MCC House Championship

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Contents

Part I

Fall 2021

Chapter 1

Championship Rules and Regulations

House Championship is an activity of Math, Chess, and Coding Club (MCC). All students must participate. The students are selected into houses. The championship consists of a number of weekly contests between the teams at various levels. The contest content contains problems and puzzles (including visual, language and chess puzzles) that can be solved by using math, coding, and chess. The houses are ranked after each round. The final standing of the championship is revealed at the end of each semester.

1.1 Rules and Regulations

- **Qualification**: *All students* are *qualified* and *required* to participate.
- **Houses**: Initially an even number of $2n$ houses are established depending on the number of students.
  1. Each house consists of 4 to 6 students. Initially, house members are selected by the club. Each house will have a captain, a lieutenant, and members.
  2. The members can name their house, design new coats-of-arm, slogan, song, cloths, etc if they wish. Common sense ethics are required.
  3. New members of the club will be assigned to a house at the beginning of each semester. The houses can be reorganized depending on the total number of students.
- **Format**: The championship is contested via $2n-2$ rounds:
  1. In each round all the houses are organized into distinct pairs. The houses in a pair participate in a *round contest*. A round contest contains of a number of problems, grouped by *levels of difficulty* (beginner, intermediate, and advanced). In a few days before the contest, the topics of the contest and the number of difficulty levels.
  2. A house organizes a number of *troops* depending on the number of difficulty levels of the coming round contest. Each troop has an approximately equal number of students. No house member can participate in more than one troop.
  3. A round contest consists of a number of troop games. Each game will be played between *same-level troops*. Each troop receive the same set of problems to be solved.
- **Hit, score, and rank**: The championship is contested via $2n-2$ rounds:
  1. For each problem in the troop game, each troop will submit a number (i.e 3), a set of numbers (i.e. $\{3, 4\}$), or a sequence of letters (i.e *T-Rex dinosaur* or *babyface*), or in a format stated by the problem.
  2. There are two different ways how a troop can score: (i) for some problems if the answer is correct, the troop scores 1 hit; (ii) for some problems, the troop provided the best answer scores 1 hit, the

troop that cannot provide the best answer score 0 point. Obviously two equal best answers result in 1 point for both troops.

3. Each troop can post up to 3 answers until the end of the game. Any answer submitted after the third is not taken into consideration.
4. For each troop game, if both troops have the same number of hits, the game results in a draw. Otherwise, the troop with more hits is the winning troop while the other becomes the ultimate loser. *For example, House A meets House B and the results of troop games are:* $A1 - B1 : 2 - 1, A2 - B2 : 3 - 1, A3 - B3 : 0 : 3$.
5. For each game, a house receives 3 points for a game, 1 point for a drawn, 0 point for a loss. The round score of a house is the sum of the points from the three games it plays. *In the example above, House A receives* $3 + 3 = 6$ *points because they won two games and lost one game, House B receives* 3 *points because they lost two games and won one game. So A is the round winner and the score is* $A - B : 6 - 3$.
6. After each round, all house standings are updated. *Note: It is important to note that since a round consists of three 1-to-1 games, it is wise for a house to organize its troop according to not only the expertises of its members but also the capabilities of the troops of the other team. A relatively weaker, but wiser house can win if it knows itself and it knows its opponent.*
7. The *champion* house of the club is the one who has the most points after all rounds are concluded. In case the two top houses have the same number of points, a final contest will be held to decide the ultimate winner.

- **Problems**: Each troop receives a number of problems for each game. The problems the house receives can be based on any of the following areas:
  1. Logical, math, visual, language, or chess puzzles.
  2. Problems that can be solved by a computer program designed in the contest.
  3. Problems that can be solved with competitive math knowledge and skills.
- **Time**:
  1. All contests start at 10:00 PM Eastern Time (EST) on a Saturday and lasts for 90 minutes.
  2. The house must announce its troop compositions (what are the troops and who are the members of which troops) to the Contest Organizer. All participants are required to be present 10 minutes before the contest starts.
  3. A house that does not have a troop for a game shall automatically lose, earning 0 points for that game. If the opposing team's troop is present, they shall receive maximum possible points for that game.
- **Contest organization**: The Contest Organizers (CO) are responsible for overseeing the troop games.
  1. The COs allow the troops participating in that game to access the game's problems. The troops can update the answers to the problems according to instructions from the COs.
  2. All game scores between the troops are publicly available at any point in the contest, automatically updated when a team has made a hit. The round scores are also automatically updated accordingly.
  3. The contest is carried out via Zoom. All team member must show up on the video. Only the team captains can communicate to the COs: raise your hand when you want to talk; mute your microphone for the rest of the time; use the chat if your audio is not working. Please inform the COs that your mic is not working immediately at the start of the contest.
- **Tools**:
  1. The contest is carried out via Zoom. All team member must show up on the video. Only the team captains can communicate to the COs: raise your hand when you want to talk; mute your microphone for the rest of the time; use the chat if your audio is not working. Please inform the COs that your mic is not working immediately at the start of the contest.
  2. Paper-based books, hand-written notes, calculators are allowed during the contest.

3. Computers are allowed for programming (coding) only.
4. Using the Internet to search for similar problems, answers, communicate with outside people, etc, is strictly prohibited.
5. Help from anyone outside the house is not allowed.

Any violation to the rules, especially using external help (results from the Internet, information from people outside of the team, etc.) can cause the house to lose all of their games in that round.

1.2 Sample Topics

1. (*Beginner level*) Counting with number parities, alternate counting, simple logical reasoning.
2. (*Beginner level*) Equations with integer solutions, comparison of exponentiation.
3. (*Intermediate level*) Retrograde chess analysis: what were the last moves?
4. (*Intermediate level*) Geometric shape division, area, perimeter.
5. (*Advanced level*) Playing games, winning positions, winning strategy.

1.3 Sample Problems

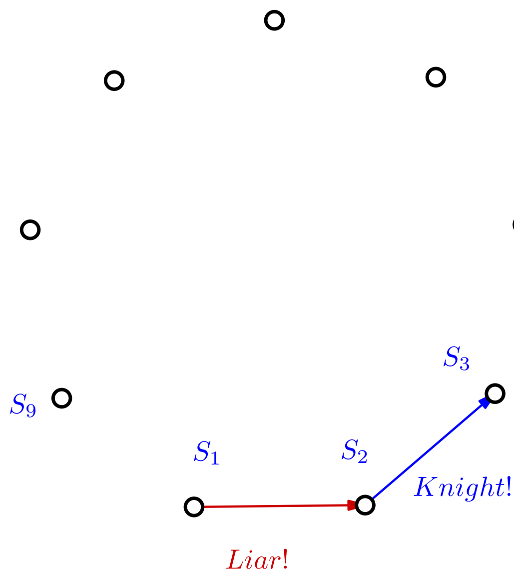


Figure 1.1: Example $S_1 \to S_2 =$ *Liar!*

Problem 1.3.1 (HC-2021-SM2-R0-P1)**.** There are two different people on the Island of Knights and Liars: the Knights, who always tell the truth, and the Liars, who always lie.

During the visit to the island, Albert saw the inhabitants played a game. 9 people $S_1, S_2, \ldots, S_9$ stand in a circle, anticlockwise. Every person pointed at the neighbour in the anticlockwise direction, and said *Liar!* or *Knight!*, indicating what person is the neighbour. See Figure 1.1 for an example.

In this example, the persons S_1 pointed at the person S_2 and said *Liar!*, then the person S_1 probably is a knight and the person S_2 is a liar, but the person S_1 can also be a liar and then the person S_2 is a knight.

After the game ended, Albert heard *Liar!* exactly 8 times. How many liars were there in the game?

How to provide your answer:

- If you think that the number of liars can only be 3, submit the set of a single number $\{3\}$, inditcating that there is only one possibility, namely 3, for the number of liars.
- If you think that the number of liars can be 3, 4, or 5, submit the set of numbers $\{3, 4, 5\}$, inditcating that there are three possibilities, namely 3, 4, and 5, for the number of liars.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is a *strictly smaller subset* of your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit an answer that is *strictly larger superset* of your answer.

Note that, the set $\{3\}$ is a *strictly smaller subset* of the set $\{3, 4\}$, while $\{3, 4\}$ is a *strictly larger superset* of $\{3\}$.

Problem 1.3.2 (HC-2021-SM2-R0-P2)**.** Let x, y, z be positive integers such that $0 \leq x, y, z \leq 5$. How many triples (x, y, z) are there such that $2x^x = y^y + z^z$? Note that: $0^0 1$.

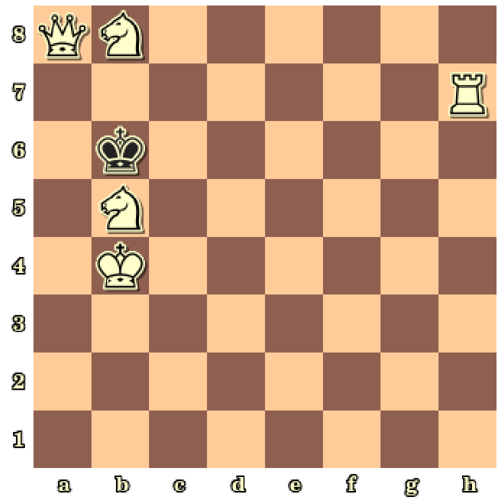
How to provide your answer:

- If you think that the number of triples can be 5, submit 5.
- If you think that there are no such triple, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 1.3.3 (HC-2021-SM2-R0-P3)**.** Below is a situation of a chess game. Both White and Black plays for win, not for draw. What were the last two moves?



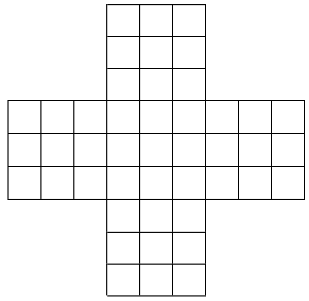
How to provide your answer:

- If you think that the last two moves started with the Black moving its King from $a6$ to $b6$, and White moves its Rook from $a7$ to $h7$, submit the string K$a6b6$ R$a7h7$.
- If you think that there are no such possible moves, submit the string *Impossible*.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 1.3.4 (HC-2021-SM2-R0-P4)**.** Cut the cross along the grid lines into three polygons equal in *area* and *perimeter.*



How to provide your answer:

- If you think that it is possible, submit the URL of an image showing how you can cut the cross in the diagram above. You can draw by hand, capture the image, upload to some storage, make sure that it is accessible by other people, and submit the URL.
- If you think that there are no such possible moves, submit the string *Impossible*.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 1.3.5 (HC-2021-SM2-R0-P5)**.** Following is a two-player game.

There is a pile of some stones. Two players take turns alternately to play the following game,

1. The current player can take 1, 2, 3, 4 or 5 stones from the pile.
2. The current player cannot take the exact same amount of stones as the other took in the previous turn.
3. The loser is the player who cannot continue.

Now, Bianca and Catherine play this game 14 times, each time with a different number of initial stones $1, 2, \ldots, 14$. Both players are smart, so they have winning strategy if the situation allows. Bianca always goes first. Determine the games that Bianca wins.

How to provide your answer:

- If you think that out of 14 games Bianca has a winning strategy to win for the games starting with $1, 2, 3$ stones, submit the set of numbers $\{1, 2, 3\}$.
- If you think that Bianca does not have a winning strategy for any of those 14 games, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is a *strictly smaller subset* of your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit an answer that is *strictly larger superset* of your answer.

Note that, the set $\{3\}$ is a *strictly smaller subset* of the set $\{3, 4\}$, while $\{3, 4\}$ is a *strictly larger superset* of $\{3\}$.

1.4 Solutions

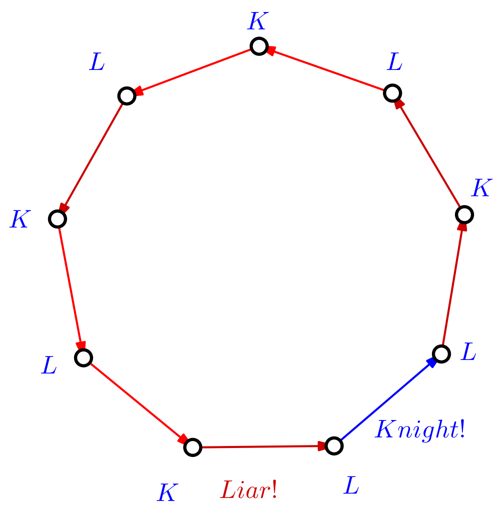


Figure 1.2: 5 Liars

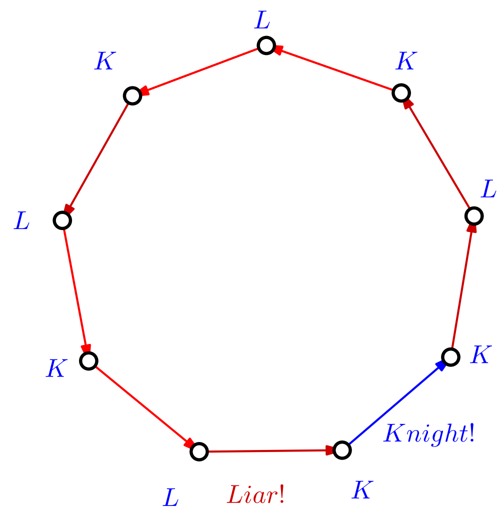


Figure 1.3: 4 Liars

Solution. HC-2021-SM2-R0-P1 Consider two neighboring persons S_i and S_{i+1}, see Table 1.1. There are four different situations. Note that S_i said *Liar!* if and only if one of them is a Knight and the other one is a Liar. Since Albert heard *Liar!* 8 times, so the Knights and Liars *alternately* stand around the circle, anticlockwise.

S_i	S_{i+1}	$S_i \to S_{i+1}$
$Knight$	$Knight$	$Knight$
$Knight$	$Liar$	$Liar$
$Liar$	$Knight$	$Liar$
$Liar$	$Liar$	$Knight$

Table 1.1: $S_i,\ S_{i+1}$

Because there are 9 people, so there are two people of the same kind stand next to each other. They can both be Liars, see Figure 1.2, meaning there are 5 $Liars$; or they can both be Knights, see Figure 1.3, meaning there are 4 $Liars$. Thus, the set of the possible numbers of liars is $\boxed{\{4, 5\}.}$ □

Solution. [First solution] HC-2021-SM2-R0-P2 Let assume that the triple (x, y, z) is a solution to $2x^x = y^y + z^z$.

If $x = 0$ or 1, $2x^x = 2$, so $y^y + z^z = 2$, or $(y, z) = (0, 0), (0, 1), (1, 0), (1, 1)$. There are $2 \cdot 4 = 8$ solutions.

If $x > 1$, and if $y > x > 1$, then $y \geq x + 1$, therefore,

$$y^y \geq (x+1)^{x+1} = (x+1)^x(x+1) = x(x+1)^x + (x+1)^x > 2x^x, \text{ thus } y \leq x.$$

Similarly $z \leq x$, so $2x^x \geq y^y + z^z$. Thus, $x = y = z$. $2 \leq x, y, z \leq 5$, so there are 4 such triples.

Therefore, in total $\boxed{\text{there are } 8 + 4 = 12 \text{ triples.}}$ □

Solution. [Second solution] HC-2021-SM2-R0-P2 Below is a solution based on Python code.

```python
def compare(x,y,z):
    return 2 * pow(x,x) == pow(y, y) + pow(z,z)

if __name__ == "__main__":
    for x in range(0,6):
        for y in range(0,6):
            for z in range(0,6):
                if compare(x,y,z):
                    print('(x,y,z)=(%s,%s,%s)' % (x,y,z))
```

The output shows that $\boxed{\text{there are } 8 + 4 = 12 \text{ triples.}}$

```
(x,y,z)=(0,0,0)
(x,y,z)=(0,0,1)
(x,y,z)=(0,1,0)
(x,y,z)=(0,1,1)
(x,y,z)=(1,0,0)
(x,y,z)=(1,0,1)
(x,y,z)=(1,1,0)
(x,y,z)=(1,1,1)
(x,y,z)=(2,2,2)
(x,y,z)=(3,3,3)
(x,y,z)=(4,4,4)
(x,y,z)=(5,5,5)
```

□

Solution. HC-2021-SM2-R0-P3 If it was Black's move for the situation shown, then the game should have ended in a draw, which was not the intention of White. Therefore it was a White's move. The Black king could have came from $a6$ or $c6$.

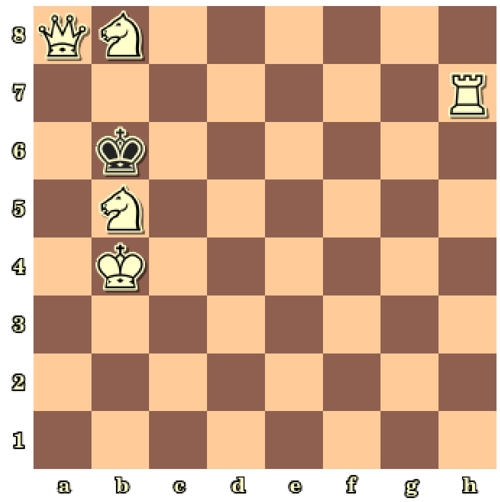


Figure 1.4: HC-2021-SM2-R0-P3

Case 1: if the Black king could have came from $a6$*,* then there should neither be a knight on $b8$, nor a white piece on $a7$, each of which could have checked the Black king. This piece moved to $b8$, became a knight on that square, and allowed the White queen to check the Black king. Thus, it was a pawn, that was promoted to knight after the pawn captured some piece at $b8$. So the two moves in this case were $a7 \times b8$ =N K$a6b6$.

Case 2: if the Black king could have came from $c6$*,* then there should neither be a knight on $b8$, nor a white piece on $b7$, each of which could have checked the Black king. then there should have been a white piece on $b7$, which could not check the Black king. This piece moved to $b8$, became a knight on that square, and allowed the White queen to check the Black king. Thus, it was a pawn, that was promoted to knight while moving to $b8$. So the two moves in this case were $b7b8$ =N K$c6b6$.

Thus, the solutions are $\boxed{a7 \times b8 \text{ =N K}a6b6 \text{ and } b7b8 \text{ =N K}c6b6.}$ □

Solution. HC-2021-SM2-R0-P4 Below, in Table 1.2, is one of the solutions.

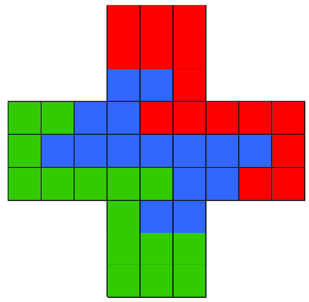


Table 1.2: HC-2021-SM2-R0-P4

Each polygon of the three polygons has an $\boxed{\text{area of 15 unit squares}}$ and a $\boxed{\text{perimeter of 28 unit distance.}}$ □

Solution. HC-2021-SM2-R0-P5 Consider the graph above in Figure 1.5. The positions are indicated by the residues modulo 13, $\{0, 3, 5, 7\}$ of the remaining stones. The green and orange arrows indicate the number of stones taken by the first and second players, respectively, in two consecutive turns. Note that in positions 3 and 5, the first players cannot take 3 or 5 stone because that are the numbers of stones taken in the previous turn. It is easy to see that, these are **winning positions** for *the second player.* The game finishes only the second player reach position 0.

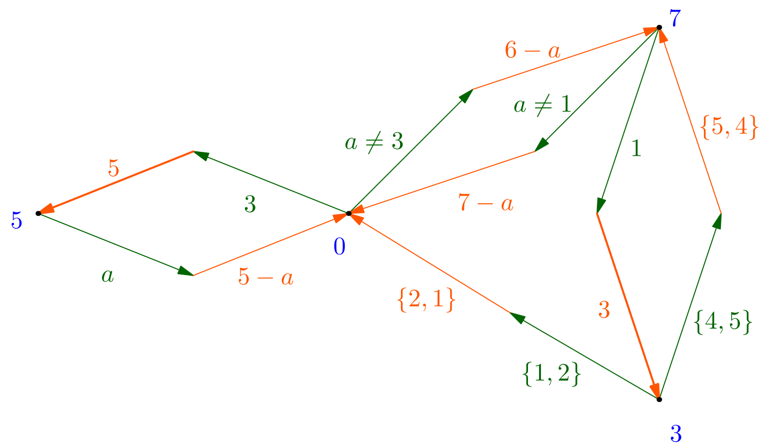


Figure 1.5: HC-2021-SM2-R0-P5

If the number of stones is one of $1, 2, \ldots, 5$, by taking all the stones, Bianca wins the game. For 6, by taking 3 stones, she gets the game into her winning position 3. For any number $8, 9, \ldots, 12$, she takes $1, 2, \ldots, 5$ stones, respectively, to get the game to her winning position 7. For 14 stones, by taking 1 stones, she get the games to 13, which is her winning position. She loses only if the number of initial stones is 7 or 13.

Thus, Bianca wins the game starting with $\boxed{1, 2, \ldots, 6, 8, \ldots, 12, 14.}$ □

Chapter 2

HC HST

2.1 Problems

Rules

- The total time to complete the test is 90 minutes.
- The test consists of 15 problems at 3 levels (*Beginner, Intermediate, Advanced*), 5 problems per levels. To answer each problem, you can select *atmost one* choice among A, B, C, D, and E.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a **correct answer**, i.e. if you select the correct choice, you get 4**,** 6, or 10 points for a *Beginner-, Intermediate-*, or *Advanced*-level problem, respectively.
  2. For an **unanswered** problem, i.e. if you select no choice, you get 1**,** 1.5, or 2.5 points for a *Beginner-, Intermediate-*, or *Advanced*-level problem, respectively.
  3. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point in any case.
- Ranking for house selection: students are ranked based on their total scores. Two students having the same scores will be ranked depending on whose score is better in regard to the *Advanced* problems, then the *Intermediate* problems, and the *Beginner* problems, in this order.

Problem 2.1.1 (HC-2021-SM2-HST-P1)**.** (*4 points*) Anna has to figure out the number puzzle below.

$$\begin{array}{ccccc} & & M & E & O \\ + & & & E & O \\ \hline & 1 & 0 & 1 & 2 \end{array}$$

Table 2.1: HC-2021-SM2-HST-P1

What is the value of $M + E + O$?

(A) 12

(B) 15

(C) 13

(D) 22

(E) 20

Problem 2.1.2 (HC-2021-SM2-HST-P2). (*4 points*) Tiger always tells the truth on Monday, Tuesday, Wednesday, and Thursday. Lion always tells the truth on Monday, Friday, Saturday, and Sunday. On the rest of the days, they may tell the truth or they may lie. On one day last week, both of them said *"I lied yesterday."*.

On what day of the week did they make those statements?

(A) Tuesday

(B) Wednesday

(C) Thursday

(D) Friday

(E) Sunday

Problem 2.1.3 (HC-2021-SM2-HST-P3). (*4 points*) Paul and Remy take turn to play a two-player game. Initially there is one number on the board. In a turn,

a. The player of that turn can choose any number on the board.

b. The player finds a positive divisor of that the chosen number, which is larger than 1 and less than the chosen number, and is not yet written on the board.

c. The player writes it on the board.

For example if the number 9 *is on the board and the player chooses it. He then chooses the number* 3 *because* 3 *is a divisor of* 9*,* 3 *is larger than* 1 *and* 3 *less than 9. Assume that* 3 *is not on the board, the player writes* 3 *on the board.* The firtst player who cannot legally complete a turn loses the game, then the game ends.

If the initial number on the board was 144, in total how many turns does the game end?

(A) 13

(B) 14

(C) 15

(D) 16

(E) It never ends.

Problem 2.1.4 (HC-2021-SM2-HST-P4)**.** (*4 points*) What is the value of the expression below,

$$\underbrace{1-(1-(1-\ldots-(1-1)))}_{\text{The number 1 appears 2021 times}}$$

(A) -1

(B) 0

(C) 1

(D) 2

(E) 1011

Problem 2.1.5 (HC-2021-SM2-HST-P5)**.** (*4 points*) Bill fits exactly six round pizzas into his newly built oven with diameter 12, as show below in the Figure 2.1.

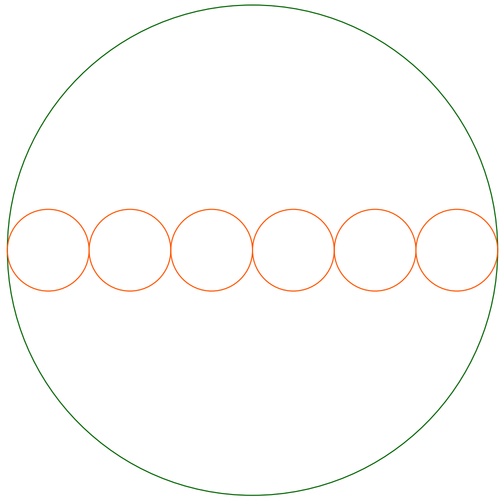


Figure 2.1: HC-2021-SM2-HST-P5

If a total of twenty pizzas are placed in this oven without overlap, what fraction of the oven is covered by the pizzas?

(A) $\frac{5}{9}$

(B) $\frac{7}{9}$

(C) $\frac{5}{6}$

(D) $\frac{23}{36}$

(E) $\frac{25}{36}$

Problem 2.1.6 (HC-2021-SM2-HST-P6). (*6 points*) The symbols $\heartsuit$, $\diamondsuit$, and $\spadesuit$ represent missing digits of n, a multiple of 792, as shown below,

$$n = \overline{1\heartsuit\diamondsuit\spadesuit 6}.$$

Find the largest value of the product $\heartsuit \times \diamondsuit \times \spadesuit$.

(A) 60

(B) 40

(C) 36

(D) 28

(E) 16

Problem 2.1.7 (HC-2021-SM2-HST-P7). (*6 points*) In each square of the 8×8 chessboard, Antoine writes a number according to the following rules:

i. The number in each square is one of $1, 2, \ldots, 64$.

ii. The two numbers, in two neighbouring squares that share the same side, differ less than 3.

iii. The number 3 is written in one of the square.

What is the largest number could Antoine writes on the board?

(A) 17

(B) 28

(C) 31

(D) 42

(E) 64

Problem 2.1.8 (HC-2021-SM2-HST-P8)**.** (*6 points*) It would take 6 hours for Amelie and Benny the magicians to cast all the frogs in the pond into beautiful princesses. Each one of Amelie and Benny casts one frog at a time, and at own pace. Amelie and Benny worked together for some time. During this time, Amelie transformed $\frac{5}{9}$, and Benny transformed $\frac{5}{18}$ of the frogs. After that, Amelie went to do some homework for school, and Benny had to finish transforming all the remaining frogs. How long did it take for Benny to finish the job?

(A) 3 hours.

(B) 3 hours and 30 minutes.

(C) 4 hours.

(D) 4 hours and 15 minutes.

(E) 4 hours and 45 minutes.

Problem 2.1.9 (HC-2021-SM2-HST-P9)**.** (*6 points*) The Pythagorean triples (a, b, c) with $c = b + 1$ in the increasing order as shown in the table below.

a	b	c
3	4	5
5	12	13
7	24	25
9	40	41
11	60	61
	...	

Table 2.2: $a^2 + b^2 = c^2, \; c = b + 1$

Find the value of c when $a = 15$.

(A) 107

(B) 109

(C) 111

(D) 113

(E) 115

Problem 2.1.10 (HC-2021-SM2-HST-P10). (*6 points*) Points E and F are selected on the sides CD and BC of the square $ABCD$ such that $\angle BAF = \angle DAE = 15°$, see Figure 2.2

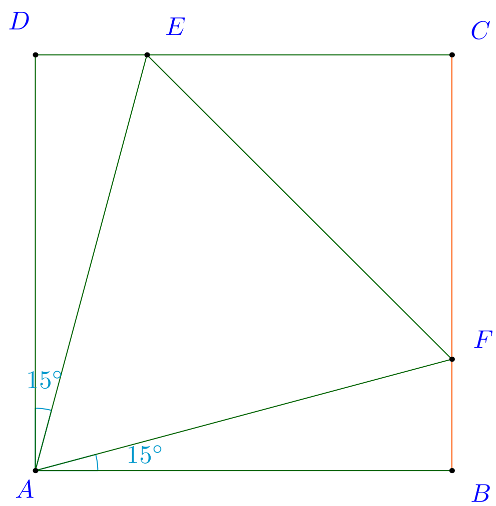


Figure 2.2: HC-2021-SM2-HST-P10

Find the value of the $\frac{CF}{FB}$.

(A) $2+\sqrt{3}$

(B) $\sqrt{2}+\sqrt{3}$

(C) $2\sqrt{2}$

(D) $2\sqrt{3}-2$

(E) $\sqrt{3}+1$

Problem 2.1.11 (HC-2021-SM2-HST-P11). (*10 points*) (x, y, z) is triple of integers such that,

$$\begin{cases} x - yz &= 11 \\ xz + y &= 13 \end{cases}$$

What is the minimal value of $x + y + z$?

(A) 24

(B) 14

(C) 6

(D) 0

(E) -11

Problem 2.1.12 (HC-2021-SM2-HST-P12)**.** (*10 points*) A pirate ship has 2021 treasure chests (all chests are closed). Each chest contains some amount of gold but it may be empty. To distribute the gold the pirates are going to do the following.

i. The captain is going to decide first how many chests he wants to keep and tell that number to the rest of the pirates.
ii. Then he is going to open all the chests and decide which ones he wants to keep (he can only choose as many as he said before opening them).

The captain wants to make sure he can keep *at least half of the total gold.* However, he wants to say the smallest possible number of chests to keep the rest of the pirates as happy as he can. What number should the captain say?

(A) 1010
(B) 1011
(C) 1101
(D) 1501
(E) 2001

Problem 2.1.13 (HC-2021-SM2-HST-P13). (*10 points*) Thanh filled a triangle of squares with the letters of her name, as shown below in the Table 2.3. She counted all the 5−letter paths that form her name T-H-A-N-H, each starts from the T letter in the middle of the bottom row, then goes left, right, or up. An example is shown in the diagram above.

				H				
			H	N	H			
		H	N	A	N	H		
	H	N	A	H	A	N	H	
H	N	A	H	T	H	A	N	H

Table 2.3: HC-2021-SM2-HST-P13

What number did she get?

(A) 8

(B) 15

(C) 16

(D) 31

(E) 32

Problem 2.1.14 (HC-2021-SM2-HST-P14). (*10 points*) Each of three squares with side length 12 is cut into two pieces, a red isosceles right triangle and a blue pentagon, as shown on the left of the Figure 2.3 below. These 6 pieces are then attached to a regular hexagon, as shown on the right, and folded into a polyhedron (a three-dimensional shape with flat polygonal faces, straight edges and sharp corners or vertices).

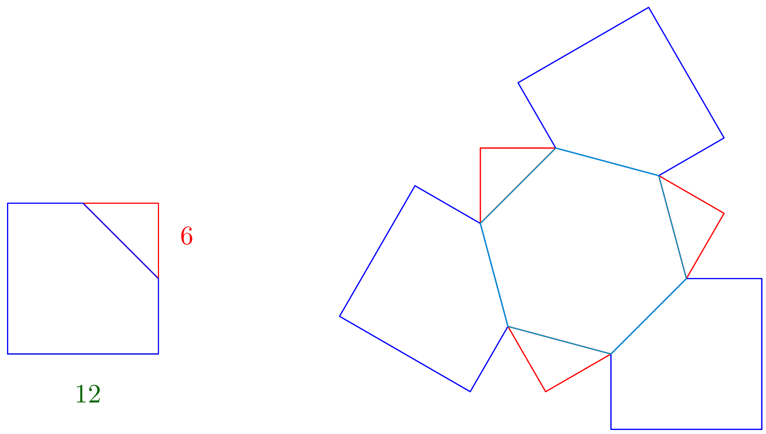


Figure 2.3: HC-2021-SM2-HST-P14

What is the volume of this polyhedron?

(A) 432

(B) 864

(C) 1248

(D) $432\sqrt{2}$

(E) 1296

Problem 2.1.15 (HC-2021-SM2-HST-P15)**.** (*10 points*) The figure, shown below in Table 2.4, is called a 135−*domino* because it contains 1, 3, and 5 squares on the first, second, and third row, respectively.

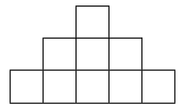


Table 2.4: A 135 domino

Harry tiles $n \times n$ board with 135−*dominoes*. For how many values of n between 1 and 20 can he succeed?

For example if you think Harry can tiles the 1×1 *and* 2×2 *boards with the* 135−*dominoes, answer* 2.

(A) 0

(B) 2

(C) 3

(D) 5

(E) 6

2.2 Answer Keys

Problem 1: E

Problem 2: D

Problem 3: A

Problem 4: C

Problem 5: A

Problem 6: B

Problem 7: C

Problem 8: A

Problem 9: D

Problem 10: E

Problem 11: E

Problem 12: B

Problem 13: D

Problem 14: B

Problem 15: C

2.3 Solutions

Solution. [First solution] HC-2021-SM2-HST-P1 First, M has to be 9, otherwise there will not be enough to carry over to have the result as a four digit number. Therefore,

$$\overline{MEO} + \overline{EO} = 900 + 2\overline{EO} \Rightarrow \overline{EO} = \frac{1012 - 900}{2} = 56.$$

Thus, the sum $M + E + O$ is $\boxed{9 + 5 + 6 = 20.}$ The answer is $\boxed{E.}$ □

Solution. [Second solution] HC-2021-SM2-HST-P1

```
if __name__ == "__main__":
    for m in range(1, 10):
        for e in range(0, 10):
            for o in range(0, 10):
                if m*100 + e*10 + o + e * 10 + o == 1012:
                    print('%d%d%d' %(m, e, o))
```

The output shows that the number is $\boxed{956.}$ The answer is $\boxed{E.}$

```
956
```

□

Solution. HC-2021-SM2-HST-P2 Neither Tiger, nor Lion can make the statement *"I lied yesterday"* on the day which the truth must be told *if the previous day* was also one of the days on which they must tell the truth. Therefore Tiger cannot make the statement on Tuesday, Wednesday, or Thursday. Lion cannot make such statement on Saturday, Sunday, or Monday.

Thus, the day of the week when they made those statements is $\boxed{\text{Friday.}}$ The answer is $\boxed{D.}$ □

Solution. HC-2021-SM2-HST-P3 It is easy to see that the players take turn to write all *distinct* divisors of 144, which are larger than 1 and less than 144, there are 13 such numbers:

$$2,\ 3,\ 4,\ 6,\ 8,\ 9,\ 12,\ 16,\ 18,\ 24,\ 36,\ 48,\ 72$$

Thus, the number of turns the game has is $\boxed{13.}$ The answer is $\boxed{A.}$ □

Solution. [First solution] HC-2021-SM2-HST-P4 Let observe a few cases,

$$\begin{aligned}
&1-1=0\\
&1-(1-1)=1-0=1\\
&1-(1-(1-1))=1-1=0\\
&1-(1-(1-(1-1))=1-0=1\\
&\ldots
\end{aligned}$$

Now, it is easy to see that the result is 0 if there is an *even* number of 0, and 1 if there is an *odd* number of 0.

Since 2021 is an odd number, the value of the expression is $\boxed{1.}$ The answer is $\boxed{C.}$ □

Solution. [Second solution] HC-2021-SM2-HST-P4

```
if __name__ == "__main__":
    s = 1
    for n in range(2, 2022):
        s = 1 - s
    print(s)
```

The output shows that the number is $\boxed{1.}$ The answer is $\boxed{C.}$

```
1
```

□

Solution. HC-2021-SM2-HST-P5 It is easy to see that if 6 pizzas fits in the oven diameter 12, then the radius of each pizza is 1. Therefore, their area is 20π.

Thus, the fraction of the oven they covered is $\boxed{\frac{20}{36} = \frac{5}{9}.}$ The answer is $\boxed{A.}$ □

Solution. [First solution] HC-2021-SM2-HST-P6 Since $792 = 8 \cdot 9 \cdot 11$, therefore,

$$\begin{aligned} 9 \mid n &\Rightarrow 1 + \heartsuit + \diamondsuit + \spadesuit + 6 \equiv 0 \pmod 9 \Rightarrow \heartsuit + \diamondsuit + \spadesuit \equiv 2 \pmod 9 \\ 11 \mid n &\Rightarrow 1 + \diamondsuit + 6 = \heartsuit + \spadesuit \\ \Rightarrow \heartsuit + \diamondsuit + \spadesuit &= 7 + 2\diamondsuit \equiv 2 \pmod 9 \Rightarrow \diamondsuit = 2, \heartsuit + \spadesuit = 9 \end{aligned}$$

Since $8 \mid n$, $\overline{\spadesuit 6} \in \{16, 36, 56, 76, 96\}$, thus $(\heartsuit, \spadesuit) \in \{(8,1), (6,3), (4,5), (2,7), (0,9)\}$.

Therefore, the largest value of $\heartsuit * \diamondsuit * \spadesuit$ $\boxed{4 \cdot 2 \cdot 5 = 40.}$ The answer is $\boxed{B.}$ □

Solution. [Second solution] HC-2021-SM2-HST-P6

```
if __name__ == "__main__":
    max_product = [0]
    for n in range(10000//792, 20000//792):
        q, r = divmod(n*792-10000, 10)
        if r == 6:
            p, spade = divmod(q, 10)
            heart, diamond = divmod(p, 10)
            s = heart * diamond * spade
            if s > max_product[0]:
                max_product = [s, heart, diamond, spade]
    print(max_product)
```

The output shows that the number is $\boxed{40.}$ The answer is $\boxed{B.}$

```
[40, 4, 2, 5]
```

□

Solution. HC-2021-SM2-HST-P7 Let m be the maximal number that can be written on the board. In the Table 2.5 below, there are atmost 7 squares, on the row of 3, from the number 3 to the intersection with the column of m. Similarly there are atmost 7 squares, on the column of m, from the intersection with the column of m to the number m. Thus the difference between m and 3 can not be more than $2 \times 7 + 2 \times 7 = 28$.

	3						
						m	

Table 2.5: Row of 3 and column of m

Thus, the largest value of m can be $\boxed{3 + 28 = 31.}$ The answer is $\boxed{C.}$ □

Solution. HC-2021-SM2-HST-P8 First, we can observe that Amelie works twice as fast as Benny ($\frac{5}{9}$ of all and $\frac{5}{18}$ of all). Next, note that by the time Amelie went elsewhere, $1 - \frac{5}{9} - \frac{5}{18} = \frac{1}{6}$ of all the frogs were not transformed yet. If the two work together, they transform $\frac{1}{6}$ of all the frogs in one hour. However, since Amelie is twice as fast, Benny transforms $\frac{1}{3}$ of this amount in one hour, and Amelie transforms $\frac{2}{3}$ of this amount. Thus, Benny transforms $\frac{1}{18}$ of all the frogs in one hour.

Since Benny is left with $\frac{1}{6}$ of all the frogs, it will take him $\boxed{\text{3 hours to finish.}}$ The answer is $\boxed{A.}$ □

Solution. HC-2021-SM2-HST-P9 First, the sequence of b values in black, their differences in red, and the differences of their differences in blue shown below Table 2.6.

4		12		24		40		60
	8		12		16		20	
		4		4		4		

Table 2.6: Sequence of differences

Continuing this pattern, the sequence of difference is extended by adding 24 and 28, see Table 2.7, thus we receive 112 as the 7^{th} term of the sequence of b values, which corresponds with $1 + 7 \cdot 2 = 15$ value for a.

4		12		24		40		60		84		**112**
	8		12		16		20		24		28	
		4		4		4		4		4		

Table 2.7: $20 \to 24 \to 28$, $60 \to 84 \to 112$

Thus, the value of $\boxed{c = 113.}$ The answer is $\boxed{D.}$ □

Solution. HC-2021-SM2-HST-P10 First $\triangle DAE \cong \triangle BAF$ (ASA), so $DE = BF$, then $EC = CF$. Thus $\angle EFC = \angle FEC = 45°$, so $\angle AEF = \angle AFE = 60°$, or $\triangle AEF$ is equilateral.

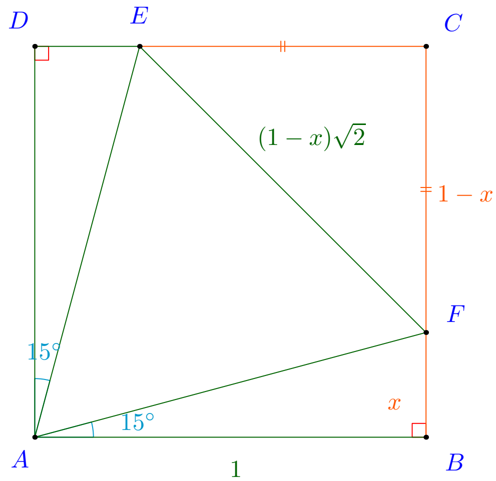


Figure 2.4: $\triangle AEF$ is equilateral

Now let $x = BF$ $(x < 1)$, then $CF = 1 - x$, and $FA = EF = (1-x)\sqrt{2}$, then

$$FA^2 = EF^2 \Rightarrow 1 + x^2 = 2(1-x)^2 \Rightarrow x^2 - 4x + 1 = 0 \Rightarrow x = 2 - \sqrt{3},\ 1 - x = \sqrt{3} - 1$$
$$\Rightarrow \frac{CF}{FB} = \frac{1-x}{x} = \frac{\sqrt{3}-1}{2-\sqrt{3}} = \frac{2(\sqrt{3}-1)}{4-2\sqrt{3}} = \frac{2(\sqrt{3}-1)}{(\sqrt{3}-1)^2} = \frac{2}{\sqrt{3}-1} = \frac{2(\sqrt{3}+1)}{2} = \sqrt{3}+1$$

Thus, the value of the fraction $\boxed{\frac{CF}{FB} = \sqrt{3}+1.}$ The answer is $\boxed{E.}$ □

Solution. HC-2021-SM2-HST-P11 Note that $x \cdot yz = xz \cdot y$, so, by squaring and adding the equalities together,

$$(x - yz)^2 + (xz + y)^2 = 11^2 + 13^2 = 290$$
$$(x - yz)^2 + (xz + y)^2 = x^2 + (yz)^2 + (xz)^2 + y^2 = (x^2 + y^2)(z^2 + 1)$$

$z^2 + 1$ is a factor of 290 that is one more then a square, therefore,

$$z^2 + 1 \in \{1, 2, 5, 10, 145, 290\} \Rightarrow z^2 \in \{0, 1, 4, 9, 144, 289\} \Rightarrow z \in \{0, \pm 1, \pm 2, \pm 3, \pm 12, \pm 17\}.$$

For a value of z, the remaining work is to substitute it into the given system of equations and solve it. The triples are $\boxed{(11, 13, 0), (12, 1, 1), (1, 12, -1), (7, -3, 2), (-2, 5, -3), (1, -1, 12), (-1, 1, -12).}$ The answer is $\boxed{E.}$ □

Solution. HC-2021-SM2-HST-P12 Note that the chests could all contain the same amount of gold, so to keep at least of half of the gold the captain should keep at least half of the chests. With this in mind, the number he must say is at least $\left\lfloor \frac{2021}{2} \right\rfloor = 1011$.

Let us see that regardless of the gold distribution, 1011 of the chests are enough to keep half of the gold. To do this it is enough to open the chests and order them according to the amount of gold they have

$$n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_{2021}.$$

If the captain keeps the chests $n_1, n_2, \ldots, n_{1011}$ then he has at least half of the gold.

The answer is $\boxed{B.}$ □

Solution. HC-2021-SM2-HST-P13 Consider the *"half"* triangle shown in the Table 2.8 below. It is easy to see that in each path, there is two choices at each steps. One example is shown in the figure, when two As can be chosen after a H. Therefore there are $2^4 = 16$ paths for a half triangle. In total there are $2 \cdot 16 - 1 = 31$,

				H
			H	N
		H	N	A
	H	N	A	H
H	N	A	H	T

Table 2.8: Two choices of A from H

because the vertical path formed by all the squares in the T column is shared by both half triangles.

Thus, the number that Thanh got is $\boxed{31.}$ The answer is $\boxed{D.}$ □

Solution. HC-2021-SM2-HST-P14 It is easy to see that the polyhedron is actually half of a cube with side length 12, cut by a plane thorugh six midpoints of its sides as shown in the Figure 2.5 below.

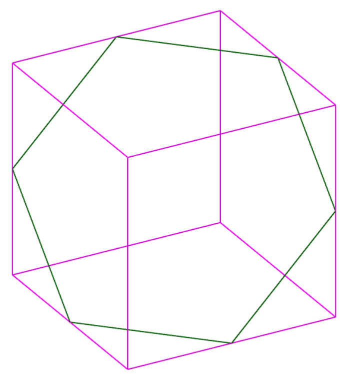


Figure 2.5: Half of a cube side length 12

Thus, the volume of the polyhedron is $\boxed{\frac{12^3}{2} = 864.}$ The answer is $\boxed{B.}$ □

Solution. HC-2021-SM2-HST-P15 Assume that a $n \times n$ board can be tiled by a number of $135-$*dominoes*. Since a $135-$*domino* contains 9 squares, so n is divisible by 3. Furthermore, by colouring alternately, as shown in Table 2.9, it is easy to see that the number of black squares and the number of white squares covered by a $135-$*domino* are divisible by 3. Thus, their difference in the $n \times n$ board, tiled by $135-$*dominoes*, is a multiple of 3.

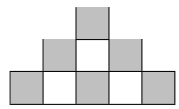
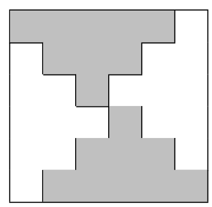


Table 2.9: Alternate colouring

Furthremore, if n is an odd number, the positive difference of black and white coloured squares in a $n \times n$ board is 1. Thus, the tiling can be done if and only if n is divisible by 2. Therefore n is divisible by 6.

Below is a tiling for a 6×6 board with 4 pieces of $135-$*dominoes*. Any $6k \times 6k$ board can be divided into k^2 of 6×6 board.



The values of n between 1 and 20 that Harry can tile $n \times n$ board are $\boxed{6, 12, 18.}$ The answer is $\boxed{C.}$ □

Chapter 3

HC R1

3.1 Topics

Chess

1. Moves: capture, check, double check, promotion, stalemate, checkmate
2. Analysis: whose turn is it.
3. Retrograde: last move, last two moves.

Coding

1. Prime divisors
2. Integer factoring
3. Permutations

Algebra

1. Sorting numbers.
2. Basic identities.
3. Telescopic sums/products.

Combinatorics

1. Counting: Permutations. Counting in two ways.
2. Process: Invariant (thing that does not change). Start state, end state in comparison with invariant.
3. Games: Winning strategy. Winning positions (where the current player wins or shall win if the game continues). Losing positions.
4. Algorithms: Extremal principle (there is always a minimal and maximal numbers in a finite set of numbers).

Geometry

1. Midpoints. Midsegment. Median triangle. Right triangles. Isosceles triangles. Equilateral triangles.
2. Trapezoids. Isosceles trapezoid.
3. Squares.
4. Regular polygons.
5. Incirles, excircles of a triangles.
6. Symmetry. Reflections.

Logic

1. Casework: what if A is true, what if A is false.
2. Process of elimination: If A is not true, B is not true, then C should be true.
3. Reverse argument: if there is at least ..., then there is atmost ...
4. Implication from truth: if A told the truth and A said X, then X is true.
5. Conflict of truth: if B said A lied, then both cannot be truth tellers.

Number Theory

1. Remainders, when divided by a prime, of additions and subtractions.
2. Parity.
3. Divisibility. Prime factorization.
4. Number digits.
5. Pythagorean triples.
6. Different ways to factorize of a number. Different sums result in the same number.
7. Integer and fractional parts of a number.

3.2 Problems

Problem 3.2.1 (HC-2021-SM2-R1-P1)**.** (*Beginner Level*) Minh created a sequence of 5−digit numbers. Each number contains exactly one of the digits $1, 2, 3, 4$, and 5. He sorted them into a sequence by increasing order from left to right. The first term of the sequence is 12345, the second term is 12354, and so on, and the last term is 54321, as shown below.

$$\underbrace{12345,}_{1^{\text{st}}}\ \underbrace{12354,}_{2^{\text{nd}}}\ldots,\ \underbrace{?,}_{91^{\text{th}}}\ \ldots,\ 54321.$$

What is the 91^{th} term of the sequence?

How to provide your answer:

- If you think that the 91^{th} term of the sequence is 12543, submit 12543.
- If you think that there are no such term, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 3.2.2 (HC-2021-SM2-R1-P2)**.** (*Beginner Level*) D, E, and F are midpoints of the sides BC, CA, and AB, respectively, of $\triangle ABC$. P is a point on AB such that $PFDE$ is a isosceles trapezoid, which means $DE \parallel FP$ and $FD = PE$.

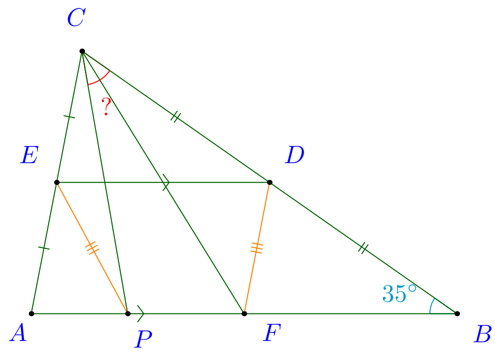


Figure 3.1: HC-2021-SM2-R1-P2

It is known that $\angle FBC = 35°$, what is the measure of the angle $\angle PCB$?

How to provide your answer:

- If you think that the measure of the angle $\angle PCB$? is $25°$, submit 25.
- If you cannot determine that, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 3.2.3 (HC-2021-SM2-R1-P3). (*Beginner Level*) Black just moved, see Figure 3.2.

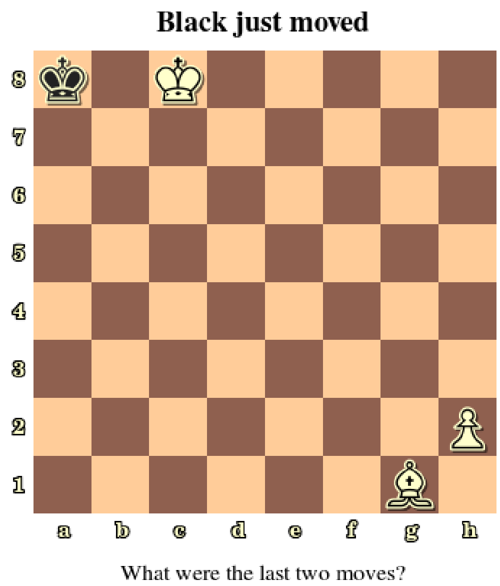


Figure 3.2: HC-2021-SM2-R1-P3

What were the last two moves?

How to provide your answer:

- If you think that the last two moves started with the White moving its king from c7 to c8, and Black moves its king from a7 to a8, then submit: *Kc7c8, Ka7a8.*
- If you think that there are no such possible moves, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 3.2.4 (HC-2021-SM2-R1-P4)**.** (*Beginner Level*) Some cookies were stolen from the kitchen. Anna and Hannah were questioned by their mother. Here were their answers,

- Anna: Hannah did not steal the cookies.
- Hannah: Anna stole the cookies.

Their mother knew that *only one of them stole the cookies* and *that one was lying.*

Which one stole the cookies? Did Anna tell the truth? How about Hannah?

How to provide your answer:

- If you think that Anna stole the cookies, Anna told the truth, and Hannah lied, then submit A10.
- If you think that Hannah stole the cookies, Anna lied, and Hannah told the truth, then submit H01.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 3.2.5 (HC-2021-SM2-R1-P5)**.** (*Beginner Level*) Berry and Cherry take alternate turns in playing a two-player game removing marbles from a pile as follows:

- Berry always goes first.
- The player whose turn it is, must remove exactly 2, 4, or 5 marbles from the pile.
- The player who at some point is unable to make a move (cannot remove 2, 4, or 5 marbles from the pile) loses the game.

They play 14 games with $\{8, 9, \ldots, 21\}$ as the initial number of marbles in the pile.

What games does Cherry win, regardless of what Berry does?

How to provide your answer:

- If you think that Cherry wins the games starting with $10, 11, 12$ marbles, then submit the set $\{10, 11, 12\}$.
- If you think that Cherry cannot win any game, then submit the empty set $\{\}$.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is a *strictly smaller subset* of your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct* answer that is *strictly larger superset* of your answer.

Problem 3.2.6 (HC-2021-SM2-R1-P6). (*Intermediate Level*) $\triangle ABC$ is an isosceles triangle, $AB = AC = 12$, and $\angle CAB = 108°$. Two rays $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ starting from A, trisect the $\angle CAB$ into three equal angles. Points D and E are the feet of the perpendiculars from C and B to α and β, respectively.

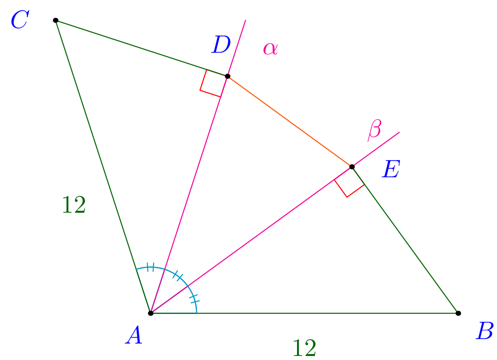


Figure 3.3: HC-2021-SM2-R1-P6

Find DE.

How to provide your answer:

- If you think that the length of DE is 5, submit 5.
- If you cannot determine that, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 3.2.7 (HC-2021-SM2-R1-P7)**.** (*Intermediate Level*) Ha Anh and a friend of hers are playing a two-player game using a number of coins. The players take turns to divide the coins into a number of piles of at least 1 coin each. In each turn, the player of that turns chooses as many piles as they want and divides each of them into two smaller piles. It is obvious that a pile consisting of only 1 coin cannot be chosen. The loser is the one who cannot move.

Initially there are 20 coins are in one pile. Ha Anh has a strategy to win the game. Should she go first or second? *Atmost* how many moves does Ha Anh need to make in order to *always* win the game?

How to provide your answer:

- If you think that Ha Anh wins by going first and she can win in atmost 10 moves, regardless whatever moves her friend makes, then submit F10.
- If you think that Ha Anh wins by going second and she can win in atmost 9 moves, regardless whatever moves her friend makes, then submit S9.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that has the number of moves *larger then* your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has the number of moves *smaller* of your answer.

Problem 3.2.8 (HC-2021-SM2-R1-P8)**.** (*Intermediate Level*) In the morning, the school headmaster brought a cake into her office. The cake was for her lunch made by her husband. The year-end award ceremony finished at noon, after which the headmaster returned to her office and discovered that the cake is gone. She questioned Fish, Shrimp, and Crab, who were assigned to help out with some duties for the ceremony and had access to her office. The answers were as below,

- Fish: Shrimp ate the cake.
- Shrimp: Yes, I ate the cake!
- Crab: I never ate the cake!

From the evidence, the headmaster knew that *only one of them ate the cake.* She also knew that *at least one of them lied* and *at least one told the truth.*

Which one ate the cake? Did Fish tell the truth? Did Shrimp lie? How about Crab?

How to provide your answer:

- If you think that Fish ate the cake, Fish told the truth, Shrimp told the truth, and Crab lied, then submit F110.
- If you think that Crab ate the cake, Fish lied, Shrimp lied, and Crab told the truth, then submit C001.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 3.2.9 (HC-2021-SM2-R1-P9)**.** (*Intermediate Level*) Samuel glue the top and bottom edges, then the left and the right sides of a chessboard together, creating a torus. See Figure 3.4.

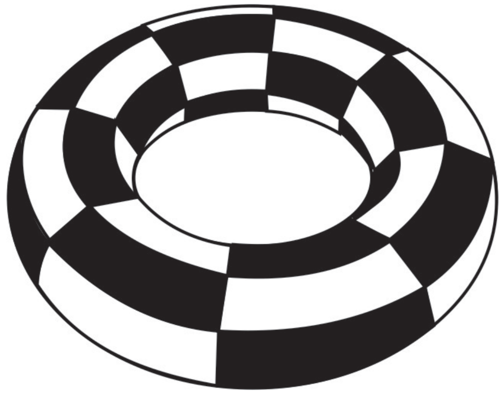


Figure 3.4: HC-2021-SM2-R1-P9

How many knights which can be placed so that *no two attack each other?*

How to provide your answer:

- If you think that Samuel can put 8 knights so that no two attack each other, then submit 8.
- If you think that it is impossible to determine, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *smaller than* your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* is *larger than* of your answer.

Problem 3.2.10 (HC-2021-SM2-R1-P10)**.** (*Intermediate Level*) The three non-zero digits of a $3-digit$ number are the lengths of the sides of a right triangle.

What the largest prime that divides the $3-digit$ number?

How to provide your answer:

- If you think that the largest prime that divides the $3-digit$ number is 7, then submit 7.
- If you think that it is impossible to determine, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *smaller than* your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* is *larger than* of your answer.

Problem 3.2.11 (HC-2021-SM2-R1-P11)**.** (*Advanced Level*) A pair of positive integers m, n is called *allies* if each is a product of two positive integers $m = ab$, $n = cd$ such that $a + b = c + d$. For example 8, 9 are *allies* because $8 = 4 \cdot 2$, $9 = 3 \cdot 3$ and $4 + 2 = 3 + 3 = 6$.

The numbers $1, 2, 3, \ldots, 100$ are written on a board. Anthony erased the numbers according to the following rules,

1. First, he erases the numbers 3 and 5.
2. Then, if m and n are *allies*, one of them was erased, and the other one was still on the board, then he can erase the remaining one.
3. He stops if he cannot erase any other number or there was no number on the board.

How many numbers could Anthony erase?

How to provide your answer:

- If you think that Anthony can erase 9 numbers, then submit 9.
- If you think that it is impossible to determine, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *smaller than* your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* is *larger than* of your answer.

Problem 3.2.12 (HC-2021-SM2-R1-P12)**.** (*Advanced Level*) $ABCD$ is a unit square. Points P and Q are on AD and AB such that $\angle PCQ = 45°$.

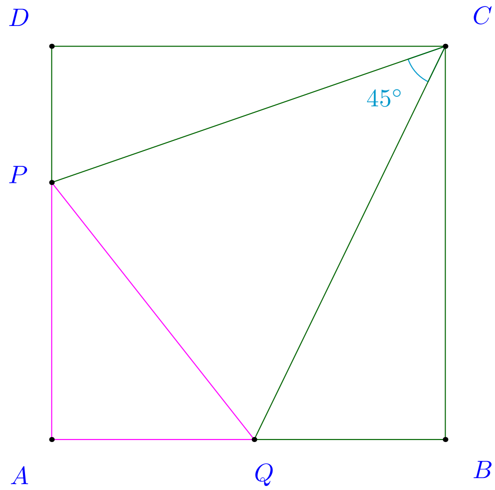


Figure 3.5: HC-2021-SM2-R1-P12

What is the perimeter of $\triangle PAQ$?

How to provide your answer:

- If you think that the perimeter is $\sqrt{2}$, then submit $\sqrt{2}$.
- If you think that it is impossible to determined, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 3.2.13 (HC-2021-SM2-R1-P13)**.** (*Advanced Level*) Once during a summer vacation. Harry got lost in the deep forest. He found a nearby village. In the village's only pub, he interviewed several people to find some guides, who can help him to get out of the forest. The pub owner secretly told him that there were exactly two werewolves in the village. They just did not know who.

There were five people at the interview, and these were what they said,

- A: E is not a werewolf.
- B: I am not a werewolf.
- C: E is lying.
- D: A is a werewolf.
- E: Each of A, B, C, and D was lying.

Now, obviously traveling with werewolves in the deep forest is an enourmous risk that he would not willing to take, which two of the five people should Harry choose as guides in order to minimize the risk?

How to provide your answer:

- If you think that Harry should choose A and B, then submit AB.
- If you think that it is impossible to determined, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 3.2.14 (HC-2021-SM2-R1-P14)**.** (*Advanced Level*) The Black pieces in this game, see Figure 3.6, are wrongly shown in white. Colour them back!

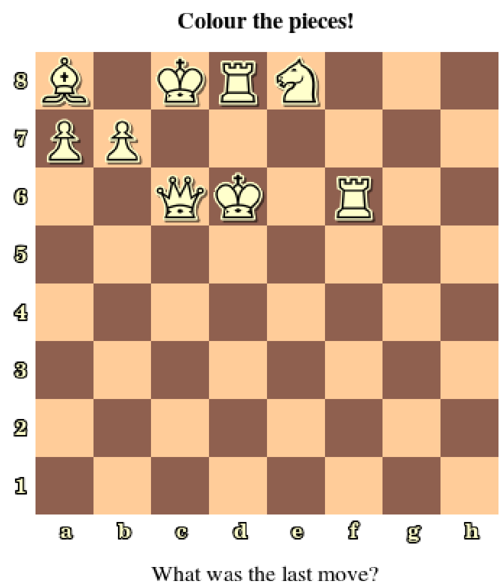


Figure 3.6: HC-2021-SM2-R1-P14

What was the last move?

How to provide your answer:

- If you think that the Black pieces, from column a to column h and from row 1 to row 8, are the pawn at b7, the queen at c6, the king at d6, and the knight at e8; and the last move was a move of White rook from d7 to d8 then submit: b7-c6-d6-e8, Rd7d8.
- If you think that there are no such possible colouring, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 3.2.15 (HC-2021-SM2-R1-P15)**.** (*Advanced Level*) Let S be the sum,

$$S = \frac{2^2+1}{2^2-1} + \frac{3^2+1}{3^2-1} + \ldots + \frac{2021^2+1}{2021^2-1}$$

Find $\lfloor S \rfloor$, the integer part of S. Estimate if $\{S\}$, the fractional part of S, is less than, equal to, or larger than $\frac{1}{2}$. *Note that* $\lfloor 1.2 \rfloor = 1,\ \{1.2\} = 0.2,\ \lfloor 3 \rfloor = 3, \{3\} = 0$.

How to provide your answer:

- If you think that $\lfloor S \rfloor = 1000$ and $\{S\}$ is less than $\frac{1}{2}$, then submit 1000, -1.
- If you think that $\lfloor S \rfloor = 1001$ and $\{S\}$ is equal to $\frac{1}{2}$, then submit 1001, 0.
- If you think that $\lfloor S \rfloor = 1002$ and $\{S\}$ is greater than $\frac{1}{2}$, then submit 1002, 1.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

3.3 Answers

Problem 1: 45123

Problem 2: 55°

Problem 3: Nb6a8+ Ka7×a8

Problem 4: H00

Problem 5: $\{8, 14, 15, 21\}$

Problem 6: 6

Problem 7: F10

Problem 8: F001

Problem 9: 32

Problem 10: 181

Problem 11: 98

Problem 12: 2

Problem 13: CD

Problem 14: b7−c8, c7×d8=R+

Problem 15: 2021, -1

3.4 Solutions

Solution. [First solution] HC-2021-SM2-R1-P1 Note that in the reasoning below, the order of 5−digit numbers in the sequence is considered to be increasing from left to right.

First, there are $5! = 120$ numbers in the sequence. If 5 is the first digit of a 5−digit number, then $1, 2, 3, 4$ are the last four digits of that number. Therefore, there are $4! = 24$ numbers with 5 as the first digit. Since $120 - 24 = 96$, so the 97^{th} term is the first 5−digit number in the sequence that starts with 5, which is 51234; and the 96^{th} term is the last 5−digit number that starts with 4, which is 45321.

$$45321,\ \underbrace{51234,\ 51243, \ldots,\ 54321}_{24\ \text{terms}}.$$

There are $3! = 6$ ways to arrange the digits of 123, so the $96 - 6 + 1 = 91^{\text{th}}$ term is the first 5−digit number starting with 45, which is 45123.

$$\underbrace{45123,\ 45132, \ldots,\ 45321,}_{6\ \text{terms}}\ \underbrace{51234,\ 51243, \ldots,\ 54321}_{24\ \text{terms}}.$$

Thus, the 91^{th} term is $\boxed{45123.}$ □

Solution. [Second solution] HC-2021-SM2-R1-P1 Note that in Python, the indexes of a sequence start at 0.

```
from itertools import permutations

if __name__ == "__main__":
    sequence = permutations('12345',5)
    print(''.join(sorted(sequence)[90]))
```

The output shows that the 91^{th} term is $\boxed{45123.}$

```
45123
```

□

Solution. HC-2021-SM2-R1-P2 First, because D and E are midpoints of BC and BA, respectively, then DF is the midsegment of $\triangle BCA$, so $DF = \frac{1}{2}CA$. $DFPE$ is an isosceles trapezoid, therefore $EP = DF$. Thus, $EP = \frac{1}{2}CA = CE = EA$. $EP = EC = EA$ implies that P is on the circle centred at E, diameter CA.

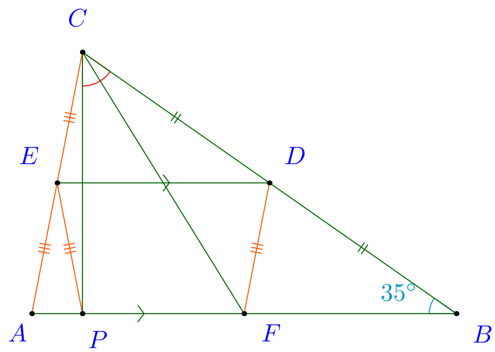


Figure 3.7: $\triangle CPA$ is a right triangle

Therefore, $\triangle CPA$ is a right triangle, and $\angle CPA = 90°$.

Thus, $\boxed{\angle PCB = 90° - \angle PBC = 55°.}$ □

Solution. HC-2021-SM2-R1-P3 First, it is easy to see that the Black king must come to a8 from a7, Ka7a8.

In its move, White could not have moved its bishop from any where outside of the diagonal g1-a7 to g1. Therefore, the Black king would have been checked by the bishop if there was nothing on the diagonal g1-a7. This piece should have been the one that moved away and this should also have been the one that disappeared after the last two moves. Thus, Black has captured this piece at a8 with its king after the piece got there from a square on the diagonal g1-a7 (not including both ends). This could only be possible if it were a white knight, originally on b6.

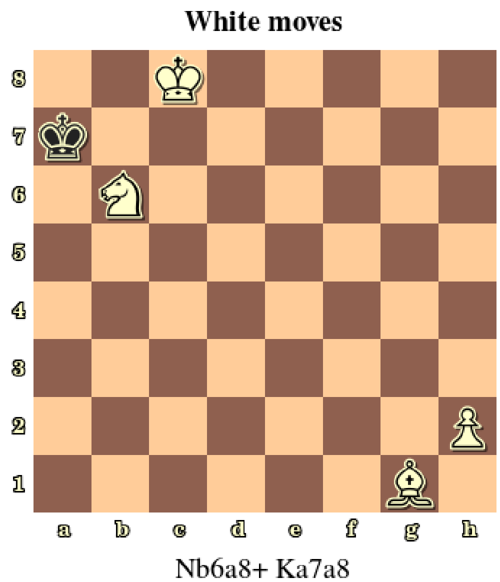


Figure 3.8: Before the last two moves

Therefore, the last two moves were $\boxed{\text{Nb6a8+ Ka7}\times\text{a8.}}$ □

Solution. HC-2021-SM2-R1-P4 Suppose that Anna stole the cookies. The she lied, which means that what she said about Hanna not stealing the cookies was false. In other words, Hannah stole the cookies. It is impossible, because only one of them stole the cookies. Therefore, Anna did not stole the cookies. Hannah did. Furthermore Anna lied, and Hannah lied, too.

Thus, the answer is $\boxed{\text{H00.}}$ □

Solution. HC-2021-SM2-R1-P5 The positive integers from 0 to 21 can be divided into 7 groups of numbers based on their remainders when divided by 7,

$$G_0 = \{0, 7, 14, 21\},\ G_1 = \{1, 8, 15\},\ G_2 = \{2, 9, 16\},$$
$$G_3 = \{3, 10, 17\},\ G_4 = \{4, 11, 18\},\ G_5 = \{5, 12, 19\},\ G_6 = \{6, 13, 20\}$$

It is easy to verify that,

Claim — If n is a number in G_0 and G_1 then $n-2$, $n-4$, and $n-5$ are in G_2, G_3, G_4, G_5, or G_6.

Now, by the rules of the game, it is obvious that $\{0, 1\}$ are *losing* positions and $\{2, 4, 5\}$ are *winning* positions. Furthermore, because $3-2 = 6-5 = 1$, so $\{3, 6\}$ are also *winning* positions, too.

Therefore G_0 and G_1 contain all *losing* positions, while G_2, G_3, G_4, G_5, and G_6 containing all *winning* positions. A player, who is in a *winning* position, can always force the game into a *losing* position. Thus, Cherry will win the game if the game starts with a *losing* position.

Thus, the games that Cherry wins are $\boxed{\{8, 14, 15, 21\}.}$ □

Solution. HC-2021-SM2-R1-P6 Let B' and C' be the reflections of A over lines BE and CD, respectively. It is easy to see that $ABB'C'C$ is a pentagon with all equal interior angles. Since $DA = DC'$, $EA = EB'$, thus triangles $\triangle ACC'$, $\triangle ABB'$ are isosceles. Thus, $CC' = CA = AB = BB'$, then $ABB'C'C$ is a regular pentagon. Therefore $B'C' = AB = 12$.

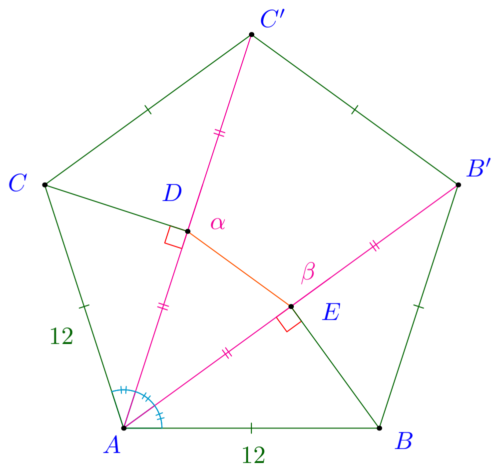


Figure 3.9: $B'C' = 12$

DE is the midsegment of $\triangle AB'C'$, thus $\boxed{DE = 6.}$ □

Solution. HC-2021-SM2-R1-P7 A strategy that guarantees a win for Ha Anh is as follows. In her turn, she splits every pile with an even number of coins (say $2k$) in two piles with an odd number of coins: the 1 coin pile and the $2k-1$ coin pile respectively.

$$\underbrace{2k}_{\text{even}} \to \underbrace{1}_{\text{odd}} + \underbrace{2k-1}_{\text{odd}}.$$

So, in her first turn, she created one pile with 1 coin and one with 19 coins. When her friend gets to make a move, all piles will have an odd number of coins. Her friend is therefore forced to split an odd pile, creating a new pile with an even number of coins.

$$\text{odd} \to \text{odd} + \text{even}.$$

This implies that Ha Anh, in the next turn, can continue her strategy, since there will be at least one even pile. She does not touch the piles with an odd number of coins. With each turn, the number of piles increases, so after at most 19 turns, the game is over. Since her friend always creates an even pile, the game cannot end during that turn. Therefore, it will be Ha Anh who wins the game.

Thus, Ha Anh should go first, and she will win after at most 10 moves. The answer is $\boxed{\text{F10.}}$ □

Solution. HC-2021-SM2-R1-P8 If Crab ate the cake, then all of them were lying. This is not possible since at least one of them told the truth. If Shrimp ate the cake, then all of them told the truth, which also is impossible. Therefore Fish ate the cake. Fish lied, Shrimp lied, and Crab told the truth.

Thus, the answer is $\boxed{\text{F001.}}$ □

Solution. HC-2021-SM2-R1-P9 It is easy to see that 32 knights can be placed on 32 black squares of the torus so that no two attack each other.

This cannot be improved. Let suppose that n knights can be placed so that no two attack each other. Since each knight attacks 8 squares, and an unoccupied square can be attacked by no more than 8 knights. Therefore,

$$8n \leq 8(64 - n) \Rightarrow n \leq 32$$

Thus, the answer is $\boxed{32.}$ □

Solution. [First solution] HC-2021-SM2-R1-P10 The only three non-zero digits as side lengths of a right triangle is 3, 4, and 5. Since,

$$345 = 3 \cdot 5 \cdot 23,\ 354 = 2 \cdot 3 \cdot 29,\ 435 = 3 \cdot 5 \cdot 29,\ 453 = 3 \cdot 151,\ 534 = 2 \cdot 3 \cdot 89, 543 = 3 \cdot 181.$$

Thus, the answer is $\boxed{181.}$ □

Solution. [Second solution] HC-2021-SM2-R1-P10

```
def largest_prime_factor(n):
i = 2
while i * i <= n:
    if n % i:
        i += 1
    else:
        n //= i
return n

if __name__ == "__main__":
    q = 0
    for a in range(1,10):
        for b in range(1,10):
            for c in range(0,10):
                if a*a == b*b + c*c:
                    p_1 = largest_prime_factor(100*a+10*b+c)
                    p_2 = largest_prime_factor(100*a+10*c+b)
                    p_3 = largest_prime_factor(100*b+10*a+c)
                    p_4 = largest_prime_factor(100*b+10*c+a)
                    p_5 = largest_prime_factor(100*c+10*a+b)
                    p_6 = largest_prime_factor(100*c+10*b+a)
                    p = max(p_1,p_2,p_3,p_4,p_5,p_6)
                    if p > q:
                        q = p
    print(q)
```

The output shows that $\boxed{\text{the largest prime factor is 181.}}$

```
181
```

□

Solution. HC-2021-SM2-R1-P11 First, for positive integers a and b, it is easy to see that, $2a+b-1 = (2a+b-1)(1)$, $2ab = (2a)(b)$ and $2a+b-1+1 = 2a+b$, so $(2a+b-1, 2ab)$ are *allies*. Similarly $(2ab, a+2b-1)$ are *allies*, and therefore $(2a+b-1, a+2b-1)$ are *allies* too. Now, by subsituting with $(2a+b-1, a+2b-1)$,

$$b = a+1 \Rightarrow 3a = 2a + \underbrace{(a+1)}_{b} - 1,\ 3a+1 = 2\underbrace{(a+1)}_{b} + a - 1$$
$$b = a+2 \Rightarrow 3a+1 = 2a + \underbrace{(a+2)}_{b} - 1,\ 3a+3 = 2\underbrace{(a+2)}_{b} + a - 1$$
$$b = a+3 \Rightarrow 3a+2 = 2a + \underbrace{(a+3)}_{b} - 1,\ 3a+5 = 2\underbrace{(a+3)}_{b} + a - 1$$

Therefore, if $3a$, then $3a+1$, and subsequently $3a+3$ was too, if $3a+2$ was also removed, then $3a+5$ was too. By starting from 3 and 5, the numbers are removed as shown in the two chains below,

$$3 \to 4 \to 6 \to 7 \ldots \qquad 5 \to 8 \to 11 \ldots$$

Thus, all positive integers from 3 that have remainders 0 or 1 modulo 3 are removed by the left chain, all positive integers from 5 that have remainders 2 modulo 3 are removed by the right chain.

All numbers except $\{1, 2\}$ can be removed, thus atmost $\boxed{98}$ number can be removed. □

Solution. HC-2021-SM2-R1-P12 First, it is easy to prove the following claim,

> **Claim —** If C is on the angle bisector of $\angle PAQ$, and $\angle PCQ = \frac{1}{2}\angle PAQ$, then C is the excentre of $\triangle PAQ$ on opposite site of A in regard to PQ.

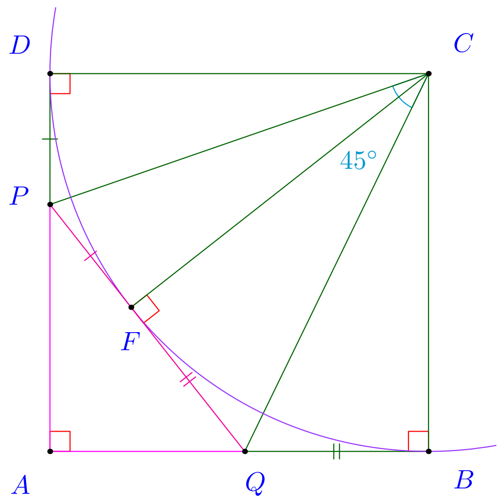


Figure 3.10: HC-2021-SM2-R1-P12

Now, since $PF = PD$ and $QF = QB$, so $AP + PQ + QA = AP + PF + FQ + QA = AD + AB = 2$.

Thus, $\boxed{\text{the perimeter of } \triangle PAQ \text{ is } 2.}$ ☐

Solution. HC-2021-SM2-R1-P13 First, if E was telling the truth, then A, B, C, and D were liars. A lied, so E was a werewolf. B lied, so B was a werewolf. C lied. D lied, so A was not a werewolf. Therefore B and E were werewolves.

Now, if E lied, then A, B, C, and D told the truth. Therefore, B was not a werewolf, and neither was E. According to D, A was a werewolf.

Thus, Harry must avoid B, E, and A. He should pick C and D as companions. The answer is $\boxed{\text{CD.}}$. □

Solution. HC-2021-SM2-R1-P14 Here one of the two kings is in double check by Rd8 and Qc6.

The only possible explanation is that the last move was −1.c7×d8=R+. Thus we conclude that Rd8, Qc6 and Kd6 are White; Kc8 is black. Then further impossible checks are avoided by setting Ne8 and Rf6 White; b7 Black. A black b7 entails that Ba8 is promoted, so Ba8 and a7 are White.

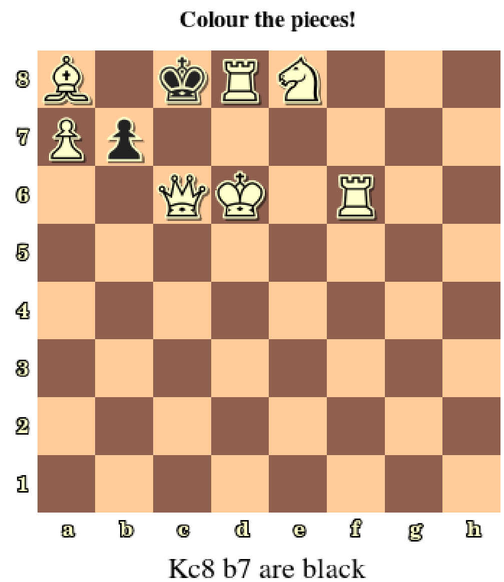


Figure 3.11: Recolouring

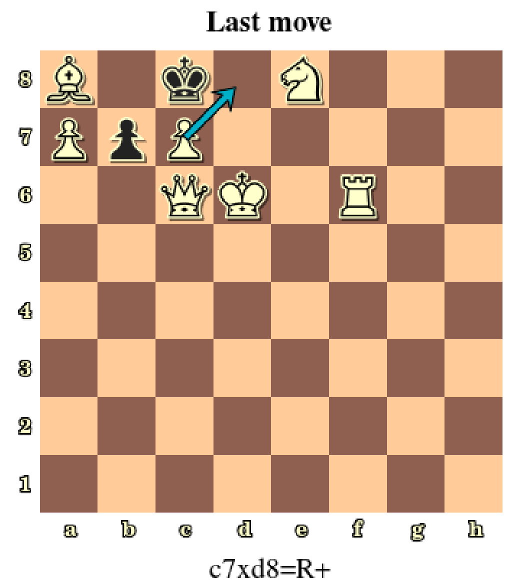


Figure 3.12: White moved last

Therefore, the answer is $\boxed{\text{b7−c8, c7×d8=R+.}}$ □

Solution. HC-2021-SM2-R1-P15 First, note that for all positive integer $n > 1$,

$$\frac{n^2+1}{n^2-1} = 1 + \frac{2}{n^2-1} = 1 + \frac{1}{n-1} - \frac{1}{n+1}.$$

Therefore,

$$\begin{aligned}
S &= \left(1 + \frac{1}{1} - \frac{1}{3}\right) + \left(1 + \frac{1}{2} - \frac{1}{4}\right) + \ldots + \left(1 + \frac{1}{2020} - \frac{1}{2022}\right) \\
&= 2020 + \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \ldots + \frac{1}{2020} - \frac{1}{2022} \\
&= 2020 + \left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{2020}\right) - \left(\frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2022}\right) \\
&= 2021 + \frac{1}{2} - \frac{1}{2021} - \frac{1}{2022}
\end{aligned}$$

Now, it is easy to see thart $\lfloor S \rfloor = 2021$, and $\{S\} < \frac{1}{2}$.

Therefore, the answer is $\boxed{2021,\ -1.}$ □

Chapter 4

HC R2

4.1 Topics

Chess

1. Moves: capture, check, double check, promotion, stalemate, checkmate
2. Analysis: whose turn is it.
3. Retrograde: last move, last two moves.

Coding

1. Same number having different remainders when divided by different primes.
2. Number digits
3. Quotient and remainder

Algebra

1. Single variable linear equation.
2. Basic identities.
3. Telescopic sums/products.

Combinatorics

1. Counting: Counting in two ways.
2. Alternate colouring: colouring with two colours.
3. Number as same sum of different integers.
4. Recurrent relations.

Geometry

1. Midpoints. Midsegment. Median triangle. Isosceles triangles. Equilateral triangles.
2. Right triangles. Congruent triangles. Similar triangles.
3. Angle chasing. Sum of angles.
4. Regular hexagons.
5. Area ratio of similar triangles, triangles with same base/height.
6. Symmetry. Reflections.

Logic

1. Casework: what if A is true, what if A is false.

2. Process of elimination: If A is not true, B is not true, then C should be true.
3. Reverse argument: if there is at least ..., then there is atmost ...
4. Implication from truth: if A told the truth and A said X, then X is true.
5. Conflict of truth: if B said A lied, then both cannot be truth tellers.

Number Theory

1. Remainders, when divided by a prime, of additions and subtractions.
2. Greatest common divisor. Least common multiple.
3. Divisibility. Prime factorization.
4. Sum of digits.
5. Estimation, upper bound, lower bound.

4.2 Problems

Problem 4.2.1 (HC-2021-SM2-R2-P1). (*Beginner Level*)

A knight is placed on intersection of the 7^{th} row and the 7^{th} column on a 13×13 chessboard as shown below in the Figure 4.1.

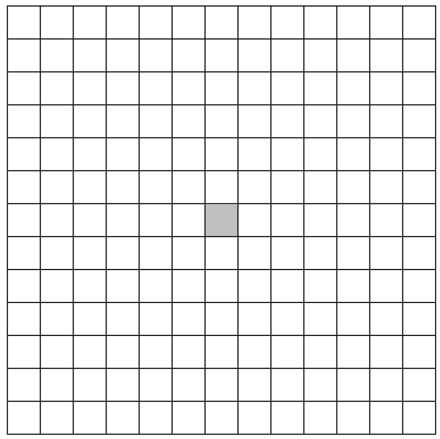


Figure 4.1: HC-2021-SM2-R2-P1

Within 3 moves, *atmost* how many squares of the chessboard the knight *cannot* reach?

How to provide your answer:

- If you think that there are atmost 5 squares the knight cannot reach within 3 moves, submit 5.
- If you think that there are no such square, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *smaller than* your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *larger than* your answer.

Problem 4.2.2 (HC-2021-SM2-R2-P2)**.** (*Beginner Level*)

Karl was captured while sneaking into the Kingdom of the Hungry Tigers. He was lead in front of three rooms, each with a separate door marked with a sign, as shown below in Figure 4.2.

I	II	III
Room III is empty	The tiger is in Room I	This room is empty

Figure 4.2: HC-2021-SM2-R2-P2

A large treasure chest was placed in one of the rooms and a hungry tiger in another. There is no princess in any of the rooms. A beautiful girl told him that,

- the sign on the door of the room containing the treasure was true,
- the sign on the door of the room with the tiger was false, and
- the sign on the door of the empty room could be either true or false.

If he opens the room with the tiger, he will be eaten. If he opens the room with the chest, they will set him free and give him the chest.

Which room has the tiger? Are the signs true or false?

How to provide your answer:

- If you think that the room I containing the tiger, the signs on room I, II, and III are true, true, and false, respectively, then submit 1110.
- If you think that the room II containing the tiger, the signs on room I, II, and III are false, false, and true, respectively, then submit 2001.
- If you cannot determine that, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 4.2.3 (HC-2021-SM2-R2-P3)**.** (*Beginner Level*)

In the right triangle ABC, O is the midpoint of the hypotenuse AC. Points M and N are chosen on sides BC and BA such that $\angle MON = 90°$, $BM = 4$, and $BN = 3$.

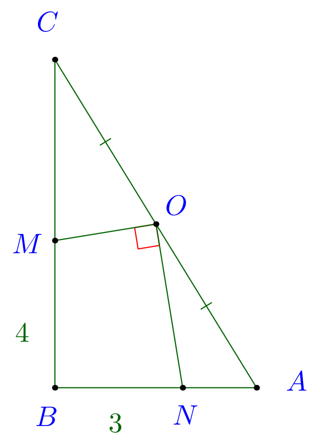


Figure 4.3: HC-2021-SM2-R2-P3

What is the value of $AN^2 + CM^2$.

How to provide your answer:

- If you think that $AN^2 + CM^2$ is 14, submit 14.
- If you cannot determine that, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 4.2.4 (HC-2021-SM2-R2-P4)**.** (*Beginner Level*)

My Linh has some \$10 bills and \$20 bills.

- If she uses all her \$10 bills, she is \$60 short of buying 4 guinea pigs.
- If she uses all her \$20 bills, she is \$60 short of buying 5 guinea pigs.
- If she uses all her \$10 bills and \$20 bills, she is \$60 short of buying 6 guinea pigs.

What is the price of one guinea pig?

How to provide your answer:

- If you think that the price of one guinea pig is \$5, then submit 5.
- If you think that it is not possible to determine, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 4.2.5 (HC-2021-SM2-R2-P5). (*Beginner Level*)

A palace has a number of rooms in equilateral triangle shapes. You can move from one room to an adjacent room if the two rooms share a common wall. The Figure 4.4 below shows a room and its three adjacent rooms.

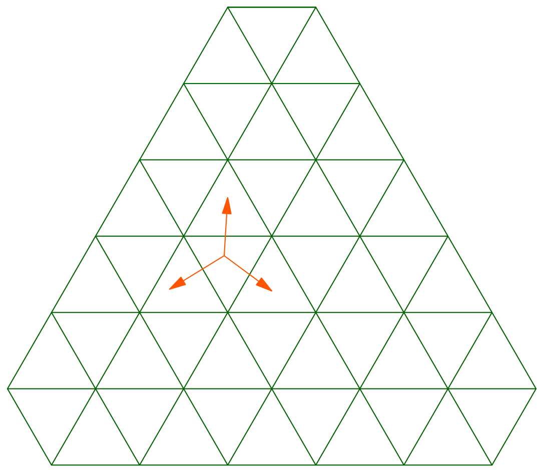


Figure 4.4: HC-2021-SM2-R2-P5

What is the *maximum* number of rooms in a path you can travel on, never visiting a room twice, and returning to the starting room at the end?

How to provide your answer:

- If you think that you can draw such a path with 40 rooms, submit 40.
- If you think that you cannot make such a path, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *smaller than* your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *larger than* your answer.

Problem 4.2.6 (HC-2021-SM2-R2-P6)**.** (*Intermediate Level*)

In the Figure 4.5 below White has the move.

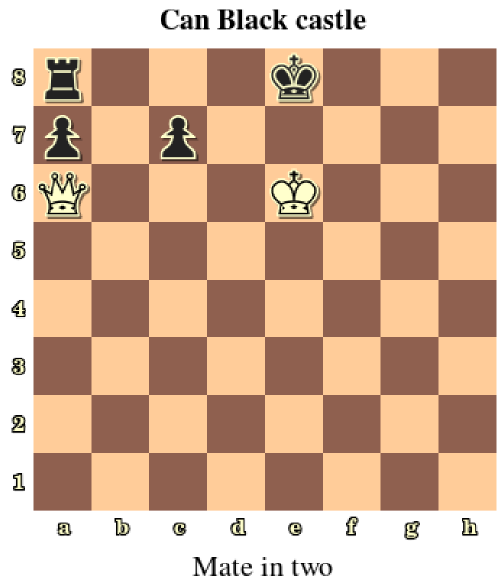


Figure 4.5: HC-2021-SM2-R2-P6

Can Black castle? Show that White can mate in two moves.

How to provide your answer:

- If you think that Black can castle and White can mate by moving its queen from a6 to d6 and then checkmate at e7, regardless of what Black does, then submit: Y, Qa6d6, Qd6e7.
- If you think that Black cannot castle and White can mate by moving its queen from a6 to d6 and then checkmate at e7, regardless of what Black does, then submit: N, Qa6d6, Qd6e7.
- If you think that there are no such possible moves, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 4.2.7 (HC-2021-SM2-R2-P7)**.** (*Intermediate Level*)

In the Mental Hospital of Geniuses there are only doctors and patients. Each of these inhabitants is sane or insane, but cannot be both. The sane people were a hundred percent accurate in all of their beliefs, and the insane people were a hundred percent inaccurate in all of their beliefs. After being in the hospital for a while, some patients got cured and became sane. Unfortunately some doctors lost their minds and became insane. The sane patients and insane doctors, if found, were removed from the hospital.

One day Inspector Melanie visited the hospital in order to determine who should be removed. She interviewed three people A, B, and C:

- A said B was insane.
- B said A is a doctor.
- C said B is a patient and A is insane.

Did Inspector Melanie remove any of A or B or both of them?

How to provide your answer:

- If you think that Inspector Melanie removed A but did not remove B because A was a insane doctor, and B was a sane doctor, then submit A, ID, SD.
- If you think that Inspector Melanie did not remove A but removed B, because A is a sane doctor, and B was a sane patient, then submit B, SD, SP.
- If you think that Inspector Melanie remove both A and B, because A is an insane doctor, and B was a sane patient, then submit AB, ID, SP.
- If you think that Inspector Melanie did not remove either A nor B, because A is a sane doctor, and B was a insane patient, then submit NO, SD, IP.
- If you think that it is impossible to determine A or B, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 4.2.8 (HC-2021-SM2-R2-P8). (*Intermediate Level*)

Line ℓ cuts the $\triangle AGH$ off the regular hexagon $ABCDEF$ and $AG + AH = AB$ (see the Figure 4.6.)

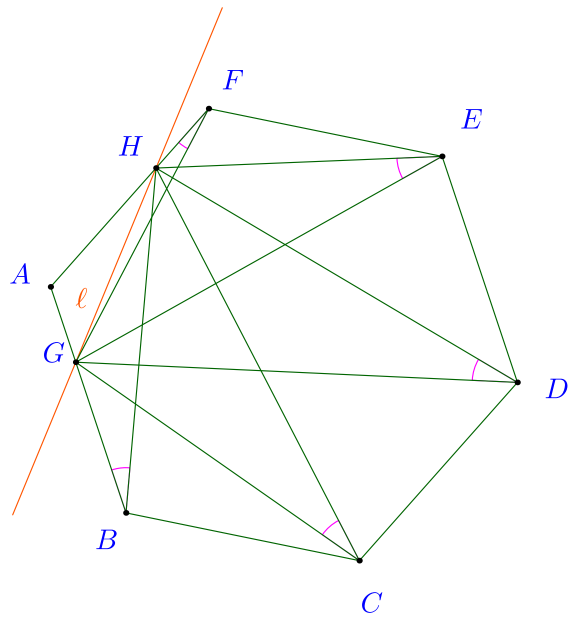


Figure 4.6: HC-2021-SM2-R2-P8

What is the sum in degree of

$$\angle GBH + \angle GCH + \angle GDH + \angle GEH + \angle GFH?$$

How to provide your answer:

- If you think that the sum $\angle GBH + \angle GCH + \angle GDH + \angle GEH + \angle GFH$ is 65°, then submit 65.
- If you think that that it is not a constant number or it is impossible to determine, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 4.2.9 (HC-2021-SM2-R2-P9). (*Intermediate Level*)

From the first 99 positive integers

$$1,\ 2, \ldots,\ 99,$$

Khoa chooses 50 numbers such that no two of them sum up to 99 or 100. *For example only two numbers can be chosen among the numbers* $99, 1, 98$.

What is the sum of those 50 numbers?

How to provide your answer:

- If you think that the sum of 50 selected numbers is 5000, then submit 5000.
- If you think that it is not possible to determine, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 4.2.10 (HC-2021-SM2-R2-P10)**.** (*Intermediate Level*)

Two circular tracks ω_a and ω_b touch each other at point S.

- Amy starts at A, where the length of the minor arc $\widehat{SA}$ is *one-fifth* of the ω_a circumference. She walks along the ω_a track *clockwise* at a constant speed and completes a round in 5 minutes.
- Bill starts at B, where the length of the minor arc $\widehat{BS}$ is *three-seventh* of the ω_b circumference. He walks along the ω_b track *anti-clockwise* at a constant speed and completes a round in 7 minutes.

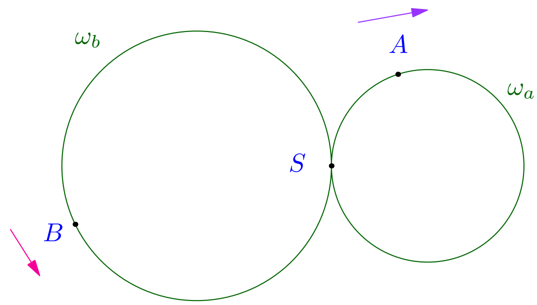


Figure 4.7: HC-2021-SM2-R2-P10

Amy and Bill both begin to run at 8:00 AM and stop running at 10:00 AM. Do they meet each other at some point? If yes, when are those meetings?

How to provide your answer:

- If you think Amy and Bill meet at 8:10 AM and 9:30 AM then submit the set $\{810, 930\}$.
- If you think that they do not meet, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is a *strictly smaller subset* of your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger superset* of your answer.

Problem 4.2.11 (HC-2021-SM2-R2-P11). (*Advanced Level*)

The White king just moved from f5 to e5. The situation is shown in the Figure 4.8 below.

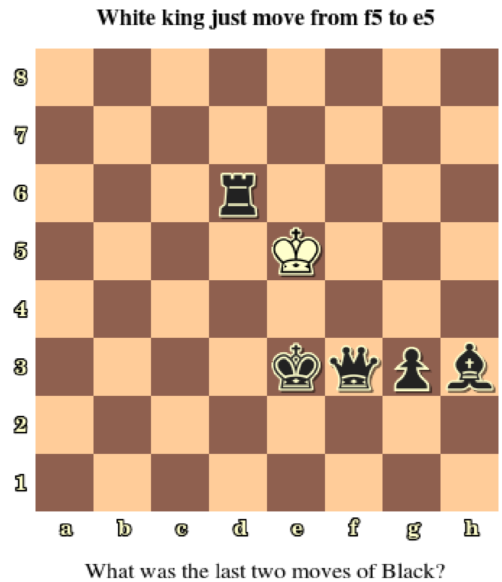


Figure 4.8: HC-2021-SM2-R2-P11

What were the last two moves of Black?

How to provide your answer:

- If you think the last two moves of Black were that the Black queen moved from e1 to e2 and then e2 to f3, then submit: Qe1e2, Qe2f3.
- If you think that the situation is impossible to be created with legal moves, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 4.2.12 (HC-2021-SM2-R2-P12)**.** (*Advanced Level*)

On the Island of Knights and Liars, there are two types of people: the knights who always tell the truth and the liars who always lie. A rare disease spread to the island. Some of the inhabitants were affected and became insane. Inhabitants who were not affected, were still sane. Sane people behaved as they used to: knights tell the truth and liars lie. Insane people behaved in a total opposite way: insane knights now lie and insane liars tell the truth.

Tung met two brothers, Nguyen the Elder and Nguyen the Younger. Here were what they said.

- Nguyen the Elder: I am a knight.
- Nguyen the Younger: I am a knight.
- Nguyen the Elder: My brother is sane.

Determine who were what if you know that *one was a knight* and *the other was a liar.*

How to provide your answer:

- If you think that Nguyen the Elder was an insane knight, Nguyen the Younger was a sane liar, then submit: IK, SL.
- If you think that Nguyen the Elder was a sane knight, Nguyen the Younger was an insane liar, then submit: SK, IL.
- If you think that it is impossible to determined, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 4.2.13 (HC-2021-SM2-R2-P13). (*Advanced Level*)

In $\triangle ABC$, A', B', and C' are points on segments BC, CA, and AB, respectively. Furthermore,

$$AB = 4,\ BC = 5,\ CA = 6,\ AB' \cdot BC' \cdot CA' + AC' \cdot CB' \cdot BA' = 24.$$

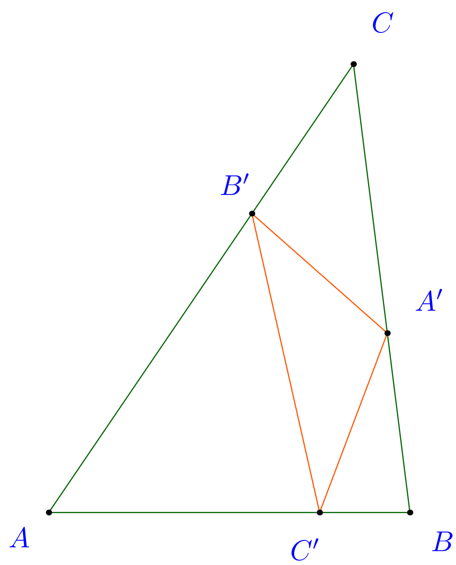


Figure 4.9: HC-2021-SM2-R2-P13

Find the ratio $\dfrac{[A'B'C']}{[ABC]}$ of the areas of $\triangle A'B'C'$ and $\triangle ABC$.

How to provide your answer:

- If you think the ratio $\dfrac{[A'B'C']}{[ABC]}$ is $\dfrac{5}{12}$, then submit $\dfrac{5}{12}$.
- If you think that it is impossible to determined, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 4.2.14 (HC-2021-SM2-R2-P14)**.** (*Advanced Level*)

For a positive integer n, let $\sigma(n)$ denote the sum of its digits. For example

$$\sigma(5) = 5,\ \sigma(15) = 1 + 5 = 6,\ \sigma(203) = 2 + 0 + 3 = 5.$$

Find all 4−digit positive integer n (starting with a non-zero digit) such that

$$n + \sigma(n) + \sigma(\sigma(n)) = 2022.$$

How to provide your answer:

- If you think that the numbers 1234 and 5678 then submit the set $\{1234,\ 5678\}$.
- If you think that there are no such number, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is a *strictly smaller subset* of your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger superset* of your answer.

Problem 4.2.15 (HC-2021-SM2-R2-P15)**.** (*Advanced Level*)

25 flies are resting on the outdoor table in the garden, waiting for lunch to be served.

- It is known that for any 3 of them, 2 are at a distance less than 20 cm.
- There are at least a pair of flies that are further than 20 cm from each other.

Minh's mother gave him a fly swatter, shown in Figure 4.10, with a hoop of radius 20 cm, With a single strike he can swat the flies where the hoop landed.

Figure 4.10: HC-2021-SM2-R2-P15

In *at least* how many strikes can he swat all of them? *Assume that Minh is so fast that the flies do not have time for reaction during and between his lightning strikes.*

How to provide your answer:

- If you think that he needs 20 strikes, then submit 20.
- If you think that it is impossible to determine, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *larger than* your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *smaller than* your answer.

4.3 Answers

Problem 1: 60

Problem 2: 2101

Problem 3: 25

Problem 4: 20

Problem 5: 42

Problem 6: N, Qa6a1, Qa1h8++

Problem 7: AB, SP, ID or AB, ID, SP

Problem 8: 120

Problem 9: 3725

Problem 10: $\{824,\ 859,\ 934\}$

Problem 11: Ng4e5, f4×g3

Problem 12: SK, SL

Problem 13: $\frac{1}{5}$

Problem 14: $\{1988,\ 1994,\ 2006,\ 2009,\ 2012\}$

Problem 15: 2

4.4 Solutions

Solution. HC-2021-SM2-R2-P1 Below in Table 4.1 the moves are shown. It is easy to see that the 7$^{\text{th}}$ row and the 7$^{\text{th}}$ column divide the board into four quadrants. Within a quadrant, not counting the squares on the 7$^{\text{th}}$ row and the 7$^{\text{th}}$ column, there are 14 (shaded) squares that cannot be reached within three moves. Furthermore, there are 4 squares on the 7$^{\text{th}}$ row and the 7$^{\text{th}}$ column, that cannot be reached within three moves.

			3		3		3		3			
		3		3		3		3		3		
	3		3	2	3	2	3	2	3		3	
3		3	2	3	2	3	2	3	2	3		3
	3	2	3		1	2	1		3	2	3	
3		3	2	1	2	3	2	1	2	3		3
	3	2	3	2	3	0	3	2	3	2	3	
3		3	2	1	2	3	2	1	2	3		3
	3	2	3		1	2	1		3	2	3	
3		3	2	3	2	3	2	3	2	3		3
	3		3	2	3	2	3	2	3		3	
		3		3		3		3		3		
			3		3		3		3			

Table 4.1: 60 squares

Thus, in total there are $\boxed{14 \times 4 + 4 = 60 \text{ squares.}}$ □

Solution. HC-2021-SM2-R2-P2 Instead of looking for the tiger, we look for the treasure, because the sign on its room is true.

It cannot be in room II, because then according to the sign, room I has the tiger, room III is empty. Thus, the sign on room I is true, which contradicts that the sign on the room with the tiger is false.

It cannot be in room III, because the according to the sign, room III is empty. Therefore the treasure is in room I, room III is empty, so the tiger is in room II.

The answer is $\boxed{2101.}$ □

Solution. HC-2021-SM2-R2-P3 Let N' be the reflection of N over O. It is easy to see that $\triangle AON \cong \triangle CON'$,

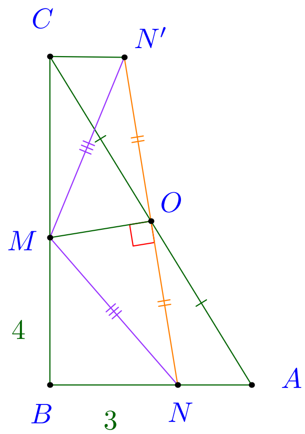


Figure 4.11: $ON = ON'$

so $AN = CN'$. Furthermore $\angle MCN' = \angle MCO + \angle OCN' = \angle BCA + \angle CAB = 90°$, thus

$$AN^2 + CM^2 = CN'^2 + CN^2 = MN'^2 = MN^2 = BM^2 + BN^2 = 25.$$

The answer is $\boxed{25.}$ □

Solution. HC-2021-SM2-R2-P4 Let the price of a guinea pig be a.

Then the amount she has in \$10 bills is equal to $4a - 60$. The amount she has in \$20 bills is $5a - 60$. The total amount she has in 10's and 20's is $6a - 60$. The sum of the first two quantities must be equal to the third one.

Thus, we have the equation,

$$6a - 60 = 5a - 60 + 4a - 60 \Rightarrow 3a = 60 \Rightarrow a = 20.$$

Thus, the price of a guinea pig is $\boxed{\$20.}$ □

Solution. HC-2021-SM2-R2-P5 We alternate colour the rooms as shown in Figure 4.12. It is easy to see that the consecutive rooms on the path have alternate colourings. Since it is a close path, so the number of shaded and unshaded rooms are the same. Because there are 21 shaded rooms, so the longest possible path can have ony 42 rooms. The figure Figure 4.13 shows one of such paths with maximal length.

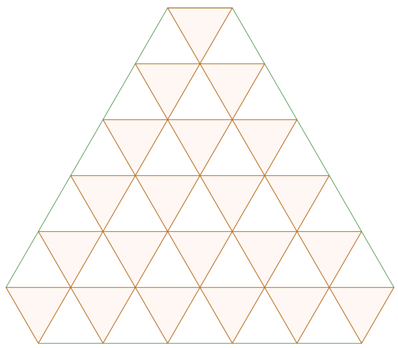


Figure 4.12: Alternate colouring

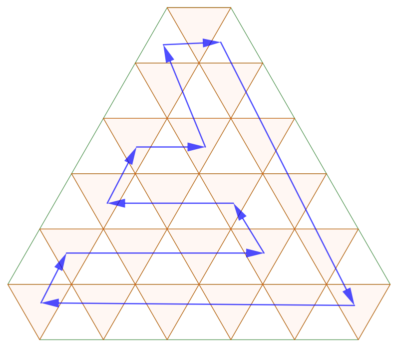
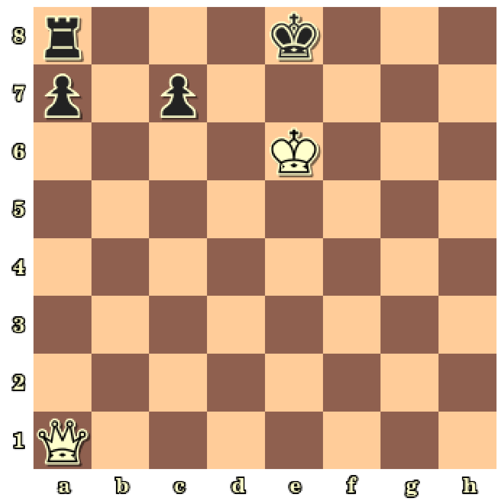


Figure 4.13: Maximal path

The answer is $\boxed{42.}$ □

Solution. HC-2021-SM2-R2-P6 Since it is White's move, so Black just moved its king or rook. In any case it cannot castle. The rest is simple by moving the White queen to a1 then h8. 1. Qa6a1 .. 2. Qa1h8++.

Black king or rook just moved



1. Qa1 .. 2. Qh8++

Figure 4.14: Mate in two

Thus, Black cannot castle, the answer is $\boxed{\text{N, Qa6a1, Qa1h8++}}$. ☐

Solution. HC-2021-SM2-R2-P7 Suppose that A was sane, then what A said was true, so B was insane, therefore B's belief that A was a doctor was false, thus, A is a sane patient and should be removed. C said A was insane, so C was insane, thus B was a doctor, thus B should be removed as well. Now, if A was insane, then B was sane, therefore A was a insane doctor and should be removed. C said A was insane, so C was sane, thus B was patient, thus B should also be removed.

Therefore, there are two possible answers $\boxed{\text{AB, SP, ID}}$ and $\boxed{\text{AB, ID, SP}}$. □

Solution. HC-2021-SM2-R2-P8 Let O be the centre of the regular hexagon. The $60°$ rotation around O

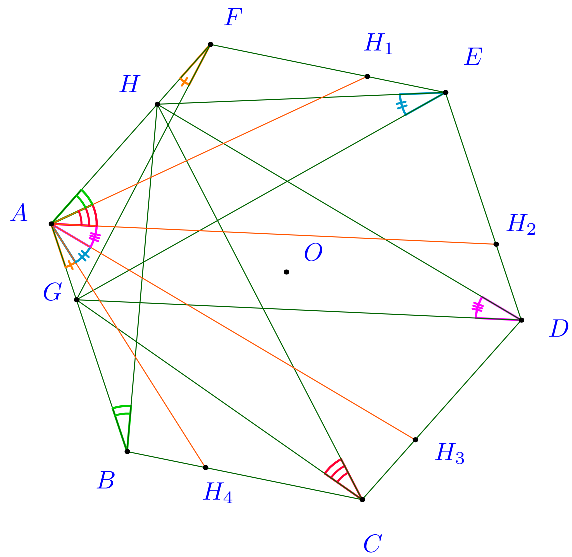


Figure 4.15: $60°$ rotation around O

moves G to H, and similarly $H \to H_1$, $H_1 \to H_2$, $H_2 \to H_3$, $H_3 \to H_4$, and $H_4 \to G$. Furthermore $GA = HF = H_1E = H_2D = H_3C = H_4B$ and $GB = H_4C = H_3D = H_2E = H_1F = HA$. Therefore,

$$\angle GBH = \angle FAH_1,\ \angle GCH = \angle H_1AH_2,\ \angle GDH = \angle H_2AH_3,\ \angle GEH = \angle H_3AH_4,\ \angle GFH = \angle H_4AB$$

Thus, the sum $\angle GBH + \angle GCH + \angle GDH + \angle GEH + \angle GFH$ is $\boxed{\angle BAF = 120°.}$ □

Solution. HC-2021-SM2-R2-P9 The first 99 positive integers can be listed as a sequence, shown below,

$$99, 1, 98, 2, 97, 3, \ldots, 51, 49, 50.$$

Note that we have any pair of two adjacent numbers add up to 99 or 100, thus only one of them can be selected from each pair and no two consecutive number can be selected. Grouping the numbers in 49 pairs and 50 as an extra single number, we can choose at most 50 numbers and 50 is one of them. Clearly the number $50, 51, 52, \ldots, 99$ are the ones to be selected. Thus their sum $50 + 51 + \ldots + 99 = (1 + 2 + \ldots + 99) - (1 + 2 + \ldots + 49) = 99 \cdot 50 - 49 \cdot 25 = 3725$.

The answer is $\boxed{3725.}$ □

Solution. [First solution] HC-2021-SM2-R2-P10 It is easy to see that Amy and Bill can only meet at S. Amy needs to complete $\widehat{AS}$ clockwise to reach S, Bill needs to completes $\widehat{BS}$ anti-clockwise to reach S. Therefore when they meet Amy completes some rounds of ω_1 plus 4 minutes, while Bill completes some rounds of ω_2 plus 3 minutes. Let a and b are the number of rounds, which can be zero, that Amy and Bill completes when they meet.

$$5a + 4 = 7b + 3 \Rightarrow 5a + 1 = 7b.$$

The smallest such pair of (a, b) is $(4, 3)$, since $5 \cdot 4 + 1 = 7 \cdot 3$. Thus, they first meet at $5 \cdot 4 + 4 = 24$, or $8 : 24$ AM.

From the moment they first meet, Amy completes a number of rounds that is a multiple of 7, Bill completes a number of rounds that is a multiple of 5. So they will meet after every $5 \cdot 7 = 35$ minutes. The other two possible times before they stop at $10 : 00$ AM is $8 : 59$, and $9 : 34$ AM.

Thus, Amy and Bill meet at $8 : 24,\ 8 : 59$, and $9 : 34$ AM. The answer is $\boxed{\{824,\ 859,\ 934\}.}$ □

Solution. [Second solution] HC-2021-SM2-R2-P10 In the code below a and b are the number of rounds, which can be zero, that Amy and Bill completes when they meet.

```
if __name__ == "__main__":
    for b in range(0, (120-3)//7):
        a, r = divmod(7 * b + 3, 5)
        if r == 4:
            t = 5 * a + 4
            print(a, b, '8:%02d AM' % t if t < 60 else '9:%02d AM' % (t-60))
```

The output shows that Amy and Bill meet at $8 : 24,\ 8 : 59$, and $9 : 34$ AM. The answer is $\boxed{\{824,\ 859,\ 934\}.}$

```
(4, 3, '8:24 AM')
(11, 8, '8:59 AM')
(18, 13, '9:34 AM')
```

□

Solution. HC-2021-SM2-R2-P11 The White king moved from f5 to e5, so previously it was in a double check. It means that two pieces on f4 between the White king and the Black queen and on g4 between the White king and the Black bishop disappeared at the same time. This can only be possible if a Black pawn on f4 captured a White pawn moved from g2 en passant. See Figure 4.17. The White pawn moved into g4 in order

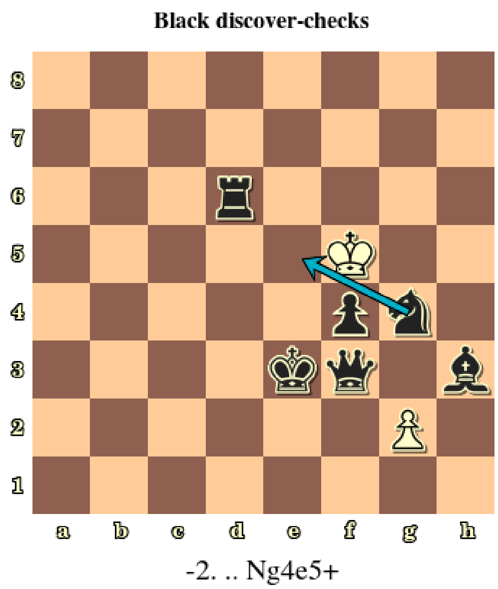


Figure 4.16: White pawn was captured

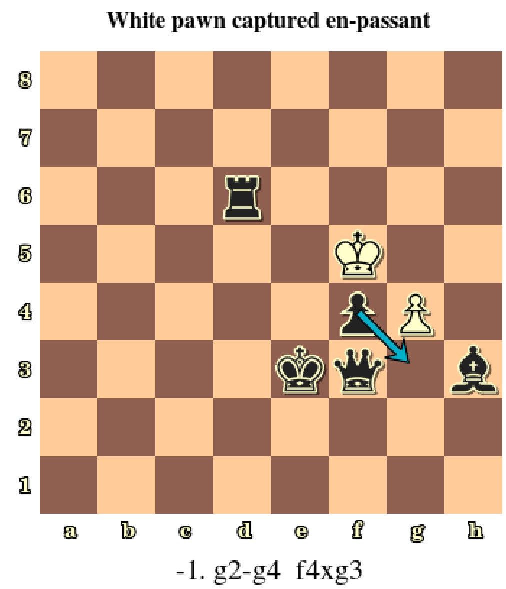


Figure 4.17: Black knight moved

to prevent the check by the Black bishop, thus there were another Black piece on g4 that moved away and disappeared after the White king's move from f5 to e5. Therefore it was a Black knight. See Figure 4.16.

Thus, −2.. Ng4e5 −1. g2g4 f4xg3 0. Kf5e5 and the correct last two moves of Black were $\boxed{\text{Ng4e5, f4×g3}}$. □

Solution. HC-2021-SM2-R2-P12 First, it is easy to verify that any islander who said she was a knight must be sane and any one who said she was a liar must be insane. Therefore both Nguyen brothers are sane. Thus, bcause Nguyen the Elder said Nguyen the Younger was sane, he made a true statement, so he was a sane knight. Because one of them is a knight and the other one was a liar, therefore, Nguyen the Younger was an sane liar.

Thus, the answer is $\boxed{\text{SK, SL.}}$ □

Solution. HC-2021-SM2-R2-P13 Let $x = \frac{BA'}{BC},\ y = \frac{CB'}{CA},\ z = \frac{AC'}{AB}$, then

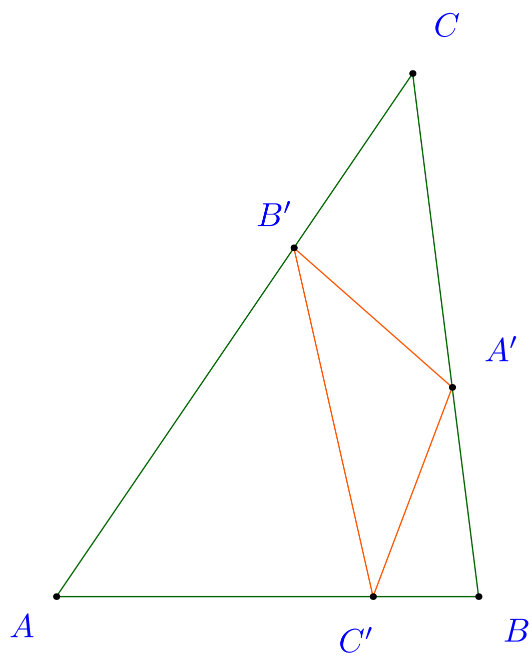


Figure 4.18: HC-2021-SM2-R2-P13

$$\frac{[AB'C']}{[ABC]} = \frac{[AB'C']}{[AB'B]} \frac{[AB'B]}{[ABC]} = \frac{AC'}{AB} \frac{AB'}{AC} = z(1-y), \Rightarrow \frac{[BC'A']}{[ABC]} = x(1-z),\ \frac{[CA'B']}{[ABC]} = y(1-x)$$
$$\Rightarrow \frac{[A'B'C']}{[ABC]} = 1 - x(1-z) - y(1-x) - z(1-y) = 1 - (x+y+z) + (xy+yz+zx)$$
$$\Rightarrow \frac{[A'B'C']}{[ABC]} = xyz + (1-x)(1-y)(1-z) = \frac{AB' \cdot BC' \cdot CA' + AC' \cdot CB' \cdot BA'}{AB \cdot BC \cdot AB} = \frac{24}{4 \cdot 5 \cdot 6} = \frac{1}{5}$$

Thus, the answer is $\boxed{\frac{1}{5}.}$ $\square$

Solution. HC-2021-SM2-R2-P14 First, it is clear that the number $1900 < n < 2022$.

Case 1: if $n = \overline{202x}$*,* then because $202x + (2 + 2 + x) > 2022$, so there no such n.

Case 2: if $n = \overline{201x}$*,* then $2022 - (\overline{201x} + (2 + 1 + x)) = 9 - 2x$, which is a single digit number and also is the sum of digits of $2 + 1 + x$. Thus, $x = 2$, and $n = 2012$.

Case 3: if $n = \overline{200x}$*,* then $2022 - (\overline{200x} + (2 + x)) = 20 - 2x$, which is a one- or two-digit number and also is the sum of digits of $2 + x$. If it is a single digit number then $20 - 2x = 2 + x$, so $x = 6$, and $n = 2006$. If it is a two-digit number, then

$$\begin{cases} 2 + x = \overline{ab} \\ 20 - 2x = a + b \end{cases} \quad \Rightarrow x \geq 8,\ (2 + x) - (20 - 2x) = 9a \Rightarrow 9 \mid 3x - 18 \Rightarrow 3 \mid x \Rightarrow x = 9$$

Thus, $n = 2009$.

Case 4: if $n = \overline{19xy}$*,* then $2022 - (\overline{19xy} + (1 + 9 + x + y)) = 112 - (11x + 2y)$, so if $x \leq 7$, then $\overline{19xy} + (1 + 9 + x + y) \leq 1979 + (1 + 9 + 7 + 9) = 2005$, and the two-digit number $1 + 9 + x + y$ cannot exceed 28, so $n + \sigma(n) + \sigma(\sigma(n))$ cannot reach more than $2005 + 2 + 8 = 2015 < 2022$. Thus, $x \geq 8$. By case work for $n = 198y$ and $n = 199y$ like the two cases above, we obtain $1988,\ 1994$ as two possible values for n.

Therefore, the values for n are $\boxed{\{1988,\ 1994,\ 2006,\ 2009,\ 2012\}.}$ □

Solution. [Second solution] HC-2021-SM2-R2-P14 In the code below *sigma*(n) is the sum of all digits of n.

```
def sigma(n4):
    d4, n3 = divmod(n4, 1000)
    d3, n2 = divmod(n3, 100)
    d2, d1 = divmod(n2, 10)
    return d4 + d3 + d2 + d1

if __name__ == "__main__":
    for n in range(1000, 10000):
        if n + sigma(n) + sigma(sigma(n)) == 2022:
            print(n)
```

The output shows that there are 5 such numbers $\boxed{\{1988,\ 1994,\ 2006,\ 2009,\ 2012\}.}$

```
1988
1994
2006
2009
2012
```

□

Solution. HC-2021-SM2-R2-P15 If no 2 flies are further than 20 cm from each other, Minh can strike them all in 1 strike by aiming the center of the swatter at any fly.

But this is not the case, so let's assume there are 2 flies, A and B, that are more than 20 cm apart. Then, every other fly is either in a 20 cm radius of A or in a 20 cm radius of B. Out of the 23 remaining flies either at least 12 will be in the 20 cm radius of A or 12 will be in the 20 cm radius of B. Swatting that the A or B fly with the center of the swatter kills at least 13.

Thus, by $\boxed{2}$ strikes, he can swat them all. $\square$

Chapter 5

MIC R1

5.1 Topics

Algebra

1. Second-degree Identities. Simplifyfing expressions. Factoring expressions.
2. Radicals. Powers.
3. Comparing radicals. Comparing powers.
4. System of Equations.
5. Quadratic. Condition for root existence.
6. Inequality by comparison.
7. Trigonometry functions sin and cos with double-, triple-, and special angles $30°$, $45°$, $60°$, $90°, \ldots$

Combinatorics

1. Probability. Number of favorable outcomes. Probability of independent events. Probabilities of consecutives events. Casework.
2. Combinatorial geometry. Dividing polygon into triangles.
3. Counting in two ways.
4. Euler's formula for a planar graph.
5. Sets and subsets.
6. Games. Winning positions. Symmetric Strategy.

Geometry

1. Triangles. Altitudes. Medians.
2. Areas.
3. Circumcircle.
4. Triangle trigonometry.
5. Ceva and Menelaus theorem.

Number Theory

1. Divisibility. Prime Factorization.
2. Perfect Powers. Exponents.
3. Factorials.

4. Diophantine equation. Condition for integer solution.
5. Integer inequality.
6. Number partitions $3 = 2 + 1 = 1 + 1 + 1$.
7. Perfect squares.
8. Nunber systems.

5.2 Rules

- The total time to complete the test is 90 minutes.
- The test consists of 10 multiple-choice and 4 show-you-work problems. To answer each of 10 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 6 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.

5.3 Problems

Problem 5.3.1 (MIC-2021-SM2-R1-P1). (*6 points*) m and n are two positive integers such that the product $m^n n^m$ can be factorized as,

$$m^n n^m = 2^{15}3^{21}5^6$$

What is the value of $m + n$?

(A) 20

(B) 21

(C) 14

(D) 15

(E) 12

Problem 5.3.2 (MIC-2021-SM2-R1-P2). (*6 points*) $\triangle ABC$ is an acute triangle, $AC < BC$ and $AB = 12$. D is the foot of the altitude from C to AB and $CD = 8$. O is the circumcenter of $\triangle ABC$.

What is the area of the quadrilateral $ACOD$?

(A) 36

(B) 18

(C) 24

(D) 42

(E) Cannot be determined.

Problem 5.3.3 (MIC-2021-SM2-R1-P3)**.** (*6 points*) Henry has two piles of marbles. The first pile contains 4 red and 5 blue marbles. The second pile contains 7 red and 11 blue marbles. He randomly chooses one of the pile and randomly picks one marble from that pile.

What is the probability that the chosen marble is red, if the probability that he chooses the first pile is $\frac{1}{3}$.

(A) $\frac{5}{12}$

(B) $\frac{1}{2}$

(C) $\frac{25}{34}$

(D) $\frac{11}{27}$

(E) $\frac{13}{34}$

Problem 5.3.4 (MIC-2021-SM2-R1-P4)**.** (*6 points*) If a and b are nonzero real numbers such that $|a| \neq |b|$, compute the value of the expression

$$\left(\frac{b^2}{a^2} + \frac{a^2}{b^2} - 2\right) \times \left(\frac{a+b}{b-a} + \frac{b-a}{a+b}\right) \times \left(\frac{\frac{1}{a^2} + \frac{1}{b^2}}{\frac{1}{b^2} - \frac{1}{a^2}} - \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{a^2} + \frac{1}{b^2}}\right).$$

(A) 1

(B) -1

(C) -2

(D) -4

(E) -8

Problem 5.3.5 (MIC-2021-SM2-R1-P5). (*6 points*) Sort the following numbers in ascending order

$$A = \sqrt[21]{21!},\ B = \sqrt[19]{19!},\ C = \sqrt[20]{20!},$$

(A) $A < B < C$

(B) $A < C < B$

(C) $B < A < C$

(D) $B < C < A$

(E) $C < B < A$

Problem 5.3.6 (MIC-2021-SM2-R1-P6)**.** (*6 points*) x, y, and z are three integers, not necessarily positive, such that

$$\begin{cases} x - yz = 2 \\ xz + y = 2 \end{cases}$$

What are the possible values of $x + y + z$?

(A) 1, 3 and 4

(B) -1, 3 and 4

(C) 0, 2 and 3

(D) 1 and 4

(E) 1 and 3

Problem 5.3.7 (MIC-2021-SM2-R1-P7). (*6 points*) 10 vertices are placed on a circle.

What is the maximum number of line segments, connecting two of the given vertices, that can be drawn such that no two intersect each other, except at the vertices.

(A) 15

(B) 17

(C) 18

(D) 19

(E) 45

Problem 5.3.8 (MIC-2021-SM2-R1-P8)**.** (*6 points*) a, b, c, d, e, f are six chosen digits $(0, 1, \ldots, 9)$, not necessarily different, such that

$$a \geq b \geq c \geq d \geq e \geq f.$$

In which reorderings below is the difference between the sum of the first three and the sum of the last three digits at most 9?

(A) e, c, b, a, d, f

(B) a, b, d, e, f, c

(C) c, d, a, f, e, a

(D) d, a, c, e, c, d

(E) a, c, e, b, d, f

Problem 5.3.9 (MIC-2021-SM2-R1-P9)**.** (*6 points*) Minh is playing the game *paper - rock - scissors* with her teammates. Because her mental focus is on the upcoming game of the MCC House Championship, so she divises a simple strategy as follow,

1. She plays *scissors* first.
2. On any subsequent turn, she plays a different move than the previous one, each with a probability of $\frac{1}{2}$.

Albert aims to defeat her by playing *rock* at the 5^{th} turn. What is the probability that he succeeds?

(A) $\frac{1}{2}$

(B) $\frac{3}{8}$

(C) $\frac{3}{16}$

(D) $\frac{7}{32}$

(E) $\frac{15}{64}$

Problem 5.3.10 (MIC-2021-SM2-R1-P10). (*6 points*) Clemence wants to divide the set of numbers $\{1, 2, 3, \ldots, 6\}$ into two subsets such that in any subset there are no three numbers a, b, c such that one is the sum of the other two numbers $a + b = c$.

For example, she can have two subsets $\{1, 2, 4\}$ *and* $\{3, 5, 6\}$*, but not* $\{1, 3, 4\}$ *and* $\{2, 5, 6\}$ *(because* $1 + 3 = 4$.*)* Furthermore, note that the order of the elements in a set or the order of the two subsets does not count.

In how many ways can she divide the set of numbers $\{1, 2, 3, \ldots, 6\}$ into such two subsets?

(A) 2

(B) 3

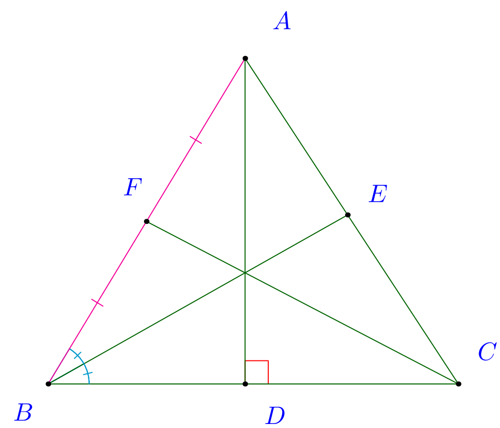
(C) 4

(D) 5

(E) 6

Problem 5.3.11 (MIC-2021-SM2-R1-P11)**.** (*10 points*) Let AD is the altitude, BE the angle bisector, and CF the median of a triangle ABC, where $D \in BC$, $E \in CA$, and $F \in AB$. Let $AB = c$, $BC = a$, and $CA = b$.

1. (*4 points*) Prove that $\frac{BD}{DC} = \frac{a^2+c^2-b^2}{a^2+b^2-c^2}$.
2. (*6 points*) Prove that AD, BE, and CF are concurrent if and only if $a^2(a-c) = (b^2-c^2)(a+c)$.



Problem 5.3.12 (MIC-2021-SM2-R1-P12)**.** (*10 points*) Prove that

1. (*4 points*) For any x real number
$$\sin 3x = 3\sin x - 4\sin^3 x.$$

2. (*6 points*)
$$16\sin^4\frac{\pi}{18} + 8\sin^3\frac{\pi}{18} - 12\sin^2\frac{\pi}{8} - 4\sin\frac{\pi}{8} + 1 = 0.$$

Problem 5.3.13 (MIC-2021-SM2-R1-P13). (*10 points*) $a_1a_2a_3$ and $a_3a_2a_1$ are two three-digit decimal numbers, with a_1 and a_3 different non-zero digits. Squares of these numbers are five-digit numbers $b_1b_2b_3b_4b_5$ and $b_5b_4b_3b_2b_1$ respectively.

Find all such three-digit numbers.

Problem 5.3.14 (MIC-2021-SM2-R1-P14). (*10 points*) Samuel and his friend take turns playing a two-player game on an 8×8 chessboard. In the first turn of the game, the player who starts the game can put a bishop on any of the squares. In any later turn,

- If there are some empty squares that are *not* under attack by any bishop on the board, the player of the turn can choose one of those squares and put a bishop on that square.
- If there is no such square, the player of the turn loses, and the game ends.

What would be the *winning strategy* for Samuel?

5.4 Grading

Answers for multiple-choice problems.

Problem 1: B 21

Problem 2: C 24

Problem 3: D $\frac{11}{27}$

Problem 4: E -8

Problem 5: D $B < C < A$

Problem 6: A $\{1, 3, 4\}$

Problem 7: B 17

Problem 8: E a, c, e, b, d, f

Problem 9: B $\frac{3}{8}$

Problem 10: D 5

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 11: Separately grading for each part,

(a) 2 point if Law of Cosines is used to find BD, 2 point for DC. 1 more points if the fraction is correctly established.

(b) 5 points if Ceva's Theorem is used to establish that $\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = 1$, 2 points if found $\frac{CE}{EA} = \frac{a}{c}$, 2 more points correctly establish that $\frac{a^2+c^2-b^2}{a^2+b^2-c^2} \cdot \frac{a}{c} = 1$.

Problem 12: Separately grading for each part,

(a) 2 point if can establish $\sin 3x = \sin(x + 2x) = \sin x \cos 2x + \cos x \sin 2x$.

(b) 5 points if can apply the first part for some parts of the expression.

Problem 13: 5 points if can estimate that both $a_1a_2a_3$ and $a_3a_2a_1$ are in the interval $[100,\ 316]$. 5 more points if can casework for (a_1, a_3) in $(1, 2),\ (1, 3)$, and $(2, 3)$.

Problem 14: 10 points if symmetry strategy is used. 5 points if proven that the game will end after a finite number of moves.

5.5 Solutions

Solution. MIC-2021-SM2-R1-P1 WLOG, assume that $2 \mid m$, then the exponent of the power n^m is even, thus $5 \mid n$. Since the exponent of the power 3^{21} is larger than the exponent of 2^{15} and of 5^6, thus $3 \mid m$, and $3 \mid n$. Hence $6 \mid m$, $15 \mid n$. Note that $6^{15}15^6 = 2^{12}3^{21}5^6$.

Thus, $\boxed{m + n = 6 + 15 = 21.}$ The answer is $\boxed{B.}$ □

Solution. MIC-2021-SM2-R1-P2 Let M be the midpoint of AB, then $OM \perp AB$, thus $[OCD] = [MCD]$, then

$$[ACOD] = [ACD] + [COD] = [ACD] + [MCD] = [ACM] = \frac{1}{2}[ACB] = \frac{1}{2} \cdot \frac{12 \cdot 8}{2} = 24$$

Thus, $\boxed{[ACOD] = 24.}$ The answer is $\boxed{C.}$ □

Solution. MIC-2021-SM2-R1-P3 There are two different cases.

Case 1: If Henry chooses the first pile, then the probability that he chooses a red marble is $\frac{4}{4+5} = \frac{4}{9}$.

Case 2: If Henry chooses the second pile, then the probability that he chooses a red marble is $\frac{7}{7+11} = \frac{7}{18}$.

Thus, the probability that he chooses a red marble is

$$\frac{1}{3}\frac{4}{9} + \frac{2}{3}\frac{7}{18} = \frac{11}{27}.$$

Thus, the probability is $\boxed{\frac{11}{27}.}$ The answer is $\boxed{D.}$ □

Solution. MIC-2021-SM2-R1-P4 First,

$$\begin{aligned}
\frac{b^2}{a^2} + \frac{a^2}{b^2} - 2 &= \left(\frac{b}{a} - \frac{a}{b}\right)^2 = \frac{(b^2 - a^2)^2}{(ab)^2} = \frac{(a-b)^2(a+b)^2}{(ab)^2} \\
\frac{a+b}{b-a} + \frac{b-a}{a+b} &= \frac{(a+b)^2 + (a-b)^2}{(b-a)(b+a)} = \frac{2(a^2+b^2)}{(b-a)(b+a)} \\
\frac{\frac{1}{a^2} + \frac{1}{b^2}}{\frac{1}{b^2} - \frac{1}{a^2}} - \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{a^2} + \frac{1}{b^2}} &= \frac{b^2 + a^2}{a^2 - b^2} - \frac{a^2 - b^2}{a^2 + b^2} = \frac{4a^2b^2}{(a^2 - b^2)(a^2 + b^2)}
\end{aligned}$$

Thus, the given product is,

$$\frac{(a-b)^2(a+b)^2}{(ab)^2} \frac{2(a^2+b^2)}{(b-a)(b+a)} \frac{4a^2b^2}{(a^2-b^2)(a^2+b^2)} = -8$$

Thus, it is $\boxed{-8.}$ The answer is $\boxed{E.}$ □

Solution. MIC-2021-SM2-R1-P4 For any n positive integer,

$$((n+1)!)^n = (n!)^n (n+1)^n > (n!)^n n! = (n!)^{n+1} \Rightarrow \sqrt[n+1]{(n+1)!} > \sqrt[n]{n!}.$$

Thus, $B = \sqrt[19]{19!} < C = \sqrt[20]{20!} < A = \sqrt[21]{21!}$. The answer is $\boxed{D.}$ □

Solution. MIC-2021-SM2-R1-P6

$$x - yz = 2 \Rightarrow x = yz + 2 \Rightarrow (yz+2)z + y = 2 \Rightarrow f(z) = yz^2 + 2z + (y-2) = 0$$
$$\Rightarrow \Delta_f = 2^2 - 4y(y-2) = 4(1 - y^2 + 2y) = 4(2 - (y-1)^2) \text{ is a perfect square}$$

Thus, $(y-1)^2 = 1$, so $y = 0$ or $y = 2$.

Case 1: $y = 0$, $0 = yz^2 + 2z + (y-2) = 2z - 2$, so $z = 1$, then $x = 2$.

Case 2: $y = 2$, $0 = yz^2 + 2z + (y-2) = 2z^2 + 2z$, so $z = -1$ or $z = 0$, then $x = 0$ or $x = 2$.

Thus, $\boxed{x + y + z \in \{1, 3, 4\}.}$ The answer is $\boxed{A.}$ □

Solution. MIC-2021-SM2-R1-P7 The 10 sides and d diagonals divide the $10-$gon into k triangles. If we count the segments by the triangles, each triangles has three segments as sides, each segment is counted once if it is a side of the $10-$gon, and twice if it is a diagonal (shared by two triangles), so

$$3k = 10 + 2d \quad (1)$$

On the other hand, by connecting the segments, we create a so-called *planar graph*, by Euler's graph formula for 10 vertices, $d + 10$ edges, and $k + 1$ faces:

$$10 - (d + 10) + (k + 1) = 2 \Rightarrow k = d + 1 \quad (2)$$

Therefore, from (1) and (2), $d = 7,\ k = 8$.

Thus, the number of segment is $\boxed{d + 10 = 17.}$ The answer is $\boxed{B.}$ □

Solution. MIC-2021-SM2-R1-P8 Note that

$$(a + c + e) - (b + d + f) = (a - f) + (c - b) + (e - d) \leq (9 - 0) + 0 + 0 = 9.$$

Thus,the answer is $\boxed{E.}$ □

Solution. MIC-2021-SM2-R1-P9 If Albert is to defeat her by play rock at the 5^{th} turn, then Minh should play *scissors* for that turn.

Her first move is *scissors*, so the second move is *paper* or *rock*. It does not matter what it is. For the third move, we have two cases.

Case 1: it is *scissors*. This occure with probability $\frac{1}{2}$. The fourth move is *paper* or *rock*, and it does not matter what it is. The fifth move is *scissors* with probability $\frac{1}{2}$. Thus for this case the probability is $\frac{1}{2}\frac{1}{2} = \frac{1}{4}$.

Case 2: it is *paper* or *rock*. This occure with probability $\frac{1}{2}$. Since her fifth move cannot be *scissors*, the fourth move cannot be *scissors*, so it should be *paper* or *rock* with a probability of $\frac{1}{2}$. Then the probability for the fifth move to be *scissors* is $\frac{1}{2}$.

Thus, the probability for this case is $\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$.

Thus, $\boxed{\frac{1}{4} + \frac{1}{8} = \frac{3}{8}.}$ The answer is $\boxed{B.}$ □

Solution. MIC-2021-SM2-R1-P10 WLOG, let $6 \in A$,

Case 1: both $1, 5 \in B$, then $4 \in A$, so $2 \in B$ and $3 \in A$. In this case $\boxed{A = \{3,4,6\},\ B = \{1,2,5\}.}$

Case 2: if $1 \in A$, then $5 \in B$.

Subcase 2a: if both 2 and 4 are in B. Then $3 \in A$. In this case $\boxed{A = \{1,3,6\},\ B = \{2,4,5\}.}$

Subcase 2b: if $2 \in A$, then $4 \in B$. Then $3 \in B$. In this case $\boxed{A = \{1,2,6\},\ B = \{3,4,5\}.}$

Subcase 2c: if $4 \in A$, then $2 \in B$. Then 3 can neither be in A nor B.

Case 3: if $5 \in A$, then $1 \in B$.

Subcase 3a: if both 2 and 4 are in B. Then $3 \in A$. In this case $\boxed{A = \{3,5,6\},\ B = \{1,2,4\}.}$

Subcase 2b: if $2 \in A$, then $4 \in B$. Then 3 can neither be in A nor B.

Subcase 2c: if $4 \in A$, then $2 \in B$. Then $3 \in A$. In this case $\boxed{A = \{3,4,5,6\},\ B = \{1,2\}.}$

Thus, there are $\boxed{\text{5 possible divisions.}}$ The answer is $\boxed{D.}$ □

Solution. MIC-2021-SM2-R1-P11 For the first part, by the Law of Cosines,

$$\frac{BD}{DC} = \frac{c\cos B}{b\cos C} = c\frac{a^2+c^2-b^2)}{2ac} \cdot \frac{1}{b}\frac{2ab}{a^2+b^2-c^2} = \frac{a^2+c^2-b^2}{a^2+b^2-c^2}.$$

For the second part, by Ceva's Theorem, AD, BE, and CF are concurrent if and only if

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = 1,$$

Since $\frac{AF}{FB} = 1$, $\frac{BD}{DC} = \frac{a^2+c^2-b^2}{a^2+b^2-c^2}$, and $\frac{CE}{EA} = \frac{a}{c}$, therefore AD, BE, and CF are concurrent if and only if,

$$\frac{a^2+c^2-b^2}{a^2+b^2-c^2} \cdot \frac{a}{c} = 1 \Leftrightarrow a(a^2+c^2-b^2) = c(a^2+b^2-c^2) \Leftrightarrow a^2(a-c) = c(b^2-c^2) - a(c^2-b^2) = (b^2-c^2)(a+c).$$

□

Solution. MIC-2021-SM2-R1-P12 For the first question,

$$\begin{aligned}\sin 3x = &\ \sin(x+2x) = \sin x\cos 2x + \cos x \sin 2x = \sin x(1-2\sin^2 x) + 2\sin x\cos^2 x \\ = &\ \sin x(1-2\sin^2 x) + 2\sin x(1-\sin^2 x) = 3\sin x - 4\sin^3 x.\end{aligned}$$

For the second question,

$$3\sin\frac{\pi}{18} - 4\sin^3\frac{\pi}{18} = \sin\frac{3\pi}{18} = \sin 30^\circ = \frac{1}{2}.$$

Therefore,

$$\begin{aligned}&16\sin^4\frac{\pi}{18} + 8\sin^3\frac{\pi}{18} - 12\sin^2\frac{\pi}{8} - 4\sin\frac{\pi}{8} \\ = &\ 4\sin\frac{\pi}{8}\left(4\sin^3\frac{\pi}{18} - 3\sin\frac{\pi}{18}\right) + 2\left(4\sin^3\frac{\pi}{18} - 3\sin\frac{\pi}{18}\right) + 2\sin\frac{\pi}{18} = -2\sin\frac{\pi}{8} - 1 + 2\sin\frac{\pi}{18} = -1\end{aligned}$$

Thus, $\boxed{16\sin^4\frac{\pi}{18} + 8\sin^3\frac{\pi}{18} - 12\sin^2\frac{\pi}{8} - 4\sin\frac{\pi}{8} + 1 = 0.}$ □

Solution. MIC-2021-SM2-R1-P13 Both $(a_1a_2a_3)^2$ and $(a_3a_2a_1)^2$ are in $[10000, 99999]$ and so both numbers are in $[100, 316]$ and $a_1,\ a_3 \in \{1,2,3\}$

WLOG say $a_1 < a_3$,

Case 1: $(a_1, a_3) = (1, 2)$, $a_1a_2a_3 = \overline{1a2}$ and $a_3a_2a_1 = \overline{2a1}$ and their squares are less than or equal to 49991, so these numbers less than 224, thus $a \in \{0, 1, 2\}$. By testing we got $\{102,\ 112,\ 122\}$.

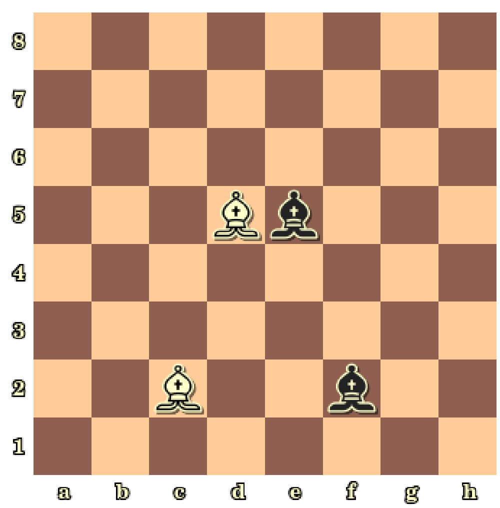
Case 2: $(a_1, a_3) = (1, 3)$, $a_1a_2a_3 = \overline{1a3}$ and $a_3a_2a_1 = \overline{3a1}$ and their squares are less than or equal to 99991, so these numbers less than 317, thus $a \in \{0, 1\}$. By testing we got $\{103,\ 113\}$

Case 3: $(a_1, a_3) = (2, 3)$, $a_1a_2a_3 = \overline{2a3}$ and $a_3a_2a_1 = \overline{3a2}$ and their squares are less than or equal to 99994, so these numbers less than 317, thus $a \in \{0, 1\}$. By testing we got no solutions.

Hence the solutions are $\boxed{a_1a_2a_3 \in \{102,\ 103,\ 112,\ 113,\ 122\}.}$ □

Solution. MIC-2021-SM2-R1-P14 Use the line ℓ separating the d and e column as a line of symmetry. Anytime a player places a bishop on a square s, the other players can place the other bishop on the square symmetric to s in regard to ℓ. In the diagram below, White starts by placing a white bishop on d5, Black put its black bishop on e5, then White with c2 and Black with f2. Note that the two squares of consecutives move by White and Black have different colours. Obviously White does not put its bishop on any square where Black bishops can attack, which means that the symmetric position should neither be under attack, thus Black can always make a move if White can make a move.

White turns 1. d5 2. c2



Black turns 1. e5 2. f2

Black, the second player could always go, regardless of what the first player does. The number of squares on the board is 64, or finite, thus after the game ends after at most 64 moves. Hence White, the first player, soon or later will not find a move. White will lose eventually.

Thus, $\boxed{\text{Samuel should go second}}$ with this strategy of exploing symmetry. □

Chapter 6

IC Level Test - October

6.1 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you 4 or 6 points, based on the number of points associated to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth 10 points.
     - A problem has one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.2 Problems for Level 1

Problem 6.2.1 (ICLT-2021-SM2-10-L1-P11). *(4 points)*

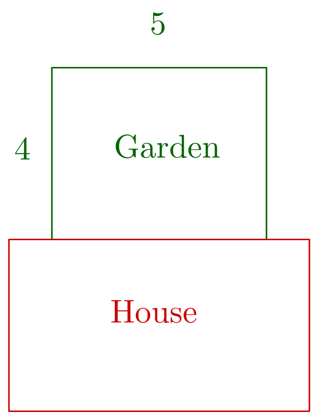
Evaluate the following expression,

$$\frac{(12+13)\times(26-23+2\times 4-9)}{\left(11-5\times\frac{3}{1}+6\right)\times 5^{\left(4-\frac{4}{6-2}\right)}}$$

$(A)\quad 25 \qquad (B)\quad 5 \qquad (C)\quad \frac{1}{5} \qquad (D)\quad -2 \qquad (E)\quad -5$

Problem 6.2.2 (ICLT-2021-SM2-10-L1-P21). *(4 points)*

Anna has a $4m \times 5m$ rectangular garden. One side of it is a part of the wall of her house, as show below. She wants to plant trees on the perimeter of the garden so that the distance (along the perimeter) between any two trees is at least $1m$.



At most how many trees can Anna plant? *She is not allowed to plant any tree on the wall of her house.*

$(A)\quad 10 \qquad (B)\quad 11 \qquad (C)\quad 12 \qquad (D)\quad 13 \qquad (E)\quad 14$

Problem 6.2.3 (ICLT-2021-SM2-10-L1-P91). *(4 points)*

What is the result by simplifying the below expression?

$$(2x-1)(x+2)+(1-3x)(x+3)-(x-1)(5-x)$$

$(A)\quad 16x-11 \qquad (B)\quad 11x-16 \qquad (C)\quad -11x-6 \qquad (D)\quad -11x+6 \qquad (E)\quad x^2-11x+6$

Problem 6.2.4 (ICLT-2021-SM2-10-L1-P61). (*6 points*)

Benny has three piles of marbles. The first pile has 3 red marbles, the second has 4 blue marbles, and the third pile has some green marbles. He divides the marbles into pairs such that no pair has marbles of the same colour.

At most how many marbles are there in the third pile, if one of the pairs contains a red and a blue marble?

(A) 3 (B) 4 (C) 5 (D) 7 (E) 9

Problem 6.2.5 (ICLT-2021-SM2-10-L1-P81). (*6 points*)

During practice hours, the teacher tried to divide the students in the class into two or more groups, each group has at least two members, and every group has the same number of students.

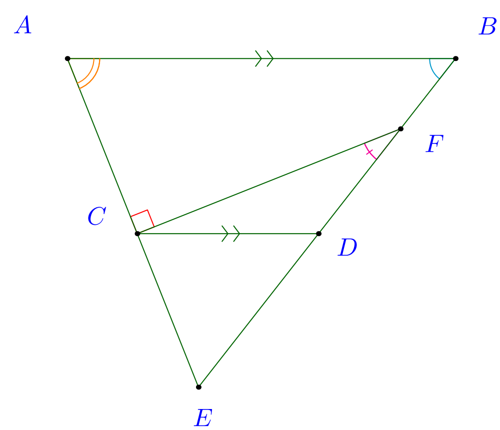
What was the least number of students are in the class if she could divide the students in exactly 4 ways?

For example if there were 6 *students, she could divide the students into* 2 *groups of* 3 *students or* 3 *groups of* 2 *students. In total, there were exactly* 2 *ways for her to divide the students into groups as required.*

(A) 6 (B) 8 (C) 9 (D) 10 (E) 12

Problem 6.2.6 (ICLT-2021-SM2-10-L1-P41). (*6 points*)

In the diagram below, AB and CD are parallel, FC is perpendicular to AE.



If $\angle ABF = 52°$ and $\angle CAB = 60°$, what is $\angle CFD$?

(A) 22° (B) 25° (C) 28° (D) 30° (E) 38°

Problem 6.2.7 (ICLT-2021-SM2-10-L1-P0). (*10 points*)

a and b are two prime numbers and $a > b$. The sum $a + b$ and the difference $a - b$ are both prime numbers.

1. (*5 points*) Prove that $b = 2$.
2. (*5 points*) Find all pairs (a, b).

Problem 6.2.8 (ICLT-2021-SM2-10-L1-P1). (*10 points*)

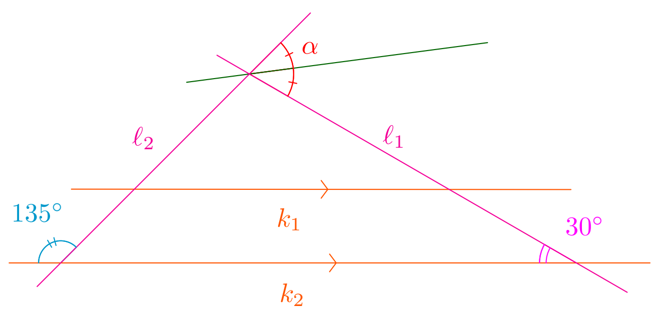
Quan places a regular $6-$side dice on a table.

1. (*5 points*) In how many ways can he place the dice such that the product of the two numbers on the top and the front face is 6?
2. (*5 points*) In how many ways can he place the dice such that the difference of the two numbers on the top and the front face is 2?

Note that a way to place a dice such that 6 *on the top face and* 1 *on the front face is different from placing way to place the dice such that* 1 *on the top face and* 6 *on the front face.*

Problem 6.2.9 (ICLT-2021-SM2-10-L1-P2). (*10 points*)

Lines k_1 and k_2 are parallel. Lines l_1 and l_2 intersect k_2 at angles $30°$ and $135°$, respectively. A line through the intersection of l_1 and l_2 bisects their angle $\angle(l_1, l_2)$ as shown below.



1. (*5 points*) Find α.
2. (*5 points*) Let a and b be the measures of the angles instead of $135°$ and $30°$. Find the formula to calculate α based on a and b.

6.3 Grading for Level 1

Answers for multiple-choice problems.

Problem 1: C $\frac{1}{5}$

Problem 2: D 13

Problem 3: D $-11x+6$

Problem 4: C 5

Problem 5: E 12

Problem 6: A 22

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately grading for each part,

(a) 2 point if determined that one of the primes is even.

(b) 1 point by stating $a = 3, b = 2$ without any reasoning. 3 point if determined that one of $a - b$, a, $a + b$ is divisible by 3.

Problem 8: Separately grading for each part,

(a) 2 point if found two pairs $(1, 6)$ and $(2, 3)$. No additional points without permutations.

(b) 3 point if found four pairs $(1, 3), (2, 4), (3, 5)$, and $(4, 6)$. No additional points without permutations.

Problem 9: Separately grading for each part,

(a) 2 if found the exterior opposite angle of k_2 and l_2 is $180° - 135° = 45°$.

(b) 3 if found the exterior opposite angle of k_2 and l_2 is $180° - b$.

6.4 Solutions for Level 1

Solution. ICLT-2021-SM2-10-L1-P11

$$(12+13)\times(26-23+2\times4-9)=25\times(3+8-9)=25\times2$$
$$11-5\times\frac{3}{1}+6=11-5\times3+6=11-15+6=2$$
$$5^{\left(4-\frac{4}{6-2}\right)}=5^{4-\frac{4}{4}}=5^{4-1}=5^3=125$$

Thus, the given fraction is $\boxed{\dfrac{25\times 2}{2\times 125}=\dfrac{1}{5}.}$ The answer is $\boxed{C.}$ □

Solution. ICLT-2021-SM2-10-L1-P21 The total length of the perimeter of the garden where Anna can plant her trees is $2\cdot4+1\cdot5=13$m. Because no tree can be planted on the wall so the total distance to plant tree is *less* than 13m. Since the distance between any two tree is at least $1m$, at most she can plant $\boxed{13}$ of them within a $(13-1)\times1=12$m. The answer is $\boxed{D.}$ □

Solution. ICLT-2021-SM2-10-L1-P91 $\boxed{(2x-1)(x+2)+(1-3x)(x+3)-(x-1)(5-x)=-11x+6.}$ The answer is $\boxed{D.}$ □

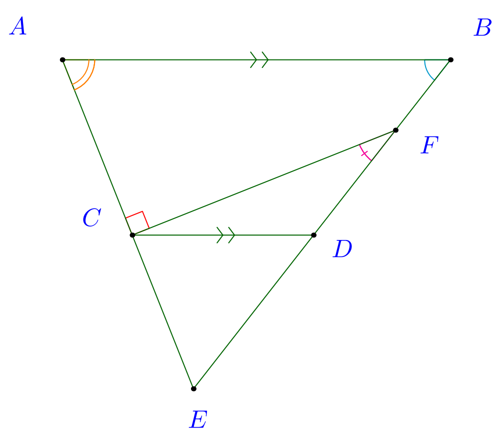
Solution. ICLT-2021-SM2-10-L1-P61 The largest number of green marbles can be achieved if beside a single pair of (red, blue) marble, he pairs only red marbles with green marbles, and blue marbles with green marbles. This way, he has $\boxed{3+4-2=5}$ green marbles. The answer is $\boxed{C.}$ □

Solution. ICLT-2021-SM2-10-L1-P81 In order to divide the students into 4 ways, their number should has exactly 4 divisors not including 1 and itself. In other words, it is a number that has exactly 6 divisors. The smallest such number is 12, so there are $\boxed{12}$ students. The answer is $\boxed{E.}$ □

Solution. ICLT-2021-SM2-10-L1-P41 First, $\angle CFD=180^\circ-\angle CDF-\angle DCF$. Since

$$180^\circ-\angle CDF=\angle ABF,\ \angle DCF=\angle DCA-90^\circ=(180^\circ-\angle CAB)-90^\circ=90^\circ-\angle CAB$$

So $\angle CFD=\angle ABF+\angle CAB-90^\circ$.



Thus, $\boxed{\angle CFD=52^\circ+60^\circ-90^\circ=22^\circ.}$ The answer is $\boxed{A.}$ □

Solution. ICLT-2021-SM2-10-L1-P92 Since there are 4 different letters A, O, T, and W, so there are 4 ways to select a triple of 3 different letters. (Each selection keeps one of the four letters A, O, T, and W out of the triple.) For each triples, there are $3! = 6$ ways to permute the letters to form a new 3−letter word.

Thus, there are $\boxed{4 \times 6 = 24.}$ The answer is $\boxed{B.}$ □

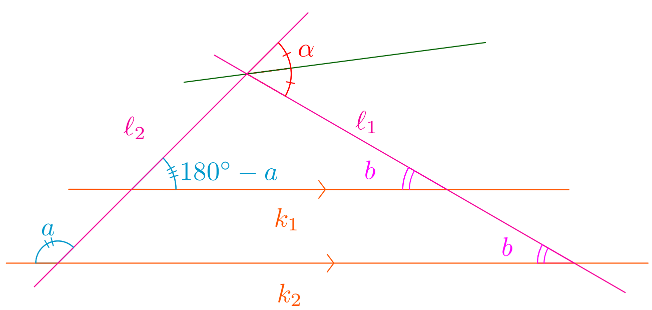
Solution. ICLT-2021-SM2-10-L1-P0 If both a and b are odd prime numbers, then $a - b$ and $a + b$ are both even and cannot be prime numbers. Thus, one of a and b is even prime, or 2. Since $a > b$, therefore $\boxed{b = 2.}$

Now $a - 2$, a, and $a + 2$ are prime numbers. These are three consecutive odd numbers, so one of them is divisible by 3, thus $a - 2 = 3$. Therefore $\boxed{a = 5 \text{ and } b = 2.}$ It is easy to see that $a - b = 3,\ a + b = 7$ are both prime numbers. □

Solution. ICLT-2021-SM2-10-L1-P1 Since $6 = 1 \times 6 = 2 \times 3$, so there are two pairs of faces. For each pair there are $2! = 2$ ways to place one number on the top face and the other number on the front face. Thus, there are $\boxed{2 \times 2 = 4}$ ways.

Now $2 = 3 - 1 = 4 - 2 = 5 - 3 = 6 - 4$. By following exactly the same reasoning as above, there are $\boxed{4 \times 2! = 8}$ such ways. □

Solution. ICLT-2021-SM2-10-L1-P2 Denote the diagram with a and b instead of 135° and 30° as below,



It is easy to see that $2\alpha = (180° - a) + b$, thus $\boxed{\alpha = 90° - \frac{1}{2}(a - b).}$

For $a = 135°,\ b = 30°$, then $\boxed{\alpha = 90 - \frac{1}{2} \cdot 105° = 37.5°.}$ □

Solution. **??** For the first question, if the hundred digit is 1, then the sum of the other two digits is 8. There are 9 ways to choose a pair of digits to sum up to 8, $8 = 8 + 0 = 7 + 1 = \ldots = 0 + 8$. Thus, there are $\boxed{9}$ such three-digit numbers.

Similarly for the first digit as $2, \ldots, 9$, the sum of the two remaining digits is $7, \ldots, 0$, respectively. There are $8, 7, \ldots, 1$ ways for the other two digits, respectively. In total, there are $\boxed{9 + 8 + \ldots + 1 = \frac{9 \cdot 10}{2} = 45}$ such numbers. □

6.5 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice,you 4 or 6 points, based on the number of points associated to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth 10 points.
     - A problem has one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.6 Problems for Level 2

Problem 6.6.1 (ICLT-2021-SM2-10-L1-P12)**.** (*4 points*)

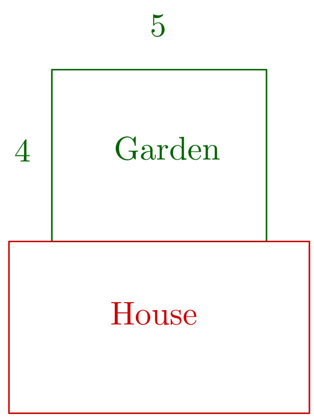
Evaluate the following expression,

$$\frac{(12+13)\times(26-23+2\times4-9)}{\left(11-5\times\frac{3}{1}+6\right)\times5^{\left(1-\frac{4}{6-2}\right)}}$$

(A) 25 (B) 5 (C) $\frac{1}{5}$ (D) -2 (E) -5

Problem 6.6.2 (ICLT-2021-SM2-10-L1-P22)**.** (*4 points*)

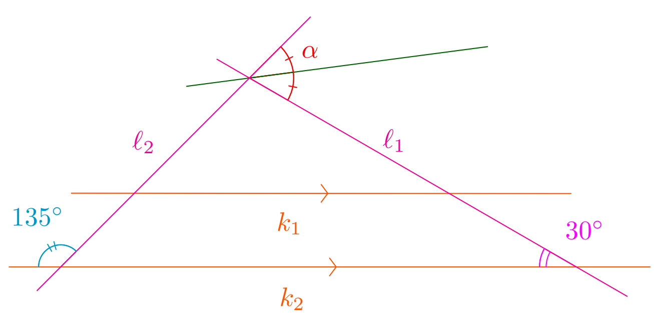
Anna has a $4m\times5m$ rectangular garden. One long side of it is a part of the wall of her house, as show below. She wants to plant trees on the perimeter of the garden so that the distance (along the perimeter) between any two trees is at least 2m.



At most how many trees can Anna plant? *She is not allowed to plant any tree on the wall of her house.*

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Problem 6.6.3 (ICLT-2021-SM2-10-L1-P52)**.** (*4 points*) Lines k_1 and k_2 are parallel. Lines l_1 and l_2 intersect k_2 at angles $30°$ and $135°$, respectively. A line through the intersection of l_1 and l_2 bisec the angle $\angle(l_1, l_2)$ as shown below.



Find α.

(A) $35°$ (B) $37.5°$ (C) $42.5°$ (D) $47.5°$ (E) Cannot be determined

Problem 6.6.4 (ICLT-2021-SM2-10-L2-P0). (*6 points*)

What is the value of $xy + \frac{1}{xy}$ if

$$\begin{cases} x + \frac{3}{y} = 3 \\ y + \frac{1}{3x} = 3 \end{cases}$$

(A) 2 $\quad$ (B) $\dfrac{11}{3}$ $\quad$ (C) 4 $\quad$ (D) $\dfrac{14}{3}$ $\quad$ (E) $\dfrac{17}{3}$

Problem 6.6.5 (ICLT-2021-SM2-10-L1-P72). (*6 points*)

ABC is a triangle such that two of its interior angles are the same, and one of its interior angle is three times another interior angle.

What would be the smallest interior angle of $\triangle ABC$?

(A) $36°$ $\quad$ (B) $\dfrac{180°}{7}$ $\quad$ (C) $\dfrac{540°}{7}$ $\quad$ (D) $72°$ $\quad$ (E) $108°$

Problem 6.6.6 (ICLT-2021-SM2-10-L1-P32). (*6 points*) Antoine has 8 marbles. He wants to divide the marbles into some piles such that the number of marbles in each pile is a prime number. (It is possible that two piles contain the same number of marbles.) *For example he can divide them into three piles: one pile of* 2 *marbles, one pile of* 3 *marbles, and one pile of* 3 *marbles.*

In how many ways can he do it?

(A) 2 $\quad$ (B) 3 $\quad$ (C) 4 $\quad$ (D) 5 $\quad$ (E) It is not possible

Problem 6.6.7 (ICLT-2021-SM2-10-L2-P3). (*10 points*)

Larry's age is 5 more than Barry's age. 10 years ago, Larry's age was twice Barry's age.

1. (*5 points*) What will Larry's age be 10 years from now?
2. (*5 points*) How old was Barry when Larry's age was four times Barry's age?

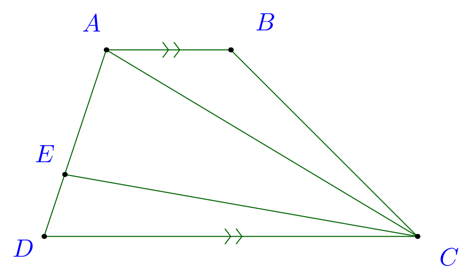
Problem 6.6.8 (ICLT-2021-SM2-10-L2-P4). (*10 points*)

Let $A = 20! = 1 \cdot 2 \cdot 3 \cdots 20$, and $B = 10! = 1 \cdot 2 \cdot 3 \cdots 10$.

1. (*5 points*) Prove that $\frac{A}{B^2}$ is an integer.
2. (*5 points*) Is $\frac{A}{B^2}$ divisible by 7?

Problem 6.6.9 (ICLT-2021-SM2-10-L2-P21). (*10 points*)

$ABCD$ is a trapezoid such that $AB \parallel CD$, $AB = 2$, $CD = 6$. E is on AD such that $DE = 2EA$.



1. (*4 points*) Prove that $[ADC] = 3[ABC]$.
2. (*5 points*) Prove that $[DEC] = [ABC]$.

6.7 Grading for Level 2

Answers for multiple-choice problems.

Problem 1: A 25

Problem 2: B 7

Problem 3: B 37.5

Problem 4: E $\frac{17}{3}$

Problem 5: B $\frac{180^\circ}{7}$

Problem 6: B 3

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately grading for each part,

(a) 2 point if can establish a system of equations for the ages.

(b) 3 point if can consider a variable n for the number of years ago.

Problem 8: Separately grading for each part,

(a) 2 if found the factoring by canceling 10! from both A and B^2.

(b) 3 if can eliminate all multiples of 7 from the numerator A.

Problem 9: Separately grading for each part,

(a) 2 if found that the distance from C to AB and A to CD is the same.

(b) 3 if determined the ratio of AD and ED.

6.8 Solutions for Level 2

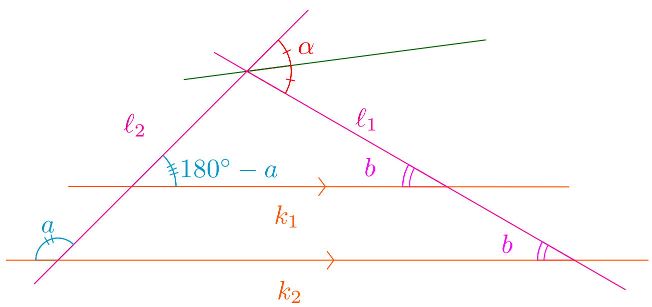
Solution. ICLT-2021-SM2-10-L1-P12

$$(12+13)\times(26-23+2\times4-9)=25\times(3+8-9)=25\times2$$
$$11-5\times\frac{3}{1}+6=11-5\times3+6=11-15+6=2$$
$$5^{\left(1-\frac{4}{6-2}\right)}=5^{1-\frac{4}{4}}=5^{1-1}=5^0=1$$

Thus, the given fraction is $\boxed{\dfrac{25\times2}{2\times1}=25.}$ The answer is $\boxed{A.}$ □

Solution. ICLT-2021-SM2-10-L1-P22 The total length of the perimeter of the garden where Anna can plant her trees is $2\cdot4+1\cdot5=13$m. Because no tree can be planted on the wall so the total distance to plant tree is *less* than 13m. Since the distance between any two tree is at least 2m, at most she can plant $\boxed{7}$ of them within $(7-1)\times2=12$m. The answer is $\boxed{B.}$ □

Solution. ICLT-2021-SM2-10-L1-P52 Denote the diagram with a and b instead of $135°$ and $30°$ as below,



It is easy to see that $2\alpha=(180°-a)+b$, thus $\boxed{\alpha=90°+\frac{1}{2}(b-a).}$

For $a=135°,\ b=30°$, then $\boxed{\alpha=90-\frac{1}{2}\cdot105°=37.5°.}$ The answer is $\boxed{B.}$ □

Solution. ICLT-2021-SM2-10-L2-P0

$$\begin{cases}x+\frac{3}{y}=3\\ y+\frac{1}{3x}=3\end{cases}\Rightarrow\left(x+\frac{3}{y}\right)\left(y+\frac{1}{3x}\right)=9\Rightarrow xy+\frac{1}{xy}+3+\frac{1}{3}=9\Rightarrow xy+\frac{1}{xy}=\frac{17}{3}$$

Therefore, $\boxed{xy+\frac{1}{xy}=\frac{17}{3}.}$ The answer is $\boxed{E.}$ □

Solution. ICLT-2021-SM2-10-L1-P72 There are two cases.

Case 1: the three angles are a, a, and $3a$. $a+a+3a=180°$, so $5a=180$, $a=36°$. They are $36°, 36°, 108°$.

Case 1: the three angles are $3a$, $3a$, and a. $7a=180$, $a=\frac{180°}{7}$. The angles are $\frac{540°}{7}$, $\frac{540°}{7}$, $\frac{180°}{7}$.

The smallest angle of both cases is $\boxed{\frac{180°}{7}.}$ The answer is $\boxed{B.}$ □

Solution. ICLT-2021-SM2-10-L1-P32 We do casework.

Case 1: if there is no pile with 2 marbles, then the only possible division is $3+5=8$. We got $8=3+5$.

Case 2: if there is a pile with 2 marbles, then with 6 marbles left, it can be divided as follow,

$$6=2+2+2=3+3 \Rightarrow 8=2+2+2+2=2+3+3.$$

Thus, there are $\boxed{3}$ ways, $8=2+2+2+2=2+3+3=3+5$. The answer is $\boxed{B.}$ □

Solution. ICLT-2021-SM2-10-L2-P3 Let Larry's age now be ℓ, Barry's age now b, then

$$\begin{cases} \ell = b+5 \\ \ell - 10 = 2(b-10) \end{cases} \Rightarrow \ell - 10 = 2(\ell-5-10) = 2\ell - 30 \Rightarrow \ell = 20,\ b = 15$$

Therefore, 10 years from now, Larry will be $\boxed{\text{30 years old.}}$ Let assume that n year ago, Larry's age was four times Barry's age, then

$$20-n = 4(15-n) \Rightarrow 20-n = 60-4n \Rightarrow 3n = 60-20 = 40$$

Since 40 is not divisible by 3, so there is $\boxed{\text{no such year}}$ where Larry's age is exactly 4 times Barry's. □

Solution. ICLT-2021-SM2-10-L2-P4 $A = 1\cdot 2\cdot 3\cdots 20$, and $B = 1\cdot 2\cdot 3\cdots 10$, so

$$\frac{A}{B\cdot B} = \frac{20!}{10!10!} = \frac{11\cdot 12\cdot 13\cdot 14\cdot 15\cdot 16\cdot 17\cdot 18\cdot 19\cdot 20}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}$$

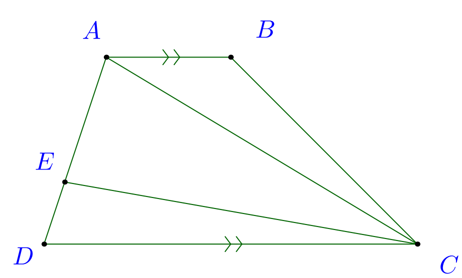
Since,

$$2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10 = 2\cdot 3\cdot 4\cdot 5\cdot (2\cdot 3)\cdot 7\cdot 8\cdot 9\cdot 10 = (3\cdot 4)(3\cdot 5)(2\cdot 7)(2\cdot 8)\cdot 9\cdot 10 = 12\cdot 15\cdot 14\cdot 16\cdot 9\cdot 10$$

$$\frac{A}{B\cdot B} = 11\cdot 13\cdot 17\cdot 19\cdot 2\cdot 2$$

Therefore $\boxed{\frac{A}{B^2}}$ is an integer and it is $\boxed{\text{not divisible by 7.}}$ □

Solution. ICLT-2021-SM2-10-L2-P21 First, since $AB \parallel CD$, so the distance from C to AB is the same as the distance from A to CD, therefore the altitudes of $\triangle ABC$ and $\triangle ACD$ from C and A, respectively, are the same. Because $CD = 3AB$, so $\boxed{[ACD] = 3[ABC].}$



For the second question, $EA = 2ED$, thus $AD = 3ED$, then $[ADC] = 3[EDC]$, and $\boxed{[ECD] = [ABC].}$ □

6.9 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice,you 4 or 6 points, based on the number of points associated to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth 10 points.
     - A problem has one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.10 Problems for Level 3

Problem 6.10.1 (ICLT-2021-SM2-10-L1-P13). (*4 points*)

Evaluate the following expression,

$$\frac{(12+13)\times(26-23+2\times4-9)}{\left(11-5\times\frac{3}{1}+6\right)\times5^{\left(2-\frac{4}{6-2}\right)}}$$

(A) 25 (B) 5 (C) $\dfrac{1}{5}$ (D) -2 (E) -5

Problem 6.10.2 (ICLT-2021-SM2-10-L2-P32). (*4 points*)

Four girls and three boys form two teams for the championship. Alice and Belice, two of the girls, want to be on the same team. Tinh and Minh, two of the boys, want to be in separate teams.

In how many ways they can form two teams if each team should have at least one boy and one girl.

(A) 2 (B) 4 (C) 6 (D) 8 (E) 12

Problem 6.10.3 (ICLT-2021-SM2-10-L3-P1). (*4 points*)

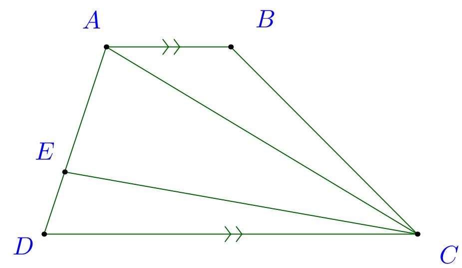
After a recent diet, Kale's weight was 67.5 kg. This means that he lost 10% of his weight. He put on the favorite hockey suit, which was 5% of his weight before the diet.

How much did he weigh after putting on the hockey suit?

(A) 64.125 (B) 71.25 (C) 72 (D) 74.25 (E) 70.875

Problem 6.10.4 (ICLT-2021-SM2-10-L2-P31). (*6 points*)

$ABCD$ is a trapezoid such that $AB \parallel CD$, $AB = 2$, $CD = 6$. E is on AD such that $2DE = EA$.



What is the ratio of $\dfrac{[ABC]}{[CDE]}$?

(A) $\frac{5}{4}$ (B) $\frac{6}{5}$ (C) 1 (D) $\frac{5}{6}$ (E) $\frac{4}{5}$

Problem 6.10.5 (ICLT-2021-SM2-10-L3-P2). (*6 points*)

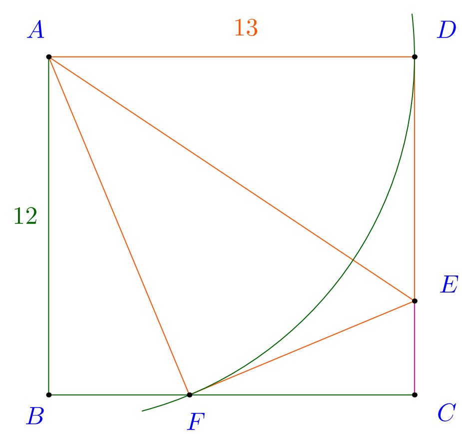
Bi, Chi, and Ni jog around in a circular path. Bi can jog 120 meters per minute, Chi can jog 80 meters per minute, and Ni can jog 70 meters per minute. They started at 9:00 AM at the same starting point S. The length of the circular path is 400 meters.

What time in the morning will they first all be together again at point S?

(A) $10:30$ (B) $10:15$ (C) $9:40$ (D) $9:35$ (E) $9:30$

Problem 6.10.6 (ICLT-2021-SM2-10-L3-P0). (*6 points*)

The square $ABCD$ is folded along the line AE such that D becomes point F on BC. $AB = 12$, $BC = 13$.



Find EC.

(A) 2 (B) $\frac{5}{2}$ (C) $\frac{10}{3}$ (D) 3 (E) $\frac{7}{2}$

Problem 6.10.7 (ICLT-2021-SM2-10-L3-P32). (*10 points*)

Lam and Lan take turns playing a two-player game. In each turn, the player of that turn write a divisor of 900, which could include 1 and 900, on the table if it is not yet written. The player who cannot make a move will lose and the game ends.

1. (*5 points*) Prove that the number of turns does not depend on what the players do.
2. (*5 points*) If Lan wants to win and if she can choose to go first or second, what is her winning startegy?

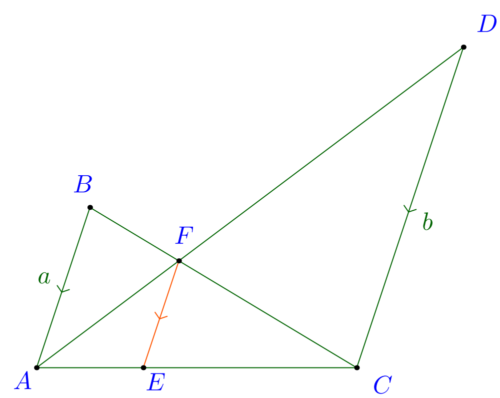
Problem 6.10.8 (ICLT-2021-SM2-10-L3-P4). (*10 points*)

The value of a land lot is proportional to the square of its area. The lot is divided into three smaller lots whose area are in the ratio $2:3:5$.

1. (*5 points*) If x is the area of the original lot, what are the area of the smaller lots?
2. (*5 points*) The total value of the three smaller lots is \$7600. What is the value of the original lot?

Problem 6.10.9 (ICLT-2021-SM2-10-L3-P31). (*10 points*)

In the diagram below, $AB \parallel CD \parallel EF$.



1. (*5 points*) Find for EF if $AB = a$ and $CD = b$.
2. (*5 points*) Prove that $\sqrt{\frac{[ECF]}{[ABC]}} + \sqrt{\frac{[EAF]}{[CAD]}} = 1$.

6.11 Grading for Level 3

Answers for multiple-choice problems.

Problem 1: B 5

Problem 2: E 12

Problem 3: B 71.25kg

Problem 4: C 1

Problem 5: C $9:40$ AM

Problem 6: C $\frac{10}{3}$

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately grading for each part,

(a) 2 point if found that the number of turns is the number of divisors of 900.

(b) 2 point if found the parity of the number of turns.

Problem 8: Separately grading for each part,

(a) 2 if found any ratios between x and the lots.

(b) 3 if can add up the ratios.

Problem 9: Separately grading for each part,

(a) 2 if found similar triangles ABC, EFC or ADC, AFE.

(b) 3 if determined the ratio of area is square of the ratio of similarity.

6.12 Solutions for Level 3

Solution. ICLT-2021-SM2-10-L1-P13

$$(12+13)\times(26-23+2\times 4-9)=25\times(3+8-9)=25\times 2$$
$$11-5\times\frac{3}{1}+6=11-5\times 3+6=11-15+6=2$$
$$5^{\left(2-\frac{4}{6-2}\right)}=5^{2-\frac{4}{4}}=5^{2-1}=5^1=5$$

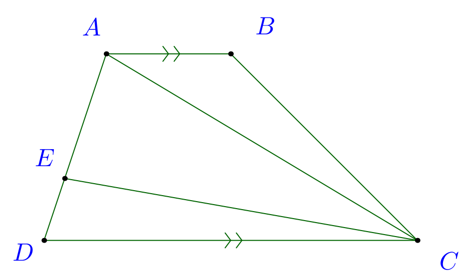
Thus, the given fraction is $\boxed{\dfrac{25\times 2}{2\times 5}=5.}$ The answer is $\boxed{B.}$ ☐

Solution. ICLT-2021-SM2-10-L2-P32 If Alice and Belice in one team, then there are 3 ways to choose to put the other two girls into the other team: none of them (1 way), or one of them (2 ways). Both of of the remaining girls cannot join because then the other team has no girl. Alice and Belice team can have Tinh or Minh (2 ways). There are two ways to put the remaining boy.

So, all togethere there are $\boxed{3\times 2\times 2=12 \text{ ways.}}$ The answer is $\boxed{E.}$ ☐

Solution. ICLT-2021-SM2-10-L3-P1 His weight before the diet was $\frac{67.5}{0.9}=75$ kg, so 5% of the weight was $\frac{1}{2}(75-67.5)=3.75$ kg. After putting on the suit, he weighed $\boxed{67.5+3.75=71.25 \text{ kg.}}$ The answer is $\boxed{B.}$ ☐

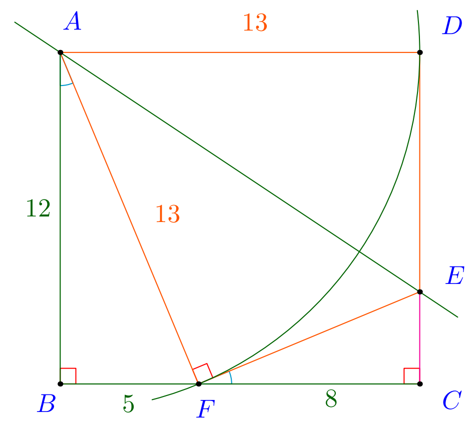
Solution. ICLT-2021-SM2-10-L2-P31 First, since $AB \parallel CD$, so the distance from C to AB is the same as the distance from A to CD, therefore the altitudes of $\triangle ABC$ and $\triangle ACD$ from C and A, respectively, are the same. Because $CD=3AB$, so $[ACD]=3[ABC]$, or $[ABC]=\frac{1}{3}[ADC]$. Since $EA=2ED$, thus $AD=3ED$,



then $[ADC]=3[EDC]$. Thus, $\boxed{\frac{[ABC]}{[ECD]}=1.}$ The answer is $\boxed{C.}$ ☐

Solution. ICLT-2021-SM2-10-L3-P2 Let n be the number of minutes from the starting time $9:00$ AM when they all first meet at S. Bi runs $\frac{120n}{400}=\frac{3n}{10}$, Chi runs $\frac{80n}{400}=\frac{n}{5}$, and Ni runs $\frac{70n}{400}=\frac{7n}{40}$ number of rounds. Each of these numbers of rounds must be an integer. So n is the least common multiple $\operatorname{lcm}(5,10,40)=40$. Therefore, they will first meet again at the starting point S at $\boxed{9:40 \text{ AM.}}$ The answer is $\boxed{C.}$ ☐

Solution. ICLT-2021-SM2-10-L3-P0 Since $\triangle AFE$ is a reflection of $\triangle ADE$ over the line AE, so $\triangle AFE \cong \triangle AFD$. Thus, $AF = AD$. Now, $BF = \sqrt{AF^2 - AB^2} = \sqrt{13^2 - 12^2} = 5 \Rightarrow FC = BC - BF = 13 - 5 = 8$.



Since $\angle AFE = 90°$, so $\angle CFE = 90° - \angle BFA = \angle BAF$, thus $\triangle BAF \sim \triangle CFE$.

$$\frac{EC}{FC} = \frac{FB}{AB} \Rightarrow EC = \frac{FB \cdot FC}{AB} = \frac{8 \cdot 5}{12} = \frac{10}{3}.$$

Therefore, $\boxed{EC = \frac{10}{3}.}$ The answer is $\boxed{C.}$ □

Solution. ICLT-2021-SM2-10-L3-P32 The numbers that the players taking turns to write on the table are the divisors of 900. The game ends when they run out of the divisors. Since

$$900 = 2^2 3^2 5^2 \quad \text{the number of divisors of 900 } is\ (2+1)(2+1)(2+1) = (2+1)^3 = 27.$$

So the game always end after exactly $\boxed{27}$ turns.

Thus, 27 is odd, the first player will win. Hence, $\boxed{\text{Lan goes first.}}$ □

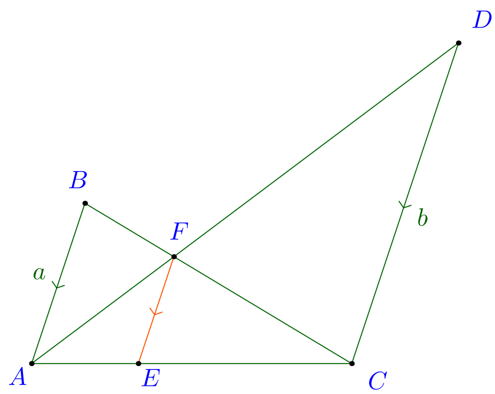
Solution. ICLT-2021-SM2-10-L3-P4 Let x be the area of the original lot. The area of the smaller lots after division are in the ratio 2 : 3 : 5. So they are $\boxed{\frac{2x}{10}, \frac{3x}{10}, \text{ and } \frac{5x}{10}.}$ Let be u, v, and w be the value of the smaller lots, since the value of a land lot is proportional to the square of its area,

$$\frac{u}{\left(\frac{2x}{10}\right)^2} = \frac{v}{\left(\frac{3x}{10}\right)^2} = \frac{u}{\left(\frac{5x}{10}\right)^2} = \frac{u+v+w}{x^2\,\frac{2^2+3^2+5^2}{10^2}} = \frac{7600}{\frac{38x^2}{100}} = \frac{20000}{x^2}$$

The value of the original lot is $\boxed{20000.}$ □

Solution. ICLT-2021-SM2-10-L3-P31 First, $\triangle ECF \sim \triangle ACB$, $\triangle EAF \sim \triangle CAD$, so

$$\frac{EF}{AB} = \frac{EC}{AC},\ \frac{EF}{CD} = \frac{AE}{AC} \Rightarrow \frac{EF}{AB} + \frac{EF}{CD} = 1 \Rightarrow EF = \frac{AB \cdot CD}{AB + CD}.$$



Thus, $\boxed{EF = \frac{ab}{a+b}}$.

Now, $\frac{[ECF]}{[ABC]} = \left(\frac{EF}{AB}\right)^2 = \left(\frac{b}{a+b}\right)^2 \Rightarrow \sqrt{\frac{[ECF]}{[ABC]}} = \frac{b}{a+b}$, similarly $\sqrt{\frac{[EAF]}{[CAD]}} = \frac{a}{a+b}$.

Thus, $\boxed{\sqrt{\frac{[ECF]}{[ABC]}} + \sqrt{\frac{[EAF]}{[CAD]}} = 1.}$ □

6.13 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
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     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth 10 points.
     - A problem has one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.14 Problems for Level 4

Problem 6.14.1 (ICLT-2021-SM2-10-L1-P14). (*4 points*)

Evaluate the following expression,

$$\frac{(12+13)\times(26-23+2\times4-9)}{\left(11-5\times\frac{3}{1}+2\right)\times5^{\left(2-\frac{4}{6-2}\right)}}$$

(A) 25 (B) 5 (C) $\dfrac{1}{5}$ (D) -2 (E) -5

Problem 6.14.2 (ICLT-2021-SM2-10-L2-P42). (*4 points*)

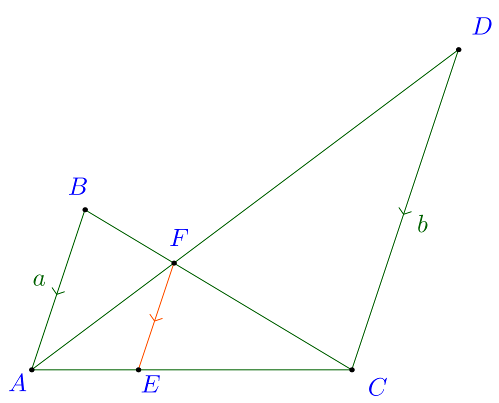
Four girls and three boys form two teams for the championship. Alice and Belice, two of the girls, want to be on the same team. Tinh and Minh, two of the boys, want to be in separate teams.

In how many ways they can form two teams if each team should have at least one boy and one girl.

(A) 3 (B) 4 (C) 6 (D) 8 (E) 12

Problem 6.14.3 (ICLT-2021-SM2-10-L3-P41). (*4 points*)

In the diagram below, $AB \parallel CD \parallel EF$.



Find for EF if $AB = a$ and $CD = b$.

(A) $\dfrac{a}{b}$ (B) $\dfrac{b}{a}$ (C) $\dfrac{a+b}{ab}$ (D) $\dfrac{ab}{a+b}$ (E) $\dfrac{b-a}{ab}$

Problem 6.14.4 (ICLT-2021-SM2-10-L4-P1). (*6 points*)

If $2x > y + z$ and $2y > z + x$, then

(A) $2z > x+y$ (B) $x-z > y-x > z-y$ (C) $x+y+z > 3x$ (D) $x+y+z > 3y$ (E) $\frac{y}{x}+\frac{z}{y}+\frac{z}{x}+\frac{x}{y} > 4$

Problem 6.14.5 (ICLT-2021-SM2-10-L4-P0). (*6 points*)

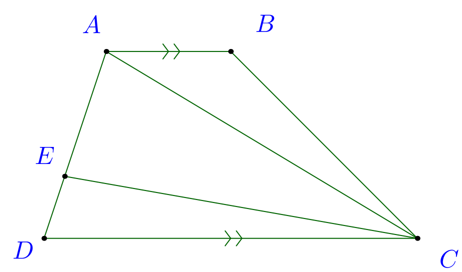
How many three-digit numbers $\overline{abc}$ are there such that

$$\begin{cases} \overline{bc} = 9a + b + c \\ b = 2c + 1 \end{cases}$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 6.14.6 (ICLT-2021-SM2-10-L2-P41). (*6 points*)

$ABCD$ is a trapezoid such that $AB \parallel CD$, $AB = 2$, $CD = 6$. E is on AD such that $3DE = 2EA$.



What is the ratio of $\dfrac{[CDE]}{[ABC]}$?

(A) $\frac{5}{4}$ (B) $\frac{6}{5}$ (C) 1 (D) $\frac{5}{6}$ (E) $\frac{4}{5}$

Problem 6.14.7 (ICLT-2021-SM2-10-L4-P42). (*10 points*)

Given the equation,

$$\frac{1}{a} + \frac{1}{b} = \frac{2020}{ab} - 1$$

1. (*5 points*) Find all solutions if a and b are positive integers.
2. (*5 points*) How many ordered integer, not necessarily positive, pairs (a, b) are solutions to the equation? *Note that for ordered pairs* $(1, 2) \neq (2, 1)$.

Problem 6.14.8 (ICLT-2021-SM2-10-L4-P2). (*10 points*)

Clemence selects 2 rows and 2 columns from the 8×8 chessboard such that at least the 2 rows or the 2 columns are adjacent. For example, she can select row 1, row 3, column 4 and column 5, but she cannot select row 1, row 3, column 4 and column 6.

The intersections of the rows and the columns form a rectangle. For example with row 1, row 3, column 4 and column 5, the rectangle has 3 rows and 2 columns, thus it is a 3×2 rectangle.

1. (*5 points*) How many 3×2 rectangles can Clemence make?
2. (*5 points*) In total how many rectangles can Clemence make?

Problem 6.14.9 (ICLT-2021-SM2-10-L4-P3). (*10 points*)

The length of the segment AB is 10.

1. (*5 points*) Find all point P on the plan such that $\angle APB = 90°$.
2. (*5 points*) Find all point P satisfying the previous questions and $PA^2 - PB^2 = 20$.

6.15 Grading for Level 4

Answers for multiple-choice problems.

Problem 1: E $\quad -5$

Problem 2: E $\quad 12$

Problem 3: D $\quad \frac{ab}{a+b}$

Problem 4: B $\quad x-z>y-x>z-y$

Problem 5: E $\quad 5$

Problem 6: B $\quad \frac{6}{5}$

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately grading for each part,

(a) 2 point if can figure out two cases for a, b.

(b) 3 point if can figure out four ways to factor 2021.

Problem 8: Separately grading for each part,

(a) 2 if found 7 choices to select a pair of neighbouring columns.

(b) 3 if consider complementary counting.

Problem 9: Separately grading for each part,

(a) 2 if found the circle O, diameter AB.

(b) 3 if found D on AB, $DA=6,\ DB=4$.

6.16 Solutions for Level 4

Solution. ICLT-2021-SM2-10-L1-P14

$$(12+13)\times(26-23+2\times 4-9)=25\times(3+8-9)=25\times 2$$
$$11-5\times\frac{3}{1}+2=11-5\times 3+2=11-15+2=-2$$
$$5^{\left(2-\frac{4}{6-2}\right)}=5^{2-\frac{4}{4}}=5^{2-1}=5^1=5$$

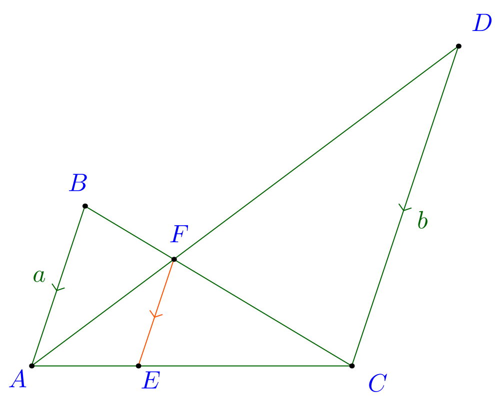
Thus, the given fraction is $\boxed{\dfrac{25\times 2}{(-2)\times 5}=-5.}$ The answer is $\boxed{E.}$ □

Solution. ICLT-2021-SM2-10-L2-P42 If Alice and Belice in one team, then there are 3 ways to choose to put the other two girls into the other team: none of them (1 way), or one of them (2 ways). Both of of the remaining girls cannot join because then the other team has no girl. Alice and Belice team can have Tinh or Minh (2 ways). There are two ways to put the remaining boy. So, all togethere there are $\boxed{3\times 2\times 2=12 \text{ ways.}}$ The answer is $\boxed{E.}$ □

Solution. ICLT-2021-SM2-10-L3-P41 First, $\triangle ECF\sim\triangle ACB$, $\triangle EAF\sim\triangle CAD$, so

$$\frac{EF}{AB}=\frac{EC}{AC},\ \frac{EF}{CD}=\frac{AE}{AC}\Rightarrow\frac{EF}{AB}+\frac{EF}{CD}=1\Rightarrow EF=\frac{AB\cdot CD}{AB+CD}.$$

Thus, $\boxed{EF=\frac{ab}{a+b}.}$ The answer is $\boxed{D.}$ □



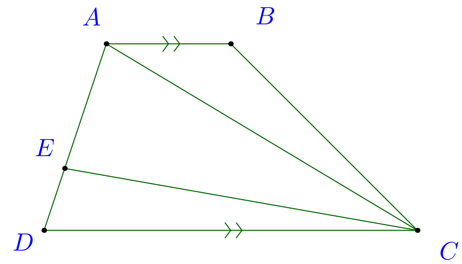
Solution. ICLT-2021-SM2-10-L4-P1 $2x>y+z\Rightarrow 2x-z>z\Rightarrow x+x-z>y\Rightarrow x-z>y-x$. Similarly $2y>z+x$ implies $y-x>z-y$. Therefore $\boxed{x-z>y-x>z-y.}$ The answer is $\boxed{B.}$ □

Solution. ICLT-2021-SM2-10-L4-P0 $\overline{bc}=9a+b+c\Rightarrow 10b+c=9a+b+c\Rightarrow 9b=9a\Rightarrow a=b$. Since $b=2c+1$, so there are $\boxed{5}$ such numbers $110, 331, 552, 773, 994$. The answer is $\boxed{E.}$ □

Solution. ICLT-2021-SM2-10-L2-P41 First, since $AB\parallel CD$, so the distance from C to AB is the same as the distance from A to CD, therefore the altitudes of $\triangle ABC$ and $\triangle ACD$ from C and A, respectively, are the same. Because $CD=3AB$, so $[ACD]=3[ABC]$, or $[ABC]=\frac{1}{3}[ADC]$.

Since $2EA=3ED$, thus $AD=\frac{5}{2}ED$, then $[ADC]=\frac{5}{2}[EDC]$, or $[EDC]=\frac{2[ADC]}{5}$.

Thus, $\boxed{\frac{[ECD]}{[ABC]}=\frac{2[ACD]}{5}\cdot\frac{3}{[ACD]}=\frac{6}{5}.}$ The answer is $\boxed{B.}$ □



Solution. ICLT-2021-SM2-10-L4-P42 For the first question,

$$\frac{1}{a}+\frac{1}{b}=\frac{2020}{ab}-1 \Rightarrow \frac{b+a}{ab}=\frac{2020-ab}{ab} \Rightarrow ab+a+b=2020 \Rightarrow (a+1)(b+1)=2021$$

If a and b are positive integers, $2021 = 43 \cdot 47$, thus $a+1 = 43, b+1 = 47$, or $a+1 = 74,\ b+1 = 43$. So $\boxed{(a,b) \in \{(42,46), 46, 42).}$

Now, if $a+1$ and $b+1$ are integers, then $2021 = 1 \cdot 2021 = (-1) \cdot (-2021) = 43 \cdot 47 = (-43) \cdot (-47)$. Note that $a \neq 0$ and $b \neq 0$, thus only three cases of $(-1) \cdot (-2021)$, $43 \cdot 47$, $(-43) \cdot (-47)$ can be considered. There are $(-2022, -2), (-48, -44), (42, 46)$ and the permutations of the two numbers in each pair.

Therefore, there are $\boxed{3 \cdot 2 = 6}$ such pairs of (a,b). □

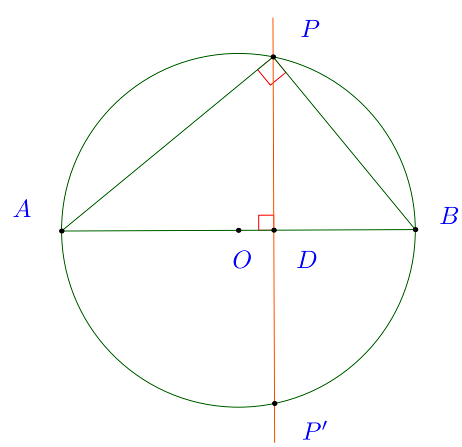
Solution. ICLT-2021-SM2-10-L4-P2 For the first question, it is easy to see that Clemence has 7 choices to select a pair of neighbouring columns. There are 6 ways to select a block of 3 rows $1-3, 2-4, \ldots, 6-8$. So in total, there are $\boxed{7 \cdot 6 = 42.}$

Now, for a pair of neighbouring columns out of 7, there are $\binom{8}{2}$ ways to chose a pair of rows. Similarly with 7 pairs of neighbouring rows. The cases with 2 neighbouring columns and 2 neighbouring rows are counted twice, so the total amount is $2 \cdot 7 \cdot \binom{8}{2} - 7 \cdot 7 = 343$. Thus, the number of desired rectangles is $\boxed{343.}$ □

Solution. ICLT-2021-SM2-10-L4-P3 First, $\angle APB = 90$ means that P is on $\boxed{\text{the circle } O\text{, diameter } AB.}$

For the second question, let D be the foot of the altitude from D to AB, then

$$PA^2 = PD^2 + DA^2,\ PB^2 = PD^2 + DB^2 \Rightarrow DA^2 - DB^2 = 20$$
$$\Rightarrow (DA+DB)(DA-DB) = 20 \Rightarrow DA - DB = 2 \Rightarrow DA = 6,\ DB = 4$$



Therefore, the positions of P are $\boxed{\text{the two intersections}}$ of the perpendicular line through D, where $DA = 6,\ DB = 4$, with the circle O diameter AB. □

6.17 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice,you 4 or 6 points, based on the number of points associated to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth 10 points.
     - A problem has one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

6.18 Problems for Level 5

Problem 6.18.1 (ICLT-2021-SM2-10-L1-P15). (*4 points*)

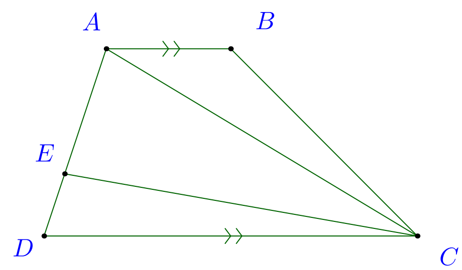
Evaluate the following expression,

$$\frac{(12+13)\times(26-23+2\times4-9)}{\left(11-5\times\frac{3}{1}+3\right)\times5^{\left(3-\frac{4}{6-2}\right)}}$$

(A) 25 (B) 5 (C) $\frac{1}{5}$ (D) -2 (E) -5

Problem 6.18.2 (ICLT-2021-SM2-10-L2-P51). (*4 points*)

$ABCD$ is a trapezoid such that $AB \parallel CD$, $AB=2$, $CD=6$. E is on AD such that $5DE=3EA$.



What is the ratio of $\frac{[ABC]}{[CDE]}$?

(A) $\frac{7}{6}$ (B) $\frac{4}{5}$ (C) 1 (D) $\frac{8}{9}$ (E) $\frac{5}{6}$

Problem 6.18.3 (ICLT-2021-SM2-10-L3-P52). (*4 points*)

n is a positive integer. Lam and Lan take turns playing a two-player game. In each turn, the player of that turn write a divisor of 30^n, which could include 1 and 30^n, on the table if it is not yet written. The player who cannot make a move will lose and the game ends.

The game ends after m turns, where $50<m<100$, what is n?

(A) 8 (B) 4 (C) 3 (D) 2 (E) 12

Problem 6.18.4 (ICLT-2021-SM2-10-L4-P51). (*6 points*)

What is the probability that when we roll 3 fair 6−sided dice, they won't show any 5 or 6?

$(A)\ \frac{215}{216}$ $(B)\ \frac{2}{3}$ $(C)\ \frac{1}{3}$ $(D)\ \frac{53}{54}$ $(E)\ \frac{26}{27}$

Problem 6.18.5 (ICLT-2021-SM2-10-L1-P35). (*6 points*) Antoine has 11 marbles. He wants to divide the marbles into some piles such that the number of marbles in each pile is a prime number. (It is possible that two piles contain the same number of marbles.) *For example he can divide them into three piles: one pile of* 3 *marbles, one pile of* 3 *marbles, and one pile of* 5 *marbles.*

In how many ways can he do it?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 6.18.6 (ICLT-2021-SM2-10-L4-P52). (*6 points*)

How many ordered integer, not necessarily positive, pairs (a, b) are solutions to the equation?

$$\frac{1}{a} + \frac{1}{b} = \frac{2020}{ab} - 1$$

Note that for ordered pairs $(1, 2) \neq (2, 1)$.

(A) 2 (B) 4 (C) 5 (D) 6 (E) 8

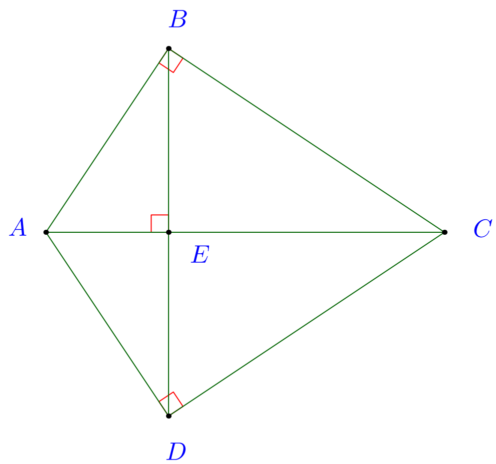
Problem 6.18.7 (ICLT-2021-SM2-10-L5-P4). (*10 points*)

Consider the positive integers written in based 3 with 4 digits (the leading digit is not 0.) For example $\overline{1201}_3$ is such a number, but $\overline{101}_3$ and $\overline{12001}_3$ are not.

1. (*5 points*) How many such numbers are there?
2. (*5 points*) What is the sum of these numbers?

Problem 6.18.8 (ICLT-2021-SM2-10-L5-P3). (*10 points*)

In the quadrilateral $ABCD$, diagonals AC and BD are perpendicular $AC \perp BD$. E is the intersection of AC and BD. $\angle ABC = \angle CDA = 90°$.



1. (*5 points*) Prove that $BE^2 = AE \cdot EC$.
2. (*5 points*) Prove that $AB \cdot CD + AD \cdot BC = BD \cdot AC$.

Problem 6.18.9 (ICLT-2021-SM2-10-L5-P1). (*10 points*)

Let a be a positive real number. Let r and s be the roots of the equation $ax^2 - (a^2+1)x + a = 0$. The equation $x^2 - px + q = 0$ has two roots, $r + \frac{1}{s}$, and $s + \frac{1}{r}$.

1. (*5 points*) Find q.
2. (*5 points*) Prove that $p \geq 4$.

6.19 Grading for Level 5

Answers for multiple-choice problems.

Problem 1: D -2

Problem 2: D $\frac{8}{9}$

Problem 3: C 3

Problem 4: E $\frac{26}{27}$

Problem 5: C 5

Problem 6: D 6

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately grading for each part,

(a) 2 point if can figure out the possible digits for each position is $0, 1, 2$, excep the leading one.

(b) 3 point if can determine the smallest and highest numbers.

Problem 8: Separately grading for each part,

(a) 2 if found pair of similar triangles $\triangle AEB \sim \triangle BEC$.

(b) 3 if found pair of similar triangles $\triangle AEB \sim \triangle ABC$.

Problem 9: Separately grading for each part,

(a) 2 if use Vieta's theorem for the sum and product of the roots.

(b) 3 if can express p based on a only.

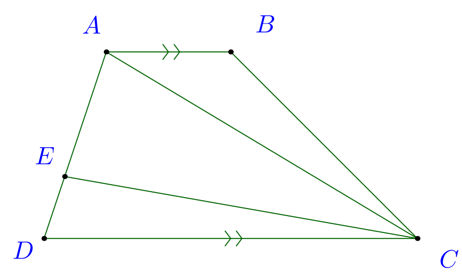
6.20 Solutions for Level 5

Solution. ICLT-2021-SM2-10-L1-P15

$$(12+13)\times(26-23+2\times4-9) = 25\times(3+8-9) = 25\times 2$$
$$11-5\times\frac{3}{1}+3 = 11-5\times3+3 = 11-15+3 = -1$$
$$5^{\left(3-\frac{4}{6-2}\right)} = 5^{3-\frac{4}{4}} = 5^{3-1} = 5^2 = 25$$

Thus, the given fraction is $\boxed{\dfrac{25\times 2}{(-1)\times 25} = -2.}$ The answer is $\boxed{D.}$ □

Solution. ICLT-2021-SM2-10-L2-P51 First, since $AB \parallel CD$, so the distance from C to AB is the same as the distance from A to CD, therefore the altitudes of $\triangle ABC$ and $\triangle ACD$ from C and A, respectively, are the same. Because $CD = 3AB$, so $[ACD] = 3[ABC]$, or $[ABC] = \frac{1}{3}[ADC]$.



Since $3EA = 5ED$, thus $AD = \frac{8}{3}ED$, then $[ADC] = \frac{8}{3}[EDC]$, or $[EDC] = \frac{3[ADC]}{8}$.

Thus, $\boxed{\frac{[ABC]}{[ECD]} = \frac{[ACD]}{3}\cdot\frac{8}{3[ACD]} = \frac{8}{9}.}$ The answer is $\boxed{D.}$ □

Solution. ICLT-2021-SM2-10-L3-P52 The numbers that the players taking turns to write on the table are the divisors of 30^n. The game ends when they run out of the divisors. Since

$$30^n = 2^n3^n5^n \quad \text{the number of divisors of } 30^n \textit{ is } (n+1)(n+1)(n+1) = (n+1)^3.$$

So the game always end after exactly $\boxed{(n+1)^3}$ turns. If $(n+1)^3$ is between 50 and 100, then $n+1=4$, so $\boxed{n=3.}$ The answer is $\boxed{C.}$ □

Solution. ICLT-2021-SM2-10-L4-P51 There are 8 cases if 3 dice show only 5 or 6, from 555 to 666. Thus the probability that none of the dice shows 5 or 6 is $\boxed{1-\frac{8}{216}=\frac{26}{27}.}$ The answer is $\boxed{E.}$ □

Solution. ICLT-2021-SM2-10-L1-P35 We do casework.

Case 1: if there is no pile with 2 or 3 marbles, then $5+5=10$ and $5+7=12$. So there is no solution.

Case 2: if there is no pile with 2 and one pile with 3 marble, then there are 8 marbles left. The only possible division for piles with at least 3 marbles is $3+5=8$. We got $11=3+3+5$.

Case 3: if there is one pile with 2, then there are 9 marbles left.

Subcase 3a: if there is no other pile with 2 marbles, then there is at least one pile with 3 marbles, otherwise $5+5=10>9$. In this case 6 marbles left, it can be divided into $3+3=6$. We got $11=2+3+3+3$.

Subcase 3b: if there another pile with 2 marbles, then there are 7 marbles left, it can be divided as follow,

$$7=2+2+3=2+5=7 \Rightarrow 11=2+2+2+2+3=2+2+2+5=2+2+7$$

There are $\boxed{5}$ ways

$$11=2+2+2+2+3=2+2+2+5=2+2+7=2+3+3+3=3+3+5.$$

The answer is $\boxed{C.}$ □

Solution. ICLT-2021-SM2-10-L4-P52

$$\frac{1}{a}+\frac{1}{b}=\frac{2020}{ab}-1 \Rightarrow \frac{b+a}{ab}=\frac{2020-ab}{ab} \Rightarrow ab+a+b=2020 \Rightarrow (a+1)(b+1)=2021$$

Now, if $a+1$ and $b+1$ are integers, then $2021=1\cdot 2021=(-1)\cdot(-2021)=43\cdot 47=(-43)\cdot(-47)$. Note that $a\neq 0$ and $b\neq 0$, thus only three cases of $(-1)\cdot(-2021)$, $43\cdot 47$, $(-43)\cdot(-47)$ can be considered. There are $(-2022,-2),(-48,-44),(42,46)$ and the permutations of the two numbers in each pair. Therefore, there are $\boxed{3\cdot 2=6}$ such pairs of (a,b).

The answer is $\boxed{D.}$ □

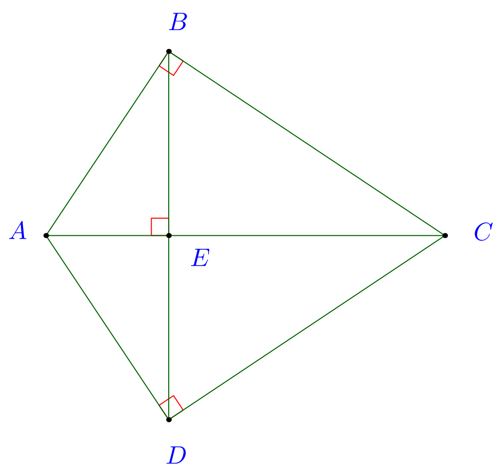
Solution. ICLT-2021-SM2-10-L5-P4 Since an $4-$ digit number in base 3 should have the leading digit as 1 or 2 and any digit $0,1,2$ for the remaining 3 digits, thus there are $\boxed{2\cdot 3^3=54}$ such numbers.

The sum of these digits is basicaly

$$1000_3+1001_3+\ldots+2222_3=3^3+\ldots+(3^4-1)=27+\ldots+80=2889.$$

Thus, the sum of those number is $\boxed{2889.}$ □

Solution. ICLT-2021-SM2-10-L5-P3 For the first question, $\triangle AEB \sim \triangle BEC$ since both are right triangles $\angle EAB = 90^\circ - \angle ABE = \angle EBC$. Thus, $\frac{AE}{EB} = \frac{BE}{EC}$, so $\boxed{BE^2 = AE \cdot EC.}$



Now, it is easy to see that $DE = BE$.

$$\triangle AEB \sim \triangle ABC \Rightarrow \frac{AB}{BE} = \frac{AC}{CB} \Rightarrow AB \cdot CB = BE \cdot AC$$
$$\Rightarrow AB \cdot CD + AD \cdot BC = 2AB \cdot CB = 2BE \cdot AC = BD \cdot AC$$

Therefore, the sum of those number is $\boxed{AB \cdot CD + AD \cdot BC = BD \cdot AC}$ □

Solution. ICLT-2021-SM2-10-L5-P1 First, since r and s are the roots of $ax^2 - (a^2+1)x + a = 0$, thus

$$\begin{cases} r + s = \frac{a^2+1}{a} \\ rs = \frac{a}{a} = 1 \end{cases}$$

For the first question,

$$q = \left(r + \frac{1}{s}\right)\left(s + \frac{1}{r}\right) = rs + \frac{1}{rs} + 2 = 4.$$

Therefore, $\boxed{q = 4.}$

For the second question, since $r + \frac{1}{s}$ and $s + \frac{1}{r}$ are the roots of $x^2 - px + q$, so

$$p = \left(r + \frac{1}{s} + s + \frac{1}{r}\right) = (r+s) + \frac{r+s}{rs} = 2(r+s) = \frac{2(a^2+1)}{a}$$
$$\Rightarrow p - 4 = \frac{2(a^2+1)}{a} - 4 = \frac{2(a^2-2a+1)}{a} = \frac{2(a-1)^2}{a} \geq 0.$$

Thus $\boxed{p \geq 4.}$ □

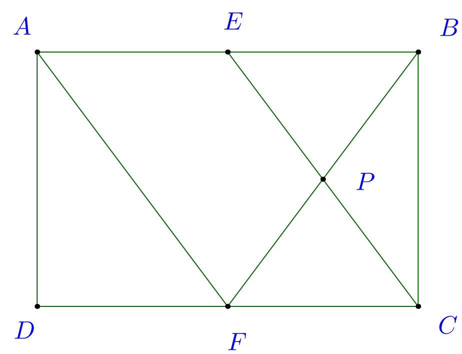
6.21 Reserved

Problem 6.21.1 (ICLT-2021-SM2-10-L2-P2)**.** (*5 points*)

In $ABCD$ rectangle, E is the midpoint of AB and F is the midpoint of CD. BF intersects CE at P. Find the ratio of the areas of $\frac{[AEPF]}{[ABCD]}$.

(A) $\dfrac{1}{2}$ (B) $\dfrac{3}{5}$ (C) $\dfrac{4}{7}$ (D) $\dfrac{5}{11}$ (E) $\dfrac{3}{8}$

Solution. ICLT-2021-SM2-10-L2-P2 It is easy to see that $[AEFD] = \frac{1}{2}[ABCD]$, so $[AEF] = \frac{1}{4}[ABCD]$. Furtheremore $[EPF] = \frac{1}{4}[EBDF] = \frac{1}{8}[ABCD]$.



Thus, $\boxed{[AEPF] = \frac{3}{8}[ABCD].}$ The answer is $\boxed{E.}$ □

Problem 6.21.2 (ICLT-2021-SM2-10-L1-P51). (*4 points*)

A dog has about 319 to 321 bones. A dinosaur has approximately 200 bones. In an excavation, the team found 1039 bones of dogs and dinosaurs.

How many dinosaurs did the team find?

(*A*) 2 (*B*) 3 (*C*) 4 (*D*) 5 (*E*) 6

Solution. ICLT-2021-SM2-10-L1-P51 Since 1039 can only be $3 \times 200 + 319 + 320$, so there are $\boxed{3}$ dinosaurs. The answer is $\boxed{B.}$ □

Problem 6.21.3 (ICLT-2021-SM2-10-L1-P92). (*5 points*)

How many distinct 3−letter words can be formed by selecting different letters from the word $OTTAWA$?

For example three letters A, T, and O are different. Three letters A, O, and A are not. Three letters A, T, and O form the 3−*letter word ATO. They also form the* 3−*letter word OTA.*

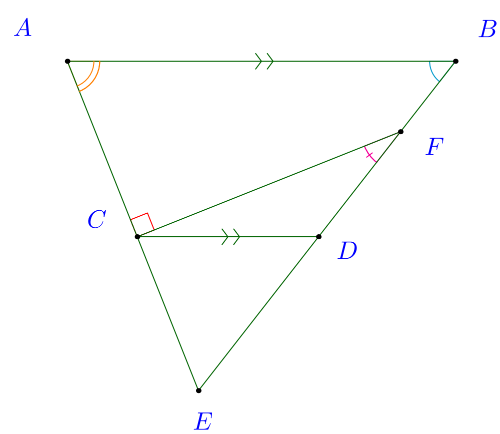
(*A*) 18 (*B*) 24 (*C*) 6 (*D*) 12 (*E*) 36

Solution. ICLT-2021-SM2-10-L1-P92 Since there are 4 different letters A, O, T, and W, so there are 4 ways to select a triple of 3 different letters. (Each selection keeps one of the four letters A, O, T, and W out of the triple.) For each triples, there are $3! = 6$ ways to permute the letters to form a new 3−letter word.

Thus, there are $\boxed{4 \times 6 = 24.}$ The answer is $\boxed{B.}$ □

Problem 6.21.4 (ICLT-2021-SM2-10-L1-P42). (*5 points*)

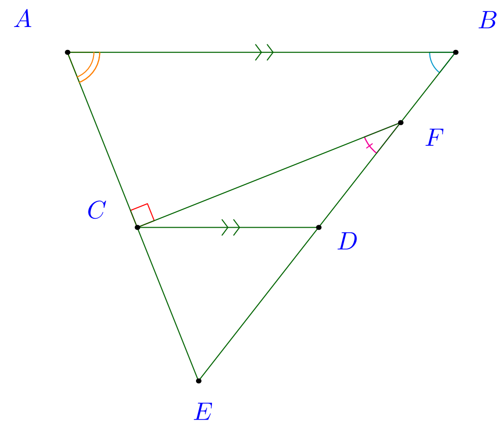
In the diagram below, AB and CD are parallel, FC is perpendicular to AE.



If $\angle ABF = 54°$ and $\angle CAB = 60°$, what is $\angle CFD$?

(*A*) 22° (*B*) 24° (*C*) 26° (*D*) 32° (*E*) 36°

Solution. ICLT-2021-SM2-10-L1-P42 First, $\angle CFD = 180^\circ - \angle CDF - \angle DCF$.



Since $180^\circ - \angle CDF = \angle ABF$, $\angle DCF = \angle DCA - 90^\circ = (180^\circ - \angle CAB) - 90^\circ = 90^\circ - \angle CAB$. So $\angle CFD = \angle ABF + \angle CAB - 90^\circ$. Thus, $\boxed{\angle CFD = 54^\circ + 60^\circ - 90^\circ = 24^\circ.}$ The answer is $\boxed{B.}$ □

Problem 6.21.5 (ICLT-2021-SM2-10-L1-P82)**.** (*5 points*)

During practice hours, the teacher tried to divide the students in the class into two or more groups, each group has at least two members, and every group has the same number of students.

What was the least number of students are in the class if she could divide the students in exactly 4 ways?

For example if there were 6 *students, she could divide the students into* 2 *groups of* 3 *students or* 3 *groups of* 2 *students. In total, there were exactly* 2 *ways for her to divide the students into groups as required.*

(A) 6 (B) 8 (C) 9 (D) 12 (E) 13

Solution. ICLT-2021-SM2-10-L1-P82 In order to divide the students into 4 ways, their number should has exactly 4 divisors not including 1 and itself. In other words, it is a number that has exactly 6 divisors. The smallest such number is 12, so there are $\boxed{12}$ students. The answer is $\boxed{D.}$ □

Problem 6.21.6 (ICLT-2021-SM2-10-L2-P22)**.** (*10 points*)

Four girls and three boys form two teams for the championship. Alice and Belice, two of the girls, want to be on the same team. Tinh and Minh, two of the boys, want to be in separate teams.

Note: the two questions below should be consider independently.

In how many ways they can form two teams if

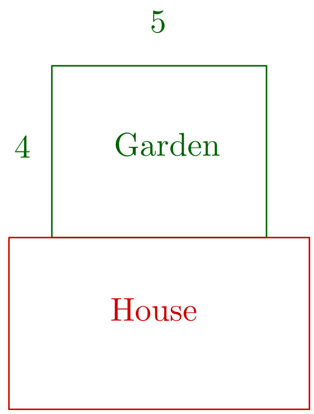
1. (*5 points*) Each team should have at least one boy and one girl, and no team have more than two girls.
2. (*5 points*) Each team should have at least one boy and one girl.

Solution. ICLT-2021-SM2-10-L2-P22 For the first question, if Alice and Belice in one team, then the other team should have the other two girls. Alice and Belice team can have Tinh or Minh (2 ways). There are two ways to put the remaining boy. So all togethere there are $\boxed{2 \times 2 = 4 \text{ ways.}}$

For the second question, if Alice and Belice in one team, then there are 3 ways to choose to put the other two girls into the other team: none of them (1 way), or one of them (2 ways). Both of of the remaining girls cannot join because then the other team has no girl. Alice and Belice team can have Tinh or Minh (2 ways). There are 2 ways to put the remaining boy. So all togethere there are $\boxed{3 \times 2 \times 2 = 12 \text{ ways.}}$ □

Problem 6.21.7 (ICLT-2021-SM2-10-L1-P23)**.** (*5 points*)

Anna has a $4m \times 5m$ rectangular garden. One long side of it is a part of the wall of her house, as show below. She wants to plant trees on the perimeter of the garden so that the distance (along the perimeter) between any two trees is at least $1.5m$.



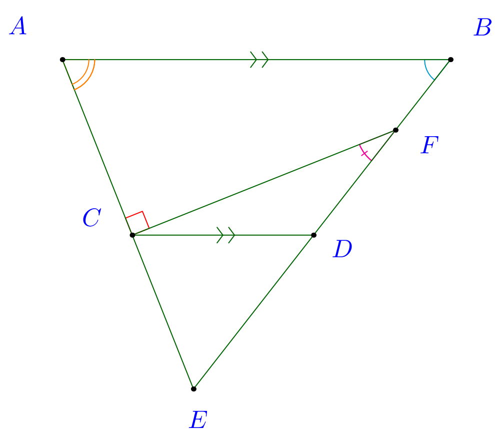
At most how many trees can Anna plant? *She is not allowed to plant any tree on the wall of her house.*

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Solution. ICLT-2021-SM2-10-L1-P23 The total length of the perimeter of the garden where Anna can plant her trees is $2 \cdot 4 + 1 \cdot 5 = 13$m. Because no tree can be planted on the wall so the total distance to plant tree is *less* than 13m. Since the distance between any two tree is at least 1.5m, at most she can plant $\boxed{9}$ of them within a $(9 - 1) \times 1.5 = 12$m. The answer is $\boxed{C.}$ □

Problem 6.21.8 (ICLT-2021-SM2-10-L1-P43)**.** (*5 points*)

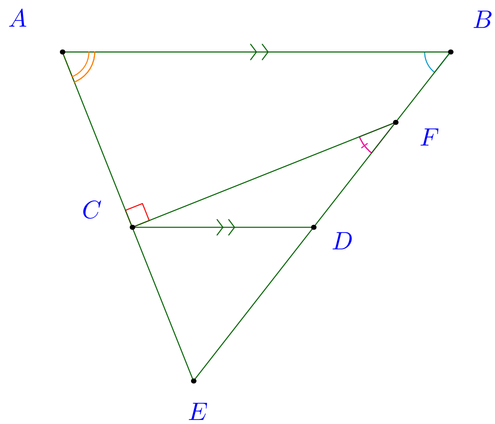
In the diagram below, AB and CD are parallel, FC is perpendicular to AE.



If $\angle ABF = 42°$ and $\angle CAB = 60°$, what is $\angle CFD$?

(A) $22°$ (B) $24°$ (C) $12°$ (D) $18°$ (E) $26°$

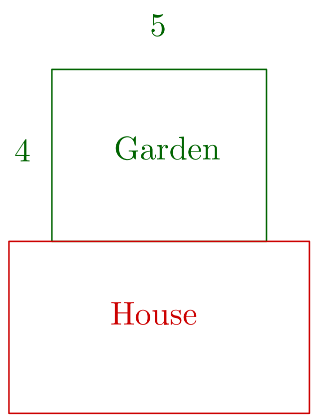
Solution. ICLT-2021-SM2-10-L1-P43 First, $\angle CFD = 180^\circ - \angle CDF - \angle DCF$. Since $180^\circ - \angle CDF = \angle ABF$, $\angle DCF = \angle DCA - 90^\circ = (180^\circ - \angle CAB) - 90^\circ = 90^\circ - \angle CAB$, so $\angle CFD = \angle ABF + \angle CAB - 90^\circ$.



Thus, $\boxed{\angle CFD = 42^\circ + 60^\circ - 90^\circ = 12^\circ.}$ The answer is $\boxed{C.}$ ☐

Problem 6.21.9 (ICLT-2021-SM2-10-L1-P24). (*5 points*)

Anna has a $4m \times 5m$ rectangular garden. One long side of it is a part of the wall of her house, as show below. She wants to plant trees on the perimeter of the garden so that the distance (along the perimeter) between any two trees is at least 1.3m.



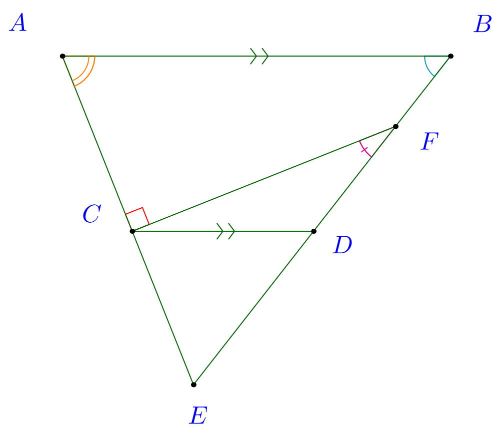
At most how many trees can Anna plant? *She is not allowed to plant any tree on the wall of her house.*

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Solution. ICLT-2021-SM2-10-L1-P24 The total length of the perimeter of the garden where Anna can plant her trees is $2 \cdot 4 + 1 \cdot 5 = 13$m. Because no tree can be planted on the wall so the total distance to plant tree is *less* than 13m. Since the distance between any two tree is at least 1.3m, at most she can plant $\boxed{10}$ of them within $(10 - 1) \times 1.3 = 11.7$m distance The answer is $\boxed{D.}$ ☐

Problem 6.21.10 (ICLT-2021-SM2-10-L1-P44). (*5 points*)

In the diagram below, AB and CD are parallel, FC is perpendicular to AE.



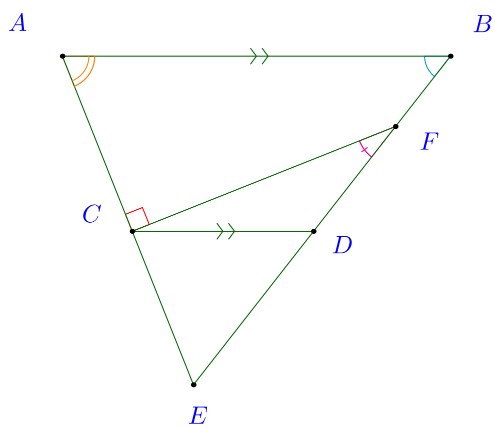
If $\angle ABF = 52°$ and $\angle CAB = 54°$, what is $\angle CFD$?

(A) 22° (B) 24° (C) 20° (D) 16° (E) 18°

Solution. ICLT-2021-SM2-10-L1-P44 First, $\angle CFD = 180° - \angle CDF - \angle DCF$. Since

$$180° - \angle CDF = \angle ABF, \ \angle DCF = \angle DCA - 90° = (180° - \angle CAB) - 90° = 90° - \angle CAB$$

So $\angle CFD = \angle ABF + \angle CAB - 90°$.



Thus, $\boxed{\angle CFD = 52° + 54° - 90° = 16°.}$ The answer is $\boxed{D.}$ □

Problem 6.21.11 (ICLT-2021-SM2-10-L3-P42)**.** (*5 points*)

Lam and Lan take turns playing a two-player game. In each turn, the player of that turn write a divisor of 30^3, which could include 1 and 30^3, on the table if it is not yet written. The player who cannot make a move will lose and the game ends.

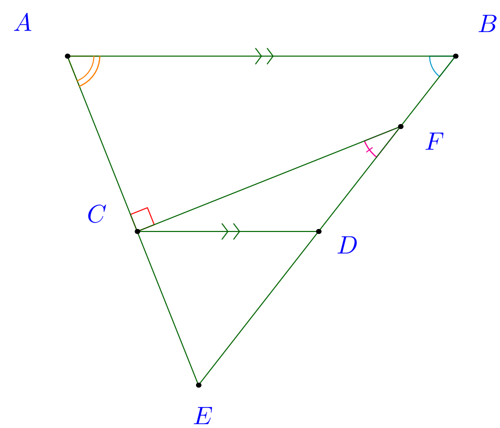
The game ends after m turns, where $7 < m < 100$, what is n?

(A) 8 $\quad$ (B) 16 $\quad$ (C) 25 $\quad$ (D) 27 $\quad$ (E) 64

Solution. ICLT-2021-SM2-10-L3-P42 The numbers that the players taking turns to write on the table are the divisors of 30^3. The game ends when they run out of the divisors. Since $30^3 = 2^3 3^3 5^3$, the number of divisors of 30^3 $(3+1)(3+1)(3+1) = 4^3 = 64$. So the game always end after exactly $\boxed{64}$ turns. The answer is $\boxed{E.}$ ☐

Problem 6.21.12 (ICLT-2021-SM2-10-L1-P45)**.** (*5 points*)

In the diagram below, AB and CD are parallel, FC is perpendicular to AE.



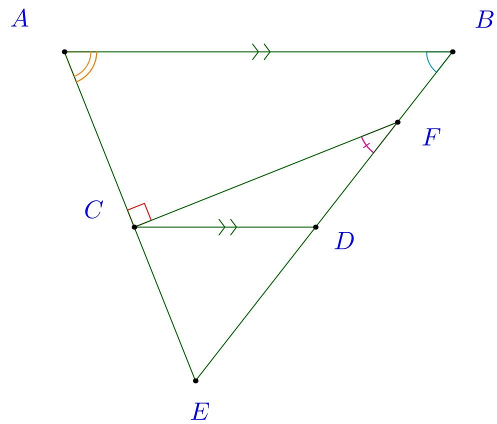
If $\angle ABF = 52°$ and $\angle CAB = 59°$, what is $\angle CFD$?

(A) 11° $\quad$ (B) 19° $\quad$ (C) 21° $\quad$ (D) 29° $\quad$ (E) 36°

Solution. ICLT-2021-SM2-10-L1-P45 First, $\angle CFD = 180° - \angle CDF - \angle DCF$. Since

$$180° - \angle CDF = \angle ABF,\ \angle DCF = \angle DCA - 90° = (180° - \angle CAB) - 90° = 90° - \angle CAB$$

So $\angle CFD = \angle ABF + \angle CAB - 90°$.



Thus, $\boxed{\angle CFD = 52° + 59° - 90° = 21°.}$ The answer is $\boxed{C.}$ □

Chapter 7

HC R3

7.1 Topics

Coding

1. Express sums in multiple ways.
2. Finding power of 2.

Algebra

1. Inequality. Comparison method. Estimation for upper bounds and lower bounds of sums of numbers.
2. Speed-Time-Distance. Rate.

Combinatorics

1. Colourings. Casework. Alternate colouring. Least number of colours to be used.
2. Counting in two ways. Sums that can be expressed in multiple ways.
3. Magic squares.
4. Sets. Sum of numbers in subsets.
5. Combinatorial Geometry. Division of shapes.
6. Games. Winning positions.
7. Algorithms and Processes. Invariant.
8. Grids. Tiling.

Geometry

1. Lines and shapes. Intersections.

Logic

1. Casework: what if A is true, what if A is false.
2. Process of elimination: If A is not true, B is not true, then C should be true.
3. Reverse argument: if there is at least ..., then there is atmost ...
4. Implication from truth: if A told the truth and A said X, then X is true.
5. Conflict of truth: if B said A lied, then both cannot be truth tellers.

Number Theory

1. Divisibility.
2. Diophantine equation. Power of 2.

7.2 Problems

Problem 7.2.1 (HC-2021-SM2-R3-P1). (*Beginner Level*)

Quan has a $3 \times 3 \times 3$ cube. Each face is divided into 9 squares 1×1. Any two squares that *share a*

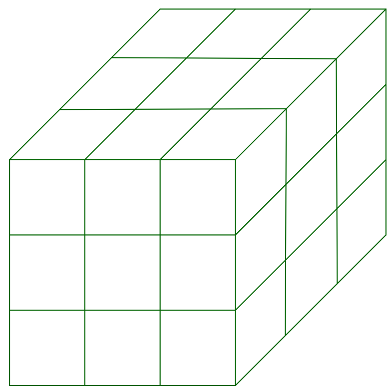


Figure 7.1: HC-2021-SM2-R3-P1

common side are called *neighboring squares*, regardless whether they are on the same face or not. Quan wants to colour the squares such that *any two neighboring squares have different colours.*

How many colours does he need?

How to provide your answer:

- If you think that Quan needs 9 colours, then submit 9 and a diagram depicting the colouring with 9 colours. Figure 7.2 shows the top, front, and right faces, other faces cannot be seen. Figure 7.3 shows the left, bottom, and back faces as if the top, front, and right faces were transparent.
- If you cannot determine that, submit 0 and give a detailed reasoning.

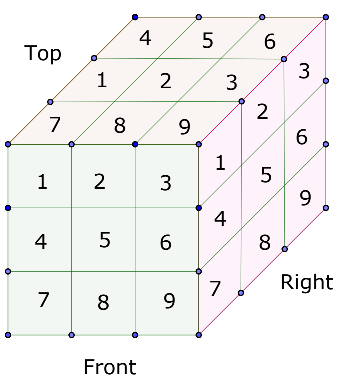


Figure 7.2: Top, front, right

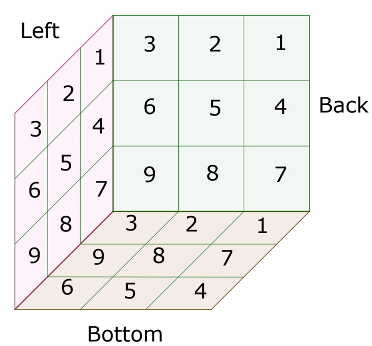


Figure 7.3: Bottom, back, left

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly larger* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is a *strictly smaller* than your answer.

Problem 7.2.2 (HC-2021-SM2-R3-P2)**.** (*Beginner Level*)

Whenever Rob the bandit is asleep, everything he believes is wrong. In other words, everything Rob believes in his sleep is false. On the other hand, everything he believes while he is awake is true. Last night at 10 o'clock sharp, while guarding the newly captured treasure chest, Rob believed that he and Sam the thug were asleep at that time.

Was Sam asleep or awake at that time? How about Rob?

How to provide your answer:

- If you think that Sam is asleep as well as Rob, submit SS.
- If you think that Sam is awake and Rob is also awake, submit WW.
- If you cannot determine that, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.2.3 (HC-2021-SM2-R3-P3)**.** (*Beginner Level*)

Below is a map of the Fortress of the Caribbean Pirates. The 8 circles are heavily armoured towers where the treasures are stored. The towers are connected by 12 roads $a, b, \ldots, \ell$, see Figure 7.4. Each tower is guarded by all pirates patrolling on all three roads connected to that tower. There are 12 groups of pirates, containing $1, 2, 3, \ldots, 11$, and 12 pirates, with each group patrolling exactly one road of $a, b, \ldots, \ell$.

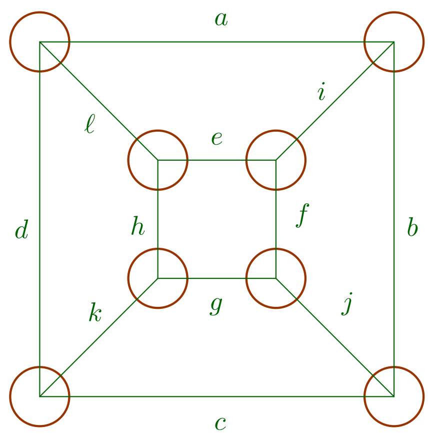


Figure 7.4: HC-2021-SM2-R3-P3

Can these 12 groups be placed on the roads so that every tower would be guarded by the same number of pirates? If yes, then what are the numbers of pirates on roads a, b, c, and d?

How to provide your answer:

- If you think that roads a, b, c, and d are patrolled by the groups having 9, 10, 11, and 12 pirates, submit the list 9, 10, 11, and 12.
- If you think that it cannot be determine, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.2.4 (HC-2021-SM2-R3-P4). (*Beginner Level*)

Tung has a balance scale and nine weights with the respective values 1kg, 2kg, ..., 9kg.

In how many ways can he place six weights in the left-hand side and three weights in the right-hand side of the scale such that the right side is heavier than the left one?

Note that selecting the weights 1*kg,* 5*kg, and* 7*kg is the same as selecting the weights* 5*kg,* 1*kg, and* 7*kg.*

How to provide your answer:

- If you think that the weights on the right-hand side can be $\{1, 2, 3\}$, $\{2, 4, 5\}$, $\{2, 6, 8\}$, or $\{1, 5, 9\}$, then submit the set $\{123, 245, 268, 159.\}$
- If you cannot determine that, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that has *strictly smaller number of elements* than the set of your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has *strictly larger number of elements* than the set of your answer.

Problem 7.2.5 (HC-2021-SM2-R3-P5). (*Beginner Level*)

The rectangle on the left contains three identical rectangles painted in three different colours. Can you cut it into exactly 4 pieces and reassemble them into the rectangle on the right?

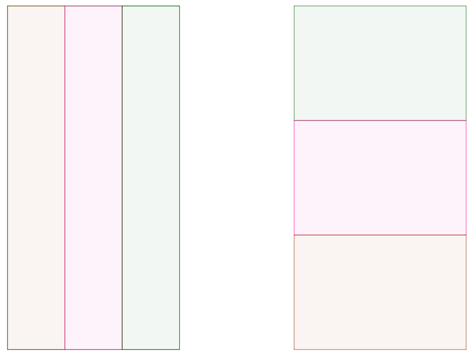


Figure 7.5: HC-2021-SM2-R3-P5

How to provide your answer:

- If you can do it, submit a diagram showing your cuts and reasssembling.
- If you cannot determine that, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.2.6 (HC-2021-SM2-R3-P6)**.** (*Intermediate Level*)

Can you fill the regions of the map below using the *least number* of colours? Regions with the same colour can meet at a point but they cannot share any border.

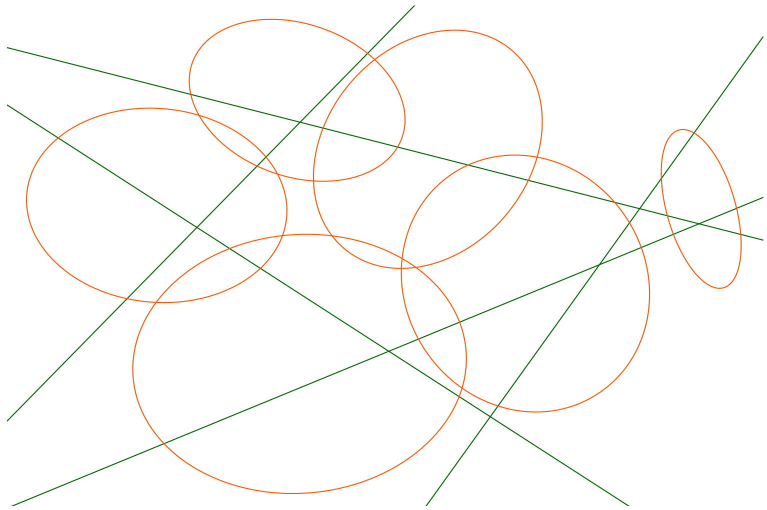


Figure 7.6: HC-2021-SM2-R3-P6

How to provide your answer:

- If you think that you needs 8 colours, then submit 8 and a diagram depicting your way of colouring as in Figure 7.7 . *You can put a number on a region to indicate the colour instead colouring the region.*
- If you cannot determine that, submit 0 and give a detailed reasoning.

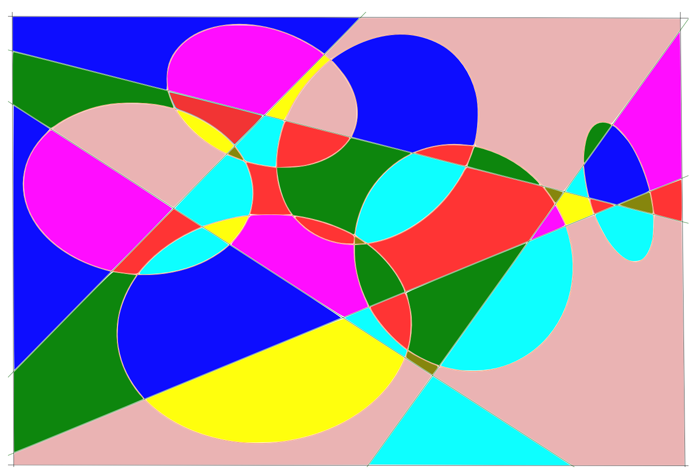


Figure 7.7: 8 colours

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly larger* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is a *strictly smaller* than your answer.

Problem 7.2.7 (HC-2021-SM2-R3-P7)**.** (*Intermediate Level*)

There are two different people on the Island of Knights and Liars: the Knights, who always tell the truth, and the Liars, who always lie.

Agent 007 was sent to the Island on a secret mission. Things gone wrong and he was arrested in a company of a Knight and a Liar. The police only knew them as Bale, Dale, and Gale. At the trial, they made the following statements in this order,

- Bale: I am 007.
- Dale: That is true.
- Gale: I am not 007.

What are they?

How to provide your answer:

- If you think that Bale is 007, Dale is the Knight, and Gale is the Liar, submit $7KL$.
- If you think that Bale is the Knight, Dale is the Liar, and Gale is 007, submit $KL7$.
- If you cannot determine that, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.2.8 (HC-2021-SM2-R3-P8)**.** (*Intermediate Level*)

A magic square is a square grid filled with numbers, where the numbers in *each column*, *each row*, and *each diagonal* all add up to *the same number*. This number is called *the magic constant* of the square. Below Figure 7.8 shows an example of a 3×3 magic square filled with numbers $1, 2, \ldots, 9$, with 15 as the magic constant.

4	9	2
3	5	7
8	1	6

Figure 7.8: HC-2021-SM2-R3-P8

Chi created a 4×4 magic square with the numbers $1, 2, \ldots, 16$, and jotted down by pencil. When she went for a swim, her friend erased 11 numbers. Figure 7.9 shows what Chi saw when she got back.

4	15		5
		8	
7			

Figure 7.9: 11 missing numbers

1	2		6
4	15	9	5
	11	8	
7			16

Figure 7.10: Restored magic square

Restore as many missing numbers in the magic square as you could.

How to provide your answer:

- Figure 7.10 above shows an example for submission of an incomplete restoration of the magic squares, with 6 numbers $1, 2, 6, 9, 11$, and 16 are filled in.
- If you cannot determine that, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or an answer that has *strictly smaller amount of correctly restored numbers* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has *strictly larger amount of correctly restored numbers* than your answer.

Problem 7.2.9 (HC-2021-SM2-R3-P9)**.** (*Intermediate Level*)

For a round of the MCC House Championship, 15 problems are given to 15 students. For each problem there is exactly one of the students who knows the solution. Every time student A phones student B, student A tells student B everything he knows, while student B tells student A nothing.

What is *the minimum* of phone calls between pairs of students needed for everyone to know everything?

How to provide your answer:

- If you think that the number of phone calls is 300, submit 300.
- If you cannot determine that, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly larger* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is a *strictly smaller* than your answer.

Problem 7.2.10 (HC-2021-SM2-R3-P10)**.** (*Intermediate Level*)

The numbers 1, 2, 3, and 4 are placed clockwise around the circle (in this order). Tan and Lan plays a two-player game. At each turn, the player of the turn can add 1 to any pair of numbers that are next to each other. *For example, after the first turn the two numbers* 1 *and* 2 *can become* 2 *and* 3*.*

The player, who can make all the numbers equal, wins the game.

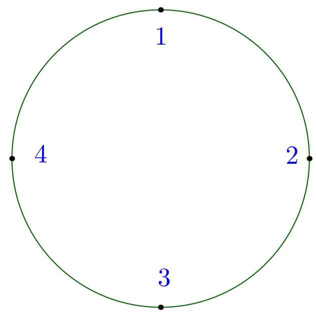


Figure 7.11: HC-2021-SM2-R3-P10

If Tan can choose, should he *go first* or *second?* In *at least how many turns* does the game end?

How to provide your answer:

- If you think that Tan should go first, and there is a way he can end the game in 5 turns, submit $F5$
- If you think that Tan should go second, and there is a way he can end the game in 10 turns, submit $S10$
- If you think it cannot be determined, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.2.11 (HC-2021-SM2-R3-P11)**.** (*Advanced Level*)

On the blackboard, Minh draws six segments. Every two segments intersect each other at a single point. The 1^{st} segment contains three of the intersections, the 2^{nd} segment contain four of the intersections, the 3^{rd}, 4^{th}, and 5^{th} segments each contains five intersections.

What segments does the 6^{th} segment intersect with?

How to provide your answer:

- If you think that the six segment intersects with the 1^{st}, 2^{nd}, and 3^{rd}, submit 123.
- If you think it cannot be determined, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.2.12 (HC-2021-SM2-R3-P12)**.** (*Advanced Level*)

p, q, r are positive integers such that $2^p + 2^q + 2^r = 2336$.

Find $p + q + r$.

How to provide your answer:

- If you think that $p + q + r = 20$, submit 20.
- If you think it cannot be determined, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.2.13 (HC-2021-SM2-R3-P13)**.** (*Advanced Level*)

Huy tiles (i.e. completely cover without overlapping) a 8×8 chessboard with 1×3 triminos and $T-$shaped tetrominos, shown below in Figure 7.12.

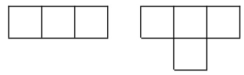


Figure 7.12: HC-2021-SM2-R3-P13

Note that a 1×3 *trimino covers exactly three squares of the chessboard, and a* $T-$*shaped tetromino covers exactly four squares of the chessboard.*

At most how many pieces can he use?

How to provide your answer:

- If you think that he can use 0 pieces of 1×3 triminoes and 16 pieces of $T-$shaped tetrominoes, then submit 16 and a tiling diagram as shown in Figure 7.13. *You can use your own way of drawing as far as it is understandable and correct.*
- If you cannot determine that, submit 0 and give a detailed reasoning.

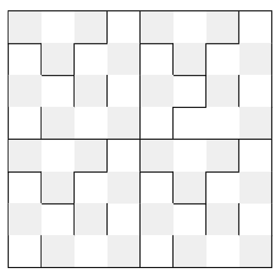


Figure 7.13: 0 triminoes and 16 tetrominoes

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly smaller* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is a *strictly larger* than your answer.

Problem 7.2.14 (HC-2021-SM2-R3-P14)**.** (*Advanced Level*)

Every weekday at 8 o'clock in the morning a merchant boat goes at a constant speed downriver from the Elven Town to the Village of Traders. It spends 2 hours at the dock of the Village of Traders for loading goods, then it goes at the same constant speed back on the same route, this time up the river.

One day, when Bilbo Baggins sneaked onto the boat during its usual morning departure, he accidentally knocked down a barrel. The barrel floated down the river, got stuck several times in some tree branches for a total of 2 hours, then floated again, and met the boat on its way back at 16:00.

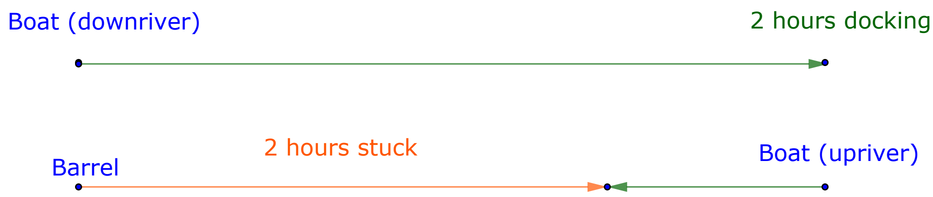


Figure 7.14: Boat meets barrel

When did the boat arrive at the dock of the Village of Traders?

How to provide your answer:

- If you think that the boat arrive at the dock of the Village of Traders at 13:15 (1:15 PM), submit 13:15.
- If you cannot determine that, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.2.15 (HC-2021-SM2-R3-P15)**.** (*Advanced Level*)

The Tale of Kieu is an epic poem written in Vietnamese by *Nguyễn Du* in 1820, which details the life of the talented young woman *Thuý Kiều*. The poem is composed of 1627 *couplets*, or groups of two lines. Here are five (05) couplets from *The Tale of Kieu* in no particular order,

1. Trăm năm trong cõi người ta,
   Chữ tài chữ mệnh khéo là ghét nhau.
2. Đầu lòng hai ả tố nga,
   Thuý kiều là chị, em là Thuý Vân.
3. Nửa năm hơi tiếng vừa quen,
   Sâu ngô cành biếc đã chen lá vàng.
4. Vân rằng: Chị cũng nực cười,
   Khéo dư nước mắt khóc người đời xưa.
5. Người đâu gặp gỡ làm chi,
   Trăm năm biết có duyên gì hay không?

In Vietnamese, the diacritics (accent marks) *a, á, à, ả, ạ, ã* indicate 5 of the 6 Vietnamese tones, known as *sắc, huyền, hỏi, nặng* and *ngã*; and the absence of a diacritic indicates the sixth tone, *ngang.* More importantly, the diacritics *â, ă, ê, ơ, ư* do not represent tones; instead, they represent slight changes to the pronunciation of the vowels. Traditionally, Vietnamese tones are divided into two groups:

- *sharp* tones, which includes *sắc, hỏi, nặng* and *ngã*, for example, *tố, tiếng, thẻ, ả, gặp, gỡ,* and *cũng.*
- *flat* tones, which includes *huyền* and *ngang.* for example, *người, ta, nga,* and *vàng.*

Each couplet of the Tale is written in the poetic meter known as *lục bát*, which is based on the positions in a couplet of the sharp and flat tones. These rules can identify whether a couplet has been *corrupted.* For example, here is a *corrupted* version of the first couplet,

Trăm năm trong **đời** người ta,
Chữ tài chữ mệnh khéo là ghét nhau.

Below are five other couplets from the Tale. Only two of them are *corrupted.* Which two?

A. Mấy lòng hạ cố đến nhau,
   Mấy lời hạ tứ ném châu gieo vàng.
B. Cớ sao trằn trọc canh sáng,
   Màu hoa lê hãy dầm dề giọt mưa?
C. Sầu đong lắc càng đầy càng,
   Ba thu dồn lại một ngày dài ghê.
D. Trong như tiếng hạc bay qua,
   Đục như tiếng suối mới sa nửa vời.
E. Trai anh hùng, gái thuyền nguyên,
   Phỉ nguyền sánh phượng, đẹp duyên cưỡi rồng.

How to provide your answer:

- If you think that A and B couplets are corrupted, submit $\{A, B\}$.
- If you cannot find them, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is a *strictly smaller subset* of your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is a *strictly larger superset* of your answer.

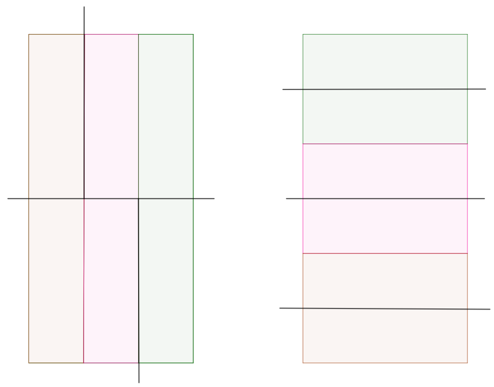
7.3 Answers

Problem 1: 3

Problem 2: WS

Problem 3: 0

Problem 4: $\{689,\ 789\}$



Problem 5:

Problem 6: 2

Problem 7: $L7K$

9	6	3	16
4	15	10	5
14	1	8	11
7	12	13	2

Problem 8:

Problem 9: 28

Problem 10: 0

Problem 11: 2345

Problem 12: 24

Problem 13: 21

Problem 14: $11:00$

Problem 15: $\{B, C\}$

7.4 Solutions

Solution. HC-2021-SM2-R3-P1 Note that a corner of the cube requires $\boxed{3}$ colours. This is where the most colours is needed to colour the neighboring squares. Thus, the minimal amount of colour is at least 3. We prove that 3 colours will be enough. In order to do that we show a concrete coloring uses 2 out of 3 colours to alternate-colour 9 squares each 3×3 face of the 6 faces.

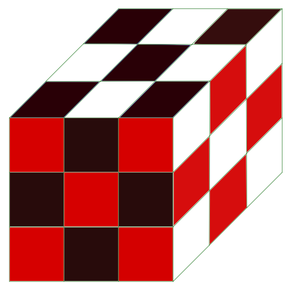


Figure 7.15: 3 colours

For example we can colour the top with (black, white) with black at 4 corners and the center squares, the front with (red, black) with red at 4 corners and the center, the left face with (white, red) with white at 4 corner and the center. The remaining faces have the same colouring as its opposite face. □

Solution. HC-2021-SM2-R3-P2 If Rob the bandit was awake at the time, he could not have had the false belief that both he and Sam the thug were asleep. Therefore he was asleep. This means that his belief was false, so it is not true that both were asleep. Therefore Sam was awake. The answer is $\boxed{WS.}$ □

Solution. HC-2021-SM2-R3-P3 First, the number of pirates is

$$1 + 2 + \ldots + 12 = \frac{12 \cdot 13}{2} = 78.$$

For every tower, the number of pirates to guard that tower is the same, so let it be n. By summing up all

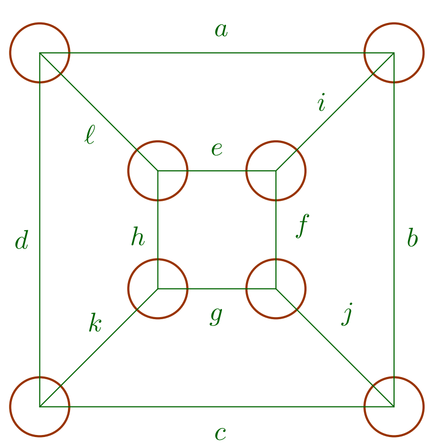


Figure 7.16: HC-2021-SM2-R3-P3

these numbers for all 8 towers, we get $8n$. On the other hand by summing up all the numbers of pirates guarding all 8 towers, each group of pirates patrolling the 12 roads is counted twice, thus

$$8n = 2 \cdot 78 \Rightarrow n = \frac{39}{2}, \text{ which is not an integer.}$$

Thus, there is $\boxed{\text{no such way}}$ to place the 12 groups. ☐

Solution. [First solution] HC-2021-SM2-R3-P4 Since $1 + 2 + \ldots 9 = 45$, thus the three weights on the right-hand side should be at least 23. The only two possible total sum of the weights are $24 = 7 + 8 + 9$, and $23 = 6 + 8 + 9$. Thus, the solutions are $\boxed{\{689,\ 789\}.}$ ☐

Solution. [Second solution] HC-2021-SM2-R3-P4

```
if __name__ == "__main__":
    s = [9,8,7,6,5,4,3,2,1]
    for w1 in range(0, 9):
        for w2 in range(w1+1, 9):
            for w3 in range(w2+1, 9):
                if 2 * (s[w1] + s[w2] + s[w3]) > sum(s):
                    print('%s%s%s' % (s[w3], s[w2], s[w1]))
```

The output shows that the solutions are $\boxed{\{689,\ 789\}.}$

```
789
689
```

☐

Solution. HC-2021-SM2-R3-P5 ☐

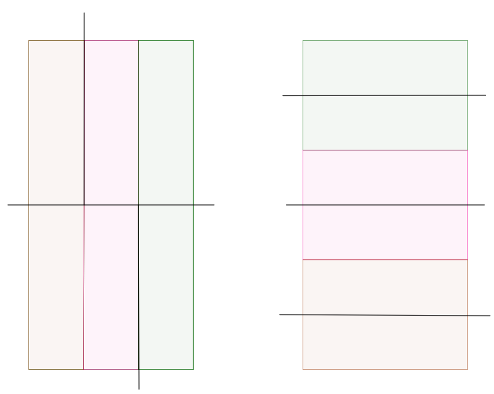


Figure 7.17: HC-2021-SM2-R3-P5

Solution. HC-2021-SM2-R3-P6 By the Two-Colour Theorem, any map on a plane can be coloured with just **two colours** if and only if all its junctions have an *even* number of edges. Thus, the given map can be

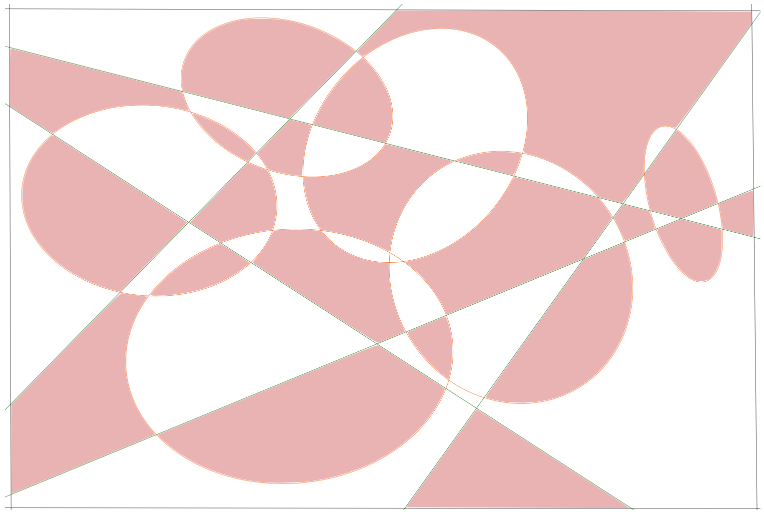


Figure 7.18: 2 colours

coloured by just $\boxed{2}$ colours. □

Solution. HC-2021-SM2-R3-P7 If Bale is 007, then all three statements are true, this is impossible since one of them is a Liar. If Gale is 007, then all three statements are false, this is impossible since one of them is a Knight. Therefore Dale is 007, Bale is the Liar and Gale is the Knight. The answer is $\boxed{L7K.}$ □

Solution. HC-2021-SM2-R3-P8 First, let c denotes the magic constant. We explore some important way to sum the numbers in the magic square. Let colour the 4 central squares red, 4 corner squares white, and the rest 8 square blue. Note that the two main diagonals consist of the 4 central red squares and the 4 corner white squares. So the sum of the squares in these diagonals, which is twice the magic constant, is equal to the sum S_r of the 4 central red squares plus the sum S_w of the 4 corner white squares. $S_r + S_w = 2c$.

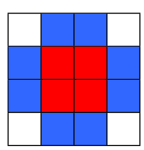


Figure 7.19: 4 red, 4 white, 8 blue squares

The sum S_b of the 8 blue squares is equal to the total sum of all numbers in the magic square, which is 4 times the magic constant, subtracts the sum of the values in the two main diagonals, $S_b = 4c - 2c = 2c$.

Thus, the 2 middle rows plus the 2 middle columns, whose sum is 4 times the magic constant, include the 8 blue squares and *twice* the 4 central red squares, $S_b + 2S_r = 4c$, so $2S_r = 2c$, or $S_r = c$, therefore, the sum of the 4 central red squares is equal to the magic constant.

Now, we calculate the magic constant. There are four rows with the same sum, which is $\frac{1+2+\ldots 16}{4} = 34$, thus the missing number in the second row is $34 - 4 - 15 - 5 = 10$. Since the sum of the 4 central squares is 34, so the missing number in the 4 central squares is $34 - 15 - 10 - 8 = 1$. The number in the top-right corner is $34 - 10 - 7 - 1 = 16$.

The sum of the two missing numbers in the corners is 11, which can only be formed by $(1, 10)$, $(2, 9)$, $(3, 8)$, $(4, 7)$, and $(5, 6)$. Out of these pairs, only $(2, 9)$ can be used. If 2 is in the top-left corner, then the

4	15	10	5
		8	
7			

			16
4	15	10	5
	1	8	
7			

9			16
4	15	10	5
14	1	8	11
7			2

Figure 7.20: Restoration steps

missing number in the first column is $34-4-7=23$, which exceeds 16. So 9 is in the top-left corner and 2 is in the bottom-right corner. It is easy to see that the two missing numbers on the first and last columns are $34-9-4-7=14$, and $34-15-5-2=11$.

The sum of the bottom two unknown numbers must be equal to $34-7-2=25$. We have only four numbers left, which is $3,6,12,13$, so $12,13$ must be on the bottom row. Furthermore the sum of the two missing numbers on the second column must be equal to $34-15-1=18$. So these two numbers must be 6 and 12. 3 and 13 go into the top and bottom cells of the third column. Below is the complete restored magic square.

9	6	3	16
4	15	10	5
14	1	8	11
7	12	13	2

Figure 7.21: The original magic square

$\square$

Solution. HC-2021-SM2-R3-P9 Let n be the number of students. We prove that after $2n-2$ calls, everybody knows all the solutions. Let everyone call a student A, then A calls everyone else. It is easy to see that after the first $n-1$ calls, A knows all the solutions. Then after $n-1$ calls by A, everyone else knows all the solutions.

Now, we prove that it is the minimum. Let B be a student who is the first to know all the solutions, after the p^{th} call. Then until the p^{th}, each of the $n-1$ students must made at least one call so that B knows all the solutions that they know. After the p^{th} call, each students must have received at least one call in order to know all the solutions. Thus, we need $2(n-1)=2n-2$.

For $n=15$, the number of calls if $\boxed{2\cdot 14=28.}$ $\square$

Solution. HC-2021-SM2-R3-P10 Let's look at the sums of two numbers opposite each other: $1+3=4,\ 2+4=6$.

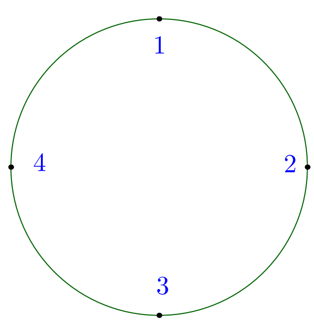


Figure 7.22: $1+3=4,\ 2+4=6$.

At every turn, these sums are both increased by 1, thus they will never be equal. Therefore the four numbers can never be equal. The game never ends. $\square$

Solution. HC-2021-SM2-R3-P11 Each of the 3$^{\text{rd}}$, 4$^{\text{th}}$, and 5$^{\text{th}}$ are to intersect all the remaining segments. Therefore the 1$^{\text{st}}$ segment intersects with the 3$^{\text{rd}}$, 4$^{\text{th}}$, and 5$^{\text{th}}$ segments only. Thus, the 2$^{\text{nd}}$ segment intersect with the 3$^{\text{rd}}$, 4$^{\text{th}}$, 5$^{\text{th}}$, and 6$^{\text{th}}$ segments.

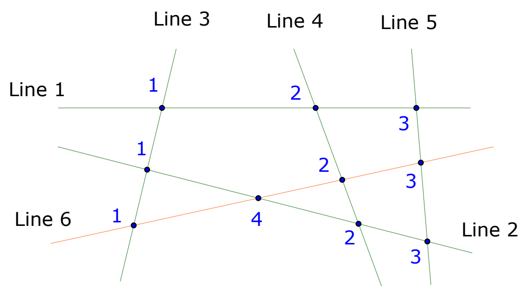


Figure 7.23: HC-2021-SM2-R3-P11

Thus, the 6$^{\text{th}}$ intersects with the $\boxed{2^{\text{nd}}, 3^{\text{rd}}, 4^{\text{th}}, \text{and } 5^{\text{th}}}$ segments. □

Solution. HC-2021-SM2-R3-P12 WLOG, $p \geq q \geq r$, $2^p + 2^q + 2^r = 2^r(2^{p-r} + 2^{q-r} + 1) = 2^5 \cdot 73$.

Case 1: $q = r$. If $p = r$, then $2^{p-r} + 2^{q-r} + 1 = 3$, but $2^5 \cdot 73$ is not divisible by 3. If $p > r$, then $2^{r+1}(2^{p-r-1} + 1) = 2^5 \cdot 73$, but 72 cannot be a power of 2. So there is no solution for this case.

Case 2: $q > r$. If $p = q$, then $2^{p-r} + 2^{q-r} + 1 = 2 \cdot 2^{p-r} + 1 = 73$, but 36 cannot be a power of 2. If $p > q$, then $2^{p-r} + 2^{q-r} = 72$ is the sum of two powers of 2, that can only be possible with $64 + 8 = 72$.

Thus, $p = 11, q = 8, r = 5$ and $\boxed{p + q + r = 24.}$ □

Solution. [Second solution] HC-2021-SM2-R3-P12

```
if __name__ == "__main__":
    for p in range(0, 12):
        for q in range(p, 12):
            for r in range(q, 12):
                if 2**p + 2**q + 2**r == 2336:
                    print(p, q, r, p+q+r)
```

The output shows that the sum of all exponents is $\boxed{24.}$

```
5 8 11 24
```

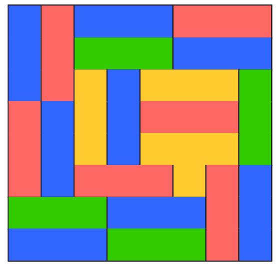
□

Solution. [Third solution] HC-2021-SM2-R3-P12 Note that p, q, and r all cannot be the same, because 2336 is not divisible by 3. If any two of p, q, and r are the same, WLOG $p = q$, then $2^{p+1} + 2^r = 2336$. In any case $2^p + 2^q + 2^r = 2336$ is basically equivalent to converting the number 2336 into base 2

$$2336 = \overline{100100100000}_2$$

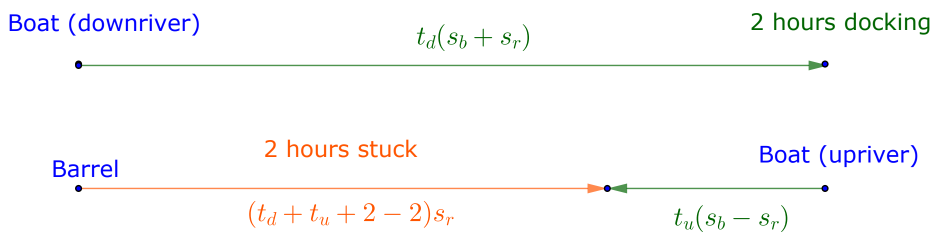
Thus if WLOG, $p \geq q \geq r$, then $p = 11, q = 8, r = 5$ and $\boxed{p + q + r = 24.}$ □

Solution. HC-2021-SM2-R3-P13 Since 64 is not divisible by 3 so the least number of $T-$shape tetrominoes is 1. We show one possible solution with 20 pieces of 1×3 triminoes and 1 $T-$shape tetromino. □



Solution. [First solution] HC-2021-SM2-R3-P14 We solve this problem by comparing the distances that the boat and the barrel travelled. Let s_b and s_r be the speed of the boat and the river, respectively. Let t_d and t_u be the time the boat traveled downriver to the Village of Traders and upriver until it met the barrel.

Since the barrel floated, its speed is the speed of the river downstream s_r and the total time it travelled is the total time the boat travelled plus the boat docking time minus the time when the barrel stuck $t_d + t_u + 2 - 2 = t_d + t_u$, thus the distance the barrel travelled is $(t_d + t_u)s_r$.



This is the same as the difference of the distances that the boat travelled downstream $t_d(s_b + s_r)$ and upstream $t_u(s_b - s_r)$ (the boat goes downstream and upstream at speeds $s_b + s_r$ and $s_b - s_r$, respectively).

$$(t_d + t_u)s_r = t_d(s_b + s_r) - t_u(s_b - s_r) \Leftrightarrow t_d s_r + t_u s_r = t_d s_b + t_d s_r - t_u s_b + t_u s_r \Leftrightarrow t_d s_b - t_u s_b = 0 \Rightarrow t_d = t_u$$

8 hours passed since the morning departure and 2 hours docking time mean that $t_d + t_u = 8 - 2 = 6$, so $t_d = t_u = 3$. Therefore the boat arrived at the dock of the Village of Traders at $\boxed{11{:}00}$ o'clock in the morning. □

Solution. [Second solution] HC-2021-SM2-R3-P14 In this solution, we *thinking physics,* namely *observe an event and use gained insights to answer questions.* WLOG, we can assume that the river did not flow, the water was still. The event now is simple. The barrel stays in one place, the boat move to the village, docked for 2 hours and came back with the same speed, thus the time it spent on each direction is the same. Therefore the boat travelled $\frac{8-2}{2} = 3$ hours, so it arrived at the dock of the Village of Traders at $\boxed{11 : 00}$ o'clock in the morning. □

Solution. HC-2021-SM2-R3-P15 By observing the five couplets, you can find one of the basic rules of a *lục bát* couplet. The *second (2), sixth (6)* and *eight (8)* syllables are *flat* (F) tones, and the *fourth (4)* syllable is a *sharp* (S) tone. For example,

Trăm $\underbrace{\text{năm}}_{F}$ trong $\underbrace{\text{cõi}}_{S}$ người $\underbrace{\text{ta}}_{F}$,
Chữ $\underbrace{\text{tài}}_{F}$ chữ $\underbrace{\text{mệnh}}_{S}$ khéo $\underbrace{\text{là}}_{F}$ ghét $\underbrace{\text{nhau.}}_{F}$
Nửa $\underbrace{\text{năm}}_{F}$ hơi $\underbrace{\text{tiếng}}_{S}$ vừa $\underbrace{\text{quen}}_{F}$,
Sâu $\underbrace{\text{ngô}}_{F}$ cành $\underbrace{\text{biếc}}_{S}$ đã $\underbrace{\text{chen}}_{F}$ lá $\underbrace{\text{vàng.}}_{F}$
Người $\underbrace{\text{đâu}}_{F}$ gặp $\underbrace{\text{gỡ}}_{S}$ làm $\underbrace{\text{chi}}_{F}$,
Trăm $\underbrace{\text{năm}}_{F}$ biết $\underbrace{\text{có}}_{S}$ duyên $\underbrace{\text{gì}}_{F}$ hay $\underbrace{\text{không?}}_{F}$

By this rule, you can find that $\boxed{B}$ couplet is corrupted because the *sixth (06)* syllable is a *sharp* tone.

Cớ sao trằn trọc canh $\underbrace{\textbf{sáng}}_{S}$,
Màu hoa lê hãy dầm dề giọt mưa?

Similarly $\boxed{C}$ couplet is corrupted because the *fourth (04)* syllable is a *flat* tone.

Sầu đong lắc $\underbrace{\textbf{càng}}_{F}$ đầy càng,
Ba thu dồn lại một ngày dài ghê.

Their original versions are,

Cớ sao trằn trọc canh **khuya**,
Màu hoa lê hãy dầm dề giọt mưa?
Sầu đong **càng lắc càng đầy**,
Ba thu dồn lại một ngày dài ghê.

□

Chapter 8

MIC R2

8.1 Topics

Algebra

1. Precent. Average of percent.
2. Decimal representations.
3. Sequences.
4. Sum and products
5. Trigonometry functions sin and cos with double-, triple-, and special angles $30°$, $45°$, $60°$, $90°$, ...

Combinatorics

1. Counting. Complementary Counting.
2. Probability. Complementary Probability.
3. Permutations.
4. Combinatorial Geometry.
5. Sets.

Geometry

1. Triangles.
2. Squares
3. Areas.
4. Circumcircle.
5. Triangle trigonometry.
6. Geometric Inequalities.
7. Reflections. Rotations. Symmetry.

Number Theory

1. Divisibility. Prime Factorization.
2. Number bases.
3. Perfect squares.
4. Diophantine equation. Condition for integer solution.

8.2 Rules

- The total time to complete the test is 90 minutes.
- The test consists of 10 multiple-choice and 4 show-you-work problems. To answer each of 10 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 6 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.

8.3 Problems

Problem 8.3.1 (MIC-2021-SM2-R2-P1). (*6 points*) To complete the grid Figure 8.1 below, each of the digits 1 through 4 must occur once in each row and once in each column. What number will occupy the *lower right-hand* square?

(A) 4 (B) 3 (C) 2 (D) 1 (E) cannot be determined

1		2	
			4
2	3		

Figure 8.1: MIC-2021-SM2-R2-P1

Problem 8.3.2 (MIC-2021-SM2-R2-P2). (*6 points*) At the Wonder Footwear County Fair a vendor is offering a s*super special deal* on shoes. For anyone who purchased a pair of shoes at the regular price of 60, a second pair can be purchased at only half the regular price, and a third pair at a 70% discount,

Danny took advantage of the *super special deal* to buy 3 pairs of shoes. What percentage of the regular price did Danny save?

(A) 40% (B) 45.6% (C) 52.5% (D) 55% (E) cannot be determined

Problem 8.3.3 (MIC-2021-SM2-R2-P3). (*6 points*) When the decimal point of a certain positive decimal number is moved four places to the right, the new number is nine times the reciprocal of the original number. What is the original number?

(A) 0.0002 (B) 0.002 (C) 0.02 (D) 0.03 (E) 0.3

Problem 8.3.4 (MIC-2021-SM2-R2-P4). (*6 points*) A 3×4 rectangle and a 2×3 rectangle are placed inside a square without overlapping at any point, except on their borders. The sides of the two given rectangles are parallel to some sides of the square. What is the smallest possible area of the square?

(A) 25 (B) 36 (C) 49 (D) 64 (E) 81

Problem 8.3.5 (MIC-2021-SM2-R2-P5). (*6 points*) If this path is to continue in the same pattern as in Figure 8.2 below, then which sequence of arrows goes from point 2020 to point 2022? Choose the answer according to Figure 8.3.

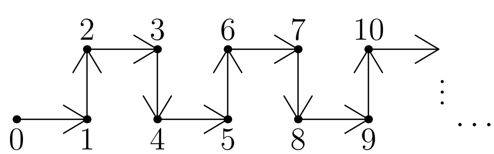


Figure 8.2: MIC-2021-SM2-R2-P5

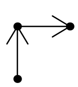
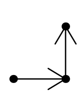
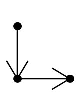
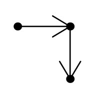
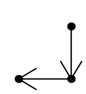
(A)  (B)  (C)  (D)  (E) 

Figure 8.3: Choose at most one

Problem 8.3.6 (MIC-2021-SM2-R2-P6). (*6 points*) What is the value of

$$2\left(1-\frac{1}{2}\right)+3\left(1-\frac{1}{3}\right)+4\left(1-\frac{1}{4}\right)+\cdots+2021\left(1-\frac{1}{2021}\right)$$

(A) 2040200 (B) 2041209 (C) 2041210 (D) 2041212 (E) 2043231

Problem 8.3.7 (MIC-2021-SM2-R2-P7). (*6 points*) In the sequence 2021, 2022, 2023, $\ldots$, each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. *For example, the fourth term is* $2021+2022-2023=2020$. What is the 2024^{th} term in this sequence?

(A) -2024 (B) 0 (C) 2 (D) 2024 (E) 4043

Problem 8.3.8 (MIC-2021-SM2-R2-P8). (*6 points*) $ADEH$ is a rectangle. Points B and C are on AD such that $AB=BC=CD$. Points G and F are on HE such that $HG=GF=FE$. In addition, $AH=AC=4$, and $AD=6$. What is the area of quadrilateral $WXYZ$ shown in Figure 8.4?

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{2}}{2}$ (C) 1 (D) $\sqrt{2}$ (E) 2

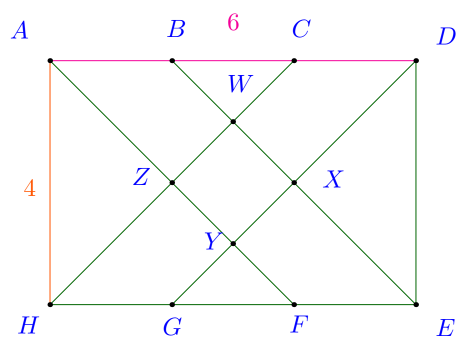


Figure 8.4: MIC-2021-SM2-R2-P8

Problem 8.3.9 (MIC-2021-SM2-R2-P9). (*6 points*) n is a positive integer with the following properties: (i) it has exactly 4 digits (ii) its leading digit is not a zero, (iii) it is a multiple of 5, (iv) it is an even number, and (v) it is divisible by both 9 and 11. How many such numbers are there?

(A) 8 (B) 9 (C) 10 (D) 12 (E) 24

Problem 8.3.10 (MIC-2021-SM2-R2-P10). (*6 points*) A particular 12-hour digital clock displays the hour and minute of a day. *For example it shows 8:15 at 15 minutes after 8 o'clock in the morning and 3:25 at 25 minutes after 3 o'clock in the afternoon.* Unfortunately, whenever it is supposed to display a 2, it mistakenly displays a 9. *For example, when it is 2:20 the clock incorrectly shows 9:90.* What fraction of the day will the clock show the correct time?

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{5}$ (D) $\frac{4}{7}$ (E) $\frac{5}{8}$

Problem 8.3.11 (MIC-2021-SM2-R2-P11). (*10 points*) A circle of radius r is cut into four congruent arcs. The four arcs are joined to form the star figure shown, see Figure 8.5.

- (*5 points*) If $r = 2$, what is the ratio of the area of the star figure to the area of the original circle?
- (*5 points*) Now, with what value of r can the ratio be $\frac{2}{3}$?

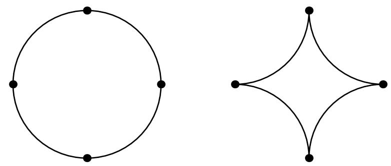


Figure 8.5: MIC-2021-SM2-R2-P11

Problem 8.3.12 (MIC-2021-SM2-R2-P12). (*10 points*) A is the set of digits $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and B is the set of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ (same as A without the digits 9.) Mai randomly picks 3 distinct digits from A and arranges them in descending order to form a $3-$digit number, *for example when Mai chooses 3, 4, and 9, she gets the number* 943. Mui randomly picks 3 distinct digits from B and also arranges them in descending order to form a $3-$digit number, *for example when Mui chooses 3, 4, ad 5, he gets the number* 543.

- (*5 points*) What is the probability that Mai's number is larger than Mui's number, so that Mai's number has 9 as an digit?
- (*5 points*) Overall, what is the probability that Mai's number is larger than Mui's number?

Problem 8.3.13 (MIC-2021-SM2-R2-P13). (*10 points*) In 2001 the populations of the town was a perfect square. 10 years later, after an increase of 91 people, the population was again a perfect square. Now, in 2021, after a descrease of 180 people, the population was again a perfect square. How many people are now in the town?

Problem 8.3.14 (MIC-2021-SM2-R2-P14). (*10 points*) In $\triangle BAC$, $\angle BAC = 40°$, $AB = 9$, and $AC = 4$. Points D and E lie on $\overline{AB}$ and $\overline{AC}$ respectively. What is the minimum possible value of $BE + DE + CD$?

8.4 Grading

Answers for multiple-choice problems.

Problem 1: C

Problem 2: A

Problem 3: D

Problem 4: A

Problem 5: B

Problem 6: C

Problem 7: B

Problem 8: E

Problem 9: B

Problem 10: E

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 11: Separately grading for each part,

(a) 2 points if can find a way to reassemble the cirle or can find the surrounding square.

(b) 2 points if can deduce a correct formula based on r.

Problem 12: Separately grading for each part,

(a) 2 points if can find the number of ways to pick any three distinct digits.

(b) 2 points if can found that Mai and Mui now picks from the same set.

Problem 13: 2 points if can factor the difference of squares.

Problem 14: 2 points if can use the Law of Consines.

8.5 Solutions

Solution. MIC-2021-SM2-R2-P1 *This is a variation of 2007 AMC 8 Problems/Problem 9.* The number in the first row, last column must be a 3 due to the fact if a 3 was in the first row, second column, there would be two threes in that column. By the same reasoning, the number in the third row, last column has to be a 1.

1		2	3
			4
2	3		1
			2

Therefore the number in the lower right-hand square is 2. The anwer is $\boxed{C.}$ □

Solution. MIC-2021-SM2-R2-P2 *This is a variation of 2013 AMC 8 Problems/Problem 12.* First, the amount of money Danny paid for three shoes without the discount is $60 \cdot 3 = 180$. Then, the amount of money Danny paid using the discount: $60 + \frac{1}{2} \cdot 60 + 0.3 \cdot 60 = 108$. Finding the percentage yields $\frac{108}{180} = 60\%$.

Thus the percentage saved is $100\% - 60\% = 40\%$. The anwer is $\boxed{A.}$ □

Solution. MIC-2021-SM2-R2-P3 *This is a variation of 2001 AMC 10 Problems/Problem 7.* Let x be the positive number, then moving the decimal point four places to the right is the same as multiplying x by 10000.

$$10000x = 9 \cdot \frac{1}{x} \Rightarrow x^2 = \frac{9}{10000} \Rightarrow x = \frac{3}{100} \Rightarrow x = 0.03$$

The anwer is $\boxed{D.}$ □

Solution. MIC-2021-SM2-R2-P4 *This is a variation of 2006 AMC 10B Problems/Problem 5.* By placing the 2×3 rectangle adjacent to the 3×4 rectangle with the 3 side of the 2×3 rectangle next to the 4 side of the 3×4 rectangle, we get a figure that can be completely enclosed in a square with a side length of 5. The area of this square is $5^2 = 25$. Thus, the smallest possible area of the square is 25.

The anwer is $\boxed{A.}$ □

Solution. MIC-2021-SM2-R2-P5 *This is a variation of 1994 AJHSME Problems/Problem 15.* Notice the pattern from 0 to 4 repeats for every four arrows. Any number that has a remainder of 0 when divided by 4 corresponds to 0. The remainder when 2020 is divided by 4 is 0. The arrows from point 2020 to point 2022 correspond to points 4 and 6, which have the same pattern as $\boxed{B.}$ □

Solution. MIC-2021-SM2-R2-P6 *This is a variation of 1998 AJHSME Problems/Problem 12.*

$$2\left(1-\frac{1}{2}\right)+3\left(1-\frac{1}{3}\right)+4\left(1-\frac{1}{4}\right)+\cdots+2021\left(1-\frac{1}{2021}\right)=2-1+3-1+\ldots+2021-1$$
$$=1+2+\ldots+2020=\frac{2020\cdot 2021}{2}=1010\cdot 2021=2041210$$

The answer is $\boxed{C.}$ □

Solution. MIC-2021-SM2-R2-P7 *This is a variation of 2004 AMC 12B Problems/Problem 12.*

$$a_1 = 2021,\ a_2 = 2022,\ a_3 = 2023, a_4 = 2020$$
$$\Rightarrow a_5 = 2022 + 2023 - 2020 = 2025, a_6 = 2023 + 2020 - 2025 = 2018, a_7 = 2020 + 2025 - 2018 = 2027$$

Now, the claim below is easily proved by induction.

Claim — $a_{2k+1} = 2021 + 2k$ and $a_{2k} = 2024 - 2k$.

It follows that $a_{2024} = a_{2\cdot 1012} = 2024 - 2 \cdot 1012$.

The answer is $\boxed{B.}$ □

Solution. MIC-2021-SM2-R2-P8 *This is a variation of 2006 AMC 10A Problems/Problem 17.* Drawing lines as shown above and piecing together the triangles, we see that $ABCD$ is made up of 12 squares congruent to $WXYZ$. Hence $[WXYZ] = \frac{4\cdot 6}{12} = 2$.

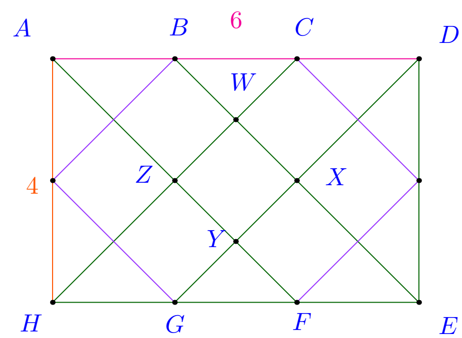


Figure 8.6: 12 squares

The answer is $\boxed{E.}$ □

Solution. MIC-2021-SM2-R2-P9 *This is a variation of 2011 AMC 8 Problems/Problem 23.*

Note that the original solution does not include 9900 *as a possible number. Thus the answer A is wrong. The solution below is the correct one. The answers is B.*

Note that the unit digit must be 0 since the number is even and divisible by 5. Since the number is divisible by 11, so the sum of the first and third digit is the same as the second digit. Then the second digit cannot be 0 because the first digit cannot be 0. Furthermore, the number is divisible by 9, so the sum of all digits, or twice the second digit, is divisible by 9. Thus, the second digit is 9, therefore the sum of the first and third digits is 9. There are 9 such possibilities $(1, 8)$, $(2, 7)$, $\ldots$ $(8, 1)$, and $(9, 0)$. The numbers are $\{1980, 2970, \ldots, 8910, 9900\}$.

Thus, the answers is $\boxed{B.}$ □

Solution. MIC-2021-SM2-R2-P10 *This is a variation of 2009 AMC 12B Problems/Problem 10.* The clock will display the incorrect time for the entire hours of 2 and 12. So the correct hour is displayed $\frac{12-2}{12} = \frac{5}{6}$ of the time. The minutes will not display correctly whenever either the tens digit or the ones digit is a 2, so the minutes that will not display correctly are $20, 21, 22, \ldots, 29$ and $02, 12, 32, 42$, and 52. This amounts to fifteen $10 + 5 = 15$ out of the sixty possible minutes for any given hour. Hence the fraction of the day that the clock shows the correct time is

$$\frac{5}{6} \cdot \left(1 - \frac{15}{60}\right) = \frac{5}{6} \cdot \frac{3}{4} = \frac{5}{8}.$$

The answer is $\boxed{E.}$ □

Solution. MIC-2021-SM2-R2-P11 *This is a variation of 2012 AMC 8 Problems/Problem 24.* Draw a square around the star figure. The sidelength of this square is $2r$, because the sidelength is the diameter of the circle. The square forms 4-quarter circles around the star figure. This is the equivalent of one large circle with radius r, meaning that the total area of the quarter circles is πr^2. The area of the square is $4r^2$. Thus, the area of the star figure is $4r^2 - \pi r^2$. The area of the circle is πr^2.

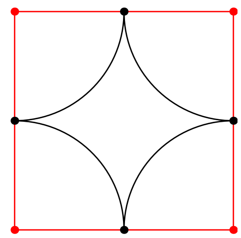


Figure 8.7: Draw a square

Taking the ratio of the two areas, we have $\boxed{\frac{4r^2-\pi r^2}{\pi r^2} = \frac{4-\pi}{\pi}.}$ This ratio does not depends on r. □

Solution. MIC-2021-SM2-R2-P12 *This is a variation of 2010 AMC 12A Problems/Problem 16.*

Note that the original problem text for the first question is ambigous: *it should ask for the probability when Mai picks* 9 *as a digit and as the consequence, her number is larger than Mui's number.*

If Mai picks 9, then her number is alwasy larger than Mui's number. The number of ways she pick 9 is she picks 9 and any other two distinct digits, which is $1 \cdot \binom{8}{2}$. There are $\binom{9}{3}$ ways for her to pick any three distinct digits. So the probability for the first question is

$$\frac{1 \cdot \binom{8}{2}}{\binom{9}{3}} = \frac{\frac{8 \cdot 7}{2}}{\frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1}} = \frac{1}{3}.$$

The probability that Mai does not pick 9 is $1 - \frac{1}{3} = \frac{2}{3}$. Now, let ignore the digit 9. Mai and Mui now picks from the same set. We have two important observations. First, the probability that Mai's number is larger is equal to the probability that Mui's number is larger. Second, the probability that they will pick the same number is the number of ways to pick Mai's digits dividing the number of ways to pick any three digits, which is $\frac{3!}{8 \cdot 7 \cdot 6} = \frac{1}{56}$. Thus, the probability that Mai's number is larger than Mui's number (when Mai does not pick the digit 9) is

$$\frac{2}{3} \frac{1 - \frac{1}{56}}{2} = \frac{2}{3} \frac{55}{112} = \frac{55}{168}$$

Therefore the probability that Mai's number is larger than Mui's number is $\boxed{\frac{1}{3} + \frac{55}{168} = \frac{37}{56}.}$ □

Solution. MIC-2021-SM2-R2-P13 *This is a variation of 2011 AMC 10A Problems/Problem 19.*

Note that the original problem text stated that: *Now, in* 2021*, after a descrease of* 273 *people, the population was again a perfect square. This was wrong and should be changed to: Now, in* 2021*, after a descrease of* 180 *people, the population was again a perfect square.*

Let a^2, b^2, and c^2 be the populations in 2001, 2011, and 2021.

$$\begin{cases} b^2 - a^2 = 91 \Rightarrow (b-a)(b+a) = 1 \cdot 91 = 7 \cdot 13 \\ b^2 - c^2 = 180 \Rightarrow (b-c)(b+c) = 2 \cdot 90 = 3 \cdot 60 = 4 \cdot 45 = 5 \cdot 36 = 6 \cdot 30 = 9 \cdot 20 = 10 \cdot 18 = 12 \cdot 15 \end{cases}$$

From the factorization, it is easy to see that $b - a = 1,\ b + a = 91$, thus $b = 46, a = 45$, then $c = 44$. The population now is $\boxed{44^2 = 1936.}$ □

Solution. MIC-2021-SM2-R2-P14 *This is a variation of 2014 AMC 12A Problems/Problem 20.*

Reflect C across AB to C'. Similarly, reflect B across AC to B'. Clearly, $BE = B'E$ and $CD = C'D$. Thus, the sum $BE+DE+CD = B'E+DE+C'D$. This value is minimized when B', C', D and E are collinear. To finish, we use the law of cosines on the triangle $AB'C'$, $\boxed{B'C' = \sqrt{5^2 + 9^2 - 2(5)(9)\cos 120^\circ} = \sqrt{151}.}$ □

Chapter 9

MIC R3

9.1 Topics

Algebra

1. Equations. Absolute values and casework.
2. Sums and Products. Average. Sums in two ways.
3. Inequalities. Comparison Method.
4. Functional Equations.

Combinatorics

1. Counting.
2. Permutations.
3. Counting in two ways.
4. Grids. Counting paths on grids.
5. Probability. Number of favorable outcomes. Independent events.
6. Sets. Principle of Inclusion-Exclusion.
7. Recurrent Relations.

Geometry

1. Triangles. Right triangles. $3-4-5$ right triangles.
2. Congruent triangles. Similar triangles.
3. Equilateral triangles.
4. Trapezoids.
5. Area. Areas as sum of sub-areas.
6. Computational Geometry. Line equation. Circle equation. Area bounded by equations. Parabola equations. Graphs of equations with absolute values.

Number Theory

1. Perfect Powers.
2. Divisibility.
3. Modular Arithmetic.

9.2 Rules

- The total time to complete the test is 90 minutes.
- The test consists of 10 multiple-choice and 4 show-you-work problems. To answer each of 10 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 6 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.

9.3 Problems

Problem 9.3.1 (MIC-2021-SM2-R3-P1). (*6 points*)

In the arrangement of letters and numerals below, by how many different paths can one spell AMC8? Beginning at the A in the middle, a path allows only moves from one letter to an adjacent (above, below, left, or right, but not diagonal) letter. One example of such a path is traced in the picture.

	8	C	8	
8	C	M	C	8
C	M	A	M	C
8	C	M	C	8
	8	C	8	

Figure 9.1: MIC-2021-SM2-R3-P1

(A) 12 (B) 18 (C) 24 (D) 36 (E) 48

Problem 9.3.2 (MIC-2021-SM2-R3-P2). (*6 points*)

Duy holds 2 green and 1 red jelly bean in his hand. Quy holds 1 green, 2 yellow, and 3 red jelly beans in her hand. Each randomly picks a jelly bean to show the other.

What is the probability that the colors match?

(A) $\frac{1}{4}$ (B) $\frac{3}{8}$ (C) $\frac{2}{15}$ (D) $\frac{5}{18}$ (E) $\frac{3}{4}$

Problem 9.3.3 (MIC-2021-SM2-R3-P3). (*6 points*)

How many perfect cubes lie between $2^7 + 1$ and $2^{24} + 1$? *For example* $2^3 = 8$, $3^3 = 27$ *are perfect cubes.*

(A) 256 (B) 251 (C) 250 (D) 123 (E) 122

Problem 9.3.4 (MIC-2021-SM2-R3-P4). (*6 points*)

$ABCD$ is a trapezoid, $AB = 12$, $BC = 20$, $DA = 13$, and the altitude is 12.

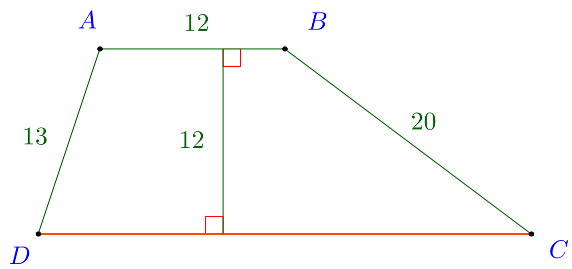


Figure 9.2: MIC-2021-SM2-R3-P4

What is the area of the trapezoid?

(A) 270 (B) 276 (C) 282 (D) 320 (E) 326

Problem 9.3.5 (MIC-2021-SM2-R3-P5). (*6 points*)

Define a function on the positive integers recursively by

- $f(1) = 2$,
- $f(n) = f(n-1) + 1$ if n is even, and
- $f(n) = f(n-2) + 2$ if n is odd and greater than 1.

What is the value of $\frac{f(2020)}{f(2021)}$?

(A) $\dfrac{2020}{2021}$ (B) $\dfrac{2021}{2022}$ (C) 1 (D) $\dfrac{2022}{2021}$ (E) $\dfrac{2021}{2020}$

Problem 9.3.6 (MIC-2021-SM2-R3-P6). (*6 points*)

All three vertices of $\triangle ABC$ are lying on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x-axis. The area of the triangle is 216.

What is the length of BC?

(A) 4 (B) 6 (C) 8 (D) 12 (E) 16

Problem 9.3.7 (MIC-2021-SM2-R3-P7). (*6 points*)

The school offers after-hour programs in robotics, coding, and painting. Each of 20 students registers to one, two, or all three programs. The robotics program has 11, the coding program has 12, and the painting program has 13 registrations, respectively. There are 9 students taking at least two programs.

How many students are taking all three programs?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Problem 9.3.8 (MIC-2021-SM2-R3-P8). (*6 points*)

Let ABC be an equilateral triangle with side length 1. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$.

What is the area of $\triangle A'B'C'$?

(A) $\dfrac{17}{8}$ (B) $\dfrac{25}{4}$ (C) $\dfrac{26\sqrt{3}}{4}$ (D) $\dfrac{36\sqrt{3}}{4}$ (E) $\dfrac{37\sqrt{3}}{4}$

Problem 9.3.9 (MIC-2021-SM2-R3-P9). (*6 points*)

When 2019^{2021} is divided by 5, the remainder is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 9.3.10 (MIC-2021-SM2-R3-P10). (*6 points*)

A sequence has 49 numbers. The average of the sequence if 2019. The average of the first 25 numbers of the sequence is 2020. The average of the last 25 numbers of the sequence is 2021.

What is the value of the 25^{th} number of the sequence?

(A) 2018 (B) 2019 (C) 2020 (D) 2064 (E) 2094

Problem 9.3.11 (MIC-2021-SM2-R3-P11). (*10 points*)

1. What is the area of the region bounded by the equations $x^2 + y^2 = 2x + 2y$, $x \geq 0$, and $y \geq 0$?
2. What is the area of the region enclosed by the graph of the equation $x^2 + y^2 = 2|x| + 2|y|$?

Problem 9.3.12 (MIC-2021-SM2-R3-P12). (*10 points*)

Rectangle $ABCD$ has $AB = 6$ and $BC = 8$. Point E is the foot of the perpendicular from B to AC.

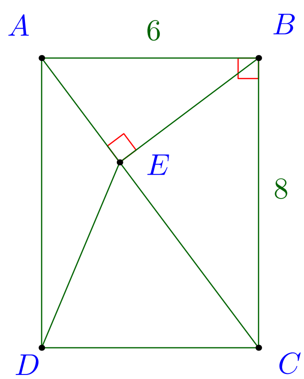


Figure 9.3: MIC-2021-SM2-R3-P12

1. Determine AE.
2. Find the area of $\triangle ADE$.

Problem 9.3.13 (MIC-2021-SM2-R3-P13). (*10 points*)

There are 24 four-digit whole numbers that use each of the four digits 2, 4, 5 and 7 exactly once. Only one of these four-digit numbers is a multiple of another one.

Determine that number.

Problem 9.3.14 (MIC-2021-SM2-R3-P14). (*10 points*)

For a particular peculiar pair of dice, the probabilities of rolling 1, 2, 3, 4, 5 and 6 on each die are in the ratio $1 : 2 : 3 : 4 : 5 : 6$.

What is the probability of rolling a sum of 8 on the two dice?

9.4 Grading

Answers for multiple-choice problems.

Problem 1: C

Problem 2: D

Problem 3: B

Problem 4: A

Problem 5: B

Problem 6: D

Problem 7: C

Problem 8: E

Problem 9: E

Problem 10: E

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 11: Separately grading for each part,

(a) 2 points if can find that the graph of describes a circle centred at O_1 radius $\sqrt{2}$.

(b) 2 points if can divise 4 cases based on signs of x and y.

Problem 12: Separately grading for each part,

(a) 2 points if can find similar triangles or calculate area in two ways.

(b) 2 points if can use area formula with sin function or using complementatry areas.

Problem 13: 2 points if can estimate the quotient to be 2, or 3.

Problem 14: 2 points if can use determine the probability for x ($1 \leq x \leq 6$), by adding outcomes and combine with probability for each outcome.

9.5 Solutions

Solution. MIC-2021-SM2-R3-P1 *This is a variation of 2017 AMC 8 Problems/Problem 15.*

From the A, there are four ways to go to an M. From any one of the Ms, then you have three ways to get a C, and from any C, two ways to get an 8.

	8	C	8	
8	C	M	C	8
C	M	A	M	C
8	C	M	C	8
	8	C	8	

Thus, there are $\boxed{4 \cdot 3 \cdot 2 = 4! = 24}$ paths and the answer is $\boxed{C.}$ □

Solution. MIC-2021-SM2-R3-P2 *This is a variation of 2013 AMC 8 Problems/Problem 14.*

The probability that both show a green bean is $\frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}$. The probability that both show a red bean is $\frac{1}{3} \cdot \frac{3}{6} = \frac{1}{6}$. Therefore the probability is $\boxed{\frac{1}{9} + \frac{1}{6} = \frac{5}{18}.}$ The answer is $\boxed{D.}$ □

Solution. MIC-2021-SM2-R3-P3 *This is a variation of 2018 AMC 8 Problems/Problem 25.*

It is easy to see that $2^7 + 1 = 129 > 125 = 5^3$. Furthermore $2^{24} = (2^8)^3 = 256^3$, thus

$$5^3 < 2^7 + 1 < 6^3 < 7^3 < \ldots < 256^3 < 2^{24} + 1 < 257^3$$

There are $\boxed{256 - 6 + 1 = 251}$ perfect cubes between $2^7 + 1$ and $2^{24} + 1$, inclusive. The answer is $\boxed{B.}$ □

Solution. MIC-2021-SM2-R3-P4 *This is a variation of 2011 AMC 8 Problems/Problem 20.*

Let G and H be the feet of the altitudes from A and B to CD, respectively. Then $DG = \sqrt{AD^2 - AG^2} = \sqrt{13^2 - 12^2} = 5$, $CH = \sqrt{BC^2 - BH^2} = \sqrt{20^2 - 12^2} = 16$, thus $CD = CH + HG + GD = 16 + 12 + 5 = 33$.

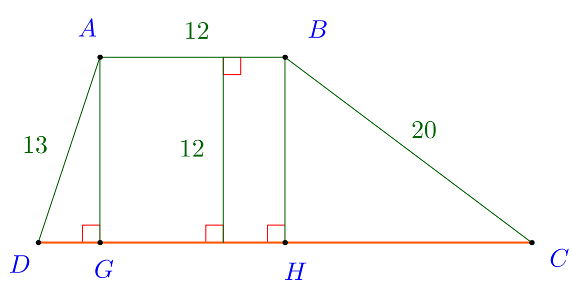


Figure 9.4: $AG \perp CD$, $BH \perp CD$

Therefore the area of $ABCD$ is $\boxed{\frac{12+33}{2} \cdot 12 = 270.}$ The answer is $\boxed{A.}$ □

Solution. MIC-2021-SM2-R3-P5 *This is a variation of 2017 AMC 12A Problems/Problem 7.*

First, $f(2n+1) = f(2n-1) + 2 \cdot 1 = f(2n-3) + 2 \cdot 2 = \ldots = f(1) + 2 \cdot n = 2 + 2n$. Then $f(2n) = f(2n-1) + 1 = 2n + 1$, thus $\frac{f(2n)}{f(2n+1)} = \frac{2n+1}{2n+2}$. Therefore $\boxed{\frac{f(2020)}{f(2021)} = \frac{2021}{2022},}$ the answer is $\boxed{B.}$ □

Solution. MIC-2021-SM2-R3-P6 *This is a variation of 2016 AMC 10B Problems/Problem 9.*

Since BC is parallel to the $x-$axis, thus let $B(-b, b^2)$ and $C(b, b^2)$, then

$$[ABC] = \frac{1}{2}b^2 \cdot 2b = b^3 \Rightarrow b^3 = 216 \Rightarrow b = 6.$$

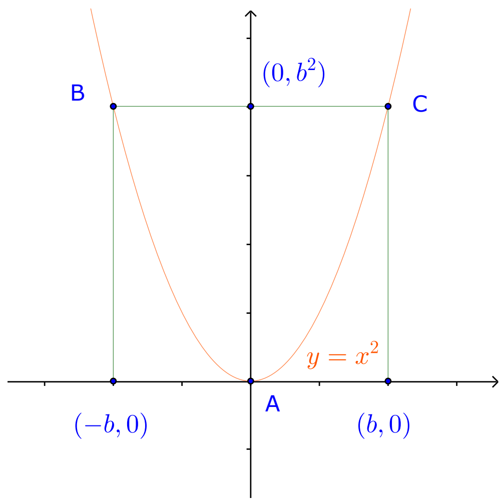


Figure 9.5: MIC-2021-SM2-R3-P6

Thus, $\boxed{BC = 2b = 12,}$ the answer is $\boxed{D.}$ □

Solution. MIC-2021-SM2-R3-P7 *This is a variation of 2017 AMC 10B Problems/Problem 13.*

Let A_1, A_2, and A_3 be the sets of students registered for robotics, coding, and painting programs, respectively. Let x be the number of students registered for all three programs.

The number of students registered for at least two programs is

$$\sum |A_i \cap A_j| - 2|A_1 \cap A_2 \cap A_3| = 9 \Rightarrow \sum |A_i \cap A_j| = 9 + 2x$$

Now, by the Principle of Inclusion-Exclusion,

$$|A_1 \cup A_2 \cup A_3| = \sum |A_i| - \sum |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3| \Rightarrow 20 = (11 + 12 + 13) - (9 + 2x) + x \Rightarrow x = 7$$

Thus, the number of students registered for all three programs is $\boxed{7.}$ The answer is $\boxed{C.}$ □

Solution. MIC-2021-SM2-R3-P8 *This is a variation of 2017 AMC 12B Problems/Problem 15.*

First, comparing bases yields that

$$[BA'B'] = 3[AA'B] = 9[ABC] \Rightarrow [AA'B'] = 12$$

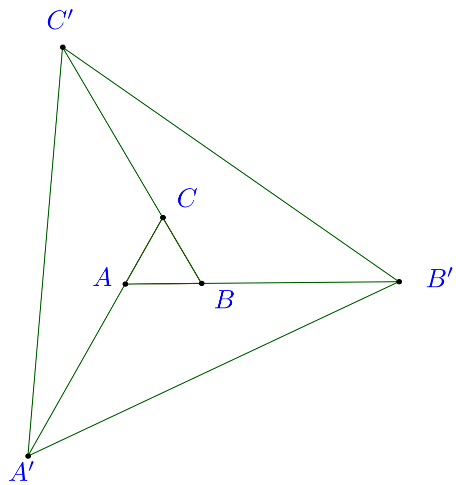


Figure 9.6: MIC-2021-SM2-R3-P8

By congruent triangles,

$$[AA'B'] = [BB'C'] = [CC'A'] \Rightarrow [A'B'C'] = (12 + 12 + 12 + 1)[ABC],$$

so $\boxed{[A'B'C'] = 37[ABC] = \frac{37\sqrt{3}}{4},}$ and the answer is $\boxed{E.}$ $\square$

Solution. MIC-2021-SM2-R3-P9 *This is a variation of 1999 AMC 8 Problems/Problem 24.*

Since $2019 \equiv -1 \pmod 5$, we have $2019^{2021} \equiv (-1)^{2021} \equiv -1 \equiv 4 \pmod 5$. Thus, the remainder of 2019^{2021} when divided by 5 is $\boxed{4,}$ and the answer is $\boxed{E.}$ $\square$

Solution. MIC-2021-SM2-R3-P10 *This is a variation of 2000 AMC 8 Problems/Problem 23.*

Let the sequence be $a_1, a_2, \ldots, a_{49}$, then

$$\begin{aligned}
&a_1 + a_2 + \ldots + a_{49} = 49 \cdot 2019,\ a_1 + a_2 + \ldots + a_{25} = 25 \cdot 2020,\ a_{25} + a_{26} + \ldots + a_{49} = 25 \cdot 2021 \\
&\Rightarrow a_{25} = (a_1 + a_2 + \ldots + a_{25}) + (a_{25} + a_{26} + \ldots + a_{49}) - (a_1 + a_2 + \ldots + a_{49}) \\
&= 25 \cdot 2020 + 25 \cdot 2021 - 49 \cdot 2019 = 2094
\end{aligned}$$

Thus, the 25^{th} number in the sequence is $\boxed{2094,}$ and the answer is $\boxed{E.}$ $\square$

Solution. MIC-2021-SM2-R3-P11 *This is a variation of 2016 AMC 10B Problems/Problem 21.*

First, the equation

$$x^2 + y^2 = 2x + 2y \Rightarrow (x-1)^2 + (y-1)^2 = 2,$$

describes a circle centred at O_1 radius $\sqrt{2}$.

Thus, the area bounded by the graphs of the equations $x^2 + y^2 = 2x + 2y$, $x = 0$, and $y = 0$ is half the area of the circle (O_1) plus the area of $\triangle AOB$, which is $\frac{2\cdot 2}{2} + \frac{\pi\cdot 2}{2}$, or $\boxed{2+\pi.}$

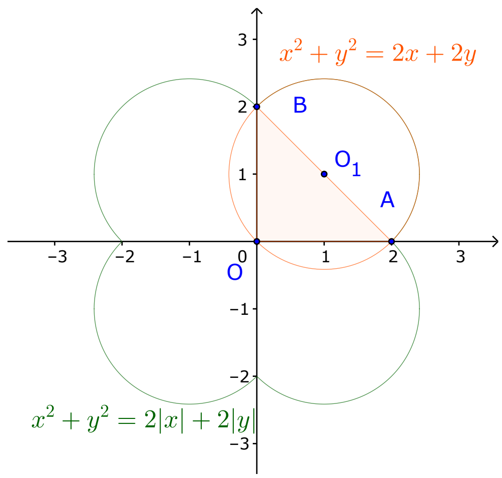


Figure 9.7: MIC-2021-SM2-R3-P11

Thus, the area bounded by the graphs of the equations $x^2 + y^2 = 2|x| + 2|y|$ is $\boxed{4(2+\pi) = 4\pi + 8.}$ □

Solution. MIC-2021-SM2-R3-P12 *This is a variation of 2017 AMC 10B Problems/Problem 15.*

First, note that $AC = 10$ because ABC is a right triangle. In addition, we have $AB \cdot BC = 2[ABC] = AC \cdot BE$, so $BE = \frac{48}{10} = \frac{24}{5}$. $\triangle AEB \sim \triangle ABC$, so $\frac{AE}{BE} = \frac{AB}{CB} = \frac{3}{4}$, so $AE = \frac{24}{5} \cdot \frac{3}{4} = \frac{18}{5}$. Thus, $\boxed{AE = \frac{18}{5}.}$

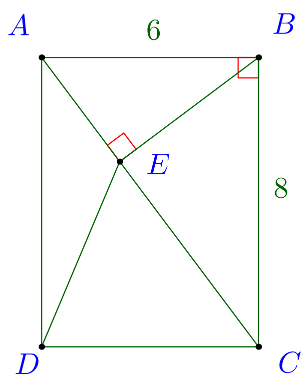


Figure 9.8: $AB \cdot BC = AC \cdot BE$

Now, $\sin\angle EAD = \sin\angle CAD = \frac{3}{5}$, thus

$$[AED] = \frac{1}{2}AE \cdot AD \cdot \sin\angle EAD = \frac{1}{2} \cdot \frac{18}{5} \cdot 8 \cdot \frac{3}{5} = \frac{216}{25}.$$

Thus $\boxed{[AED] = \frac{216}{25}.}$ □

Solution. MIC-2021-SM2-R3-P13 *This is a variation of 2001 AMC 8 Problems/Problem 25.*

Since the greatest number possible, 7542, divided by the smallest number possible, 2457, is slightly greater than 3, thus we can divide all of the permutations of the digits in the number 7542 by 2, then 3, and see if the resulting answer contains 2, 4, 5 and 7.

Doing so, you find that $\boxed{7425 = 3 \cdot 2475.}$ □

Solution. MIC-2021-SM2-R3-P14 *This is a variation of 2006 AMC 12B Problems/Problem 17.* Since $1 + 2 + \ldots + 6 = 21$, so the probability of getting an x $(1 \le x \le 6)$ on one of these dice is $\frac{x}{21}$. The probability of getting 2 on the first and 6 on the second die is $\frac{2}{21} \cdot \frac{6}{21}$ Similarly we can express the probabilities for the other ways how we can get a total 8.

Summing these, the probability of getting a total 8 is,

$$\frac{2}{21} \cdot \frac{6}{21} + \frac{3}{21} \cdot \frac{5}{21} + \frac{4}{21} \cdot \frac{4}{21} + \frac{5}{21} \cdot \frac{3}{21} + \frac{6}{21} \cdot \frac{2}{21} = \frac{2(12 + 15) + 16}{441} = \frac{70}{441}$$

Thus, the probability is $\boxed{\frac{70}{441}.}$ □

Chapter 10

HC R4

10.1 Topics

Chess

1. Moves: capture, check, double check, promotion, stalemate, checkmate
2. Analysis: whose turn is it.
3. Retrograde: last move, last two moves, last multiple of moves.
4. Proof games: White and Black, instead of making the best moves, they co-operate to make the game legally into a set position.
5. Symmetric position with odd number of moves.
6. Switchback: a piece moved out and came back into that same square.

Coding

1. String operations
2. Permutations.
3. Lists. Dictionaries. Sets.
4. Matrix. Rows. Columns.
5. Random. Choice. Shuffle.
6. Comparing multiple conditions at once.
7. File IO: read, write, save, load

Logic

1. Casework: what if A is true, what if A is false.
2. Process of elimination: If A is not true, B is not true, then C should be true.
3. Reverse argument: if there is at least ..., then there is atmost ...
4. Implication from truth: if A told the truth and A said X, then X is true.
5. Conflict of truth: if B said A lied, then both cannot be truth tellers.

Algebra

1. Sums and Products
2. Functional Equations.
3. Integer Functional Equations.

Combinatorics

1. Counting. Combinations. Correction over Counting.
2. Permutations. Comparison of neighbours in a permutation. Correction over Counting.
3. Algorithms and Processes.
4. Combinatorial Geometry.
5. Geometric Probability.

Geometry

1. Circle. Diameters. Area. Inscribed Circle.
2. Tangent lines.

Number Theory

1. Divisibility
2. Modular Arithmetic. Prime Divisors.
3. Number bases.
4. Diophantine Equations.

10.2 Problems

Problem 10.2.1 (HC-2021-SM2-R4-P1)**.** (*Beginner Level*)

A *block move* is a way to pick a block of numbers from a sequence and move it to a different position.

For example, here are two block moves for the sequence 1234

$$1\underline{23}4 \to \underline{23}14,\ \ 12\underline{3}4 \to 1\underline{3}24$$

In *at least* how many block moves does Khanh need to change the sequence 12345 into 54321?

How to provide your answer:

- If you think that Khanh needs at least 7 block moves, then submit 7 and a sequence depicting the moves, underlining the blocks that are moved, *for example*

$$1234\underline{5} \to 123\underline{5}4 \to 12\underline{5}34 \to 1\underline{5}234 \to 512\underline{3}4 \to 5412\underline{3} \to 54\underline{3}12 \to 54321$$

- If you cannot determine that, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly larger* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is a *strictly smaller* than your answer.

Problem 10.2.2 (HC-2021-SM2-R4-P2). (*Beginner Level*)

Julie has just raided an ancient tomb. On her way out, she found herself in a long corridor. In front of her are three traps and three doors. See Figure 10.1.

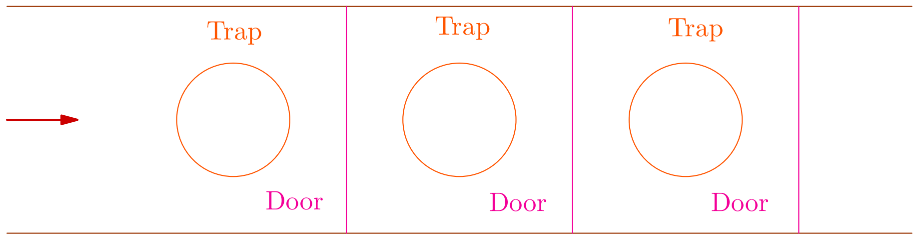


Figure 10.1: HC-2021-SM2-R4-P2

There were no way to go around them and there were no other exit. Luckily, Julie saw something similar to a control panel on a nearby wall. There were three stones and some instructions written beneath them,

- Pushing the small stone arms one unarmed trap.
- Pushing the medium stone disarms any two arm traps and locks one unlocked door.
- Pushing the large stone unlocks two locked doors.

Julie escaped by pushing the stones a *minimal number of times*. How did she do it?

How to provide your answer:

- If you think that Julie had to push at least 7 times, then submit 7 and a sequence describing which stones she pushed, *for example* $SSMLMLL$ if she pushed the small stones twice, then the medium one, then the large one, then the medium again, and finally the large one twice, in this order.
- If you cannot determine that, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly larger* than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is a *strictly smaller* than your answer.

Problem 10.2.3 (HC-2021-SM2-R4-P3)**.** (*Beginner Level*)

Amy draws a semicircle with diameter 4. Inside this semicircle, she draws a smaller semicircle so that (i) the two diameters are parallel, (ii) both ends of the small diameter are on the perimeter of the large semicircle, and (iii) the small semicircle touches the large diameter at a single point, see Figure 10.2.

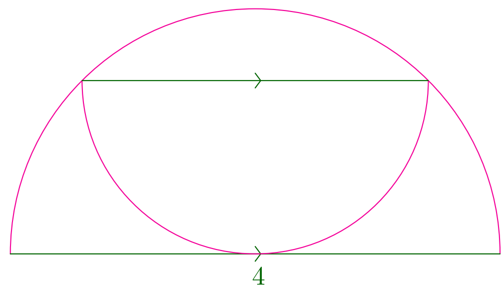


Figure 10.2: HC-2021-SM2-R4-P3

What is the *radius* of the small semicircle?

How to provide your answer:

- If you think that the radius is 2, then submit 2.
- If you cannot determine that, submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 10.2.4 (HC-2021-SM2-R4-P4)**.** (*Beginner Level*)

The Figure 10.3 shows the situation after a total of 7 legal moves made by both White and Black since the beginning of the game.

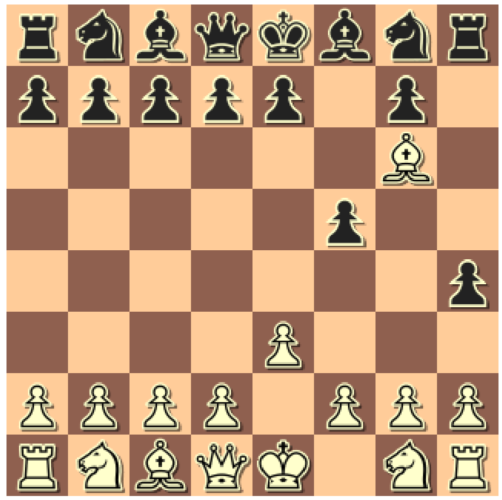


Figure 10.3: HC-2021-SM2-R4-P4

What were the moves?

How to provide your answer:

- If you think that the moves were 1.e4 e5 2.d4 d5 3.c4 c5 4.f4, then submit 1.e4 e5 2.d4 d5 3.c4 c5 4.f4.
- If you think that there are no such possible moves, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 10.2.5 (HC-2021-SM2-R4-P5)**.** (*Beginner Level*)

Father was not happy when he heard that Mother could not make his favourite cake because butter, eggs and milk all were stolen. Mother recalled seeing Chipmunk, Groundhog, and Sparrow sneaking out of the kitchen when she came into it. Everyone was carrying something, but she couldn't tell who was carrying what. After a brief investigation all the ingredients were found at the homes of Chipmunk, Groundhog, and Sparrow. Here were what they said,

- Chipmunk: Groundhog stole the butter.
- Groundhog: Sparrow stole the eggs.
- Sparrow: I stole the milk.

As it happened, the one who stole the butter told the truth and the one who stole the eggs lied.

Who stole what?

How to provide your answer:

- If you think that Chipmunk stole the butter, Groundhog stole the eggs, and Sparrow stole the milk, submit $CB\ GE\ SM$.
- If you think that Chipmunk stole the eggs, Groundhog stole the milk, and Sparrow stole the butter, submit $CE\ GM\ SB$.
- If you think that it cannot be determined, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 10.2.6 (HC-2021-SM2-R4-P6)**.** (*Intermediate Level*)

Detective Antoine intercepted an encrypted missive sent by an alien to her spaceship, shows in Figure 10.4.

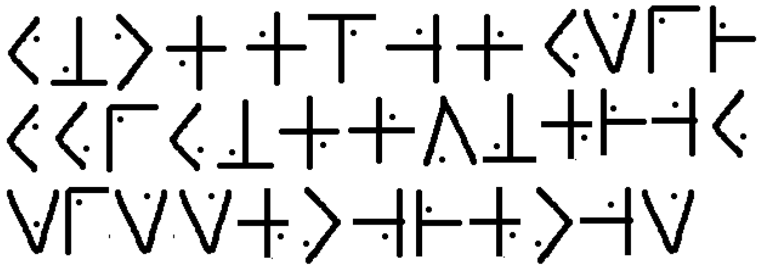


Figure 10.4: Encrypted missive

In an earlier incident, when leaving in a hurry, the aliens left a piece of paper, shown in Figure 10.5. When comparing the encrypted missive with the piece of paper, he recognized that some of the letters were missing from the paper. By recovering the missing letters and used it as cypher he was able to decode the encrypted missive.

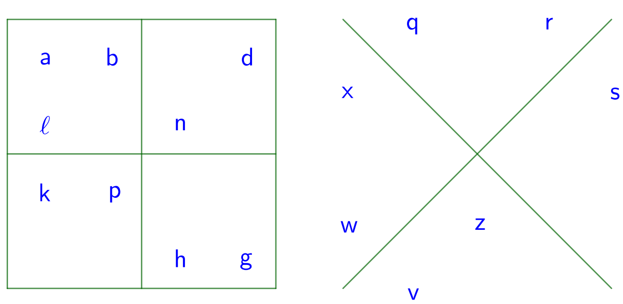


Figure 10.5: Incomplete cypher

What was the original content of the encrypted missive?

How to provide your answer:

- If you think that the original content of the encrypted missive is *"I enjoyed having vacation on Earth"*, then submit *I enjoyed having vacation on Earth.*
- If you think that it cannot be determined, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 10.2.7 (HC-2021-SM2-R4-P7). (*Intermediate Level*)

The $\star$ operator is defined for every pair of a and b real numbers, where $ab \neq 1$, then

$$a \star b = \frac{a-b}{1-ab}$$

What is the value of

$$1 \star (2 \star (3 \star \ldots (2019 \star (2020 \star 2021))))?$$

How to provide your answer:

- If you think that the value if $\frac{1023}{2021}$, then submit $\frac{1023}{2021}$.
- If you think that it cannot be determined, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 10.2.8 (HC-2021-SM2-R4-P8). (*Intermediate Level*)

Ha Anh throws a dart at a square dartboard of side length 2. What is the probability that the dart hits closer to the center O than any corner, but within a distance 1 of a corner? The Figure 10.6 shows some hits that were closer to the center O and within a distance 1 of the corner A.

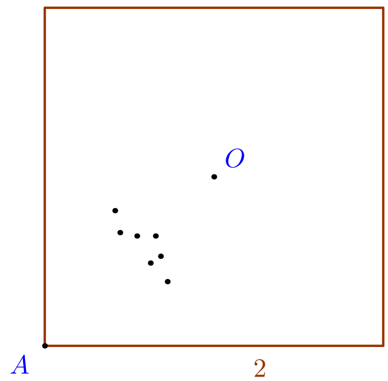


Figure 10.6: HC-2021-SM2-R4-P8

How to provide your answer:

- If you think that the probability that Ha Anh can hit as she desired is $\frac{1}{2}$, submit $\frac{1}{2}$.
- If you think that it cannot be determined, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 10.2.9 (HC-2021-SM2-R4-P9). (*Intermediate Level*)

The Figure 10.7 shows the situation after a total of 7 legal moves made by both White and Black since the beginning of the game.

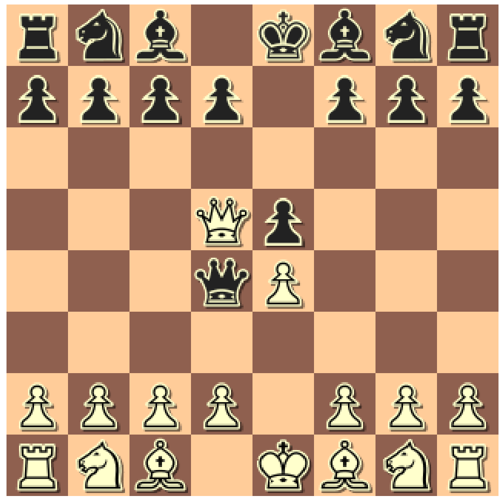


Figure 10.7: HC-2021-SM2-R4-P9

What were the moves?

How to provide your answer:

- If you think that the moves were 1.e4 e5 2.d4 d5 3.c4 c5 4.f4, then submit 1.e4 e5 2.d4 d5 3.c4 c5 4.f4.
- If you think that there are no such possible moves, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 10.2.10 (HC-2021-SM2-R4-P10). (*Intermediate Level*)

In the Kingdom of the Veganworld, humans and vampires happily live together. They are all vegetarian and look pretty much alike. The only difference is the distinct behavior in belief and truth-telling: (i) sane humans make only true statements, (ii) insane humans uncontrollably lie, (iii) sane vampires always lie, and (iv) insane vampires always tell the truth.

For example, if you ask the inhabitants of the Kingdom whether the earth is round, a sane human knows the earth is round and truthfully say so, an insane human believes the earth is not round and says it is not round, a sane vampire knows the earth is round, but will then lie and say it isn't, and an insane vampire believes the earth is not round and then lies and say it is round.

Inspector Linh was invited to the kingdom to solve some crime. Here was the first interview with the first couple, *Sylvia and Sylvan,*

- Inspector Linh (to Sylvia): Tell me something about yourselves.
- Sylvia: My husband is human.
- Sylvan: My wife is a vampire.
- Sylvia: One of us is sane and one of us is not.

Here was the second interview with the second couple, *Gloria and George,*

- Inspector Linh (to Gloria): Tell me something about yourselves.
- Gloria: Whatever my husband said is true.
- George: My wife is insane.

Now, knowing that it is *illegal for humans and vampires to intermarry,* meaning humans should marry only humans and vampires should marry only vampires, could Inspector Linh determine what are they?

How to provide your answer:

- If you think that for the first couple, Sylvan is a sane human and Sylvia is an insane human, and for the second couple, Gloria is a sane vampire and George is an insane vampire, then submit $SH\ IH\ SV\ IV$.
- If you think nothing can be determined about them, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 10.2.11 (HC-2021-SM2-R4-P11). (*Advanced Level*)

Find all *ordered* triples of non-negative integers (a, b, c) such that,

$$2^a + 2^b = c!$$

Note that the ordered triple of $(0, 1, 0)$ *is different from the ordered triple* $(1, 0, 0)$.

How to provide your answer:

- If you think that there are two triples $(0, 0, 0)$ and $(1, 1, 1)$, then submit $(0, 0, 0)$, $(1, 1, 1)$.
- If you think that they cannot be determined, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is a *strictly smaller subset* of your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger superset* of your answer.

Problem 10.2.12 (HC-2021-SM2-R4-P12). (*Advanced Level*)

The Figure 10.8 shows the situation after a total of 8 legal moves made by both White and Black since the beginning of the game.

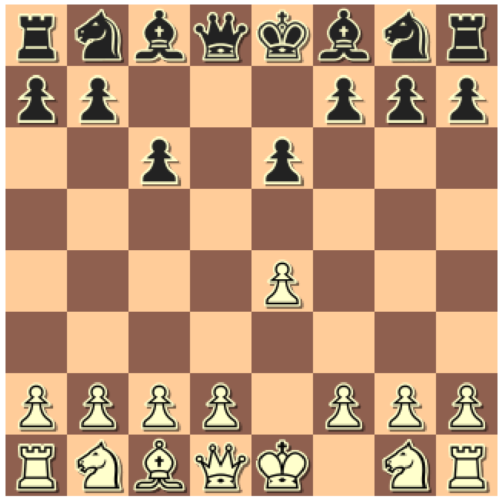


Figure 10.8: HC-2021-SM2-R4-P12

What were the moves?

How to provide your answer:

- If you think that the moves were 1.e4 e5 2.d4 d5 3.c4 c5 4.f4 f5, submit 1.e4 e5 2.d4 d5 3.c4 c5 4.f4 f5.
- If you think that there are no such possible moves, submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 10.2.13 (HC-2021-SM2-R4-P13). (*Advanced Level*)

In a far far faraway land, there is place called the Village of Dreamers. The villagers dream as if they were awake. Their life while asleep continues from night to night while their life while awake goes from day to day. Thus, it is difficult for some of them to actually know if they were asleep or awake at that time.

Now, each of the villagers is either an *early-bird* or a *night-owl*.

- An *early-bird* is a person that everything he believes while he is awake is true, and everything he believes while he is asleep is false.
- A *night-owl* is the opposite, everything he believes while he is asleep is true, and everything he believes while he is awake is false.

Inspector Albert was tasked to find out who are the *early-birds* and who are the *night-owls*. He interviewed a Anna and Anthony. Here were what they said,

- Andrew: I am an *early-bird*.
- Anna: I am asleep at all times.
- Anthony: I am awake at all times.

What are they?

How to provide your answer:

- If you think that Andrew is an *early-bird*, nothing can be determined about Anna, and Anthony is a *night-owl*, submit $E0N$.
- If you think that Andrew is a *night-owl*, Anna is an *early-bird*, and nothing can be determined about Anthony, submit $NE0$.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 10.2.14 (HC-2021-SM2-R4-P14)**.** (*Advanced Level*)

Braille is a tactile writing system used by people who are visually impaired. It is not a language. This method of writting uses a system of raised dots that can be read with the fingers. Braille is adapted to many languages. Here is the Japanese word for **karaoke** (written in Latin letters) written in a modified version of Tenji (Japanese Braille). The large • represents a raised bump, the tiny · represents an empty position.

```
● ·   ● ·   · ●   ● ●
· ●   · ●   ● ·   ● ●
● ●   · ●   · ·   ● ●
```

The words **(A) atari, (B) haiku, (C) katana, (D) kimono, (E) koi,** and **(F) sake** are represented by six Tenji words numbered $1, 2, \ldots, 6$ as below,

(1)
```
● ·   ● ·   ● ●
· ·   ● ·   · ●
· ●   · ·   ● ●
```

(2)
```
● ·   · ●   · ●
● ●   ● ·   ● ●
● ●   · ●   ● ·
```

(3)
```
● ·   ● ·   ● ·
· ●   · ·   · ●
● ●   ● ·   ● ·
```

(4)
```
● ·   ● ·   ● ·
· ·   · ·   ● ●
· ·   ● ·   · ●
```

(5)
```
· ●   ● ·
● ●   ● ·
● ●   · ·
```

(6)
```
● ·   ● ●
· ●   ● ●
· ·   ● ●
```

Which is which?

How to provide your answer:

- If you think that the correct pairs of words are $1A$ $2B$ $3C$ $4D$ $5E$, then submit $1A$ $2B$ $3C$ $4D$ $5E$.
- If you think that they cannot be determined, submit 0 and give a detailed reason.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is a *strictly smaller subset* of your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger superset* of your answer.

Problem 10.2.15 (HC-2021-SM2-R4-P15). (*Advanced Level*)

In the diagram below, nine circles are placed on a 9 × 9 board such that there are no row, column, or diagonal with more than one circle.

	a	b	c	d	e	f	g	h	i
1			●						
2					●				
3								●	
4		●							
5									●
6						●			
7	●								
8							●		
9				●					

Can you move *exactly three circles*, each to a neighbouring cell in a horizontally, vertically, or diagonally direction such that there are still no row, column, or diagonal with more than one circle.

How to provide your answer:

- If you think that you can move the circles at a7, b4, c1 to the cells b7 (horizontally neighbour to a7), a5 (diagonally neighbour to b4), and c2 (vertically neighbour to c1), then submit a7b7 b4a5 c1c2. *Note that you can move the three selected circles in any direction you like, as long as they end up on neighbouring squares.*
- If you think that they cannot be determined, submit 0 and give a detailed reason.

How your troop's performance is graded:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

10.3 Answers

Problem 1: 3 $12\underline{34}5 \to 3\underline{41}25 \to \underline{32}541 \to 54321$

Problem 2: 5 $MLSML$

Problem 3: $\sqrt{2}$

Problem 4: 1.e3 h5 2.Bd3 h4 3.Bh7 f5 4.Bg6++

Problem 5: $CM\ GB\ SE$

Problem 6: sixpmcentralstationvioletyarrowflower

Problem 7: 1

Problem 8: $\frac{\pi-2}{4}$

Problem 9: 1.e3 e5 2.Qf3 Qh4 3.Qd5 Qd4 4.e4.

Problem 10: $IH\ SH\ IV\ IV$

Problem 11: $\{(0,0,2),(1,2,3),(2,1,3),(3,4,4),(4,3,4)\}$

Problem 12: 1.e4 e6 2.Bb5 Ke7 3.B×d7 c6 4.Be8 K×e8.

Problem 13: $0EN$

Problem 14: $1B\ 2D\ 3C\ 4A\ 5E\ 6F$

Problem 15: d9d8 f6g6 g8f9

10.4 Solutions

Solution. HC-2021-SM2-R4-P1 Below is a set of $\boxed{3}$ block moves that can change 54321 to 12345,

$$12\underline{34}5 \to 3\underline{41}25 \to \underline{32}541 \to 54321$$

Now, we prove that 3 moves is the minimum. Given a permutation of $\{1, 2, 3, 4, 5\}$, a *descent* is an adjacent pair of numbers in the permutation such that the left number is greater than the right one. *For example,* 12345*,* 34215*, and* 54321 *have* 0*,* 2*, and* 4 *descents, respectively.*

Any permutation obtained from 12345 by one block move has (at most) one descent, at the left edge of the moved block. Similarly, any permutation obtained from 54321 by one block move has (at least) three descents, so that we can't get from 54321 to 12345 by two block moves. □

Solution. HC-2021-SM2-R4-P2 At the beginning, by pushing ML, Julie (i) disarmed two armed traps and locked no door (because all doors was locked), (ii) unlocked two locked doors.

Thus, there remained one armed trap and one locked door. By the same sequence SML, Julie (i) armed one trap (so the number of armed traps was 2) (ii) disarmed two armed traps (thus all traps disarmed) and locked one door (so the number of locked doors was 2), (iii) unlocked two locked doors (thus all doors unlocked).

Therefore she needs $\boxed{\text{5 pushes}}$ by the sequence $MLSML$. □

Solution. HC-2021-SM2-R4-P3 Let O_1 and O_2 be the centres of the semicircles. Let P be one end of the small diameter and let r be the radius of the small semicircle.

$$O_2O_1 = r,\ O_2P = r,\ \angle PO_2O_1 = 90^\circ,\ O_1P = 2 \Rightarrow 2r^2 = 4 \Rightarrow r = \sqrt{2}$$

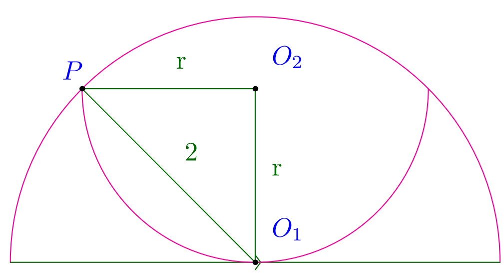
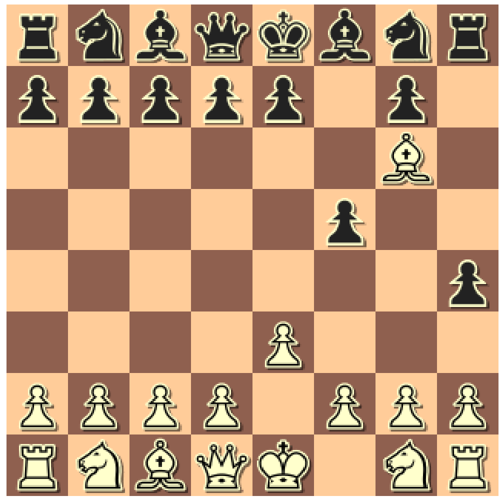


Figure 10.9: $O_2P = O_2O_1 = r$

Thus, the radius of the small semicircle is $\boxed{\sqrt{2}.}$ □

Solution. HC-2021-SM2-R4-P4 It is obvious that Black can only moves the pawns along the f and h column.



The key is to *hide* the white bishop in h7 before the final attack, 1.e3 h5 2.Bd3 h4 3.Bh7 f5 4.Bg6++. □

Solution. HC-2021-SM2-R4-P5 First, Sparrow cannot have stolen the butter because otherwise she would be the truth teller but she said she stole the milk. Thus, Chipmunk or Groundhog must have stolen the butter.

If Chipmunk stole the butter, then what she said about Groundhog is true, thus Groundhog stole the butter, which is a contradiction. Therefore Groundhog must have stolen the butter. Thus, what Groundhog said was true, so Sparrow stole the eggs, therefore Chipmunk stole the milk.

The answer is *CM GB SE.* □

Solution. HC-2021-SM2-R4-P6 The cypher can be completed by filling the letters in alphabetical order from a to z, first clockwise in the square, and then in the cross, see Figure 10.10. Now, a symbol has some the

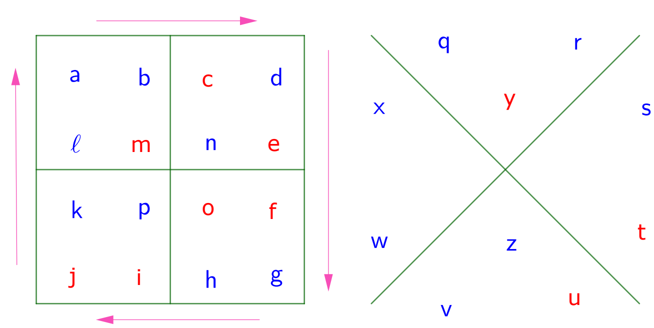


Figure 10.10: Complete cypher

visible segments and a dot. These segments represent the segments in the cypher that are closest to the alphabet and the dot represent the position of the alphabet to the segment. *For example, the first symbol in the missive is the letter s, the second is the letter i, the third one is the letter x, and so on.*

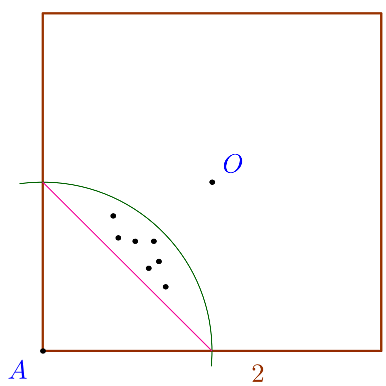
The original content of the missive is sixpmcentralstationvioletyarrowflower, and in a more readable format **six pm central station violet yarrow flower.** □

Solution. HC-2021-SM2-R4-P7 Note that

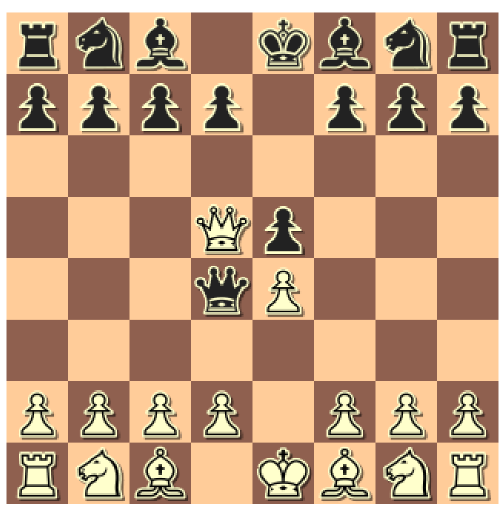
$$a \star b = \frac{a-b}{1-ab} \Rightarrow 1 \star c = \frac{1-c}{1-1\cdot c} = 1 \Rightarrow 1 \star \underbrace{(2 \star (3 \star \ldots (2019 \star (2020 \star 2021))))}_{c} = 1$$

Thus, the answer is $\boxed{1.}$ □

Solution. HC-2021-SM2-R4-P8 Therefore the probability is the area of the desired region in this square. The desired region is the part of the circle of radius 1 centered at the corner A that is closer to the opposite corner. The points closer to the opposite corner are those that are on the other side of the diagonal through the other two corners, so the desired region is a quarter of a circle of radius 1 minus a right triangle with legs of length 1. Therefore the area is $\frac{4\cdot\left(\frac{\pi}{4}-\frac{1}{2}\right)}{4} = \frac{\pi-2}{4}$. Thus, the probability is $\boxed{\frac{\pi-2}{4}.}$ □



Solution. HC-2021-SM2-R4-P9 It is obvious that the white and black queens must move out at the second and fourth moves. Their move cannot be symmetric because of the pawns. Thus the white queen move to f3, while the black queen to h4. Thus, $\boxed{\text{1.e3 e5 2.Qf3 Qh4 3.Qd5 Qd4 4.e4.}}$ □



Solution. HC-2021-SM2-R4-P10 First, both Sylvan's and Sylvia's statements cannot either be true, nor false. Thus, one of them is true and another is false. So one of them is sane and the other is insane (otherwise if both are sane then both of their statements must be right if they are humans, or both of their statements are wrong if they both are vampires). Now, this is exactly what Sylvia said, thus she is sane and Sylvan is insane. Furthermore, she said her husband is human, so both of them are human.

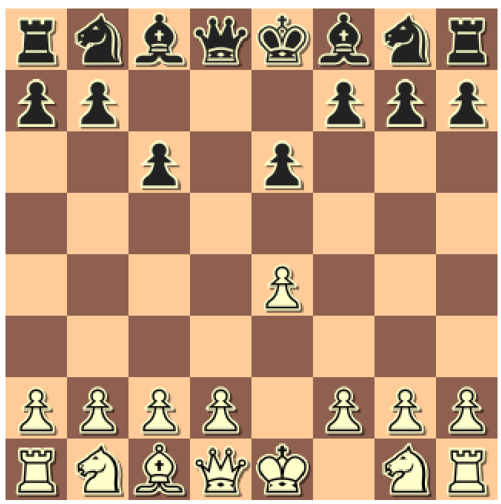
It is easier for the second couple, Gloria indirectly claims that she is insane by saying whatever her husband saying is true. Only vampires can claim that they are insane. Thus she is a vampire, therefore both of them are vampire. Since George said that Gloria is insane, so he is insane too.

The answer is $\boxed{IH\ SH\ IV\ IV.}$ □

Solution. HC-2021-SM2-R4-P11 It is easy to verify that the remainders of 2^n when divided by 7 are 1, 2, and 4. Therefore $2^a + 2^b$ cannot be dividible by 7. Thus $c \leq 6$. Now, WLOG assume that $a > b$, $c! = 2^a + 2^b$ means that $c! = \overline{\underbrace{1}_{a}0\ldots0\underbrace{1}_{b}0\ldots0}_{2}$. By direct testing, the only possibilities are $c = 2, 3, 4$.

Thus, $\boxed{(a, b, c) \in \{(0, 0, 2), (1, 2, 3), (2, 1, 3), (3, 4, 4), (4, 3, 4)\}.}$ □

Solution. HC-2021-SM2-R4-P12 In this game the so-called *switchback* is applied by moving the black king out in order to allow the white bishop to get to the king's place e8 and then reoccupy it with the black king.



Thus, $\boxed{\text{1.e4 e6 2.Bb5 Ke7 3.B×d7 c6 4.Be8 K×e8.}}$ □

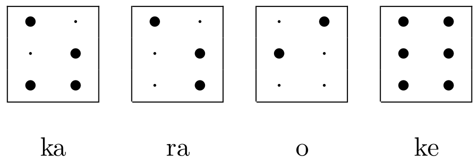
Solution. HC-2021-SM2-R4-P13 First, a person who is an *early-bird* will correctly believe that he is an *early-bird* while awake. A *night-owl* will incorrectly believe that he is an *early-bird* while awake. So nothing can be determined about Andrew.

Second, an *early-bird* while awake correctly believes that he is awake. When asleep the person incorrectly believes that he is awake. Thus a person who believes that he is awake at all time is an *early-bird*.

Similarly, a person who believes that he is asleep at all time is a *night-owl*.

The answer is $\boxed{0EN.}$ □

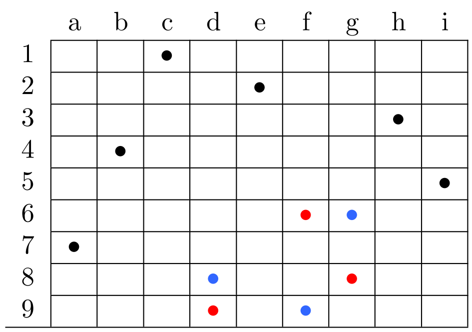
Solution. HC-2021-SM2-R4-P14 First, Japanese Braille is a vowel-based *abugida*. That is, the glyphs are syllabic, but unlike *kana* they contain separate symbols for consonant and vowel, and the vowel takes primacy. The word **karaoke** is broken down into a four vowels **ka-ra-o-ke**, thus



The rest is simple, word 3 starts with the same vowel **ka** as karaoke, so word 3 is (C) katana. Word 6 ends with the same vowel **ke** as karaoke, so word 6 is (F) sake. Word 4 and 3 have the same **ta** as second vowel, so word 4 is (A) atari. Word 5 and 4 have the same **i** as last vowel, so word 5 is (E) koi. Word 5 and 1 have the same **i** as last and second vowels, respectively, so word 1 is (B) haiku. The remaining word 2 is (D) kimono.

The answer is $\boxed{1B\ 2D\ 3C\ 4A\ 5E\ 6F.}$ □

Solution. HC-2021-SM2-R4-P15 Below is one solution, the three red circles are the ones to be moved. The blue circles are the new neighbouring placements of the red circles.



The answer is $\boxed{\text{d9d8 f6g6 g8f9.}}$ □

10.5 Reserved

Problem 10.5.1 (HC-2021-SM2-R4-P16)**.** (*Advanced Level*)

What is puzzle?

An $n \times n$ $(3 \leq n \leq 10)$ board is called *good* board if

- In each of its n^2 cells, exactly one of the digits $\{0, 1, \ldots, n-1\}$ is written.
- No digit appear twice in any row.
- No digit appear twice in any column.

A $n \times n$ $(3 \leq n \leq 10)$ **puzzle** is obtained from a *good* board by selecting two distinct digits that are on different rows and columns then *switch* them. It is important to note that the rows and columns are numbered from 0 (as in most of programming languages).

Figure 10.11 *shows an example, where the digit* 2 *at the intersection of row* 1 *and column* 1 *and the digit* 3 *at the intersection of row* 3 *and column* 3 *are switched.*

	0	1	2	3
0	0	2	3	1
1	2	**3**	1	0
2	1	0	2	3
3	3	1	0	**2**

$\rightarrow$

	0	1	2	3
0	0	2	3	1
1	2	**2**	1	0
2	1	0	3	2
3	3	2	0	**3**

Figure 10.11: HC-2021-SM2-R4-P15

Generating a puzzle

In order to generate a puzzle, an **input string** is given by the jury. This is a list of n digits from $\{0, 1, \ldots, n-1\}$, all digits are distinct, and they are not in any particular order. This list should be used as the row 0 of the puzzle to generate an $n \times n$ puzzle. The rest of the rows for the puzzle can be generated arbitrary, but have to obey the rules described above.

For example, 0231 *the given input string to generate the puzzle above. The puzzle generator must make sure that the puzzle size is* 4×4 *and it contains only the digits* $0, 1, 2, 3$.

Storing a puzzle

The generated puzzle then is stored in a *text file*. The file storing an $n \times n$ puzzle contains n lines, each line corresponds to a row of the puzzle. Note that each line in a text file ends with a linefeed character ($\backslash n$)

Below is the content of the file storing the generated puzzle, with the first row is the input string 0123*,*

```
0123
2110
1032
3203
```

Solve a puzzle

Now, solving puzzle is to read the content of the puzzle stored in the text file, and then try to figure out **the two (row, column) pairs** containing the swiched digits.

For example, the solution to the puzzle above is the pair of row 1 *and column* 1 *and the pair of row* 3 *and column* 3*. They should be submitted as* 11 33*. Note that there is an empty space character between* 11 *and* 33*.*

How to your troop can compete in this Coding Challenge?

- **Preparation:** first you have to study the problem specified and should fully understand the problem. If you have any questions, please ask. You have one week before the contest to make preparation based on the description of the challenge. You have to create a **generator** and a **solver**. You have to test them thoroughly to make sure that they work correctly.
- **Puzzle generating task:** At the beginning of the contest, your troop will receive **three** (03) **input strings** from the jury. You have to **generate three** (03) **puzzles** based on these inputs.

  For example if you received inputs that are 0231, 34015*, and* 021*, then you have to generate a* 4×4*, a* 5×5 *and* 3×3 *board with the digits* $\{0, 1, 2, 3\}$*;* $\{0, 1, 2, 3, 4, 5\}$*; and* $\{0, 1, 2\}$*, respectively; and using* 0231*,* 34015*, and* 021 *as the first row for each of these boards.*

  The generated boars must be stored in text files as specified above and submitted to shared folders as instructed in the contest. Your troop have 30 **minutes to generate all three** (03) **puzzles.** The jury verify your submissions. Any submissions that contain errors will be discarded. No resubmission is allowed. The correct puzzles are automatically placed to shared folders for your opponent. They are notified. **The number of correctly generated puzzles** that have been successfully submitted is recorded.
- **Puzzle solving task:** your troop always **receive three** (03) **puzzles** in the designated shared folders. Some are the correctly generated puzzles from your opponents If some puzzles generated by the opponent are errorneous, the jury will provide you with the correct puzzles generated with the same given input. Your troop have 30 **minutes to solve all** (03) **puzzles.** You submit the solutions for the puzzles in any usual way. The jury will then verify your solutions. **The number of correct solutions** will be recorded.

How your troop's performance is graded:

- The *coding score* of your troop is the sum of the number of correct puzzles generated by your troop and the number of correct solutions provided by your troop.
- If your troop's *coding score* is higher or equal to the *coding score* of you opponent then your troop scores 1 hit, otherwise 0 hit.

Solution. HC-2021-SM2-R4-P16 For the puzzle generator,

```
from random import choice, randint, sample, shuffle
from itertools import permutations


def no_fixed_point(row1, row2):
    return all([i1 != i2 for i1, i2 in zip(row1, row2)])


def get_different_digit(size, value):
    return choice(list(set(range(0, size)).difference({value})))


def save_board(r, m):
    name = 'puzzle-' + ''.join('%s' % e for e in r) + '.txt'
    with open(name, 'wt') as f:
        s = '\n'.join(''.join('%s' % e for e in r) for r in m)
        f.write(s)
        print(s)


if __name__ == "__main__":
    size = choice([3,4,5,6,7,8,9,10])

    first_row = list(range(0, size))
    shuffle(first_row)
    board = [first_row]
    permutation_list = permutations(first_row)

    while len(board) < size:
        for permutation in permutation_list:
            has_overlapping = False
            for row in board:
                if not no_fixed_point(permutation, row):
                    has_overlapping = True
                    break
            if not has_overlapping:
                board.append(list(permutation))
                break

    row1, col1 = randint(0, size-1), randint(0, size-1)
    while True:
        row2, col2 = get_different_digit(size, row1), get_different_digit(size, col1)
        if board[row1][col1] != board[row2][col2]:
            board[row1][col1], board[row2][col2] = board[row2][col2], board[row1][col1]
            break

    save_board(first_row, board)
```

For the puzzle solver,

```
import sys


def read_board(n):
    with open(n, 'rt') as f:
        return [[int(e) for e in r.strip()] for r in f.readlines()]


def print_board(m):
    print('\n'.join(''.join('%s' % e for e in r) for r in m))


if __name__ == "__main__":
    board = read_board(sys.argv[1])
    size = len(board)

    pos = []
    for i in range(0, size):
        if len(set(board[i])) < size:
            bug = sum([j if board[i].count(j) == 2 else 0 for j in range(0, size)])
            fix = set(range(0, size)).difference(set(board[i])).pop()
            for j in range(0, size):
                if  board[i][j] == bug and fix not in [board[i][j] for i in range(0, size)]:
                    pos.append([i,j])

    row1, col1 = pos[0]
    row2, col2 = pos[1]
    board[row1][col1], board[row2][col2] = board[row2][col2], board[row1][col1]
    print_board(board)
    print('%s%s %s%s' % (row1, col1, row2, col2))
```

□

Chapter 11

MIC R4

11.1 Topics

Algebra

1. Second-degree Identities. Higher-degree Identities.
2. Inequalities. Comparison Method. Least Value of Sum of Squares. AM-GM Inequality.
3. Sequences.
4. Powers and Exponents.
5. Fractions. Compare fractions.

Combinatorics

1. Counting.
2. Permutation. Correction over Counting.
3. Recurrence Relations. Periodic Sequences.
4. Counting in two ways.
5. Sets.

Geometry

1. Angle Chasing.
2. Triangles. Similar Triangles.
3. Paralellograms.
4. Length and Ratios.
5. Geometric Inequalities. Median of a triangle.

Number Theory

1. Divisibility. Euclidean Algorithm. Greatest Common Divisor. Least Common Multiple.
2. Prime Factorization.
3. Diophantine Equations.
4. Modular Arithmetic. Perfect Powers.

11.2 Rules

- The total time to complete the test is 90 minutes.
- The test consists of 10 multiple-choice and 4 show-you-work problems. To answer each of 10 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 6 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.

11.3 Problems

Problem 11.3.1 (MIC-2021-SM2-R4-P1). (*6 points*)

How many pairs of positive integers (x, y) are there such that $x^2 - y^2 = 45$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) Cannot be determined

Problem 11.3.2 (MIC-2021-SM2-R4-P2). (*6 points*)

Which number is the largest one?

(A) 2^{50} (B) 3^{30} (C) 4^{24} (D) 5^{20} (E) 6^{18}

Problem 11.3.3 (MIC-2021-SM2-R4-P3). (*6 points*)

How many ways are there to sit four boys and three girls in a row if no two boys can sit next to each other? Note that all boys and all girls are considered different.

(A) 48 (B) 96 (C) 144 (D) 576 (E) 5040

Problem 11.3.4 (MIC-2021-SM2-R4-P4). (*6 points*)

Two same-size regular heptagon (regular $7-$gon, or a regular polygon with 7 sides) overlap each other. The overlapping region is a convex polygon. *For example, the Figure 11.1 shows an overlapping hexagon.*

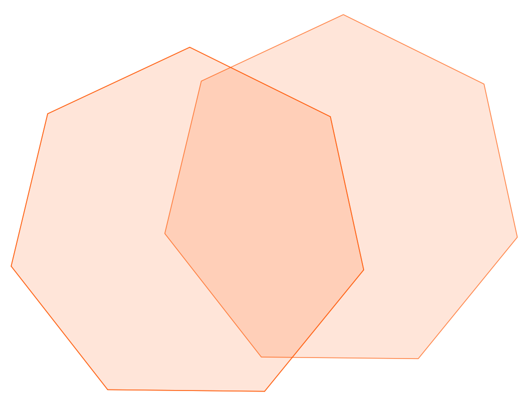


Figure 11.1: MIC-2021-SM2-R4-P4

What is the *largest* number of sides can the overlapping polygon have?

(A) 7 (B) 9 (C) 11 (D) 12 (E) 14

Problem 11.3.5 (MIC-2021-SM2-R4-P5). (*6 points*)

Anna carries six fruit baskets. The fruits in the baskets are apples, pears, and plums. The number of apples in each basket is equal to the total number of pears in all other baskets combined. The number of plums in each basket is equal to the total number of apples in all other baskets combined.

What would the number of fruits in all six baskets be if it is less then 50?

(A) 25 (B) 28 (C) 30 (D) 31 (E) 33

Problem 11.3.6 (MIC-2021-SM2-R4-P6). (*6 points*)

Let n be the *smallest positive integer* that can be written as $n = 21a + 35b$, where a and b are non-zero integers (a and b can be positive or negative integers but not zero).

What is the *least value* of $a^2 + b^2$ for such n?

(A) 2 (B) 3 (C) 5 (D) 19 (E) 64

Problem 11.3.7 (MIC-2021-SM2-R4-P7). (*6 points*)

What is the unit digit of n if $2^n + 1$ is divisible by 11?

(A) 1 (B) 2 (C) 4 (D) 5 (E) 7

Problem 11.3.8 (MIC-2021-SM2-R4-P8). (*6 points*)

In $\triangle ABC$, $AB = 5, BC = 6, CA = 7$. D, E, and F are the midpoints of BC, CA, and AB, respectively. Find

$$\frac{1}{3}(AD^2 + BE^2 + CF^2).$$

(A) 35 (B) 45 (C) 49 (D) 52 (E) 55

Problem 11.3.9 (MIC-2021-SM2-R4-P9). (*6 points*)

n is the *smallest* possible positive integer such that m and n are relatively prime, and

$$\frac{11}{25} < \frac{m}{n} < \frac{4}{9}$$

What is the value of $m + n$?

(A) 34 (B) 37 (C) 42 (D) 49 (E) 52

Problem 11.3.10 (MIC-2021-SM2-R4-P10). (*6 points*)

AB is the diameter of the circle centred at O with radius 3. C is a point on line AB, outside of the circle O, such that $CA = 2$. CT is tangent with the circle at T, and D is the intersection of line AT and the line through C perpendicular to AB. See Figure 11.2.

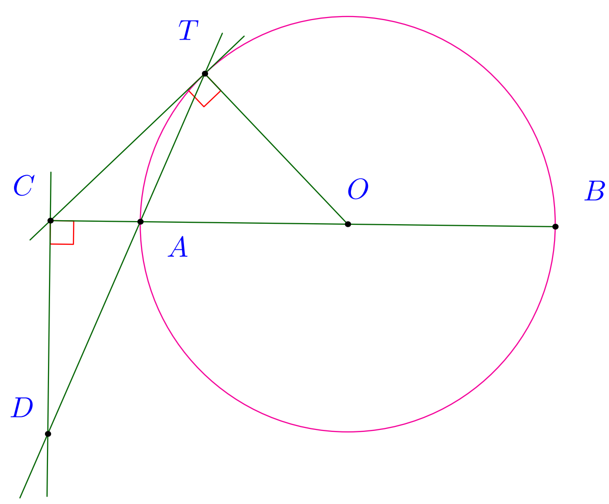


Figure 11.2: MIC-2021-SM2-R4-P10

Find the length of CD.

(A) 4 (B) 4.5 (C) 5 (D) $\frac{28}{5}$ (E) 6

Problem 11.3.11 (MIC-2021-SM2-R4-P11)**.** (*10 points*)

(*5 points*) For $0 < x < 1$, prove that
$$x(1-x) \leq \frac{1}{4}.$$

(*5 points*) For $0 < x,\ y,\ z < 1$ such that $xyz = (1-x)(1-y)(1-z)$, prove that
$$xyz \leq \frac{1}{8}.$$

Problem 11.3.12 (MIC-2021-SM2-R4-P12)**.** (*10 points*)

$ABCD$ is a paralellogram. P and Q are points outside of $ABCD$ such that $\triangle PAB \sim \triangle BCQ$, as shown in Figure 11.3.

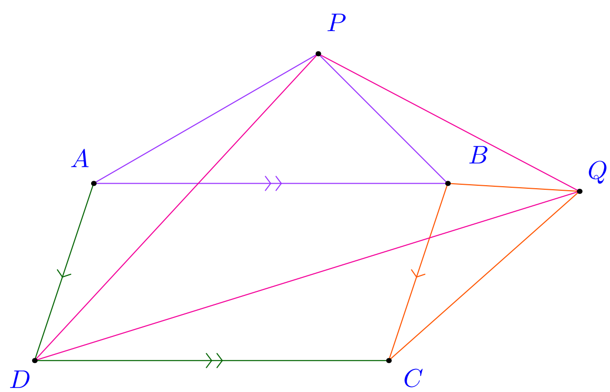


Figure 11.3: MIC-2021-SM2-R4-P12

(*5 points*) Prove that $\triangle PAD \sim \triangle DCQ$.

(*5 points*) Prove that $\triangle PDQ \sim \triangle PAB \sim \triangle BCQ$.

Problem 11.3.13 (MIC-2021-SM2-R4-P13)**.** (*10 points*)

(*5 points*) Prove that $1900^n + 121^n - 25^n - (-4)^n$ is divisible by 25.

(*5 points*) Prove that $1900^n + 121^n - 25^n - (-4)^n$ is divisible by 2000.

Problem 11.3.14 (MIC-2021-SM2-R4-P14)**.** (*10 points*)

Three pairs of twins participate in house competition. They have to be divided into different houses so that no pair of twins can be in the same house.

(*5 points*) How many ways are there to divide them into two houses of three?

(*5 points*) How many ways are there to divide them into three houses of two?

11.4 Grading

Answers for multiple-choice problems.

Problem 1: C

Problem 2: A

Problem 3: B

Problem 4: E

Problem 5: D

Problem 6: C

Problem 7: D

Problem 8: E

Problem 9: D

Problem 10: A

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 11: Separately grading for each part,

(a) 2 points if can complete the square $\left(x - \frac{1}{2}\right)^2$.

(b) 2 points if can find $(xyz)^2 = xyz(1-x)(1-y)(1-z)$.

Problem 12: Separately grading for each part,

(a) 2 points if can find the common ratio $\frac{PA}{AD} = \frac{DC}{CQ}$.

(b) 2 points if can find the corresponding pair of angles $\angle PDQ = \angle PAB$.

Problem 13: Separately grading for each part,

(a) 2 points if can use $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \ldots + y^{n-1})$.

(b) 2 points if can recognize $16 \cdot 25 = 2000$, $\gcd(16, 25) = 1$

Problem 14: Separately grading for each part,

(a) 2 points if recognize 2^3 ways to split the twins.

(b) 2 points if can find 12 ways for the first house.

11.5 Solutions

Solution. MIC-2021-SM2-R4-P1

$$x^2 - y^2 = (x-y)(x+y) = 45 \cdot 1 = 15 \cdot 3 = 9 \cdot 5 \Rightarrow \begin{cases} x+y=45, x-y=1 \Rightarrow x=32, y=22 \\ x+y=15, x-y=3 \Rightarrow x=9, y=6 \\ x+y=9, x-y=5 \Rightarrow x=7, y=2 \end{cases}$$

Thus, there are 3 such pairs. The answer is $\boxed{C.}$ □

Solution. MIC-2021-SM2-R4-P2 It is easy to see that,

$$\begin{aligned} &5^2 = 25 < 3^3 = 27 < 2^5 = 32 \Rightarrow 5^{20} < 3^{30} < 2^{50} \\ &4^{24} = 2^{48} < 2^{50} \\ &6^3 = 216 < 256 = 2^8 \Rightarrow 6^{18} = (6^3)^6 < (2^8)^6 = 2^{48} < 2^{50} \end{aligned}$$

The answer is $\boxed{A.}$ □

Solution. MIC-2021-SM2-R4-P3 Since seven persons sit on a row, and no two boys can sit next to each other, so the four boys must occupy positions $1, 3, 5$, and 7

$$B\ G\ B\ G\ B\ G\ B$$

There are $4! = 24$ ways to sit the boys, $3! = 6$ ways to sit the girls, in total $24 \cdot 6 = 144$ ways. The answer is $\boxed{C.}$ □

Solution. MIC-2021-SM2-R4-P4 The sides of the overlapping polygon are the sides of the two regular heptagons. In other words, its sides are (part of) the sides of the two polygons. So the largest number of sides it can have is the sum of the numbers of sides of both polygons. See Figure 11.4 for an example.

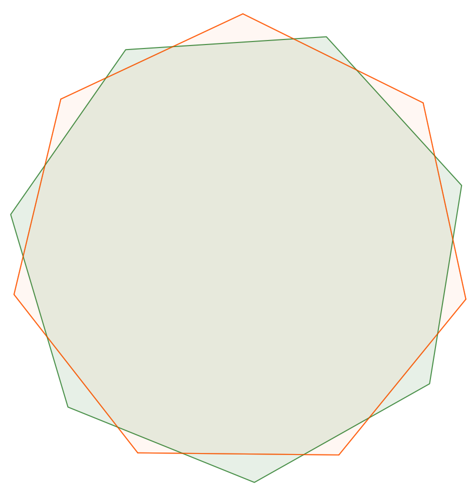


Figure 11.4: 14−gon

Thus, the largest number of sides is $7 + 7 = 14$. The answer is $\boxed{E.}$ □

Solution. MIC-2021-SM2-R4-P5 If we add the number of apples together, we count the numbers of pears in each basket 5 times. Thus, the total number of apples is 5 times the total number of pears. Similarly, the total number of plumbs is 5 times the total number of apples. So the total number of fruits is $25+5+1=31$ times the total number of pears. Since it is smaller than 50, so it has to be 31. Here is an example,

$$\begin{aligned}&\text{Basket } 1: 1 \text{ pear}, 0 \text{ apple}, 5 \text{ plums}\\&\text{Basket } 2: 0 \text{ pear}, 1 \text{ apple}, 4 \text{ plums}\\&\text{Basket } 3: 0 \text{ pear}, 1 \text{ apple}, 4 \text{ plums}\\&\text{Basket } 4: 0 \text{ pear}, 1 \text{ apple}, 4 \text{ plums}\\&\text{Basket } 5: 0 \text{ pear}, 1 \text{ apple}, 4 \text{ plums}\\&\text{Basket } 6: 0 \text{ pear}, 1 \text{ apple}, 4 \text{ plums}\end{aligned}$$

The answer is $\boxed{D.}$ □

Solution. MIC-2021-SM2-R4-P6 Note that, $n=21a+35b=7(3a+5b)$, so n is always divisible by 7, thus the least value for n is 7. Therefore $3a+5b=1$. It is easy to see that the least value for a^2+b^2 is 5, where $a=2, b=-1$.

The answer is $\boxed{C.}$ □

Solution. MIC-2021-SM2-R4-P7 The remainders when divided 2^n+1 $(n \geq 0)$ by 11 is a periodic sequence

$$\underbrace{2,3,5,9,6,0,10,8,4,7}_{\text{period } 10},2,3,\ldots$$

Thus, for every 5^{th} out of 10 in the sequence $0,1,2,\ldots$ of n, $11 \mid 2^n+1$, or $n \equiv 5 \pmod{10}$.

The answer is $\boxed{D.}$ □

Solution. MIC-2021-SM2-R4-P8 Let $AB=c, BC=a, CA=b$, then by the theorem of median in a triangle, $AD^2=\frac{1}{2}(2b^2+2c^2-a^2)$, $BE^2=\frac{1}{2}(2c^2+2a^2-b^2)$, and $CF^2=\frac{1}{2}(2a^2+2b^2-c^2)$, thus

$$\frac{1}{3}(AD^2+BE^2+CF^2)=\frac{1}{3}\frac{1}{2}(3a^2+3b^2+3c^2)=\frac{1}{2}(a^2+b^2+c^2)=\frac{1}{2}(25+36+49)=55$$

The answer is $\boxed{E.}$ □

Solution. MIC-2021-SM2-R4-P9 By the well-known fact that,

> **Claim —** If $\frac{a}{b}<\frac{c}{d}$, $bc-ad=1$, then the fraction with least denominator between $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$.

Since $25\cdot 4-9\cdot 11=1$, the fraction $\frac{m}{n}=\frac{11+4}{25+9}=\frac{15}{34}$ with the least denominator n, where

$$\frac{11}{25}<\frac{m}{n}=\frac{15}{34}<\frac{4}{9},\ m+n=49$$

The answer is $\boxed{D.}$ □

Solution. MIC-2021-SM2-R4-P10 Connect BT, see Figure 11.5. It is easy to see that

$$\angle TBA = 90^\circ - \angle TAB = 90^\circ - \angle CAD = \angle CDA,\ \angle CTA = \frac{1}{2}\overset{\frown}{TA} = \angle TBA \Rightarrow \angle CDT = \angle CTD$$

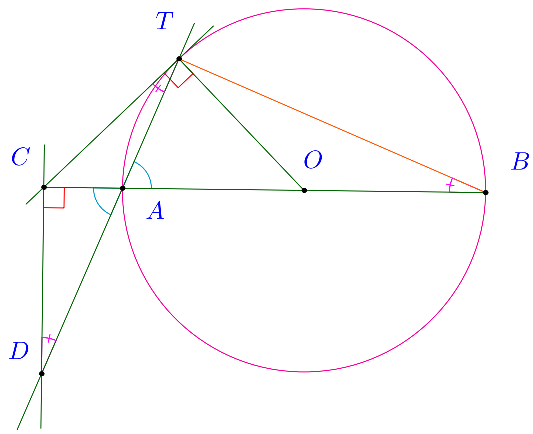


Figure 11.5: $\angle CDT = \angle CTD$

Thus $\triangle CTD$ is isosceles, so $CD = CT = \sqrt{CO^2 - OT^2} = \sqrt{(3+2)^2 - 3^2} = 4$.

The answer is $\boxed{A.}$ □

Solution. MIC-2021-SM2-R4-P11 For the first question,

$$x(1-x) \leq \frac{1}{4} \Leftrightarrow x^2 - x + \frac{1}{4} \geq 0 \Leftrightarrow x^2 - 2\cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 \geq 0 \Leftrightarrow \left(x - \frac{1}{2}\right)^2 \geq 0.$$

Applying the result of the first question, and note that $(xyz)^2 = xyz(1-x)(1-y)(1-z)$,

$$(xyz)^2 = x(1-x)y(1-y)z(1-z) \leq \left(\frac{1}{4}\right)^3 = \left(\frac{1}{8}\right)^2 \Rightarrow (xyz)^2 \leq \left(\frac{1}{8}\right)^2 \Rightarrow xyz \leq \frac{1}{8}.$$

□

Solution. MIC-2021-SM2-R4-P12 For the first question, see Figure 11.6.

$$\left.\begin{aligned}\triangle PAB \sim \triangle BCQ \Rightarrow \frac{PA}{BC} = \frac{AB}{CQ} \Rightarrow \frac{PA}{AD} = \frac{DC}{CQ} \\ \angle PAD = \angle PAB + \angle BAD = \angle BCQ + \angle BCD = \angle DCQ\end{aligned}\right\} \Rightarrow \triangle PAD \sim \triangle DCQ.$$

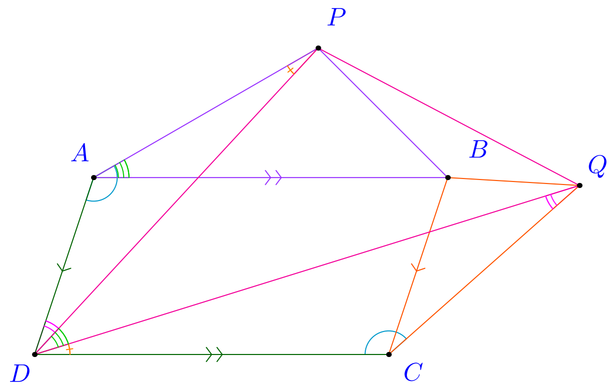


Figure 11.6: $\triangle PAD \sim \triangle DCQ$

Now, for the second question,

$$\triangle PAD \sim \triangle DCQ \Rightarrow \angle PDA = \angle DQC,\ \angle APD = \angle QDC,\ \frac{PD}{PQ} = \frac{PA}{DC} = \frac{PA}{AB} \quad (1)$$
$$\angle PDQ = \angle ADC - \angle PDA - \angle QDC = \angle ADC + \angle DAP - 180^\circ = \angle PAB \Rightarrow \angle PDQ = \angle PAB \quad (2)$$
$$(1)\ \&\ (2) \Rightarrow \triangle PDQ \sim \triangle PAB.$$

□

Solution. MIC-2021-SM2-R4-P13 Since, $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \ldots + y^{n-1})$, let $S = 1900^n - 25^n + 121^n - (-4)^n$, then

$$1900^n - 25^n + 121^n - (-4)^n = (1875)(1900^{n-1} + \ldots + 25^{n-1}) + (125)(121^{n-1} + \ldots + (-4)^{n-1}) \Rightarrow 25 \mid S$$
$$1900^n - (-4)^n + 121^n - 25^n = (1904)(1900^{n-1} + \ldots + (-4)^{n-1}) + (96)(121^{n-1} + \ldots + (25)^{n-1}) \Rightarrow 16 \mid S$$

Because $\gcd(16, 25) = 1$, thus $\boxed{2000 = 16 \cdot 25 \mid S.}$ □

Solution. MIC-2021-SM2-R4-P14 For each pair of twins, there are 2 ways to put one twin in the first house and the other into the other house. The two houses can be swapped, so the number of ways is $\boxed{\frac{1}{2}2^3 = 4}$ ways.

Now, for the first house, there is 6 ways to chose the first person, 4 ways to chose the second person. Since the order of selection does not matter, so $\frac{6 \cdot 4}{2!} = 12$ ways. For second house, there remain 4 persons, a pair of twins and two other people. There are $\binom{4}{2} - 1 = 5$ ways to choose two persons who do not belong to the pair of twins. The remaining two persons form the last house. Now, since there are $3! = 6$ ways to permute the houses, thus the total number is $\boxed{\frac{12 \cdot 5}{6} = 10}$ ways. □

Chapter 12

HC R5

12.1 Topics

Chess

1. Moves: capture, check, double check, promotion, stalemate, checkmate
2. Analysis: whose turn is it.
3. Retrograde: last move, last two moves, last multiple of moves.
4. Proof games: White and Black, instead of making the best moves, they co-operate to make the game legally into a set position.
5. Symmetric position with odd number of moves.
6. Switchback: a piece moved out and came back into that same square.

Logic

1. Casework: what if A is true, what if A is false.
2. Process of elimination: If A is not true, B is not true, then C should be true.
3. Reverse argument: if there is at least ..., then there is atmost ...
4. Implication from truth: if A told the truth and A said X, then X is true.
5. Conflict of truth: if B said A lied, then both cannot be truth tellers.

Algebra

1. Equations. Quadratic equations.
2. Third-degree identities.
3. Inequality. Comparison Method. Percents and Ratios.
4. Functional Equations. Sequence of functions.

Combinatorics

1. Combinatorial Geometry.
2. Grids. Tilings.
3. Sets. Sums of numbers in a set.
4. Algorithms and Processes.
5. Colourings.
6. Games. Winning positions. Losing positions. Invariants.

Geometry

1. Triangles. Right Triangles. Pythagoream Theorem.
2. Triangles and circumcircle. Computing radius of circumcircle.
3. Areas. Multiple ways to compute area of a triangle.
4. Geometric Inequality. Triangle has maximal area with two given sides.
5. Same-perimeter geometric figures with different areas.
6. Tangent circles.

Number Theory

1. Sums and differences of several integers to produce different numbers.
2. Number bases. Base-3.
3. Consecutive integers.
4. Perfect powers.

12.2 Problems

Problem 12.2.1 (HC-2021-SM2-R5-P1)**.** (*Beginner Level*)

In the Figure 12.1 below, a pentagon (5−sided polygon), a hexagon (6−sided polygon), and a heptagon (7−sided polygon) are created by using several equilateral triangles and squares with sides of the same length.

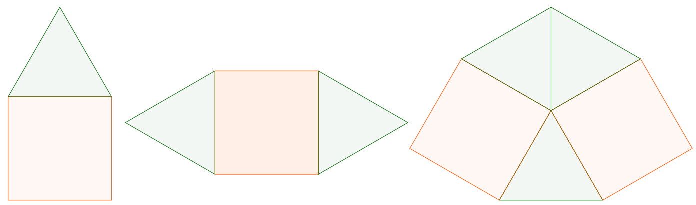


Figure 12.1: HC-2021-SM2-R5-P1

Is it possible to make an *octagon* (8−sided polygon), a *nonagon* (9−sided polygon), and a *decagon* (10−sided polygon) by using same size equilateral triangles and squares?

How to provide your answer:

- If you think you can, show a diagram containing your solutions for some or all desired polygons.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that has *strictly larger* number of polygons than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has *strictly smaller* number of polygons than your answer.

Problem 12.2.2 (HC-2021-SM2-R5-P2)**.** (*Beginner Level*)

Chi has two 10−liter jars. In the first jar, there was 6 litres of wine. In the second jar, there was 6 litres of water. In every five minutes,

- She poured half of the content of the first jar into the second jar (so the second jar has 9 litres of mixture), mixed it well, then
- She poured third of the content of the second jar into the first jar (so each of both jars has exactly 6 lites of mixture).

She started at 4:00 PM. *At what time* the amounts of wine in both jar are the same?

How to provide your answer:

- If you think she can achieved the balance of wine in both jars art 5:00 PM, then submit 5:00 PM.
- If you think that it cannot be determined, then submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.3 (HC-2021-SM2-R5-P3). (*Beginner Level*)

In triangle ABC, $AB = 9$, $BC = 40$. The area of $\triangle ABC$ is 180.

What is the *length* of CA?

How to provide your answer:

- If you think the length of CA is 20, then submit 20.
- If you think that it cannot be determined, then submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.4 (HC-2021-SM2-R5-P4). (*Beginner Level*)

There are bags with a total weight of 9 kgs that should be transported by carts. None of the bags is heavier than 1 kg and each cart has a capacity of 3 kgs.

What is the *minimal number* of necessary carts such that the bags can be transported at the same time *for sure.* Give *an example of bags* to show that a smaller number of carts cannot perform so.

How to provide your answer:

- If you think that 2 carts are enough and your example is $1, 0.9, 0.8, 1, \ldots$ (*you have to complete the list* and make sure that their sum equal to 9kg) as the weights of the bags that 1 cart cannot carry, then submit $2C$, $\{1, 0.9, 0.8, 1, \ldots\}$ (*you have to complete the list* and make sure that their sum equal to 9kg)
- If you think that it cannot be determined, then submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that has *strictly larger* number than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has *strictly smaller* number than your answer.

Problem 12.2.5 (HC-2021-SM2-R5-P5). (*Beginner Level*)

One main diagonal of the 8×8 chessboard is removed, leaving two triangular parts, as shown in Figure 12.2 below.

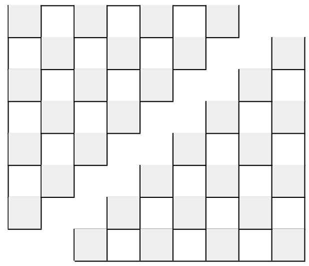


Figure 12.2: HC-2021-SM2-R5-P5

What is the *largest number* of 1×2 dominoes can be used to tile the two triangular parts?

How to provide your answer:

- If you think that at most 20 dominoes can be used to tile board, then submit 20.
- If you think that it cannot be determined, then submit 0 and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that has *strictly smaller* number than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has *strictly larger* number than your answer.

Problem 12.2.6 (HC-2021-SM2-R5-P6). (*Intermediate Level*)

In the Figure 12.3 below, 12 match sticks are used to make a square with area of 9, and the same 12 match sticks can be reused to make a cross with area of 5.

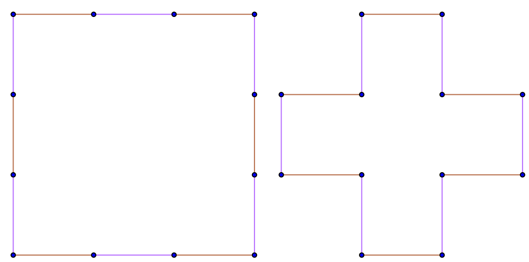


Figure 12.3: HC-2021-SM2-R5-P6

Is it possible to make three figures, each must use all 12 match sticks, so that they have an area of 6, 4, and $3\sqrt{3}$, respectively.

How to provide your answer:

- If you think you can, show a diagram containing your solutions for some or all figures with desired aera.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that has *strictly larger* number of figures than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has *strictly smaller* number of figures than your answer.

Problem 12.2.7 (HC-2021-SM2-R5-P7)**.** (*Intermediate Level*)

On the Planet of the Apes, there live the *green* apes and *red* apes. Furthermore these inhabitants were born in the northern or southern hemisphere. It is known that,

- the green northeners and the red southerners always tell the truth, and
- the red northeners and the green southerners always lie.

Melanie visited the Planet of the Apes on a bright day. At the center plaza, she met a native, who said that *"I am a green northerner"*. In broad daylight, she saw what colour the native was, but from the answer, she did not know where the native was from.

What *colour* was he? Was he a *northerner* or a *southerner*?

How to provide your answer:

- If you think he was red, and he was from the northern hemisphere, then submit RN.
- If you think he was green, and he was from the southern hemisphere, then submit GS.
- If you cannot determine what colour he was, but you know that he was from the northern hemisphere, then submit $0N$.
- If you can neither determine what colour he was, nor where he was from, then submit 00.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.8 (HC-2021-SM2-R5-P8)**.** (*Intermediate Level*)

Albert needs to make some sticks to use as ruler for the construction of his sandcastle. He wants to measure distances in centimetres. The sticks have very hard surface, so he cannot make any marks on them. However, he can use multiple sticks at once when measuring distances.

For example if he can measure a distance of 50 *centimetres by using two sticks with length* 20 *and* 30 *centimetres, or by using three sticks with lengths of* 20, 30, *and* 40 *as shown below in the Figure 12.4.*

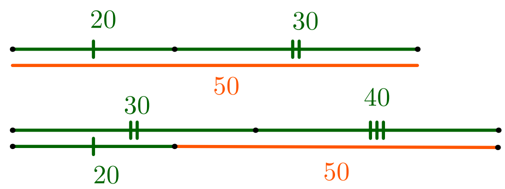


Figure 12.4: Measuring 50 cm in two ways

What is the *least number of the sticks* and with what *lengths* can Albert use to measure any distances of $10, 20, 30, \ldots, 120$, and 130 centimetres.

How to provide your answer:

- If you think he needs the $10, 20$, and 30 centimeter long sticks, submit $\{10, 20, 30\}$ with a detailed list how can you measure of **all** of $10, 20, 30, \ldots, 120$, and 130 centimetres, *for example*

$$10 = 10,\ 20 = 20,\ 30 = 30,\ 40 = 10 + 30,\ 50 = 20 + 30,\ 60 = 10 + 20 + 30.$$

- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that has *strictly larger* number of sticks than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has *strictly smaller* number of sticks than your answer.

Problem 12.2.9 (HC-2021-SM2-R5-P9)**.** (*Intermediate Level*)

$ABCD$ is a rectangle, $AD = 5$, $DC = 7$. $(O_1), (O_2)$, and (O_3) are three unit circles inside $ABCD$, where O_1 is at the distance of 1 from both AD and AB, O_2 is at the distances of 1 and 2 from DC and CB, respectively, and O_2 is at the distance of 1 from both CB and BA. Circle (O) is tangent to all three circles $(O_1), (O_2)$, and (O_3).

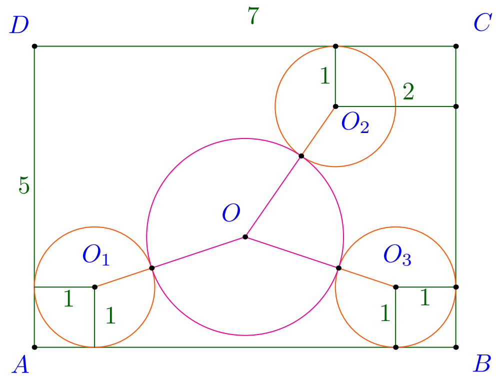


Figure 12.5: HC-2021-SM2-R5-P9

What is the *radius* of the circle (O)?

How to provide your answer:

- If you think the radius is 2, submit 2.
- If you cannot determine that, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.10 (HC-2021-SM2-R5-P10)**.** (*Intermediate Level*)

n, m, and p are three consecutive positive integers ($m < n < p$). Find all the three-digit numbers $\overline{ABC}$ (the leading digit A is not zero) such that,

$$\begin{cases} \overline{ABC} = p^3 - n^3 \\ \overline{ACB} = n^3 - m^3 \\ \overline{ABC} + \overline{ACB} = 488 \end{cases}$$

How to provide your answer:

- If you think all three-digit numbers ABC can be $\{123, 456\}$, submit $\{123, 456\}$.
- If you cannot determine that, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.11 (HC-2021-SM2-R5-P11). (*Advanced Level*)

In the Figure 12.6 below, which symbol is *the most* different?

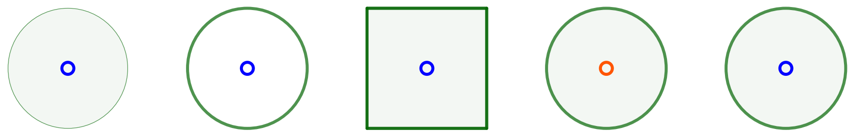


Figure 12.6: HC-2021-SM2-R5-P11

How to provide your answer:

- If you think it is the second symbol from from the left, submit 2.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.12 (HC-2021-SM2-R5-P12). (*Advanced Level*)

Inspector Minh examined the dossier of the newest crime. Here were what discovered so far,

- If Alpha is guilty and Beta is innocent, then Gamma is guilty.
- Gamma never works alone.
- Alpha nevers works with Gamma.
- Beta never works alone.
- No one other than Alpha, Beta, or Gamma was involved, and at least one of them is guilty.

Find all the *guilty persons* for each possible scenario.

How to provide your answer:

- If you think Alpha is guilty, then submit A.
- If you think Alpha or Alpha and Beta are guilty, then submit $A,\ AB$.
- If you cannot determine who are guilty, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that has *strictly larger* number of possiblities than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that has *strictly smaller* number of possiblities than your answer.

Problem 12.2.13 (HC-2021-SM2-R5-P13)**.** (*Advanced Level*)

For all real number $x \neq 1$, let

$$f(x) = \frac{1}{1-x}.$$

For k positive integer, let

$$f_1(x) = f(x),\ f_{k+1}(x) = f(f_k(x)).$$

Find $f_{2021}(2021)$.

How to provide your answer:

- If you think that $f_{2021}(2021) = \frac{123}{23}$, then submit $\frac{123}{23}$.
- If you cannot determine that, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.14 (HC-2021-SM2-R5-P14)**.** (*Advanced Level*)

Clemence was kidnapped by Baba Yaga, the fearsome witch. She was put in a maximum-security prison.

The prison cell that hold her has some dozens doors from left to right. The doors on the left seem weaker, and the guards who brought foods used to come through them. To the right, the doors seem to be hardened, she did not remember seeing anybody coming through them. She knows that the door leading to the garden, where from she can escape, is *the last weak* door from the left. Unfortunately she does not know exactly where at the weak doors end or where from the hardened doors start.

Figure 12.7 shows an example. The green doors are the weak, the brown doors are the hardened ones.

Figure 12.7: HC-2021-SM2-R5-P14

She gathered the bones in the cell and made *two bone keys*, and she knew that,

- The bone keys are weak, *they would break instantly* on *the hardened doors.*
- A bone key would break after *three times of use* on *the weak doors.*
- The last weak door is somewhere *among the first* 12 *doors* from the left.

Can she escape the prison? *How would she use the keys?*

How to provide your answer:

- If you think that she had an escape strategy, submit a detailed solution explaining how would that work. *For example,*

  She use the first key on the first door, then the second door, and so on until it breaks.

  Then she use the second key on the next door, then the one after that door, and so on until it breaks.

  In the worse case she can open all the first 6 *doors.*

  (This solution works only if the door to the garden is within the first 6 *doors, thus it is incorrect).*

- If you cannot determine that, then submit 0.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 12.2.15 (HC-2021-SM2-R5-P15). (*Advanced Level*)

Linh and Minh play a two-player game by taking turns removing some marbles from a pile. In each turn the player of the turn is permitted to remove a number of marbles that is a *divisor of the number of marbles* remaining in the pile. The one that removes the last marble loses.

They decided to play 6 games, each starts with $5, 6, 7, 8, 9$, or 10 marbles in the pile, such that

- If the number of initial marbles in the pile is $5, 7, 10$, then Linh starts the game.
- Minh starts all the other games.

Who *won* which game?

How to provide your answer:

- If you think Linh won the games $5, 7, 10$, and Minh won the rest, submit $5L, 6M, 7L, 8M, 9M, 10L$.
- If you cannot determine that, submit 0, and give a detailed reasoning.

How your answer is graded for this problem:

- When your troop submit an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *answer* that contains *strictly smaller* number of correct outcomes than your answer, then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that contains *strictly larger* number of correct outcomes than your answer.

12.3 Answers

Problem 1: See the solution.

Problem 2: 0

Problem 3: 41

Problem 4: $4C,\ \{0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9\}$

Problem 5: 24

Problem 6: See the solution.

Problem 7: $G0$

Problem 8: $10, 30, 90$

Problem 9: $\frac{5\sqrt{10}}{6} - 1$

Problem 10: $\{271\}$

Problem 11: 5

Problem 12: $AB,\ BC$

Problem 13: $\frac{2020}{2021}$

Problem 14: See the solution.

Problem 15: $\{5M, 6M, 7M, 8M, 9L, 10L\}$

12.4 Solutions

Solution. HC-2021-SM2-R5-P1 The Figure 12.8 below shows an example of an octagon (8−sided polygon), a nonagon (9−sided polygon), and a decagon (10−sided polygon) created by using same-size equilateral triangles and squares. □

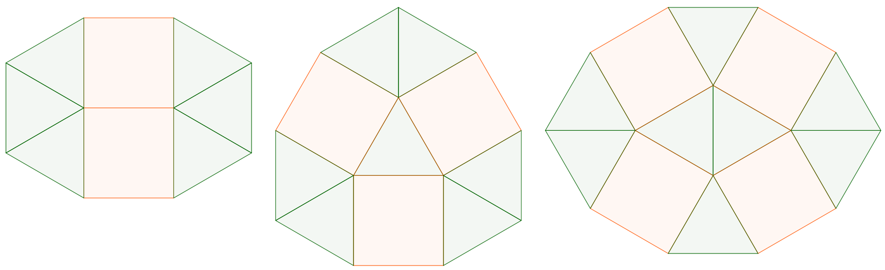


Figure 12.8: An octagon, a nonagon, and a decagon

Solution. HC-2021-SM2-R5-P2 At the beginning, the concentration of wine (the amount of wine over the whole content) in the first jar was higher than the second jar. After the first pouring, the concentration of wine in the second jar was weaker then in the first jar (because of more water). After the second pouring, the concentration of wine in the first jar was weaker, but still higher than the second jar. It does not matter how many turns Chi tried, the concentration of wine in both jar can never be the same.

The answer was $\boxed{0.}$ □

Solution. HC-2021-SM2-R5-P3 The area of $\triangle ABC$ cannot exceed $\frac{1}{2} \cdot AB \cdot BC = \frac{1}{2} \cdot 9 \cdot 40 = 180$. This equality $[ABC] = 180$ means that $AB \perp BC$, thus $\triangle ABC$ is a right triangle at B.

Therefore, $\boxed{AC = \sqrt{AB^2 + BC^2} = \sqrt{9^2 + 40^2} = 41.}$ □

Solution. HC-2021-SM2-R5-P4 Consider 10 bags of 0.9 kg each; clearly these cannot be packed into 3 carts.

$$\underbrace{0.9, 0.9, 0.9}_{\text{cart 1}}\ \underbrace{0.9, 0.9, 0.9}_{\text{cart 2}}\ \underbrace{0.9, 0.9, 0.9}_{\text{cart 3}}\ \underbrace{0.9}_{\text{cart 4}}$$

Now, we prove that 4 carts are enough. First, place bags on the first cart using a *greedy algorithm,* i.e. we choose the bags to maximize the weight in the first cart, noticing that each bag can be packed with greater than 2 kgs else we can pack more. Therefore, the weight on the first three carts is greater 6, meaning the remaining mass of less than 3 kgs can be packed on the 4^{th} truck.

Thus the answer is $\boxed{4C.}$ □

Solution. HC-2021-SM2-R5-P5 Note that each triangular part contains 16 square of one colour and 12 of the other. A domino contains one square for each colours, so the total numbers of squares for each colour the dominoes cover are the same. Thus, at most 12 dominoes can be used to tile it. Below Figure 12.9 shows an example of tiling with 12 dominoes.

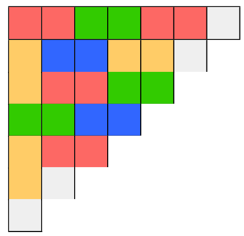


Figure 12.9: Tiling with 12 dominoes

Therefore, the maximum number of dominoes can be used for both part is $\boxed{2 \cdot 12 = 24.}$ □

Solution. HC-2021-SM2-R5-P6 The Figure 12.10 below shows an example of three figures with areas 6, 4, and $3\sqrt{3}$. Note that the rightmost figure is made of 12 equilateral triangles with side length 1, so its area is $12 \cdot \frac{1^2 \cdot \sqrt{3}}{4} = 3\sqrt{3}$. □

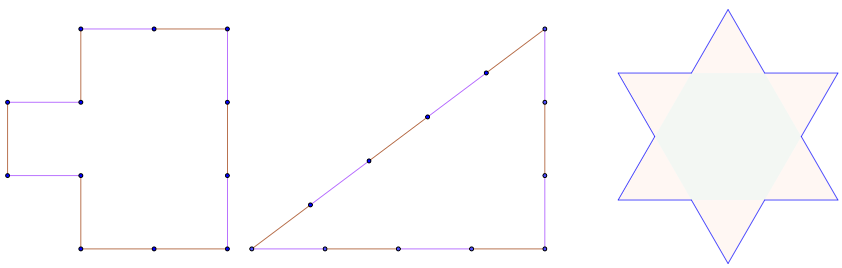


Figure 12.10: Three figures with areas 6, 4, and $3\sqrt{3}$

Solution. HC-2021-SM2-R5-P7 If the native had been red, then Melanie would have known that he was a northerner, since the red southerner would never claimed to be a green northerner. Because Melanie would not know where the native was from, so the native was green. It is impossible to know if he was a truth teller (green northerner) or a liar (green southerner).

The answer is $\boxed{G0.}$ □

Solution. HC-2021-SM2-R5-P8 We show how to construct the number $1, 2, 3, \ldots, 13$ from the numbers $1, 3, 9$ using additions and subtraction,

$$1 = 1, 2 = 3 - 1, 3 = 3, 4 = 3 + 1, 5 = 9 - 3 - 1, 6 = 9 - 3, 7 = 9 - (3 - 1),$$
$$8 = 9 - 1, 9 = 9, 10 = 9 + 1, 11 = 9 + (3 - 1), 12 = 9 + 3, 13 = 9 + 3 + 1$$

To explain the solution to this problem we need the following interesting property of the base 3 system. Every natural number can be represented as the difference of two numbers whose base 3 representations contain only 0's and 1's. We can prove this property by writing the original number in the base 3 system and constructing the required numbers digit by digit from right to left.

The answer is $\boxed{10, 30, 90.}$ □

Solution. HC-2021-SM2-R5-P9 First, it is easy to see that O is also the centre of the circle through O_1, O_2, and O_3. So it is the circumcenter of $\triangle O_1O_2O_3$, see Figure 12.11.

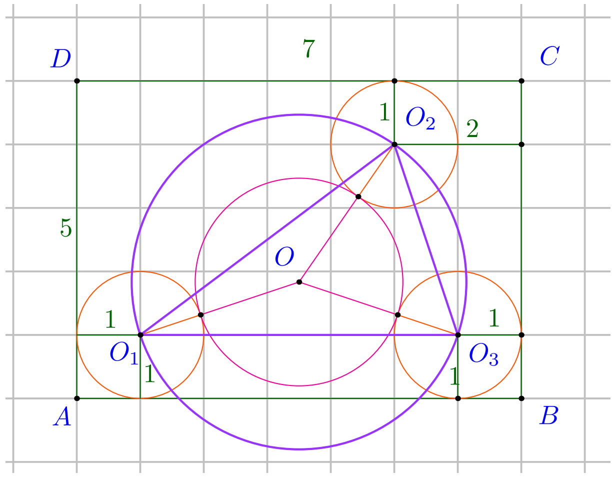


Figure 12.11: Circle $(O_1O_2O_3)$

Since $O_1O_2 = 5$, $O_2O_3 = \sqrt{10}$, $O_3O_1 = 5$, and $[O_1O_2O_3] = \frac{15}{2}$ thus according to the formula $[ABC] = \frac{abc}{4R}$,

$$R_{(O_1O_2O_3)} = \frac{O_1O_2 \cdot O_2O_3 \cdot O_3O_1}{4[O_1O_2O_3]} = \frac{25\sqrt{10}}{4 \cdot \frac{15}{2}} = \frac{5\sqrt{10}}{6}$$

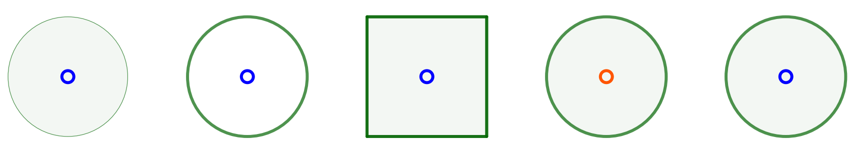
The answer is $\boxed{\frac{5\sqrt{10}}{6} - 1}$ □

Solution. HC-2021-SM2-R5-P10 First, since m, n, p are three (increasing) consecutive positive integers, so $p = m + 2, n = m + 1$, thus

$$\begin{cases} \overline{ABC} = (m+2)^3 - (m+1)^3 \\ \overline{ACB} = (m+1)^3 - m^3 \\ \overline{ABC} + \overline{ACB} = 488 \end{cases} \Rightarrow (m+2)^3 - m^3 = 488 \Rightarrow 6m^2 + 12m - 480 = 0 \Rightarrow 6(m+10)(m-8) = 0$$

Therefore $m = 8$, so $ABC = 10^3 - 9^3 = 271$. The answer is $\{271\}$. □

Solution. HC-2021-SM2-R5-P11 The first symbol from the left differs from the others because it has a *light* border, the second one because *not being shaded*, the third one because *not being a circle*, and the fourth one because it has a *red center*.



The last one, $\boxed{\text{fifth from the left}}$ is not unique with respect to any property, therefore it is the *most different*. □

Solution. HC-2021-SM2-R5-P12 Assume that Beta is innocent, then if Alpha is guilty, then Gamma too (according to the first statement), but then it means Alpha and Gamma worked together, which is not possible (according to the third statement). Thus, Alpha is innocent, but then Gamma is the only one who is guilty. This is impossible because it contradicts the second statement. Therefore Beta is guilty. Since Beta never work alone (according to the fourth statement), one of Alpha and Gamma is involved, but not both (according to the third statement).

Thus, the possible cases are $\boxed{AB,\ BC.}$ □

Solution. HC-2021-SM2-R5-P13 Note that,

$$f_2(x) = \frac{1}{1-f_1(x)} = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x} \Rightarrow f_3(x) = \frac{1}{1-\frac{x-1}{x}} = x$$

thus, the sequence $\{f_k(x)\}$

$$\underbrace{\frac{1}{1-x},\ \frac{x-1}{x},\ x,}_{\text{period 3}}\ \frac{1}{1-x}, \ldots,$$

is a periodic sequence with period 3. Therefore $f_{2021}(x) = f_2(x) = \frac{x-1}{x}$, thus $\boxed{f_{2021}(2021) = \frac{2020}{2021}.}$ □

Solution. HC-2021-SM2-R5-P14 *The key idea is to use the first key to anchor at doors* $4, 8,$ *and* 12*. If the first key successful, there is nothing to try. If it is broken, then we reach the hardened doors or all* 12 *doors, then the first key accomplished its task. In anycase, when the first key has done its job then we know that the door to the garden is on the left of the last tried door. Thus the door to the garden is between a pair of anchors, for example* 4 *and* 8*, once we know the door to the garden is between them, then there are exactly three doors* $5, 6, 7$ *for the second key to try.*

Clemence tries the first key on the sequence of the doors $4, 8,$ and 12.

Each time there are several possiblities, (i) the door opens, and it is the door to the garden, then it's done. (ii) if the key is broken before or when reaching 12, then the door to the garden is on the left of it. (iii) if she can open the door 4, but it does not lead to the garden, then the door to the garden is on the right of it.

For case (iii), she should go next on the sequence of the doors $4, 8,$ and 12. For case (ii), she should tries the second key on the three doors on the left of the door she just open ($4, 8,$ and 12), For 4, they are $1, 2, 3$, for 8, they are $5, 6, 7$, for 12, they are $10, 11, 12$. □

Solution. HC-2021-SM2-R5-P15 It is easy to see that the *losing positions* are the odd numbers. We know that every divisor of an odd number is odd. So, if we have an odd number left, for the next step we leave an even number. If we have an even number left, we can remove exactly one marble. In this way we leave an odd number and we have not lost the game.

If the game starts with an *odd* number of marbles, then the *second player wins*, otherwise the first player.

Thus, the answer is $\boxed{\{5M, 6M, 7M, 8M, 9L, 10L\}.}$ □

Chapter 13

AMC8 R1

13.1 Selected Problems

Problems

1. Practice 1, Problem 24: Gauss Grade 8 2009/24
2. Practice 1, Problem 25: Gauss Grade 8 2019/25
3. Practice 2, Problem 24: Gauss Grade 8 2014/24
4. Practice 2, Problem 25: Gauss Grade 8 2016/25
5. Practice 3, Problem 24: Gauss Grade 8 2007/24
6. Practice 3, Problem 25: Gauss Grade 8 2014/25
7. Practice 4, Problem 24: Gauss Grade 8 2008/24
8. Practice 4, Problem 25: Gauss Grade 8 2008/25
9. Practice 5, Problem 24: Gauss Grade 8 2018/24
10. Practice 5, Problem 25: Gauss Grade 8 2012/25

13.2 Problems

Example 13.2.1 (Gauss G8 2009/24)

Starting at point P, Breenah constructs a straight sided spiral so that

1. all angles are $90°$,
2. after starting with a line segment of length 1, each side is 1 longer than the previous side.

After completing the side with length 21, what will Breenah's distance be from her original starting point P?

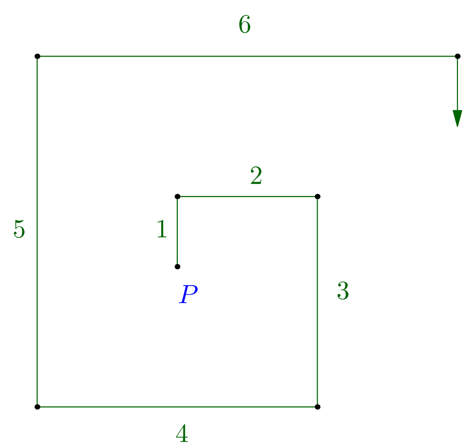


Figure 13.1: Gauss G8 2009/24

> **Remark** (First strategy). Gauss G8 2009/24
>
> Let P_i be the endpoint of the spiral after the construction step i^{th}. *We compare the positions of the endpoints comparing to the initial point P.* It is easy to see that the point P_4 is reached after four moves: up $+1$, right $+2$, down -3, left -4. So the position of P_4 is *two* units down $((+1)+(-3)=-2)$ and *two* units left $((+2)+(-4)=-2)$ of P. Similarly P_8 is reached from P_4 after four moves: up $+5$, right $+6$, down -7, left -8, so it is *two* units down and *two* units left of P_4, in other words P_8 is *four* units down and *four* units left of P_4.

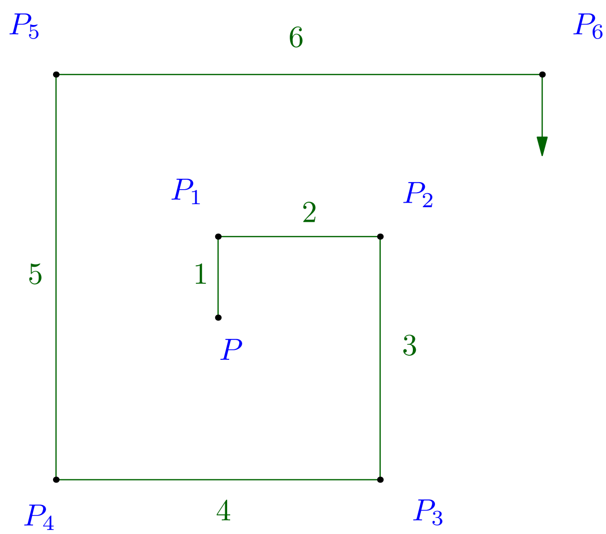


Figure 13.2: P_4 and P_0

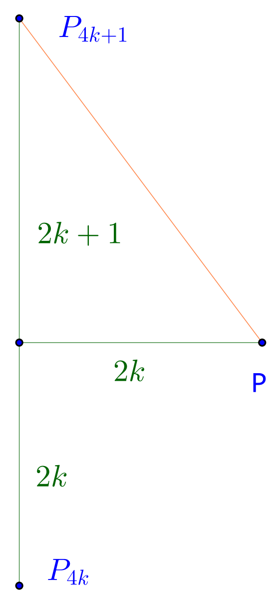


Figure 13.3: P_{4k+1} and P

First proof. Gauss G8 2009/24 Let P_i denote the endpoint of the spiral after the construction step i^{th}, where $P_0 \equiv P$. Note that the point P_{4k+4} $(k \geq 0)$ is reached from eached from P_{4k} after four moves: up $+(4k+1)$, right $+(4k+2)$, down $-(4k+3)$, left $-(4k+4)$, thus it is *two* units down and *two* units left of P_{4k}. Therefore P_{4k} is $2k$ units down and $2k$ units left of P_0, or P. Thus P_{4k+1} is at $(-2k)+(4k+1)=2k+1$ units up and $2k$ unit left of P.

Therefore $P_{4k+1}P$ is the hypotenuse of a right triangles with two sides with length of $2k$ and $2k+1$,

$$P_{4k+1}P = \sqrt{(2k+1)^2 + (2k)^2}$$

For $4k+1=21$, $k=5$, thus $\boxed{P_{21}P = \sqrt{11^2+10^2} = \sqrt{221}.}$ □

Remark (Second strategy)**.** Gauss G8 2009/24

In this strategy, we set P as the origin in an $x-$ and $y-$ coordinate system. Let P_i be the endpoint of the spiral after the construction step i^{th}. Instead of manualy computing the coordinates of P_i, *we spot patterns in positions of* P_i. By continuing drawing a few more points, it is easy to see that points $P_1, P_5, P_9, \ldots$ or P_{4k+1} $(k \geq 0)$ all lie on the ray ℓ_4 starting from $P_1(0,1)$ then $P_5(-2,3), P_9(-4,5), \ldots$.

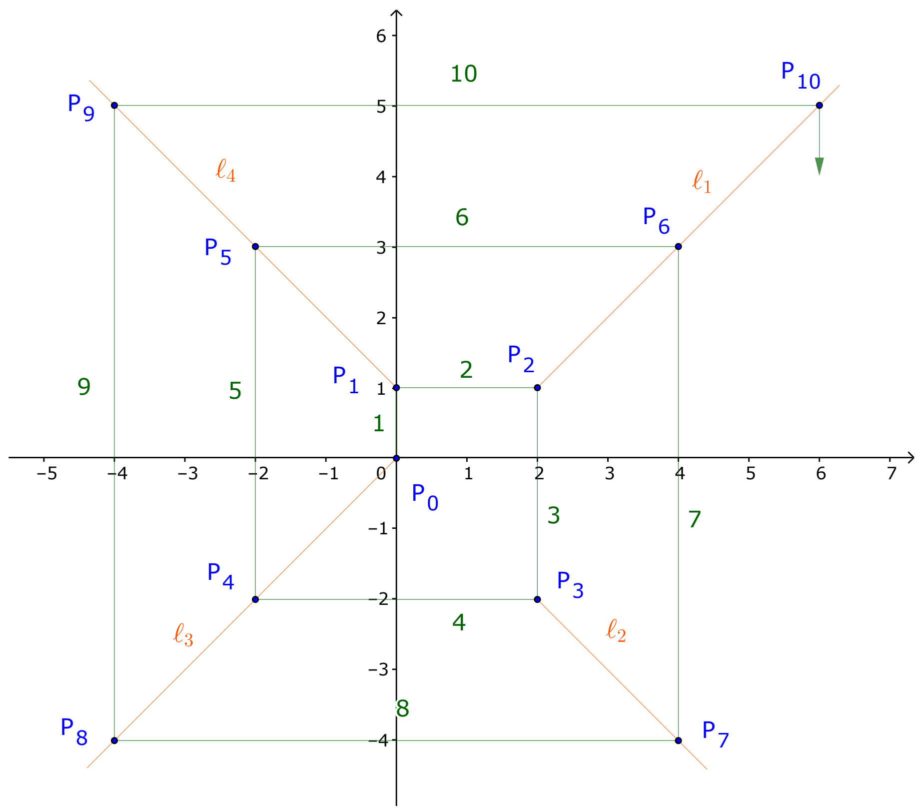


Figure 13.4: Ray $l_1, \ldots, l_4$

Second proof. Gauss G8 2009/24 Let P_i denote the endpoint of the spiral after the construction step i^{th}, where $P_0 \equiv P$. Furthermore let $P(0,0)$ be the origin, each move up, right, down, left is equivalent to moving the point along the $x-$ or $y-$axis in natural direction.

It is easy to see that the set of points $\{P_{4k+1},\ k \geq 0\}$ lie on the ray ℓ_4, similarly $\{P_{4k+2},\ k \geq 0\}$ on ℓ_1, $\{P_{4k+3},\ k \geq 0\}$ on ℓ_2, and $\{P_{4k+4},\ k \geq 0\}$ on l_3,

$$P_1(0,1),\ P_5(-2,3),\ P_9(-4,5),\ P_{4k+1}(-2k, 2k+1), \ldots$$

Therefore the distance $P_{4k+1}P$ is

$$P_{4k+1}P = \sqrt{(x_{P_{4k+1}} - x_P)^2 + (y_{P_{4k+1}} - y_P)^2} = \sqrt{((-2k)-0)^2 + ((2k+1)-0)^2} = \sqrt{(2k+1)^2 + (2k)^2}$$

For $4k+1 = 21$, $k = 5$, thus $\boxed{P_{21}P = \sqrt{11^2 + 10^2} = \sqrt{221}.}$ □

Example 13.2.2 (Gauss G8 2019/25)

In quadrilateral $PQRS$, diagonals PR and QS intersect at O inside $PQRS$, $SP = SQ = SR = 1$, and $\angle QSR = 2\angle QSP$. Marc determines the measure of the twelve angles that are the interior angles of $\triangle POS$, $\triangle POQ$, $\triangle ROS$, and $\triangle ROQ$. The measure of each of these angles, in degrees, is a positive integer, and exactly six of these integers are prime numbers. See Figure 13.5.

How many different quadrilaterals have these properties and are not rotations or translations of each other?

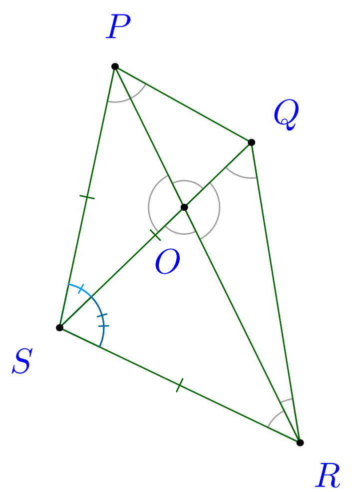


Figure 13.5: Gauss G8 2019/25

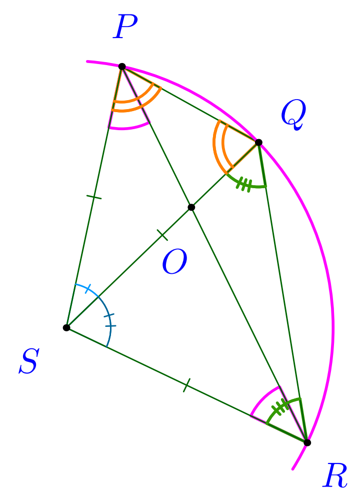


Figure 13.6: What do we know?

Remark. Gauss G8 2019/25 Let analyze what are given by the problem.

First, the two diagonals PR and QS intersect at a point O inside $PQRS$, so the quadrilateral $PQRS$ is *convex*. Thus any of the angles $\angle PQR$, $\angle QRS$, $\angle RSP$, and $\angle SPQ$ does not exceed $180°$, so the angles of $\triangle POS$, $\triangle POQ$, $\triangle ROS$, and $\triangle ROQ$, at P, Q, R, and S are parts of the interior angles of the quadrilateral $PQRS$, for example for the angles with Q as their vertex,

$$\angle PQR = \angle PQO + \angle OQR.$$

This seemingly unimportant fact is actually automatically assumed to be true when we look at the diagram.

Furthermore, $SP = SQ = SR = 1$ implies that P, Q, and R are on the circle radius 1 centred at S. In addition, $\triangle PSQ, \triangle QSR$, and $\triangle PSR$ are isosceles, see Figure 13.6, therefore

$$\angle SPQ = \angle SRQ,\ \angle OQR = \angle ORQ,\ \angle SPO = \angle SRQ.$$

But there are so many angles involved! What should we do? Intuitively, the angles seems to be related to each other, because *the sum of the angles in a triangle is* $180°$, so we would

1. denote the given angles in the relations $\angle QSR = 2\angle QSP$ as *initial* variables, and
2. use them to *establish some relations among other angles as equations*, and then
3. by *solving the equations to determine the values of the angles.*

Note that $\angle SPQ = \angle SQP = 90° - \frac{1}{2}\angle PSQ$, thus the measure of $\angle PSQ$ should be an **even integer**, therefore by denoting $\angle PSQ = 2\theta$, then $\angle QSR = 4\theta$, and $\angle SPQ = \angle SQP = 90° - \theta$. This is a good starting point.

Proof. Gauss G8 2019/25 Let $\angle PSQ = 2\theta$, then $\angle QSR = 2\angle PSQ = 4\theta$,

$$\angle SPQ = \angle SQP = 90^\circ - \theta,\ \angle SQR = \angle SRQ = 90^\circ - 2\theta,\ \angle SPR = \angle SRP = 90^\circ - 3\theta$$
$$\angle OPQ = \angle SPQ - \angle RPQ = (90^\circ - \theta) - (90^\circ - 3\theta) = 2\theta$$
$$\angle POQ = 180^\circ - \angle OPQ - \angle OQP = 180^\circ - 2\theta - (90^\circ - \theta) = 90^\circ - \theta$$
$$\angle ORQ = \angle SRQ - \angle PRQ = (90^\circ - 2\theta) - (90^\circ - 3\theta) = \theta$$
$$\angle ROQ = 180^\circ - \angle ORQ - \angle OQR = 180^\circ - \theta - (90^\circ - 2\theta) = 90^\circ + \theta$$
$$\angle SQP = \angle RQO = 90^\circ + \theta,\ \angle SOR = \angle POQ = 90^\circ - \theta$$

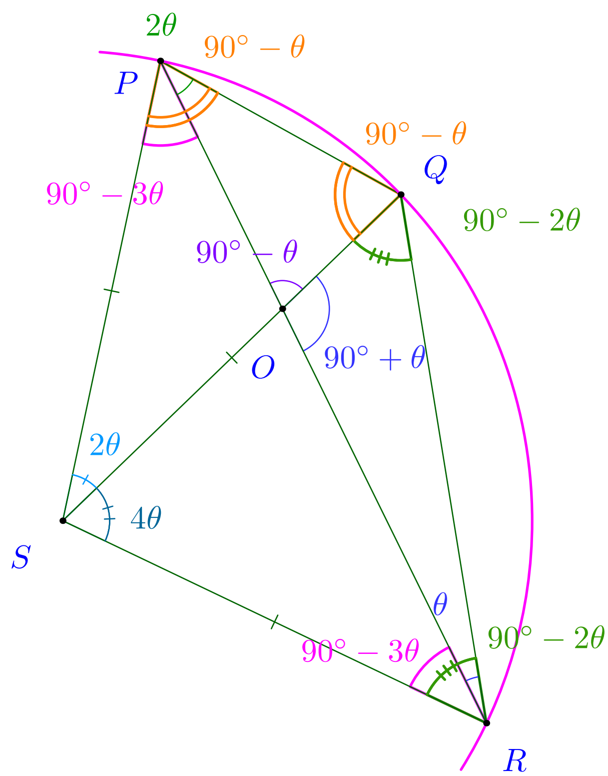


Figure 13.7: $\angle PSQ = 2\theta$

Thus the measures of the twelve angles,

$$\begin{cases} \angle ORQ & = \theta \\ \angle OPQ = \angle OSP & = 2\theta \\ \angle OSR & = 4\theta \\ \angle SRO = \angle SPO & = 90^\circ - 3\theta \\ \angle PQO & = 90^\circ - 2\theta \\ \angle POQ = \angle PQO = \angle SOR & = 90^\circ - \theta \\ \angle POS = \angle QOR & = 90^\circ + \theta \end{cases}$$

The measures of the twelve angles are integers and they are elements of the sequence below,

$$\theta,\ 2\theta,\ 2\theta,\ 4\theta,\ 90^\circ - 3\theta,\ 90^\circ - 3\theta,\ 90^\circ - 2\theta,\ 90^\circ - \theta,\ 90^\circ - \theta,\ 90^\circ - \theta,\ 90^\circ + \theta,\ 90^\circ + \theta.$$

We need to determine with what value of θ, there are exactly six angles have measures as prime numbers.

Case 1: 2θ is a prime, $2\theta = 2^\circ$, in this case, the sequence becomes

$$1^\circ,\ 2^\circ,\ 2^\circ,\ 4^\circ,\ 87^\circ,\ 87^\circ,\ 88^\circ,\ 89^\circ,\ 89^\circ,\ 89^\circ,\ 91^\circ,\ 91^\circ,$$

where only two copies of 2 and three copies of 89 are prime numbers.

Case 2: $\theta > 1$ (thus 2θ is not a prime). Therefore $2\theta,\ 2\theta,\ 4\theta$ are all composite. Since $\angle SRO = \angle SPO = 90^\circ - 3\theta > 0$, thus $\theta < 30^\circ$.

Case 2a: $90^\circ - 3\theta$ is a prime number. It means that $90^\circ - 3\theta = 3(30^\circ - \theta)$, so $\theta = 29^\circ$. In this case, the sequence becomes

$$29^\circ,\ 58^\circ,\ 58^\circ,\ 116^\circ,\ 3^\circ,\ 3^\circ,\ 29^\circ,\ 32^\circ,\ 61^\circ,\ 61^\circ,\ 61^\circ,\ 119^\circ,\ 119^\circ,$$

where there are exactly six prime numbers: one copy of 29, two copies of 3, and three copies of 61.

Case 2b: $90^\circ - 3\theta$ is not a prime number, then $1^\circ < \theta < 29^\circ$. Since $90^\circ - 2\theta = 2(45^\circ - \theta)$ is an even number and larger than 2, so it is a composite number. In this case six angles with composite measureas are

$$2\theta,\ 2\theta,\ 4\theta,\ 90^\circ - 3\theta,\ 90^\circ - 3\theta,\ 90^\circ - 2\theta.$$

That means all six angles with measureas as prime numbers are

$$\theta,\ 90^\circ - \theta,\ 90^\circ - \theta,\ 90^\circ - \theta,\ 90^\circ + \theta,\ 90^\circ + \theta.$$

By testing with all θ as prime numbers where $1^\circ < \theta < 29^\circ$, only $7^\circ, 11^\circ, 17^\circ, 19^\circ$, and 23° satisfy the given conditions. Thus, there are $\boxed{6}$ possible cases, where $\theta = \frac{1}{2}\angle PSQ \in \{7^\circ, 11^\circ, 17^\circ, 19^\circ, 23^\circ, 29^\circ\}$. $\square$

Example 13.2.3 (Gauss G8 2014/24)

Grid are formed using 1×1 squares. The grid shown in Figure 13.8 contains squares of sizes $1 \times 1, 2 \times 2, 3 \times 3$, and 4×4, for a total of exactly 30 squares.

Which of the following grids in Figure 13.9 contains exactly 24 squares?

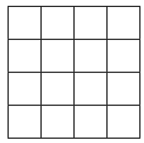


Figure 13.8: 30 squares in a 4×4 grid

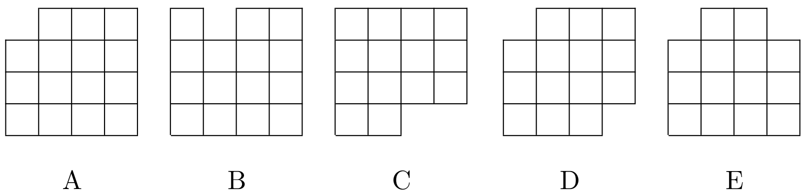


Figure 13.9: Which grid among A, B, C, D and E contains exactly 24 squares?

Remark. Gauss G8 2014/24

This problem is a classic problem that can be solved with *complementary counting* technique.

1. We count *all the possible squares* of sizes $1 \times 1, 2 \times 2, 3 \times 3$, and 4×4, which already given by the problem as a total of exactly 30 squares, then
2. Subtract the number of squares of sizes $1 \times 1, 2 \times 2, 3 \times 3$, and 4×4 *covering part of or the whole missing region.*

For example, the Figure 13.10 shows three possible squares covering the missing square at the top-left corner.

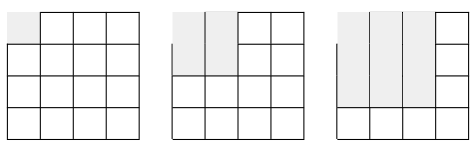


Figure 13.10: Three squares 1×1, 2×2, 3×3 cover the missing square

Proof. Gauss G8 2014/24

First, the number of *all the possible squares* of sizes $1 \times 1, 2 \times 2, 3 \times 3$, and 4×4, is a total of 30 squares.

By direct counting the number of missing squares of sizes $1 \times 1,\ 2 \times 2,\ 3 \times 3$, and 4×4, for case A and B are $1 + 1 + 1 + 1 = 4$ and $1 + 2 + 2 + 1 = 6$, resulting in the total number of squares are $30 - 4 = 26$, and $30 - 6 = 24$, respectively.

For each of C, D, and E case the number of missing squares is $2 + 2 + 2 + 1 = 7$, so the total number of squares is $30 - 7 = 23$. Thus, the answer is $\boxed{B.}$

The figure Figure 13.11 below shows six missing squares for the case B, □

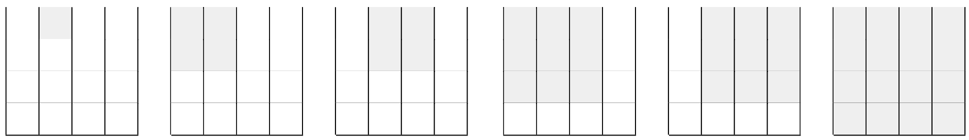


Figure 13.11: 6 missing squares

Example 13.2.4 (Gauss G8 2016/25)

In the table, the numbers in each row form an arithmetic sequence when read from left to right. Similarly, the numbers in each column form an arithmetic sequence when read from top to bottom.

What is the sum of the digits of the value of x?

				18
	43			
		40		
x			26	

Figure 13.12: Gauss G8 2016/25

Remark. Gauss G8 2016/25

The key idea to solve this problem is to use the fact that *every three consecutive cells in a row or in a column from an arithmetic sequence,*

1. we can *start at an empty cell* next to some cells with existing value, use *a variable to denote its value*,
2. then use the fact to find the value of the neighbour cells, and continue until,
3. the relations of the cells lead to an equation that we can solve in order to obtain the value of the variable

In the Figure 13.13, the cells above and below of the number 40, together with 40, form an arithmetic sequence, so let d be the *common difference* of that sequence, then the value in cell above 40 is $40-d$, the value in the cell immediately below 40 is $40+d$, and the one below that is $40+2d$. We continue to populate the cells in the order shows in Figure 13.14. The numbers $40+2d$, 26, and ⑤ form an arithmetic sequence, this is what we use to determine the value of d.

				18
	43	$40-d$		
		40		
		$40+d$		
x		$40+2d$	26	

Figure 13.13: Column containing 40

				18
	43	$40-d$	①	②
		40		③
		$40+d$		④
x		$40+2d$	26	⑤

Figure 13.14: Order for populating

Proof. Gauss G8 2016/25

The cells above and below of the number 40, together with 40, form an arithmetic sequence, so let d be the *common difference* of that sequence, then the value in cell ⓐ above 40 is $40 - d$, the value in the cell ⓑ immediately below 40 is $40 + d$, and the one ⓒ below that is $40 + 2d$.

Similarly we populate the cells from ① to ⑤ as shown below.

				18
	43	$40 - d$ ⓐ	$37 - 2d$ ①	$34 - 3d$ ②
		40		$50 - 6d$ ③
		$40 + d$ ⓑ		$66 - 9d$ ④
x		$40 + 2d$ ⓒ	26	$82 - 12d$ ⑤

Figure 13.15: Populating the cells

The numbers $40 + 2d$, 26, and $82 - 12d$ form an arithmetic sequence, thus

$$(40 + 2d) + (82 - 12d) = 2 \cdot 26 \Rightarrow 10d = 70 \Rightarrow d = 7.$$

By substituting $d = 7$ and calculate the rest of the bottom row,

				18
	43	33	23	13
		40		8
		47		3
110	82	54	26	-2

Thus, $x = 110$ and the sum of its digits is $1 + 1 + 0 = 2$. □

Example 13.2.5 (Gauss G8 2007/24)

A lattice point is a point (x, y), with x and y both integers. For example, $(2, 3)$ is a lattice point but $(4, \frac{1}{3})$ is not.

In the diagram, how many lattice points lie on the perimeter of the triangle?

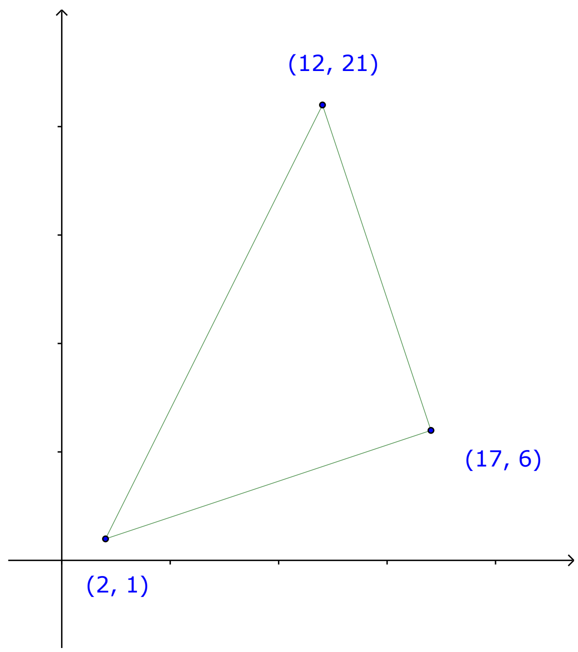


Figure 13.16: Gauss G8 2007/24

Remark. Gauss G8 2007/24

Drawing lines through the given points and parallel to the $x-$ and $y-$axis. Assume that E is a lattice point on the segment connecting $(2,1)$ and $(17,6)$. Let the distance from E to the axis-parallel lines be p and q as shown in Figure 13.17, then

1. Both p and q are positive integers, $1 \le p \le 14$ and $1 \le q \le 4$,
2. By similar triangles $\frac{p}{15} = \frac{5-q}{5}$, or $p = 15 - 3q$.

Now, it is simple to find how many such pairs of integers (p,q) are there to satisfy the inequalities and equality above.

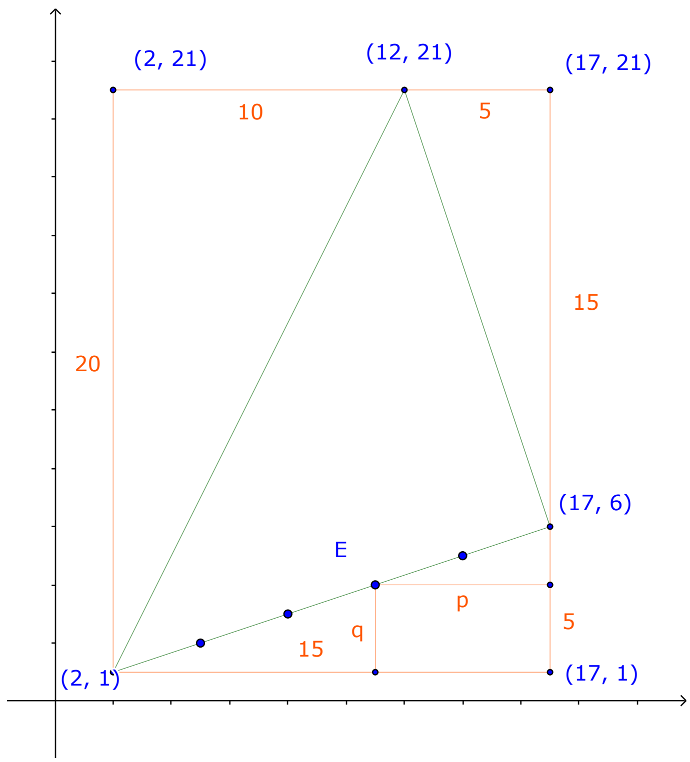


Figure 13.17: 4 lattice points on segment connecting $(2,1)$ and $(17,6)$

Proof. Gauss G8 2007/24

Drawing lines through the given points and parallel to the $x-$ and $y-$axis. Let E be a lattice point on the segment connecting $(2,1)$ and $(17,6)$, and let the distances from E to the axis-parallel lines be p and q as shown in Figure 13.17, then

$$p \in \{1,2,\ldots,15\},\ q \in \{1,2,\ldots,5\},\ \frac{p}{15} = \frac{5-q}{5}$$
$$\Rightarrow p = 15 - 3q \Rightarrow (p,q) \in \{(12,1),(9,2),(6,3),(3,4)\}$$

Similarly, there are 4 lattice points on the segment connecting $(17,6)$ and $(12,21)$; and 9 lattice points on the segment connecting $(2,1)$ and $(12,21)$. Thus, there are all together $\boxed{4+4+9+3=20}$ lattice points on the perimeter of the triangle. □

Example 13.2.6 (Gauss G8 2014/25)

Residents were surveyed in order to determine which flowers to plant in the new Public Garden. A total of N people participated in the survey. Exactly $\frac{9}{14}$ of those surveyed said that the colour of the flower was important. Exactly $\frac{7}{12}$ of those surveyed said that the smell of the flower was important. In total, 753 people said that both the colour and smell were important.

How many possible values are there for N?

Remark. Gauss G8 2014/25

The difficulty of the problem is that the quantities are given as fractions of unknown value N. Our approach is to *turn N into a multiple of the least common multiple of the denominators* of the fractions. This ways the fractions can be transformed into integer variables. The rest is based *using a Venn-diagram to estimate*,

1. the lowest value of k based on the number of common people in both sets, and
2. the highest value of k based on the number of people not in any of the sets.

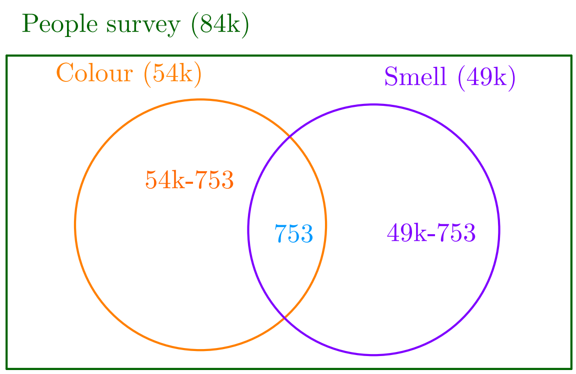


Figure 13.18: Gauss G8 2014/25

Proof. Gauss G8 2014/25 Since the least common multiple of the denimonator $\text{lcm}(12, 14) = 84$, thus $84 \mid N$. Let $N = 84k$, where k is a positive integer.

The minimum value of k, since 753 is the number of peole in both sets, so

$$\begin{cases} 54k - 753 \geq 0 \\ 49k - 753 \geq 0 \end{cases} \Rightarrow 49k - 753 \geq 0 \Rightarrow k \geq 16$$

The maximum value of k, the number of people outside of both sets is

$$84k - (54k - 753) - (49k - 753) - 753 = 753 - 19k \geq 0 \Rightarrow k \leq 39$$

It is easy to see that all values of k between 16 and 39, including both 16 and 39, satisfy the given conditions. Thus there are $\boxed{39 - 16 + 1 = 24}$ such values. □

Example 13.2.7 (Gauss G8 2008/24)
The sum of all of the digits of the integers from 98 to 101 is

$$9+8+9+9+1+0+0+1+0+1=38$$

What was the sum all of the digits of the integers from 1 to 2008?

Remark. Gauss G8 2008/24

The key of the problem is to *to recognize the digits are repeated in a sequence of* $0, 1, 2, \ldots, 9$ regardless what position (unit, tens, hundreds, ...) it might be, for example,

$$01, 02, \ldots, 09, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, \ldots, 29, 30, \ldots, 90, 91, \ldots, 99$$

The unit digits repeat as $0, 1, 2, \ldots, 9$. The tens digits in a block of 10 digits 0, and then 10 digits 1, ..., and 10 digits 9. This pattern helps to make the counting easier and with less error.

Proof. Gauss G8 2008/25 First, for the single-digit numbers from 1 to 9, the sum of all digits of them is

$$1+2+\ldots+9=45.$$

Case 1: Now, the sum of all digits of the numbers from 2000 to 2008 is $2 \cdot 9+45-9=54$.

Case 2: If we look at the numbers from 1 to 1999 as all four-digit numbers such that any number with less than 4 digit will have some *padding* of 0, for example

$$2=0002,\ 34=0034,\ 763=0763.$$

This way it is easier to count the digits in the block of 2000 numbers from 0001 to 1999, as follow

1. Out of 2000 numbers, 200 of them has 0 as unit digit, 200 of them has 1 as unit digit, ..., and 200 of them has 9 as unit digit, so the sum of all unit digits in these 2000 numbers is

$$200(1+2+\ldots+9)=200 \cdot 45=9000.$$

2. Similarly, the sum of all tens digits in these 2000 numbers is also 9000, and of course
3. The sum of all hundreds digits in these 2000 numbers is also 9000.
4. Finally, the sum of all thousands digits in these 2000 numbers is 1000 (from 1000 to 1999).

Thus the desired sum is $\boxed{54+3 \cdot 9000+1000=28054.}$ □

Example 13.2.8 (Gauss G8 2008/25)

Chantelle had two candles, one of which was 32 cm longer than the other. She lit the longer one at 3 p.m. and lit the shorter one at 7 p.m. At 9 p.m., they were both the same length. The longer one was completely burned out at 10 p.m. and the shorter one was completely burned at midnight. The two candles burned at different, but constant, rates.

What was the sum of the original lengths of the two candles?

Remark. Gauss G8 2008/25

The key of the problem is to *find the way to compare the rate of burnings,* which is given by the problem. At 9 p.m. they have equal length and burned out at different times.

Proof. Gauss G8 2008/25 Since the length of the candles was equal at 9 p.m., the longer one burned out at 10 p.m., and the shorter one burned out at midnight, then it took 1 hour for the longer candle and 3 hours for the shorter candle to burn this equal length.

Therefore, the longer candle burned 3 times as quickly as the shorter candle Suppose that the shorter candle burned x cm per hour, then the longer candle burned $3x$ cm per hour.

From its lighting at 3 p.m. to 9 p.m., the longer candles burned for 6 hours, so burned $6 \cdot 3x = 18x$ cm From its lighting at 7 p.m. to 9 p.m., the shorter candle burned for 2 hours, so burns $2 \cdot x = 2x$ cm. But, up to 9 p.m., the longer candle burned 32 cm more than the shorter candle, since it began 32 cm longer. Therefore,

$$18x - 2x = 32 \Rightarrow 16x = 32 \Rightarrow x = 2$$

In summary, the shorter candle burned for 5 hours at 2 cm per hour, so its initial length was 10 cm. Also, the longer candle burned for 7 hours at 6 cm per hour, so its initial length was 42 cm. Thus, the sum of the original lengths is $\boxed{10 + 42 = 52}$ cm. □

Example 13.2.9 (Gauss G8 2018/24)

Lynne chooses four distinct digits from 1 to 9 and arranges them to form the 24 possible four-digit numbers. These 24 numbers are added together giving the result N.

For all possible choices of the four distinct digits, what is the largest sum of the distinct prime factors of N?

Remark. Gauss G8 2018/24

The simplest approach is to use variables. Let a, b, c, d denote four different digits that Lynne chooses from $1, 2, \ldots, 9$. Then each number, for example $\overline{abcd}$, can be written as

$$\overline{abcd} = 1000a + 100b + 10c + d$$

For a digit, says a, to be as a thousands digit, there are $3! = 6$ ways to permute b, c, d as hundreds, tens, and unit digits. Thus the sum of all numbers is

$$6(a + b + c + d) \cdot (1000 + 100 + 10 + 1).$$

Proof. Gauss G8 2018/24 Let a, b, c, d denote four different digits such that the sum of distinct prime factors of N is the largest. Easy to see that the sum of all permutations of $\overline{abcd}$ is the sum of all permutations of a, b, c, d in $\overline{abcd} = 1000a + 100b + 10c + d$, thus

$$N = 6(a + b + c + d) \cdot (1000 + 100 + 10 + 1) = 2 \cdot 3 \cdot 11 \cdot 101 \cdot (a + b + c + d)$$

Note that $2, 3, 11, 101$ are already prime factors of N, so $a + b + c + d$ should not have any of those as a factor. $1 + 2 + 3 + 4 = 10 \leq a + b + c + d \leq 6 + 7 + 8 + 9 = 30$, thus 29 is the largest possible prime factor.

Therefore the largest sum of all prime factors of N is $\boxed{2 + 3 + 11 + 29 + 101 = 146.}$ □

Example 13.2.10 (Gauss G8 2012/25)
In rectangle $WXYZ$, the parallelogram $PQRS$ is formed as shown. The segment PT is perpendicular to SR.

What is the length of ST?

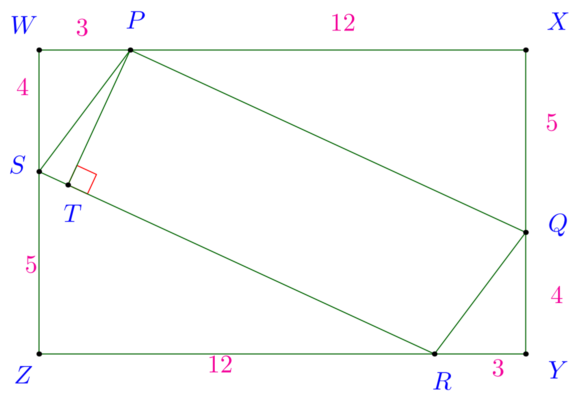


Figure 13.19: Gauss G8 2012/25

Remark. Gauss G8 2012/25

There are several approaches to solve this problem. Here, we consider an approach based on *finding the length of* PT *based on the area of* $PQRS$.

Proof. Gauss G8 2012/25 The area of $PQRS$ is

$$[WXYZ] - 2[WPS] - 2[PXQ] = 9 \cdot 15 - 3 \cdot 4 - 5 \cdot 12 = 63$$

Since $\triangle PXQ$ is right triangle, so $PQ = \sqrt{12^2 + 5^2} = 13$. Similarly $PS = 5$. Therefore $PT = \frac{63}{13}$, and $ST = \sqrt{5^2 - \left(\frac{63}{13}\right)^2} = \frac{16}{13}$. $\square$

Chapter 14

IC Level Test - November

14.1 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

14.2 Problems for Level 1

Problem 14.2.1 (ICLT-2021-SM2-11-L1-P1). (*4 points*)

What is the smallest positive integer that does not divide 2520?

(A) 7 (B) 9 (C) 11 (D) 13 (E) 17

Problem 14.2.2 (ICLT-2021-SM2-11-L1-P2). (*4 points*)

Compute

$$\frac{15 \times 6^{(2^2-3)}}{3^2 + 3 \times 2}$$

(A) 5 (B) 6 (C) 8 (D) 9 (E) 15

Problem 14.2.3 (ICLT-2021-SM2-11-L1-P3). (*4 points*)

The angles in a triangle are in the ratio $1 : 3 : 8$. What is the difference in measure of the largest and smallest angles?

(A) 60° (B) 75° (C) 80° (D) 90° (E) 105°

Problem 14.2.4 (ICLT-2021-SM2-11-L1-P4). (*6 points*)

Points A, B, C, D, and E are five points on a line segment with endpoints A and E. The points are in the order listed above from left to right such that $CD = \frac{1}{2}AB$, $BC = \frac{1}{2}CD$, $AB = \frac{1}{2}AE$, and $AE = 10$.

What is the length of AD?

(A) 6 (B) 6.25 (C) 8.25 (D) 8.75 (E) 9

Problem 14.2.5 (ICLT-2021-SM2-11-L1-P5). (*6 points*)

How many sets of four consecutive positive integers are there such that the product of the four integers is less than 10000?

(A) 8 (B) 9 (C) 10 (D) 16 (E) 17

Problem 14.2.6 (ICLT-2021-SM2-11-L1-P6). (*6 points*)

The grid shown in Figure 14.1 contains squares of sizes $1 \times 1, 2 \times 2, 3 \times 3$, and 4×4.

In total how many squares are there?

(A) 24 (B) 28 (C) 30 (D) 36 (E) 48

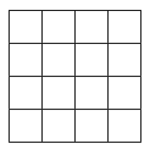


Figure 14.1: ICLT-2021-SM2-11-L1-P6

Problem 14.2.7 (ICLT-2021-SM2-11-L1-P7). (*10 points*)

At the end of 1998 Oanh was half as old as her sister. The sum of the years in which they were born is 3990.

1. (*5 points*) Let Oanh be y years old in 1998. Establish the equation expressing the sum of the years in which Oanh and her sister were born to be 3990.
2. (*5 points*) How old will her sister be at the end of 2021?

Problem 14.2.8 (ICLT-2021-SM2-11-L1-P8). (*10 points*)

p is a prime factor of the positive integer n if p is a divisor of n and p is a prime number. For example 3 is a prime factor of 24, but 6 is not.

1. (*5 points*) How many prime factors does 24 have?
2. (*5 points*) Find the smallest composite number that has no prime factor less than 5.

Problem 14.2.9 (ICLT-2021-SM2-11-L1-P9). (*10 points*)

Anna has 6 books.

1. (*5 points*) Considering that the books are all different, in how many ways she can arrange them on the shelf?
2. (*5 points*) Looking closely, she realizes that three of them are identical copies of the Sleeping Beauty book, and the other three are identical copies of the How to Teach Your Dragon book. Now, in how many ways she can arrange them on the shelf?

14.3 Grading for Level 1

Answers for multiple-choice problems.

Problem 1: C 11

Problem 2: B 6

Problem 3: E 105°

Problem 4: D 8.75

Problem 5: A 8

Problem 6: C 30

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 1 point if can determine the year of her was born was $1994 - y$

(b) 1 point if can determine the age of her siter is $2y$

Problem 8: Separately for each part,

(a) 2 points if can determine 2 as a prime factor.

(b) 2 points if can determine 5 as a factor.

Problem 9: Separately for each part,

(a) 2 points if can determine 6! as the number of ways to permute.

(b) 2 points if can determine 3! as the number of ways to permute a set of 3 identical books.

14.4 Solutions for Level 1

Solution. ICLT-2021-SM2-11-L1-P1 By testing, it is easy to see that $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ all divide 2520. $\boxed{11}$ is the first positive integer that does not divide 2520. The answer is $\boxed{C.}$ □

Solution. ICLT-2021-SM2-11-L1-P2

$$\frac{15 \times 6^{(2^2-3)}}{3^2 + 3 \times 2} = \frac{15 \times 6^1}{9+6} = 6$$

The answer is $\boxed{B.}$ □

Solution. ICLT-2021-SM2-11-L1-P3 Let x be the measure of the smallest angle, then the three angles have the measures $x, 3x$, and $8x$. Since

$$x + 3x + 8x = 180^\circ \Rightarrow 12x = 180^\circ \Rightarrow x = 15^\circ$$

The difference of the largest and smallest angles is $8x - x = 7x = 7 \cdot 15^\circ = 105^\circ$ The answer is $\boxed{E.}$ □

Solution. ICLT-2021-SM2-11-L1-P4

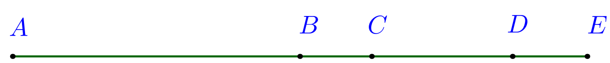


Figure 14.2: ICLT-2021-SM2-11-L1-P4

$$AB = \frac{AE}{2} \Rightarrow AB = 5 \Rightarrow CD = \frac{AB}{2} = 2.5 \Rightarrow BC = \frac{CD}{2} = 1.25$$
$$\Rightarrow AC = AB + BC = 5 + 1.25 = 6.25 \Rightarrow AD = AC + CD = 6.25 + 2.5 = 8.75$$

Thus, $\boxed{AD = 8.75.}$ The answer is $\boxed{D.}$ □

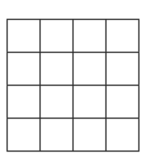
Solution. ICLT-2021-SM2-11-L1-P5 The set of four consecutive positive integers whose product is smallest is the set $\{1, 2, 3, 4\}$. Since $10000 = 10 \cdot 10 \cdot 10 \cdot 10$, thus the product of the integers in the set $\{10, 11, 12, 13\}$ exceeds 10000. Now

$$8 \cdot 9 \cdot 10 \cdot 11 = 7920 < 10000 < 9 \cdot 10 \cdot 11 \cdot 12 = 11180$$

Thus the set $\{8, 9, 10, 11\}$ still satisfies the required condition, but the set $\{9, 10, 11, 12\}$ does not. Therefore, there are $\boxed{8}$ such sets. The answer is $\boxed{A.}$ □

Solution. ICLT-2021-SM2-11-L1-P6 Lets count the number of squares by their sizes. There are

1. $4 \cdot 4 = 16$ squares of size 1×1,
2. $3 \cdot 3 = 9$ squares of size 2×2,
3. $2 \cdot 2 = 4$ squares of size 3×3,
4. $1 \cdot 1 = 1$ squares of size 3×3.



In total there are $\boxed{16 + 9 + 4 + 1 = 30}$ squares. The answer is $\boxed{C.}$ □

Solution. ICLT-2021-SM2-11-L1-P7 Let Oanh be y years old at the end of 1998. Then her sister was $2y$ years old at that time. The years they were born were $1998 - y$ and $1998 - 2y$, thus

$$\boxed{\text{(1998-y) + (1998-2y) = 3990}} \Rightarrow 3996 - 3y = 3990 \Rightarrow y = 2$$

Thus Oanh'sister was born in $1998 - 2y = 1994$. At the end of 2021, she is $\boxed{2021 - 1994 = 27}$ years old. $\square$

Solution. ICLT-2021-SM2-11-L1-P8 Since $24 = 2^3 \cdot 3$, thus it has $\boxed{\text{2 and 3}}$ as prime factors. For the second question, the smallest prime factor that desired number must have is 5. Since the desired number is composite, so it must have at least two factors, therefore the smallest one is $\boxed{5 \times 5 = 25.}$ $\square$

Solution. ICLT-2021-SM2-11-L1-P9 If all 6 books are considered different, then there are $\boxed{6! = 720}$ ways to arrange them on the shelf. If 3 of them are identical copies of Sleeping Beauty, then there are 3! ways to permute these three copies. Similarly for the How to Teach Your Dragon, thus by correction of overcounting the number of different arrangement is $\boxed{\frac{6!}{3!3!} = 20.}$ $\square$

14.5 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

14.6 Problems for Level 2

Problem 14.6.1 (ICLT-2021-SM2-11-L2-P1). (*4 points*)

Points A, B, C, D, and E are five points on a line segment with endpoints A and E. The points are in the order listed above from left to right such that $CD = \frac{1}{2}AB$, $BC = \frac{1}{2}CD$, $AB = \frac{1}{2}AE$, and $AE = 10$.

What is the length of AD?

(A) 6 (B) 6.25 (C) 8.25 (D) 8.75 (E) 9

Problem 14.6.2 (ICLT-2021-SM2-11-L2-P2). (*4 points*)

How many sets of four consecutive positive integers are there such that the product of the four integers is less than 10000?

(A) 8 (B) 9 (C) 10 (D) 16 (E) 17

Problem 14.6.3 (ICLT-2021-SM2-11-L2-P3). (*4 points*)

The grid shown in Figure 14.3 contains squares of sizes $1 \times 1, 2 \times 2, 3 \times 3$, and 4×4.

In total how many squares are there?

(A) 24 (B) 28 (C) 30 (D) 36 (E) 48

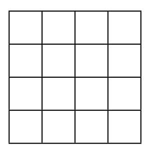


Figure 14.3: ICLT-2021-SM2-11-L2-P3

Problem 14.6.4 (ICLT-2021-SM2-11-L2-P4). (*6 points*)

Simplify the following

$$2(t^2 + 2ts - s^2) - 3(t^2 - 2ts - s^2)$$

(A) $-t^2+10ts+s^2$ (B) $t^2+10ts-s^2$ (C) $-t^2-10ts+s^2$ (D) $-t^2+10ts-s^2$ (E) $t^2+10ts+s^2$

Problem 14.6.5 (ICLT-2021-SM2-11-L2-P5). (*6 points*)

In the Figure 14.4 $AB = BC = 5$, $AD = DC = 8$, and $\angle BAC = 26°$.

What is the measure of the angle $\angle CBD$?

(A) 63 (B) 64 (C) 70 (D) 72 (E) 74

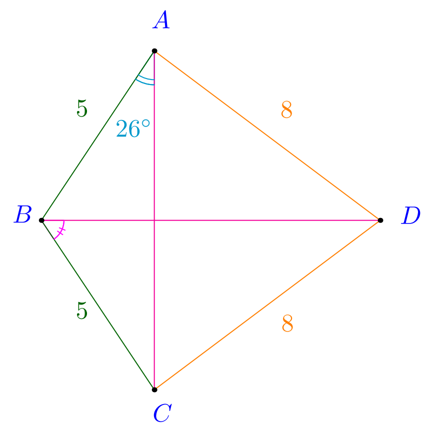


Figure 14.4: ICLT-2021-SM2-11-L2-P5

Problem 14.6.6 (ICLT-2021-SM2-11-L2-P6). (*6 points*)

Let n be the smallest number of students that can be broken up both into 15 groups of equal membership into 21 groups of equal membership, and into 48 groups of equal membership.

What is the sum of the digits of n?

(A) 6 (B) 8 (C) 10 (D) 12 (E) 14

Problem 14.6.7 (ICLT-2021-SM2-11-L2-P7). (*10 points*)

Anna has 6 books.

1. (*5 points*) Considering that the books are all different, in how many ways she can arrange them on the shelf?
2. (*5 points*) Looking closely, she realizes that three of them are identical copies of the Sleeping Beauty book, and the other three are identical copies of the How to Teach Your Dragon book. Now, in how many ways she can arrange them on the shelf?

Problem 14.6.8 (ICLT-2021-SM2-11-L2-P8). (*10 points*)

(x, y) is a pair of real numbers that satisfy the following system of equations

$$\begin{cases} 5x - 6y &= 1 \\ 15x - 19y &= 4 \end{cases}$$

1. (*5 points*) Eliminate x from the equation. What is the value of y? *Note that using substitution will not help you to earn full points.*
2. (*5 points*) Now the number 19 in the second equation is changed to 18. Solve the modified system of equations.

Problem 14.6.9 (ICLT-2021-SM2-11-L2-P9). (*10 points*)

Club Firebird makes preparation for the upcoming swimming competition. Sonny and his brother Winne are members of the club. There are six other people who are also members of the club. On the competition day, because of family duty, Sonny or Winne (but not both) will be absent. Five members must be chosen to make the team that represents the club at the competition.

1. (*5 points*) If Sony or Winnie is chosen to represent the club, how many ways are there to select the five members to make the team?
2. (*5 points*) In how many ways five club members can be chosen to make the team?

14.7 Grading for Level 2

Answers for multiple-choice problems.

Problem 1: D 8.75

Problem 2: A 8

Problem 3: C 30

Problem 4: A $-t^2 + 10ts + s^2$

Problem 5: B 64

Problem 6: D 12

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately for each part,

(a) 2 points if can determine 6! as the number of ways to permute.

(b) 2 points if can determine 3! as the number of ways to permute a set of 3 identical books.

Problem 8: Separately for each part,

(a) 2 points if can combine both equations to eliminate x. Only 2 points can be given for this first question if substitution is used instead of elimination.

(b) 2 points if can spot any pattern that leads to non-existing solution.

Problem 9: Separately for each part,

(a) 2 points if can find the number of ways to make team if Sonny or Winnie is on the team.

(b) 2 points if can find the number of ways to make team if neither Sonny nor Winnie on the team.

14.8 Solutions for Level 2

Solution. ICLT-2021-SM2-11-L2-P1

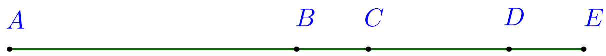


Figure 14.5: ICLT-2021-SM2-11-L2-P1

$$AB = \frac{AE}{2} \Rightarrow AB = 5 \Rightarrow CD = \frac{AB}{2} = 2.5 \Rightarrow BC = \frac{CD}{2} = 1.25$$
$$\Rightarrow AC = AB + BC = 5 + 1.25 = 6.25 \Rightarrow AD = AC + CD = 6.25 + 2.5 = 8.75$$

Thus, $\boxed{AD = 8.75.}$ The answer is $\boxed{D.}$ □

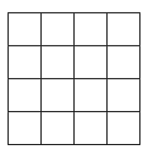
Solution. ICLT-2021-SM2-11-L2-P2 The set of four consecutive positive integers whose product is smallest is the set $\{1, 2, 3, 4\}$. Since $10000 = 10 \cdot 10 \cdot 10 \cdot 10$, thus the product of the integers in the set $\{10, 11, 12, 13\}$ exceeds 10000. Now

$$8 \cdot 9 \cdot 10 \cdot 11 = 7920 < 10000 < 9 \cdot 10 \cdot 11 \cdot 12 = 11180$$

Thus the set $\{8, 9, 10, 11\}$ still satisfies the required condition, but the set $\{9, 10, 11, 12\}$ does not. Therefore, there are $\boxed{8}$ such sets. The answer is $\boxed{A.}$ □

Solution. ICLT-2021-SM2-11-L2-P3 Lets count the number of squares by their sizes. There are

1. $4 \cdot 4 = 16$ squares of size 1×1,
2. $3 \cdot 3 = 9$ squares of size 2×2,
3. $2 \cdot 2 = 4$ squares of size 3×3,
4. $1 \cdot 1 = 1$ squares of size 3×3.



In total there are $\boxed{16 + 9 + 4 + 1 = 30}$ squares. The answer is $\boxed{C.}$ □

Solution. ICLT-2021-SM2-11-L2-P4

$$2(t^2 + 2ts - s^2) - 3(t^2 - 2ts - s^2) = -t^2 + 10ts + s^2$$

The answer is $\boxed{A.}$ □

Solution. ICLT-2021-SM2-11-L2-P5 First, $BA = BC, DA = DC, BD = BD$, so $\triangle BAD \cong \triangle BCD$ (SSS).

Therefore $\angle CBD = \angle ABD = \frac{1}{2}\angle ABC$. Furthermore $\triangle ABC$ is isosceles at B. Thus $\angle ABC = 180 - 2\angle BAC = 128°$, so $\boxed{\angle CBD = 64°.}$ The answer is $\boxed{B.}$ □

Solution. ICLT-2021-SM2-11-L2-P6 Let n be the smallest number of students that can be broken into 15, 21, and 48 groups of equal membership. This means that n is the least common multiple of $15, 21$, and 48. Since

$$15 = 3 \cdot 5,\ \ 21 = 3 \cdot 7,\ \ 48 = 2^3 \cdot 3 \Rightarrow \text{lcm}[15, 21, 48] = 2^3 \cdot 3 \cdot 5 \cdot 7 = 840$$

The sum of digits of x is $\boxed{8 + 4 + 0 = 12.}$ The answer is $\boxed{D.}$ □

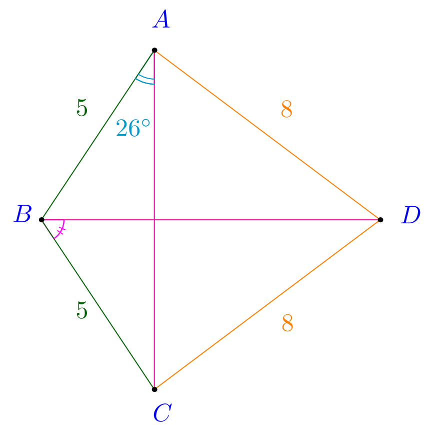


Figure 14.6: $\triangle BAD \cong \triangle BCD$

Solution. ICLT-2021-SM2-11-L2-P7 If all 6 books are considered different, then there are $\boxed{6! = 720}$ ways to arrange them on the shelf. If 3 of them are identical copies of Sleeping Beauty, then there are 3! ways to permute these three copies. Similarly for the How to Teach Your Dragon, thus by correction of overcounting the number of different arrangement is $\boxed{\frac{6!}{3!3!} = 20.}$ □

Solution. ICLT-2021-SM2-11-L2-P8

$$\begin{cases} 5x - 6y & = 1 \\ 15x - 19y & = 4 \end{cases} \Rightarrow (15x - 19y) - 3(5x - 6y) = 4 - 3 \cdot 1 \Rightarrow -y = 1$$

Thus, for the first question $\boxed{y = -1.}$ For the second question,

$$\begin{cases} 5x - 6y & = 1 \\ 15x - 18y & = 4 \end{cases} \Rightarrow \begin{cases} 3(5x - 6y) & = 3 \\ 15x - 18y & = 4 \end{cases} \Rightarrow \begin{cases} 15x - 18y & = 3 \\ 15x - 18y & = 4 \end{cases}$$

It is easy to see that there is $\boxed{\text{no solutions}}$ for this question. □

Solution. ICLT-2021-SM2-11-L2-P9 If Sonny or Winne is chosen to be in the team, then 4 people have to be select from the remaining 6 members of the club. Thus $\boxed{2 \cdot \binom{6}{4} = 30}$ ways to make the team.

For the second question, if neither Sonny nor Winne is chosen, then 5 people have to be select from the remaining 6 members of the club. Thus $\boxed{30 + \binom{6}{5} = 36}$ ways to make the team. □

14.9 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

14.10 Problems for Level 3

Problem 14.10.1 (ICLT-2021-SM2-11-L3-P1). (*4 points*)

Simplify the following

$$2(t^2 + 2ts - s^2) - 3(t^2 - 2ts - s^2)$$

(A) $-t^2+10ts+s^2$ (B) $t^2+10ts-s^2$ (C) $-t^2-10ts+s^2$ (D) $-t^2+10ts-s^2$ (E) $t^2+10ts+s^2$

Problem 14.10.2 (ICLT-2021-SM2-11-L3-P2). (*4 points*)

In the Figure 14.7 diagram $AB = BC = 5$, $AD = DC = 8$, and $\angle BAC = 26°$.

What is the measure of the angle $\angle CBD$?

(A) 63 (B) 64 (C) 70 (D) 72 (E) 74

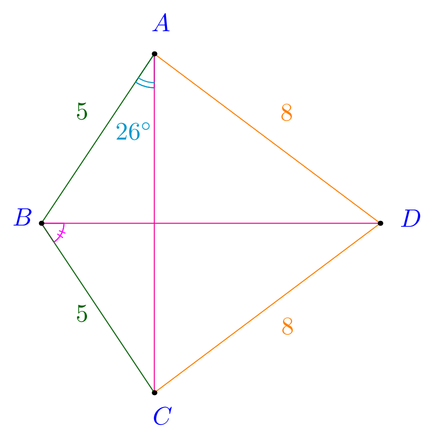


Figure 14.7: ICLT-2021-SM2-11-L3-P2

Problem 14.10.3 (ICLT-2021-SM2-11-L3-P3). (*4 points*)

Let n be the smallest number of students that can be broken up both into 15 groups of equal membership into 21 groups of equal membership, and into 48 groups of equal membership.

What is the sum of the digits of n?

(A) 6 (B) 8 (C) 10 (D) 12 (E) 14

Problem 14.10.4 (ICLT-2021-SM2-11-L3-P4). (*6 points*)

How many of the positive divisors of 900 have 6 positive divisors?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 8

Problem 14.10.5 (ICLT-2021-SM2-11-L3-P5). (*6 points*)

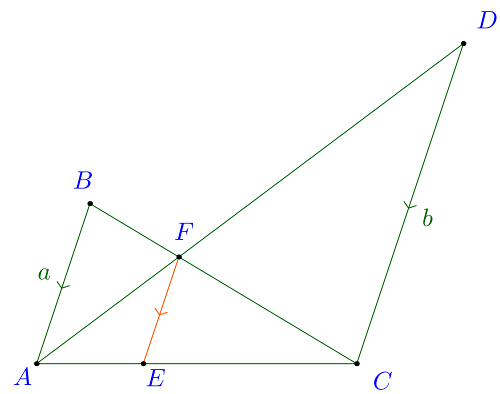
Albert has 4 bags of golds. The ratio of the first bag's weight to the second one's is $9:8$. The ratio of the second bag's weight to the third one's is $5:3$. The ratio of the first bag's weight to the fourth one's is $12:5$. The heaviest bag weighs 120 kgs.

What is the weight of the lightest one?

(A) 30 (B) 36 (C) 42 (D) 48 (E) 50

Problem 14.10.6 (ICLT-2021-SM2-11-L3-P6). (*6 points*)

In the diagram below, $AB \parallel CD \parallel EF$.



Find for EF if $AB = a$ and $CD = b$.

(A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{a+b}{ab}$ (D) $\frac{ab}{a+b}$ (E) $\frac{b-a}{ab}$

Problem 14.10.7 (ICLT-2021-SM2-11-L3-P7). (*10 points*)

Club Firebird makes preparation for the upcoming swimming competition. Sonny and his brother Winne are members of the club. There are six other people who are also members of the club. On the competition day, because of family duty, Sonny or Winne (but not both) will be absent. Five members must be chosen to make the team that represents the club at the competition.

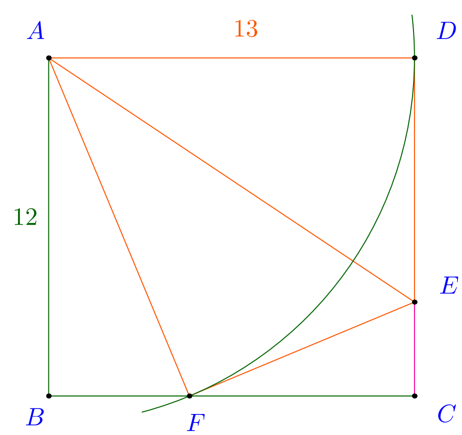
1. (*5 points*) If Sony or Winnie is chosen to represent the club, how many ways are there to select the five members to make the team?
2. (*5 points*) In how many ways five club members can be chosen to make the team?

Problem 14.10.8 (ICLT-2021-SM2-11-L3-P8). (*10 points*) Antoine has 9 marbles. He wants to divide the marbles into some piles such that the number of marbles in each pile is a prime number. (It is possible that two piles contain the same number of marbles.) *For example he can divide them into three piles: one pile of* 2 *marbles, one pile of* 2 *marbles, and one pile of* 5 *marbles.*

1. (*5 points*) If there is no pile containing 2 or 3 marbles, in how many ways can he do it?
2. (*5 points*) In how many ways in total can he do it?

Problem 14.10.9 (ICLT-2021-SM2-11-L3-P9). (*10 points*)

The square $ABCD$ is folded along the line AE such that D becomes point F on BC. $AB = 12$, $BC = 13$.



1. (*5 points*) Find FC.
2. (*5 points*) Find EC.

14.11 Grading for Level 3

Answers for multiple-choice problems.

Problem 1: A $\quad -t^2 + 10ts + s^2$

Problem 2: B $\quad 64$

Problem 3: D $\quad 12$

Problem 4: D $\quad 6$

Problem 5: E $\quad 50$

Problem 6: D $\quad \frac{ab}{a+b}$

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately for each part,

(a) 2 points if can find the number of ways to make team if Sonny or Winnie is on the team.

(b) 2 points if can find the number of ways to make team if neither Sonny nor Winnie on the team.

Problem 8: Separately for each part,

(a) 2 points if can determine the least number of marbles in a pile must be 5.

(b) 2 points if can determine casework based on the least or most number of marbles in a pile.

Problem 9: Separately for each part,

(a) 2 points if find $\triangle AFE \cong \triangle AFD$ or $AF = AD$.

(b) 2 points if can determine $\angle AFB + \angle CFE = 90°$, or $\triangle BAF \sim \triangle CFE$.

14.12 Solutions for Level 3

Solution. ICLT-2021-SM2-11-L3-P1

$$2(t^2 + 2ts - s^2) - 3(t^2 - 2ts - s^2) = -t^2 + 10ts + s^2$$

The answer is $\boxed{A.}$ □

Solution. ICLT-2021-SM2-11-L3-P2 First, $BA = BC, DA = DC, BD = BD$, so $\triangle BAD \cong \triangle BCD$ (SSS).

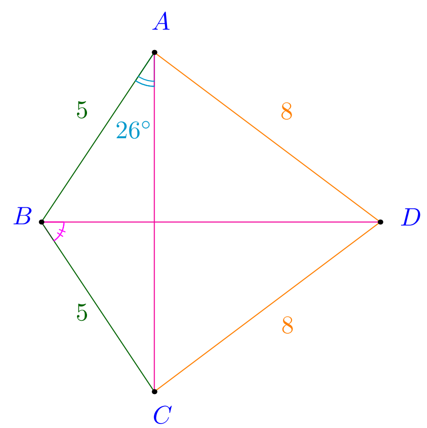


Figure 14.8: $\triangle BAD \cong \triangle BCD$

Therefore $\angle CBD = \angle ABD = \frac{1}{2}\angle ABC$. Furthermore $\triangle ABC$ is isosceles at B. Thus $\angle ABC = 180 - 2 \cdot \angle BAC = 128°$, so $\boxed{\angle CBD = 64°.}$ The answer is $\boxed{B.}$ □

Solution. ICLT-2021-SM2-11-L3-P3 Let n be the smallest number of students that can be broken into 15, 21, and 48 groups of equal membership. This means that n is the least common multiple of 15, 21, and 48. Since

$$15 = 3 \cdot 5,\ 21 = 3 \cdot 7,\ 48 = 2^3 \cdot 3 \Rightarrow \text{lcm}[15, 21, 48] = 2^3 \cdot 3 \cdot 5 \cdot 7 = 840$$

The sum of digits of x is $\boxed{8 + 4 + 0 = 12.}$ The answer is $\boxed{D.}$ □

Solution. ICLT-2021-SM2-11-L3-P4 First, $900 = 2^2 \cdot 3^2 \cdot 5^2$, a divisor d of 900 has the form $d = 2^a \cdot 3^b \cdot 5^c$, and its number of divisors is $(a+1)(b+1)(c+1)$, where $0 \le a, b, c \le 2$. Thus,

$$(a+1)(b+1)(c+1) = 6 = 1 \cdot 1 \cdot 6 = 1 \cdot 2 \cdot 3.$$

Because $0 \le a, b, c \le 2$, thus it is only possible if (a, b, c) is one of permutations of $(0, 1, 2)$. There are $\boxed{6}$ such permutations. The answer is $\boxed{D.}$

What are those 6 numbers?

$$2^2 \cdot 3^1 = 12,\ 2^1 \cdot 3^2 = 18,\ 2^2 \cdot 5^1 = 20,\ 2^1 \cdot 5^2 = 50,\ 3^2 \cdot 5^1 = 45,\ 3^1 \cdot 5^2 = 75$$

□

Solution. ICLT-2021-SM2-11-L3-P5 Let the weight of the first bag be w.

The ratio of the first bag's weight to the second one's is $9:8$, so the weight of the second bag is $\frac{8}{9}w$. The ratio of the second bag's weight to the third one's is $5:3$, so the weight of the third bag is $\frac{3}{5}\cdot\frac{8}{9}w = \frac{8}{15}w$. The ratio of the first bag's weight to the fourth one's is $12:5$, so the weight of the fourth bag is $\frac{5}{12}w$. Then the weight of four bags are

$$w,\ \frac{8}{9}w,\ \frac{3}{5}\cdot\frac{8}{9}w = \frac{8}{15}w,\ \frac{5}{12}w$$

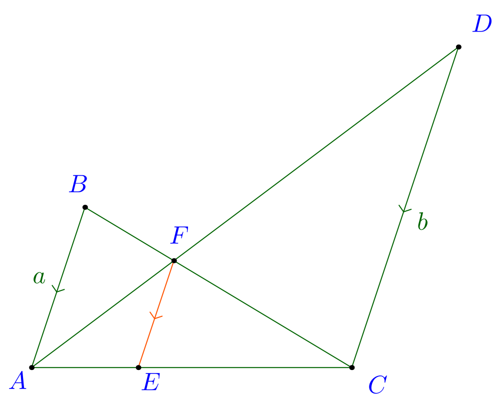
Since $\frac{5}{12} < \frac{1}{2} < \frac{8}{15} < \frac{8}{9} < 1$, thus the heaviest one is the first bag, and the lightest one is the last bag, whose weight is $\boxed{\frac{5}{12}w = 50}$ kg.

The answer is $\boxed{E.}$ □

Solution. ICLT-2021-SM2-11-L3-P6 First, $\triangle ECF \sim \triangle ACB$, $\triangle EAF \sim \triangle CAD$, so

$$\frac{EF}{AB} = \frac{EC}{AC},\ \frac{EF}{CD} = \frac{AE}{AC} \Rightarrow \frac{EF}{AB} + \frac{EF}{CD} = 1 \Rightarrow EF = \frac{AB\cdot CD}{AB+CD}.$$

Thus, $\boxed{EF = \frac{ab}{a+b}.}$ The answer is $\boxed{D.}$ □



Solution. ICLT-2021-SM2-11-L2-P7 If Sonny or Winne is chosen to be in the team, then 4 people have to be select from the remaining 6 members of the club. Thus $\boxed{2\cdot\binom{6}{4} = 30}$ ways to make the team.

For the second question, if neither Sonny nor Winne is chosen, then 5 people have to be select from the remaining 6 members of the club. Thus $\boxed{30+\binom{6}{5} = 36}$ ways to make the team. □

Solution. ICLT-2021-SM2-11-L3-P8 For the first question, if there is no pile containing 2 or 3 marbles, then any pipe must contain at least 5 marbles. This is not possible, since Antoine has only 9 marbles.

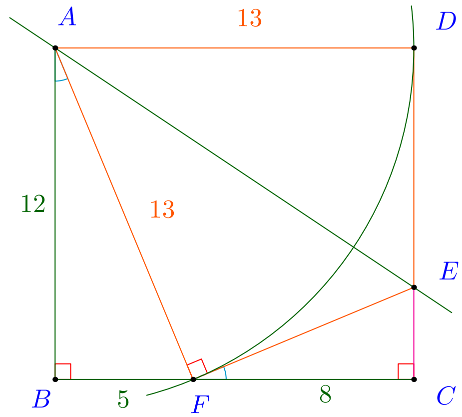
For the second question, we do a bit of case work.

Case 1: if there is no pile with 2 marbles, then there is at least one pile with 3 marbles (from the first question). In this case 6 marbles left, it can be divided into piles with at least 3 marble in only one way, $3+3=6$. We got $9 = 3+3+3$.

Case 2: if there is a pile with 2 marbles, then there are 7 marbles left, it can be divided as follow,

$$7 = 2+2+3 = 2+5 = 7 \Rightarrow 9 = 2+2+2+3 = 2+2+5 = 2+7.$$

Thus, there are $\boxed{4}$ ways, $9 = 2+2+2+3 = 2+2+5 = 2+7 = 3+3+3$. □



Solution. ICLT-2021-SM2-11-L3-P9

Since $\triangle AFE$ is a reflection of $\triangle ADE$ over the line AE, so $\triangle AFE \cong \triangle AFD$. Thus, $AF = AD$. Now, $BF = \sqrt{AF^2 - AB^2} = \sqrt{13^2 - 12^2} = 5$. Therefore $\boxed{FC = BC - BF = 13 - 5 = 8.}$

Since $\angle AFE = 90°$, so $\angle CFE = 90° - \angle BFA = \angle BAF$, thus $\triangle BAF \sim \triangle CFE$.

$$\frac{EC}{FC} = \frac{FB}{AB} \Rightarrow EC = \frac{FB \cdot FC}{AB} = \frac{8 \cdot 5}{12} = \frac{10}{3}.$$

Therefore, $\boxed{EC = \frac{10}{3}.}$ □

14.13 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

14.14 Problems for Level 4

Problem 14.14.1 (ICLT-2021-SM2-11-L4-P1). (*4 points*)

How many of the positive divisors of 900 have 6 positive divisors?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 8

Problem 14.14.2 (ICLT-2021-SM2-11-L4-P2). (*4 points*)

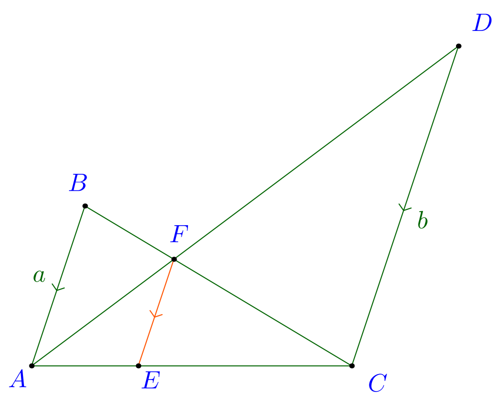
Albert has 4 bags of golds. The ratio of the first bag's weight to the second one's is $9:8$. The ratio of the second bag's weight to the third one's is $5:3$. The ratio of the first bag's weight to the fourth one's is $12:5$. The heaviest bag weighs 120 kgs.

What is the weight of the lightest one?

(A) 30 (B) 36 (C) 42 (D) 48 (E) 50

Problem 14.14.3 (ICLT-2021-SM2-11-L4-P3). (*4 points*)

In the diagram below, $AB \parallel CD \parallel EF$.



Find for EF if $AB = a$ and $CD = b$.

(A) $\dfrac{a}{b}$ (B) $\dfrac{b}{a}$ (C) $\dfrac{a+b}{ab}$ (D) $\dfrac{ab}{a+b}$ (E) $\dfrac{b-a}{ab}$

Problem 14.14.4 (ICLT-2021-SM2-11-L4-P4). (*6 points*)

Let x and y be positive integers such that

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{5}$$

Find the sum of all possible values of x.

(A) 20 (B) 36 (C) 46 (D) 50 (E) 56

Problem 14.14.5 (ICLT-2021-SM2-11-L4-P5). (*6 points*)

Khang is rowing on a river. The river flows at 5 kilometres per hour. Khang decides to start rowing downstream, with the current, at noon. He wants to return to his house at 3:30 p.m. At what time should Barry turn around and row home if he normally rows 7 kilometres per hour in water that has no current?

(A) 12:30 pm (B) 12:45 pm (C) 1:00 pm (D) 1:15 pm (E) 1:25 pm

Problem 14.14.6 (ICLT-2021-SM2-11-L4-P6). (*6 points*)

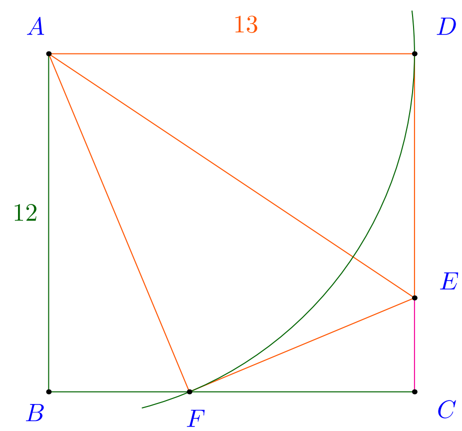
Bu, Chu, and Fu each sit in a row of 5 chairs. They choose their seats at random.

What is the probability that at least two of them sit next to each other?

(A) $\frac{11}{12}$ (B) $\frac{9}{10}$ (C) $\frac{25}{36}$ (D) $\frac{24}{35}$ (E) $\frac{3}{4}$

Problem 14.14.7 (ICLT-2021-SM2-11-L4-P7). (*10 points*)

The square $ABCD$ is folded along the line AE such that D becomes point F on BC. $AB = 12$, $BC = 13$.



1. (*5 points*) Find FC.
2. (*5 points*) Find EC.

Problem 14.14.8 (ICLT-2021-SM2-11-L4-P8). (*10 points*)

Let $A(1, 3)$, $B(-2, -1)$, and $C(3, -2)$ be points on the Cartesian plane.

1. (*5 points*) Determine the length of the longest segment among AB, BC, and CA.
2. (*5 points*) Determine the area of $\triangle ABC$.

Problem 14.14.9 (ICLT-2021-SM2-11-L4-P9). (*10 points*)

Minh has three fair $6-$sided dice (numbers from 1 to 6). He rolls all three of them at the same time.

1. (*5 points*) What is the probability that the three numbers shown sum up to a multiple of 3, and each of them they has the remainder 1 when divided by 3.
2. (*5 points*) What is the probability that the three numbers shown sum up to a multiple of 3, and none of them is divisible by 3.

Note that each dice is distinguishable, for example rolling 2 for the first dice, 4 for the second dice, and 4 for the third dice is different from 4 for the first dice, 2 for the second dice and 4 for the third dice.

14.15 Grading for Level 4

Answers for multiple-choice problems.

Problem 1: D 6

Problem 2: E 50

Problem 3: D $\frac{ab}{a+b}$

Problem 4: C 46

Problem 5: A 12 : 30 p.m.

Problem 6: B $\frac{9}{10}$

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately for each part,

(a) 2 points if find $\triangle AFE \cong \triangle AFD$ or $AF = AD$.

(b) 2 points if can determine $\angle AFB + \angle CFE = 90°$, or $\triangle BAF \sim \triangle CFE$.

Problem 8: Separately for each part,

(a) 2 points if can plot the points and use the formula to determine the distance between two points.

(b) 2 points if can use complementary counting to determine the area or use Heron formula.

Problem 9: Separately for each part,

(a) 2 points if can determine $\{1, 4\}$ is the set of integers to chose from.

(b) 2 points if can determine the second set of $\{2, 5\}$.

14.16 Solutions for Level 4

Solution. ICLT-2021-SM2-11-L4-P1 First, $900 = 2^2 \cdot 3^2 \cdot 5^2$, a divisor d of 900 has the form $d = 2^a \cdot 3^b \cdot 5^c$, and its number of divisors is $(a+1)(b+1)(c+1)$, where $0 \le a, b, c \le 2$. Thus,

$$(a+1)(b+1)(c+1) = 6 = 1 \cdot 1 \cdot 6 = 1 \cdot 2 \cdot 3.$$

Because $0 \le a, b, c \le 2$, thus it is only possible if (a, b, c) is one of permutations of $(0, 1, 2)$. There are $\boxed{6}$ such permutations. The answer is $\boxed{D.}$

What are those 6 numbers?

$$2^2 \cdot 3^1 = 12,\ 2^1 \cdot 3^2 = 18,\ 2^2 \cdot 5^1 = 20,\ 2^1 \cdot 5^2 = 50,\ 3^2 \cdot 5^1 = 45,\ 3^1 \cdot 5^2 = 75$$

□

Solution. ICLT-2021-SM2-11-L4-P2 Let the weight of the first bag be w.

The ratio of the first bag's weight to the second one's is $9 : 8$, so the weight of the second bag is $\frac{8}{9}w$. The ratio of the second bag's weight to the third one's is $5 : 3$, so the weight of the third bag is $\frac{3}{5} \cdot \frac{8}{9}w = \frac{8}{15}w$. The ratio of the first bag's weight to the fourth one's is $12 : 5$, so the weight of the fourth bag is $\frac{5}{12}w$. Then the weight of four bags are

$$w,\ \frac{8}{9}w,\ \frac{3}{5} \cdot \frac{8}{9}w = \frac{8}{15}w,\ \frac{5}{12}w$$

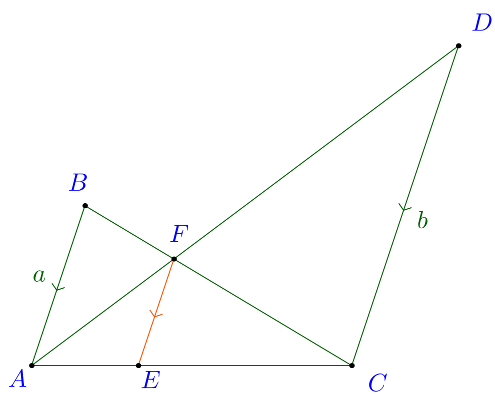
Since $\frac{5}{12} < \frac{1}{2} < \frac{8}{15} < \frac{8}{9} < 1$, thus the heaviest one is the first bag, and the lightest one is the last bag, whose weight is $\boxed{\frac{5}{12}w = 50}$ kg.

The answer is $\boxed{E.}$ □

Solution. ICLT-2021-SM2-11-L4-P3 First, $\triangle ECF \sim \triangle ACB$, $\triangle EAF \sim \triangle CAD$, so

$$\frac{EF}{AB} = \frac{EC}{AC},\ \frac{EF}{CD} = \frac{AE}{AC} \Rightarrow \frac{EF}{AB} + \frac{EF}{CD} = 1 \Rightarrow EF = \frac{AB \cdot CD}{AB + CD}.$$

Thus, $\boxed{EF = \frac{ab}{a+b}.}$ The answer is $\boxed{D.}$ □



Solution. ICLT-2021-SM2-11-L4-P4

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{5} \Rightarrow 5x + 5y = xy \Rightarrow xy - 5x - 5y = 0 \Rightarrow (x-5)(y-5) = 25$$

Since $25 = 1 \cdot 25 = 5 \cdot 5$, thus $(x, y) \in \{(6, 30), (10, 10), (30, 6)\}$.

Therefore all possible values of x is $\boxed{6 + 10 + 30 = 46.}$ The answer is $\boxed{C.}$ □

Solution. ICLT-2021-SM2-11-L4-P5 Since Khang rows 7 km/h and the river flows at 5 km/h, so he is going downstream at $7+5=12$ km/h, and upstream at $7-5=2$ km/h. Let d and u be the amount of time he rows down- and upstream, then

$$\begin{cases} d+u &= 3.5 \\ 12d &= 2u \end{cases} \Rightarrow u = 6d \Rightarrow 7d = 3.5 \Rightarrow d = 0.5,\ u = 3$$

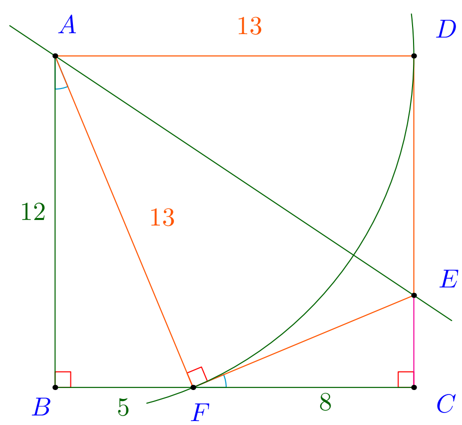
Thus, he turns around at $\boxed{\text{12:30 p.m}}$. The answer is $\boxed{A.}$ □

Solution. ICLT-2021-SM2-11-L4-P6 Let assume the opposite when hey do not sit next to each other. Since there are 5 chairs, so Bu, Chu, and Fu must sit on the chairs 1, 3, 5, so that they do not sit next to each other. There are $3! = 6$ permutations for them to sit on those chairs. In total there are $\binom{5}{3} = 10$ ways to choose three seats for them and $3! = 6$ ways to permutate. Thus the probability is $\frac{6}{10\cdot 6} = \frac{1}{10}$.

Therefore the probability that at least two of them sit next to each other is $1 - \frac{1}{10} = \frac{9}{10}$. The answer is $\boxed{B.}$ □

Solution. ICLT-2021-SM2-11-L4-P7

Since $\triangle AFE$ is a reflection of $\triangle ADE$ over the line AE, so $\triangle AFE \cong \triangle AFD$. Thus, $AF = AD$. Now,



$BF = \sqrt{AF^2 - AB^2} = \sqrt{13^2 - 12^2} = 5$. Therefore $\boxed{FC = BC - BF = 13 - 5 = 8.}$

Since $\angle AFE = 90°$, so $\angle CFE = 90° - \angle BFA = \angle BAF$, thus $\triangle BAF \sim \triangle CFE$.

$$\frac{EC}{FC} = \frac{FB}{AB} \Rightarrow EC = \frac{FB \cdot FC}{AB} = \frac{8\cdot 5}{12} = \frac{10}{3}.$$

Therefore, $\boxed{EC = \frac{10}{3}.}$ □

Solution. ICLT-2021-SM2-11-L4-P8 First, lets plot the points on the plane using the given coordinates

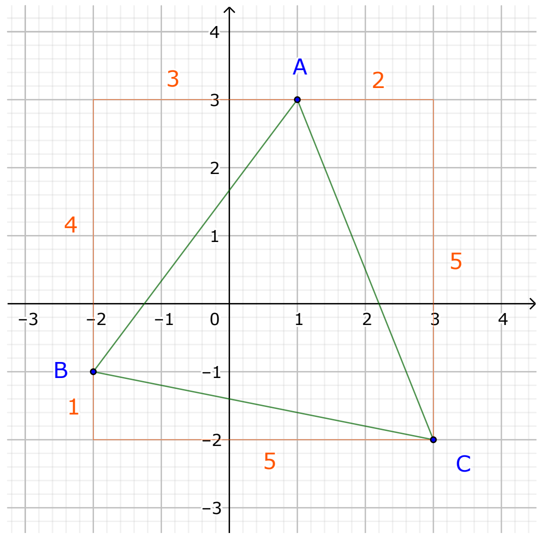


Figure 14.9: $\triangle ABC$

$AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} = \sqrt{(1-(-2))^2 + (3-(-1))^2} = \sqrt{3^2+4^2} = 5,\ BC = \sqrt{26},\ CA = \sqrt{29}$
$\Rightarrow CA > BC > AB$

Thus, the longest segment is $\boxed{CA = \sqrt{29}.}$

The area of $\triangle ABC$ can be determined based on the area of the surrrounding rectangle subtracting the three corner right triangles as shown in Figure 14.9,

$$5 \cdot 5 - \frac{1}{2}(3 \cdot 4 + 2 \cdot 5 + 1 \cdot 5) = 25 - \frac{27}{2} = \frac{23}{2}.$$

□

Solution. ICLT-2021-SM2-11-L4-P9 Note that if none of the dice shows 3 or 6 then it can shows only $1, 2, 4, 5$, which are the numbers that are not divisible by 3. Furthermore all the three numbers must have the same remainders when divided by 3. Therefore they all must be chosen from $\{1, 4\}$ or all from $\{2, 5\}$. In each case there are $\boxed{2^3 = 8}$ ways to choose.

Thus, the probability is $\boxed{\frac{2 \cdot 2^3}{6^3} = \frac{2}{27}.}$ □

14.17 Rules

- The total time to complete the test is 90 **minutes.**
- The test consists of 6 *multiple-choice* and 3 *show-you-work problems.* To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 3 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed.
- Grading:
  1. For a**multiple-choice problem**,
     - For a *correct answer*, i.e. if you select the correct choice, you get 4 or 6 points, based on the number of points assigned to the problem.
     - For an *unanswered* problem, i.e. if you select no choice, you get 1 point.
     - For a *wrong answer*, i.e. when you select the wrong choice, you get 0 point.
  2. For a **show-you-work problem**,
     - A problem is worth a total of 10 points.
     - A problem can have one or more questions.
     - For a complete solution for each question, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{31.}$

14.18 Problems for Level 5

Problem 14.18.1 (ICLT-2021-SM2-11-L5-P1). (*4 points*)

Let x and y be positive integers such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{5}$$

Find the sum of all possible values of x.

(A) 20 (B) 36 (C) 46 (D) 50 (E) 56

Problem 14.18.2 (ICLT-2021-SM2-11-L5-P2). (*4 points*)

Khang is rowing on a river. The river flows at 5 kilometres per hour. Khang decides to start rowing downstream, with the current, at noon. He wants to return to his house at 3:30 p.m. At what time should Barry turn around and row home if he normally rows 7 kilometres per hour in water that has no current?

(A) 12:30 pm (B) 12:45 pm (C) 1:00 pm (D) 1:15 pm (E) 1:25 pm

Problem 14.18.3 (ICLT-2021-SM2-11-L5-P3). (*4 points*)

Bu, Chu, and Fu each sit in a row of 5 chairs. They choose their seats at random.

What is the probability that at least two of them sit next to each other?

(A) $\frac{11}{12}$ (B) $\frac{9}{10}$ (C) $\frac{25}{36}$ (D) $\frac{24}{35}$ (E) $\frac{3}{4}$

Problem 14.18.4 (ICLT-2021-SM2-11-L5-P4). (*6 points*)

One leg of a right triangle is 2 cm more than 2 times the other leg, and the hypotenuse is 1 cm longer than the longest leg.

Find the area of the triangle.

$$(A)\quad 24 \qquad (B)\quad 15\sqrt{2} \qquad (C)\quad 28 \qquad (D)\quad 30 \qquad (E)\quad 32\sqrt{2}$$

Problem 14.18.5 (ICLT-2021-SM2-11-L5-P5). (*6 points*)

Let x and y be real numbers such that

$$\begin{cases} \sqrt{z^2+7} = z + x \\ \sqrt{z^2+7} = 7y - z \end{cases}$$

Find y if $x = \frac{\sqrt{7}}{7}$.

$$(A)\quad 11 \qquad (B)\quad 7\sqrt{7} \qquad (C)\quad 7 \qquad (D)\quad 2\sqrt{7} \qquad (E)\quad \sqrt{7}$$

Problem 14.18.6 (ICLT-2021-SM2-11-L5-P6). (*10 points*)

x and y are two real numbers such that $0 < x < 1$ and $0 < y < 3$.

What is the probability that $y > x + 1$?

$$(A)\quad \frac{4}{5} \qquad (B)\quad \frac{2}{3} \qquad (C)\quad \frac{1}{2} \qquad (D)\quad \frac{2}{5} \qquad (E)\quad \frac{1}{3}$$

Problem 14.18.7 (ICLT-2021-SM2-11-L5-P7). (*10 points*)

Minh has three fair 6−sided dice (numbers from 1 to 6). He rolls all three of them at the same time.

1. (*5 points*) What is the probability that the three numbers shown sum up to a multiple of 3, and each of them they has the remainder 1 when divided by 3.
2. (*5 points*) What is the probability that the three numbers shown sum up to a multiple of 3, and none of them is divisible by 3.

Note that each dice is distinguishable, for example rolling 2 *for the first dice,* 4 *for the second dice, and* 4 *for the third dice is different from* 4 *for the first dice,* 2 *for the second dice and* 4 *for the third dice.*

Problem 14.18.8 (ICLT-2021-SM2-11-L5-P8). (*10 points*)

1. (*5 points*) What is remainder of the number $\overline{123}_7$, which is in base 7, when divided by 6?
2. (*5 points*) The number $\overline{x123456}_7$ in the number base 7 is divisible by 6. What is the sum of all possible value of x?

Problem 14.18.9 (ICLT-2021-SM2-11-L5-P9). (*10 points*)

What is the probability that a random rearrangement of the letters in the word OTTAWA will NOT begin with the letters OTT?

14.19 Grading for Level 5

Answers for multiple-choice problems.

Problem 1: C 46

Problem 2: A 12 : 30 p.m.

Problem 3: B $\frac{9}{10}$

Problem 4: D 30

Problem 5: E $\sqrt{7}$

Problem 6: C $\frac{1}{2}$

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: Separately for each part,

(a) 2 points if can determine $\{1, 4\}$ is the set of integers to chose from.

(b) 2 points if can determine the second set of $\{2, 5\}$.

Problem 8: Separately for each part,

(a) 2 points if can determine any power of 7 has remainder 1 when divided by 6.

(b) 2 points if can determine that $x + 3$ is divisible by 6.

Problem 9: 2 points if can determine the probability to choose O as the first letter. 2 more points if can determine the probability to choose T as the second letter.

14.20 Solutions for Level 5

Solution. ICLT-2021-SM2-11-L5-P1

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{5} \Rightarrow 5x+5y=xy \Rightarrow xy-5x-5y=0 \Rightarrow (x-5)(y-5)=25$$

Since $25=1\cdot 25=5\cdot 5$, thus $(x,y)\in\{(6,30),(10,10),(30,6)\}$.

Therefore all possible values of x is $\boxed{6+10+30=46.}$ The answer is $\boxed{D.}$ □

Solution. ICLT-2021-SM2-11-L5-P2 Since Khang rows 7 km/h and the river flows at 5 km/h, so he is going downstream at $7+5=12$ km/h, and upstream at $7-5=2$ km/h. Let d and u be the amount of time he rows down- and upstream, then

$$\begin{cases} d+u &= 3.5 \\ 12d &= 2u \end{cases} \Rightarrow u=6d \Rightarrow 7d=3.5 \Rightarrow d=0.5,\ u=3$$

Thus, he turns around at $\boxed{\text{12:30 p.m}}$. The answer is $\boxed{A.}$ □

Solution. ICLT-2021-SM2-11-L5-P3 Let assume the opposite when hey do not sit next to each other. Since there are 5 chairs, so Bu, Chu, and Fu must sit on the chairs 1, 3, 5, so that they do not sit next to each other. There are $3!=6$ permutations for them to sit on those chairs. In total there are $\binom{5}{3}=10$ ways to choose three seats for them and $3!=6$ ways to permutate. Thus the probability is $\frac{6}{10\cdot 6}=\frac{1}{10}$.

Therefore the probability that at least two of them sit next to each other is $1-\frac{1}{10}=\frac{9}{10}$. The answer is $\boxed{B.}$ □

Solution. ICLT-2021-SM2-11-L5-P4 Let x be the length of the shortest leg. Then the length of the other leg is $2x+2$, and the length of the hypotenuse is $2x+3$. By Pythagorean Theorem,

$$x^2+(2x+2)^2=(2x+3)^2 \Rightarrow x^2-4x-5=0 \Rightarrow (x-5)(x+1)=0 \Rightarrow x=5$$

Thus the area is $\boxed{\frac{1}{2}\cdot 5\cdot(2\cdot 5+2)=30.}$ The answer is $\boxed{D.}$ □

Solution. ICLT-2021-SM2-11-L5-P5

$$\begin{cases} \sqrt{z^2+7}=z+x \\ \sqrt{z^2+7}=7y-z \end{cases} \Rightarrow x=\sqrt{z^2+7}-z,\ 7y=\sqrt{z^2+7}+z$$
$$\Rightarrow 7xy=(\sqrt{z^2+7}-z)(\sqrt{z^2+7}+z)=(z^2+7)-z^2=7 \Rightarrow xy=1$$

Thus, if $x=\frac{\sqrt{7}}{7}$, then $\boxed{y=\frac{1}{x}=\sqrt{7}.}$ The answer is $\boxed{E.}$ □

Solution. ICLT-2021-SM2-11-L5-P6 The area determined by $x=0, y=0, x=1, y=3$ is the rectangle $ABCD$. Line $y=x+1$ split it into two equal parts. All points in the upper part, the quadrilateral $BCEF$, satisfies all required inequalities,

$$0<x<1,\ 0<y<3,\ y>x+1$$

Thus the probability is $\boxed{\frac{1}{2}.}$ The answer is $\boxed{C.}$ □

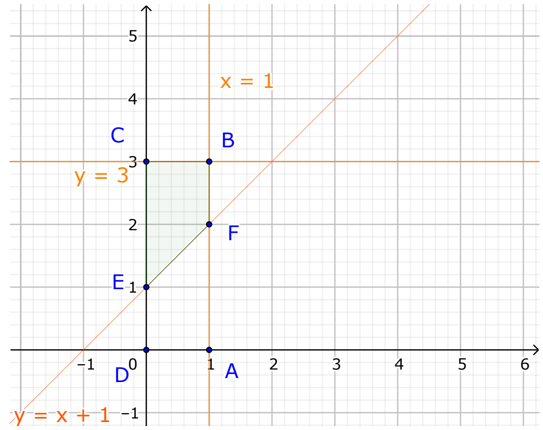


Figure 14.10: $\triangle ABC$

Solution. ICLT-2021-SM2-11-L5-P7 Note that if none of the dice shows 3 or 6 then it can shows only $1, 2, 4, 5$, which are the numbers that are not divisible by 3. Furthermore all the three numbers must have the same remainders when divided by 3. Therefore they all must be chosen from $\{1, 4\}$ or all from $\{2, 5\}$. In each case there are $\boxed{2^3 = 8}$ ways to choose.

Thus, the probability is $\boxed{\frac{2 \cdot 2^3}{6^3} = \frac{2}{27}.}$ □

Solution. ICLT-2021-SM2-11-L5-P8 First $7 \equiv 1 \pmod 6$, thus $7^6 \equiv 7^5 \equiv \ldots \equiv 1 \pmod 6$, thus

$$\overline{123}_7 = 1 \cdot 7^2 + 2 \cdot 7^1 + 3 \equiv 1 + 2 + 3 \equiv 0 \pmod 6$$

Similarly

$$\overline{x123456}_7 = x \cdot 7^6 + 1 \cdot 7^5 + 2 \cdot 7^4 + 3 \cdot 7^3 + 4 \cdot 7^2 + 5 \cdot 7^1 + 6 \equiv x + 1 + 2 + 3 + 4 + 5 + 6 \equiv x + 3 \pmod 6$$

If $6 \mid \overline{x123456}_7$, then $6 \mid x + 3$. The only such digit for x is $\boxed{x = 3.}$ The answer is $\boxed{C.}$ □

Solution. ICLT-2021-SM2-11-L5-P9 First we determine the probability when a permutation of OTTAWA starts with OTT. The probability to choose O is $\frac{1}{6}$. The probability to choose the first T is $\frac{2}{5}$. The probability to choose the second T is $\frac{1}{4}$. Therefore the probability when a permutation of OTTAWA starts with OTT is

$$\frac{1}{6} \cdot \frac{2}{5} \cdot 14 = \frac{1}{60}$$

Thus, the probability when a permutation of OTTAWA does NOT start with OTT is $\boxed{\frac{59}{60}.}$ □

Chapter 15

AMC8 R2

15.1 Selected Problems

Problems

1. Practice 6, Problem 24: Gauss Grade 8 2019/24
2. Practice 6, Problem 25: Gauss Grade 8 2013/25
3. Practice 7, Problem 24: already discussed in Practice 1, Problem 24: Gauss Grade 8 2009/24
4. Practice 7, Problem 25: already discussed in Practice 3, Problem 25: Gauss Grade 8 2014/25
5. Practice 8, Problem 24: Gauss Grade 8 2013/24
6. Practice 8, Problem 25: Gauss Grade 8 2007/25
7. Practice 9, Problem 24: Gauss Grade 8 2017/24
8. Practice 9, Problem 25: Gauss Grade 8 2020/25
9. Practice 10, Problem 24: Gauss Grade 8 2020/24
10. Practice 10, Problem 25: already discussed in Practice 1, Problem 25: Gauss Grade 8 2019/25

15.2 Problems

Example 15.2.1 (Gauss G8 2019/24)

There are many ways in which the list $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ can be separated into groups. For example, this list could be separated into the four groups $0, 3, 4, 8$ and $1, 2, 7$ and 6 and $5, 9$. The sum of the numbers in each of these four groups is $15, 10, 6$, and 14, respectively.

In how many ways can the list $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ be separated into at least two groups so that the sum of the numbers in each group is the same?

Proof. Gauss G8 2019/24 First, notice that the given list of 10 numbers has a sum of $0+1+2+3+4+5+6+7+8+9 = 45$. If n is the number of groups and m is the total in each group, then we must have,

$$mn = 45.$$

This means 45 is *a multiple of the number of groups.* Thus the pair (the number of groups, the sum of the numbers in a group) can be,

$$(1, 45), (3, 15), (5, 9), (9, 5), (15, 3), (45, 1).$$

Note that,

1. *the number of groups n should be at least 2, and cannot be more than 10*
2. *the sum of the numbers in a group is at least 9.*

Therefore the number of groups is one of 3 and 5, and the sum of the numbers in each group is 15 and 9, respectively.

Case 1: the number of the groups is 5, then the sum of the numbers is 9. It is easy to see that 9 has to be in a group of itself, 8 with 1, 7 with 2, 6 with 3, and 5 with 4.

$$\{9\}, \{8, 1\}, \{7, 2\}, \{6, 3\}, \{5, 4\}$$

There are $\boxed{\text{5 ways}}$ to put 0 into one of the groups.

Case 2: the number of the groups is 3, then the sum of the numbers is 15. There are four ways to construct a group with 9 and other numbers,

$$A\{9, 6\},\ B\{9, 5, 1\},\ C\{9, 4, 2\},\ D\{9, 3, 2, 1\}$$

There are eleven of other groups that also having sum to 15.

$$\begin{aligned}
&E\{8, 7\},\ F\{8, 6, 1\},\ G\{8, 5, 2\},\ H\{8, 4, 3\},\ I\{8, 4, 2, 1\}\\
&J\{7, 6, 2\},\ K\{7, 5, 3\},\ L\{7, 5, 2, 1\},\ M\{7, 4, 3, 1\}\\
&N\{6, 5, 4\},\ O\{6, 5, 3, 1\},\ P\{6, 4, 3, 2\},\ Q\{5, 4, 3, 2, 1\}
\end{aligned}$$

Case 2a: consider group $A\{9, 6\}$. The other two groups cannot contain 6 or 9, so they can only be E, G, H, I, K, L, M, and Q. It is easy to verify that if one of the two groups is E, then the remaining is Q, if one of the two groups is G, then the remaining is M, similarly H and L, I and K. So we have four combinations

$$(A, E, Q),\ (A, G, M),\ (A, H, L),\ (A, I, K),$$

Case 2b: consider group $B\{9, 5, 1\}$. Analyzing similarly we find two combinations

$$(B, E, P),\ (B, H, J)$$

Case 2c: consider group $C\{9,4,2\}$. Analyzing similarly we find two combinations

$$(C,E,O),\ (C,F,K)$$

Case 2d: consider group $D\{9,3,2,1\}$. Analyzing similarly we find one combination

$$(D,E,N)$$

All together $4+2+2+1=9$. Since 0 can be put into any of the three groups, so we have $\boxed{9\cdot 3=27 \text{ ways.}}$

Thus, the total number of ways is $\boxed{27+5=32.}$ □

Example 15.2.2 (Gauss G8 2013/25)

At the beginning of the winter, there were at least 66 students registered in a ski class. After the class started, eleven boys transferred into this class and thirteen girls transferred out. The ratio of boys to girls in the class was then 1 : 1.

Which of the following is not a possible ratio of boys to girls before the transfers?

$(A)\quad 4:7 \qquad (B)\quad 1:2 \qquad (C)\quad 9:13 \qquad (D)\quad 5:11 \qquad (E)\quad 3:5$

Proof. Gauss G8 2013/25 Let b represent the number of boys initially registered in the class. Let g represent the number of girls initially registered in the class. When 11 boys transferred into the class, the number of boys in the class was $b+11$. When 13 girls transferred out of the class, the number of girls in the class was $g-13$. The ratio of boys to girls in the class at this point was $1:1$, thus

$$b+11=g-13 \Rightarrow g=b+24.$$

Since there were at least 66 students initially registered in the class, then

$$66 \le b+g=b+(b+24)=2b+24 \Rightarrow 21 \le b \quad (*)$$

Now, instead of finding what value that the ratio $b:g$ can be,

1. we find the value of b based on a given assumption (choice)

2. we test if that value of b satisfy the condition (*)

It is easy to verify that

$$\begin{cases} (A)\ b:(b+24)=4:7 \Rightarrow 7b=4b+96 \Rightarrow b=32 \\ (B)\ b:(b+24)=1:2 \Rightarrow 2b=b+24 \Rightarrow b=24 \\ (C)\ b:(b+24)=9:13 \Rightarrow 13b=9b+216 \Rightarrow b=54 \\ (D)\ b:(b+24)=5:11 \Rightarrow 11b=5b+120 \Rightarrow b=20 \\ (E)\ b:(b+24)=3:5 \Rightarrow 5b=3b+72 \Rightarrow b=36 \end{cases}$$

The case $\boxed{D}$ is the right answer because it does not satisfy the condition $b \ge 21$. □

Example 15.2.3 (Gauss G8 2013/24)

In any triangle, the length of the longest side is less than half of the perimeter. All triangles with perimeter 57 and integer side lengths x, y, z, such that $x < y < z$ are constructed.

How many such triangles are there?

Proof. Gauss G8 2013/24 The length of the longest side, z, is less than half of the perimeter 57, thus $z \leq 28$.

Now, this seems to be a long case-work, but it is not. So,

1. we just need to find out the answer for a case, then

2. try to find a pattern to list the answers in an easily-verifiable way

Case 1: $z = 28$, then $x + y = 57 - 28 = 29$,

$$(x, y) \in \{(2, 27), (3, 26), \dots (14, 15)\}.$$

Case 2: $z = 27$, then $x + y = 57 - 27 = 30$,

$$(x, y) \in \{(4, 26), (5, 25), \dots (14, 16)\}.$$

Case 3: $z = 26$, then $x + y = 57 - 26 = 31$,

$$(x, y) \in \{(6, 25), (7, 26), \dots (15, 16)\}.$$

Case 4: $z = 25$, then $x + y = 57 - 25 = 32$,

$$(x, y) \in \{(8, 24), (9, 23), \dots (15, 17)\}.$$

Case 5: $z = 24$, then $x + y = 57 - 24 = 33$,

$$(x, y) \in \{(10, 23), (11, 22), \dots (16, 17)\}.$$

Case 6: $z = 23$, then $x + y = 57 - 23 = 34$,

$$(x, y) \in \{(12, 22), (13, 21), \dots (16, 18)\}.$$

Case 7: $z = 22$, then $x + y = 57 - 22 = 35$,

$$(x, y) \in \{(14, 21), (15, 20), \dots (17, 18)\}.$$

Case 8: $z = 21$, then $x + y = 57 - 21 = 36$,

$$(x, y) \in \{(16, 20), (17, 19)\}.$$

Case 9: $z = 20$, then $x + y = 57 - 20 = 37$,

$$(x, y) \in \{(18, 19)\}.$$

The total number of possible cases is $\boxed{13 + 11 + 10 + 8 + 7 + 5 + 4 + 2 + 1 = 61}$. □

Example 15.2.4 (Gauss G8 2007/25)

A rectangular piece of paper $ABCD$ is folded so that edge CD lies along edge AD, making a crease DP. It is unfolded, and then folded again so that edge AB lies along edge AD, making a second crease AQ. The two creases meet at R, forming triangles PQR and ADR, as shown.

If $AB = 5$cm and $AD = 8$cm, what is the area of quadrilateral $DRQC$ in cm^2?

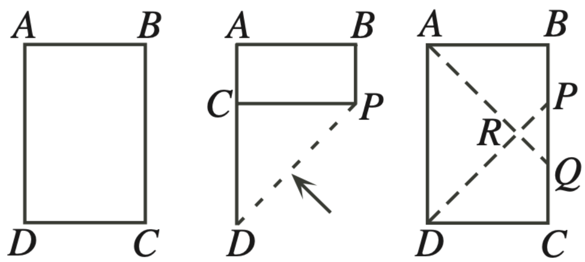


Figure 15.1: Gauss G8 2007/25

Proof. Gauss G8 2007/25

Instead to compute directly the area of $DRQC$, we need to find

1. *the area of the whole figure,*
2. *then subtract the parts outside of $DRQC$, namely $\triangle ARD$ and $\triangle ABQ$*

Notice that $\angle CDP = \frac{1}{2}\angle ADC = 45°$, similarly $\angle DAQ = 45°$. Thus $\triangle ARD$ is an isosceles right triangle,

$$AD^2 = AR^2 + DR^2 = 2AR^2 \Rightarrow AR^2 = \frac{1}{2}64 = 32 \Rightarrow [ARD] = \frac{1}{2}AR^2 = 16$$

Similarly $\triangle ABQ$ is an isosceles right triangle, so $[ABQ] = \frac{1}{2}AB^2 = \frac{25}{2}$.

Therefore $\boxed{[DRQC] = AB \cdot AD - [ARD] - [ABQ] = 40 - 16 - \frac{25}{2} = \frac{23}{2}.}$ □

Example 15.2.5 (Gauss G8 2017/24)

In the diagram, ABC is a quarter of a circle with radius 8. A semi-circle with diameter AB is drawn, as shown. A second semi-circle with diameter BC is also drawn.

What is the area of the shaded region?

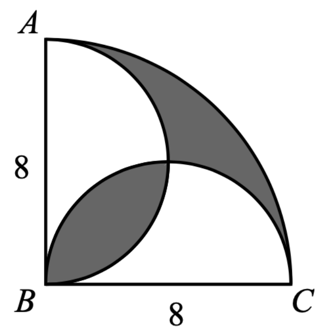


Figure 15.2: Gauss G8 2017/24

Proof. Gauss G8 2017/24

Let the area of the shaded region that lies outside of both semi-circles be X. Let the area of the shaded region that lies inside of both semi-circles be Y. See Figure 15.3.

Now, we try to compute directly the area of the shaded area, base on the facts that

1. *The sum of the areas of both semi-circles counts the shaded area Y twice,*
2. *The area that both semi-circles cover plus X is the area of quarter-circle ABC.*

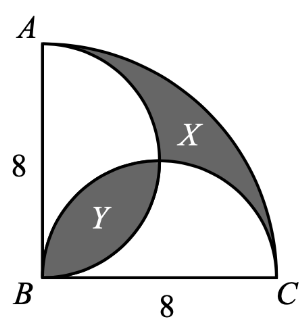


Figure 15.3: X, Y regions

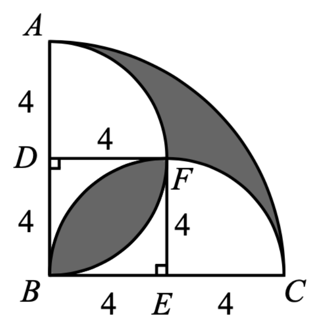


Figure 15.4: $BDFE$ square

$$\frac{\pi \cdot 8^2}{4} = (2 \cdot \frac{\pi \cdot 4^2}{2} - Y) + X \Rightarrow X = Y$$

Now lets construct points D, E, F as shown in the second diagram, Figure 15.4. By symmetry, BF divides the shaded area Y into two equal areas,

$$\frac{Y}{2} = \frac{\pi \cdot 4^2}{4} - [BEF] = \frac{\pi \cdot 4^2}{4} - \frac{4^2}{2} \Rightarrow Y = 8\pi - 16$$

Therefore $\boxed{X + Y = 16\pi - 32.}$ □

Example 15.2.6 (Gauss G8 2020/25)

A sequence of positive integers with 2020 terms is called an *FTsequence* if each term after the second is the sum of the previous two terms. For example, if the first two terms of an *FTsequence* are 8 and 7, the sequence would begin $8, 7, 15, 22, 37, \ldots$. For some positive integer m, there are exactly 2415 *FTsequences* where the first two terms are each less than $2m$ and the number of odd-valued terms is more than twice the number of even-valued terms.

What is the value of m?

Proof. Gauss G8 2020/25

First, we note two important facts that

1. *The result of adding two integers that have the same parity is an even integer.*
2. *The result of adding two integers that have different parity is an odd integer.*

Thus, the parity of each term of an FT sequence (after the second term) is determined by the parity of the first two terms in the sequence.

For example, if each of the first two terms of an FT sequence is odd, then the third term is even (since odd plus odd is even), the fourth term is odd (since odd plus even is odd), the fifth term is odd (since even plus odd is odd), and so on.

There are 4 possibilities for the parities of the first two terms of an *FT sequence.* The sequence could begin (odd, odd), or (even, even), or (odd, even), or (even, odd). In the table below, we write the parity of the first few terms of the *FT sequences* that begin in each of the 4 possible ways.

Term Number	1	2	3	4	5	6	7	8	9	10
Parity #1	odd	odd	even	odd	odd	even	odd	odd	even	odd
Parity #2	even	even	even	even	even	even	even	even	even	even
Parity #3	odd	even	odd	odd	even	odd	odd	even	odd	odd
Parity #4	even	odd	odd	even	odd	odd	even	odd	odd	even

Figure 15.5: 4 parity cases

Case 1: The *FT sequence* beginning (odd, odd) (Parity #1) continues to repeat (odd, odd, even). Since the parity of each term is dependent on the parity of the two terms preceding it, this (odd, odd, even) pattern will continue throughout the entire sequence. That is, in each successive group of three terms beginning at the first term, one out of three terms will be even and two out of three terms will be odd.

The (odd, odd, even) pattern ends at term numbers that are multiples of 3 (the even-valued terms are terms $3, 6, 9, 12$, and so on). Since 2019 is a multiple of 3 ($2019 = 3 \cdot 673$), $\frac{1}{3}$ of the first 2019 terms will be even-valued and $\frac{2}{3}$ will be odd-valued, and so there are twice as many odd-valued terms as there are even-valued terms in the first 2019 terms.

The 2020^{th} term is odd (since the pattern begins with an odd-valued term), and so there are more than twice as many odd-valued terms as there are even-valued terms in every *FT sequence* that begins (odd, odd). This is exactly the required condition for the *FT sequences* that we are interested in.

How many *FT sequences* begin with two odd-valued terms, each of which is a positive integer less than $2m$? The list of integers from 1 to $2m - 1$, namely $1, 2, 3, 4, \ldots, 2m - 1$ contains m odd integers and $m - 1$ even integers. The first term in the sequence is odd-valued and so there are m choices for it. Similarly, the second term in the sequence is also odd-valued and so there are also m choices for it.

Thus, there are a total of $\boxed{m \cdot m = m^2}$ *FT sequences* that begin with two odd-valued terms.

Case 2: It is easy to see that the *FT sequence* beginning (even, even) (Parity #2) does not satisfy.

Case 3: The *FT sequence* beginning (odd, even) (Parity #3) continues to repeat (odd, even, odd). Similarly $\frac{1}{3}$ of the first 2019 terms will be even-valued and $\frac{2}{3}$ will be odd-valued. The 2020$^{\text{th}}$ term is odd (since the pattern begins with an odd-valued term) and so there are more than twice as many odd-valued terms as there are even-valued terms in every *FT sequence* that begins (odd, even). As shown before, the list of integers from 1 to $2m-1$ contains m odd integers and $m-1$ even integers. The first term in the sequence is odd-valued and so there are m choices for it, the second term in the sequence is also even-valued and so there are also $m-1$ choices for it.

Thus, there are a total of $\boxed{m \cdot (m-1) = m(m-1)}$ *FT sequences* that begin with (odd, even) terms.

Case 4: The *FT sequence* beginning (even, odd) (Parity #3) continues to repeat (even, odd, odd). Again, $\frac{1}{3}$ of the first 2019 terms will be even-valued and $\frac{2}{3}$ will be odd-valued. The 2020$^{\text{th}}$ term is even (since the pattern begins with an even-valued term) and so there are *fewer* than twice (!) as many odd-valued terms as there are even-valued terms in every *FT sequence* that begins (odd, even).

Thus, there are $m^2 + m(m-1) = 2m^2 - m$ such *FT sequences.*

$$2m^2 - m = 2415 \Rightarrow (2m+69)(m-35) = 0 \Rightarrow m = 35.$$

The answer is $\boxed{m = 35.}$ □

Example 15.2.7 (Gauss G8 2020/24)

Every 12 minutes, Bus A completes a trip from P to X to S to X to P. Every 20 minutes, Bus B completes a trip from Q to X to T to X to Q. Every 28 minutes, Bus C completes a trip from R to X to U to X to R. At 1 : 00 p.m., Buses A, B, and C depart from P, Q, and R, respectively, each driving at a constant speed, and each turning around instantly at the endpoint of its route. Each bus runs until 11 : 00 p.m.

At how many times between 5 : 00 p.m. and 10 : 00 p.m. will two or more buses arrive at X at the same time?

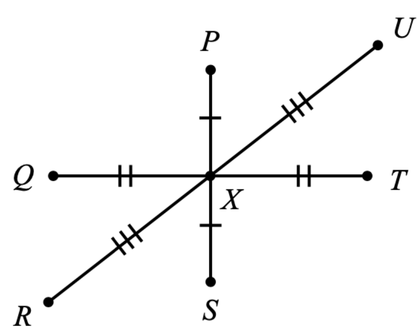


Figure 15.6: Gauss G8 2020/24

Proof. Gauss G8 2020/24

Bus A takes 12 minutes to complete one round trip that begins and ends at P. Since $PX = XS$, it takes Bus A $12 \div 4 = 3$ minutes to travel from P to X, 6 minutes to travel from X to S to X (3 minutes from X to S and 3 minutes from S to X), and 6 minutes to travel from X to P to X.

That is, Bus A first arrives at X at $1:03$ and then continues to return to X every 6 minutes. We write times that Bus A arrives at X as below,

$$(A) \quad 1:03\ 1:09\ 1:15\ 1:21\ 1:27\ 1:33\ 1:39\ 1:45\ 1:51\ 1:57\ 2:03$$

This makes sense since Bus A returns to X every 6 minutes and 60 minutes (one hour) is divisible by 6. This tells us that Bus A will continue to arrive at the same number of minutes past each hour, or

$$2:03\ 2:09\ 2:15\ \ldots\ 3:03\ 3:09\ \ldots\ 5:03\ 5:09\ \ldots\ 9:03\ 9:09\ \ldots\ 9:51\ 9:57.$$

Similarly, Bus B takes 20 minutes to complete one round trip that begins and ends at Q. Since $QX = XT$, it takes Bus B $20 \div 4 = 5$ minutes to travel from Q to X, 10 minutes to travel from X to T to X (5 minutes from X to T and 5 minutes from T to X), and 10 minutes to travel from X to Q to X.

That is, Bus B first arrives at X at $1:05$ and then continues to return to X every 10 minutes. We write times that Bus B arrives at X as below,

$$(B) \quad 1:05\ 1:15\ 1:25\ 1:35\ 1:45\ 1:55\ 2:05$$

Similarly, Bus B will continue to arrive at the same number of minutes past each hour,

$$2:05\ 2:15\ 2:25\ \ldots\ 3:05\ 3:15\ \ldots\ 5:05\ 5:15\ \ldots\ 9:05\ 9:15\ \ldots\ 9:45\ 9:55.$$

Thus, A and B meet at

$$(AB)\ 5:15\ 5:45\ 6:15\ 6:45\ 7:15\ 7:45\ 8:15\ 8:45\ 9:15\ 9:45.$$

In a similar way, Bus C takes 28 minutes to complete one round trip that begins and ends at R. Since $RX = XU$, it takes Bus C $28 \div 4 = 7$ minutes to travel from R to X $2 \cdot 7 = 14$ minutes to travel from X to U to X, and 14 minutes to travel from X to R to X. That is, Bus C first arrives at X at $1:07$ and then continues to return to X every 14 minutes. Unlike Bus A and Bus B, Bus C will not arrive at X at consistent times past each hour since 60 is not divisible by 14.

What is the first time after $5:00$ p.m. that Bus C arrives at X? Since $14 \cdot 17 = 238$, Bus C will arrive at X 238 minutes after first arriving at X at $1:07$ p.m. Since 238 minutes is 2 minutes less than 4 hours $(4 \cdot 60 = 240)$, Bus C will arrive at X at $5:05$ p.m. (This is the first time after $5:00$ p.m. that Bus C arrives at X.) Bus B also arrives at X at $5:05$ p.m. Now, Bus B arrives at X every 10 minutes and Bus C arrives at X every 14 minutes. Since the lowest common multiple of 10 and 14 is 70, then Bus B and Bus C will each arrive at X every 70 minutes after $5:05$ p.m, so they meet

$$(BC) \quad 5:05\ 6:15\ 7:25\ 8:35\ 9:45.$$

Bus C arrives at X every 14 minutes after $5:05$ p.m., or $5:19$ p.m., $5:33$ p.m., and so on. Bus A also arrives at X at $5:33$ p.m. Now, Bus A arrives at X every 6 minutes and Bus C arrives at X every 14 minutes. Since the lowest common multiple of 6 and 14 is 42, then Bus A and Bus C will each arrive at X every 42 minutes after $5:33$ p.m, so they meet at

$$(AC) \quad 5:33\ 6:15\ 6:57\ 7:39\ 8:21\ 9:03\ 9:45.$$

The number of times that two or more buses arrive at X between $5:00$ p.m. and $10:00$ p.m. is $\boxed{10+5+7-4=18}$ (since each of $6:15$ and $9:45$ are counted three times). ☐