Summer Camp 2023

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Contents

Introduction

Welcome to the MCC Summer Camp 2023!

The program consists 12 weekly session from June 04 until August 20.

Each week you will receive the material for studying. There are separate ones for MS and HS categories. If you are a MS student but feel like ready for more challenge, then read the HS material and compete with HS students! The material would already be included in this online course book, but you might need to read the club's official online book series Learning Problem Solving Vol 1-3 to obtain more knowledge, sharpen your skills, and be ready for competitions. The material would discuss a number of problems, how to solve them, some additional guidance on how to approach problems, building your heuristics, or how to venture further.

Each weekly session is accompanied by a number of problems, given at the end of the examples. For each problem there are a number of points you would be awarded if your submitted solution is correct. You have 4 weeks to work on the problems. Any questions, issues, concerns, can be sent to this email group or to the club organizers, to the COs, or to me as you see appropriate.

Note that submission without any written solutions, e.g. only answers, will be rejected. The better the solution is, the more points it can be awarded. In addition, if you manage to submit multiple solutions, and if they are truly different in nature, methods, approaches, or ideas, you can be awarded up to the maximum number of points again. For three different correct solutions for a 5-point problem, you can get up to 15 points(!). Points are accumulated and ranking will be posted after each grading. Rankings will be done separately for MS and HS, also per grade in each category.

It is an an ongoing one, so you can download for offline reading but revisit every week for updates. Solutions will be posted after the grading is done.

Have a great summer!

Your teacher,

Nghia Doan

Part I

Middle School

Chapter 1

Sums of Sequence

1.1 Examples

Example 1.1.1 (SC-23-MS-1-E1)

The sum of numbers from 1 to n is $\frac{n(n+1)}{2}$

Solution. The mathematician Gauss, sometimes referred as the *Prince of Mathematics*, at the age of 7, showed instead to sum of the numbers with in the sequence $1, 2, \ldots, n$, he could use *two sequences* $1, 2, \ldots, n$ *and* $n, n-1, \ldots, 1$ and pair the terms up to obtain the same sum fir any pair

$$\begin{aligned}
&(1+2+\ldots+n)+(n+(n-1)+\ldots+1)= \\
&\underbrace{(1+n)}_{n+1}+\underbrace{(2+(n-1))}_{n+1}+\ldots+\underbrace{(n+1)}_{n+1}=(n+1)n \\
&\Rightarrow 1+2+\ldots+n=\boxed{\frac{n(n+1)}{2}.}
\end{aligned}$$

□

Example 1.1.2 (SC-23-MS-1-E2)

Find the sum of the first n odd positive integers $1+3+\ldots+(2n-1)$.

Solution. We show two approaches.

The first is to use the *Gauss paring technique*, shown in the previous example,

$$\begin{aligned}
&(1+3+\ldots+(2n-1))+((2n-1)+(2n-3)+\ldots+1) \\
&=\underbrace{(1+(2n-1))}_{2n}+\underbrace{(2+(2n-3))}_{2n}+\ldots+\underbrace{((2n-1)+1)}_{2n}=(2n)n \\
&\Rightarrow 1+3+\ldots+(2n-1)=\frac{2n^2}{2}=\boxed{n^2.}
\end{aligned}$$

The second uses the result of the previous example. *The n^{th} odd number can be written as* $2\cdot n-1$, thus

$$\begin{aligned}
&1+3+\ldots+(2n-1)=(2\cdot 1-1)+(2\cdot 2-1)+\ldots+(2\cdot n-1) \\
&=2(1+2+\ldots+n)-n=2\cdot\frac{n(n+1)}{2}-n=\boxed{n^2}
\end{aligned}$$

□

Example 1.1.3 (SC-23-MS-1-E3)

Divide the number $1, 2, \ldots, 12$ into three groups of four numbers, each with equal sum.

Remark. The term *equal sum* reminds us the *Gauss paring technique* When pairing, we got two pairs with the same sums. Thus if we want a group of four numbers with the same sum we just need two pairs for each group.

Solution. Let use *Gauss paring technique* to divide the twelves numbers into six pairs, each with equal sum.

$$(1, 12),\ (2, 11),\ (3, 10),\ (4, 9),\ (5, 8),\ (6, 7).$$

Then we can divide these six pairs into three groups in whatever way we want, for example

$$(1, 12, 2, 11),\ (3, 10, 4, 9),\ (5, 8, 6, 7).$$

Since the sum of four numbers in any two pairs is equal to $2 \cdot 13 = 26$, thus these are the desired groups. □

Example 1.1.4 (SC-23-MS-1-E4)

Divide the number $1, 2, \ldots, 15$ into four groups of numbers, each with equal sum.

Remark. The *Gauss paring technique* would not work out-of-the-box, but if you have followed closely the discussion in previous examples, you would have seen that the key question is to form the same sum. Thus, we can calculate the sum and then try to find groups of numbers with equal sum.

Solution. Note that $1 + 2 + \ldots + 15 = \frac{15 \cdot 16}{2} = 120$, thus the sum of numbers in a desired group is $\frac{120}{4} = 30$. Now, since the sum is 30, beside the number 15, we can pair up numbers to have the same sum of 15, then any two pairs will have the total sum of 30,

$$\underbrace{15 = 14 + 1}_{1^{\text{st}}} = \underbrace{13 + 2 = 12 + 3}_{2^{\text{nd}}} = \underbrace{11 + 4 = 10 + 5}_{3^{\text{rd}}} = \underbrace{9 + 6 = 8 + 7}_{4^{\text{th}}}.$$

Thus, we can divide the number into four groups of $(15, 14, 1),\ (13, 2, 12, 3),\ (11, 4, 10, 5),\ (9, 6, 8, 7)$. □

Example 1.1.5 (SC-23-MS-1-E5)

Observe the following sequence

$$\underbrace{1}_{1}, \underbrace{2, 2}_{2}, \underbrace{3, 3, 3}_{3}, \underbrace{4, 4, 4, 4}_{4}, \underbrace{5, 5, 5, 5, 5}_{5}, 6, \ldots,$$

where each number appears the same number of times as its value. Find which number occupies the 100^{th} position in this sequence.

Remark. Note that the copies of each the consecutive numbers $1, 2, 3, \ldots$ form *subsequence* or *blocks* of same numbers. The length of a block is the value of a number in the block. Thus, we need to find which block contains the 100^{th} position.

Solution. Let assume the n number occupies the 100^{th} position in the given sequence. Preceding n are $n-1$ blocks of numbers $1, 2, 3, \ldots, n-1$,

$$\underbrace{1}_{1}, \underbrace{2,2}_{2}, \underbrace{3,3,3}_{3}, \ldots, \underbrace{n-1, n-1, \ldots, n-1}_{n-1}$$

altogether $1+2+\ldots+(n-1) = \frac{(n-1)n}{2}$ numbers.

These blocks are followed by a block of n numbers $n, n, \ldots, n$, and somewhere in this block is the 100^{th} position. It means that we need to find an n positive integer such that:

$$\frac{(n-1)n}{2} < 100 \leq \frac{(n-1)n}{2} + n = \frac{n(n+1)}{2}.$$

It is easy to test that $n = \boxed{14}$ is the answer. □

1.2 Problems

Submission deadline: June 28, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 1.2.1 (SC-23-MS-1-P6)**.** (*5 points*) n is an arbitrary positive integer. Find the formula for the sum of the first n even numbers $2+4+\ldots+2n$.

Problem 1.2.2 (SC-23-MS-1-P7)**.** (*5 points*) Divide the number $1, 2, \ldots, 8$ into three groups, not necessarily of the same size, such that the sum in each group is the same.

Problem 1.2.3 (SC-23-MS-1-P8)**.** (*10 points*) The sum of fifteen consecutive integers is 105. Find their product.

Problem 1.2.4 (SC-23-MS-1-P9)**.** (*10 points*) n is an arbitrary positive integer. Prove that $8n+16n+24n+\ldots+8000n$ is one unit apart from a perfect square.

Problem 1.2.5 (SC-23-MS-1-P10)**.** Anna writes the sequence of perfect squares

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$$

Then after the number 1, she then alternately *negates* the terms by *making two terms negative* followed by *leaving two terms positive*. Here's what she's got

$$1, -4, -9, 16, 25, -36, -49, 64, 81, -100, \ldots$$

(*20 points*) What is the sum of the first 2023 terms in this sequence?

1.3 Solutions

Problem 1.3.1 (SC-23-MS-1-P6)**.** (*5 points*) n is an arbitrary positive integer. Find the formula for the sum of the first n even numbers $2 + 4 + \cdots + 2n$.

Solution. $2 + 4 + \cdots + 2n = 2(1 + 2 + \cdots + n) = 2\frac{n(n+1)}{2} = \boxed{n(n + 1).}$ □

Problem 1.3.2 (SC-23-MS-1-P7)**.** (*5 points*) Divide the number $1, 2, \ldots, 8$ into three groups, not necessarily of the same size, such that the sum in each group is the same.

Solution. Since $1 + 2 + \cdots + 8 = \frac{8 \cdot 9}{2} = 36$, so the sum of the numbers in each of the groups is $\frac{36}{3} = 12$. That help to group the numbers, for example $\boxed{\{8, 4\},\ \{7, 5\},\ \{6, 3, 2, 1\}.}$ □

Problem 1.3.3 (SC-23-MS-1-P8)**.** (*10 points*) The sum of fifteen consecutive integers is 105. Find their product.

Solution. Let the fifteen consecutive integers be $n, n + 1, \ldots, n + 14$, their sum is 105, thus

$$105 = n + (n + 1) + \cdots + (n + 14) = 15n + (1 + 2 + \cdots + 14) = 15n + \frac{14 \cdot 15}{2} = 15n + 105 \Rightarrow n = 0$$
$$\Rightarrow n(n + 1) \cdots (n + 14) = \boxed{0}$$

□

Problem 1.3.4 (SC-23-MS-1-P9)**.** (*10 points*) n is an arbitrary positive integer. Prove that $n^2 + 8n^2 + 16n^2 + 24n^2 + \ldots + 8000n^2$ is a perfect square.

> **Remark.** The original text requires to prove
>
> $8n + 16n + 24n + \ldots + 8000n$ is one unit apart from a perfect square.

Solution.

$$n^2 + 8n^2 + 16n^2 + 24n^2 + \ldots + 8000n^2 = n^2 (1 + 8(1 + 2 + 3 + 10000)) = n^2 \left(1 + 8 \cdot \frac{1000 \cdot 10001}{2}\right)$$
$$= 4004001n^2 = \boxed{(2001n)^2.}$$

□

Problem 1.3.5 (SC-23-MS-1-P10)**.** Anna writes the sequence of perfect squares

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$$

Then after the number 1, she then alternately *negates* the terms by *making two terms negative* followed by *leaving two terms positive*. Here's what she's got

$$1, -4, -9, 16, 25, -36, -49, 64, 81, -100, \ldots$$

(*20 points*) What is the sum of the first 2023 terms in this sequence?

Solution. Lets take a group of first four terms $1, -4, -9, 16$. Their sum is:

$$1 - 4 - 9 + 16 = 4.$$

For the next four terms $25, -36, -49, 64$, it is:

$$25 - 36 - 49 + 64 = 4.$$

Now, let generalize: for four consecutive terms $(4n+1)^2, -(4n+2)^2, -(4n+3)^2, (4n+4)^2$, we have:

$$(4n+1)^2-(4n+2)^2-(4n+3)^2+(4n+4)^2 = 16n^2+8n+1-16n^2-16n-4-16n^2-24n-9+16n^2+32n+16 = 4.$$

Thus, by grouping the sum of the first 2023 terms into group of four consecutive terms:

$$(1 - 2^2 - 3^2 + 4^2) + \cdots + (2017^2 - 2018^2 - 2019^2 + 2020^2) + (2021^2 - 2022^2 - 2023^2) = \boxed{-4094552}.$$

$\square$

Chapter 2

Proof by Contradiction

2.1 Examples

Proof by Contradiction: We start by assuming that some fact or statement is true. Next, we demonstrate that the consequences of this assumption lead to inconsistency. Therefore, we can conclude that the original assumption is incorrect. This approach of thinking about a problem is called *proof by contradiction*.

Example 2.1.1 (SC-23-MS-2-E1)

Together, 5 soccer players together scored 14 goals, with every player scoring at least 1. Prove that at least 2 of them scored the same number of goals.

> **Remark.** Let say if a student comes up with a solution like this: *"If the first player scored 1 goal, the second player scored 2, and so on, then the total number of goals would be 15, which is more than 14. Problem solved."*
>
> What's wrong with this? The main issue with this argument is that it deals with *one specific scenario*: 1 goal by the first player, 2 by the second, and so on. What if no one scored exactly 1 goal, or the third player scored 7? **We cannot use one case to prove the entire problem.**

Solution. Proof by contradiction provides a simple framework for a proof. Let's start by **assuming that no two players scored the same number of goals.** If we order the players by their scores, then the first player scored *at least 1*, the second one *at least 2*, the third one *at least 3*, the fourth one *at least 4*, and the fifth one *at least 5*. (Note that we never claim that a player achieved some specific score: we always use the word *at least*.) The players altogether scored at least $1 + 2 + 3 + 4 + 5 = 15$. However, the total score is 14. **This contradiction** proves that *there must have been at least two players with the same score.* ☐

Example 2.1.2 (SC-23-MS-2-E2)

The parliament of a certain country is formed by representatives from 8 provinces. Fifty of these parliamentarians decide to form a committee. Prove that this committee will include 8 people from the same province or people from all 8 provinces.

Solution. To the contrary, assume that we were able to find a group of 50 parliamentarians such that no more than 7 people are from the same province and at least 1 province is not represented. Then, since 1 province is not there, at most 7 provinces are included. At the same time, each province is represented by at most 7 parliamentarians. So, there should be no more than $7 \times 7 = 49$ representatives altogether. This conclusion *contradicts the fact* that the group has 50 people. Therefore, such a committee cannot be formed. ☐

Example 2.1.3 (SC-23-MS-2-E3)

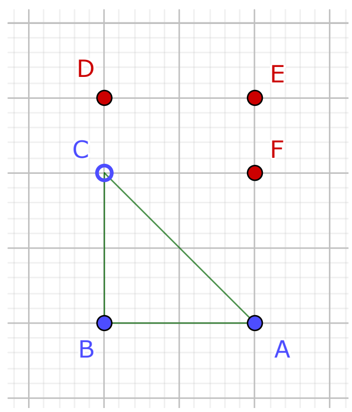
Can you find five odd numbers whose sum is 100?

Solution. Let assume that you can, then after subtracting one of the numbers from 100. We know that the sum of any two odd numbers is an even number, so the sum of any two three numbers is an odd number, thus the sum of any two four numbers is an even number, and finally the sum of any two five numbers is an odd number. This contradicts the fact that 100 is an even number. □

Example 2.1.4 (SC-23-MS-2-E4)

Each node of a square grid is coloured either black or white. Prove that it is possible to find 3 nodes of the same colour that are located at the vertices of a right triangle.

Solution. Let assume the opposite, so no right triangle can have all three vertices with the same colour. Let $\triangle ABC$ be a right triangle such that A and B with the same colour, while C has the other. Since we can always rorate the grid, let $\triangle ABC$ be coloured as shown in the diagram below.



Now, consider points D, E, and F temporarily shown as red nodes. None of them can be the same colour of A (and B). Thus all of them has the same colour and they form a right triangle with three vertices of the same colour. This contracicts the assumption, thus it is possible to find 3 nodes of the same colour that are located at the vertices of a right triangle. □

Example 2.1.5 (SC-23-MS-2-E5)

A closed path is made up of 11 line segments. Can one line, not containing a vertex of the path, intersect each of its segments?

Solution. No, it cannot. Suppose we did have such a line. If we trace the path, each time we intersect the given line we pass from the half-plane on one side of the line to the half-plane on the other side (any line divides a plane into two half-planes) Since the path is closed, we begin and end on the same side of the line. The sides of the line alternate, so the close path would have evenly many vertices.

This contradicts the fact that the close path has 11 segments. □

2.2 Problems

Submission deadline: July 05, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 2.2.1 (SC-23-MS-2-P6)**.** (*5 points*) Twenty-five persons from 8 different provinces elected to the National Congress. Prove that at least 4 of them are from the same province.

Problem 2.2.2 (SC-23-MS-2-P7)**.** (*5 points*) At the graduation event of the School of Wizardry, 5 outstanding students called to the podium. They standing in a row. Altogether, these 5 students know 300 different spells. Prove that there are 2 students standing next to each other who, if combined, know at least 100 spells.

Problem 2.2.3 (SC-23-MS-2-P8)**.** (*10 points*) The only way to travel in the Kingdom-of-so-many-swamps is to use magic carpets. Twenty-one carpet-transportation lines serve the capital. A single carpet-transportation line goes to Tinyville, and every other city is served by exactly 20 carpet-transportation lines. Show that it is possible to travel by magic carpet from the capital to Tinyville (perhaps by transferring from one carpet line to another).

Problem 2.2.4 (SC-23-MS-2-P9)**.** (*10 points*) The game of Trick-a-Troll is played with 10 players and a deck of 20 cards: 2 through 10 and an ace of spades, and 2 through 10 and an ace of clubs. Each player gets 1 club and 1 spade and adds his cards (aces count as 1). Prove that there will be at least 2 players with sums that end in the same digit.

Problem 2.2.5 (SC-23-MS-2-P10)**.** (*20 points*) At each of the vertices of a regular hexagon there stands a grasshopper. At the same time, all six grasshoppers jump off the ground. They land at the same time, each at one of the vertices. No two grasshoppers land at the same vertex. Each of the grasshoppers does not necessarily land at a vertex different from the one it jumps off. Prove that there exist three grasshoppers jump off vertices A, B, and C, and land at A', B' and C', such that $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

2.3 Solutions

Problem 2.3.1 (SC-23-MS-2-P6)**.** (*5 points*) Twenty-five persons from 8 different provinces elected to the National Congress. Prove that at least 4 of them are from the same province.

Solution. Assume that this is not true. Then no 4 of them are from the same province. In this case, not more than 3 persons are from the 1^{st} province, not more than 3 from the 2^{nd}, and so on. Altogether, there are not more than $3 \times 8 = 24$ persons. This contradicts the fact that there are 25 persons. □

Problem 2.3.2 (SC-23-MS-2-P7)**.** (*5 points*) At the graduation event of the School of Wizardry, 5 outstanding students called to the podium. They standing in a row. Altogether, these 5 students know 300 different spells. Prove that there are 2 students standing next to each other who, if combined, know at least 100 spells.

Solution. Suppose that this is not true. Then the 1^{st} and 2^{nd} students together know less than 100 spells, and the 4^{th} and 5^{th} students together know less than 100 spells. Then these four know less than 200 spells together. In this case, the 3^{rd} student knows more than 100 spells. □

Problem 2.3.3 (SC-23-MS-2-P8)**.** (*10 points*) The only way to travel in the Kingdom-of-so-many-swamps is to use magic carpets. Twenty-one carpet-transportation lines serve the capital. A single carpet-transportation line goes to Tinyville, and every other city is served by exactly 20 carpet-transportation lines. Show that it is possible to travel by magic carpet from the capital to Tinyville (perhaps by transferring from one carpet line to another).

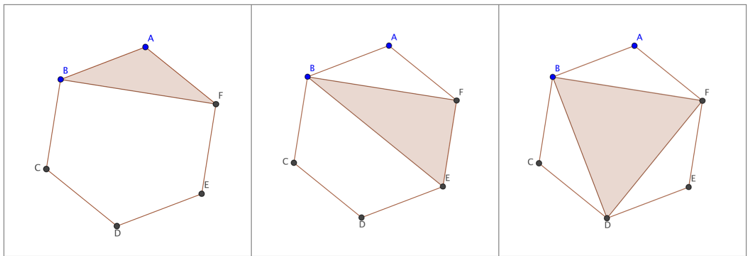
Solution. Let's take a look at all the cities that are accessible by magic carpet from Tinyville. The capital does not belong to this group. In this group there is a number of cities, each has 20 lines leading to it, and one city (Tinyville) with only one line leading to it. Thus the sum of all lines leading to the cities is an odd number. But this is not true. If we count the lines among these cities, then they have to be counted twice, thus it is an even number. This contradiction means Tinyville is connected to the capital. □

Problem 2.3.4 (SC-23-MS-2-P9)**.** (*10 points*) The game of Trick-a-Troll is played with 10 players and a deck of 20 cards: 2 through 10 and an ace of spades, and 2 through 10 and an ace of clubs. Each player gets 1 club and 1 spade and adds his cards (aces count as 1). Prove that there will be at least 2 players with sums that end in the same digit.

Solution. Let's consider the sum of the values of all the cards. If we assume that all players had different last digits for their sums, then all 10 digits would be present, and the sum of them all would end with a 5. On the other hand, the sum of the values of all cards in play ends with a 0, a contradiction. □

Problem 2.3.5 (SC-23-MS-2-P10)**.** (*20 points*) At each of the vertices of a regular hexagon there stands a grasshopper. At the same time, all six grasshoppers jump off the ground. They land at the same time, each at one of the vertices. No two grasshoppers land at the same vertex. Each of the grasshoppers does not necessarily land at a vertex different from the one it jumps off. Prove that there exist three grasshoppers jump off vertices A, B, and C, and land at A', B' and C', such that $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

Solution. First, we find out how many types of triangles there are. See the diagrams below.



Type 1 | Type 2 | Type 3

For Type 1, there are 6 triangles. Type 2, 12 triangles. Type 3, 2 triangles. Each triangle represents a combination of butterflies. There's 20 in total. There's 12 for Type 2, which means after max $6 + 2 = 8$ combinations can change to other types, there are always some that return to the same combination type. $\square$

Chapter 3

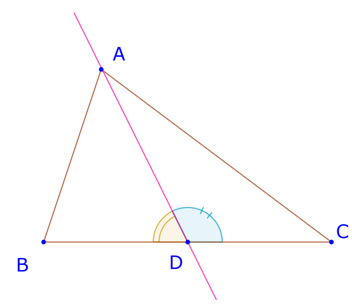
Divide and Conquer

3.1 Examples

Dissection Problems: Partitioning a geometry shape into (usually finite) number of parts that can be assembled into a different shape. Translations, rotations, and sometime reflections, are allowed.

Example 3.1.1 (SC-23-MS-3-E1)

Cut an equilateral triangle into two non-congruent triangles with equal area. *Note: cut means that you need to draw a single line to divide the triangle into two parts.*



Remark. First, **the dissecting line must go through one of the vertices of the triangle**, otherwise one of the two shapes cannot be a triangle. *Why? If the dissecting line is not through any of the vertices of the triangle, then it shall cut two sides of the triangle, divides it into a triangle and a quadrilateral.* If it goes through one of the vertices of the triangle, then it divides the opposite side in two segments.

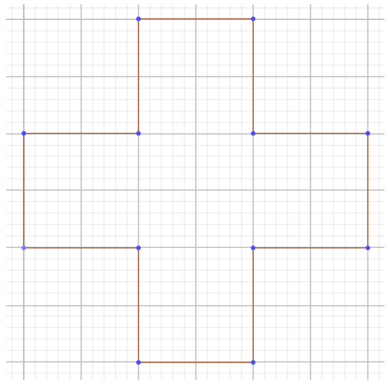
Solution. The dissecting line must go through one of the vertices of the $\triangle ABC$ triangle, let that be A. Assume that the $\triangle ABD$ and $\triangle ACD$ have equal area, then since they have the same height from A to BC, their respective bases BD and CD must be equal, thus D is mid-point of BC. (*)

Furthermore, $\angle ADB = 180° - \angle ADC = \angle ACD + \angle CAD$, which is large than any of $\angle ACD$ and $\angle CAD$. Since $\angle ADB$ must be equal to one of the angles in $\triangle ACD$, thus $\angle ADB = \angle ADC$, or $\angle ADB = \angle ADC = 90°$, thus AD is the altitude of $\triangle ABC$. (**)

This means that if $\triangle ABC$ is equilateral, then from (*) and (**) $\triangle ADB$ and $\triangle ADC$ are congruent, which contradicts to the required condition that the triangle must be divided into two non-congruent triangles. Hence, **it is impossible** cut an equilateral triangle into two non-congruent triangles with equal area. $\square$

Example 3.1.2 (SC-23-MS-3-E2)

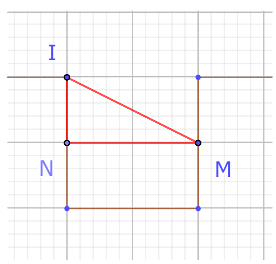
Dissect and assemble the following shape (of five equal squares) into one square.



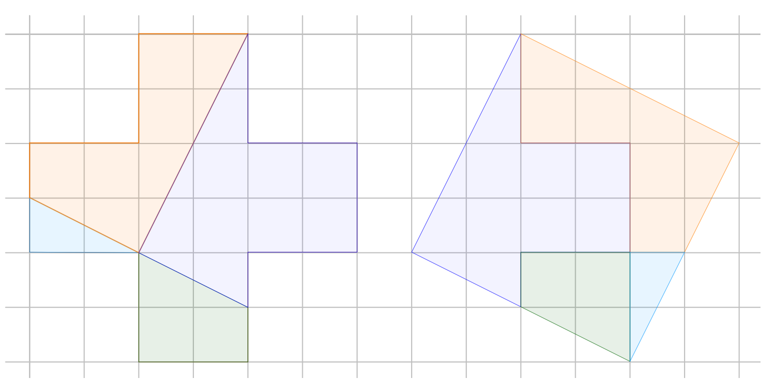
Remark. First, the area of the shape is $5 \times 4 = 20$, thus if it is possible dissect the shape and assemble it into a square, then the side of the square is $\sqrt{20} = 2\sqrt{5}$. How can we make **a segment with length** $\sqrt{5}$? By the Pythagorean theorem a segment with length $\sqrt{5}$ is a hypotenuse of a right triangle with leg lengths 1 and 2:

$$1^2 + 2^2 = 5 = (\sqrt{5})^2$$

The diagram below shows an example of a segment IM with length $\sqrt{5}$.



Solution. The area of the shape is $5 \times 4 = 20$, thus the side length of the desired square is $\sqrt{20} = 2\sqrt{5}$. Using Pythagorean theorem, we can construct a such a segment as a cut on the given shape, then assemble the parts as shown below.

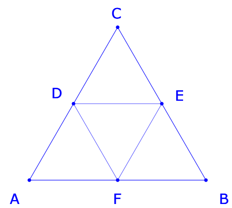


□

Example 3.1.3 (SC-23-MS-3-E3)

Prove that an equilateral triangle can be dissected into n^2 congruent triangles for all positive integer n.

Remark. It is easy to see that if $n = 1$ then the result triangle is the original one. For $n = 2$, we need 4 congruent triangles. This perhaps remind us of triangles made by **drawing mig-segments** (line segments connecting the midpoints of two sides) as shown below.



Solution. Following the idea of *mid-segments*, let divide each of the sides AB, BC, and CA into n equal parts. Connect corresponding lines across the sides (as we would with mid-segments.) By counting from the top, we have the following number of triangles:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

□

Example 3.1.4 (SC-23-MS-3-E4)

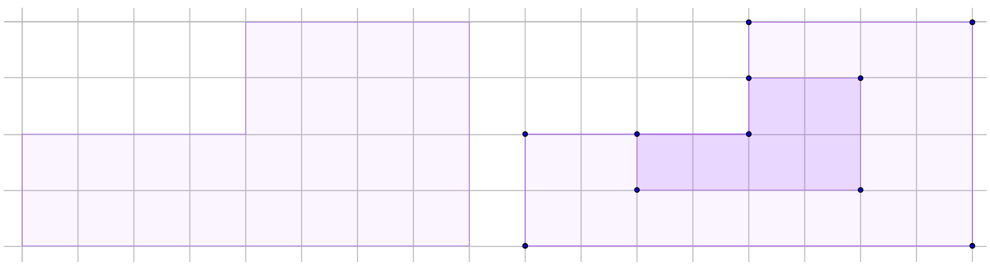
Dissect a regular hexagon into 6 congruent triangles.

Remark. It might be quite easy if **the center of the hexagon** comes up as a point of interest.

Solution. Let $ABCDEF$ be the regular hexagon. Let O be its center, then the triangles $\triangle AOB$, $\triangle BOC$, $\triangle COD$, $\triangle DOE$, $\triangle EOF$, and $\triangle FOA$ are congruent. □

Example 3.1.5 (SC-23-MS-3-E5)

Dissect the figure below on the left (of three congruent rectangles) into four congruent pieces.



Remark. The idea is similar to the problem SC-23-MS-3-E3. How can we create a figure **similar to the original** with **similarity ratio 1:2**? In other words, each side of a new figure must be half of the original one. The figure above on the right is an example, you can easily make it into a solution.

3.2 Problems

Submission deadline: July 12, 2023.

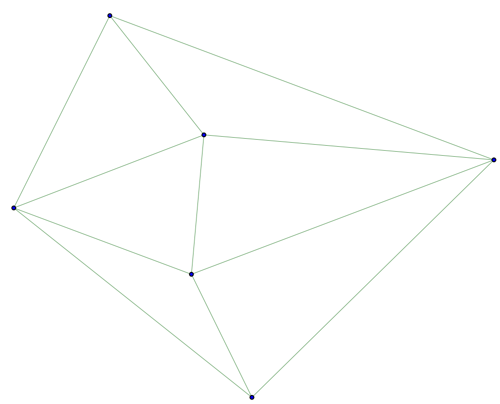
Note that for every problem you earn an additional same number of points for every different solution.

Problem 3.2.1 (SC-23-MS-3-P6)**.** (*5 points*) Divide an equilateral triangle into three triangles that have equal areas and no two among them are congruent.

Problem 3.2.2 (SC-23-MS-3-P7)**.** (*5 points*) Given six points in the plane, we form triangles using the points as vertices. Which of the configurations of these six points will give us the greatest number of triangles?

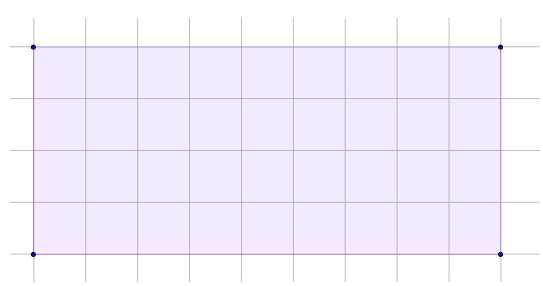
1. Six point as vertices of a hexagon,
2. Five points as vertices of a pentagon and the sixth one is inside the pentagon.
3. Four points as vertices of a quadrilateral and the remaining two points are inside the quadrilateral.
4. Three points as vertices of a triangle and the remaining three points are inside the triangle.

The diagram below show an example for the third configuration.

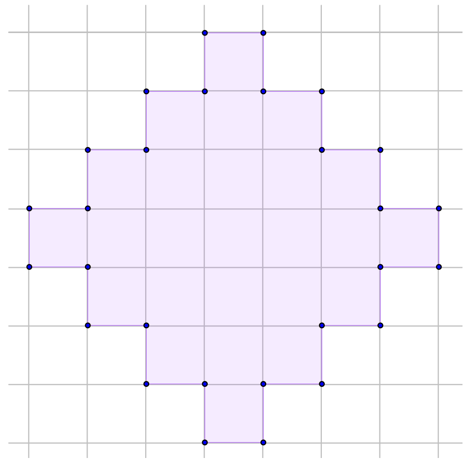


Problem 3.2.3 (SC-23-MS-3-P8)**.** (*10 points*) Dissect a regular hexagon into three rhombuses and assemble them into a parallelogram.

Problem 3.2.4 (SC-23-MS-3-P9)**.** (*10 points*) Dissect a 4×9 rectangle into two pieces which could then be assembled to form a square.



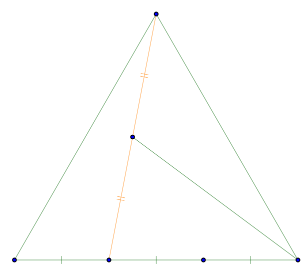
Problem 3.2.5 (SC-23-MS-3-P10)**.** (*20 points*) Dissect and assemble the figure below into an isosceles triangle.



3.3 Solutions

Problem 3.3.1 (SC-23-MS-3-P6)**.** (*5 points*) Divide an equilateral triangle into three triangles that have equal areas and no two among them are congruent.

Solution. Below is one solution. First, we divide the base in three equal segments, then the newly established side into two equal segments.



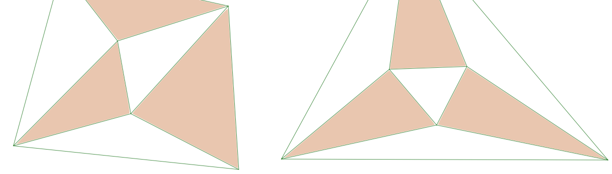
□

Problem 3.3.2 (SC-23-MS-3-P7)**.** (*5 points*) Given six points in the plane, we form triangles using the points as vertices. Which of the configurations of these six points will give us the greatest number of triangles?

1. Six point as vertices of a hexagon,
2. Five points as vertices of a pentagon and the sixth one is inside the pentagon.
3. Four points as vertices of a quadrilateral and the remaining two points are inside the quadrilateral.
4. Three points as vertices of a triangle and the remaining three points are inside the triangle.

The diagram below show an example for the third configuration.

Solution. Below, there is an example with the greatest number of triangles for each of the configuration. It is interesting to see that the numbers of triangles in the configurations are $4, 5, 6$, and 7.

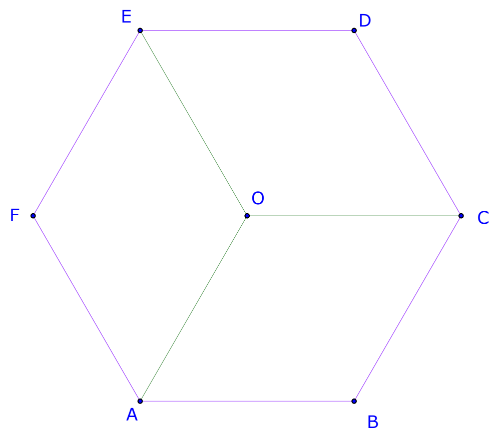


□

Problem 3.3.3 (SC-23-MS-3-P8). (*10 points*) Dissect a regular hexagon into three rhombuses and assemble them into a parallelogram.

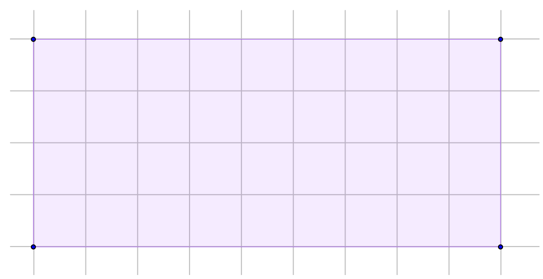
> **Remark.** It is always worth to investigate the role of the *centre* of a *regular* shapes.

Solution. Let O be the centre of $ABCDEF$ regular hexagon. It is easy to see that $OABC$, $OCDE$, and $OEFA$ are rhombuses. It is also easy assemble them into a parallelogram.



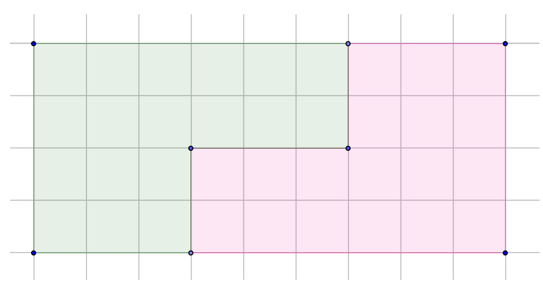
□

Problem 3.3.4 (SC-23-MS-3-P9). (*10 points*) Dissect a 4×9 rectangle into two pieces which could then be assembled to form a square.



> **Remark.** The area of the square is $4 \times 9 = 36$, thus it must have side length of 6.

Solution. Below is one solution.



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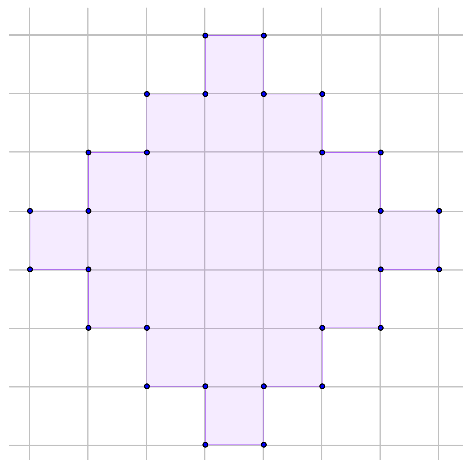
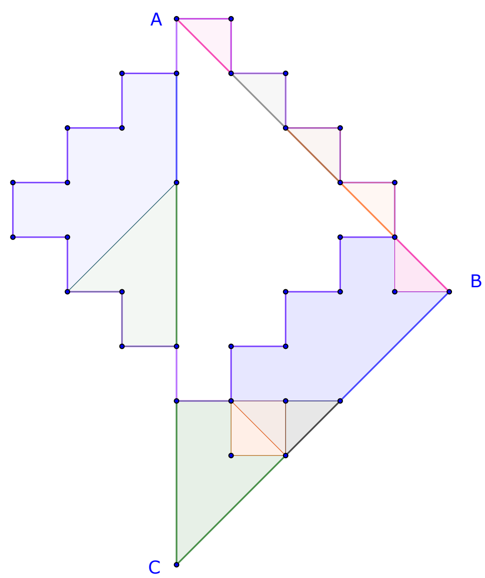
Problem 3.3.5 (SC-23-MS-3-P10). (*20 points*) Dissect and assemble the figure below into an isosceles triangle.



> **Remark.** The area of the isosceles triangle is 25, thus it gives a hint to construct a triangle with base 10 and height 5.

Solution. Below is one of the solutions.



□

Chapter 4

Egyptian Fractions

4.1 Examples

The Egyptian fractions were particularly useful when dividing a number of objects equally for more number of people. This was practically important because many of the Egyptian structures required massive labor work. Any uneven distribution of food ration among the labors could easily kindle dispute and disrupt their work process. For example, if the Egyptian boss were to give 5 *Kubz* (Arabic pancake-like bread) rations for 5 workmen, and then he just realized there was an extra man in the site, how would he divide 5 Kubz among 6 men? The answer is, as we know, $\frac{5}{6}$. Using common sense, we would simply *divide each Kubz into 6 equal portions, resulting in 30 pieces, and each workman would happily pick 5 of them.* Surely, the problem was solved, but the Egyptian mathematician would apply a more elegant way to divide the Kubz by following the Egyptian fractions:

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$$

This way, *each workman would get half a Kubz plus one-third of a Kubz.* To do so, first we take 3 Kubz and divide each in halves, resulting in 6 equal half portions, and each man would obtain each half. Then we take the remaining 2 Kubz and divide each into 3 equal portions, resulting in 6 equal one-third portions, and again each man would get one. By using this method, only a few cuts are needed for each Kubz without any need to go through all the same 6-denominator!

Definition. An Egyptian fraction is the sum of finitely many rational numbers, each of which can be expressed in the form of *unit fraction*, or $\frac{1}{q}$, where q is a positive integer. For example, the Egyptian fraction $\frac{61}{66}$ can be written as

$$\frac{61}{66} = \frac{1}{2} + \frac{1}{3} + \frac{1}{11}.$$

A simple algorithm to understand for finding the Egyptian fraction representation of a number is the as below:

Algorithm — Takes a fraction $\frac{a}{b}$ and continues to subtract off the largest fraction $\frac{1}{n}$ until he/she is left only with a set of Egyptian fractions.

For example, lets find the Egyptian fraction representation of $\frac{8}{9}$,

- The greatest unit fraction less than $\frac{8}{9}$ is $\frac{1}{2}$ $(\frac{1}{2} < \frac{8}{9} < \frac{1}{1})$, $\frac{8}{9} - \frac{1}{2} = \frac{7}{18}$,
- The greatest unit fraction less than $\frac{7}{18}$ is $\frac{1}{2}$ $(\frac{1}{3} < \frac{7}{18} < \frac{1}{2})$, $\frac{7}{18} - \frac{1}{3} = \frac{1}{18}$.
- $\frac{1}{18}$ is a unit fraction, thus the algorithm stops.

Therefore $\frac{8}{9} = \frac{1}{2} + \frac{1}{3}\frac{1}{18}$.

In 1202, Leonardo of Pisa (better known as Fibonacci!) showed that every positive rational number is an Egyptian fraction.

Theorem

For r rational number ($r \in \mathbb{Q}$), there exist $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$ positive integers such that:

$$r = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}(*)$$

Remark. Two important notes for the theorem above:

- the sequence $a_1 < a_2 < \cdots < a_k$ is strictly increasing, so for any two numbers $a_i \neq a_j$!.
- r can be any rational number, thus every positive integer is an Egyptian fraction!

Example 4.1.1 (SC-23-MS-4-E1)

Find four positive integers a, b, c, and d so that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{20}{21}.$$

Remark. A straight forward application of the can help.

Solution.

$$\begin{gathered}
\frac{1}{2} < \frac{20}{21} < \frac{1}{1}, \ \frac{20}{21} - \frac{1}{2} = \frac{19}{42} \\
\frac{1}{3} < \frac{19}{42} < \frac{1}{2}, \ \frac{19}{42} - \frac{1}{3} = \frac{5}{42} \\
\frac{1}{9} < \frac{5}{42} < \frac{1}{8}, \ \frac{5}{42} - \frac{1}{9} = \frac{1}{126} \\
\Rightarrow \frac{20}{21} = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{126}
\end{gathered}$$

□

Example 4.1.2 (SC-23-MS-4-E2)

Find positive integers a and b, $(a < b)$ if
$$\frac{1}{3} = \frac{1}{a} + \frac{1}{b}$$

Remark. First, we find the Egyptian fraction representation of $\frac{1}{3}$, hopefully it is a sum of two unit fractions. Then, since a number can be represented by a sum of two unit fractions in only one way (why?) then a and b are the denominators of found fractions.

Solution. First, $\frac{1}{4} < \frac{1}{3} < \frac{1}{2}$, and $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, thus
$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12} \Rightarrow a = 4, b = 12.$$
□

Example 4.1.3 (SC-23-MS-4-E3)

Positive a, b, and c satisfy that:

- $1 < a < b < c$,
- Any two of a, b, and c are relatively primes (or co-prime), meaning that the pair have no other common divisor than 1, and
- $abc = 10(a + b + c)$.

What is the minimum value of $ab + bc + ca$?

Solution. By cross division the given equality $abc = 10(a + b + c)$:
$$\frac{1}{10} = \frac{a+b+c}{abc} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}.$$

Now, let try to find the Egyptian fraction representation of $\frac{1}{10}$, note that since $1 < a < b < c$, so no unit fraction with a prime denominator can be accepted.
$$\begin{cases} a \geq 6 \Rightarrow \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \leq \frac{3}{36} = \frac{1}{12} < \frac{1}{10} \\ a = 5, a < b \leq c \Rightarrow \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \leq \frac{2}{30} + \frac{1}{36} < \frac{1}{10} \\ a = 5 = b = c \Rightarrow \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \leq \frac{3}{25} \neq \frac{1}{10} \end{cases}$$

Thus a one of the positive integers of $\{2, 3, 4\}$.

Case 1: $a = 2$, then
$$2bc = 10(2 + b + c) \Rightarrow (b - 5)(c - 5) = 35 \Rightarrow (b, c) \in \{(6, 40), (10, 12)\}.$$

Case 2: $a = 3$, then
$$3bc = 10(3 + b + c) \Rightarrow (3b - 10)(3c - 10) = 190 \Rightarrow (b, c) \in \{(4, 35), (5, 16)\}.$$

Case 3: $a = 4$, then
$$4bc = 10(4 + b + c) \Rightarrow (2b - 5)(2c - 5) = 65 \Rightarrow (b, c) \in \{(5, 9)\}.$$

By direct testing, the minimal value of $ab + bc + ca = \boxed{101}$, where $(a, b, c) = (4, 5, 9)$. □

Definition. n is called **beautiful number** if it can be written as a sum of some positive integers

$$n = a_1 + a_2 + \cdots + a_k,$$

where the sequence $a_1 \leq a_2 le \cdots \leq a_k$ is non-strictly increasing, so for there would be two same numbers $a_i = a_j$!.

such that

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} (**)$$

Example 4.1.4 (SC-23-MS-4-E4)

Any perfect square is a beautiful number.

Solution. It is quite easy to see that

$$n^2 = \underbrace{n + n + \cdots + n}_{n \text{ times}}, \; \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ times}} = 1$$

□

Example 4.1.5 (SC-23-MS-4-E5)

Prove that if n is a beautiful number, then $2n + 2$ too.

Solution. Let $n = a_1 + a_2 + \cdots + a_k$, where

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}$$

Then $2n + 2 = 2a_1 + 2a_2 + \cdots + 2a_k + 2$, and

$$\frac{1}{2a_1} + \frac{1}{2a_2} + \cdots + \frac{1}{2a_k} + \frac{1}{2} = \frac{1}{2}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}\right) + \frac{1}{2} = 1.$$

□

4.2 Problems

Submission deadline: July 19, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 4.2.1 (SC-23-MS-4-P6)**.** (*5 points*) Show that $10, 20, 22, 24$, and 34 are beautiful numbers.

Problem 4.2.2 (SC-23-MS-4-P7)**.** (*5 points*) Show that if n is a beautiful number, then $2n+8$ and $3n+8$ are beautiful numbers, too.

Problem 4.2.3 (SC-23-MS-4-P8)**.** (*10 points*) For $k = 2, 3, 4$, find all possible ways to represent $\frac{1}{k}$ as a sum of two unit fractions, that is

$$\frac{1}{k} = \frac{1}{a} + \frac{1}{b},\ a \leq b$$

Problem 4.2.4 (SC-23-MS-4-P9)**.** (*10 points*) Silverster's sequence $s_1, s_2, \ldots$ is defined as below:

$$s_1 = 2, s_2 = s_1 + 1 = 3, s_3 = s_1 s_2 + 1 = 7, \text{ and in general } s_n = s_1 s_2 \cdots s_{n-1} + 1.$$

For any $n \geq 1$, find the value of the sum:

$$\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_n} + \frac{1}{s_{n+1} - 1}.$$

Problem 4.2.5 (SC-23-MS-4-P10)**.** (*20 points*) For which positive integer n, there exist n perfect squares $a_1^2, a_2^2, \ldots, a_n^2$, such that

$$1 = \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \quad (***)$$

> **Remark.** It would be adequate for each value of n to show an example for n perfect squares satisfying (***). For example
>
> $n = 1$, then $1 = \frac{1}{1^2}$.
>
> $n = 4$, then $1 = \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2}$.
>
> However, the proof requires to prove or disprove for all possible positive integers n.

4.3 Solutions

Problem 4.3.1 (SC-23-MS-4-P6). (*5 points*) Show that $10, 20, 22, 24$, and 34 are beautiful numbers.

Solution. 4 is a beautiful number, since

$$4 = 2 + 2,\ 1 = \frac{1}{2} + \frac{1}{2}.$$

By SC-23-MS-4-E5, $2 * 4 + 2 = \boxed{10}$ is a beautiful number. By following the proof of SC-23-MS-4-E5,

$$1 = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}.\right) + \frac{1}{2} = \frac{1}{4} + \frac{1}{4} + \frac{1}{2},\ \text{or} 10 = 4 + 4 + 2,\ 1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{2}$$

$$20 = 6 + 6 + 6 + 2, 1 = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + 2 \Rightarrow \boxed{20} \text{ is a beautiful number.}$$

Since 10 is a beautiful number, thus $2 \times 10 + 2 = \boxed{22}$ is.

$$22 = 8 + 8 + 4 + 2,\ 1 = \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2}$$

$$24 = 2 + 4 + 6 + 12, 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} \Rightarrow \boxed{24} \text{ is a beautiful number.}$$

16 is a beautiful number, because $16 = 4 + 4 + 4 + 4, 1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$. Thus,

$$34 = 8 + 8 + 8 + 8 + 2, 1 = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{2} \Rightarrow \boxed{34} \text{ is a beautiful number.}$$

□

Problem 4.3.2 (SC-23-MS-4-P7). (*5 points*) Show that if n is a beautiful number, then $2n + 8$ and $3n + 8$ are beautiful numbers, too.

Solution. Let $n = a_1 + a_2 + \cdots + a_k$, where

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}$$

Then $\boxed{2n + 8} = 2a_1 + 2a_2 + \cdots + 2a_k + 4 + 4$ is beautiful, since

$$\frac{1}{2a_1} + \frac{1}{2a_2} + \cdots + \frac{1}{2a_k} + \frac{1}{4} + \frac{4}{=2}\frac{1}{2}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}\right) + \frac{1}{4} + \frac{1}{4} = 1.$$

Ans $\boxed{3n + 8} = 3a_1 + 3a_2 + \cdots + 3a_k + 6 + 2$ is beautiful, since

$$\frac{1}{3a_1} + \frac{1}{3a_2} + \cdots + \frac{1}{3a_k} + \frac{1}{6} + \frac{2}{=3}\frac{1}{3}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}\right) + \frac{1}{6} + \frac{1}{2} = 1.$$

□

Problem 4.3.3 (SC-23-MS-4-P8). (*10 points*) For $k = 2, 3, 4$, find all possible ways to represent $\frac{1}{k}$ as a sum of two unit fractions, that is

$$\frac{1}{k} = \frac{1}{a} + \frac{1}{b},\ a \leq b$$

Solution. Lets casework by k.

Case 1: $k = 2$.

$$\frac{1}{2} = \frac{1}{a} + \frac{1}{b} \Rightarrow ab = 2(a+b) \Rightarrow (a-2)(b-2) = 4 \Rightarrow \begin{cases} a-2=4, b-2=1 \\ a-2=2, b-2=2 \end{cases}$$
$$\Rightarrow (a,b) \in \{(6,3),(4,4).\}$$

Case 1: $k = 3$.

$$\frac{1}{3} = \frac{1}{a} + \frac{1}{b} \Rightarrow ab = 3(a+b) \Rightarrow (a-3)(b-3) = 9 \Rightarrow \begin{cases} a-3=9, b-3=1 \\ a-3=3, b-3=3 \end{cases}$$
$$\Rightarrow (a,b) \in \{(12,4),(6,6).\}$$

Case 1: $k = 4$.

$$\frac{1}{4} = \frac{1}{a} + \frac{1}{b} \Rightarrow ab = 4(a+b) \Rightarrow (a-4)(b-4) = 16 \Rightarrow \begin{cases} a-4=16, b-4=1 \\ a-4=8, b-4=2 \\ a-4=4, b-4=4 \end{cases}$$
$$\Rightarrow (a,b) \in \{(20,5),(12,6),(8,8).\}$$

□

Problem 4.3.4 (SC-23-MS-4-P9)**.** (*10 points*) Silverster's sequence $s_1, s_2, \dots$ is defined as below:

$$s_1 = 2, s_2 = s_1 + 1 = 3, s_3 = s_1 s_2 + 1 = 7, \text{ and in general } s_n = s_1 s_2 \cdot s_{n-1} + 1.$$

For any $n \geq 1$, find the value of the sum:

$$\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_n} + \frac{1}{s_{n+1} - 1}.$$

Solution. Note that for $k \geq 2$,

$$\frac{1}{s_k - 1} - \frac{1}{s_{k+1} - 1} = \frac{1}{s_1 s_2 \cdots s_{k-1}} - \frac{1}{s_1 s_2 \cdots s_k} = \frac{1}{s_1 s_2 \cdots s_{k-1}} \left(1 - \frac{1}{s_k}\right) = \frac{1}{s_1 s_2 \cdots s_{k-1}} \frac{s_1 s_2 \cdots s_{k-1}}{s_k} = \frac{1}{s_k}$$

Thus
$$\begin{aligned} & \frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_n} + \frac{1}{s_{n+1} - 1} \\ =& \frac{1}{s_1} + \left(\frac{1}{s_2 - 1} - \frac{1}{s_3 - 1}\right) + + \cdots + \left(\frac{1}{s_n - 1} - \frac{1}{s_{n+1} - 1}\right) + \frac{1}{s_{n+1} - 1} \\ =& \frac{1}{s_1} + \frac{1}{s_2 - 1} = \boxed{1} \end{aligned}$$

□

Problem 4.3.5 (SC-23-MS-4-P10)**.** (*20 points*) For which positive integer n, there exist n perfect squares $a_1^2, a_2^2, \dots, a_n^2$, such that

$$1 = \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \quad (***)$$

Solution. Case 1: For $n = 1$ the equation has one solution $x_1 = 1$.

Case 2: For $n = 2$ we have $1 = \frac{1}{x_1^2} + \frac{1}{x_2^2}$.

If, without loss of generality, $x_1 \geq x_2$, then $1 = \frac{1}{x_1^2} + \frac{1}{x_2^2} \leq \frac{2}{x_2^2}$, which gives $x_2^2 \leq 2$ and so $x_2^2 = 1$. But then $0 = \frac{1}{x_1^2}$, a contradiction.

Case 3: For $n = 3$ we have $1 = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2}$.

If, without loss of generality $x_1 \geq x_2 \geq x_3$, we have $1 = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} \leq \frac{3}{x_3^2}$ which gives $x_3^2 \leq 3$ and so $x_3 = 1$. The equation does not admit a solution in this case.

Case 4: For $n = 4$ the only solution is $x_1 = x_2 = x_3 = x_4 = 2$.

Case 5: For $n = 5$, if we suppose, without loss of generality, that $x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5$, we have

$$1 = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2} + \frac{1}{x_5^2} \leq \frac{5}{x_5^2}$$

If $x_5 = 1$, the equation has no solution.

If $x_5 = 2$, we have $\frac{3}{4} = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2} \leq \frac{4}{x_4^2}$, which gives $x_4^2 \leq \frac{16}{3}$ so $x_4 = 1$ or $x_4 = 2$.

If $x_4 = 1$ the equation has no solution, because $x_4 \geq 2$.

If $x_4 = 2$ we have $\frac{1}{2} = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2}$.

Since $x_1 \geq x_2 \geq x_3$, we have $\frac{1}{2} \leq \frac{3}{x_3^2}$ which leads to $x_3 \leq 3$, so $x_3 = 1$ or $x_3 = 2$.

If $x_3 = 1$ the equation has no solution, because $x_3 \geq 2$.

If $x_3 = 2$ we have $\frac{1}{4} = \frac{1}{x_1^2} + \frac{1}{x_2^2}$.

Since $x_1 \geq 2$ and $x_2 \geq 2$, the equation has no solution.

Case 6: For $n = 6$ the equation has one solution $x_1 = x_2 = x_3 = 2; x_4 = x_5 = 6; x_6 = 6$ because

$$\frac{3}{4} = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2}, \ \frac{1}{4} = \frac{1}{x_4^2} + \frac{1}{x_5^2} + \frac{1}{x_6^2}.$$

Case 7: For $n = 7$ the equation has one solution $x_1 = x_2 = x_3 = 2; x_4 = x_5 = x_6 = x_7 = 4$.

Case 7: For $n = 8$ the equation has one solution $x_1 = x_2 = x_3 = 2; x_4 = x_5 = 3; x_6 = 7; x_7 = 14; x_8 = 21$.

Now, let $A_n = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_{n-1}^2} + \frac{1}{x_n^2} = 1$ and we show by induction the following statement

$$(P_n): \text{ if } A_n = 1 \text{ has a solution then } A_{n+3} = 1 \text{ also has a solution }, \ n \geq 6.$$

It is clear that P_6 is true. If $A_n = 1$ for a certain $n \geq 6$, let $(y_1, y_2, \dots, y_n)$ be a solution of $A_n = 1$ and let

$$y_1 = x_1, y_2 = x_2, \dots, y_{n-1} = x_{n-1}, x_n = x_{n+1} = x_{n+2} = x_{n+3} = 2y_n.$$

Then

$$\begin{aligned}
\frac{1}{x_1^2}+\frac{1}{x_2^2}+\cdots+\frac{1}{x_{n+2}^2}+\frac{1}{x_{n+3}^2} &= \frac{1}{y_1^2}+\frac{1}{y_2^2}+\cdots+\frac{1}{y_{n-1}^2}+\frac{1}{(2y_n)^2}+\frac{1}{(2y_n)^2}+\frac{1}{(2y_n)^2}+\frac{1}{(2y_n)^2}\\
&= \frac{1}{y_1^2}+\frac{1}{y_2^2}+\cdots+\frac{1}{y_{n-1}^2}+\frac{1}{4y_n^2}+\frac{1}{4y_n^2}+\frac{1}{4y_n^2}+\frac{1}{4y_n^2}\\
&= \frac{1}{y_1^2}+\cdots+\frac{1}{y_{n-1}^2}+\frac{4}{4y_n^2}\\
&= \frac{1}{y_1^2}+\frac{1}{y_2^2}+\cdots+\frac{1}{y_{n-1}^2}+\frac{1}{y_n^2}\\
&= 1.
\end{aligned}$$

Therefore the equation $A_{n+3}=1$ has a solution, and this completes the induction.

For $\boxed{n=1, n=4 \text{ and } n\geq 6}$ natural numbers the equation $\frac{1}{x_1^2}+\frac{1}{x_2^2}+\cdots+\frac{1}{x_{n-1}^2}+\frac{1}{x_n^2}=1$ has a solution. $\square$

Chapter 5

Binaries

5.1 Examples

In the far-far-away Kingdom of Powers of 2, when people trade of buying and selling, they use only special weights for weighing goods on their two-pan balance scales. Those weights are in powers of two: $1, 2, 4, 8, 16, 32$, and so on. It is interesting that they can weight any goods with an integral amount, for example between 1 and 7,

- Weighing 2 kilograms? We can use a 2-kg weight. - Weighing 3 kilograms? We can use 2-kg and 1-kg weights. - Weighing 4 kilograms? We can use a 4-kg weight. - Weighing 5 kilograms? We can use 4-kg and 1-kg weights. - Weighing 6 kilograms? We can use 4-kg and 2-kg weights. - Weighing 7 kilograms? We can use 4-kg, 2-kg, and 1-pound weights.

Or as shown on the left of the equations as below:

$$\begin{cases} 1 = 1 \\ 2 = 2 \\ 3 = 1 + 2 \\ 4 = 4 \\ 5 = 1 + 4 \\ 6 = 2 + 4 \\ 7 = 1 + 2 + 4 \end{cases}$$

Will this pattern work for bigger numbers? Would weights from 1 to 64 add up to 127? Would weights from 1 to 128 add up to 255? Etc.? This pattern is indeed true, any integer quantity can be balanced with some combination of these weights.

Writing an integer as a sum of powers of 2 in a simple manner which is called **binary representation** of the number. First, we express a decimal number as a combination of distinct powers of 2. Next, we write this combination down using two symbols 0 and 1. Symbols 1 are used to mark the powers that go into the number, symbols 0—those that are not. This way, we get the corresponding binary number. For example

$$9 = 1 + 8 = 2^0 + 2^3 = \underline{1}00\underline{1}$$

And of course we can convert the binary representation back to decimal, for example:

$$10101 = \underline{1}0\underline{1}0\underline{1} = 2^4 + 2^2 + 2^0 = 16 + 4 + 1 = 21.$$

Now, suppose that a teacher writes the number 101 on the board. Being familiar with binary numbers, we can interpret 101 in 2 different ways. It is a *regular* base−10 number or as a binary number, which would be equal to 5. How do we know which is right? To get rid of this ambiguity, mathematicians came up with a special notation: a *subscript* that indicates the base of a number: for example:

$$101_2 \text{ and } 11111_2 \text{ are binary numbers.}$$

For other bases, 101_3 would mean a base−3 number; 1014 would mean a base−4 number; etc. Our regular base−10 numbers can be marked with subscript 10 as well. However, by default base−10 subscript can be omitted: $101 = 101_{10}$.

Example 5.1.1 (SC-23-MS-5-E1)

Which binary number is 1 bigger than 100000000_2? Which is 1 smaller? Which binary number is 1 smaller than 1111111111_2? Which is 2 smaller? Which is 2 bigger?

Solution. $100000000_2 = 2^8$, the number that is one bigger is $2^8 + 1 = \boxed{100000001_2.}$

Now, remember that

$$\left.\begin{aligned} 2^0 + 2^0 &= 2^1 \\ 2^0 + 2^0 + 2^1 &= 2^2 \\ 2^0 + 2^0 + 2^1 + 2^2 &= 2^3 \\ &\cdots \\ 2^0 + 2^0 + 2^1 + \cdots + 2^7 &= 2^8 \end{aligned}\right\} \Rightarrow 2^8 - 1 = 2^8 - 2^0 = 2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = \boxed{11111111_2}$$

The binary number 1 smaller than 1111111111_2, by subtracting 1 from the last digit:

$$1111111111_2 - 1 = \boxed{1111111110_2.}$$

Therefore, the binary number 2 smaller than 1111111111_2 is the result of subtracting 1 from the last digit of 1111111110_2 with a carry over:

$$1111111110_2 - 1 = \boxed{1111111101_2.}$$

For the binary number that is 1 bigger than 1111111111_2:

$$\begin{aligned} 1111111111_2 &= 2^9 + 2^8 + \cdots + 2^0 \\ 1111111111_2 + 1 &= 2^9 + 2^8 + \cdots + 2^0 + 1 = 2^{10} = 10000000000_2. \end{aligned}$$

The binary number that is 1 bigger than 10000000000_2 is $\boxed{10000000001_2.}$ □

Example 5.1.2 (SC-23-MS-5-E2)

Julie the Gold Smith has 30 golden nuggets worth $1, 2, \ldots, 30$ dollars. Which 5 of these nuggets should she take with her if she wants to be able to pay any integer amount ranging from 1 dollar to 30 dollars?

Solution. These are the powers of 2, if she take $\boxed{1, 2, 4, 8, \text{ and } 16.}$ Since $1 + 2 + 4 + 8 + 16 = 31$, she should be able to pay any amount between 1 and 30. □

Example 5.1.3 (SC-23-MS-5-E3)

Albert has a sack of flour, a 1−gram weight, and a box of light plastic bags. Can he measure 1 kg (1000 grams) of flour in 10 weighings on a balance scale?

Solution. Albert performs the following steps, by imitating the way he can construct 2^n from a sum of smaller powers of 2:

$$2^n = 2^{n-1} + \cdots + 2^1 + 2^0 + 2^0.$$

Step 1: He measure 1 gram of flour using the 1−gram weight.

Step 2: Using 1 gram of flour and the 1−gram weight, he can measure 2 grams of flour.

Step 3: Using 2 grams and 1 gram of flour and the 1−gram weight, he can measure 4 (2^2) grams of flour.

Step 4: Using $4, 2$, and 1 grams of flour and the 1−gram weight, he can measure 8 (2^3) grams.

...

Step 10: Using all measures of flour he already has and the 1-gram weight, he can measure 512 (2^9) grams.

Step 11: This steps does not involve using the scale. He chooses a combination of weights from $1, 2, 4, 8, 16, 32, 64, 128, 256$, and 512 that adds up to 1000. These are $\boxed{512, 256, 128, 64, 32, 8.}$ ☐

Example 5.1.4 (SC-23-MS-5-E4)

An eight-bit binary word is a sequence of eight digits each of which is either 0 or 1. Find the number of eight-bit binary words.

Solution. An eight-bit binary word is a nonnegative integer with fewer than 9 digits when expressed in binary. This means we are counting the integers from 0 to $2^8 - 1$ inclusive, of which there are $2^8 = \boxed{256}$. We could also view each digit as a choice. There are 2 choices for each of 8 digits, so 2^8 eight-bit binary words are possible. ☐

Example 5.1.5 (SC-23-MS-5-E5)

Find the largest power of 2 that divides 694!.

Solution. We could solve this problem by finding the prime factorization of 694!. However, we are only interested in the powers of 2, so we hope to find a more direct solution. Since 694! is the product of the 694 smallest positive integers, we can count the powers of 2 in the prime factorizations of each of those positive integers to determine the power of 2 in the prime factorization of 694!.

Let's try counting the powers of 2 in an organized way. We first search among the smallest 30 positive integers for those that include at least 1 power of 2 in their prime factorizations - the even integers. Every second positive integer includes at least 1 power of 2 in its prime factorization. Furthermore, every fourth positive integer includes at least 2 powers of 2 in its prime factorization. Each eighth positive integer includes at least 3 powers, and so on. This leads us to dividing 694! up into parts based on the power of 2 that each part contributes to the final prime factorization.

$$\frac{694}{2} = 347,\ \frac{347}{2} = 173.5,\ \frac{173}{2} = 86.5,\ \frac{86}{2} = 43,\ \frac{43}{2} = 21.5,\ \frac{21}{2} = 10.5,\ \frac{10}{2} = 5,\ \frac{5}{2} = 2.5,\ \frac{2}{2} = 1$$
$$347 + 173 + 86 + 43 + 21 + 10 + 5 + 2 + 1 = \boxed{688.}$$

☐

5.2 Problems

Submission deadline: July 26, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 5.2.1 (SC-23-MS-5-P6)**.** (*5 points*) Write these binary numbers as decimals:

$$\overline{110011}_2, \overline{10000000}_2, \overline{1000101}_2, \overline{111111}_2, \overline{111110}_2.$$

Problem 5.2.2 (SC-23-MS-5-P7)**.** Each of the questions below can be solved independently.

1. (*5 points*) How can one tell if a binary number is even or odd?
2. (*5 points*) How can one tell if a binary number is divisible by 4?
3. (*5 points*) If a binary number ends with 10, what is the remainder when this number is divided by 4?

Problem 5.2.3 (SC-23-MS-5-P8)**.** (*10 points*) Chi is buying a new computer. She has 260 of her own.

Her father, her mother, and her brother give her the rest of the sum. Anthony, her brother gave her one-eighth of the price of the laptop. Her mother gave her one-fourth of the price, and her father gave her half of the price of the laptop.

What is the price of the new computer?

Problem 5.2.4 (SC-23-MS-5-P9)**.** (*10 points*) Five pirates — Chi the Captain, Jade the First Mate, Khoa the Gunner, Luong the Rigger, and Tuan the Cook — got a chest of gold coins.

Chi took half of all the coins and half of a coin; Jade received half of the remaining coins and half of a coin; Khoa share was half of the remaining coins and half of a coin, and Luong got half of the remaining coins and half of a coin. Finally, when Tuan took half of what was left and half of a coin, there was no gold left.

How many gold coins did the pirates originally have? (The pirates did not cut any coins in half.)

Problem 5.2.5 (SC-23-MS-5-P10)**.** (*20 points*) Karl the Wizard has built his castle next to the dwelling place of the mighty Dragon Smough. Every morning with the first rays of the rising sun the Dragon flies out of his lair, heading toward one of the four nearby towns, which are located to the north, south, east, and west of the castle.

Karl wants to warn the townspeople where to expect Smough today. The castle has three towers topped with magic spheres; one is made out of ruby, the other of emerald, and the third one is made out of sapphire. When the magic spheres are lit up, their radiance is seen from the farthest corners of the land. So it has been agreed that by these lights Karl sends his signal, using some code to indicate where the Dragon is heading.

The spheres's lights, however, tend to flicker randomly at night, and the morning may find any combination of them turned on or off. And, because of his age (he's only in middle school!) Karl will have the strength to climb only one tower to manipulate its light (to turn it on or off). Thus, starting from a random combination, Karl can change the state of no more than one sphere. After that, the townspeople should be able to read where the Dragon is heading by seeing which lights are on.

Can you help Karl come up with such a code?

5.3 Solutions

Problem 5.3.1 (SC-23-MS-5-P6)**.** (*5 points*) Write these binary numbers as decimals:

$$\overline{110011}_2, \overline{10000000}_2, \overline{1000101}_2, \overline{111111}_2, \overline{111110}_2.$$

Solution.

$$\begin{aligned}
\overline{110011}_2 &= 2^5 + 2^4 + 2^1 + 2^0 = 32 + 16 + 2 + 1 = \boxed{51} \\
\overline{10000000}_2 &= 2^7 = \boxed{128} \\
\overline{1000101}_2 &= 2^6 + 2^2 + 2^0 = 64 + 4 + 1 = \boxed{69} \\
\overline{111111}_2 &= \overline{1000000}_2 - 1 = 2^6 - 1 = \boxed{63} \\
\overline{111110}_2 &= \overline{111111}_2 - 1 = \boxed{62}
\end{aligned}$$

□

Problem 5.3.2 (SC-23-MS-5-P7)**.** Each of the questions below can be solved independently.

1. (*5 points*) How can one tell if a binary number is even or odd?
2. (*5 points*) How can one tell if a binary number is divisible by 4?
3. (*5 points*) If a binary number ends with 10, what is the remainder when this number is divided by 4?

Solution. A binary number is even if it is divisible by 2, thus it ends with $\boxed{0.}$ A binary number is odd if it is not divisible by 2, thus it ends with $\boxed{1.}$ If a binary number ends with 10, then the remainder when this number is divided by 4 is $\overline{10}_2 = \boxed{2.}$ □

Problem 5.3.3 (SC-23-MS-5-P8)**.** (*10 points*) Chi is buying a new computer. She has 260 of her own.

Her father, her mother, and her brother give her the rest of the sum. Anthony, her brother gave her one-eighth of the price of the laptop. Her mother gave her one-fourth of the price, and her father gave her half of the price of the laptop.

What is the price of the new computer?

Solution. Let p be the price of the laptop, then

$$260 + \frac{p}{8} + \frac{p}{4} + \frac{p}{2} = p \Rightarrow 260 + \frac{7p}{8} = p \Rightarrow \frac{p}{8} = 260 \Rightarrow p = 260 \times 8 = \boxed{2080.}$$

□

Problem 5.3.4 (SC-23-MS-5-P9)**.** (*10 points*) Five pirates — Chi the Captain, Jade the First Mate, Khoa the Gunner, Luong the Rigger, and Tuan the Cook — got a chest of gold coins.

Chi took half of all the coins and half of a coin; Jade received half of the remaining coins and half of a coin; Khoa share was half of the remaining coins and half of a coin, and Luong got half of the remaining coins and half of a coin. Finally, when Tuan took half of what was left and half of a coin, there was no gold left.

How many gold coins did the pirates originally have? (The pirates did not cut any coins in half.)

Solution. Let's start working on this problem backward: Tuan's share x equals half of his share and a half-coin:

$$x = 0.5 + 0.5x \Rightarrow x = 1.$$

Thus, Tuan received 1 gold coin.

Take a look at Luong: from what was left (y coins), he received half of the amount and a half-coin. The rest (1 coin) was given to Tuan. Therefore,

$$y = 0.5 + 0.5y + 1 \Rightarrow 0.5y = 1.5 \Rightarrow y = 3.$$

Take a look at Khoa: from what was left (z coins), he received half of the amount and half-a-coin. The rest (3 coins) was given to Luong. Therefore,

$$z = 0.5 + 0.5z + 3 \Rightarrow 0.5z = 3.5 \Rightarrow z = 7$$

Thus, after Jade received his share (t coins), there were 7 coins left. So, we can write

$$t = 0.5 + 0.5t + 7\ Rightarrow 0.5t = 7.5 \Rightarrow t = 15.$$

Finally, the entire sum (m coins) was divided between Chi and the rest a sfollows:

$$m = 0.5 + 0.5m + 15 \Rightarrow 0.5m = 15.5 \Rightarrow m = \boxed{31.}$$

□

Problem 5.3.5 (SC-23-MS-5-P10)**.** (*20 points*) Karl the Wizard has built his castle next to the dwelling place of the mighty Dragon Smough. Every morning with the first rays of the rising sun the Dragon flies out of his lair, heading toward one of the four nearby towns, which are located to the north, south, east, and west of the castle.

Karl wants to warn the townspeople where to expect Smough today. The castle has three towers topped with magic spheres; one is made out of ruby, the other of emerald, and the third one is made out of sapphire. When the magic spheres are lit up, their radiance is seen from the farthest corners of the land. So it has been agreed that by these lights Karl sends his signal, using some code to indicate where the Dragon is heading.

The spheres's lights, however, tend to flicker randomly at night, and the morning may find any combination of them turned on or off. And, because of his age (he's only in middle school!) Karl will have the strength to climb only one tower to manipulate its light (to turn it on or off). Thus, starting from a random combination, Karl can change the state of no more than one sphere. After that, the townspeople should be able to read where the Dragon is heading by seeing which lights are on.

Can you help Karl come up with such a code?

Solution. Lets agree to mark a sphere that is "on" by 1 and that is off by "0" and to list the spheres in this order: ruby, emerald, sapphire. For example, the string 101 would mean that the ruby sphere shines, the emerald is not shiny, and the sapphire sphere shines.

Here is one possible code:

1. The dragon is heading north if the spheres are set into one of the two states 000 or 111.
2. The dragon is heading south if the spheres are set into one of the two states 100 or 011.
3. The dragon is heading east if the spheres are set into one of the two states 010 or 101.
4. The dragon is heading west if the spheres are set into one of the two states 001 or 110.

Now, lets prove that the code works. Indeed, suppose that we have a random three-digit string of 0's and 1's and a pair of codes for a direction.

Case 1: If this three-digit string matches one of the codes for this direction, the problem is solved.

Case 2: If the difference between the string and the first code is exactly 1 symbol, then we can change the sphere that corresponds to this symbol.

Case 3: If the difference between the string and the first code is 2 symbols, then the difference with the second code must be 1 symbol. (Indeed, the two codes are opposites of each other.)

Case 4: If the difference between the string and the first code is 3 symbols, then the string is a perfect match with the second code. □

Chapter 6

Counting and Probabilities

6.1 Examples

Example 6.1.1 (SC-23-MS-6-E1)

A 51 cm rod is built from 5 cm rods and 2 cm rods. All of the 5 cm rods must come first, and are followed by the 2 cm rods. For example, the rod could be made from seven 5 cm rods followed by eight 2 cm rods. How many ways are there to build the 51 cm rod?

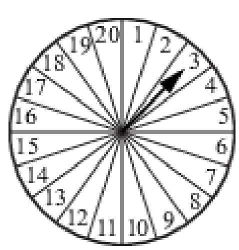
Solution. Any number of 2 cm rods add to give a rod having an even length. Since we need an odd length, 51 cm, then we must combine an odd length from the 5 cm rods with the even length from the 2 cm rods to achieve this. An odd length using 5 cm rods can only be obtained by taking an odd number of them. All possible combinations are shown in the table below.

Number of 5 cm rods 1	Length in 5 cm rods 5	Length in 2 cm rods ($51 - 5 = 46$)	Number of 2 cm rods ($46 \div 2 = 23$)
3	15	$51 - 15 = 36$	$36 \div 2 = 18$
5	25	$51 - 25 = 26$	$26 \div 2 = 13$
7	35	$51 - 35 = 16$	$16 \div 2 = 8$
9	45	$51 - 45 = 6$	$6 \div 2 = 3$

Note that attempting to use 11 (or more) 5 cm rods gives more than the 51 cm length required. Thus, there are exactly $\boxed{5}$ possible combinations that add to 51 cm using 5 cm rods first followed by 2 cm rods. □

Example 6.1.2 (SC-23-MS-6-E2)

A circular spinner is divided into 20 equal sections, as shown. An arrow is attached to the centre of the spinner. The arrow is spun once. What is the probability that the arrow stops in a section containing a number that is a divisor of 20?



Solution. The positive divisors of 20 are: $1, 2, 4, 5, 10, 20$. Of the 20 numbers on the spinner, 6 of the numbers are divisors of 20. It is equally likely that the spinner lands on any of the 20 numbers. Therefore, the probability that the spinner lands on a number that is a divisor of 20 is $\frac{6}{20} = \boxed{\frac{3}{10}}$. □

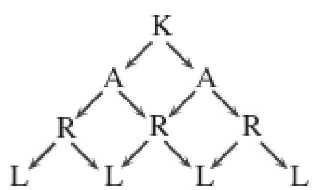
Example 6.1.3 (SC-23-MS-6-E3)

At the Gaussland Olympics there are 480 student participants. Each student is participating in 4 different events. Each event has 20 students participating and is supervised by 1 adult coach. There are 16 adult coaches and each coach supervises the same number of events. How many events does each coach supervise?

Solution. Since there are 480 student participants and each student is participating in 4 events then across all events the total number of (non-unique) participants is $480 \times 4 = 1920$. Each event has 20 students participating. Thus, the number of different events is $1920 \div 20 = 96$. Each event is supervised by 1 adult coach, and there are 16 adult coaches each supervising the same number of events. Therefore, the number of events supervised by each coach is $96 \div 16 = \boxed{6}$. □

Example 6.1.4 (SC-23-MS-6-E4)

In the diagram, how many paths can be taken to spell *KARL*?



Solution. [Solution 1] Starting at the letter K there are two possible paths that can be taken. At each A, there are again two possible paths that can be taken. Similarly, at each R there are two possible paths that can be taken. Therefore, the total number of paths is $2 \times 2 \times 2 = 8$. (*We can check this by actually tracing out the paths.*) □

Solution. [Solution 2] Each path from the K at the top to one of the L's at the bottom has to spell $KARL$.

There is 1 path that ends at the first L from the left. This path passes through the first A and the first R.

There are 3 paths that end at the second L. The first of theses passes through the first A and the first R. The second of theses passes through the first A and the second R. The third of these passes through the second A and the second R.

There are 3 paths that end at the third L. The first of theses passes through the first A and the second R. The second of theses passes through the second A and the second R. The third of these passes through the second A and the third R.

There is 1 path that ends at the last L. This path passes through the last A and the last R.

So the total number of paths to get to the bottom row is $1 + 3 + 3 + 1 = \boxed{8}$, which is the number of paths that can spell $KARL$. □

Example 6.1.5 (SC-23-MS-6-E5)

The ratio of green balls to yellow balls in a bag is $3:7$. When 9 balls of each colour are removed, the ratio of green balls to yellow balls becomes $1:3$. How many balls were originally in the bag?

Solution. Originally, the ratio of green balls to yellow balls in the bag was $3:7$. This means that for every 3 green balls in the bag, there were 7 yellow balls. Equivalently, if there were $3n$ green balls, then there were $7n$ yellow balls where n is a positive integer.

After 9 balls of each colour are removed, the number of green balls in the bag is $3n-9$ and the number of yellow balls is $7n-9$. At this point, the ratio of green balls to yellow balls is $1:3$, and so 3 times the number of green balls is equal to the number of yellow balls.

Multiplying the number of green balls by 3, we get $3(3n-9) = 9n-27$ green balls. Solving the equation

$$9n - 27 = 7n - 9 \Rightarrow 2n = 18 \Rightarrow n = 9.$$

Originally, there were $3n$ green balls and $7n$ yellow balls, for a total of $3n + 7n = 10n = \boxed{90}$ balls.

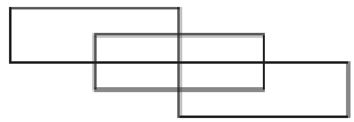
Note: If there were 90 balls, then 27 were green and 63 were yellow (since $27:63 = 3:7$ *and* $27+63 = 90$*). After 9 balls of each colour are removed, the ratio of green balls to yellow balls becomes* $18:54 = 1:3$*, as required.* ∎

6.2 Problems

Submission deadline: August 2, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

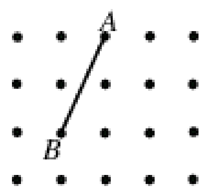
Problem 6.2.1 (SC-23-MS-6-P6)**.** (*5 points*) Count the number of rectangles of all sizes in the diagram shown below.



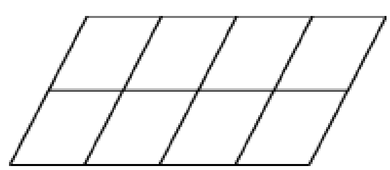
Problem 6.2.2 (SC-23-MS-6-P7)**.** (*5 points*) The Math Club has sold 120 tickets for a lottery. One winning ticket will be drawn. If the probability of one of Leah's tickets being drawn is $\frac{1}{15}$, how many tickets did she buy?

Problem 6.2.3 (SC-23-MS-6-P8)**.** (*10 points*) In how many ways can 101 be expressed as the sum of two integers, both greater than zero, with the second integer greater than the first?

Problem 6.2.4 (SC-23-MS-6-P9)**.** (*15 points*) In the diagram, the points are evenly spaced vertically and horizontally. A segment AB is drawn using two of the points, as shown below Point C is chosen to be one of the remaining 18 points. For how many of these 18 possible points is triangle $\triangle ABC$ isosceles?

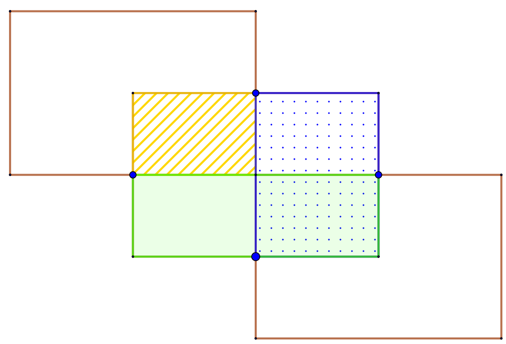


Problem 6.2.5 (SC-23-MS-6-P10)**.** (*15 points*) The figure consists of 8 identical small parallelograms, joined as shown below. Including these 8 small parallelograms, how many parallelograms appear in this figure?



6.3 Solutions

Problem 6.3.1 (SC-23-MS-6-P6)**.** (*5 points*) Count the number of rectangles of all sizes in the diagram shown below.



Solution. It is easy to see that there are three large rectangles with brown borders. There are two rectangles coloured green. There are two rectangles with blue dots. There are four rectangles with yellow hatching. Altogether $3 + 2 + 2 + 4 = \boxed{11}$ rectangles. □

Problem 6.3.2 (SC-23-MS-6-P7)**.** (*5 points*) The Math Club has sold 120 tickets for a lottery. One winning ticket will be drawn. If the probability of one of Leah's tickets being drawn is $\frac{1}{15}$, how many tickets did she buy?

Solution. The probability that Leah wins the lottery is equal to the number of tickets that Leah bought divided by the total number of tickets in the lottery. Let n be the number of tickets that she bought, the probability that one of her tickets to be a winning one is

$$\frac{n}{120} = \frac{1}{15} \Rightarrow n = \frac{120}{15} = \boxed{8.}$$

□

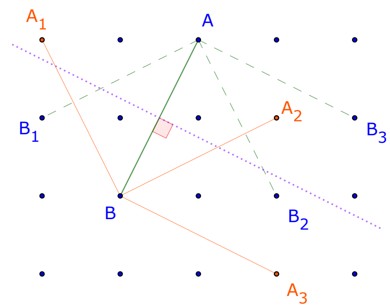
Problem 6.3.3 (SC-23-MS-6-P8)**.** (*10 points*) In how many ways can 101 be expressed as the sum of two integers, both greater than zero, with the second integer greater than the first?

Solution. Beginning with the positive integer 1 as a number in the rst pair, we get the sum $101 = 1 + 100$. From this point we can continue to increase the first number by one while decreasing the second number by one, keeping the sum equal to 101. The list of possible sums is:

$$101 = 1 + 100 = 2 + 99 = 3 + 98 = \ldots = 50 + 51.$$

After this point, the first number will no longer be smaller than the second if we continue to add 1 to the first number and subtract one from the second number. Thus, there are $\boxed{50}$ sums in total. □

Problem 6.3.4 (SC-23-MS-6-P9)**.** (*15 points*) In the diagram, the points are evenly spaced vertically and horizontally. A segment AB is drawn using two of the points, as shown below Point C is chosen to be one of the remaining 18 points. For how many of these 18 possible points is triangle $\triangle ABC$ isosceles?



Solution. There are three possible cases:

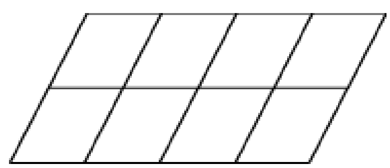
Case 1: $AB = AC$, then C is one of B_1, B_2, and B_3.

Case 2: $AB = BC$, then C is one of A_1, A_2, and A_3.

Case 3: $AC = BC$, then C should be on the perpendicular bisector (purple dotted line). Since this line contains none of given points, thus there is no solution for this case.

Altogether, there are $3 + 3 + 0 = \boxed{6}$ possible such points for C. □

Problem 6.3.5 (SC-23-MS-6-P10). (*15 points*) The figure consists of 8 identical small parallelograms, joined as shown below. Including these 8 small parallelograms, how many parallelograms appear in this figure?



Solution. Note that the figure formed by combining a pair of adjacent small parallelograms, is also a parallelogram. By counting the number of parallelograms with size 1×1, 1×2, 2×1, 1×3, 3×1, 1×4, 2×2, 2×3, 2×4, we receive $8, 6, 4, 4, 2, 3, 2$, and 1 parallelograms, respectively. Thus, the total number of them is $8 + 6 + 4 + 4 + 2 + 3 + 2 + 1 = \boxed{30.}$ □

Chapter 7

Area

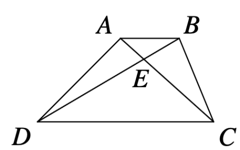
7.1 Examples

Theorem (Ratio of similarity)

Tf two triangles are similar and their sides have common ratio k, the ratio of their areas is k^2. This is true of any two similar figures.

Example 7.1.1 (SC-23-MS-7-E1)

In trapezoid $ABCD$, $AB \parallel CD$ and the diagonals meet at E. If $AB = 4$ and $CD = 12$, show that the area of $\triangle CDE$ is 9 times the area of $\triangle ABE$.



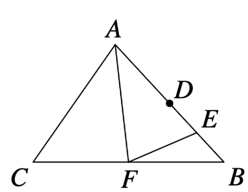
Solution. First, since $AB \parallel CD$, we have $\angle BAE = \angle DCE$ and $\angle ABE = \angle CDE$ as shown. Thus, by *AA Similarity* we get $\triangle ABE \sim \triangle CDE$. Since $\frac{CD}{AB} = 3$, we find

$$\frac{[CDE]}{[ABE]} = \left(\frac{CD}{AB}\right)^2 = \boxed{9.}$$

□

Example 7.1.2 (AHSME 1976)

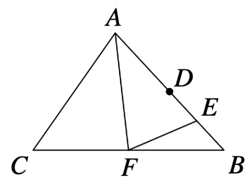
In $\triangle ABC$, D is the midpoint of AB, E is the midpoint of DB, and F is the midpoint of BC. If the area of $\triangle ABC$ is 96, then find the area of $\triangle AEF$.



Solution. Since $\triangle ABF$ has the same altitude as $\triangle ABC$ and $\frac{1}{2}$ the base, it has $\frac{1}{2}$ the area of $\triangle ABC$. Thus, $[ABF] = \frac{[ABC]}{2} = 48$. Now, $\triangle AEF$ has the same altitude (from F) as $\triangle ABF$. The base of $\triangle AEF$ is $\frac{3}{4}$ that of $\triangle ABF$ ($AE = \frac{3}{4}AB$), so $[AEF] = \frac{3}{4}[ABF] = \boxed{36.}$ □

Example 7.1.3 (M&IQ 1992)

If the diagonal AC of quadrilateral $ABCD$ divides the diagonal BD into two equal segments, prove that $[ACD] = [ACB]$.



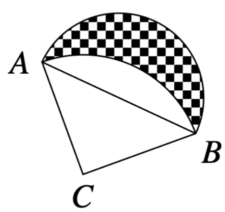
Solution. As described in the problem, X, the intersection of the diagonals, is the midpoint of BD. Since $\triangle ACD$ and $\triangle ABC$ share base AC, we can prove the areas of the triangles are equal if we show that the altitudes of the triangles to this segment are equal. Thus, we draw altitudes BY and DZ. Since $DX = BX$ and $\angle DXZ = \angle BXY$, we have $\triangle DZX \cong \triangle BYX$ by SA for right triangles, so $DZ = BY$. Hence,

$$[ABC] = \frac{1}{2}(AC)(BY) = \frac{1}{2}(AC)(DZ) = [ACD].$$

□

Example 7.1.4 (MAϕ 1990)

Find the shaded area, given that $\triangle ABC$ is an isosceles right triangle. The midpoint of AB is the center of semicircle $\widehat{AB}$, point C is the center of quarter circle $\widehat{AB}$, and $AB = 2\sqrt{2}$.

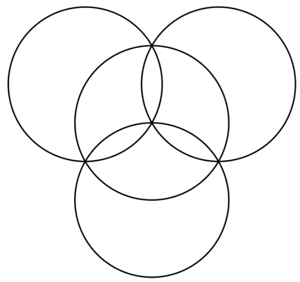


Remark. *We are given three pieces, the triangle, the semicircle, and the quarter circle, which we must add or subtract to form the shaded region.* How can we combine these pieces to get the shaded area? This is where these problems become like puzzles. This requires some intuition and practice. Here, *we add together the triangle and the semicircle, then subtract the quarter circle to leave the shaded region.* Make sure you see this. This is how we do all problems of this sort. We find the area of the simple figures in the diagram and determine how these figures can be added together or subtracted from each other to find the desired (usually *shaded*) region.

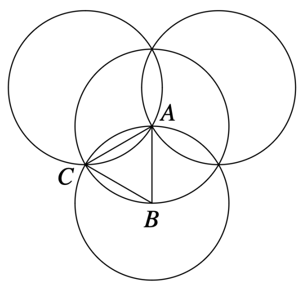
Solution. What simple areas can we find in this figure? Since $AB = 2\sqrt{2}$ and ABC is an isosceles right triangle, we have $AC = CB = 2$ and $[ABC] = \frac{1}{2}(2)(2) = 2$. We can also find the area of sector ABC and the semicircle with diameter AB. The area of quarter circle ABC is $1/4$ that of the circle with radius BC. Thus, it has area $\frac{(2^2)\pi}{4} = \pi$. The semicircle is half the area of the circle with diameter AB, or $\frac{(\sqrt{2})^2\pi}{2} = \pi$. By adding together the triangle and the semicircle, then subtracting the quarter circle to leave the shaded region, the desired area is $\pi + 2 - \pi = \boxed{2.}$ □

Example 7.1.5 (MATHCOUNTS 1992)

Each of the circles shown has a radius of 6 cm. The three outer circles have centers that are equally spaced on the original circle. Find the area, in square centimeters, of the sum of the three regions which are common to three of the four circles.



Remark. Our pieces in this problem are four circles which we unfortunately cannot puzzle together to make the desired region as we have done in prior examples. Thus, we must add lines to the diagram to give us more pieces. In problems involving intersecting circles, the best lines to add are radii and lines which divide the regions of intersection in half, forming segments and sectors as mentioned in our tips.



Solution. In the diagram, we have drawn AC to divide one of the *leaves* in half and we have drawn radii AB and BC of the lowest circle. Since AC is also a radius of the circle A, which has the same radius as the circle B, we have $AB = BC = AC$, and $\triangle ABC$ is equilateral.

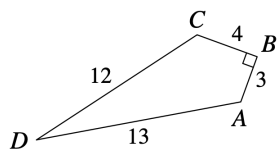
Now we can find the area of circular segment AC, since it is the area of sector ABC minus the area of $\triangle ABC$. Since this triangle is equilateral, the sector is $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of the circle. Thus, the sector has area $\frac{6^2\pi}{6} = 6\pi$. The area of the segment then is $6\pi - \frac{6^2\sqrt{3}}{4} = 6\pi - 9\sqrt{3}$. Since the three leaves together consist of 6 such segments, the total area is $\boxed{36\pi - 54\sqrt{3}}$. $\square$

7.2 Problems

Submission deadline: August 9, 2023.

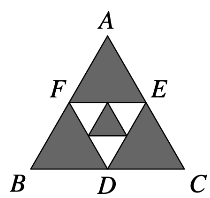
Note that for every problem you earn an additional same number of points for every different solution.

Problem 7.2.1 (SC-23-MS-7-P6)**.** (*5 points*) Sides AB, BC, CD, and DA of convex quadrilateral $ABCD$ have lengths 3, 4, 12, and 13, respectively; and $\angle CBA$ is a right angle. What is the area of the quadrilateral?

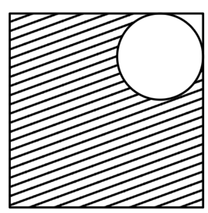


Problem 7.2.2 (SC-23-MS-7-P7)**.** (*5 points*) Find the ratio of the area of an equilateral triangle inscribed in a circle to the area of a square circumscribed about the same circle.

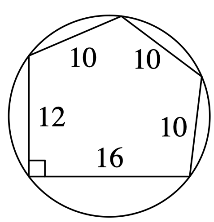
Problem 7.2.3 (SC-23-MS-7-P8)**.** (*10 points*) Points D, E, and F are midpoints of the sides of equilateral triangle ABC. The shaded central triangle is formed by connecting the midpoints of the sides of $\triangle DEF$. What fraction of the total area of ABC is shaded?



Problem 7.2.4 (SC-23-MS-7-P9)**.** (*15 points*) The square in the figure has sides with length 9 centimeters. The radius of the circle is 2 centimeters. What is the area of the shaded region?

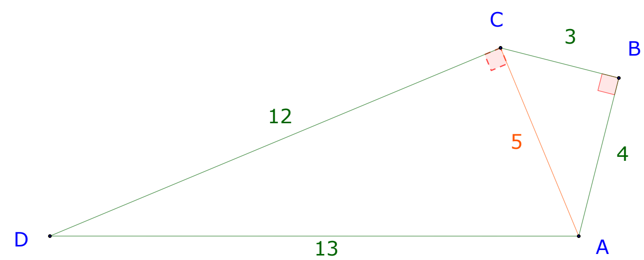


Problem 7.2.5 (SC-23-MS-7-P10)**.** (*15 points*) Find the number of square units in the area of the inscribed pentagon with right angle and dimensions as shown.



7.3 Solutions

Problem 7.3.1 (SC-23-MS-7-P6)**.** (AHSME 1980) (*5 points*) Sides AB, BC, CD, and DA of convex quadrilateral $ABCD$ have lengths 3, 4, 12, and 13, respectively; and $\angle CBA$ is a right angle. What is the area of the quadrilateral?

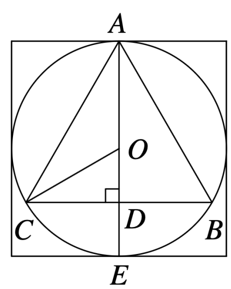


Solution. Since $\triangle ABC$ is right at B, thus

$$\begin{aligned}
&AC^2 = AB^2 + BC^2 = 3^2 + 4^2 = 5^2 \Rightarrow AC = 5\\
&\Rightarrow AD^2 - DC^2 = 13^2 - 12^2 = 5^2 = AC^2 \angle ACD = 90^\circ\\
&\Rightarrow [ABCD] = [ABC] + [ADC] = \frac{1}{2}AB \cdot BC + \frac{1}{2}AC \cdot CD = \frac{1}{2}(12+60) = \boxed{36.}
\end{aligned}$$

□

Problem 7.3.2 (SC-23-MS-7-P7)**.** (MAϕ 1987) (*5 points*) Find the ratio of the area of an equilateral triangle inscribed in a circle to the area of a square circumscribed about the same circle.



Solution. Let the radius of the circle be r. Since a side of the square equals AE, the square has side length $2r$ and hence area $4r^2$. To find the area of the triangle, we must find a side of the triangle. Thus we draw radius OC to a vertex of the triangle. Triangle OCD is a 30°-60°-90° triangle because OC is an angle bisector of $\angle ACB$ (any line through a vertex and the center of an equilateral triangle is an angle bisector). Thus $\angle OCD = 30^\circ, OD = \frac{r}{2}$, and $CD = \frac{r\sqrt{3}}{2}$. Thus $CB = 2CD = r\sqrt{3}$ and

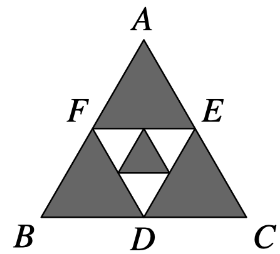
$$[ABC] = \frac{(r\sqrt{3})^2\sqrt{3}}{4} = \frac{3r^2\sqrt{3}}{4}.$$

Hence the ratio of the area of the triangle to that of the square is

$$\frac{\frac{3r^2\sqrt{3}}{4}}{4r^2} = \boxed{\frac{3\sqrt{3}}{16}}.$$

□

Problem 7.3.3 (SC-23-MS-7-P8). (*10 points*) Points D, E, and F are midpoints of the sides of equilateral triangle ABC. The shaded central triangle is formed by connecting the midpoints of the sides of $\triangle DEF$. What fraction of the total area of ABC is shaded?



Solution. Triangle AEF is similar to $\triangle ABC$ and its sides are half those of ABC. Thus $[AEF] = \frac{1}{4}[ABC]$. Similarly,

$$[DEF] = [BDF] = [DEC] = \frac{1}{4}[ABC].$$

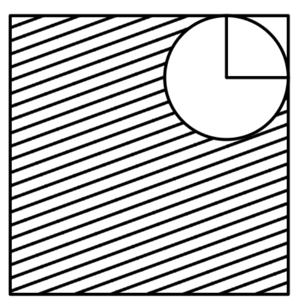
Since the inner shaded triangle is formed by connecting the midpoints of equilateral triangle DEF, it is similar to $\triangle DEF$ and has sides $\frac{1}{2}$ the length of the sides of $\triangle DEF$. Thus its area is $\frac{1}{4}[DEF]$, or $\frac{1}{16}[ABC]$.

Thus the sum of the shaded areas, S, is the sum

$$\frac{1}{4}[ABC] + \frac{1}{4}[ABC] + \frac{1}{4}[ABC] + \frac{1}{16}[ABC] = \frac{13}{16}[ABC].$$

Thus the ratio of the shaded area to $[ABC]$ is $\boxed{\frac{13}{16}}$. □

Problem 7.3.4 (SC-23-MS-7-P9). (*15 points*) The square in the figure has sides with length 9 centimeters. The radius of the circle is 2 centimeters. What is the area of the shaded region?

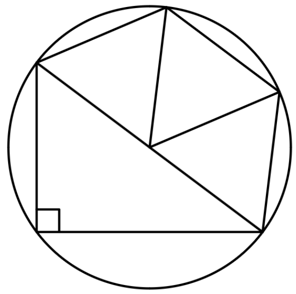


Solution. By drawing the two shown radii of the circle, we divide the unshaded area into a $\frac{3}{4}$ circle of radius 2 and a square of side length 2. The shaded area is then the difference of the area of the large square and the total unshaded area, or

$$9^2 - \left[\left(\frac{3}{4}\right)(2^2\pi) + 2^2\right] = \boxed{77 - 3\pi}.$$

□

Problem 7.3.5 (SC-23-MS-7-P10). (*15 points*) Find the number of square units in the area of the inscribed pentagon with right angle and dimensions as shown.



Solution. First we draw the hypotenuse of the right triangle with legs 12 and 16. This hypotenuse has length 20 and is a diameter of the circle (because a right angle subtends it). Drawing the radii to the other two vertices of the pentagon, we form 3 equilateral triangles of side length 10. These each have area $\frac{1}{4}10^2\sqrt{3} = 25\sqrt{3}$. Thus the total area of the pentagon is

$$\frac{1}{2}12(16) + 3(25\sqrt{3}) = \boxed{96 + 75\sqrt{3}}.$$

□

Chapter 8

Divisibility & Number Sense

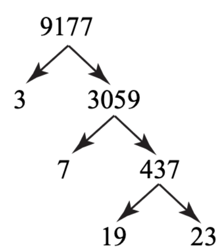
8.1 Examples

Example 8.1.1 (SC-23-MS-8-E1)

The product of three consecutive odd numbers is 9177. What is the sum of the numbers?

Solution. [Solution 1] Since $20 \times 20 \times 20 = 8000$ and $30 \times 30 \times 30 = 27000$, then we might guess that the three consecutive odd numbers whose product is 9177 are closer to 20 than they are to 30. Using trial and error, we determine that $21 \times 23 \times 25 = 12075$, which is too large. The next smallest set of three consecutive odd numbers is $19, 21, 23$ and the product of these three numbers is $19 \times 21 \times 23 = 9177$, as required. Thus, the sum of the three consecutive odd numbers whose product is 9177 is $19 + 21 + 23 = \boxed{63.}$ □

Solution. [Solution 2] We begin by determining the prime numbers whose product is 9177. (This is called the prime factorization of 9177.) This prime factorization of 9177 is shown in the factor tree below.



That is, $9177 = 3 \times 3059 = 3 \times 7 \times 437 = 3 \times 7 \times 19 \times 23$. Since $3 \times 7 = 21$, then $9177 = 21 \times 19 \times 23$ and so the three consecutive numbers whose product is 9177 are $19, 21, 23$. Thus, the sum of the three consecutive odd numbers whose product is 9177 is $19 + 21 + 23 = \boxed{63.}$ □

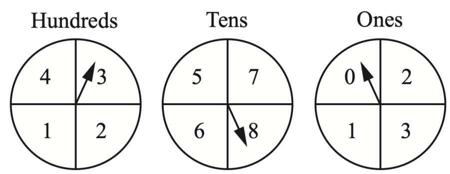
Example 8.1.2 (SC-23-MS-8-E2)

Dalia's birthday is on a Wednesday and Bruce's birthday is 60 days after Dalia's. On what day of the week is Bruce's birthday?

Solution. Since Dalia'ss birthday is on a Wednesday, then any exact number of weeks after Dalia'ss birthday will also be a Wednesday. Therefore, exactly 8 weeks after Dalia's birthday is also a Wednesday. Since there are 7 days in each week, then $7 \times 8 = 56$ days after Dalia's birthday is a Wednesday. Since 56 days after Dalia's birthday is a Wednesday, then 60 days after Dalia's birthday is a Sunday (since 4 days after Wednesday is Sunday). Therefore, Bruce's birthday is on a $\boxed{\text{Sunday.}}$ □

Example 8.1.3 (SC-23-MS-8-E3)

Three spinners are shown. The spinners are used to determine the hundreds, tens and ones digits of a three-digit number. How many possible three-digit numbers that can be formed in this way are divisible by 6?



Solution. A number is divisible by 6 if it is divisible by both 2 and 3. To be divisible by 2, the three-digit number that is formed must be even and so the ones digit must be 0 or 2. To be divisible by 3, the sum of the digits of the number must be a multiple of 3.

Case 1: Consider the possible tens and hundreds digits when the ones digit is 0. In this case, the sum of the tens and hundreds digits must be a multiple of 3 (since the ones digit does not add anything to the sum of the digits). We determine the possible sums of the tens and hundreds digits in the table below. The sums which are a multiple of 3 are circled. When the ones digit is 0, the possible three-digit numbers are: 150, 180, 270, 360, 450, and 480.

The Tens Digit

The Hundreds Digit

100s \ 10s	5	6	7	8
1	6	7	8	9
2	7	8	9	10
3	8	9	10	11
4	9	10	11	12

Case 2: Consider the possible tens and hundreds digits when the ones digit is 2. In this case, the sum of the tens and hundreds digits must be 2 less than a multiple of 3 (since the ones digit adds 2 to the sum of the digits). When the ones digit is 2, the possible three-digit numbers are: 162, 252, 282, 372, and 462.

The number of three-digit numbers that can be formed that are divisible by 6 is $\boxed{11.}$ □

Example 8.1.4 (SC-23-MS-8-E4)

The smallest positive integer n for which $n(n+1)(n+2)$ is a multiple of 5 is $n = 3$. All positive integers, n, for which $n(n+1)(n+2)$ is a multiple of 5 are listed in increasing order.

$$3, 4, 5, \ldots$$

What is the 2018$^{\text{th}}$ integer in the list?

Solution. In general, because 5 is a prime number, the product $n(n+1)(n+2)$ is a multiple of 5 exactly when at least one of its factors $n, n+1$, and $n+2$ is a multiple of 5. A positive integer is a multiple of 5 when its units (ones) digit is either 0 or 5. Next, we make a table that lists the units digits of $n+1$ and $n+2$ depending on the units digit of n:

Units digit of n	Units digit of $n+1$	Units digit of $n+2$
1	2	3
2	3	4
3	4	5
4	5	6
5	6	7
6	7	8
7	8	9
9	0	1
0	1	2

From the table, one of the three factors has a units digit of 0 or 5 exactly when the units digit of n is one of 3, 4, 5, 8, 9, 0. (Notice that this agrees with the first table above.) This means that 6 out of each block of 10 values of n ending at a multiple of 10 give a value for $n(n+1)(n+2)$ that is a multiple of 5.

We are asked for the 2018$^{\text{th}}$ positive integer n for which $n(n+1)(n+2)$ is a multiple of 5. Note that $2018 = 336 \times 6 + 2$. This means that, in the first $336 \times 10 = 3360$ positive integers, there are $336 \times 6 = 2016$ integers n for which $n(n+1)(n+2)$ is a multiple of 5. (Six out of every ten integers have this property.) We need to count two more integers along the list. The next two integers n for which $n(n+1)(n+2)$ is a multiple of 5 will have units digits 3 and 4, and so are 3363 and 3364. This means that $\boxed{3364}$ is the 2018$^{\text{th}}$ integer with this property. □

Example 8.1.5 (SC-23-MS-8-E5)

Two 5-digit positive integers are formed using each of the digits from 0 through 9 once. What is the smallest possible positive difference between the two integers if it is less than 1000?

Solution. If the ten thousands digits of the two numbers differ by more than 1, then the two numbers will differ by more than 10 000. (For example, a number of the form $5x\ xxx$ is at least 50 000 and a number of the form $3x\ xxx$ is less than 40 000 so these numbers differ by more than 10 000.) Since the answer must be less than 1000 and since the two ten thousands digits cannot be equal, then the ten thousands digits must differ by 1. We will determine the exact ten thousands digits later, so we let the smaller of the two ten thousands digits be d and the larger be D.

To make the difference between $Dxxxx$ and $dxxxx$ as small as possible, we try to simultaneously make $Dx\ xxx$ as close to $D0\ 000$ as possible and $dx\ xxx$ as close to $d9\ 999$ as possible while using all different digits. In other words, we try to make $Dxxxx$ as small as possible and $dxxxx$ as large as possible while using all different digits.

To make $Dx\ xxx$ as small as possible, we use the smallest possible digits in the places with the highest value. Since all of the digits must be different, then the minimum possible value of $Dx\ xxx$ is $D0123$. To make $dxxxx$ as large as possible, we use the largest possible digits in the places with the highest value. Since all of the digits must be different, then the maximum possible value of $dx\ xxx$ is $d9876$. Since we have made $Dx\ xxx$ as small as possible and $dx\ xxx$ as large as possible and used completely different sets of digits, then doing these two things will make their difference as small as possible, assuming that there are digits remaining to assign to D and d that differ by 1. The digits that have not been used are 5 and 4; thus, we set $D = 5$ and $d = 4$. This gives numbers 50 123 and 49 876. Their difference is $50\ 123 - 49\ 876 = \boxed{247,}$ which is the minimum possible difference. □

8.2 Problems

Submission deadline: August 16, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 8.2.1 (SC-23-MS-8-P6). (*5 points*) Let $n = 2^5 \times 3^2 \times 7 \times m$. If 100 is a divisor of n then what is the least value of m?

Problem 8.2.2 (SC-23-MS-8-P7). (*5 points*) Sophia did push-ups every day for 7 days. Each day after the first day, she did 5 more push-ups than the day before. In total she did 175 push-ups. How many push-ups did Sophia do on the last day?

Problem 8.2.3 (SC-23-MS-8-P8). (*10 points*) Lynne chooses four distinct digits from 1 to 9 and arranges them to form the 24 possible four-digit numbers. These 24 numbers are added together giving the result s. For all possible choices of the four distinct digits, what is the largest sum of the distinct prime factors of s?

Problem 8.2.4 (SC-23-MS-8-P9). (*15 points*) In the triangle shown, the first diagonal line, $1, 2, 3, \ldots$ begins at 1 and each number after the first is one larger than the previous number. The second diagonal line, $2, 4, 6, \ldots$ begins at 2 and each number after the first is two larger than the previous number. The n^{th} diagonal line begins at n and each number after the first is n larger than the previous number. In which horizontal row does the number 2016 first appear?

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 2 & & 2 & & & & \\
 & & & 3 & & 4 & & 3 & & & \\
 & & 4 & & 6 & & 6 & & 4 & & \\
 & 5 & & 8 & & 9 & & 8 & & 5 & \\
6 & & 10 & & 12 & & 12 & & 10 & & 6 \\
 & \therefore & & \therefore & & \vdots & & \ddots & & \ddots &
\end{array}$$

Problem 8.2.5 (SC-23-MS-8-P10). (*15 points*) The sum of all of the digits of the integers from 98 to 101 is

$$9 + 8 + 9 + 9 + 1 + 0 + 0 + 1 + 0 + 1 = 38.$$

What is the sum of all of the digits of the integers from 1 to 2023?

8.3 Solutions

Problem 8.3.1 (SC-23-MS-8-P6)**.** (*5 points*) Let $n = 2^5 \times 3^2 \times 7 \times m$. If 100 is a divisor of n then what is the least value of m?

Solution. Since $100 = 2^2 \cdot 5^2$, thus 5^2 should be a divisor of m. Hence its smallest value is $\boxed{25.}$ □

Problem 8.3.2 (SC-23-MS-8-P7)**.** (*5 points*) Sophia did push-ups every day for 7 days. Each day after the first day, she did 5 more push-ups than the day before. In total she did 175 push-ups. How many push-ups did Sophia do on the last day?

Solution. Sophia did push-ups for 7 days (an odd number of days), and on each day she did an equal number of push-ups more than the day before (5 more). Therefore, the number of push-ups that Sophia did on the middle day (day 4) is equal to the average number of push-ups that she completed each day. Sophia did 175 push-ups in total over the 7 days, and thus on average she did $175 \div 7 = 25$ push-ups each day. Therefore, on day 4 Sophia did 25 push-ups, and so on day 5 she did $25 + 5 = 30$ push-ups, on day 6 she did $30 + 5 = 35$ push-ups, and on the last day she did $35 + 5 = \boxed{40}$ push-ups.

(Note: We can check that $10 + 15 + 20 + 25 + 30 + 35 + 40 = 175$, as required.) □

Problem 8.3.3 (SC-23-MS-8-P8)**.** (*10 points*) Lynne chooses four distinct digits from 1 to 9 and arranges them to form the 24 possible four-digit numbers. These 24 numbers are added together giving the result s. For all possible choices of the four distinct digits, what is the largest sum of the distinct prime factors of s?

Solution. A number with four digit $abcd = 1000a + 100b + 10c + d$. Note that a digit a appear as a thousands, hundreds, tens, or unit digit exactly 6 times. Thus, when adding them up we have

$$6(a + b + c + d)(1000 + 100 + 10 + 1) = 6666(a + b + c + d) = 2 \times 3 \times 11 \times 101 \times (a + b + c + d).$$

Since $10 = 1+2+3+4 \leq a+b+c+d \leq 6+7+8+9 = 30$, and it is easdy to see that for any value of the sum $a + b + c + d$ which is not prime, then the sum of its prime factors is less then itself. Thus $a + b + c + d$ must be the larget possible prime within 10 and 30, which is 29.

Hence the largest sum of the distinct prime factors of s is $2 + 3 + 11 + 29 + 101 = \boxed{146.}$ □

Problem 8.3.4 (SC-23-MS-8-P9). (*15 points*) In the triangle shown, the first diagonal line, $1, 2, 3, \ldots$ begins at 1 and each number after the first is one larger than the previous number. The second diagonal line, $2, 4, 6, \ldots$ begins at 2 and each number after the first is two larger than the previous number. The n^{th} diagonal line begins at n and each number after the first is n larger than the previous number. In which horizontal row does the number 2016 first appear?

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 2 & & 2 & & & & \\
 & & & 3 & & 4 & & 3 & & & \\
 & & 4 & & 6 & & 6 & & 4 & & \\
 & 5 & & 8 & & 9 & & 8 & & 5 & \\
6 & & 10 & & 12 & & 12 & & 10 & & 6 \\
 & ⋰ & & ⋰ & & \vdots & & \ddots & & \ddots &
\end{array}
$$

Solution. The first number in the n^{th} diagonal line is n, and it lies in the n^{th} horizontal row. The second number in the n^{th} diagonal line is $n + n$ or $2n$ and it lies in the horizontal row numbered $n + 1$. The third number in the n^{th} diagonal line is $n + n + n$ or $3n$ and it lies in the horizontal row numbered $n + 2$ (each number lies one row below the previous number in the diagonal line). Following this pattern, the m^{th} number in the n^{th} diagonal line is equal to $m \times n$ and it lies in the horizontal row numbered $n + (m - 1)$.

The number 2016 lies in some diagonal line(s). To determine which diagonal lines 2016 lies in, we express 2016 as a product $mtimesn$ for positive integers m and n. We want the horizontal row in which 2016 first appears, and so we must find positive integers m and n so that $m \times n = 2016$ and $n + m$ (and therefore $n + m - 1$) is as small as possible. Below, we summarize the factor pairs (m, n) of 2016 and the horizontal row number $n + m - 1$ in which each occurence of 2016 appears

$$
\begin{array}{l}
(1, 2016)\ 2016 \\
(2, 1008)\ 1009 \\
(3, 672)\ 674 \\
(4, 504)\ 507 \\
(6, 336)\ 341 \\
(7, 288)\ 294 \\
(8, 252)\ 259 \\
(9, 224)\ 232 \\
(12, 168)\ 179 \\
(14, 144)\ 157 \\
(16, 126)\ 141 \\
(18, 112)\ 129 \\
(21, 96)\ 116 \\
(24, 84)\ 107 \\
(28, 72)\ 99 \\
(32, 63)\ 94 \\
(36, 56)\ 91 \\
(42, 48)\ 89
\end{array}
$$

We see that 2016 will appear in 18 different locations in the triangle. However, the first appearance of 2016 occurs in the horizontal row numbered $\boxed{89.}$ □

Problem 8.3.5 (SC-23-MS-8-P10). (*15 points*) The sum of all of the digits of the integers from 98 to 101 is

$$9+8+9+9+1+0+0+1+0+1=38.$$

What is the sum of all of the digits of the integers from 1 to 2023?

Solution. First lets take a look at the numbers from 2000 to 2023:

$$\begin{aligned}&2+(2+1)+\ldots+(2+9)+(2+1+0)+\ldots+(2+1+9)+(2+2+0)+\ldots(2+2+3)\\&=20+\frac{1}{2}(9)(10)+30+\frac{1}{2}(9)(10)+16+6=20+45+30+45+22=162.\end{aligned}$$

Next, we look at the integers from 1 to 1999 Again, since we can ignore digits of 0, we consider these numbers as 0001 to 1999, and in fact as the integers from 0000 to 1999. Of these 2000 integers, 200 have a units digit of 0, 200 have a units digit of 1, and so on. Therefore, the sum of the units digits of these integers is

$$200(0)+200(1)+\cdots+200(9)=9000$$

Similarly, the sum of the tens digits of these integers is 9000. The same is true for the sum of the hundreds digits.

Furthermore, of these 2000 integers, 1000 have a thousands digit of 0 and 1000 have a thousands digits of 1, thus the sum of the thousands digits of these integers is

$$1000(0)+1000(1)=1000.$$

Overall, the sum of all of the digits of these integers is

$$162+9000+9000+9000+1000=\boxed{28\ 162.}$$

□

Chapter 9

Test

9.1 Rules

You have a total of 270 minutes (4 hours and 30 minutes) to provide solutions to the problems below. You must submit a scanned version of the solutions no later than 15 minutes of your registered time, i.e. if you registered your test from 10:00 - 14:30, then you must submit your solution latest 14:45.

This is an open book test. You can use any material that you can access.

Each correct solution is worth of 10 points. Partially correct solution can earn some, but not all 10 points. Solutions must be cleanly, clearly, and well written. Unreadable solution, even if correct, could earn 0 point. Answers, even if correct, without a solution or depicted detailed diagram, will not be considered. Note that for every problem you earn an additional same number of points for every different correct solution.

9.2 Problems

Problem 9.2.1 (SC-23-MS-T-P1)**.** Khoa wrote all multiples of 4, from 4 to 100, in an increasing order:

$$4, 8, 12, 16, \ldots, 96, 100.$$

For every two consecutive terms, he calculates a quarter of their sum,

$$\frac{4+8}{4} = 3, \quad \frac{8+12}{4} = 5, \quad \frac{12+16}{4} = 7, \quad \frac{16+20}{4} = 9, \quad \ldots, \quad \frac{96+100}{4} = 49.$$

and inserts them between each pair of the respective two consecutive terms:

$$4,\ \underline{3},\ 8,\ \underline{5},\ 12,\ \underline{7},\ 16,\ \underline{9},\ \ldots,\ 96,\ \underline{49},\ 100.$$

Once done, he continues by removing every fourth numbers from the sequence:

$$4,\ 3,\ 8,\ \cancel{5},\ 12,\ 7,\ 16,\ \cancel{9},\ \ldots,\ 47,\ 96,\ \cancel{49},\ 100.$$

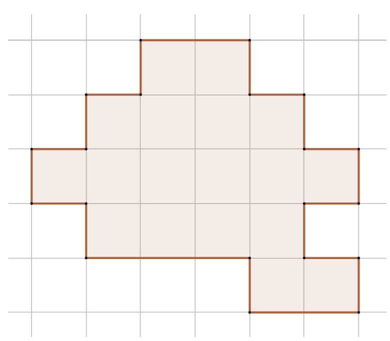
Thus this is the final sequence:

$$4,\ 3,\ 8,\ 12,\ 7,\ 16,\ \ldots,\ 47,\ 96,\ 100.$$

What is the sum of the numbers in the final sequence?

Problem 9.2.2 (SC-23-MS-T-P2)**.** At the end of the summer camp, all students are invited to a celebratory dinner. Twenty-five girls and twenty-five boys are seated at a round table. Prove that there is a students whose both neighbours are girls.

Problem 9.2.3 (SC-23-MS-T-P3)**.** Cut the figure below in three congruent parts. You can only cut along the grid lines.



Problem 9.2.4 (SC-23-MS-T-P4)**.** Find all possible ways to represent $\frac{1}{6}$ as a sum of two unit fractions:

$$\frac{1}{6} = \frac{1}{a} + \frac{1}{b}$$

where a and b are positive integers and $a < b$.

Problem 9.2.5 (SC-23-MS-T-P5)**.** In the equation below the sum of some four-digit numbers is equal to a five-digit number, where different letters represent different digits:

$$\overline{XYYYX} = \overline{ABCD} + \overline{BCDA} + \overline{CDAB} + \overline{DABC}$$

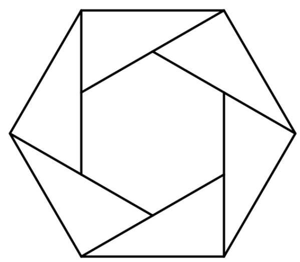
Prove that $2X = Y$.

Problem 9.2.6 (SC-23-MS-T-P6)**.** Chi, Khoa, and Samuel were counting a pile of marbles of four different colours. Each one of them counted two of the colours correctly, but confused the other two colours with each other: one of them confused red and orange, one of them confused orange and yellow, and one of them confused yellow and green. The results of their counting are provided in the table below: How many marbles

	red	orange	yellow	green
Chi	2	5	7	9
Khoa	2	4	9	8
Samuel	4	2	8	9

were there in reality?

Problem 9.2.7 (SC-23-MS-T-P7)**.** Given hexagon $ABCDEF$ with sides of length 6, six congruent $30°$-$60°$-$90°$ triangles are drawn as in the figure. Find the ratio of the area of the smaller hexagon formed to the area of the original hexagon.



Problem 9.2.8 (SC-23-MS-T-P8)**.** I have eight envelopes each containing \$1, another eight envelopes each containing \$3, and another eight envelopes each containing \$5. How can I share these envelopes among three people so that each person gets an equal number of envelopes and equal total amount of money? Find all possible ways.

Don't count repetitions by changing the roles of the people.

Problem 9.2.9 (SC-23-MS-T-P9)**.** Place 6 different numbers (not necessarily integers) around a circle so that each number is a product of its two neighbours.

Problem 9.2.10 (SC-23-MS-T-P10)**.** Rectangle $ABCD$ has all integer side lengths and its interior is divided into seven non-overlapping rectangular regions: two 8×10 regions, a 10×12 region, a 5×13 region, a 13×13 region, a 10×22 region, and a 3×16 region. These region may be rotated as needed to properly cover $ABCD$. What is the sum of the length and width of rectangle $ABCD$?

Show a tiling with such lengths.

9.3 Solutions

Problem 9.3.1 (SC-23-MS-T-P1)**.** Khoa wrote all multiples of 4, from 4 to 100, in an increasing order:

$$4, 8, 12, 16, \ldots, 96, 100.$$

For every two consecutive terms, he calculates a quarter of their sum,

$$\frac{4+8}{4} = 3,\ \frac{8+12}{4} = 5,\ \frac{12+16}{4} = 7,\ \frac{16+20}{4} = 9,\ \ldots,\ \frac{96+100}{4} = 49.$$

and inserts them between each pair of the respective two consecutive terms:

$$4,\ \underline{3},\ 8,\ \underline{5},\ 12,\ \underline{7},\ 16,\ \underline{9},\ \ldots,\ 96,\ \underline{49},\ 100.$$

Once done, he continues by removing every fourth numbers from the sequence:

$$4,\ 3,\ 8,\ \cancel{5},\ 12,\ 7,\ 16,\ \cancel{9},\ \ldots,\ 47,\ 96,\ \cancel{49},\ 100.$$

Thus this is the final sequence:

$$4,\ 3,\ 8,\ 12,\ 7,\ 16,\ \ldots,\ 47,\ 96,\ 100.$$

What is the sum of the numbers in the final sequence?

Solution. Note that the sum of the numbers in the final sequence is equal to the sum of the number in the original sequence plus the sum of the terms that are not cancelled.

Lets take a look at the terms that are cancelled

$$4,\ \frac{4+8}{4},\ 8,\ \cancel{\frac{8+12}{4}},\ 12,\ \frac{12+16}{4},\ 16,\ \cancel{\frac{16+20}{4}},\ \ldots,\ \frac{92+96}{4},\ 96,\ \cancel{\frac{96+100}{4}},\ 100.$$

Thus, he sum of the terms that are not cancelled is:

$$\frac{4+8}{4} + \frac{12+16}{4} + \cdots + \frac{92+96}{4} = \frac{1}{4}(4+8+12+16+\cdots+92+96) = 1+2+\cdots+24 = \frac{24\cdot 25}{2} = 300.$$

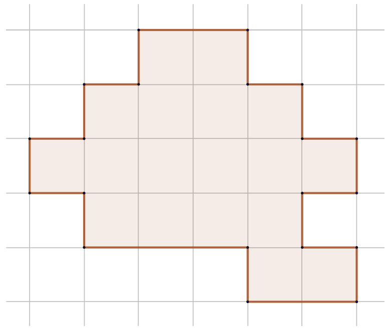
The sum of the terms in the original sequence is $4+8+\cdots+100 = 4(1+2+\cdots+25) = 25\cdot 26\cdot 2 = 1300$.

Hence, the desired sum is $1300 + 300 = \boxed{1600}$. □

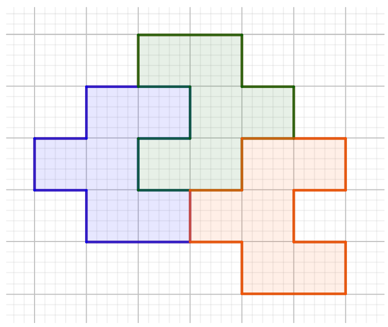
Problem 9.3.2 (SC-23-MS-T-P2)**.** At the end of the summer camp, all students are invited to a celebratory dinner. Twenty-five girls and twenty-five boys are seated at a round table. Prove that there is a student whose both neighbours are girls.

Solution. Lets assume the opposite: there is no such a student whose both neighbours are girls. Lets number the girls clockwise around the table as $g_1, g_2, \ldots, g_{25}$. Between every two consecutive girls, (g_1, g_2), (g_2, g_3), $\ldots, (g_{24}, g_{25})$, and (g_{25}, g_1), there should be at least two boys or no boys, which is an even number, thus the number of boys is the sum of all even numbers, which is an even number. This is a contradiction to the fact that there are 25 boys. □

Problem 9.3.3 (SC-23-MS-T-P3)**.** Cut the figure below in three congruent parts. You can only cut along the grid lines.



Solution. Below is one of the solutions



□

Problem 9.3.4 (SC-23-MS-T-P4)**.** Find all possible ways to represent $\frac{1}{6}$ as a sum of two unit fractions:

$$\frac{1}{6} = \frac{1}{a} + \frac{1}{b}$$

where a and b are positive integers and $a < b$.

Solution. Lets rearrange the equation

$$\frac{1}{6} = \frac{1}{a} + \frac{1}{b} \Rightarrow ab - 6(a+b) = 0 \Rightarrow b(a-6) - 6(a-6) = 36 \Rightarrow (a-6)(b-6) = 36$$

Case 1: $a - 6 < 0$. Since

$$1 - 6 = -5 \leq a - 6 \leq 5 - 6 = -1.$$

Because $a - 6$ is a negative factor of 36, it can only be $-4, -3, -2$, or -1. This means that $b - 6$ must be $-9, -12, -18$, or -36. All of these cases result in a negative b. Thus, there is no solution if $a - 6 < 0$.

Case 2: $a - 6 > 0$. $a - 6 > b - 6$ means to factor 36 as a product of two different numbers:

$$36 = 36 \cdot 1 = 18 \cdot 2 = 12 \cdot 3 = 9 \cdot 4 \Rightarrow \boxed{(a,b) \in \{(7,42),\ (8,24),\ (9,18),\ (10,15)\}}$$

□

Problem 9.3.5 (SC-23-MS-T-P5). In the equation below the sum of some four-digit numbers is equal to a five-digit number, where different letters represent different digits:

$$\overline{XYYYX} = \overline{ABCD} + \overline{BCDA} + \overline{CDAB} + \overline{DABC}$$

Prove that $2X = Y$.

Solution. Note that $\overline{ABCD} = 1000A + 100B + 10C + D$, thus

$$\overline{ABCD} + \overline{BCDA} + \overline{CDAB} + \overline{DABC} = 1111(A + B + C + D) = 11 \cdot 101 \cdot (A + B + C + D)$$

This means that $11 \mid XYYYX$, thus the alternate sum of digits $X - Y + Y - Y + X = 2X - Y$ is divisible by 11.

Furthermore,

$$A + B + C + D \leq 9 + 8 + 7 + 6 = 30 \Rightarrow \overline{XYYYX} = 1111(A + B + C + D) \leq 33330 \Rightarrow X \leq 3.$$

Therefore

$$2 \cdot 1 - 9 = -7 \leq 2X - Y \leq 2 \cdot 3 - 0 = 6 \Rightarrow 2X - Y = 0 \Rightarrow \boxed{2X = Y}.$$

□

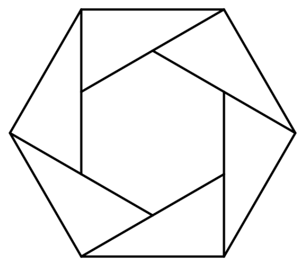
Problem 9.3.6 (SC-23-MS-T-P6). Chi, Khoa, and Samuel were counting a pile of marbles of four different colours. Each one of them counted two of the colours correctly, but confused the other two colours with each other: one of them confused red and orange, one of them confused orange and yellow, and one of them confused yellow and green. The results of their counting are provided in the table below: How many marbles

	red	orange	yellow	green
Chi	2	5	7	9
Khoa	2	4	9	8
Samuel	4	2	8	9

of each colour were there in reality?

Solution. Only one of them could have made a mistake when counting the red marbles, so two of them counted the number of red marbles correctly. Thus, there were 2 red marbles. Samuel made a mistake when counting the red marbles, so he confused the red and orange, but counted the yellow and green correctly. Thus, there were $\boxed{\text{2 red, 4 orange, 8 yellow, and 9 green}}$ marbles. □

Problem 9.3.7 (SC-23-MS-T-P7)**.** Given hexagon $ABCDEF$ with sides of length 6, six congruent 30°-60°-90° triangles are drawn as in the figure. Find the ratio of the area of the smaller hexagon formed to the area of the original hexagon.



Solution. Since the length of the long leg of a right triangle is 6 and it is $\frac{\sqrt{3}}{2}$ times the hypotenuse, therefore the length of the hypotenuse is $6 \cdot \frac{2}{\sqrt{3}} = 4\sqrt{3}$. In a 30°-60°-90° triangle, the hypotenuse is twice the short leg, thus the side length of the smaller hexagon is half of the hypotenuse of a right triangle, or $2\sqrt{3}$.

The ratio of the smaller hexagon to the area of the original hexagon is the square of their similarity ratio:

$$\left(\frac{2\sqrt{3}}{6}\right)^2 = \left(\frac{1}{\sqrt{3}}\right)^2 = \boxed{\frac{1}{3}}$$

□

Problem 9.3.8 (SC-23-MS-T-P8)**.** I have eight envelopes each containing \$1, another eight envelopes each containing \$3, and another eight envelopes each containing \$5. How can I share these envelopes among three people so that each person gets an equal number of envelopes and equal total amount of money? Find all possible ways.

Don't count the cases where the roles of the people are interchangeable.

Solution. Since there are 24 envelopes, each person must get eight envelopes. The total amount of money is $8 \cdot 1 + 8 \cdot 3 + 8 \cdot 5 = \$72$, so each person must get $\frac{72}{3} = \$24$. Let x, y, and z be the number of envelopes containing \$1, \$3, and \$5, respectively. Then

$$\left.\begin{aligned} x + 3y + 5z &= 24 \\ x + y + z &= 8 \end{aligned}\right\} \Rightarrow y + 2z = 8 \Rightarrow 2 \mid y \Rightarrow y \in \{0, 2, 4, 6, 8\}$$

Therefore $(x, y, z) \in \{(4, 0, 4),\ (3, 2, 3),\ (2, 4, 2),\ (1, 6, 1),\ (0, 8, 0)\}$.

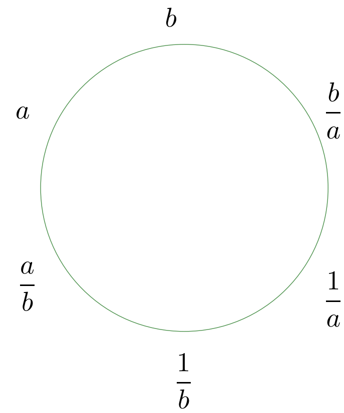
The possible cases are:

$$\boxed{\begin{array}{l} (4,0,4),\ (4,0,4),\ (0,8,0) \\ (4,0,4),\ (3,2,3),\ (1,6,1) \\ (4,0,4),\ (2,4,2),\ (2,4,2) \\ (3,2,3),\ (3,2,3),\ (2,4,2) \end{array}}$$

□

Problem 9.3.9 (SC-23-MS-T-P9)**.** Place 6 different numbers (not necessarily integers) around a circle so that each number is a product of its two neighbours.

Solution. Below is one of the solutions with a, b are arbitrary real numbers, where $a \neq b, ab \neq 1, a, b \notin \{0, 1\}$.



□

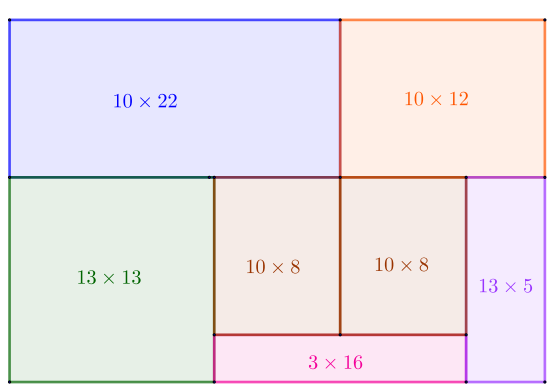
Problem 9.3.10 (SC-23-MS-T-P10)**.** Rectangle $ABCD$ has all integer side lengths and its interior is divided into seven non-overlapping rectangular regions: two 8×10 regions, a 10×12 region, a 5×13 region, a 13×13 region, a 10×22 region, and a 3×16 region. These region may be rotated as needed to properly cover $ABCD$. What is the sum of the length and width of rectangle $ABCD$? Show a tiling with such lengths.

Solution. First, we add all the given areas together. This is the area of the rectangle $ABCD$:

$$2 \cdot 8 \cdot 10 + 10 \cdot 12 + 5 \cdot 13 + 13 \cdot 13 + 10 \cdot 22 + 3 \cdot 16 = 782.$$

Now $782 = 2 \cdot 17 \cdot 23$, and since any side length of $ABCD$ should be at least 3, so we have two possible configurations: 17×46 or 23×34. The first one is impossible since 10×22 region has to be oriented with its 10 long side parallel to the 17 long side of $ABCD$, leaving a distance of $17 - 10 = 7$ to be tiled, which can not be a sum of any set of side lengths of the given pieces.

Therefore $ABCD$ is 23×34, the desired sum is $23 + 34 = \boxed{57}$, and below is a tiling.



□

Part II

High School

Chapter 10

Sums and Products

10.1 Telescoping Sums and Products in Algebra

You should review the **Sums and Products** chapter in the **Learning Problem Solving, Vol. 3** to understand the basics, practice with examples, and master a few techniques. The basic idea is as below,

Lemma (Telescoping sum)

$$(a_2 - a_1) + (a3 - a_2) + \ldots + (a_n - a_{n-1}) = a_n - a_1$$

Example 10.1.1 (SC-23-HS-1-E1)
For n arbitrary positive integer, prove that

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{2}.$$

Solution. By binominal theorem:

$$(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1.$$

Thus, by telescoping sum,

$$(n+1)^5 - 1 = \sum_{k=1}^{n}(k+1)^5 - k^5 = \sum_{k=1}^{n} 5k^4 + 10k^3 + 10k^2 + 5k + 1.$$

Since $\sum_{k=1}^{n} 1 = n$, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$, thus

$$(n+1)^5 = 5\sum_{k=1}^{n} k^4 = (n+1)^5 - 10\frac{n^2(n+1)^2}{4} - 10\frac{n(n+1)(2n+1)}{6} - 5\frac{n(n+1)}{2} - (n+1).$$

Expand the right-hand side expression, then simplify it to receive $\boxed{\frac{n(n+1)(2n+1)(3n^2+3n-1)}{2}.}$

Note that there is also a simple proof by Induction Principle. □

Example 10.1.2 (SC-23-HS-1-E2)
For n arbitrary even positive integer, prove that

$$\sum_{i=1}^{n}(-1)^{(i+1)}\frac{1}{i} = 2\sum_{i=1}^{\frac{n}{2}}\frac{1}{n+2i}$$

Solution. Note that,

$$\sum_{i=1}^{2m}(-1)^{(i+1)}\frac{1}{i} = \frac{1}{1}-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2m-1}-\frac{1}{2m} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2m-1}+\frac{1}{2m}-\frac{1}{2}-\frac{1}{4}-\cdots-\frac{1}{2m}$$

$$= \sum_{i=1}^{2m}\frac{1}{i} - 2\sum_{i=1}^{m}\frac{1}{2i} = \sum_{i=1}^{2m}\frac{1}{i} - \sum_{i=1}^{m}\frac{1}{i} = \frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{2m} = \boxed{2\sum_{i=1}^{m}\frac{1}{2m+2i}}$$

Note that there is also a simple proof by Induction Principle. □

Example 10.1.3 (SC-23-HS-1-E3)
For n arbitrary positive integer, evaluate

$$1^2\cdot 2! + 2^2\cdot 3! + \cdots + n^2(n+1)!$$

Remark. The key is to find a sequence $\{a_k\}$ such that,

$$k^2(k+1)! = a_{k+1} - a_k$$

The left-hand side has the product $(k+1)!$, thus it gives us an idea of a sequence $\{b_k\}$, where $= b_k = a_k k!$, or

$$k^2(k+1)! = b_{k+1}(k+1) - b_k k! \Rightarrow k^2(k+1) = (k+1)b_{k+1} - b_k$$

The right-hand side now is similar to a polynomial expression if we substitute b_k with a polynomial $P(k)$

$$k^2(k+1) = (k+1)P(k+1) - P(k)$$

Thus, the degree of P is 2, and its highest coefficient is 1 so let $P(x) = x^2 + ax + b$ be the polynomial, then

$$x^2(x+1) = (x+1)\left[(x+1)^2 + a(x+1) + b\right] - (x^2+ax+b) \Rightarrow x^3+x^2 = x^3+x^2 = x^3+(a+2)x^2+(a+b+3)x+a+1$$

Since the equality stands for all x, thus coefficients on both sides have to be the same, or $a = -1, b = -2$, and we have established a nice formula:

$$k^2(k+1)! = \left[(k+1)^2 - (k+1) - 2\right](k+1)! - (k^2 - k - 2)k!$$

Solution. Note that

$$k^2(k+1)! = \left[(k+1)^2 - (k+1) - 2\right](k+1)! - (k^2 - k - 2)k!$$

Therefore by telescoping sum,

$$\sum_{k=1}^{n}k^2(k+1)! = \left[(n+1)^2 - (n+1) - 2\right](n+1)! + (1^2 - 1 - 2)1! = \boxed{(n-1)(n+2)! + 2}$$

□

10.2 Telescoping Sums and Products in Trigonometry

You should review the **Trigonometry** section in **Sums and Products** chapter as well as the **Trigonometry** chapter in the **Learning Problem Solving, Vol. 3** to understand the basics, practice with examples, and master a few techniques.

Example 10.2.1 (SC-23-HS-1-E4)
Prove that
$$\tan 10^\circ = \tan 20^\circ \cdot \tan 30^\circ \cdot \tan 40^\circ$$

Remark. Note that
$$\begin{aligned}
&\tan x = \frac{\sin x}{\cos x}, \sin 2x = 2\sin x \cos x\\
&\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right],\ \sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]\\
&\sin(x+y) = \sin x\cos y + \cos x \sin y, \sin(x-y) = \sin x \cos y - \cos x \sin y
\end{aligned}$$

Solution. By applying $\tan x = \frac{\sin x}{\cos x}$, then cross-multiplying both sides,
$$\tan 10^\circ = \tan 20^\circ \cdot \tan 30^\circ \cdot \tan 40^\circ \Leftrightarrow \sin 10^\circ \cos 20^\circ \cos 30^\circ \cos 40^\circ = \cos 10^\circ \sin 20^\circ \sin 30^\circ \sin 40^\circ (*)$$

Since $\cos 40^\circ \cos 20^\circ = \frac{1}{2}(\cos 60^\circ + \cos 20^\circ),\ \sin 40^\circ \sin 20^\circ = \frac{1}{2}(\cos 20^\circ - \cos 60^\circ)$.
$$\begin{aligned}
(*) &\Leftrightarrow \sin 10^\circ \cos 30^\circ(\cos 60^\circ + \cos 20^\circ) = \cos 10^\circ \sin 30^\circ(\cos 60^\circ - \cos 20^\circ)\\
&\Leftrightarrow \cos 60^\circ(\sin 10^\circ \cos 30^\circ + \cos 10^\circ \sin 30^\circ) = \cos 20^\circ(\cos 10^\circ \sin 30^\circ - \sin 10^\circ \cos 30^\circ)\\
&\Leftrightarrow \cos 60^\circ \sin 40^\circ = \cos 20^\circ \sin 20^\circ \Leftrightarrow \frac{1}{2}\sin 40^\circ = \cos 20^\circ \sin 20^\circ (**)
\end{aligned}$$

(**) is obviously true, thus the given equality stands. ☐

Example 10.2.2 (SC-23-HS-1-E5)
Show that
$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$$

Remark. Note that if we multiply the left hand-side with $\sin\frac{\pi}{7}$, then we can turn it into a sum of products. Each product of a sin and a cos is actually a difference, which brings in telescoping sum.

Solution. Let S be the left-hand side, and note that $\sin\frac{3\pi}{7} = \sin\frac{\pi - 4\pi}{7} = \sin\frac{4\pi}{7}$,
$$\begin{aligned}
2\sin\frac{\pi}{7}\cdot S &= 2\cos\frac{\pi}{7}\sin\frac{\pi}{7} - 2\cos\frac{2\pi}{7}\sin\frac{\pi}{7} + 2\cos\frac{3\pi}{7}\sin\frac{\pi}{7}\\
&= \sin\frac{2\pi}{7} - \sin\frac{3\pi}{7} + \sin\frac{\pi}{7} + \sin\frac{4\pi}{7} - \sin\frac{2\pi}{7} = \sin\frac{\pi}{7}
\end{aligned}$$
☐

Note that in competition, in order to get partial points or to rescue a solution based on some right ideas but somehow producing wrong answer, do not forget to submit your sketches or heuristics like what we have done in the remark analysis shown above. If you write them clean and clear, there are chances you will get additional points. They would not withstand a rigorous check, but there is no such need.

10.3 Problems

Submission deadline: June 28, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 10.3.1 (SC-23-HS-1-P6). (*5 points*) For any n positive integer, let $a_n = 1 + 2 + \cdots + n$, compute

$$\frac{a_2}{a_2 - 1} \cdot \frac{a_3}{a_3 - 1} \cdots \frac{a_n}{a_n - 1}$$

Problem 10.3.2 (SC-23-HS-1-P7). (*5 points*) Compute

$$\frac{1}{1\sqrt{2} + 2\sqrt{1}} + \frac{1}{2\sqrt{3} + 3\sqrt{2}} + \frac{1}{4\sqrt{4} + 4\sqrt{3}} + \cdots + \frac{1}{4092528\sqrt{4092529} + 4092529\sqrt{4092528}}$$

Problem 10.3.3 (SC-23-HS-1-P8). (*5 points*) Prove that, for any n positive integer larger than 1,

$$\frac{1}{2\sqrt{n}} < \frac{1}{2}\frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$

Problem 10.3.4 (SC-23-HS-1-P9). (*5 points*) Prove that

$$\sin 25^\circ \sin 35^\circ \sin 60^\circ \sin 85^\circ = \sin 20^\circ \sin 40^\circ \sin 75^\circ \sin 80^\circ$$

Problem 10.3.5 (SC-23-HS-1-P10). (*5 points*) Prove that

$$\tan 50^\circ + \tan 60^\circ + \tan 70^\circ = \tan 80^\circ$$

Problem 10.3.6 (SC-23-HS-1-P11). (*10 points*) Prove that

$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \ldots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}$$

Problem 10.3.7 (SC-23-HS-1-P12). (*15 points*) Let $a_1, a_2, \ldots, a_n$ be the positive real numbers such that

$$a_1 a_2 \cdots a_n = 1$$

Prove that

$$\frac{a_1}{1+a_1} + \frac{a_2}{(1+a_1)(1+a_2)} + \frac{a_3}{(1+a_1)(1+a_2)(1+a_3)} + \cdots + \frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)} \geq \frac{2^n - 1}{2^n}.$$

10.4 Solutions

Problem 10.4.1 (SC-23-HS-1-P6). (*5 points*) For any n positive integer, let $a_n = 1+2+\cdots+n$, compute

$$\frac{a_2}{a_2-1}\cdot\frac{a_3}{a_3-1}\cdots\frac{a_n}{a_n-1}$$

> **Remark.** The key is to factor $a_k - 1 = k^2+k-2 = (k-1)(k+2)$.

Solution. Note that for $k = 1, 2, \ldots, n$:

$$a_k = \frac{k(k+1)}{2} \Rightarrow a_k - 1 = \frac{k^2+k-2}{2} = \frac{(k-1)(k+2)}{2} \Rightarrow \frac{a_k}{a_k-1} = \frac{k(k+1)}{(k-1)(k+2)} = \frac{k}{k-1}\cdot\frac{k+1}{k+2}$$

Thus,

$$\begin{aligned}\frac{a_2}{a_2-1}\cdot\frac{a_3}{a_3-1}\cdots\frac{a_n}{a_n-1} &= \prod_{k=2}^{n}\frac{a_k}{a_k-1} = \prod_{k=2}^{n}\frac{k}{k-1}\cdot\frac{k+1}{k+2} = \prod_{k=2}^{n}\frac{k}{k-1}\prod_{k=2}^{n}\frac{k+1}{k+2}\\ &= \left(\frac{2}{1}\frac{3}{2}\cdots\frac{n}{n-1}\right)\left(\frac{3}{4}\frac{4}{5}\cdots\frac{n+1}{n+2}\right) = \frac{n}{1}\cdot\frac{3}{n+2} = \boxed{\frac{3n}{n+2}}.\end{aligned}$$

□

Problem 10.4.2 (SC-23-HS-1-P7). (*5 points*) Compute

$$\frac{1}{1\sqrt{2}+2\sqrt{1}}+\frac{1}{2\sqrt{3}+3\sqrt{2}}+\frac{1}{3\sqrt{4}+4\sqrt{3}}+\cdots+\frac{1}{4092528\sqrt{4092529}+4092529\sqrt{4092528}}$$

> **Remark.** The key is to recognize that $k\sqrt{k+1}+(k+1)\sqrt{k} = \sqrt{k}\sqrt{k+1}\left(\sqrt{k}+\sqrt{k+1}\right)$.

Solution. The generic term of the given sum is: $\dfrac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}} = \dfrac{1}{\sqrt{k}\sqrt{k+1}}\cdot\dfrac{1}{\sqrt{k}+\sqrt{k+1}}$, and

$$\frac{1}{\sqrt{k}+\sqrt{k+1}} = \frac{\sqrt{k+1}-\sqrt{k}}{(\sqrt{k}+\sqrt{k+1})(\sqrt{k+1}-\sqrt{k})} = \frac{\sqrt{k+1}-\sqrt{k}}{(\sqrt{k+1})^2-(\sqrt{k})^2} = \sqrt{k+1}-\sqrt{k}.$$

Thus the generic term becomes $\dfrac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1}} = \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$, thus the given sum is a telescoping sum of

$$\sum_{k=1}^{4092528}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right) = 1 - \frac{1}{\sqrt{4092529}} = \boxed{\frac{2022}{2023}}.$$

□

Problem 10.4.3 (SC-23-HS-1-P8)**.** (*5 points*) Prove that, for any n positive integer larger than 1,

$$\frac{1}{2\sqrt{n}} < \frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$

> **Remark.** The key is to use simple identity
>
> $$(2k-2)(2k) = ((2k-1)-1)((2k-1)+1) = (2k-1)^2 - 1 < (2k-1)^2,\ (2k-1)(2k+1) = (2k)^2 - 1 < (2k)^2$$

Solution. For the left-hand side inequality, let take the square of the expression on the right:

$$\left(\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n}\right)^2 = \frac{1}{2}\left(\frac{3^2}{2\cdot 4}\right)\left(\frac{5^2}{4\cdot 6}\right)\cdots\left(\frac{(2n-1)^2}{(2n-2)\cdot(2n)}\right)\frac{1}{2n}$$

Note that for $k = 2, 3, \ldots, n$ $\dfrac{(2k-1)^2}{(2k-2)(2k)} = \dfrac{(2k-1)^2}{((2k-1)-1)((2k-1)+1)} = \dfrac{(2k-1)^2}{(2k-1)^2-1} > 1$, therefore

$\left(\dfrac{1}{2}\dfrac{3}{4}\cdots\dfrac{2n-1}{2n}\right)^2 > \dfrac{1}{2}\dfrac{1}{2n}$, and the left-hand side inequality follows.

Similarly,

$$\left(\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n}\right)^2 = \left(\frac{1\cdot 3}{2^2}\right)\left(\frac{3\cdot 5}{4^2}\right)\cdots\left(\frac{(2n-1)(2n+1)}{(2n)^2}\right)\frac{1}{2n+1}$$

And since for $k = 2, 3, \ldots, n$ $\dfrac{(2k-1)(2k+1)}{(2k)^2} = \dfrac{(2k)^2-1}{(2k)^2} < 1$, therefore

$\left(\dfrac{1}{2}\dfrac{3}{4}\cdots\dfrac{2n-1}{2n}\right)^2 < \dfrac{1}{2n+1}$, and the right-hand side inequality follows.

$\square$

Problem 10.4.4 (SC-23-HS-1-P9). (*5 points*) Prove that

$$\sin 25^\circ \sin 35^\circ \sin 60^\circ \sin 85^\circ = \sin 20^\circ \sin 40^\circ \sin 75^\circ \sin 80^\circ$$

> **Remark.** The key is to recognize the relation between $\sin x \sin(60^\circ - x)\sin(60^\circ + x)$ and $\sin 3x$.

Solution. We first prove a very useful identity, for any x,

> **Claim** — $\sin x \sin(60^\circ - x)\sin(60^\circ + x) = \frac{1}{4}\sin 3x$.

Proof.

$$\begin{aligned}
\sin(60^\circ - x)\sin(60^\circ + x) &= \frac{1}{2}\left(\cos((60^\circ - x) - (60^\circ + x)) - \cos((60^\circ - x) + (60^\circ + x))\right)\\
&= \frac{1}{2}(\cos 2x - \cos 120^\circ) = \frac{1}{2}\cos 2x + \frac{1}{4}\\
\sin 3x - \sin x &= 2\cos\frac{3x + x}{2}\sin\frac{3x - x}{2} = 2\cos 2x \sin x\\
\Rightarrow \frac{1}{4}\sin 3x &= \frac{1}{4}(2\cos 2x \sin x + \sin x) = \sin x\left(\frac{1}{2}\cos 2x + \frac{1}{4}\right)\\
&= \sin x \sin(60^\circ - x)\sin(60^\circ + x)
\end{aligned}$$

■

Thus the left-hand side is equal to: $\sin 60^\circ(\sin 25^\circ \sin 35^\circ \sin 85^\circ) = \frac{1}{4}\sin 60^\circ \sin 75^\circ$. (*)

Similarly, the right-hand side is: $\sin 75^\circ(\sin 20^\circ \sin(60^\circ - 20^\circ)\sin(60^\circ + 20^\circ)) = \frac{1}{4}\sin 60^\circ \sin 75^\circ$. (**)

By comparing (*) with (**) we receive the desired equality. □

Problem 10.4.5 (SC-23-HS-1-P10). (*5 points*) Prove that

$$\tan 50^\circ + \tan 60^\circ + \tan 70^\circ = \tan 80^\circ$$

> **Remark.** The key is to see that $50^\circ + 60^\circ + 70^\circ = 180^\circ$, then $\tan 50^\circ + \tan 60^\circ + \tan 70^\circ = \tan 50^\circ \tan 60^\circ \tan 70^\circ$.

Solution. Note that $50^\circ + 60^\circ + 70^\circ = 180^\circ$, hence

$$\tan 50^\circ + \tan 60^\circ + \tan 70^\circ = \tan 50^\circ \tan 60^\circ \tan 70^\circ.$$

$$\begin{aligned}
\tan 50^\circ \tan 70^\circ &= \tan(60^\circ - 10^\circ)\tan(60^\circ + 10^\circ) = \frac{\tan 60^\circ - \tan 10^\circ}{1 + \tan 60^\circ \tan 10^\circ} \cdot \frac{\tan 60^\circ + \tan 10^\circ}{1 - \tan 60^\circ \tan 10^\circ}\\
&= \frac{(\tan 60^\circ)^2 - (\tan 10^\circ)^2}{1 - (\tan 60^\circ \tan 10^\circ)^2} = \frac{3 - (\tan 10^\circ)^2}{1 - 3(\tan 10^\circ)^2} = \frac{1}{\tan 10^\circ} \cdot \frac{3\tan 10^\circ - (\tan 10^\circ)^3}{1 - 3(\tan 10^\circ)^2}\\
&= \frac{1}{\tan 10^\circ} \cdot \tan(3 \cdot 10^\circ) = \frac{\tan 30^\circ}{\tan 10^\circ}\\
\tan 50^\circ \tan 60^\circ \tan 70^\circ &= \frac{\tan 30^\circ \tan 60^\circ}{\tan 10^\circ} = \frac{\cot 60^\circ \tan 60^\circ}{\tan 10^\circ} = \frac{1}{\tan 10^\circ} = \tan 80^\circ
\end{aligned}$$

□

Problem 10.4.6 (SC-23-HS-1-P11). (*10 points*) Prove that

$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \ldots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}$$

Solution. It is easy to prove that for any x, y,

Claim —

$$\tan x - \tan y = \frac{\sin(x-y)}{\cos x \cos y}.$$

Note that

$$\frac{\sin 1^\circ}{\cos k^\circ \cos (k+1)^\circ} = \frac{\sin\left((k+1)^\circ - k^\circ\right)}{\cos k^\circ \cos (k+1)^\circ} = \tan (k+1)^\circ - \tan k^\circ$$

Thus, the product of the left-hand side sum with $\sin 1^\circ$ is equal to a telescoping sum:

$$\sum k = 0^{88^\circ} \tan (k+1)^\circ - \tan k^\circ = \tan 89^\circ - \tan 0^\circ = \tan 89^\circ = \cot 1^\circ$$

Hence, the left-hand side sum is equal to $\frac{\cot 1^\circ}{\sin 1^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}$. □

Problem 10.4.7 (SC-23-HS-1-P12). (*15 points*) Let $a_1, a_2, \ldots, a_n$ be the positive real numbers such that

$$a_1 a_2 \cdots a_n = 1$$

Prove that

$$\frac{a_1}{1+a_1} + \frac{a_2}{(1+a_1)(1+a_2)} + \frac{a_3}{(1+a_1)(1+a_2)(1+a_3)} + \cdots + \frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)} \geq \frac{2^n - 1}{2^n}.$$

Solution. It is fairly easy to notice that

$$\begin{aligned} \frac{a_k}{(1+a_1)(1+a_2)\cdots(1+a_k)} &= \frac{(1+a_k)-1}{(1+a_1)(1+a_2)\cdots(1+a_k)} \\ &= \frac{1}{(1+a_1)(1+a_2)\cdots(1+a_{k-1})} - \frac{1}{(1+a_1)(1+a_2)\cdots(1+a_k)}. \end{aligned}$$

Thus, the desired inequality with the left-hand side a telescoping sum, is equivalent to:

$$1 - \frac{1}{(1+a_1)(1+a_2)\cdots(1+a_n)} \geq \frac{2^n-1}{2^n} \Leftrightarrow \frac{1}{2^n} \geq \frac{1}{(1+a_1)(1+a_2)\cdots(1+a_n)} \Leftrightarrow (1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n$$

By AM-GM inequality,

$$(1+a_1)(1+a_2)\cdots(1+a_n) = \prod_{k=1}^{n}(1+a_k) \geq \prod_{k=1}^{n} 2\sqrt{a_k} = 2^n \sqrt{\prod_{k=1}^{n} a_k} = 2^n \sqrt{a_1 a_2 \cdots a_n} = 2^n.$$

□

Chapter 11

Recurrence Relations

11.1 Examples

You should review the **Recurrence Relations** chapter in the **Learning Problem Solving, Vol. 3** to understand the basics, practice with examples, and master a few techniques.

Example 11.1.1 (SC-23-HS-2-E1)
Find the number of subsets of $1, 2, \ldots, n$ that contains no two consecutive elements.

Remark. Here we employ a technique called *take a look at the last element.*

Solution. First, let call a set of numbers *good* if it contains no two consecutive integers. Let a_n denote the number of such sets. Let S be a *good* subset of the set $1, 2, \ldots, n$, and lets *take a look at* n.

Case 1: S contains n. It is obvious that $n-1$ is not in S, thus S contains a *good* subset of the set $\{1, 2, \ldots, n-2\}$ and the number n. Thus the number of such S subsets is a_{n-2}.

Case 2: S does not contain n. In this case S simply a *good* subset of the set $\{1, 2, \ldots, n-1\}$. The number of such S subsets is a_{n-1}.

Therefore $a_n = a_{n-1}+a_{n-2}$. It is easy to see that $a_1 = 1, a_2 = 1$, thus $\boxed{\{a_n\} \text{ is the Fibonacci sequence.}}$ □

Example 11.1.2 (SC-23-HS-2-E2)
Find a recurrence relation for the number of partitions of an $n-$set.

Remark. A partition of a set is a grouping of its elements into non-empty subsets, in such a way that every element is included in exactly one subset.

The set $\{1, 2, 3\}$ has these five partitions (one partition per item):

1. $\{1\}, \{2\}, \{3\}$, sometimes written as $1 \mid 2 \mid 3$.
2. $\{1, 2\}, \{3\}$, or $1\ 2 \mid 3$.
3. $\{1, 3\}, \{2\}$, or $1\ 3 \mid 2$.
4. $\{1\}, \{2, 3\}$, or $1 \mid 2\ 3$.
5. $\{1, 2, 3\}$, or 123 (in contexts where there will be no confusion with the number).

Solution. Let P_n be the number of partitions of the set $\{1, 2, \dots, n\}$. Let *take a look at the element* $n+1$.

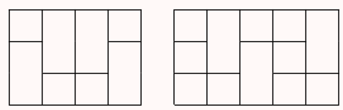
Consider a block containing $n+1$. Suppose that it contains k elements. These elements can be choosen in $\binom{n}{k}$ ways. The remaining $n-k$ elements can be partitioned into P_{n-k} blocks. Since k can be any number from 0 to n, by the rule of *product-sum rule*:

$$P_{n+1} = \sum_{k=0}^{n} \binom{n}{k} P_{n-k} = \sum_{r=0}^{n} \binom{n}{r} P_r.$$

□

Example 11.1.3 (SC-23-HS-2-E3)

We are given sufficiently many stones of the forms of a rectangle 2×1 and square 1×1. Let $n > 3$ be a natural number. In how many ways can one tile a rectangle $3 \times n$ using these stones, so that no two 2×1 rectangles have a common point, and each of them has the longer side parallel to the shorter side of the big rectangle? Below are two examples for $n = 4$ and $n = 5$.



Solution. First, let a tiling *good* if no two 2×1 rectangles have a common point. Let a_n be the number of such tilings.

We start from $n = 1$. If all 2×1 rectangles have their longer sides parallel to the shorter side of the big rectangle, then there are three ways: (i) two ways of tiling with one 2×1 rectangle and one 1×1 square, and one way with three 1×1 squares. Thus, $a_1 = 3$.

Similarly $a_2 = 5$ since there are four ways to place one 2×1 rectangle, one way with 1×1 squares only.

Now, let consider a general situation of $3 \times (n+1)$ rectangle. We have $n+1$ *vertical bands* of length 3. Lets *take a look at the* $n+1$ *band.*

Case 1: If a 2×1 rectangle is placed in the $(n+1)^{\text{th}}$ band, then there is no 2×1 rectangle in the n^{th} band, thus that band contains only 1×1 squares. There are two ways to place the 2×1 rectangle in in the $(n+1)^{\text{th}}$ band, hence the number of *good* tilings in this case is *twice* the number of *good* tilings of the $3 \times (n-1)$ rectangle, or $2a_{n-1}$.

Case 2: If a 2×1 rectangle is not placed in the $(n+1)^{\text{th}}$ band, then that band contains only 1×1 squares, and there is a single way to do so. Thus he number of *good* tilings in this case is the number of *good* tilings of the $3 \times n$ rectangle, or a_n.

Therefore

$$a_{n+1} = a_n + 2a_{n-1} \quad (*)$$

The characteristic polynomial of (*) is $P(x) = x^2 - x - 2$, with two roots $r_1 = 2, r_2 = -1$, thus the generic form of a_n is as below:

$$a_n = c_1 2^n + c_2(-1)^n \quad (**)$$

To determine the coefficients c_1 and c_2, we need to plug the values of a_1 and a_2 to obtain a system of linear equations:

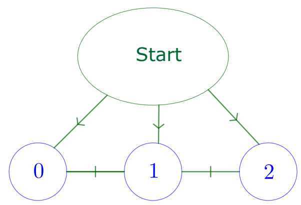
$$\begin{cases} 3 = 2c_1 - c2 \\ 5 = 4x_1 + c_2 \end{cases} \Rightarrow c_1 = \frac{4}{3}, \; c_2 = -\frac{1}{3}$$

Hence, $\boxed{a_n = \frac{1}{3}\left(2^{n+2} + (-1)^{n+1}\right).}$ □

Example 11.1.4 (SC-23-HS-2-E4)

How many $n-$words from the alphabet $\{0, 1, 2\}$ are such that neighbouring letters differ at most by 1?

Solution. In the graph show below, a word can be generated by walking from the *Start*-labeled node, then repeat within a node $0, 1$, or 2, or travel to a node connected with an edge.



Let a_n be the number of $n-$words generated in such way. Then the number of $(n-1)-$words staring from 1 is x_{n-1}. The numbers of $(n-1)-$words staring from 0 or 2 are the same, and let y_{n-1} be that number. Thus,

$$\begin{cases} x_n = x_{n-1} + 2y_{n-1} \quad (1) \\ y_n = y_{n-1} + x_{n-1} \quad (2) \end{cases}$$

From (1), we have $2y_{n-1} = x_n - x_{n-1}$, then $2y_n = x_{n+1} - x_n$. Substitute these into (2), then

$$x_{n+1} = 2x_n + x_{n-1} \quad (*)$$

By solving the characteristic polynomial of (*), we obtain

$$x_n = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n.$$

It is easy to see that $x_1 = 3, x_2 = 7$. Thus by substitution, then solving the system of linear equations as in the previous example $\boxed{x_n = \frac{1}{2}\left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right).}$ □

Example 11.1.5 (SC-23-HS-2-E5)

Let R_n be the number of ways to place n undistinguishable rooks peacefully on a $n \times n$ chessboard. Let H_n be the number of those placings, which are invariant with respect to a half-turn of the board. Find formulas for R_n and H_n.

Solution. A placing for R_n is similar to chose n cells, each from a distinct column. Then since no two cells are on the same row, so their row numbers are distinct. Thus is it the same a the number of permutations of $\{1, 2, \ldots, n\}$ or $\boxed{R_n = n!.}$

Now, consider a $2n \times 2n$ board. In the first column, a rook can be place in $2n$ ways. Then, the rook in the last column is also fixed because of the half-turn. We are left with a $(2n-2) \times (2n-2)$ board to be filled.

$$H_{2n} = (2n)H_{2n-2} \Rightarrow H_{2n} = 2^n n!.$$

In a $(2n+1) \times (2n+1)$ board, the central cell remains fixed and must be occupied by a rook. Then we are left with a $2n \times 2n$ board, thus

$$H_{2n+1} = H_{2n} = 2^n n!.$$

Therefore $\boxed{H_n = 2^{\lfloor \frac{n}{2} \rfloor} \left\lfloor \frac{n}{2} \right\rfloor !}.$ □

Note that in competition, in order to get partial points or to rescue a solution based on some right ideas but somehow producing wrong answer, do not forget to submit your sketches or heuristics like what we have done in the remark analysis shown above. If you write them clean and clear, there are chances you will get additional points. They would not withstand a rigorous check, but there is no such need.

11.2 Problems

Submission deadline: July 05, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 11.2.1 (SC-23-HS-2-P6)**.** Let R_n be the number of ways to place n undistinguishable rooks peacefully on a $n \times n$ chessboard. Let Q_n, M_n, and D_n be the number of those placings, which are invariant with respect to a quarter-turn, reflection at a diagonal, and reflection at both diagonals of the board, respectively.

(*5 points*) Find formula for Q_n.

(*5 points*) Find formula for M_n.

(*5 points*) Find formula for D_n.

Problem 11.2.2 (SC-23-HS-2-P7)**.** (*5 points*) $2n$ marbles of each of three different colours were given to Alice and Bob, so that each person gets $3n$ marbles. Prove that this can be done in $3n^2 + 3n + 1$ ways.

Problem 11.2.3 (SC-23-HS-2-P8)**.** (*10 points*) In how many ways can you select two disjoint subsets from the set $\{1, 2, \dots, n\}$?

Problem 11.2.4 (SC-23-HS-2-P9)**.** (*10 points*) Find the number of strings of n letters ($n-$words), each letter equal to A, B, C, or D such that the same letter never occurs three times in succession.

Problem 11.2.5 (SC-23-HS-2-P10)**.** (*10 points*) Find the number of functions $f : \{1, 2, \dots, n\} \to \{1, 2, 3, 4, 5\}$, such that

$$|f(k+1) - f(k) \geq 3, \ \forall k = 1, 2, \dots, n-1.$$

11.3 Solutions

Problem 11.3.1 (SC-23-HS-2-P6)**.** Let R_n be the number of ways to place n undistinguishable rooks peacefully on a $n \times n$ chessboard. Let Q_n, M_n, and D_n be the number of those placings, which are invariant with respect to a quarter-turn, reflection at a diagonal, and reflection at both diagonals of the board, respectively.

(*5 points*) Find formula for Q_n.

(*5 points*) Find formula for M_n.

(*5 points*) Find formula for D_n.

Solution. For the first question, consider a $4n \times 4n$ board. In the first column, there are $4n-2$ ways to place a rook, since the corner cells must be left free. Then 4 rows and 4 columns are eliminated, and we are left with a $(4n-4) \times (4n-4)$ board. Thus

$$Q_{4n} = (4n-2)Q_{4n-4} = (4n-2)(4n-6)Q_{4n-8} = 2(2n-1)2(2n-3)Q_{4n-4} = 2^n(2n-1)(2n-3)\cdots 3 \cdot 1.$$

In a $(4n+1) \times (4n+1)$ board with the central cell fixed and occupied, we are left with a $4n \times 4n$ board

$\boxed{Q_{4n+1} = Q_{4n}.}$ It is easy to see that $\boxed{Q_{4n+2} = Q4n+3 = 0.}$

For the second question, if a rook is placed on a diagonal in the first column, then we have a $(n-1)\times(n-1)$ board. If it is placed elsewhere in the other $(n-1)$ cells, then we are left with a $(n-2) \times (n-2)$ board. Thus

$$\boxed{M_n = M_{n-1} + (n-1)M_{n-2}.}$$

For the last question, in the first column of a $2n \times 2n$ board, there are two ways to place the rook on a diagonal and $2n-2$ other ways. In the first case we are left with a $(2n-2) \times (2n-2)$ board and in the second case, with a $(2n-4) \times (2n-4)$ board. Thus

$$\boxed{D_{2n} = 2D_{2n-2} + (2n-2)D_{2n-4}, \text{ and } D_{2n+1} = D_{2n}.}$$

□

Problem 11.3.2 (SC-23-HS-2-P7)**.** (*5 points*) $2n$ marbles of each of three different colours were given to Alice and Bob, so that each person gets $3n$ marbles. Prove that this can be done in $3n^2+3n+1$ ways.

Solution. If the Alice gets $n-p$ marbles of the first colour (where p is a non-negative integer), then she gets p to $2n$ marbles of of the second colour. The remaining marbles are of the third colour. For each p, there are $2n-p+1$ ways, thus the sum is

$$\sum_{p=0}^{n}(2n-p+1) = (n+1)(2n+1) - \frac{n(n+1)}{2} = \frac{(n+1)(3n+2)}{2}.$$

If the Alice gets $n+p$ marbles of the first colour (where p is a positive integer), then she gates 0 to $2n-p$ marbles of of the second colour. The remaining marbles are of the third colour. For each p, there are $2n-p+1$ ways, thus the sum is

$$\sum_{p=1}^{n}(2n-p+1) = n(2n+1) - \frac{n(n+1)}{2} = \frac{n(3n+1)}{2}.$$

The total sum is

$$\frac{(n+1)(3n+2)}{2} + \frac{n(3n+1)}{2} = \boxed{3n^2+2n+1.}$$

□

Problem 11.3.3 (SC-23-HS-2-P8)**.** (*10 points*) In how many ways can you select two disjoint subsets from the set $\{1, 2, \ldots, n\}$?

Solution. For an *ordered* pair (A, B) of disjoint subsets, we define a characteristic function

$$f(x) = \begin{cases} 1 \text{ if } x \in A, \\ 2 \text{ if } x \in B, \\ 0 \text{ otherwise} \end{cases}$$

Then applying the function f on each of the elements of the set $\{1, 2, \ldots, n\}$:

$$f(1)f(2)f(3)\cdots f(n)$$

is an $n-$word from the alphabet $\{0, 1, 2\}$. The number of possible functions is 3^n. There are 2^n words with letters from $\{0, 2\}$ (when A is empty), 2^n $n-$words with letters from $\{0, 1\}$ (when B is empty), and 1 $n-$word consisting entirely of zeros (when both A and B are empty). Thus, by the Principle of Inclusion-Exclusion, the number of ordered disjoint pairs is $3^n - 2^n - 2^n + 1$. Thus, the number of unordered pairs is

$$\boxed{\frac{3^n + 1}{2} - 2^n.}$$

□

Problem 11.3.4 (SC-23-HS-2-P9)**.** (*10 points*) Find the number of strings of n letters ($n-$words), each letter equal to A, B, C, or D such that the same letter never occurs three times in succession.

Solution. Lets call such string is a *good* string. Lets look for recursion when we try to add a new letter to a string of length $n - 1$. There are usually 4 choices, but if the string ends in a double letter, then we can not adjoin this letter again and there are only three choices. The idea is to define two sequences, let a_n be the number of good strings with length n, and b_n be the number of good strings with length n ending in a double letter.

First, there are $a_{n-1} - b_{n-1}$ strings of length $n - 1$, each of which does not end in a double letter. There are 4 ways we can add a letter to get a good strong of length n. For each of b strings of length $n - 1$ ending in a double letter, there are 3 ways to add a letter, thus

$$a_n = 4(an - 1 - b_{n-1}) + 3b_{n-1} = 4a_{n-1} - b_{n-1}.$$

Second, if a good string of length $n - 1$ does not end in a double letter, then by repeating the last letter we get a good string with length n ending with a double letter. If a good string of length $n - 1$ already ends in a double letter, then there is no way to add a letter to have a good string length n ending in a double letter. Therefore:

$$b_n = a_{n-1} - b_{n-1}.$$

Now

$$\begin{aligned} &b_{n-1} = 4a_{n-1} - a_n \Rightarrow a_{n-1} = b_n + b_{n-1} = 4a_n - a_{n+1} + 4a_{n-1} - a_n \\ &\Rightarrow a_{n+1} = 3a_n + 3a_{n-1}. \end{aligned}$$

The characteristic polynomial is $r^2 - 3r - 3$, with roots $r_1 = \dfrac{3 + \sqrt{21}}{2}$, $r_2 = \dfrac{3 - \sqrt{21}}{2}$. Therefore by substituting $a_1 = 4, a_2 = 16$ into $a_n = c_1 r_1^n + c_2 r_2^n$ and solving a system of two linear equations, then

$$\boxed{a_n = \frac{4\sqrt{21}}{63}\left(\left(\frac{3 + \sqrt{21}}{2}\right)^{n+1} - \left(\frac{3 - \sqrt{21}}{2}\right)^{n+1}\right).}$$

□

Problem 11.3.5 (SC-23-HS-2-P10). (*10 points*) Find the number of functions $f : \{1, 2, \ldots, n\} \to \{1, 2, 3, 4, 5\}$, such that

$$|f(k+1) - f(k) \geq 3, \ \forall k = 1, 2, \ldots, n-1.$$

Solution. The idea is quite simple: we have to split up the counting process according to the value of $f(n)$ and then obtain recursions. If $n = 1$ it is easy to see that there are 5 such functions. However, if $n \geq 2$ the function $f(n)$ can never take on the value 3 since it would contradict the inequality.

First, let a_n, b_n, c_n, and d_n be the number of such functions with $f(n) = 1$, $f(n) = 2$, $f(n) = 4$, and $f(n) = 5$, respectively. Now let $s_n = a_n + b_n + c_n + d_n$, $n \geq 2$. Note that $s_1 = 5$.

Now, for $f(n) = 1$, $f(n-1)$ can be 4 or 5, so $a_n = c_{n-1} + d_{n-1}$.

For $f(n) = 2$, $f(n-1)$ can be 5, so $b_n = d_{n-1}$.

For $f(n) = 4$, $f(n-1)$ can be 1, so $c_n = a_{n-1}$.

For $f(n) = 5$, $f(n-1)$ can be 1 or 2, so $d_n = a_{n-1} + b_{n-1}$.

Hence,

$$\begin{cases} a_n + d_n = c_{n-1} + d_{n-1} + a_{n-1} + b_{n-1} = s_{n-1} \\ b_n + c_n = d_{n-1} + a_{n-1} = s_{n-2} \end{cases} \Rightarrow s_n = s_{n-1} + s_{n-2} \ \forall n \geq 4$$

By direct calculation, $s_2 = 6, s_3 = 10$, It is easy to see that $\boxed{s_n = 2f_{n+2}}$, where $\{f_n\}$ is the Fibonacci sequence. $\square$

Chapter 12

Ceva and Menelaus

12.1 Examples

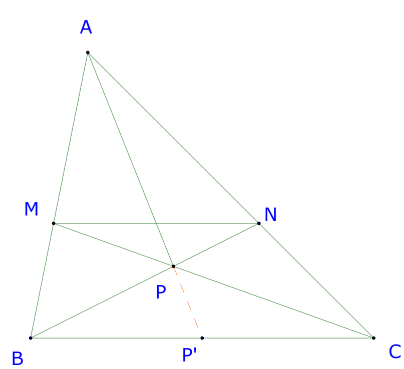
You should review the **Lengths and Ratios** and **Angle Chasing** chapters in the **Learning Problem Solving, Vol. 3** to understand the basics, practice with examples, and master a few techniques.

Definition (Cevian)**.** In a triangle, the segments joining a vertex of a triangle with a point on the opposite side is called a **cevian**.

> **Remark.** *Cevian theorem* discusses the condition if and only if the *three cevians* are *concurrent (all meet at a single point)*. *Menelaus theorem* discusses the condition if and only if the *three points*, each on a distinct side of triangle, are *collinear (all on a single line)*

Example 12.1.1 (SC-23-HS-3-E1)

Points M, N on the sides AB, AC of the $\triangle ABC$ satisfy $MN \parallel BC$. Prove that lines BN and CM intersect on the median from A of the $\triangle ABC$.



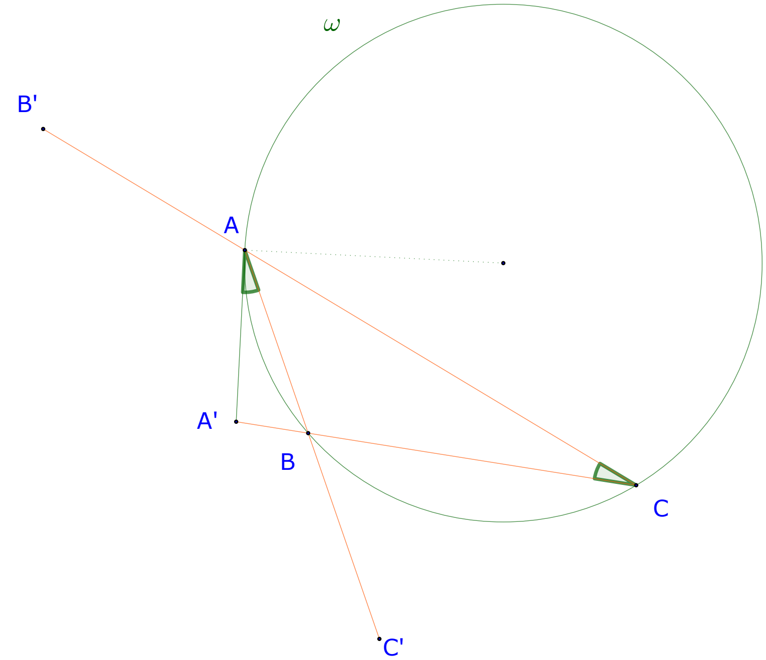
Solution. Since $MN \parallel BC$, thus $\dfrac{AM}{MB} = \dfrac{AN}{NC}$. Let P be the intersection of BN and CM, P' be the intersection of AP with BC. Then by Ceva theorem for concurrent cevians AP', BN, and CM,

$$\frac{BP'}{P'C} \cdot \frac{CN}{NA} \cdot \frac{AM}{MB} = 1 \Rightarrow \frac{BP'}{P'C} = \frac{AN}{NC} \cdot \frac{MB}{AM} = 1 \Rightarrow P' \text{ is midpoint of } BC.$$

□

Example 12.1.2 (SC-23-HS-3-E2)

Let ω be the circumcircle of $\triangle ABC$ and let the tangent to ω at A intersect BC at A'. In a similar way, define B' and C'. Prove that A', B', and C' are collinear.

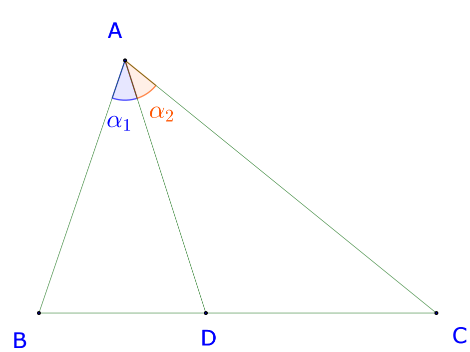


Remark. Let's get familiar with a useful lemma that can easily bve proved by using the *Law of Sines.*

Lemma (Ratio Lemma)

In $\triangle ABC$ let D be a point on segment BC (not necessarily between B and C) and let α_1 and α_2 be the angles $\angle BAD$ and $\angle DAC$, respectively, then

$$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin \alpha_1}{\sin \alpha_2}$$



Solution. By the the Ratio Lemma and note that $\angle A'AB = \angle ACB$ and $\angle A'AC = 180^\circ - \angle ABC$,

$$\frac{A'B}{A'C} = \frac{AB}{AC} \cdot \frac{\sin \angle A'AB}{\sin \angle A'AC} = \frac{AB}{AC} \cdot \frac{\sin \angle C}{\sin \angle B} = \left(\frac{AB}{AC}\right)^2.$$

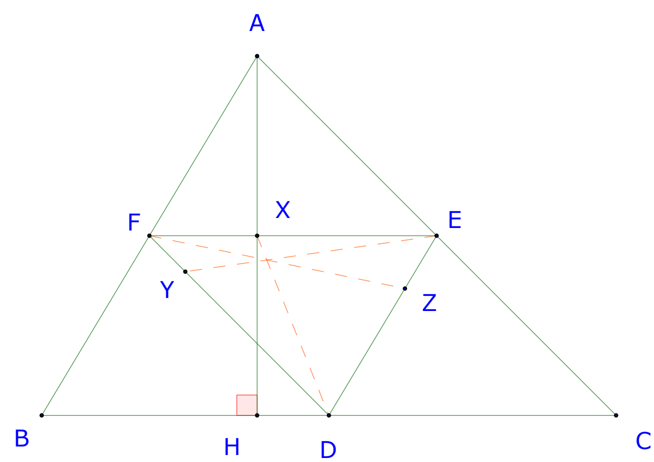
Thus

$$\frac{A'B}{A'C}\frac{B'C}{B'A}\frac{C'A}{C'B} = 1 \Rightarrow A', B', C' \text{ are collinear.}$$

□

Example 12.1.3 (SC-23-HS-3-E3)

Let ABC be a triangle. Prove that the lines joining midpoints of the sides with midpoints of the corresponding altitudes pass through a single point.



Remark. We show an **alternative way to construct** the midpoint of an altitude and use **the property derived from the construction** for the proof.

Solution. Let D, E, F be the midpoints of the sides BC, CA, AB, respectively. Let H be the foot of the altitude from A to BC, and let X be the intersection of AH and EF. It is easy to see that X is the midpoint of the AH altitude. Y and Z are constructed in similar way. Now, lets take a look at the ratio, since $EF \parallel CB$, thus,

$$\frac{EX}{XF} = \frac{CH}{HB}, \text{ and similarly } \frac{FY}{YD} = \frac{AG}{GC}, \frac{DZ}{ZE} = \frac{BI}{IA},$$

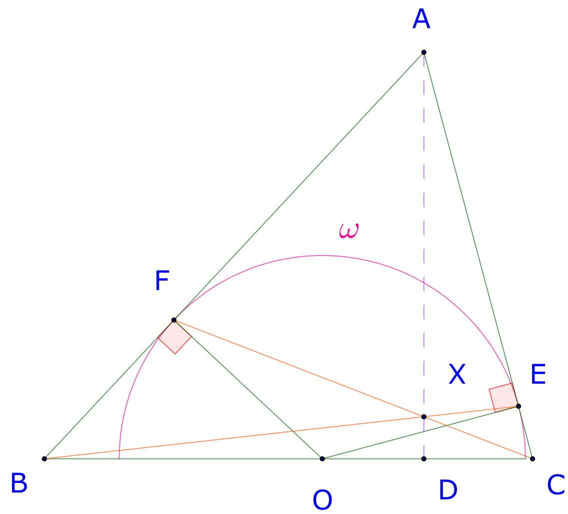
where G and I and the feet of the altitudes from B and C, respectively. Thus,

$$\frac{EX}{XF} \cdot \frac{FY}{YD} \cdot \frac{DZ}{ZE} = \frac{CH}{HB} \cdot \frac{AG}{GC} \cdot \frac{BI}{IA}$$

The right side product is 1 since the three altitudes in a triangle are concurrent, so are DX, EY, and FZ. □

Example 12.1.4 (SC-23-HS-3-E4)

In an acute triangle ABC a semicircle ω centered on the side BC is tangent to the side AB and AC at points F and E, respectively. If X is the intersection of BE and CF, show that $AX \perp BC$.



Remark. Since we have to prove a property for D, the intersection of AX with BC, where X is intersection of BE and CF, so we show another **synthetic proof** by first select D point with such a property then prove that AD, BE, CF are concurrent, then $AX \equiv AD$.

Solution. Let D be the foot of the altitude from A, lets prove that AD, BE, CF are concurrent,

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Let r be the radius of the circle, then $FB = r \cot \angle B, EC = r \cot \angle C$, thus

$$\frac{FB}{CE} = \frac{\cot \angle B}{\cot \angle C} (*)$$

On the other hand, $BD = AD \cot \angle B, DC = AD \cot \angle C$,

$$\frac{BD}{DC} = \frac{\cot \angle B}{\cot \angle C} (**)$$

From (*) and (**), and note that $AE = AF$ (equal tangent segments from A to E and F, respectively),

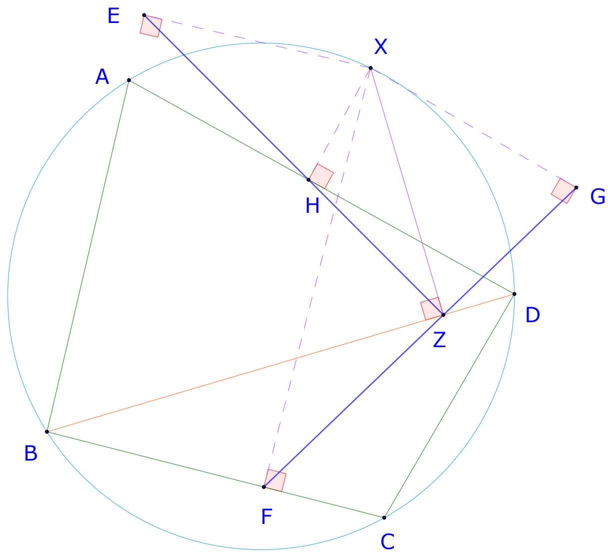
$$\frac{BD}{DC} = \frac{FB}{CE} \Rightarrow \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

□

Example 12.1.5 (SC-23-HS-3-E5)

Let X be a point on the circumcircle of a cyclic quadrilateral $ABCD$. Denote E, F, G, and H the projections of X onto lines AB, BC, CD, DA, respectively. Prove that

$$BE \cdot CF \cdot DG \cdot AH = AE \cdot BF \cdot CG \cdot DH$$



> **Remark.** In this example we use an **additional point** and the **Simson line theorem**.

Solution. Note that $ABCD$ is cyclic also mean ABD and BCD are cyclic(!).

Let Z be the foot of the altitude from X to the diagonal BD. By Simson line theorem E, Z, H are collinear. Similarly Z, G, F are collinear too.

From here by applying the Menelaus's theorem twice for $\triangle BAD$ and points E, Z, H, then for $\triangle BCD$ and points Z, G, F, we obtain

$$\frac{BZ}{ZD} \cdot \frac{DH}{HA} \cdot \frac{AE}{EB} = 1, \ \frac{BZ}{ZD} \cdot \frac{DG}{GC} \cdot \frac{CF}{FB} = 1 \Rightarrow \frac{DH}{HA} \cdot \frac{AE}{EB} = \frac{ZD}{BZ} = \frac{DG}{GC} \cdot \frac{CF}{FB}$$
$$\Rightarrow DH \cdot AE \cdot GC \cdot FB = HA \cdot EB \cdot DG \cdot CF \Rightarrow BE \cdot CF \cdot DG \cdot AH = AE \cdot BF \cdot CG \cdot DH$$

□

Note that in competition, in order to get partial points or to rescue a solution based on some right ideas but somehow producing wrong answer, do not forget to submit your sketches or heuristics like what we have done in the remark analysis shown above. If you write them clean and clear, there are chances you will get additional points. They would not withstand a rigorous check, but there is no such need.

Note that for every problem you earn an additional same number of points for every different solution.

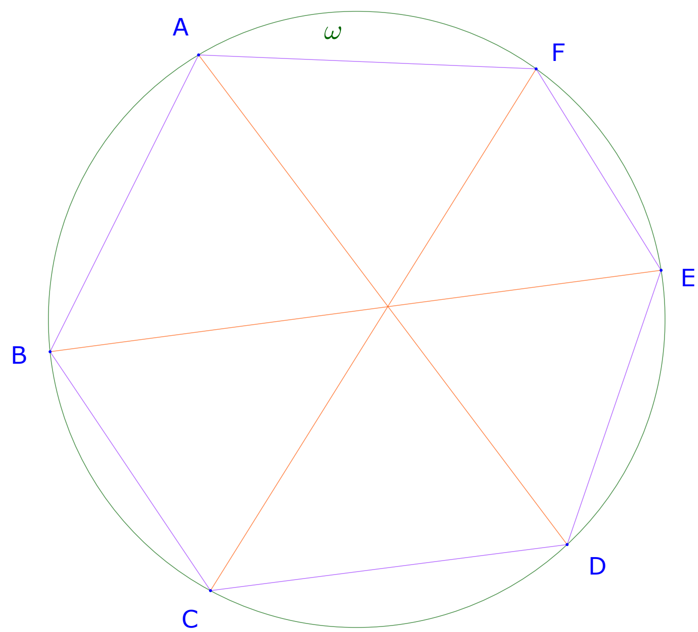
12.2 Problems

Submission deadline: July 12, 2023.

Problem 12.2.1 (SC-23-HS-3-P6)**.** (*10 points*) Let $ABCDEF$ be a hexagon inscribed in a circle ω. Show that the diagonals AD, BE, CF are concurrent if and only if

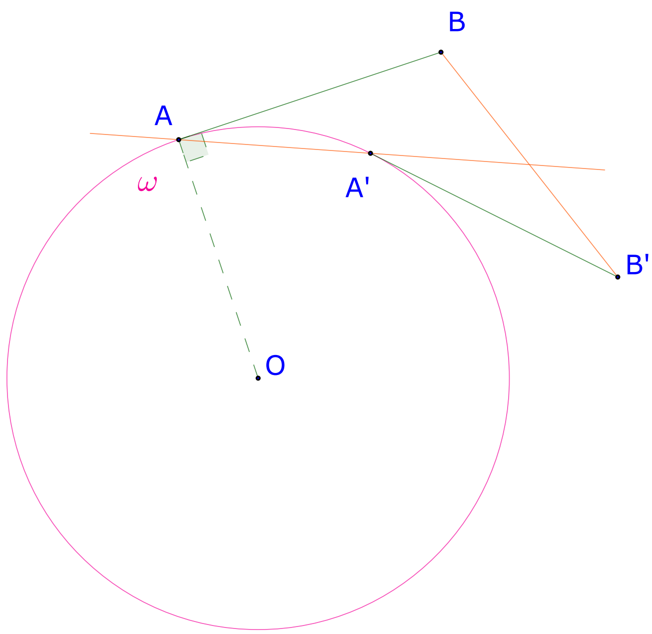
$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$$

> **Remark.** You might find *Ceva Theorem in Trigonometric Form* useful for solving this problem.



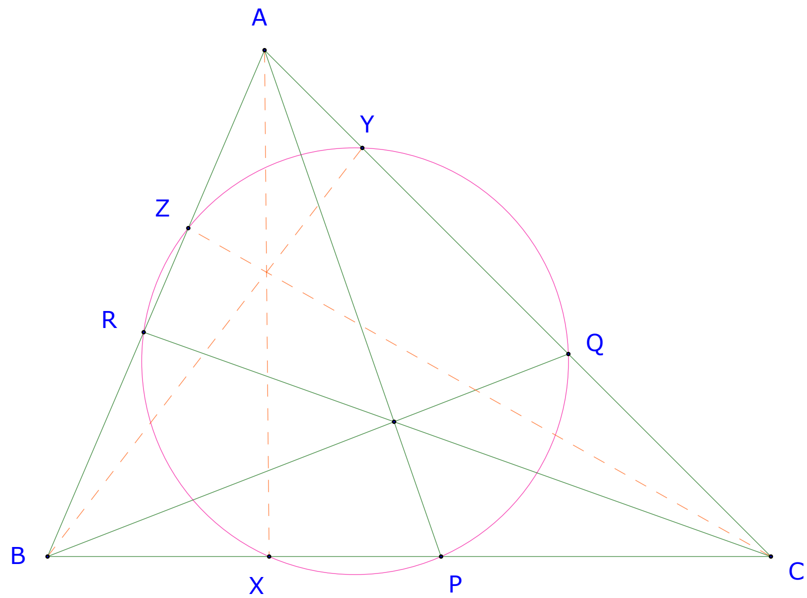
Problem 12.2.2 (SC-23-HS-3-P7)**.** (*10 points*) Point B lies on a line which is tangent to circle ω at point A. The line segment AB is rotated about the center of the circle by some angle to form segment $A'B'$. prove that the line AA' bisects the segment BB'.

> **Remark.** You might consider *Equal Tangents* of a point to both A and A'.



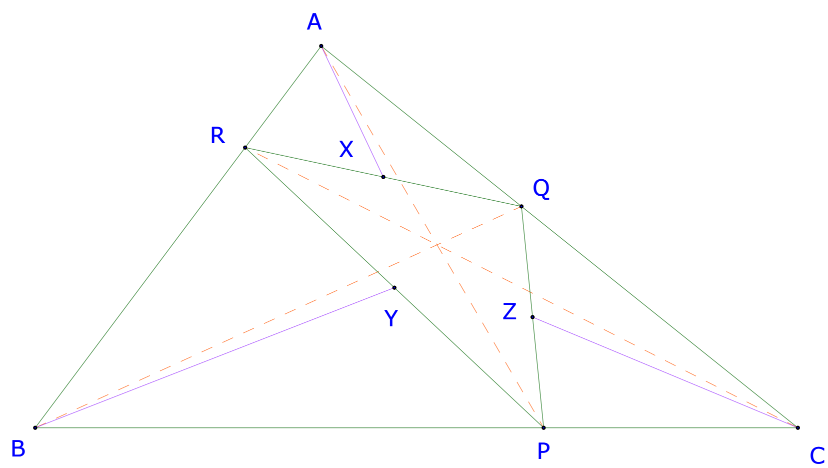
Problem 12.2.3 (SC-23-HS-3-P8)**.** (*10 points*) In triangle ABC, let AP, BQ, CR be concurrent cevians. Let the circumcircle of triangle PQR intersect the side BC, CA, AB for the second time at X, Y, Z, respectively. Prove that AX, BY, CZ also concurrent.

> **Remark.** You might consider *Power of a Point* theorem for A with respect to Z, R, Y, Q.



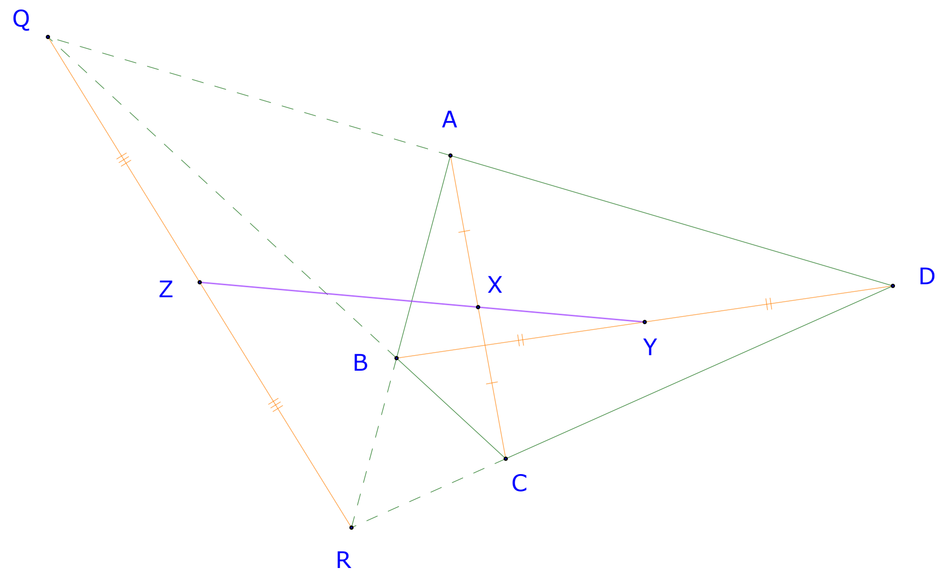
Problem 12.2.4 (SC-23-HS-3-P9)**.** (*10 points*) In triangle ABC, let AP, BQ, CR be concurrent cevians. Denote X, Y, Z the midpoints of segments QR, RP, PQ. Prove that the lines AX, BY, CZ are concurrent.

> **Remark.** You might find both *Ceva Theorem in Trigonometric Form* and *Ratio Lemma* very useful for solving this problem.



Problem 12.2.5 (SC-23-HS-3-P10)**.** (*10 points*) Let $ABCD$ be a convex quadrilateral. Denote by Q the intersection of AD and BC and by R the intersection of AB and CD. Let X, Y, and Z be the midpoints of AC, BD, and QR, respectively. Prove that X, Y, and Z lie on a single line.

> **Remark.** You might find consider $\triangle ABQ$ as a candidate for the application of Menelaus's theorem.



12.3 Solutions

Theorem (Ceva Theorem)

Let ABC be a triangle, and let D, E, F be points on lines BC, CA, AB, respectively. Lines AD, BE, CF are concurrent *if and only if*:

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Theorem (Menelaus Theorem)

Let ABC be a triangle, and let D, F be points on lines BC, AB, respectively. E is on the extension of CA. Points D, E, F are collinear *if and only if*:

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

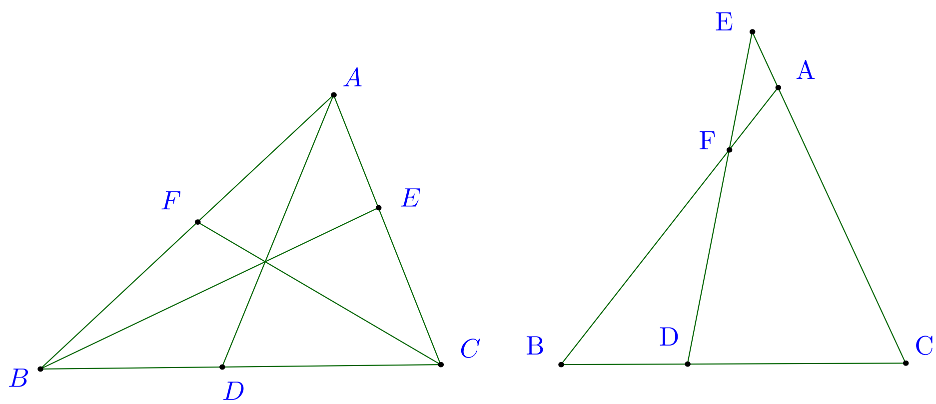


Figure 12.1: Ceva (left) and Menelaus (right) theorems

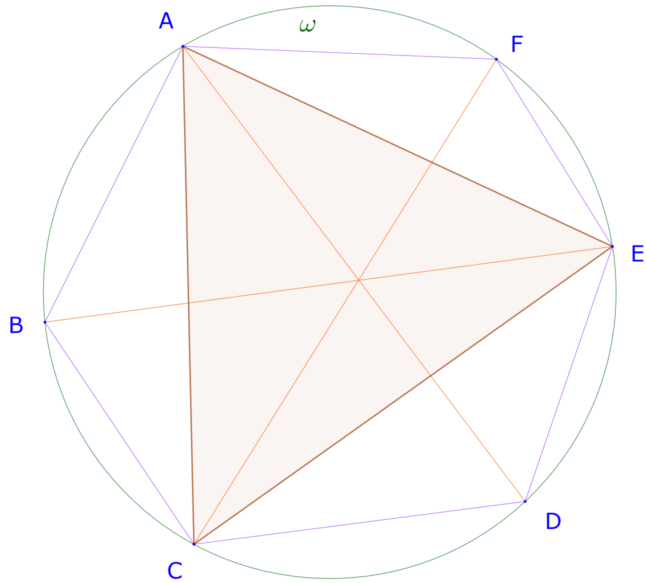
Theorem (Ceva Theorem in Trigonometric Form)

Cevians AD, BE, CF concur if and only if

$$\frac{\sin \angle BAD}{\sin \angle DAC} \cdot \frac{\sin \angle CBE}{\sin \angle EBA} \cdot \frac{\sin \angle ACF}{\sin \angle FCB} = 1.$$

Problem 12.3.1 (SC-23-HS-3-P6). (*10 points*) Let $ABCDEF$ be a hexagon inscribed in a circle ω. Show that the diagonals AD, BE, CF are concurrent if and only if

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$$



> **Remark.** The key idea here is to turn the sides $AB, BC, CD, \ldots$ into the angles $\angle AEB, \angle BEC, \angle CAD, \ldots$ and then using the Ceva Theorem in Trigonometric Form to investigate their relations.

Solution. By the Ceva Theorem in Trigonometric Form for $\triangle ACE$, the cevians AD, CF, EB are concurrent *if and only if* (when applying the theorem, the roles change as $B \to C$, $C \to E$, $E \to F$, $F \to E$)

$$\frac{\sin\angle CAD}{\sin\angle DAE} \cdot \frac{\sin\angle ECF}{\sin\angle FCA} \cdot \frac{\sin\angle AEB}{\sin\angle BEC} = 1 \quad (*)$$

Now, let R be the radius of the circle ω, by the Law of Sines for $\triangle ACD$, whose circumcircle is ω:

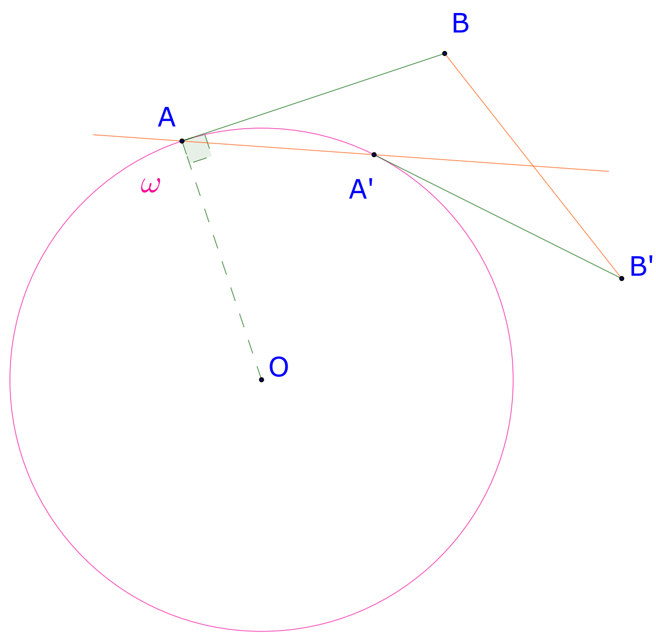
$$\begin{aligned} &CD = 2R\sin\angle CAD, \text{ similarly} \\ &AB = 2R\sin\angle AEB, BC = 2R\sin\angle BEC, \text{ and} \\ &DE = 2R\sin\angle DAE, EF = 2R\sin\angle ECF, FA = 2R\sin\angle FCA, \end{aligned}$$

Thus

$$(*) \Leftrightarrow \frac{CD}{DE} \cdot \frac{EF}{FA} \cdot \frac{AB}{BC} \Leftrightarrow \boxed{AB \cdot CD \cdot EF = BC \cdot DE \cdot FA.}$$

□

Problem 12.3.2 (SC-23-HS-3-P7)**.** (*10 points*) Point B lies on a line which is tangent to circle ω at point A. The line segment AB is rotated about the center of the circle by some angle to form segment $A'B'$. prove that the line AA' bisects the segment BB'.



> **Remark.** The key idea here are: first is to add *hidden* intersection points of AA' and BB', AB and $A'B'$ to discover a configuration where Ceva or Menelaus theorems can be used with ease; then is to use the *equal tangents* of a point to simplify Ceva/Menelaus product of fractions.

Solution. Let E be the intersections of lines through AA' and BB'. Let F be the intersections of lines through AB and $A'B'$. Now,

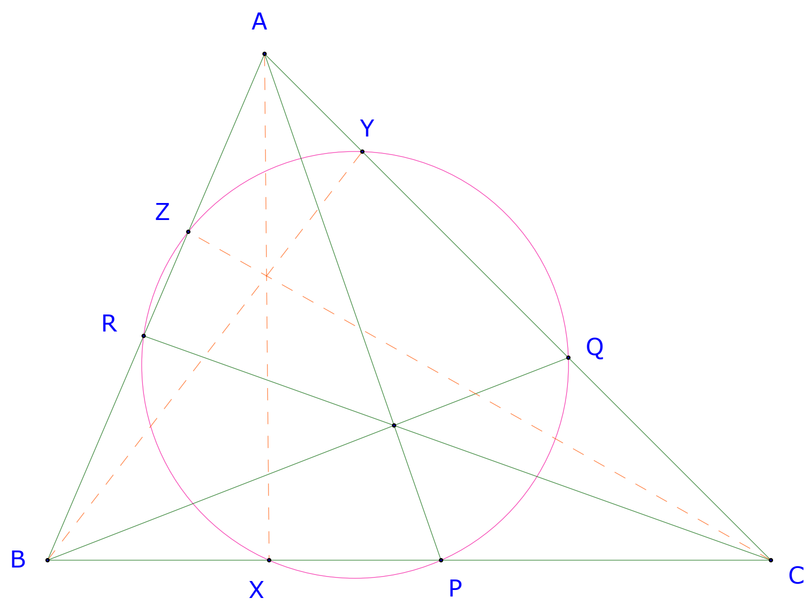
$$FAandFA' \text{ are both tangents of } \omega \Rightarrow FA = FA',$$
$$A'B' \text{ is the image of the rotation of } AB \text{ about the center of } \omega \Rightarrow A'B' = AB$$

By the Menelaus Theorem for $\triangle B'BF$:

$$\frac{B'E}{EB} \cdot \frac{BA}{AF} \cdot \frac{FA'}{A'B'} = 1 \Rightarrow \frac{B'E}{EB} = 1 \Rightarrow \boxed{AA' \text{ bisects } BB'}.$$

□

Problem 12.3.3 (SC-23-HS-3-P8)**.** (*10 points*) In triangle ABC, let AP, BQ, CR be concurrent cevians. Let the circumcircle of triangle PQR intersect the side BC, CA, AB for the second time at X, Y, Z, respectively. Prove that AX, BY, CZ also concurrent.



Remark. The key idea here is to use *power of a point* to replace unknown fractions with known fractions by an existing Ceva configuration.

Solution. AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} \quad (1)$$

By Power-of-the-point of A in respect to Y, Z, Q, R :

$$AY \cdot AQ = AZ \cdot AR \quad (*)$$

Similarly

$$BR \cdot BZ = BX \cdot BP,\ CP \cdot CX = CQ \cdot CY \quad (**)$$

From (*) and (**)

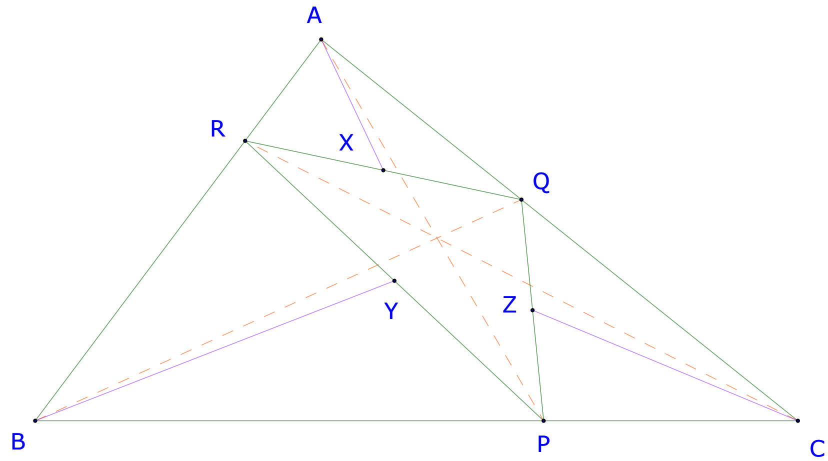
$$\frac{AZ}{AY} = \frac{AQ}{AR}, \frac{BX}{ZB} = \frac{BR}{BP}, \frac{CY}{XC} = \frac{CP}{CQ}$$

Therefore

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{AQ}{AR} \cdot \frac{BR}{BP} \cdot \frac{CP}{CQ} = \frac{AQ}{QC} \cdot \frac{CP}{PB} \cdot \frac{BR}{RA} = 1$$

Hence, $\boxed{AX, BY, CZ \text{ are also concurrent.}}$ □

Problem 12.3.4 (SC-23-HS-3-P9). (*10 points*) In triangle ABC, let AP, BQ, CR be concurrent cevians. Denote X, Y, Z the midpoints of segments QR, RP, PQ. Prove that the lines AX, BY, CZ are concurrent.



Remark. First, since it is difficult to determine the intersections of AX, BY, CZ with BC, CA, AB, respectively, thus Ceva Theorem in Trigonometric Form can be considered as a good alternative.

$$AX, BY, CZ \text{ are concurrent } \Leftrightarrow \frac{\sin\angle RAX}{\sin\angle XAQ}\cdot\frac{\sin\angle QCZ}{\sin\angle ZCP}\cdot\frac{\sin\angle PBY}{\sin\angle YBR} = 1.$$

Second, the ratio $\frac{\sin\angle RAX}{\sin\angle XAQ}$ can be determined by the Ratio Lemma.

Solution. First, by the Ratio Lemma for $\triangle ARQ$ and line AX:

$$\frac{RX}{XQ} = \frac{AR}{AQ}\cdot\frac{\sin\angle RAX}{\sin\angle XAQ} \quad (1)$$

Similarly

$$\frac{QZ}{ZP} = \frac{CQ}{CP}\cdot\frac{\sin\angle QCZ}{\sin\angle ZCP},\ \frac{PY}{YR} = \frac{BP}{BR}\cdot\frac{\sin\angle PBY}{\sin\angle YBR} \quad (2)$$

From (1) and (2)

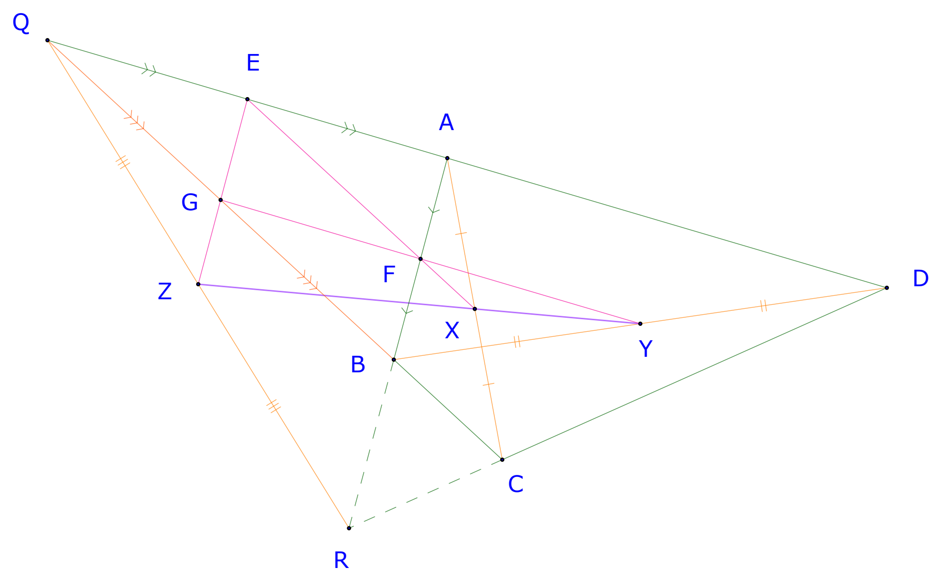
$$\frac{RX}{XQ}\cdot\frac{QZ}{ZP}\cdot\frac{PY}{YR} = \left(\frac{AR}{AQ}\cdot\frac{CQ}{CP}\cdot\frac{BP}{BR}\right)\left(\frac{\sin\angle RAX}{\sin\angle XAQ}\cdot\frac{\sin\angle QCZ}{\sin\angle ZCP}\cdot\frac{\sin\angle PBY}{\sin\angle YBR}\right)$$

Since AP, BQ, CR are concurrent cevians and X, Y, Z are the midpoints of segments QR, RP, PQ, thus

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA} = \frac{AR}{AQ}\cdot\frac{CQ}{CP}\cdot\frac{BP}{BR} = 1,\ \frac{RX}{XQ}\cdot\frac{QZ}{ZP}\cdot\frac{PY}{YR} = 1.$$

Hence, $\frac{\sin\angle RAX}{\sin\angle XAQ}\cdot\frac{\sin\angle QCZ}{\sin\angle ZCP}\cdot\frac{\sin\angle PBY}{\sin\angle YBR} = 1$, or $\boxed{AX, BY, CZ \text{ are concurrent.}}$ $\square$

Problem 12.3.5 (SC-23-HS-3-P10)**.** (*10 points*) Let $ABCD$ be a convex quadrilateral. Denote by Q the intersection of AD and BC and by R the intersection of AB and CD. Let X, Y, and Z be the midpoints of AC, BD, and QR, respectively. Prove that X, Y, and Z lie on a single line.



Definition (Complete quadrilateral)**.** A **complete quadrilateral** is a system of four lines, no three of which pass through the same point, and the six points of intersection of these lines.

Note that the given $ABCD$ convex quadrilateral in the problem SC-23-HS-3-P10 is a complete quadrilateral.

Theorem (Newton-Gauss line)

The three midpoints of the diagonals of a complete quadrilateral are collinear.

In other words, let $ABCD$ be a convex quadrilateral. Denote by Q the intersection of AD and BC and by R the intersection of AB and CD. Let X, Y, and Z be the midpoints of AC, BD, and QR, respectively, then X, Y, and Z lie on a single line.

Solution. [Solution by Menelaus Theorem] Let E, F, and G be the midpoints of QA, AB, and BQ, respectively.

It is easy to see that E, F, X are collinear since in $\triangle AQC$, they are midpoints of AQ, AB, and AC, respectively, and Q, B, C are collinear. Similarly G, F, Y are collinear as well as E, G, Z.

Now, by Menelaus Theorem for $\triangle ABQ$ and line RCD

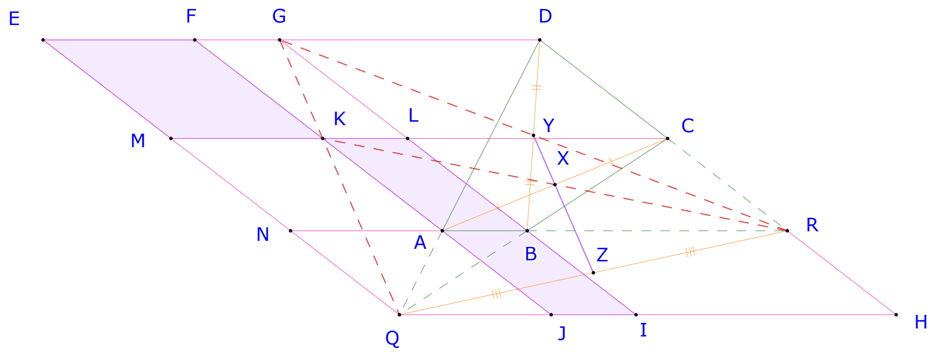
$$1 = \frac{AR}{RB} \cdot \frac{BC}{CQ} \cdot \frac{QD}{DA} = \frac{EZ}{ZG} \cdot \frac{FX}{XE} \cdot \frac{QY}{YF}$$

Thus by Menelaus Theorem for $\triangle EFG$, points $\boxed{X, Y, \text{ and } Z \text{ are collinear.}}$ □

Lemma (Parallelograms with equal areas)

In a parallelogram $ABCD$, lines EF and GH are parallel to the sides AB and AD, respectively. EF, GH, and AC are concurrent if and only if $[ABEF] = [AGHD]$ (and $[GBEK] = [FKHD]$.)

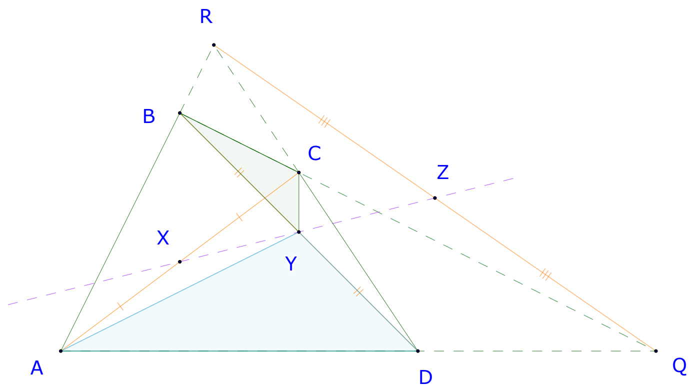
The proof is very simple based on triangles with same base and height should have equal areas.



Solution. [Solution by Parallelograms with equal areas] Draw several lines parallel to AB and CD as shown above. By the Parallelograms with equal areas for parallelogram $EDHQ$ and lines NR, FJ; then the parallelogram $MCHQ$ and lines MC, LI:

$$[EFJQ] = [NRHQ] = [MLIQ] \Rightarrow [EFKM] = [KLIJ] \Rightarrow Q, K, G \text{ collinear.}$$

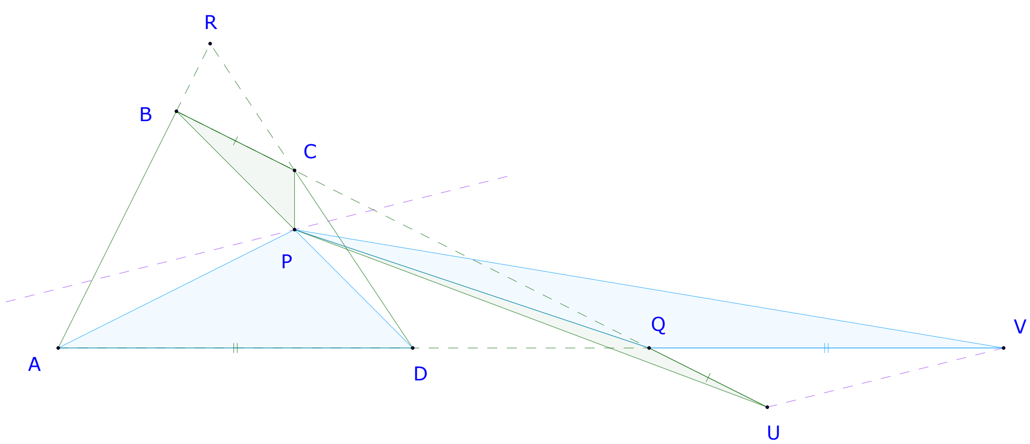
Therefore since Z is a midpoint of RQ, Z and the midpoints of RK and RG are collinear. But the midpoint of RK is the same as the midpoint of AC, which is X, because both are diagonals of parallelogram $ARCK$; the midpoint of RG is the same as the midpoint of BD, which is Y, because both are diagonals of parallelogram $BRDG$. Hence points $\boxed{X, Y, \text{ and } Z \text{ are collinear.}}$ □



Remark. Lets investigate the problem from a different approach. In below diagram we can easily see that

$$[AYD] + [BYC] = \frac{1}{2}[ABD] + \frac{1}{2}[BCA] = \frac{1}{2}[ABCD], \ text\ this is true for\ X, Z \text{ too.}$$

Thus, is it true that the **locus** of all points P, such that $[APD] + [BPC] = \frac{1}{2}[ABCD]$, is a line?



Solution. [Solution by finding locus of point P where $[APD] + [BPC]$ is a constant] Let translate AD to QV and BC to QU, as show above. It is easy to see that

$$[AYD] + [BYC] = [QPV] + [QPU] = [UPV] - [UQV].$$

Now, $[QUV]$ is a constant, thus the locus of points P, where $[AYD] + [BYC]$ is a constant, is the locus of points P, where $[UPV]$ is a constant or a line parallel with UV. Obviously it is the lint through X, Y. Hence the midpoints of QR, Z, should lies on this line. Hence points $\boxed{X, Y, \text{ and } Z \text{ are collinear.}}$ □

Chapter 13

Divisors

13.1 Examples

You should review the **Divisibility** and **Arithmetic Functions** chapters in the **Learning Problem Solving, Vol. 3** to understand the basics, practice with examples, and master a few techniques.

Example 13.1.1 (SC-23-HS-4-E1)

Let $d(n)$ the number of divisors of n, then for any $n \geq 1$,

1. $\sum_{m=1}^{n} d(m) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor$,
2. $d(n) = \sum_{k=1}^{n} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor \right)$.

Solution. First, since k is a divisor of exactly $\left\lfloor \frac{n}{k} \right\rfloor$ of the numbers $\{1, 2, \ldots, n\}$. Thus

$$\sum_{m=1}^{n} d(m) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor.$$

Now, note that

$$\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor = \begin{cases} 1 \text{ if } k \mid n, \\ 0 \text{ otherwise} \end{cases}$$

thus

$$\sum_{k=1}^{n} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor \right) = \sum_{k|n} 1 = d(n).$$

□

Example 13.1.2 (Irish Mathematical Olympiad 1998)

Find all positive integers d that have exactly 16 positive integer divisors $d_1, d_2, \ldots, d_{16}$ such that

$$1 = d_1 < d_2 < d_3 < \ldots < d_{16} = d, \ d_6 = 18, d_9 - d_8 = 17.$$

Solution. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes, then d has $(1+a_1)(1+a_2)\cdots(1+a_k)$ divisors. $18 = 2 \cdot 3^2$, which has 6 divisors: $1, 2, 3, 6, 9$, and 18. $d_6 = 18$, which means the largest power of 2 that divides d can only be 2 (and not 4),

$$\begin{cases} d = 2 \cdot 3^7, \text{ or} \\ d = 2 \cdot 3^3 \cdot p, \ p \text{ prime }, \ p > 18 \end{cases}$$

If $d = 2 \cdot 3^7$ then $d_8 = 54, d_9 = 81$, and $d_9 - d_8 = 27 \neq 17$. Therefore $d = d = 2 \cdot 3^3 \cdot p$.

Case 1: if $d < 27$, then $d_7 = p, d_8 = 27, d_9 = 2p = 27 + 17 \Rightarrow p = 22$, not possible for a prime.

Case 2: if $27 < d < 54$, then $d_7 = 27, d_8 = p, d_9 = 54 = p + 17 \Rightarrow p = 37$.

Case 3: if $54 < d$, then $d_7 = 27, d_8 = 54, d_9 = p = 54 + 17 \Rightarrow p = 71$.

We have two solutions $\{2 \cdot 3^3 \cdot 37, 2 \cdot 3^3 \cdot 71\}$. □

Example 13.1.3 (Russian Mathematical Olympiad 2001)

Does there exists a positive integer such thar the product of its *proper* divisors ends with exactly 2001 zeros?

Solution. Note that the product of n divisors is

$$\sqrt{\left(\prod d \mid nd\right)\left(\prod d \mid n\frac{n}{d}\right)} = \sqrt{\prod d \mid nd\frac{n}{d}} = \sqrt{n^{d(n)}}.$$

Thus, the product of all proper divisors of n is:

$$\sqrt{n^{d(n)} - 2} = n^{\frac{1}{2}d(n)-1}.$$

Since this ends in exactly 2001 zeros, thus $2001 \mid \frac{1}{2}d(n) - 1$. Now let $\frac{1}{2}d(n) - 1 = 2001$, then $10 \mid n$, but $100 \nmid n$, and $d(n) = 4004 = 2 \cdot 2 \cdot 7 \cdot 11 \cdot 13$.

For example we can choose $n = 2^1 \cdot 5^1 \cdot 7^6 \cdot 11^{10} \cdot 13^{12}$. □

Example 13.1.4 (SC-23-HS-4-E4)

(1995 Greece Math Olympiad) Given 81 natural numbers whose prime divisors belong to the set $\{2, 3, 5\}$, prove there exist 4 numbers whose product is the fourth power of an integer.

Solution. It suffices to take 25 such numbers. To each number, associate the triple (x_2, x_3, x_5) recording the parity of the exponents of 2, 3, and 5 in its prime factorization. Two numbers have the same triple if and only if their product is a perfect square. As long as there are 9 numbers left, we can select two whose product is a square; in so doing, we obtain 9 such pairs. Repeating the process with the square roots of the products of the pairs, we obtain four numbers whose product is a fourth power. □

Example 13.1.5 (Czech-Slovak Match 1995)

Show that an integer $p > 3$ is prime if and only if for any non-zero integers a, b ($a \neq 0, b \neq 0$) exactly one of the numbers

$$\begin{cases} N_1 = a + b - 6ab + \frac{p-1}{6}, \\ N_2 = a + b + 6ab + \frac{p+1}{6} \end{cases}$$

is a non-zero integer.

Solution. If $N_1 = 0$, then $p = (6a-1)(6b-1)$ is a composite number. Similarly, $N_2 = 0$ implies that $p = -(6a+1)(6b+1)$ is composite.

Conversely, if $p \equiv 0, 2, 4$ or $4 \pmod 6$, then N_1 and N_2 are not integers. Otherwise, all divisors of p are congruent to $\pm 1 \pmod 6$, so there exist natural numbers c, d such that

$$\begin{cases} p = (6c+1)(6d+1) \Rightarrow \text{ if } a = -c, b = -d \Rightarrow N_1 = 0, N_2 \text{ is not an integer} \\ p = (6c-1)(6d-1) \Rightarrow \text{ if } a = c, b = d \Rightarrow N_1 = 0, N_2 \text{ is not an integer} \\ p = (6c+1)(6d-1) \Rightarrow \text{ if } a = c, b = -d \Rightarrow N_2 = 0, N_1 \text{ is not an integer} \end{cases}$$

□

Note that in competition, in order to get partial points or to rescue a solution based on some right ideas but somehow producing wrong answer, do not forget to submit your sketches or heuristics like what we have done in the remark analysis shown above. If you write them clean and clear, there are chances you will get additional points. They would not withstand a rigorous check, but there is no such need.

Note that for every problem you earn an additional same number of points for every different solution.

13.2 Problems

Submission deadline: July 21, 2023.

Problem 13.2.1 (SC-23-HS-4-P6). (*5 points*) Prove that n is a prime if and only if

$$\sum_{k=1}^{n}\left(\left\lfloor\frac{n}{k}\right\rfloor-\left\lfloor\frac{n-1}{k}\right\rfloor\right)=2.$$

Problem 13.2.2 (SC-23-HS-4-P7). For how many

1. (*5 points*) *even* numbers n,
2. (*5 points*) *odd* numbers n,

does n divide $3^{12}-1$, yet n does not divide 3^k-1, for $k=1,2,\ldots,11$.

Problem 13.2.3 (SC-23-HS-4-P8). (*10 points*) Find all integers $n>1$ such that for all integer a, $a^{25}-a$ is divisible by n.

> **Remark.** Prove that if p prime is a divisor of n then p^2 is not a divisor of n. Note that for all integer a, $a^{25}-a$ is divisible by n, thus $2^{25}-2$ is also divisible by n.

Problem 13.2.4 (SC-23-HS-4-P9). (*10 points*) Two players play the following game. They in turn write on a blackboard different divisors of 100! (except 1). A player loses if after his turn, the greatest common divisor of the all the numbers written becomes 1. Which of the players has a winning strategy?

> **Remark.** Notice that every prime $p<100$ divides an even number of factors of 100!: the factors it divides can be split into disjoint pairs $(k,97k)$ — or, if $p=97$, into the pairs $(k,89k)$.

Problem 13.2.5 (SC-23-HS-4-P10). (*15 points*) Let p be a prime number. Find all integer k such that $\sqrt{k^2-pk}$ is a positive integer.

> **Remark.** First, find out if $p=2$ then what value k can take. Then, assume that $p\mid k$, or $k=np$, what value of k such that k^2-kp is a perfect square? If p is not a divisor of k, what value of k and $k-p$ such that k^2-kp is a perfect square?

13.3 Solutions

Problem 13.3.1 (SC-23-HS-4-P6)**.** (*5 points*) Prove that n is a prime if and only if

$$\sum_{k=1}^{n}\left(\left\lfloor\frac{n}{k}\right\rfloor-\left\lfloor\frac{n-1}{k}\right\rfloor\right)=2.$$

Solution. From SC-23-HS-4-E1,

$$d(n)=\sum_{k=1}^{n}\left(\left\lfloor\frac{n}{k}\right\rfloor-\left\lfloor\frac{n-1}{k}\right\rfloor\right)\Rightarrow d(n)=2\Rightarrow n \text{ is a prime.}$$

□

Problem 13.3.2 (SC-23-HS-4-P7)**.** For how many

1. (*5 points*) *even* numbers n,
2. (*5 points*) *odd* numbers n,

does n divide $3^{12}-1$, yet n does not divide 3^k-1, for $k=1,2,\ldots,11$.

Solution. Note that $3^{12}-1=2^4\cdot5\cdot7\cdot13\cdot73$, and for $k=1,2,\ldots,11$, 3^k-1 are factored as below:

$$\begin{cases} 3-1 \qquad\qquad\qquad\qquad\quad =2 \\ 3^2-1=2^3 \\ 3^3-1=2\cdot13 \\ 3^4-1=2^4\cdot5 \\ 3^5-1=2\cdot11^2 \\ 3^6-1=2^3\cdot7\cdot13 \\ 3^7-1=2\cdot1093 \\ 3^8-1=2^5\cdot5\cdot41 \\ 3^9-1=2\cdot13\cdot757 \\ 3^10-1=2^3\cdot11^2\cdot61 \\ 3^11-1=2\cdot23\cdot3851 \end{cases}$$

Thus in order for a number m be a divisor of $3^{12}-1$ but not a divisor of any of 3^k-1, for $k=1,2,\ldots,11$, For odd m, it must have $5,7,13,73$ as divisors, except

1. only $5,7$, or 13
2. 7 and 13

Altogether $2^4-1-4=\boxed{11}$ such numbers.

For even m, let the highest power of 2 divides it is 2^k, then

1. $k=1$, then $\frac{m}{2}$ is odd, and there is 11 such numbers,
2. $k=2$, then $\frac{m}{2^2}$ is odd, and there is 11 such numbers,
3. $k=3$, then $\frac{m}{2^3}$ is odd, and there is 11 such numbers,
4. $k=4$, then $\frac{m}{2^4}$ is odd, and there is 14 such numbers, since only $2^4\cdot5$ is a divisor of any of 3^k-1, for $k=1,2,\ldots,11$,

Altogether $\boxed{47}$ such even numbers.

□

Problem 13.3.3 (SC-23-HS-4-P8)**.** Find all integers $n > 1$ such that for all integer a, $a^{25} - a$ is divisible by n.

Solution. Let p be a prime divisor of n, since p^2 is not a divisor of $p^{25} - p$, thus p^2 is not a divisor of n. Thus n can be a product of pairwise different prime numbers.

However $2^25 - 2 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$, and since $3^{25} \equiv -3 \pmod{17}$ and $3^{25} \equiv 32 \pmod{241}$, so n is not divisible by 17 or 241.

On the other hand, by Fermat theorem $a^{25} - a \equiv 0 \pmod p$, for $p \in \{2, 3, 5, 7, 13\}$, thus all the prime factors of n are $\{2, 3, 5, 7, 13\}$. There are $2^5 - 1 = \boxed{31}$ such products of these prime numbers $2, 3, \ldots, 13, 2 \cdot 3, \ldots, 2 \cdot 3 \cdots 13$. □

Problem 13.3.4 (SC-23-HS-4-P9)**.** (*10 points*) Two players play the following game. They in turn write on a blackboard different divisors of 100! (except 1). A player loses if after his turn, the greatest common divisor of the all the numbers written becomes 1. Which of the players has a winning strategy?

Solution. The second player has a winning strategy. Notice that every prime $p < 100$ divides an even number of factors of 100!: the factors it divides can be split into disjoint pairs $(k, 97k)$ — or, if $p = 97$, into the pairs $(k, 89k)$. (Note that none of these factors is 1, since 1 is not divisible by p.)

If the first player writes down a prime p, the second player can write down any other number divisible by p; if the first player writes down a composite number, the second player can write down a prime p dividing that number.

Either way, from now on the players can write down a new number $q \mid 100!$ without losing if and only if it is divisible by p. Since there are an even number of such q, the second player will write down the last acceptable number and the first player will lose. □

Problem 13.3.5 (SC-23-HS-4-P10)**.** (*15 points*) Let p be a prime number. Find all integer k such that $\sqrt{k^2 - pk}$ is a positive integer.

Solution. First, if $p = 2$, then $k^2 - kp = k^2 - 2k = (k-1)^2 - 1$ cannot be a perfect square. So p is an odd prime.

Case 1: assume that p is a divisor of k, or $k = np$.

$$k^2 - kp = (np)^2 - np^2 = p^2 n(n-1).$$

Since n and $n - 1$ are consecutive numbers, they cannot both be perfect square, and since they are relatively primes, so $n(n-1)$ cannot be a perfect square.

Case 2: now p is not a divisor of k. Thus $k(k-p)$ can be a perfect square if and only if k and $k - p$ are perfect square. Let $k = m^2, k - p = n^2$, then $p = m^2 - n^2 = (m-n)(m+n)$, thus $m = n + 1$, and $\boxed{k = \frac{(p+1)^2}{4}.}$

□

Chapter 14

Complex Numbers

14.1 Building our understanding from scratch

You should also review the **Complex Numbers** and **Sets** chapters in the **Learning Problem Solving, Vol. 3** to understand the basics, practice with examples, and master a few techniques.

It is advised that *you attempt to prove any of the theorems, propositions, or corollaries right after you read them.* They are quite basic and easy to prove. Understanding a theorem well will lead you to enjoy seeing why the corollaries naturally follow.

Let $\mathbb{R}$ be the set of all real numbers. Consider the set

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) | x, y \in \mathbb{R}\}.$$

Two elements (x_1, y_1) and (x_2, y_2) of $\mathbb{R}^2$ are equal *if and only if* $x_1 = x_2$ and $y_1 = y_2$.

The **addition** and **multiplication** operations on $\mathbb{R}^2$ are defined as below:

$$\begin{array}{r} z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2 \\ z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \in \mathbb{R}^2 \end{array} \forall z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathbb{R}^2$$

The element $z_1 + z_2 \in \mathbb{R}^2$ is called the **sum** of z_1, z_2 and $z_1 \cdot z_2 \in \mathbb{R}^2$ is called the **product** of z_1, z_2.

Examples

$z_1 = (-5, 6), z_2 = (1, -2) \Rightarrow z_1 + z_2 = (-5, 6) + (1, -2) = (4, -4), z_1 \cdot z_2 = (-5, 6) + (1, -2) = (7, 16)$

$z_1 = (-\frac{1}{2}, 1), z_2 = (-\frac{1}{3}, \frac{1}{2}) \Rightarrow z_1 + z_2 = (-\frac{5}{6}, \frac{3}{2}), z_1 \cdot z_2 = (-\frac{1}{3}, -\frac{7}{12})$

$z_1 = (x_1, 0), z_2 = (x_2, 0) \Rightarrow z_1 \cdot z_2 = (x_1 x_2, 0)$

$z_1 = (0, y_1), z_2 = (0, y_2) \Rightarrow z_1 \cdot z_2 = (-y_1 y_2, 0)$

Definition. The set $\mathbb{R}^2$, together with the addition and multiplication operations (as defined above) is called the **set of complex numbers**, denoted by $\mathbb{C}$. Any element $z \in \mathbb{C}$ is called a **complex number**.

The notation $\mathbb{C}^*$ is used to indicate the set $\mathbb{C} \setminus \{(0,0)\}$.

Theorem (Properties of addition)

For all $z = (x, y)$, $z_1, z_2, z_3 \in \mathbb{C}$:

1. **Commutative laws** $z_1 + z_2 = z_2 + z_1$.
2. **Associative laws** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
3. **Additive identity** There is a unique complex number $0 = (0, 0) \in \mathbb{C}$, such that $z + 0 = 0 + z = z$.
4. **Additive inverse** There is a unique $-z = (-x, -y) \in \mathbb{C})$, such that $z + (-z) = (-z) + z = 0$.

Theorem (Properties of multiplication)

For all $z = (x, y)$, $z_1, z_2, z_3 \in \mathbb{C}$, for all $w = (u, v) \in \mathbb{C}^*$, and integers m, n:

1. **Commutative laws** $z_1 \cdot z_2 = z_2 \cdot z_1$.
2. **Associative laws** $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$.
3. **Multiplicative identity** There is a unique complex number $1 = (1, 0) \in \mathbb{C}$, such that $z \cdot 1 = 1 \cdot z = z$.
4. **Multiplicative inverse** There is a unique $w^{-1} = (u', v') \in \mathbb{C}^*)$, such that $w \cdot w^{-1} = (1, 0) = w^{-1} \cdot w$, where $u' = \frac{u}{u^2+v^2}, v' = -\frac{v}{u^2+v^2}$.

   Furthermore $\frac{z_1}{z} = z_1 \cdot z^{-1}$ is called the **quotient** of z_1 over z.
5. **Distributive laws** $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$.

Fact. By denoting $z^0 = 1, z^1 = z, z^m = \underbrace{z \cdot z \cdots z}_{m}, z^{-n} = (z^{-1})^n$, then for all $z, z_1, z_2 \in \mathbb{C}^*$:

1. $z^m \cdot z^n = z^{m+n}$.
2. $\frac{z^m}{z^n} = z^{m-n}$.
3. $(z^m)^n = z^{mn}$.
4. $(z_1 \cdot z_2)^m = z_1^m \cdot z_2^m$.
5. $\left(\frac{z_1}{z_2}\right)^m = \frac{z_1^m}{z_2^m}$.

Note that for $0 = (0, 0) \in \mathbb{C}$, then $0^n = 0$, for all integer $n > 0$.

Theorem (Algebraic form)

Denote $i = (0, 1) \in CC$, any complex number $z = (x, y) \in \mathbb{C}$ can be uniquely represented in the form

$$z = x + iy, \text{ where } x, y \in \mathbb{R}.$$

The relation $i^2 = -1$ holds.

Definition. For the complex number $z = (x, y) \in \mathbb{C}$, $x = Re(z)$ is called the **real part** of z, $y = Im(z)$ is called the **imaginary part** of z. Complex numbers of the form iy (whose real part is 0) is called *imaginary* Complex numbers of the form iy, where $y \in \mathbb{R}$ $(y \neq 0)$ is called *purely imaginary.* The complex number i is called **imaginary unit.**

Corollary (Bijective properties)

For all $z, z_1, z_2 \in \mathbb{C}$:

1. $z_1 = z_2$ *if and only if* $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.
2. $z \in \mathbb{R}$ *if and only if* $Im(z) = 0$.
3. $z \in \mathbb{C} \setminus \mathbb{R}$ *if and only if* $Im(z) \neq 0$.

Corollary (Algebraic form properties)

For all $z = x + iy, z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$, $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$:

1. $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, or $Re(z_1 + z_2) = Re(z_1) + Re(z_2)$, $Im(z_1 + z_2) = Im(z_1) + Im(z_2)$.
2. $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$, or $Re(z_1 - z_2) = Re(z_1) - Re(z_2)$, $Im(z_1 - z_2) = Im(z_1) - Im(z_2)$.
3. $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$, or
$$Re(z_1 \cdot z_2) = Re(z_1)Re(z_2) - Im(z_1)Im(z_2), Im(z_1 \cdot z_2) = Im(z_1)Re(z_2) + Re(z_1)Im(z_2).$$
4. $\lambda \cdot z = \lambda \cdot (x + iy) = \lambda x + i\lambda y$.
5. $\lambda(z_1 + z_2) = \lambda z_1 + \lambda z_2$.
6. $(\lambda_1 z + \lambda_2)z = \lambda_1 z + \lambda_2 z$.
7. $\lambda_1(\lambda_2 z) = (\lambda_1\lambda_2)z$.

Definition (Conjugate of a complex number)**.** For the complex number $z = (x, y) \in \mathbb{C}$, the number $\overline{z} = x - iy$ is called the **complex conjugate** of z.

Proposition (Properties of conjugate)

For all $z, z_1, z_2 \in \mathbb{C}$:

1. $z = \overline{z}$ *if and only if* $z \in \mathbb{R}$.
2. $z = \overline{\overline{z}}$.
3. $z \cdot \overline{z} \in \mathbb{R}^+$ (a nonnegative real number).
4. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ (the conjugate of a sum is the sum of the conjugates).
5. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ (the conjugate of a product is the product of the conjugates).
6. $\overline{z^{-1}} = (\overline{z})^{-1}$, if $z \in \mathbb{C}^*$.
7. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, if $z_2 \in \mathbb{C}^*$, (the conjugate of a quotient is the quotient of the conjugates).
8. $Re(z) = \frac{1}{2}\frac{z+\overline{z}}{,}Im(z) = \frac{1}{2}\frac{z-\overline{z}}{.}$

Corollary (Conjugates of a sum and a product)

For all $z_1, z_2, \ldots, z_n \in \mathbb{C}$:

1. $\overline{\sum_{k=1}^{n} z_k} = \sum_{k=1}^{n} \overline{z_k}$.
2. $\overline{\prod_{k=1}^{n} z_k} = \prod_{k=1}^{n} \overline{z_k}$.
3. $\overline{z^n} = (\overline{z})^n$.

Definition (Modulus of a complex number)**.** For the complex number $z = (x, y) \in \mathbb{C}$, the number $|z| = \sqrt{x^2 + y^2}$ is called the **modulus** or the **absolute value** of z.

Proposition (Properties of the modulus)

For all $z, z_1, z_2 \in \mathbb{C}$:

1. $-|z| \leq Re(z) \leq |z|, -|z| \leq Im(z) \leq |z|$
2. $|z| \geq 0$.
3. $|z| = |-z| = |\overline{z}|$.
4. $z \cdot \overline{z} = |z|^2$
5. $|z_1 \cdot z_2| = |z_1||z_2|$.
6. $|z_1| - |z_2| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$.
7. $|z^{-1}| = |z|^{-1}$.
8. $\left\|\frac{z_1}{z_2}\right\| = \frac{|z_1|}{|z_2|}$.

Corollary (Moduli of a sum and a product)

For all $z_1, z_2, \ldots, z_n \in \mathbb{C}$:

1. $|\prod_{k=1}^n z_k| = \prod_{k=1}^n |z_k|$.
2. $|\sum_{k=1}^n z_k| = \sum_{k=1}^n |z_k|$.

14.2 Problems

Submission deadline: July 26, 2023.

Problem 14.2.1 (SC-23-HS-5-P6). (*5 points*)

Theorem (Properties of addition)

For all $z = (x, y)$, $z_1, z_2, z_3 \in \mathbb{C}$:

1. **Commutative laws** $z_1 + z_2 = z_2 + z_1$.
2. **Associative laws** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
3. **Additive identity** There is a unique complex number $0 = (0, 0) \in \mathbb{C}$, such that $z + 0 = 0 + z = z$.
4. **Additive inverse** There is a unique $-z = (-x, -y) \in \mathbb{C})$, such that $z + (-z) = (-z) + z = 0$.

Problem 14.2.2 (SC-23-HS-5-P7). (*5 points*)

Theorem (Properties of multiplication)

For all $z = (x, y)$, $z_1, z_2, z_3 \in \mathbb{C}$, for all $w = (u, v) \in \mathbb{C}^*$, and integers m, n:

1. **Commutative laws** $z_1 \cdot z_2 = z_2 \cdot z_1$.
2. **Associative laws** $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$.
3. **Multiplicative identity** There is a unique complex number $1 = (1, 0) \in \mathbb{C}$, such that $z \cdot 1 = 1 \cdot z = z$.
4. **Multiplicative inverse** There is a unique $w^{-1} = (u', v') \in \mathbb{C}^*)$, such that $w \cdot w^{-1} = (1, 0) = w^{-1} \cdot w$, where $u' = \frac{u}{u^2+v^2}, v' = -\frac{v}{u^2+v^2}$.

   Furthermore $\frac{z_1}{z} = z_1 \cdot z^{-1}$ is called the **quotient** of z_1 over z.
5. **Distributive laws** $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$.

Problem 14.2.3 (SC-23-HS-5-P8). (*5 points*)

Corollary (Bijective properties)

(*5 points*) For all $z, z_1, z_2 \in \mathbb{C}$:

1. $z_1 = z_2$ *if and only if* $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.
2. $z \in \mathbb{R}$ *if and only if* $Im(z) = 0$.
3. $z \in \mathbb{C} \setminus \mathbb{R}$ *if and only if* $Im(z) \neq 0$.

Problem 14.2.4 (SC-23-HS-5-P9). (*5 points*)

Corollary (Algebraic form properties)

(*5 points*) For all $z = x + iy, z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$, $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$:

1. $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, or $Re(z_1 + z_2) = Re(z_1) + Re(z_2)$, $Im(z_1 + z_2) = Im(z_1) + Im(z_2)$.
2. $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$, or $Re(z_1 - z_2) = Re(z_1) - Re(z_2)$, $Im(z_1 - z_2) = Im(z_1) - Im(z_2)$.
3. $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$, or
$$Re(z_1 \cdot z_2) = Re(z_1)Re(z_2) - Im(z_1)Im(z_2), Im(z_1 \cdot z_2) = Im(z_1)Re(z_2) + Re(z_1)Im(z_2).$$
4. $\lambda \cdot z = \lambda \cdot (x + iy) = \lambda x + i\lambda y$.
5. $\lambda(z_1 + z_2) = \lambda z_1 + \lambda z_2$.
6. $(\lambda_1 z + \lambda_2)z = \lambda_1 z + \lambda_2 z$.
7. $\lambda_1(\lambda_2 z) = (\lambda_1\lambda_2)z$.

Problem 14.2.5 (SC-23-HS-5-P10). (*5 points*)

Proposition (Properties of conjugate)

For all $z, z_1, z_2 \in \mathbb{C}$:

1. $z = \overline{z}$ *if and only if* $z \in \mathbb{R}$.
2. $z = \overline{\overline{z}}$.
3. $z \cdot \overline{z} \in \mathbb{R}^+$ (a nonnegative real number).
4. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ (the conjugate of a sum is the sum of the conjugates).
5. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ (the conjugate of a product is the product of the conjugates).
6. $\overline{z^{-1}} = (\overline{z})^{-1}$, if $z \in \mathbb{C}^*$.
7. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, if $z_2 \in \mathbb{C}^*$, (the conjugate of a quotient is the quotient of the conjugates).
8. $Re(z) = \frac{1}{2}\frac{z+\overline{z}}{,}Im(z) = \frac{1}{2}\frac{z-\overline{z}}{.}$

Problem 14.2.6 (SC-23-HS-5-P11). (*5 points*)

Proposition (Properties of the modulus)

For all $z, z_1, z_2 \in \mathbb{C}$:

1. $-|z| \le Re(z) \le |z|, -|z| \le Im(z) \le |z|$
2. $|z| \ge 0$.
3. $|z| = |-z| = |\overline{z}|$.
4. $z \cdot \overline{z} = |z|^2$
5. $|z_1 \cdot z_2| = |z_1||z_2|$.
6. $|z_1| - |z_2| \le |z_1 \pm z_2| \le |z_1| + |z_2|$.
7. $|z^{-1}| = |z|^{-1}$.
8. $\left\|\frac{z_1}{z_2}\right\| = \frac{|z_1|}{|z_2|}$.

Problem 14.2.7 (1999 Romanian Mathematical Olympiad - Final Round)**.** (*10 points*) Let p and q be complex numbers with $q \neq 0$. Prove that if the roots of the quadratic equation $x^2 + px + q^2 = 0$ have the same absolute value, then $\dfrac{p}{q}$ is a real number.

Problem 14.2.8 (SC-23-HS-5-P13)**.** (*10 points*) Find all positive integers n such that

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = 2.$$

Note that in competition, in order to get partial points or to rescue a solution based on some right ideas but somehow producing wrong answer, do not forget to submit your sketches or heuristics like what we have done in the remark analysis shown above. If you write them clean and clear, there are chances you will get additional points. They would not withstand a rigorous check, but there is no such need.

Note that for every problem you earn an additional same number of points for every different solution.

14.3 Solutions

Problem 14.3.1 (SC-23-HS-5-P6)**.** (*5 points*) For all $z = (x, y)$, $z_1, z_2, z_3 \in \mathbb{C}$:

1. **Commutative laws** $z_1 + z_2 = z_2 + z_1$.
2. **Associative laws** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
3. **Additive identity** There is a unique complex number $0 = (0, 0) \in \mathbb{C}$, such that $z + 0 = 0 + z = z$.
4. **Additive inverse** There is a unique $-z = (-x, -y) \in \mathbb{C})$, such that $z + (-z) = (-z) + z = 0$.

Solution.

$$\begin{aligned}
z_1 + z_2 &= (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1 \\
(z_1 + z_2) + z_3 &= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3) = (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)) = z_1 + (z_2 + z_3) \\
0 = (0, 0) &\Rightarrow z + 0 = (x + 0, y + 0) = z = (0 + x, 0 + y) = 0 + z \\
-z = (-x, -y) &\Rightarrow z + (-z) = (x + (-x), y + (-y)) = 0 = ((-x) + x, (-y) + y) = (-z) + z
\end{aligned}$$

□

Problem 14.3.2 (SC-23-HS-5-P7)**.** (*5 points*) For all $z = (x, y)$, $z_1, z_2, z_3 \in \mathbb{C}$, for all $w = (u, v) \in \mathbb{C}^*$, and integers m, n:

1. **Commutative laws** $z_1 \cdot z_2 = z_2 \cdot z_1$.
2. **Associative laws** $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$.
3. **Multiplicative identity** There is a unique complex number $1 = (1, 0) \in \mathbb{C}$, such that $z \cdot 1 = 1 \cdot z = z$.
4. **Multiplicative inverse** There is a unique $w^{-1} = (u', v') \in \mathbb{C}^*)$, such that $w \cdot w^{-1} = (1, 0) = w^{-1} \cdot w$, where $u' = \frac{u}{u^2+v^2}, v' = -\frac{v}{u^2+v^2}$.

   Furthermore $\frac{z_1}{z} = z_1 \cdot z^{-1}$ is called the **quotient** of z_1 over z.
5. **Distributive laws** $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$.

Solution.

$$\begin{aligned}
z_1 \cdot z_2 &= (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1) = (x_2x_1 - y_2y_1,\ x_2y_1 + x_1y_2) = z_2 \cdot z_1 \\
(z_1 \cdot z_2) \cdot z_3 &= (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1) \cdot (x_3, y_3) \\
&= ((x_1x_2 - y_1y_2)x_3 - (x_1y_2 + x_2y_1)y_3,\ (x_1x_2 - y_1y_2)y_3 + (x_1y_2 + x_2y_1)x_3) \\
&= (x_1x_2 - x_1y_2y_3 - y_1x_2y_3 - y_1y_2x_3,\ x_1x_2y_3 + x_1y_2x_3 + y_1x_2x_3 - y_1y_2y_3) \\
&= x_1(x_2x_3 - y_2y_3) - y_1(x_2y_3 + x_3y_2),\ x_1(x_2y_3 + x_3y_2) + (x_2x_3 - y_2y_3)y_1 \\
&= (x_1, y_1) \cdot (x_2x_3 - y_2y_3,\ x_2y_3 + x_3y_2) = z_1 \cdot (z_2 \cdot z_3)
\end{aligned}$$

$$\begin{aligned}
1 = (1, 0) &\Rightarrow z \cdot 1 (= 1 \cdot z) = (x \cdot 1 - y \cdot 0, x \cdot 0 + y \cdot 1) = (x, y) = z \\
w^{-1} = \left(\frac{u}{u^2+v^2}, -\frac{v}{u^2+v^2}\right) &\Rightarrow w \cdot w^{-1} = \left(u\frac{u}{u^2+v^2} + v\frac{v}{u^2+v^2}, u\frac{v}{u^2+v^2} - v\frac{u}{u^2+v^2}\right) = (1, 0) \\
w^{-1} \cdot w &= w \cdot w^{-1} = (1, 0) \\
z_1 \cdot (z_2 + z_3) &= (x_1, y_1) \cdot (x_2 + x_3,\ y_2 + y_3) = (x_1(x_2 + x_3) - y_1(y_2 + y_3),\ x_1(y_2 + y_3) + (x_2 + x_3)y_1) \\
&= (x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3,\ x_1y_2 + x_1y_3 + y_1x_2 + y_1x_3) \\
&= ((x_1x_2 - y_1y_2) + (x_1x_3 - y_1y_3),\ (x_1y_2 + x_2y_1) + (x_1y_3 + x_3y_1)) \\
&= (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1) + (x_1x_3 - y_1y_3,\ x_1y_3 + x_3y_1) = z_1 \cdot z_2 + z_1 \cdot z_3.
\end{aligned}$$

□

Problem 14.3.3 (SC-23-HS-5-P8)**.** (*5 points*) For all $z, z_1, z_2 \in \mathbb{C}$:

1. $z_1 = z_2$ *if and only if* $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.
2. $z \in \mathbb{R}$ *if and only if* $Im(z) = 0$.
3. $z \in \mathbb{C} \setminus \mathbb{R}$ *if and only if* $Im(z) \neq 0$.

Solution. $z_1 = (x_1, y_1), z_2 = (x_2, y_2), x_1 = Re(z_1), x_2 = Re(z_2), y_1 = Im(z_1), y_2 = Im(z_2)$, then

$$z_1 = z_2 \Leftrightarrow x_1 = x_2,\ y_1 = y_2 \Leftrightarrow Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$$

$$z = (x, y), x = Re(z), y = Im(z),\ z \in \mathbb{R} \Leftrightarrow y = 0 \Leftrightarrow Im(z) = 0,\ z \in \mathbb{C} \setminus \mathbb{R} \Leftrightarrow y \neq 0 \Leftrightarrow Im(z) \neq 0.$$ □

Problem 14.3.4 (SC-23-HS-5-P9)**.** (*5 points*) For all $z = x + iy, z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$, $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$:

1. $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, or $Re(z_1 + z_2) = Re(z_1) + Re(z_2)$, $Im(z_1 + z_2) = Im(z_1) + Im(z_2)$.
2. $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$, or $Re(z_1 - z_2) = Re(z_1) - Re(z_2)$, $Im(z_1 - z_2) = Im(z_1) - Im(z_2)$.
3. $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$, or
$$Re(z_1 \cdot z_2) = Re(z_1)Re(z_2) - Im(z_1)Im(z_2), Im(z_1 \cdot z_2) = Im(z_1)Re(z_2) + Re(z_1)Im(z_2).$$
4. $\lambda \cdot z = \lambda \cdot (x + iy) = \lambda x + i\lambda y$.
5. $\lambda(z_1 + z_2) = \lambda z_1 + \lambda z_2$.
6. $(\lambda_1 + \lambda_2)z = \lambda_1 z + \lambda_2 z$.
7. $\lambda_1(\lambda_2 z) = (\lambda_1\lambda_2)z$.

Solution.

$$\begin{aligned}
z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)\\
&\Rightarrow Re(z_1 + z_2) = Re(z_1) + Re(z_2),\ Im(z_1 + z_2) = Im(z_1) + Im(z_2)\\
z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) = (x_1 + x_2) - i(y_1 + y_2)\\
&\Rightarrow Re(z_1 - z_2) = Re(z_1) - Re(z_2),\ Im(z_1 - z_2) = Im(z_1) - Im(z_2)\\
z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\\
&\Rightarrow Re(z_1 \cdot z_2) = Re(z_1)Re(z_2) - Im(z_1)Im(z_2), Im(z_1 \cdot z_2) = Im(z_1)Re(z_2) + Re(z_1)Im(z_2)\\
\lambda \cdot z &= \lambda \cdot (x + iy) = \lambda x + i\lambda y\\
\lambda(z_1 + z_2) &= \lambda((x_1 + x_2) + i(y_1 + y_2)) = \lambda(x_1 + iy_1) + \lambda(x_2 + iy_2)\\
(\lambda_1 + \lambda_2)z &= (\lambda_1 + \lambda_2)(x + iy) = \lambda_1 z + \lambda_2 z\\
\lambda_1(\lambda_2 z) &= \lambda_1\lambda_2(x + iy) = (\lambda_1\lambda_2)z
\end{aligned}$$

□

Problem 14.3.5 (SC-23-HS-5-P10)**.** (*5 points*) For all $z, z_1, z_2 \in \mathbb{C}$:

1. $z = \overline{z}$ *if and only if* $z \in \mathbb{R}$.
2. $z = \overline{\overline{z}}$.
3. $z \cdot \overline{z} \in \mathbb{R}^+$ (a nonnegative real number).
4. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ (the conjugate of a sum is the sum of the conjugates).
5. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ (the conjugate of a product is the product of the conjugates).
6. $\overline{z^{-1}} = (\overline{z})^{-1}$, if $z \in \mathbb{C}^*$.
7. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, if $z_2 \in \mathbb{C}^*$, (the conjugate of a quotient is the quotient of the conjugates).
8. $Re(z) = \frac{1}{2}(z + \overline{z}), Im(z) = \frac{1}{2}(z - \overline{z})$.

Solution.

$$\begin{aligned}
z = \overline{z} &\Leftrightarrow x + iy = x - iy \Leftrightarrow y = 0 \Leftrightarrow z \in \mathbb{R} \\
\overline{\overline{z}} &= \overline{\overline{x + iy}} = \overline{x + i(-y)} = x - i(-y) = x + iy \\
z \cdot \overline{z} &= (x + iy)(x - iy) = x^2 + y^2 \\
\overline{z_1 + z_2} &= (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \overline{z_1} + \overline{z_2} \\
\overline{z_1 \cdot z_2} &= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)} = x_1x_2 - y_1y_2 - ix_1y_2 - ix_2y_1 \\
&= x_1(x_2 - iy_2) - iy_1(x_2 - iy_2) = (x_1 - iy_1)(x_2 - iy_2) = \overline{z_1} \cdot \overline{z_2} \\
z \cdot \frac{1}{z} &= 1 \Rightarrow \overline{z \cdot \frac{1}{z}} = \overline{1} \Rightarrow \overline{z} \cdot \overline{\frac{1}{z}} = 1 \Rightarrow \overline{z^{-1}} = (\overline{z})^{-1} \\
\overline{\left(\frac{z_1}{z_2}\right)} &= \overline{z_1 \cdot \frac{1}{z_2}} = \overline{z_1} \cdot \overline{\frac{1}{z_2}} = \overline{z_1}\frac{1}{\overline{z_2}} = \frac{\overline{z_1}}{\overline{z_2}} \\
z + \overline{z} &= 2x = 2Re(z),\ z - \overline{z} = 2y = 2Im(z)
\end{aligned}$$

□

Problem 14.3.6 (SC-23-HS-5-P11). (*5 points*) For all $z, z_1, z_2 \in \mathbb{C}$:

1. $-|z| \le Re(z) \le |z|, -|z| \le Im(z) \le |z|$
2. $|z| \ge 0$.
3. $|z| = |-z| = |\overline{z}|$.
4. $z \cdot \overline{z} = |z|^2$
5. $|z_1 \cdot z_2| = |z_1||z_2|$.
6. $|z_1| - |z_2| \le |z_1 \pm z_2| \le |z_1| + |z_2|$.
7. $|z^{-1}| = |z|^{-1}$.
8. $\left\|\frac{z_1}{z_2}\right\| = \frac{|z_1|}{|z_2|}$.

Solution.

$$-|z| = -\sqrt{x^2+y^2} \le x = Re(z) \le \sqrt{x^2+y^2},\ -|z| = -\sqrt{x^2+y^2} \le y = Im(z) \le \sqrt{x^2+y^2}$$

$$|z| = \sqrt{x^2+y^2} \ge 0$$

$$|z| = \sqrt{x^2+y^2} = \sqrt{(-x)^2+(-y)^2} = |-z| = \sqrt{(x)^2+(-y)^2} = |\overline{z}|$$

$$z \cdot \overline{z} = x^2+y^2 = |z|^2$$

$$|z_1 \cdot z_2| = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 = (x_1^2+y_1)^2(x_2^2+y_2^2) = |z_1||z_2|$$

$$|z_1| - |z_2| = \sqrt{x_1^2+y_1^2} - \sqrt{x_2^2+y_2^2} \le \sqrt{(x_1 \pm x_2)^2 + (y_1 \pm y_2)^2} \le \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$$

$$z \cdot \frac{1}{z} = 1 \Rightarrow |z| \cdot \left|\frac{1}{z}\right| \Rightarrow \left|\frac{1}{z}\right| = \frac{1}{|z|} \Rightarrow |z^{-1}| = |z|^{-1}.$$

$$\left|\frac{z_1}{z_2}\right| = \left|z_1 \cdot \frac{1}{z_2}\right| = |z_1 \cdot z_2^{-1}| = |z_1| \cdot |z_2^{-1}| = |z_1| \cdot |z_2|^{-1} = \frac{|z_1|}{|z_2|}$$

□

Problem 14.3.7 (1999 Romanian Mathematical Olympiad - Final Round). (*10 points*) Let p and q be complex numbers with $q \ne 0$. Prove that if the roots of the quadratic equation $x^2+px+q^2=0$ have the same absolute value, then $\dfrac{p}{q}$ is a real number.

Solution. Let $r = |x_1| = |x_2|$. Then

$$\frac{p^2}{q^2} = \frac{(x_1+x_2)^2}{x_1x_2} = \frac{x_1}{x_2} + \frac{x_2}{x_1} + 2 = \frac{x_1\overline{x_2}}{r^2} + \frac{\overline{x_1}x_2}{r^2} + 2 = 2 + 2\frac{Re(x_1\overline{x_2})}{r^2} \in \mathbb{R}$$

$$Re(x_1\overline{x_2}) \ge -|x_1\overline{x_2}| = -r^2 \Rightarrow \frac{p^2}{q^2} \ge 0 \Rightarrow \boxed{\frac{p}{q} \in \mathbb{R}.}$$

□

Problem 14.3.8 (SC-23-HS-5-P13). (*10 points*) Find all positive integers n such that

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = 2.$$

Solution. Let $\epsilon = -\frac{1}{2} + i\sqrt{3}2$, then

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = \epsilon^n + (\overline{\epsilon})^n = \epsilon^n + \overline{\epsilon}^n = 2Re(\epsilon^n)$$

$$\epsilon^3 = 1,\ \epsilon^{3k} = 1,\ \epsilon^{3k+1} = \epsilon,\ \epsilon^{3k+2} = \epsilon^2,\ 2Re(\epsilon^n) = 2 \Leftrightarrow \boxed{n = 3k,\ \forall k \in \mathbb{Z}^+.}$$

□

Chapter 15

Counting & Sets

15.1 Examples

You should review the **Counting**, **Counting in two ways**, **Permutations**, and **Sets** chapters in the **Learning Problem Solving, Vol. 3** to understand the basics, practice with examples, and master a few techniques.

Example 15.1.1 (Canada MO 1996)

Given a finite number of closed intervals of length 1, whose union is the closed interval $[0, 50]$, prove that there exists a subset of the intervals, any two of whose members are disjoint, whose union has total length at least 25. (Two intervals with a common endpoint are not disjoint.)

Remark. Note that we have 50 unit intervals $[0, 1],\ [1, 2],\ \ldots,\ [49, 50]$. The idea here is to choose the first one $[0, 1]$ and then move each of 24 intervals

$$[1, 2],\ [3, 4],\ \ldots, [47, 48].$$

a little bit so that they still non-overlapping and still not exceeding 50.

Solution. Consider 25 *unit-length* intervals

$$[0, 1],\ [1, 2],\ [3, 4],\ \ldots, [47, 48]$$

By moving the intervals $[1, 2],\ [3, 4],\ \ldots, [47, 48]$ with distance of lengths $\ell, 2\ell, \ldots, 24\ell$ to the right, we obtain

$$\boxed{[1 + \ell, 2 + \ell],\ [3 + 2\ell, 4 + 2\ell]\ \ldots,\ [47 + 24\ell, 48 + 24\ell]}$$

It is easy to see that these intervals are non-overlapping if $\ell > 0$ and $48 + 24\ell < 50$, thus $0 < \ell < \frac{1}{12}$. There exists such ℓ, hence the intervals described above are the desired ones. □

Example 15.1.2 (Hungarian Mathematical Olympiad 1996)

Each member of a committee ranks applicants A, B, C in some order. It is given that the majority of the committee ranks A higher than B, and also that the majority of the committee ranks B higher than C. Does it follow that the majority of the committee ranks A higher than C?

Remark. This is a remarkable example of circular permutations retains order in relations.

Solution. No. Suppose the committee has three members, one who ranks $A > B > C$, one who ranks $B > C > A$, and one who ranks $C > A > B$. Then the first and third both prefer A to B, and the first and second both prefer B to C, but only the first prefers $AtoC$. □

Example 15.1.3 (All-Russian Mathematical Olympiad 1995)

In the Duma there are 1600 delegates, who have formed 16000 committees of 80 persons each. Prove that one can find two committees having no fewer than four common members.

Solution. Suppose any two committees have at most three common members. Lets have two deputies count the possible ways to choose a chairman for each of three sessions of the Duma. The first deputy assumes that any deputy can chair any session, and so gets 16003 possible choices. The second deputy makes the additional re- striction that all of the chairmen belong to a single committee. Each of the 16000 committees yields 80^3 choices, but this is an overcount; each of the $\frac{16000(16000-1)}{2}$ pairs of committees give at most 3^3 overlapping choices. Since the first deputy counts no fewer possibilities than the second, we have the inequality:

$$1600^3 \geq 16000 \cdot 80^3 - \frac{16000(16000-1)}{2} \cdot 3^3$$

However

$$16000 \cdot 80^3 - \frac{16000 \cdot 15999}{2} \cdot 3^3 = 4\ 736\ 216\ 000 > 4\ 096\ 000\ 000 = 16000^3, \text{ a contradiction!}$$

□

Example 15.1.4

(1995 Asian Pacific Mathematics Olympiad) The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

1. All members of the group must be of the same sex, i.e. they are either all male or all female.
2. The difference in the size of any two groups is either 0 or 1.
3. All groups have at least one member.
4. Each person must belong to one and only one group.

Find all values of $n, n \leq 1996$, for which this is possible. Justify your answer.

Solution. Clearly $n \geq 9$ since each of 17 groups must contain at least one member. Suppose there are k groups of men and $17 - k$ groups of women; without loss of generality, we assume $k \leq 8$. If m is the minimum number of members in a group, then the number of men in the groups is at most $k(m+1)$, while the number of women is at least $(k+1)m$. As there are the same number as men as women, we have

$$k(m+1) \geq (k+1)m \Rightarrow m \leq k \leq 8,$$

and the maximum number of couples is $k(k+1) \leq 72$. In fact, any number of couples between 9 and 72 can be distributed: divide the men as evenly as possible into 8 groups, and divide the women as evenly as possible into 9 groups. Thus $\boxed{9 \leq n \leq 72}$ is the set of acceptable numbers of couples. □

Example 15.1.5 (Ireland MO 1997)

Let A be a subset of $\{0, 1, \ldots, 1997\}$ containing more than 1000 elements. Prove that A contains either a power of 2, or two distinct integers whose sum is a power of 2.

Solution. Suppose A did not verify the conclusion. Then A would contain at most half of the integers from 51 to 1997, since they can be divided into pairs whose sum is 2048 (with 1024 left over); likewise, A contains at most half of the integers from 14 to 50, at most half of the integers from 3 to 13, and possibly 0, for a total of

$$973 + 18 + 5 + 1 = 997 \text{ integers, which is a contradiction!}$$

$\square$

Note that in competition, in order to get partial points or to rescue a solution based on some right ideas but somehow producing wrong answer, do not forget to submit your sketches or heuristics like what we have done in the remark analysis shown above. If you write them clean and clear, there are chances you will get additional points. They would not withstand a rigorous check, but there is no such need.

Note that for every problem you earn an additional same number of points for every different solution.

15.2 Problems

Submission deadline: August 2, 2023.

Problem 15.2.1 (SC-23-HS-6-P6)**.** (*10 points*) Show that there exist infinitely many positive integers n such that the numbers $1, 2, \ldots, 3n$ can be labeled as

$$a_1,\ a_2,\ \ldots,\ a_n,\ b_1,\ b_2,\ \ldots,\ b_n,\ c_1,\ c_2,\ \ldots,\ c_n \textit{ such that}$$

1. $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = \ldots = a_n + b_n + c_n$ is a multiple of 6.
2. $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n = c_1 + c_2 + \cdots + c_n$ is also a multiple of 6.

> **Remark.** Note the sum of all the number is $\frac{3n(3n+1)}{2}$ which has to be a multiple of both $6n$ and 9. For $n = 9$, consider the table below.

$\{a_i\}$	8	1	6	17	10	15	26	19	24
$\{b_i\}$	21	23	25	3	5	7	12	14	16
$\{b_i\}$	13	18	11	22	27	20	4	9	2

Problem 15.2.2 (SC-23-HS-6-P7)**.** (*10 points*) Eight singers participate in an art festival where m songs are performed. Each song is performed by 4 singers, and each pair of singers performs together in the same number of songs. Find the smallest m for which this is possible.

> **Remark.** Let s be the number of songs each pair of singers performs together. Use *counting in two ways* method to establish the number of times each pair of singers performs together.

Problem 15.2.3 (SC-23-HS-6-P8)**.** (*10 points*) There are 16 secret agents in the Spy's Casino. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered.

> **Remark.** Lets call two agents *allies* if neither of them watches the other. First, prove that each agent is allied with at most one other.

Problem 15.2.4 (SC-23-HS-6-P9)**.** (*10 points*) Determine, as a function of n, the number of permutations of the set $\{1, 2, \ldots, n\}$ such that no three of $1, 2, 3, 4$ appear consecutively.

Problem 15.2.5 (SC-23-HS-6-P10)**.** (*10 points*) No three diagonals of a convex 1996−gon meet in a point. Prove that the number of triangles lying in the interior of the 1996−gon and having sides on its diagonals is divisible by 11.

15.3 Solutions

Problem 15.3.1 (SC-23-HS-6-P6)**.** (*10 points*) Show that there exist infinitely many positive integers n such that the numbers $1, 2, \ldots, 3n$ can be labeled as

$$a_1,\ a_2,\ \ldots,\ a_n,\ b_1,\ b_2,\ \ldots,\ b_n,\ c_1,\ c_2,\ \ldots,\ c_n \text{ such that}$$

1. $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = \ldots = a_n + b_n + c_n$ is a multiple of 6.
2. $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n = c_1 + c_2 + \cdots + c_n$ is also a multiple of 6.

Solution. Note that the sum of all the number is $\frac{3n(3n+1)}{2}$ which has to be a multiple of both $6n$ and 9. Thus n must be a multiple of 3 congruent to 1 modulo 4. We will show that the desired arrangement exists for $n = 9m$. For $n = 9$, use the arrangement below:

$\{a_i\}$	8	1	6	17	10	15	26	19	24
$\{b_i\}$	21	23	25	3	5	7	12	14	16
$\{b_i\}$	13	18	11	22	27	20	4	9	2

(in which the first row is $a_1, a_2, \ldots$ and so on). It suffices to produce from arrangements for m (without primes) and n (with primes) an arrangement for mn (with double primes):

$$a''_{i+(j-1)m} = a_i + (m-1)a'_j \quad (1 \le i \le m,\ 1 \le j \le n)$$

and likewise for b_i and c_i. □

Problem 15.3.2 (SC-23-HS-6-P7)**.** (*10 points*) Eight singers participate in an art festival where m songs are performed. Each song is performed by 4 singers, and each pair of singers performs together in the same number of songs. Find the smallest m for which this is possible.

> **Remark.** Let s be the number of songs each pair of singers performs together. Use *counting in two ways* method to establish the number of times each pair of singers performs together.

Solution. Let s be the number of songs each pair of singers performs together.

$$m\binom{4}{2} = s\binom{8}{2}$$

and so $m = \frac{14s}{3}$, so $m \ge 14$.

Here is one example with $m = \boxed{14}$,

$$\begin{array}{llll} 1,2,3,4 & 5,6,7,8 & 1,2,5,6 & 3,4,7,8 \\ 3,4,5,6 & 1,3,5,7 & 2,4,6,8 & 1,3,6,8 \\ 2,4,5,7 & 1,4,5,8 & 2,3,6,7 & 1,4,6,7 \\ 1,2,7,8 & 2,3,5,8. & & \end{array}$$

□

Problem 15.3.3 (SC-23-HS-6-P8)**.** (*10 points*) There are 16 secret agents in the Spy's Casino. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered.

> **Remark.** Lets call two agents if neither of them watches the other. First, prove that each agent is allied with at most one other.

Solution. We say two agents are *allies* if neither watches the other.

First note that each agent watches at least 7 others; if an agent were watching 6 or fewer others, we could take away 6 agents and leave a group of 10 which could not be arranged in a circle. Similarly, each agent is watched by at least 7 others. Hence each agent is allied with at most one other.

Now, given a group of 11 agents, there must be one agent x who is not allied with any of the others in the group (since allies come in pairs). Remove that agent and arrange the other 10 in a circle. The removed agent watches at least one of the other 10 and is watched by at least one. Thus there exists a pair u, v of agents with u watching v, u watching x, and x watching v (move around the circle until the direction of the arrow to x changes); thus x can be spliced into the loop between u and v. ☐

Problem 15.3.4 (SC-23-HS-6-P9)**.** (*10 points*) Determine, as a function of n, the number of permutations of the set $\{1, 2, \ldots, n\}$ such that no three of $1, 2, 3, 4$ appear consecutively.

Solution. There are $n!$ permutations in all. Of those, we exclude $(n-2)!$ permutations for each arrangement of $1, 2, 3, 4$ into an ordered triple and one remaining element, or $24(n-2)!$ in all. However, we have twice excluded each of the $24(n-3)!$ permutations in which all four of $1, 2, 3, 4$ occur in a block. Thus the number of permutations of the desired form is

$$\boxed{n! - 24(n-2)! + 24(n-3)!.}$$

☐

Problem 15.3.5 (SC-23-HS-6-P10)**.** (*10 points*) No three diagonals of a convex 1996−gon meet in a point. Prove that the number of triangles lying in the interior of the 1996−gon and having sides on its diagonals is divisible by 11.

Solution. There is exactly one such triangle for each choice of six vertices of the 1996−gon: if A, B, C, D, E, F are the six vertices in order, the corresponding triangle is formed by the lines AD, BE, CF. Hence the number of triangles is $\binom{1996}{6}$, which is a multiple of 1991, and since 1991 is a multiple of 11, so is the number of triangles. ☐

Chapter 16

A point and a triangle

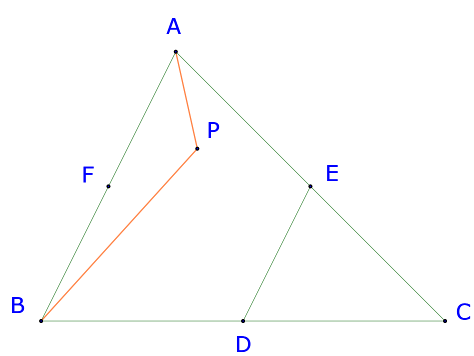
16.1 Examples

You should review the **Geometric Inequalities** chapter in the **Learning Problem Solving, Vol. 3** to understand the basics, practice with examples, and master a few techniques.

Example 16.1.1 (IMO SL 1999/G1)

Let ABC be a triangle and P be an interior point. Prove that

$$\min\{PA, PB, PC\} + PA + PB + PC < AB + AC + BC.$$

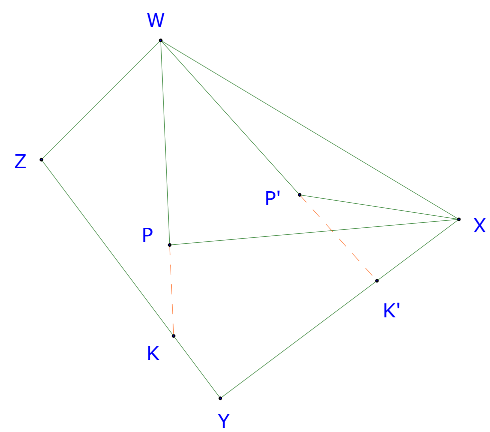


Remark. The idea is to investigate the position of P in regard to the mid-segments (the segments connecting pairs of midpoints of any two sides of a triangle). For example, as show above, since the $AEDB$ path is *outside* of the path APD, it seems that:

$$AP + PB < AE + ED + DB = \frac{1}{2}(CA + AB + BC).$$

Lemma (Perimeters of convex polygons)
If a point P lies inside a convex quadrilateral $WXYZ$, then $WP + PX < WZ + ZY + YX$.

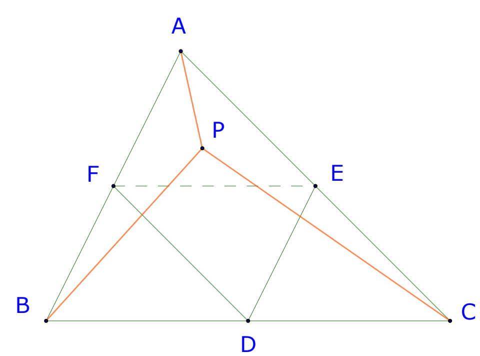
Proof. Line WP must intersect the segment XY or the segment YZ,



$$WZ + ZY + YX = (WZ + ZY + YK) + KX > WK + KX = WP + (PK + KX) > WP + PX.$$
$$\begin{aligned} WZ + YC + YX &= (WZ + ZK') + K'Y + YX > WK' + K'Y + YX \\ &= WP' + (P'K' + K'Y + YX) > WP' + P'X. \end{aligned}$$

□

Solution. Now, let D, E, and F be the midpoints of BC, CA, and AB. Note that P is inside at least two of $ABDE, BCEF$, and $CAFD$ convex quadrilaterals.



Suppose that then P is in $ABDE$ and $CAFD$, then by the lemma Perimeters of convex polygons:

$$\left.\begin{aligned} AP + PB < AE + ED + DB, \\ AP + PC < AF + FD + DB, \end{aligned}\right\} \Rightarrow AP + PA + PB + PC < AB + BC + CA.$$

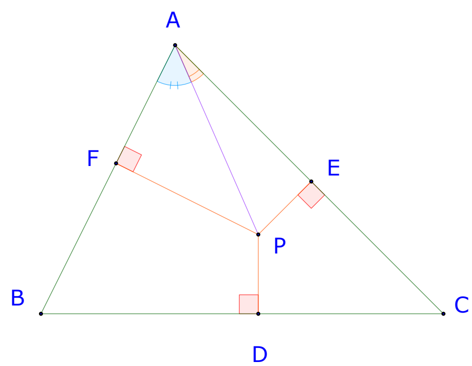
Finally, $\min\{PA, PB, PC\} \leq PA$, thus $\boxed{\min\{PA, PB, PC\} + PA + PB + PC < AB + BC + CA.}$ □

Example 16.1.2 (IMO SL 2001/G4)

Let P be a point in the interior of triangle ABC. Let D, E, and F be the feet of the perpendiculars from P to BC, CA, and AB, respectively. Find the point P which maximizes

$$f(P) = \frac{PD \cdot PE \cdot PF}{PA \cdot PB \cdot PC}.$$

Which triangles give f the maximum value?



Remark. Note that:

$$\frac{PF}{PA} \cdot \frac{PE}{PA} = \sin \angle PAF \cdot \sin \angle PAE = \frac{1}{2} \left[\cos\left(\angle PAF - \angle PAE\right) - \cos\left(\angle PAF + \angle PAE\right)\right].$$

Note that the last quantity on the right **cannot exceed** $\frac{1}{2}(1 - \cos \angle A)$, which is **free of** P. This is how we often try to find the maximal value.

Solution. By definition of sin functions

$$\sin \angle PAF = \frac{PF}{PA}, \sin \angle PAE = \frac{PE}{PA}.$$

Furthermore, by trigonometric identities:

$$\sin \angle PAF \sin \angle PAE = \frac{1}{2} \left[\cos\left(\angle PAF - \angle PAE\right) - \cos\left(\angle PAF + \angle PAE\right)\right].$$

Note that $\cos \alpha \leq 1$, for any value of α,

$$\frac{PF}{PA}\frac{PE}{PA} \leq \frac{1}{2}(1 - \cos \angle A) = \sin^2 \frac{\angle A}{2}$$

Similarly for the other pairs of fractions in $f(P)$,

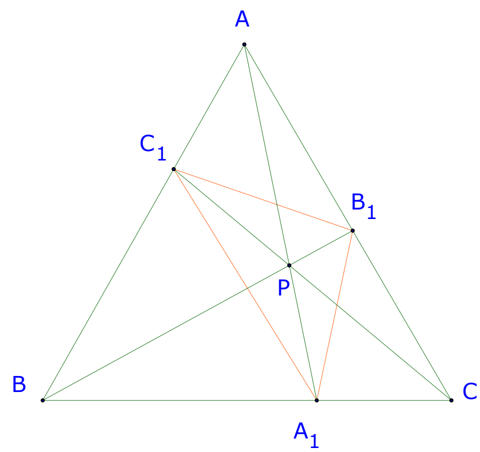
$$(f(P))^2 \leq \left(\sin 2 \frac{\angle A}{2}\right)^2 \left(\sin 2 \frac{\angle B}{2}\right)^2 \left(\sin 2 \frac{\angle C}{2}\right)^2 \Rightarrow f(P) \leq \sin \frac{\angle A}{2} \sin \frac{\angle B}{2} \sin \frac{\angle C}{2}.$$

The equality stands if and only if $A = B = C$, or $\boxed{\triangle ABC \text{ is equilateral}}$ and $f(P) = (\sin 30^\circ)^3 = \boxed{\frac{1}{8}.}$ $\square$

Example 16.1.3 (IMO SL 1996/4)

Let ABC be an equilateral triangle and let P be a point in its interior. Let the lines AP, BP, CP meet the sides BC, CA, AB at the points A_1, B_1, C_1, respectively. Prove that

$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A.$$



Remark. It is obvious that Ceva's theorem would be the first one to come to mind.

Solution. From Ceva's Theorem, we obtain that

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1 \Leftrightarrow AC_1 \cdot BA_1 \cdot CB_1 = C_1B \cdot A_1C \cdot B_1A.$$

By the Law of Cosines and the trivial inequality,

$$A_1C_1^2 = C_1C^2 + A_1B^2 - C_1B \cdot A_1B \geq C_1B \cdot A_1B$$

Thus, by applying the above similarly:

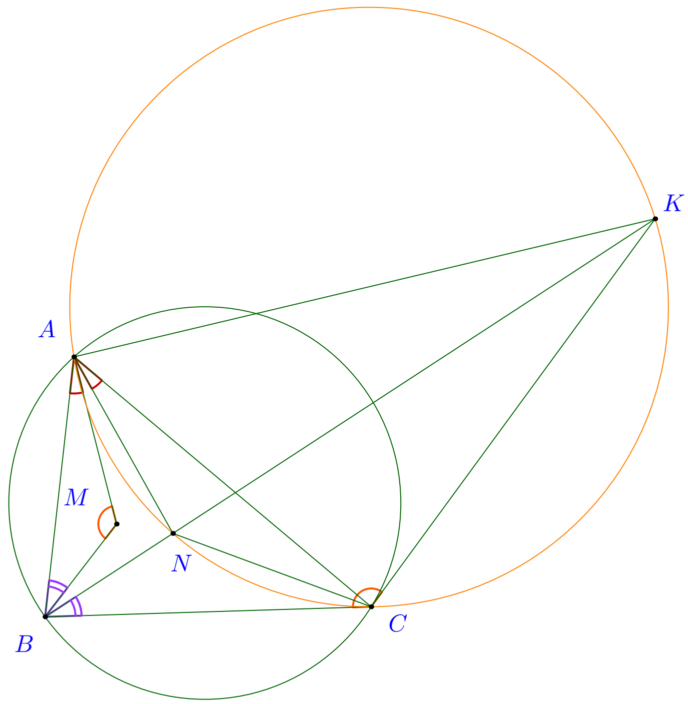
$$\begin{aligned}
&A_1C_1^2 = C_1C^2 + A_1B^2 - C_1B \cdot A_1B \geq C_1B \cdot A_1B \\
&A_1B_1^2 = B_1C^2 + A_1C^2 - B_1C \cdot A_1C \geq B_1C \cdot A_1C \\
&B_1C_1^2 = B_1A^2 + C_1A^2 - B_1A \cdot C_1A \geq B_1A \cdot C_1A \\
&\Rightarrow (A_1B_1 \cdot B_1C_1 \cdot C_1A_1)^2 \geq A_1B \cdot C_1B \cdot B_1C \cdot A_1C \cdot C_1A \cdot B_1A \\
&= (A_1B \cdot B_1C \cdot C_1A) \cdot (C_1B \cdot A_1C \cdot B_1A) = (A_1B \cdot B_1C \cdot C_1A)^2 \\
&\Rightarrow \boxed{A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A.}
\end{aligned}$$

$\square$

Example 16.1.4 (IMO SL 1998/G4)

Let M and N be two points inside triangle ABC such that $\angle MAB = \angle NAC$ and $\angle MBA = \angle NBC$. Prove that:

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$



Solution. The inequality to be proven is equivalent to:

$$AM \cdot AN \cdot BC + BM \cdot BN \cdot CA + CM \cdot CN \cdot AB = AB \cdot BC \cdot CA \quad (*).$$

Let K be the point on the ray BN such that $\angle BCK = \angle BMA$, see the figure above.

$$\triangle BCK \sim \triangle BMA \Rightarrow \angle BKC = \angle BAM = \angle NAC$$

Thus, $ANCK$ is cyclic. By Ptolemy theorem, and note that $BK = BN + NK$:

$$AC \cdot NK = AN \cdot CK + AK \cdot CN$$
$$\Rightarrow AC \cdot BN + AC \cdot NK = AC \cdot BN + AN \cdot CK + AK \cdot CN$$
$$\Rightarrow AC \cdot BK = AC \cdot BN + AN \cdot CK + AK \cdot CN \quad (**)$$

Let compute the terms AK, BK, CK. Because $\triangle BCK \sim \triangle BMA$, therefore:

$$\frac{BC}{BK} = \frac{BM}{BA} \Rightarrow \frac{BC}{BM} = \frac{BK}{BA} \Rightarrow \triangle BCM \sim \triangle BKA \Rightarrow \frac{AK}{AB} = \frac{CM}{BM} \Rightarrow AK = \frac{AB \cdot CM}{BM} \quad (1)$$

Similarly $CK = \dfrac{BC \cdot AM}{BM}$, $BK = \dfrac{AB \cdot BC}{BM}$(2). Substituting AK, BK, CK from (1) and (2) into (**)

$$AC \cdot \frac{AB \cdot BC}{BM} = AC \cdot BN + AN \cdot \frac{AB \cdot AM}{BM} + \frac{AB \cdot CM}{BM} \cdot CN$$
$$\Rightarrow AC \cdot AB \cdot BC = AC \cdot BN \cdot BM + AN \cdot AB \cdot AM + AB \cdot CM \cdot CN, \text{ which is } (*)$$

□

Example 16.1.5 (IMO SL 2003/G3)

Let P be a point in the interior of $\triangle ABC$. Let D, E, F be the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that $AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2$. Let I_a, I_b, I_c be the excenters of the $\triangle ABC$. Prove that P is the circumcenter of the $\triangle I_aI_bI_c$.

Solution. First, see Figure 16.1, $BD^2 = BP^2 - PD^2 = AP^2 - PE^2 = AE^2$, so $BD = AE$. Similarly $AF = CD$, $BF = CE$. Furthermore $BD + DC = a$, $CE + EA = b$, $AF + FB = c$, so $a + b - c = 2BD$, so $BD = \frac{1}{2}(a + b - c)$.

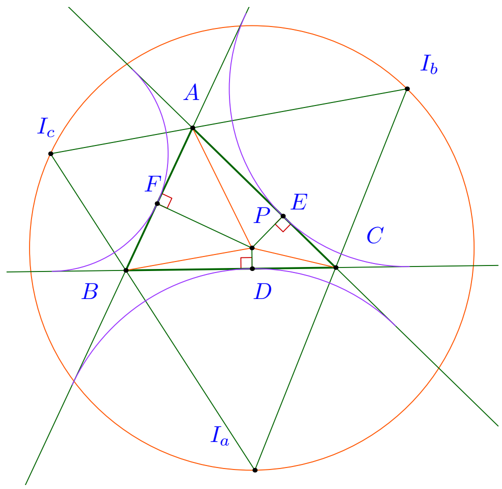


Figure 16.1: IMO SL 2003/G3

This means that D is the tangent point of BC with the excircle I_a. Similarly E and F are the tangent points of CA and AB with the excircle I_b and I_c. respectively. This means that P, D, I_a are collinear, $PI_a \perp BD$. Therefore $\angle PI_aB = 90^\circ - \angle I_aBD = \frac{1}{2}\angle B$, similarly $\angle PI_cB = \frac{1}{2}\angle B$. Thus, $\triangle PI_aI_c$ is isosceles, $PI_a = PI_c$, similarly $PI_a = PI_c$. Hence, $\boxed{P \text{ is the circumcenter of the } \triangle I_aI_bI_c.}$ □

Note that in competition, in order to get partial points or to rescue a solution based on some right ideas but somehow producing wrong answer, do not forget to submit your sketches or heuristics like what we have done in the remark analysis shown above. If you write them clean and clear, there are chances you will get additional points. They would not withstand a rigorous check, but there is no such need.

Note that for every problem you earn an additional same number of points for every different solution.

16.2 Problems

Submission deadline: August 9, 2023.

Problem 16.2.1 (SC-23-HS-7-P6)**.** Let ABC be a triangle and P be an interior point in $\triangle ABC$. Prove that

1. (*5 points*)
$$\frac{AB+BC+AC}{2} < PA+PB+PC < AB+BC+AC.$$
2. (*5 points*)
$$PA+PB+PC < AB+BC+AC-\min\{AB,BC,CA\}.$$

> **Remark.** For the second question, let shortest side of $\triangle ABC$ be BC. Draw the parallel to BC passing through P that cuts AC and AB at X, Y, respectively. Draw the altitude AH. P can be in $\triangle AHB$ or $\triangle AHC$.

Problem 16.2.2 (SC-23-HS-7-P7)**.** (*10 points*) Let P be a point outside of the triangle ABC. Suppose the lines AP, BP, CP meet the sides BC, CA, AB (or extensions thereof) in D, E, F, respectively. The triangles $\triangle PAF, \triangle PBD$, and $\triangle PCE$ all have equal area. Show that their area must equal that of $\triangle ABC$.

> **Remark.** In principle, there are two possible configurations:
> 1. P can lie between the two rays BA and BC (on the opposite side of AC to B), or similarly between the rays AB and AC or between the rays CA and CB; or
> 2. P can lie between the rays AB and CB (on the infinite side, opposite to A and C), or the two similar arrangements.
>
> Why the second configuration is not possible?
>
> For the first one, let assume P lies between the two rays BA and BC, on the opposite side of AC to B. Let $[PAF]=[PBD]=[PCE]=x$, and $[APE]=u$, $[ABE]=v$, and $[CBE]=w$. Compute the ratios $\frac{BD}{DC}$, $\frac{CE}{EA}$, and $\frac{AF}{FB}$ in regards to x, u, v, and w.

Problem 16.2.3 (SC-23-HS-7-P8)**.** (*10 points*) Let AX, BY, CZ be three cevians concurrent at an interior point D of a triangle ABC. Prove that if two of the quadrilaterals $DYAZ, DZBX, DXCY$ are cyclic, so is the third.

Problem 16.2.4 (SC-23-HS-7-P9)**.** (*10 points*) Let A, B and C be non-collinear points. Prove that there is a unique point P in the plane of ABC such that

$$PA^2+PB^2+AB^2 = PB^2+PC^2+BC^2 = PC^2+PA^2+CA^2.$$

> **Remark.** The first condition $PA^2+PB^2+AB^2 = PB^2+PC^2+BC^2$ is equivalent to $PA^2-PC^2 = BC^2-AB^2$. This means that P must lie on a fixed line perpendicular to AC.

Problem 16.2.5 (SC-23-HS-7-P10)**.** (*10 points*) The point M is inside the convex quadrilateral $ABCD$, such that

$$MA = MC,\ \angle AMB = \angle MAD + \angle MCD,\ \angle CMD = \angle MCB + \angle MAB.$$

Prove that $AB \cdot CM = BC \cdot MD$ and $BP \cdot AD = MA \cdot CD$.

Remark. Construct the convex quadrilateral $PQRS$ and the interior point T such that

$$\triangle PTQ \equiv \triangle AMB,\ \triangle QTR \sim \triangle AMD,\ \triangle PTS \sim \triangle CMD.$$

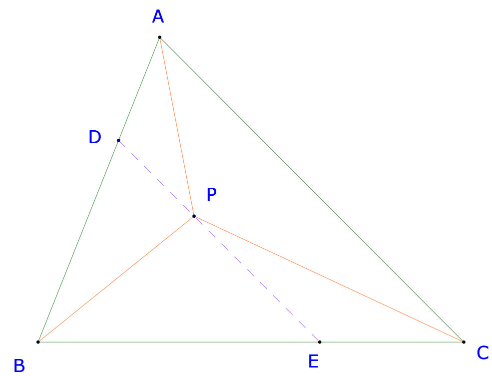
Prove that $\triangle RTS \sim \triangle BMC$.

16.3 Solutions

Problem 16.3.1 (SC-23-HS-7-P6)**.** Let ABC be a triangle and P be an interior point in $\triangle ABC$. Prove that

1. (*5 points*)
$$\frac{AB+BC+AC}{2} < PA+PB+PC < AB+BC+AC.$$

2. (*5 points*)
$$PA+PB+PC < AB+BC+AC-\min\{AB,BC,CA\}.$$



Solution. Let line parallel with AC intersects BA and BC at D and E, respectively.

For the left inequality in the first question, by triangle inequality:

$$PA+PB>AB,\ PB+PC>BC,\ PC+PA>CA \Rightarrow \boxed{PA+PB+PC>\frac{1}{2}(AB+BC+CA).}$$

For the right inequality, by triangle inequality:

$$AB+BC = AD+(DB+BE)+EC > AD+DE+EC = (AD+DP)+(PE+EC) > AP+PC \quad (1)$$

Similarly, $BC+CA > BP+PA,\ CA+AB > CP+PB$ (2)

From (1) and (2)

$$(AB+BC)+(BC+CA)+(CA+AB) > (AP+PC)+(BP+PA)+(CP+PB)$$
$$\Rightarrow \boxed{AB+BC+CA > PA+PB+PC.}$$

Now, for the second question, let $BC = \min\{AB,BC,CA\}$. Draw altitude AH to BC.

WLOG, let P be inside $\triangle AHB$, then $AD > AP$ (*)

Now since $\triangle ADE \sim \triangle ABC$ and $AC > CB$, thus $AE > ED$, or $AC = AE+EC > ED+EC$ (**)

By triangle inequality $PD+DB > PB, PE+EC > PC$, thus $BD+DE+EC > PB+PC$ (***)

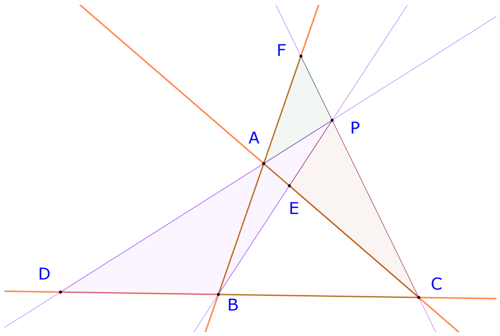
From (*), (**), and (***),

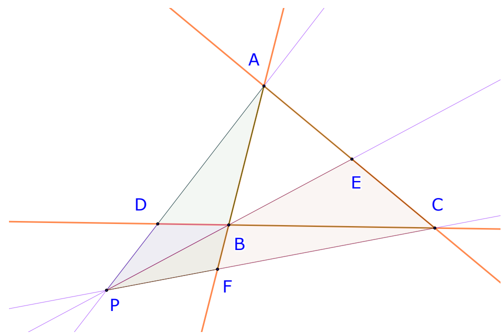
$$AD+AC+(BD+DE+EC) > AP+(ED+EC)+(PB+PC) \Rightarrow \boxed{AB+AC > PA+PB+BC.}$$

$\square$

Problem 16.3.2 (SC-23-HS-7-P7)**.** (*10 points*) Let P be a point outside of the triangle ABC. Suppose the lines AP, BP, CP meet the sides BC, CA, AB (or extensions thereof) in D, E, F, respectively. The triangles $\triangle PAF, \triangle PBD$, and $\triangle PCE$ all have equal area. Show that their area must equal that of $\triangle ABC$.

Solution. In principle, there are two possible configurations:

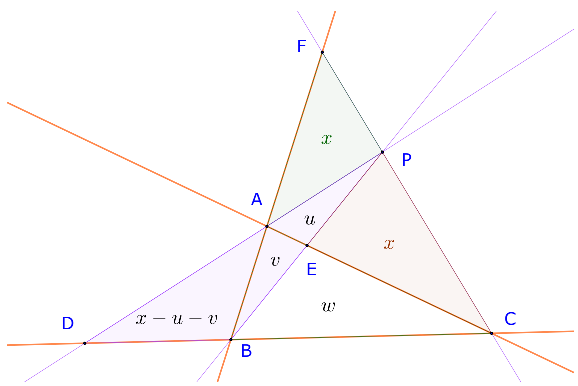




1. P can lie between the two rays BA and BC (on the opposite side of AC to B), or similarly between the rays AB and AC or between the rays CA and CB; or
2. P can lie between the rays AB and CB (on the infinite side, opposite to C and to A), or the two similar arrangements.

The second configuration is not possible since one of the triangles $\triangle PAF, \triangle PBD$, and $\triangle PCE$, is proper subset of another.

For the first one, let assume P lies between the two rays BA and BC, on the opposite side of AC to B. Let $[PAF] = [PBD] = [PCE] = x$, and $[APE] = u$, $[ABE] = v$, and $[CBE] = w$.



$$\frac{[ABD]}{[ADC]} = \frac{BD}{DC} = \frac{[PDB]}{[PDC]} \Rightarrow \frac{[PDB]-[APE]-[ABE]}{[PDB]-[APE]+[CBE]} = \frac{[PDB]}{[PDB]+[PCE]+[CBE]}$$
$$\Rightarrow \frac{x-u-v}{x-u+w} = \frac{x}{2x+w} \Rightarrow (x-u-v)(2x+w) = x(x-u+w)$$
$$\Rightarrow 2x^2 + x(w-2u-2v) - w(u+v) = x^2 + (w-u)x \Rightarrow x^2 - x(u+2v) - w(u+v) = 0 \quad (*)$$

Similarly,

$$\frac{[CBE]}{[EBA]} = \frac{CE}{EA} = \frac{[CPE]}{[EPA]} \Rightarrow \frac{w}{v} = \frac{x}{u} \Rightarrow u = \frac{vx}{w} \quad (**)$$

Substitute u in (**) into (*):

$$x^2 - x\left(\frac{vx}{w} + 2v\right) - w\left(\frac{vx}{w} + v\right) = 0 \Rightarrow x^2\left(1 - \frac{v}{w}\right) - 3vx - vw = 0 \Rightarrow x^2(w - v) = vw(3x + w) \quad (1)$$

Again, similarly

$$\begin{aligned} &\frac{[AFP]}{[BFP]} = \frac{AF}{BF} = \frac{[AFC]}{[BFC]} \Rightarrow \frac{x}{x+u+v} = \frac{2x+u}{2x+u+v+w} \\ &\Rightarrow x(2x+u+v+w) = (x+u+v)(2x+u) \Rightarrow x = \frac{u(u+v)}{w-2u-v} \quad (***) \end{aligned}$$

By substituting u in (**) into (***):

$$\begin{aligned} &x = \frac{\frac{vx}{w}\left(\frac{vx}{w} + v\right)}{w - 2\frac{vx}{w} - v} \Rightarrow 1 = \frac{x\left(\frac{v}{w}\right)^2 + \frac{v^2}{w}}{w - 2\frac{vx}{w} - v} \Rightarrow (w - v) - 2x\frac{v}{w} = x\left(\frac{v}{w}\right)^2 + \frac{v^2}{w} \\ &\Rightarrow x\frac{v}{w}\left(\frac{v}{w} + 2\right) = \frac{-v^2 - vw + w^2}{w} \Rightarrow x = \frac{w(w^2 - vw - v^2)}{v(v+2w)} \quad (2) \end{aligned}$$

Substitute x in (2) into (1):

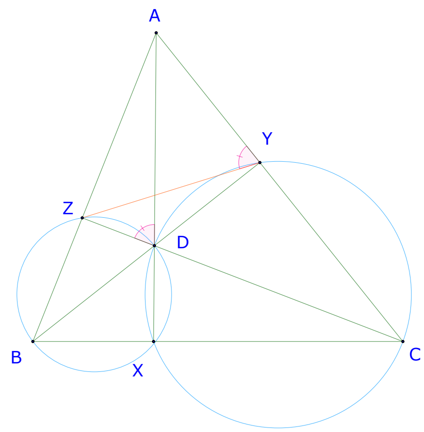
$$\begin{aligned} &\left(\frac{w(-v^2 - vw + w^2)}{v(v+2w)}\right)^2 (w - v) = vw\left(3\frac{w(-v^2 - vw + w^2)}{v(v+2w)} + w\right) \\ &\Rightarrow \frac{w^2(-v^2 - vw + w^2)^2(w - v)}{v^2(v+2w)^2} = \frac{vw^2(2v+3w)(w-v)}{v(v+2w)} \\ &w \neq 0, v \neq 0 \Rightarrow (w - v)\left[(-v^2 - vw + w^2)^2 - v^2(2v+3w)(v+2w)\right] = 0 \\ &\Rightarrow (w - v)(v + w)(v^3 + 4v^2w + 3vw^2 - w^3) = 0 \end{aligned}$$

Now if $w = v$, then from (2) $x < 0$, which is impossible. Thus

$$\begin{aligned} &v^3 + 4v^2w + 3vw^2 - w^3 = 0 or w^3 = v(v^2 + 4vw + 3w^2) = v(v+w)(v+3w) \\ &\Rightarrow x = \frac{w(w^2 - vw - v^2)}{v(v+2w)} = \frac{w^3 - vw^2 - v^2w}{v(v+2w)} = \frac{v(v+w)(v+3w) - vw(v+w)}{v(v+2w)} = v + w \\ &\Rightarrow [PAF] = [PBD] = [PCE] = x = v + w = [ABE] + [CBE] = [ABC]. \end{aligned}$$

$\square$

Problem 16.3.3 (SC-23-HS-7-P8)**.** (*10 points*) Let AX, BY, CZ be three cevians concurrent at an interior point D of a triangle ABC. Prove that if two of the quadrilaterals $DYAZ, DZBX, DXCY$ are cyclic, so is the third.



Solution. WOLG, suppose that the quadrilaterals $DXBZ$ and $DXCY$ are cyclic and we will prove that $AZDY$ is also cyclic. Its enough to prove that $\angle ADZ = \angle AYZ$ but we have:

$$AZ \cdot AB = AD \cdot AX = AY \cdot AC.$$

So $BZYC$ is also cyclic and we get:

$$\angle AYZ = \angle B = \angle ADZ.$$

□

Problem 16.3.4 (SC-23-HS-7-P9). (*10 points*) Let A, B and C be non-collinear points. Prove that there is a unique point P in the plane of ABC such that

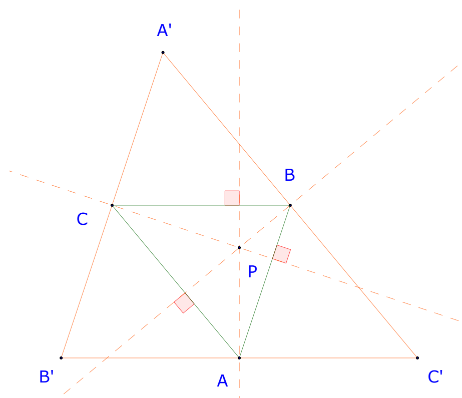
$$PA^2 + PB^2 + AB^2 = PB^2 + PC^2 + BC^2 = PC^2 + PA^2 + CA^2.$$

Solution. It is simple to prove the following claim:

Claim — Let the perpendicular from P meet AC at K. Then

$$PA^2 - PC^2 = (PK^2 + AK^2) - (PK^2 + CK^2) \Rightarrow AK^2 - CK^2 = \text{ constant}$$

Thus P must lie on a fixed line perpendicular to AC.



Now, let A, B, C be the midpoints of the sides of $B'C'$, $C'A'$, and $A'B'$. Then $B'A^2 - B'C^2 = BC^2 - BA^2$ (since $B'A = BC, B'C = BA$).

Since

$$PA^2 + PB^2 + AB^2 = PB^2 + PC^2 + BC^2 \Rightarrow PA^2 - PC^2 = BC^2 - AB^2.$$

so the fixed line passes through B'.

Similarly, P must lie on the line through A' perpendicular to BC.

So P must be $\boxed{\text{the orthocenter of } \triangle A'B'C'.}$ □

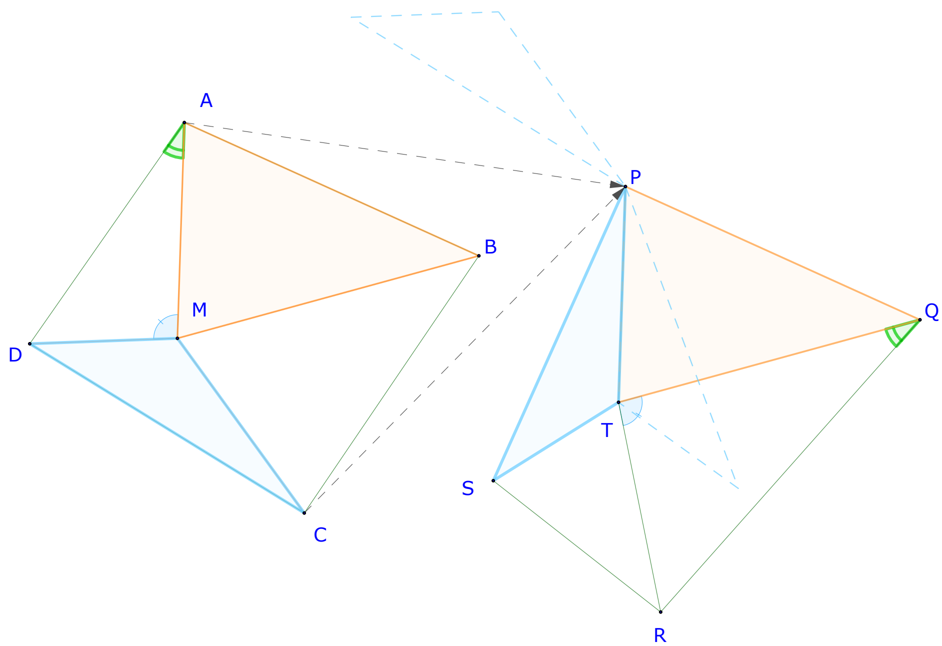
Problem 16.3.5 (SC-23-HS-7-P10)**.** (*10 points*) The point M is inside the convex quadrilateral $ABCD$, such that

$$MA = MC,\ \angle AMB = \angle MAD + \angle MCD,\ \angle CMD = \angle MCB + \angle MAB.$$

Prove that $AB \cdot CM = BC \cdot MD$ and $BP \cdot AD = MA \cdot CD$.

Solution. Construct the convex quadrilateral $PQRS$ and the interior point T such that

$$\triangle PTQ \equiv \triangle AMB,\ \triangle QTR \sim \triangle AMD,\ \triangle PTS \sim \triangle CMD.$$



It follows that

$$TS = \frac{MD \cdot PT}{MC} = MD, \frac{TR}{TS} = \frac{MD \cdot MB}{MA \cdot MD} = \frac{MB}{MC}$$

and $\angle STR = \angle BMC$, therefore $\triangle RTS \sim \triangle BMC$. The assumption on angles leads to

$$\angle QPS + \angle RSP = \angle QPT + \angle TPS + \angle TSP + \angle TSR = \angle PTS + \angle TPS + \angle TSP = 180^\circ$$

and

$$\angle RQP + \angle SPQ = \angle RQT + \angle TQP + \angle TPQ + \angle TRS = \angle QTP + \angle TQP + \angle TPQ = 180^\circ,$$

so $PQRS$ is a parallelogram. Hence $PQ = RS$ and $QR = PS$, that is

$$AB = \frac{BC \cdot TS}{MC} = \frac{BC \cdot MD}{MC}, \frac{AD \cdot QT}{AM} = \frac{CD \cdot TS}{MD} = CD.$$

The conclusion is now obvious. $\square$

Chapter 17

Integer Powers

17.1 Examples

You should review the **Number Bases** and **Integer Polynomials** chapter in the **Learning Problem Solving, Vol. 3** to understand the basics, practice with examples, and master a few techniques.

Example 17.1.1 (1986 AIME/7)
The increasing sequence $1, 3, 4, 9, 10, 12, 13 \cdots$ consists of all those positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100^{th} term of this sequence.

Solution. [Solution 1] Rewrite all of the terms in base 3. Since the numbers are sums of distinct powers of 3, in base 3 each number is a sequence of 1s and 0s (if there is a 2, then it is no longer the sum of distinct powers of 3).

Therefore, we can recast this into base 2 (binary) in order to determine the 100th number. 100 is equal to $64 + 32 + 4$, so in binary form we get 1100100. However, we must change it back to base 10 for the answer, which is $3^6 + 3^5 + 3^2 = 729 + 243 + 9 = \boxed{981}$. □

Solution. [Solution 2] Notice that the first term of the sequence is 1, the second is 3, the fourth is 9, and so on. Thus the $64th$ term of the sequence is 729.

Now out of 64 terms which are of the form $729 + \cdots$, 32 of them include 243 and 32 do not. The smallest term that includes 243, i.e. 972, is greater than the largest term which does not, or 854. So the 96th term will be 972, then 973, then 975, then 976, and finally $\boxed{981}$ □

Solution. [Solution 3] After the nth power of 3 in the sequence, the number of terms after that power but before the $(n + 1)$th power of 3 is equal to the number of terms before the nth power, because those terms after the nth power are just the nth power plus all the distinct combinations of powers of 3 before it, which is just all the terms before it.

Adding the powers of 3 and the terms that come after them, we see that the 100th term is after 729, which is the 64th term. Also, note that the kth term after the nth power of 3 is equal to the power plus the kth term in the entire sequence. Thus, the 100th term is 729 plus the 36th term. Using the same logic, the 36th term is 243 plus the 4th term, 9. We now have $729 + 243 + 9 = \boxed{981}$ □

Solution. [Solution 4] Writing out a few terms of the sequence until we reach the next power of 3 (27), we see that the 2^{nth} term is equal to 3^n. From here, we can ballpark the range of the 100th term.

The 64th term is $3^6 = 729$ and the 128th term is $3^7 = 2187$. Writing out more terms of the sequence until the next power of 3 again (81) we can see that the $(2^n + 2^{n+1})/2$ term is equal to $3^n + 3^{n-1}$. From here, we know that the 96th term is $3^6 + 3^5 = 972$. From here, we can construct the 100th term by following the sequence in increasing order.

The 97th term is $972 + 1 = 973$, the 98th term is $972 + 3 = 975$, the 99th term is $972 + 3 + 1 = 976$, and finally the 100th term is $972 + 9 = \boxed{981}$ □

Solution. [Solution 5] The number of terms 3^n produces includes each power of 3 $(1, 3^1, ..., 3^n)$, the sums of two power of 3s(ex. $3^1 + 1$), three power of 3s (ex. $3^1 + 1 + 3^n$), all the way to the sum of them all.

Since there are $n+1$ powers of 3, the one number sum gives us $\binom{n+1}{1}$ terms, the two number $\binom{n+1}{2}$ terms, all the way to the sum of all the powers which gives us $\binom{n+1}{n+1}$ terms. Summing all these up gives us $2^{n+1} - 1^*$ according to the theorem

$$\sum_{k=0}^{N} \binom{N}{k} = 2^N.$$

Since 2^6 is the greatest power < 100, then $n = 5$ and the sequence would look like $3^0, ..., 3^5$, where 3^5 or 243 would be the $2^5 - 1 = 63$rd number. The next largest power 729 would be the 64th number. However, its terms contributed extends beyond 100, so we break it to smaller pieces. Noting that 729 plus any combination of lower powers $1, 3^1 ... 3^4$ is ¡ $729 + 243$, so we can add all those terms$(2^5 - 1 = 31)$ into our sequence:

$$\{3^0,\ \ldots 3^5,\ 729,\ 729+1, \ldots,\ 729 + (1 + 3^1 + \cdots + 3^4)\}$$

Our sequence now has $63 + 1 + 31 = 95$ terms. The remaining 5 would just be the smallest sums starting with $729 + 243$ or 972:

$$972,\ 972+1,\ 972+3,\ 972+1+3,\ 972+9$$

Hence the 100th term would be $972 + 9 = \boxed{981}$. □

Example 17.1.2 (1960 IMO/1)

Determine all three-digit numbers N having the property that N is divisible by 11, and $\dfrac{N}{11}$ is equal to the sum of the squares of the digits of N.

Solution. [Solution 1] Let $N = 100a + 10b + c$ for some digits a, b, and c. Then

$$100a + 10b + c = 11m$$

for some m.

We also have $m = a^2 + b^2 + c^2$. Substituting this into the first equation and simplification, we get

$$100a + 10b + c = 11a^2 + 11b^2 + 11c^2$$

For an integer divisible by 11, the sum of digits in the odd positions minus the sum of digits in the even positions is divisible by 11. Thus we get: $b = a + c$ or $b = a + c - 11$.

Case 1*:* Let $b = a + c$. We get

$$100a + c + 10a + 10c = 11a^2 + 11c^2 + 11(a+c)^2$$

$$10a + c = 2a^2 + 2ac + 2c^2$$

Since the right side is even, the left side must also be even. Let $c = 2q$ for some $q = 0, 1, 2, 3, 4$. Then

$$10a + 2q = 2a^2 + 4aq + 8q^2$$

$$5a + q = a^2 + 2aq + 4q^2$$

Substitute $q = 0, 1, 2, 3, 4$ into the last equation and then solve for a.

Sub-case 1: When $q = 0$, we get $a = 5$. Thus $c = 0$ and $b = 5$. We get that $N = 550$ which works.

Sub-case 2: When $q = 1$, we get that a is not an integer. There is no N for this case.

Sub-case 3: When $q = 2$, we get that a is not an integer. There is no N for this case.

Sub-case 4: When $q = 3$, we get that a is not an integer. There is no N for this case.

Sub-case 5: When $q = 4$, we get that a is not an integer. There is no N for this case.

Case 2: Let $b = a + c - 11$. We get

$$\begin{aligned}
&100a + c + 10a + 10c - 110 = 11(a^2 + (a+c)^2 - 22(a+c) + c^2 + 121) \\
&10a + c = 2a^2 + 2c^2 + 2ac - 22a - 22c + 131 \\
&2(a-8)^2 + 2(c - \frac{23}{4})^2 + 2ac - \frac{505}{8} = 0
\end{aligned}$$

Now we test all $c = 0 \to 10$. When $c = 0, 1, 2, 4, 5, 6, 7, 8, 9$, we get no integer solution to a. Thus, for these values of c, there is no valid N.

However, when $c = 3$, we get

$$\begin{aligned}
&2(a-8)^2 + 2(3 - \frac{23}{4})^2 + 6a - \frac{505}{8} = 0 \\
&2(a-8)^2 + 6a - 48 = 0
\end{aligned}$$

We get that $a = 8$ is a valid solution. For this case, we get $a = 8, b = 0, c = 3$, so $N = 803$, and this is a valid value. Thus, the answers are $\boxed{N = 550, 803}$. □

Solution. [Solution 2] Define a **ten** to be all ten positive integers which begin with a fixed tens digit. We can make a systematic approach to this:

By inspection, $\frac{N}{11}$ must be between 10 and 90 inclusive. That gives us 8 tens to check, and 90 as well. For a given ten, the sum of the squares of the digits of N increases faster than $\frac{N}{11}$, so we can have at most one number in every ten that works.

We check the first ten:

$$11 \times 11 = 121,\ 1^2 + 2^2 + 1^2 = 4$$

$$12 \times 11 = 132,\ 1^2 + 3^2 + 2^2 = 14$$

11 is too small and 12 is too large, so all numbers below 11 will be too small and all numbers above 12 will be too large, so no numbers in the first ten work.

We try the second ten:

$$21 \times 11 = 231,\ 2^2 + 3^2 + 1^2 = 14$$

$$22 \times 11 = 242,\ 2^2 + 4^2 + 2^2 = 24 = 14$$

Therefore, no numbers in the second ten work. We continue, to find out that 50 and 73 are the only ones that works. $N = 50 * 11 = \boxed{550}$, $N = 73 * 11 = \boxed{803}$ so there are two N that works. □

Solution. [Solution 3]

```
for n in range(110, 990, 11):
    unitsDigit = n%10
    tensDigit = (n//10)%10
    hundredsDigit = n//100
    if n//11 == hundredsDigit**2+tensDigit**2+unitsDigit**2:
        print(n)
```

□

Example 17.1.3 (IMO SL 1989/30)

Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

Solution. [Solution 1] It is easy to prove the following two claims

Claim — There is no m, k positive integer such that

$$m^k = ((n+1)!)^2 + \ell, \ \forall 2 \le \ell \le n+1.$$

Consider the set of n consecutive integers:

$$\{((n+1)!)^2 + 2, ((n+1)!)^2 + 3, \dots, ((n+1)!)^2 + (n+1)\}.$$

Lets assume that there exists i such that:

$$((n+1)!)^2 + i = p^k, \text{ where } p \text{ is a prime}, 2 \le i \le n+1.$$

Then

$$i \mid (n+1)! \Rightarrow i \mid (n+1)!^2 + i \Rightarrow i \mid p^k \Rightarrow \exists \ell \ge 1 : \ i = p^\ell \Rightarrow p \le i \le n+1.$$

The highest exponent of p in the factorization of $((n+1)!)^2 + i$ is the minimal value of the highest exponents of p in the factorizations of $((n+1)!)^2$ and i, which is the highest exponent of p in i, which is l, thus $k = l$.

Therefore,

$$i = p^l = p^k = [(n+1)!]^2 + i, \text{ a contradiction!}$$

□

Solution. [Solution 2] By Chinese Remainder theorem, there exists x such that:

$$\begin{cases} x \equiv -1 \pmod{p_1 q_1} \\ x \equiv -1 \pmod{p_2 q_2} \\ \dots x \equiv -1 \pmod{p_n q_n} \end{cases}$$

Where $p_1, p_2, \dots, p_n, q_1, q_2, \dots q_n$ are distinct primes. The n consecutive numbers $x+1, x+2, \dots, x+n$ each have at least two prime factors, so none of them can be expressed as an integral power of a prime. □

Example 17.1.4 (1999 Canada MO/3)

Determine all positive integers n such that $n = d(n)^2$, where $d(n)$ denotes the number of positive divisors of n (including 1 and n).

Theorem (Bernoulli's inequality)

For all $x, r > 0 : (1+x)^r > 1 + rx$.

Solution. Label the prime numbers $p_1 = 2, p_2 = 3, \ldots$ Since n is a perfect square, we have

$$n = \prod_{i=1}^{\infty} {p_i}^{2a_i},\ d(n) = \prod_{i=1}^{\infty}(2a_i + 1).$$

Then $d(n)$ is odd and so is n, thus $a_1 = 0$. $n = d(n)^2$, or $\frac{d(n)}{\sqrt{n}} = 1$, thus

$$\prod_{i=1}^{\infty} \frac{2a_i + 1}{{p_i}^{a_i}} = 1 \quad (1)$$

By Bernoulli's inequality, ${p_i}^{a_i} \geq 1 + (p_i - 1)a_i > 1 + 2a_i,\ \forall p_i \geq 5 \quad (*)$

It is also easy to see that $3^{a_2} \geq 2a_2 + 1 \quad (**)$, for all non-negative a_2, and the equality stands if and only if $a_2 \in \{0, 1\}$

From (*) and (**), (1) can stands if and only if

$$\left.\begin{aligned} a_1 = a_3 = a_4 = \ldots = 0 \\ a_2 \in \{0,1\} \end{aligned}\right\} \Rightarrow n \in \boxed{\{1, 9\}}$$

□

Example 17.1.5 (1991 USA MO/3)

Show that, for any fixed integer $n \geq 1$, the sequence

$$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \ldots \pmod n$$

is eventually constant.

Solution. Suppose that the problem statement is false for some integer $n \geq 1$. Then there is a least n, which we call b, for which the statement is false.

Since all integers are equivalent mod 1, $b \neq 1$. Note that for all integers b, the sequence $2^0, 2^1, 2^2, \ldots$ eventually becomes cyclic mod b. Let k be the period of this cycle. Since there are $k-1$ nonzero residues mod b. $1 \leq k \leq b - 1 < b$. Since

$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots$$

does not become constant mod b, it follows the sequence of exponents of these terms, i.e., the sequence

$$1, 2, 2^2, 2^{2^2}, \ldots$$

does not become constant mod k. Then the problem statement is false for $n = k$. Since $k < b$, this is a contradiction. Therefore the problem statement is true.

Note that we may replace 2 with any other positive integer, and both the problem and this solution remain valid. □

Note that in competition, in order to get partial points or to rescue a solution based on some right ideas but somehow producing wrong answer, do not forget to submit your sketches or heuristics like what we have done in the remark analysis shown above. If you write them clean and clear, there are chances you will get additional points. They would not withstand a rigorous check, but there is no such need.

Note that for every problem you earn an additional same number of points for every different solution.

17.2 Problems

Submission deadline: August 16, 2023.

Problem 17.2.1 (SC-23-HS-8-P6). (*10 points*) For every natural number n, evaluate the sum

$$\sum_{k=0}^{\infty}\left[\frac{n+2^k}{2^{k+1}}\right]=\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\cdots+\left[\frac{n+2^k}{2^{k+1}}\right]+\cdots$$

(The symbol $[x]$ denotes the greatest integer not exceeding x.)

Problem 17.2.2 (SC-23-HS-8-P7). (*10 points*) Let N be the number of positive integers that are less than or equal to 2003 and whose base-2 representation has more 1's than 0's. Find the remainder when N is divided by 1000.

Problem 17.2.3 (SC-23-HS-8-P8). (*10 points*) N is an integer whose representation in base b is 777. Find the smallest positive integer b for which N is the fourth power of an integer.

Problem 17.2.4 (SC-23-HS-8-P9). (*10 points*) An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to $0,1,0$ in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to $0,0,1,1$ or $1,1,0,0$ in that order. Prove that

$$b_{n+1}=2a_n,\ \forall n>0.$$

Remark. Given any binary sequence $B=(b_1,b_2,b_3,\ldots,b_k)$, define

$$f(B)=(|b_2-b_1|,|b_3-b_2|,\ldots,|b_k-b_{k-1}|).$$

The operator f basically takes pairs of consecutive terms and returns 0 if the terms are the same and 1 otherwise. Prove that for every sequence S of length n there exist exactly two binary sequences B of length $n+1$ such that $f(B)=S$.

Problem 17.2.5 (SC-23-HS-8-P10). (*10 points*) Prove that

$$\frac{(5m)!(5n)!}{m!n!(3m+n)!(3n+m)!}$$

is integral for all positive integral m and n.

Remark. Prove that

$$[5x]+[5y]\geq[3x+y]+[3y+x]$$

where $x,y\geq 0$ and $[u]$ denotes the greatest integer $\leq u$ (e.g., $[\sqrt{2}]=1$).

17.3 Solutions

Problem 17.3.1 (1968 IMO/6)**.** (*10 points*) For every natural number n, evaluate the sum

$$\sum_{k=0}^{\infty}\left[\frac{n+2^k}{2^{k+1}}\right]=\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\cdots+\left[\frac{n+2^k}{2^{k+1}}\right]+\cdots$$

(The symbol $[x]$ denotes the greatest integer not exceeding x.)

Solution. [Solution 1] Since $\lfloor x+\frac{1}{2}\rfloor = \lfloor 2x\rfloor - \lfloor x\rfloor$, $\forall x\in\mathbb{R}$. Let $x=\frac{n}{2}$, the left-hand side is:

$$\left\lfloor x+\frac{1}{2}\right\rfloor+\left\lfloor \frac{x}{2}+\frac{1}{2}\right\rfloor+\cdots+\left\lfloor \frac{x}{2^{n-1}}+\frac{1}{2}\right\rfloor=(\lfloor 2x\rfloor-\lfloor x\rfloor)+\left(\lfloor x\rfloor-\left\lfloor\frac{x}{2}\right\rfloor\right)+\ldots+\left(\left\lfloor\frac{x}{2^{n-2}}\right\rfloor-\left\lfloor\frac{x}{2^{n-1}}\right\rfloor\right)$$
$$=\lfloor 2x\rfloor-\left\lfloor\frac{x}{2^{n-1}}\right\rfloor=n-\left\lfloor\frac{n}{2^n}\right\rfloor=n$$

Hence, the value of the sum is $\boxed{n.}$ □

Problem 17.3.2 (2003 AIME I/13)**.** (*10 points*) Let N be the number of positive integers that are less than or equal to 2003 and whose base-2 representation has more 1's than 0's. Find the remainder when N is divided by 1000.

Solution. [Solution 1] In base-2 representation, all positive numbers have a leftmost digit of 1. Thus there are $\binom{n}{k}$ numbers that have $n+1$ digits in base 2 notation, with $k+1$ of the digits being 1's.

In order for there to be more 1's than 0's, we must have $k+1>\frac{n+1}{2}\Rightarrow k>\frac{n-1}{2}\Rightarrow k\geq\frac{n}{2}$. Therefore, the number of such numbers corresponds to the sum of all numbers on or to the right of the vertical line of symmetry in Pascal's Triangle, from rows 0 to 10 (as $2003<2^{11}-1$). Since the sum of the elements of the rth row is 2^r, it follows that the sum of all elements in rows 0 through 10 is

$$2^0+2^1+\cdots+2^{10}=2^{11}-1=2047$$

.

The center elements are in the form $\binom{2i}{i}$, so the sum of these elements is $\sum_{i=0}^{5}\binom{2i}{i}=1+2+6+20+70+252=351$.

The sum of the elements on or to the right of the line of symmetry is thus $\frac{2047+351}{2}=1199$. However, we also counted the 44 numbers from 2004 to $2^{11}-1=2047$. Indeed, all of these numbers have at least 6 1's in their base-2 representation, as all of them are greater than $1984=11111000000_2$, which has 5 1's. Therefore, our answer is $1199-44=1155$, and the remainder is $\boxed{155}$. □

Solution. [Solution 2] We seek the number of allowed numbers which have k 1's, not including the leading 1, for $k=0,1,2,\ldots,10$.

For $k=0,\ldots,4$, this number is

$$\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{2k}{k},$$

which, by the Hockey Stick Identity, is equal to $\binom{2k+1}{k+1}$. So we get

$$\binom{1}{1}+\binom{3}{2}+\binom{5}{3}+\binom{7}{4}+\binom{9}{5}=175.$$

For $k = 5, \ldots, 10$, we end on $\binom{10}{k}$ - we don't want to consider numbers with more than 11 digits. So for each k we get

$$\binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{10}{k} = \binom{11}{k+1}$$

again by the Hockey Stick Identity. So we get

$$\binom{11}{6} + \binom{11}{7} + \binom{11}{8} + \binom{11}{9} + \binom{11}{10} + \binom{11}{11} = \frac{2^{11}}{2} = 2^{10} = 1024.$$

The total is $1024 + 175 = 1199$. Subtracting out the 44 numbers between 2003 and 2048 gives 1155. Thus the answer is $\boxed{155}$. $\square$

Solution. [Solution 3] We will count the number of which $< 2^{11} = 2048$ instead of 2003 In other words, the length of the base-2 representation is at most 11. If there are even digits, $2n$, then the leftmost digit is 1, the rest, $2n-1$, has odd number of digits. In order for the base-2 representation to have more 1's, we will need more 1 in the remaining $2n-1$ than 0's. Using symmetry, this is equal to

$$\frac{2^9 + 2^7 + \cdots + 2^1}{2}.$$

Using similar argument where there are odd amount of digits. The remaining even amount of digit must contains the number of 1's at least as the number of 0's. So it's equal to

$$\frac{\binom{10}{5} + 2^{10} + \binom{8}{4} + 2^8 + \binom{6}{3} + 2^6 + \cdots + \binom{0}{0} + 2^0}{2}.$$

Summing both cases, we have

$$\frac{2^0 + 2^1 + .. + 2^{10} + \binom{0}{0} + .. + \binom{10}{5}}{2} = 1199.$$

There are 44 numbers between 2004 and 2047 inclusive that satisfy it. So the answer is $1199 - 44 = \boxed{155}$ □

Problem 17.3.3 (1977 Canadian MO/3)**.** (*10 points*) N is an integer whose representation in base b is 777. Find the smallest positive integer b for which N is the fourth power of an integer.

Solution. Rewriting N in base 10, $N = 7(b^2 + b + 1) = a^4$ for some integer a. Because $7 \mid a^4$ and 7 is prime, $a^4 \geq 7^4$. Since we want to minimize b, we check to see if $a = 7$ works. When $a = 7$, $b^2 + b + 1 = 7^3$. Solving this quadratic, $b = \boxed{18}$. □

Problem 17.3.4 (1996 USA MO/4)**.** (*10 points*) An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to $0, 1, 0$ in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to $0, 0, 1, 1$ or $1, 1, 0, 0$ in that order. Prove that

$$b_{n+1} = 2a_n, \ \forall n > 0.$$

Solution. Given any binary sequence $B = (b_1, b_2, b_3, \ldots, b_k)$, define

$$f(B) = (|b_2 - b_1|, |b_3 - b_2|, \ldots, |b_k - b_{k-1}|).$$

The operator f basically takes pairs of consecutive terms and returns 0 if the terms are the same and 1 otherwise. Note that for every sequence S of length n there exist exactly two binary sequences B of length $n+1$ such that $f(B) = S$.

If $f(B)$ does not contain the string $0, 1, 0$, B cannot contain either of the strings $0, 0, 1, 1$ or $1, 1, 0, 0$. Conversely, if B does not contain the sequences $0, 0, 1, 1$ or $1, 1, 0, 0$, $f(B)$ cannot contain $0, 1, 0$. There are a_n such $f(B)$ and b_{n+1} such B. Since each S corresponds with two B, there are twice as many such B as such S; thus, $\boxed{b_{n+1} = 2a_n}$. □

Problem 17.3.5 (1975 USA MO/1). (*10 points*) Prove that

$$\frac{(5m)!(5n)!}{m!n!(3m+n)!(3n+m)!}$$

is integral for all positive integral m and n.

Lemma

Prove that

$$[5x] + [5y] \geq [3x+y] + [3y+x]$$

where $x, y \geq 0$ and $[u]$ denotes the greatest integer $\leq u$ (e.g., $[\sqrt{2}] = 1$).

Proof. We shall first prove the lemma statement for $0 \leq x, y < 1$. Then $[x] = [y] = 0$, and so we have to prove that

$$[5x] + [5y] \geq [3x+y] + [3y+x].$$

Let $[5x] = a, [5y] = b$, for integers a and b. Then $5x < a+1$ and $5y < b+1$, and so $x < \frac{a+1}{5}$ and $y < \frac{b+1}{5}$.

Define a new function, the ceiling function of x, to be the least integer greater than or equal to x. Also, define the trun-ceil function, $[[x]]$, to be the value of the ceiling function minus one. Thus, $[[a]] = a-1$ if a is an integer, and $[[a]] = [a]$ otherwise. It is not difficult to verify that if a and b are real numbers with $a < b$, then

$$[[a]] \leq [a] \leq [[b]].$$

(The only new thing we have to consider here is the case where b is integral, which is trivial.)

Therefore,

$$[3x+y] + [3y+x] \leq \left[\left[\frac{3a+b+4}{5}\right]\right] + \left[\left[\frac{3b+a+4}{5}\right]\right] = S.$$

We shall prove that $S \leq a+b = T$; to do so, we list cases. Without loss of generality, let $a \leq b$. Because x and y are less than one, we have $a \leq b \leq 4$. Then, we find, for all 15 cases:

$a = 0, b = 0 \to S = 0, T = 0.$

$a = 0, b = 1 \to S = 1, T = 1.$

$a = 0, b = 2 \to S = 2, T = 2.$

$a = 0, b = 3 \to S = 3, T = 3.$

$a = 0, b = 4 \to S = 4, T = 4.$

$a = 1, b = 1 \to S = 2, T = 2.$

$a = 1, b = 2 \to S = 3, T = 3.$

$a = 1, b = 3 \to S = 3, T = 4.$

$a = 1, b = 4 \to S = 5, T = 5.$

$a = 2, b = 2 \to S = 4, T = 4.$

$a = 2, b = 3 \to S = 4, T = 5.$

$a = 2, b = 4 \to S = 5, T = 6.$

$a = 3, b = 3 \to S = 6, T = 6.$

$a = 3, b = 4 \to S = 6, T = 7.$

$a = 4, b = 4 \to S = 6, T = 8.$

Thus, we have proved for all x and y in the range $[0, 1)$,

$$[5x] + [5y] = T \geq S \geq [3x + y] + [3y + x].$$

Now, we prove the lemma statement without restrictions on x and y. Let $x = [x] + \{x\}$, and $y = [y] + \{y\}$, where $\{x\}$, the fractional part of x, is defined to be $x - [x]$. Note that $\{x\} < 1$ as a result. Substituting gives the equivalent inequality

$$[5[x] + 5\{x\}] + [5[y] + 5\{y\}] \geq [x] + [y] + [3[x] + 3\{x\} + [y] + \{y\}] + [3[y] + 3\{y\} + [x] + \{x\}].$$

But, because $[x] + a = [x + a]$ for any integer a, this is obtained from simplifications following the adding of $5[x] + 5[y]$ to both sides of

$$[5\{x\}] + [5\{y\}] \geq [3\{x\} + \{y\}] + [3\{y\} + \{x\}],$$

which we have already proved (as $0 \leq \{x\}, \{y\} < 1$). Thus, the lemma is proved. □

Solution. We have to prove the exponent of p in

$$I = \frac{(5m)!(5n)!}{m!n!(3m + n)!(3n + m)!}$$

is non-negative, or equivalently that

$$\sum_{k=1}^{\infty} \left(\left[\frac{5m}{p^k} \right] + \left[\frac{5n}{p^k} \right] \right) \geq \sum_{k=1}^{\infty} \left(\left[\frac{m}{p^k} \right] + \left[\frac{n}{p^k} \right] + \left[\frac{3m + n}{p^k} \right] + \left[\frac{3n + m}{p^k} \right] \right).$$

But, the right-hand side minus the left-hand side of this inequality is

$$\sum_{k=1}^{\infty} \left(\left[\frac{5m}{p^k} \right] + \left[\frac{5n}{p^k} \right] - \left[\frac{m}{p^k} \right] - \left[\frac{n}{p^k} \right] - \left[\frac{3m + n}{p^k} \right] - \left[\frac{3n + m}{p^k} \right] \right),$$

which is the sum of non-negative terms by the lemma.

Thus, the inequality is proved, and so, by considering all primes p, we deduce that the exponents of all primes in I are non-negative. This proves the integrality of I (i.e. I is an integer). □

Chapter 18

Test

18.1 Rules

You have a total of 270 minutes (4 hours and 30 minutes) to provide solutions to the problems below. You must submit a scanned version of the solutions no later than 15 minutes of your registered time, i.e. if you registered your test from 10:00 - 14:30, then you must submit your solution latest 14:45.

This is an open book test. You can use any material that you can access.

Each correct solution is worth of 10 points. Partially correct solution can earn some, but not all 10 points. Solutions must be cleanly, clearly, and well written. Unreadable solution, even if correct, could earn 0 point. Answers, even if correct, without a solution or depicted detailed diagram, will not be considered. Note that for every problem you earn an additional same number of points for every different correct solution.

18.2 Problems

Problem 18.2.1 (SC-23-HS-T-P1)**.** Compute

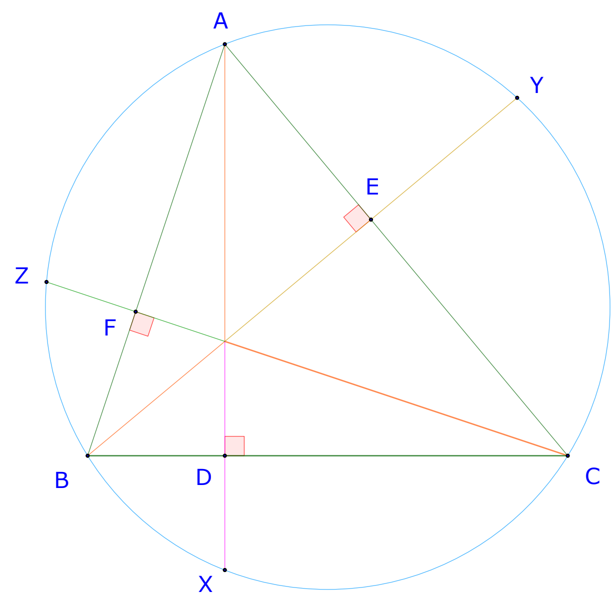
$$4\sum_{n=1}^{30} n\left[\cos^2(30^\circ - n) - \cos(30^\circ - n)\cos(30^\circ + n) + \cos^2(30^\circ + n)\right]$$

Problem 18.2.2 (SC-23-HS-T-P2)**.** Let $(a_1, a_2, \ldots, a_{10})$ be a list of the first 10 positive integers such that for each $2 \le i \le 10$ either $a_i + 1$ or $a_i - 1$ or both appear somewhere before a_i in the list. How many such list are there?

Problem 18.2.3 (SC-23-HS-T-P3)**.** Let ω be a given circle. Points A, B, and C lie on ω such that ABC is an acute triangle. Points X, Y, and X are also on ω such thar $AX \perp BC$ at D, $BY \perp CA$ at E, and $CZ \perp AB$ at F. Show that the value

$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF}.$$

does not depend on the choice of A, B, and C.



Problem 18.2.4 (SC-23-HS-T-P4)**.** Using Euler's $e^{ix} = \cos x + i\sin x$ and de Moivre's formula $\cos nx + i\sin nx = (\cos x + i\sin x)^n$, where i is the imaginary unit, prove that

$$\cos^n x = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left((n-2k)x\right).$$

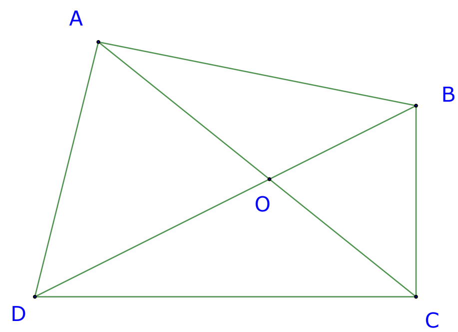
Hint: prove that $e^{ix} + e^{i(-x)} = 2\cos x$.

Problem 18.2.5 (SC-23-HS-T-P5)**.** Let $S = \{1, 2, 3, 4, 5\}$. How many functions $f : S \to S$ satisfy

$$f(f(x)) = f(x),\ \forall x \in S?$$

Problem 18.2.6 (SC-23-HS-T-P6)**.** Let $ABCD$ be a quadrilateral and O be the intersection of its diagonals AC and BD. Prove that

$$\sqrt{[ABCD]} \geq \sqrt{[AOB]} + \sqrt{[COD]}.$$



Problem 18.2.7 (SC-23-HS-T-P7)**.** Let a, b be two positive integers and $a > b$. We know that $\gcd(a - b, ab+1) = 1$ and $\gcd(a+b, ab-1) = 1$. Prove that $(a-b)^2 + (ab+1)^2$ is not a perfect square.

Problem 18.2.8 (SC-23-HS-T-P8)**.** Let AC and BD be two segments in the plane, then $AC \perp BD$ if and only if $AB^2 + CD^2 = AD^2 + BC^2$.

Problem 18.2.9 (SC-23-HS-T-P9)**.** Each of the students in a class writes a different 2-digit number on the whiteboard. The teacher claims that no matter what the students write, there will be at least three numbers on the whiteboard whose digits have the same sum. What is the smallest number of the students in the class for the teacher to be correct?

Show a counter example for the number of students is one less than that smallest number

Problem 18.2.10 (SC-23-HS-T-P10)**.** a, b, and c are real numbers such that $a > -\frac{1}{2}$, $b > -\frac{1}{2}$, $c > -\frac{1}{2}$. Prove that

$$\frac{a^2+2}{b+c+1} + \frac{b^2+2}{c+a+1} + \frac{c^2+2}{a+b+1} \geq 3.$$

When does the equality hold?

18.3 Solutions

Problem 18.3.1 (Purple Comet 2016 HS/P25)**.** Compute

$$4\sum_{n=1}^{30} n\left[\cos^2(30^\circ - n) - \cos(30^\circ - n)\cos(30^\circ + n) + \cos^2(30^\circ + n)\right]$$

Solution. Let $f(n) = \cos^2(30^\circ - n) - \cos(30^\circ - n)\cos(30^\circ + n) + \cos^2(30^\circ + n)$, then

$$\begin{cases} 2\cos^2(30^\circ - n) = 1 + \cos(60^\circ - 2n) \\ 2\cos(30^\circ - n)\cos(30^\circ + n) = \cos(60^\circ) + \cos(2n) \\ 2\cos^2(30^\circ + n) = 1 + \cos(60^\circ + 2n) \end{cases}$$
$$\Rightarrow 2f(n) = 2 + \cos(60^\circ - 2n) + \cos(60^\circ + 2n) - \cos(60^\circ) - \cos(2n)$$
$$\Rightarrow 2f(n) = 2 + 2\cos(60^\circ)\cos(2n) - \cos(60^\circ) - \cos(2n) = \frac{3}{2}$$

Thus $4\sum_{n=1}^{30} nf(n) = 3\sum_{n=1}^{30} n = 3 \cdot \frac{30\cdot 31}{2} = \boxed{1395.}$ □

Problem 18.3.2 (2012 AMC 12B/P18)**.** Let $(a_1, a_2, \ldots, a_{10})$ be a list of the first 10 positive integers such that for each $2 \le i \le 10$ either $a_i + 1$ or $a_i - 1$ or both appear somewhere before a_i in the list. How many such list are there?

Solution. [Solution 1] For a list $a_1, a_2, \ldots, a_k$ with k terms, 2 valid lists with $k+1$ terms can be created by 2 ways:

1. Add a_{k+1} to the end of the list, making a new list $a_1, a_2, \ldots, a_k, a_{k+1}$

2. Increase the value of all existing terms by one, making a new list $a_2, a_3, \ldots, a_{k+1}$. Then add a_1 to the end of the list, making a new list $a_2, a_3, \ldots, a_{k+1}, a_1$

Let $F(n)$ be the number of lists with n elements, $F(n) = 2 \cdot F(n-1)$. As $F(2) = 2$, $F(n) = 2^{n-1}$, $F(10) = 2^9 = \boxed{512.}$ □

Solution. [Solution 2] Let $1 \le k \le 10$. Assume that $a_1 = k$. If $k < 10$, the first number appear after k that is greater than k must be $k+1$, otherwise if it is any number x larger than $k+1$, there will be neither $x-1$ nor $x+1$ appearing before x. Similarly, one can conclude that if $k+1 < 10$, the first number appear after $k+1$ that is larger than $k+1$ must be $k+2$, and so forth.

On the other hand, if $k > 1$, the first number appear after k that is less than k must be $k-1$, and then $k-2$, and so forth.

To count the number of possibilities when $a_1 = k$ is given, we set up 9 spots after k, and assign $k-1$ of them to the numbers less than k and the rest to the numbers greater than k. The number of ways in doing so is 9 choose $k-1$.

Therefore, when summing up the cases from $k = 1$ to 10, we get

$$\binom{9}{0} + \binom{9}{1} + \cdots + \binom{9}{9} = 2^9 = \boxed{512.}$$

□

Solution. [Solution 3] The problem can be formulated as: *In how many ways can you order numbers 1-10 so that each number is one above or below some previous term?*

Then, the method becomes clear. For some initial number, WLOG examine the numbers greater than it. They always must appear in ascending order later in the list, though not necessarily as adjacent terms. Then, for some initial number, the number of possible lists is just the number of combination where this number of terms can be placed in 9 slots. For 9, that's 1 number in 9 potential slots. For 8, that's 2 numbers in 9 potential slots.

$$\binom{9}{0} + \binom{9}{1} + \cdots + \binom{9}{9} = \boxed{512}$$

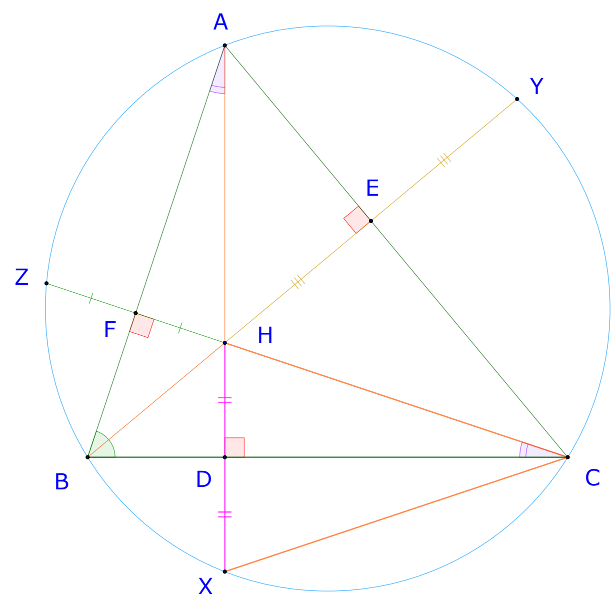
□

Problem 18.3.3 (USA MTS 2005 R3/P3)**.** Let ω be a given circle. Points A, B, and C lie on ω such that ABC is an acute triangle. Points X, Y, and X are also on ω such thar $AX \perp BC$ at D, $BY \perp CA$ at E, and $CZ \perp AB$ at F. Show that the value

$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF}.$$

does not depend on the choice of A, B, and C.

Solution. Let H be the ortho center of $\triangle ABC$. It is easy to see that $\angle HCD = 90° - \angle CBF = \angle BAX = \angle BCX$, thus $\triangle HCD \cong \triangle XCD$, or $DX = DH$. similarly $EY = EH$ and $FZ = FH$.



Now,

$$\frac{AX}{AD} = 1 + \frac{DX}{AD} = 1 + \frac{DH}{AD} = 1 + \frac{[BHC]}{[ABC]}, \text{ and similarly}$$
$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} = 3 + \frac{[BHC]}{[ABC]} + \frac{[CHA]}{[ABC]} + \frac{[AHB]}{[ABC]} = \boxed{4.}$$

□

Problem 18.3.4 (SC-23-HS-T-P4)**.** Using Euler's $e^{ix} = \cos x + i \sin x$ and de Moivre's formula $\cos nx + i \sin nx = (\cos x + i \sin x)^n$, where i is the imaginary unit, prove that

$$\cos^n x = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos\left((n-2k)x\right).$$

Hint: prove that $e^{ix} + e^{i(-x)} = 2 \cos x$*.*

Solution. Since $e^{i(-x)} = \cos(-x) + i \sin(-x) = \cos x - i \sin x$, thus by the binominal theorem:

$$\begin{aligned}\cos^n x &= \left(\frac{e^{ix} + e^{-ix}}{2}\right) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(e^{ix}\right)^{n-k} \left(e^{-ix}\right)^{k} \\ &= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} e^{ix(n-2k)} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos\left((n-2k)x\right) + i \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \sin\left((n-2k)x\right)\end{aligned}$$

Now, since $\cos^n x$ is a real number, thus the imaginary part of the last sum must be zero, thus

$$\cos^n x = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos\left((n-2k)x\right)$$

□

Problem 18.3.5 (SC-23-HS-T-P5)**.** Let $S = \{1, 2, 3, 4, 5\}$. How many functions $f : S \to S$ satisfy

$$f(f(x)) = f(x), \ \forall x \in S?$$

Solution. Note that if $s \in S = \{1, 2, 3, 4, 5\}$, i.e. s is in the image of f, then $f(s) = s$, since we must have $f(f(x)) = f(x)$, for all $x \in S$. This give us an idea to do casework base on the side of $f(S)$, i.e. how many elements are there in the image of f.

Case 1: if all elements of S are in $f(S)$, then there is only one such function $f(x) = x, \forall x \in S$.

Case 2: if 4 elements of S are in $f(S)$. There are $\binom{5}{4} = 5$ ways to choose which four elements of S to be in $f(S)$. Each of these elements must map to itself, which means there are 4 choices for the remaining element since it cannot be mapped to itself. Thus, the number of functions in this case is $5\dot{4} = 20$.

Case 3: if 3 elements of S are in $f(S)$. There are $\binom{5}{3} = 10$ ways to choose which three elements of S to be in $f(S)$. Each of these elements must map to itself, which means there are 3 choices for each of the remaining 2 elements. Thus, the number of functions in this case is $10\dot{3}^2 = 90$.

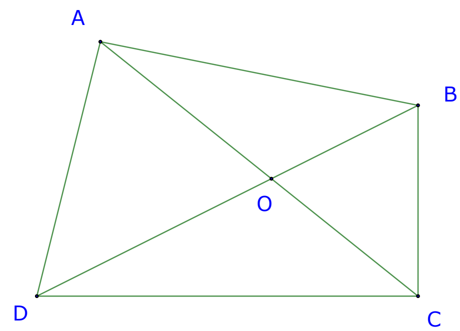
Case 4: if 2 elements of S are in $f(S)$. There are $\binom{5}{2} = 10$ ways to choose which two elements of S to be in $f(S)$. Each of these elements must map to itself, which means there are 2 choices for each the remaining 3 elements. Thus, the number of functions in this case is $10\dot{2}^3 = 80$.

Case 5: if only 1 element of S are in $f(S)$. There are $\binom{5}{1} = 5$ ways to choose which one element of S to be in $f(S)$. All elements must be map to this, so the number of functions in this case is 5.

Hence, the total number of functions is $1 + 20 + 90 + 80 + 5 = \boxed{196}$. □

Problem 18.3.6 (SC-23-HS-T-P6)**.** Let $ABCD$ be a quadrilateral and O be the intersection of its diagonals AC and BD. Prove that

$$\sqrt{[ABCD]} \geq \sqrt{[AOB]} + \sqrt{[COD]}.$$



Solution. First, note that

$$\frac{[AOB]}{[AOD]} = \frac{BO}{DO} = \frac{[COB]}{[COD]} \Rightarrow [AOB] \cdot [COD] = [COB] \cdot [AOD].$$

Now,

$$\begin{aligned}[ABCD] &= [AOB] + [BOC] + [COD] + [DOA] \geq [AOB] + [COD] + 2\sqrt{[BOC][DOA]} \\ &= [AOB] + [COD] + 2\sqrt{[AOB] \cdot [COD]} = \left(\sqrt{[AOB]} + \sqrt{[COD]}\right)^2.\end{aligned}$$

By taking the square root of both sides, we receive the desired inequality. □

Problem 18.3.7 (Iran MO 2010 R2/P1)**.** Let a, b be two positive integers and $a > b$. We know that $\gcd(a-b, ab+1) = 1$ and $\gcd(a+b, ab-1) = 1$. Prove that $(a-b)^2 + (ab+1)^2$ is not a perfect square.

Solution. Note that

$$(a-b)^2 + (ab+1)^2 = (a^2+1)(b^2+1) = (a+b)^2 + (ab-1)^2$$

Now, it is easy to see that, each of a^2+1, b^2+1 is not a perfect square. If we can prove that they are relatively primes, then their product is not a perfect square.

Let assume there exist prime p such that $p \mid gcd(a^2+1, b^2+1)$, then

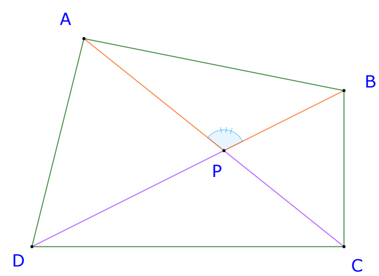
$$p \mid (a^2+1) - (b^2+1) = (a-b)(a+b) \Rightarrow p \mid a-b \text{ or } p \mid a+b.$$

If $p \mid a-b$, then $p \mid ab+1$, which contradicts the given condition that $\gcd(a-b, ab+1) = 1$. Similarly $p \mid a+b$ also leads to contradiction. Thus $\gcd(a^2+1, b^2+1)$, hence the given expression is not a perfect square. □

Problem 18.3.8 (SC-23-HS-T-P8)**.** Let AC and BD be two segments in the plane, then $AC \perp BD$ if and only if $AB^2 + CD^2 = AD^2 + BC^2$.

Solution. There are two possible configurations.

Case 1: $ABCD$ is a convex quadrilateral. An example is shown as below.



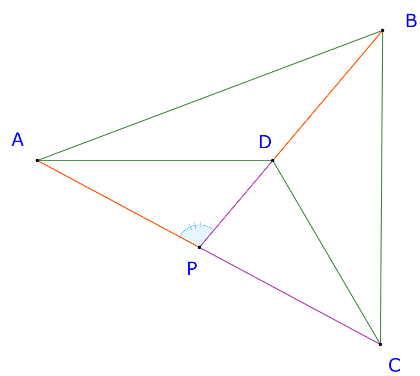
Note that, by the Law of Cosines,

$$\left.\begin{aligned} AB^2 &= AP^2 + BP^2 - 2 \cdot AP \cdot BP \cdot \cos\angle APB \\ CD^2 &= CP^2 + DP^2 - 2 \cdot CP \cdot DP \cdot \cos\angle CPD \end{aligned}\right\}$$
$$\Rightarrow AB^2 + CD^2 = (AP^2 + BP^2 + CP^2 + DP^2) - 2\cos\angle APB(AP \cdot BP + CP \cdot DP) \quad (*)$$

Similarly $AD^2 + BC^2 = ((AP^2 + DP^2 + BP^2 + CP^2) - 2\cos\angle APD(AP \cdot DP + BP \cdot CP) \quad (**)$.

Since $\angle APB + \angle APD = \pi$, so $\cos\angle APB = -\cos\angle APD$, thus the two quantities (*) and (**) can only be equal if and only if $\cos\angle APB = \cos\angle APD = 0$, or $\boxed{AC \perp BD.}$

Case 2: $ABCD$ is a concave quadrilateral. An example is shown as below.



Note that, by the Law of Cosines,

$$\left.\begin{aligned} AB^2 &= AP^2 + BP^2 - 2 \cdot AP \cdot BP \cdot \cos\angle APB \\ CD^2 &= CP^2 + DP^2 - 2 \cdot CP \cdot DP \cdot \cos\angle CPD \end{aligned}\right\}$$
$$\Rightarrow AB^2 + CD^2 = (AP^2 + BP^2 + CP^2 + DP^2) - 2\cos\angle APB(AP \cdot BP - CP \cdot DP) \quad (*)$$

Similarly $AD^2 + BC^2 = (AP^2 + DP^2 + BP^2 + CP^2) - 2\cos\angle APD(AP \cdot DP - BP \cdot CP) \quad (**)$.

Since $AP \cdot BP \geq AP \cdot DP$ and $CP \cdot DP \leq BP \cdot CP$, $\angle APB = \angle APD$, thus the two quantities (*) and (**) can only be equal if and only if $\cos\angle APB = \cos\angle APD = 0$, or $\boxed{AC \perp BD.}$ □

Problem 18.3.9 (SC-23-HS-T-P9)**.** Each of the students in a class writes a different 2−digit number on the whiteboard. The teacher claims that no matter what the students write, there will be at least three numbers on the whiteboard whose digits have the same sum. What is the smallest number of the students in the class for the teacher to be correct?

Show a counter example for the number of students is one less than that smallest number.

Solution. Let n be the number of students. The 2−digit numbers can be put into the sets A_k where the digit sum equal to k as follow

$$\begin{aligned}
A_1 &= \{10\},\\
A_2 &= \{11, 20\},\\
A_3 &= \{12, 21, 30\},\\
&\dots\\
A_{17} &= \{89, 98\},\\
A_{18} &= \{99\}
\end{aligned}$$

At most one student can write the number 10 and one student can write the number 99. The remaining $n-2$ students must write one of the numbers in the 16 sets $A_2, A_3, \dots, A_{17}$. If $n-2 > 32$, then there exists a set where at least three students wrote some of those numbers. Thus the smallest number of students in the class for the teacher to be correct is $33 + 2 = \boxed{35.}$

We show a counter example for $n = 34$

$$10, 11, 12, \dots, 29, 38, 39, 48, 49, \dots, 98, 99.$$

□

Problem 18.3.10 (SC-23-HS-T-P10)**.** a, b, and c are real numbers such that $a > -\frac{1}{2}$, $b > -\frac{1}{2}$, $c > -\frac{1}{2}$. Prove that

$$\frac{a^2+2}{b+c+1} + \frac{b^2+2}{c+a+1} + \frac{c^2+2}{a+b+1} \geq 3.$$

When does the equality hold?

Solution. Let $x = a + \frac{1}{2}, y = b + \frac{1}{2}, z = c + \frac{1}{2}$, then x, y, z are positive real numbers.

$$\frac{a^2+2}{b+c+1} = \frac{\left(x-\frac{1}{2}\right)^2+2}{y+z} = \frac{\left(x-\frac{3}{2}\right)^2+2x}{y+z} \geq \frac{2x}{y+z}.$$

Thus,

$$\frac{a^2+2}{b+c+1} + \frac{b^2+2}{c+a+1} + \frac{c^2+2}{a+b+1} \geq 2\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right).$$

By Nesbitt inequality, the right hand-side is at least $2 \cdot \frac{3}{2} = 3$. □