ and $x^4 = 3^4$, so $x = 3, y = 2$. Therefore $c = kx^2y^2 = 1 \cdot 3^2 \cdot 2^2 = \boxed{36}$.

The answer is $\boxed{B.}$ □

Solution. **??** The number of possible handshakes are $\frac{20 \cdot 19}{2} = 190$. Out of this $m \cdot w$ pairs did not shake hand, thus $mw = 190 - 94 = 96$. We obtain a system of equations

$$\begin{cases} m + w = 20 \\ mw = 96 \end{cases}$$

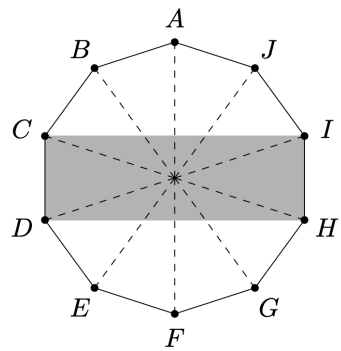
Thus, m and w are roots of the quadratic equation $x^2 - 20x + 96 = 0$,

$$\Delta = 20^2 - 4 \cdot 96 = 16 = 4^2 \Rightarrow x_{1,2} = \frac{20 \pm 4}{2} \Rightarrow x_1 = 8, x_2 = 12$$

Therefore $(m, w) \in \{(8, 12), (12, 8)\}$, or $|m - w| = \boxed{4.}$

The answer is $\boxed{D.}$ □

Solution. **??** If O is the centre of the decagon and we cut the decagon into equal isosceles triangles, as shown, we can see that the area of the decagon is 10 times the area of one triangle.



Since diagonals CH and DI are equal in length and bisect each other, quadrilateral $CDHI$ is a rectangle, with the same base as each of the triangles, but twice the height. Thus the area of $CDHI$ is 4 times the area of each of the triangles and hence 40% of the area of the decagon or $0.4 \cdot 2022 = \boxed{808.8}$ square units.

The answer is $\boxed{E.}$ □

Solution. **??** Let us colour the squares of the grid with a chessboard pattern of black and white, starting with black in the bottom left-hand corner. A path in the grid will alternate between black and white squares, beginning and ending on black. Thus, any path cannot contain all 8 white squares.

4	8	12	16
3	6	9	12
2	4	6	8
1	2	3	4

The path with highest possible score should contains all squares except for the white square with the lowest point value 2, The sum of the point values for all the squares is

$$(1+2+3+4)+(2+4+6+8)+(3+6+9+12)+(4+8+12+16) = (1+2+3+4)(1+2+3+4) = 10^2 = 100,$$

Thus, the score of the path is $100 - 2 = \boxed{98}$.

The answer is $\boxed{D.}$ □

Solution. **??** To choose a chord, we know that two points must be chosen. This implies that for three chords to create a triangle and not intersect at a single point, six points need to be chosen. We also know that for any six points we pick, there is only 1 way to connect the points such that a triangle is formed in the circle's interior (this is because we want no two chords to be parallel). Therefore, there are $\binom{7}{6} = \boxed{7}$ such triangles.

The answer is $\boxed{E.}$ □

Solution. **??** We observe that because $P(1) = P(3) = P(5) = P(7) = a$, if we define a new polynomial $R(x)$ such that $R(x) = P(x) - a$, $R(x)$ has 4 roots where $P(x) = a$; namely, at $x = 1, 3, 5, 7$.

Since $R(x)$ has $x = 1, 3, 5$, and 7 as roots, thus it can be factored into a product of $(x-1)(x-3)(x-5)(x-7)$ and a new polynomial $Q(x)$, such that

$$P(x) - a = R(x) = (x-1)(x-3)(x-5)(x-7)Q(x)$$

Then, plugging in values of $2, 4, 6, 8$, we get

$$\begin{aligned}
P(2) - a &= (2-1)(2-3)(2-5)(2-7)Q(2) = -15Q(2) = -2a \\
P(4) - a &= (4-1)(4-3)(4-5)(4-7)Q(4) = 9Q(4) = -2a \\
P(6) - a &= (6-1)(6-3)(6-5)(6-7)Q(6) = -15Q(6) = -2a \\
P(8) - a &= (8-1)(8-3)(8-5)(8-7)Q(8) = 105Q(8) = -2a \\
\Rightarrow \ -2a &= -15Q(2) = 9Q(4) = -15Q(6) = 105Q(8)
\end{aligned}$$

Now, it is easy to see that a is a multiple of $15, 9$, and 105. Thus, the least value of a must be the $\text{lcm}(15, 9, 105) = \boxed{315}$.

The answers is $\boxed{D.}$ □

Solution. **??** Let the number of tickets sold for section A be a. The combined revenue for sections A and C is $12a + 6a = 18a$, and the revenue for section B is $10(120 - 2a)$, thus

$$18a + 10(120 - 2a) = 1150 \Rightarrow a = 25$$

Thus the number of tickets sold for sections A, B, and C are $\boxed{25, 70, \text{ and } 25}$. □

Solution. **??** Let $a = BX, b = CY$.

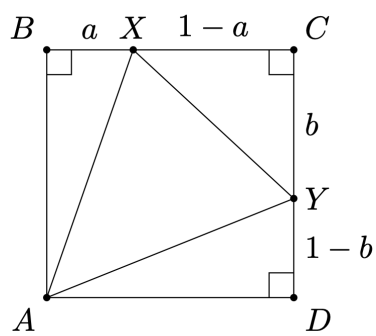


Figure 18.5: **??**

$$[ABX] = [XCY] = [YDA] \Rightarrow \frac{a}{2} = \frac{(1-a)b}{2} = \frac{1-b}{2} \Rightarrow \begin{cases} a + b = 1 \\ b^2 + b - 1 = 0 \end{cases} \Rightarrow b = \frac{-1+\sqrt{5}}{2}, a = \frac{3-\sqrt{5}}{2}$$

Since

$$[AXY] = 1 - 3[ABX] = 1 - \frac{3a}{2} = \frac{3\sqrt{5}-5}{4}, \ [CXY] = \frac{3-\sqrt{5}}{4} \Rightarrow \frac{[AXY]}{[CXY]} = \sqrt{5}$$

Therefore $\left(\frac{[CXY]}{[AXY]}\right)^2 = \boxed{\frac{1}{5}.}$ □

Solution. **??** Let casework to eliminate trivial cases.

Case 1: $x = y$. Then the first equation is always true. For the second equation:

$$x^3 + y^3 = 2(x+y) \Rightarrow 2x^3 = 4x \Rightarrow \begin{cases} x = 0 \\ x^2 = 2 \end{cases} \Rightarrow x = y \in \{0, \pm\sqrt{2}\}$$

Thus, $(x, y) \in \{(0,0), (\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})\}$.

Case 2: $x = -y$. Then the second equation is always true. For the first equation:

$$x^3 - y^3 = 4(x-y) \Rightarrow 2x^3 = 8x \Rightarrow \begin{cases} x = 0 \\ x^2 = 4 \end{cases} \Rightarrow x = -y \in \{0, \pm 2\}$$

Thus, $(x, y) \in \{(0,0), (2,-2), (-2,2)\}$.

Case 3: now assume that $x \neq \pm y$. Simplify both equations

$$\begin{cases} x^3 - y^3 = 4(x-y) \\ x^3 + y^3 = 2(x+y) \end{cases} \Rightarrow \begin{cases} x^2 + xy + y^2 = 4 \\ x^2 - xy + y^2 = 2 \end{cases} \Rightarrow \begin{cases} x^2 + y^2 = 3 \\ xy = 1 \end{cases} \Rightarrow \begin{cases} x + y & = \pm\sqrt{5} \\ xy & = 1 \end{cases}$$

Thus, x and y are roots of the quadratic equations $t^2 \mp \sqrt{5} + 1 = 0$. Solving these, we obtain $(x, y) \in \{(\frac{\mp\sqrt{5}\pm 1}{2}, \frac{\mp\sqrt{5}\mp 1}{2})\}$.

Thus the given system of equations has $3 + 3 + 4 = \boxed{10}$ solutions. □

Solution. **??** Consider the function

$$f : (0, +\infty) \to (-1, 1),$$
$$f(x) = \frac{1+x}{1-x}$$

Claim — $f(x \star y) = f(x)f(y)$.

Proof.

$$f(x \star y) = \frac{1 + \frac{x+y}{1+xy}}{1 - \frac{x+y}{1+xy}} = \frac{(1+x)(1+y)}{(1-x)(1-y)} = f(x)f(y).$$

■

Let $a = (\ldots(((2 \star 3)\star)4) \star \ldots) \star 12022$, then by the claim,

$$a = (\ldots(((2 \star 3)\star)4) \star \ldots) \star 12022 \Rightarrow f(a) = f(2) \cdot f(3) \cdots f(2022)$$
$$= \frac{3}{-1} \cdot \frac{4}{-2} \cdot \frac{5}{-3} \cdots 2021 - 2019 \cdot \frac{2022}{-2020} \cdot \frac{2023}{-2021} = \boxed{\frac{2022 \cdot 2023}{2}}$$

□

Chapter 19

HC R10

19.1 Problems

Problem 19.1.1 (HC-2022-SM1-R10-P1)**.** (*Beginner Level*)

At most how many numbers can be chosen from the set $\{1, 2, \ldots, 20\}$ such that *no chosen number can be twice any other chosen number*? *For example, if* 2 *is chosen then* 3 *can be chosen but* 4 *cannot be chosen.*

How to provide your answer:

- If you think it is 10, then submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly smaller* than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger* than your answer.

Problem 19.1.2 (HC-2022-SM1-R10-P2)**.** (*Beginner Level*)

The diagram below shows two 4×4 grids. The grid on the right shows a closed polygon whose edges do not intersect each other (except at their endpoints) and whose 7 vertices are also vertices of the grid. No vertex of the grid is used more than once as part of the polygon.

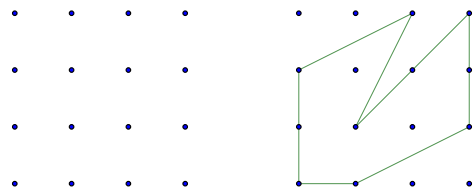


Figure 19.1: **??**

What is the maximum number of vertices of the left grid can be used to create such closed polygon?

How to provide your answer:

- If you think that 10 is such a number, submit 10 and a diagram to justify your answer.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly smaller* than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger* than your answer.

Problem 19.1.3 (HC-2022-SM1-R10-P3)**.** (*Beginner Level*)

The letters A, B, C, D, E, F, and G denote seven different digits from $0, 1, 2, \ldots, 9$, such that,

$$A \times B \times C = C \times D \times E = E \times F \times G$$

What is the value of D?

How to provide your answer:

- If you have think that $D = 5$ is a solution, submit 5.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 19.1.4 (HC-2022-SM1-R10-P4)**.** (*Beginner Level*)

Last night, 9 mafia bosses came together. In a single question, you can find out the total wealth of any 2 of them. *For example in a single question you can know the sum $w_1 + w_2$ of Boss 1 and Boss 2 wealth.*

What is the least number of questions do you need to find out the total wealth of all 9 bosses?

How to provide your answer:

- If you think it is 10, submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly larger* than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly smaller* than your answer.

Problem 19.1.5 (HC-2022-SM1-R10-P5)**.** (*Beginner Level*)

A and B takes turn to play a game on a standard 8×8 chessboard. First, A can choose any square of the board to put the king.

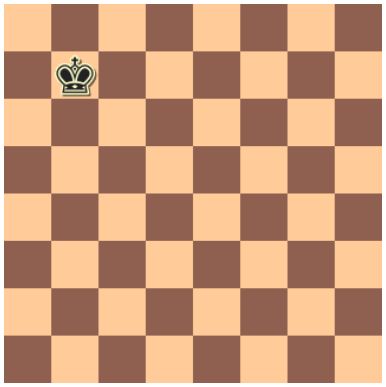


Figure 19.2: **??**

Once the king is on the board, the player of the turn can move the king by a legal move to a square *not occupied earlier.* The loser is the one who cannot move anymore.

Who has the strategy to win the game and in at most how many moves?

How to provide your answer:

- If you think that A wins in 10 moves, submit $A,\ 10$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 19.1.6 (HC-2022-SM1-R10-P6)**.** (*Intermediate Level*)

A *close path* $A_1A_2 \ldots A_5$ contains 5 segments $A_1A_2, A_2A_3, \ldots, A_5A_1$ that has 5 *self-intersection* points . This close path has the *maximum* number of self-intersection points for any 5−segment close path.

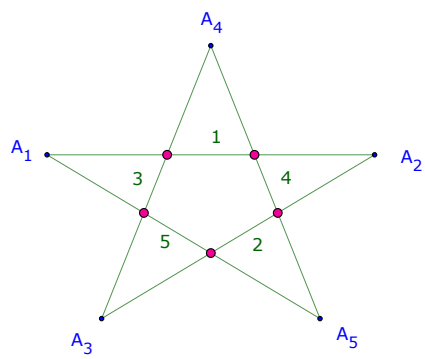


Figure 19.3: **??**

Your task is to make a guess to find the *maximum* number of self-intersection points for a 9−segment close path $A_1A_2 \ldots A_9$.

How to provide your answer:

- If you think that the maximum value is 10, submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly smaller* than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger* than your answer.

Problem 19.1.7 (HC-2022-SM1-R10-P7)**.** (*Intermediate Level*)

There are several ways to put the numbers $0, 1, \ldots, 9$ into the ten circles shown the the diagram below, such that the sums of the numbers at the vertices of each triangle A, B, C, D, E, and F are all equal.

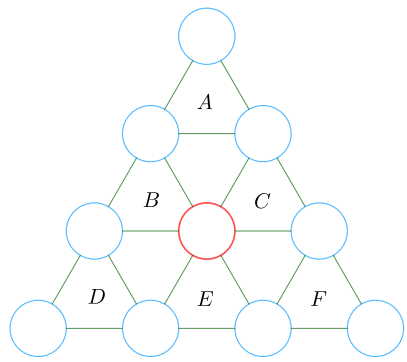


Figure 19.4: **??**

What would be the largest possible value of the sum of the numbers at the vertices of the triangle A?

In that case what is the number in the red circle in the middle of the diagram?

How to provide your answer:

- If you think that the largest possible value of the sum is 10, and the number in the red circle is 5, then submit 10, 5.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 19.1.8 (HC-2022-SM1-R10-P8)**.** (*Intermediate Level*)

Khoa was swimming in a river, against the current, when he lost his goggles. He continued for 10 minutes, then decided to turn around and retrieve them. He caught up with the lost goggles half a kilometer from the point where he lost them.

What is the speed of the current?

How to provide your answer:

- If you think that the speed of the current is 1 kilometer per hour, submit 1.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 19.1.9 (HC-2022-SM1-R10-P9)**.** (*Intermediate Level*)

In at most how many ways can one go from the city A to the city B, if it is possible to pass through each of the places $1, 2, \ldots, 14$ not more than once.

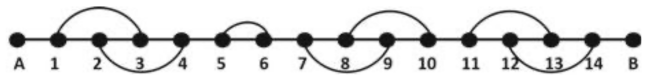


Figure 19.5: **??**

How to provide your answer:

- If you think the maximum number of ways is 10, then submit 10.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly smaller* than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger* than your answer.

Problem 19.1.10 (HC-2022-SM1-R10-P10)**.** (*Intermediate Level*)

The Coding Challenge

Find the number of all six-digit positive integers with the distinct digits which are divisible by 11 and consist of the digits $1, 2, 3, 4, 5$, and 8.

How to provide your answer:

- If your code shows the answer is 100, then submit 100. You have to also submit the code for verification. If your code does not produce the result you submit, then the submission will not be qualified.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 19.1.11 (HC-2022-SM1-R10-P11)**.** (*Advanced Level*)

Amy rolls 2 dice 4 times. What is the probability that the dice show the same number exactly twice? *For example* $(1,2),(3,3),(4,4),(5,6)$, *or* $(2,2),(1,3),(2,2),(4,6)$, *are successful sequences of* 4 *rolls, but* $(1,1),(3,2),(4,6),(5,3)$ *is not.*

How to provide your answer:

- If you think that the probability is $\frac{1}{2}$, submit $\frac{1}{2}$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 19.1.12 (HC-2022-SM1-R10-P12)**.** (*Advanced Level*)

Five friends live on the same street at A, B, C, D, and E. Place F is not a ideal place to meet because the sum of distances from A, B, C, D, and E to F is $7+5+4+2+1=19$. As exercise to the actual problem, try

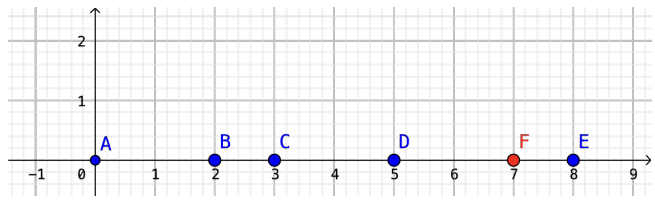


Figure 19.6: Friends on the same street

to figure out *where* should they meet so that the *sum of distances from that place to their houses is minimal*. Now, in the diagram below, the red dots indicate houses of friends, who can only walk along streets and

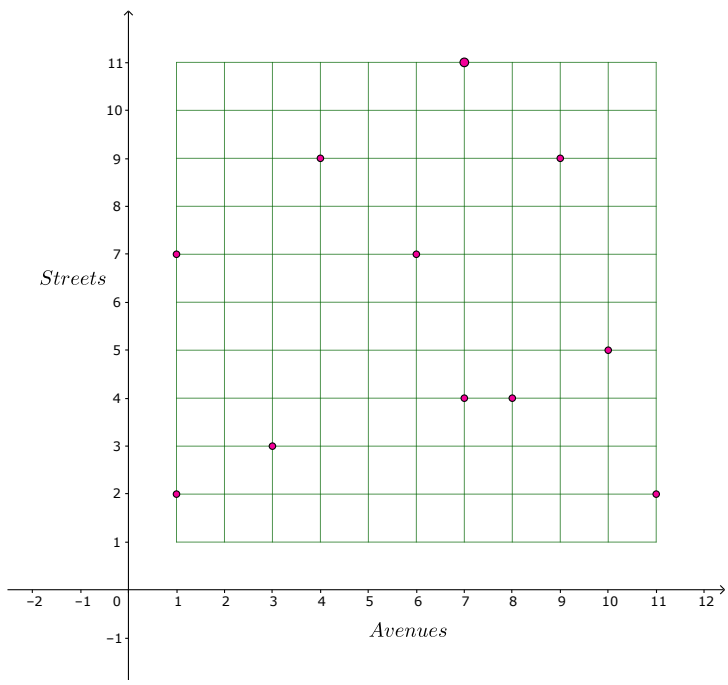


Figure 19.7: **??**

avenues. Where is the place that has the minimal sum of distances to their houses?

How to provide your answer:

- If you think the place is the intersection $(1,1)$, submit $(1,1)$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 19.1.13 (HC-2022-SM1-R10-P13)**.** (*Advanced Level*)

It is not difficult to cover a 16×16 grid with 1×4 tiles in such a way that 32 of them are horizontal and 32 vertical. See the diagram below,

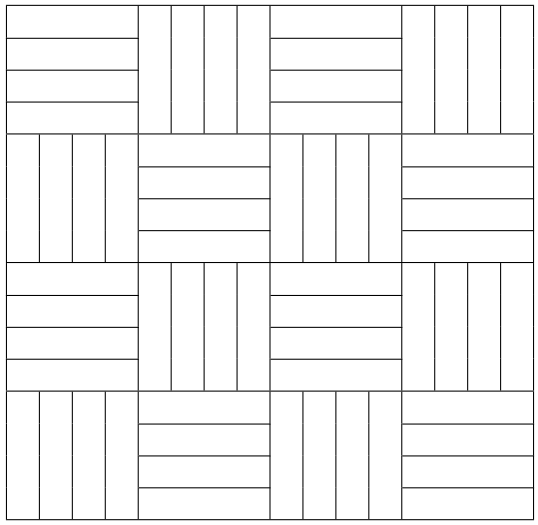


Figure 19.8: **??**

Now, rearrange some of the tiles so that the difference in the number of horizontal and vertical tiles is 4.

How to provide your answer:

- If you have a solution, then submit your diagram.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 19.1.14 (HC-2022-SM1-R10-P14)**.** (*Advanced Level*)

The diagrams below shows 8 queens on a 8×8 chessboard such that *no two attack one another.*

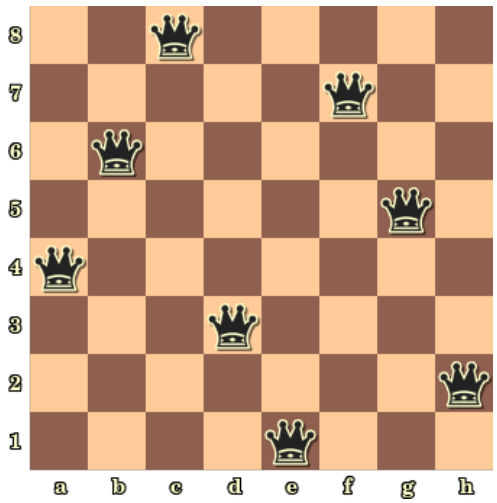


Figure 19.9: 8 queens

What is the *minimmal* number of queens needed to *occupy* or *attack all squares* of a 7×7 chessboard?

How to provide your answer:

- If you think 10 is the maximum number, submit 10 and a diagram depicting how do you solve it.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, an answer *without a diagram*, or a *correct answer* that is *strictly larger* than yours then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer with diagram* that is *strictly smaller* than your answer.

Problem 19.1.15 (HC-2022-SM1-R10-P15)**.** (*Advanced Level*)

The Coding Challenge

You have two sentences as shown below. You need to return two lists of words

- the first list contains all the words that are in the first sentence, but not in the second,
- the second list contains all the words that are in both sentences,
- Both lists are sorted alphabetically (with uppercase preceeds lowercase: $A, B, \ldots, Z, a, b, \ldots, z$.)

Input

```
The long sentence stretches out piling up its clauses as if description were a kind of
    respiratory need before sinking in breathless defeat
Afterwards some argue criminals tended to do so as a matter of course because they
    would face a long sentence in any case
```

How to provide your answer:

- Submit both lists. You have to also submit the code for verification. If your code does not produce the result you submit, then the submission will not be qualified.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

19.2 Answers

Problem 1: 14

Problem 2: 16

Problem 3: 2

Problem 4: 6

Problem 5: B, 32

Problem 6: 27

Problem 7: 14, 3

Problem 8: 1.5

Problem 9: 128

Problem 10: 72

Problem 11: $\frac{25}{256}$

Problem 12: $(7, 5)$

Problem 13: 0

Problem 14: 16

Problem 15: Output as below

```
The before breathless clauses defeat description if its kind need out piling
    respiratory sinking stretches up were
a as in long of sentence
```

19.3 Solutions

Solution. **??** Let $A_1 = \{1, 2, 4, 8, 16\}$, $A_2 = \{3, 6, 12\}$, $A_3 = \{5, 10, 20\}$, $A_4 = \{7, 14\}$, $A_5 = \{9, 18\}$, $A_6 = \{11\}$, $A_7 = \{13\}$, $A_8 = \{15\}$, $A_9 = \{17\}$, $A_{10} = \{19\}$.

We can take at most 3 numbers from A_1, 2 numbers from A_2, 2 numbers from A_3, and 1 number from each of $A_4, A_5, \ldots, A_{10}$.

An example of a set of $3 + 2 + 2 + \underbrace{1 + 1 + \ldots + 1}_{7} = \boxed{14}$ chosen numbers is,

$$1, 4, 16, 3, 12, 5, 20, 7, 9, 11, 13, 15, 17, 19.$$

□

Solution. **??** It turns out that there is such a closed $\boxed{16}$−gon with non-intersecting edges and whose vertices are the vertices of the grid as shown in **??**. □

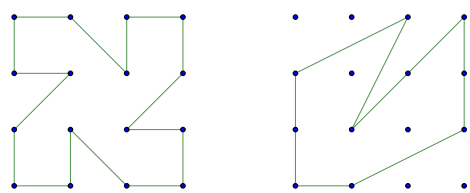


Figure 19.10: The 16−gon

Solution. **??** It is easy to see that if any of the digits $0, 5$, or 7, appear in $A, B, \ldots, G$, then each of the products should also contains another $0, 5$, or 7, which is impossible.

So $\{A, B, \ldots, G\} = \{1, 2, 3, 4, 6, 8, 9\}$. Now let P be the value of any product,

$$P = A \times B \times C = C \times D \times E = E \times F \times G.$$

Because 1 is a factor of P, so $P \leq 1 \times 8 \times 9 = 72$. Since $8 \mid P$, and $9 \mid P$, so $72 \mid P$, or $P = 72$. Now, 72 can only be expressed in three different ways

$$1 \times 8 \times 9 = 9 \times 2 \times 4 = 4 \times 6 \times 3.$$

Since D appears in the triple (C, D, E), where C and E appear second time in other product, so $D = \boxed{2}$. □

Solution. **??** Let the wealth of the bosess be $w_1, w_2, \ldots, w_9$. Lets use the first three questions to determine $w_1 + w_2, w_3 + w_4, w_5 + w_6$. Lets use the last three questions to determine $w_7 + w_8, w_8 + w_9, w_9 + w_7$. The total weath of them can be determined by,

$$(w_1 + w_2) + (w_3 + w_4) + (w_5 + w_6) + \frac{1}{2}\left[(w_7 + w_8) + (w_8 + w_9) + (w_9 + w_7)\right]$$

Thus in $\boxed{6}$ questions, the total wealth of the bosses can be determined. Five questions can cover the wealth of 9 bosses, but they overcount at least the wealth of one boss. □

Solution. **??** Let divide the board into 2×1 dominoes, see **??**.

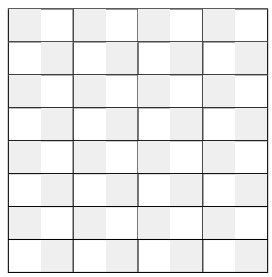


Figure 19.11: 32 dominoes

Player A can put the king anywhere he wants. The king is on one of 32 dominoes. Player B then moves the king to the remaining square of that dominoe. Since there is no square left on that dominoe, A must then move the king to a new dominoe. Again, B then moves the king to the remaining square of that dominoe. Following this strategy, $\boxed{B}$ can always make his move and A will ran out of dominoe after $\boxed{32}$ moves. □

Solution. **??** If we take two points, says A_1 and A_2, then except A_nA_1, A_1A_2, and A_2A_3 the remaining $n-3$ segments of the path $A_1A_2\ldots A_nA_1$ can intersect with A_1A_2. There are n such consecutive pairs like (A_1, A_2), and since each intersection of two segments is counted twice, so the number of intersections is atmost $\frac{n(n-3)}{2}$.

For a 9−segment close path, the maximum number of intersection points is $\frac{9\cdot 6}{2} = \boxed{27,}$ see **??**.

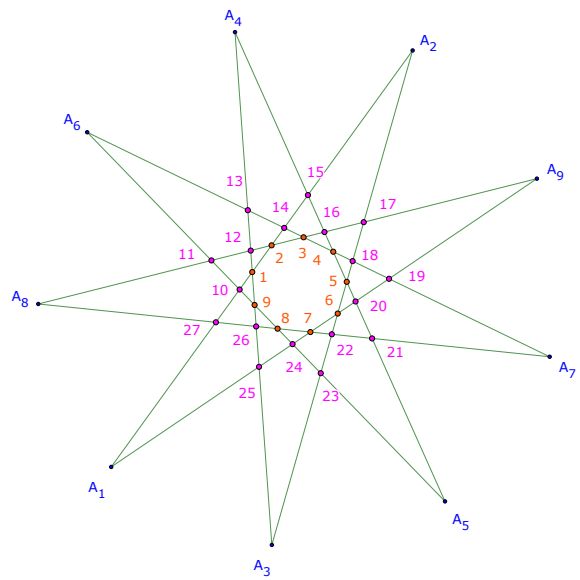


Figure 19.12: 9−segment close path

Note that if we walk along the points $A_1, A_2, \ldots, A_n, A_1$, then for $i = 1, 2, \ldots, n-1$, the ith−segment can intersect at most $i-2$ previous segments on the path. Finally the nth−segment intersects at most $n-3$ previous segments. Therefore the number of intersection points cannot exceed,

$$1 + 2 + \ldots + (n-3) + (n-3) = \frac{(n-3)(n-2)}{2} + (n-3) = \frac{n(n-3)}{2}.$$

□

Solution. **??** Let s be the sum of the numbers at vertices of a triangle and m be the number in the red circle in the middle. It is easy to see that by adding up all the sums of A, D, and F triangles,

$$3s + m = 1 + 2 + \ldots + 9 = 45 \Rightarrow 3 \mid m \Rightarrow m \in \{0, 3, 6, 9\},\ \ s \in \{15, 14, 13, 12\}$$

First if $s = 15, m = 0$, then the triples containing m and summing up to 15 are only $(0, 7, 8)$ and $(0, 6, 9)$, this contradicts the fact that m is a vertex of three triangles B, C, and E. Furthermore, note that *if we have a solution, then for any number x we can replace it with $9-x$ and have a solution.* Thus the case $s = 12, m = 9$ has to be eliminated as well. The two cases $\boxed{s = 14, m = 3}$ (**??**) and $s = 13, m = 6$ (**??**) have solutions as shown. □

Solution. [First solution] **??** In this solution, we assume that the speed of Khoa is s_k and the speed of the current is s_c. The distance he took under 10 minutes, or $\frac{1}{6}$ hour, is $\frac{s_k - s_c}{6}$. During this time, the goggles did a distance of $\frac{s_c}{6}$. Thus the distance between them when he turns back was $\frac{s_k - s_c}{6} + \frac{s_c}{6} = \frac{s_k}{6}$. Now he swims towards to goggle with the speed $s_k + s_c$ and the goggles continue to float with speed s_c, so it took him $\frac{1}{6}$ hour to reach the goggles. Therefore the goggles floated $\frac{1}{2}$ kilometer in $2 \cdot \frac{1}{6} = \frac{1}{3}$ hour, thus the speed of the current is $\frac{1}{2} \div \frac{1}{3} = \boxed{1.5}$ kilometer per hour. □

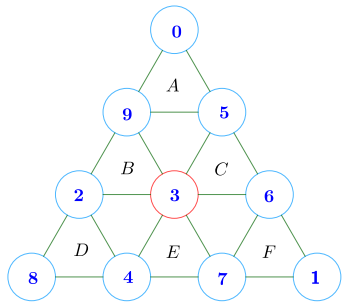


Figure 19.13: $s = 14, m = 3$

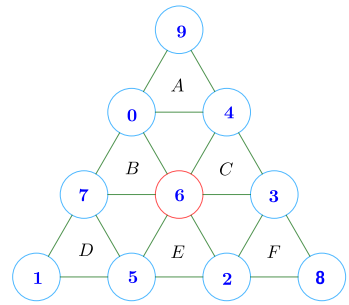
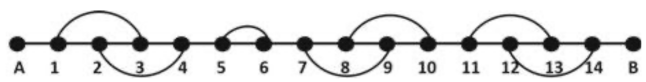


Figure 19.14: $s = 13, m = 6$

Solution. [Second solution] **??** In this solution, we assume there is an observer on a raft next to the goggles. From her point of view, Khoa swam away in 10 minutes and swam back under the same amount of time. Thus, she or the goggles floated $\frac{1}{2}$ kilometer in $2 \cdot \frac{1}{6} = \frac{1}{3}$ hour, thus the speed of the current is $\frac{1}{2} \div \frac{1}{3} = \boxed{1.5}$ kilometer per hour. □

Solution. **??**



Assume we go from the city A to the city B. Let us write on each of the curves $(1, 3)$, $(2, 4)$, $(5, 6)$, $(7, 9)$, $(8, 10)$, $(11, 13)$, $(12, 14)$ the corresponding number, which indicates how many times did we pass through that curve.

Therefore, it will be a $7-$tuple of zeros and ones. It holds true also the opposite statement, that is, every $7-$tuple determines one way for going from the city A to the city B. Therefore, the total number of all the ways is equal to $2^7 = \boxed{128.}$ □

Solution. **??** Following is an example solution in Python.

```
from itertools import permutations

count = 0
for p in permutations([1, 2, 3, 4, 5, 8]):
    if int(''.join('%s' % e for e in p)) % 11 == 0:
        count += 1

print(count)
```

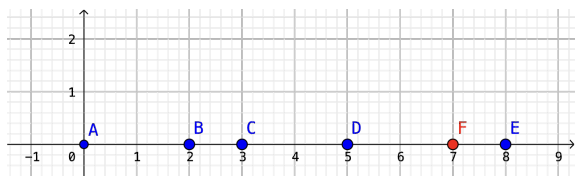
□

Solution. **??** Note that there are 6 *favorable* outcomes of the event for the dice to have the same number $(1,1),(2,2),\ldots,(6,6)$. When rolling two dice, there are 36 outcomes from $(1,1),(1,2),\ldots,$ to $(6,6)$. Thus for a roll, the probability getting the same number for two dice is $\frac{1}{6}$, and $\frac{5}{6}$ otherwise. Since the number of times that favorable outcome happens is exactly two, so out of four rolls, two rolls have the probability $\frac{1}{6}$, and the other two have $\frac{5}{6}$. Now, there are $\binom{4}{2}$ ways for the two favorable outcomes happen in four rolls, thus, the desired probability is,

$$\binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 = \boxed{\frac{25}{256}}.$$

□

Solution. **??** For the example, we can easily figure out that the place should be some where between A and E, otherwise it would increase the sum of the distances. Similarly, the place should be somewhere between B and D. Finally, in between B and D, C is the place where the distance to C is 0. This place, C, actually is the *median* or A, B, C, D, and E.



Now, lets project the locations to the two axis, see **??**.

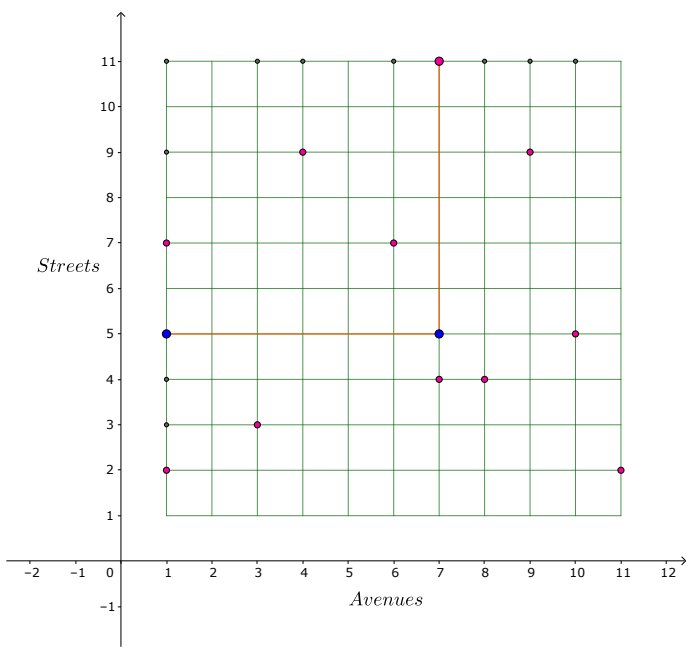


Figure 19.15: Project the locations to the axis

By applying the result of the one-dimensional problem, it is easy to see that the intersection of Avenue 1 and Street 5 at $(1,5)$ is ideal if all the friends would have been living on the Avenue 1. Similarly, the intersection of Avenue 7 and Street 11 at $(7,11)$ is ideal if the friends would have been living on the Street 11.

Since *the sum of distances from the friend houses consist of the two sums of distances along each axis* from the friend houses to the meeting place. The intersection $\boxed{(7,5)}$ minimizes both sums. □

Solution. **??** We show that it is impossible to have the difference of 4 between the numbers of horizontal and vertical tiles.

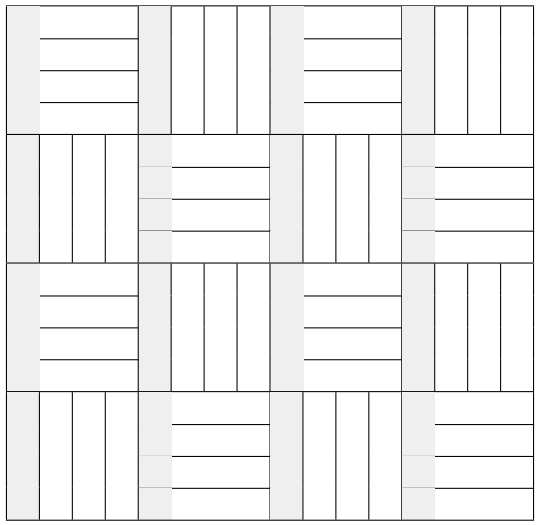


Figure 19.16: Colour every fourth column

Let $2d$ be the difference between the number of horizontal and vertical tiles. So there are $32 \pm d$ horizontal and $32 \mp d$ vertical tiles. Let colour every fourth column black. A horizontal tile covers a single black square, so the number of black squares covered by vertical tiles is,

$$64 - (32 \pm d) = 32 \mp d$$

Since a vertical tile covers none or four black squares, thus $4 \mid d$, so $8 \mid 2d$, therefore $\boxed{2d \neq 4.}$ $\square$

Solution. **??** A queen can cover $2 \cdot 7 - 1 = 13$ squares in its column and row. 3 queens can cover at most $3 \cdot 13 - 3 \cdot 2 = 33$ squares in their columns and rows. A queen, let say A, with longest diagonals can cover in addition at most $2 \cdot 6 - 4 = 8$ squares (not including the squares covered by other two queens in their columns and rows), The other two queens, says B and C can cover diagonally $2 \cdot 5 - 4 = 4$ squares, (not including the squares covered by other two queens in their columns and rows). Note that at least two squares covered diagonally by these B and C queens will overlap if they cover diagonally the same coloured squares or overlap with squares covered diagonally by the A queeen if they cover diagonally different coloured squares, thus the total number of squares the three queens can covered is

$$33 + 8 + 2 \cdot 4 - 1 = 48 < 49.$$

Thus, it is impossible to cover the whole board with 3 queens. A solution with $\boxed{4}$ queens is shown as below. □

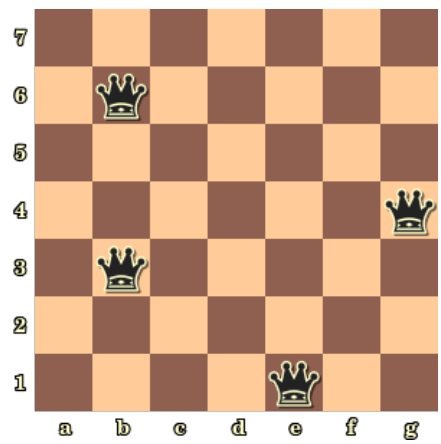


Figure 19.17: 4 queens

Solution. **??** Following is an example solution in Python.

```
if __name__ == "__main__":
with open('hc-2022-1-10-15-input.txt', 'rt') as i_file:
    lines = [s.strip() for s in i_file.readlines()]
    word_set_1 = set(lines[0].split())
    word_set_2 = set(lines[1].split())

    list_1 = word_set_1.difference(word_set_2)
    list_2 = word_set_1.intersection(word_set_2)

    with open('hc-2022-1-10-15-output.txt', 'wt') as o_file:
        o_file.write("%s\n" % ' '.join(sorted(list_1)))
        o_file.write("%s\n" % ' '.join(sorted(list_2)))
```

□

Chapter 20

ICLT May

20.1 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

20.2 Problems for Level 1

Problem 20.2.1 (ICLT-2022-SM1-5-L1-P1). (*4 points*)

Shauna did a number trick with Zach. She told him to pick an even number, double it, add 44, divide by 4, subtract 7, multiply by 2, and subtract his original number. She then told him the result he should have attained. What number did Shauna tell Zach?

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Problem 20.2.2 (ICLT-2022-SM1-5-L1-P2). (*4 points*)

It is easy to see that one straight line can intersect one circle at most two points. What is the maximum intersections if two circles and one straight line intersect each other?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 20.2.3 (ICLT-2022-SM1-5-L1-P3). (*4 points*)

A grocer puts 12 apples in n bags such that each bag contains the same number of apples. If there is more than one apple in each bag and fewer than 12, How many possible values of n are there?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 20.2.4 (ICLT-2022-SM1-5-L1-P4). (*6 points*)

The Smith family has 3 sons and 3 daughters. In how many ways can they be seated in a row of 6 chairs such that at least 2 boys are next to each other?

(A) 128 (B) 248 (C) 441 (D) 546 (E) 648

Problem 20.2.5 (ICLT-2022-SM1-5-L1-P5). (*6 points*)

Solve the equation $\frac{3}{x} - 3 = 8 + \frac{2}{x}$.

(A) $\frac{1}{7}$ (B) $\frac{1}{8}$ (C) $\frac{1}{9}$ (D) $\frac{1}{10}$ (E) $\frac{1}{11}$

Problem 20.2.6 (ICLT-2022-SM1-5-L1-P6). (*6 points*)

In the diagram, $m \parallel n$, $AB \perp m$, $\angle ADC = \angle BCE = 3x$, $\angle CEB = 50°$, and $\angle BCD = x$. Find x.

(A) $35°$ (B) $40°$ (C) $45°$ (D) $50°$ (E) $55°$

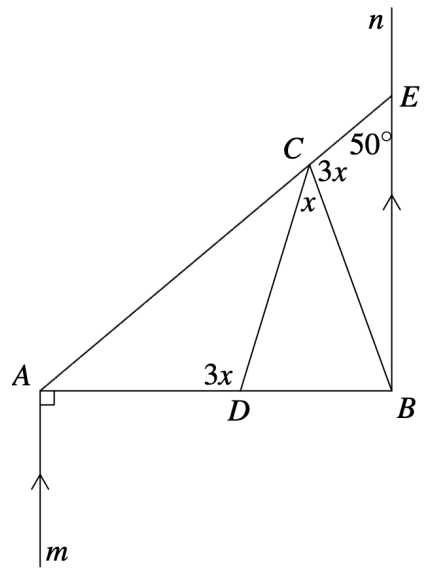


Figure 20.1: **??**

Problem 20.2.7 (ICLT-2022-SM1-5-L1-P7). (*10 points*)

1. (*5 points*) What is the smallest three-digit prime number?
2. (*5 points*) What is the smallest three-digit prime number whose digits are also each prime?

Problem 20.2.8 (ICLT-2022-SM1-5-L1-P8). (*10 points*)

A sports conference has 16 teams divided into two divisions, each division has 8 teams.

If each team must play every other team in its own division twice and every team in the other division once,

1. (*5 points*) how many game a team must play?
2. (*5 points*) how many game are to be played in total?

Problem 20.2.9 (ICLT-2022-SM1-5-L1-P9). (*10 points*)

x is a real number such that $\dfrac{1}{x-1} + \dfrac{2x}{x-1} = 5$.

1. (*5 points*) Combine the two terms on the left-hand side of the equation into a single fraction.
2. (*5 points*) Solve it by cross-multiplying both sides of the equation.

20.3 Grading for Level 1

Answers for multiple-choice problems.

Problem 1:	A	8
Problem 2:	C	6
Problem 3:	D	4
Problem 4:	E	648
Problem 5:	E	$\frac{1}{11}$
Problem 6:	B	40

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the second part 2 if can find out that the hundreds digit must be at least 2.

Problem 8: Separately for each part,

(a) 2 points if can count correctly the number of the games with in its own division

(b) 2 points if can correct the overcounting by dividing by 2.

Problem 9: Separately for each part,

(a) 2 points if combine correctly the numerators.

(b) 2 points if cross-multiplying correctly.

20.4 Solutions for Level 1

Solution. **??** Let x be the original number,

$$2\left(\frac{2x+44}{4}-7\right)-x=2\left(\frac{x+22}{2}-7\right)-x=2\left(\frac{x+22-14}{2}\right)-x=x+8-x=\boxed{8}$$

The answer is $\boxed{A.}$ □

Solution. **??** The maximum number of intersections of a circle and a straight is two. Similarly two circles can have at most 2 intersections. Therefore the maximum intersections of two circles and one straight line is $3\times 2=\boxed{6.}$

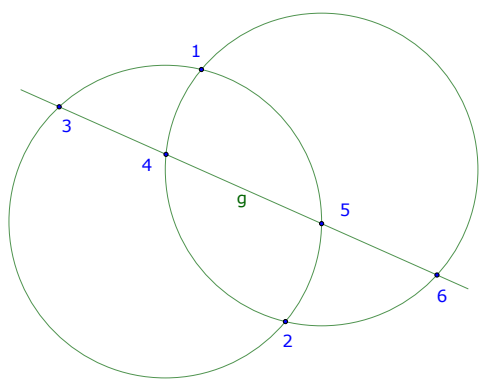


Figure 20.2: 6 intersection points

The answer is $\boxed{C.}$ □

Solution. **??** There are $\frac{12}{n}$ apples in each bag, so n must be a divisor of 12. We are told that there are between 1 and 12 apples in each bag, so n could be $2, 3, 4,$, or 6. Therefore, there are $\boxed{4}$ such numbers.

The answer is $\boxed{D.}$ □

Solution. **??** There are two ways that no boys sit next to each other

$$BGBGBG,\ GBGBGB.$$

In each case there are 3! ways to permute the boys and 3! ways to permute the girls. So in total, there are $2\cdot 3!\cdot 3!$ ways to sit the children so that no boys sit next to each other. There are 6! ways in total to sit 6 children, so the number of desired ways is $6!-2\cdot 3!\cdot 3!=\boxed{648}$.

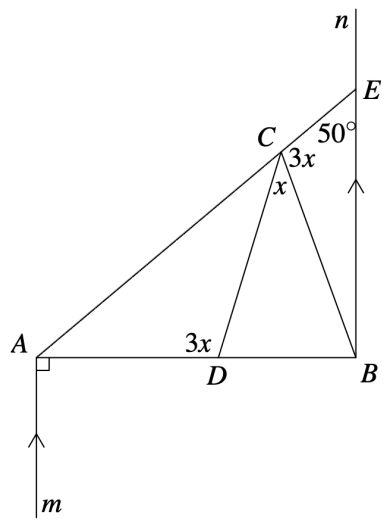
The answer is $\boxed{E.}$ □

Solution. **??** Subtracting $\frac{2}{x}$ from both sides,

$$\frac{3}{x}-3=8+\frac{2}{x}\Rightarrow\frac{3}{x}-\frac{2}{x}-3=8\Rightarrow\frac{3}{x}-3=8\Rightarrow\frac{3}{x}=11\Rightarrow x=\boxed{\frac{1}{11}}$$

The answer is $\boxed{E.}$ □

Solution. **??** We start with our parallel lines, and see that $\angle ABE = 90^\circ$ because $m \parallel n$ and $m \perp AB$. Then,



from $\triangle ABE$ we have,

$$\angle EAB + \angle ABE + \angle AEB = 180^\circ, \Rightarrow \angle EAB = 180^\circ - \angle ABE - \angle AEB = 40^\circ.$$

By $\triangle ACD$

$$\angle ACD = 180^\circ - \angle DAC - \angle ADC = 180^\circ - 40^\circ - 3x = 140^\circ - 3x.$$

Three angles with vertices at C that together make a straight line, so we have

$$\angle ACD + \angle DCB + \angle BCE = 180^\circ \Rightarrow 140^\circ - 3x + x + 3x = 180^\circ \Rightarrow x = \boxed{40^\circ.}$$

The answer is $\boxed{B.}$ □

Solution. **??** For the first question, it is easy to verify that $\boxed{101}$ is a prime number.

For the second question, the hundreds and ten digits must be at least 2, and the unit digit must be at least 3. Luckily $\boxed{223}$ is a prime number, □

Solution. **??** For the first question, if each team must play every other team in its own division twice and every team in the other division once, then a team must play $2 \cdot (8-1) + 1 \cdot 8 = \boxed{22.}$

For the second question, by counting all the games per team and then divide that by two (because a game is played by two team, so it is counted twice,) we have $\frac{1}{2} \cdot 22 \cdot 16 = \boxed{176}$. □

Solution. **??** For the first question,

$$\frac{1}{x-1} + \frac{2x}{x-1} = 5 \Rightarrow \frac{2x+1}{x-1} = 5$$

For the second question,

$$\frac{2x+1}{x-1} = 5 \Rightarrow 2x+1 = 5(x-1) \Rightarrow 2x+1 = 5x-5 \Rightarrow 3x = 6 \Rightarrow x = \boxed{2.}$$

□

20.5 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
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     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
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     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

20.6 Problems for Level 2

Problem 20.6.1 (ICLT-2022-SM1-5-L2-P1). (*4 points*)

Solve the equation $\frac{3}{x} - 3 = 8 + \frac{2}{x}$.

$(A)\ \frac{1}{7}$ $(B)\ \frac{1}{8}$ $(C)\ \frac{1}{9}$ $(D)\ \frac{1}{10}$ $(E)\ \frac{1}{11}$

Problem 20.6.2 (ICLT-2022-SM1-5-L2-P2). (*4 points*)

In the diagram, $m \parallel n$, $AB \perp m$, $\angle ADC = \angle BCE = 3x$, $\angle CEB = 50°$, and $\angle BCD = x$. Find x.

$(A)\ 35°$ $(B)\ 40°$ $(C)\ 45°$ $(D)\ 50°$ $(E)\ 55°$

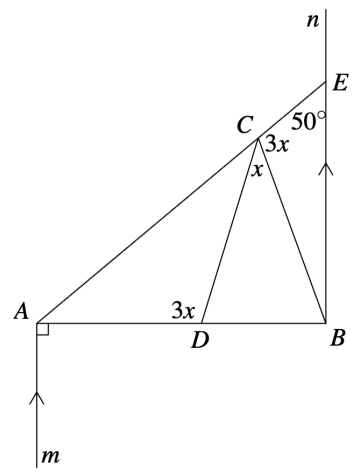


Figure 20.3: **??**

Problem 20.6.3 (ICLT-2022-SM1-5-L2-P3). (*4 points*)

What is the smallest three-digit prime number whose digits are also each prime?

$(A)\ 101$ $(B)\ 103$ $(C)\ 223$ $(D)\ 227$ $(E)\ 257$

Problem 20.6.4 (ICLT-2022-SM1-5-L2-P4). (*6 points*)

A sports conference has 16 teams divided into two divisions, each division has 8 teams. If each team must play every other team in its own division twice and every team in the other division once, how many game are to be played in total?

$(A)\ 96$ $(B)\ 128$ $(C)\ 133$ $(D)\ 176$ $(E)\ 288$

Problem 20.6.5 (ICLT-2022-SM1-5-L2-P5). (*6 points*)

Simplify the expression

$$\left(3ab - 4cd + \frac{3}{2}\right) + \left(2cd - \frac{ab}{2} + 3\right) + (2 - ab)$$

$(A)\ 5ab - 2$ $(B)\ 10ab - 2cd + 1$ $(C)\ \frac{3}{2}$ $(D)\ 5$ $(E)\ \frac{3ab}{2} - 2cd + \frac{13}{2}$

Problem 20.6.6 (ICLT-2022-SM1-5-L2-P6). (*6 points*)

In triangle NOK below, $KO = KN$. Let I be the point on KN such that $OI = ON$. Let A be the point on NO such that $IA = IN$. What is the difference of $\angle KON - \angle OIN$.

(A) -10° (B) -16° (C) 0° (D) 10° (E) 16°

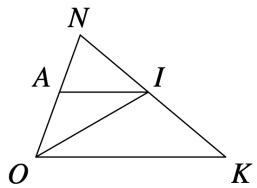


Figure 20.4: **??**

Problem 20.6.7 (ICLT-2022-SM1-5-L2-P7). (*10 points*)

1. (*5 points*) Using Euclidean algorithm, determine $\gcd(1938, 3306)$?
2. (*5 points*) What is the least common multiple of $12, 18$, and 30?

Problem 20.6.8 (ICLT-2022-SM1-5-L2-P8). (*10 points*)

A lock has 5 buttons numbered $1 - 5$.

1. (*5 points*) How many combinations are possible, if the lock is opened by pushing two buttons simultaneously and then pushing one button alone?
2. (*5 points*) How many combinations are possible, if the lock is opened by pushing three buttons simultaneously and then pushing two other button simultaneously?

Problem 20.6.9 (ICLT-2022-SM1-5-L2-P9). (*10 points*)

Consider the grid in the figure below,

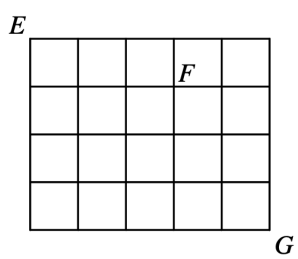


Figure 20.5: **??**

1. (*5 points*) How many $4-$step paths are there from E to F (every step must be down or to the right)?
2. (*5 points*) How many $9-$step paths are there from E to G through F (every step must be down or to the right)?

20.7 Grading for Level 2

Answers for multiple-choice problems.

Problem 1:	E	$\frac{1}{11}$
Problem 2:	B	40
Problem 3:	C	223
Problem 4:	D	176
Problem 5:	E	$\frac{3ab}{2} - 2cd + \frac{13}{2}$
Problem 6:	C	0

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately for each part,

(a) 2 points if can use Euclidean algorithm for some steps correctly.

(b) 2 points if can determine lcm of two numbers

Problem 8: Separately for each part,

(a) 2 points if can determine the ways to choose two objects out of five.

(b) 2 points if can determine the ways to choose three objects out of five.

Problem 9: Separately for each part,

(a) 2 points if can use combinations.

(b) 2 points if know to multiply result of two combinations.

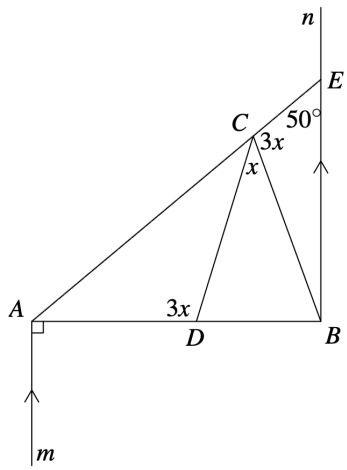
20.8 Solutions for Level 2

Solution. **??** Subtracting $\frac{2}{x}$ from both sides,

$$\frac{3}{x} - 3 = 8 + \frac{2}{x} \Rightarrow \frac{3}{x} - \frac{2}{x} - 3 = 8 \Rightarrow \frac{3}{x} - 3 = 8 \Rightarrow \frac{3}{x} = 11 \Rightarrow x = \boxed{\frac{1}{11}}$$

The answer is $\boxed{E.}$ □

Solution. **??** We start with our parallel lines, and see that $\angle ABE = 90^\circ$ because $m \parallel n$ and $m \perp AB$. Then,



from $\triangle ABE$ we have,

$$\angle EAB + \angle ABE + \angle AEB = 180^\circ, \Rightarrow \angle EAB = 180^\circ - \angle ABE - \angle AEB = 40^\circ.$$

By $\triangle ACD$

$$\angle ACD = 180^\circ - \angle DAC - \angle ADC = 180^\circ - 40^\circ - 3x = 140^\circ - 3x.$$

Three angles with vertices at C that together make a straight line, so we have

$$\angle ACD + \angle DCB + \angle BCE = 180^\circ \Rightarrow 140^\circ - 3x + x + 3x = 180^\circ \Rightarrow x = \boxed{40^\circ.}$$

The answer is $\boxed{B.}$ □

Solution. **??** The hundreds and ten digits must be at least 2, and the unit digit must be at least 3. Luckily $\boxed{223}$ is a prime number,

The answer is $\boxed{C.}$ □

Solution. **??** If each team must play every other team in its own division twice and every team in the other division once, then a team must play $2 \cdot (8-1) + 1 \cdot 8 = 22$. Now, by counting all the games per team and then divide that by two (because a game is played by two team, so it is counted twice,) we have $\frac{1}{2} \cdot 22 \cdot 16 = \boxed{176}$.

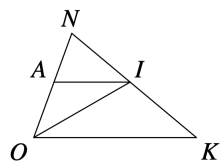
The answer is $\boxed{D.}$ □

Solution. **??**

$$\left(3ab - 4cd + \frac{3}{2}\right) + \left(2cd - \frac{ab}{2} + 3\right) + (2 - ab) = \left(3ab - \frac{ab}{2} - ab\right) + (-4cd + 2cd) + \left(\frac{3}{2} + 3 + 2\right)$$
$$= \left(\frac{6ab}{2} - \frac{ab}{2} - \frac{2ab}{2}\right) + (-2cd) + \left(\frac{13}{2}\right) = \frac{3ab}{2} - 2cd + \frac{13}{2}.$$

The answer is $\boxed{E.}$ □

Solution. **??** Since $KO = KN$, $\triangle KNO$ is isosceles with $\angle KON = \angle N$. Since $IA = IN$, we have $\angle IAN =$



$\angle N$. Since $OI = ON$, we have $\angle N = \angle OIN$. Thus $\angle KON - \angle OIN = \boxed{0^\circ}$.

The answer is $\boxed{C.}$ □

Solution. **??** For the first question,

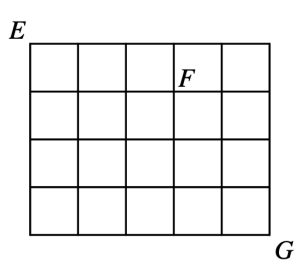
$$\gcd(1938, 3306) = \gcd(1938, 1368) = \gcd(570, 1368) = \gcd(570, 228) = \gcd(114, 228) = \gcd(114, 0) = \boxed{114.}$$

For the second question, it is probably easier to first find the lcm of two of the integers. $\text{lcm}(12, 18) = 36$. Every common multiple of 12 and 18 will be a multiple of their lcm, so the lcm of all three integers must be a multiple of 36. The lcm of all three integers is the lcm of 36 and 30, which is $\boxed{180}$. □

Solution. **??** For the first question, there are $\binom{5}{2} = 10$ ways to choose the first two buttons, and then 5 choices for the last button. So there are $10 \times 5 = \boxed{50}$ possible combinations.

For the second question, there are $\binom{5}{3} = 10$ ways to choose the first three buttons, and then $\binom{5}{2} = 10$ choices for the last button. So there are $10 \times 10 = \boxed{100}$ possible combinations. □

Solution. **??** For the first question, from E to F, it is 3 steps to the right and 1 step down, for a total of $\binom{4}{1} = \frac{4}{1} = \boxed{4}$ different paths.



For the second question, from F to G, it is 2 steps to the right and 3 steps down, for a total of $\binom{5}{2} = \frac{5 \times 4}{2 \times 1} = 10$ different paths. Combine with the result from the first question, there are $4 \times 10 = \boxed{40}$ paths from E to G that pass through F. □

20.9 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

20.10 Problems for Level 3

Problem 20.10.1 (ICLT-2022-SM1-5-L3-P1). (*4 points*)

Simplify the expression

$$\left(3ab - 4cd + \frac{3}{2}\right) + \left(2cd - \frac{ab}{2} + 3\right) + (2 - ab)$$

(A) $5ab - 2$ (B) $10ab - 2cd + 1$ (C) $\frac{3}{2}$ (D) 5 (E) $\frac{3ab}{2} - 2cd + \frac{13}{2}$

Problem 20.10.2 (ICLT-2022-SM1-5-L3-P2). (*4 points*)

In triangle NOK below, $KO = KN$. Let I be the point on KN such that $OI = ON$. Let A be the point on NO such that $IA = IN$. What is the difference of $\angle KON - \angle OIN$.

(A) -10° (B) -16° (C) 0° (D) 10° (E) 16°

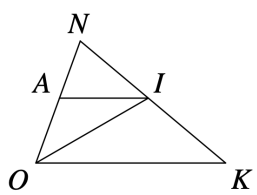


Figure 20.6: **??**

Problem 20.10.3 (ICLT-2022-SM1-5-L3-P3). (*4 points*)

Determine $\gcd(1938, 3306)$ and $12, 18$, and 30.

(A) 144 and 180 (B) 144 and 90 (C) 72 and 180 (D) 36 and 30 (E) 72 and 60

Problem 20.10.4 (ICLT-2022-SM1-5-L3-P4). (*6 points*)

A lock has 5 buttons numbered $1 - 5$. How many combinations are possible, if the lock is opened by pushing three buttons simultaneously and then pushing two other button simultaneously?

(A) 20 (B) 40 (C) 80 (D) 100 (E) 120

Problem 20.10.5 (ICLT-2022-SM1-5-L3-P5). (*6 points*)

Suppose a is jointly proportional to b and c. If $a = 4$ when $b = 8$ and $c = 9$, then what is a when $b = 3$ and $c = 18$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 20.10.6 (ICLT-2022-SM1-5-L3-P6). (*6 points*)

The side lengths of a triangle are 4 centimeters, 6 centimeters, and 9 centimeters. One of the side lengths of a similar triangle is 36 centimeters. What is the minimal number of centimeters possible in the perimeter of the second triangle?

(A) 76 (B) 114 (C) 171 (D) 85 (E) 96

Problem 20.10.7 (ICLT-2022-SM1-5-L3-P7)**.** (*10 points*)

Find the number of odd three-digit palindromes.

Problem 20.10.8 (ICLT-2022-SM1-5-L3-P8)**.** (*10 points*)

How many quadrilaterals can be formed using the vertices of a regular decagon (a $10-$sided polygon)?

Problem 20.10.9 (ICLT-2022-SM1-5-L3-P9)**.** (*10 points*)

Let

$$x = 0.1234567891011121314\ldots997998999,$$

where x is formed by writing all of the integers from 1 to 999 after the decimal point. What is the 2022th digit to the right of the decimal point?

20.11 Grading for Level 3

Answers for multiple-choice problems.

Problem 1:	E	$\frac{3ab}{2} - 2cd + \frac{13}{2}$
Problem 2:	C	0
Problem 3:	A	144 and 180
Problem 4:	D	100
Problem 5:	B	3
Problem 6:	A	76

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: 2 points if can find 5 choices for the unit digits.

Problem 8: 2 points if can use combination to choose 4 points.

Problem 9: 2 points if can divise a process of counting based on the number of digits.

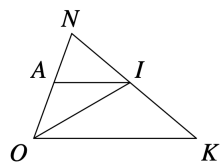
20.12 Solutions for Level 3

Solution. **??**

$$\left(3ab - 4cd + \frac{3}{2}\right) + \left(2cd - \frac{ab}{2} + 3\right) + (2 - ab) = \left(3ab - \frac{ab}{2} - ab\right) + (-4cd + 2cd) + \left(\frac{3}{2} + 3 + 2\right)$$
$$= \left(\frac{6ab}{2} - \frac{ab}{2} - \frac{2ab}{2}\right) + (-2cd) + \left(\frac{13}{2}\right) = \frac{3ab}{2} - 2cd + \frac{13}{2}.$$

The answer is $\boxed{E.}$ □

Solution. **??** Since $KO = KN$, $\triangle KNO$ is isosceles with $\angle KON = \angle N$. Since $IA = IN$, we have $\angle IAN =$



$\angle N$. Since $OI = ON$, we have $\angle N = \angle OIN$. Thus $\angle KON - \angle OIN = \boxed{0^\circ}$.

The answer is $\boxed{C.}$ □

Solution. **??** For the first part,

$$\gcd(1938, 3306) = \gcd(1938, 1368) = \gcd(570, 1368) = \gcd(570, 228) = \gcd(114, 228) = \gcd(114, 0) = \boxed{114.}$$

For the second part, it is probably easier to first find the lcm of two of the integers. $\text{lcm}(12, 18) = 36$. Every common multiple of 12 and 18 will be a multiple of their lcm, so the lcm of all three integers must be a multiple of 36. The lcm of all three integers is the lcm of 36 and 30, which is $\boxed{180}$.

The answer is $\boxed{A.}$ □

Solution. **??** There are $\binom{5}{3} = 10$ ways to choose the first three buttons, and then $\binom{5}{2} = 10$ choices for the last button. So there are $10 \times 10 = \boxed{100}$ possible combinations.

The answer is $\boxed{D.}$ □

Solution. **??** Because a is jointly proportional to b and c, there is some constant k for which $a = kbc$. Since $a = 4$ when $b = 8$ and $c = 9$, we have $4 = k(8)(9)$, so $k = \frac{4}{72} = \frac{1}{18}$. So, when $b = 3$ and $c = 18$, we have $a = kbc = \left(\frac{1}{18}\right)(3)(18) = \boxed{3}$.

The answer is $\boxed{B.}$ □

Solution. **??** To minimize the perimeter of the second triangle, we should minimize the ratio of corresponding side lengths between the two triangles. If the side of length 9 cm in the first triangle corresponds to the side of length 36 cm in the second triangle, then this ratio is $\frac{36}{9} = 4$. If it is the side of length 6 cm, then this ratio is $\frac{36}{6} = 6$, and if it is the side of length 4 cm, then this ratio is $\frac{36}{4} = 9$. Hence, the minimal ratio is 4, and the minimal perimeter of the second triangle is $4 \cdot (4 + 6 + 9) = \boxed{76}$.

The answer is $\boxed{A.}$ □

Solution. **??** In an odd three-digit palindrome, the first and the last digit are the same odd digit There are 5 such digits. For each of these 5 digits, the middle digit can be any of the 10 digits from 0 to 9 inclusive. The total number of odd three-digit palindromes is $5 \cdot 10 = \boxed{50}$. □

Solution. **??** No three vertices are collinear, so any combination of 4 vertices will make a quadrilaterals. Choosing 4 out of 10 is $\binom{10}{4} = \boxed{210}$. □

Solution. **??** The numbers 1 through 9 each use up 1 decimal place, so after writing 1 through 9 we've used up 9 total decimal places.

The numbers 10 through 99 each use up 2 decimal places, and there are 90 numbers from 10 through 99, so all together the numbers 10 through 99 use up $2 \times 90 = 180$ decimal places. So, after writing 1 through 99, we've used $9 + 180 = 189$ decimal places in total.

The numbers 100 through 999 each use up 3 decimal places, and there are 900 numbers from 100 through 999, so all together the numbers 100 through 999 use up $3 \times 900 = 2700$ decimal places. That's too many. We only want the 2022th digit. So we need to count in smaller chunks.

The numbers 100 through 199 each use up 3 decimal places, and there are 100 numbers from 100 through 199, so all together the numbers 100 through 199 use up $3 \times 100 = 300$ decimal places. So, after writing 1 through 199, we've used $189 + 300 = 489$ decimal places in total.

Similarly, writing up through 299 uses another 300 decimal places on top of what we've already written, so after writing 1 through 299, we've used $489 + 300 = 789$ decimal places.

After writing 1 through 399, we've used $789 + 300 = 1089$ decimal places.

After writing 1 through 499, we've used $1089 + 300 = 1389$ decimal places.

After writing 1 through 599, we've used $1389 + 300 = 1689$ decimal places.

After writing 1 through 699, we've used $1689 + 300 = 1989$ decimal places.

The 1990th digit is the first digit of 700, and we want the $(2022 - 1989) = 33^{\text{th}}$ digit of the part of the decimal that starts at 700,

$$\underbrace{700}_{1-3}\ \underbrace{701}_{4-6}\ \cdots\ \underbrace{709}_{28-30}\ \underbrace{710}_{31-33}$$

Easy to see that it is the last digit of 710, which is $\boxed{0}$. □

20.13 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

20.14 Problems for Level 4

Problem 20.14.1 (ICLT-2022-SM1-5-L4-P1). (*4 points*)

Suppose a is jointly proportional to b and c. If $a = 4$ when $b = 8$ and $c = 9$, then what is a when $b = 3$ and $c = 18$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 20.14.2 (ICLT-2022-SM1-5-L4-P2). (*4 points*)

The side lengths of a triangle are 4 centimeters, 6 centimeters, and 9 centimeters. One of the side lengths of a similar triangle is 36 centimeters. What is the minimal number of centimeters possible in the perimeter of the second triangle?

(A) 76 (B) 114 (C) 171 (D) 85 (E) 96

Problem 20.14.3 (ICLT-2022-SM1-5-L4-P3). (*4 points*)

Find the number of odd three-digit palindromes.

(A) 20 (B) 30 (C) 36 (D) 45 (E) 50

Problem 20.14.4 (ICLT-2022-SM1-5-L4-P4). (*6 points*)

How many quadrilaterals can be formed using the vertices of a regular decagon (a $10-$sided polygon)?

(A) 60 (B) 120 (C) 210 (D) 220 (E) 495

Problem 20.14.5 (ICLT-2022-SM1-5-L4-P5). (*6 points*)

Let $a = 2^{600},\ b = 3^{400},\ c = 4^{250},\ d = 5^{200}$. Which one below is correct?

$(A)\ a < b < c < d$ $(B)\ b < a < d < c$ $(C)\ c < d < b < a$ $(D)\ c < b < d < a$ $(E)\ d < c < a < b$

Problem 20.14.6 (ICLT-2022-SM1-5-L4-P6). (*6 points*)

Medians AD, BE, and CF meet at G. The area of $\triangle ABC$ is 48. Find the areas of $AEGF$.

(A) 10 (B) 16 (C) 18 (D) 24 (E) 28

Problem 20.14.7 (ICLT-2022-SM1-5-L4-P7). (*10 points*)

A six-digit number is formed by repeating a two-digit number; for example, 252525, or 676767, etc. Find the least common divisor of all numbers of this form.

Problem 20.14.8 (ICLT-2022-SM1-5-L4-P8). (*10 points*)

Suppose we have 2 fair $6-$sided dice numbered from 1 to 6. If we roll them both, what is the probability the sum of the numbers shown up is a multiple of 5?

Problem 20.14.9 (ICLT-2022-SM1-5-L4-P9). (*10 points*)

I have a weighted coin that lands on heads with probability $\frac{1}{3}$ (and thus lands on tails with probability $\frac{2}{3}$).

1. E_1 is the event that I flip my coin 2022 times and exactly 1012 of the flips come up heads.
2. E_2 is the event that I flip my coin 2022 times and exactly 1012 of the flips come up tails.

What is the ratio of the probability of the event E_1 to the probability of the event E_2?

20.15 Grading for Level 4

Answers for multiple-choice problems.

Problem 1:	B	3
Problem 2:	A	76
Problem 3:	E	50
Problem 4:	C	210
Problem 5:	E	$d < c < a < b$
Problem 6:	B	16

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: 2 points if can factor 10101 out of any number with the given form.

Problem 8: 2 points if can find 4 ways to sum up to 5 or 3 ways to sum up to 10.

Problem 9: 2 points if can determine $\binom{2022}{1012}$ ways to choose the coins to land on heads for the first event or $\binom{2022}{1012}$ ways to choose the coins to land on tails for the second event.

20.16 Solutions for Level 4

Solution. **??** Because a is jointly proportional to b and c, there is some constant k for which $a = kbc$. Since $a = 4$ when $b = 8$ and $c = 9$, we have $4 = k(8)(9)$, so $k = \frac{4}{72} = \frac{1}{18}$. So, when $b = 3$ and $c = 18$, we have $a = kbc = \left(\frac{1}{18}\right)(3)(18) = \boxed{3}$.

The answer is $\boxed{B.}$ □

Solution. **??** To minimize the perimeter of the second triangle, we should minimize the ratio of corresponding side lengths between the two triangles. If the side of length 9 cm in the first triangle corresponds to the side of length 36 cm in the second triangle, then this ratio is $\frac{36}{9} = 4$. If it is the side of length 6 cm, then this ratio is $\frac{36}{6} = 6$, and if it is the side of length 4 cm, then this ratio is $\frac{36}{4} = 9$. Hence, the minimal ratio is 4, and the minimal perimeter of the second triangle is $4 \cdot (4 + 6 + 9) = \boxed{76}$.

The answer is $\boxed{A.}$ □

Solution. **??** In an odd three-digit palindrome, the first and the last digit are the same odd digit There are 5 such digits. For each of these 5 digits, the middle digit can be any of the 10 digits from 0 to 9 inclusive. The total number of odd three-digit palindromes is $5 \cdot 10 = \boxed{50}$.

The answer is $\boxed{E.}$ □

Solution. **??** No three vertices are collinear, so any combination of 4 vertices will make a quadrilaterals. Choosing 4 out of 10 is $\binom{10}{4} = \boxed{210}$.

The answer is $\boxed{C.}$ □

Solution. **??** Reduce the power of all numbers $a = 2^{600},\ b = 3^{400},\ c = 4^{250},\ d = 5^{200}$ by taking to the root 50th,

$$\sqrt[50]{2^{600}} = \left(2^{600}\right)^{\frac{1}{50}} = 2^{600\cdot\frac{1}{50}} = 2^{12},\ \sqrt[50]{3^{400}} = 3^8,\ \sqrt[50]{4^{250}} = 4^5,\ \sqrt[50]{5^{200}} = 5^4$$

It is easy to see that

$$2^{12} = 4096,\ 3^8 = 6561,\ 4^5 = 1024,\ 5^4 = 625 \Rightarrow 5^4 < 4^5 < 2^{12} < 3^8 \Rightarrow \boxed{d < c < a < b}$$

The answer is $\boxed{E.}$ □

Solution. **??** $AEGF$ consist of $\triangle AEG$ and $\triangle AFG$. $\triangle AEG$ has half the base and the same height as $\triangle AGC$. $\triangle AGC$ has the same base as $\triangle ABC$ and since $\frac{GE}{BE} = \frac{1}{3}$, so the altitude from G to AC is a third of the altitude from B to AC, therefore

$$[AGC] = \frac{[ABC]}{3} \Rightarrow [AEG] = \frac{[ABC]}{6},\ \text{similarly}$$
$$[AFG] = \frac{[ABC]}{6} \Rightarrow [AEGF] = [AEG] + [AFG] = \frac{[ABC]}{3} = \boxed{16}$$

The answer is $\boxed{B.}$ □

Solution. **??** Note that

$$252525 = 25 \cdot 10^4 + 25 \cdot 10^2 + 12 \cdot 1 = 25(10^4 + 10^2 + 1) = 25 \cdot 10101$$

It is easy to see that 10101 is a common divisor of all such numbers. Furthermore $\gcd(25\cdot 10101, 36\cdot 10101) = 10101$, so $\boxed{10101}$ is the least common divisor of all such numbers. □

Solution. **??** In order for the sum of two numbers between 1 and 6 to be a multiple of 5, the sum of the two dice must be 5 or 10. There are 4 ways to for two dice sum up to 5, $(1,4),(2,3),(3,2),(4,1)$. There are 3 ways to for two dice sum up to 10, $(4,6),(5,5),(6,4)$. There are $6 \cdot 6$ ways to cast 2 dice. Thus, the probability is $\frac{4+3}{36} = \boxed{\frac{7}{36}.}$ □

Solution. **??** There are $\binom{2022}{1012}$ ways for 1012 of the coins to land on heads and 1010 of the coins to land on tails. Each of these sequences has a probability $\left(\frac{1}{3}\right)^{1012}\left(\frac{2}{3}\right)^{1010}$, since heads are flipped 2020 times and tails are flipped 2 times. Thus, the probability that exactly 1012 of the coins come up heads is

$$\binom{2022}{1012}\left(\frac{1}{3}\right)^{1012}\left(\frac{2}{3}\right)^{1010}.$$

Similarly, thus, the probability that exactly 1012 of the coins come up tails is

$$\binom{2022}{1010}\left(\frac{1}{3}\right)^{1010}\left(\frac{2}{3}\right)^{1012}.$$

Since $\binom{2022}{1012} = \binom{2022}{1010}$, their ratio is

$$\frac{\binom{2022}{1012}\left(\frac{1}{3}\right)^{1012}\left(\frac{2}{3}\right)^{1010}}{\binom{2022}{1010}\left(\frac{1}{3}\right)^{1010}\left(\frac{2}{3}\right)^{1012}} = \boxed{\frac{1}{4}.}$$

□

20.17 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

20.18 Problems for Level 5

Problem 20.18.1 (ICLT-2022-SM1-5-L5-P1). (*4 points*)

Let $a = 2^{600},\ b = 3^{400},\ c = 4^{250},\ d = 5^{200}$. Which one below is correct?

$(A)\ a < b < c < d \qquad (B)\ b < a < d < c \qquad (C)\ c < d < b < a \qquad (D)\ c < b < d < a \qquad (E)\ d < c < a < b$

Problem 20.18.2 (ICLT-2022-SM1-5-L5-P2). (*4 points*)

Medians AD, BE, and CF meet at G. The area of $\triangle ABC$ is 48. Find the areas of $AEGF$.

$(A)\ 10 \qquad (B)\ 16 \qquad (C)\ 18 \qquad (D)\ 24 \qquad (E)\ 28$

Problem 20.18.3 (ICLT-2022-SM1-5-L5-P3). (*4 points*)

A six-digit number is formed by repeating a two-digit number; for example, 252525, or 676767, etc. Find the least common divisor of all numbers of this form.

$(A)\ 10001 \qquad (B)\ 11001 \qquad (C)\ 10111 \qquad (D)\ 10101 \qquad (E)\ 1111$

Problem 20.18.4 (ICLT-2022-SM1-5-L5-P4). (*6 points*)

Suppose we have 2 fair 6−sided dice numbered from 1 to 6. If we roll them both, what is the probability the sum of the numbers shown up is a multiple of 5?

$(A)\ \frac{11}{36} \qquad (B)\ \frac{5}{36} \qquad (C)\ \frac{5}{18} \qquad (D)\ \frac{1}{9} \qquad (E)\ \frac{7}{36}$

Problem 20.18.5 (ICLT-2022-SM1-5-L5-P5). (*6 points*)

i is the imaginary number, $i = \sqrt{-1}$. Find the value of the sum

$$1 - i + i^2 - i^3 + i^4 - i^5 + i^6 - i^7 + i^8 + \ldots - i^{2021} + i^{2022}.$$

$(A)\ i \qquad (B)\ 1 \qquad (C)\ 0 \qquad (D)\ -1 \qquad (E)\ -i$

Problem 20.18.6 (ICLT-2022-SM1-5-L5-P6). (*6 points*)

$EFGH$ is a rhombus. Given that $EF = EG = 2\sqrt{3}$, find the area of the rhombus.

$(A)\ 6 \qquad (B)\ 6\sqrt{3} \qquad (C)\ 18 \qquad (D)\ 18\sqrt{3} \qquad (E)\ 24\sqrt{3}$

Problem 20.18.7 (ICLT-2022-SM1-5-L5-P7). (*10 points*)

Which of the following numbers, in different bases as indicated, are a multiple of 5? Show your proof.

$(A)\ 117_9 \qquad (B)\ 111101_2 \qquad (C)\ 111101_4 \qquad (D)\ 4105_6 \qquad (E)\ A1BA_{15}$

Problem 20.18.8 (ICLT-2022-SM1-5-L5-P8). (*10 points*)

What is the probability that a random rearrangement of the letters in the word BOOKKEEPER will begin with the BEEK?

Problem 20.18.9 (ICLT-2022-SM1-5-L5-P9). (*10 points*)

Let x and y be real numbers such that

$$\begin{cases} -2 \le x \le 4 \\ -1 \le y \le 5 \end{cases}$$

What is the probability to choose a pair of (x, y) such that xy is positive?

20.19 Grading for Level 5

Answers for multiple-choice problems.

Problem 1:	E	$d < c < a < b$
Problem 2:	B	16
Problem 3:	D	10101
Problem 4:	E	$\frac{7}{36}$
Problem 5:	E	$-i$
Problem 6:	B	$6\sqrt{3}$

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: 2 points if use conversion to base 10 to solve the problem.

Problem 8: 2 points if can use multiplication of probabilities.

Problem 9: 2 points if can determine the regions of valid x, y.

20.20 Solutions for Level 5

Solution. **??** Reduce the power of all numbers $a = 2^{600},\ b = 3^{400},\ c = 4^{250},\ d = 5^{200}$ by taking to the root 50th,

$$\sqrt[50]{2^{600}} = \left(2^{600}\right)^{\frac{1}{50}} = 2^{600\cdot\frac{1}{50}} = 2^{12},\ \sqrt[50]{3^{400}} = 3^8,\ \sqrt[50]{4^{250}} = 4^5,\ \sqrt[50]{5^{200}} = 5^4$$

It is easy to see that

$$2^{12} = 4096,\ 3^8 = 6561,\ 4^5 = 1024,\ 5^4 = 625 \Rightarrow 5^4 < 4^5 < 2^{12} < 3^8 \Rightarrow \boxed{d < c < a < b}$$

The answer is $\boxed{E.}$ □

Solution. **??** $AEGF$ consist of $\triangle AEG$ and $\triangle AFG$. $\triangle AEG$ has half the base and the same height as $\triangle AGC$. $\triangle AGC$ has the same base as $\triangle ABC$ and since $\frac{GE}{BE} = \frac{1}{3}$, so the altitude from G to AC is a third of the altitude from B to AC, therefore

$$[AGC] = \frac{[ABC]}{3} \Rightarrow [AEG] = \frac{[ABC]}{6},\ \text{similarly}$$
$$[AFG] = \frac{[ABC]}{6} \Rightarrow [AEGF] = [AEG] + [AFG] = \frac{[ABC]}{3} = \boxed{16}$$

The answer is $\boxed{B.}$ □

Solution. **??** Note that

$$252525 = 25 \cdot 10^4 + 25 \cdot 10^2 + 12 \cdot 1 = 25(10^4 + 10^2 + 1) = 25 \cdot 10101$$

It is easy to see that 10101 is a common divisor of all such numbers. Furthermore $\gcd(25\cdot 10101, 36\cdot 10101) = 10101$, so $\boxed{10101}$ is the least common divisor of all such numbers.

The answer is $\boxed{D.}$ □

Solution. **??** In order for the sum of two numbers between 1 and 6 to be a multiple of 5, the sum of the two dice must be 5 or 10. There are 4 ways to for two dice sum up to 5, $(1,4),(2,3),(3,2),(4,1)$. There are 3 ways to for two dice sum up to 10, $(4,6),(5,5),(6,4)$. There are $6 \cdot 6$ ways to cast 2 dice. Thus, the probability is $\frac{4+3}{36} = \boxed{\frac{7}{36}.}$

The answer is $\boxed{E.}$ □

Solution. **??** Note that, $i^2 = -1$, so

$$1 - i + i^2 - i^3 = 1 - i + (-1) + i = 0$$
$$i^4 - i^5 + i^6 - i^7 = i^4(1 - i + i^2 - i^3) = 0$$
$$\cdots$$
$$i^{2016} - i^{2017} + i^{2018} - i^{2019} = i^{2016}(1 - i + i^2 - i^3) = 0$$

Thus the first 2020 terms of the given sum cancel each other, and the remaining sum is

$$i^{2020} - i^{2021} + i^{2022} = i^{2020}(1 - i + i^2) = (i^4)^{505}(-i) = \boxed{-i}$$

The answer is $\boxed{E.}$ □

Solution. **??** Since $EFGH$ is a rhombus, we have $FG = EF$. Therefore, $FG = EF = EG$, so EFG is equilateral with side length $2\sqrt{3}$. Since $EH = HG = EF = EG$, $\triangle EHG$ is equilateral with side length $2\sqrt{3}$, too. Hence,

$$[EFGH] = 2[EFG] = 2\left(\frac{(2\sqrt{3})^2\sqrt{3}}{4}\right) = \boxed{6\sqrt{3}}.$$

The answer is $\boxed{B.}$ □

Solution. **??** Converting to base 10, $117_9 = 97,\ 111101_2 = 61,\ 111101_4 = 1361,\ 4105_6 = \boxed{905},\ A1BC_{15} = 34152.$ □

Solution. **??** The probability to choose B from BOOKKEEPER as the first letter is $\frac{1}{10}$, (since there are 1 B among the 10 letters that we can choose). The probability to choose E from BOOKKEEPER as the second letter is $\frac{3}{9}$, (since there are 3 Es among the remaining 9 letters that we can choose). The probability to choose E from BOOKKEEPER as the third letter is $\frac{2}{8}$, (since there are 2 Es among the remaining 8 letters that we can choose). The probability to choose K from BOOKKEEPER as the fourth letter is $\frac{2}{7}$. (since there are 2 Ks among the remaining 7 letters that we can choose).

Thus, the probability that a random arrangement of of the letters in the word BOOKKEEPER will begin with BEEK is

$$\frac{1}{10}\cdot\frac{3}{9}\cdot\frac{2}{8}\cdot\frac{2}{7} = \boxed{\frac{1}{420}.}$$

□

Solution. **??** The region of (x, y) satisfying

$$\begin{cases} -2 \le x \le 4 \\ -1 \le y \le 5 \end{cases}$$

is the rectangle determined by four lines $x = -2, x = -4$, and $y = -1, y = 5$. For (x, y) such that $xy > 0$,

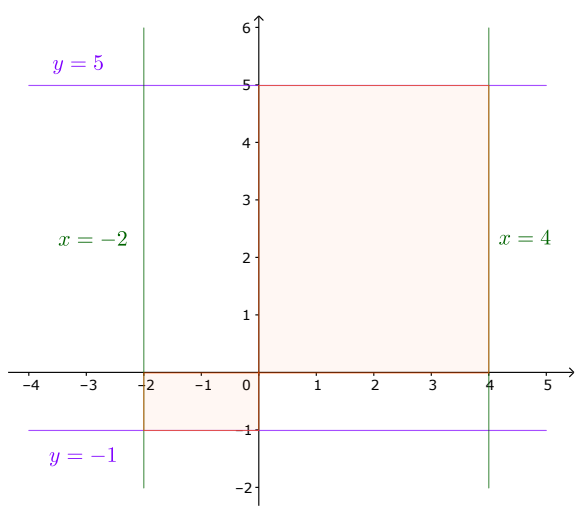


Figure 20.7: Two regions for $xy > 0$

then (x, y) must be a point in the first or third quadrants, which is the region consists of the two shaded rectangles. The probability is $\frac{2\cdot 1+4\cdot 5}{36} = \boxed{\frac{11}{36}.}$ □

20.21 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

20.22 Problems for Level 6

Problem 20.22.1 (ICLT-2022-SM1-5-L6-P1). (*4 points*)

i is the imaginary number, $i = \sqrt{-1}$. Find the value of the sum

$$1 - i + i^2 - i^3 + i^4 - i^5 + i^6 - i^7 + i^8 + \ldots - i^{2021} + i^{2022}.$$

(A) i (B) 1 (C) 0 (D) -1 (E) $-i$

Problem 20.22.2 (ICLT-2022-SM1-5-L6-P2). (*4 points*)

$EFGH$ is a rhombus. Given that $EF = EG = 2\sqrt{3}$, find the area of the rhombus.

(A) 6 (B) $6\sqrt{3}$ (C) 18 (D) $18\sqrt{3}$ (E) $24\sqrt{3}$

Problem 20.22.3 (ICLT-2022-SM1-5-L6-P3). (*4 points*)

Which of the following numbers, in different bases as indicated, are a multiple of 5?

(A) 117_9 (B) 111101_2 (C) 111101_4 (D) 4105_6 (E) $A1BA_{15}$

Problem 20.22.4 (ICLT-2022-SM1-5-L6-P4). (*6 points*)

What is the probability that a random rearrangement of the letters in the word BOOKKEEPER will begin with the BEEK?

(A) $\frac{1}{72}$ (B) $\frac{1}{180}$ (C) $\frac{1}{360}$ (D) $\frac{1}{420}$ (E) $\frac{1}{560}$

Problem 20.22.5 (ICLT-2022-SM1-5-L6-P5). (*6 points*)

One root of the quadratic equation $y^2 + by + c = 0$, where b and c are real constants, is $5 - i\sqrt{5}$. What is the value of $b + c$?

(A) 20 (B) $20 + 2\sqrt{5}$ (C) $30 - 2\sqrt{5}$ (D) 15 (E) $45 - 2\sqrt{5}$

Problem 20.22.6 (ICLT-2022-SM1-5-L6-P6). (*6 points*)

The perimeter of an isosceles triangle is 34 centimeters and two sides of the triangle are whole numbers in the ratio $3 : 7$. What is the number of centimeters in the length of the longest side?

(A) 6 (B) 8 (C) 12 (D) 14 (E) 15

Problem 20.22.7 (ICLT-2022-SM1-5-L6-P7). (*10 points*)

When the product of two consecutive positive integers is written in base 7, the units digit is 5. Find the units digit of their base-7 sum.

Problem 20.22.8 (ICLT-2022-SM1-5-L6-P8). (*10 points*)

Bin A has one white ball and four black balls. Bin B has three balls labeled "\$1" and one ball labeled "\$7". Bin W has five balls labeled "\$8" and one ball labeled "\$500". A game is played as follows: a ball is randomly selected from bin A. If it is black, then a ball is randomly selected from bin B; otherwise, if the original ball is white, then a ball is randomly selected from bin W. You win the amount printed on the second ball selected.

What is your expected win?

Problem 20.22.9 (ICLT-2022-SM1-5-L6-P9). (*10 points*)

Solve the inequality $(2x - 3)(x^2 + 7x + 10) \leq 0$.

20.23 Grading for Level 6

Answers for multiple-choice problems.

Problem 1: E $\quad -i$

Problem 2: B $\quad 6\sqrt{3}$

Problem 3: D $\quad 4105_6$

Problem 4: D $\quad \frac{1}{420}$

Problem 5: A $\quad 20$

Problem 6: D $\quad 14$

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: 2 points if list the cases of products of two consecutive digits in base 7.

Problem 8: 2 points if can figure out how to determine any of the expected values of bin W or B or A.

Problem 9: 2 points if can casework on the signs of the factors.

20.24 Solutions for Level 6

Solution. **??** Note that, $i^2 = -1$, so

$$\begin{aligned}
&1 - i + i^2 - i^3 = 1 - i + (-1) + i = 0\\
&i^4 - i^5 + i^6 - i^7 = i^4(1 - i + i^2 - i^3) = 0\\
&\dots\\
&i^{2016} - i^{2017} + i^{2018} - i^{2019} = i^{2016}(1 - i + i^2 - i^3) = 0
\end{aligned}$$

Thus the first 2020 terms of the given sum cancel each other, and the remaining sum is

$$i^{2020} - i^{2021} + i^{2022} = i^{2020}(1 - i + i^2) = (i^4)^{505}(-i) = \boxed{-i}$$

The answer is $\boxed{E.}$ □

Solution. **??** Since $EFGH$ is a rhombus, we have $FG = EF$. Therefore, $FG = EF = EG$, so EFG is equilateral with side length $2\sqrt{3}$. Since $EH = HG = EF = EG$, $\triangle EHG$ is equilateral with side length $2\sqrt{3}$, too. Hence,

$$[EFGH] = 2[EFG] = 2\left(\frac{(2\sqrt{3})^2\sqrt{3}}{4}\right) = \boxed{6\sqrt{3}}.$$

The answer is $\boxed{B.}$ □

Solution. **??** Converting to base 10, $117_9 = 97,\ 111101_2 = 61,\ 111101_4 = 1361,\ 4105_6 = \boxed{905},\ A1BC_{15} = 34152.$

The answer is $\boxed{D.}$ □

Solution. **??** The probability to choose B from BOOKKEEPER as the first letter is $\frac{1}{10}$, (since there are 1 B among the 10 letters that we can choose). The probability to choose E from BOOKKEEPER as the second letter is $\frac{3}{9}$, (since there are 3 Es among the remaining 9 letters that we can choose). The probability to choose E from BOOKKEEPER as the third letter is $\frac{2}{8}$, (since there are 2 Es among the remaining 8 letters that we can choose). The probability to choose K from BOOKKEEPER as the fourth letter is $\frac{2}{7}$. (since there are 2 Ks among the remaining 7 letters that we can choose).

Thus, the probability that a random arrangement of of the letters in the word BOOKKEEPER will begin with BEEK is

$$\frac{1}{10}\cdot\frac{3}{9}\cdot\frac{2}{8}\cdot\frac{2}{7} = \boxed{\frac{1}{420}.}$$

The answer is $\boxed{D.}$ □

Solution. **??** Because the quadratic has real coefficients, we know that if one root is not real, then the roots are complex conjugates. One root of the quadratic is $5 - i\sqrt{5}$, so the other root is $5 + i\sqrt{5}$. The sum of the roots of the quadratic is $-b$ and the product is c, so we have

$$\begin{aligned}
-b &= (5 - i\sqrt{5}) + (5 + i\sqrt{5}) = 10 \Rightarrow b = -10\\
c &= (5 - i\sqrt{5})(5 + i\sqrt{5}) = 25 - (-1)5 = 30
\end{aligned}$$

Therefore $b + c = \boxed{20}$.

The answer is $\boxed{A.}$ □

Solution. **??** There are two possibilities: The sides are of the form $3x$, $3x$, and $7x$ for some number x, or they are of the form $3x$, $7x$, and $7x$ for some number x. In the first case, the sides do not satisfy the Triangle Inequality because $3x + 3x = 6x < 7x$. In the second case, the sum of the sides is $3x + 7x + 7x = 17x = 34$, so $x = 2$. Hence, the side lengths are $6, 14$, and 14, so the longest side length is $\boxed{14 \text{ cm.}}$

The answer is $\boxed{D.}$ □

Solution. **??** We know that neither of the units digits is 0, so we test the other 5 cases of consecutive units digits in base 7 and find that the two integers must have units digits of 3 and 4.

$$1_7 \cdot 2_7 = 2_7,\ 2_7 \cdot 3_7 = 6_7,\ 3_7 \cdot 4_7 = 15_7,\ 4_7 \cdot 5_7 = 26_7,\ 5_7 \cdot 6_7 = 42_7.$$

Since $3_7 + 4_7 = 10_7$, the units digit of their sum of $\boxed{0}$. □

Solution. **??** Since Bin A has one white ball and four black balls, the money ball has a $\frac{1}{5}$ chance of coming from Bin W and a $\frac{4}{5}$ chance of coming from Bin B. The total expected value therefore is

$$E = \frac{1}{5}E_W + \frac{4}{5}E_B,$$

where E_W and E_B are the expected values of a ball drawn from bins W and B, respectively.

Since Bin W has five \$8 balls and 1 \$500 ball, its expected value is

$$E_W = \frac{5}{6} \times \$8 + \frac{1}{6} \times \$500 = \$90.$$

Since Bin B has three \$1 balls and one \$7 ball, its expected value is

$$E_B = \frac{3}{4} \times \$1 + \frac{1}{4} \times \$7 = \$2.5.$$

Therefore

$$E = \frac{1}{5}E_W + \frac{4}{5}E_B = \frac{1}{5}(\$90) + \frac{4}{5}(\$2.5) = \boxed{\$20}.$$

□

Solution. **??** We factor the quadratic to find $(2x-3)(x+2)(x+5) \le 0$. The product on the left side is negative when all three factors are negative, and when one factor is negative and the other two are positive. We have $2x - 3 < 0$ when $x < 3/2$, $x + 2 < 0$ when $x < -2$, and $x + 5 < 0$ when $x < -5$. We can combine this information in a table:

	$2x-3$	$x+2$	$x+5$	$(2x-3)(x+2)(x+5)$
$x < \frac{3}{2}$	$+$	$+$	$+$	$+$
$-2 < x < \frac{3}{2}$	$-$	$+$	$+$	$-$
$-5 < x < -2$	$-$	$-$	$+$	$+$
$x < -5$	$-$	$-$	$-$	$-$

We see that one factor is negative when $-2 < x < \frac{3}{2}$ and all three are negative when $x < -5$. Since the inequality is nonstrict, the values $x = \frac{3}{2}$, $x = -2$, and $x = -5$ are also solutions, so the values of x that satisfy the inequality are $-2 \le x \le \frac{3}{2}$ or $x \le -5$. In interval notation, this is $\boxed{x \in (-\infty, -5] \cup \left[-2, \frac{3}{2}\right]}$. □

20.25 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

20.26 Problems for Level 7

Problem 20.26.1 (ICLT-2022-SM1-5-L7-P1). (*4 points*)

One root of the quadratic equation $y^2 + by + c = 0$, where b and c are real constants, is $5 - i\sqrt{5}$. What is the value of $b + c$?

(A) 20 (B) $20 + 2\sqrt{5}$ (C) $30 - 2\sqrt{5}$ (D) 15 (E) $45 - 2\sqrt{5}$

Problem 20.26.2 (ICLT-2022-SM1-5-L7-P2). (*4 points*)

The perimeter of an isosceles triangle is 34 centimeters and two sides of the triangle are whole numbers in the ratio $3 : 7$. What is the number of centimeters in the length of the longest side?

(A) 6 (B) 8 (C) 12 (D) 14 (E) 15

Problem 20.26.3 (ICLT-2022-SM1-5-L7-P3). (*4 points*)

When the product of two consecutive positive integers is written in base 7, the units digit is 5. Find the units digit of their base-7 sum.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 6

Problem 20.26.4 (ICLT-2022-SM1-5-L7-P4). (*6 points*)

Bin A has one white ball and four black balls. Bin B has three balls labeled "\$1" and one ball labeled "\$7". Bin W has five balls labeled "\$8" and one ball labeled "\$500". A game is played as follows: a ball is randomly selected from bin A. If it is black, then a ball is randomly selected from bin B; otherwise, if the original ball is white, then a ball is randomly selected from bin W. You win the amount printed on the second ball selected.

What is your expected win?

(A) 10 (B) 15.5 (C) 20 (D) 22.25 (E) 30

Problem 20.26.5 (ICLT-2022-SM1-5-L7-P5). (*6 points*)

Let $f(x) = 3x - 8$ and $g(f(x)) = 2x^2 + 5x - 3$. Find $g(-5)$.

(A) 2 (B) 3 (C) 4 (D) 6 (E) 8

Problem 20.26.6 (ICLT-2022-SM1-5-L7-P6). (*6 points*)

In the diagram, O is the center of the circle and $\angle ACB = 36°$. Find $\angle ABF$.

(A) $100°$ (B) $108°$ (C) $120°$ (D) $126°$ (E) $132°$

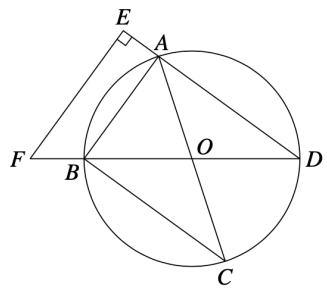


Figure 20.8: **??**

Problem 20.26.7 (ICLT-2022-SM1-5-L7-P7). (*10 points*)

Find the 2022st digit past the decimal point in the expansion of $\dfrac{1}{39}$.

Problem 20.26.8 (ICLT-2022-SM1-5-L7-P8). (*10 points*)

n is a positive integer. Prove that,

$$\binom{2n}{n} + \binom{2n}{n-1} = \frac{1}{2}\binom{2n+2}{n+1}$$

Problem 20.26.9 (ICLT-2022-SM1-5-L7-P9). (*10 points*)

P is a point outside of the circle (O). Line ℓ_1 through P intersects the circle at A and B, where B lies between A and P, such that $AB = 10, BP = 8$. Line ℓ_2 through P intersects the circle at C and D, where C lies between D and P, such that $CD = 7, \angle APC = 60°$.

Find the area of the circle.

20.27 Grading for Level 7

Answers for multiple-choice problems.

Problem 1: A 20

Problem 2: D 14

Problem 3: A 0

Problem 4: C 20

Problem 5: C 4

Problem 6: D 126

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: 2 points if made correct long division to obtain the repeating decimal.

Problem 8: 2 points if can use $\binom{2n}{n} + \binom{2n}{n-1} = \binom{2n+1}{n}$.

Problem 9: 2 points if can use power of the point to find the length of PC.

20.28 Solutions for Level 7

Solution. **??** Because the quadratic has real coefficients, we know that if one root is not real, then the roots are complex conjugates. One root of the quadratic is $5 - i\sqrt{5}$, so the other root is $5 + i\sqrt{5}$. The sum of the roots of the quadratic is $-b$ and the product is c, so we have

$$-b = (5 - i\sqrt{5}) + (5 + i\sqrt{5}) = 10 \Rightarrow b = -10$$
$$c = (5 - i\sqrt{5})(5 + i\sqrt{5}) = 25 - (-1)5 = 30$$

Therefore $b + c = \boxed{20}$.

The answer is $\boxed{A.}$ □

Solution. **??** There are two possibilities: The sides are of the form $3x$, $3x$, and $7x$ for some number x, or they are of the form $3x$, $7x$, and $7x$ for some number x. In the first case, the sides do not satisfy the Triangle Inequality because $3x + 3x = 6x < 7x$. In the second case, the sum of the sides is $3x + 7x + 7x = 17x = 34$, so $x = 2$. Hence, the side lengths are $6, 14$, and 14, so the longest side length is $\boxed{14 \text{ cm.}}$

The answer is $\boxed{D.}$ □

Solution. **??** We know that neither of the units digits is 0, so we test the other 5 cases of consecutive units digits in base 7 and find that the two integers must have units digits of 3 and 4.

$$1_7 \cdot 2_7 = 2_7,\ 2_7 \cdot 3_7 = 6_7,\ 3_7 \cdot 4_7 = 15_7,\ 4_7 \cdot 5_7 = 26_7,\ 5_7 \cdot 6_7 = 42_7.$$

Since $3_7 + 4_7 = 10_7$, the units digit of their sum of $\boxed{0}$.

The answer is $\boxed{A.}$ □

Solution. **??** Since Bin A has one white ball and four black balls, the money ball has a $\frac{1}{5}$ chance of coming from Bin W and a $\frac{4}{5}$ chance of coming from Bin B. The total expected value therefore is

$$E = \frac{1}{5}E_W + \frac{4}{5}E_B,$$

where E_W and E_B are the expected values of a ball drawn from bins W and B, respectively.

Since Bin W has five \$8 balls and 1 \$500 ball, its expected value is

$$E_W = \frac{5}{6} \times \$8 + \frac{1}{6} \times \$500 = \$90.$$

Since Bin B has three \$1 balls and one \$7 ball, its expected value is

$$E_B = \frac{3}{4} \times \$1 + \frac{1}{4} \times \$7 = \$2.5.$$

Therefore

$$E = \frac{1}{5}E_W + \frac{4}{5}E_B = \frac{1}{5}(\$90) + \frac{4}{5}(\$2.5) = \boxed{\$20}.$$

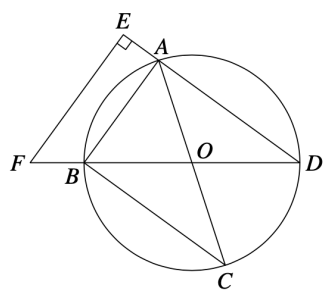
The answer is $\boxed{C.}$ □

Solution. **??** If $f(x) = -5$, we have $3x - 8 = -5$, so $x = 1$. Therefore, letting $x = 1$ in $g(f(x)) = 2x^2 + 5x - 3$ gives

$$g(f(1)) = 2 \cdot 1^2 + 5 \cdot 1 - 3 \Rightarrow g(-5) = \boxed{4}.$$

The answer is $\boxed{C.}$ □

Solution. **??** First, $\angle ADB = \angle ACB = 36°$. Thus



$$\widehat{AB} = 2\angle ADB = 72° \Rightarrow \angle ABD = \frac{1}{2}\widehat{AD} = \frac{1}{2}(180° - \widehat{AB}) = 54° \Rightarrow \angle ABF = 180° - \angle ABD = \boxed{126°.}$$

The answer is $\boxed{D.}$ □

Solution. **??** By long division $\frac{1}{39} = 0.\overline{025641}$. $2022 = 6 \cdot 337$, the $2022^{\text{th}}-$digit is $\boxed{1.}$ □

Solution. **??** Note that

$$\begin{aligned}
&\binom{2n}{n} + \binom{2n}{n-1} = \binom{2n+1}{n} \\
&\binom{2n+1}{n} = \binom{2n+1}{(2n+1)-n} = \binom{2n+1}{n+1} \\
&\Rightarrow 2\binom{2n+1}{n} = \binom{2n+1}{n} + \binom{2n+1}{n+1} = \binom{2n+2}{n+1} \\
&\Rightarrow \binom{2n}{n} + \binom{2n}{n-1} = \frac{1}{2}\binom{2n+2}{n+1}
\end{aligned}$$

□

Solution. **??** By power of the point

$$PA \cdot PB = PC \cdot PD \Rightarrow PC(PC + 7) = 144 \Rightarrow PC = 9.$$

Since $\angle APC = 60°, PC = \frac{AP}{2}$, so PAC is a $30 - 60 - 90$ triangle, where $\angle PCA = 90°$, thus $AC = PC\sqrt{3} = 9\sqrt{3}$. Now, $\angle ACD = 90°$, and AD is the diameter, so

$$AD = \sqrt{AC^2 + CD^2} = \sqrt{243 + 49} = 2\sqrt{73}$$

Thus the area of the circle is $\boxed{73\pi}$. □

20.29 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

20.30 Problems for Level 8

Problem 20.30.1 (ICLT-2022-SM1-5-L8-P1). (*4 points*)

Let $f(x) = 3x - 8$ and $g(f(x)) = 2x^2 + 5x - 3$. Find $g(-5)$.

(A) 2 (B) 3 (C) 4 (D) 6 (E) 8

Problem 20.30.2 (ICLT-2022-SM1-5-L8-P2). (*4 points*)

In the diagram, O is the center of the circle and $\angle ACB = 36°$. Find $\angle ABF$.

(A) 100° (B) 108° (C) 120° (D) 126° (E) 132°

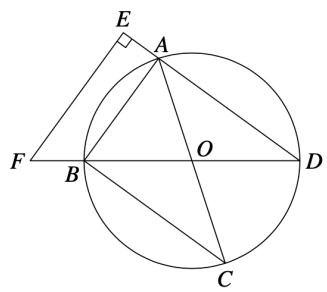


Figure 20.9: **??**

Problem 20.30.3 (ICLT-2022-SM1-5-L8-P3). (*4 points*)

Find the 2022st digit past the decimal point in the expansion of $\frac{1}{39}$.

(A) 1 (B) 2 (C) 4 (D) 5 (E) 6

Problem 20.30.4 (ICLT-2022-SM1-5-L8-P4). (*6 points*)

In how many ways can we distribute 13 pieces of candy to 5 kids, if the oldest kid must receive at least 2 pieces?

(A) 126 (B) 356 (C) 476 (D) 1256 (E) 1356

Problem 20.30.5 (ICLT-2022-SM1-5-L8-P5). (*6 points*)

Claire has borrowed \$5000. She will repay the loan entirely by making one payment in 3 years, then another payment in 6 years. The second payment will be exactly double the amount of the first payment. How much is the first payment if the interest rate of the loan is 7.5%, compounded annually?

(A) 2489.05 (B) 2379.95 (C) 2439.75 (D) 2756.65 (E) 3452.55

Problem 20.30.6 (ICLT-2022-SM1-5-L8-P6). (*6 points*)

Bridget places a box that is $4 \times 6 \times 8$ on the floor. She then places a $3 \times 5 \cdot 7$ box on top of the first box, forming a two-box tower. She will then paint all the surfaces she can paint without moving either of the boxes. She wants to paint as little as possible, so she places the boxes in a way that minimizes the amount she'll have to paint.

What is the total area she has to paint?

(A) 232 (B) 218 (C) 206 (D) 192 (E) 178

Problem 20.30.7 (ICLT-2022-SM1-5-L8-P7). (*10 points*)

How many integers between 1000 and 3000 inclusive are congruent to 7 (mod 11)?

Problem 20.30.8 (ICLT-2022-SM1-5-L8-P8). (*10 points*)

n is a positive integer. Prove that,

$$\binom{2n}{n} + \binom{2n}{n-1} = \frac{1}{2}\binom{2n+2}{n+1}$$

Problem 20.30.9 (ICLT-2022-SM1-5-L8-P9). (*10 points*)

P is a point outside of the circle (O). Line ℓ_1 through P intersects the circle at A and B, where B lies between A and P, such that $AB = 10, BP = 8$. Line ℓ_2 through P intersects the circle at C and D, where C lies between D and P, such that $CD = 7, \angle APC = 60°$.

Find the area of the circle.

20.31 Grading for Level 8

Answers for multiple-choice problems.

Problem 1:	C	4
Problem 2:	D	126
Problem 3:	A	1
Problem 4:	E	1356
Problem 5:	B	2379.95
Problem 6:	A	232

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: 2 points if can establish such number in a form of $11n + 7$ for estimation.

Problem 8: 2 points if can use $\binom{2n}{n} + \binom{2n}{n-1} = \binom{2n+1}{n}$.

Problem 9: 2 points if can use power of the point to find the length of PC.

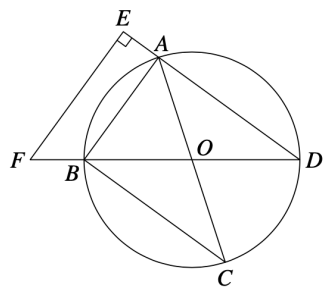
20.32 Solutions for Level 8

Solution. **??** If $f(x) = -5$, we have $3x - 8 = -5$, so $x = 1$. Therefore, letting $x = 1$ in $g(f(x)) = 2x^2 + 5x - 3$ gives

$$g(f(1)) = 2 \cdot 1^2 + 5 \cdot 1 - 3 \Rightarrow g(-5) = \boxed{4}.$$

The answer is $\boxed{C.}$ □

Solution. **??** First, $\angle ADB = \angle ACB = 36°$. Thus



$$\widehat{AB} = 2\angle ADB = 72° \Rightarrow \angle ABD = \frac{1}{2}\widehat{AD} = \frac{1}{2}(180° - \widehat{AB}) = 54° \Rightarrow \angle ABF = 180° - \angle ABD = \boxed{126°.}$$

The answer is $\boxed{D.}$ □

Solution. **??** By long division $\frac{1}{39} = 0.\overline{025641}$. $2022 = 6 \cdot 337$, the $2022^{\text{th}}-$digit is $\boxed{1.}$

The answer is $\boxed{A.}$ □

Solution. **??** We first give 2 pieces of candy to the oldest kid, then distribute the remaining 11 pieces of candy to the 5 kids. The answer is

$$\binom{11+5-1}{5-1} = \binom{15}{4} = \boxed{1365}.$$

The answer is $\boxed{E.}$ □

Solution. **??** Let Claire's first payment be x, so her second one is $2x$. The present value of the first payment is $\frac{x}{1.075^3}$ and the present value of the second is $\frac{2x}{1.075^6}$. The sum of these present values must equal \$5000, the amount Claire borrows today:

$$\frac{x}{1.075^3} + \frac{2x}{1.075^6} = 5000 \Rightarrow x = \boxed{\$2379.95}.$$

The answer is $\boxed{B.}$ □

Solution. **??** The total surface area of the two boxes is

$$2(4 \cdot 6 + 6 \cdot 8 + 8 \cdot 4) + 2(3 \cdot 5 + 5 \cdot 7 + 7 \cdot 3) = 350.$$

Now we subtract twice the area of overlap to get the final surface area. This is because the area of overlap must be subtracted once from the big box and once from the small box. The largest area of overlap cannot exceed $5 \cdot 7 = 35$, which is the largest area among all six faces of the smaller box. Bridget also will not paint the face that rests on the ground, so she places the first box with one of its two 6×8 faces on the ground. This reduces the amount she must paint by $(6)(8) = 48$. Therefore, she is left with $350 - 2 \cdot 35 - 48 = \boxed{232}$ to paint.

The answer is $\boxed{A.}$ □

Solution. **??** Integers congruent to 7 (mod 11) are the ones that can be written as $11n + 7$ for some integer n. We are counting the values of n such that

$$1000 \leq 11n + 7 \leq 3000 \Rightarrow 91 \leq n \leq 272$$

There are $272 - 91 + 1 = \boxed{182}$ such values of n. □

Solution. **??** Note that

$$\begin{aligned}
&\binom{2n}{n} + \binom{2n}{n-1} = \binom{2n+1}{n} \\
&\binom{2n+1}{n} = \binom{2n+1}{(2n+1)-n} = \binom{2n+1}{n+1} \\
&\Rightarrow 2\binom{2n+1}{n} = \binom{2n+1}{n} + \binom{2n+1}{n+1} = \binom{2n+2}{n+1} \\
&\Rightarrow \binom{2n}{n} + \binom{2n}{n-1} = \frac{1}{2}\binom{2n+2}{n+1}
\end{aligned}$$

□

Solution. **??** By power of the point

$$PA \cdot PB = PC \cdot PD \Rightarrow PC(PC + 7) = 144 \Rightarrow PC = 9.$$

Since $\angle APC = 60°, PC = \frac{AP}{2}$, so PAC is a $30 - 60 - 90$ triangle, where $\angle PCA = 90°$, thus $AC = PC\sqrt{3} = 9\sqrt{3}$. Now, $\angle ACD = 90°$, and AD is the diameter, so

$$AD = \sqrt{AC^2 + CD^2} = \sqrt{243 + 49} = 2\sqrt{73}$$

Thus the area of the circle is $\boxed{73\pi}$. □

20.33 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

20.34 Problems for Level 9

Problem 20.34.1 (ICLT-2022-SM1-5-L9-P1). (*4 points*)

Claire has borrowed \$5000. She will repay the loan entirely by making one payment in 3 years, then another payment in 6 years. The second payment will be exactly double the amount of the first payment. How much is the first payment if the interest rate of the loan is 7.5%, compounded annually?

(A) 2489.05 (B) 2379.95 (C) 2439.75 (D) 2756.65 (E) 3452.55

Problem 20.34.2 (ICLT-2022-SM1-5-L9-P2). (*4 points*)

Bridget places a box that is $4 \times 6 \times 8$ on the floor. She then places a $3 \times 5 \cdot 7$ box on top of the first box, forming a two-box tower. She will then paint all the surfaces she can paint without moving either of the boxes. She wants to paint as little as possible, so she places the boxes in a way that minimizes the amount she'll have to paint.

What is the total area she has to paint?

(A) 232 (B) 218 (C) 206 (D) 192 (E) 178

Problem 20.34.3 (ICLT-2022-SM1-5-L9-P3). (*4 points*)

How many integers between 1000 and 3000 inclusive are congruent to 7 (mod 11)?

(A) 82 (B) 118 (C) 144 (D) 156 (E) 182

Problem 20.34.4 (ICLT-2022-SM1-5-L9-P4). (*6 points*)

I have 6 marbles numbered 1 through 6 in a bag. Suppose I take out two different marbles at random. What is the expected value of the sum of the numbers on the marbles?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Problem 20.34.5 (ICLT-2022-SM1-5-L9-P5). (*6 points*)

Evaluate the sum

$$4 - \frac{8}{3} + \frac{16}{9} - \frac{32}{27} + \cdots$$

(A) $\frac{12}{5}$ (B) $\frac{12}{7}$ (C) $\frac{17}{5}$ (D) $\frac{15}{2}$ (E) $\frac{5}{12}$

Problem 20.34.6 (ICLT-2022-SM1-5-L9-P6). (*6 points*)

Let point X be $(10, 1)$, and let $\mathcal{C}$ be the graph of $(x-1)^2 + (y+2)^2 = 10$.

Let Y be the point on $\mathcal{C}$ that is farthest to X. Find the length XY.

(A) $\sqrt{10}$ (B) $2\sqrt{10}$ (C) $3\sqrt{10}$ (D) $4\sqrt{10}$ (E) $5\sqrt{10}$

Problem 20.34.7 (ICLT-2022-SM1-5-L9-P7). (*10 points*)

Solve the system of linear congruences:

$$\begin{cases} 5x \equiv 1 \pmod 3 \\ 12x \equiv 1 \pmod 7 \end{cases}$$

Problem 20.34.8 (ICLT-2022-SM1-5-L9-P8). (*10 points*)

What is the coefficients of the x^9 term in

$$\left(\frac{x^2}{2} - 3x\right)^7$$

Problem 20.34.9 (ICLT-2022-SM1-5-L9-P9). (*10 points*)

If A and B represent digits of the base ten number $60A5B$, which is a multiple of 72, what are A and B?

20.35 Grading for Level 9

Answers for multiple-choice problems.

Problem 1:	B	2379.95
Problem 2:	A	232
Problem 3:	E	182
Problem 4:	C	7
Problem 5:	A	$\frac{12}{5}$
Problem 6:	D	$4\sqrt{10}$

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: 2 points if can reduce to $x \equiv 2 \pmod 3,\ x \equiv 3 \pmod 7$.

Problem 8: 2 points if can find the generic term $\binom{7}{k}\left(\frac{x^2}{2}\right)^{7-k}(-3x)^k$.

Problem 9: 2 points if can find $A + B + 2 \equiv 0 \pmod 9$, or $4A + 2 + B \equiv 0 \pmod 8$.

20.36 Solutions for Level 9

Solution. **??** Let Claire's first payment be x, so her second one is $2x$. The present value of the first payment is $\frac{x}{1.075^3}$ and the present value of the second is $\frac{2x}{1.075^6}$. The sum of these present values must equal \$5000, the amount Claire borrows today:

$$\frac{x}{1.075^3} + \frac{2x}{1.075^6} = 5000 \Rightarrow x = \boxed{\$2379.95}.$$

The answer is $\boxed{B.}$ □

Solution. **??** The total surface area of the two boxes is

$$2(4 \cdot 6 + 6 \cdot 8 + 8 \cdot 4) + 2(3 \cdot 5 + 5 \cdot 7 + 7 \cdot 3) = 350.$$

Now we subtract twice the area of overlap to get the final surface area. This is because the area of overlap must be subtracted once from the big box and once from the small box. The largest area of overlap cannot exceed $5 \cdot 7 = 35$, which is the largest area among all six faces of the smaller box. Bridget also will not paint the face that rests on the ground, so she places the first box with one of its two 6×8 faces on the ground. This reduces the amount she must paint by $(6)(8) = 48$. Therefore, she is left with $350 - 2 \cdot 35 - 48 = \boxed{232}$ to paint.

The answer is $\boxed{A.}$ □

Solution. **??** Integers congruent to 7 (mod 11) are the ones that can be written as $11n + 7$ for some integer n. We are counting the values of n such that

$$1000 \leq 11n + 7 \leq 3000 \Rightarrow 91 \leq n \leq 272$$

There are $272 - 91 + 1 = \boxed{182}$ such values of n.

The answer is $\boxed{E.}$ □

Solution. **??** There are $\binom{6}{2} = 15$ different pairs of marbles can be drawn, and the expected value of the sum is the average of the sums of each pair. This is

$$\begin{aligned}\frac{1}{15}\big((1+2) + (1+3) + \ldots + (5+6)\big) \\ = \frac{5(1+2+\ldots+6)}{15} = \boxed{7}\end{aligned}$$

The answer is $\boxed{C.}$ □

Solution. **??** First, the common ratio is,

$$\frac{-\frac{8}{3}}{4} = -\frac{2}{3}.$$

Hence,

$$4 - \frac{8}{3} + \frac{16}{9} - \frac{32}{27} + \cdots = \frac{4}{1-\left(-\frac{2}{3}\right)} = \frac{4}{\frac{5}{3}} = \boxed{\frac{12}{5}.}$$

The answer is $\boxed{A.}$ □

Solution. **??** The circle has radius $\sqrt{10}$ and center $(1, -2)$. Let the center be point Z. The distance formula gives us $XZ = 3\sqrt{10}$, so X is outside the circle. The point on the circle that is farthest to the circle is the second intersection of $\overline{XZ}$ with the circle. Let this point be Y. Since $XZ = 3\sqrt{10}$ and $YZ = \sqrt{10}$, we have $XY = XZ + YZ = \boxed{4\sqrt{10}}$.

The answer is $\boxed{D.}$ □

Solution. **??** We begin by solving each of the linear congruences in the system to get

$$\begin{cases} x \equiv 2 \pmod 3 \\ x \equiv 3 \pmod 7 \end{cases}$$

From this system of congruences, we know that for a, b integers,

$$x = 3a + 2 = 7b + 5$$

Looking at $3a + 2 = 7b + 5$ in modulo 3, we have

$$2 \equiv b + 5 \pmod 3 \ \Rightarrow b \equiv 0 \pmod 3.$$

This means that $b = 3c$ for some integer c. Thus

$$x = 7b + 5 = 7(3c) + 5 = 21c + 5.$$

The solutions are $\boxed{x \equiv 5 \pmod{21}.}$ □

Solution. **??** Each term of $\left(\frac{x^2}{2} - 3x\right)^7$ is the form $\binom{7}{k}\left(\frac{x^2}{2}\right)^{7-k}(-3x)^k$. The term containing x^9 is the term with

$$2(7-k) + k = 9 \Rightarrow k = 5$$

That term is

$$\binom{7}{5}\left(\frac{x^2}{2}\right)^2(-3x)^5 = \boxed{-\frac{5103}{4}}x^9.$$

□

Solution. **??** Since $72 = 8 \cdot 9$ and $\gcd(8, 9) = 1$, we use the divisibility rules for 8 and 9 to find the digits A and B. First, using the divisibility rule for 9,

$$60A5B \equiv A + B + 11 \equiv A + B + 2 \equiv 0 \pmod 9.$$

This means that $60A5B$ is a multiple of 9 if and only if $A + B \equiv 7 \pmod 9$, so either $A + B = 7$ or $A + B = 16$.

Now, using the divisibility rule for 8,

$$60A5B \equiv A5B \equiv 100A + 50 + B \equiv 4A + 2 + B \equiv 0 \pmod 8.$$

Since $4A$ is a multiple of 4, $2 + B$ must also be a multiple of 4. So, either $B = 2$ or $B = 6$. Also, since $B \leq 6$, $A + B \leq 15$, meaning $A + B$ is 7, not 16. The only remaining possibilities for (A, B) are $(5, 2)$ and $(1, 6)$. Plugging these in to $4A + 2 + B \equiv 0 \pmod 8$, we find that $\boxed{(5, 2)}$ is the only solution (and 60552 is a multiple of 72). $\square$

Chapter 21

PC MS/HS 2022

21.1 Problems

Problem 21.1.1 (PCT-2022-SM1-MS-P19)**.** Given $a_1, a_2, a_3, \ldots, a_{99}$ is a permutation of $1, 2, 3, \ldots, 99$, find the maximum possible value of

$$|a_1 - 1| + |a_2 - 2| + |a_3 - 3| + \ldots + |a_{99} - 99|.$$

Solution. **??** Note that $|a_k - k|$ is either $a_k - k$ or $k - a_k$. Thus, for each permutation, there is a subset $S \subseteq \{1, 2, 3, \ldots, 99\}$, such that

$$|a_1 - 1| + |a_2 - 2| + |a_3 - 3| + \ldots + |a_{99} - 99| = \left(\sum_{k \in S} a_k - \sum_{k \notin S} a_k\right) + \left(\sum_{k \notin S} k - \sum_{k \in S} k\right)$$

If S contains m elements, the maximum possible value for $\sum_{k \in S} a_k - \sum_{k \notin S} a_k$ is,

$$\sum_{k=100-m}^{99} k - \sum_{k=1}^{99-m} k = \sum_{k=1}^{99} k - 2\left(\sum_{k=1}^{99-m} k\right) = 99 \cdot 50 - (99 - m)(100 - m).$$

Furthermore, if S has m elements, then the complement of S, $\{1, 2, 3, \ldots, 99\} \backslash S$, has $99 - m$ elements, thus for a given value of m, the maximum for the given sum is

$$[99 \cdot 50 - (99 - m)(100 - m)] + [99 \cdot 50 - (m)(m + 1)] = 2m(99 - m).$$

This $2m(99 - m)$ reach its maximum when m is as close as possible to $\frac{99}{2}$, which is 49 or 50. The maximum value is $2 \cdot 49 \cdot 50 = \boxed{\mathbf{4900}.}$

For example, this is achieved when the permutation $a_1, a_2, a_3, \ldots, a_{99}$ *is* $50, 51, \ldots, 99, 1, 2, \ldots, 49$. $\square$

Problem 21.1.2 (PCT-2022-SM1-HS-P30)**.** **??** There is a positive integer s such that there are s solutions to the equation

$$64\sin^2(2x) + \tan^2(x) + \cot^2(x) = 46$$

in the interval $\left[0, \frac{\pi}{2}\right]$ all of the form $\frac{m_k}{n_k}\pi$, where m_k and n_k are relatively prime positive integers, for $k = 1, 2, 3, \ldots, s$. Find $(m_1 + n_1) + (m_2 + n_2) + (m_3 + n_3) + \ldots + (m_s + n_s)$.

Solution. **??**

$$\tan^2(x)+\cot^2(x)+2 = (\tan x + \cot x)^2 = \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)^2 = \left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right)^2 = \left(\frac{1}{\frac{\sin(2x)}{2}}\right)^2 = \frac{4}{\sin^2(2x)}$$

Now, the given equation can be rewritten as $64\sin^2(2x) + \frac{4}{\sin^2(2x)} = 48$, and therefore

$$\begin{aligned}
&\Leftrightarrow 16\sin^2(2x) + \frac{1}{\sin^2(2x)} = 12 \\
&\Leftrightarrow 16\left(\sin^2(2x)\right)^2 - 12\sin^2(2x) + 1 = 0 \\
&\Leftrightarrow \left[16\left(\sin^2(2x)\right)^2 - 8\sin^2(2x) + 1\right] - 4\sin^2(2x) = 0 \\
&\Leftrightarrow (4\sin^2(2x) + 2\sin(2x) - 1)(4\sin^2(2x) - 2\sin(2x) - 1) = 0 \\
&\Leftrightarrow \begin{cases} 4\sin^2(2x) + 2\sin(2x) - 1 = 0, \text{ or} \\ 4\sin^2(2x) - 2\sin(2x) - 1 = 0 \end{cases}
\end{aligned}$$

Case 1: $4\sin^2(2x) + 2\sin(2x) - 1 = 0$. Note that $\cos 2x \neq 0$ (otherwise $x = \frac{\pi}{4}$, thus $3 - 4\cos^2(2x) + 2\sin(2x) = 5 > 0$.) By the identity $\cos(3a) = 4\cos^3(a) - 3\cos(a)$, we obtain that,

$$\begin{aligned}
&4\sin^2(2x) + 2\sin(2x) - 1 = 4(1 - \cos^2(2x)) + 2\sin(2x) - 1 = 3 - 4\cos^2(2x) + 2\sin(2x) \\
&= \frac{3\cos(2x) - 4\cos^3(2x) + 2\sin(2x)\cos(2x)}{\cos(2x)} = \frac{\sin(4x) - \cos(6x)}{\cos 2x} \Rightarrow \sin(4x) = \cos 6x.
\end{aligned}$$

Case 2: $4\sin^2(2x) - 2\sin(2x) - 1 = 0$. Similarly we can obtain that $\sin(4x) = -\cos 6x$.

From both cases, the sum of the two angles $4x$ and $6x$ must be *odd multiple of* $\frac{\pi}{2}$ (make sure that you see this by using the unit circle.) Therefore,

$$10x = 4x + 6x = \frac{(2k+1)\pi}{2}, \ k \in \mathbb{Z}.$$

Since $0 \leq 10x \leq 5\pi$, thus $10x \in \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}\}$, or $x \in \{\frac{\pi}{20}, \frac{3\pi}{20}, \frac{7\pi}{20}, \frac{9\pi}{20}\}$ ($x \neq \frac{\pi}{4}$.)

The answer is $(1 + 20) + (3 + 20) + (7 + 20) + (9 + 20) = \boxed{\mathbf{100}.}$ □