Learning Problem Solving

for Middle- and High School Students

10 Strategies, 5 Principles, and 10 Counting Techniques

Math & Chess Club in Ottawa

Nghia Doan & Catherine Doan

Problems

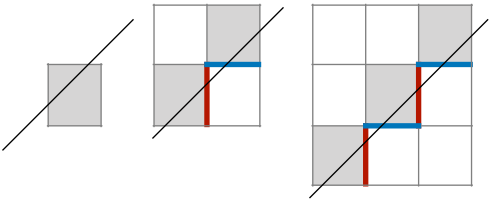
Chapter 1 - Ten Strategies

Ten strategies

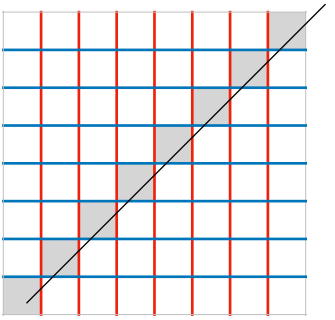
1. Solve a simpler problem

Example 1.1*: A straight line is drawn across an* 8×8 *chessboard. It is said to pierce a square if it passes through an interior point of the square. At most how many of the 64 squares can this line pierce?*

Solution: Let consider some simpler problems: 1×1, 2×2, and 3×3chessboard as in the diagrams below.



It is easy to see that for an $n \times n$ chessboard, $n = 1, 2, 3,$ the largest number of pierced square is $2n - 1$. This is achievable when the line alternately cross the squares in one of the main diagonals. It shows that each black square shares at most one vertical side (red) and one horizontal side (blue) with a white square.So *it leads to the idea of colouring the internal grid lines*.



In the figure on the left, the internal vertical grid lines are coloured red and the horizontal ones are coloured blue. Any line can cross at most 7 red and 7 blue grid lines. 7 red and 7 blue grid lines with alternate pairing (red, blue, red, blue, ...) as shared sides of $7 + 7 + 1 = 15$ squares pierced squares.

2. Search for a pattern

Example 1.2*: Prove that for all positive integers n:*

$$\frac{1}{1.2} + \frac{1}{3.4} + \ldots + \frac{1}{(2n-1)2n} = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$$

Solution: the left side of the identity is a *sum of similar fractions:*

1/each denominator is *a product of two consecutive integers*

2/each numerator is *the difference of those two consecutive integers*

$$\frac{1}{k(k+1)} = \frac{(k+1)-k}{k(k+1)} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

Therefore the left side can be rewritten as:

$$(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + \ldots + (\frac{1}{2n-1} - \frac{1}{2n}) \qquad (*)$$

If we add the sum $\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}$ to the right side, it is now a more complete sum of $\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{2n}$, so let see what we get if we add this sum $\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}$ to the left side in rewritten form (*):

$$(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + \ldots + (\frac{1}{2n-1} - \frac{1}{2n}) + (\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n})$$
$$= (\frac{1}{1} - \frac{1}{2} + \frac{1}{1}) + (\frac{1}{3} - \frac{1}{4} + \frac{1}{2}) + \ldots + (\frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{n})$$
$$= (\frac{1}{1} + \frac{1}{2}) + (\frac{1}{3} + \frac{1}{4}) + \ldots + (\frac{1}{2n-1} + \frac{1}{2n}) = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{2n}$$

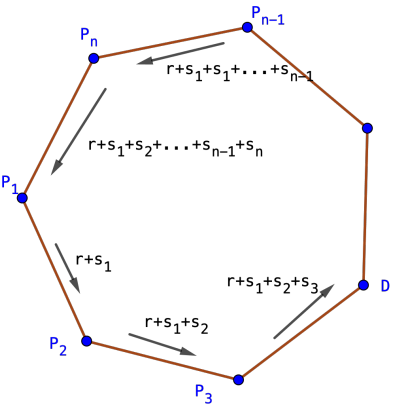
Hence both sides of the original identity are equal.

3. Think outside-of-the-box

Example 1.3: *A group of jealous professors is locked up in a room. There is nothing else in the room but pencils and one tiny scrap of paper per person. The professors want to determine their average (mean, not median) salary so that each one can gloat or grieve over his or her personal situation compared to their peers. However, they are secretive people, and do not want to give away any personal salary information to anyone else.*

Can they determine the average salary in such a way that no professor can discover any fact about the salary of anyone but herself? For example, even facts such as "three people earn more than $40,000" or "no one earns more than $90,000" are not allowed.

Solution: The problem sounds hard because it is difficult to find a starting point. The idea here is somehow *the sum of all salary amounts must be formed but that should not be known by any one*. It leads to the idea of someone *adding a random number to the sum*, which is known by only that person. If that person subtract the random number, the person is able to obtain the sum of all salaries.



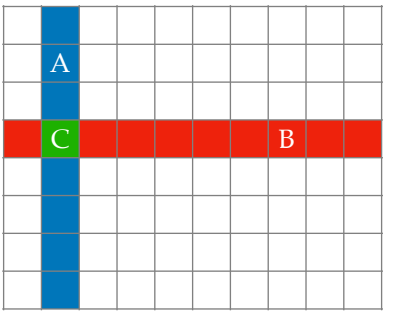
On the diagram, the first professor P_1 gives the second professor P_2 the sum $r + s_1$, where s_1 is the salary of P_1 and r is a random number. The second professor P_2 adds his salary s_2 to this number and send the new sum $r + s_1 + s_2$ to the professor P_3 and so on. After the last professor P_n have added his salary, the sum is $r + s_1 + s_2 + \ldots + s_n$. So the professor P_1 just need to subtract r from it and divides the remaining by n. Now the average can be announced to everybody.

4. Draw a picture

Example 1.4: *Two hundred students are positioned in 10 rows, each contained 20 students. From each of the 20 columns thus formed the shortest student is selected, and the tallest of these 20 students is tagged as A. These students now return to their initial places. Next the tallest student from each rows is selected, and from these 10 students the shortest is tagged as B.*

Which of the two tagged students is the taller if they are two different people?

Solution



In the diagram on the left: A is shortest in the blue column, since A is selected from the shortest ones. Similarly B is tallest in the red row.

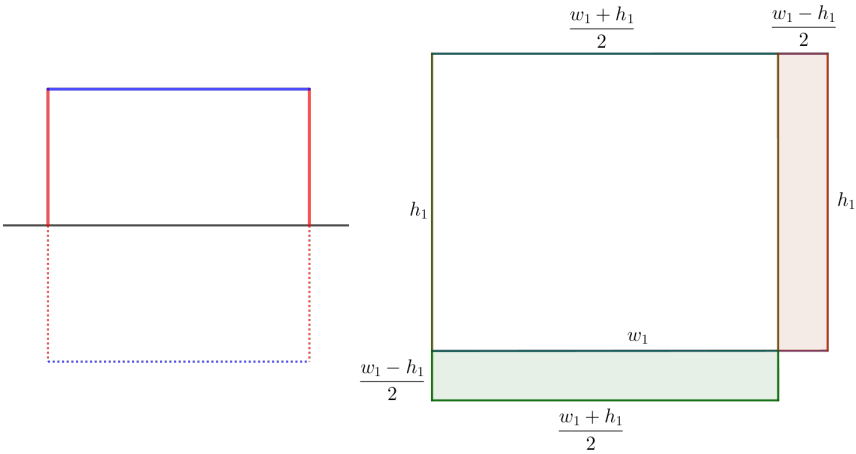
Let C stands on the intersection. B is taller than C, C is taller than A. This is also true if C is A or C is B.

5. Cast the problem in another way

***Example 1.5**: A farmer wishes to fence in a rectangular field using 300 meters of fence. One side of the field needs no fencing since it is alongside an already existing fence.*

What is the largest possible area of the field and what are the dimensions?

Solution: Let h be the height (red) and w be the width (blue) of the fence. The question here is if $2h + w = 300$, what value of h and w can make the rectangle area to be largest. One of the solution might be try into find the largest value of the area by maximizing $hw = h(300 - 2h)$ which requires knowledge of quadratics.



Here, we consider application of geometric reflection of the field over the long fence. The result is a rectangle is double in height and the new rectangle area, obviously is twice of the original one. It is shown on the left diagram above. Let h_1 be the height and w_1 be the width of the new rectangle. We know $h_1 + w_1 = 2h + w = 300$. So the question now is among those rectangles with fix perimeter, which one has largest area. Our guess is that the square area is largest among rectangles with same perimeter. Assume $w_1 \geq h_1$, let construct a square with side $\dfrac{w_1 + h_1}{2}$, which has the same perimeter as the rectangle. The difference in area is between the green and brown rectangles as in the right diagram. above. Easy to see that the green rectangle $\dfrac{w_1 - h_1}{2} \times \dfrac{w_1 + h_1}{2}$ has larger area than the brown rectangle $\dfrac{w_1 - h_1}{2} \times h_1$, because $w_1 \geq h_1$.

6. Change your point of view

***Example 1.6**: What is the unit digit of:*

$$\underbrace{2^{2^{2^{\cdots^{2}}}}}_{222}$$

***Solution**:* It is not likely that you have enough time or paper to calculate the number above. *Instead of making an approach to calculate the number, we shall investigate what is the question.*

The unit digit is the remainder when divided a number by 10. If, for example the remainder of a power 2^4 when divided by 10 is 6, then the remainder of 2^5 when divided by 10 is the same as 2×6 divided by 10, which is 2. It is easy to see that the remainders of the powers $2, 2^2, 2^3, \ldots$ form *a periodic sequence of* 2, 4, 8, 6, 2, 4, 8, 6,... Since the exponent of $\underbrace{2^{2^{2^{\cdots^{2}}}}}_{222}$ is a power of 2 and thus it is divisible by 4, so the number has a remainder of 6.

Hence the unit digit of the given number is 6.

7. Proof by contradiction

***Example 1.7**: Show that $b^2 + b + 1 = a^2$ has no integer solutions.*

***Solution**:* Assume the opposite, there are a, b integers such that $b^2 + b + 1 = a^2$. Easy to see that $a > b$. In addition:

$$b + 1 = a^2 - b^2 = (a - b)(a + b) \ (*)$$

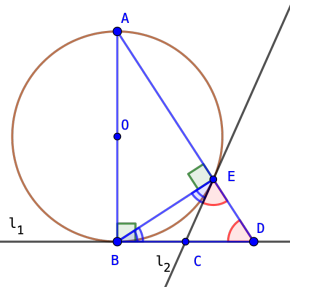
Since $a - b \geq 1$, $a + b > b + 1$, the right side of (*) is larger than the left side. It is contradictory to the existence of a, b.

Hence, the assumption is false: there is no integer solutions

8. Working backwards

***Example 1.8**: AOB is the diameter of the circle O. l_1 is tangent with the circle at B. D is a point on l_1. AD intersects the circle at E. l_2 is tangent with the circle at E and intersects BD at C.*

Prove that $BC = CD$.



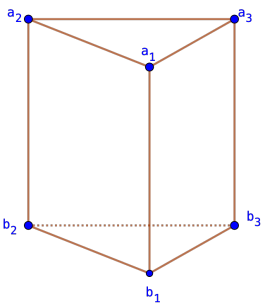
Discussion: Let assume that $BC = CD$. AB is the diameter, so $\triangle AEB$ is a right triangle at E. $\triangle BED$ is also a right triangle at E.C is the midpoint of the hypotenuse BD, so $BC = CE = CD$. This leads to $\triangle ECD$ and $\triangle ECB$ are isosceles with equal red and blue angle pair as above. The blue angle at E thus complementary to the red angle at E, therefore is complementary with the red angle at D, which is complementary to $\angle A$. Thus the blue angle at E is equal to $\angle A$. This is true since both angles cut off the arc BE.

Solution: Since $\angle BEC$ and $\angle A$ cut off the same arc BE, so they are equal. $\angle BDA = 90^\circ - \angle A = 90^\circ - \angle BEC = \angle CED \Rightarrow CE = CD$. Similarly $BC = CE$. So $BC = CD$.

9. Effective notations

Example 1.9*: A real number is assigned to each vertex of a triangular prism such that the number on any vertex is the arithmetic mean of the numbers on the three adjacent vertices.*

Prove that all six numbers are equal.



Let a_1, a_2, a_3 and b_1, b_2, b_3 be the numbers on the vertices as shown.

$$\begin{cases} 3a_1 = a_2 + a_3 + b_1 & (1) \\ 3a_2 = a_1 + a_3 + b_2 & (2) \\ 3a_3 = a_1 + a_2 + b_3 & (3) \\ 3b_1 = a_1 + b_2 + b_3 & (4) \\ 3b_2 = a_2 + b_1 + b_3 & (5) \end{cases}$$

Group (1)+(2)+(3), (1)+(4), (2)+(5):

$$\begin{cases} a_1 + a_2 + a_3 = b_1 + b_2 + b_3 & (6) \\ 2(a_1 + b_1) = (a_2 + b_2) + (a_3 + b_3) & (7) \\ 2(a_2 + b_2) = (a_1 + b_1) + (a_3 + b_3) & (8) \end{cases}$$

Subtract (7)-(8): $3(a_1 + b_1) = 3(a_2 + b_2) \Rightarrow a_1 - a_2 = b_2 - b_1$ (9)

Finally (1)-(2)+(9):

$$3(a_1 - a_2) = (a_2 - a_1) - (a_1 - a_2) \Rightarrow a_1 = a_2$$

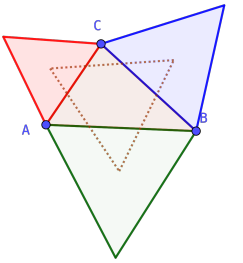
Because of the symmetry: $a_1 = a_2 = a_3$. Similarly $b_1 = b_2 = b_3$. Using (1): $a_1 = a_2 = a_3 = b_1 = b_2 = b_3$.

Hence, all six numbers are equal.

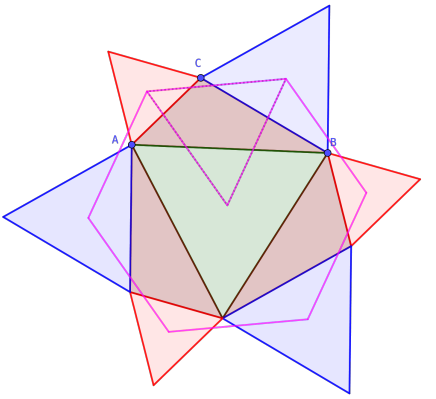
10. Exploit symmetry

Example 1.10: *(Napoleon's Theorem) Three equilateral triangles are constructed outwardly on sides AB, BC, CA of $\triangle ABC$. The centres of the three triangles are connected to form a new triangle.*

The new triangle is an equilateral triangle.



Solution: On the sides of the green triangle, we construct another two triangles, both are congruent with $\triangle ABC$. Outwardly of those, we construct similar equilateral triangles as below.



Easy to see that the newly formed figure is symmetric. The centres now form a regular hexagon. This means that the triangle in question must be an equilateral one.

Problems

Problem 1.11

Prove that $\dfrac{2020!}{1010!.2^{1010}}$ is an integer.

Problem 1.12

Using proof by contradiction, prove that there are infinitely many primes. Assume that $p_1, p_2, \ldots, p_n$ are the only prime numbers, investigate prime factors of $p_1 . p_2 \ldots . p_n + 1$.

Problem 1.13

What is the sum of digit of P if $P = (\underbrace{3...3}_{333})^2$?

Problem 1.14

Is it possible to place 2019 different positive integers along a circle so that the ratio of the greatest to the least of any two neighbouring numbers is a prime.

Problem 1.15

Let S be a set of numbers, $\circ$ is an operator such that:

a/$x \circ x = x$, for all $x \in S$.

b/$(x \circ y) \circ z = (y \circ z) \circ x$, for all $x, y, z \in S$.

Prove that for all $x, y \in S$: $x \circ y = y \circ x$

Problem 1.16

Find the number of quadruples (a, b, c, d) such that:

$$0 \leq a \leq b \leq c \leq d \leq 2019$$

Problem 1.17

n is a natural number, $3n + 1$ is a perfect square. Prove that $n + 1$ is the sum of three perfect squares.

Problem 1.18

Prove that $(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq 8a^2b^2c^2$

Problem 1.19

Two towns, A and B, are connected by a road. At sunrise, Pat begins biking from A to B along this road, while simultaneously Dana begins biking from B to A. Each person bikes at a constant speed, and they cross paths at noon. Pat reaches B at 5PM while Dana reaches A at 11:15PM.

When was sunrise?

Problem 1.20

Let f be a function such that:

For every n natural number:

$$f(1) = 1, f(2n) = f(n), f(2n+1) = f(2n) + 1$$

Can you summarize in a single sentence, what is $f(n)$?

Solutions

Solution 1.11

$$\frac{2020!}{1010!.2^{1010}} = \frac{1.2.3...2020}{1.2...1010.2^{1010}} = \frac{1.2.3...2020}{2.4...2020} = 1.3...2019$$

Solution 1.12

Assume that $p_1, p_2, \ldots, p_n$ are the only prime numbers. The number $p_1 . p_2 \ldots . p_n + 1$ is not divisible by any of $p_1, p_2, \ldots, p_n$. This is contradiction because any number larger than 1 must be factorable into product of primes.

Solution 1.13

$$P = (\underbrace{3...3}_{333})^2 = \underbrace{3...3}_{333} . \underbrace{3...3}_{333} = \frac{1}{9} \underbrace{9...9}_{333} . \underbrace{9...9}_{333}$$

$$= \frac{1}{9}(10^{333} - 1)(10^{333} - 1) = \frac{1}{9}(10^{666} - 2.10^{333} + 1) = \frac{1}{9}(\underbrace{9...9}_{332} 8 \underbrace{0...0}_{332} 1)$$

$$= \underbrace{1...1}_{332} 0 \underbrace{8...8}_{332} 9.$$

Hence the sum of digits of P is $332 + 8.332 + 9 = 2997$.

Solution 1.14

No, it is not possible. Without loss of generality, let there be 3 (which is odd) numbers placed on the circle; call them a, b, c. Now consider the ratios $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$. Each is either a prime or the reciprocal of a prime, while the product is equal to 1. The product will then be a fraction, whose numerator (before simplifying) is a product of primes, and whose denominator is a product of primes. Since there are a total of 3 primes in the numerator and denominator, we have a contradiction: the numerator and denominator will end up with different numbers of primes, and there is no way that the two quantities can be equal, so that the fraction will equal 1. As long as there are an odd number of numbers placed on the circle, it cannot be done.

Solution 1.15

Solution 1

For all $x, y \in S$:

Rule a/with $x \circ y$ as x:

$$x \circ y = (x \circ y) \circ (x \circ y)$$

Rule b/with $(x, y, x \circ y)$ as (x, y, z):

$$(x \circ y) \circ (x \circ y) = [y \circ (x \circ y)] \circ x$$

Rule b/with $(y, x \circ y, x)$as (x, y, z):

$$[y \circ (x \circ y)] \circ x = [(x \circ y) \circ x] \circ y$$

Rule b/with (x, y, x)as (x, y, z):

$$[(x \circ y) \circ x] \circ y = [(y \circ x) \circ x)] \circ y$$

Rule b/with (y, x, x)as (x, y, z):

$$[(y \circ x) \circ x)] \circ y = [(x \circ x) \circ y] \circ y$$

Rule b/with $(x \circ x, y, y)$as (x, y, z):

$$[(x \circ x) \circ y] \circ y = (y \circ y) \circ (x \circ x)$$

Rule a/twice with y as x, and x as x:

$$(y \circ y) \circ (x \circ x) = y \circ x$$

Solution 2

Rule a/with $x \circ x$ as x, $y \circ y$ as y:

$$x \circ y = (x \circ x) \circ (y \circ y)$$

Rule b/with $(x, x, y \circ y)$ as (x, y, z):

$$(x \circ x) \circ (y \circ y) = [(y \circ y) \circ x] \circ x$$

Rule b/with (y, y, x) as (x, y, z):

$$[(y \circ y) \circ x] \circ x = [(y \circ x) \circ y] \circ x$$

Rule b/with $(y \circ x, y, x)$ as (x, y, z):

$$[(y \circ x) \circ y] \circ x = (y \circ x) \circ (y \circ x)$$

Rule a/with $y \circ x$ as x:

$$(y \circ x) \circ (y \circ x) = (y \circ x) = y \circ x$$

Solution 1.16

Since $0 \leq a \leq b \leq c \leq d \leq 2019$, so they can be equal to each other. Let make them distinct:

$$0 \leq a < b + 1 < c + 2 < d + 3 \leq 2019 + 3.$$

So the problem now is how many ways we can choose 4 different integers between 0 and 2022. It is $\binom{2023}{4}$.

Solution 1.17

$3n + 1$ is a perfect square, so let $m^2 = 3n + 1$. Since m is not divisible by 3, let $m = 3k \pm 1$:

$9k^2 \pm 6k + 1 = 3n + 1 \Rightarrow n + 1 = 3k^2 \pm 2k + 1 = k^2 + k^2 + (k \pm 1)^2$

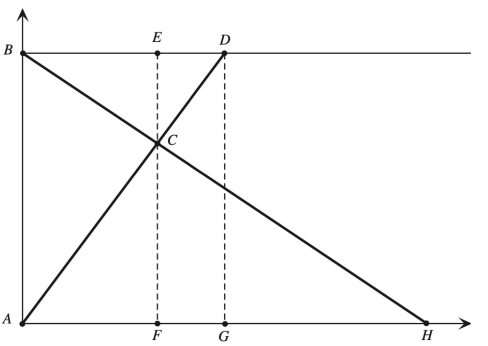
Solution 1.18

Easy to verify $a^2 + b^2 \geq 2ab$. Apply this for b, c and c, a. The symmetry leads to the desired inequality.

Solution 1.19

Below is the time-distance graph, the horizontal axis measures time starting with sunrise and the vertical axis represents the distance along the road from A to B.

The points F,G, and H are, respectively; noon, 5 PM and 11:15 PM, and the line segments AD and BH represent the time-distance paths of Pat and Dana, respectively, which meet at C. We can determine the time of sunrise once we compute the length AF.



Triangles $ACH$and DCB are similar, so $\dfrac{BE}{ED} = \dfrac{FH}{AF}$

Since $ED = 5$, $FH = 11.25$ and $BE = AF$, so

$AF^2 = \frac{45}{4}.5 \Rightarrow AF = \frac{15}{2}$. Hence sunrise was at 4:30 AM.

Solution 1.20

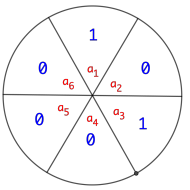
$f(n)$ is the number of digit 1's of n in binary representation.

Chapter 2 - Five principles

Five principles

1. The Invariance Principle

***Example 2.1**: A circle is divided into six sectors. The numbers* $1, 0, 1, 0, 0, 0$ *are written into them. In a step you might increase any two neighbouring numbers by 1. Can all the numbers be equal after some steps?*



Let $a_1, a_2, \ldots, a_6$ *are the numbers inside the sectors. Initially they are* $1, 0, 1, 0, 0, 0$. *Note that initially* $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 2$ *(*) If we increase any two neighbours by 1 (or any number), the equation(*) is not changed. This is* ***the invariant****.*

Obviously the invariant means that the six numbers can not be equal at any time, not matter how many steps have been taken.

If there is repetition, look for what does not change!

Exploring a few states from the initial state, answer the questions:

i. Can a given end-state be reached?

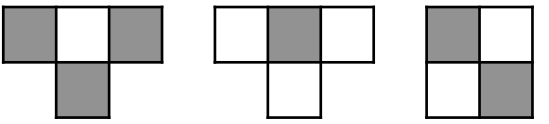
ii. What are all the end-states?

iii. Will an end-state be reached , no matter what steps are taken?

iv. Is there a periodic sequence of states?

2. Colouring proofs

***Example 2.2**: Prove that the* 8×8 *chessboard cannot be covered by 15* $T-$*tetrominoes and 1 square tetromino.*



Assume that it is possible. A $T-$*tetromino has three black squares and one white square or three white squares and one black square. A square tetromino contains two black and two white squares. So the number of black squares in 15* $T-$*tetrominoes and 1 square tetromino is an odd number. This is not possible because the* 8×8 *chessboard has an even number of black squares.*

Colouring proofs are useful for partitioning a set into a finite number of subsets. The partitioning is done by *colouring* the elements of each set by a different colour. It is very useful to investigate the problem based on properties such as parity, divisibility, invariant, and so on.

3. The Pigeonhole Principle

Pigeons in holes. Here there are $n = 10$ pigeons in $m = 9$ holes. Since 10 is greater than 9, the pigeonhole principle says that at least one hole has more than one pigeon.

If there are $n + 1$ pigeons, n holes, then two pigeons are in a hole.

Example* 2.3: 10 people attended a party. Some people shake hand with others. Any person shakes hand with another only once. Nobody shakes hand with himself.*

Prove that there are two people who shake hand with the same number of people.

Assume the opposite: each person has a different number of handshakes. *These 10 numbers are between 0 and 9. Because both 0 (a person who shake no hand) and 9 (a person who shake hand with everybody) cannot happen at the same time, so there are only 9 possible values for 10 different numbers. This is contradictory to the assumption. Hence, there are two people who have the same number of handshakes.*

4. The Extremal Principle

- Any *finite nonempty* set of numbers always has a *minimal* and a *maximal* elements, which need not to be unique.
- *Well-ordering principle*: Any set of *positive integers* always have a *minimal* (smallest) number.

Example* 2.4: In the Parliament of Sikinia, each member has at most three enemies. Prove that the house can be separated into two houses, so that each member has at most one enemy in his own house.*

Let us consider all possible ways to divide the members of parliament into two houses (two groups). For each way, let us count the number of enemies each person has in his own house, and let E be the sum of those numbers.

Since the number of ways to divide the members of parliament into two houses is finite, so there is a way that E has a minimal value. This is the desired way so that each member has at most one enemy in his own house.

Why? Let us assume that is not true: there is a person who has at least two enemies in his own house. By placing him into the other house, his number of enemies shall be at most one - because he has at most three enemies and at least two are in the other house.

5. The Induction Principle

Example 2.5*: Prove that for every n positive integer*

$$1 + 2 + 3 + \ldots + (n - 1) + n = \frac{n(n+1)}{2} \quad (*)$$

Basis of induction: It is easy to verify that (*) *is correct if* $n = 1$.

Induction hypothesis: assume that (*) is correct for n.

Inductive step: now let us prove that it is true for $n + 1$ *by using the induction hypothesis:*

$$1 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

Induction principle is a technique to prove statements about natural numbers. ***The key idea here is similar to domino fall***:

i. *Basis of induction*: Starting by proving the statement for the smallest possible number, so the first domino falls

ii. *Induction hypothesis:* Assuming that the statement is true for an n natural number, in another word the nth domino falls

iii. *Inductive step*: prove that the statement is true $n + 1$, hence the $(n+1)^{th}$ domino also falls.

Exercises

Exercise 2.6

There are 2019 marbles on the table. Alice and Bob play by turn to remove marbles from the table. At each turn, they must remove 1, 3, 4, 5 or 7 marbles. Whoever removes the last marble wins.

If Alice plays first, who has the winning strategy?

Discussion

Note that $1 + 7 = 3 + 7 = 4 + 4 = 8$. It seems like *the number of marbles of two consecutive of turns would be a constant, which is 8.*

Solution - Invariant as the sum of marbles in a pair of turns

If player A, takes a number of marbles according to the rules of the game, player B can takes a number of marbles so that their sum is 8. So player B always win if the number of marbles is a multiple of 8.

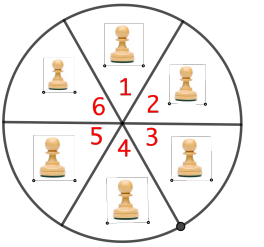
The invariant of this problem is to make the number of marbles of a pair of turns to be 8. Since Alice goes first, she can take 3 marbles off the table. Because 2016 is a multiple of 8 and it is Bob's turn now.

Alice will win this game.

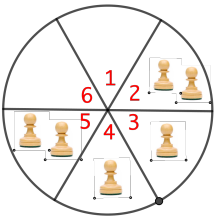
Exercise 2.7

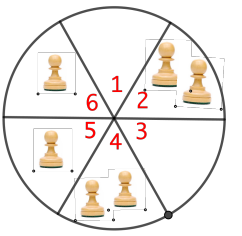
A circle is divided into 6 sectors, and a pawn stands in each of them as in the diagram on the right. It is allowed to shift any two pawns to sectors bordering those they stand on at the moment.

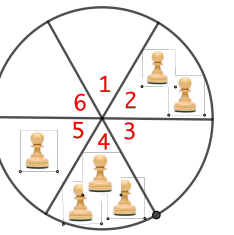
Is it possible to gather all pawns in one sector using such operations?



Solution - Invariant as the sum of sectors where the pawns are







Let us take the look at the diagrams above.

- The left: the pawns in sector 1 and 6 move to sector 2 and 5.
- The middle: the pawns in sector 1, 3 move to sector 2 and 4.
- The right: the middle with pawns 5 to 4 and 6 to 5.

The *sums of the sector numbers where the pawns stand* are *odds*:

- The left: $2 + 2 + 3 + 4 + 5 + 5 = 21$
- The middle: $2 + 2 + 4 + 4 + 5 + 6 = 15$
- The right: $2 + 2 + 4 + 4 + 4 + 5 = 21$

It is easy to see that originally the sum is $1 + 2 + \ldots + 6 = 21$, which is an odd number. Each operation moves two pawns to two neighbouring sectors, changing the parities of the sectors where they stand, resulting in the *same parity of the sums of the sector numbers where the pawns stand*. If all pawns would be in the same sector at some point, this sum would be $6n$, where n is the number of the sector where they stand. This is an even number and a contradiction to the fact that the sum is odd.

Hence, it is not possible for such a situation.

Definition: $n \equiv r \ (mod \ q)$ means r is the remainder of n when divided by q. For example $5 \equiv 2 \ (mod \ 3)$, $15 \equiv 3 \ (mod \ 4)$

Exercise 2.8

Let $d(n)$ be the digital sum - the sum of all digits - of n. Find n:

$$n + d(n) + d(d(n)) = 2020$$

Discussion:

$$234 \equiv 2 \times 1 + 3 \times 1 + 4 = 9 \equiv 0 \ (mod \ 3).$$

Solution - Invariant as any number and its digital sum have the same remainder when divided by 3

$10^k \equiv 10^{k-1} \equiv \ldots 10^2 \equiv 10 \equiv 1 \ (mod \ 3)$, so n *and its digital sum* $d(n)$ *has the same remainder when divided by 3:* $d(n) \equiv n \ (mod \ 3)$. Hence:

$$n + d(n) + d(d(n)) \equiv n + n + d(n) \equiv 3n \equiv 0 \ (mod \ 3)$$

Because $2020 \equiv 1 \ (mod \ 3)$, the given equation has no solution.

Exercise 2.9

$a_1, a_2, \ldots, a_{2019}$ be a permutation of $1, 2, \ldots, 2019$. Prove that

$$P = (a_1 - 1)(a_2 - 2)\ldots(a_{2019} - 2019)$$

is an even number.

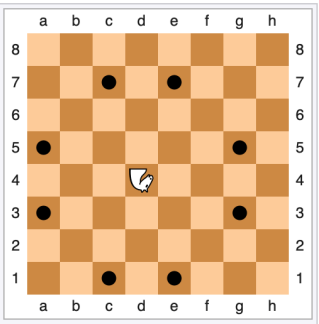
Solution - Invariant as the sum of number in any permutation

Let us assume that P is an odd number. All factors of P: $a_1 - 1$, $a_2 - 2$, ..., $a_{2019} - 2019$ must be odd numbers. Because there are an odd number - 2019 - of them, so the sum of all factors must be odd as well.

Since $a_1, a_2, \ldots, a_{2019}$ is a permutation of $1, 2, \ldots, 2019$, and $(a_1 - 1) + \ldots + (a_{2019} - 2019) = (a_1 + \ldots + a_{2019}) - (1 + \ldots + 2019) = 0$ This is contradictory to the assumption.

Hence P is even.

Exercise 2.10



The camel (represented as an inverted knight) may move to any of eight squares (black dots).

A special chess piece called a *camel* moves along a 10×10 board in almost the same way as a knight does: *it moves to any adjacent square and then move three squares in any particular direction* (up, down, left, right).

Is it possible for a *camel* to go from some square to an adjacent square?

Solution - Chessboard colouring

It is easy to see that a *camel* moves to the same coloured squares. An adjacent square has a different colour.

Hence it is not possible.

Exercise 2.11

In the right figures, two squares are neighbours if they have a common boundary. O is an operation: chooses two neighbouring squares and adds a same integer to both of them.

1	2	3
4	5	6
7	8	9

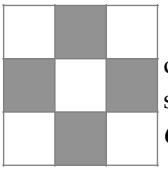
Figure 1

7	8	9
6	2	4
3	5	1

Figure 2

Can you use operation O to transform Figure 1 into Figure 2?

Solution - Alternate colouring + Invariant



Let colour the given figures with alternate colouring as in the diagram on the left. Let W and B be the sum of the numbers in white and black squares. Operation O does not change the difference of W and B.

The value of $W - B$ in Figure 1 is 5 , in Figure 2 it is -1.

Hence,. it is not possible to transform Figure 1 into Figure 2.

Exercise 2.12

Suppose you have an art gallery containing priceless paintings and sculptures. You would like it to be supervised by security guards, and you want to employ enough of them so that at any one time the guards can between them oversee the whole gallery.

How many guards will you need if the gallery has a shape of a simple $n-$gon and not necessarily convex? Below on the left are some examples of various polygon shapes. On the right is an example of a gallery with 4 guards, all of them see

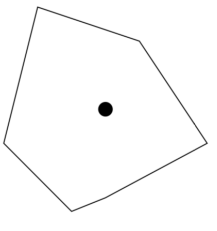


Figure 1: Convex Polygon

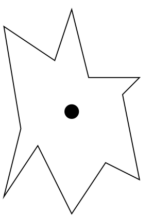


Figure 2: Star

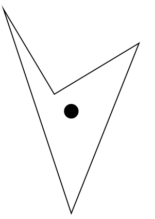


Figure 3: 4-sided polygon

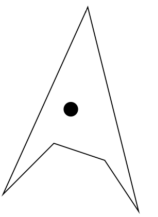
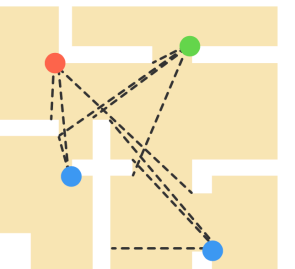


Figure 4: 5-sided polygon

Discussion

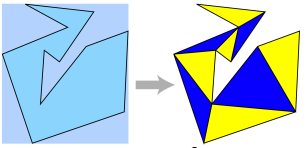
First, let's define what we mean to guard a museum. In this problem, guards must stay at fixed positions but are able to see every angle from their position by rotating. Second the guard cannot see past the wall.

The figure on the right shows an example of a gallery with 4 guards.

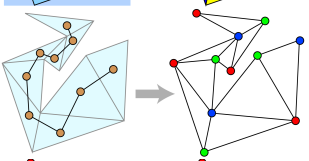


Solution - Triangulation + Different colouring for each triangle vertex

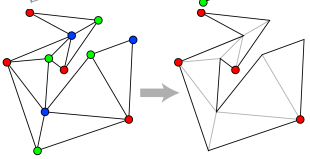
First, we draw non-intersecting diagonals to divide the polygon into triangles.



Second, we colour the vertices by triangle.



Third, we select the vertices with the least frequently used colour.

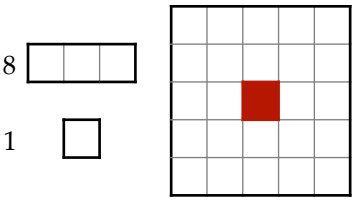


Since there are n vertices and 3 colours, the result is $\lfloor \frac{n}{3} \rfloor$ - the integer that less or equal $\frac{n}{3}$.

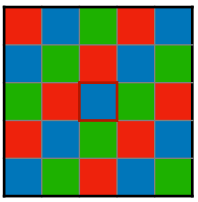
Exercise 2.13

Eight 1×3 rectangles and one 1×1 square cover a 5×5 board.

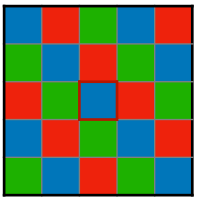
Prove that the square must be at the centre of the board.



Solution - Multi-colouring + Colour-rotation



Let paint the 25 squares of the board with colours red, blue, and green, as on the left. Easy to see that there are 8 red, 9 blue, and 8 green squares. Since a 1×3 covers 1 red, 1 blue, and 1 green square, so the 1×1 square must cover a blue square.



Now we rotate the colouring of the board (not the board itself) clockwise 90° around its centre. Using the same reasoning, the 1×1 square must also cover a blue square.

The only square that is coloured blue in both colouring is the centre unit square of the board.

Exercise 2.14

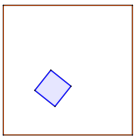
Given 8 *different* positive integers, none greater than 15. Show that there are at least three pairs of them have the same positive difference. Note that the pairs might share some common numbers.

Solution - How many differences are there?

There are 8 *different* positive integers, so $\binom{8}{2} = 28$ ways to choose two integers to form a pair. A positive difference of any pair can be from 1 to 14, but 14 can be made by one pair only: $14 = 15 - 1$.

Hence there should be at least three pairs of them having the same difference.

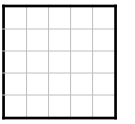
Exercise 2.15



Fifty-one points are scattered inside a square with a side of 5m.

Prove that there are three different points can be covered by a square with side 1m.

Solution - How to divide the square in to smaller squares?



Let divide the 5×5 square into 25 1×1 squares. Since there are 51 points, so at least three of them must be inside one of the 1×1 squares.

Exercise 2.16

A chess master has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more than 132 games.

Prove that there is a sequence of successive days on which he plays exactly 21 games.

Solution - Sequence of numbers

Let $a_1, a_2, \ldots, a_{77}$ be the accumulative number of games played until the day #1, 2, . . . , 77.

$$1 \le a_1 < a_2 < \ldots < a_{77} \le 132 \text{ (*)}$$

$$22 \le a_1 + 21 < a_2 + 21 < \ldots < a_{77} + 21 \le 153 \text{ (**)}$$

Now consider $a_1, a_2, \ldots, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$. These are $77 + 77 = 154$ numbers between 1 and 153. Therefore there must be two same numbers.

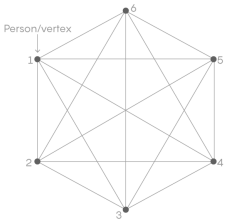
According to (*) all a_i are different, and from (**) all $a_i + 21$ are also different, there exist an i and j such that $a_i = a_j + 21$. It means the number games on day a_i is 21 more than that number on day a_j.

On these days #$j + 1$, $j + 2$, $\ldots$, i, he plays exactly 21 games.

Exercise 2.17

Among six persons, there are always three who know each other or three who are complete strangers.

Solution - Pigeonhole principle twice!



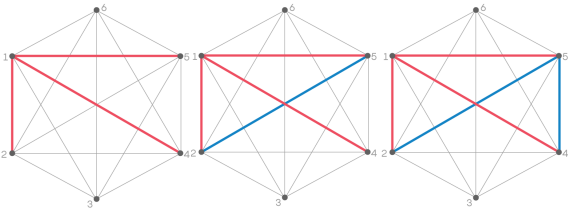
First, let number the persons as 1, 2, $\ldots$, 6, assign these numbers to the vertices of a hexagon. We will colour the edge between two persons by red if they know each other and blue if they are strangers.

Let start with person #1.

Because there are 5 other persons, so without loss of generality, we can assume this person know 3 of them.

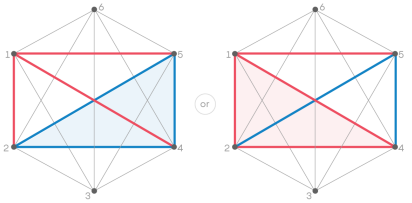
Let them be #2, #4, and #5.

Diagram A shows the three persons that #1 knows connected by red segments.



A B C

If #2 and #5 are connected by a red segment then we find the three persons who know each other. Assume that they don't know each other. Diagram B shows this situation. Similarly #4 and #5 would not be connected by a red segment, so they should be connected by a blue segment, as shown in C.

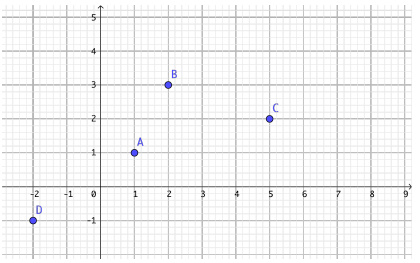


If #2 and #4 are connected by a blue segment, then #2, #4, and #5 are strangers.

If #2 and #4 are connected by a red segment, then #1, #2, and #4 all know each other.

Hence, in any case there are three persons who know each other or there are three persons who are complete strangers.

Definition: A lattice point is a point with integer coordinates.



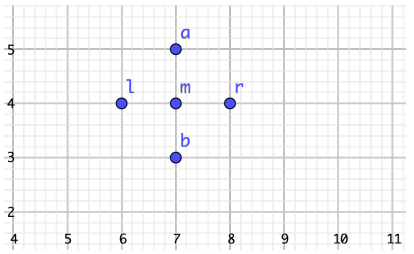
For example: in the diagram on the left $A(1,1)$, $B(2,3)$, $C(5,2)$, and $D(-2,-1)$ are lattice points.

Exercise 2.18

Each lattice point of the plane is labeled by a positive integer. Each of these number is the arithmetic mean (average) of its four neighbours (above, below, left, right).

Show that all the labels are equal.

Solution - Choose the label with minimal value!



Since the labels are positive integers, so there must be a label $m$that has the minimal value, meaning all other labels are greater than or equal it.

Let a, b, l, r be the labels of its above, below, left, right neighbours.

Since $a, b, l, r \geq m$ and $m = \dfrac{a+b+l+r}{4} \Rightarrow a = b = l = r = m$.

From this it is easy to see that all the labels are equal.

Exercise 2.19

10 students are standing in a field such that the distance between each pair is distinct. Each student is holding a ball, and when the teacher blows a whistle, each student throws their ball to the nearest student.

Prove that there is a pair of students that throw their balls to each other.

Solution - Choose the smallest distance!

Since there are only finite number of pairs of student, so there are a finite number of distances between students. According to the extremal principle, there is a minimal distance.

Because all distances are different, so there is only one minimal distance. Let it be the distance between A and B students. Since B is the nearest to A, A must throw the ball to B. It is also true that A is the nearest to B, B must throw the ball to A. They are the pair that throw balls to each other.

Exercise 2.20

There are 2019 red and blue points on a plane with the following interesting property: every line segment that joins two points of the same colour contains a point of another colour.

Prove that all the points lie on a single straight line.

Solution - Proof by contradiction using triangle with smallest area!

If the points were not a single straight line, different triangles can be formed with the points as vertices. Let $\triangle ABC$ be the triangle with smallest area.

At least two of the vertices of this triangle have the same colour, let them be A, C. Between them is a point D of different colour. Both $\triangle ABD, BDC$ have smaller area than $\triangle ABC$. This is contradiction!

Hence, all the points must lie on a single straight line.

Exercise 2.21

Prove that $n\sqrt{2}$ is not an integer for any positive integer n.

Solution - Proof by contradiction with least element!

Assume $k\sqrt{2}$ is the smallest such integer (*).

Because $k\sqrt{2}-k$ and $2k-k\sqrt{2}$ are positive difference of integers, so they are also integers. $2k-k\sqrt{2}=(k\sqrt{2}-k)\sqrt{2}$, so $(k\sqrt{2}-k)\sqrt{2}$ is in $n\sqrt{2}$ format, and since $k\sqrt{2}-k<k$, so $(k\sqrt{2}-k)\sqrt{2}<k\sqrt{2}$. It is contradictory to (*).

Hence, $n\sqrt{2}$ is not an integer for any positive integer n.

Exercise 2.22

Show that for $n \geq 2$:

$$(1-\frac{1}{4})(1-\frac{1}{9})\ldots(1-\frac{1}{n^2}) = \frac{n+1}{2n} \ (*)$$

Solution - Induction

It is easy to verify that (*) is true for $n = 2$. Assume that it is true for n. We prove that it is also true for $n+1$:

$$(1-\frac{1}{4})(1-\frac{1}{9})\ldots(1-\frac{1}{n^2})(1-\frac{1}{(n+1)^2}) = \frac{n+1}{2n}(1-\frac{1}{(n+1)^2})$$

$$= \frac{n+1}{2n} \cdot \frac{(n+1)^2-1}{(n+1)^2} = \frac{n+1}{2n} \cdot \frac{n(n+2)}{(n+1)^2} = \frac{n+2}{2(n+1)}$$

So it is also true for $n+1$. Hence it is true for all $n \geq 2$.

Exercise 2.23

A country has n cities. Any two cities are connected by a one-way road.

Show that there is a route that passes through every city.

Solution - Induction

The statement is obviously true for $n = 1$ or $n = 2$. Assume that it is true for n. Let prove it for $n+1$.

For cities $C_1, C_2, \ldots, C_n$, without loss of generality, let assume that there is a route: $C_1 \to C_2 \to \ldots \to C_n$. Because there is a road between C_n and C_{n+1}. There are two cases:

a/ $C_n \to C_{n+1}$: we are done $C_1 \to C_2 \to \ldots \to C_n \to C_{n+1}$

b/ $C_{n+1} \to C_n$: there are two possibilities:

- Going from $n-1, n-2, \ldots, 2, 1$, there is a first k such that $C_k \to C_{n+1}$, then $C_{n+1} \to C_{k+1}$ and we have

$C_1 \to C_2 \to \ldots \to C_k \to C_{n+1} \to C_{k+1} \to \ldots \to C_n$

- There is no such a k, so $C_{n+1} \to C_1$, in this case we have:

$C_{n+1} \to C_1 \to C_2 \to \ldots \to C_n$

Hence, there is a route that passes through every city.

Exercise 2.24

Prove that for all $n \geq 0$, $120\underbrace{3...3}_{n}08 \vdots 19$ (*)

Solution - Induction

If $n = 0$: $12008 = 19 \times 632$, so (*) is true. Assume that it is true for n. Let prove that it is also true for $n + 1$.

$$120\underbrace{3...3}_{n+1}08 - 120\underbrace{3...3}_{n}08 = (1203 - 120) \times 10^{n+2} = 1083 \times 10^{n+2}$$

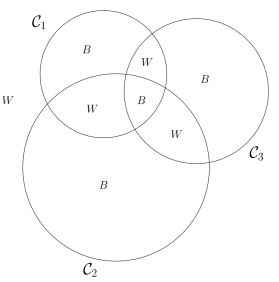
Since $1803 = 19 \times 57$, so their difference is divisible by 19, therefore $120\underbrace{3...3}_{n+1}08 \vdots 19$. Hence the statement is true for all $n \geq 0$.

Exercise 2.25

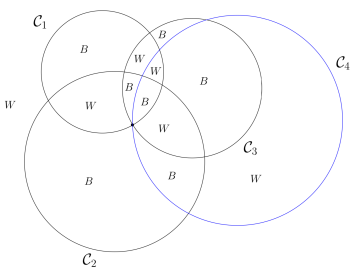
$n \geq 1$circles are given in the plane. They divide the plane into regions. Show that it is possible to colour the plane using two colours, so that no two regions with a common boundary line are assigned the same colour.

Solution - Induction

Let the two colours be B and W. For $n = 1$, if there is only one circle, we may colour the inside B and the outside W, and this colouring satisfies the conditions of the problem.



Assume it is true for k circles; so we know that for any k circles there is a colouring which satisfies the conditions of the problem.



Next consider $k + 1$ circles $C_1, C_2, \ldots, C_{k+1}$ Ignore the circleC_{k+1}, for k circles $C_1, C_2, \ldots, C_k$ by the assumption, there is a colouring of the plane which satisfies the conditions of the problem. Now add the circle C_{k+1} back into the picture.

To obtain a new colouring: a/for any region inside C_{k+1}, do not change its colour; b/for any region outside C_{k+1}, colour it into the opposite colour.

Problems

Problem 2.26

In the mid-term party of the Magical School each student wears a coloured wizard cloak: 13 black, 15 red, and 17 white. When two students with different coloured cloaks meet, their cloaks change to the third colour.

Is it possible that after some time all student cloaks have the same colour?

Problem 2.27

The numbers 1, 1, 2, 2, 3, 3, 4, 4 can be rearranged into such that for $k = 1,2,3,4$ between two number k's there are exactly $k - 1$ numbers (no number between two 1's, one number between two 2's, two numbers between two 3's, and three numbers between two 4's).

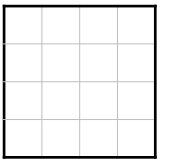
Here is one solution: 2, 3, 2, 4, 3, 1, 1, 4.

a/Can you solve a similar problem with 1, 1, 2, 2, . . . , 5, 5?

b/Is it possible to rearrange 1, 1, 2, 2, . . . , 2019, 2019?

Problem 2.28

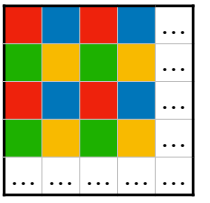
a/Can you colour a 4×4 board by two colours black and white such that any 1×4 rectangle has 2 black and 2 white squares; any 2×2 square has 1 black and 3 white or 3 black and



b/Using the result of a/ can a 8×8 board be covered by fifteen 1×4 rectangles and one 2×2 square without overlapping?

Problem 2.29

On a 9×9 board coloured as on the right, 65 insects are placed in the centre of some of the squares. The insects start moving at the same time and speed to a square that shares a side with the one they were in.



When they reach the centre of that square, they make a 90° degrees turn they can turn either to the right or to the left - and keep walking, without leaving the board.

Prove that at some moment of time there are two insects in the same square.

Problem 2.30

17 students gathered and discussed 3 assignments with all the others. Any two students discussed only one assignment.

Prove that there are at least three students, who discussed with each other about the same assignment.

Problem 2.31

a is a positive integer and it is relative prime to 2 and 5.

Prove that for any n positive integer, there is a power of a ending with $\underbrace{00...0}_{n}1$.

Problem 2.32

For each convex pentagon, three diagonals can be chosen to form a triangle.

Problem 2.33

Seven lines on the plane, any two lines intersect each other.

Prove that there are two lines intersecting each other with an angle less than 26°.

Problem 2.34

Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet at a single point.

Easy to see that $f(1) = 2$, $f(2) = 4$

Find a formula for $f(n)$. Prove by induction.

Problem 2.35

Prove that:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$$

Problem 2.36

There are no quadruples of positive integers (x, y, z, w) satisfy:

$$x^2 + y^2 = 3(z^2 + w^2)$$

Problem 2.37

n tennis players participated in a round-robin competition, i.e. each player played with every other player once.

Prove that when the competition ended, the players can be sorted into a line such that the one stands in front of the other is the one who won their game.

Problem 2.38

$ABCD$ is a convex quadrilateral.

Prove that four circles with the quadrilateral sides as diameters together cover the quadrilateral.

Challenges

Challenge 2.39

23 friends want to play soccer. For this they choose a referee and the other split into two teams of 11 people each. They want to do this so that the total weight of each team is the same. We know that they all have integer weights and regardless of who is the referee, it is possible to make the two teams.

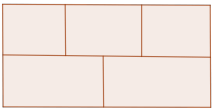
Prove that they all have the same weight.

Challenge 2.40

There is one stone at each vertex of a square. We are allowed to change the number of stones according to the following rule: we might take away any number of the stones at one vertex and add twice as many stones to the pile at one of the adjacent vertices.

Is it possible to get 1989, 1988, 1990, and 1989 stones at consecutive vertices after a finite number of moves?

Challenge 2.41



Show that there is no curve that intersects every segment of the diagram on the left exactly once.

Challenge 2.42

Every participant of a tournament plays with every other participant exactly once. No game is draw. After the tournament, every player makes a list with the names of all players, who were beaten by him or were beaten by the players beaten by him.

Prove that there is a list of a player contains all other players.

Challenge 2.43

The sequence $\{a_n\}$ is defined as follow:

$$a_0 = 9,\ a_{n+1} = 3a_n^4 + 4a_n^3,\ n > 0$$

Prove that a_{10} contains more than 1000 nines in decimal notation.

Challenge 2.44

Prove that there exist three *consecutive* vertices A, B, C in every convex n-gon, $n \geq 3$, such that the circumcircle of $\triangle ABC$ cover the whole n-gon.

Solutions

Solution 2.26 - Invariant as the remainders when divided by 3

Let (b, r, w) be the number of black, red, and white cloaks. When two students with different colours cloaks meet, there are three possible outcomes:

$$(b+2, r-1, w-1), (b-1, r+2, w-1), (b-1, r-1, w+2) \ (*)$$

The remainders of $(13, 15, 17)$ *when divided by 3 is* $(1, 0, 2)$*, meaning they are three different remainders*. Any operation in (*) does not reduce the number of different remainders. Two of the colours have no cloaks meaning their numbers are the same and have the same remainders when divided by 3.

This is contradictory to the invariance.

Solution 2.27 - Invariant as the sum of positions

a/There are many different solutions, below is one of them:

$$2, 3, 2, 5, 3, 4, 1, 1, 5, 4$$

b/Assume there is a permutation of $1, 1, 2, 2, \ldots, n, n$ satisfy the problem. For $k = 1, 2, \ldots, n$, let p_k be the position in the row of the first number k, the second number k shall have the position $p_k + k$. Since all positions are different integers between 1 and n:

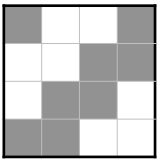
$$(p_1 + p_1 + 1) + (p_2 + p_2 + 2) + \ldots + (p_n + p_n + n) = 1 + 2 + .. + 2n$$

$$\Rightarrow p_1 + p_2 + \ldots + p_n = \frac{1}{2}\left(\frac{2n(2n+1)}{2} - \frac{n(n+1)}{2}\right) = \frac{n(3n+1)}{4}$$

The fraction can be an integer if and only if $n(3n+1)$ is divisible by 4. Hence $n \equiv 0, 1 \ (mod\ 4)$.

Since $2019 \equiv 3 \ (mod\ 4)$, so it is not possible.

Solution 2.28 - Colouring two consecutive squares



a/By colouring as shown on the left, it is simple to verify that any 1×4 rectangle has 2 black and 2 white squares; any 2×2 square has 1 black and 3 white or 3 black and 1 white squares.

b/Extend this colouring to an 8×8 board. From the previous result: fifteen 1×4 and one 2×2 cover 31 black squares and 33 white squares or 33 black squares and 31 white squares. It means that they cannot cover the 8×8 board.

Solution 2.29 - Multi-colourings + different patterns

Since the board is 9×9, so there are 25 red, 20 blue, 20 green, and 16 yellow squares.

Because there are 65 insects, so there should be: a/at least 33 insects in red or yellow squares, or b/at least 33 insects in blue and green squares. If b/ is true, then after moving, there should be *at least 33 insects in red or yellow squares at any moment* (*).

We also know that *if an insect in a red square, then it will be in a yellow square after two moves* (**). Combining (*) and (**), for any moment, there will be at least 17 insects in red or at least 17 insects in yellow squares.

Because there are only 16 yellow squares, so there are two insects at a yellow square.

Solution 2.30 - Pigeonhole principle twice!

Let S be a student. Since S discussed three assignments with 16 other students, there should be at least 6 students that S discussed the same assignment A.

Now we have 6 other students. If any of two of them discuss the assignment A, then S and these two students all together discussed A. So they are the three students we look for. If there is no such a pair, we have 6 students discussing among themselves two different assignments B, C only. This is exactly the Exercise 2.17 in disguise.

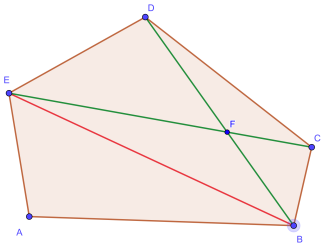
Hence, there are always three students discussing a single assignment among themselves.

Solution 2.31 - Pigeonhole principle with remainders!

a is relative prime to 2, 5, so none of $a, a^2, a^3, \ldots, a^{10^n}$(*) is divisible by 10^n. There are $10^n - 1$ different non-zero remainders: $1, 2, 3, \ldots, 10^n - 1$. So two terms $a^k, a^l\ (k > l)$ of the 10^n- term sequence (*) have same remainder, their difference is divisible by 10^n: $a^k - a^l \vdots 10^n \Rightarrow a^k - a^l = a^l(a^{k-l} - 1) \vdots 10^n$and $GCD(a^l, 10^n) = 1$ so $a^{k-l} - 1 \vdots 10^n$.

This means that a^{k-l} has an ending of $\underbrace{00...01}_{n}$.

Solution 2.32 - Choose the longest diagonal!



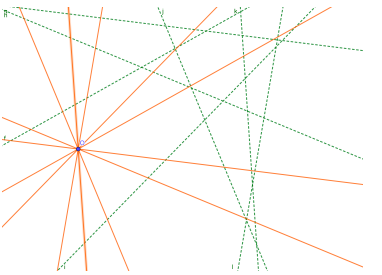
Let BE be the longest diagonal of the pentagon $ABCDE$, F be the intersection of BD and CE.

$BD + CE$
$= (BF + FD) + (CF + FE)$
$= (BF + FE) + (CF + FD)$
$> BE + CD > BE \Rightarrow BD + CE > BE.$

Since BE is the largest among (BD, CE, BE), these three diagonals form a triangle.

Solution 2.33 - Align the lines and choose the smallest angle!

7 original lines are coloured as green in the diagram below. Let O be a point on the plane. Seven orange lines are drawn through O and parallel with the original lines.



These 7 orange lines are dividing the full angle at O into 14 smaller angles. Because of the extremal principle, one of these angles is smaller or equal $\frac{360^\circ}{14} < 26^\circ$.

This is an angle between two orange lines, which is also an angle between two original green lines.

Therefore, there are two lines intersecting each other with an angle less than 26°.

Solution 2.34 - Induction

The nth line will be cut into n parts by $n - 1$ lines, adding n new parts to the plane. By induction: $f(n) = \frac{n(n+1)}{2} + 1$

Solution 2.35 - Induction

Easy to verify it is true for $n = 1$. Assume that is it true for n:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3) = \frac{1}{4}n^2(n+1)^2 + (n+1)^3$$
$$= \frac{1}{4}(n+1)^2(n^2 + 4(n+1)) = \frac{1}{4}(n+1)^2(n+2)^2$$

So it is also true for $n+1$. Hence, it is true for all $n \geq 1$.

Solution 2.36

Suppose that there is such a quadruple. We choose the solution with the smallest value for $x^2 + y^2$. Let (a, b, c, d) be such a solution.

$$a^2 + b^2 = 3(c^2 + d^2) \Rightarrow 3 \mid a^2 + b^2 \Rightarrow 3 \mid a, 3 \mid b \Rightarrow a = 3a_1, b = 3b_1$$

$$3(c^2 + d^2) = a^2 + b^2 = 9(a_1^2 + b_1^2) \Rightarrow c^2 + d^2 = 3(a_1^2 + b_1^2)$$

So, we found a new solution (c, d, a_1, b_1) of (*) with $c^2 + d^2 < a_1^2 + b_1^2$. This is contradiction.

Solution 2.37

Consider all ways to line up some players such that anyone stands in front of the other won the game both of them played. It is easy to see that there is a finite number of ways.

Let L be the length of the line. The largest L is when such line contains all n players. Assume that for the largest L there is only $k < n$ players $P_1, P_2, \ldots, P_k$ in the line. There should be a player P among n players, who is not in the line.

P did not win against P_1, because otherwise $P, P_1, P_2, \ldots, P_k$ is a longer line ($L = k + 1$) with required property. So P lost against P_1. P did not win against P_2, because then $P_1, P, P_2, \ldots, P_k$. So P lost against P_2. Continue the reasoning, P lost against all $P_1, P_2, \ldots, P_k$. Hence $P_1, P_2, \ldots, P_k, P$ is a longer line with required property.

This is contradictory with the assumption. So there is a way to line up *all* players such that anyone stands in front of the other won their game.

Solution 2.38

Consider a point M on the perimeter of $ABCD$. We can assume M is on AB. M is covered by the circle with the diameter AB.

Now let consider a point M inside $ABCD$, because:

$$\angle AMB + \angle BMC + \angle CMD + \angle DMA = 360^\circ$$

so one of these four angles is not exceeding 90°. Let that be AB. Because $\angle AMB \leq 90^\circ$, the circle with diameter AB covers M.

Hence any point M inside or on perimeter of $ABCD$ is covered by the four circles with the quadrilateral sides as diameters.

Therefore the four circles with the quadrilateral sides as diameters together cover the quadrilateral.

Solution 2.39 - Invariant as parity of all numbers and infinite descent

Let $a_1, a_2, \ldots, a_{23}$ be the weights of the people, Let S be their sum: $S = a_1 + a_2 + \ldots + a_{23}$.

He given condition assumes that for any person a_i, the remaining 22 people can be split into 2 teams with equal weight sum, so $S - a_i = 2W$, where W is the weight sum of a team.

This means that for any a_i: S and a_i have the same parity. Therefore S *and all* $a_1, a_2, \ldots, a_{23}$ *has the same parity*.

Now, consider an operation: if all a_i is even, let us replace them by $\frac{a_i}{2}$, otherwise $a_i - 1$. It is easy to see that the new weights are also satisfy the condition of the problem. Each operation strictly reduces the weight sum of everyone, which is a non-negative integer. The operation repetition is stop only if all weights are 0.

Hence, all person have the same weight.

Solution 2.40 - Invariant as the remainder when divided by 3

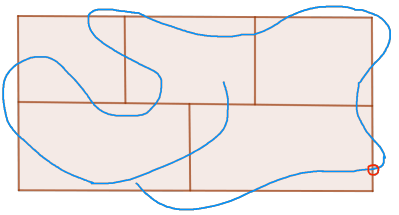
Notice that the difference between the total number of stones at the first and third vertices and the total number of stones at the second and fourth vertices changes by a multiple of 3 under the given operation.

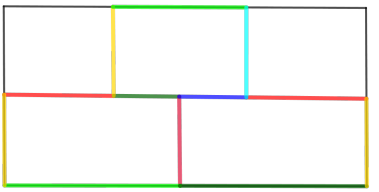
So the difference of these two total numbers has an invariant remainder when divided by 3.

Since (1, 1, 1, 1) has a remainder 0 when divided by 3, and (1989, 1988, 1990, 1989) has a remainder 2 when divided by 3, it is not possible to get to the desired situation.

Hence, it is not possible to obtain the desired position.

Solution 2.41 - Colouring the segments of a boundary differently





The diagram on the left shows an *almost successful* attempt, notice the *blue* curve cuts a segment *twice*, the second cut is *red*-circled.

Now, we colour the middle rectangle in the top row and all others in the second row. For a rectangle, each segment on it sides has a different colour.

It is now easy to see that we have three rectangle, each has 5 different segments.

For these rectangle, if a curve intersects each segment once, then it should start or end inside the rectangle - in another word it should have an endpoint inside the rectangle.

It is impossible since a curve can have two or zero endpoints.

Hence it is not possible to draw such a curve.

Solution 2.42 - Chose the player who won maximum number of games!

Consider a player P who *won the maximum number of games*.

Assume P is not such a player with required property.. There is a player Q who has beaten P and all players that P has beaten. Thus, Q has beaten more players than P. In another word, *the number of won games* of Q is greater than P. This is a contradiction.

Hence, P is the player whose list contains the name of all other players.

Solution 2.43 - Induction

Let prove that a_n is in $a \times 10^{2^n} - 1$ format. $a_0 = 1.10^{2^0} - 1 = 9$. Assume $a_n = a \times 10^{2^n} - 1$. $a_{n+1} = 3(a \times 10^{2^n} - 1)^4 + 4(a \times 10^{2^n} - 1)^3$

$= 3a^4 10^{4.2^n} - 12a^3 10^{3.2^n} + 18a^2 10^{2.2^n} - 12a10^{2^n} + 3$

$+4a^3 10^{3.2^n} - 12a^2 10^{2.2^n} + 12a10^{2^n} - 4 = b.10^{2^{n+1}} - 1$

Solution 2.44

Consider all circles through three vertices of the $n-$gon. There is a finite number of such circles. Let Ω be the *maximal circle*. We will prove the following statements:

(i) Ω covers the whole $n-$gon.

(ii) Ω passes through three *consecutive* vertices.

Prove (i) *indirectly* - it means we prove (i) by contradiction. Suppose that vertex A' lies outside Ω. Because Ω is the circumcircle of $\triangle ABC$ where A, B, C are denoted such that A, B, C, A' form convex quadrilateral. Easy to see that the circumcircle of $\triangle A'BC$ is larger than Ω. This is contradictory.

Now we prove (ii) *indirectly*. Let A, B, C be vertices on Ω. Suppose there exists vertex A' between B, C, and A' not on Ω. Because of (i), Ω covers the whole $n-$ gon, so A' is inside Ω. Hence the circumcircle of $\triangle A'BC$ is larger than Ω. This is a contradiction.

From (i) and (ii), we can obtain the desired result.

Chapter 3 - Ten counting techniques

Counting techniques

1. Sum rule

Example 3.1*: In a drawer Ben has 5 red shirts and 3 blue shirts. In how many ways can he pick one shirt from the drawer?*

He has 5 choices to pick a red shirt and 3 choices to pick a blue one. So in total, there are $5 + 3 = 8$ ways for Ben to pick a shirt.

If set A has m elements while set B has n elements, and they have no elements in common, then their union has $m + n$ elements.

2. Product rule

Example 3.2*: In a drawer Ben has 5 shirts and 3 pairs of jeans. In how many ways can he choose a shirt and a pair of jeans?*

He has 5 choices to pick a shirt and 3 choices to pick a pair of jeans. So in total, there are $5 \times 3 = 15$ ways for Ben to choose.

If set A has m while set B has n elements, the number of pairs that consist of one element from A and one element from B is $m \times n$.

3. Recursion

Example 3.3*: In how many ways can Ben make a 9-digit number with the digits* $1, 2, \ldots, 9$*, such that no digit is chosen twice?*

There is 9 ways to choose the first digit. After choosing the first digit, the problem is reduce to a smaller copy of the same problem: how many ways to make an 8-digit number with 8 digits, such that no digit can be chosen twice. Continuing this way, it is easy to see that the answer is $9 \times 8 \times \ldots \times 1 = 9!$

A problem can be split into smaller copies of the original problem. Those problems, in turn, are split into smaller parts until they become trivial. The partial solutions, if combined, can provide a full solution.

4. Counting the same objects in two different ways

Example 3.4*: In how many ways can Ben choose some amount of marbles from a bag containing one red, one blue, and one green marble?*

Using casework with the number of marbles makes a more elaborate solution. In each case, the red marble can be chosen or not chosen, the blue marble can be chosen or not chosen, and the same is applied to the green one. Hence, there are $2 \times 2 \times 2 = 2^3 = 8$ ways to choose them.

Oftentimes, it is hard to count the given objects in the way the problem offers them. Sometimes, the solution can be more easily found by using another way to count the same objects.

5. Counting by bijection

***Example 3.5**: There are 10 people at a party. Everyone has shaken hands with exactly two other people. How many handshakes are there in total? Each handshake can be counted as a one-to-one correspondence to a pair of two people who shake hand. For each person, there are two such pairs and each pair is counted twice. For example: A shaking hands with B is same as B shaking hands with A. Hence the number of handshaking pairs is* $\frac{10 \times 2}{2} = 10$, *which is also the number of handshakes.*

If we find a one-to-one correspondence between the elements in set A to the elements in B, then instead of counting the elements of set A, we can count the elements of B.

6. Permutation of distinguishable objects

***Example 3.6**: How many 5-letter words can be formed by using the letters* A, B, C, D, E, *each once? This can be easily obtained by applying the product rule repeatedly.*

The number of permutations of n distinguishable objects is $n!$.

7. Permutation of some indistinguishable objects

***Example 3.7**: How many ways Ben can shuffle the letters in the word OTTAWA?*

Notice that the letter A, as well as letter T, appears twice. Let assign a number for each copy of A, and T. Now the word becomes $OT_1T_2A_1WA_2$. Since all these six letters now are distinguishable, there are $6!$ permutations. However because the permutations $\ldots A_1 \ldots A_2 \ldots$ and $\ldots A_2 \ldots A_1 \ldots$ are effectively same. So every permutation is repeated by the number of permutations of A_1A_2, which is $2!$. Similarly, the number of permutations of T_1T_2, which is also $2!$, is applied. The number of permutations is $\frac{6!}{2!2!}$.

If there are n marbles in m different colours so that n_1 has the first colour, n_2 has the second colour, ..., until n_m has the last colour, then $n_1 + n_2 + \ldots + n_m = n$.

The number of permutations is $\frac{n!}{n_1!n_2!\ldots n_m!}$.

8. Combination

***Example 3.8**: In how many ways can Ben choose two marbles from a bag containing five different marbles: R, B, G, Y, W?*

There are five ways to select the first marble, and four ways to choose the second one. Since the order of the marbles do not matter (i.e (R, W) is same as (W, R)), the selection would be repeated by the number of permutation of any pair, which is $2!$. Hence the number of combinations is $\frac{5 \times 4}{2!} = \frac{5!}{2!3!} = \binom{5}{2}$.

A combination is a way to choose a m of objects from n given objects, where the order in which the objects are chosen does not matter. We express this as $\binom{n}{m} = \frac{n!}{m!(n-m)!}$.

9. Sticks and stones

***Example 3.9**: Six stones can be divided into three piles placing two sticks between any pair of stones:*

* * * | * | * *

How many ways are there to split this pile of rocks in the method described above? The number of slots is $6 - 1 = 5$, so there are $\binom{5}{2}$ ways to divide.

n stones can be divided into m piles by lining up the stones in a row and put the $m - 1$ sticks between stones. The number of ways is the number of ways to choose $m - 1$ slots between two stones from $n - 1$ slots, which is $\binom{n-1}{m-1}$.

10. The Exclusion - Inclusion Principle

Definition: $|A|$ is the number of elements of A

For example $A = \{1,2,3\}$, $|A| = 3$.

For any two sets A, B: $|A \cup B| = |A| + |B| - |A \cap B|$.

For any three sets A, B, C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

***Example 3.10**: If x is element of B, C, and not element of A, we can verify how many times it is counted on each side of the equation above:*

$1 = 0 + 1 + 1 - 0 - 1 - 0 + 0$

Exercises

Exercise 3.11

Mr. and Mrs. Zeta wants to name their new baby Zeta so that its monogram (first, middle, and last initials) will be in *alphabetical order,* with no letters repeated. In how many ways are such monograms possible?

Discussion:

What is a desired name? Note that the last initial is always Z because the baby last name must be Zeta. The first initial and middle initial must be in alphabetical order, for example A(lan) D(ennis), and not T(alon) Q(in).

Let us solve this problem by using two different techniques.

Solution 1 - Sum rule

Note that the last initial is always Z. All the possible monograms can be listed in alphabetical order as below:

$$ABZ, ACZ, \ldots, AYZ, BCZ \ldots, WXZ, WYZ, XYZ \qquad (*)$$

If the first initial is A, then the second initial can be any of the following set: B, C, $\ldots$, Y, so there are 24 possible choices in this case.

If the first initial is B, then the second initial an be any of the following set: C, D, $\ldots$, Y, so there are 23 possible choices in this case.

Continuing with this way while noting that all of the choices are different, we can use the *Sum rule* to find that the number of possible monograms is:

$$24 + 23 + \ldots + 1 = \frac{24 \times 25}{2} = 300$$

Solution 2 - Combination

The first two initials of a suitable monogram is a two-element subset of distinct letters from A, B, $\ldots$, Y, for example $\{I, J\}$.

It means that each pair is *a way to choose two different letters from 25 letters and the order in which the letters are chosen does not matter*.

This is exactly the number of combinations as if we were choosing 2 different letters from 25 different letters, which is:

$$\binom{25}{2} = 300$$

Exercise 3.12

$2n$ players are participating in a tennis tournament. Let P_n be the number of possible parings for the first round. Find a formula to compute P_n.

Discussion:

Let $n = 1$, making two players: A, B. There is only one way to pair them, so $P_1 = 1$.

Let $n = 2$, making four players A, B, C, D, there are $\binom{4}{2}$ ways to choose the first pair and $\binom{2}{2}$ ways to choose the second pair. Therefore, the total is $\binom{4}{2}\binom{2}{2} = \frac{4!}{2!2!} = \frac{4!}{2^2}$ ways to choose. However, in choosing this way, the order of the pairs do matter. For example a selection of $(A, B), (C, D)$ differs from $(C, D), (A, B)$. So in order to obtain P_2, we have to divide the total by the number of permutations of these 2 chosen pairs, which is $2!$. Hence $P_2 = \frac{4!}{2^2 \times 2!}$.

Solution 1 - Product rule + Combination:

Our strategy here is to generalize to the $n = 2$ case in the discussion. We find the number of ways to choose the first pair from all $2n$ players, then the number of ways to choose the second pair from the remaining $2n - 2$ players, and so on. Since the choices are independent according to the product rule , we can find the total number of ways by multiplying the number of ways each case has together, then dividing by the number of permutations, which is $n!$.

The number of ways to choose the first pair from $2n$ players is $\binom{2n}{2}$, the second pair from the remaining $2n - 2$ is $\binom{2n-2}{2}$, ..., the last pair from the remaining 2 is $\binom{2}{2}$. The product is $\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2}$.

So $P_n = \frac{(2n)!}{2!(2n-2)!}\frac{(2n-2!}{2!(2n-4)!}\cdots\frac{2!}{2!}\frac{1}{n!} = \frac{(2n)!}{2^n n!}$.

Solution 2 - Product rule + Recursion:

We can choose a P player, and his partner can be chosen from the remaining $2n-1$ players. Now, we need to find the number of pairings of $2n-2$ players, which is P_{n-1}. Therefore:

$$P_n = (2n-1)P_{n-1} = (2n-1)(2n-3)\ldots 1 = \frac{(2n)!}{(2n)(2n-2)\ldots 2} = \frac{(2n)!}{2^n n!}$$

Solution 3 - Permutation:

Line up $2n$ players in a row and pair up the neighbours as below

$$(P_1, P_2), (P_3, P_4), \ldots, (P_{2n-1}, P_{2n})$$

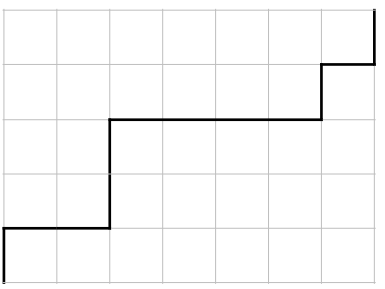
There are $(2n)!$ ways to line up (permute) the players. Let us count the duplicates by ordering

- each pair, for example (P_1, P_2) has an ordering duplicate (P_2, P_1). The number of duplicates is 2.
- n pairs $(P_1, P_2), (P_3, P_4), \ldots, (P_{2n-1}, P_{2n})$ can exchange (permute) their positions. The number of duplicates is $n!$

Hence: $P_n = \dfrac{(2n)!}{\underbrace{2.2\ldots 2}_{n}.\,n!} = \dfrac{(2n)!}{2^n n!}$

Exercise 3.13

The diagram below shows a path that a bug can use to move from lower left corner to the upper right corner of a 5×7 rectangle following the grid lines. Note that the bug can only move in the *up* or *right* directions. How many paths are possible for a $m \times n$ rectangle?



Solution - Bijection + Combination:

The moves of the bugs can be written in a row: U if it is an "up" move, R if it is a "right" move. For example, $URRUURRRRURU$ (pronounced incoherently) is the sequence of moves described by the diagram above.

Any possible path can be mapped by using *one-to-one correspondence* to a row of letters. The number of letters in the row is $m+n$; m letters U, n letters R. Different paths have different permutations of the letters. The number of permutations of m letters U is the number of ways to choose m positions from $m+n$ places in a row to put the U letters.

Hence, the number of paths possible is $\binom{m+n}{m}$. Additionally, note that $\binom{m+n}{m} = \binom{m+n}{n}$.

Exercise 3.14

Find the number of ordered quadruples (a, b, c, d), where a, b, c, d are positive odd integers with sum 98?

Solution - Bijection + Sticks and stones:

Since a, b, c, d are positive odd integers, we can substitute them with $2x-1, 2y-1, 2z-1, 2w-1$, where x, y, z, w are positive integers:

$$(2x-1)+(2y-1)+(2z-1)+(2w-1) = 98 \Rightarrow x+y+z+w = 51$$

Each quadruple (a, b, c, d) is a one-to-one correspondent to a row of 51 stones that are separated into four groups. Now, we use the Sticks and Stones method.

$$\underbrace{**\ldots*}_{x} \mid \underbrace{**\ldots*}_{y} \mid \underbrace{**\ldots*}_{z} \mid \underbrace{**\ldots*}_{w}$$

Hence, the number of the ordered quadruples is $\binom{50}{3} = 19600$ ways.

Exercise 3.15

A sequence of numbers have the following properties:

a) The sum of any 3 consecutive terms is negative (smaller than zero)

b) The sum of any 4 consecutive terms is positive (larger than zero)

What is the maximum number of terms the sequence can have?

Solution - Counting twice

Let us assume that there are at least six terms in the sequence:

$$a_1, a_2, a_3, a_4, a_5, a_6$$

(1) Let us investigate the *first three sums of four consecutive terms*:

$a_1 + a_2 + a_3 + a_4 > 0$

$a_2 + a_3 + a_4 + a_5 > 0$

$a_3 + a_4 + a_5 + a_6 > 0$

(2) Each column of these sums forms a *sum of three consecutive terms*:

$a_1 + a_2 + a_3 < 0$

$a_2 + a_3 + a_4 < 0$

$a_3 + a_4 + a_5 < 0$

$a_4 + a_5 + a_6 < 0$

(1) and (2) are contradictory: a sum of negative numbers is equal a sum of positive numbers. Therefore the maximum number of terms is 5. Here's an example: $2, 2, -5, 2, 2$.

Problems

Problem 3.16

How many positive integers, not exceeding 2019, are multiples of 3 or 4 but not 5?

Problem 3.17

A positive integer is said to be bi-digital if it uses two different digits, with each digit used exactly twice. For example, 1331 is bi-digital, whereas 1113, 1111, 1333, and 303 are not. Determine the exact value of integer b, the number of bi-digital positive integers.

Problem 3.18

A local high school math club has 12 students in it. Each week, 6 students go on a field trip.

a. Jeffrey, a student in the math club, has been on a trip with every other student in the math club. Determine the minimum number of trips Jeffrey could have been on.

b. If every pair of students has been on at least one field trip together, determine the minimum number of field trips that could have happened.

Problem 3.19

At a party, each man danced with exactly three women and each woman danced with exactly two men. We know that twelve men attended the party. How many women attended the party?

Problem 3.20

The student lockers at Clever Gals and Guys school are numbered consecutively, beginning with locker number 1. The plastic digits used to number the locker costs 2 cents a piece. Thus, it costs two cents to label locker number 9 and 4 cents to label locker number 10. If it costs \$137.94 to label all the lockers, how many lockers are there in this school?

Problem 3.21

P is a convex 30−polygon. Compute the following:

a. d_{30}, the number of diagonals P has.

b. s_{30}, the number of intersections of diagonals P has.

Problem 3.22

There are 12 books on a shelf. In how many ways can one choose five of them so that no pair of chosen books used to stand next to each other?

Problem 3.23

Each face of a cube is coloured by a different colour of six given colours. How many colourings are distinct?

Problem 3.24

Burger Haven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onion. A customer can choose one, two, or three meat patties and any number of condiments. How many different kinds of hamburgers can be ordered?

Problem 3.25

At the end of a professional bowling tournament, the top 5 bowlers have a playoff. First #5 bowls #4. The loser receives 5th prize and the winner bowls #3 in another game. The loser of this game receives 4th prize and the winner bowls #2. The loser of this game receives 3rd prize and the winner bowls #1. The winner of this game gets 1st prize and the loser gets 2nd prize. In how many orders can bowlers #1 through #5 receive the prizes?

Challenges

Challenge 3.26

In how many ways can one take an odd number of marbles from a bag of 10 marbles?

Challenge 3.27

10 people sit around a circular table. How many of the 10! arrangements are distinct, i.e. do not have the same neighbouring relations?

Challenge 3.28

a. m, n are positive integers so that $n \geq m$. Prove that:

$$\binom{n+m}{m} = \binom{n}{0}\binom{m}{m} + \binom{n}{1}\binom{m}{m-1} + \binom{n}{2}\binom{m}{m-2} + \ldots + \binom{n}{m}\binom{m}{0}$$

b. n is a positive integer. Prove that:

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2$$

Challenge 3.29

How many ways are there to represent the number 1000000 as the product of three factors, if we consider the order of factors as a way to differentiate the representations?

Challenge 3.30

P is a convex $n-$polygon. No three diagonals pass through a point. The diagonals of P divide P into T_n parts. Compute T_n

Solutions

Definition: $\lfloor x \rfloor$ is the largest integer less than or equal x. In other words, if n is an integer and $n = \lfloor x \rfloor$ then $n \leq x < n + 1$.

Lemma: the number of multiples of a positive integer m that are less than or equal to a positive integer n is $\lfloor \frac{n}{m} \rfloor$. For example, the only multiples of 3 not exceeding 10 are 3,6,9. There are $\lfloor \frac{10}{3} \rfloor = 3$ of them.

Solution 3.16

Solution 1 -Exclusion - Inclusion Principle

First we count the positive integers that are do not exceed 2019 and are multiples of 3 or 4. According to the Exclusion - Inclusion principle:

The number of multiples of 3 or 4 = the number multiples of 3 + the number multiples of 4 - the number multiples of 3 and 4.

For numbers not exceeding 2019: the number of multiples of 3 is $\lfloor \frac{2019}{3} \rfloor = 673$, the number of multiples of 4 is $\lfloor \frac{2019}{4} \rfloor = 504$, the number of multiples of 3 and 4 is $\lfloor \frac{2019}{12} \rfloor = 168$. Therefore, the number of multiples of 3 or 4 is $673 + 504 - 168 = 1009$.

Now, we need to eliminate the multiples of 5 from these 1009 numbers. We need to count the multiples of 3 or 4, which are also the multiples of 5. Applying the Exclusion - Inclusion principle again:

The number of multiples of 3 or 4 that are also the multiples of 5 = the number of multiples of 3 and 5 + the number of multiples of 4 and 5 - the number of multiple of 3, 4, and 5

For numbers not exceeding 2019: the number of multiples of 3 and 5 is $\lfloor \frac{2019}{15} \rfloor = 134$, the number of multiples of 4 and 5 is $\lfloor \frac{2019}{20} \rfloor = 100$, the number of multiples of 3, 4 and 5 is $\lfloor \frac{2019}{60} \rfloor = 33$, so the number of multiples of 3 or 4, also multiples of 5 is $134 + 100 - 33 = 201$.

Hence, the answer is $1009 - 201 = 808$.

Solution 2 - Exclusion - Inclusion Principle + Sum rule

First we use the Exclusion - Inclusion principle to count the number of multiples of 3 or 4 in the set $\{1,2,3,...,29,30\}$ as in Solution 1:

$$\lfloor \frac{30}{3} \rfloor + \lfloor \frac{30}{4} \rfloor - \lfloor \frac{30}{12} \rfloor = 10 + 7 - 2 = 15$$

It is easy to see that if we remove the numbers 15, 20, 30 from those 15 numbers then we have *12 numbers not exceeding 30 that are multiples of 3 or 4, but not 5.*

So the numbers 1, 2,..., 2010 can be divided into $\frac{2010}{30} = 67$ sets of 30 numbers: $\{1,2,...,30\}$, $\{31,32,...,60\}$,…,$\{1981,1982,...,2010\}$. Each set contains 12 numbers that are multiples of 3 or 4, but not 5, so there are $67 \times 12 = 804$ such numbers. Since 2012, 2013, 2016, 2019 are also such numbers, the total number of those are $804 + 4 = 808$.

Solution 3.17

Solution 1 - Product rule

There are 9 choices for what the left-most digit of the number is (it cannot be 0) and there are 3 choices for where the second copy of this digit is. There are 9 possibilities for the digit that fills the remaining positions. Thus, $b = 9 \times 3 \times 9 = 243$ choices.

Solution 2 - Permutation of some indistinguishable objects

We consider two cases. Either 0 is one of the digits, or it is not.

If 0 is not one of the digits, then we have $\boxed{?}\binom{9}{2} = 36$ ways to choose 2 digits which are not 0. These two digits will form a 4-digit number for example as $XYYX$. There are $\frac{4!}{2!2!} = 6$ ways to arrange these digits, for a total of $36 \times 4 = 216$ numbers.

If 0 is one of the digits, it cannot be the first digit of the number since then the number would have fewer than 4 digits. In this case, there are $\boxed{?}9\boxed{?}$ ways to choose the other digit. The first digit must be the non-zero digit and there are 3 places for the other non-zero digit, so there are $9 \times 3 = 27$ such numbers.

Therefore, $b = 216 + 27 = 243$

Definition: $\lceil x \rceil$ is the smallest integer larger than or equal x. In other words, if n is an integer and $n = \lceil x \rceil$ then $n - 1 < x \leq n$.

Solution 3.18 - Counting twice

a/ There are 11 students in the club other than Jeffrey, and each field trip that Jeffrey is on has 5 other students. In order for Jeffrey to go on a field trip with each other student, he must go on at least $\lceil \frac{11}{5} \rceil = \lceil 2.2 \rceil = 3$. field trips. To see that 3 trips is sufficient, we let the other students be$\{s1, s2, ..., s11\}$ and consider an example.

On the first trip, Jeffrey can go with $\{s1, s2, ..., s5\}$, on the second trip with $\{s6, s7, ..., s10\}$ and on the third trip with $s11$ and any other 4 students.

b/Each student must go on at least 3 trips and there are 12 students in total. If we count the number of trips that each student went on, we would get a minimum of $12 \times 3 = 36$ trips from the point of view of every student. Since 6 students attend each field trip, that means there must be at least $\frac{36}{6} = 6$ trips.

We now show that it is possible to do this with exactly 6 trips. Divide the students into 4 groups of 3 students each: A, B, C, D. There are 6 different pairs of groups AB, AC, AD, BC, BD, CD. Let these pairs of groups be our 6 field trips. We see that since each group goes on a trip with each other group, that each pair of students goes on a trip together.

Hence, this can be done with 6 trips.

Solution 3.19 - Counting twice

First, let us count the number of parings from men's point of view, For each man, there are three women of whom he danced with. Therefore, the number of pairings is $12 \times 3 = 36$. Coincidentally , this is also the number of parings from women's view. Let w be the number of women: $36 = w \times 2 \Rightarrow w = 18$.

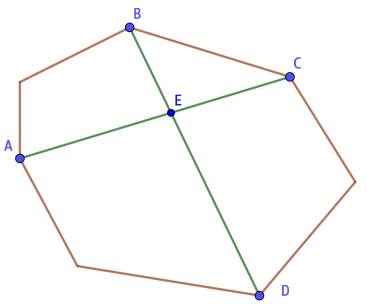
Solution 3.20 - Sum rule

The total amount of digits is $\dfrac{137.94}{0.02} = 6897$. Since the numbers from 1 to 999 requires $9 \times 1 + 90 \times 2 + 900 \times 3 = 2889$ digits, the rest $6897 - 2889 = 4008$ are used for the 4-digit lockers, or $\dfrac{4008}{4} = 1002$ lockers. It means that the number of lockers is $999 + 1002 = 2001$.

Solution 3.21 - Bijection + Combination

a/For each vertex, a diagonal is a segment connecting it with another non-neighbouring vertex. Therefore, there are $30 - 2 - 1 = 27$ diagonals for a vertex.

Since there are 30 vertices, d_{30} is $\dfrac{30 \times 27}{2} = 405$.



b/The intersection E of two diagonals AC, BD is the intersection of two diagonals of quadrilateral $ABCD$.

For quadruple (A, B, C, D), because A, C are not neighbours, the segment connecting them must be a diagonal. This is also true with B, D. This means that intersection E is also an intersection of two diagonals of the original polygon.

The two sentences above mean that there is a *one-to-one correspondence between an intersection of two diagonals with a group of four vertices of the polygon*. Therefore, the number of intersection is the number of ways to choose 4 different vertices from 30 vertices.

Hence, $s_{30} = \binom{30}{4} = 27405$.

Solution 3.22 - Sticks and stones

Let us remove the chosen books from the shelf. 7 books remain. There are 8 *open gaps*, 6 between the 7 books and 2 at both ends. These are the slots the chosen books can be put back in, making it so that none of the books are neighbours. Hence, there are $\binom{8}{5} = 56$ ways.

Solution 3.23 - Combination + Permutation

Since six different colours are used to colour six faces of the cube, there are $6!$ ways to choose them for colouring.

Let us chose a face and let it be the top of the cube, no matter what. There are 6 ways to choose a face. The cube can be rotated, as show in the diagram on the left, in 4 different positions.

This means that there are $6 \times 4 = 24$ repetitions for each colouring. Hence, the number of distinct colourings is $\frac{6!}{24} = 30$.

Solution 3.24 - Product rule

There are 3 ways to choose meat patties: 1, 2, or 3. For each condiment, there are 2 possible choices for the customer: choose it or don't choose it. Therefore, there are $3 \times \underbrace{2 \times 2 \times \ldots \times 2}_{8} = 768$ possibility.

Solution 3.25 - Product rule

If #5 wins the first game, and then loose the second game, #3 wins the third game, but loses the fourth game, then *the ranking from last to first positions would be: #4, #5, #2, #3, #1*. So for each game, there are two possible outcomes. It is easy to see that a sequence of 4 games with different results will result in different rankings. Hence the number of different rankings is actually the product of the possible outcomes of each games, which is $2^4 = 16$.

Solution 3.26 - Bijection + Product rule

If we number the marbles with 1, 2, 3,..., 9, 10, then the choices resulting in the sets below each have an odd number of marbles:

- Sets with 1 marble: $\{1\}, \{2\}, \ldots, \{10\}$
- Sets with 3 marbles: $\{1,2,3\}, \{1,2,4\}, \ldots, \{8,9,10\}$
- …
- Sets with 9 marbles: $\{1,2,3,...,9\}, \{1,3,...,10\}, \ldots, \{2,3,...,10\}$

If we remove the number 10 from the sets above:

- Sets with 1 marble become:
  - empty set $\{\}$ (after removing 10 from $\{10\}$), and
  - 1-element sets $\{1\}, \{2\}, \ldots, \{9\}$ with element from 1,2,...,9
- Sets with 3 marbles become:
  - 2-element sets $\{1,2\}, \{1,3\}, \ldots, \{8,9\}$ (after removing 10)
  - 3-element sets $\{1,2,3\}, \ldots, \{7,8,9\}$with element from 1,2,...,9
- …
- Sets with 9 marbles become:
  - 8-element sets $\{1,2,...,8\}, \ldots, \{2,3,...,9\}$ (after removing 10)
  - 9-element sets with element from 1,2,...,9

Basically, these are all subsets of a 9-element set. These are all the number of ways to choose *some* elements of set A. Since one can either choose or not choose an element of A, there are 2^9 possible ways to make a subset. In other words, the number of subsets is 2^9.

Theorem: from a set of n elements

i. The number of ways to choose *some* elements from it is 2^n

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

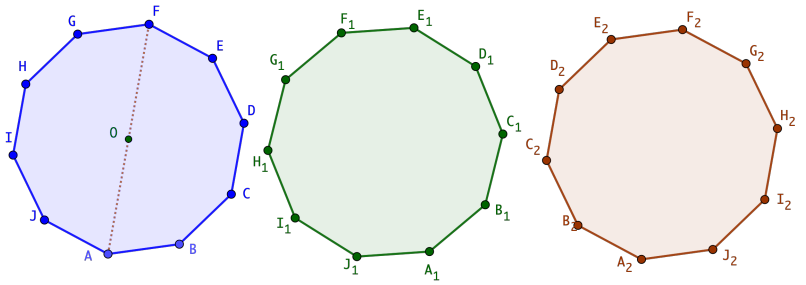
ii. The number of its subsets is 2^n

iii. The number of its subsets with *odd number of elements* is equal to the number of its subsets with *even number of elements*, and equal to 2^{n-1}

$$\sum_{k\ odd}^{n} \binom{n}{k} = \sum_{k\ even}^{n} \binom{n}{k} = 2^{n-1}$$

Solution 3.27 - Combination + Permutation

Let the people A, B, C, $\ldots$, I, J sit as shown in the figure below.



Since there are 10 people and 10 positions, the total permutation is $10!$. Now let us investigate the sittings that have the same neighbouring relations. The green figure is obtained from the blue one by it rotating around its centre O and angle of $\dfrac{360^\circ}{10} = 36^\circ$, which means A moves to B position: A becomes A_1, B becomes B_1. There are 10 such rotations. The brown figure is obtained by reflect the blue figure over the main diagonal AF. Hence, the number of possible sitting arrangements with different neighbouring relations is $\dfrac{10!}{10 \times 2} = \dfrac{9!}{2}$.

Solution 3.28 - Bijection

a/ Assume that we have n blue marbles and m red marbles.

$$\binom{n+m}{m} = \binom{n}{0}\binom{m}{m} + \binom{n}{1}\binom{m}{m-1} + \binom{n}{2}\binom{m}{m-2} + \ldots + \binom{n}{m}\binom{m}{0}$$

The formula on the left is the ways to choose a set of m marbles from 2 marble piles: a pile of n blue marbles and another pile of m red marbles. A term of the sum on the right, $\binom{n}{k}\binom{m}{m-k}$, is the number of ways to choose k marbles from n blue marbles and $m-k$ marbles from m red marbles, where k takes all possible values between $0, 1, \ldots, m$. Therefore, the formula on the right is equal to the sum on the left.

b/ Use the result of a/ and note that $\binom{n}{k} = \binom{n}{n-k}$.

Solution 3.29 - Bijection + Sticks and stones

Since $n = 1000000 = 10^6 = 2^6.5^6$. A factor of n is determined by the number of 2's and 5's in its factored state. For example:

$$10 = 2^1 \times 5^1,\ 25 = 5^2,\ 40 = 2^3 \times 5^1$$

Let a_1, a_2, a_3, and b_1, b_2, b_3 be the number of 2's, and 5's in the first, second, and third factors. We have:

$$2^6.5^6 = (2^{a_1}5^{b_1})(2^{a_2}5^{b_2})(2^{a_3}5^{b_3}) = 2^{a_1+a_2+a_3}.5^{b_1+b_2+b_3}$$

$$\Rightarrow a_1 + a_2 + a_3 = 6\ (*),\ b_1 + b_2 + b_3 = 6\ (**)$$

So *the number of different factors will be the number of solutions for (*) and (**) combined*.

Since a_1, a_2, a_3 are non-negative integers, the *sticks and stones rule* cannot be applied directly for $a_1 + a_2 + a_3 = 6$, because *the rule can be use with non-empty piles of stones only*. Here we turn the problem into an equivalent one by using bijection technique:

If $c_1 = a_1 + 1$, $c_2 = a_2 + 1$, $c_3 = a_3 + 1$, then $c_1 + c_2 + c_3 = 9$. For any triple of (a_1, a_2, a_3) a one-to-one correspondence of (c_1, c_2, c_3) can be made. Since $c_1, c_2, c_3 \geq 1$, the *sticks and stone rule* say that number of (c_1, c_2, c_3) triples is $\binom{9-1}{3-1} = \binom{8}{2}$. So the number of (a_1, a_2, a_3) triples is $\binom{8}{2}$.

Solving the same equation for (b_1, b_2, b_3), the number of triples for the number of 2's and 5's in the first, second, and third factors of 10^6 is $\binom{8}{2}\binom{8}{2} = 28^2 = 784$.

Solution 3.30 - Sum rule

Let T_i be the number of triangles with i vertices among the vertices of P. The sum rule gives:

$$P_n = T_0 + T_1 + T_2 + T_3 = \binom{n}{6} + 5\binom{n}{5} + 4\binom{n}{4} + \binom{n}{3}.$$