School year 2023 - 2024

Nghia Doan

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Contents

Introduction

Welcome to the second semester of the school year 2023 - 2024!

For the second semester of the 2023-2024 school year, the club plans to organize the program as follows:

- Objective 1: maintain current levels of O, A, I, IC, Pre-algebra.
- Objective 2: strong emphasis on working under pressure.
- Objective 3: solving very challenging problems.

A. Biweekly activity

- On Friday, students will receive 10 problems to solve in that week.
- Detailed solutions must be submitted no later than the following Monday.
- On the following Saturday/Sunday, students receive exam questions already available in the common online folder. The test consists of 10 exercises that are almost identical to the 10 exercises received last week, with some minor changes in the test conditions. Students only need to submit the answer (an integer) for each problem. After 30 minutes, students only need to fill in 10 answers and a file is available in their personal folder.
- If the official solutions of the previous week are available, students must study them and make sure that they master the solutions in order to know how to reuse them, especially when working under duress.
- The submissions, both solutions on Friday and answers on Saturday, are graded, each contributes 50% to the final mark of the problem.

For example:

- *Friday, January 19: Students receive assignments of Session 1*
- *Friday, January 26: Students submit detailed solutions for assignments of Session 1*
- *Saturday, January 27: Students receive the exam of Session 1 (similar problems as received on Friday, January 19), take the exam for 30 minutes, and submit their answers.*
- *Monday, January 29: Students receive the official solutions of Session 1 for both the assignment and the exam Problem. They should study them carefully.*
- *Friday, February 2: Students receive markings of both solutions (submitted on Friday) and answers (submitted on Saturday)*
- *Friday, February 2: Students receive assignments of Session 2*

Note that: students who read and do well on the exercises during the week will have the opportunity to do the exercises correctly during the Purple Comet Competition (with an estimated duration of 3 minutes per exercise).

Below is an example of a problem in a weekly assignment. You will need to submit a solution, similar like the one below.

Example (Problem 1 - Weekly)

(*5 points*) Find the number of integers n for which $\sqrt{\dfrac{(2020-n)^2}{2020-n^2}}$ is a real number.

Solution. The square root is a real number if the expression under the square root sign is non-negative.

$$\frac{(2020-n)^2}{2020-n^2} \geq 0.$$

There are two cases.

Case 1:

$$\frac{(2020-n)^2}{2020-n^2} = 0 \Rightarrow n = 2020.$$

Case 1:

$$\frac{(2020-n)^2}{2020-n^2} > 0 \Rightarrow 2020-n^2 > 0 \Rightarrow |n| < \sqrt{2020} < 45^2 = 2025 \Rightarrow -44 \leq n \leq 44.$$

There are $1 + 2 \cdot 44 = 89$ such integers between -44 and 44.

Hence, in total there are $1 + 89 = \boxed{90}$ such numbers. □

Then on the exam day, you get a variation of the problem. The correct answer of this problem is $\boxed{92.}$ *Below is the official* **generic solution** *that you don't have to submit.*

Example (Problem 1 - Exam)

(*5 points*) Find the number of integers n for which $\sqrt{\dfrac{(2048-n)^2}{2048-n^2}}$ is a real number.

Solution. Let $m > 1$ is a positive integer.

$$\sqrt{\frac{(m-n)^2}{m-n^2}} \text{ is real number } \Rightarrow \frac{(m-n)^2}{m-n^2} \geq 0.$$

We have two cases:

$$\begin{cases} \dfrac{(m-n)^2}{m-n^2} = 0 \Rightarrow n = m. \\ \dfrac{(m-n)^2}{m-n^2} > 0 \Rightarrow m-n^2 > 0 \Rightarrow |n| < \sqrt{m}. \end{cases} \Rightarrow n \in \{m, -\lfloor\sqrt{m}\rfloor, \ldots, -1, 0, 1, \ldots, \lfloor\sqrt{m}\rfloor\}.$$

There are $2 + 2\lfloor\sqrt{m}\rfloor$ such numbers. For $m = 2048$, there are $2(1 + \left\lfloor\sqrt{2048}\right\rfloor) = \boxed{92.}$ □

If you have successfully solved the problem in time then you will have an advantage at the exam. If you solve the problem in a generic way, the exam will be a breeze.

Solution based on coding can accepted only if

- The code is written in C, C++, Python, MatLab, or Java. No other languages are allowed.
- The code should not use any non-standard libraries.
- The code should compile without extra settings, should result in no compilation errors, or runtime errors.
- Execution time should not exceed 10 minutes.
- Submission must contain both source code, input (if any) and output (printout).

Failure to comply to any of the requirements will lead to rejection of the solution.

B. Semester open competition: Every months for each level there will be 4 problems to solve. Students must submit solutions before given deadlines (will be stated clearly in the text).

C. Purple Comet competition: This year's Purple Comet exam schedule is as follow:

- April 6: middle-school (MS) teams - grade 8 (US, CA), grade 9 (FR, UK, VN) and younger
- April 7: high-school teams - grade 9 (US, CA), grade 10 (FR, UK, VN) and above

Each team has 6 members. Students will be selected based on their weekly activity before April.

- MS team: 60 minutes, 20 problems, middle school math, a team is divided into two groups:
  - M1 - the first ten lessons (1-10): corresponds to level I;
  - M2 - the last ten lessons (11-20): corresponds to level A.
- HS team: 90 minutes, 30 problems, high school math, a team is divided into three groups:
  - H1 - first ten lessons (1-10): corresponds to level I;
  - H2 - the middle ten lessons (11-20): corresponds to level A;
  - H3 - last ten lessons (20-30): corresponds to level O.

Have a great year!

Your teacher,

Nghia Doan

Part I

2024

Chapter 1

Session 1: Jan 19 - Jan 27

1.1 Middle School - Assignment

Middle school students: grade 8 (US, CA), grade 9 (FR, UK, VN) and younger.

- **Submission deadline: Friday, January 26**
- **Test: Saturday, January 27**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 5-14**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 1.1.1 (Problem 1). (*5 points*) Alex launches his boat into a river and heads upstream at a constant speed. At the same time at a point 8 miles upstream from Alex, Alice launches her boat and heads downstream at a constant speed. Both boats move at 6 miles per hour in still water, but the river is flowing downstream at $2\frac{3}{10}$ miles per hour. Alex and Alice will meet at a point that is $\frac{m}{n}$ miles from Alex's starting point, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Relative to the moving water, Alex and Alice are moving at 6 miles per hour, so they are approaching each other at 12 miles per hour. Thus, they meet after $\dfrac{8}{12}=\dfrac{2}{3}$ hours has travel $\dfrac{8}{2}=4$ relative to the moving water. But the water is moving Alex downstream at r miles per hour, so Alex and Alice meet after Alex has traveled:

$$\frac{m}{n}=4-\frac{2}{3}\left(2\frac{3}{10}\right)=\frac{37}{15}.$$

Hence, $m+n=37+15=\boxed{52}$. □

Problem 1.1.2 (Problem 2). (*5 points*) Find a positive integer n such that there is a polygon with n sides where each of its interior angles measures $177°$.

Solution. The sum of the measures of the interior angles in a polygon with n sides is $(n-2)180$ in degree.

$$177n=(n-2)180\Rightarrow n(180-177)=360\Rightarrow n=\frac{360}{3}=\boxed{120.}$$

□

Problem 1.1.3 (Problem 3). (*5 points*) Patrick started walking at a constant rate along a straight road from school to the park. One hour after Patrick left, Tanya started running along the same road from school to the park. One hour after Tanya left, Jose started bicycling along the same road from school to the park. Tanya ran at a constant rate of 2 miles per hour faster than Patrick walked, Jose bicycled at a constant rate of 7 miles per hour faster than Tanya ran, and all three arrived at the park at the same time. The distance from the school to the park is $\frac{m}{n}$ miles, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let t be the time it took Patrick to walk from school to the park in hours and let s be the speed that Patrick walked in miles per hour. Then the distance from the school to the park can be measured in different ways:

$$st=(s+2)(t-1)=(s+9)(t-2)\Rightarrow st=st-s+2t-2=st-2s+9t-18$$
$$-s+2t-2=-2s+9t-18=0\Rightarrow\begin{cases}s=2t-2\\2s=9t-18\end{cases}\Rightarrow s=\frac{18}{5},t=\frac{14}{5}\Rightarrow st=\frac{252}{25}$$

Therefore $\frac{m}{n}=\frac{252}{25}$, so $m+n=252+25=\boxed{277.}$ □

Problem 1.1.4 (Problem 4). (*5 points*) Find the number of positive integers less than or equal to 2020 that are relatively prime to 588.

Solution. A number is relatively prime to $588=2^2\cdot3\cdot7^2$ if and only if it is not divisible by $2,3$, or 7. Now, we count the number of positive integers less than or equal to 2020 that are divisible by $2,3$, or 7, then find the total of those who is not divisible by $2,3$, or 7 by complementary counting.

There are $\frac{2020}{2}=1010$ positive integers that are divisible by 2.

There are $\left\lfloor\frac{2020}{3}\right\rfloor=673$ positive integers that are divisible by 3.

There are $\left\lfloor \frac{2020}{7} \right\rfloor = 288$ positive integers that are divisible by 7.

There are $\left\lfloor \frac{2020}{2\cdot 3} \right\rfloor = 336$ positive integers that are divisible by $6 = 2 \cdot 3$.

There are $\left\lfloor \frac{2020}{2\cdot 7} \right\rfloor = 144$ positive integers that are divisible by $14 = 2 \cdot 7$.

There are $\left\lfloor \frac{2020}{3\cdot 7} \right\rfloor = 96$ positive integers that are divisible by $21 = 3 \cdot 7$.

There are $\left\lfloor \frac{2020}{2\cdot 3\cdot 7} \right\rfloor = 48$ positive integers that are divisible by $42 = 2 \cdot 3 \cdot 7$.

By the Inclusion-Exclusion principle, there are $(1010 + 673 + 288) - (336 + 144 + 96) + 48 = 1443$ positive integers less than or equal to 2020 that are divisible by 2, 3, or 7. Hence, there are $2020 - 1443 = \boxed{577}$ positive integers less than or equal to 2020 that are relatively prime to 588. ☐

Problem 1.1.5 (Problem 5)**.** (*5 points*) Given that a, b, and c are distinct positive integers such that $a \cdot b \cdot c = 2020$, the minimum possible positive value of $\frac{1}{a} - \frac{1}{b} - \frac{1}{c}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Note that $2020 = 2^2 \cdot 5 \cdot 101$, thus we can assume that c is a multiple of 101, otherwise the expression $\frac{1}{a} - \frac{1}{b} - \frac{1}{c}$ cannot be positive. Furthermore,

$$\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$$

In order for $\frac{1}{a} - \frac{1}{b}$ be as small as possible we make $b - a$ as small as possible and ab as large as possible.

Note that c is a multiple of 101 so $ab \leq 20$ thus its largest value is 20. In addition $1 \leq b - a$, so the smallest possible value for $b - a$ is 1. Both cases happen when $b = 5, a = 4, c = 101$.

Hence

$$\frac{m}{n} = \frac{1}{4} - \frac{1}{5} - \frac{1}{101} = \frac{81}{2020} \Rightarrow m + n = \boxed{2101.}$$

☐

Problem 1.1.6 (Problem 6)**.** (*5 points*) Mary mixes 2 gallons of a solution that is 40 percent alcohol with 3 gallons of a solution that is 60 percent alcohol. Sandra mixes 4 gallons of a solution that is 30 percent alcohol with $\frac{m}{n}$ gallons of a solution that is 80 percent alcohol, where m and n are relatively prime positive integers. Mary and Sandra end up with solutions that are the same percent alcohol. Find $m + n$.

Solution. Let $x = \frac{m}{n}$, then the fraction of each mixture that is alcohol is

$$\frac{0.4(2) + 0.6(3)}{5} = \frac{0.3(4) + 0.8x}{4 + x} \Rightarrow 0.52 = \frac{0.8x + 1.2}{x + 4} \Rightarrow x = \frac{22}{7} \Rightarrow m + n = \boxed{29.}$$

☐

Problem 1.1.7 (Problem 7)**.** (*5 points*) Let a and b be positive integers such that $(a^3 - a^2 + 1)(b^3 - b^2 + 2) = 2020$. Find $10a + b$.

Solution. *Case 1:* $b = 1$, then $a^3 - a^2 + 1 = 1010 \Rightarrow a^2(a - 1) = 1009$, impossible since 1009 is prime.

Case 2: $b = 2$, then $b^3 - b^2 + 2 = 6 \not| 2020$.

Case 3: $b = 3$, then $b^3 - b^2 + 2 = 20 \Rightarrow a^2(a - 1) = 100 \Rightarrow a = 5$.

Case 4: $b = 4$, then $b^3 - b^2 + 2 = 50 \not| 2020$.

Case 5: $b \geq 5$, then $b^3 - b^2 = b^2(b - 1) + 2 \geq 25 \cdot 4 + 2 = 102$.

If $a \geq 4$ then $a^3 - a^2 + 1 = a^2(a - 1) + 1 \geq 16 \cdot 3 + 1 = 49$, thus $1 \leq a \leq 3$. It is easy to test that there is no such positive integer value for a.

Thus $a = 5, b = 3 \Rightarrow 10a + b = \boxed{53.}$ ☐

Problem 1.1.8 (Problem 8)**.** (*5 points*) Find the number of three-digit palindromes that are divisible by 3. Recall that a palindrome is a number that reads the same forward and backward like 727 or 905509.

Solution. Let aba be the three-digit palindromes that are divisible by 3. Since $3 \mid 2a+b$, thus we can choose b based on the remainder of a when divided by 3.

Case 1: $a \in \{3,6,9\}$, then $b = 0,3,6,9$.

Case 2: $a \in \{1,4,7\}$, then $b = 1,4,7$.

Case 3: $a \in \{2,5,8\}$, then $b = 2,5,8$.

Thus the number of three-digit palindromes that are divisible by 3 is $3 \cdot 4 + 3 \cdot 3 + 3 \cdot 3 = \boxed{30.}$ □

Problem 1.1.9 (Problem 9)**.** (*5 points*) Six different small books and three different large books sit on a shelf. Three children may each take either two small books or one large book. Find the number of ways the three children can select their books.

Solution. Consider the number of children who take large books.

Case 1: No child take a large book. Each child can select two small books in $\binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2} = 90$ ways.

Case 2: One child takes one large book. There are 3 ways to select a child who takes a large book, and there are 3 ways for that child to select which large book. Two child can select two small books in $\binom{6}{2} \cdot \binom{4}{2} = 90$ ways. In total there are $3 \cdot 3 \cdot 90 = 810$ ways.

Case 3: Two children takes two large books. There are 3 ways to select a child who takes two small books, and there are $\binom{6}{2} = 15$ ways for that child to select which two small books. There are $\binom{3}{1} \cdot \binom{2}{1} = 6$ ways for the other two children to select two large books. In total there are $3 \cdot 15 \cdot 6 = 270$ ways.

Case 4: There are $\binom{3}{1} \cdot \binom{2}{1} \cdot \binom{1}{1} = 6$ ways for each of the children to select one large book.

Thus the number ways is $90 + 810 + 270 + 6 = \boxed{1176.}$ □

Problem 1.1.10 (Problem 10)**.** (*5 points*) Daniel had a string that formed the perimeter of a square with area 98. Daniel cut the string into two pieces. With one piece he formed the perimeter of a rectangle whose width and length are in the ratio $2:3$. With the other piece he formed the perimeter of a rectangle whose width and length are in the ratio $3:8$. The two rectangles that Daniel formed have the same area, and each of those areas is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let s be the side length of the square then $s^2 = 98$, so $s = 7\sqrt{2}$. So the length of the string is $28\sqrt{2}$. Let the two rectangles made by Daniel be $2x \times 3x$ and $3y \times 8y$, then

$$2x \cdot 3x = 3y \cdot 8y \Rightarrow x^2 = 4y^2 \Rightarrow x = 2y.$$

The total perimeter of the two rectangles is $10x + 22y = 21x = 28\sqrt{2}$, so

$$x = \frac{4\sqrt{2}}{3}, \text{ and the area is } 6\left(\frac{4\sqrt{2}}{3}\right)^2 = \frac{64}{3}.$$

Hence, $m + n = 64 + 3 = \boxed{67.}$ □

Problem 1.1.11 (Problem 11)**.** (*5 points*) Find the number of permutations of the letters ABCDE where the letters A and B are not adjacent and the letters C and D are not adjacent. For example, count the permutations ACBDE and DEBCA but not ABCED or EDCBA.

Solution. There are $5! = 120$ permutations of the letters ABCDE.

The number of permutations where A is adjacent to B can be counted considering the permutations of XCDE, where X represents AB or BA. There are $4! = 24$ permutations of XCDE and 2 possibilities for X. Thus, there are $24 \cdot 2 = 48$ permutations where A is adjacent to B.

Similarly there are 48 permutations where C is adjacent to D.

The number of permutations where A is adjacent to B and C is adjacent to D can be counted considering the permutations of XYE, where X represents AB or BA, Y represents CD or DC. There are $3! = 6$ permutations of XYE, 2 possibilities for X, 2 possibilities for Y. Thus, there are $6 \cdot 2 \cdot 2 = 24$ permutations where A is adjacent to B and C is adjacent to D.

Therefore there are $48 + 48 - 24 = 72$ permutations where A is adjacent to B or C is adjacent to D.

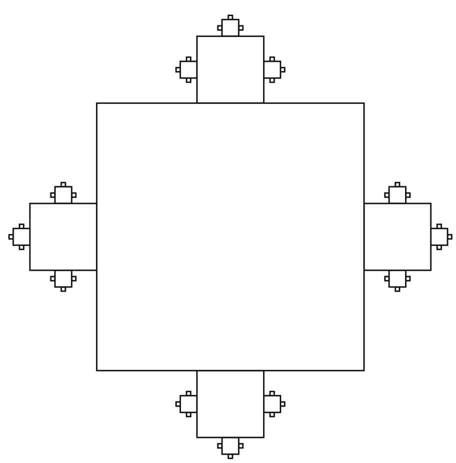
Hence, the number of permutations of the letters ABCDE where the letters A and B are not adjacent and the letters C and D are not adjacent is $120 - 72 = \boxed{48.}$ □

Problem 1.1.12 (Problem 12)**.** (*5 points*) Construct a geometric figure in a sequence of steps. In step 1, begin with a 4×4 square.

In step 2, attach a 1×1 square onto the each side of the original square so that the new squares are on the outside of the original square, have a side along the side of the original square, and the midpoints of the sides of the original square and the attached square coincide.

In step 3, attach a $\frac{1}{4} \times \frac{1}{4}$ square onto the centers of each of the 3 exposed sides of each of the 4 squares attached in step 2. For each positive integer n, in step $n + 1$, attach squares whose sides are $\frac{1}{4}$ as long as the sides of the squares attached in step n placing them at the centers of the 3 exposed sides of the squares attached in step n. The diagram shows the figure after step 4.

If this is continued for all positive integers n, the area covered by all the squares attached in all the steps is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. The square in the first step has an area of $4 \cdot 4 = 16$. There are 4 squares added in the second step, each with area $1 \cdot 1 = 1$ for a total of 4. After this, there are 3 times as many squares added in each step, and each square has area $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$ as large as the areas of the squares added in the previous step.

Thus, each step adds $\dfrac{3}{16}$ the area that was added in the previous step. The total area of all the squares added in every step is:

$$16+4+4\cdot\frac{3}{16}+4\cdot\left(\frac{3}{16}\right)^2+\cdots=16+4\left(1+\left(\frac{3}{16}\right)+\left(\frac{3}{16}\right)^2+\cdots\right)=16+4\frac{1}{1-\left(\frac{3}{16}\right)}=16+\frac{64}{13}=\frac{272}{13}.$$

Hence, $m+n=272+13=\boxed{285.}$ □

Problem 1.1.13 (Problem 13)**.** (*5 points*) Wendy randomly chooses a positive integer less than or equal to 2020. The probability that the digits in Wendy's number add up to 10 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. There are 2020 equally likely choices.

Case 1: For the number of choices less than 1000. Selecting a number less than 1000 whose digits add up to 10 is the same as to place 10 stones into three piles. We use the sticks-and-stones method with 10 stones and 2 sticks. The number of ways to place 10 stones into three order piles is given by $\binom{10+2}{2}=66$. This number count the 3 cases when all ten stones end up in the same pile, which means some digit can be equal 10. Therefore the number of possible cases is $66-3=63$.

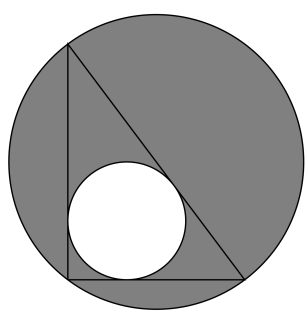
Case 2: For the number of choices at least 1000 and less than 2000. Similarly there are $\binom{9+2}{2}=55$.

Case 3: For the number of choices at least 2000. There are two such numbers: 2008 and 2017.

Therefore there are $63+55+2=120$ such numbers. The required probability is $\frac{120}{2020}=\frac{6}{101}$.

Hence, $m+n=6+101=\boxed{107.}$ □

Problem 1.1.14 (Problem 14)**.** (*5 points*) Right $\triangle ABC$ has side lengths 6, 8, and 10. Find the positive integer n such that the area of the region inside the circumcircle but outside the incircle of $\triangle ABC$ is $n\pi$.



Solution. The hypotenuse of the right $\triangle ABC$ is a diameter of the circumcircle, so the area of the circle is $\pi\left(\frac{10}{2}\right)^2=25\pi$. The area of the right $\triangle ABC$ is $\frac{1}{2}(6\cdot 8)=24$, its semi-perimeter is $\frac{1}{2}(6+8+10)=12$, its inradius is $\dfrac{24}{12}=2$. Thus, the area of the incircle is 4π. The area of the region inside the circumcircle but outside the incircle of $\triangle ABC$ is $25\pi-4\pi=21\pi$. Hence $n=\boxed{21.}$ □

1.2 Middle School - Test

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 5-14**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 1.2.1 (Problem 1)**.** (*5 points*) Alex launches his boat into a river and heads upstream at a constant speed. At the same time at a point 10 miles upstream from Alex, Alice launches her boat and heads downstream at a constant speed. Both boats move at 6 miles per hour in still water, but the river is flowing downstream at 3 miles per hour. Alex and Alice will meet at a point that is $\frac{m}{n}$ miles from Alex's starting point, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Relative to the moving water, Alex and Alice are moving at 6 miles per hour, so they are approaching each other at 12 miles per hour. Thus, they meet after $\dfrac{10}{12} = \dfrac{5}{6}$ hours has travel $\dfrac{10}{2} = 5$ relative to the moving water. But the water is moving Alex downstream at r miles per hour, so Alex and Alice meet after Alex has traveled:

$$\frac{m}{n} = 5 - \frac{5}{6}(3) = \frac{5}{2}.$$

Hence, $m+n = 5+2 = \boxed{7}$. □

Problem 1.2.2 (Problem 2)**.** (*5 points*) Find a positive integer n such that there is a polygon with n sides where each of its interior angles measures $165°$.

Solution. The sum of the measures of the interior angles in a polygon with n sides is $(n-2)180$ in degree.

$$165n = (n-2)180 \Rightarrow n(180-165) = 360 \Rightarrow n = \frac{360}{15} = \boxed{24.}$$

□

Problem 1.2.3 (Problem 3)**.** (*5 points*) Patrick started walking at a constant rate along a straight road from school to the park. Two hour after Patrick left, Tanya started running along the same road from school to the park. One hour after Tanya left, Jose started bicycling along the same road from school to the park. Tanya ran at a constant rate of 1 miles per hour faster than Patrick walked, Jose bicycled at a constant rate of 6 miles per hour faster than Tanya ran, and all three arrived at the park at the same time. The distance from the school to the park is $\frac{m}{n}$ miles, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let t be the time it took Patrick to walk from school to the park in hours and let s be the speed that Patrick walked in miles per hour. Then the distance from the school to the park can be measured in different ways:

$$st = (s+1)(t-2) = (s+7)(t-3) \Rightarrow st = st - 2s + t - 2 = st - 3s + 7t - 21$$
$$-2s + t - 2 = -3s + 7t - 21 = 0 \Rightarrow \begin{cases} t = 2s+2 \\ 7t = 3s + 21 \end{cases} \Rightarrow s = \frac{7}{11}, t = \frac{36}{11} \Rightarrow st = \frac{252}{121}$$

Therefore $\frac{m}{n} = \frac{252}{121}$, so $m+n = 252+121 = \boxed{373.}$ □

Problem 1.2.4 (Problem 4)**.** (*5 points*) Find the number of positive **even** integers less than or equal to 2024 that are relatively prime to 15.

Solution. A number is relatively prime to $15 = 3 \cdot 5$ if and only if it is not divisible by 3 or 5.

There are $\frac{2024}{2} = 1011$ positive integers that are divisible by 2.

Now we find the number of positive **even** integers that are divisible by 3 or 5

There are $\left\lfloor \frac{2024}{2\cdot 3} \right\rfloor = 337$ positive integers that are divisible by $6 = 2 \cdot 3$.

There are $\left\lfloor \frac{2024}{2\cdot 5} \right\rfloor = 202$ positive integers that are divisible by $10 = 2 \cdot 5$.

There are $\left\lfloor \frac{2024}{2\cdot 3\cdot 5} \right\rfloor = 67$ positive integers that are divisible by $30 = 2 \cdot 3 \cdot 5$.

By the Inclusion-Exclusion principle, there are $1012 - (337+202) + 67 = \boxed{540}$ positive even integers less than or equal to 2024 that are relatively prime to 15. □

Problem 1.2.5 (Problem 5)**.** (*5 points*) Given that a, b, and c are distinct positive integers such that $a \cdot b \cdot c = 1236$, the minimum possible positive value of $\frac{1}{a} - \frac{1}{b} - \frac{1}{c}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Note that $1236 = 2^2 \cdot 3 \cdot 103$, thus we can assume that c is a multiple of 103, otherwise the expression $\frac{1}{a} - \frac{1}{b} - \frac{1}{c}$ cannot be positive. Furthermore,

$$\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$$

In order for $\frac{1}{a} - \frac{1}{b}$ be as small as possible we make $b - a$ as small as possible and ab as large as possible.

Note that c is a multiple of 103 so $ab \leq 12$ thus its largest value is 12. In addition $1 \leq b - a$, so the smallest possible value for $b - a$ is 1. Both cases happen when $b = 4, a = 3, c = 101$.

Hence

$$\frac{m}{n} = \frac{1}{3} - \frac{1}{4} - \frac{1}{103} = \frac{91}{1236} \Rightarrow m + n = \boxed{1327.}$$

□

Problem 1.2.6 (Problem 6)**.** (*5 points*) Mary mixes 2 gallons of a solution that is 30 percent alcohol with 3 gallons of a solution that is 70 percent alcohol. Sandra mixes 4 gallons of a solution that is 20 percent alcohol with $\frac{m}{n}$ gallons of a solution that is 90 percent alcohol, where m and n are relatively prime positive integers. Mary and Sandra end up with solutions that are the same percent alcohol. Find $m + n$.

Solution. Let $x = \frac{m}{n}$, then the fraction of each mixture that is alcohol is

$$\frac{0.3(2) + 0.7(3)}{5} = \frac{0.2(4) + 0.9x}{4 + x} \Rightarrow 0.54 = \frac{0.9x + 0.8}{x + 4} \Rightarrow x = \frac{34}{9} \Rightarrow m + n = \boxed{43.}$$

□

Problem 1.2.7 (Problem 7)**.** (*5 points*) Let a and b be positive integers such that $(a^3 - a^2 + 1)(b^3 - b^2 + 2) = 1938$. Find $10a + b$.

Solution. Case 1: $b = 1$, then $a^3 - a^2 + 1 = 969 \Rightarrow a^2(a - 1) = 968 = 2^3 \cdot 11^2$. No solution.

Case 2: $b = 2$, then $b^3 - b^2 + 2 = 6 \Rightarrow a^2(a - 1) = 322 = 2 \cdot 6 \cdot 23$. No solution.

Case 3: $b = 3$, then $b^3 - b^2 + 2 = 20 \not| 1938$.

Case 4: $b = 4$, then $b^3 - b^2 + 2 = 50 \not| 1938$.

Case 5: $b = 5$, then $b^3 - b^2 + 2 = 102 \Rightarrow a^2(a - 1) = 18 \Rightarrow a = 3$.

Case 6: $b \geq 6$, then $b^3 - b^2 + 2 = b^2(b - 1) + 3 \geq 36 \cdot 5 + 2 = 182$. Thus $a^3 - a^2 + 1 = \leq \frac{1938}{182} < 11$, thus $a \leq 2$. It is easy to test that there is no such positive integer value for a.

Thus $a = 3, b = 5 \Rightarrow 10a + b = \boxed{35.}$ □

Problem 1.2.8 (Problem 8)**.** (*5 points*) Find the number of *four-digit* palindromes that are divisible by 3. Recall that a palindrome is a number that reads the same forward and backward like 727 or 905509.

Solution. Let $abba$ be the four-digit palindromes that are divisible by 3. Since $3 \mid 2a + 2b$, so $3 \mid a + b$ thus we can choose b based on the remainder of a when divided by 3.

Case 1: $a \in \{3, 6, 9\}$, then $b = 0, 3, 6, 9$.

Case 2: $a \in \{1, 4, 7\}$, then $b = 2, 5, 8$.

Case 3: $a \in \{2, 5, 8\}$, then $b = 1, 4, 7$.

Thus the number of four-digit palindromes that are divisible by 3 is $3 \cdot 4 + 3 \cdot 3 + 3 \cdot 3 = \boxed{30.}$ □

Problem 1.2.9 (Problem 9)**.** (*5 points*) Four different small books and four different large books sit on a shelf. Three children may each take either two small books or one large book. Find the number of ways the three children can select their books.

Solution. Consider the number of children who take large books. Since there are only four different small books, there should be at least one child who takes a large book.

Case 1: One child takes one large book. There are 3 ways to select a child who takes a large book, and there are 4 ways for that child to select which large book. Two child can select two small books in $\binom{4}{2} \cdot \binom{2}{2} = 6$ ways. In total there are $3 \cdot 4 \cdot 6 = 72$ ways.

Case 3: Two children takes two large books. There are 3 ways to select a child who takes two small books, and there are $\binom{4}{2} = 6$ ways for that child to select which two small books. There are $\binom{4}{1} \cdot \binom{3}{1} = 12$ ways for the other two children to select two large books. In total there are $3 \cdot 6 \cdot 12 = 216$ ways.

Case 3: There are $\binom{4}{1} \cdot \binom{3}{1} \cdot \binom{2}{1} = 24$ ways for each of the children to select one large book.

Thus the number ways is $72 + 216 + 24 = \boxed{312.}$ □

Problem 1.2.10 (Problem 10)**.** (*5 points*) Daniel had a string that formed the perimeter of a square. Daniel cut the string into two pieces. With one piece he formed the perimeter of a rectangle whose width and length are in the ratio $2 : 3$. With the other piece he formed the perimeter of a rectangle whose width and length are in the ratio $3 : 8$. The two rectangles that Daniel formed have the same area, and the measure of each of those areas is 2.

The length of the string is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by any perfect square larger than 1. Find $m + n$.

Solution. Let s be the side length of the original square, the length of the string is $4s$.

Let the two rectangles made by Daniel be $2x \times 3x$ and $3y \times 8y$, then

$$2x \cdot 3x = 3y \cdot 8y \Rightarrow x^2 = 4y^2 \Rightarrow x = 2y.$$

The total perimeter of the two rectangles is $10x + 22y = 21x = 4s$. Thus , and the area is $6 \cdot \left(\frac{4s}{21}\right)^2$. Since

$$x = \frac{4s}{21}, \text{ and the area is } 6\left(\frac{4s}{21}\right)^2 = \frac{32s^2}{147} = 2 \Rightarrow s = \frac{7\sqrt{3}}{4} \Rightarrow 4s = 7\sqrt{3}.$$

Hence, $m + n = 7 + 3 = \boxed{10.}$ □

Problem 1.2.11 (Problem 11)**.** (*5 points*) Find the number of permutations of the letters ABCDEF where the letters A and B are not adjacent, the letters C and D are not adjacent, and the letters E and F are not adjacent. For example, count the permutations FACBDE and DEBCAF but not ABCDEF or EDCBAF.

Solution. There are $6! = 720$ permutations of the letters ABCDEF.

The number of permutations where A is adjacent to B can be counted considering the permutations of XCDEF, where X represents AB or BA. There are $5! = 120$ permutations of XCDEF and 2 possibilities for X. Thus, there are $120 \cdot 2 = 240$ permutations where A is adjacent to B.

Similarly there are 240 permutations where C is adjacent to D, 240 permutations where E is adjacent to F.

The number of permutations where A is adjacent to B and C is adjacent to D can be counted considering the permutations of XYEF, where X represents AB or BA, Y represents CD or DC. There are $4! = 24$ permutations of XYEF, 2 possibilities for X, 2 possibilities for Y. Thus, there are $24 \cdot 2 \cdot 2 = 96$ permutations where A is adjacent to B and C is adjacent to D.

Similarly there are 96 permutations where A is adjacent to B and E is adjacent to F. and 240 permutations where C is adjacent to D and E is adjacent to F.

Finally, the number of permutations where A is adjacent to B, C is adjacent to D, and E is adjacent to F can be counted considering the permutations of XYZ, where X represents AB or BA, Y represents CD or DC, Z represents EF. There are $3! = 6$ permutations of XYZ, 2 possibilities for X, 2 possibilities for Y, and 2 possibilities for Z. Thus, there are $6 \cdot 2 \cdot 2 \cdot 2 = 48$ permutations where A is adjacent to B, C is adjacent to D, and E is adjacent to F

Therefore there are $(240 + 240 + 240) - (96 + 96 + 96) + 48 = 480$ permutations where A is adjacent to B, C is adjacent to D, or E is adjacent to F.

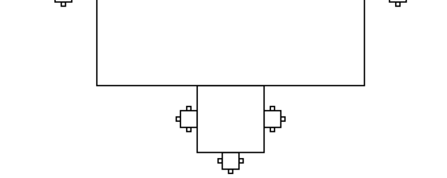
Hence, the number of permutations of the letters ABCDE where the letters A and B are not adjacent, the letters C and D are not adjacent, and the letters E and F are not adjacent is $720 - 480 = \boxed{240.}$ $\square$

Problem 1.2.12 (Problem 12)**.** (*5 points*) Construct a geometric figure in a sequence of steps. In step 1, begin with a 4×4 square.

In step 2, attach a 1×1 square onto the each side of the original square so that the new squares are on the outside of the original square, have a side along the side of the original square, and the midpoints of the sides of the original square and the attached square coincide.

In step 3, attach a $\frac{1}{4} \times \frac{1}{4}$ square onto the centers of each of the 3 exposed sides of each of the 4 squares attached in step 2. For each positive integer n, in step $n + 1$, attach squares whose sides are $\frac{1}{4}$ as long as the sides of the squares attached in step n placing them at the centers of the 3 exposed sides of the squares attached in step n. The diagram shows the figure after step 4.

If this is continued for all positive integers n, let $\frac{m}{n}$, where m and n are relatively prime positive integers, be the area covered by all the squares attached in all the **odd** steps: step 1, step 3, step 5, etc. Find $m + n$.



Solution. The square in the first step has an area of $4 \cdot 4 = 16$. There are 4 squares added in the second step, each with area $1 \cdot 1 = 1$ for a total of 4. After this, there are 3 times as many squares added in each step, and each square has area $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$ as large as the areas of the squares added in the previous step.

Thus, each step adds $\frac{3}{16}$ the area that was added in the previous step. The areas of all the squares added are:

$$16,\ 4,\ 4 \cdot \frac{3}{16},\ 4 \cdot \left(\frac{3}{16}\right)^2,\ \ldots$$

Now, the total area of all the squares added in every **odd** step is:

$$16+4\cdot\frac{3}{16}+4\cdot\left(\frac{3}{16}\right)^3+4\cdot\left(\frac{3}{16}\right)^5+\cdots=16+4\cdot\frac{3}{16}\left(1+\left(\frac{3}{16}\right)^2+\left(\frac{3}{16}\right)^3+\cdots\right)$$
$$=16+4\frac{1}{1-\left(\frac{3}{16}\right)^2}=\frac{4976}{247}$$

Hence, $m+n=4976+247=\boxed{5223.}$ □

Problem 1.2.13 (Problem 13)**.** (*5 points*) Wendy randomly chooses a positive integer less than or equal to 2024. The probability that the digits in Wendy's number add up to 8 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. There are 2024 equally likely choices.

Case 1: For the number of choices less than 1000. Selecting a number less than 1000 whose digits add up to 8 is the same as to place 8 stones into three piles. We use the sticks-and-stones method with 8 stones and 2 sticks. The number of ways to place 8 stones into three order piles is given by $\binom{8+2}{2}=45$.

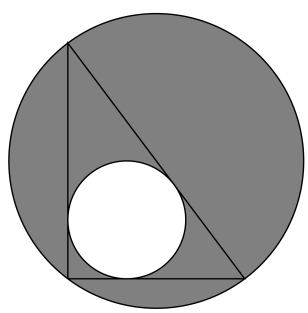
Case 2: For the number of choices at least 1000 and less than 2000. Similarly there are $\binom{7+2}{2}=36$.

Case 3: For the number of choices at least 2000. There are three such numbers: 2006, 2015, 2024.

Therefore there are $45+36+3=84$ such numbers. The required probability is $\frac{84}{2024}=\frac{21}{506}$.

Hence, $m+n=21+506=\boxed{527.}$ □

Problem 1.2.14 (Problem 14)**.** (*5 points*) Right $\triangle ABC$ has integer side lengths a, b, and c, where $\angle BAC=90°$. Let r be the radius of the incircle, R be the radius of the circumcircle, then the ratio $\frac{r+R}{a+b}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution. The hypotenuse of the right $\triangle ABC$ is the diameter of the circle, thus $R=\frac{1}{2}c=\frac{1}{2}\sqrt{a^2+b^2}$.

The area of the right $\triangle ABC$ is $\frac{1}{2}(a\cdot b)$, its semi-perimeter is $\frac{1}{2}(a+b+c)$, therefore its inradius is

$$r=\frac{ab}{a+b+c}=\frac{ab}{a+b+\sqrt{a^2+b^2}}$$
$$r+R=\frac{ab}{a+b+\sqrt{a^2+b^2}}+\frac{1}{2}\sqrt{a^2+b^2}=\frac{2ab+(a+b)\sqrt{a^2+b^2}+a^2+b^2}{2(a+b+\sqrt{a^2+b^2})}$$
$$=\frac{(a+b)^2+(a+b)\sqrt{a^2+b^2}}{2(a+b+\sqrt{a^2+b^2})}=\frac{(a+b)(a+b+\sqrt{a^2+b^2})}{2(a+b+\sqrt{a^2+b^2})}=\frac{a+b}{2}$$
$$\Rightarrow\frac{r+R}{a+b}=\frac{1}{2}\Rightarrow m+n=1+2=\boxed{3.}$$

□

1.3 High School - Assignment

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

- **Submission deadline: Friday, January 26**
- **Test: Saturday, January 27**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 4-13**
- **Olympiad (O) level: Problems 10-19**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 1.3.1 (Problem 1)**.** (*5 points*) There is a complex number K such that the quadratic polynomial $7x^2 + Kx + 12 - 5i$ has exactly one root, where $i = \sqrt{-1}$. Find $|K|^2$.

Note that for a complex number $x = a + bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2 + b^2}$.

Solution. A quadratic polynomial has exactly one root if and only if its determinant is zero, therefore for $7x^2 + Kx + 12 - 5i$, we have:

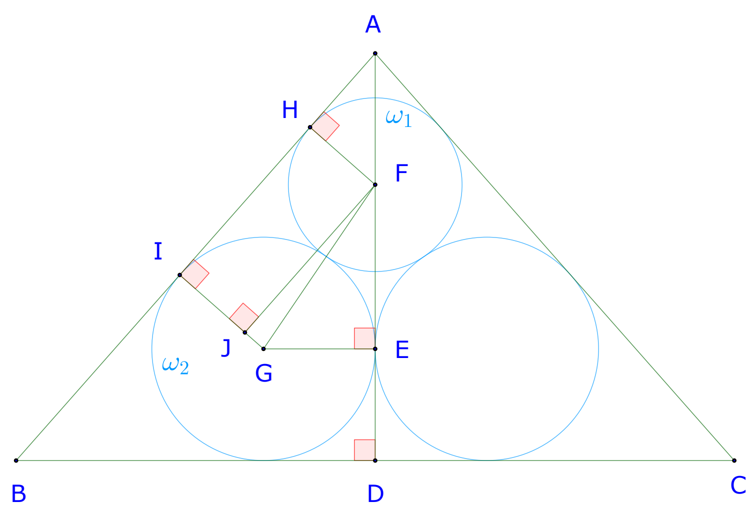
$$K^2 - 4(7)(12 - 5i) = 0 \Rightarrow K^2 = (4 \cdot 7 \cdot 12) - (4 \cdot 7 \cdot 5)i.$$

Note that for a complex number $K = a + bi$, then

$$|K| = \sqrt{a^2 + b^2} \text{ and } K^2 = (a + bi)(a + bi) = a^2 - b^2 + 2abi, \text{ note that}$$
$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2 \Rightarrow |K|^2 = a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + (2ab)^2} = |K^2|$$

Hence, $|K|^2 = |K^2| = \sqrt{(4 \cdot 7 \cdot 12)^2 + (4 \cdot 7 \cdot 5)^2} = 28\sqrt{12^2 + 5^2} = 28 \cdot 13 = \boxed{364.}$. □

Problem 1.3.2 (Problem 2)**.** (*5 points*) Two circles have radius 9, and one circle has radius 7. Each circle is externally tangent to the other two circles, and each circle is internally tangent to two sides of an isosceles triangle, as shown. The sine of the base angle of the triangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. Let ω_1 be the circle with radius 7 with center F and ω_2 be one of the circles with radius 9 with center G. Let label the vertices of the triangles to be A, B, and C as shown in the diagram above. Let AD be the altitude from A to BC, E be the tangent point of the circles with radius 9, and H, I be the feet of the altitudes from F, G to AB, respectively. Let J be the foot of the altitudes from F to GJ.

Then

$$EG = 9, FG = 7 + 9 = 16 \Rightarrow EF = \sqrt{16^2 - 9^2} = \sqrt{175}$$
$$GJ = GI - IJ = GI - FH = 9 - 7 = 2 \Rightarrow FJ = \sqrt{16^2 - 2^2} = \sqrt{252}$$
$$\sin(\angle ABC) = \cos(\angle BAD) = \cos(\angle JFE) = \cos(\angle JFG + \angle GFE)$$
$$= \cos(\angle JFG)\cos(\angle GFE) - \sin(\angle JFG)\sin(\angle GFE)$$
$$= \frac{FJ}{FG}\frac{FE}{FG} - \frac{GJ}{FG}\frac{EG}{FG} = \frac{\sqrt{175} \cdot \sqrt{252} - 2 \cdot 9}{16^2} = \frac{210 - 18}{256} = \frac{3}{4}.$$

Hence, $m + n = \boxed{7.}$ □

Problem 1.3.3 (Problem 3)**.** (*5 points*) There are two distinct pairs of positive integers $a_1 < b_1$ and $a_2 < b_2$ such that both $|(a_1+ib_1)(b_1-ia_1)|$ and $|(a_2+ib_2)(b_2-ia_2)|$ equal 2020, where $i=\sqrt{-1}$. Find $a_1+b_1+a_2+b_2$.

Note that for a complex number $x = a+bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2+b^2}$.

Solution. Note that

$$(a_1+ib_1)(b_1-ia_1) = 2a_1b_1 + i(b_1^2-a_1^2) \Rightarrow |(a_1+ib_1)(b_1-ia_1)| = \sqrt{(2a_1b_1)^2+(b_1^2-a_1^2)^2} = a_1^2+b_1^2$$

Thus $a_1^2+b_1^2 = 2020$. Since a perfect square has a remainder of 0 or 1 when divided by 4, thus both a_1 and b_1 have to be even since $4 \mid 2020$. Therefore $\frac{a_1}{2}, \frac{b_1}{2}$ are positive integers and

$$\left(\frac{a_1}{2}\right)^2 + \left(\frac{b_1}{2}\right)^2 = 505.$$

Since $505 < 23^2 = 529$, by testing we can see that $\left(\frac{a_1}{2}, \frac{b_1}{2}\right) \in \{(12,19),(8,21)\}$.

Similarly for (a_2,b_2). By the given conditions (a_1,b_1) and (a_2,b_2) are distinct pairs, so they must be $(24,38)$ and $(16,42)$.

Hence, $a_1+b_1+a_2+b_2 = 24+38+16+42 = \boxed{120.}$ □

Problem 1.3.4 (Problem 4)**.** (*5 points*) There are relatively prime positive integers s and t such that

$$\sum_{n=2}^{100}\left(\frac{n}{n^2-1}-\frac{1}{n}\right) = \frac{s}{t}.$$

Find $s+t$.

Solution. Note that

$$\frac{n}{n^2-1}-\frac{1}{n} = \frac{n}{(n-1)(n+1)}-\frac{1}{n} = \frac{1}{2}\left(\frac{1}{n-1}+\frac{1}{n+1}\right)-\frac{1}{n} = \frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n}\right)-\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$\sum_{n=2}^{100}\left(\frac{n}{n^2-1}-\frac{1}{n}\right) = \frac{1}{2}\sum_{n=2}^{100}\left(\frac{1}{n-1}-\frac{1}{n}\right)-\frac{1}{2}\sum_{n=2}^{100}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \frac{1}{2}\left(\frac{1}{1}-\frac{1}{100}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{101}\right) = \frac{5049}{20200}$$

Hence, $s+t = \boxed{25249.}$ □

Problem 1.3.5 (Problem 5)**.** (*5 points*) Let x be a real number such that $3\sin^4 x - 2\cos^6 x = -\dfrac{17}{25}$. Then $3\cos^4 x - 2\sin^6 x = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $10m+n$.

Solution.

$$-\frac{17}{25} = 3\sin^4 x - 2\cos^6 x = 3\sin^4 x - 2(1-\sin^2 x)^3 = 2\sin^6 x - 3\sin^4 x + 6\sin^2 x - 2$$
$$3\cos^4 x - 2\sin^6 x = 3(1-\sin^2 x)^2 - 2\sin^6 x = 3 - 6\sin^2 x + 3\sin^4 x - 2\sin^6 x$$

Hence, $\dfrac{m}{n} = -\left(-\dfrac{17}{25}+2\right)+3 = \frac{42}{25}$, so $10m+n = 420+25 = \boxed{445.}$ □

Problem 1.3.6 (Problem 6)**.** (*5 points*) Find the sum of all values of x such that the set

$$\{107, 122, 127, 137, 152, x\} \text{ has a mean that is equal to its median.}$$

Solution. First, let's simplify the problem by subtracting 107 from each of the elements, then the statement still stands for the set $\{0, 15, 20, 30, 45, x - 107\}$. Then, by dividing each of the elements by 5, it is the same assumption for the set $\{, y = \frac{x-107}{5}\}$. The mean of the set now is $\dfrac{0+3+4+6+9+y}{6} = \dfrac{22+y}{6}$.

Case 1: $y < 3$, then the median of the set is $\dfrac{3+4}{2} = \dfrac{7}{2}$, so $y = -1$.

Case 2: $3 \le y \le 6$, then the median of the set is $\dfrac{4+y}{2}$, so $y = 5$.

Case 3: $y < 6$, then the median of the set is $\dfrac{4+6}{2}$, so $y = 8$.

Hence, the sum of all values of x is

$$(-1 \cdot 5 + 107) + (5 \cdot 5 + 107) + (8 \cdot 5 + 107) = 102 + 132 + 147 = \boxed{381.}$$

□

Problem 1.3.7 (Problem 7)**.** (*5 points*) Find the maximum possible value of

$$\left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right)^2$$

where a, b, and c are nonzero real numbers satisfying

$$a\sqrt[3]{\frac{a}{b}} + b\sqrt[3]{\frac{b}{c}} + c\sqrt[3]{\frac{c}{a}} = 0.$$

Solution.

$$\text{Let } x = a\sqrt[3]{\frac{a}{b}},\ y = b\sqrt[3]{\frac{b}{c}},\ z = c\sqrt[3]{\frac{c}{a}} \Rightarrow x + y + z = 0$$

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx) = 3xyz$$

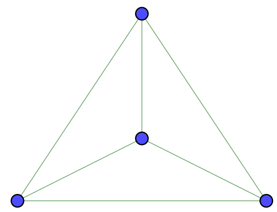
$$x^3 = \left(a\sqrt[3]{\frac{a}{b}}\right)^3 = (a^3)\left(\frac{a}{b}\right) = (abc)\left(\frac{a^3}{b^2c}\right), y^3 = (abc)\left(\frac{b^3}{c^2a}\right), z^3 = (abc)\left(\frac{c^3}{a^2b}\right)$$

$$\Rightarrow x^3 + y^3 + z^3 = (abc)\left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right),\ 3xyz = 3a\sqrt[3]{\frac{a}{b}} \cdot b\sqrt[3]{\frac{b}{c}} \cdot c\sqrt[3]{\frac{c}{a}} = 3abc$$

$$\Rightarrow \left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right)^2 = \left(\frac{x^3+y^3+z^3}{abc}\right)^2 = \left(\frac{3xyz}{abc}\right)^2 = \left(\frac{3abc}{abc}\right)^2 = \boxed{9.}$$

□

Problem 1.3.8 (Problem 8). (*5 points*) The following diagram shows four vertices connected by six edges. Suppose that each of the edges is randomly painted either red, white, or blue. The probability that the three edges adjacent to at least one vertex are colored with all three colors is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. There are six edges in the diagrams, and there are 3 equally likely ways to paint each edge. Thus, there are 3^6 equally likely ways to paint all the edges.

To count the number of ways to paint the edges so that at least one vertex is adjacent to one edge of each colour, number the vertices 1, 2, 3, and 4 and let A_1, A_2, A_3, and A_4 be the sets of colouring patterns, where each of the vertices 1, 2, 3, and 4, respectively, is adjacent to edges of all three colours. Then, the number of colouring patterns in $A_1 \cup A_2 \cup A_3 \cup A_4$ is the number of ways to colouring the edges such that at least one vertex are colored with all three colors.

There are $3! = 6$ ways to paint the 3 edges adjacent to a particular vertex so that there is one edge of each colour, and there are 3^3 ways to paint the other three edges showing that for each j, the cardinality of A_j is $6 \cdot 3^3$.

A pattern is in $A_j \cap A_k$, for $j \neq k$ if the edges adjacent to vertex j are painted in one of 6 ways, the two edges adjacent to vertex k not adjacent to vertex j are painted in one of 2 ways, and the one edge not adjacent to either vertex j or vertex k is painted in one of 3 ways. This shows that $A_j \cap A_k$ containing $6 \cdot 2 \cdot 3 = 36$ patterns. This is true for all $\binom{4}{2} = 6$ pairs of j and k.

To obtain a pattern in $A_i \cap A_j \cap A_k$ for distinct i, j, and k, the edges adjacent to vertex i can be painted in one of 6 ways, then there is only one way to paint the edge between vertex j and vertex k, and this fixes how all the other edges must be painted, so there are only 6 patterns in $A_i \cap A_j \cap A_k$ for each of the $\binom{4}{3} = 4$ choices of i, j, k.

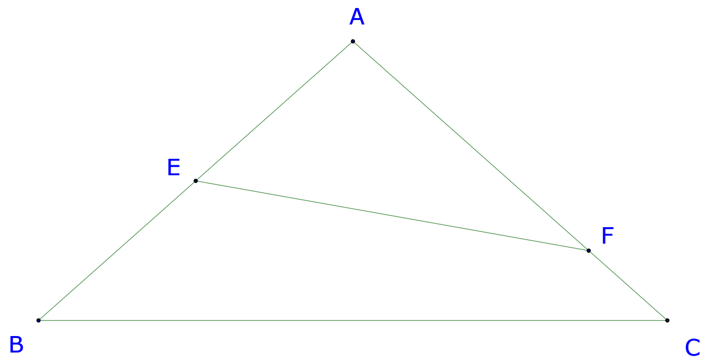
Similarly, there are 6 patterns in $A_1 \cap A_2 \cap A_3 \cap A_4$.

By the Inclusion - Exclusion Principle then the number of colouring patterns in $A_1 \cup A_2 \cup A_3 \cup A_4$ is:

$$\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| \\
&\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\
&\quad + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\
&\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= 4(6 \cdot 3^3) - 6(6 \cdot 2 \cdot 3) + 4 \cdot 6 - 6 = 6 \cdot 75
\end{aligned}$$

The required probability is $\dfrac{6 \cdot 75}{3^6} = \frac{50}{81}$. Hence, $m + n = 50 + 81 = \boxed{131.}$ □

Problem 1.3.9 (Problem 9)**.** (*5 points*) In isosceles $\triangle ABC$, $AB = AC$, $\angle BAC$ is obtuse, and points E and F lie on sides AB and AC, respectively, so that $AE = 10$, $AF = 15$. The area of $\triangle AEF$ is 60, and the area of quadrilateral $BEFC$ is 102. Find BC.



Solution. Keep in mind that $\angle BAC > 90°$, so $\cos(\angle BAC) < 0$,

$$60 = [AEF] = \frac{1}{2}(AE \cdot AF)\sin(\angle EAF) = \frac{1}{2}(10 \cdot 15)\sin(\angle EAF) \Rightarrow \sin(\angle EAF) = \frac{4}{5}$$

$$[AEF] + [BEFC] = [ABC] = \frac{1}{2}(AB \cdot AC)\sin(\angle BAC) = \frac{1}{2}(AB^2)\frac{4}{5} \Rightarrow AB^2 = \frac{162 \cdot 2 \cdot 5}{4} = 405$$

$$BC^2 = AB^2 + AC^2 - 2(AB \cdot AC)\cos(\angle BAC) = 405(2)(1 - \cos(\angle BAC)) = 405(2)\left(1 + \frac{3}{5}\right) = 1296 = 36^2.$$

Hence, $BC = \boxed{36.}$ □

Problem 1.3.10 (Problem 10)**.** (*5 points*) Find the least prime number greater than 1000 that divides $2^{1010} \cdot 23^{2020} + 1$.

Solution. First, let $n = 2^{1010} \cdot 23^{2020}$, then

$$n = 2^{1010} \cdot 23^{2020} + 1 = (4 \cdot 23^4)^{505} + 1 \Rightarrow 4 \cdot 23^4 + 1 \mid n.$$

By Sophie Germain identity $a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab)$ for $a = 1, b = 23$,

$$4 \cdot 23^4 + 1 = (1 + 2 \cdot 23^2 - 2 \cdot 23)(1 + 2 \cdot 23^2 + 2 \cdot 23) = 1013 \cdot 1105 = 5 \cdot 13 \cdot 17 \cdot 1013.$$

Now, since $7 \mid 1001, 17 \mid 1003, 19 \mid 1007, 3 \mid 1011$, thus the only prime number larger than 1000 and less than 1013 is 1009. We shall prove that it is not a factor of n,

By Fermat's Little Theorem:

$$2^{1010} \equiv 2^2 \pmod{1009},\ 23^{2020} \equiv 23^4 \pmod{1009} \Rightarrow n \equiv 4 \cdot 23^4 + 1 \pmod{1009}.$$

From the factorization of $4 \cdot 23^4 + 1$ shown above, $1009 \nmid n$, hence the required prime is $\boxed{1013.}$ □

Problem 1.3.11 (Problem 11)**.** (*5 points*) Find the maximum possible value of

$$9\sqrt{x} + 8\sqrt{y} + 5\sqrt{z},$$

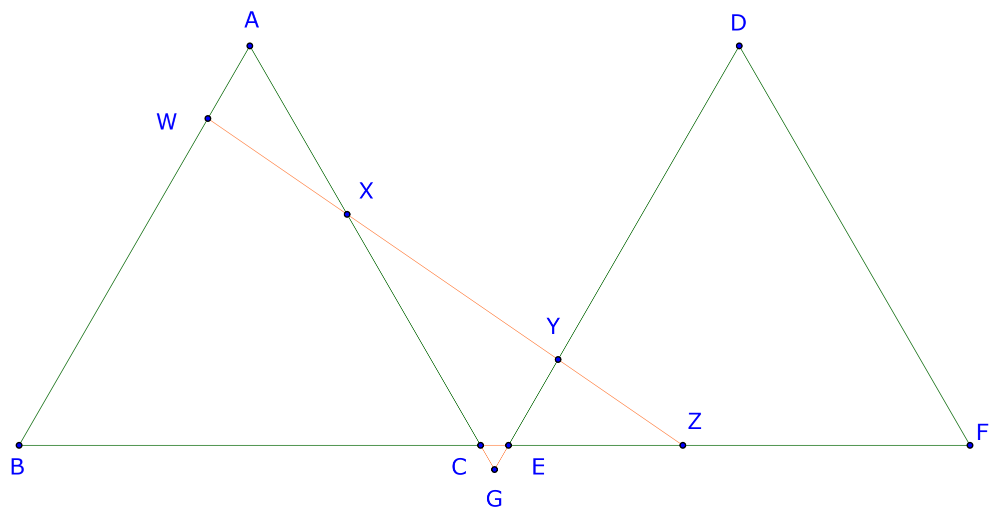
where x, y, and z are positive real numbers satisfying $9x + 4y + z = 128$.

Solution. By the Cauchy-Schwarz inequality,

$$9\sqrt{x} + 8\sqrt{y} + 5\sqrt{z} = (3)(\sqrt{9x}) + (4)(\sqrt{4y}) + (5)(\sqrt{z}) \le \sqrt{(3^2 + 4^2 + 5^2)(9x + 4y + z)} = \sqrt{50 \cdot 128} = \boxed{80.}$$

□

Problem 1.3.12 (Problem 12). (*5 points*) Two congruent equilateral triangles $\triangle ABC$ and $\triangle DEF$ lie on the same side of line BC so that B, C, E, and F are collinear as shown. A line intersects AB, AC, DE, and EF at W, X, Y, and Z, respectively, such that $\dfrac{AW}{BW} = \frac{2}{9}$, $\dfrac{AX}{CX} = \frac{5}{6}$, and $\dfrac{DY}{EY} = \frac{9}{2}$. The ratio $\dfrac{EZ}{FZ}$ can then be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. Without loss of generality, let the triangles have side length 11, so

$$AW = 2, BW = 9, AX = 5, CX = 6, DY = 9, EY = 2.$$

Let G be the intersection of lines AC and DE, then $\triangle AWX \sim GYX$, so

$$\frac{GE + EY}{GC + CX} = \frac{GY}{GX} = \frac{AW}{AX} = \frac{2}{5} \Rightarrow \frac{GE + 2}{GC + 6} = \frac{2}{5} \Rightarrow GC = GE = \frac{2}{3}.$$

By Menelaus' Theorem,

$$1 = \frac{AW}{BW} \cdot \frac{BZ}{CZ} \cdot \frac{CX}{AX} = \frac{2}{9} \cdot \frac{11 + CZ}{CZ} \cdot \frac{6}{5} \Rightarrow CZ = 4.$$

$$EZ = 4 - CE = 4 - \frac{2}{3} = \frac{10}{3} \Rightarrow \frac{EZ}{FZ} = \frac{\frac{10}{3}}{11 - \frac{10}{3}} = \frac{10}{23}$$

Thus the desired sum $m + n = 10 + 23 = \boxed{33.}$ □

Problem 1.3.13 (Problem 13)**.** (*5 points*) Find the number of permutations of the letters AAAABBBCC where no letter is next to another letter of the same type. For example, count ABCABCABA and ABABCABCA but not ABCCBABAA.

Solution. Let first consider the letters AAAABBB. There are three cases that could lead to permutations with no repeated letters.

Case 1: there is 1 permutation of As and Bs such that no two like letters are next to each other: $ABABABA$. For these, there are 8 positions around the 7 letters to place the two letters C, thus there are $\binom{8}{2} = 28$ ways to choose two distinct positions for Cs. Thus, in total, there are $1 \cdot 28 = 28$ ways for this case.

Case 2: there is 6 permutations of As and Bs that have one pair of like letters are adjacent. These are obtained by starting with $ABABABA$, removing the beginning or ending A, then placing that A next to one of another A (there are two ways to *insert* right before of or right after it.) Into this permutation, a letter C must be used to insert in between the two neighbouring letters A. For the second letter C there are $9 - 2 = 7$ positions not immediately next to the first letter C. Thus, in total, there are $6 \cdot 7 = 42$ ways for this case.

Case 3: there is 9 permutations of As and Bs such that there are two set of like letters adjacent.

- There are 2 permutations where three letters A appear together: $BAAABAB$ and $BABAAAB$.
- There is 1 permutations where four letters A appear in pairs: $BAABAAB$.
- There are 6 permutations where there is a pair of adjacent letters A and a pair of adjacent letters B:

$$AABBABA,\ AABABBA,\ ABBAABA,\ ABAABBA,\ ABBABAA,\ ABABBAA.$$

Into each of these permutation, two letter C must be used to insert in between the two neighbouring like letters. Thus, in total, there are $(2 + 1 + 6) \cdot 1 = 9$ ways for this case.

Therefore, altogether, there are $28 + 42 + 9 = \boxed{79}$ permutations. □

Problem 1.3.14 (Problem 14)**.** (*5 points*) There is a real number x between 0 and $\dfrac{\pi}{2}$ such that

$$\frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{12}{11}$$

and $\sin x + \cos x = \frac{\sqrt{m}}{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Solution. Let $u = \sin(x), v = \cos(x)$, then $\sin^3 x + \cos^3 x = u^3 + v^3, \sin^5 x + \cos^5 x = u^5 + v^5$, and $u^2 + v^2 = 1$, furthermore

$$\begin{aligned}
&u^3 + v^3 = (u + v)(u^2 - uv + v^2) = (u + v)(1 - uv)\\
&u^5 + v^5 = (u + v)(u^4 - u^3v + u^2v^2 + uv^3 + u^4) = (u + v)((u^2 + v^2)^2 - u^2v^2 - uv(u^2 + v^2))\\
&\ = (u + v)(1 - uv - u^2u^2)
\end{aligned}$$

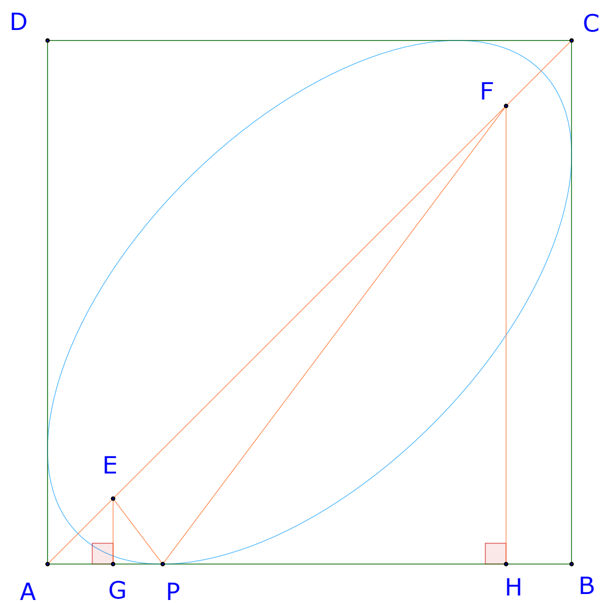
Now, let $s = u + v$, then $uv = \frac{s^2-1}{2}$, therefore

$$\begin{aligned}
&\frac{12}{11} = \frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{s\left(1 - \frac{s^2-1}{2}\right)}{s\left(1 - \frac{s^2-1}{2} - \left(\frac{s^2-1}{2}\right)^2\right)} = \frac{2(3 - s^2)}{5 - s^4}\\
&\Rightarrow 6s^4 - 11s^2 + 3 = 0 \Rightarrow s^2 = \frac{3}{2} \text{ or } \frac{1}{2}.
\end{aligned}$$

Since $0 \leq x \leq \frac{\pi}{2}$, so $u, v \geq 0$, thus $s^2 = 1 + 2uv \geq 1$, therefore $s^2 = \frac{3}{2} \Rightarrow s = \frac{\sqrt{6}}{2}$. Hence, the desired sum $m + n = 6 + 2 = \boxed{8.}$ □

Problem 1.3.15 (Problem 15)**.** (*5 points*) Points E and F lie on diagonal AC of square $ABCD$ with side length 24, such that $AE = CF = 3\sqrt{2}$. An ellipse with foci at E and F is tangent to the sides of the square. Find the sum of the distances from any point on the ellipse to the two foci.

Hint: Let P be the tangent point on AB. By the properties of the eclipse, a light ray passing from E to P that reflects of the eclipse will pass through point F. Because line AB is tangent to the eclipse, $\angle EPA = \angle FPB$.



Solution. Let P be the tangent point on AB. By the properties of the eclipse, a light ray passing from E to P that reflects of the eclipse will pass through point F. Because line AB is tangent to the eclipse, $\angle EPA = \angle FPB$. Let points G and H be the projections onto AB of points E and F, respectively. Then $\triangle EGP \sim \triangle FHP$,

$$AE = CF = 3\sqrt{2}, \angle CAB = 45^\circ \Rightarrow AG = BH = 3, FH = 24 - 3 = 21, GH = 24 - 2 \cdot 3 = 18$$

$$\frac{PG}{PH} = \frac{EG}{FH} = \frac{3}{21} = \frac{1}{7} \Rightarrow PG = GH \cdot \frac{PG}{PG + PH} = 18 \cdot \frac{PG}{PG + 7PG} = \frac{9}{4}$$

$$EP = \sqrt{EG^2 + PG^2} = \sqrt{3^2 + \left(\frac{9}{4}\right)^2} = \frac{15}{4}$$

$$FP = 7EP \Rightarrow EP + FP = 8EP = \boxed{30.}$$

□

Problem 1.3.16 (Problem 16)**.** (*5 points*) A deck of eight cards has cards numbered $1, 2, 3, 4, 5, 6, 7, 8$, in that order, and a deck of five cards has cards numbered $1, 2, 3, 4, 5$, in that order. The two decks are riffle-shuffled together to form a deck with 13 cards with the cards from each deck in the same order as they were originally. Thus, numbers on the cards might end up in the order 1122334455678 or 1234512345678 but not 1223144553678. Find the number of possible sequences of the 13 numbers.

Solution. There is a one-on-one correspondence between possible orderings and the paths of length 13 in the coordinate plane from $(0, 0)$ to $(8, 5)$ where each step from point (x, y) is one unit to the right to $(x + 1, y)$ or one unit up to $(x, y + 1)$, and there is no point (x, y) on the path where $y > x$.

Indeed, given a possible ordering, construct a path by reading the numbers in order, and each time a number is seen for the first time, have the path take one step right (R), and each time a number is seen for the second time, have the path take one step up (U). This process take every possible number sequence and convert it into a path of the correct type, and the process is reversible showing that this correspondence is one-to-one.

It remains to count the number of paths from $(0, 0)$ to (8.5) that make unit steps to the right or upward that avoid points with $y > x$. Each such path make 8 steps right and 5 steps upward in some order.

Suppose one of these paths crosses a point (x, y) with $y > x$. For example the path $RURRUUURRURRR$ reaches $(3, 4)$ after making the steps $RURRUUU$. If, after this points all U steps are changed to R steps and all R steps are changed to U steps, the path become $RURUUUUURUUU$ which ends up at $(4, 9)$.

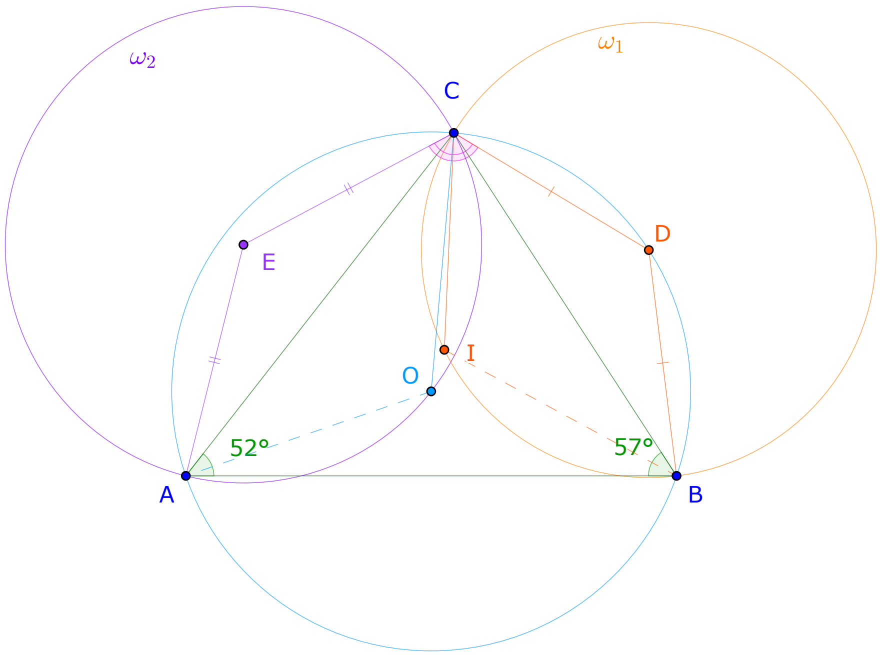
In fact, if any path from $(0, 0)$ to $(8, 5)$ that reaches a point $(x.y)$ where $y > x$ is transformed by interchanging subsequent steps U and R, the path will reach the point $(4, 9)$ instead of point $(8, 5)$. The number of paths from $(0, 0)$ to $(4, 9)$ is $\binom{4+9}{4} = \binom{13}{4}$. This shows that the number of paths to $(8, 5)$ that do not contain point (x, y) with $y > x$ is:

$$\binom{5+8}{5} - \binom{4+9}{4} = \boxed{572.}$$

$\square$

Problem 1.3.17 (Problem 17). (*5 points*) In $\triangle ABC$, $\angle A = 52°$, and $\angle B = 57°$. One circle centred at D passes through the points B, C, and the incenter I of $\triangle ABC$, and a second circle centred at E passes through the points A, C, and the circumcenter O of $\triangle ABC$. Find the degree measure of the acute angle at which the two circles intersect.

Hint: find the measure of the complementary angle of the angle $\angle DCE$.



Solution. First $\angle C = 180° - \angle A - \angle B = 71°$. Furthermore

$$\widehat{BI} = 2\angle BCI = \angle ACB,\ \widehat{CI} = 2\angle CBI = \angle ABC \Rightarrow \angle BDC = \widehat{BC} = \widehat{BI} + \widehat{CI} = \angle ACB + \angle ABC.$$

Since $\triangle BDC$ isosceles, in circle ω_1,

$$\angle BCD = \frac{1}{2}(180° - \angle BDC) = \frac{1}{2}(180° - \angle ACB + \angle ABC) = \frac{1}{2}\angle BAC.$$

On the other hand, in circle $O, \angle AOC = 2\angle AB$, so in circle ω_2,

$$\text{(major) } \widehat{AC} = 2\angle AOC = 4\angle ABC \Rightarrow \angle AEC = 360° - \widehat{AC} = 360° - 4\angle ABC.$$

Since $\triangle AEC$ is isosceles, in circle ω_2,

$$\angle ACE = \frac{1}{2}(180° - \angle AEC) = 2\angle ABC - 90°.$$

The diagram at the right shows both circles ω_1 and ω_2. Because the line tangent to a circle is perpendicular to the radius of the circle that ends at the point of tangency, it follows that the two circles intersect at an angle θ which satisfies $\angle DCE = 180° - \theta$, thus,

$$\theta = 180° - \angle DCE = 180° - (\angle BCD + \angle ACB + \angle ACE) = 180° - \left(\frac{1}{2}\angle BAC + \angle ACB + 2\angle ABC - 90°\right)$$

$$= 270° - (\frac{1}{2}52° + 71° + 2 \cdot 57°) = \boxed{59°.}$$

□

Problem 1.3.18 (Problem 18)**.** (*5 points*) Three doctors, four nurses, and three patients stand in a line in random order. The probability that there is at least one doctor and at least one nurse between each pair of patients is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. The required probability is equal to the probability that if the ten letters $DDDNNNNPPP$ are arranged in random order, then there is at least one D and one N between each pair of Ps. There are

$$\binom{10}{3,4,3} = \frac{10!}{3!4!3!}$$

equally likely ways to arrange the ten letters.

Consider first the possible arrangements of the Ps and Ds that result in at least one D between each pair of Ps. There are four possible arrangements:

$$DPDPDP, PDDPDP, PDPDDP, \text{ and } PDPDPD.$$

Now consider how many ways four Ns can be inserted into one of these lists of six letters.

Without regard to whether there are Ns between each pair of Ps, the number of ways of inserting four Ns into a sequence of six other letters is given by the sticks-and-stones technique as $\binom{4+6}{6} = 210$.

Let X be the set of such arrangements that leave no Ns between the first P and second P, and let Y be the set of such arrangements that leave no Ns between the second P and the third P.

There are now four cases to consider.

Case 1: In the case of inserting four Ns into $DPDPDP$, the sizes of X, Y, and $X \cap Y$ are given by

$$\binom{4+4}{4} = 70, \ \binom{4+4}{4} = 70, \ \binom{4+2}{2} = 15, \text{ respectively.}$$

Thus, by the Inclusion/Exclusion Principle, there are $70 + 70 - 15 = 125$ ways to insert four Ns into the sequence and not have at least one N between each pair of Ps. In each of these two cases there are, therefore, $210 - 125 = 85$ ways of inserting four Ns so that there is at least one N between each pair of Ps.

Case 2: There are also 85 arrangements associated with inserting four Ns into $PDPDPD$.

Case 3: In the case of inserting four Ns into $PDDPDP$, the sizes of X, Y, and $X \cap Y$ are given by

$$\binom{4+3}{3} = 35, \ \binom{4+4}{4} = 70, \ \binom{4+1}{1} = 5, \text{ respectively.}$$

Thus, by the Inclusion/Exclusion Principle, there are $435 + 70 - 5 = 100$ ways to insert four Ns into the sequence and not have at least one N between each pair of Ps. In each of these two cases there are, therefore, $210 - 100 = 110$ ways of inserting four Ns so that there is at least one N between each pair of Ps.

Case 4: There are also 110 arrangements associated with inserting four Ns into $PDPDDP$.

Thus, there are $2(85 + 110) = 390$ arrangements of the ten letters so that there is at least one D and one N between each pair of Ps. The required probability is

$$\frac{390}{\frac{10!}{3!4!3!}} = \frac{13}{140}.$$

Hence, the desired sum is $13 + 140 = \boxed{153.}$ □

Problem 1.3.19 (Problem 19)**.** (*5 points*) Let p, q, and r be prime numbers such that $2pqr+p+q+r = 2020$. Find $pq + qr + rp$.

Solution. Note that $p + q + r = 2020 - 2pqr$, thus it is even, so at least one prime is even. WLOG, let $p = 2$.

$$4qr + q + r = 2018 \Rightarrow (4q + 1)(4r + 1) = 4 \cdot 2018 + 1 = 8073 = 3^3 \cdot 13 \cdot 23.$$

Thus, 8073 has 16 divisors. We are looking for those divisors in the form of $4k + 1$ and they are $1, 9, 13, 3 \cdot 23 = 69$. Easy to test and verify that the pair q, r is $17, 29$.

Hence, the desired sum is $pq + qr + rp = 2 \cdot 17 + 17 \cdot 29 + 29 \cdot 2 = \boxed{585.}$ □

1.4 High School - Test

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 4-13**
- **Olympiad (O) level: Problems 10-19**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 1.4.1 (Problem 1)**.** (*5 points*) There is a complex number K such that the quadratic polynomial $5x^2 + Kx + 4 - 3i$ has exactly one root, where $i = \sqrt{-1}$. Find $|K|^2$.

Note that for a complex number $x = a + bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2 + b^2}$.

Solution. A quadratic polynomial has exactly one root if and only if its determinant is zero, therefore for $5x^2 + Kx + 4 - 3i$, we have:

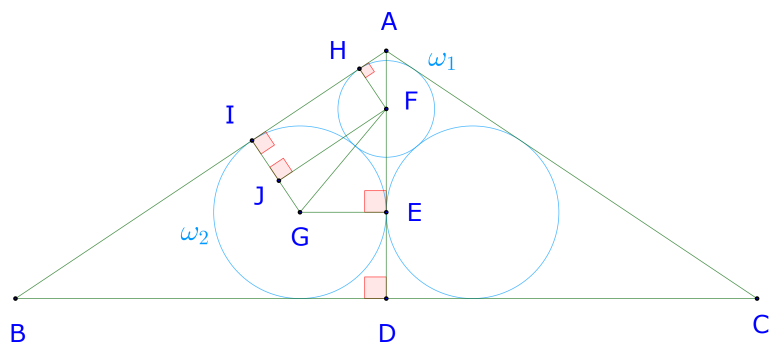
$$K^2 - 4(5)(4 - 3i) = 0 \Rightarrow K^2 = (4 \cdot 5 \cdot 4) - (4 \cdot 5 \cdot 3)i.$$

Note that for a complex number $K = a + bi$, then

$$|K| = \sqrt{a^2 + b^2} \text{ and } K^2 = (a + bi)(a + bi) = a^2 - b^2 + 2abi, \text{ note that}$$
$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2 \Rightarrow |K|^2 = a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + (2ab)^2} = |K^2|$$

Hence, $|K|^2 = |K^2| = \sqrt{(4 \cdot 5 \cdot 4)^2 + (4 \cdot 5 \cdot 3)^2} = 20\sqrt{4^2 + 3^2} = 20 \cdot 5 = \boxed{100.}$. □

Problem 1.4.2 (Problem 2)**.** (*5 points*) Two circles have radius 25, and one circle has radius 14. Each circle is externally tangent to the other two circles, and each circle is internally tangent to two sides of an isosceles triangle, as shown. The sine of the base angle of the triangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. Let ω_1 be the circle with radius 10 with center F and ω_2 be one of the circles with radius 6 with center G. Let label the vertices of the triangles to be A, B, and C as shown in the diagram above. Let AD be the altitude from A to BC, E be the tangent point of the circles with radius 10, and H, I be the feet of the altitudes from F, G to AB, respectively. Let J be the foot of the altitudes from F to GJ.

Then

$$EG = 25, FG = 14 + 25 = 39 \Rightarrow EF = \sqrt{39^2 - 25^2} = \sqrt{896}$$
$$GJ = GI - IJ = GI - FH = 25 - 14 = 11 \Rightarrow FJ = \sqrt{39^2 - 11^2} = \sqrt{1400}$$
$$\sin(\angle ABC) = \cos(\angle BAD) = \cos(\angle JFE) = \cos(\angle JFG + \angle GFE)$$
$$= \cos(\angle JFG)\cos(\angle GFE) - \sin(\angle JFG)\sin(\angle GFE)$$
$$= \frac{FJ}{FG}\frac{FE}{FG} - \frac{GJ}{FG}\frac{EG}{FG} = \frac{\sqrt{1400} \cdot \sqrt{896} - 11 \cdot 25}{39^2} = \frac{5}{9}.$$

Hence, $m + n = 5 + 9 = \boxed{14.}$ □

Problem 1.4.3 (Problem 3)**.** (*5 points*) There are three distinct pairs of positive integers $a_1 < b_1$, $a_2 < b_2$, and $a_3 < b_3$ such that

$$|(a_1 + ib_1)(b_1 - ia_1)| = |(a_2 + ib_2)(b_2 - ia_2)| = |(a_3 + ib_3)(b_3 - ia_3)| = 1025, \text{ where } i = \sqrt{-1}.$$

Find $a_1 + b_1 + a_2 + b_2 + a_3 + b_3$.

Note that for a complex number $x = a + bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2 + b^2}$.

Solution. Note that

$$(a_1 + ib_1)(b_1 - ia_1) = 2a_1b_1 + i(b_1^2 - a_1^2) \Rightarrow |(a_1 + ib_1)(b_1 - ia_1)| = \sqrt{(2a_1b_1)^2 + (b_1^2 - a_1^2)^2} = a_1^2 + b_1^2$$

Thus $a_1^2 + b_1^2 = 1025$. Since $1024 = 32^2$, by testing we can see that $(a_1, b_1) \in \{(1, 32), (8, 31), (20, 25)\}$.

Similarly for (a_2, b_2) and (a_3, b_3). By the given conditions they are distinct pairs, so they must be all these three pairs.

Hence, $a_1 + b_1 + a_2 + b_2 + a_3 + b_3 = 1 + 32 + 8 + 31 + 20 + 25 = \boxed{117.}$ □

Problem 1.4.4 (Problem 4)**.** (*5 points*) There are relatively prime positive integers s and t such that

$$\sum_{n=2}^{101} \left(\frac{n}{n^2 - 1} - \frac{1}{n}\right) = \frac{s}{t}.$$

Find $s + t$.

Solution. Note that

$$\frac{n}{n^2 - 1} - \frac{1}{n} = \frac{n}{(n-1)(n+1)} - \frac{1}{n} = \frac{1}{2}\left(\frac{1}{n-1} + \frac{1}{n+1}\right) - \frac{1}{n} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right) - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$\sum_{n=2}^{101} \left(\frac{n}{n^2 - 1} - \frac{1}{n}\right) = \frac{1}{2}\sum_{n=2}^{101}\left(\frac{1}{n-1} - \frac{1}{n}\right) - \frac{1}{2}\sum_{n=2}^{101}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{101}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{102}\right) = \frac{2575}{10302}$$

Hence, $s + t = \boxed{12877.}$ □

Problem 1.4.5 (Problem 5)**.** (*5 points*) Let x be a real number such that $3\sin^4 x - 2\cos^6 x = \frac{17}{25}$. Then $3\cos^4 x - 2\sin^6 x = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $10m + n$.

Solution.

$$\frac{17}{25} = 3\sin^4 x - 2\cos^6 x = 3\sin^4 x - 2(1 - \sin^2 x)^3 = 2\sin^6 x - 3\sin^4 x + 6\sin^2 x - 2$$
$$3\cos^4 x - 2\sin^6 x = 3(1 - \sin^2 x)^2 - 2\sin^6 x = 3 - 6\sin^2 x + 3\sin^4 x - 2\sin^6 x$$

Hence, $\frac{m}{n} = -\left(\frac{17}{25} + 2\right) + 3 = \frac{8}{25}$, so $10m + n = 80 + 25 = \boxed{105.}$ □

Problem 1.4.6 (Problem 6)**.** (*5 points*) Find the sum of all values of x such that the set

$$\{108, 123, 128, 138, 153, x\} \text{ has a mean that is equal to its median.}$$

Solution. First, let's simplify the problem by subtracting 108 from each of the elements, then the statement still stands for the set $\{0, 15, 20, 30, 45, x - 108\}$. Then, by dividing each of the elements by 5, it is the same assumption for the set $\{, y = \frac{x-108}{5}\}$. The mean of the set now is $\dfrac{0+3+4+6+9+y}{6} = \dfrac{22+y}{6}$.

Case 1: $y < 3$, then the median of the set is $\dfrac{3+4}{2} = \dfrac{7}{2}$, so $y = -1$.

Case 2: $3 \le y \le 6$, then the median of the set is $\dfrac{4+y}{2}$, so $y = 5$.

Case 3: $y < 6$, then the median of the set is $\dfrac{4+6}{2}$, so $y = 8$.

Hence, the sum of all values of x is

$$(-1 \cdot 5 + 108) + (5 \cdot 5 + 108) + (8 \cdot 5 + 108) = 103 + 133 + 148 = \boxed{384.}$$

$\square$

Problem 1.4.7 (Problem 7)**.** (*5 points*) Find the **minimum** possible value of

$$\left(\frac{a^2}{b^3c^2} + \frac{b^2}{c^3a^2} + \frac{c^2}{a^3b^2}\right)^2$$

where $a > b > 0 > c$ are real numbers satisfying

$$a\sqrt[3]{\frac{a}{b}} + b\sqrt[3]{\frac{b}{c}} + c\sqrt[3]{\frac{c}{a}} = 0.$$

Solution.

$$\text{Let } x = a\sqrt[3]{\frac{a}{b}},\ y = b\sqrt[3]{\frac{b}{c}},\ z = c\sqrt[3]{\frac{c}{a}} \Rightarrow x + y + z = 0$$

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx) = 3xyz$$

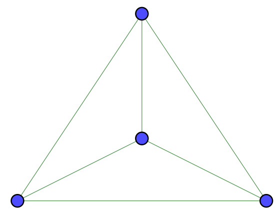
$$x^3 = \left(a\sqrt[3]{\frac{a}{b}}\right)^3 = (a^3)\left(\frac{a}{b}\right) = (abc)\left(\frac{a^3}{b^2c}\right), y^3 = (abc)\left(\frac{b^3}{c^2a}\right), z^3 = (abc)\left(\frac{c^3}{a^2b}\right)$$

$$\Rightarrow x^3 + y^3 + z^3 = (abc)\left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right),\ 3xyz = 3a\sqrt[3]{\frac{a}{b}} \cdot b\sqrt[3]{\frac{b}{c}} \cdot c\sqrt[3]{\frac{c}{a}} = 3abc$$

$$\Rightarrow \left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right)^2 = \left(\frac{x^3+y^3+z^3}{abc}\right)^2 = \left(\frac{3xyz}{abc}\right)^2 = \left(\frac{3abc}{abc}\right)^2 = \boxed{9.}$$

$\square$

Problem 1.4.8 (Problem 8)**.** (*5 points*) The following diagram shows four vertices connected by six edges. Suppose that each of the edges is randomly painted either red, white, or blue. The probability that the three edges adjacent to **at least two** vertices are colored with all three colors is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution. There are six edges in the diagrams, and there are 3 equally likely ways to paint each edge. Thus, there are 3^6 equally likely ways to paint all the edges.

To count the number of ways to paint the edges so that at least one vertex is adjacent to one edge of each colour, number the vertices 1, 2, 3, and 4 and let A_1, A_2, A_3, and A_4 be the sets of colouring patterns, where each of the vertices 1, 2, 3, and 4, respectively, is adjacent to edges of all three colours.

There are $3! = 6$ ways to paint the 3 edges adjacent to a particular vertex so that there is one edge of each colour, and there are 3^3 ways to paint the other three edges showing that for each j, the cardinality of A_j is $6 \cdot 3^3$.

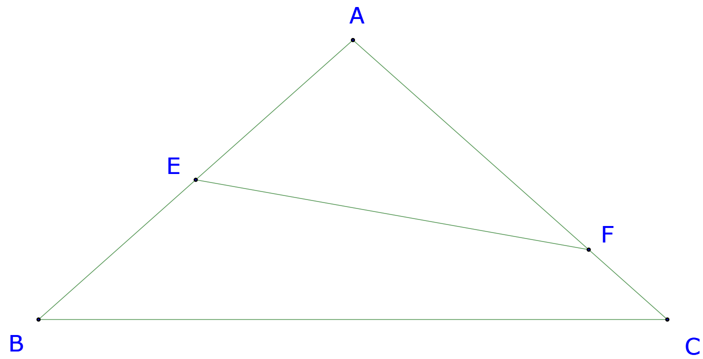
A pattern is in $A_j \cap A_k$, for $j \neq k$ if the edges adjacent to vertex j are painted in one of 6 ways, the two edges adjacent to vertex k not adjacent to vertex j are painted in one of 2 ways, and the one edge not adjacent to either vertex j or vertex k is painted in one of 3 ways. This shows that $A_j \cap A_k$ containing $6 \cdot 2 \cdot 3 = 36$ patterns. This is true for all $\binom{4}{2} = 6$ pairs of j and k.

By the Inclusion - Exclusion Principle, the number of ways to colouring the edges such that *at least two vertices* are colored with all three colors is:

$$\begin{aligned}
&|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4| \\
&- |A_1 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_4| - |A_1 \cap A_3 \cap A_4| - |A_2 \cap A_3 \cap A_4| \\
&+ |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= 6(6 \cdot 2 \cdot 3) - 4 \cdot 6 + 6 = 198
\end{aligned}$$

The required probability is $\dfrac{198}{3^6} = \frac{22}{81}$. Hence, $m + n = 22 + 81 = \boxed{103.}$ □

Problem 1.4.9 (Problem 9)**.** (*5 points*) In isosceles $\triangle ABC, AB = AC, \angle BAC$ is obtuse, and points E and F lie on sides AB and AC, respectively, so that $AE = 20, AF = 30$. The area of $\triangle AEF$ is 240, and the area of quadrilateral $BEFC$ is 408. Find BC.



Solution. Keep in mind that $\angle BAC > 90°$, so $\cos(\angle BAC) < 0$,

$$240 = [AEF] = \frac{1}{2}(AE \cdot AF)\sin(\angle EAF) = \frac{1}{2}(20 \cdot 30)\sin(\angle EAF) \Rightarrow \sin(\angle EAF) = \frac{4}{5}$$
$$[AEF] + [BEFC] = [ABC] = \frac{1}{2}(AB \cdot AC)\sin(\angle BAC) = \frac{1}{2}(AB^2)\frac{4}{5} \Rightarrow AB^2 = \frac{648 \cdot 2 \cdot 5}{4} = 1620$$
$$BC^2 = AB^2 + AC^2 - 2(AB \cdot AC)\cos(\angle BAC) = 1620(2)(1 - \cos(\angle BAC)) = 1620(2)\left(1 + \frac{3}{5}\right) = 5184.$$

Hence, $BC = \sqrt{5184} = \boxed{72.}$ ☐

Problem 1.4.10 (Problem 10)**.** (*5 points*) Find the least prime number greater than 1000 that divides $2^{1010} \cdot 235^{2020} + 1$.

Solution. First, let $n = 2^{1010} \cdot 235^{2020}$, then

$$n = 2^{1010} \cdot 23^{2020} + 1 = (4 \cdot 235^4)^{505} + 1 \Rightarrow 4 \cdot 235^4 + 1 \mid n.$$

By Sophie Germain identity $a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab)$ for $a = 1, b = 235$,

$$4 \cdot 235^4 + 1 = (1 + 2 \cdot 235^2 - 2 \cdot 235)(1 + 2 \cdot 235^2 + 2 \cdot 235) = (109 \cdot 1009) \cdot (110921).$$

Now, since $7 \mid 1001, 17 \mid 1003, 5 \mid 1005, 19 \mid 1007$, thus $\boxed{1009}$ is the first prime number larger than 1000 that divides n. ☐

Problem 1.4.11 (Problem 11)**.** (*5 points*) Find the maximum possible value of

$$8\sqrt{x} + 9\sqrt{y} + 12\sqrt{z},$$

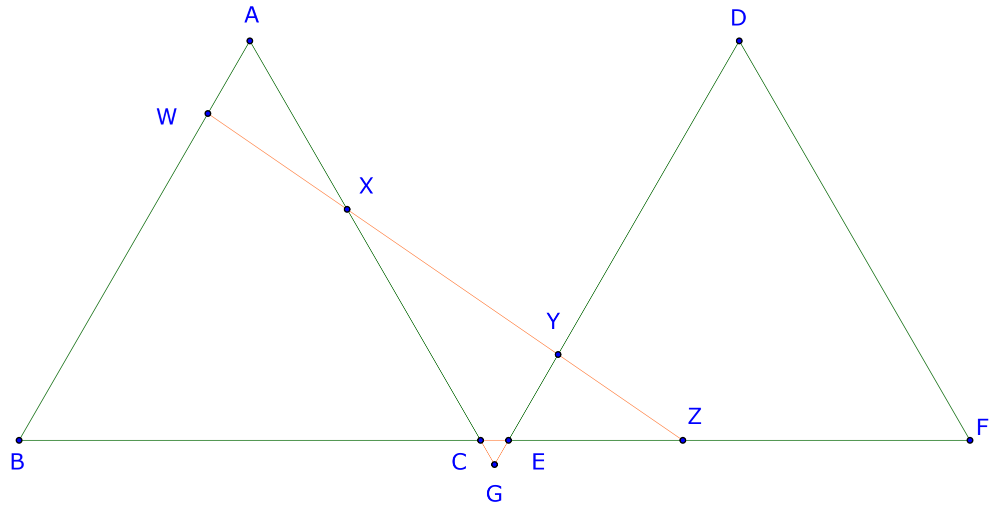
where x, y, and z are positive real numbers satisfying $4x + 9y + z = 256$.

Solution. By the Cauchy-Schwarz inequality,

$$8\sqrt{x}+9\sqrt{y}+12\sqrt{z} = (4)(\sqrt{4x})+(3)(\sqrt{9y})+(12)(\sqrt{z}) \le \sqrt{(4^2 + 3^2 + 12^2)(4x + 9y + z)} = \sqrt{169 \cdot 256} = \boxed{208.}$$

☐

Problem 1.4.12 (Problem 12). (*5 points*) Two congruent equilateral triangles $\triangle ABC$ and $\triangle DEF$ lie on the same side of line BC so that B, C, E, and F are collinear as shown. A line intersects AB, AC, DE, and EF at W, X, Y, and Z, respectively, such that $\dfrac{AW}{BW} = \frac{2}{9}$, $\dfrac{AX}{CX} = \frac{5}{6}$, and $\dfrac{DY}{EY} = \frac{9}{2}$. The ratio $\dfrac{XW}{XZ}$ can then be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution. Without loss of generality, let the triangles have side length 11, so

$$AW = 2, BW = 9, AX = 5, CX = 6, DY = 9, EY = 2.$$

Let G be the intersection of lines AC and DE, then $\triangle AWX \sim GYX$, so

$$\frac{GE + EY}{GC + CX} = \frac{GY}{GX} = \frac{AW}{AX} = \frac{2}{5} \Rightarrow \frac{GE + 2}{GC + 6} = \frac{2}{5} \Rightarrow GC = GE = \frac{2}{3}.$$

By Menelaus' Theorem for $\triangle ABC$ and line WXZ,

$$1 = \frac{AW}{BW} \cdot \frac{BZ}{CZ} \cdot \frac{CX}{AX} = \frac{2}{9} \cdot \frac{11 + CZ}{CZ} \cdot \frac{6}{5} \Rightarrow CZ = 4.$$

Again by Menelaus' Theorem for $\triangle BWZ$ and line AXC,

$$1 = \frac{BA}{AW} \cdot \frac{WX}{XZ} \cdot ZCCB = \frac{11}{2} \cdot \frac{WX}{XZ} \cdot \frac{4}{11} \Rightarrow \frac{WX}{XZ} = \frac{1}{2}.$$

Thus the desired sum $m + n = 1 + 2 = \boxed{3.}$ □

Problem 1.4.13 (Problem 13)**.** (*5 points*) Find the number of permutations of the letters AAABBCC where there are no two like letters adjacent. For example, count ABCABCA and ABABCAC but not ABCCBAA.

Solution. Let first consider the letters $AAABB$. There are three cases that could lead to permutations with no repeated letters.

Case 1: there is 1 permutation of As and Bs such that no two like letters are next to each other: $ABABA$. For these, there are 6 positions around the 5 letters to place the two letters C, thus there are $\binom{6}{2} = 15$ ways to choose two distinct positions for Cs. Thus, in total, there are $1 \cdot 15 = 15$ ways for this case.

Case 2: there is 4 permutations of As and Bs that have one pair of like letters are adjacent. These are obtained by starting with $ABABA$, removing the beginning or ending A, then placing that A next to one of another A (there are two ways to *insert* right before of or right after it.) Into this permutation, a letter C must be used to insert in between the two neighbouring letters A. For the second letter C there are $7 - 2 = 5$ positions not immediately next to the first letter C. Thus, in total, there are $4 \cdot 5 = 20$ ways for this case.

Case 3: there is 9 permutations of As and Bs such that there are two set of like letters adjacent.

- There are 1 permutations where three letters A appear together: $BAAAB$.
- There are 2 permutations where there is a pair of adjacent letters A and a pair of adjacent letters B:

$$AABBA, \ ABBAA.$$

Into each of these permutation, two letter C must be used to insert in between the two neighbouring like letters. Thus, in total, there are $(1 + 2) \cdot 1 = 3$ ways for this case.

Therefore, altogether, there are $15 + 20 + 3 = \boxed{38}$ permutations. □

Problem 1.4.14 (Problem 14)**.** (*5 points*) There is a real number x between 0 and $\dfrac{\pi}{2}$ such that

$$\frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{20}{19}$$

and $\sin x + \cos x = \frac{\sqrt{m}}{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Solution. Let $u = \sin(x), v = \cos(x)$, then $\sin^3 x + \cos^3 x = u^3 + v^3, \sin^5 x + \cos^5 x = u^5 + v^5$, and $u^2 + v^2 = 1$, furthermore

$$\begin{aligned}
&u^3 + v^3 = (u + v)(u^2 - uv + v^2) = (u + v)(1 - uv)\\
&u^5 + v^5 = (u + v)(u^4 - u^3v + u^2v^2 + uv^3 + u^4) = (u + v)((u^2 + v^2)^2 - u^2v^2 - uv(u^2 + v^2))\\
&\ = (u + v)(1 - uv - u^2u^2)
\end{aligned}$$

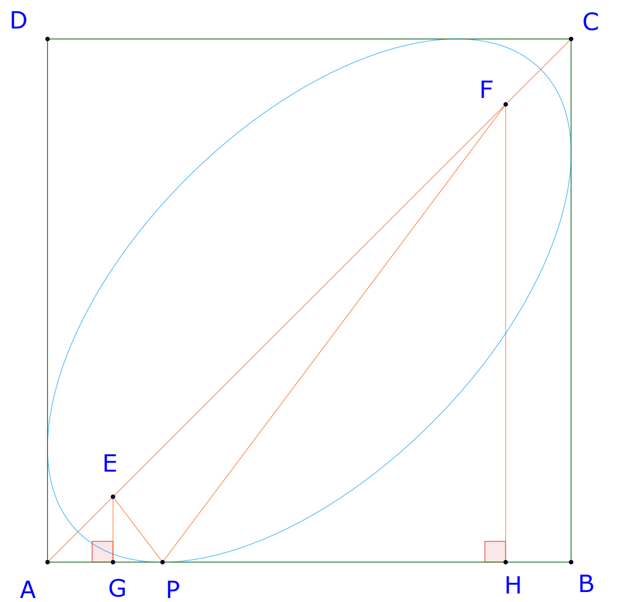
Now, let $s = u + v$, then $uv = \frac{s^2-1}{2}$, therefore

$$\begin{aligned}
&\frac{20}{19} = \frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{s\left(1 - \frac{s^2-1}{2}\right)}{s\left(1 - \frac{s^2-1}{2} - \left(\frac{s^2-1}{2}\right)^2\right)} = \frac{2(3 - s^2)}{5 - s^4}\\
&\Rightarrow 10s^4 - 19s^2 + 7 = 0 \Rightarrow s^2 = \frac{7}{5} \text{ or } \frac{1}{2}.
\end{aligned}$$

Since $0 \le x \le \frac{\pi}{2}$, so $u, v \ge 0$, thus $s^2 = 1 + 2uv \ge 1$, therefore $s^2 = \frac{7}{5} \Rightarrow s = \frac{\sqrt{35}}{5}$. Hence, the desired sum $m + n = 35 + 5 = \boxed{40.}$ □

Problem 1.4.15 (Problem 15). (*5 points*) Points E and F lie on diagonal AC of square $ABCD$ with side length 24, such that $AE = CF = 3\sqrt{2}$. An ellipse with foci at E and F is tangent to the sides of the square. Let P be the tangent point of the eclipse with AB. Let points G and H be the projections onto AB of points E and F, respectively. Let $\frac{m}{n}$ be the value of $\cos EPF$, where m and n are relatively prime positive integers. Find $m + n$.

Hint: By the properties of the eclipse, a light ray passing from E to P that reflects of the eclipse will pass through point F. Because line AB is tangent to the eclipse, $\angle EPA = \angle FPB$.



Solution. Then $\triangle EGP \sim \triangle FHP$,

$$AE = CF = 3\sqrt{2}, \angle CAB = 45^\circ \Rightarrow AG = BH = 3, FH = 24 - 3 = 21, GH = 24 - 2 \cdot 3 = 18$$

$$\frac{PG}{PH} = \frac{EG}{FH} = \frac{3}{21} = \frac{1}{7} \Rightarrow PG = GH \cdot \frac{PG}{PG + PH} = 18 \cdot \frac{PG}{PG + 7PG} = \frac{9}{4}$$

$$EP = \sqrt{EG^2 + PG^2} = \sqrt{3^2 + \left(\frac{9}{4}\right)^2} = \frac{15}{4}, FP = 7EP = 7\left(\frac{15}{4}\right)$$

$$EF^2 = EP^2 + PF^2 - 2(EP \cdot PF)\cos(\angle EPF) \Leftrightarrow 648 = 50\left(\frac{15}{4}\right)^2 - 14\left(\frac{15}{4}\right)^2 \cos(\angle EPF)$$

$$\Leftrightarrow 648 = \left(\frac{15}{4}\right)^2 (50 - 14\cos(\angle EPF)) \Rightarrow \cos(\angle EPF) = \frac{1}{14}\left(50 - \frac{18^2 \cdot 2 \cdot 4^2}{15^2}\right) = \frac{7}{25}.$$

Hence, $m + n = 7 + 25 = \boxed{32.}$ □

Problem 1.4.16 (Problem 16)**.** (*5 points*) A deck of eight cards has cards numbered $1, 2, 3, 4, 5, 6, 7$, in that order, and a deck of five cards has cards numbered $1, 2, 3, 4, 5, 6$ in that order. The two decks are riffle-shuffled together to form a deck with 13 cards with the cards from each deck in the same order as they were originally. Thus, numbers on the cards might end up in the order 1122334455667 or 1234561234567 but not 1223144553667. Find the number of possible sequences of the 13 numbers.

Solution. We show a coding solution. The 13-element *sequence* array is the one that holds a valid sequence. We use 7 loops, each to identify the indexes of the number $1, 2, 3, 4, 5, 6, 7$, when put them into the *sequence*. As you might see, we use the for loops in away that

$$0 \leq i_1 < i_2 < \ldots < i_7 \leq 12.$$

The indexes and the number $1, 2, 3, 4, 5, 6, 7$ are held temporarily in a dictionary d :

```
    d = {i0: 1, i1:2, i2: 3, i3:4, i4: 5, i5: 6, i6:7}   # Python dictionary
```

Now, we go through a loop to fill the 13-element *sequence* array with the numbers $1, 2, 3, 4, 5, 6, 7$ by using the dictionary d, the rest of the positions in the array is filled with the numbers $1, 2, 3, 4, 5, 6$ whenever we find an empty slot

```
    c = 1
    for i in range(0, 13):
        if i in d:  # find an index that is mapped to a number in the dictionary
            sequence[i] = d[i]
        else:   # the dictionary does not has such index, fill it with the current number from 1, 2, 3, 4, 5, 6
            sequence[i] = c
            c += 1
```

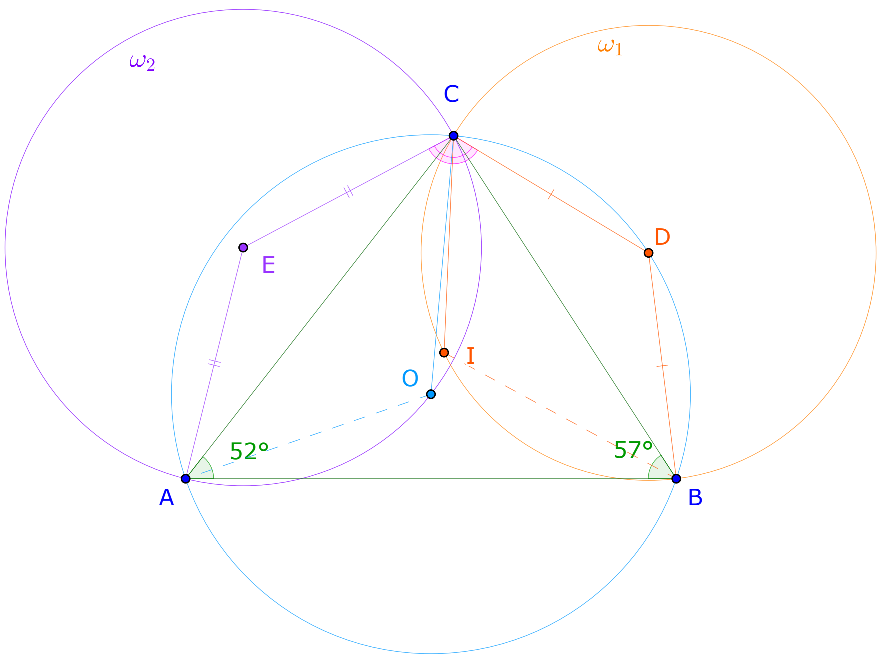
Once the sequence is generated correctly, then we put it into the set *all_sequences* containing all strings representing the sequences. Because it is a set, *all_sequences* does not contain duplicates can will hold correctly the number of found sequences.

```
    sequence = [None for i in range(0, 13)]
    all_sequences = set()
    for i0 in range(0, 6):
        for i1 in range(i0+1, 7):
            for i2 in range(i1+1, 8):
                for i3 in range(i2+1, 9):
                    for i4 in range(i3+1, 10):
                        for i5 in range(i4+1, 11):
                            for i6 in range(i5+1, 12):
                                d = {i0: 1, i1:2, i2: 3, i3:4, i4: 5, i5: 6, i6:7}
                                c = 1
                                for i in range(0, 13):
                                    if i in d:
                                        sequence[i] = d[i]
                                    else:
                                        sequence[i] = c
                                        c += 1
                                all_sequences.add(’’.join(str(e) for e in sequence))
    print(len(all_sequences))
```

The answer is $\boxed{297.}$ □

Problem 1.4.17 (Problem 17)**.** (*5 points*) In $\triangle ABC$, $\angle A = 50°$, and $\angle B = 59°$. One circle centred at D passes through the points B, C, and the incenter I of $\triangle ABC$, and a second circle centred at E passes through the points A, C, and the circumcenter O of $\triangle ABC$. Find the degree measure of the acute angle at which the two circles intersect.

Hint: find the measure of the complementary angle of the angle $\angle DCE$.



Solution. First $\angle C = 180° - \angle A - \angle B = 71°$. Furthermore

$$\widehat{BI} = 2\angle BCI = \angle ACB,\ \widehat{CI} = 2\angle CBI = \angle ABC \Rightarrow \angle BDC = \widehat{BC} = \widehat{BI} + \widehat{CI} = \angle ACB + \angle ABC.$$

Since $\triangle BDC$ isosceles, in circle ω_1,

$$\angle BCD = \frac{1}{2}(180° - \angle BDC) = \frac{1}{2}(180° - \angle ACB + \angle ABC) = \frac{1}{2}\angle BAC.$$

On the other hand, in circle $O, \angle AOC = 2\angle AB$, so in circle ω_2,

$$\text{(major) } \widehat{AC} = 2\angle AOC = 4\angle ABC \Rightarrow \angle AEC = 360° - \widehat{AC} = 360° - 4\angle ABC.$$

Since $\triangle AEC$ is isosceles, in circle ω_2,

$$\angle ACE = \frac{1}{2}(180° - \angle AEC) = 2\angle ABC - 90°.$$

Therefore

$$\angle ICO = \angle ACB - (\angle BCI + \angle ACO) = 180° - \left(\frac{1}{2}\angle BAC + \angle ACB + 2\angle ABC - 90°\right)$$
$$= 270° - (\frac{1}{2}50° + 71° + 2 \cdot 59°) = \boxed{56°.}$$

□

Problem 1.4.18 (Problem 18)**.** (*5 points*) Three doctors, three nurses, and three patients stand in a line in random order. The probability that there is at least one doctor and at least one nurse between each pair of patients is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. We show a coding solution. The idea is to permute the given string 'DDDNNNPPP', then split it into smaller chunks, using the character P as a separator. The first and last chunks should be ignored. Then we verify if all chunks containing a D and an N letter.

```
from itertools import permutations
sequence = [e for e in 'DDDNNNPPP']
results = set()
for permutation in permutations(sequence):
    found = True
    chunks = (''.join(permutation)).split('P')
    chunks = chunks[1:-1] if len(chunks) == 4 else chunks[:-1] if permutation[0] ==
        'P' else chunks[1:]
    for s in chunks:
        if 'D' not in s or 'N' not in s:
            found = False
            break
    if found:
        results.add(''.join(permutation))
print(len(results))
```

The answer is $\boxed{114.}$ □

Problem 1.4.19 (Problem 19)**.** (*5 points*) Let p, q, and r be prime numbers such that $2pqr + p + q + r = 916$. Find $pq + qr + rp$.

Solution. Note that $p + q + r = 916 - 2pqr$, thus it is even, so at least one prime is even. WLOG, let $p = 2$.

$$4qr + q + r = 914 \Rightarrow (4q + 1)(4r + 1) = 4 \cdot 914 + 1 = 3657 = 3 \cdot 23 \cdot 53.$$

We are looking for pair of factors in the form of $4k + 1$ and they are $3 \cdot 23 = 69$. and 53. Easy to test and verify that the pair q, r is $13, 17$.

Hence, the desired sum is $pq + qr + rp = 2 \cdot 13 + 13 \cdot 17 + 17 \cdot 2 = \boxed{281.}$ □

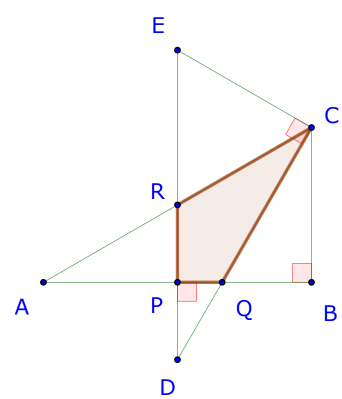
Chapter 2

January Challenges

2.1 I-Level

For both middle school ang high school students. No solution based on coding is allowed.

Problem 2.1.1 (Problem 1)**.** (*10 points*) We divide an equilateral triangle with area 600 into two halves and then arrange them as shown below. Find the area of the $PQCR$.



Solution. First let a be the side length of the equilateral triangle, then $BC = \dfrac{a}{2}, AB = \dfrac{a\sqrt{3}}{2}$. Since $\angle EDC = \angle DCB$, so $\triangle QBC \sim \triangle CBA$, thus

$$\frac{[QBC]}{[CBA]} = \left(\frac{CB}{AB}\right)^2 = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} \Rightarrow [QBC] = \frac{[CBA]}{3}.$$

Second, $\angle ECR = \angle PRC - \angle REC = \angle RAP + 90^\circ - 60^\circ = 60^\circ$. Thus $\triangle ECR$ is equilateral, and $RC = EC = CB = \frac{1}{2}AC$. because $RP \parallel CB$, so P is midpoint of AB, thus $[ARP] = \frac{1}{4}[ACB]$.

Therefore

$$[PQCR] = [ACB] - [ARP] - [QBC] = [ACB]\left(1 - \frac{1}{4} - \frac{1}{3}\right) = \frac{5}{12}[ACB] = \frac{5}{12}(300) = \boxed{125.}$$

$\square$

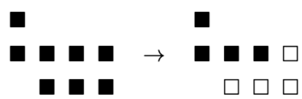
Problem 2.1.2 (Problem 2)**.** (*10 points*) Find all pairs of positive intergers (m, n) such that

$$1 \cdot 2 \cdots (m-1)(m) + 76 = n^2$$

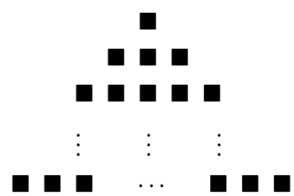
Solution. For $m \geq 7$, we have that $1 \cdot 2 \cdots (m-1)(m) = m! \equiv 0 \pmod 7$, $m! + 76 \equiv 6 \pmod 7$. For any perfect square $n^2 \equiv 0, 1, 2, 4 \pmod 7$, so there is no solution for $m \geq 7$.

For $m \leq 7$, by testing $4! + 76 = 10^2, 5! + 76 = 14^2$, and there is no solution for other value of m. Hence the solutions are $\boxed{(4, 10), (5, 14).}$ □

Problem 2.1.3 (Problem 3)**.** (*10 points*) Eight black squares are place on the plane, see the diagram below on the left. Without moving any of them, four of the squares can be coloured white, so that they are divided into *two congruent sets of different coloured squares*: a set of four black squares and a set of four white squares, as show below. *Note that one set is the mirror image of the other.*



Now, 100 black squares are placed on the plane as shown below *(19 black squares in the 10^{th} row.)*



In how many ways, without moving any of them, 50 of the squares can be can be coloured white, so that they are divided into two congruent sets of different coloured squares?

Solution. First, the bottom two corners are the only two tiles with one coordinate equal and the other differing by 18. Second, the only pairs in which both coordinates differ by 9 consists of the top corner and one of the bottom corners.

Now, suppose for a contradiction that a partition exists. By the Pigeonhole Principle, one set contains two of the three corners. By the two facts above, the other set cannot contain two tiles equivalently situated. □

Problem 2.1.4 (Problem 4)**.** (*10 points*) Let s be the sum of some integers, one of them has at least two digits. Two digits of one of the integers in s is chosen and they change places. The new sum is r. Which prime numbers 2, 3, 5, 7, 9, or 11 are always divisors of the difference $s - r$?

For example

$$s = 1 + 34 + 6 + 752 = 793,\ \underline{7}5\underline{2} \to \underline{2}5\underline{7} \Rightarrow r = 1 + 34 + 6 + 257 = 298 \Rightarrow s - r = 793 - 298 = 495.$$

Solution. First, since all different sums of integers are question, we can try a simple one to narrow down the possibility. Let $s = 21$, then $r = 12$ so $s - r = 9$. Thus the only possible primes are 3 and 9.

Now, for let n be the number in s whose two digits change their places:

$$n = \overline{\dots a \underbrace{\dots b \underbrace{\dots}_{n}}_{m}} \to \overline{\dots b \underbrace{\dots a \underbrace{\dots}_{n}}_{m}} \Rightarrow s - r = \overline{\dots a \underbrace{\dots b \underbrace{\dots}_{n}}_{m}} - \overline{\dots b \underbrace{\dots a \underbrace{\dots}_{n}}_{m}}$$

$$(a10^m + b10^n) - (b10^m + a10^n) = (10^m - 10^n)(a - b) = 10^n(9)(10^{m-n-1} + \dots + 1).$$

Thus the difference is always divisible by $\boxed{3 \text{ and } 9.}$ □

2.2 A-Level

For both middle school ang high school students. No solution based on coding is allowed.

2.3 O-Level

For both middle school ang high school students. No solution based on coding is allowed.

Chapter 3

Entrance Tests

3.1 I-Level Entrance Test

For both middle school ang high school students. You are allowed to use books and calculator. No solution based on coding is allowed. No searching on the Internet for hints or solutions is permitted. **Full and detailed solution is required for every problem.**

Problem 3.1.1 (Problem 1)**.** (*10 points*) You are given a set of 10 positive integers. Summing nine of them in ten possible ways we get only nine different sums

$$86, 87, 88, 89, 90, 91, 93, 94, 95.$$

Find those numbers.

Solution. Let S be the sum of all ten positive integers and suppose x is the repeated sum. Call the elements $a_1, a_2, \ldots, a_10$, Then we have

$$S - a_1 = 86, S - a_2 = 87, \ldots, S - a_9 = 95, S - a_10 = x.$$

Adding these, $9S = 813 + x$. The only value of x from $86, 87, 88, \ldots, 95$ which makes $813 + x$ divisible by 9 is $x = 87$ and then $S = 100$. It follows that the ten numbers are respectively $\boxed{14, 13, 12, 11, 10, 9, 7, 6, 5, \text{ and } 13.}$ $\square$

Problem 3.1.2 (Problem 2). (*10 points*) Replace the asterisks in the equilateral triangle by the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ so that, starting from the second line, each number is equal to the absolute value of the difference of the nearest two numbers in the line above.

```
* * * * * * * * *
 * * * * * * * *
  * * * * * * *
   * * * * * *
    * * * * *
     * * * *
      * * *
       * *
        *
```

Note that: for this problem, submitting the equilateral triangle with all the asterisks replaced bu the number $1, 2, \ldots, 9$ *is considered as a solution. No additional explanation is required.*

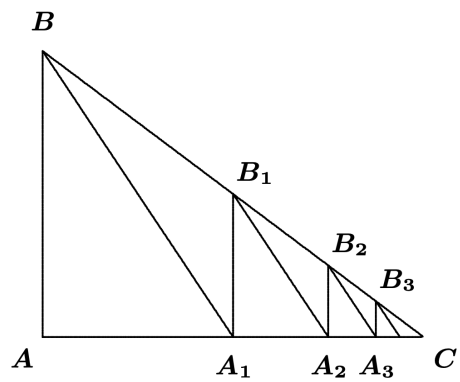
Solution. Below is one of possible solutions.

```
1   7   8   1   9   8   1   7   6
  6   1   7   8   1   7   6   1
    5   6   1   7   6   1   5
      1   5   6   1   5   4
        4   1   5   4   1
          3   4   1   3
            1   3   2
              2   1
                1
```

□

Problem 3.1.3 (Problem 3)**.** (*10 points*) In the diagram below, ABC is a right triangle, where $AB = 4, AC = 4$. Furthermore, each of the line segments $A_1B_1, A_2B_2, \dots$ is perpendicular to AC, A_1 bisects segment AC, A_2 bisects segment A_1C, A_3 bisects segment A_2C, and so on. Find the sum of the lengths:

$$BA_1 + B_1A_2 + B_2A_3 + \dots$$



Solution. Since A_1 bisects segment AC, $AA_1 = \frac{1}{2}AC = 2$, and $B_1A_1 = \frac{1}{2}BA = \frac{3}{2}$. By Pythagorean theorem,

$$BA_1 = \sqrt{BA^2 + AA_1^2} = \sqrt{3^2 + 2^2} = \sqrt{13}.$$

Each of the triangles $B_1A_1A_2$, $B_2A_2A_3, \dots$ is similar to the triangle BAA_1, with the similarity ratio is $\frac{1}{2}$:

$$\frac{1}{2} = \frac{B_1A_1}{BA} = \frac{B_2A_2}{B_1A_1} = \frac{B_3A_3}{B_2A_2} = \dots \Rightarrow \frac{1}{2} = \frac{B_1A_2}{BA_1} = \frac{B_2A_3}{B_1A_2} = \cdots$$

Therefore

$$BA_1 + B_1A_2 + B_2A_3 + \dots = BA_1 + \left(\frac{1}{2}\right) BA_1 + \left(\frac{1}{2}\right)^2 BA_1 + \dots = BA_1 \frac{1}{1 - \frac{1}{2}} = \boxed{2\sqrt{13}.}$$

□

Problem 3.1.4 (Problem 4)**.** (*10 points*) A four-digit number which is a perfect square is created by writing Lan's age in years followed by Nam's age in years. Similarly, after 31 years, their ages in the same order will again form a four-digit perfect square. Determine the present ages of Lan and Nam.

Note: for example if Lan's is 19 years old and Name is 11 years old then the four-digit number which is a perfect square is created by writing Lan's age in years followed by Nam's age in years is 1911.

Solution. After 31 years, each of these ages is a number with at least two digits, and then their ages in the same order will again form a four-digit perfect square, thus their ages then should be two-digit numbers. Since currently their ages in the same order form a four-digit perfect square, thus their present ages are also two-digit numbers.

Now let $\ell_1\ell_2$ and n_1n_2 be the their present ages, then $\ell_3\ell_4 = \ell_1\ell_2 + 31$, $n_3n_4 = n_1n_2 + 31$ are their ages after 31 years, then

$$\overline{\ell_3\ell_4n_3n_4} - \overline{\ell_1\ell_2n_1n_2} = (\overline{\ell_3\ell_4}\cdot 100 + \overline{n_3n_4}) - (\overline{\ell_1\ell_2}\cdot 100 + \overline{n_1n_2}) = (\overline{\ell_3\ell_4} - \overline{\ell_1\ell_2})\cdot 100 + (\overline{n_3n_4}) - \overline{n_1n_2}) = 31\cdot 100 + 31$$

Now, let the two perfect squares be $x^2 = \overline{\ell_1\ell_2n_1n_2}$, $y^2 = \overline{\ell_3\ell_4n_3n_4}$, then

$$y^2 - x^2 = 3131 = 31 \cdot 101 \Rightarrow \begin{cases} y - x = 31 \\ y + x = 101 \end{cases} \Rightarrow x = 35, y = 66 \Rightarrow x^2 = 1225, y^2 = 4356.$$

Thus, Lan's age now is 12, Nam's age now is 25. □

Chapter 4

Session 2: Feb 02 - Feb 10

4.1 Middle School - Assignment

Middle school students: grade 8 (US, CA), grade 9 (FR, UK, VN) and younger.

- **Submission deadline: Friday, February 9**
- **Test: Saturday, February 10**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 4.1.1 (Problem 1)**.** (*5 points*) Find the number of three-digit positive integers where the digits are three different prime numbers. For example, count 235 but not 553

Solution. There are 4 prime numbers that are digits: 2, 3, 5, and 7. They are all distinct. Thus, there are $4 \cdot 3 \cdot 2 = \boxed{24}$ three-digit positive integers where the digits are three different prime numbers. □

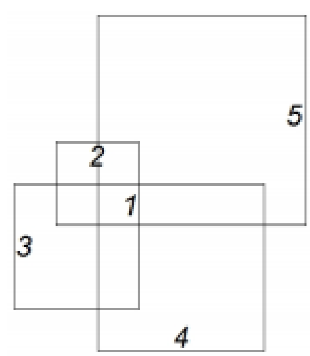
Problem 4.1.2 (Problem 2)**.** (*5 points*) Melanie has $4\frac{2}{5}$ cups of flour. The recipe for one batch of cookies calls for $1\frac{1}{2}$ cups of flour. Melanie plans to make $2\frac{1}{2}$ batches of cookies. When she is done, she will have $\frac{m}{n}$ cups of flour remaining, where m and n are relatively prime positive integers. Find $m + n$.

Solution. The number of cups of flour remaining is:

$$4\frac{2}{5} - 2\frac{1}{2} \cdot 1\frac{1}{2} = \frac{22}{5} - \frac{5}{2} \cdot \frac{3}{2} = \frac{13}{20}.$$

Hence, $m + n = 13 + 20 = \boxed{33.}$ □

Problem 4.1.3 (Problem 3)**.** (*5 points*) The figure below has a 1×1 square, a 2×2 square, a 3×3 square, a 4×4 square, and a 5×5 square. Each of the larger squares shares a corner with the 1×1 square. Find the area of the region covered by the 1×1, 2×2, 3×3, and 4×4 squares, and the 5×5 square.



Solution. Let $S_1, S_2, \ldots, S_5$ denote the area of the squares 1×1, 2×2, $\ldots$, and 5×5.

It is easy to see that the area cover by the squares 1×1, 2×2 is the square 2×2.

If we add the area cover by the squares 3×3 to it we can see that they overlap a 1×2 area, thus three of them cover $4 + 9 - 2 = 11$.

If we add the area cover by the squares 4×4 to it we can see that they overlap a 1×3 area, thus four of them cover $16 + 11 - 3 = 24$.

If we add the area cover by the squares 5×5 to it we can see that they overlap the squares 1×1, 1×1, and 1×3 thus five of them cover $24 + 25 - (1 + 1 + 3) = \boxed{44.}$ □

Problem 4.1.4 (Problem 4)**.** (*5 points*) Find the value of x such that $2^{x+3} - 2^{x-3} = 2016$.

Solution.

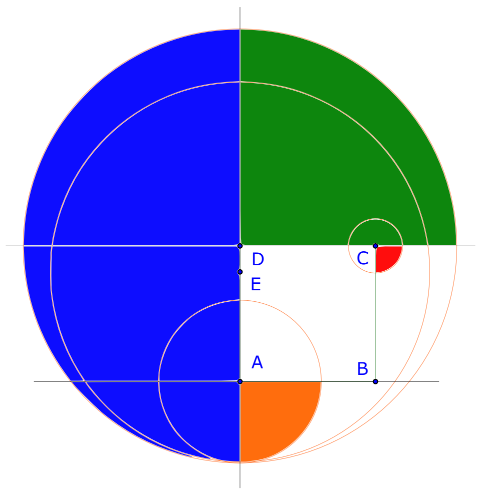
$$2016 = 2^{x+3} - 2^{x-3} = 2^{x-3}(2^6 - 1) \Rightarrow 2^{x-3} = 32 \Rightarrow x = \boxed{8}.$$

□

Problem 4.1.5 (Problem 5)**.** (*5 points*) Mildred the cow is tied with a rope to the side of a square shed with side length 10 meters. The rope is attached to the shed at a point two meters from one corner of the shed. The rope is 14 meters long. The area of grass growing around the shed that Mildred can reach is given by $n\pi$ square meters, where n is a positive integer. Find n.

Solution. The total area is the sum of the areas of:

- half of the blue circle centred at E, where the rope is tied to, with radius 14;
- a quarter of the orange circle centred at A with radius $14 - 8 = 6$;
- a quarter of the green circle centred at D with radius $14 - 2 = 12$;
- a quarter of the red circle centred at C with radius $14 - 2 - 10 = 2$.



Thus, $\left(\frac{14^2}{2} + \frac{6^2}{4} + \frac{12^2}{4} + \frac{2^2}{4}\right)\pi = \boxed{144}\pi.$ □

Problem 4.1.6 (Problem 6)**.** (*5 points*) One evening a theater sold 300 tickets for a concert. Each ticket sold for \$40, and all tickets were purchased using \$5, \$10, and \$20 bills. At the end of the evening the theater had received twice as many \$10 bills as \$20 bills, and 20 more \$5 bills than \$10 bills. How many bills did the theater receive altogether?

Solution. Let n be the number of \$20 bills the theater received. Then the number of \$10 bills they received is $2n$, and the number of \$5 bills they received is $2n + 20$, for a total amount collected equalling

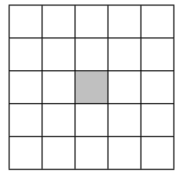
$$5(2n + 20) + 10(2n) + 20n = 300 \cdot 40 \Rightarrow 50n + 100 = 12000.$$

The amount of bills is

$$n + 2n + 2n + 20 = 5n + 20 = \frac{12000}{10} + 10 = \boxed{1210.}$$

□

Problem 4.1.7 (Problem 7)**.** (*5 points*) Find the number of squares such that the sides of the square are segments in the following diagram and where the shaded area is inside the square.



Solution. To count the square let's seletc one of the three vertical line segments to the left of the shaded square and one of the 3 vertical line segments to the right.

Of these $3 \cdot 3 = 9$, ways to select vertical line segments. 1 selection gives segments a distance 1 apart, 2 selections give segments a distance 2 apart, 3 selections give segments a distance 3 apart, 2 selections give segments a distance 4 apart, and 1 selection gives segments a distance 5 apart.

There is the same number of ways of selecting horizontal line segments a distance k apart that make up the side of a square containing the shaded square as there are selecting vertical line segments a distance k apart, so the total number of ways of selecting line segments that make up the sides of a square containing the shaded area is $1^2 + 2^2 + 3^2 + 2^2 + 1^2 = \boxed{19.}$ □

Problem 4.1.8 (Problem 8)**.** (*5 points*) One afternoon Elizabeth noticed that twice as many cars on the expressway carried only a driver as compared to the number of cars that carried a driver and one passenger.

She also noted that twice as many cars carried a driver and one passenger as those that carried a driver and two passengers.

10% of the cars carried a driver and three passengers, and no car carried more than four people.

Any car containing at least three people was allowed to use the fast lane. Elizabeth calculated that $\frac{m}{n}$ of the people in cars on the expressway were allowed to ride in the fast lane, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Let n be the number of the cars that have a driver and two passengers, then

Car type	Number of cars	Number of people
Only a driver	$4n$	$1 \cdot 4n$
A driver and one passenger	$2n$	$2 \cdot 2n$
A driver and two passengers	n	$3 \cdot n$
A driver and three passengers	$\frac{n+2n+4n}{9}$	$4 \cdot \frac{7n}{9}$

The number of people in cars that can use the fast lane is the number of people in cars carrying a driver and two or three passengers, which is $3n + \frac{28n}{9} = \frac{55n}{9}$. The total number of people is $4n + 4n + 3n + \frac{28n}{9} = \frac{127n}{9}$.

The ratio of the people in cars were allowed to ride in the fast lane is $\frac{55n}{9} : \frac{127n}{9} = \frac{55}{127}$.

Hence, $m + n = 55 + 127 = \boxed{182.}$ □

Problem 4.1.9 (Problem 9)**.** (*5 points*) Find the number of positive integers n such that a regular polygon with n sides has internal angles with measures equal to an integer number of degrees.

Solution. The measure of an internal angle, in degrees, of a regular polygon with n sides is:

$$\frac{(n-2)180}{n} = 180 - \frac{360}{n}.$$

This measure is an integer if and only if n is at least 3 and it is a divisor of $360 = 2^3 \cdot 3^2 \cdot 5$.

Since 360 has $(3+1)(2+1)(1+1) = 24$ divisors, and $\boxed{22}$ of them are at least 3. □

Problem 4.1.10 (Problem 10)**.** (*5 points*) The real numbers x, y, and z satisfy the system of equations

$$\begin{cases} x^2 + 27 = -8y + 10z \\ y^2 + 196 = 18z + 13x \\ z^2 + 119 = -3x + 30y. \end{cases}$$

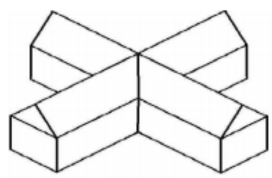
Find $x + 3y + 5z$.

Solution. Adding the equations yields

$$x^2 + y^2 + z^2 + 342 = 10x + 22y + 28z \Rightarrow (x-5)^2 + (y-11)^2 + (z-14)^2 = 0 \Rightarrow x = 5, y = 11, z = 14.$$

Hence, $x + 3y + 5z = 5 + 3 \cdot 11 + 5 \cdot 14 = \boxed{108.}$ □

Problem 4.1.11 (Problem 11)**.** (*5 points*) The figure below shows a barn in the shape of two congruent pentagonal prisms that intersect at right angles and have a common center. The ends of the prisms are made of a 12 foot by 7 foot rectangle surmounted by an isosceles triangle with sides 10 feet, 10 feet, and 12 feet. Each prism is 40 feet long. Find the volume of the barn in cubic feet.



Solution. The volume is the sum of the volume of the 2 pentagonal prisms minus the common center area.

Lets split the pentagonal prism into 2 parts: the top part which is where the isoceles triangle is and the bottom part which is the rectangle underneath the isoceles triangle. The top part is a triangular prism with a triangular base $10, 10, 12$, height 40, so its volume is 1920. The bottom part is a box with side lengths $7, 12, 40$, thus its volume is $7 \cdot 12 \cdot 40 = 3360$. Consequently, the volume of a pentagonal prism is $1920 + 3360 = 5280$.

Now all we need to find the volume of the common center. We can split the common center place into 2 parts as well. The top part which is a pyramid and the bottom which is a box. The top part volume is a pyramid with base side length 12, height 8, so its volume is $\frac{12 \cdot 12 \cdot 8}{3} = 384$. The bottom part is a box which has side lengths $12, 12, 7$ thus its volume is $12 \cdot 12 \cdot 7 = 1008$ Therefore, the total volume of the common center is $1008 + 384 = 1392$.

Thus, the answer is $2 \cdot 5280 - 1392 = \boxed{9168}$. □

Problem 4.1.12 (Problem 12)**.** (*5 points*) Suzie flips a fair coin 6 times. The probability that Suzie flips 3 heads in a row but not 4 heads in a row is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Three consecutive head, but not four:

$$\text{HHHTXX}, \ \text{THHHTX}, \ \text{XTHHHT}, \ \text{XXTHHH}$$

where H denote head, T denote tail and X denote either head or tail.

There are 2 string of the format THHHTX, because X can be fill in 2 ways, 2 string of the format XTHHHT, because X can be fill in 2 ways, $2 \cdot 2$ strings of HHHTXX, and $2 \cdot 2$ strings of XXTHHH. Thus total number of favourable cases are $2 + 2 + 4 + 4 = 12$.

Total possible outcome is $2^6 = 64$. Hence probability is $\frac{12}{64} = \frac{3}{16}$.

Hence $m = 3, n = 16$ and $m + n = \boxed{19}$. □

Problem 4.1.13 (Problem 13)**.** (*5 points*) Find the least positive integer N that is 50 times the number of positive integer divisors that N has.

Solution. Let $N = p_1^a p_2^b \cdots$ for distinct prime numbers $p_1, p_2, \ldots$ and $a, b, \ldots$ positive integers. Then N has $(a+1)(b+1)\cdots$ divisors. Since $50 = 2 \cdot 5^2$ so the *least* N must have

$$p_1^a p_2^b \cdots = 2^1 5^2 (a+1)(b+1)\cdots$$

Thus two of the primes $p_1, p_2, \ldots$ must be 2 and 5.

Case 1: $N = 2^a 5^b$.

$$N = 2^a 5^b = 2^1 5^2 (a+1)(b+1) \Rightarrow b \geq 2.$$

Case 1a: $b = 2$. Then $3 \mid 2^a 5^b$, impossible.

Case 1b: $b = 3$. Then $5^b = 5^3$, thus $5 \mid a+1$, or $a \geq 4$, so $N \geq 2^4 5^3 = 2000$.

Case 1c: $b \geq 4$. It is easy to prove that $5^b \geq 5(b+1)$, thus $a + 1 \geq 10$, or $a \geq 9$, so $N \geq 2^9 5^2 > 2000$.

Case 2: $N = 2^a 3^b 5^c$.

$$N = 2^a 3^b 5^c = 2^1 5^2 (a+1)(b+1)(c+1).$$

With the least possible of $c = 2$, then $b = 2, a = 3$, then $N = \boxed{1800}$ is a solution.

For $N = 2^a 5^b p^c$, where $p \geq 7$ leads to $N > 2000$. □

Problem 4.1.14 (Problem 14)**.** (*5 points*) Find the positive integer n such that the least common multiple of n and $n - 30$ is $n + 1320$.

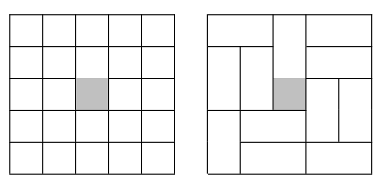
Solution. First, since $n + 1320 = \text{lcm}(n, n-30)$, so $n(n-30) \geq n + 1320$, or $n \geq 55$.

Second, $n \mid n + 1320$, so $n \mid 1320$, the divisors of 1320 that are at least 55 are 55, 60, 66, 88, 110, 120, 132, 165, 220, 264, 330, 440, 660, and 1320.

Third, $n - 30 \mid (n + 1320) - (n - 30) = 1350$. Only $n = 55, 60, 120$, and 165 satisfies $n - 30 \mid 1350$.

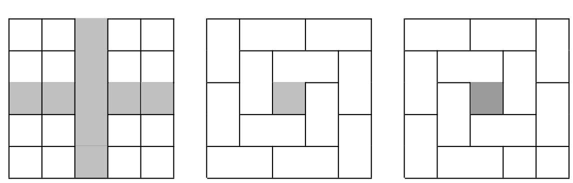
By testing, it is easy to see that only $n = \boxed{165}$ satisfies $n + 1320 = \text{lcm}(n, n-30)$. □

Problem 4.1.15 (Problem 15)**.** (*5 points*) The 24 unshaded squares in the 5×5 grid below can be tiled with twelve 1×2 tiles. One such tiling is shown below.



Find the number of ways the grid can be tiled.

Solution. Call the two-squares pairs that align with the shaded central square *bridges* which are shaded in the left diagram below. An organized way to count the tilings is to consider how many of these four bridges are tiled by a single tile.

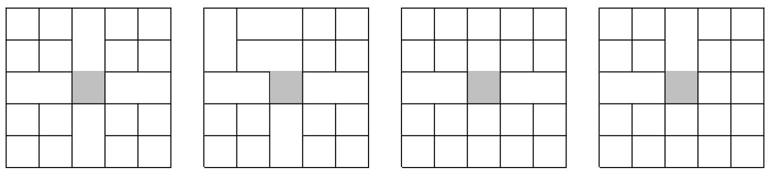


Case 1: First, note that if any one bridge is tiled by two different tiles that do not aligned, then there is only one way to complete the tiling of remaining unshaded squares. Thus, there are only two tilings of this type, both shown above.

Case 2: If all four bridge are each tiled by a single tile as in the first diagram below, it leaves four 2×2 grids which can each be tiled in 2 different ways. Thus, this accounts for $2^4 = 16$ tilings.

Case 3: If exactly three bridges are each tiled by a single tile as shown in the second diagram below, then there are 4 ways to select the one bridge that is not tiled by a single tile. There are two ways to tile the bridge that is tiled by two tiles. This leaves three 2×2 grids that can each be tiled in 2 ways. Thus, this accounts for $4 \cdot 2 \cdot 2^3 = 64$ tilings.

Case 4: If exactly two bridges are each tiled by a single tile, then the two bridges tiled by a single tile can be across the center from each other as shown in the third diagram below, or the two bridges tiled by a single tile can be adjacent bridges as shown in the fourth diagram below.



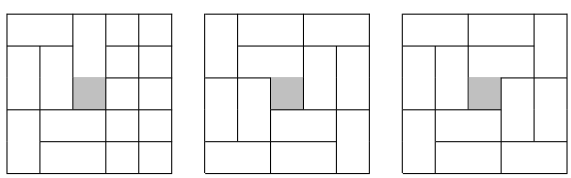
In the first case, there are 2 ways to select the two bridges each tiled by a single tile, and that leaves two 5×2 grids which can each be tiled in one of 4 ways (2 ways to tile the included bridge and 2 ways to tile a 2×2 grid that remains.) This accounts for $2 \cdot 4^2 = 32$ tilings.

In the second case, there are 4 ways to select the two bridges each tile by a single bridge. That leaves 3 ways to tile the remaining two bridges, and each of these 3 ways leaves two 2×2 grids, each to be tiled in one of 2 ways. This accounts for $4 \cdot 3 \cdot 2^2 = 48$. tilings.

Thus, there are $32 + 48 = 80$ tilings where exactly two bridges are each tiled by a single tile.

Case 5: If exactly one bridge is tile by a single tile as shown in the first diagram below, then there are 4 ways to select the one bridge that is tiled by a single tile. There are then two ways to tile the bridge on the opposite side of the center square. This determines how to tile the grid, except for on 5×2 grid that can be tiles in one of 4 ways. This accounts for $4 \cdot 2 \cdot 4 = 32$ tilings.

Case 6: If none of the bridges are tiled by a single tile, there are two ways to tile the grid as shown in the second and third diagrams below accounting for 2 tilings.



Hence, the number of tilings is $2 + 16 + 64 + 80 + 32 + 2 = \boxed{196.}$ □

4.2 Middle School - Test

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 4.2.1 (Problem 1). (*5 points*) Find the number of four-digit positive integers where the digits are four different prime numbers. For example, count 2357 but not 5537

Solution. There are 4 prime numbers that are digits: 2, 3, 5, and 7. They are all distinct. Thus, there are $4 \cdot 3 \cdot 2 \cdot 1 = \boxed{24}$ three-digit positive integers where the digits are three different prime numbers. □

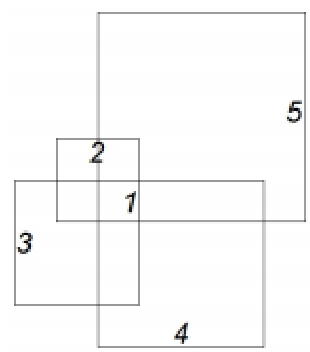
Problem 4.2.2 (Problem 2). (*5 points*) Melanie has $3\frac{4}{5}$ cups of flour. The recipe for one batch of cookies calls for $1\frac{1}{2}$ cups of flour. Melanie plans to make $2\frac{1}{2}$ batches of cookies. When she is done, she will have $\frac{m}{n}$ cups of flour remaining, where m and n are relatively prime positive integers. Find $m + n$.

Solution. The number of cups of flour remaining is:

$$3\frac{4}{5} - 2\frac{1}{2} \cdot 1\frac{1}{2} = \frac{19}{5} - \frac{5}{2} \cdot \frac{3}{2} = \frac{1}{20}.$$

Hence, $m + n = 1 + 20 = \boxed{21.}$ □

Problem 4.2.3 (Problem 3). (*5 points*) The figure below has a 1×1 square, a 2×2 square, a 3×3 square, a 4×4 square, and a 5×5 square. Each of the larger squares shares a corner with the 1×1 square. Find the area of the region covered by the 1×1, 2×2, 3×3, and 4×4 squares, but not covered by the 5×5 square.



Solution. Let $S_1, S_2, \ldots, S_5$ denote the area of the squares 1×1, 2×2, ..., and 5×5.

It is easy to see that the area cover by the squares 1×1, 2×2 is the square 2×2.

If we add the area cover by the squares 3×3 to it we can see that they overlap a 1×2 area, thus three of them cover $4 + 9 - 2 = 11$.

If we add the area cover by the squares 4×4 to it we can see that they overlap a 1×3 area, thus four of them cover $16 + 11 - 3 = 24$.

Now, we need to remove the area covered by the square 5×5 overlapping with it, and it is a 1×1, a 1×1, and a 1×3 thus $24 - 1 - 1 - 3 = \boxed{19.}$ □

Problem 4.2.4 (Problem 4). (*5 points*) Find the value of x such that $2^{x+5} - 3 \cdot 2^{x-3} = 2024$.

Solution.

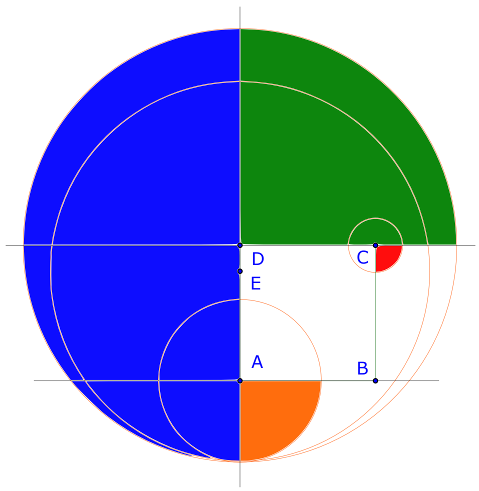
$$2024 = 2^{x+5} - 3 \cdot 2^{x-3} = 2^{x-3}(2^8 - 3) \Rightarrow 2^{x-3} = 8 \Rightarrow x = \boxed{6}.$$

□

Problem 4.2.5 (Problem 5). (*5 points*) Mildred the cow is tied with a rope to the side of a square shed with side length 10 meters. The rope is attached to the shed at a point two meters from one corner of the shed. The rope is 13 meters long. The area of grass growing around the shed that Mildred can reach is given by $\frac{m\pi}{n}$ square meters, where m and n are relatively prime positive integers. Find $m + n$.

Solution. The total area is the sum of the areas of:

- half of the blue circle centred at E, where the rope is tied to, with radius 13;
- a quarter of the orange circle centred at A with radius $13 - 8 = 5$;
- a quarter of the green circle centred at D with radius $13 - 2 = 11$;
- a quarter of the red circle centred at C with radius $13 - 2 - 10 = 1$.



Thus, $\left(\frac{13^2}{2} + \frac{5^2}{4} + \frac{11^2}{4} + \frac{1^2}{4}\right)\pi = \frac{485\pi}{4}$.

Hence $m + n = 485 + 4 = \boxed{489.}$ □

Problem 4.2.6 (Problem 6). (*5 points*) One evening a theater sold 300 tickets for a concert. Each ticket sold for \$40, and all tickets were purchased using \$5, \$10, and \$20 bills. At the end of the evening the theater had received twice as many \$10 bills as \$20 bills, and 10 more \$5 bills than \$10 bills. How many bills did the theater receive altogether?

Solution. Let n be the number of \$20 bills the theater received. Then the number of \$10 bills they received is $2n$, and the number of \$5 bills they received is $2n + 10$, for a total amount collected equalling

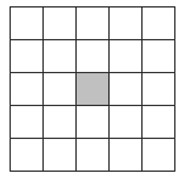
$$5(2n + 10) + 10(2n) + 20n = 300 \cdot 40 \Rightarrow 50n + 50 = 12000.$$

The amount of bills is

$$n + 2n + 2n + 10 = 5n + 10 = \frac{12000}{10} + 5 = \boxed{1205.}$$

□

Problem 4.2.7 (Problem 7)**.** (*5 points*) The 5×5 grid below is made up by *unit* squares. The center square is shaded. Find the number of squares with side lengths 1, 3, or 5, such that the sides of the square are segments in the grid, and the shaded area is inside the square.



Solution. To count the square let's select one of the three vertical line segments to the left of the shaded square and one of the 3 vertical line segments to the right.

Of these $3 \cdot 3 = 9$, ways to select vertical line segments. 1 selection gives segments a distance 1 apart, 2 selections give segments a distance 2 apart, 3 selections give segments a distance 3 apart, 2 selections give segments a distance 4 apart, and 1 selection gives segments a distance 5 apart.

There is the same number of ways of selecting horizontal line segments a distance k apart that make up the side of a square containing the shaded square as there are selecting vertical line segments a distance k apart, so the total number of ways of selecting line segments that make up the sides of a square with **odd area** containing the shaded area is $1^2 + 3^2 + 1^2 = \boxed{11.}$ □

Problem 4.2.8 (Problem 8)**.** (*5 points*) One afternoon Elizabeth noticed that twice as many cars on the expressway carried only a driver as compared to the number of cars that carried a driver and one passenger.

She also noted that twice as many cars carried a driver and one passenger as those that carried a driver and two passengers.

10% of the cars carried a driver and three passengers, and no car carried more than four people.

Any car containing at least three people was allowed to use the fast lane. Elizabeth calculated that $\frac{m}{n}$ of the people in cars on the expressway were **not** allowed to ride in the fast lane, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Let n be the number of the cars that have a driver and two passengers, then

Car type	Number of cars	Number of people
Only a driver	$4n$	$1 \cdot 4n$
A driver and one passenger	$2n$	$2 \cdot 2n$
A driver and two passengers	n	$3 \cdot n$
A driver and three passengers	$\frac{n+2n+4n}{9}$	$4 \cdot \frac{7n}{9}$

The number of people in cars that can not use the fast lane is the number of people in cars carrying a driver and at most one passenger, which is $4n + 4n = 8n$. The total number of people is $4n + 4n + 3n + \frac{28n}{9} = \frac{127n}{9}$.

The ratio of the people in cars were allowed to ride in the fast lane is $8n : \frac{127n}{9} = \frac{72}{127}$.

Hence, $m + n = 72 + 127 = \boxed{199.}$ □

Problem 4.2.9 (Problem 9)**.** (*5 points*) Find the number of **odd** positive integers n such that a regular polygon with n sides has internal angles with measures equal to an integer number of degrees.

Solution. The measure of an internal angle, in degrees, of a regular polygon with n sides is:

$$\frac{(n-2)180}{n} = 180 - \frac{360}{n}.$$

This measure is an integer if and only if n is at least 3 and it is a divisor of $360 = 2^3 \cdot 3^2 \cdot 5$.

Since n must be odd, so n is a divisor of $3^2 \cdot 5$, and there are $(2+1)(1+1) = 6$ such divisors, and $\boxed{5}$ of them are at least 3: $3, 5, 9, 15$, and 45. $\square$

Problem 4.2.10 (Problem 10)**.** (*5 points*) The real numbers x, y, and z satisfy the system of equations

$$\begin{cases} x^2 + 27 = -8y + 10z \\ y^2 + 196 = 18z + 13x \\ z^2 + 119 = -3x + 30y. \end{cases}$$

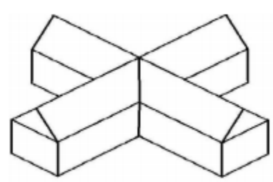
Find $5x + 3y + z$.

Solution. Adding the equations yields

$$x^2 + y^2 + z^2 + 342 = 10x + 22y + 28z \Rightarrow (x-5)^2 + (y-11)^2 + (z-14)^2 = 0 \Rightarrow x = 5, y = 11, z = 14.$$

Hence, $5x + 3y + z = 5 \cdot 5 + 3 \cdot 11 + 14 = \boxed{72.}$ $\square$

Problem 4.2.11 (Problem 11)**.** (*5 points*) The figure below shows a barn in the shape of two congruent pentagonal prisms that intersect at right angles and have a common center. The ends of the prisms are made of a 6 foot by 4 foot rectangle surmounted by an isosceles triangle with sides 5 feet, 5 feet, and 6 feet. Each prism is 40 feet long. Find the volume of the barn in cubic feet.



Solution. The volume is the sum of the volume of the 2 pentagonal prisms minus the common center area.

Lets split the pentagonal prism into 2 parts: the top part which is where the isoceles triangle is and the bottom part which is the rectangle underneath the isoceles triangle. The top part is a triangular prism with a triangular base $5, 5, 6$, height 40, so the area of the triangle is 12, its volume is 480.

The bottom part is a box with side lengths $4, 6, 40$, thus its volume is $4 \cdot 6 \cdot 40 = 960$. Consequently, the volume of a pentagonal prism is $480 + 960 = 1440$.

Now all we need to find the volume of the common center. We can split the common center place into 2 parts as well. The top part which is a pyramid and the bottom which is a box. The top part volume is a pyramid with base side length 6, height 4, so its volume is $\frac{6 \cdot 6 \cdot 4}{3} = 48$. The bottom part is a box which has side lengths $6, 6, 4$ thus its volume is $6 \cdot 6 \cdot 4 = 144$ Therefore, the total volume of the common center is $48 + 144 = 192$.

Thus, the answer is $2 \cdot 1440 - 192 = \boxed{2688}$. $\square$

Problem 4.2.12 (Problem 12)**.** (*5 points*) Suzie flips a fair coin 7 times. The probability that Suzie flips 4 heads in a row but not 5 heads in a row is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Four consecutive heads, but not five:

$$\text{HHHHTXX},\ \text{THHHHTX},\ \text{XTHHHHT},\ \text{XXTHHHH},$$

where H denote head, T denote tail and X denote either head or tail.

There are 4 string of the format HHHHTXX because XX can be fill in 4 ways, 2 string of the format THHHHTX because X can be fill in 2 ways, 2 strings of XTHHHHT and 4 strings of XXTHHHH. Thus total number of favourable cases are $4 + 2 + 2 + 4 = 12$.

Total possible outcome is $2^7 = 128$. Thus, the probability is $\frac{12}{128} = \frac{3}{32}$.

Hence $m = 3, n = 32$ and $m + n = \boxed{35}$. □

Problem 4.2.13 (Problem 13)**.** (*5 points*) Find the least positive integer N that is 49 times the number of positive integer divisors that N has.

Solution. [Solution 1] Let $N = p_1^a p_2^b \cdots$ for distinct prime numbers $p_1, p_2, \ldots$ and $a, b, \ldots$ positive integers. Then N has $(a+1)(b+1)\cdots$ divisors. Since $50 = 2 \cdot 5^2$ so the *least* N must have

$$p_1^a p_2^b \cdots = 7^2(a+1)(b+1)\cdots$$

Thus one of the primes $p_1, p_2, \ldots$ must be 7. *Case 1*: $N = 2^a 7^b$.

$$N = 2^a 7^b = 7^2(a+1)(b+1).$$

Case 1a: $b = 2$. Then $3 \mid 2^a 7^b$, impossible.

Case 1b: $b = 3$. Then $7^b = 7^3$, thus $7 \mid a + 1$, or $a \geq 6$, With $a = 6$ then $N \geq 2^6 7^3 > 2000$.

Case 1c: $b \geq 4$. Then $N > 2000$.

Case 2: $N = 3^a 7^b$.

$$N = 3^a 7^b = 7^2(a+1)(b+1).$$

For $b = 2 : 3^a = (a+1)3$, so $a = 2$ thus $N = \boxed{441}$. □

Solution. [Solution 2] We show a coding solution. The *count_divisor* function count all divisors of n between 2 and $\left\lfloor \frac{n}{2} \right\rfloor$.

```
def count_divisors(n):
    c = 2
    for i in range(2, n+1 // 2):
        if n % i == 0:
            c += 1
    return c

n = 1
while True:
    n_dc = count_divisors(n)
    if n_dc * 49 == n:
        print(n)
        break
    n += 1
```

Thus $N = \boxed{441}$. □

Problem 4.2.14 (Problem 14)**.** (*5 points*) Find the positive integer n such that the least common multiple of n and $n-30$ is $n+4900$.

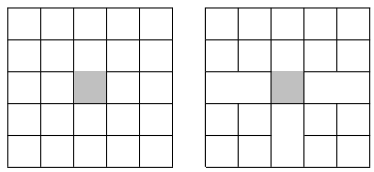
Solution. First, since $n+4900 = \text{lcm}(n, n-30)$, so $n(n-30) \geq n+4900$, or $n \geq 87$.

Second, $n \mid n+4900$, so $n \mid 4900$, the divisors of 4900 that are at least 87 are 98, 100, 140, 175, 196, 245, 350, 490, 700, 980, 1225, 2450, and 4900.

Third, $n-30 \mid (n+4900)-(n-30) = 4930$. The divisors of 4930 that are at least 57 are 58, 85, 145, 170, 290, 493, 986, 3465, and 4930.

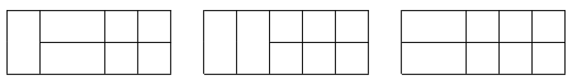
Only $n = \boxed{175}$ satisfies $n-30 \mid 4930$. □

Problem 4.2.15 (Problem 15)**.** (*5 points*) The 24 unshaded squares in the 5×5 grid below (the left diagram) are already tiled by three 1×2 tiles (the right diagram).



Find the number of ways the rest of the grid can be tiled by nine 1×2 tiles.

Solution. For the upper left 2×5 grid as shown below, we can consider three cases, as show below.



The first case has 2 ways to tile the right 2×2 grid. For each of the second and third cases there are 3 ways to tile the right 2×3 grid. Thus, there 8 ways to tile this 2×5 grid. Each of the 2×2 grids on the bottom left and bottom right can each be tiled in 2 ways. Hence, the number of tilings is $8 \times 2 \times 2 = \boxed{32.}$ □

4.3 High School - Assignment

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

- **Submission deadline: Friday, February 9**
- **Test: Saturday, February 10**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**
- **Olympiad (O) level: Problems 12-21**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 4.3.1 (Problem 1)**.** (*5 points*) Jeremy wrote all the three-digit integers from 100 to 999 on a blackboard. Then Allison erased each of the 2700 digits Jeremy wrote and replaced each digit with the square of that digit. Thus, Allison replaced every 1 with a 1, every 2 with a 4, every 3 with a 9, every 4 with a 16, and so forth. The proportion of all the digits Allison wrote that were ones is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Jeremy wrote 2700 digits. If he had also written $000, 001, 002, \ldots, 099$, he would have written 3000 digits in all which would have consisted of 300 of each of the 10 digits. In the list of those 100 numbers that he did not write, each nonzero digit appears $\frac{200}{10} = 20$ times, and 0 appear $20 + 100 = 120$ times.

Thus, among the digits Jeremy actually wrote, the digit 0 appears $300 - 120 = 180$ times, and each of the digits each appear $300 - 20 = 280$ times. Allison replaces each $0, 1, 2,$ and 3 with one-digit numbers, and each of $4, 5, 6, 7, 8, 9$ with a two-digit number, so Allison wrote a total of

$$180 + 3 \cdot 280 + 2 \cdot 6 \cdot 280 = 180 + 15 \cdot 280 \text{ digits.}$$

Allison wrote one digit 1 each time she squared 1, 4, and 9 accounting for $3 \cdot 280$ digits that were 1. The required proportion is,

$$\frac{3 \cdot 280}{180 + 15 \cdot 280} = \frac{28}{6 + 5 \cdot 28} = \frac{14}{73}.$$

Hence, $m + n = 14 + 73 = \boxed{87.}$ □

Problem 4.3.2 (Problem 2)**.** (*5 points*) Find the number of three-digit positive integers which have three distinct digits where the sum of the digits is an even number such as 925 and 824.

Solution. If the hundred digit is even, then the other two digit must be both odd or even. There are 4 choices for the hundred digit and $5 \cdot 4 + 4 \cdot 3 = 32$ ways to choose the following digits, (5 choices for the odd tens then 4 choices for the odd unit digit or 4 choices for the even tens and 3 choices for the even unit digit), altogether $4 \cdot 32 = 128$ numbers.

If the hundred digit is odd, then it is followed by one even and one odd digit. There are 5 choices for the hundred digit and $5 \cdot 4 + 4 \cdot 5 = 40$ ways to choose the following digits, altogether $5 \cdot 40 = 200$.

Hence, there are $128 + 200 = \boxed{328}$ such numbers. □

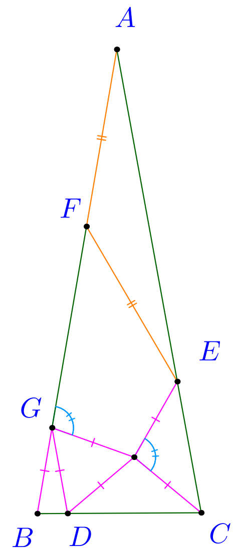
Problem 4.3.3 (Problem 3)**.** (*5 points*) Find the number whose reciprocal is the sum of the reciprocal of $9 + 15i$ and the reciprocal of $9 - 15i$.

Solution.

$$\frac{1}{9 + 15i} + \frac{1}{9 - 15i} = \frac{18}{9^2 + 15^2} = \frac{1}{17}.$$

The number is $\boxed{17.}$ □

Problem 4.3.4 (Problem 4)**.** (*5 points*) In $\triangle ABC$, $AB = AC$, $AF = EF$, $\angle CHE = \angle FGH$, and $EH = CH = DH = GH = DG = BG$. Find $\angle BAC$.



Solution. Let $x = \angle ABC$,

$$\triangle BAC \sim \triangle BGD \Rightarrow \angle BDG = \angle DBG = x\angle BGD + \angle GDH + \angle HDC = 180^\circ \Rightarrow \angle HDC = 120^\circ - x$$
$$\angle HCD = \angle HDC,\ \angle ECH = \angle ACB - \angle HCD \Rightarrow \angle ECH = x - (120^\circ - x) = 2x - 120^\circ$$
$$\angle EHC = 180^\circ - 2(2x - 120^\circ) = 420^\circ - 4x,\ \angle FGH = 180^\circ - \angle HGD - \angle DGB$$
$$\Rightarrow \angle HGF = 180^\circ - 60^\circ - (180^\circ - 2x) = 2x - 60^\circ$$
$$\angle CHE = \angle FGH \Rightarrow 420^\circ - 4x = 2x - 60^\circ \Rightarrow x = 80^\circ$$

Therefore, $\boxed{\angle BAC = 180^\circ - 2 \cdot 80^\circ = 20^\circ.}$ □

Problem 4.3.5 (Problem 5)**.** (*5 points*) Find the greatest possible value of $pq + r$, where p, q, and r are (not necessarily distinct) prime numbers satisfying $pq + qr + rp = 2016$.

Solution. If each of p, q, and r is an odd prime, then $pq + qr + rp$ is odd, thus cannot be equal to 2016.

If $p = 2$, then $qr + 2(q + r) = 2016$, so qr is even, so q or r is even. If $q = 2$ then $4 + 4r = 2016 \Rightarrow r = 503$.

Now, in order to maximize $pq + r$, p or q must be 503, and $pq + r = 2(503 + 1) = \boxed{1008.}$ □

Problem 4.3.6 (Problem 6)**.** (*5 points*) Find the least positive integer of the form $\overline{abaaba}$, where a and b are distinct digits, such that the integer can be written as a product of six distinct primes.

Solution. Let $n = \overline{aba}$, then the given number is $1001n = 7 \cdot 11 \cdot 13 \cdot n$. Thus n must be the least three-digit number that is a product of three distinct primes other than 7, 11, 13.

Now if $a = 1$, then n is odd and $n \geq 3 \cdot 5 \cdot 17 = 255 > \overline{1b1} = n$.

If $a = 2$, then n is even so one of the prime factor of n is 2. 5 cannot be a factor of n since it ends in 2. So $n \geq 2 \cdot 17 \cdot 19 = 646 > \overline{2n2}$. It leaves the only possible case if n has 2 and 3 as prime factors. Now $\frac{300}{6} = 50$, $\frac{200}{6} > 33$, so we can test the primes between 37 and 47. It turns out that $282 = 2 \cdot 3 \cdot 47$ is what we are looking for.

Thus $n = \boxed{282282.}$ □

Problem 4.3.7 (Problem 7)**.** (*5 points*) Henry rolls a fair die. If the die shows the number k, Henry will then roll the die k more times. The probability that Henry will never roll a 3 or a 6 either on his first roll or on one of the k subsequent rolls is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. In order to never roll a 3 or 6, Henry needs to roll $k = 1, 2, 4$, or 5 on his first roll, and then no 3 or 6, on any of the k subsequent rolls.

The probability that Henry does not roll a 3 or 6, on a single throw is $\frac{4}{6} = \frac{2}{3}$. Thus, the probability that Henry will never roll a 3 or a 6 on any of his roll is

$$\frac{1}{6}\left(\frac{2}{3}\right)^1 + \frac{1}{6}\left(\frac{2}{3}\right)^2 + \frac{1}{6}\left(\frac{2}{3}\right)^4 + \frac{1}{6}\left(\frac{2}{3}\right)^5 = \frac{175}{729}.$$

Hence, $m + n = 175 + 729 = \boxed{904.}$ □

Problem 4.3.8 (Problem 8)**.** (*5 points*) The cubic polynomials $p(x)$ and $q(x)$ satisfy

$$\begin{cases} p(1) = q(2) \\ p(3) = q(4) \\ p(5) = q(6) \\ p(7) = q(8) + 13 \end{cases}$$

Find $p(9) - q(10)$.

Solution. Since $p(x)$ and $q(x)$ are cubic polynomials, thus $r(x) = p(x) - q(x+1)$ is also a cubic polynomial. Furthermore $r(1) = r(3) = r(5) = 0$, thus $1, 3$, and 5 are all of its roots, therefore:

$$r(x) = a(x-1)(x-3)(x-5), \text{ where } a \text{ is a real constant.}$$

$r(7) = 13 \Rightarrow a(6)(4)(2) = 13 \Rightarrow a = \frac{13}{48}$.

Hence, $p(9) - q(10) = r(10) = \frac{13}{48}(8)(6)(4) = \boxed{52.}$ □

Problem 4.3.9 (Problem 9)**.** (*5 points*) The Tasty Candy Company always puts the same number of pieces of candy into each one-pound bag of candy they sell. Mike bought 4 one-pound bags and gave each person in his class 15 pieces of candy. Mike had 23 pieces of candy left over. Betsy bought 5 one-pound bags and gave 23 pieces of candy to each teacher in her school. Betsy had 15 pieces of candy left over. Find the least number of pieces of candy the Tasty Candy Company could have placed in each one-pound bag.

Solution. Let n be the number of pieces of candies in each bag. Then Mike's purchase shows that $4n \equiv 23 \pmod{15}$, and Betsy purchase shows that $5n \equiv 15 \pmod{23}$. Multiplying the first congruence by 4 shows that

$$16n = 4 \cdot 4n \equiv 4 \cdot 23 = 92 \pmod{15} \Rightarrow n \equiv 2 \pmod{15}.$$

Dividing the second congruence by 5,

$$n \equiv 3 \pmod{23}.$$

There exists integer k such that

$$n = 3 + 23k \equiv 3 + 8k \pmod{15},\ n \equiv 2 \pmod{15} \Rightarrow 8k + 3 \equiv 2 \pmod{15} \Rightarrow k \equiv 13 \pmod{15}$$

With $k = 13$, then $n = 3 + 23 \cdot 13$ satisfy the required conditions. This solution is unique modulo $15 \cdot 23$ (modulo 345), thus the least number of bad is $3 + 23 \cdot 13 = \boxed{302.}$ □

Problem 4.3.10 (Problem 10)**.** (*5 points*) Jar #1 contains five red marbles, three blue marbles, and one green marble. Jar #2 contains five blue marbles, three green marbles, and one red marble. Jar #3 contains five green marbles, three red marbles, and one blue marble.

You randomly select one marble from each jar.

Given that you select one marble of each color, the probability that the red marble came from jar #1, the blue marble came from jar #2, and the green marble came from jar #3 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let the triple (a,b,c) represent the event that the red marble came from jar $\#a$, the blue marble came from jar $\#b$, and the green marble came from jar $\#c$, then the probability of $(1,2,3)$ is $\left(\frac{5}{9}\right)^3$, the probabilities of $(1,3,2)$, $(2,1,3)$, and $(3,2,1)$ are each $\frac{5}{9}\frac{3}{9}\frac{1}{9}$, the probability of $(3,1,2)$ is $\left(\frac{3}{9}\right)^3$, and the probability of $(2,3,1)$ is $\left(\frac{1}{9}\right)^3$.

Thus, the required probability is

$$\frac{\left(\frac{5}{9}\right)^3}{\left(\frac{5}{9}\right)^3+3\frac{5}{9}\frac{3}{9}\frac{1}{9}+\left(\frac{3}{9}\right)^3+\left(\frac{1}{9}\right)^3}=\frac{125}{198}.$$

Thus $m+n=125+198=\boxed{323.}$ □

Problem 4.3.11 (Problem 11)**.** (*5 points*) Positive integers a,b,c,d, and e satisfy the equations

$$\begin{cases} (a+1)(3bc+1)=d+3e+1 \\ (b+1)(3ca+1)=3d+e+13 \\ (c+1)(3ab+1)=4(26-d-e)-1 \end{cases}$$

Find d^2+e^2.

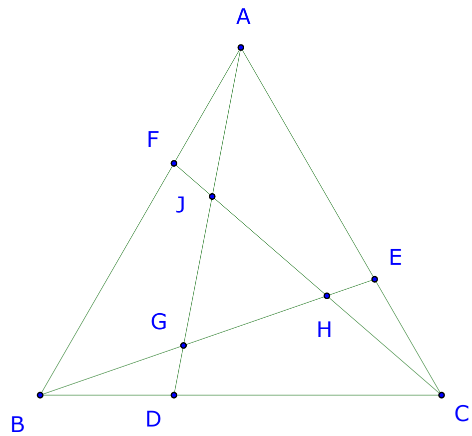
Solution. Adding the three equations together yields

$$9abc+3(ab+bc+ca)+(a+b+c)+3=117 \Rightarrow 27abc+9(ab+bc+ca)+3(a+b+c)+1=343$$
$$\Rightarrow (3a+1)(3b+1)(3c+1)=7^3.$$

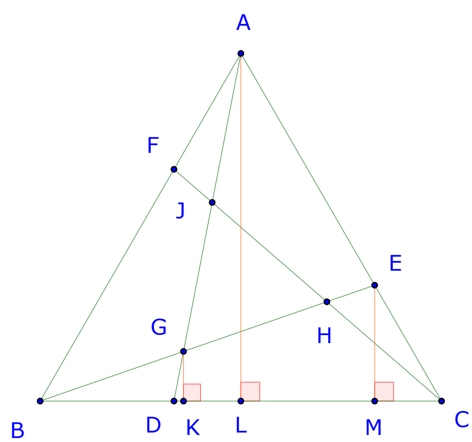
Since a,b,c are positive integers, thus $3a+1,3b+1,3c+1\geq 4$, so the only possible factorization of 7^3 into a product of three factors each at least 4 isosceles $3a+1=3b+1=3c+1=7$, so $a=b=c=4$.

Thus $d+3e+1=38, 3d+e+13=38$, so $d=5,e=11$. Hence $d^2+e^2=25+121=\boxed{146.}$ □

Problem 4.3.12 (Problem 12)**.** (*5 points*) On equilateral $\triangle ABC$ point D lies on BC a distance 1 from B, point E lies on AC a distance 1 from C, and point F lies on AB a distance 1 from A. Segment AD, BE, and CF intersect in pairs at points G, H, and J which are the vertices of another equilateral triangle. The area of $\triangle ABC$ is twice the area of $\triangle GHJ$. The side length of $\triangle ABC$ can be written $\dfrac{r+\sqrt{s}}{t}$, where r,s, and t are relatively prime positive integers. Find $r+s+t$.



Solution. Let d be the side length of $\triangle ABC$. Let point K, L, and M be the projections onto side BC of points G, A, and E, respectively, as shown. Let x and y be the length of BK and GK.



Because $\triangle ECM$ is $30-60-90$ with hypotenuse $CE = 1$, so $EM = \frac{\sqrt{3}}{2}$, and $CM = \frac{1}{2}$.

$$\triangle EMB \sim \triangle GKB \Rightarrow \frac{EM}{MB} = \frac{GK}{KB} \Rightarrow \frac{\frac{\sqrt{3}}{2}}{d-\frac{1}{2}} = \frac{y}{x}$$

$$\triangle ALD \sim \triangle GKD \Rightarrow \frac{AL}{LD} = \frac{GK}{KD} \Rightarrow \frac{\frac{d\sqrt{3}}{2}}{\frac{d}{2}-1} = \frac{y}{x-1}$$

$$\Rightarrow \frac{x\sqrt{3}}{2d-1} = \frac{d\sqrt{3}(x-1)}{d-2} \Rightarrow x = \frac{d(2d-1)}{2(d^2-d+1)},\ y = \frac{d\sqrt{3}}{2(d^2-d+1)}.$$

$$[ABC] = \frac{d^2\sqrt{3}}{4} \Rightarrow [GHJ] = \frac{d^2\sqrt{3}}{8}$$

$$[BCE] = [CAF] = [ABD] = \frac{d\sqrt{3}}{4},\ [BDG] = [CEH] = [AFJ] = \frac{y}{2} = \frac{d\sqrt{3}}{4(d^2-d+1)}$$

$$\Rightarrow \frac{d^2\sqrt{3}}{8} = 3\frac{d\sqrt{3}}{4} - 3\frac{d\sqrt{3}}{4(d^2-d+1)} \Rightarrow d(d^2-7d+7) = 0,\ d > 1 \Rightarrow d = \frac{7+\sqrt{21}}{2}$$

Thus the desired sum $7 + 21 + 2 = \boxed{30.}$ $\square$

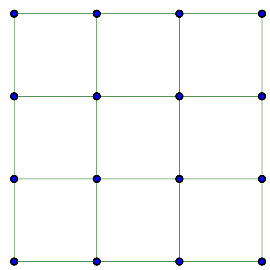
Problem 4.3.13 (Problem 13)**.** (*5 points*) In $\triangle ABC$, $\cos(\angle A) = \frac{2}{3}$, $\cos(\angle B) = \frac{1}{9}$, and $BC = 24$. Find the length AC.

Solution. It is easy to see that $\sin(\angle A) = \sqrt{1-\cos^2(\angle A)} = \frac{\sqrt{5}}{3}$ (note that $\sin(\angle A) > 0$), similarly $\sin(\angle B) = \frac{4\sqrt{5}}{9}$. By the Law of Sines:

$$\frac{AC}{\sin(\angle B)} = \frac{BC}{\sin(\angle A)} \Rightarrow AC = \frac{4\sqrt{5}}{9} \cdot \frac{24}{\frac{\sqrt{5}}{3}} = \boxed{32.}$$

□

Problem 4.3.14 (Problem 14)**.** (*5 points*) Sixteen dots are arranged in a four by four grid as shown. The distance between any two dots in the grid is the minimum number of horizontal and vertical steps along the grid lines it takes to get from one dot to the other. For example, two adjacent dots are a distance 1 apart, and two dots at opposite corners of the grid are a distance 6 apart.



The mean distance between two distinct dots in the grid is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. There are $\binom{16}{2} = 120$ pairs of points in the grid.

For $4 \cdot \binom{4}{2} = 24$ of the pairs of points (the same column), the horizontal distance between the points is 0.

For $3 \cdot 4 \cdot 4 = 48$ of the pairs of points, the horizontal distance between the points is 1.

For $2 \cdot 4 \cdot 4 = 32$ of the pairs of points, the horizontal distance between the points is 2.

For $4 \cdot 4 = 16$ of the pairs of points, the horizontal distance between the points is 3.

Thus, the sum of all the horizontal distances between pairs of points for all the pairs is

$$0 \cdot 24 + 1 \cdot 48 + 2 \cdot 32 + 3 \cdot 16 = 160.$$

This also is the sum of all the vertical distances between pairs of points for all the pairs.

Thus, the mean distances between points is $\dfrac{2 \cdot 160}{120} = \dfrac{8}{3}$. Hence $m+n = 8+3 = \boxed{11.}$ □

Problem 4.3.15 (Problem 15)**.** (*5 points*) Find the largest prime p such that p divides $2^{p+1}+3^{p+1}+5^{p+1}+7^{p+1}$.

Solution. Fermat's Little Theorem states that if p is a prime that does not divides a, then $a^{p-1} \equiv 1 \pmod{p}$.

Let's assume that $p > 7$, then

$$2^{p+1} + 3^{p+1} + 5^{p+1} + 7^{p+1} \equiv 2^2 + 3^2 + 5^2 + 7^2 = 87 \pmod{p}.$$

Thus $p \mid 87$. The largest such prime is $\boxed{29}$. □

Problem 4.3.16 (Problem 16)**.** (*5 points*) For n measured in degrees,

$$\text{let } T(n) = \cos^2(30^\circ - n) - \cos(30^\circ - n)\cos(30^\circ + n) + \cos^2(30^\circ + n). \text{ Evaluate } 4\sum_{n=1}^{30} n \cdot T(n).$$

Solution. By the Double Angle formula for cosine and the Prosthaphaeresis formulas, and $\cos(60^\circ) = \frac{1}{2}$, thus

$$\begin{cases} 2\cos^2(30^\circ - n) = 1 + \cos(60^\circ - 2n) = 1 + \cos(60^\circ)\cos(2n) + \sin(60^\circ)\sin(2n) \\ 2\cos(30^\circ - n)\cos(30^\circ + n) = \cos(60^\circ) + \cos(2n) \\ 2\cos^2(30^\circ + n) = 1 + \cos(60^\circ + 2n) = 1 + \cos(60^\circ)\cos(2n) - \sin(60^\circ)\sin(2n) \end{cases}$$
$$\Rightarrow 2T(n) = 2 + 2\cos(60^\circ)\cos(2n) - \cos(60^\circ) - \cos(2n) = \frac{3}{2}$$

Hence

$$4\sum_{n=1}^{30} n \cdot T(n) = 4 \cdot \sum_{n=1}^{30} n \cdot \frac{3}{4} = 3 \cdot \frac{30 \cdot 31}{2} = \boxed{1395.}$$

□

Problem 4.3.17 (Problem 17)**.** (*5 points*) Find the sum of all values of a such that there are positive integers a and b satisfying $(a - b)\sqrt{ab} = 2016$.

Solution. Let $d = \gcd(a, b)$, then $d \mid \sqrt{ab}, d \mid a - b \Rightarrow d^2 \mid 2016 = 2^5 \cdot 3^2 \cdot 7$. Thus d is $1, 2, 3, 4, 6$, or 12.

Furthermore, let $a = dr, b = ds$, then $\gcd(r, s) = 1$. Since $d^2(r - s)\sqrt{rs} = 2016$, thus each of r and s is a perfect square. In other words, there exist positive integers m and n,

$$a = dm^2,\ b = dn^2,\ \gcd(m, n) = 1,\ d^2(m^2 - n^2)mn = 2^5 \cdot 3^2 \cdot 7.$$

Case 1: $d = 1$.

$$(m - n)(m + n)mn = 2^5 \cdot 3^2 \cdot 7.$$

If one of m or n is even, then the other has to be odd and both $m - n, m + n$ are odd, thus 2^5 divides m or n. In that case $mn \geq 2^5 = 32$, $m + n \geq 33$, thus

$$2016 = (m - n)(m + n)mn \geq (m - n)(33)(32) = 1056(m - n) \Rightarrow m - n = 1$$

It is easy to test with m as multiple of 32 that there is no solution.

If both m or n are odd, then one of them is divisible by 7 and the other is divisible by 9. In fact $m = 9, n = 7$ satisfy above equality and $(a, b) = (81, 49)$ is a solution.

Case 2: $d = 2$.

$$(m - n)(m + n)mn = 2^3 \cdot 3^2 \cdot 7.$$

If both m or n are odd, it is easy to test that there is no solution.

If m is divisible by 2^3 then $m + n \geq 9$ and $(m - n)(m + n) \leq 63$, thus $m = 8, n = 1$ and $(a, b) = (128, 2)$ is a solution.

Case 3: $d = 3$.

$$(m - n)(m + n)mn = 2^5 \cdot 7.$$

If both m or n are odd, then $m = 7, n = 1$ leading to no solution.

If m is divisible by 2^5 then $m + n \geq 33$ and $(m - n)(m + n)$ is too large.

Case 4: $d \geq 4$.

$$(m-n)(m+n)mn \leq 2 \cdot 3^2 \cdot 7 = 126.$$

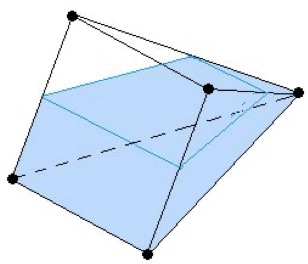
One of $m-n, m+n, m$, or n is divisible by 7.

If $m \geq 7$, then $m+n \geq 8$, $(m-n)n \geq 3$, thus $(m-n)(m+n)mn \geq 8 \cdot 7 \cdot 3 > 126$.

If $m \leq 6$, then $m+n = 7$, $(m-n)mn \leq \frac{126}{7} = 18$. Thus $(m,n) = (4,3)$. It is easy to test that there is no such solution.

Hence, there are two solutions $\{(81,49),(128,2)\}$, and the sum of all values of a is $81 + 128 = \boxed{209.}$ □

Problem 4.3.18 (Problem 18)**.** (*5 points*) A container the shape of a pyramid has a 12×12 square base, and the other four edges each have length 11. The container is partially filled with liquid so that when one of its triangular faces is lying on a flat surface, the level of the liquid is half the distance from the surface to the top edge of the container.

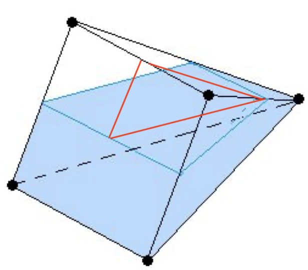


Find the volume of the liquid in the container.

Solution. A square pyramid with base side b and base to the apex height h has a volume $\frac{1}{3}b^2h$. Let s be the length of the other four edges. The height to the apex can be calculated considering the cross section of the pyramid that contains the apex, and two opposite corners of the square base. This cross section is an isosceles triangle with base $b\sqrt{2}$ and two sides equal to s,

$$h^2 = s^2 - \left(\frac{1}{2}b\sqrt{2}\right)^2 = s^2 - \frac{b^2}{2} \Rightarrow h = 7 \Rightarrow V = \frac{1}{3}b^2h = 336.$$

When lying on its side, the portion of the pyramid containing no liquid can be partitioned into two sections, one shaped like the original pyramid by half its size and a skewed triangular prism.



The volume of the pyramid is $\frac{1}{2^3}336 = 42$. The prism has a triangular cross section with base equal to $\frac{b}{2} = 6$ and height equal to $\frac{h}{2} = \frac{7}{2}$ so the area of the triangle is $\frac{21}{2}$. The length of the prism is $\frac{b}{2} = 6$, so the volume of the prism is $6 \cdot \frac{21}{2} = 63$.

This means that the volume of the portion of the container that does not contain liquid is $42 + 63 = 105$. Hence, the volume of the liquid is $336 - 105 = \boxed{231.}$ □

Problem 4.3.19 (Problem 19). (*5 points*) Find the sum of all the possible values of the product xy such that x and y are positive integers satisfying

$$(x^2+1)(y^2+1)+2(x-y)(1-xy)=4(1+xy)+140.$$

Solution. Note that

$$\begin{aligned}(x^2+1)(y^2+1) &= (xy)^2+x^2+y^2+1=(xy)^2-2xy+1+x^2-2xy+y^2+4xy\\ &= (1-xy)^2+(x-y)^2+4xy=(1-xy+x-y)^2-2(x-y)(1-xy)+4xy\\ \Rightarrow 144 &= (x^2+1)(y^2+1)+2(x-y)(1-xy)-4xy=(1-xy+x-y)^2 \Rightarrow (1+x)^2(1-y)^2=144.\end{aligned}$$

Since x, y are positive integers, thus

$$(x+1, y-1)\in\{(2,6),(3,4),(4,3),(6,2),(12,1)\} \Rightarrow (x,y)\in\{(1,7),(2,5),(3,4),(5,3),(11,2)\}.$$

Hence, $xy \in \{7, 10, 12, 15, 22\}$, and the desired sum is $7+10+12+15+22=\boxed{66.}$ □

Problem 4.3.20 (Problem 20). (*5 points*) Ten square tiles are placed in a row, and each can be painted with one of the four colors red (R), yellow (Y), blue (B), and white (W). Find the number of ways this can be done so that each block of five adjacent tiles contains at least one tile of each color. That is, count the patterns RWBWYRRBWY and WWBYRWYBWR but not RWBYYBWWRY because the five adjacent tiles colored BYYBW does not include the color red.

Solution. Suppose that the colouring $C_1C_2\dots C_9C_{10}$ is an acceptable colouring. Then each sequence of five colours $C_kC_{k+1}C_{k+2}C_{k+3}C_{k+4}$ contains exactly one colour that is repeated.

Call a tile *repeated* if its colour appears among one of the following four tiles immediately to its right. For example, in the pattern $RWBWYRRBWY$, tiles 2 and 6 are repeated tiles:

$$R\underline{W}B\underline{W}YRRBWY,\ \ RWBWY\underline{RR}BWY.$$

In the pattern $WWBYRWYBWR$, tiles 1, 2, 4, and 6 are repeated tiles:

$$\underline{WW}BYRWYBWR,\ \ W\underline{W}BYR\underline{W}YBWR, WWB\underline{Y}RW\underline{Y}BWR, WWBYR\underline{W}YB\underline{W}R.$$

Note that each sequence of the four tiles: 1234, 2345, 3456, 4567, 5678, and 6789 must include exactly one repeated tile. It follows that the sequence of repeated tiles is an increasing sequence of k positive integers $a_1 < a_2 < \cdots < a_k$, where a_1 is one of 1, 2, 3, 4, the difference between two adjacent terms $a_{j+1}-a_j$ cannot exceed 4, the final term a_k is one of 6, 7, 8, 9, and there can be only one term with value greater than 5.

Given any sequence of repeated values that satisfy these conditions, it is easy to construct an acceptable colouring with that sequence of repeated values. For example, the repeated tile sequence 1, 2, 5, 9 comes from the acceptable pattern $\underline{RB}WY\underline{RBR}WW\underline{YY}$.

Also note that if the colour are known for the first tiles, and the repeated sequence is known, then the complete sequence of ten tile colours can be determined. For example, if the colour pattern for the first five tiles is $RBWYR$, and the repeated sequence is 1,2,5,9, then the sixth rile by be a B so that the second tile becomes repeated. Then the seventh tile must be R so the tile 5 becomes repeated. Then the eight tile must be W so the tiles 5678 has a W, the ninth tile must be Y so tiles 6789 has an Y, and finally the tenth tile must be Y so the ninth tile becomes repeated.

Let m_j be the number of ways a repeated sequence containing the number j can be completed by adding terms to the right of j. Then $m_9 = m_8 = m_7 = m_6 = 1$ because any repeated sequence containing a 6, 7, 8, 9 must end with that term and have no terms to the right of it. For j between 1 and 5, a repeated sequence that includes a term equal to j must be follow by a term with value $j+1, j+2, j+3$, or $j+4$ showing that $m_j = m_{j+1} + m_{j+2} + m_{j+3} + m_{j+4}$. Thus $m_5 = m_6 + m_7 + m_8 + m_9 = 4$, $m_4 = 7$, $m_3 = 13$, $m_2 = 25$, $m_1 = 49$.

Because every repeated sequence must begin with exactly one of 1, 2, 3, or 4, it follows that there are

$$m_1 + m_2 + m_3 + m_4 = 49 + 25 + 13 + 7 = 94 \text{ repeated sequences.}$$

A repeated sequence does not determine a colour pattern. A repeated sequence together with the first five colours of an acceptable colour pattern does determine a full pattern.

If a repeated sequence begins with 4, it means that tile 4 and 5 are coloured the same in the associated colour pattern. There are 4! ways to set a sequence of five tiles using 24 colours when tiles 4 and 5 are colour the same. So there are $4! \cdot a_4 = 24 \cdot 7$ acceptable colour patterns where tiles 4 and 5 are colour the same.

Similarly, a repeated sequence begins with 3, then tile 3 must be coloured the same as either tile 4 or tile 5. Thus, there are $24 \cdot 2 \cdot a_3 = 24 \cdot 26$ acceptable colour patterns where a repeated sequence begins with 3.

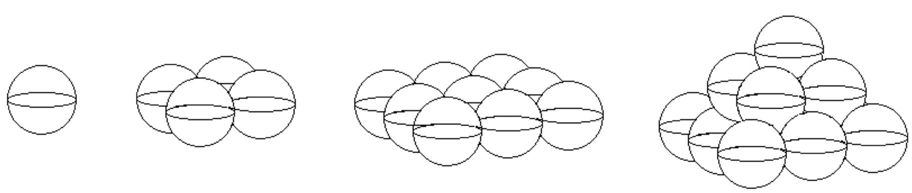
The number of acceptable colour patterns where a repeated sequence begins with 2 is $24 \cdot 3 \cdot a_2 = 24 \cdot 75$.

The number of acceptable colour patterns where a repeated sequence begins with 1 is $24 \cdot 4 \cdot a_1 = 24 \cdot 196$.

Hence, the total number of acceptable colour patterns is $24 \cdot (196 + 75 + 26 + 7) = \boxed{7296.}$ □

Problem 4.3.21 (Problem 21)**.** (*5 points*) Some identically sized spheres are piled in n layers in the shape of a square pyramid with one sphere in the top layer, 4 spheres in the second layer, 9 spheres in the third layer, and so forth so that the bottom layer has a square array of n^2 spheres. In each layer the centers of the spheres form a square grid so that each sphere is tangent to any sphere adjacent to it on the grid. Each sphere in an upper level is tangent to the four spheres directly below it.

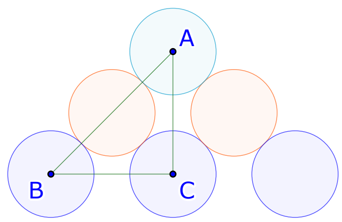
The diagram shows how the first three layers of spheres are stacked.

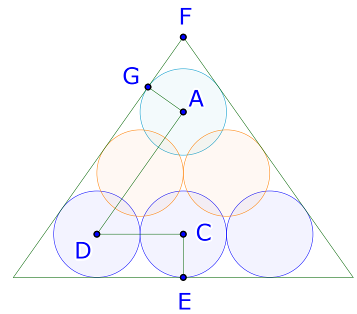


A square pyramid is built around the pile of spheres so that the sides of the pyramid are tangent to the spheres on the outside of the pile. There is a positive integer m such that as n gets large, the ratio of the volume of the pyramid to the total volume inside all of the spheres approaches $\frac{\sqrt{m}}{\pi}$. Find m.

Solution. Let assume that each of the spheres has a radius equal to 1. Then the center of any two adjacent spheres are a distance 2 apart. If four spheres in one layer have their centers at four vertices of a 2×2 square, then the centers of the spheres across the square from each other are a distance of $2\sqrt{2}$ apart.

Consider the diagram below on the left showing a cross section of the pile of spheres passing through the center of the sphere on the top layer labelled A, the center of the sphere in the third layer directly below the top sphere labeled C, and the center of a sphere across a 2×2 square from there labeled B. Because AC is vertical, BC is horizontal, $\triangle ABC$ is right triangle with $AB = 4$, $BC = 2\sqrt{2}.$, thus $AC = 2\sqrt{2}$. It follows that the vertical distance between the plane containing the centers of spheres at one level of the pile, and the plane containing the center of spheres at the next lower level of the pile is $\sqrt{2}$.





Now, consider the diagram above on the right showing a cross section of the pile of spheres and the enclosing pyramid that passes through the top vertex of the pyramid, the center of the top sphere labeled A, the center of the sphere directly below A in the third level labeled C, and the center of an adjacent sphere in the third level labeled D. Then ACD is a right triangle with legs $AC = 2\sqrt{2}$, $CD = 2$, thus $AD = 2\sqrt{3}$. Let G be a point where the top sphere is tangent to the side of the pyramid. Since $\triangle AFG \sim \triangle CAD$ and $AG = 1$, thus $AF = \sqrt{3}$.

If the pile of sphere has n layers, then the distance between a horizontal plane passing through point A and a plane passing through the centers of the bottom layers is $(n-1)\sqrt{2}$. Since $AF = \sqrt{3}$, and the centers of the spheres are a distance 1 from the bottom of the pyramid, the height of the pyramid is given by

$$(n-1)\sqrt{2} + 1 + \sqrt{3}.$$

Because the side of the pyramid is parallel to AD, by similarity, so the side of the base square of the pyramid is:

$$2 \cdot \frac{\sqrt{2}}{2}\left((n-1)\sqrt{2} + 1 + \sqrt{3}\right) = 2(n-1) + \sqrt{2} + \sqrt{6}.$$

The volume of the pyramid is equal to $\frac{1}{3}$ times the area of its base, times its height, so the volume is:

$$\frac{1}{3}\left(2(n-1) + \sqrt{2} + \sqrt{6}\right)^2 \cdot \left((n-1)\sqrt{2} + 1 + \sqrt{3}\right).$$

The number of spheres in this pile is given by:

$$1^2 + 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

So the volume of all the spheres is:

$$\frac{4\pi}{3}\frac{n(n+1)(2n+1)}{6}.$$

Now, since the limit of ratio of volumes is:

$$\lim_{n\to+\infty} \frac{\frac{1}{3}\left(2(n-1) + \sqrt{2} + \sqrt{6}\right)^2 \cdot \left((n-1)\sqrt{2} + 1 + \sqrt{3}\right)}{\frac{4\pi}{3}\frac{n(n+1)(2n+1)}{6}} = \frac{\frac{2^2\sqrt{2}}{3}}{\frac{4\pi}{3}\frac{2}{6}} = \frac{\frac{4\sqrt{2}}{3}}{\frac{4\pi}{9}} = \frac{3\sqrt{2}}{\pi} = \frac{\sqrt{18}}{\pi}.$$

Thus $m = \boxed{18.}$ $\square$

4.4 High School - Test

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**
- **Olympiad (O) level: Problems 12-21**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 4.4.1 (Problem 1)**.** (*5 points*) Jeremy wrote all the three-digit integers from 100 to 999 on a blackboard. Then Allison erased each of the 2700 digits Jeremy wrote and replaced each digit with the square of that digit. Thus, Allison replaced every 1 with a 1, every 2 with a 4, every 3 with a 9, every 4 with a 16, and so forth. The proportion of all the digits Allison wrote that were **twos** is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Jeremy wrote 2700 digits. If he had also written $000, 001, 002, \ldots, 099$, he would have written 3000 digits in all which would have consisted of 300 of each of the 10 digits. In the list of those 100 numbers that he did not write, each nonzero digit appears $\frac{200}{10} = 20$ times, and 0 appear $20 + 100 = 120$ times.

Thus, among the digits Jeremy actually wrote, the digit 0 appears $300 - 120 = 180$ times, and each of the digits each appear $300 - 20 = 280$ times. Allison replaces each $0, 1, 2,$ and 3 with one-digit numbers, and each of $4, 5, 6, 7, 8, 9$ with a two-digit number, so Allison wrote a total of

$$180 + 3 \cdot 280 + 2 \cdot 6 \cdot 280 = 180 + 15 \cdot 280 \text{ digits.}$$

Allison wrote one digit 2 each time she squared 5 accounting for $1 \cdot 280$ digits that were 5. The required proportion is,

$$\frac{1 \cdot 280}{180 + 15 \cdot 280} = \frac{14}{219}.$$

Hence, $m + n = 14 + 219 = \boxed{233.}$ □

Problem 4.4.2 (Problem 2)**.** (*5 points*) Find the number of three-digit positive integers which have three distinct digits where the sum of the digits is an odd number such as 935 and 834.

Solution. If the hundred digit is even, then it is followed by one even and one odd digit. There are 4 choices for the hundred digit and $5 \cdot 4 + 4 \cdot 5 = 40$ ways to choose the following digits, (5 choices for the odd tens then 4 choices for the even unit digit or 4 choices for the even tens and 5 choices for the odd unit digit), altogether $4 \cdot 40 = 160$.

If the hundred digit is odd, then the other two digit must be both odd or even. There are 5 choices for the hundred digit and $4 \cdot 3 + 5 \cdot 4 = 32$ ways to choose the following digits, (4 choices for the odd tens then 3 choices for the odd unit digit or 5 choices for the even tens and 4 choices for the even unit digit), altogether $5 \cdot 32 = 160$.

Hence, there are $160 + 160 = \boxed{320}$ such numbers. □

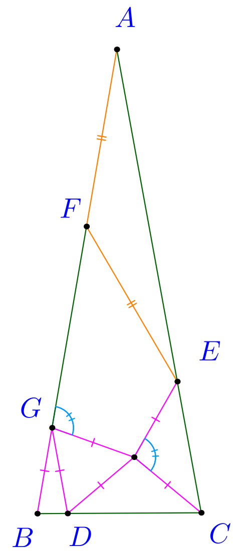
Problem 4.4.3 (Problem 3)**.** (*5 points*) The sum of the reciprocal of $5 + 12i$ and the reciprocal of $5 - 12i$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution.

$$\frac{1}{5 + 12i} + \frac{1}{5 - 12i} = \frac{10}{5^2 + 12^2} = \frac{10}{169}.$$

Hence, $m + n = 10 + 169 = \boxed{179.}$ □

Problem 4.4.4 (Problem 4)**.** (*5 points*) In $\triangle ABC$, $AB = AC$, $AF = EF$, $\angle CHE = \angle FGH$, and $EH = CH = DH = GH = DG = BG$. Find $\angle GFE$.



Solution. Let $x = \angle ABC$,

$$\triangle BAC \sim \triangle BGD \Rightarrow \angle BDG = \angle DBG = x \angle BGD + \angle GDH + \angle HDC = 180^\circ \Rightarrow \angle HDC = 120^\circ - x$$
$$\angle HCD = \angle HDC,\ \angle ECH = \angle ACB - \angle HCD \Rightarrow \angle ECH = x - (120^\circ - x) = 2x - 120^\circ$$
$$\angle EHC = 180^\circ - 2(2x - 120^\circ) = 420^\circ - 4x,\ \angle FGH = 180^\circ - \angle HGD - \angle DGB$$
$$\Rightarrow \angle HGF = 180^\circ - 60^\circ - (180^\circ - 2x) = 2x - 60^\circ$$
$$\angle CHE = \angle FGH \Rightarrow 420^\circ - 4x = 2x - 60^\circ \Rightarrow x = 80^\circ$$

Therefore, $\angle BAC = 180^\circ - 2 \cdot 80^\circ = 20^\circ$, so $\angle GFE = 2\angle FAE = \boxed{40^\circ.}$ □

Problem 4.4.5 (Problem 5)**.** (*5 points*) Find the greatest possible value of $pq + r$, where p, q, and r are (not necessarily distinct) prime numbers satisfying $pq + qr + rp = 2024$.

Solution. If each of p, q, and r is an odd prime, then $pq + qr + rp$ is odd, thus cannot be equal to 2016.

If $p = 2$, then $qr + 2(q + r) = 2024$, so qr is even, so q or r is even. If $q = 2$ then $4 + 4r = 2024 \Rightarrow r = 505$.

But then r is not a prime number. Thus there is $\boxed{\text{no solution.}}$ □

Problem 4.4.6 (Problem 6)**.** (*5 points*) n the least positive integer of the form $\overline{abbabb}$, where a and b are distinct digits, such that the integer can be written as a product of six distinct primes. Find the sum of these primes.

Solution. Let $n = \overline{abb}$, then the given number is $1001n = 7 \cdot 11 \cdot 13 \cdot n$. Thus n must be the least three-digit number that is a product of three distinct primes other than 7, 11, 13.

Now if $a = 1$, then n is odd and $n \geq 3 \cdot 5 \cdot 17 = 255 > \overline{1b1} = n$.

If $a = 2$, and since 3, 5, 17 are the least primes that satisfy $3 \cdot 5 \cdot 17 = 255$, Thus $n = 255255$.

Hence $3 + 5 + 7 + 11 + 13 + 17 = \boxed{56.}$ □

Problem 4.4.7 (Problem 7)**.** (*5 points*) Henry rolls a fair die. If the die shows the number k, Henry will then roll the die k more times. The probability that Henry will never roll a 1, 3 or a 6 either on his first roll or on one of the k subsequent rolls is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. In order to never roll a $1, 3$ or 6, Henry needs to roll $k = 2, 4$, or 5 on his first roll, and then no $1, 3$ or 6, on any of the k subsequent rolls.

The probability that Henry does not roll a $1, 3$ or 6, on a single throw is $\frac{3}{6} = \frac{1}{2}$. Thus, the probability that Henry will never roll a 1, 3 or a 6 on any of his roll is

$$\frac{1}{6}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\left(\frac{1}{2}\right)^4 + \frac{1}{6}\left(\frac{1}{2}\right)^5 = \frac{11}{192}.$$

Hence, $m + n = 11 + 192 = \boxed{203.}$ □

Problem 4.4.8 (Problem 8)**.** (*5 points*) The cubic polynomials $p(x)$ and $q(x)$ satisfy

$$\begin{cases} p(1) = q(2) \\ p(3) = q(4) \\ p(5) = q(6) \\ p(7) = q(8) + 13 \end{cases}$$

Find $p(11) - q(12)$.

Solution. Since $p(x)$ and $q(x)$ are cubic polynomials, thus $r(x) = p(x) - q(x+1)$ is also a cubic polynomial. Furthermore $r(1) = r(3) = r(5) = 0$, thus $1, 3$, and 5 are all of its roots, therefore:

$$r(x) = a(x-1)(x-3)(x-5), \text{ where } a \text{ is a real constant.}$$

$r(7) = 13 \Rightarrow a(6)(4)(2) = 13 \Rightarrow a = \frac{13}{48}$.

Hence, $p(11) - q(12) = r(11) = \frac{13}{48}(10)(8)(6) = \boxed{130.}$ □

Problem 4.4.9 (Problem 9)**.** (*5 points*) The Tasty Candy Company always puts the same number of pieces of candy into each one-pound bag of candy they sell. Mike bought 4 one-pound bags and gave each person in his class 15 pieces of candy. Mike had 19 pieces of candy left over. Betsy bought 5 one-pound bags and gave 19 pieces of candy to each teacher in her school. Betsy had 15 pieces of candy left over. Find the least number of pieces of candy the Tasty Candy Company could have placed in each one-pound bag.

Solution. We show a coding solution. The problem is equivalent to look for the first positive integer n in ascending order such that:

$$\begin{cases} 4n \equiv 19 \pmod{15} \\ 5n \equiv 15 \pmod{19} \end{cases}$$

```
n = 1
while True:
    if (4*n - 19 )% 15 == 0 and (5*n - 15 )% 19 == 0:
        print(n)
        break
    n +=1
```

The answer is $\boxed{136.}$ □

Problem 4.4.10 (Problem 10)**.** (*5 points*) Jar #1 contains five red marbles, three blue marbles, and one green marble. Jar #2 contains five blue marbles, three green marbles, and one red marble. Jar #3 contains five green marbles, three red marbles, and one blue marble.

You randomly select one marble from each jar.

Given that you select one marble of each color, the probability that the red marble came from jar #2, the blue marble came from jar #3, and the green marble came from jar #1 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let the triple (a,b,c) represent the event that the red marble came from jar $\#a$, the blue marble came from jar $\#b$, and the green marble came from jar $\#c$, then the probability of $(1,2,3)$ is $\left(\frac{5}{9}\right)^3$, the probabilities of $(1,3,2)$, $(2,1,3)$, and $(3,2,1)$ are each $\frac{5}{9}\frac{3}{9}\frac{1}{9}$, the probability of $(3,1,2)$ is $\left(\frac{3}{9}\right)^3$, and the probability of $(2,3,1)$ is $\left(\frac{1}{9}\right)^3$.

Thus, the required probability is

$$\frac{\left(\frac{1}{9}\right)^3}{\left(\frac{5}{9}\right)^3+3\frac{5}{9}\frac{3}{9}\frac{1}{9}+\left(\frac{3}{9}\right)^3+\left(\frac{1}{9}\right)^3}=\frac{1}{198}.$$

Thus $m+n=1+198=\boxed{199.}$ □

Problem 4.4.11 (Problem 11)**.** (*5 points*) Positive integers a,b,c,d, and e satisfy the equations

$$\begin{cases} (a+1)(3bc+1)=d+3e+1 \\ (b+1)(3ca+1)=3d+e+13 \\ (c+1)(3ab+1)=4(26-d-e)-1 \end{cases}$$

Find $a+b+c+d+e$.

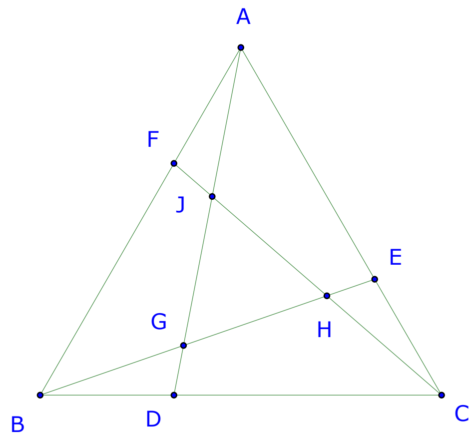
Solution. Adding the three equations together yields

$$\begin{aligned} &9abc+3(ab+bc+ca)+(a+b+c)+3=117 \Rightarrow 27abc+9(ab+bc+ca)+3(a+b+c)+1=343 \\ &\Rightarrow (3a+1)(3b+1)(3c+1)=7^3. \end{aligned}$$

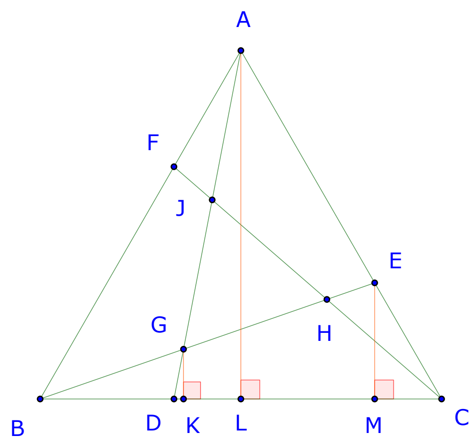
Since a,b,c are positive integers, thus $3a+1,3b+1,3c+1\geq 4$, so the only possible factorization of 7^3 into a product of three factors each at least 4 isosceles $3a+1=3b+1=3c+1=7$, so $a=b=c=4$.

Thus $d+3e+1=38, 3d+e+13=38$, so $d=5, e=11$. Hence $a+b+c+d+e=2+2+2+5+11=\boxed{24.}$ □

Problem 4.4.12 (Problem 12)**.** (*5 points*) On equilateral $\triangle ABC$ point D lies on BC a distance 1 from B, point E lies on AC a distance 1 from C, and point F lies on AB a distance 1 from A. Segment AD, BE, and CF intersect in pairs at points G, H, and J which are the vertices of another equilateral triangle. The area of $\triangle ABC$ is twice the area of $\triangle GHJ$. The side length of $\triangle ABC$ can be written $\frac{r+\sqrt{s}}{t}$, where r,s, and t are relatively prime positive integers. Find the value of the product rst.



Solution. Let d be the side length of $\triangle ABC$. Let point K, L, and M be the projections onto side BC of points G, A, and E, respectively, as shown. Let x and y be the length of BK and GK.



Because $\triangle ECM$ is $30-60-90$ with hypotenuse $CE = 1$, so $EM = \frac{\sqrt{3}}{2}$, and $CM = \frac{1}{2}$.

$$\triangle EMB \sim \triangle GKB \Rightarrow \frac{EM}{MB} = \frac{GK}{KB} \Rightarrow \frac{\frac{\sqrt{3}}{2}}{d-\frac{1}{2}} = \frac{y}{x}$$

$$\triangle ALD \sim \triangle GKD \Rightarrow \frac{AL}{LD} = \frac{GK}{KD} \Rightarrow \frac{\frac{d\sqrt{3}}{2}}{\frac{d}{2}-1} = \frac{y}{x-1}$$

$$\Rightarrow \frac{x\sqrt{3}}{2d-1} = \frac{d\sqrt{3}(x-1)}{d-2} \Rightarrow x = \frac{d(2d-1)}{2(d^2-d+1)},\ y = \frac{d\sqrt{3}}{2(d^2-d+1)}.$$

$$[ABC] = \frac{d^2\sqrt{3}}{4} \Rightarrow [GHJ] = \frac{d^2\sqrt{3}}{8}$$

$$[BCE] = [CAF] = [ABD] = \frac{d\sqrt{3}}{4},\ [BDG] = [CEH] = [AFJ] = \frac{y}{2} = \frac{d\sqrt{3}}{4(d^2-d+1)}$$

$$\Rightarrow \frac{d^2\sqrt{3}}{8} = 3\frac{d\sqrt{3}}{4} - 3\frac{d\sqrt{3}}{4(d^2-d+1)} \Rightarrow d(d^2-7d+7) = 0,\ d > 1 \Rightarrow d = \frac{7+\sqrt{21}}{2}$$

Thus the desired product $7 \cdot 21 \cdot 2 = \boxed{294.}$ □

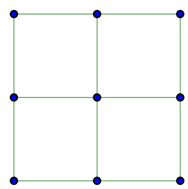
Problem 4.4.13 (Problem 13)**.** (*5 points*) In $\triangle ABC$, $\cos(\angle A) = \frac{\sqrt{5}}{5}$, $\cos(\angle B) = \frac{3}{5}$, $BC = 17$. Find AB.

Solution. It is easy to see that $\sin(\angle A) = \sqrt{1 - \cos^2(\angle A)} = \frac{2\sqrt{5}}{5}$ $(\sin(\angle A) > 0)$, similarly $\sin(\angle B) = \frac{4}{5}$.

$$\sin(\angle C) = \sin(\angle A + \angle B) = \sin(\angle A)\cos(\angle B) + \sin(\angle B)\cos(\angle A) = \frac{2\sqrt{5}}{5} \cdot \frac{3}{5} + \frac{4}{5} \cdot \frac{\sqrt{5}}{5} = \frac{2\sqrt{5}}{5}$$

Thus $\angle C = \angle A$, so $AB = BC = \boxed{17.}$ □

Problem 4.4.14 (Problem 14)**.** (*5 points*) Nine dots are arranged in a three by three grid as shown. The distance between any two dots in the grid is the minimum number of horizontal and vertical steps along the grid lines it takes to get from one dot to the other. For example, two adjacent dots are a distance 1 apart, and two dots at opposite corners of the grid are a distance 6 apart.



The mean distance between two distinct dots in the grid is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. There are $\binom{9}{2} = 36$ pairs of points in the grid.

For $3 \cdot \binom{3}{2} = 9$ of the pairs of points (the same column), the horizontal distance between the points is 0.

For $2 \cdot 3 \cdot 3 = 18$ of the pairs of points, the horizontal distance between the points is 1.

For $1 \cdot 3 \cdot 3 = 9$ of the pairs of points, the horizontal distance between the points is 2.

Thus, the sum of all the horizontal distances between pairs of points for all the pairs is

$$0 \cdot 9 + 1 \cdot 18 + 2 \cdot 9 = 36.$$

This also is the sum of all the vertical distances between pairs of points for all the pairs.

Thus, the mean distances between points is $\dfrac{2 \cdot 36}{36} = \dfrac{2}{1}$. Hence $m + n = 2 + 1 = \boxed{3.}$ □

Problem 4.4.15 (Problem 15)**.** (*5 points*) Find the largest prime p such that p divides $2^{p+2}+3^{p+2}+5^{p+2}+7^{p+2}$.

Solution. Fermat's Little Theorem states that if p is a prime that does not divides a, then $a^{p-1} \equiv 1 \pmod{p}$.

Let's assume that $p > 7$, then

$$2^{p+1} + 3^{p+1} + 5^{p+1} + 7^{p+1} \equiv 2^3 + 3^3 + 5^3 + 7^3 = 503 \pmod{p}.$$

Thus $p \mid 503$. The largest such prime is $\boxed{503}$. □

Problem 4.4.16 (Problem 16)**.** (*5 points*) For n measured in degrees,

$$\text{let } T(n) = \cos^2(30^\circ - n) - \cos(30^\circ - n)\cos(30^\circ + n) + \cos^2(30^\circ + n). \text{ Evaluate } 4\sum_{n=1}^{2024} n \cdot T(n).$$

Solution. By the Double Angle formula for cosine and the Prosthaphaeresis formulas, and $\cos(60^\circ) = \frac{1}{2}$, thus

$$\begin{cases} 2\cos^2(30^\circ - n) = 1 + \cos(60^\circ - 2n) = 1 + \cos(60^\circ)\cos(2n) + \sin(60^\circ)\sin(2n) \\ 2\cos(30^\circ - n)\cos(30^\circ + n) = \cos(60^\circ) + \cos(2n) \\ 2\cos^2(30^\circ + n) = 1 + \cos(60^\circ + 2n) = 1 + \cos(60^\circ)\cos(2n) - \sin(60^\circ)\sin(2n) \end{cases}$$

$$\Rightarrow 2T(n) = 2 + 2\cos(60^\circ)\cos(2n) - \cos(60^\circ) - \cos(2n) = \frac{3}{2}$$

$$\Rightarrow 4\sum_{n=1}^{2024} n \cdot T(n) = 4 \cdot \sum_{n=1}^{2024} n \cdot \frac{3}{4} = 3 \cdot \frac{2024 \cdot 2025}{2} = \boxed{6147900.}$$

□

Problem 4.4.17 (Problem 17)**.** (*5 points*) Find the sum of all values of b such that there are positive integers a and b satisfying $(a-b)\sqrt{ab} = 840$.

Solution. Let $d = \gcd(a, b)$, then $d \mid \sqrt{ab}, d \mid a - b \Rightarrow d^2 \mid 840 = 2^3 \cdot 3 \cdot 5 \cdot 7$. This means that d is 1 or 2.

Furthermore, let $a = dr, b = ds$, then $\gcd(r, s) = 1$. Since $d^2(r-s)\sqrt{rs} = 840$, thus each of r and s is a perfect square. In other words, there exist positive integers m and n,

$$a = dm^2,\ b = dn^2,\ \gcd(m, n) = 1,\ d^2(m^2 - n^2)mn = 2^3 \cdot 3 \cdot 5 \cdot 7.$$

Case 1: $d = 1$.

$$(m-n)(m+n)mn = 2^3 \cdot 3 \cdot 5 \cdot 7.$$

If one of m or n is even, then the other has to be odd and both $m - n, m + n$ are odd. Thus one of m or n is divisible by 8.

If n is divisible by 8, then $m - n \geq 1$, $mn \geq 72$, $m + n \geq 15$, which is too large.

Thus m is divisible by 8, then if m is divisible by 3, 5, or 7, then $mn \geq 24$, $m + n \geq 25$ and $(m-n)(m+n)mn \geq 24 \cdot 25(m-n) = 600(m-n)$, so $m - n = 1$, thus

$$(2m-1)m(m-1) = 840 > 2(m-1)^3 \Rightarrow 7 \geq m.$$

Thus m is divisible by 8, and it cannot be divisible by 3, 5, or 7, and so $m = 8$. In that case $n = 7$, and $(a, b) = (64, 49)$ is a solution.

If both m and n are odd, it is easy to see that $n > 1$. So if $m > 7$, then $m \geq 3 \cdot 5 = 15$, thus

$$(m-n)(m+n)mn \geq 2 \cdot (3 + 15) \cdot 15 \cdot 3 = 1620 > 840.$$

Thus $m = 7$, then

$$(7-n)(7+n)7(n) = 840 \Rightarrow n = 5 \text{ or } 3.$$

And $(a, b) = (49, 25)$, or $(a, b) = (49, 9)$.

Case 2: $d = 2$.

$$(m-n)(m+n)mn = 2 \cdot 3 \cdot 5 \cdot 7.$$

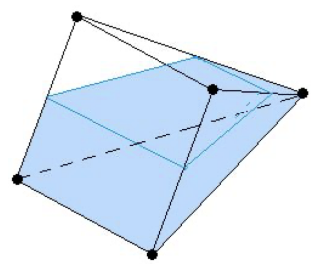
Both m or n cannot be even, and since both cannot be odd, so one is even and the other is odd.

If $n = 1$ then $(m^2 - 1)m = 210 \Rightarrow m = 6$, then $(a, b) = (72, 2)$.

If $n > 1$ then $n, m - n, m, m + n$ are four different pair-wise relatively prime numbers, whose product is $2 \cdot 3 \cdot 5 \cdot 7$. It is easy to see that $m - n$ can only be 3 there for $n = 2$, thus $m = 5$, and $(a, b) = (50, 8)$.

Hence, there are four solutions $\{(64, 49), (49, 25), (49, 9), (72, 2), (50, 8)\}$, and the sum of all values of a is $49 + 25 + 9 + 2 + 8 = \boxed{93.}$ □

Problem 4.4.18 (Problem 18)**.** (*5 points*) A container the shape of a pyramid has a 8×8 square base, and the other four edges each have length 9. The container is partially filled with liquid so that when one of its triangular faces is lying on a flat surface, the level of the liquid is half the distance from the surface to the top edge of the container.

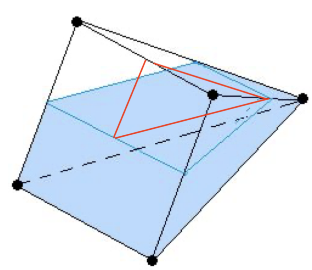


The volume of the liquid in the container is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. A square pyramid with base side b and base to the apex height h has a volume $\frac{1}{3}b^2h$. Let s be the length of the other four edges. The height to the apex can be calculated considering the cross section of the pyramid that contains the apex, and two opposite corners of the square base. This cross section is an isosceles triangle with base $b\sqrt{2}$ and two sides equal to s,

$$h^2 = s^2 - \left(\frac{1}{2}b\sqrt{2}\right)^2 = s^2 - \frac{b^2}{2} = 9^2 - \frac{8^2}{2} \Rightarrow h = 7 \Rightarrow V = \frac{1}{3}b^2h = \frac{448}{3}.$$

When lying on its side, the portion of the pyramid containing no liquid can be partitioned into two sections, one shaped like the original pyramid by half its size and a skewed triangular prism.



The volume of the pyramid is $\frac{1}{2^3}\frac{448}{3} = \frac{56}{3}$. The prism has a triangular cross section with base equal to $\frac{b}{2} = 4$, and height equal to $\frac{h}{2} = \frac{7}{2}$ so the area of the triangle is $\frac{1}{2} \cdot 4 \cdot \frac{7}{2} = 7$. The length of the prism is $\frac{b}{2} = 4$, so the volume of the prism is $4 \cdot 7 = 28$.

This means that the volume of the portion of the container that does not contain liquid is $\frac{56}{3} + 28 = \frac{140}{3}$. Hence, the volume of the liquid is $\frac{448}{3} - \frac{140}{3} = \frac{136}{1}$, and $m + n = 136 + 1 = \boxed{137.}$ □

Problem 4.4.19 (Problem 19)**.** (*5 points*) Find the sum of all the possible values of the product xy such that x and y are positive integers satisfying

$$(x^2+1)(y^2+1)+2(x-y)(1-xy)=4(1+xy)+96.$$

Solution. Note that

$$\begin{aligned}
&(x^2+1)(y^2+1)=(xy)^2+x^2+y^2+1=(xy)^2-2xy+1+x^2-2xy+y^2+4xy\\
&=(1-xy)^2+(x-y)^2+4xy=(1-xy+x-y)^2-2(x-y)(1-xy)+4xy\\
\Rightarrow\ &100=(x^2+1)(y^2+1)+2(x-y)(1-xy)-4xy=(1-xy+x-y)^2\Rightarrow(1+x)^2(1-y)^2=100.
\end{aligned}$$

Since x, y are positive integers, thus

$$(x+1, y-1)\in\{(2,5),(5,2),(10,1)\}\Rightarrow(x,y)\in\{(1,6),(4,3),(9,2)\}\Rightarrow xy\in\{6,12,18\}$$

Hence, the desired sum is $6+12+18=\boxed{36.}$ ☐

Problem 4.4.20 (Problem 20)**.** (*5 points*) Eight square tiles are placed in a row, and each can be painted with one of the three colors red (R), yellow (Y), and blue (B). Find the number of ways this can be done so that each block of four adjacent tiles contains at least one tile of each color. That is, count the patterns RBYRBYB and BYRYBRY but not RBYYBRYR because the four adjacent tiles colored BYYB does not include the color red.

Solution. We show a coding solution. The problem is equivalent to count all a base-3 number between 0 and 3^8. For the numbers has less then 8 digits, we add as many digits 0 to the left of the numbers as needed to create a 8-digit base-3 numbers Then for each of these base-3 number, we look at every block of 3 consecutive digit to make sure that it contains three different digits.

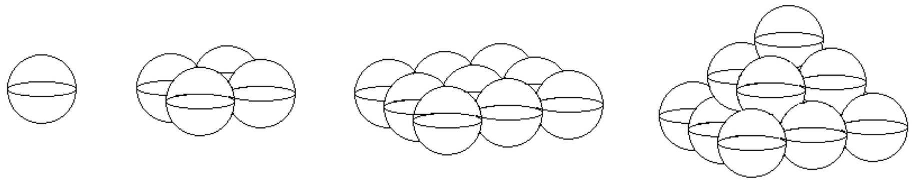
```
def to_base(number, base):
    digits = []
    while number:
        digits.append(number % base)
        number //= base
    return list(reversed(digits))

count = 0
for i in range(0, 3**10):
    s = to_base(i, 3)
    if len(s) < 8:
        s = [0 for j in range(len(s), 8)] + s
    found = True
    for j in range(0, 5):
        if len(set([e for e in s[j: j+4]])) < 3:
            found = False
            break
    if found:
        count +=1
print(count)
```

The answer is $\boxed{444.}$ ☐

Problem 4.4.21 (Problem 21)**.** (*5 points*) Some **unit** spheres are piled in 10 layers in the shape of a square pyramid with one sphere in the top layer, 4 spheres in the second layer, 9 spheres in the third layer, and so forth so that the bottom layer has a square array of 100 spheres. In each layer the centers of the spheres form a square grid so that each sphere is tangent to any sphere adjacent to it on the grid. Each sphere in an upper level is tangent to the four spheres directly below it.

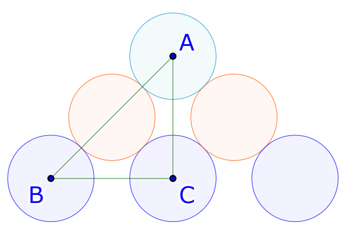
The diagram shows how the first three layers of spheres are stacked.

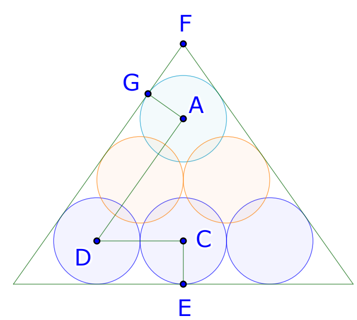


A square pyramid is built around the pile of spheres so that the sides of the pyramid are tangent to the spheres on the outside of the pile. The height of the pyramid is $a + b\sqrt{2} + c\sqrt{3}$, where a, b, and c are positive integers. Find $a + b + c$.

Solution. Let'ss assume that there are n layers. Then the center of any two adjacent spheres are a distance 2 apart. If four spheres in one layer have their centers at four vertices of a 2×2 square, then the centers of the spheres across the square from each other are a distance of $2\sqrt{2}$ apart.

Consider the diagram below on the left showing a cross section of the pile of spheres passing through the center of the sphere on the top layer labelled A, the center of the sphere in the third layer directly below the top sphere labeled C, and the center of a sphere across a 2×2 square from there labeled B. Because AC is vertical, BC is horizontal, $\triangle ABC$ is right triangle with $AB = 4$, $BC = 2\sqrt{2}$., thus $AC = 2\sqrt{2}$. It follows that the vertical distance between the plane containing the centers of spheres at one level of the pile, and the plane containing the center of spheres at the next lower level of the pile is $\sqrt{2}$.





Now, consider the diagram above on the right showing a cross section of the pile of spheres and the enclosing pyramid that passes through the top vertex of the pyramid, the center of the top sphere labeled A, the center of the sphere directly below A in the third level labeled C, and the center of an adjacent sphere in the third level labeled D. Then ACD is a right triangle with legs $AC = 2\sqrt{2}$, $CD = 2$, thus $AD = 2\sqrt{3}$. Let G be a point where the top sphere is tangent to the side of the pyramid. Since $\triangle AFG \sim \triangle CAD$ and $AG = 1$, thus $AF = \sqrt{3}$.

If the pile of sphere has n layers, then the distance between a horizontal plane passing through point A and a plane passing through the centers of the bottom layers is $(n-1)\sqrt{2}$. Since $AF = \sqrt{3}$, and the centers of the spheres are a distance 1 from the bottom of the pyramid, the height of the pyramid is given by

$$(n-1)\sqrt{2} + 1 + \sqrt{3}.$$

Now, since $n = 10$, thus the height is

$$a + b\sqrt{2} + c\sqrt{3} = 1 + 9\sqrt{2} + \sqrt{3} \Rightarrow a = 1, b = 9, c = 1.$$

Thus $a + b + c = \boxed{11.}$ $\square$

Chapter 5

Session 3: Feb 16 - Feb 24

5.1 Middle School - Assignment

Middle school students: grade 8 (US, CA), grade 9 (FR, UK, VN) and younger.

- **Submission deadline: Friday, February 23**
- **Test: Saturday, February 24**
- **Official solutions: Monday, February 26**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 5.1.1 (Problem 1)**.** (*5 points*) A picture with an area of 160 square inches is surrounded by a 2 inch border. The picture with its border is a rectangle twice as long as it is wide. How many inches long is that rectangle?

Solution. Let the width and length of the rectangle be w and $2w$. Then the picture is $w-4$ wide and $2w-4$ long. Its area $(w-4)(2w-4)$ is 160, so $w^2-6w-72=0$, so $w=12$. Thus $2w=\boxed{24.}$ □

Problem 5.1.2 (Problem 2)**.** (*5 points*) Pete's research shows that the number of nuts collected by the squirrels in any park is proportional to the square of the number of squirrels in that park. If Pete notes that four squirrels in a park collect 60 nuts, how many nuts are collected by 20 squirrels in a park?

Solution. The ratio of the number of nuts collected by 20 squirrels to the number collected by four squirrels is $\left(\frac{20}{4}\right)^2=25$. Thus 20 squirrels will collect $60\cdot 25=\boxed{1500.}$ □

Problem 5.1.3 (Problem 3)**.** (*5 points*) How many seven-digit positive integers do not either start or end with 7?

Solution. When selecting the seven-digit positive integer which does not either start or end with 7, there are 8 choices for the first digit (neither 0 or 7), 9 choices for the last digit (not 7), and 10 choices for each of the other five digits. Thus, there are $8\cdot 9\cdot 10^5=\boxed{7\ 200\ 000}$ choices. □

Problem 5.1.4 (Problem 4)**.** (*5 points*) Asheville, Bakersfield, Charter, and Darlington are four small towns along a straight road in that order. The distance from Bakersfield to Charter is one-third the distance from Asheville to Charter and one-quarter the distance from Bakersfield to Darlington. If it is 12 miles from Bakersfield to Charter, how many miles is it from Asheville to Darlington?

Solution. Since the distance from Bakersfield to Charter is one-third the distance from Asheville to Charter, the distance from Asheville to Charter is $3\cdot 12=36$. Similarly, the distance from Bakersfield to Darlington is $4\cdot 12=48$. It follows that the distance from Asheville to Darlington is $36+48-12=\boxed{72.}$ □

Problem 5.1.5 (Problem 5)**.** (*5 points*) Find the sum of all four-digit integers whose digits are a rearrangement of the digits 1, 2, 3, 4, such as 1234, 1432, or 3124.

Solution. There are 4! permutations of the digits 1234. In these permutations each of the four digits appear in each location (thousands, hundreds, tens, unit) exactly 6 times. Thus the sum of all the numbers is $6(1+2+3+4)(1111)=\boxed{66660.}$ □

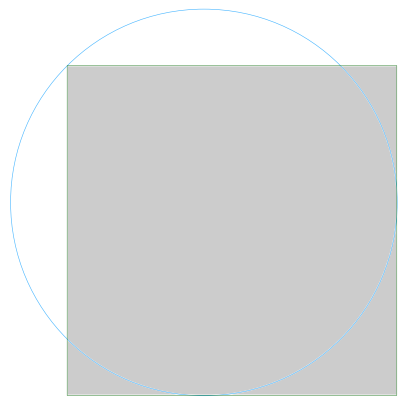
Problem 5.1.6 (Problem 6)**.** (*5 points*) Find the least positive integer k so that the mean of the numbers $k, k+1, k+2, k+3, \ldots, 2k$ exceeds 200.

Solution. The numbers $k, k+1, k+2, k+3, \ldots, 2k$ form an arithmetic sequence, thus the mean is the average of the first and last terms:

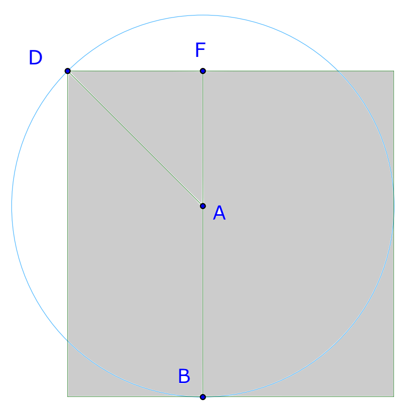
$$\frac{k+2k}{2}=\frac{3k}{2}.$$

If this exceeds 200, then $k>\frac{2\cdot 200}{3}$, or $k\geq\boxed{134.}$ □

Problem 5.1.7 (Problem 7)**.** (*5 points*) In the following diagram two sides of a square are tangent to a circle with diameter 8. One corner of the square lies on the circle. There are positive integers m and n so that the area of the square is $m + \sqrt{n}$. Find $m + n$.



Solution. Let the center of the circle be A, and the corner of the square on the circle be D. Let B and F be points on the sides of the squares so BF perpendicular to the sides of the square so that A lies on BF as shown.



$\triangle ADF$ is isosceles, AD is the radius of length 4, thus $AF = \frac{4}{\sqrt{2}} = 2\sqrt{2}$.

$$BF = AB + AF = 4 + 2\sqrt{2}.$$

So the area of the square is $(4 + 2\sqrt{2})^2 = 24 + 16\sqrt{2} = 24 + \sqrt{512}$. Thus, $m + n = 24 + 512 = \boxed{536.}$ □

Problem 5.1.8 (Problem 8)**.** (*5 points*) Find integer n such that both $n - 86$ and $n + 86$ are perfect squares.

Solution. Let $k^2 = n + 86, \ell^2 = n - 86$, then $k^2 - \ell^2 = 172 = 2^2 \cdot 43$. Thus $(k - \ell)(k + \ell) = 2^2 \cdot 43$. Since both $k - \ell, k + \ell$ have the same parity and their product is even, so both are even: $k - \ell = 2, k + \ell = 86$, so $k = 44, \ell = 42$.

Hence, $n = 42^2 + 86 = \boxed{1850.}$ □

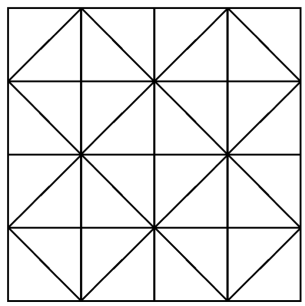
Problem 5.1.9 (Problem 9)**.** (*5 points*) Find n so that $4^{4^{4^2}} = 2^{8^n}$.

Solution.

$$4^{4^{4^2}} = 2^{2\cdot 4^{4^2}},\ 8^n = 2^{3n} \Rightarrow 2\cdot 4^{4^2} = 2^{3n} \Rightarrow 2^{2\cdot 4^2} = 2^{3n-1} \Rightarrow 2\cdot 4^2 = 3n-1 \Rightarrow n = \boxed{11.}$$

□

Problem 5.1.10 (Problem 10)**.** (*5 points*) How many triangles appear in the diagram below?



Solution. First, note that all of the triangles in the diagram are isosceles right triangles whose hypotenuses are either horizontal, vertical, of on an 45° diagonal.

The diagram is made up of 16 small squares which each contain two small triangles. This accounts for 32 small triangles.

Now, count the larger triangles with horizontal hypotenuses, 12 have hypotenuses made up of sides of two of the small squares, and 2 have hypotenuses made up of side of four of small squares for a total of 14 triangles.

Similarly, there are also 14 triangles with vertical hypotenuses.

There are 16 triangles whose hypotenuses are diagonal lines made up of the diagonals from two small squares, and 8 triangles whose hypotenuses are diagonal lines made up of the diagonals from three small squares.

Hence, the number of triangles is $32 + 14 + 14 + 16 + 8 = \boxed{84.}$ □

Problem 5.1.11 (Problem 11)**.** (*5 points*) Let a, b, and c be positive real numbers such that $a^2+b^2+c^2 = 989$ and $(a+b)^2 + (b+c)^2 + (c+a)^2 = 2013$. Find $a+b+c$.

Solution.

$$1024 = 2013-989 = \left[(a+b)^2 + (b+c)^2 + (c+a)^2\right] - \left(a^2+b^2+c^2\right) = a^2+b^2+c^2+2ab+2bc+2ca = (a+b+c)^2$$

Thus, $a+b+c = \boxed{32.}$ □

Problem 5.1.12 (Problem 12)**.** (*5 points*) A quarry wants to sell a large pile of gravel. At full price, the gravel would sell for 3200 dollars. But during the first week the quarry only sells 60% of the gravel at full price. The following week the quarry drops the price by 10%, and, again, it sells 60% of the remaining gravel. Each week, thereafter, the quarry reduces the price by another 10% and sells 60% of the remaining gravel. This continues until there is only a handful of gravel left. How many dollars does the quarry collect for the sale of all its gravel?

Solution. The first week 60% of the gravel is sold for the revenue of 3200(0.6)

The following week the revenue is 3200(0.4)(0.9)(0.6). Each week, 40% of the gravel from the previous week remains, and 60% of it is sold at 90% of the previous price. Thus, during week k, the quarry has a revenue of $3200(0.6)[(0.9)(0.4)]^{k-1}$.

It follows that the total revenue to the quarry is given by the geometric series

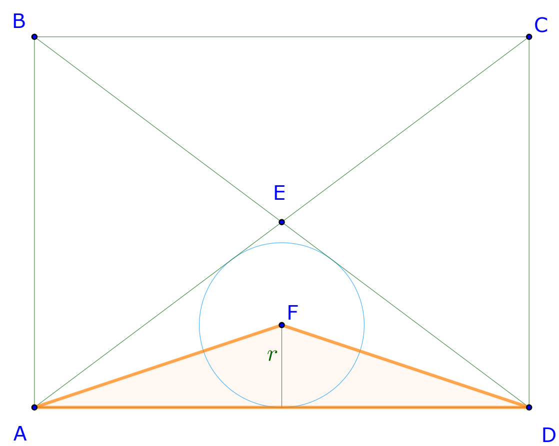
$$\sum_{k=1}^{\infty} 3200(0.6)[(0.9)(0.4)]^{k-1} = 3200(0.6)\sum_{k=0}^{\infty}(0.36)^k = 3200(0.6)\frac{1}{1-0.36} = \frac{3200(0.6)}{0.64} = \boxed{3000.}$$

□

Problem 5.1.13 (Problem 13)**.** (*5 points*) A rectangle has side lengths 6 and 8. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that a point randomly selected from the inside of the rectangle is closer to a side of the rectangle than to either diagonal of the rectangle. Find $m+n$.

Solution. Let the vertices of the rectangle be A, B, C, and D with side $AB = 6$ and $AD = 8$. Let E be the center of the rectangle. Then

$$AE = BE = CE = DE = \frac{1}{2}AC = \frac{1}{2}\sqrt{6^2+8^2} = 5.$$



Let F and r be the incenter and inradius of the $\triangle AED$. Then

$$[AED] = \frac{1}{2}(8\cdot 3) = \frac{8r+5r+5r}{2} \Rightarrow r = \frac{4}{3}.$$

The region with in $\triangle AED$ where the points are closer to side AD to either diagonal AE or DE is the $\triangle AFD$. The area of $\triangle AFD$ is $\frac{1}{2}\cdot 8\cdot\frac{4}{3} = \frac{16}{3}$.

Similarly the area of the region of $\triangle AEB$ of points closer to AB than to either diagonal AE or BE is $\frac{9}{2}$.

The desired probability is

$$\frac{2\frac{16}{3}+2\frac{9}{2}}{6\cdot 8} = \frac{59}{144} \Rightarrow m+n = 59+144 = \boxed{203.}$$

□

Problem 5.1.14 (Problem 14)**.** (*5 points*) Six children stand in a line outside their classroom. When they enter the classroom, they sit in a circle in random order. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that no two children who stood next to each other in the line end up sitting next to each other in the circle. Find $m + n$.

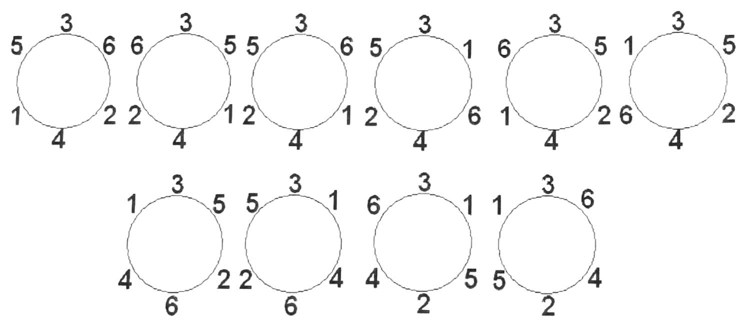
Solution. Let the children be numbered $1, 2, 3, 4, 5$, and 6 in order that they were standing in the line. There are $5! = 120$ equally likely arrangements of these children when they sit in a circle.

To count the number of arrangements where no child sits next to a child that he/she was next to in the original line, consider where the children number 3 and 4 sit.

If child 3 sits opposite child 4, then children 2 and 5 can either sit opposite each other (2 ways) or next to each other (4 ways).

If child 3 and 4 sit with one child between them, then the child between them must be either 1 (2 ways) or 6 (2 ways).

This gives 10 possible arrangements of the children as shown in the diagram below.



The desired probability is $\frac{10}{120} = \frac{1}{12}$. The sum is $m + n = 1 + 12 = \boxed{13}$. ☐

Problem 5.1.15 (Problem 15)**.** (*5 points*) For positive integer n let an be the integer consisting of n digits of 9 followed by the digits 488. For example, $a_3 = 999,488$ and $a_7 = 9,999,999,488$. Find the value of n so that a_n is divisible by the highest power of 2.

Solution. Note that
$$a_n = 10^{n+3} - 512 = 2^{n+3}5^{n+3} - 2^9.$$

If $n < 6$, then $a_n = 2^{n+3}(5^{n+3} - 2^{6-n})$, and the highest power of 2 divides a_n is $2^{n+3} < 2^9$.

If $n > 6$, then $a_n = 2^9(2^{n-6}5^{n+3} - 1)$, and the highest power of 2 divides a_n is 2^9.

If $n = 6$, then $a_6 = 2^9(5^9 - 1)$, which is divisible by at least 2^{10}.

Hence, the value of n is $\boxed{6}$. ☐

5.2 Middle School - Test

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 5.2.1 (Problem 1)**.** (*5 points*) A picture with an area of 180 square inches is surrounded by a 1 inch border. The picture with its border is a rectangle twice as long as it is wide. How many inches long is that rectangle?

Solution. Let the width and length of the rectangle be w and $2w$. Then the picture is $w-2$ wide and $2w-2$ long. Its area $(w-2)(2w-2)$ is 180, so $w^2-3w-88=0$, so $w=11$. Thus $2w = \boxed{22.}$ □

Problem 5.2.2 (Problem 2)**.** (*5 points*) Pete's research shows that the number of nuts collected by the squirrels in any park is proportional to the square of the number of squirrels in that park. If Pete notes that four squirrels in a park collect 15 nuts, how many nuts are collected by 16 squirrels in a park?

Solution. The ratio of the number of nuts collected by 16 squirrels to the number collected by four squirrels is $\left(\frac{16}{4}\right)^2 = 16$. Thus 16 squirrels will collect $15 \cdot 16 = \boxed{240.}$ □

Problem 5.2.3 (Problem 3)**.** (*5 points*) How many six-digit positive integers do not either start or end with 7?

Solution. When selecting the six-digit positive integer which does not either start or end with 7, there are 8 choices for the first digit (neither 0 or 7), 9 choices for the last digit (not 7), and 10 choices for each of the other four digits. Thus, there are $8 \cdot 9 \cdot 10^4 = \boxed{720\ 000}$ choices. □

Problem 5.2.4 (Problem 4)**.** (*5 points*) Asheville, Bakersfield, Charter, and Darlington are four small towns along a straight road in that order. The distance from Bakersfield to Charter is one-third the distance from Asheville to Charter and one-quarter the distance from Bakersfield to Darlington. If it is 18 miles from Bakersfield to Charter, how many miles is it from Asheville to Darlington?

Solution. Since the distance from Bakersfield to Charter is one-third the distance from Asheville to Charter, the distance from Asheville to Charter is $3 \cdot 18 = 54$. Similarly, the distance from Bakersfield to Darlington is $4 \cdot 18 = 72$. It follows that the distance from Asheville to Darlington is $54 + 72 - 18 = \boxed{108.}$ □

Problem 5.2.5 (Problem 5)**.** (*5 points*) Find the sum of all three-digit integers whose digits are a rearrangement of the digits 1, 2, 3, 4, such as 123, 142, or 324.

Solution. There are $\binom{4}{3} = 4$ ways to choose three digits out of 1,2,3,4, let them be a, b, c.

There are 3! permutations of the three chosen digits a, b, c. In these permutations each of the three digits appear in each location (hundreds, tens, unit) exactly 2 times. Thus the sum of all the numbers is $2(a+b+c)(111)$.

The total is $222((1+2+3)+(1+2+4)+(1+3+4)+(2+3+4)) = \boxed{6660.}$ □

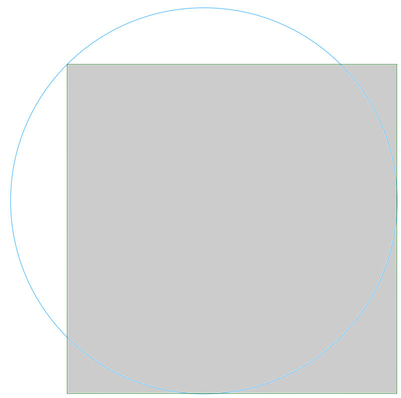
Problem 5.2.6 (Problem 6)**.** (*5 points*) Find the least positive integer k so that the mean of the numbers $k, k+1, k+2, k+3, \ldots, 2k$ exceeds 300.

Solution. The numbers $k, k+1, k+2, k+3, \ldots, 2k$ form an arithmetic sequence, thus the mean is the average of the first and last terms:

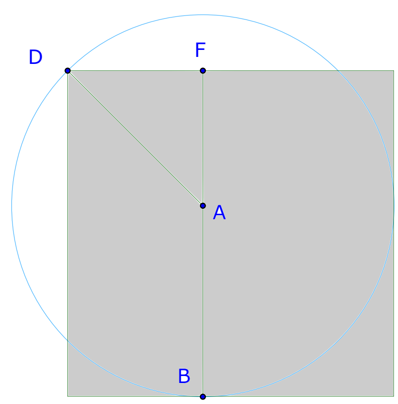
$$\frac{k+2k}{2} = \frac{3k}{2}.$$

If this exceeds 300, then $k > \frac{2 \cdot 300}{3} = 200$, or $k \geq \boxed{201.}$ □

Problem 5.2.7 (Problem 7)**.** (*5 points*) In the following diagram two sides of a square are tangent to a circle with diameter 8. One corner of the square lies on the circle. There are positive integers m and n so that the area of the square is $m + \sqrt{n}$. Find $n - m$.



Solution. Let the center of the circle be A, and the corner of the square on the circle be D. Let B and F be points on the sides of the squares so BF perpendicular to the sides of the square so that A lies on BF as shown.



$\triangle ADF$ is isosceles, AD is the radius of length 4, thus $AF = \frac{4}{\sqrt{2}} = 2\sqrt{2}$.

$$BF = AB + AF = 4 + 2\sqrt{2}.$$

So the area of the square is $(4 + 2\sqrt{2})^2 = 24 + 16\sqrt{2} = 24 + \sqrt{512}$. Thus, $n - m = 512 - 24 = \boxed{488.}$ □

Problem 5.2.8 (Problem 8)**.** (*5 points*) Find the sum of all values of n such that both $n - 70$ and $n + 70$ are perfect squares.

Solution. Let $k^2 = n + 70, \ell^2 = n - 70$, then $k^2 - \ell^2 = 140 = 2^2 \cdot 5 \cdot 7$. Thus $(k - \ell)(k + \ell) = 2^2 \cdot 5 \cdot 7$. Since both $k - \ell, k + \ell$ have the same parity and their product is even, so both are even:

$$\begin{cases} k - \ell = 2, k + \ell = 70 \Rightarrow k = 36, \ell = 34 \Rightarrow n = 1226 \\ k - \ell = 10, k + \ell = 14 \Rightarrow k = 12, \ell = 2 \Rightarrow n = 74 \end{cases}$$

Hence, the sum of all values of n is $1226 + 74 = \boxed{1300.}$ □

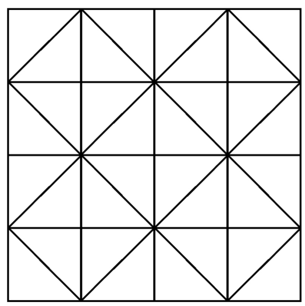
Problem 5.2.9 (Problem 9)**.** (*5 points*) Find n so that $4^{4^{4^2}} = 16^{2^n}$.

Solution.

$$4^{4^{4^2}} = 4^{4^{16}} = 4^{2^{32}} = 4^{2 \cdot 2^{31}} = 16^{2^{31}} \Rightarrow n = \boxed{31.}$$

□

Problem 5.2.10 (Problem 10)**.** (*5 points*) The diagram below is made up of 16 unit squares which each contain two small triangles. How many triangles appear in the diagram below which is **not half of a unit square**?



Solution. First, note that all of the triangles in the diagram are isosceles right triangles whose hypotenuses are either horizontal, vertical, of on an 45° diagonal.

The diagram below is made up of 16 small squares which each contain two small triangles. This accounts for 32 small triangles. These are not the ones we want.

Now, count the larger triangles with horizontal hypotenuses, 12 have hypotenuses made up of sides of two of the small squares, and 2 have hypotenuses made up of side of four of small squares for a total of 14 triangles.

Similarly, there are also 14 triangles with vertical hypotenuses.

There are 16 triangles whose hypotenuses are diagonal lines made up of the diagonals from two small squares, and 8 triangles whose hypotenuses are diagonal lines made up of the diagonals from three small squares.

Hence, the number of triangles is $14 + 14 + 16 + 8 = \boxed{52.}$ □

Problem 5.2.11 (Problem 11)**.** (*5 points*) Let a, b, and c be positive real numbers such that $ab+bc+ca = 432$ and $(a+b)^2 + (b+c)^2 + (c+a)^2 = 2024$. Find $a+b+c$.

Solution.

$$\begin{aligned} 2(a^2+b^2+c^2) &= \left[(a+b)^2 + (b+c)^2 + (c+a)^2\right] - 2(ab+bc+ca) = 1160 \\ \Rightarrow (a+b+c)^2 &= a^2+b^2+c^2+2ab+2bc+2ca = \frac{1160}{2} + 2(432) = 1444. \end{aligned}$$

Thus, $a+b+c = \boxed{38.}$ □

Problem 5.2.12 (Problem 12)**.** (*5 points*) A quarry wants to sell a large pile of gravel. At full price, the gravel would sell for 1600 dollars. But during the first week the quarry only sells 60% of the gravel at full price. The following week the quarry drops the price by 10%, and, again, it sells 60% of the remaining gravel. Each week, thereafter, the quarry reduces the price by another 10% and sells 60% of the remaining gravel. This continues until there is only a handful of gravel left. How many dollars does the quarry collect for the sale of all its gravel?

Solution. The first week 60% of the gravel is sold for the revenue of 1600(0.6)

The following week the revenue is 1600(0.4)(0.9)(0.6). Each week, 40% of the gravel from the previous week remains, and 60% of it is sold at 90% of the previous price. Thus, during week k, the quarry has a revenue of $1600(0.6)[(0.9)(0.4)]^{k-1}$.

It follows that the total revenue to the quarry is given by the geometric series

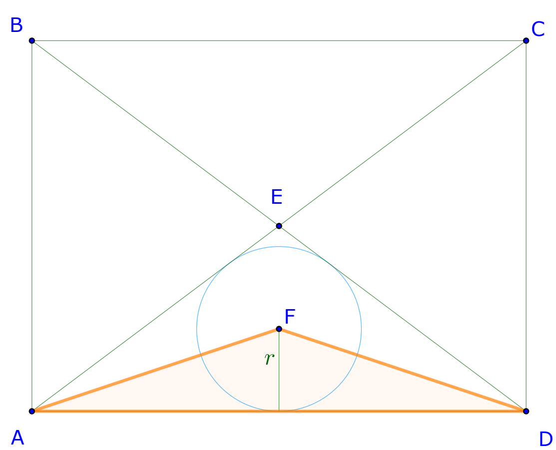
$$\sum_{k=1}^{\infty} 1600(0.6)[(0.9)(0.4)]^{k-1} = 1600(0.6)\sum_{k=0}^{\infty}(0.36)^k = 1600(0.6)\frac{1}{1-0.36} = \frac{1600(0.6)}{0.64} = \boxed{1500.}$$

□

Problem 5.2.13 (Problem 13)**.** (*5 points*) A rectangle has side lengths 6 and 8. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that a point randomly selected from the inside of the rectangle is closer to a side of the rectangle than to either diagonal of the rectangle. Find $n-m$.

Solution. Let the vertices of the rectangle be A, B, C, and D with side $AB = 6$ and $AD = 8$. Let E be the center of the rectangle. Then

$$AE = BE = CE = DE = \frac{1}{2}AC = \frac{1}{2}\sqrt{6^2+8^2} = 5.$$



Let F and r be the incenter and inradius of the $\triangle AED$. Then

$$[AED] = \frac{1}{2}(8\cdot 3) = \frac{8r+5r+5r}{2} \Rightarrow r = \frac{4}{3}.$$

The region with in $\triangle AED$ where the points are closer to side AD to either diagonal AE or DE is the $\triangle AFD$. The area of $\triangle AFD$ is $\frac{1}{2}\cdot 8\cdot\frac{4}{3} = \frac{16}{3}$.

Similarly the area of the region of $\triangle AEB$ of points closer to AB than to either diagonal AE or BE is $\frac{9}{2}$.

The desired probability is

$$\frac{2\frac{16}{3}+2\frac{9}{2}}{6\cdot 8}=\frac{59}{144}\Rightarrow n-m=144-59=\boxed{85.}$$

□

Problem 5.2.14 (Problem 14)**.** (*5 points*) Six children stand in a line outside their classroom. When they enter the classroom, they sit in a circle in random order. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that no two children who stood next to each other in the line end up sitting next to each other in the circle. Find the difference $n-m$.

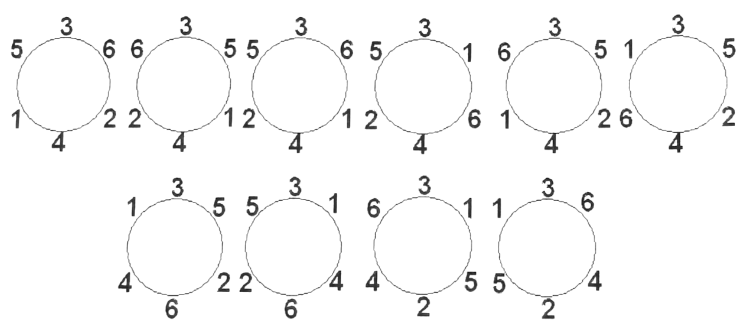
Solution. Let the children be numbered $1,2,3,4,5$, and 6 in order that they were standing in the line. There are $5!=120$ equally likely arrangements of these children when they sit in a circle.

To count the number of arrangements where no child sits next to a child that he/she was next to in the original line, consider where the children number 3 and 4 sit.

If child 3 sits opposite child 4, then children 2 and 5 can either sit opposite each other (2 ways) or next to each other (4 ways).

If child 3 and 4 sit with one child between them, then the child between them must be either 1 (2 ways) or 6 (2 ways).

This gives 10 possible arrangements of the children as shown in the diagram below.



The desired probability is $\frac{10}{120}=\frac{1}{12}$. The sum is $n-m=12-1=\boxed{11}$.

□

Problem 5.2.15 (Problem 15)**.** (*5 points*) For positive integer n let an be the integer consisting of n digits of 9 followed by the digits 1808. For example, $a_2=991,808$ and $a_6=9,999,991,808$. Find the value of n so that a_n is divisible by the highest power of 2.

Solution. Note that

$$a_n=10^{n+4}-8192=2^{n+4}5^{n+4}-2^{13}.$$

If $n<9$, then $a_n=2^{n+4}(5^{n+4}-2^{9-n})$, and the highest power of 2 divides a_n is $2^{n+4}<2^{13}$.

If $n>9$, then $a_n=2^{13}(2^{n-9}5^{n+4}-1)$, and the highest power of 2 divides a_n is 2^{13}.

If $n=9$, then $a_9=2^{13}(5^{13}-1)$, which is divisible by at least 2^{14}.

Hence, the value of n is $\boxed{9}$.

□

5.3 High School - Assignment

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

- **Submission deadline: Friday, February 23**
- **Test: Saturday, February 24**
- **Official solutions: Monday, February 26**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 8-17**
- **Olympiad (O) level: Problems 16-25**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 5.3.1 (Problem 1)**.** (*5 points*) Find the least positive integer n so that both n and $n+1$ have prime factorizations with exactly four (not necessarily distinct) prime factors.

Solution. One of n and $n+1$ is an odd number. The smallest integer with four prime factors is $3^4 = 81$. But neither 80 or 82 has 4 prime factors. The second smallest is $3^3 \cdot 5 = 135$ and $134 = 2 \cdot 67$, $136 = 2^3 \cdot 17$. The latter one has the desired property. The answer is $\boxed{135.}$ □

Problem 5.3.2 (Problem 2)**.** (*5 points*) Two convex polygons have a total of 33 sides and 243 diagonals. Find the number of diagonals in the polygon with the greater number of sides.

Solution. A convex polygon has n sides then it has $\frac{n(n-1)}{2} - n = \frac{n^2-3n}{2}$ diagonals.

Now

$$243 = \frac{n^2-3n}{2} + \frac{(33-n)^2 - 3(33-n)}{2} = n^2 - 33n + 495 \Rightarrow n^2 - 33n + 252 = 0 \Rightarrow n = 12 \text{ or } n = 21$$

The number of diagonals of the polygon with 21 sides is $\frac{21^2 - 3\cdot 21}{2} = \boxed{189.}$ □

Problem 5.3.3 (Problem 3)**.** (*5 points*) In the tribe of Zimmer, being able to hike long distances and knowing the roads through the forest are both extremely important, so a boy who reaches the age of manhood is not designated as a man by the tribe until he completes an interesting rite of passage. The man must go on a sequence of hikes. The first hike is a 5 kilometer hike down the main road. The second hike is a $5\frac{1}{4}$ kilometer hike down a secondary road. Each hike goes down a different road and is a quarter kilometer longer than the previous hike. The rite of passage is completed at the end of the hike where the cumulative distance walked by the man on all his hikes exceeds 1000 kilometers. So in the tribe of Zimmer, how many roads must a man walk down, before you call him a man?

Solution. The total distance a man walked in the n hikes is

$$5 + 5\frac{1}{4} + 5\frac{1}{2} + \cdots + \frac{19+n}{4} = \sum_{k=1}^{n} \frac{19+k}{4} = \frac{1}{4}\left(19n + \frac{n(n+1)}{2}\right) = \frac{n(n+39)}{8}$$

The smallest n for $n^2 + 39n > 8 \cdot 1000$ is $n = \boxed{73.}$ □

Problem 5.3.4 (Problem 4)**.** (*5 points*) Find the value of x that satisfies $\log_3(\log_9 x) = \log_9(\log_3 x)$.

Solution. Note that

$$\log_9 x = \frac{\log_3 x}{\log_3 9} \Rightarrow \begin{cases} \log_3(\log_9 x) = \log_3\left(\frac{\log_3 x}{\log_3 9}\right) = \log_3\left(\frac{\log_3 x}{2}\right) \\ \log_9(\log_3 x) = \frac{\log_3(\log_3 x)}{\log_3 9} = \frac{\log_3(\log_3 x)}{2} \end{cases}$$

$$\Rightarrow \log_3\left(\frac{\log_3 x}{2}\right) = \frac{\log_3(\log_3 x)}{2} \Rightarrow 2\log_3\left(\frac{\log_3 x}{2}\right) = \log_3(\log_3 x) \Rightarrow \log_3\left(\frac{\log_3 x}{2}\right)^2 = \log_3(\log_3 x)$$

Thus $\left(\frac{\log_3 x}{2}\right)^2 = \log_3 x$. Since $\log_3 x > 0$, $\log_3 x = 4 \Rightarrow x = \boxed{81.}$ □

Problem 5.3.5 (Problem 5)**.** (*5 points*) Consider a sequence of eleven squares that have side lengths $3, 6, 9, 12, \ldots, 33$. Eleven copies of a single square each with area A have the same total area as the total area of the eleven squares of the sequence. Find A.

Solution. It is easy to see that the total area of eleven squares is:

$$3 \cdot 3 + 6 \cdot 6 + \cdots + 33 \cdot 33 = 9(1^2 + 2^2 + \cdots + 11^2) = 9 \cdot \frac{11 \cdot 12 \cdot 23}{6} = 11 \cdot 414 = 11 \cdot A$$

Hence, the area $A = \boxed{414.}$ □

Problem 5.3.6 (Problem 6)**.** (*5 points*) Define $f(x) = 2x + 3$and suppose that $g(x+2) = f(f(x-1) \cdot f(x+1) + f(x))$. Find $g(6)$.

Solution.

$$g(6) = g(4+2) = f(f(4-1) \cdot f(4+1) + f(4)) = f(f(3)f(5) + f(4)) = f(9 \cdot 13 + 11) = f(128) = \boxed{259.}$$

□

Problem 5.3.7 (Problem 7)**.** (*5 points*) Ted flips seven fair coins. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that Ted flips at least two heads given that he flips at least three tails. Find $m + n$.

Solution. The total number of possibilities of flipping k tails with seven flips is $\binom{7}{k}$, thus the probability that Ted flips at least three tails is

$$\frac{\binom{7}{3} + \binom{7}{4} + \binom{7}{5}}{\binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}} = \frac{91}{99}.$$

The answer is $m + n = 91 + 99 = \boxed{190.}$ □

Problem 5.3.8 (Problem 8)**.** (*5 points*) Find the least n for which $n!(n+1)!(2n+1)! - 1$ ends in thirty digits that are all 9's.

Solution. If $n!(n+1)!(2n+1)! - 1$ ends in thirty digits that are all 9's then $n!(n+1)!(2n+1)!$ ends in thirty digits that are all 0's.

Let's try to find if the least $n < 100$. The number of trailing zeros in $n!$ is the number of factors of 5 in $n!$, which is $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor$.

So we are trying to find an n such that the following sum is at least 30:

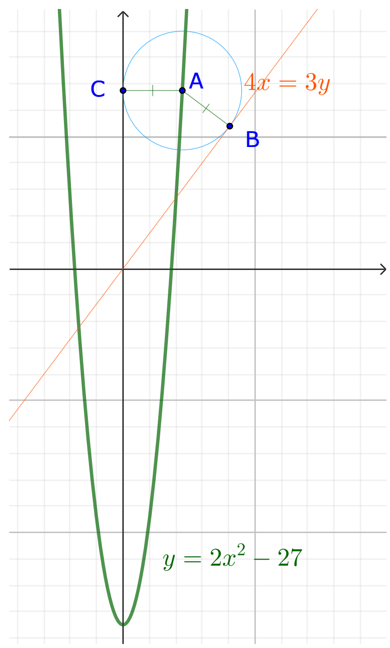
$$S(n) = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n+1}{5} \right\rfloor + \left\lfloor \frac{n+1}{25} \right\rfloor + \left\lfloor \frac{2n+1}{5} \right\rfloor + \left\lfloor \frac{2n+1}{25} \right\rfloor + \left\lfloor \frac{2n+1}{125} \right\rfloor$$

For example $S(25) = 5 + 1 + 5 + 1 + 10 + 2 = 24$, $S(27) = 6 + 6 + 11 + 2 = 25$. By continuing doing so $S(34) = 30$. Hence, the answer is $\boxed{34.}$ □

Problem 5.3.9 (Problem 9)**.** (*5 points*) A circle in the first quadrant with center on the curve $y = 2x^2 - 27$ is tangent to the $y-$axis and the line $4x = 3y$. The radius of the circle is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Solution. The distance from a point $A(x_0, y_0)$ to the line $4x = 3y$ is given by the normal form for the equation of the line $(4)x + (-3)y + 0 = 0$,

$$\frac{|4x_0 + (-3)y_0 + 0|}{\sqrt{(4^2 + (-3)^2)}} = \frac{|4x_0 - 3y_0|}{5}.$$



If point A, in the first quadrant, is on the curve $y = 2x^2 - 27$ at the same distance from the $y-$axis and the line $4x = 3y$, then the distance to the $y-$axis, or its $x-$coordinate (AC), must be equal to the distance to the line $4x = 3y$, thus

$$x_0 = \frac{|4x_0 - 3y_0|}{5} \Rightarrow x_0 = \frac{|4x_0 - 3(x_0^2 - 27)|}{5} = \frac{3(2x_0^2 - 27) - 4x_0}{5} \Rightarrow 6x_0^2 - 9x_0 - 81 = 0 \Rightarrow x_0 = \frac{9}{2} \ (x_0 > 0)$$

Hence, $m + n = 9 + 2 = \boxed{11.}$ □

Problem 5.3.10 (Problem 10)**.** (*5 points*) Let N be a positive integer whose digits add up to 23. What is the greatest possible product the digits of N can have?

Solution. First, digits 0 and 1 would not be among the digits of N.

A digit of 4 can be replaced by two digits of 2 having the same sum and same product.

Any digit k larger then or equal to 5 can be replace by a pair of 2 and $k - 2$, where $2(k - 2) = 2k - 4 > k$.

Therefore N has a digits of 2 and b digits of 3, where a and b are non-negative integers and the product of them is $2^a 3^b$, where $2a + 3b = 23$.

Now, $2 + 2 + 2 = 3 + 3$, and $2 \cdot 2 \cdot 2 < 3 \cdot 3$, thus a can be 0, 1 or 2. Only $a = 1$ gives $b = 7$ integer value.

Thus $N = 2 \cdot 3^7 = \boxed{4374.}$ □

Problem 5.3.11 (Problem 11)**.** (*5 points*) Let a, b, and c be non-zero real numbers such that

$$\frac{ab}{a+b} = 3, \frac{bc}{b+c} = 4, \frac{ca}{c+a} = 5.$$

There are relatively prime positive integers m and n so that

$$\frac{abc}{ab+bc+ca} = \frac{m}{n}.$$

Find $m+n$.

Solution.

$$\begin{aligned}
&\frac{ab}{a+b} = 3 \Rightarrow \frac{a+b}{ab} = \frac{1}{3} \Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{1}{3} \\
&\text{Similarly } \frac{1}{b} + \frac{1}{c} = \frac{1}{4}, \ \frac{1}{c} + \frac{1}{a} = \frac{1}{5} \\
&\Rightarrow \frac{ab+bc+ca}{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{47}{120}
\end{aligned}$$

Hence, $m+n = 47+120 = \boxed{167.}$ □

Problem 5.3.12 (Problem 12)**.** (*5 points*) How many positive integer solutions are there to $w+x+y+z = 20$ where $w+x \geq 5$ and $y+z \geq 5$?

Solution. Let $k = w+x$, then $y+z = 20-k$. For a value of $k \geq 2$, there are $k-1$ ways to assign positive value to w and x so that their sum is k.

Similarly with $20-k \geq 2$, there are $19-k$ ways to assign positive value to w and x so that their sum is $20-k$.

Hence, the number of ways is:

$$\sum_{k=5}^{15}(k-1)(19-k) = \sum_{k=1}^{11}(k+3)(15-k) = \sum_{k=1}^{11}(45+12k-k^2) = 11\cdot 45 + 12\frac{11\cdot 12}{2} - \frac{11\cdot 12\cdot 23}{6} = \boxed{781.}$$

□

Problem 5.3.13 (Problem 13)**.** (*5 points*) Find the number of three-digit numbers such that its first two digits are each divisible by its third digit.

Solution. It is obvious that the third digit cannot be 0. There are five cases:

3rd digit	1st digit	2nd digit	# of ways
1	1,2,3,4,5,6,7,8,9	0,1,2,3,4,5,6,7,8,9	90
2	2,4,6,8	0,2,4,6,8	20
3	3,6,9	0,3,6,9	12
4	4,8	0,4,8	6
5,6,7,8,9	same as 3rd	0 or same as 3rd	10

Hence, the number of ways is $90+20+12+6+10 = \boxed{138.}$ □

Problem 5.3.14 (Problem 14)**.** (*5 points*) Find the remainder when $2^{5^9} + 5^{9^2} + 9^{2^5}$ is divided by 11.

Solution. By Fermat's Little Theorem: $\gcd(n, p) = 1$, then $n^{p-1} \equiv 1 \pmod{p}$, so:

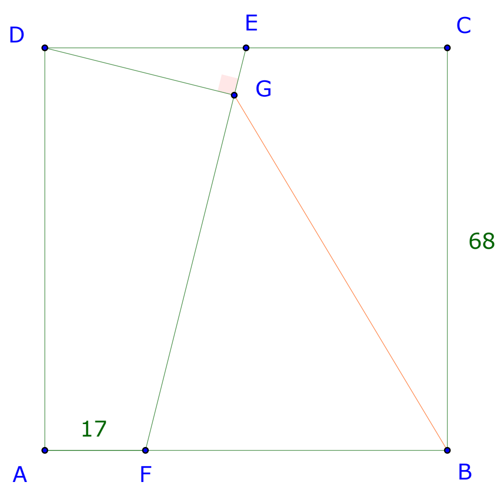
$$2^{10} \equiv 5^{10} \equiv 9^{10} \equiv 1 \pmod{11}.$$

It is easy to see remainders of $5^9, 9^2$, and 2^5 modulo 10 are 5, 1, and 2. Thus:

$$2^{5^9} + 5^{9^2} + 9^{2^5} \equiv 2^5 + 5^1 + 9^2 = 32 + 5 + 81 \equiv \boxed{8} \pmod{11}$$

$\square$

Problem 5.3.15 (Problem 15)**.** (*5 points*) Square $ABCD$ has side length 68. Let E be the midpoint of segment CD, and let F be the point on segment AB a distance 17 from point A. Point G is on segment EF so that EF is perpendicular to segment GD. The length of segment BG can be written as $m\sqrt{n}$, where m and n be are positive integers, and n is not divisible by the square of any prime. Find $m + n$.



Solution. Let vector $\overrightarrow{BC} = u$, and $\overrightarrow{BA} = v$. Then $\overrightarrow{EF} = \frac{1}{4}v - u$. Note that $\overrightarrow{EG}$ is the projection of $\overrightarrow{ED} = \frac{1}{2}v$ onto $\overrightarrow{EF}$. This projection is:

$$\left(\frac{\overrightarrow{ED} \cdot \overrightarrow{EF}}{\overrightarrow{EF} \cdot \overrightarrow{EF}}\right)\overrightarrow{EF} = \frac{\frac{1}{2}v \cdot \left(\frac{1}{4}v - u\right)}{\left(\frac{1}{4}v - u\right) \cdot \left(\frac{1}{4}v - u\right)}\left(\frac{1}{4}v - u\right) = \frac{\frac{1}{8}}{\frac{1}{16} + 1}\left(\frac{1}{4}v - u\right) = \frac{1}{34}v - \frac{2}{17}u$$

Then

$$\overrightarrow{BG} = u + \frac{1}{2}v + \overrightarrow{EG} = \frac{15}{17}u + \frac{9}{17}v$$

The length of this vector is

$$\frac{68}{17}\sqrt{15^2 + 9^2} = 12\sqrt{34} \Rightarrow m + n = 12 + 34 = \boxed{46.}$$

$\square$

Problem 5.3.16 (Problem 16)**.** (*5 points*) Each time you click a toggle switch, the switch either turns from *off* to *on* or from on to *off*. Suppose that you start with three toggle switches with one of them *on* and two of them *off*. On each move you randomly select one of the three switches and click it. Let m and n be relatively prime positive integers so that $\frac{m}{n}$ is the probability that after four such clicks, one switch will be *on* and two of them will be *off*. Find $m + n$.

Solution. Without loss of generality assume that at the beginning the first of the three toggle switches is *on*, and the second and third are *off*.

There are $3^4 = 81$ equally likely ways to select a sequence of four toggle switches to click. After four clicks, there will either be one or three switches in the *on* position.

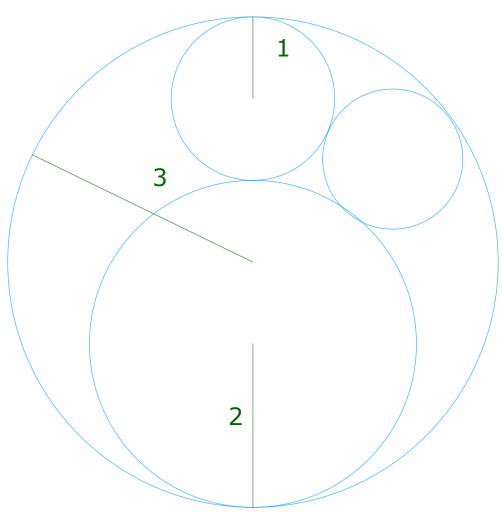
To get three toggle switches in the *on* position, one much click the first switch an even number of times and the second and third switches an odd number of times each. This can be done by clicking the first switch twice and the other switches once each, or by clicking either the second and third switches three times and the other of the two switches one time. The number of ways to do one of these is:

$$\binom{4}{2,1,1} + \binom{4}{0,1,3} + \binom{4}{0,3,1} = \frac{4!}{2!1!1!} + \frac{4!}{0!1!3!} + \frac{4!}{0!3!1!} = 12 + 4 + 4 = 20.$$

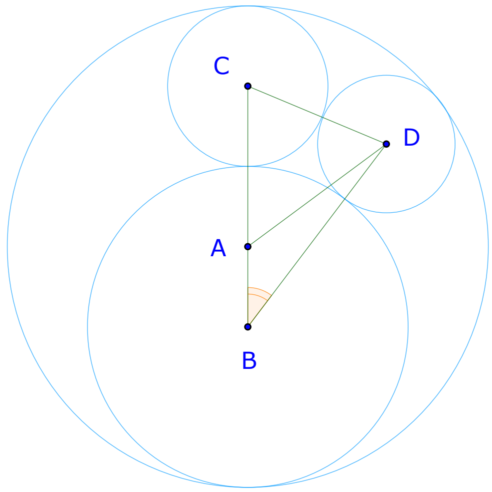
It follows that the desired probability is $\frac{81-20}{81} = \frac{61}{81}$.

Thus the desired sum $61 + 81 = \boxed{142.}$ □

Problem 5.3.17 (Problem 17)**.** (*5 points*) The diagram below shows circles radius 1 and 2 externally tangent to each other and internally tangent to a circle radius 3. There are relatively prime positive integers m and n so that a circle radius $\frac{m}{n}$ is internally tangent to the circle radius 3 and externally tangent to the other two circles as shown. Find $m + n$.



Solution. Let A, B, C, and D be the centres of the circles radius 3, 2, 1 and $r = \frac{m}{n}$.



By the Law of Cosines:

$$\begin{aligned}
&\triangle ABD: \ 1^2 + (2+r)^2 - 2\cdot 1\cdot(2+r)\cos(\angle ABD) = (3-r)^2 \\
&\triangle CBD: \ 3^2 + (2+r)^2 - 2\cdot 3\cdot(2+r)\cos(\angle ABD) = (1+r)^2 \\
&\Rightarrow 3(3-r)^2 - (1+r)^2 = -6 + 2(2+r)^2 \Rightarrow r = \frac{6}{7}.
\end{aligned}$$

Thus $m + n = 6 + 7 = \boxed{13.}$ □

Problem 5.3.18 (Problem 18)**.** (*5 points*) Find the greatest seven-digit integer divisible by 132 whose digits, in order, are $2, 0, x, y, 1, 2, z$ where x, y, and z are single digits.

Solution. [Solution 1] Let n be the greatest seven-digit integer divisible by 132, whose digits, in order, are $2, 0, x, y, 1, 2, z$ where x, y, and z are single digits.

$$\begin{aligned}
&3 \mid 132 \Rightarrow 5 + x + y + z \equiv 0 \pmod 3 \\
&11 \mid 132 \Rightarrow 1 + x - y + z \equiv 0 \pmod{11} \\
&4 \mid 132 \Rightarrow z \equiv 0 \pmod 4
\end{aligned}$$

Thus, $z \in \{0, 4, 8\}$. n would be greatest if $x = 9$, in this case $y \equiv 1 - z \pmod 3$, and $y \equiv z - 1 \pmod{11}$.

$z = 0$ means $y \equiv 10 \pmod{11}$, which is impossible.

$z = 8$ means $y \equiv 1 - 8 \equiv 2 \pmod 3$, and $y \equiv 7 \pmod{11}$, which is impossible.

$z = 4$ means $y \equiv 1 - 4 \equiv 0 \pmod 3$, $y \equiv 4 - 1 = 3 \pmod{11}$, thus $y = 3$.

Hence, the number is $\boxed{2093124.}$ □

Solution. [Solution 2] We show a coding solution. The problem is equivalent to find the largest multiple of 132 in the format $\overline{20xy12z}$.

```
for x in range(9, -1, -1):
    for y in range(9, -1, -1):
        for z in range(9, -1, -1):
            i = 2000000 + x * 10000 + y * 1000  + 120 + z
            if i % 132 == 0:
                print(i)
                exit(0)
```

The answer is $\boxed{2093124.}$ □

Problem 5.3.19 (Problem 19). (*5 points*) There are positive integers m and n so that $x = m + \sqrt{n}$ is a solution to the equation

$$x^2 - 10x + 1 = \sqrt{x}(x+1).$$

Find $m + n$.

Solution. It is easy to see that $x = 0$ is not a solution. Divide both side of the equation by x, we have

$$x - 10 + \frac{1}{x} = \sqrt{x} + \frac{1}{\sqrt{x}}.$$

Let $y = \sqrt{x} + \frac{1}{\sqrt{x}} > 0$, then $y^2 = x + \frac{1}{x} + 2$, thus

$$\begin{aligned} &y^2 - 12 = y \Rightarrow (y+3)(y-4) = 0 \Rightarrow y = 4 \Rightarrow \sqrt{x} + \frac{1}{\sqrt{x}} = 4 \Rightarrow x - 4\sqrt{x} + 1 = 0 \\ &\Rightarrow \sqrt{x} = 2 \pm 3 \Rightarrow x = (2+\sqrt{3})^2 = 7 + \sqrt{48}. \end{aligned}$$

Thus $m + n = 7 + 48 = \boxed{55.}$ □

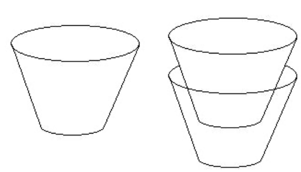
Problem 5.3.20 (Problem 20). (*5 points*) Find the largest prime that divides $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + 44 \cdot 45 \cdot 46$.

Solution. Note that $(k-1)k(k+1) = (k^2-1)k = k^3 - k$, thus the given sum is equivalent to

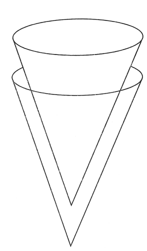
$$\sum_{k=2}^{45}\left(k^3 - k\right) = \sum_{k=1}^{45} k^3 - \sum_{k=1}^{45} k = \left(\frac{45 \cdot 46}{2}\right)^2 - \frac{45 \cdot 46}{2} = 45 \cdot 23\,(45 \cdot 23 - 1) = 2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 23 \cdot 47.$$

Thus the largest prime factor is $\boxed{47}$. □

Problem 5.3.21 (Problem 21). (*5 points*) A paper cup has a base that is a circle with radius r, a top that is a circle with radius $2r$, and sides that connect the two circles with straight line segments as shown below. This cup has height h and volume V. A second cup that is exactly the same shape as the first is held upright inside the first cup so that its base is a distance of $\frac{h}{2}$ from the base of the first cup. The volume of liquid that will fit inside the first cup and outside the second cup can be written as $\frac{m}{n} \cdot V$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. The volume between the cups does not change if the sides of each cup are extended so that the cups form a circular cones with radius $2r$ and height $2h$.



The volume of the first cone is:

$$\frac{1}{3}\pi(2r)^2(2h) = \frac{8}{3}\pi r^2 h.$$

The part of the second cone inside the first is a cone that is $\frac{3}{4}$ the height of the first cone, so the volume that it takes up within the first cone is:

$$\left(\frac{3}{4}\right)^3 \frac{8}{3}\pi r^2 h = \frac{9}{8}\pi r^2 h$$

Thus the volume inside the first cup and outside of the second cup is

$$\left(\frac{8}{3} - \frac{9}{8}\right)\pi r^2 h = \frac{37}{24}\pi r^2 h$$

Thus, the volume of the first cup is

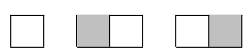
$$\left(1 - \left(\frac{1}{2}\right)^3\right)\frac{4}{3}\pi r^2(2h) = \frac{7}{3}\pi r^2 h$$

The desired fraction is:

$$\frac{\frac{37}{24}}{\frac{7}{3}} = \frac{37}{56} \Rightarrow m + n = 37 + 56 = \boxed{93}.$$

□

Problem 5.3.22 (Problem 22)**.** (*5 points*) You have some white one-by-one tiles and some black and white two-by-one tiles as shown below. There are four different color patterns that can be generated when using these tiles to cover a three-by-one rectangle by laying these tiles side by side (WWW, BWW, WBW, WWB). How many different color patterns can be generated when using these tiles to cover a ten-by-one rectangle?



Solution. Let A_j be the number of colour patterns one can obtain by laying these tiles together to tile a $j \times 1$ rectangle. It is easy to see that $A_1 = 1$, $A_2 = 3$, and $A_3 = 4$.

For $j > 2$, categorize the colouring patterns by the position of the first black square among the j square.

If the colouring patter begins with a black square in its first position, the pattern must have a 2×1 tile followed by one of the A_{j-2} patterns of length $j - 2$.

Similarly if the colouring patter begins with a white square in the first position, and a black square in the second position, the first two squares be followed by any of the A_{j-2} patterns of length $j - 2$.

If the first black square appears in the k^{th} position, where $k > 2$, it can be followed by any of the A_{j-k}, pattern of length $j - k$. Note that if the first black square appears in the last position, it constitutes a pattern. Finally, there is one pattern consisting of all white square. Thus:

$$A_j = A_{j-2} + A_{j-2} + A_{j-3} + A_{j-4} + \ldots + A_1 + 1 + 1.$$

Below is the number of patterns for each of $j \times 1$ rectangles, where $j = 1, 2, \ldots, 10$.

1	2	3	4	5	6	7	8	9	10
1	3	4	9	14	28	47	89	155	286

The answer is $\boxed{286}$.

□

Problem 5.3.23 (Problem 23)**.** (*5 points*) A bag contains 8 green candies and 4 red candies. You randomly select one candy at a time to eat. If you eat five candies, there are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that you do not eat a green candy after you eat a red candy. Find $m+n$.

Solution. The five candies can be eaten in order

$$GRRRR,\ GGRRR,\ GGGRR,\ GGGGR,\ GGGGG.$$

The probability of eating the candies in these orders are, respectively:

$$\begin{cases} GRRRR: & \frac{8}{12}\cdot\frac{4}{11}\cdot\frac{3}{10}\cdot\frac{2}{9}\cdot\frac{1}{8} \\ GGRRR: & \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{4}{10}\cdot\frac{3}{9}\cdot\frac{2}{8} \\ GGGRR: & \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{4}{9}\cdot\frac{3}{8} \\ GGGGR: & \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8} \\ GGGGG: & \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8} \end{cases}$$

Their sum is

$$\frac{8\cdot4\cdot3\cdot2\cdot1+8\cdot7\cdot4\cdot3\cdot2+8\cdot7\cdot6\cdot4\cdot3+8\cdot7\cdot6\cdot5\cdot4\cdot(2)}{12\cdot11\cdot10\cdot9\cdot8}=\frac{1}{5}.$$

Thus $m+n=1+5=\boxed{6.}$ □

Problem 5.3.24 (Problem 24)**.** (*5 points*) Let $A=1,3,5,7,9$ and $B=2,4,6,8,10$. Let f be a randomly chosen function from the set $A\cup B$ into itself. There are relatively prime positive integers m and n such that $\frac{m}{n}$ is the probability that f is a one-to-one function on $A\cup B$ given that it maps A one-to-one into $A\cup B$ and it maps B one-to-one into $A\cup B$. Find $m+n$.

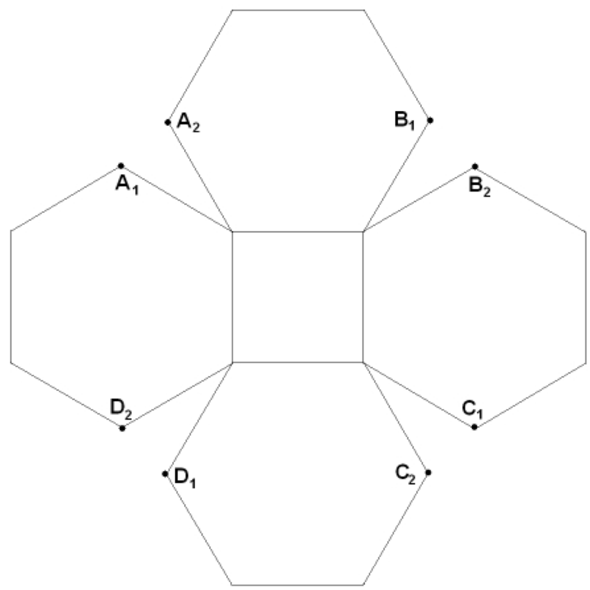
Solution. A one-to-one function f that maps $A\cup B$ into itself is just a permutation of the set $A\cup B$. Thus there are 10! such functions.

The number of one-to-one functions that map A into $A\cup B$ is the number of ways to map each element of A to a distinct element of $A\cup B$, or $10\cdot9\cdot8\cdot7\cdot6$. That is also the number of one-to-one functions that map B into $A\cup B$. Thus the desired probability isosceles

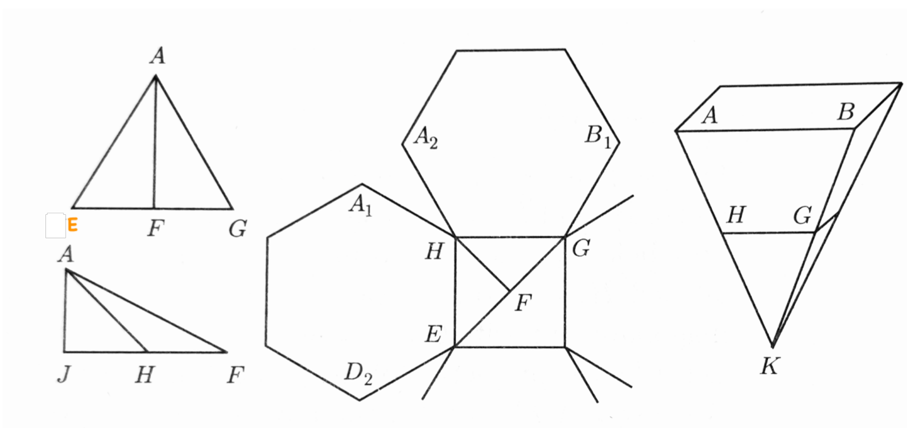
$$\frac{10!}{(10\cdot9\cdot8\cdot7\cdot6)^2}=\frac{5\cdot4\cdot3\cdot2\cdot1}{10\cdot9\cdot8\cdot7\cdot6}=\frac{1}{252}.$$

The answer is $m+n=1+252=\boxed{253.}$ □

Problem 5.3.25 (Problem 25)**.** (*5 points*) The diagram below shows four regular hexagons each with side length 1 meter attached to the sides of a square. This figure is drawn onto a thin sheet of metal and cut out. The hexagons are then bent upward along the sides of the square so that A_1 meets A_2, B_1 meets B_2, C_1 meets C_2, and D_1 meets D_2. The resulting dish is set on a table with the square lying flat on the table. If this dish is filled with water, the water will rise to the height of the corner where the A_1 and A_2 meet. There are relatively prime positive integers m and n so that the number of cubic meters of water the dish will hold is $\sqrt{\frac{m}{n}}$. Find $m+n$.



Solution. Let the point where A_1 and A_2 meet be labeled A, and the point where B_1 and B_2 meet be labeled B. Let the center of the square be labeled F. Let the corner of the square nearest point A be labeled H, the corner of nearest point B be labeled G, and the corner diagonally opposite G be labeled E as shown below.



Let J be the projection of the point A in the plane of the square. Because $\angle AHE = 120°$, the Law of Cosines gives that

$$AE = \sqrt{AH^2 + HE^2 - 2(AH)(HE)\cos(120°)} = \sqrt{1+1+1} = \sqrt{3}.$$

Since $EF = HF = \sqrt{2}2$, the Pythagorean Theorem gives that

$$AF = \sqrt{AE^2 - EF^2} = \sqrt{3 - \frac{1}{2}} = \sqrt{\frac{5}{2}}.$$

The Law of Cosines then gives

$$\cos(\angle AHF) = \frac{AF^2 - AH^2 - HF^2}{-2(AH)(HF)} = \frac{\frac{5}{2} - 1 - \frac{1}{2}}{-\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

Therefore $\angle AHF = 135°$, $\angle AHJ = 45°$. Thus $AJ = \frac{1}{\sqrt{2}}$ is the height of A from the plane of the square. Since the diameter of a regular hexagon is twice the length of one of its sides, it follows that $AB = 2$.

If the vertical edges of the dish AH and BG are extended, they will meet at a point K as shown in the diagram. Note that K is the vertex of a downward pointing square pyramid with base having side length 2 and with height $\sqrt{2}$.

The volume of this pyramid is given by $\frac{1}{3}$ the area of its base times its height $\frac{1}{3} \cdot 4 \cdot \sqrt{2}$. The lower half of the pyramid is another square pyramid with base having side length 1 and with height $\frac{\sqrt{2}}{2}$ so its volume is $\frac{1}{3} \cdot 1 \cdot \sqrt{2}$.

The desired volume is

$$\frac{4}{3} \cdot \sqrt{2} - \frac{1}{3} \cdot \frac{\sqrt{2}}{2} = \frac{7\sqrt{2}}{2} = \sqrt{\frac{49}{18}}.$$

Thus $m + n = 49 + 18 = \boxed{67.}$ $\square$

5.4 High School - Test

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 8-17**
- **Olympiad (O) level: Problems 16-25**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 5.4.1 (Problem 1)**.** (*5 points*) Find the least positive integer n so that both n and $n+1$ have prime factorizations with exactly three (not necessarily distinct) prime factors.

Solution. One of n and $n+1$ is an odd number. The smallest integer with four prime factors is $3^3 = 27$. And $28 = 2^2 \cdot 7$ has the desired property. The answer is $\boxed{27.}$ □

Problem 5.4.2 (Problem 2)**.** (*5 points*) Two convex polygons have a total of 33 sides and 279 diagonals. Find the number of diagonals in the polygon with the greater number of sides.

Solution. A convex polygon has n sides then it has $\frac{n(n-1)}{2} - n = \frac{n^2-3n}{2}$ diagonals.

Now

$$279 = \frac{n^2-3n}{2} + \frac{(33-n)^2 - 3(33-n)}{2} = n^2 - 33n + 495 \Rightarrow n^2 - 33n + 216 = 0 \Rightarrow n = 9 \text{ or } n = 24$$

The number of diagonals of the polygon with 24 sides is $\frac{24^2-3\cdot 24}{2} = \boxed{252.}$ □

Problem 5.4.3 (Problem 3)**.** (*5 points*) In the tribe of Zimmer, being able to hike long distances and knowing the roads through the forest are both extremely important, so a boy who reaches the age of manhood is not designated as a man by the tribe until he completes an interesting rite of passage. The man must go on a sequence of hikes. The first hike is a 5 kilometer hike down the main road. The second hike is a $5\frac{1}{4}$ kilometer hike down a secondary road. Each hike goes down a different road and is a quarter kilometer longer than the previous hike. The rite of passage is completed at the end of the hike where the cumulative distance walked by the man on all his hikes exceeds **500** kilometers. So in the tribe of Zimmer, how many roads must a man walk down, before you call him a man?

Solution. The total distance a man walked in the n hikes is

$$5 + 5\frac{1}{4} + 5\frac{1}{2} + \cdots + \frac{19+n}{4} = \sum_{k=1}^{n} \frac{19+k}{4} = \frac{1}{4}\left(19n + \frac{n(n+1)}{2}\right) = \frac{n(n+39)}{8}$$

The smallest n for $n^2 + 39n > 8 \cdot 500$ is $n = \boxed{47.}$ □

Problem 5.4.4 (Problem 4)**.** (*5 points*) Find the value of x that satisfies $\log_2(\log_4 x) = \log_4(\log_2 x)$.

Solution. Note that

$$\log_4 x = \frac{\log_2 x}{\log_2 4} \Rightarrow \begin{cases} \log_2(\log_4 x) = \log_2\left(\frac{\log_2 x}{\log_2 4}\right) = \log_2\left(\frac{\log_2 x}{2}\right) \\ \log_4(\log_2 x) = \frac{\log_2(\log_2 x)}{\log_2 4} = \frac{\log_2(\log_2 x)}{2} \end{cases}$$

$$\Rightarrow \log_2\left(\frac{\log_2 x}{2}\right) = \frac{\log_2(\log_2 x)}{2} \Rightarrow 2\log_2\left(\frac{\log_2 x}{2}\right) = \log_2(\log_2 x) \Rightarrow \log_2\left(\frac{\log_2 x}{2}\right)^2 = \log_2(\log_2 x)$$

Thus $\Rightarrow \left(\frac{\log_2 x}{2}\right)^2 = log_2 x$. Since $\log_2 x > 0$, $\log_2 x = 4 \Rightarrow x = \boxed{16.}$ □

Problem 5.4.5 (Problem 5)**.** (*5 points*) Consider a sequence of twelve squares that have side lengths $3, 6, 9, 12, \ldots, 33, 36$. Ten copies of a single square each with area A have the same total area as the total area of the twelve squares of the sequence. Find A.

Solution. It is easy to see that the total area of twelve squares is:

$$3\cdot 3 + 6\cdot 6 + \cdots + 36\cdot 36 = 9(1^2 + 2^2 + \cdots + 12^2) = 9\cdot\frac{12\cdot 13\cdot 25}{6} = 10\cdot 585 = 10\cdot A$$

Hence, the area $A = \boxed{585.}$ □

Problem 5.4.6 (Problem 6)**.** (*5 points*) Define $f(x) = 2x + 3$and suppose that $g(x+2) = f(f(x-1) \cdot f(x+1) + f(x))$. Find $g(5)$.

Solution.

$$g(5) = g(3+2) = f(f(3-1) \cdot f(3+1) + f(3)) = f(f(2)f(4) + f(3)) = f(7 \cdot 11 + 9) = f(86) = \boxed{175.}$$

□

Problem 5.4.7 (Problem 7)**.** (*5 points*) Ted flips seven fair coins. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that Ted flips at least one head given that he flips at least three tails. Find $m+n$.

Solution. The total number of possibilities of flipping k tails with seven flips is $\binom{7}{k}$, thus the probability that Ted flips at least three tails is

$$\frac{\binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6}}{\binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}} = \frac{98}{99}.$$

The answer is $m + n = 98 + 99 = \boxed{197.}$ □

Problem 5.4.8 (Problem 8)**.** (*5 points*) Find the least n for which $n!(n+1)!(2n+1)! - 1$ ends in thirty-five digits that are all 9's.

Solution. If $n!(n+1)!(2n+1)! - 1$ ends in thirty-five digits that are all 9's then $n!(n+1)!(2n+1)!$ ends in thirty-five digits that are all 0's.

Let's try to find if the least $n < 100$. The number of trailing zeros in $n!$ is the number of factors of 5 in $n!$, which is $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor$.

So we are trying to find an n such that the following sum is at least 35:

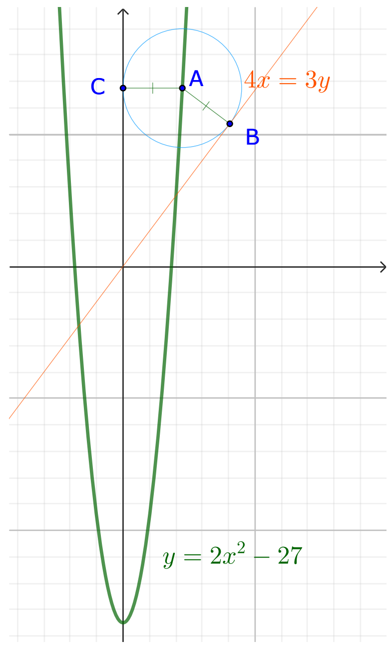
$$S(n) = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n+1}{5} \right\rfloor + \left\lfloor \frac{n+1}{25} \right\rfloor + \left\lfloor \frac{2n+1}{5} \right\rfloor + \left\lfloor \frac{2n+1}{25} \right\rfloor + \left\lfloor \frac{2n+1}{125} \right\rfloor$$

For example $S(25) = 5 + 1 + 5 + 1 + 10 + 2 = 24$, $S(27) = 6 + 6 + 11 + 2 = 25$. By continuing doing so $S(38) = 7 + 1 + 7 + 1 + 15 + 3 = 34$. $S(39) = 7 + 1 + 8 + 1 + 15 + 3 = 35$. Hence, the answer is $\boxed{39.}$ □

Problem 5.4.9 (Problem 9)**.** (*5 points*) A circle in the first quadrant with center on the curve $y = 2x^2 - 27$ is tangent to the $y-$axis and the line $4x = 3y$. The radius of the circle is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m - n$.

Solution. The distance from a point $A(x_0, y_0)$ to the line $4x = 3y$ is given by the normal form for the equation of the line $(4)x + (-3)y + 0 = 0$,

$$\frac{|4x_0 + (-3)y_0 + 0|}{\sqrt{(4^2 + (-3)^2)}} = \frac{|4x_0 - 3y_0|}{5}.$$



If point A, in the first quadrant, is on the curve $y = 2x^2 - 27$ at the same distance from the $y-$axis and the line $4x = 3y$, then the distance to the $y-$axis, or its $x-$coordinate (AC), must be equal to the distance to the line $4x = 3y$, thus

$$x_0 = \frac{|4x_0 - 3y_0|}{5} \Rightarrow x_0 = \frac{|4x_0 - 3(x_0^2 - 27)|}{5} = \frac{3(2x_0^2 - 27) - 4x_0}{5} \Rightarrow 6x_0^2 - 9x_0 - 81 = 0 \Rightarrow x_0 = \frac{9}{2}\ (x_0 > 0)$$

Hence, $m + n = 9 - 2 = \boxed{7.}$ □

Problem 5.4.10 (Problem 10)**.** (*5 points*) Among positive integers whose digits add up to 2024, N is the number whose product of digits is the greatest possible. Find the number of divisors of N.

Solution. First, digits 0 and 1 would not be among the digits of N.

A digit of 4 can be replaced by two digits of 2 having the same sum and same product.

Any digit k larger then or equal to 5 can be replace by a pair of 2 and $k - 2$, where $2(k - 2) = 2k - 4 > k$.

Therefore N has a digits of 2 and b digits of 3, where a and b are non-negative integers and the product of them is $2^a 3^b$, where $2a + 3b = 2024$.

Now, $2 + 2 + 2 = 3 + 3$, and $2 \cdot 2 \cdot 2 < 3 \cdot 3$, thus a can be 0, 1 or 2. Only $a = 1$ gives $b = 674$ integer value.

Thus $N = 2 \cdot 3^{674}$, and the number of divisors of N is $(1 + 1)(674 + 1) = \boxed{1350.}$ □

Problem 5.4.11 (Problem 11)**.** (*5 points*) Let a, b, and c be non-zero real numbers such that

$$\frac{ab}{a+b} = 3, \frac{bc}{b+c} = 4, \frac{ca}{c+a} = 5.$$

There are relatively prime positive integers m and n so that

$$\frac{abc}{ab+bc+ca} = \frac{m}{n}.$$

Find $n - m$.

Solution.

$$\frac{ab}{a+b} = 3 \Rightarrow \frac{a+b}{ab} = \frac{1}{3} \Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{1}{3}$$

$$\text{Similarly } \frac{1}{b} + \frac{1}{c} = \frac{1}{4},\ \frac{1}{c} + \frac{1}{a} = \frac{1}{5}$$

$$\Rightarrow \frac{ab+bc+ca}{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{47}{120}$$

Hence, $n - m = 120 - 47 = \boxed{73.}$ □

Problem 5.4.12 (Problem 12)**.** (*5 points*) How many positive integer solutions are there to $w+x+y+z = 21$ where $w + x \geq 5$ and $y + z \geq 5$?

Solution. Let $k = w + x$, then $y + z = 21 - k$. For a value of $k \geq 2$, there are $k - 1$ ways to assign positive value to w and x so that their sum is k.

Similarly with $21 - k \geq 2$, there are $20 - k$ ways to assign positive value to w and x so that their sum is $21 - k$.

Hence, the number of ways is:

$$\sum_{k=5}^{16}(k-1)(20-k) = \sum_{k=1}^{12}(k+3)(16-k) = \sum_{k=1}^{12}(48 + 13k - k^2) = 12 \cdot 48 + 13\frac{12 \cdot 13}{2} - \frac{12 \cdot 13 \cdot 25}{6} = \boxed{940.}$$

□

Problem 5.4.13 (Problem 13)**.** (*5 points*) Find the number of three-digit numbers such that the hundreds digit is divisible by the tens digit and the tens digit is divisible by the unit digit.

Solution. It is clear that the unit digit cannot be 0. Similarly the tens digit cannot be 0. We show a coding solution.

```
count = 0
for i in range(100, 1000):
    u = i % 10
    t = (i // 10) % 10
    h = i // 100
    print(h,t,u)
    if u != 0 and t != 0 and h % t == 0 and t % u == 0:
        count +=1
print(count)
```

The answer is $\boxed{44.}$ □

Problem 5.4.14 (Problem 14)**.** (*5 points*) Find the remainder when $2^{5^9} + 5^{9^2} + 9^{2^5}$ is divided by 13.

Solution. By Fermat's Little Theorem: $\gcd(n, p) = 1$, then $n^{p-1} \equiv 1 \pmod{p}$, so:

$$2^{12} \equiv 5^{12} \equiv 9^{12} \equiv 1 \pmod{13}.$$

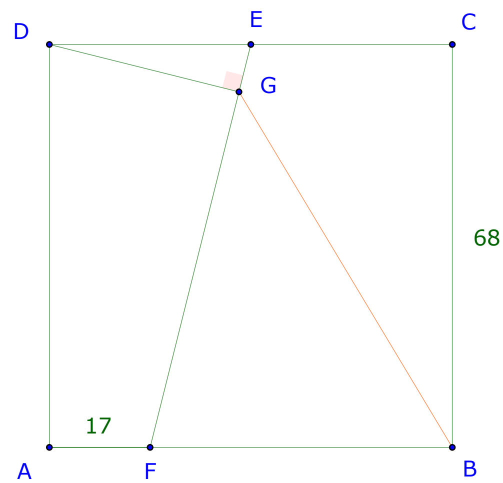
$$5^9 = 5 \cdot (5^2)^4 = 5 \cdot (25)^4 \equiv 5 \cdot 1^4 = 5 \pmod{12} \Rightarrow 2^{5^9} \equiv 2^5 = 32 \equiv 6 \pmod{13}$$

$$9^2 = 81 \equiv 9 \pmod{12} \Rightarrow 5^{9^2} \equiv 5^9 = 5 \cdot (5^2)^4 \equiv 5 \cdot (-1)^4 \equiv 5 \pmod{13}$$

$$2^5 = 32 \equiv 8 \pmod{12} \Rightarrow 9^{2^5} \equiv 9^8 \equiv 4^8 = 4^2 \cdot (4^3)^2 \equiv 3 \pmod{13}$$

Thus $2^{5^9} + 5^{9^2} + 9^{2^5} \equiv 6 + 5 + 3 \equiv \boxed{1} \pmod{13}$. □

Problem 5.4.15 (Problem 15)**.** (*5 points*) Square $ABCD$ has side length 68. Let E be the midpoint of segment CD, and let F be the point on segment AB a distance 17 from point A. Point G is on segment EF so that EF is perpendicular to segment GD. The length of segment BG can be written as $m\sqrt{n}$, where m and n be are positive integers, and n is not divisible by the square of any prime. Find $n-m$.



Solution. Let vector $\overrightarrow{BC}=u$, and $\overrightarrow{BA}=v$. Then $\overrightarrow{EF}=\frac{1}{4}v-u$. Note that $\overrightarrow{EG}$ is the projection of $\overrightarrow{ED}=\frac{1}{2}v$ onto $\overrightarrow{EF}$. This projection is:

$$\left(\frac{\overrightarrow{ED}\cdot\overrightarrow{EF}}{\overrightarrow{EF}\cdot\overrightarrow{EF}}\right)\overrightarrow{EF}=\frac{\frac{1}{2}v\cdot\left(\frac{1}{4}v-u\right)}{\left(\frac{1}{4}v-u\right)\cdot\left(\frac{1}{4}v-u\right)}\left(\frac{1}{4}v-u\right)=\frac{\frac{1}{8}}{\frac{1}{16}+1}\left(\frac{1}{4}v-u\right)=\frac{1}{34}v-\frac{2}{17}u$$

Then

$$\overrightarrow{BG}=u+\frac{1}{2}v+\overrightarrow{EG}=\frac{15}{17}u+\frac{9}{17}v$$

The length of this vector is

$$\frac{68}{17}\sqrt{15^2+9^2}=12\sqrt{34}\Rightarrow n-m=34-12=\boxed{22.}$$

□

Problem 5.4.16 (Problem 16)**.** (*5 points*) Each time you click a toggle switch, the switch either turns from *off* to *on* or from on to *off.* Suppose that you start with three toggle switches with one of them *on* and two of them *off.* On each move you randomly select one of the three switches and click it. Let m and n be relatively prime positive integers so that $\frac{m}{n}$ is the probability that after four such clicks, one switch will be *on* and two of them will be *off.* Find $n-m$.

Solution. Without loss of generality assume that at the beginning the first of the three toggle switches is *on*, and the second and third are *off.*

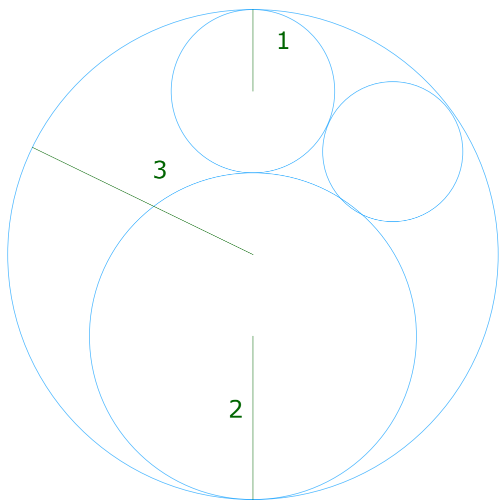
There are $3^4=81$ equally likely ways to select a sequence of four toggle switches to click. After four clicks, there will either be one or three switches in the *on* position.

To get three toggle switches in the *on* position, one much click the first switch an even number of times and the second and third switches an odd number of times each. This can be done by clicking the first switch twice and the other switches once each, or by clicking either the second and third switches three times and the other of the two switches one time. The number of ways to do one of these is:

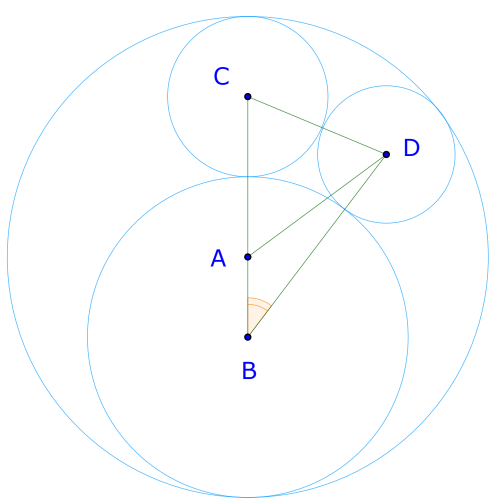
$$\binom{4}{2,1,1} + \binom{4}{0,1,3} + \binom{4}{0,3,1} = \frac{4!}{2!1!1!} + \frac{4!}{0!1!3!} + \frac{4!}{0!3!1!} = 12 + 4 + 4 = 20.$$

It follows that the desired probability is $\frac{81-20}{81} = \frac{61}{81}$. Thus the desired difference is $81 - 61 = \boxed{20.}$ $\square$

Problem 5.4.17 (Problem 17)**.** (*5 points*) The diagram below shows circles radius 1 and 2 externally tangent to each other and internally tangent to a circle radius 3. There are relatively prime positive integers m and n so that a circle radius $\frac{m}{n}$ is internally tangent to the circle radius 3 and externally tangent to the other two circles as shown. Find the product mn.



Solution. Let A, B, C, and D be the centres of the circles radius 3, 2, 1 and $r = \frac{m}{n}$.



By the Law of Cosines:

$$\begin{aligned}
&\triangle ABD: \ 1^2 + (2+r)^2 - 2\cdot 1\cdot(2+r)\cos(\angle ABD) = (3-r)^2 \\
&\triangle CBD: \ 3^2 + (2+r)^2 - 2\cdot 3\cdot(2+r)\cos(\angle ABD) = (1+r)^2 \\
&\Rightarrow 3(3-r)^2 - (1+r)^2 = -6 + 2(2+r)^2 \Rightarrow r = \frac{6}{7}.
\end{aligned}$$

Thus $mn = 6\cdot 7 = \boxed{42.}$ ☐

Problem 5.4.18 (Problem 18). (*5 points*) Find the greatest seven-digit integer divisible by 132 whose digits, in order, are $x, 0, y, 2, 1, 2, z$ where x, y, and z are single digits.

Solution. We show a coding solution. The problem is equivalent to find the largest multiple of 132 in the format $\overline{x0y212z}$.

```
for x in range(9, -1, -1):
for y in range(9, -1, -1):
    for z in range(9, -1, -1):
        i = x * 1000000 + y * 10000 +  2 * 1000 +  120 + z
        if i % 132 == 0:
            print(i)
            exit(0)
```

The answer is $\boxed{9082128.}$ ☐

Problem 5.4.19 (Problem 19). (*5 points*) There are positive integers m and n so that $x = m + \sqrt{n}$ is a solution to the equation

$$x^2 - 28x + 1 = \sqrt{x}(x+1).$$

Find $m + n$.

Solution. It is easy to see that $x = 0$ is not a solution. Divide both side of the equation by x, we have

$$x - 28 + \frac{1}{x} = \sqrt{x} + \frac{1}{\sqrt{x}}.$$

Let $y = \sqrt{x} + \frac{1}{\sqrt{x}} > 0$, then $y^2 = x + \frac{1}{x} + 2$, thus

$$\begin{aligned}
&y^2 - 30 = y \Rightarrow (y+5)(y-6) = 0 \Rightarrow y = 6 \Rightarrow \sqrt{x} + \frac{1}{\sqrt{x}} = 6 \Rightarrow x - 6\sqrt{x} + 1 = 0 \\
&\Rightarrow \sqrt{x} = 3 \pm 2\sqrt{2} \Rightarrow x = (3+2\sqrt{2})^2 = 17 + 12\sqrt{2} = 17 + \sqrt{288}
\end{aligned}$$

Thus $m + n = 17 + 288 = \boxed{305.}$ ☐

Problem 5.4.20 (Problem 20). (*5 points*) Find the largest prime that divides $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + 2023\cdot 2024\cdot 2025$.

Solution. Note that $(k-1)k(k+1) = (k^2-1)k = k^3 - k$, thus the given sum is equivalent to

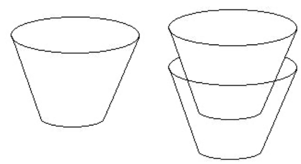
$$\begin{aligned}
&\sum_{k=2}^{n}(k-1)k(k+1) = \sum_{k=2}^{n}\left(k^3 - k\right) = \sum_{k=1}^{n}k^3 - \sum_{k=1}^{n}k = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} - 1\right) = \frac{(n-1)n(n+1)(n+2)}{4}
\end{aligned}$$

For $n = 2024$,

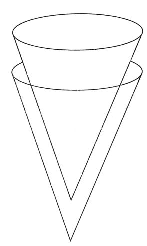
$$\frac{(n-1)n(n+1)(n+2)}{4} = \frac{(2024-1)(2024)(2024+1)(2024+2)}{4} = 2^2 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17^2 \cdot 23 \cdot 1013.$$

Thus the largest prime factor is $\boxed{1013}$. $\square$

Problem 5.4.21 (Problem 21)**.** (*5 points*) A paper cup has a base that is a circle with radius r, a top that is a circle with radius $2r$, and sides that connect the two circles with straight line segments as shown below. This cup has height h and volume V. A second cup that is exactly the same shape as the first is held upright inside the first cup so that its base is a distance of $\frac{h}{2}$ from the base of the first cup. The volume of liquid that will fit inside the first cup and outside the second cup can be written as $\frac{m}{n} \cdot V$, where m and n are relatively prime positive integers. Find $n - m$.



Solution. The volume between the cups does not change if the sides of each cup are extended so that the cups form a circular cones with radius $2r$ and height $2h$.



The volume of the first cone is:

$$\frac{1}{3}\pi(2r)^2(2h) = \frac{8}{3}\pi r^2 h.$$

The part of the second cone inside the first is a cone that is $\frac{3}{4}$ the height of the first cone, so the volume that it takes up within the first cone is:

$$\left(\frac{3}{4}\right)^3 \frac{8}{3}\pi r^2 h = \frac{9}{8}\pi r^2 h$$

Thus the volume inside the first cup and outside of the second cup is

$$\left(\frac{8}{3} - \frac{9}{8}\right)\pi r^2 h = \frac{37}{24}\pi r^2 h$$

Thus, the volume of the first cup is

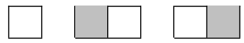
$$\left(1 - \left(\frac{1}{2}\right)^3\right)\frac{4}{3}\pi r^2 (2h) = \frac{7}{3}\pi r^2 h$$

The desired fraction is:

$$\frac{\frac{37}{24}}{\frac{7}{3}} = \frac{37}{56} \Rightarrow n - m = 56 - 37 = \boxed{19}.$$

$\square$

Problem 5.4.22 (Problem 22)**.** (*5 points*) You have some white one-by-one tiles and some black and white two-by-one tiles as shown below. There are four different color patterns that can be generated when using these tiles to cover a three-by-one rectangle by laying these tiles side by side (WWW, BWW, WBW, WWB). How many different color patterns can be generated when using these tiles to cover a eight-by-one rectangle?



Solution. Let A_j be the number of colour patterns one can obtain by laying these tiles together to tile a $j \times 1$ rectangle. It is easy to see that $A_1 = 1$, $A_2 = 3$, and $A_3 = 4$.

For $j > 2$, categorize the colouring patterns by the position of the first black square among the j square.

If the colouring patter begins with a black square in its first position, the pattern must have a 2×1 tile followed by one of the A_{j-2} patterns of length $j - 2$.

Similarly if the colouring patter begins with a white square in the first position, and a black square in the second position, the first two squares be followed by any of the A_{j-2} patterns of length $j - 2$.

If the first black square appears in the k^{th} position, where $k > 2$, it can be followed by any of the A_{j-k}, pattern of length $j - k$. Note that if the first black square appears in the last position, it constitutes a pattern. Finally, there is one pattern consisting of all white square. Thus:

$$A_j = A_{j-2} + A_{j-2} + A_{j-3} + A_{j-4} + \ldots + A_1 + 1 + 1.$$

Below is the number of patterns for each of $j \times 1$ rectangles, where $j = 1, 2, \ldots, 10$.

1	2	3	4	5	6	7	8	9	10
1	3	4	9	14	28	47	89	155	286

The answer is $\boxed{89}$. ☐

Problem 5.4.23 (Problem 23)**.** (*5 points*) A bag contains 7 green candies and 4 red candies. You randomly select one candy at a time to eat. If you eat five candies, there are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that you do not eat a green candy after you eat a red candy. Find $m + n$.

Solution. The five candies can be eaten in order

$$GRRRR,\ GGRRR,\ GGGRR,\ GGGGR,\ GGGGG.$$

The probability of eating the candies in these orders are, respectively:

$$\begin{cases} GRRRR: \frac{7}{11} \cdot \frac{4}{10} \cdot \frac{3}{9} \cdot \frac{2}{8} \cdot \frac{1}{7} \\ GGRRR: \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} \\ GGGRR: \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7} \\ GGGGR: \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{4}{7} \\ GGGGG: \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7} \end{cases}$$

Their sum is

$$\frac{7 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 7 \cdot 6 \cdot 4 \cdot 3 \cdot 2 + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 4 + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7} = \frac{19}{110}.$$

Thus $m + n = 19 + 110 = \boxed{129.}$ ☐

Problem 5.4.24 (Problem 24). (*5 points*) Let $A = 1, 3, 5, 7$ and $B = 2, 4, 6, 8$. Let f be a randomly chosen function from the set $A \cup B$ into itself. There are relatively prime positive integers m and n such that $\frac{m}{n}$ is the probability that f is a one-to-one function on $A \cup B$ given that it maps A one-to-one into $A \cup B$ and it maps B one-to-one into $A \cup B$. Find $m + n$.

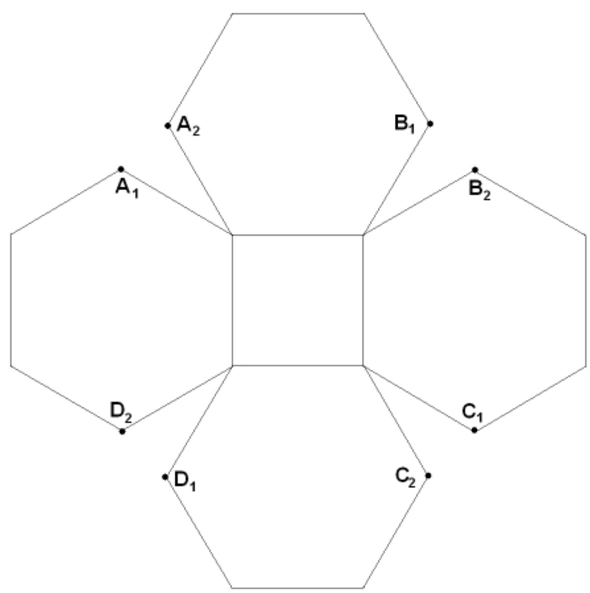
Solution. A one-to-one function f that maps $A \cup B$ into itself is just a permutation of the set $A \cup B$. Thus there are 8! such functions.

The number of one-to-one functions that map A into $A \cup B$ is the number of ways to map each element of A to a distinct element of $A \cup B$, or $8 \cdot 7 \cdot 6 \cdot 5$ That is also the number of one-to-one functions that map B into $A \cup B$. Thus the desired probability isosceles

$$\frac{8!}{(8 \cdot 7 \cdot 6 \cdot 5)^2} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5} = \frac{1}{70}.$$

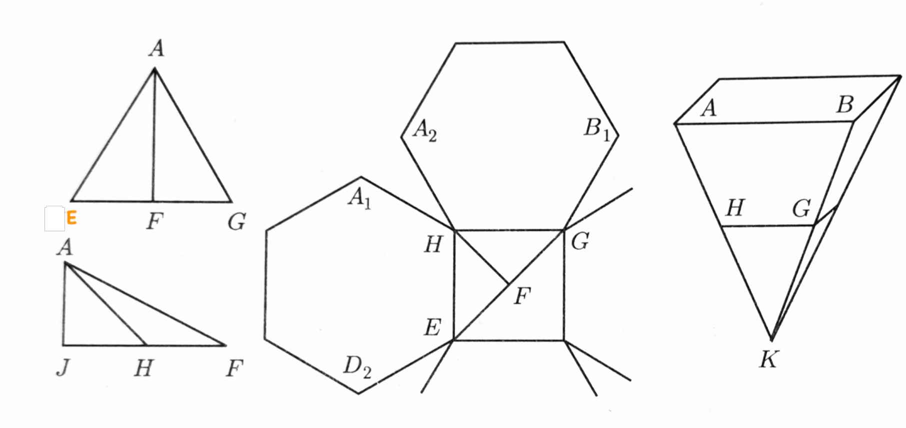
The answer is $m + n = 1 + 70 = \boxed{71.}$ □

Problem 5.4.25 (Problem 25). (*5 points*) The diagram below shows four regular hexagons each with side length 1 meter attached to the sides of a square. This figure is drawn onto a thin sheet of metal and cut out. The hexagons are then bent upward along the sides of the square so that A_1 meets A_2, B_1 meets B_2, C_1 meets C_2, and D_1 meets D_2. The resulting dish is set on a table with the square lying flat on the table. If this dish is filled with water, the water will rise to the height of the corner where the A_1 and A_2 meet.



Let the point where A_1 and A_2 meet be labeled A, the point B_1 and B_2 meet be labeled B, the point C_1 and C_2 meet be labeled C, and the point D_1 and D_2 meet be labeled D. Let the center of the square be labeled F. Let the corner of the square nearest point A be labeled H, the corner of nearest point B be labeled G, and the corner diagonally opposite G be labeled E as shown below.

Let J be the projection of the point A in the plane of the square. If the vertical edges of the dish AH and BG are extended, they will meet at a point K as shown in the diagram. Note that K is the vertex of a downward pointing square pyramid.



Find the measure of $\angle AHF$ in degrees.

Solution. Because $\angle AHE = 120°$, the Law of Cosines gives that

$$AE = \sqrt{AH^2 + HE^2 - 2(AH)(HE)\cos(120°)} = \sqrt{1+1+1} = \sqrt{3}.$$

Since $EF = HF = \frac{1}{\sqrt{2}}$, the Pythagorean Theorem gives that

$$AF = \sqrt{AE^2 - EF^2} = \sqrt{3 - \frac{1}{2}} = \sqrt{\frac{5}{2}}.$$

The Law of Cosines then gives

$$\cos(\angle AHF) = \frac{AF^2 - AH^2 - HF^2}{-2(AH)(HF)} = \frac{\frac{5}{2} - 1 - \frac{1}{2}}{-\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

Therefore $\angle AHF = \boxed{135°.}$ □

Chapter 6

Session 4

6.1 Middle School - Test 4

You have **60 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 11-20**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 6.1.1. (*1 point*) There are relatively prime positive integers m and n so that

$$\frac{\frac{1}{2}}{\frac{\frac{1}{3}}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}+\frac{\frac{1}{3}}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}}=\frac{m}{n}.$$

Find $2m+n$.

Solution.

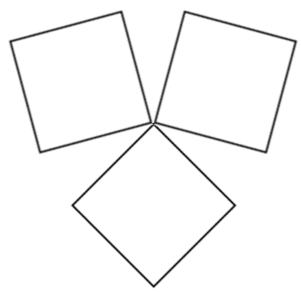
$$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4},\quad \frac{\frac{1}{3}}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}=\frac{\frac{1}{3}}{\frac{3}{4}}=\frac{1}{3}\cdot\frac{4}{3}=\frac{4}{9}$$

Thus the given fraction is

$$\frac{\frac{1}{2}}{2\cdot\frac{4}{9}}=\frac{1}{2}\cdot\frac{9}{4}=\frac{9}{16}\Rightarrow 2m+n=2\cdot 9+16=\boxed{34.}$$

□

Problem 6.1.2. (*1 point*) The diagram below shows a 12-sided figure made up of three congruent squares. The figure has total perimeter 60. Find its area.



Solution. If the total perimeter is 60, each square has a perimeter of 20, thus each square has a side length of 5. Thus the entire area is $3\cdot 25=\boxed{75.}$ □

Problem 6.1.3. (*1 point*) Find the sum of all two-digit integers which are both prime and are 1 more than a multiple of 10.

Solution. It is easy to see that these primes are $11, 31, 41, 61, 71$, and their sum $11+31+41+61+71=\boxed{215.}$ □

Problem 6.1.4. (*1 point*) Jerry buys a bottle of 150 pills. Using a standard 12 hour clock, he sees that the clock reads exactly 12 when he takes the first pill. If he takes one pill every five hours, what hour will the clock read when he takes the last pill in the bottle?

Solution. Jerry will take the last pill 5×149 hours after he takes the first pill. Since $5\times 149 =\equiv 1 \pmod{12}$, the clock will read 1 hour pass 12, or $\boxed{1}$ when Jerry takes the last pill. □

Problem 6.1.5. (*1 point*) Given that $\frac{6}{11} - \frac{10}{19} = \frac{9}{19} - \frac{n}{11}$, find n.

Solution.

$$\frac{6}{11} - \frac{10}{19} = \frac{9}{19} - \frac{n}{11} \Rightarrow \frac{6}{11} = \frac{10}{19} + \frac{9}{19} - \frac{n}{11} = 1 - \frac{n}{11}$$
$$\Rightarrow \frac{n+6}{11} = 1 \Rightarrow n = \boxed{5.}$$

□

Problem 6.1.6. (*1 point*) The following addition problem is not correct if the numbers are interpreted as base 10 numbers. In what number base is the problem correct?

$$\begin{array}{r} 66 \\ 87 \\ 85 \\ +\,48 \\ \hline 132 \end{array}$$

Solution. Let the number base for this problem be b. The ones column in the problem adds to $6+7+5+8 = 26 \equiv 2 \pmod b$. Thus b is a divisor of 24. Since the problem includes the digit 8, so $b > 8$. Therefore $b = 12$ or $b = 24$. It is easy to check that $b = \boxed{24.}$ □

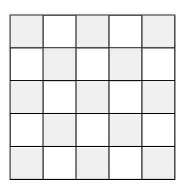
Problem 6.1.7. (*1 point*) When 12^{18} is divided by 18^{12}, the result is $\left(\frac{m}{n}\right)^3$, where m and n are relatively prime positive integers. Find mn.

Solution.

$$\frac{12^{18}}{18^{12}} = \frac{4^{18}3^{18}}{2^{12}9^{12}} = \frac{2^{36}3^{18}}{2^{12}3^{24}} = \frac{2^{24}}{3^{6}} = \left(\frac{256}{9}\right)^3$$

Thus $mn = 256 \cdot 9 = \boxed{2304.}$ □

Problem 6.1.8. (*1 point*) A square measuring 15 by 15 is partitioned into five rows of five congruent squares as shown below. The small squares are alternately colored black and white as shown. Find the total area of the part colored black.



Solution. Each of the small square has side length $\frac{15}{5} = 3$, so each has an area of $3^2 = 9$. There are 13 shaded squares, Thus the shaded area is $9 \cdot 13 = \boxed{117.}$ □

Problem 6.1.9. (*1 point*) A jar contains one quarter red jelly beans and three quarters blue jelly beans. If Chris removes three quarters of the red jelly beans and one quarter of the blue jelly beans, what percent of the jelly beans remaining in the jar will be blue?

Solution. At the start $\frac{1}{4}$ of the jelly beans are red, and $\frac{3}{4}$ of them are removed, so at the end, the number od red jelly bean will be:

$$\frac{1}{4} - \frac{3}{4} \cdot \frac{1}{4} = \frac{1}{16} \text{ of the original number of jelly beans.}$$

Similarly, at the end, the number of blue jelly beans will be

$$\frac{3}{4} - \frac{1}{4} \cdot \frac{3}{4} = \frac{9}{16} \text{ of the original number of jelly beans.}$$

Therefore the fraction of blue jelly beans is $\boxed{90\%.}$ □

Problem 6.1.10. (*1 point*) Five rays $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD}$, and $\overrightarrow{OE}$ radiate in a clockwise order from O forming four non-overlapping angles such that $\angle EOD = 2\angle COB$, $\angle COB = 2\angle BOA$, while $\angle DOC = 3\angle BOA$. If E, O, A are collinear with O between A and E, what is the degree measure of $\angle DOB$?

Solution. Let $x = \angle BOA$, then

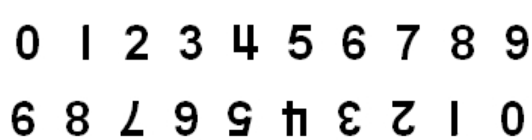
$$\angle COB = 2x, \angle DOC = 3x, \angle EOD = 4x \Rightarrow x + 2x + 3x + 4x = 10x = 180^\circ \Rightarrow x = 18^\circ.$$

Thus $\angle DOB = \angle DOC + \angle COB = 3x + 2x = 5x = \boxed{90^\circ}$. □

Problem 6.1.11. (*1 point*) How many numbers are there that appear both in the arithmetic sequence $10, 16, 22, 28, \ldots 1000$ and the arithmetic sequence $10, 21, 32, 43, \ldots 1000$?

Solution. The first sequence includes every sixth integer from 10 up to 1000. The second sequence includes every eleventh integer from 10 up to 1000. The number contained in both sequences are the every sixty-sixth number from 10 to 1000, that inscribed $10, 76, 142, \ldots, 1000$. There are $\frac{1000-10}{66} + 1 = \boxed{16}$ such numbers. □

Problem 6.1.12. (*1 point*) When Troy writes his digits, his 0, 1, and 8 look the same right-side-up and upside-down as seen in the figure below. His 6 and 9 look like upside-down images of each other. None of his other digits look like digits when they are inverted. How many different five-digit numbers (which do not begin with the digit zero) can Troy write which read the same right-side-up and upside-down?



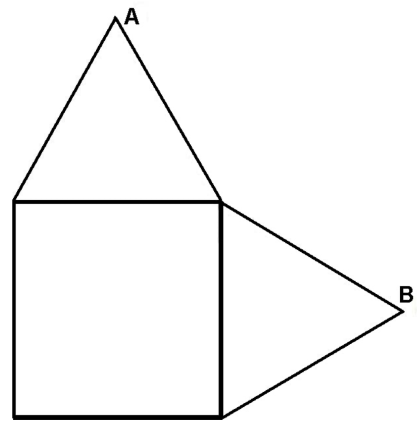
Solution. Each of the digits that Troy writes in a five-digit number must be one of the five: 0,1,8,6,9.

In addition, the numbers in positions 4 and 5 must be the up-side-down versions of the digits be written in position 2 and 1, respectively. Also, the digit in position 3 must be 0, 1, or 8 since it must look the same right-side-up and up-side-down. Finally, the digit in the first position cannot be 0 since no five-digit number can begin with a digit 0.

Thus, there are 4 possible choices for the first digit, 5 choices for the second digit, three choices for the third digit, and the last two digits are determined by the choices of the first two.

Hence, there are $4 \cdot 5 \cdot 3 = \boxed{60}$ choices. □

Problem 6.1.13. (*1 point*) The diagram shows two equilateral triangles with side length 4 mounted on two adjacent sides of a square also with side length 4. The distance between the two vertices marked A and B can be written as $\sqrt{m} + \sqrt{n}$ for two positive integers m and n. Find mn.



Solution. Let the center of the square be the point C. The distance from A to C is given by the height of the equilateral triangle $4 \cdot \frac{\sqrt{3}}{2}$ plus half the length of the side of the square, 2; or $\sqrt{12} + 2$. Triangle ACB is an isosceles right triangle, so its hypotenuse AB has a length of $\sqrt{2}(\sqrt{12} + 2) = \sqrt{24} + \sqrt{8}$.

Hence, $mn = 24 \cdot 8 = \boxed{192.}$ □

Problem 6.1.14. (*1 point*) The five-digit number 12110 is divisible by the sum of its digits $1+2+1+1+0 = 5$. Find the greatest five-digit number which is divisible by the sum of its digits.

Solution. Let n be such a number. We will try strategy by descend from 99999 to find the first number satisfies the required conditions.

If $n = \overline{9999a}$, then $36+a \mid n \Rightarrow 36+a \mid n-(36+a) = 99954$. But $99954 = 2 \cdot 3^4 \cdot 617$. Since $36 \leq 36+a \leq 45$, it is easy to verify that there is no such a so that $36 + a \mid 99954$.

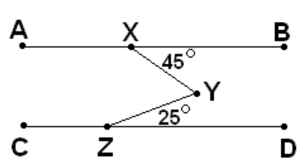
If $n = \overline{9998a}$, then $35 + a \mid n \Rightarrow 35 + a \mid n - (35 + a) = 99945 = 3^2 \cdot 5 \cdot 2221$. By the same method as above case, there is no such a.

If $n = \overline{9997a}$, then $34 + a \mid n \Rightarrow 34 + a \mid n - (34 + a) = 99936 = 2^5 \cdot 3^2 \cdot 347$. It is easy to test that for $a = 2, 36 = 2^2 \cdot 3^2 \mid 99936$ is the only solution.

Thus $n = \boxed{99972.}$ □

Problem 6.1.15. (*1 point*) In the diagram below, AB and CD are parallel, $\angle BXY = 45°$, $\angle DZY = 25°$, and $XY = YZ$. What is the degree measure of $\angle YZX$?

> **Remark.** The diagram mentioned in the text was missing. It is shown as below.



Solution. By drawing line through Y parallel to AB, it is easy to see that $\angle XYZ = \angle YXB + \angle YZD = 70°$. $\triangle XYZ$ is isosceles at Y, so $\angle YZX = \frac{1}{2}(180° - 70) = \boxed{55°.}$ □

Problem 6.1.16. (*1 point*) Let a and b be nonzero real numbers such that

$$\begin{cases} \frac{1}{3a} + \frac{1}{b} = 2011 \\ \frac{1}{a} + \frac{1}{3b} = 1 \end{cases}$$

What is the quotient when $2a + 2b$ is divided by $3ab$?

Solution.

$$\begin{cases} \frac{1}{3a} + \frac{1}{b} = 2011 \\ \frac{1}{a} + \frac{1}{3b} = 1 \end{cases} \Rightarrow \frac{3a+b}{3ab} = 2011, \frac{a+3b}{3ab} = 1 \Rightarrow \frac{4a+4b}{3ab} = 2012 \Rightarrow \frac{2a+2b}{3ab} = \frac{2012}{2} = \boxed{1006.}$$

□

Problem 6.1.17. (*1 point*) Find the number of ordered quadruples (a, b, c, d) where each of a, b, c, and d are (not necessarily distinct) elements of $\{1, 2, 3, 4, 5, 6, 7\}$ and $3abc + 4abd + 5bcd$ is even. For example, $(2, 2, 5, 1)$ and $(3, 1, 4, 6)$ satisfy the conditions.

Solution. The given sum is even exactly when the products abc and bcd are either both even or both odd. This happens either when the product bc is even and or when bc is odd and a and d are either both odd or both even.

Of the $7 \cdot 7 = 49$ ways to assign values to b and c, $4 \cdot 4 = 16$ ways result in bc being odd, and $49 - 16 = 33$ ways result in bc being even. Similarly, there are 49 ways to assign value to a and d, of these $4 \cdot 4 = 16$ ways result in both a and d being odd, and $3 \cdot 3$ in both a and d being even.

It follows that the total number of ways to assign value to a, b, c, and d so that $3abc + 4abd + 5bcd$ is even is

$$33 \cdot 49 + 16 \cdot (16 + 9) = \boxed{2017.}$$

□

Problem 6.1.18. (*1 point*) Find the positive integer n so that n^2 is the least perfect square exceeding $8 + 16 + 24 + \cdots + 8040$.

Solution.

$$8 + 16 + 24 + \cdots + 8040 = 8\left(\sum_{i=1}^{1005} i\right) = 8 \cdot \frac{1005 \cdot 1006}{2} = 2010 \cdot 2012 = 2011^2 - 1$$

The least perfect square exceeding $2011^2 - 1$ is 2011^2, thus $n = \boxed{2011.}$ □

Problem 6.1.19. (*1 point*) How many ordered pairs of sets (A, B) have the properties:

$$\begin{cases} A \subseteq \{1, 2, 3, 4, 5, 6\} \\ B \subseteq \{2, 3, 4, 5, 6, 7, 8\} \\ |A \cap B| = 3 \ (A \text{ and } B \text{ has exactly 3 elements.}) \end{cases}$$

Solution. The sets $\{1, 2, 3, 4, 5, 6\}$ and $\{2, 3, 4, 5, 6, 7, 8\}$ have five elements in common, so there are $\binom{5}{3} = 10$ ways to select three elements that the sets A and B have in common.

Having selected the three shared elements, there are two choices of what to do with the element 1 which can either be in or out of set A Similarly, there are two choices for each of the element 7 and 8 which can be in or out of set B. Of the element in $\{2, 3, 4, 5, 6\}$, three of the elements have been chosen to appear in both A and B, but there are three choices of what to do with each of the other two elements which can be placed either in the set A, in the set B, or left out of both sets.

It follows that the number of ways to choose the sets A and B is $10 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = \boxed{720.}$ □

Problem 6.1.20. (*1 point*) Let V be the set of vertices of a regular 25 sided polygon with center at point C. How many triangles have vertices in V and contain the point C in the interior of the triangle?

Solution. The number of triangles with vertices among the 25 vertices is $\binom{25}{3} = 2300$. The triangles that do not contain the center C are exactly the triangles that are obtuse.

Number the vertices of the regular 25 sided polygon in order from 1 to 25. A triangle will have an obtuse angle at the point number 11 if one of its other two vertices is numbered with a number less than 11, one is numbered with a number greater than 11, and the difference between these two number does not exceed 11. Thus one of these vertices could be numbered k for $1 \le k \le 10$ while the other could be numbered m with $12 \le m \le k + 11$.

That is, for each choice of k, there are k choice for m. It follows that the number of triangles with an obtuse angle at vertex 11 is

$$1 + 2 + 3 + \ldots + 10 = \frac{10 \cdot 11}{2} = 55.$$

An obtuse triangle could have its obtuse angle at any of the 25 vertices, thus there are $25 \cdot 55$ obtuse triangles. Hence the number of triangles that contain point C is $2300 - 25 \cdot 55 = \boxed{925.}$ □

6.2 High School - Test 4

You have **90 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 11-20**
- **Olympiad (O) level: Problems 21-30**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

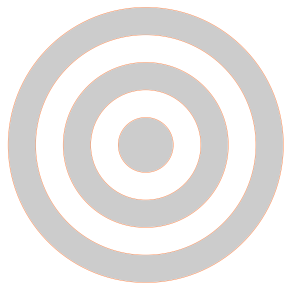
Problem 6.2.1. (*1 point*) The ratio of 3 to the positive number n is the same as the ratio of n to 192. Find n.

Solution.

$$\frac{3}{n} = \frac{n}{192} \Rightarrow n^2 = 3 \cdot 193 = 3 \cdot 3 \cdot 64 \Rightarrow n = 3 \cdot 8 = \boxed{24.}$$

□

Problem 6.2.2. (*1 point*) The target below is made up of concentric circles with diameters 3, 6, 9, 12, and 15. The area of the dark region is $n\pi$. Find n.



Solution. The area of a circle with radius r is πr^2. The area of the shaded region is

$$\pi\left[\left(\frac{15}{2}\right)^2 - \left(\frac{12}{2}\right)^2 + \left(\frac{9}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + \left(\frac{3}{2}\right)^2\right] = \pi\boxed{\frac{135}{4}.}$$

□

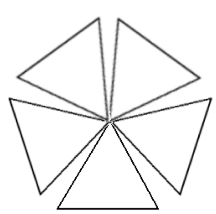
Problem 6.2.3. (*1 point*) Shirley went to the store planning to buy 160 balloons for 10 dollars. When she arrived, she was surprised to find that the balloons were on sale for 20 percent less than expected. How many balloons could Shirley buy for her 10 dollars?

Solution. Shirley will spend 10 dollars to buy balloons selling at the price $\frac{10(1-0.20)}{160}$. This will let her to purchase for \$10:

$$\frac{10}{\frac{10(1-0.20)}{160}} = \boxed{200.}$$

□

Problem 6.2.4. (*1 point*) Five non-overlapping equilateral triangles meet at a common vertex so that the angles between adjacent triangles are all congruent. What is the degree measure of the angle between two adjacent triangles?



Solution. There are 10 angles that meet at the center point of the diagram. The angles within the five triangles are all $60°$, thus the other five angles have the total measures of $360° - 5 \cdot 60° = 60°$. Thus each of them is $\frac{60°}{5} = \boxed{12°.}$ □

Problem 6.2.5. (*1 point*) Let $a_1 = 2$, and for $n \geq 1$, let $a_{n+1} = 2a_n + 1$. Find the smallest value of an $a_n > 100$ that is not a prime number.

Solution. Let just compute the terms

$$a_2 = 2 \cdot 2 + 1 = 5, a_3 = 2 \cdot 5 + 1 = 11, a_4 = 2 \cdot 11 + 1 = 23, a_5 = 2 \cdot 23 + 1 = 47$$
$$a_6 = 2 \cdot 47 + 1 = 95, a_6 = 2 \cdot 95 + 1 = 191 \text{ (prime)}, a_7 = 2 \cdot 191 + 1 = 382 \text{ (prime)}, a_8 = 767 = 13 \cdot 59$$

So the smallest value of an $a_n > 100$ that is not a prime number, is $\boxed{767.}$ □

Problem 6.2.6. (*1 point*) Working alone, the expert can paint a car in one day, the amateur can paint a car in two days, and the beginner can paint a car in three days. If the three painters work together at these speeds to paint three cars, it will take them $\frac{m}{n}$ days where m and n are relatively prime positive integers. Find mn.

Solution. The expert paints car at the rate of 1 per day, the amateur at $\frac{1}{2}$ per day, and the beginner at $\frac{1}{3}$ per day. When working together, they paint $1 + \frac{1}{2} + \frac{1}{3}$ cars per day. To paint three cars, they need:

$$\frac{3}{1 + \frac{1}{2} + \frac{1}{3}} = \frac{18}{11}.$$

Thus $mn = 18 \times 11 = \boxed{198.}$ □

Problem 6.2.7. (*1 point*) Find the prime number p such that $73p + 1$ is a perfect square.

Solution. Let $n^2 = 73p + 1$, then $73p = (n-1)(n+1)$. Since 73 and p are both primes, so

$$\begin{cases} 73 = n - 1, p = n + 1 \Rightarrow p = 75 \text{ not a prime} \\ 73 = n + 1, p = n - 1 \Rightarrow p = \boxed{71} \end{cases}$$

□

Problem 6.2.8. (*1 point*) When 126 is added to its reversal, 621, the sum is $126 + 621 = 747$. Find the greatest integer which when added to its reversal yields 1110.

Solution. Assume that the answer is a three-digit number $\overline{abc}$, then according to the given conditions:

$$(100a + 10b + c) + (100c + 10b + a) = 1110 \Rightarrow 101(a + c) + 20b = 1110 \Rightarrow 101(a + c) = 10(111 - 2b).$$

Note that 101 is a prime number, so $101 = 111 - 2b \Rightarrow b = 5$, then $a + c = 10$. The largest number is $\boxed{951.}$ □

Problem 6.2.9. (*1 point*) There are integers m and n so that $9 + \sqrt{11}$ is the root of the polynomial $x^2 + mx + n$. Find mn.

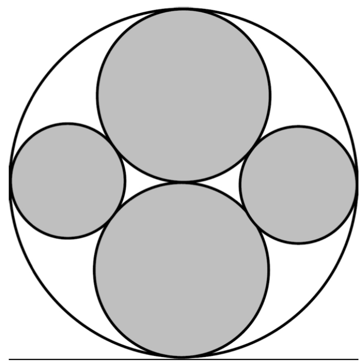
Solution. From the quadratic formula $x^2 + mx + n$, its root $9 + \sqrt{11}$ should be in the format

$$\frac{-m + \sqrt{m^2 - 4n}}{2} \Rightarrow m = -18, n = 70.$$

Thus $mn = (-18) \cdot 70 = \boxed{-1260.}$ □

Problem 6.2.10. (*1 point*) The diagram shows a large circular dart board with four smaller shaded circles each internally tangent to the larger circle. Two of the internal circles have half the radius of the large circle, and are, therefore, tangent to each other. The other two smaller circles are tangent to these circles. If a dart is thrown so that it sticks to a point randomly chosen on the dart board, then the probability that the dart sticks to a point in the shaded area is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find mn.

> **Remark.** The diagram mentioned in the text of the problem was missing. It is now shown as below.



Solution. WLOG, assume that the dart board has radius 2. Then the larger shaded circles have radius 1, and the smaller shaded circles have radius r. The Pythagorean Theorem can be used ot relate the distance between the centers of a larger and a smaller shaded circle to the distance of these two circles to the center of the dartboard:

$$1^2 + (2-r)^2 = (1+r)^2 \Rightarrow r = \frac{2}{3}$$

The probability that the dart sticks to a point in the shaded area is the ratio of the shaded area to the area of the dart board:

$$\frac{2\pi 1^2 + 2\pi \left(\frac{2}{3}\right)^2}{\pi 2^2} = \frac{13}{18}.$$

Hence, $mn = 13 \cdot 18 = \boxed{234.}$ □

Problem 6.2.11. (*1 point*) Six distinct positive integers are randomly chosen between 1 and 2023, inclusive. The probability that some pair of the six chosen integers has a difference that is a multiple of 5 is n percent. Find n.

Solution. There are only 5 different possible remainders when a number is divided by 5. If six numbers are selected, then by thee Pigeonhole Principle, at least two of the selected numbers have the same remainder when divided by 5. The difference of these two numbers will be an integer divisible by 5. Thus the probability is $\boxed{100}$ percent. □

Problem 6.2.12. (*1 point*) Find the area of the region in the coordinate plane satisfying the three conditions

$$\begin{cases} x \le 2y \\ y \le 2x \\ x + y \le 30 \end{cases}$$

Solution. Let B be the point where lines $x = 2y$ and $x + y = 30$ meet:

$$\begin{cases} x = 2y \\ x + y = 30 \end{cases} \Rightarrow x = 20, y = 10 \Rightarrow B(20, 10)$$

Let C be the point where lines $y = 2x$ and $x + y = 30$ meet:

$$\begin{cases} y = 2x \\ x + y = 30 \end{cases} \Rightarrow x = 10, y = 20 \Rightarrow C(10, 20)$$

It is obvious that $A(0, 0)$ is where lines $x = 2y$ and $y = 2x$. It is easy to see that ABC is a isosceles triangle.

The length of the base BC is $\sqrt{(20-10)^2 + (10-20)^2} = 10\sqrt{2}$. The midpoint M of BC is $M(\frac{20+10}{2}, \frac{10+20}{2})$ or $M(15, 15)$. The length of the altitude AM is $\sqrt{(0-15)^2 + (0-15)^2} = 15\sqrt{2}$.

Thus the area of $\triangle ABC$, where $x \le 2y, y \le 2x$, and $x + y \le 30$ is $\frac{1}{2}(10\sqrt{2})(15\sqrt{2}) = \boxed{150}$. □

Problem 6.2.13. (*1 point*) Let S be the set of all 10-term arithmetic progressions that include the numbers 4 and 10. For example,

$$(-2, 1, 4, 7, 10, 13, 16, 19, 22, 25) \text{ and } (10, 8\frac{1}{2}, 7, 5\frac{1}{2}, 4, 2\frac{1}{2}, 1, -\frac{1}{2}, -2, -3\frac{1}{2})$$

are both members of S. Find the sum of all values of a_{10} for each $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \in S$, that is

$$\sum_{(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10}) \in S a_{10}}.$$

Solution. An arithmetic progression containing the terms 4 and 10 is completely determine by the position in the progression of the 4 and the 10.

For each $(a1, a2, a3, a4, a5, a6, a7, a8, a9, a10) \in S$ there is exactly one $(b1, b2, b3, b4, b5, b6, b7, b8, b9, b10) \in S$ where the positions of the 4 and the 10 are reversed.

Then

$$(a_1 + b_1, a_2 + b_2, a_3 + b_3, a_3 + b_4, a_5 + b_5, a_6 + b_6, a_7 + b_7, a_8 + b_8, a_9 + b_9, a_{10} + b_{10})$$

is also an arithmetic progression, but since two of its terms are $4 + 10 = 10 + 4 = 14$, so it is a constant progression with all terms equal to 14. Therefore $a_{10} + b_{10} = 14$. It follows that the mean of all tenth terms of the progressions in S is $\frac{1}{2}14 = 7$, and the requested sums of all tenth terms is 7 times the number of elements in S.

Since an element of S is completely determined by the positions of the 4 and the 10, we just need to count those. There are 10 possible position for the 4 and 9 possible position for the 10, thus there are $10 \cdot 9 = 90$ sequences. Hence the requested sum is $7 \cdot 90 = \boxed{630.}$ □

Problem 6.2.14. (*1 point*) The lengths of the three sides of a right triangle form a geometric sequence. The sine of the smallest of the angles in the triangle is $\frac{m+\sqrt{n}}{k}$, where m, n, and k are integers, and k is not divisible by the square of any prime. Find $(m+n)k$.

Solution. There are real number $a > 0$ and $r > 1$ so that the sides of the triangle have lengths a, ar, ar^2. By the Pythagorean Theorem,

$$a^2 + (ar)^2 = (ar^2)^2 \Rightarrow 1 + r^2 = r^4 \Rightarrow r^2 = \frac{1+\sqrt{5}}{2}.$$

The sine of the smallest angle is

$$\frac{a^2}{ar^2} = \frac{1}{r^2} = \frac{2}{1+\sqrt{5}} = \frac{-1+\sqrt{5}}{2}.$$

Thus $(m+n)k = (-1+5)2 = \boxed{8.}$ □

Problem 6.2.15. (*1 point*) A pyramid has a base which is an equilateral triangle with side length 300 centimeters. The vertex of the pyramid is 100 centimeters above the center of the triangular base. A mouse starts at a corner of the base of the pyramid and walks up the edge of the pyramid toward the vertex at the top. When the mouse has walked a distance of 148 centimeters, how many centimeters above the base of the pyramid is the mouse?

Solution. An equilateral triangle with side length a has an altitude $\frac{a\sqrt{3}}{2}$, and the distance from a vertex to the center (the centroid, incenter, and circumcenter are all at the same point in an equilateral triangle) is two-thirds the altitude or $\frac{a}{\sqrt{3}}$.

Thus, for the base of the pyramid, the distance from a vertex to the center is $100\sqrt{3}$ centimeters. By the Pythagorean Theorem, the length of an edge of the pyramid going from the corner of the base to the top vertex inscribed

$$\sqrt{(100\sqrt{3})^2 + 100^2} = 200.$$

Thus. the edge is twice as long as the height of the pyramid. It follows by similar triangles that the mouse walks a distance equal to twice the height that the mouse is from the base of the pyramid. When the mouse has walked 148 centimeters, it is at a height of $\frac{1}{2}(148) = \boxed{74}$ centimeters above the base. □

Problem 6.2.16. (*1 point*) Evaluate $1^3 - 2^3 + 3^3 - 4^3 + 5^3 - \cdots + 99^3$.

Solution.

$$1^3 - 2^3 + 3^3 - 4^3 + \cdots + 99^3 = \sum_{k=1}^{99}(-1)^{k+1}k^3 = \sum_{k=1}^{99} k^3 - 2\sum_{k=1}^{49}(2k)^3 = \left(\frac{99\cdot 100}{2}\right)^2 - 16\left(\frac{49\cdot 50}{2}\right)^2 = \boxed{492\,500}$$

□

Problem 6.2.17. (*1 point*) In how many distinguishable rearrangements of the letters ABBCCDEEF does the A precede both C's, the F appears between the 2 C's, and the D appears after the F?

Solution. The conditions in the problem indicate that the letters $ACFC$ must appear in that order.

Into this sequence of four letters, there are two positions into which the D letter can be placed, either before or after the right-most C. Thus, we obtain a sequence of five letters.

Into this sequence of five letters we need insert four positions for the four letters B, B, E, and E. By star-and-bar counting method, there are $\binom{5+4}{4} = 126$ ways to find four places.

Finally, there are $\frac{4!}{2!2!} = 6$ ways to put the four letters B, B, E, and E into these four positions.

Hence the number of ways is $2 \cdot 126 \cdot 6 = \boxed{1512.}$ □

Problem 6.2.18. (*1 point*) Let a be a positive real number such that $\dfrac{a^2}{a^4-a^2+1}=\dfrac{4}{37}$. Then $\dfrac{a^2}{a^4-a^2+1}=\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find mn.

> **Remark.** The second expression is incorrect. It should be $\dfrac{a^3}{a^6-a^3+1}=\dfrac{m}{n}$,

Solution. Let $x=a+\frac{1}{a}>0$, then $x^2=a^2+\frac{1}{a^2}+2$,

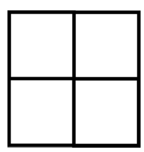
$$\frac{4}{37}=\frac{a^2}{a^4-a^2+1}=\frac{1}{a^2-1+\frac{1}{a^2}}=\frac{1}{x^2-3}\Rightarrow x^2=\frac{37}{4}+3=\frac{49}{4}\Rightarrow x=\frac{7}{2}$$

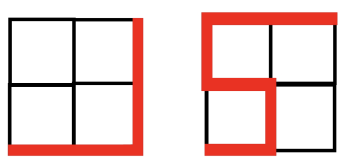
$$\Rightarrow\left(a^2+\frac{1}{a^2}\right)\left(a+\frac{1}{a}\right)=a^3+\frac{1}{a^3}+a+\frac{1}{a}\Rightarrow a^3+\frac{1}{a^3}=(x^2-2)(x)-x=\left(\frac{7}{2}\right)^3-3\cdot\frac{7}{2}=\frac{259}{8}$$

$$\Rightarrow\frac{a^3}{a^6-a^3+1}=\frac{1}{a^3-1+\frac{1}{a^3}}=\frac{8}{251}\Rightarrow mn=8\cdot 251=\boxed{2008.}$$

□

Problem 6.2.19. (*1 point*) The diagrams below shows a 2 by 2 grid made up of four 1 by 1 squares. Shown are two paths along the grid from the lower left corner to the upper right corner of the grid, one with length 4 and one with length 6. A path may not intersect itself by moving to a point where the path has already been. Find the sum of the lengths of all the paths from the lower left corner to the upper right corner of the grid.





Solution. By alternately colouring the corners of the squares black and white and noticing that each edge of a square connects a black corner to a white corner, it is seen that any path from the lower left corner to the upper right corner of the grid must be of even length.

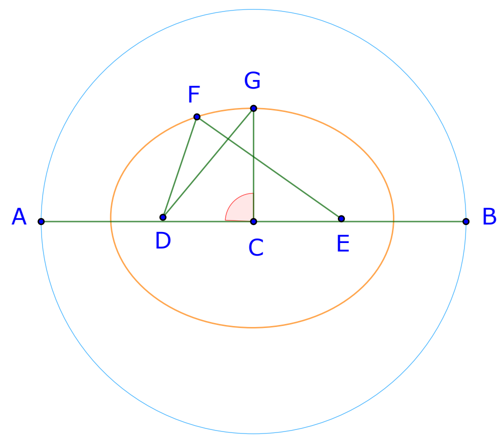
Because a path from the lower left corner of the grid to the upper right corner of the grid must move right along at least to sides of the 1×1 squares, such a path must have a length at least 4. Since there are only 9 corner of he 1×1 squares in the grid, a a path cannot cross any corner twice, a path must have length at most 8. Thus, a path from the lower left corner of the grid to the upper right corner of the grid can have length 4, 6, or 8.

A path of length 4 consists of movements along two horizontal edges and two vertical edges of 1×1 squares. The number of ways to arrange two horizontal moves and two vertical moves in a sequence of four moves in $\binom{4}{2}=6$, so there are 6 paths of length 4. There are 4 paths of length 6, a fact easily verified by considering that a path of length 6 is determined by the first two edges of the path. There are only 2 paths of length 8, a fact easily verified by noticing that a path of length 9 must reach each of the corners which can be done only by including either all horizontal edges or all vertical edges of the 1×1 squares.

Thus, the sum of the lengths of all the paths is $6\cdot 4+4\cdot 6+2\cdot 8=\boxed{64.}$ □

Problem 6.2.20. (*1 point*) Points A and B are the endpoints of a diameter of a circle with center C. Points D and E lie on the same diameter so that C bisects segment DE. Let F be a randomly chosen point within the circle. The probability that $\triangle DEF$ has a perimeter less than the length of the diameter of the circle is $\frac{17}{128}$. There are relatively prime positive integers m and n so that the ratio of DE to AB is $\frac{m}{n}$. Find mn.

Solution. WLOG, assume that the circle has radius 1 so its diameter is 2. Assume that point D and E are each a distance x from C.



If $\triangle DEF$ has a perimeter less than 2, then

$$DF + FE + 2x < 2 \Rightarrow DF + FE < 2 - 2x.$$

It follows that point F lies inside an ellipse with foci D and E where the sum of distances from a point on the ellipse to the points D and E is the constant $2 - 2x$. It follows that the endpoints of the major axis of the ellipse are each a distance $1 - x$ from C.

Let G be an endpoint of the minor axis of the ellipse. Then $DG + GE = 2 - 2x$ and $DG = EG$, so $DG = 1 - x$. Since $\triangle CDG$ is a right triangle with hypotenuse $DG = 1 - x$ and leg $CD = x$, then the leg

$$CG = \sqrt{(1-x)^2 - x^2} = \sqrt{1 - 2x}.$$

The semiminor axis of the ellipse, therefore has length $\sqrt{1-2x}$, while the semimajor axis of the ellipse has length $1 - x$.

It follows that the area of the ellipse is $\pi(1-x)\sqrt{1-2x}$. The area of the circle is π, therefore the probability that $\triangle DEF$ has a perimeter less than the length of the diameter of the circle inscribed satisfy that

$$\begin{aligned}
&\frac{17}{128} = \frac{\pi(1-x)\sqrt{1-2x}}{\pi} \Rightarrow 2x^3 - 5x^2 + 4x - 1 = \frac{289}{16384} \\
&\Rightarrow 2^{15}x^3 - 52^{14}x^2 + 2^{16}x - 16095 = 0
\end{aligned}$$

Since it is given by the problem that x is a rational root of the above equations, thus there exists m positive integer such that $x = \frac{m}{2^5}$. By substitution

$$\begin{aligned}
&m^3 - 80m^2 - 2^{11}m - 16095 = 0 \Rightarrow m \mid 16095 = 3 \cdot 5 \cdot 29 \cdot 37 \\
&\text{By testing } m = 15 \Rightarrow x = \frac{15}{32}
\end{aligned}$$

Hence $mn = 15 \cdot 32 = \boxed{480.}$ $\square$

Problem 6.2.21. (*1 point*) Let a be the sum of the numbers:

$$99 \times 0.9,\ 999 \times 0.9, 9999 \times 0.9 \ldots, 999 \cdots 9 \times 0.9$$

where the final number in the list is 0.9 times a number written as a string of 100 digits all equal to 9. Find the sum of the digits in the number a.

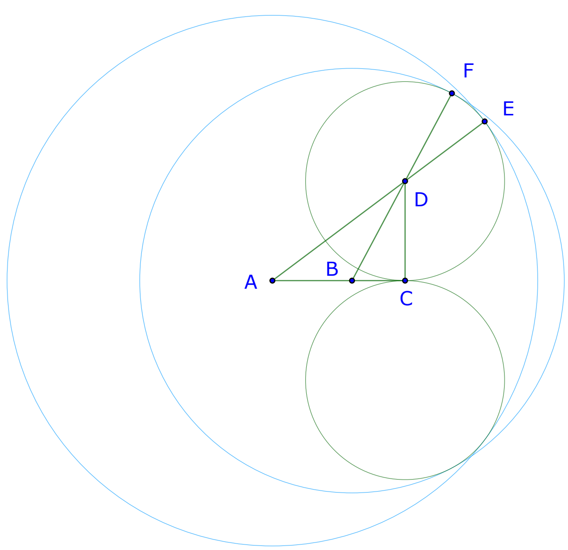
Solution. Note that the list contains 99 numbers. Their sum is:

$$\begin{aligned} a &= 0.9(99 + 999 + \cdots + 999\ldots 9) = 0.9((100-1) + (1000-1) + \cdots + (100\ldots 0 - 1)) \\ &= 0.9((1-1) + (10-1) + (100-1) + (1000-1) + \cdots + (100\ldots 0-1)) - 0.9((1-1) + (10-1)) \\ &= 0.9\left(\sum_{n=0}^{100}(10^n - 1)\right) - 0.9(9) = 0.9\left(\sum_{n=0}^{100} 10^n\right) - 0.9(101) - 0.9(9) \\ &= 0.9\left(\frac{10^{101}-1}{10-1}\right) - 0.9(110) = 0.1(10^{101}-1) - 99 = 0.1(10^{101} - 991) \end{aligned}$$

Note that the sum of digits of $10a$ and a is the same, so we just need to consider $10^{101} - 991$, this number has $101 - 3 = 98$ digits of 9, followed by 0, 0, and 9. Thus, their sum is $98 \cdot 9 + 0 + 0 + 9 = \boxed{891.}$ ☐

Problem 6.2.22. (*1 point*) A disk with radius 10 and a disk with radius 8 are drawn so that the distance between their centers is 3. Two congruent small circles lie in the intersection of the two disks so that they are tangent to each other and to each of the larger circles as shown. The radii of the smaller circles are both $\frac{m}{n}$ where m and n are relatively prime positive integers. Find mn.

Solution. Let A be the center of the circle with radius 10, and B be the center of the circle with radius 8. The two small circles are tangent to each other and the line AB at point C. Let D be the center of one of the small circles. E and F be the point where that small circle is tangent to the circles centered at A and B, respectively.



Both triangles ACD and BCD are right triangle with the right angles at C. Let r be the radius of a small circle. Since $AE = 10$, $AC = \sqrt{(10-r)^2 - r^2}$. Similarly $BC = \sqrt{(8-r)^2 - r^2}$. Furthermore

$$AC = AB + BC \Rightarrow \sqrt{(10-r)^2 - r^2} = 3 + \sqrt{(8-r)^2 - r^2}$$

$$\Rightarrow 27 - 4r = 6\sqrt{(8-r)^2 - r^2} \Rightarrow 16r^2 + 360r - 1575 = 0 \Rightarrow r = \frac{15}{4}$$

Hence $mn = 15 \cdot 4 = \boxed{60.}$ □

Problem 6.2.23. (*1 point*) Let x be a real number in the interval $\left(0, \frac{\pi}{2}\right)$ such that

$$\frac{1}{\sin x \cos x} + 2\cot 2x = \frac{1}{3}.$$

Evaluate

$$\frac{1}{\sin x \cos x} - 2\cot 2x = \frac{1}{2}.$$

> **Remark.** The given expression is incorrect. The correct expression is:
>
> $$\frac{1}{\sin x \cos x} - 2\cot 2x.$$

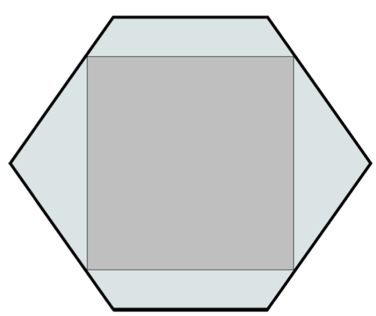
Solution. Note that

$$\frac{1}{\sin x \cos x} + 2\cot 2x = \frac{2}{\sin 2x} + 2\cot 2x, \left(\frac{2}{\sin 2x} + 2\cot 2x\right)\left(\frac{2}{\sin 2x} - \frac{2\cos 2x}{\sin 2x}\right) = \frac{4}{\sin^2 2x} - \frac{4\cos^2 2x}{\sin^2 2x}$$
$$= \frac{4}{\sin^2 2x}(1 - \cos^2 2x) = 4 \Rightarrow \frac{2}{\sin 2x} - \frac{2\cos 2x}{\sin 2x} = \frac{4}{\frac{2}{\sin 2x} + 2\cot 2x} = \boxed{12.}$$

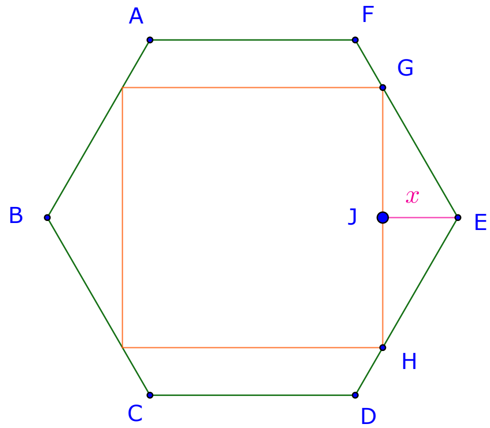
□

Problem 6.2.24. (*1 point*) The diagram below shows a regular hexagon with an inscribed square where two sides of the square are parallel to two sides of the hexagon. There are positive integers m, n, and p such that the ratio of the area of the hexagon to the area of the square can be written as $\frac{m+\sqrt{n}}{p}$ where m and p are relatively prime. Find $(m+n)p$.

> **Remark.** The diagram associated with the problem was missing.



Solution. Label the vertices of the hexagon as shown. Let G and H be the points where one side of the square touches the hexagon. Let J be the point on the side of the square closest to vertex E. Let the distance from E to J be x.



WLOG, assume that the hexagon has side length 1. Since $\angle DEF = 120°$, $DE = 1$, the distance from D to F is twice the height of a $30-60-90$ triangle with hypotenuse 1, or $2 \cdot 1 \cdot \sin(60°) = \sqrt{3}$. So the hexagon is made up of a rectangles measuring 1 by $\sqrt{3}$ and four $30-60-90$ triangle with legs measuring $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$. Thus the area of the hexagon is $\sqrt{3} + 4 \cdot \frac{\sqrt{3}}{8} = \frac{3\sqrt{3}}{2}$.

The width of the hexagon is 2, so the width of the square is $2-2x$. The height of the square is the length GH, which is $2x\sqrt{3}$. Therefore $2-2x = 2x\sqrt{3}$, so $x = \frac{1}{1+\sqrt{3}}$. The area of the square is

$$(2x\sqrt{3})^2 = \left(\frac{2\sqrt{3}}{1+\sqrt{3}}.\right) = \frac{4 \cdot 3}{4+2\sqrt{3}} = \frac{6}{2+\sqrt{3}}.$$

The ratio of the area is

$$\left(\frac{3\sqrt{3}2}{\frac{6}{2+\sqrt{3}}}\right) = \frac{3+\sqrt{12}}{4}.$$

Hence $(m+n)p = (3+12)4 = \boxed{60.}$ $\square$

Problem 6.2.25. (*1 point*) Let x_1, x_2, and x_3 be the three roots of the polynomial $x^3 + 3x + 1$. There are relatively prime positive integers m and n so that

$$\frac{m}{n} = \frac{x_1^2}{(5x_2+1)(5x_3+1)} + \frac{x_2^2}{(5x_3+1)(5x_1+1)} + \frac{x_3^2}{(5x_1+1)(5x_2+1)}.$$

Find mn.

Solution. First, note that $x^3 + 3x + 1 = (x-x_1)(x-x_2)(x-x_3)$. By Viete's Theorem:

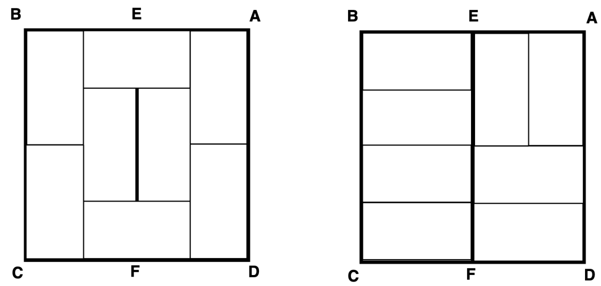
$$\begin{aligned}
&x_1 + x_2 + x_3 = 0, x_1x_2 + x_2x_3 + x_3x_1 = 3, x_1x_2x_3 = -1\\
&\Rightarrow 0 = (x_1+x_2+x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_2x_3 + x_3x_1) \Rightarrow x_1^2 + x_2^2 + x_3^2 = -6\\
&\Rightarrow x_1^3 + x_2^3 + x_3^3 = (-3x_1 - 1) + (-3x_2 - 1) + (-3x_3 - 1) = -3(x_1 + x_2 + x_3) - 3 = -3
\end{aligned}$$

Now

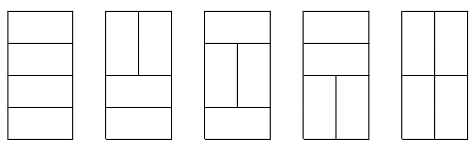
$$\frac{x_1^2}{(5x_2+1)(5x_3+1)} + \frac{x_2^2}{(5x_3+1)(5x_1+1)} + \frac{x_3^2}{(5x_1+1)(5x_2+1)} = \frac{x_1^2(5x_1+1) + x_2^2(5x_2+1) + x_2^2(5x_2+1)}{(5x_1+1)(5x_2+1)(5x_3+1)}$$
$$= \frac{5(x_1^3 + x_2^3 + x_3^3) + (x_1^2 + x_2^2 + x_3^2)}{125(x_1x_2x_3) + 25(x_1x_2 + x_2x_3 + x_3x_1) + 5(x_1 + x_2 + x_3) + 1} = \frac{5(-3) + (-6)}{125(-1) + 25(3) + 5(0) + 1} = \frac{3}{7}$$

Hence $mn = 3 \cdot 7 = \boxed{21.}$ $\square$

Problem 6.2.26. (*1 point*) Square ABCD has side length 4. Points E and F are the midpoints of sides AB and CD, respectively. Eight 1 by 2 rectangles are placed inside the square so that no two of the eight rectangles overlap (see diagram). If the arrangement of eight rectangles is chosen randomly, then there are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that none of the rectangles crosses the line segment EF (as in the arrangement on the right). Find mn.

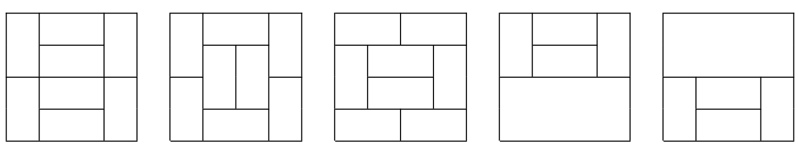


Solution. First, count the number of ways to place the rectangles so that none of them cross the segment EEF. In these arrangements four 1×2 rectangles cover the 2×4 rectangle $AEFD$, and four 1×2 rectangles cover teh 2×4 rectangle $EBCF$. Four 1×2 rectangles can cover a 2×4 rectangle in five ways, as shown below.



Thus, there are $5 \cdot 5 = 25$ ways to cover the 4×4 rectangle without crossing the segment EF.

Now, count the number of ways to cover the 4×4 rectangle while crossing the segment EF. The segment EF can be crossed by either two 1×2 rectangles or by four 1×2 rectangles. Below show the possible arrangements:



For the first three figures from the left, there is 1 way for each to cross EF with 4 rectangles. For the last two figures, there are 4 ways to cross with 2 rectangles. Thus, all together there are $1 + 1 + 4 + 4 = 11$ ways to cover the 4×4 rectangle while crossing the segment EF.

It follows that the probability that none of the rectangles crosses the line segment EF is $\frac{25}{11+25} = \frac{25}{36}$.

Hence $mn = 25 \cdot 36 = \boxed{900.}$ $\square$

Problem 6.2.27. (*1 point*) Find the smallest prime number that does not divide

$$9 + 9^1 + 9^2 + \ldots + 9^{2064}$$

> **Remark.** The given expression is incorrect, it should have been as follow
> $$9^1 + 9^2 + \ldots + 9^{2064}.$$

Solution. Let $n = 9^1 + 9^2 + \ldots + 9^{2064}$.

First, $2 \mid n$ because n is a sum of even number of odd terms, thus n is even.

Second, $3 \mid n$, because each of the terms in the sum if divisible by 9.

Third, 2064 is even, so $n = 9(1+9) + 9^3(1+9) + \ldots + 9^{2063}(1+9)$, so $5 \mid 10 = 1 + 9 \mid n$.

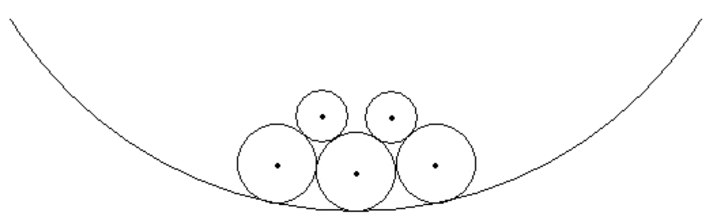
Forth, $3 \mid 2064$, so $n = 9(1+9+9^2)+9^4(1+9+9^2)+\ldots+9^{2062}(1+9+9^2)$, and since $1+9+9^2 = 91 = 7\cdot 13$, so $7 \mid n$ and $13 \mid n$.

Fifth, by Fermat's Little Theorem, $2^{10} \equiv 1 \pmod{11}$,

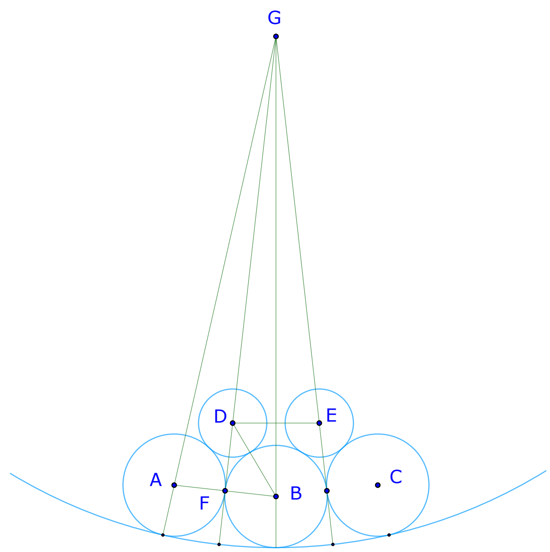
$$9 + 9^1 + 9^2 + \ldots + 9^{2064} = 9 \cdot \frac{9^{2064} - 1}{9 - 1} \Rightarrow n \equiv 9^{2064} - 1 \pmod{11}$$
$$9^{2064} \mid (-2)^{2064} \mid (2^{10})^{206} \cdot 2^4 \mid 5 \pmod{11} \Rightarrow 9^{2064} - 1 \equiv 4 \pmod{11}$$

Thus, $\boxed{11}$ is the smallest prime number that does not divide n. □

Problem 6.2.28. (*1 point*) Pictured below is part of a large circle with radius 30. There is a chain of three circles with radius 3, each internally tangent to the large circle and each tangent to its neighbors in the chain. There are two circles with radius 2 each tangent to two of the radius 3 circles. The distance between the centers of the two circles with radius 2 can be written as $d\frac{a\sqrt{b}-c}{d}$ where a, b, c, and d are positive integers, c and d are relatively prime, and b is not divisible by the square of any prime. Find $(a + b + c)d$.



Solution. Let G be the center of the circle with radius 30. A, B, and C be the centers of the circles with radius 3, and D, E be the centers of the circles with radius 2 as shown in the diagram. Also shown in the diagram are radii of the circle with radius 30 passing through point F where the circles with centers A, D, B, and E.



Line GD also passes through point F where the circles centered at A and B are tangent because point D is equally distant from points A and B. Thus GD is tangent to the circle centered at B, so $\triangle FDB$ is right with $FB = 3$, and $BD = 2 + 3 = 5$, thus $FD = 4$.

Also $GB = 30 - 3 = 27$, so it follows that $GF = \sqrt{GB^2 - BF^2} = \sqrt{27^2 - 3^2} = \sqrt{720} = 12\sqrt{5}$. Then $GD = GF - FD = 12\sqrt{5} - 4$. Now since $\triangle GAB$ is similar to $\triangle GDE$, it follows that

$$\frac{DE}{GD} = \frac{AB}{GA} \Rightarrow DE = \frac{6}{27} \cdot (12\sqrt{5} - 4) = \frac{24\sqrt{5} - 8}{9}.$$

Thus $(a + b + c)d = (24 + 5 + 8)9 = \boxed{333.}$ □

Problem 6.2.29. (*1 point*) Let S be a randomly selected four-element subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Let m and n be relatively prime positive integers so that the expected value of the maximum element in S is $\frac{m}{n}$. Find mn.

Solution. If S has a maximum element k, then it has three other elements less than k. Thus, the number of four-element subsets S with maximum element $k > 3$ is $\binom{k-1}{3}$. The expected value of the maximum element is

$$E = \frac{4\binom{3}{3} + 5\binom{4}{3} + 6\binom{5}{3} + 7\binom{6}{3} + 8\binom{7}{3}}{\binom{8}{4}}.$$

Since

$$k\binom{k-1}{3} = \frac{k(k-1)!}{3!(k-4)!} = 4 \cdot \frac{k!}{4!(k-4)!} = 4\binom{k}{k-4}, \ textthus\ E = 4 \cdot \frac{\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} + \binom{8}{4}}{\binom{8}{4}}.$$

Now, each term of the sum $\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} + \binom{8}{4}$ is the number of ways counting the number of subsets of size 5 from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with maximum element 5, 6, 7, 8, and 9. Therefore the sum is the number of ways selecting a subset of size 5 from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, or $\binom{9}{5}$.

Thus

$$E = 4 \cdot \frac{\binom{9}{5}}{\binom{8}{4}} = \frac{36}{5}.$$

Hence $mn = 36 \cdot 5 = \boxed{180}$. □

Problem 6.2.30. (*1 point*) Four congruent spheres are stacked so that each is tangent to the other three. A larger sphere, R, contains the four congruent spheres so that all four are internally tangent to R. A smaller sphere, S, sits in the space between the four congruent spheres so that all four are externally tangent to S. The ratio of the surface area of R to the surface area of S can be written $m+\sqrt{n}$ where m and n are positive integers. Find $m+n$.

Solution. The center of the four congruent spheres are at the vertices of a regular tetrahedron. Without loss of generality, three of those vertices are located in three-dimensional coordinate space at $(2,0,0)$, $(-1,\sqrt{3},0)$, and $(-1,-\sqrt{3},0)$. Note that the distance between any two of these points is $2\sqrt{3}$. The fourth vertex can be located at $(0,0,\sqrt{2}$ which is a distance $2\sqrt{3}$ from each of the other three vertices. Each of the four congruent spheres then has radius $\sqrt{3}$.

The center of both spheres R and S is equidistant from these three vertices and is located at $(0,0,h)$ where the square of its distance from $(2,0,0)$ is $2^2+h^2=4+h^2$, and its distance squared from $(0,0,2\sqrt{2})$ is $(2\sqrt{2}-h)^2=8-4h\sqrt{2}+h^2$. Equating these two squared distances and solving for h yields $h=\frac{1}{\sqrt{2}}$.

The radius of S is the distance from $\left(0,0,\frac{1}{\sqrt{2}}\right)$ to $(0,0,2\sqrt{2})$ minus the radius of one of the congruent spheres. Thus S has a radius $2\sqrt{2}-\frac{1}{\sqrt{2}}-\sqrt{3}=\frac{3}{\sqrt{2}}-\sqrt{3}$. The radius of R is the distance from $\left(0,0,\frac{1}{\sqrt{2}}\right)$ to $(0,0,2\sqrt{2})$ plus the radius of one of the congruent spheres. THus R has a radius $2\sqrt{2}-\frac{1}{\sqrt{2}}+\sqrt{3}=\frac{3}{\sqrt{2}}+\sqrt{3}$.

The ratio of surface areas is the square of the ratio of these radii or

$$\left(\frac{\frac{3}{\sqrt{2}}+\sqrt{3}}{\frac{3}{\sqrt{2}}-\sqrt{3}}\right)^2=(5+2\sqrt{6})^2=49+20\sqrt{6}=49+\sqrt{2400}.$$

Hence, $m+n=49+2400=\boxed{2449.}$ □

6.3 IC Test for Level 1

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
    1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 6.3.1. A bag containing some widgets weighs 77 kg. When 2 of the widgets are removed, the weight of the bag of widgets drops to 69 kg. The bag weighs 1 kg when empty. How many widgets are left in the bag?

Solution. Let w be the weight of one widget. Since the weight of the bag of widgets drops 8 kg when 2 widgets are removed, we have $2w = 8$. Dividing by 2 gives $w = 4$, so each widget weighs 4 kg. Since the bag weighs 1 kg, the total weight of the widgets in the final bag is $77 - 1 = 76$ kg. Suppose there are n widgets in the bag. Each widget weighs 4 kg and all of them together weigh 76 kg, so we must have $4n = 76$. Dividing both sides of this equation by 4 gives $n = 19$ widgets. The number of widgets left in the bag after removing 2 of them is $19 - 2 = \boxed{17.}$ □

Problem 6.3.2. The angle $\angle B$ in $\triangle ABC$ is $60°$. If an exterior angle at A is $170°$, what is $\angle C$?

Solution. We know that an exterior angle of a triangle is equal to the sum of its remote interior angles. Since the angles remote to the exterior angle at A are $\angle B$ and $\angle C$, we have $170° = \angle B + \angle C = 60° + \angle C$. Therefore, $\angle C = \boxed{110°}$. □

Problem 6.3.3. Each week, between 60 and 80 students show up for an archery class run by Betty and Wilma. Usually the students break up into groups of equal size for target practice. However, this week, Betty noticed that she could not break the students up into multiple groups of equal size. Wilma noticed that if Betty and Wilma joined the students in practicing, they still could not break the archers up into groups of equal size. How many students showed up to the archery class this week?

Solution. Let n be the number of students at class. Betty noticed that n has no divisors between 1 and itself, so n is prime. Wilma noticed that $n + 2$ is prime. This means we are looking for the smaller of two primes that differ by 2 that are between 60 and 80. The primes in that range are 61, 67, 71, 73, and 79. Since 71 and 73 differ by 2, $n = \boxed{71}$. □

Problem 6.3.4. In how many ways can I place 6 different beads on a bracelet if rotating or flipping the bracelet does not change the order of the beads?

Solution. There are 5! ways to arrange 6 beads in a line. Since there are 6 rotations in a circle for each of these arrangements, we must divide by 6, and since there are two matching reflections for each arrangement, we must divide by 2. So there are $\dfrac{6!}{6 \times 2} = \boxed{60}$ ways. □

6.4 IC Test for Level 2

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 6.4.1. For what value of the constant c does the system of equations below have no solutions (x, y)?

$$\begin{cases} 3x - 7y = -2.3 \\ 6x = cy + 9.3 \end{cases}$$

Solution. We collect all the x and y terms on the left and put the constants on the right:

$$\begin{cases} 3x - 7y = -2.3 \\ 6x = cy + 9.3 \end{cases}$$

We multiply the first equation by -2 to set up eliminating the x terms:

$$\begin{cases} -6x + 14y = 4.6 \\ 6x = cy + 9.3 \end{cases}$$

If we add these equations, we eliminate x. If $c = \boxed{14}$, we eliminate y, as well, and are left with $0 = 13.9$. So, when $c = 14$, we have no solutions to this system. If c is not equal to 14, then when we add the equations above, we will eliminate x but not y. We will therefore have a linear equation we can solve for y, and we can then use this value for y to solve for x. So, if c is any value besides 14, then the system of equations has one solution. □

Problem 6.4.2. Points P and Q are on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{PQ} \parallel \overline{BC}$. Given $AB = 36$, $PB = 27$, and $AC = 54$, find QA.

Solution. Since $\overline{PQ} \parallel \overline{BC}$, we have $\angle APQ = \angle ABC$ and $\angle AQP = \angle ACB$. Therefore, $\triangle APQ \sim \triangle ABC$ by AA Similarity. Since $AB = 36$, $PB = 27$, and $AP = AB - PB$, we have $AP = 9$. Our similarity gives us

$$AQ/AC = AP/AB = 9/36 = 1/4.$$

So, $AQ = AC/4 = \boxed{27/2}$. □

Problem 6.4.3. What is the smallest number of marbles that could either be divided up into bags of 12 marbles or into bags of 18 marbles or into bags of 30 marbles with no marbles leftover in each case?

Solution. Let x be this smallest number of marbles that can be broken up into bags of 12 or bags of 18 or bags of 30. This means x must be a multiple of 12, 18 and 30. The smallest such number is $\text{lcm}[12, 18, 30] = 180$, so $x = \boxed{180}$. □

Problem 6.4.4. The student council has 7 boys and 5 girls as class representatives. Two committees, each consisting of 2 boys and 2 girls, are to be created. If no student can serve on both committees, how many different pairs of committees are possible?

Solution. To make the first committee, we choose 2 boys in $\binom{7}{2} = 21$ ways and 2 girls in $\binom{5}{2} = 10$ ways, for a total of $21 \times 10 = 210$ possible committees. Then to make the second committee, we choose 2 more boys in $\binom{5}{2} = 10$ ways and 2 more girls in $\binom{3}{2} = 3$ ways, for a total of $10 \times 3 = 30$ possible committees. So the number of ways to form two committees like this is $210 \times 30 = 6300$.

But this counts each pair of committees twice (in other words, there is no "first" committee or "second" committee), so we must divide by 2 to get the final answer of $6300/2 = \boxed{3150}$ pairs of committees. □

6.5 IC Test for Level 3

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 6.5.1. Alice began peeling a pile of 52 potatoes at the rate of 3 potatoes per minute. Four minutes later Bob joined him and peeled at the rate of 5 potatoes per minute. When they finished, how many potatoes had Bob peeled?

Solution. After 4 minutes, Alice has peeled $3(4) = 12$ potatoes, so there are 40 left. Together, Alice and Bob peel $3 + 5 = 8$ potatoes each minute. So, it takes them $40/8 = 5$ minutes to peel the rest of the potatoes. In those 5 minutes, Bob peels $5(5) = \boxed{25}$ of the potatoes. □

Problem 6.5.2. In triangle ABC, let M be the midpoint of $\overline{BC}$. Prove that if $MA = MB = MC$, then $\angle BAC = 90°$.

Solution. Both $\triangle AMB$ and $\triangle AMC$ are isosceles:

$$\angle MBA = \angle MAB, \angle MCA = \angle MAC \Rightarrow \angle BAC = \angle MAB + \angle MAC = \angle MBA + \angle MCA$$
$$\angle MBA = \angle B, \angle MCA = \angle C \Rightarrow \angle BAC = \angle B + \angle C = 180° - \angle BAC$$

Thus $\angle BAC = 90°$. □

Problem 6.5.3. How many of the divisors of 10800 are perfect squares?

Solution. The perfect square divisors of $10800 = 2^4 \cdot 3^3 \cdot 5^2$ are in the form $2^{2a} \cdot 3^{2b} \cdot 5^{2c}$, where $0 \leq a \leq 2$, $0 \leq b \leq 1$, and $0 \leq c \leq 1$. This gives $3 \cdot 2 \cdot 2 = \boxed{12}$ choices for the exponents to the primes that make up the prime factorizations of the perfect square divisors of 10800. □

Problem 6.5.4. Clara is a house painter. She has been given a contract to paint any 3 of the houses that she wants on a street that contains 7 houses. Clara has 4 different colors of paint, in unlimited supply. She can use only one color on any given house, but for each house, she gets to pick which color to use. In how many different ways can Clara fulfill her contract?

Solution. First, she has to choose which 3 of the 7 houses she's going to paint. Since it doesn't matter in what order she paints the houses, the number of ways she can choose the houses is given by $\binom{7}{3}$.

Second, for each of the 3 houses, Clara has to choose one of her 4 colors to use. The choice of color for each house is independent of her choice for the other houses, so to get the total number of ways that Clara can choose colors, we multiply the choices for each individual house. Thus there are 4^3 ways to choose colors once the houses have been chosen.

Since for each of the $\binom{7}{3}$ ways to choose the houses, Clara has 4^3 ways to choose the colors, the total number of choices that she has is $\binom{7}{3} \cdot 4^3 = \boxed{2240.}$ □

6.6 IC Test for Level 4

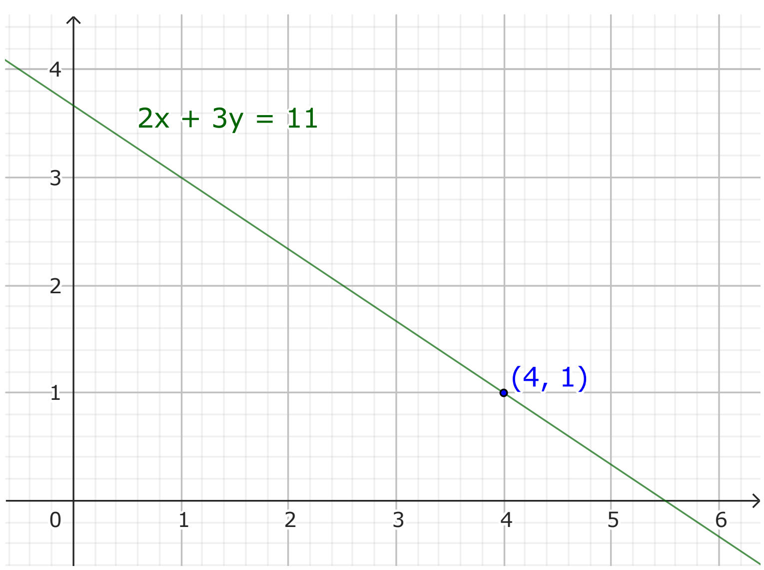
- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 6.6.1. Find the equation, in standard form, of the line that passes through $(4, 1)$ and has slope $-\frac{2}{3}$. Graph the equation.

Solution. We start with a point-slope form of the line through $(4, 1)$ with slope $-2/3$, and then convert this equation to standard form:

$$y - 1 = -\frac{2}{3}(x - 4) \Rightarrow 3(y - 1) = -2(x - 4) \Rightarrow 2x + 3y = 11$$

The graph of the line is shown below.



□

Problem 6.6.2. $\triangle XYZ$ is an equilateral triangle with side length 12. M is the midpoint of side $\overline{YZ}$ and N is the midpoint of $\overline{XZ}$. $\overline{YN}$ and $\overline{XM}$ meet at E. Find the inradius of $\triangle XYZ$.

Solution. The semiperimeter of triangle XYZ is $s = (12 + 12 + 12)/2 = 18$. Therefore, the inradius is $A/s = 36\sqrt{3}/18 = \boxed{2\sqrt{3}}$. We also might note that EM is the inradius, and $EM = XM/3 = \boxed{2\sqrt{3}}$. □

Problem 6.6.3. Find all ordered pairs of positive integers (x, y) where

$$\frac{1}{2x} + \frac{1}{3y} = \frac{1}{12}.$$

Solution. It's easier to work without fractions, so we multiply both sides of the equation by $12xy$ to get

$$6y + 4x = xy.$$

Now we reorganize the equation in order to apply Simon's Favorite Factoring Trick:

$$xy - 4x - 6y \Rightarrow xy - 4x - 6y + 24 = 24 \Rightarrow (x - 6)(y - 4) = 24$$

Since $(6 - 3) \mid 24$, we get the possible values of x by solving for equations made by equating $x - 3$ with various divisors of 24, which are

$$-24, -12, -8, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 8, 12, 24$$

Noting that $x \geq 1, y \geq 1$, meaning $x - 6 \geq -5, y - 4 \geq -3$, thus $x - 6, y - 4$ can only be positive.

$$\begin{cases} x - 6 = 1, y - 4 = 24 \Rightarrow (x, y) = (7, 28) \\ x - 6 = 2, y - 4 = 12 \Rightarrow (x, y) = (8, 16) \\ x - 6 = 3, y - 4 = 8 \Rightarrow (x, y) = (9, 12) \\ x - 6 = 4, y - 4 = 6 \Rightarrow (x, y) = (10, 10) \\ x - 6 = 6, y - 4 = 4 \Rightarrow (x, y) = (12, 8) \\ x - 6 = 8, y - 4 = 3 \Rightarrow (x, y) = (14, 7) \\ x - 6 = 12, y - 4 = 2 \Rightarrow (x, y) = (18, 6) \\ x - 6 = 24, y - 4 = 1 \Rightarrow (x, y) = (30, 5) \end{cases}$$

□

Problem 6.6.4. The box A holds 7 red balls and 5 green balls; boxes B and C each hold 2 red balls and 4 green balls. A box is selected at random and then a ball is randomly selected from that box. What is the probability that the ball selected is green?

Solution. The probability to select box A is $\frac{1}{3}$. For A, the probability to select a green ball is $\frac{5}{12}$.

The probability to select box B is $\frac{1}{3}$. For B, the probability to select a green ball is $\frac{4}{6}$.

The probability to select box C is $\frac{1}{3}$. For C, the probability to select a green ball is $\frac{4}{6}$.

Thus the probability that the ball selected is green is:

$$\frac{1}{3} \cdot \frac{5}{12} + \frac{1}{3} \cdot \frac{4}{6} + \frac{1}{3} \cdot \frac{4}{6} = \boxed{\frac{7}{12}.}$$

□

6.7 IC Test for Level 5

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 6.7.1. Let $x = 2001^{1002} - 2001^{-1002}$ and $y = 2001^{1002} + 2001^{-1002}$. Find $x^2 - y^2$.

Solution. We could square those expressions for x and y, but it's much more convenient to factor $x^2 - y^2$ as $(x+y)(x-y)$. Since $x + y = 2(2001^{1002})$ and $x - y = -2(2001^{-1002})$, we have

$$x^2 - y^2 = (x+y)(x-y) = 2(2001^{1002}) \cdot (-2)(2001^{-1002}) = [2 \cdot (-2)](2001^{1002} \cdot 2001^{-1002}) = \boxed{-4.}$$

□

Problem 6.7.2. The bases of a trapezoid are $2x$ and $4x$. The height is $2x$. If the area is 54, what is x?

Solution. In terms of x, the area of the trapezoid is $(2x + 4x)(2x)/2 = 6x^2$. Since we are given that the area is 48, we have $6x^2 = 54$, so $x = \boxed{3}$. □

Problem 6.7.3. Determine which of the following are multiples of 12.

$$717_8,\ 212021_3,\ 14202_5,\ 6234_7,\ C10B_{13}$$

Solution. Since $12 = 3 \cdot 4$, a multiple of 12 must be a multiple of 4. However,

$$717_8 = 7 \cdot 8^2 + 1 \cdot 8^1 + 7 \cdot 8^0 = 8(7 \cdot 8^1 + 1 \cdot 8^0) + 7$$

is not a multiple of 4 because the first two digit bundles are multiples of 4 and the last is not. Therefore, 717_8 is not a multiple of 12 either.

$$212021_3 = 2 \cdot 3^5 + 1 \cdot 3^4 + 2 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0$$

All of the base-3 digit bundles except the last are multiples of 3, so 212021_3 is not a multiple of 3. Since a multiple of 12 is a multiple of 3, 212021_3 is not a multiple of 12.

n order for an integer to be a multiple of 12, it must be even. However,

$$14202_5 = 1 \cdot 5^4 + 4 \cdot 5^3 + 2 \cdot 5^2 + 0 \cdot 5^1 + 2 \cdot 5^0$$

is a sum of one odd integer and four even ones (including $0 \cdot 5^1 = 0$), so 14202_5 is odd and therefore not a multiple of 12.

$$6234_7 = 6 \cdot 7^3 + 2 \cdot 7^2 + 3 \cdot 7^1 + 4 \cdot 7^0$$

is the sum of three even integers and one odd integer, so it is itself odd and not a multiple of 12.

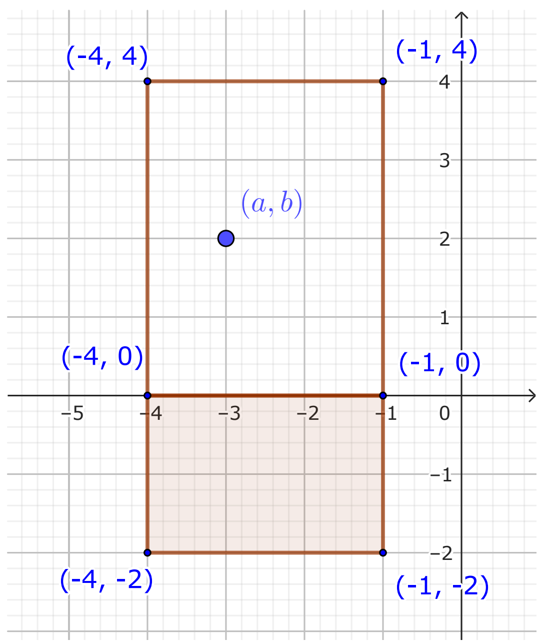
A base-13 integer is a multiple of 12 only if the sum of its digits is a multiple of 12.

$$C_{13} + 1_{13} + 0_{13} + B_{13} = 12 + 1 + 0 + 11 = 24.$$

Since $12 \mid 24$, $C10B_{13}$ is a multiple of 12. □

Problem 6.7.4. Given that a and b are real numbers such that $-4 \leq a \leq -1$ and $-2 \leq b \leq 4$, and values for a and b are chosen at random, what is the probability that the product ab is positive?

Solution. $ab > 0$ if $a < 0, b < 0$ or $a > 0, b > 0$. Since $-4 \leq a \leq -1$ so $a < 0$, thus $-2 \leq b < 0$. In the diagram below, point (a, b) can only be chosen from the shaded area.



Hence, the probability is the ratio of the shaded area to the area of the whole rectangle of $\frac{6}{18} = \boxed{\frac{1}{3}}.$

□

6.8 IC Test for Level 6

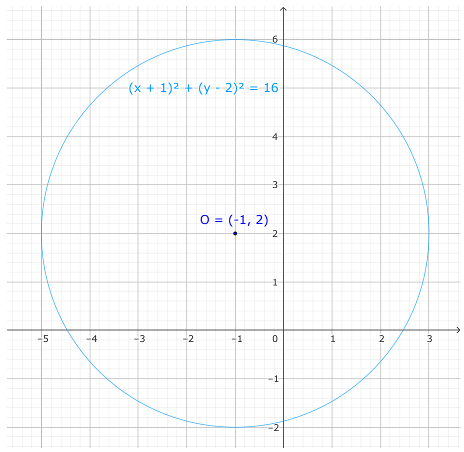
- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 6.8.1. Graph the equation $x^2 + 2x + y^2 - 4y - 11 = 0$.

Solution. We start by completing the squares in both x and y, which gives

$$x^2 + 2x + 1 + y^2 - 4y + 4 - 11 = 1 + 4 \Rightarrow (x+1)^2 + (y-2)^2 = 16.$$

The graph of this equation is a circle with center $(-1, 2)$ and radius $\sqrt{16} = 4$, as shown below.



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Problem 6.8.2. $A_1A_2A_3A_4A_5$ is a pentagon. Prove that $A_1A_3 > A_2A_4$.

> **Remark.** There is typo in the text. The correct inequality should be $A_1A_3 > A_1A_2$.

Solution. For $n > 4$, we consider the exterior angles of the polygon, each of which measures $360°/5 < 360°/4 = 90°$, so the exterior angles are acute. Since each interior angle is supplementary to each exterior angle, we know that each interior angle is obtuse. Specifically, $\angle A_1A_2A_3$ is obtuse, so it is the largest angle in $\triangle A_1A_2A_3$. Therefore, $\overline{A_1A_3}$, which is opposite the obtuse angle in $\triangle A_1A_2A_3$, is the longest side of $\triangle A_1A_2A_3$. Thus, $A_1A_3 > A_1A_2$. □

Problem 6.8.3. Find the units digit of $3^{2024} - 2^{2024}$.

Solution. The units digit of powers of 2 and 3 both repeat in cycles four long. Since $2024 \div 4$ has a remainder of 0, the units digit of $3^{2024} - 2^{2024}$ is the same as $3^4 - 2^4 = 65$ or $\boxed{5}$. □

Problem 6.8.4. The numbers on a standard six-sided die are arranged such that numbers on opposite faces always add to 7. The die is rolled, and the product of the numbers appearing on the four lateral faces of the die is calculated (ignoring the numbers on the top and bottom). What is the expected value of this product?

Solution. The products of opposite faces are $1 \times 6 = 6$, $2 \times 5 = 10$, and $3 \times 4 = 12$. Therefore, after rolling the die, the outcomes $6 \times 10 = 60$, $6 \times 12 = 72$, and $10 \times 12 = 120$ are equally likely. So the expected value is just the average of these outcomes, namely $(60 + 72 + 120)/3 = \boxed{84}$. □