Ongoing school year 2023 - 2024. Last update on

School year 2023 - 2024

Nghia Doan

January 2, 2024

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Introduction

Welcome to the MCC School year 2023 - 2024!

Levels: - Introductory (I): for anyone who start middle schools, are beginners, not confident of being good at problem solving, or just simply don't have enough time for math. They are encouraged to start, or continue, or finish the Introductory Curriculum. Expected workload: 2 hours per week - Advanced (A): for returning students, who already have a good grasp of problem solving, having done a few semesters with the club, wish to learn more, and ready to achieve higher knowledge and skill sets. Expected workload: 4 hours per week - Olympiad (O): for excellent students who are already or about to take challenges at national and international competitions. Expected workload: 8 hours per week.

Have a great year!

Your teacher,

Nghia Doan

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Part I

2023

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Chapter 1

Entrance tests

1.1 Rules

Choice of level:

- There are three test papers, each for one level.
- Each student can chose multiple test levels. If the students choose more than one level, the student must choose multiple different test times.

Registration:

- At registration, student and their parents are responsible for the correct email address and the time, where and when they wish to receive the test problems.
- The club will not verify whether the registered email address is correct.

Entrance test:

- Each level test consisting of 8 problems will be delivered by automatic scheduled email at the registered time. No test paper will be sent twice.
- Students have 270 minutes to solve the problems. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET.
- After the allowed time, the student has 15 minutes to scan the solutions. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET, and has to submit the scanner solutions before 11:45 AM. Any solution submitted later than the allowed time will not be accepted.
- All submissions must be done by email to nghia71@gmail.com.
- Solutions must be written in English. Any solution looks unreadable, or is presented badly will not be considered for grading. Answer without proper solution will not be considered as solution. Answer consists of self-explanatory and easy-to-understand diagrams are acceptable but might not be considered as full solution.
- A problem can contain more than one questions, each with an indication of the maximum number of points can be awarded if a correct solution is provided. Partially correct solution is awarded with a some points depending on its quality. Multiple solutions can be submitted for a single problem. Each different solution, if correct, will be awarded the same number of points. Partially distinct solution also counts.
- Student who performs adequately will be accepted to the registered level. Student who are accepted to different levels can choose any of those levels to attend.

1.2 Problems for Intermediate Level

Problem 1.2.1 (23-24-ET-I-P1)**.** (*10 points*) Place algebraic operations $+; -; \times; \div$ between the digits 1 to 9, in that order, so that the total equals 100. You may freely use brackets before or after any of the digits in the expression. No digits may be placed together, such as 123 and 67 are not allowed.

Here is an example for a correct answer: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9 = 100$.

Here is an example for an incorrect answer: $12 + 3 + 4 + 5 - 6 - 7 + 89 = 100$.

Do not submit any of these examples as your solution.

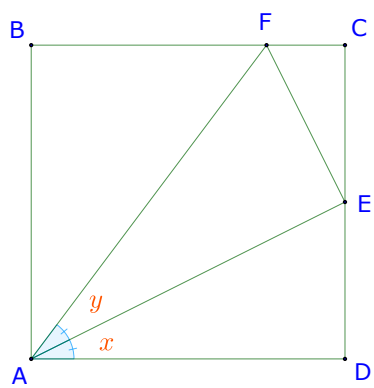
Problem 1.2.2 (23-24-ET-I-P2)**.** (*10 points*) Linh, Minh and Vinh are identical triplet sisters. Linh always tells the truth, Minh always lies and Vinh sometimes lies and sometimes tells the truth.

They are invited to a party. One of them arrives late. The party host asks this sister, who just arrived late, who she is. She answers "I am Vinh". The party host obviously cannot tell the girls apart so she asks the other two sisters the name of the sister who was late. One of them says: "Linh was late", and the other says: "Minh was late".

Which sister was late?

Problem 1.2.3 (23-24-ET-I-P3)**.** (*10 points*) $ABCD$ is a square with side length 4. E is midpoint of CD. $AE \perp EF$. Let $x = \angle EAD, y = \angle FAE$.

Prove that $x = y$.



Problem 1.2.4 (23-24-ET-I-P4)**.** (*10 points*) A pair of two-digit numbers has the following properties:

1. The sum of the four digits is 25.
2. The sum of the two numbers is 97.
3. The product of the four digits is 864.
4. The product of the two numbers is 1972.

Determine the two numbers.

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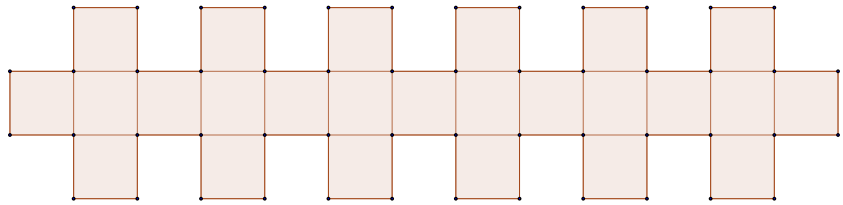
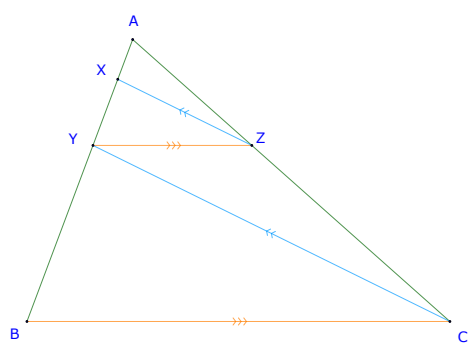
Problem 1.2.5 (23-24-ET-I-P5)**.** (*10 points*) The shape in the diagram below was created by putting together 25 unit squares. When a similar shape is created with n squares, its perimeter is 100 units. Determine n.



Problem 1.2.6 (23-24-ET-I-P6)**.** (*10 points*) N is the sum of all integers from 1 to n and M is the sum of all integers from 1 to m, where $n > m + 1$. If the difference of the two sum $N - M = 2012$, then what is the value of $n + m$?

Problem 1.2.7 (23-24-ET-I-P7)**.** (*10 points*) The area of $\triangle ABC$ is 1. X, Y are points on AB, Z is a point on AC such that $XY = 2AX$. Furthermore $XZ \parallel YC$, $YZ \parallel BC$. Find the area of $\triangle XYZ$.



Problem 1.2.8 (23-24-ET-I-P8)**.** (*10 points*) Consider a sequence of integers

$$1, 3, 2, -1, \ldots$$

where each term is equal to the term preceding it minus the term before that.

What is the sum of the first 2023 terms?

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1.3 Solutions for Introductory Level

Problem 1.3.1 (Mathematics Competitions Vol. 34, #1 2021)**.** (*10 points*) Place algebraic operations $+; -; \times; \div$ between the digits 1 to 9, in that order, so that the total equals 100. You may freely use brackets before or after any of the digits in the expression. No digits may be placed together, such as 123 and 67 are not allowed.

Here is an example for a correct answer: $1+2+3+4+5+6+7+8\times 9=100$.

Here is an example for an incorrect answer: $12+3+4+5-6-7+89=100$.

Do not submit any of these examples as your solution.

Solution. Following are five solutions, including the correct example:

$$1+2+3+4+5+6+7+8\times 9=100 \qquad 1\times 2\times 3+4+5+6+7+8\times 9=100$$
$$(1+2+3+4+5)\times 6-7+8+9=100 \qquad 1\times(-2-3-4-5)+6\times 7+8\times 9=100$$
$$1-2+3\times 4\times 5+6\times 7+8-9=100 \qquad (1-2+3)\times(4+5)\times(-6+7\times 8)\div 9=100$$
$$1\times(2+3+4-5+6)\times(-7+8+9)=100 \qquad 1\times(2+3)\times 4(-5-6+7\times 8)\div 9=100$$
$$1\times(-2+3+4+5)\times 6\times(7+8)\div 9=100 \qquad (1+2\times 3\times 4)\times 5-6\times 7+8+9$$
$$-1\times 2-3-4-5+6\times 7+8\times 9=100 \qquad 1\times 2\times(3+4)\times 5+6+7+8+9=100$$

□

Problem 1.3.2 (Problem Solving for Irish Second level Mathematicians - PRISM 2015/Problem 20)**.** (*10 points*) Linh, Minh and Vinh are identical triplet sisters. Linh always tells the truth, Minh always lies and Vinh sometimes lies and sometimes tells the truth.

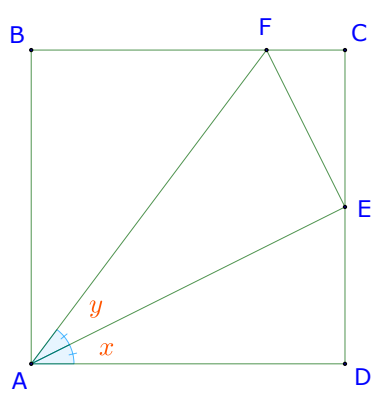
They are invited to a party. One of them arrives late. The party host asks this sister, who just arrived late, who she is. She answers "I am Vinh". The party host obviously cannot tell the girls apart so she asks the other two sisters the name of the sister who was late. One of them says: "Linh was late", and the other says: "Minh was late".

Which sister was late?

Solution. Obviously the sister in question should not be Linh, since Linh always tells the truth. Thus she could be Minh or Vinh. It means that Linh was one of the other two sisters.

One of these two sisters (both arrived in time) who said that Linh was late could have not been Linh, thus the other one was Linh. Linh told the truth, she said Minh was late, hence $\boxed{\text{Minh was late.}}$ □

Problem 1.3.3 (17th Blundon Mathematics Contest 2000/P5)**.** (*10 points*) $ABCD$ is a square with side length 4. E is midpoint of CD. $AE \perp EF$. Let $x = \angle EAD, y = \angle FAE$. Prove that $x = y$.



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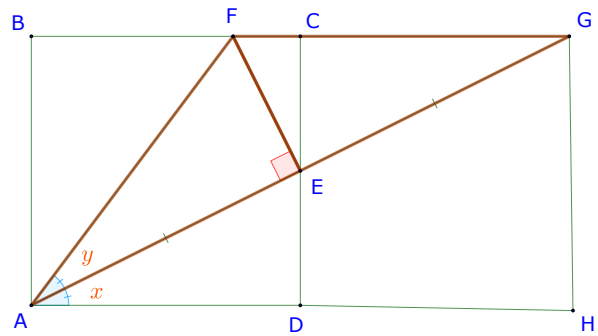
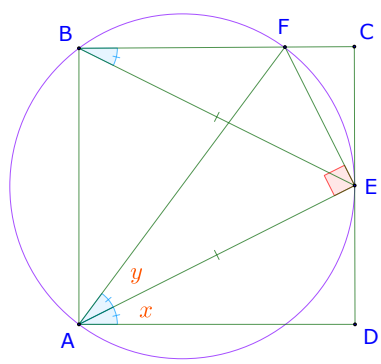
Solution. [Solution 1] Extend AE intersecting BC at G.



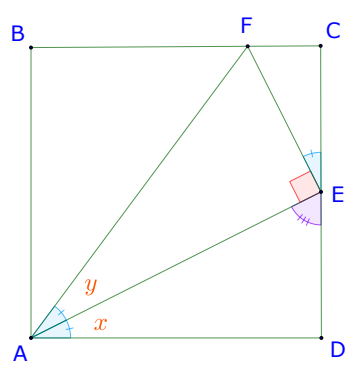
E is midpoint of CD, so $AE = EG$. Thus $\triangle AEF \cong \triangle GEF$. Thus $\angle EAF = \angle EGF = \angle EAD$. □

Solution. [Solution 2] Since both $\triangle AEF$ and ABF are right triangles with the same hypotenuse AF, thus B and E are on the circle diameter AF.



$\angle FAE = \angle EBF$(subtend $\widehat{EF}$) $= \angle EAD$ ($\triangle EAD \cong \triangle EBC$). □

Solution. [Solution 3] Since $\triangle ADE$ is a right triangle at D, so $\angle AED = 90^\circ - \angle EAD$.



Thus $\angle CEF = 90^\circ - \angle AED = \angle EAD$. Therefore $\triangle EAD \sim \triangle FEC$ (angle-angle-angle). $AD = 2EC$, so $AE = 2EF$, or $\frac{AE}{EF} = \frac{AD}{DE} = 2$, $\angle AEF = \angle ADE = 90^\circ \Rightarrow \triangle AEF \sim \triangle ADE \Rightarrow \boxed{\angle FAE = \angle EAD.}$ □

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Problem 1.3.4 (Modification to Crux Mathematicorum M266)**.** (*10 points*) A pair of two-digit numbers has the following properties:

1. The sum of the four digits is 25.
2. The sum of the two numbers is 97.
3. The product of the four digits is 864.
4. The product of the two numbers is 1972.

Determine the two numbers.

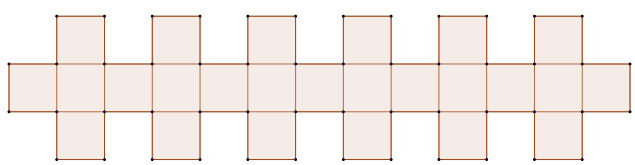
Solution. [Solution 1] Note that $1972 = 2^2 \cdot 17 \cdot 29$, thus it has 5 two-digit divisors $17, 29, 34, 58, 68$:

$$1972 = 17 \cdot 116 = 29 \cdot 68 = 34 \cdot 58.$$

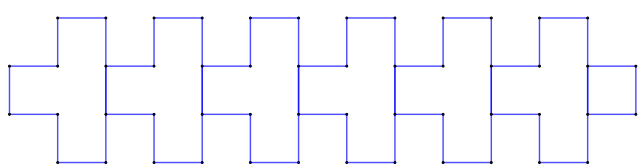
It is easy to verify that only the pair $\boxed{(29, 68)}$ satisfy other conditions. □

Solution. [Solution 2] Let the two-digit numbers be x and y, then $x + y = 97$, $xy = 1972$. Therefore x, y are the roots of $t^2 - 97t + 1972 = 0$. By solving the quadratic equation, we have $(x, y) = \boxed{(29, 68).}$ □

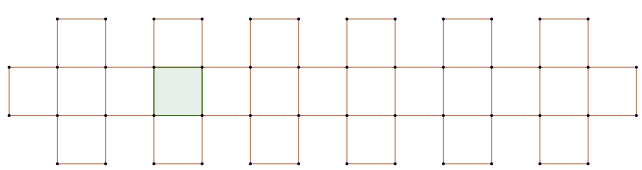
Problem 1.3.5 (2014-2015 Nova Scotia Math League/Game 1/Team Question 2)**.** (*10 points*) The shape in the diagram below was created by putting together 25 unit squares. When a similar shape is created with n squares, its perimeter is 100 units. Determine n.



Solution. The figure shown above consists of k blue block of 4 unit squares and a single unit square at the far right end. Thus altogether $4k + 1$ squares.



The perimeter of the figure is the sum of perimeters of all $4k + 1$ squares, subtracting *twice* the perimeters of all middle squares like the one coloured green.



Thus, the perimeter is:

$$4(4k + 1) - 2(4)(k) = 8k + 4.$$

Now, if $8k + 4 = 100$, then $k = 12$, hence the number of square $n = 4k + 1 = 4 \cdot 12 + 1 = \boxed{49.}$ □

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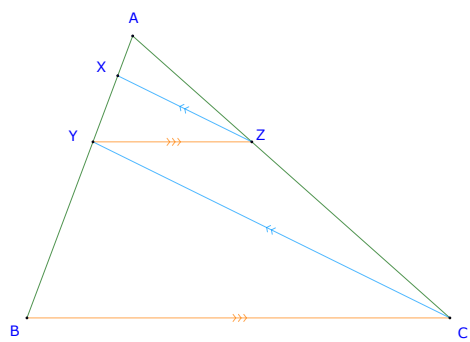
Problem 1.3.6 (2012 BC Secondary School Mathematics Contest, Senior Final, Part A/Problem 5)**.** (*10 points*) N is the sum of all integers from 1 to n and M is the sum of all integers from 1 to m, where $n > m+1$. If the difference of the two sum $N - M = 2012$, then what is the value of $n + m$?

Solution. First, note that

$$N - M = (1 + 2 + \cdots + n) - (1 + 2 + \cdots + m) = \frac{n(n+1)}{2} - \frac{m(m+1)}{2} = \frac{(n-m)(n+m+1)}{2}$$
$$N - M = 2012 \Rightarrow (n-m)(n+m+1) = 4024 \quad (*)$$

Now, $(n - m) + (n + m + 1) = 2n + 1$ is an odd number, so $n - m$ and $n + m + 1$ have different parities (one is odd and the other is even). Because their product, as shown above in (*), is an even number and since $n - m \geq 2$ and $4024 = 2^3 \cdot 503$, therefore the only possibility is $n - m = 8$, $n + m + 1 = 503$, or $n + m = \boxed{502.}$ □

Problem 1.3.7 (2010 Alberta High School Mathematics Competition, Part I/Problem 15)**.** (*10 points*) The area of $\triangle ABC$ is 1. X, Y are points on AB, Z is a point on AC such that $XY = 2AX$. Furthermore $XZ \parallel YC$, $YZ \parallel BC$. Find the area of $\triangle XYZ$.



Solution. Note that $\triangle AYZ \sim \triangle ABC$ and $XZ \parallel YC$, so

$$\frac{AY}{AB} = \frac{AZ}{AC} = \frac{AX}{AY} = \frac{1}{3}$$

Thus $[XYZ] = \frac{2}{3}[AYZ] = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 [ABC] = \boxed{\frac{2}{27}.}$ □

Problem 1.3.8 (2009 Fifth Annual Kansas Collegiate Mathematics Competition/Problem 1)**.** (*10 points*) Consider a sequence of integers

$$1, 3, 2, -1, \ldots$$

where each term is equal to the term preceding it minus the term before that.

What is the sum of the first 2023 terms?

Solution. Lets write down some more terms of the sequence

$$1, 3, 2, -1, (-1 - 2 =) - 3, (-3 - (-1) =) - 2, (-2 - (-3)) = 1, (1 - (-2) = 3), (3 - 1) = 2, (2 - 3 =) - 1, \ldots$$

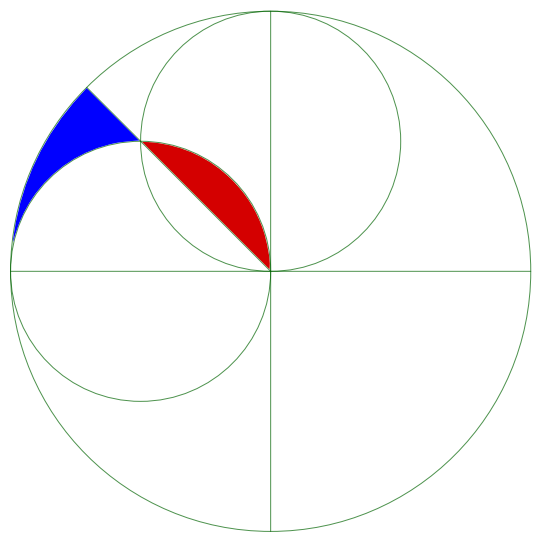
It is easy to see that the sequence consists of repeated blocks of the first six terms. The sum of these six terms is $1 + 3 + 2 + (-1) + (-3) + (-2) = 0$. Since $2023 = 2022 + 1 = 337 * 6 + 1$ thus the sum of the first 2023 terms is the first term of the block, which is $\boxed{1}$. □

1.4 Problems for Advanced Level

Problem 1.4.1 (23-24-ET-A-P1)**.** (*10 points*) Two friends agreed to meet at a club between 3:00 PM and 5:00 PM. What is the probability that they meet at the club within 15 minutes of arrival?

Problem 1.4.2 (23-24-ET-A-P2)**.** (*10 points*) A number n written in base b reads 211, but it becomes 110 when written in base $b+2$. Find n and b in base 10.

Problem 1.4.3 (23-24-ET-A-P3)**.** (*10 points*) In the diagram below two perpendicular diameters divide a circle into four parts. On each of these diameters a circle of half the diameter is drawn, tangent to the original circle and meeting at its centre. A radius to the large circle is drawn through the intersection points of these smaller circles. Show that the red and blue shaded regions are of the same area.



Problem 1.4.4 (23-24-ET-A-P4)**.** (*10 points*) Determine all pairs of integers (not necessarily positive) (x, y) such that

$$(x-8)(x-10) = 2^y.$$

Problem 1.4.5 (23-24-ET-A-P5)**.** (*10 points*) A game is played on a 7×7 board, initially blank. Chi and Mai make alternate moves, with Chi going first. In each of her moves, Chi chooses any four blank squares which form a 2×2 block, and paints these squares brown. In each of her moves, Mai chooses any blank square and paints it green. They take alternate turns until no more moves can be made by Chi. Then Mai paints the remaining blank squares green.

Which player, if either, can guarantee to be able to paint 25 or more squares in her colour, regardless of how her opponent plays?

Problem 1.4.6 (23-24-ET-A-P6)**.** (*10 points*) Determine all sets consisting of an odd number $2m+1$ of consecutive positive integers, for some integer $m \geq 1$ such that the sum of the squares of the smallest $m+1$ integers is equal to the sum of the squares of the largest m integers.

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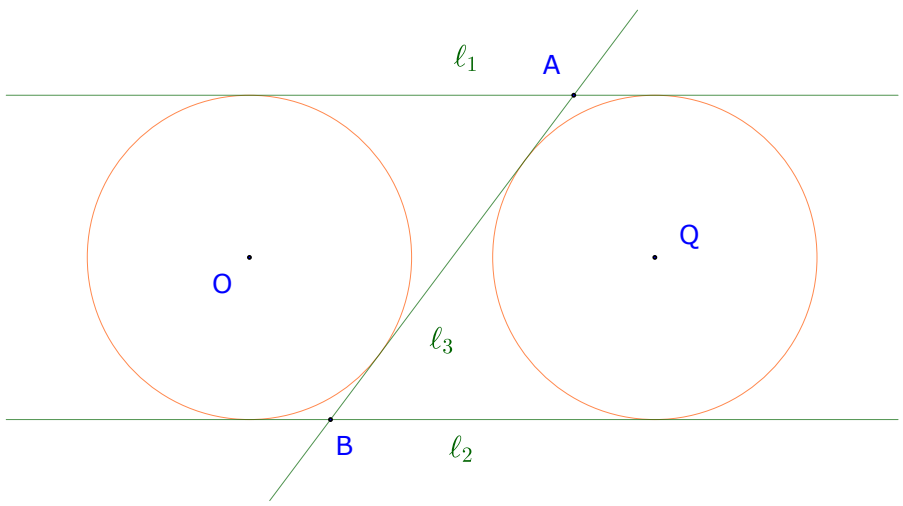
Problem 1.4.7 (23-24-ET-A-P7)**.** (*10 points*) Given two parallel lines ℓ_1 and ℓ_2, the transversal ℓ_3 intersects them at points A and B respectively. Two circles with centres at O and Q lie between the parallel lines on the left and on the right sides of the transversal such that the circles are tangent to all three lines.

Show that $OQ = AB$.



Problem 1.4.8 (23-24-ET-A-P8)**.** (*10 points*) Find the primes p, q, and r, given that one of the number pqr and $p + q + r$ is 101 times the other.

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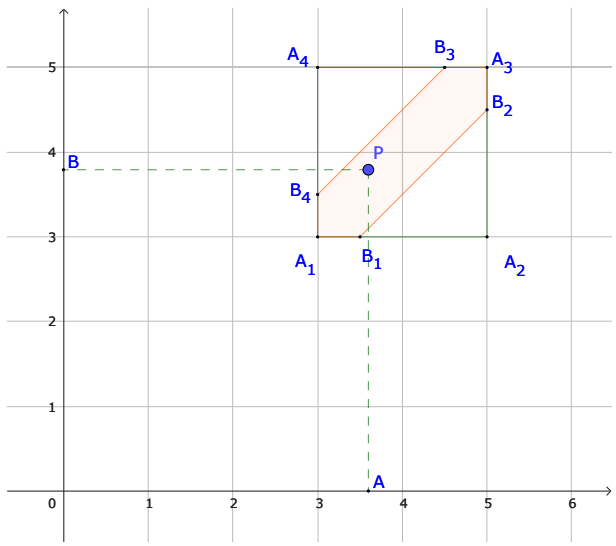
1.5 Solutions for Advanced Level

Problem 1.5.1 (2016 Kansas MAA Undergraduate Mathematics Competition/Problem 2)**.** (*10 points*) Two friends agreed to meet at a club between 3:00 PM and 5:00 PM. What is the probability that they meet at the club within 15 minutes of arrival?



Solution. Let $3 \le A \le 5$ and $3 \le B \le 5$ denote the times of arrival of the two friends, shown in the diagram above. Point $P(A, B)$ should be within the square $A_1A_2A_3A_4$ in order to satisfy the given conditions of the problem. Now, if the two friends meet within 15 minutes of arrival, then P point must be within the polygon $A_1B_1B_2A_3B_3B_4$. The probability is the ratio of the two areas, which is

$$\frac{[A_1B_1B_2A_3B_3B_4]}{[A_1A_2A_3A_4]} = 1 - \frac{2[B_1B_2A_2]}{[A_1A_2A_3A_4]} = 1 - \frac{\left(\frac{7}{4}\right)^2}{4} = \boxed{\frac{15}{64}}.$$

□

Problem 1.5.2 (Santa Clara University High School Mathematics 2001 Contest/Problem 4)**.** (*10 points*) A number n written in base b reads 211, but it becomes 110 when written in base $b+2$. Find n and b in base 10.

Solution. We have

$$\left.\begin{aligned} n &= 211_b = 2 \cdot b^2 + 1 \cdot b + 1 \\ n &= 110_{b+2} = 1 \cdot (b+2)^2 + 1 \cdot (b+2) \end{aligned}\right\} \Rightarrow 2b^2 + b + 1 = (b+2)^2 + (b+2) \Rightarrow b^2 - 4b - 5 = 0 \Rightarrow b = -1 \text{ or } 5$$

Since $b > 0$, thus $\boxed{b = 5}$. $n = 211_5 = 110_7 = \boxed{56.}$ □

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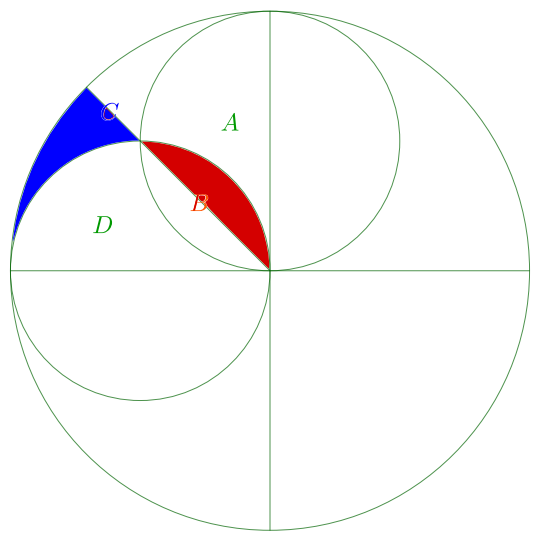
Problem 1.5.3 (2013 Manitoba Mathematical Competition/Problem 6)**.** (*10 points*) In the diagram below two perpendicular diameters divide a circle into four parts. On each of these diameters a circle of half the diameter is drawn, tangent to the original circle and meeting at its centre. A radius to the large circle is drawn through the intersection points of these smaller circles. Show that the red and blue shaded regions are of the same area.



Solution. Let A, B, C, and D denote the regions in the figure above. Let $[X]$ denotes the area of a region X.

$$[A] + [C] = \frac{1}{4}\pi(2r)^2 - ([B] + [D]) = \frac{1}{4}\pi(2r)^2 - \frac{1}{2}\pi r^2 = \frac{1}{2}\pi r^2 = [A] + [B]$$

Thus $[B] = [C]$, or $\boxed{\frac{1}{2}[B] = \frac{1}{2}[C].}$ □

Problem 1.5.4 (1962 examination, MAA Problem Book II (1961-1965)/Problem 36)**.** (*10 points*) Determine all pairs of integers (not necessarily positive) (x, y) such that

$$(x - 8)(x - 10) = 2^y.$$

Solution. First, the right hand side of the equation is an integer, thus $y \geq 0$. Second, since $2^y > 0$, thus both $x - 8$ and $x - 10$ have the same sign (both are positive or both are negative). Third, their product is a power of two, thus the absolute value of each of them is a power of two.

Since $(x - 8) - (x - 10) = 2$, and the only pair of powers of two differ by 2 is 2 and 4, therefore

$$\begin{cases} x - 10 = 2 \Rightarrow x = 12, \text{ or} \\ x - 10 = -4 \Rightarrow x = 6 \end{cases} \quad \text{in both cases } y = 3$$

Hence, the desired pairs are $(x, y) \in \{(6, 3),\ (12, 3)\}$. □

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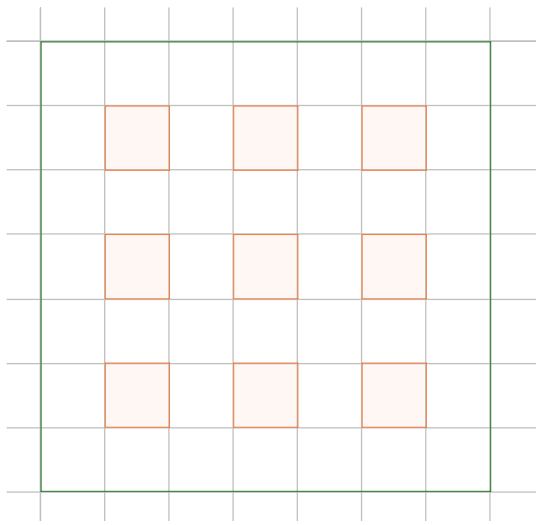
Problem 1.5.5 (2009 Alberta High School Mathematics Competi- tion, Part II/Problem 3)**.** (*10 points*) A game is played on a 7×7 board, initially blank. Chi and Mai make alternate moves, with Chi going first. In each of her moves, Chi chooses any four blank squares which form a 2×2 block, and paints these squares brown. In each of her moves, Mai chooses any blank square and paints it green. They take alternate turns until no more moves can be made by Chi. Then Mai paints the remaining blank squares green.

Which player, if either, can guarantee to be able to paint 25 or more squares in her colour, regardless of how her opponent plays?



Solution. There are nine squares at the intersections of even-numbered rows. Any block chosen by Chi will include one of these nine squares. Thus, Mai has to play in those nine squares in her first four moves. This way Chi will have at most five moves and will be able to paint at most twenty squares. Thus, Mai will win. □

Problem 1.5.6 (Crux Mathematicorum/MA 136)**.** (*10 points*) Determine all sets consisting of an odd number $2m+1$ of consecutive positive integers, for some integer $m \geq 1$ such that the sum of the squares of the smallest $m+1$ integers is equal to the sum of the squares of the largest m integers.

Solution. Let $S = \{a, a+1, \ldots, a+2m\}$ be the set with integers a, m and $m \geq 1$. Let $s_k = a^2 + (a+1)^2 + \cdots + (a+k)^2$, then

$$\begin{aligned} s_k &= a^2 + (a+1)^2 + \cdots + (a+k)^2 = (k+1)a^2 + 2a(1+2+\cdots+k) + (1^2+2^2+\cdots+k^2) \\ &= (k+1)a^2 + k(k+1)a + \frac{k(k+1)(2k+1)}{6}. \end{aligned}$$

According to the given condition:

$$\begin{aligned} &a^2 + (a+1)^2 + \cdots + (a+m)^2 = (a+m+1)^2 + \cdots + (a+2m)^2 \\ \Rightarrow &2\left(a^2 + (a+1)^2 + \cdots + (a+m)^2\right) = a^2 + (a+1)^2 + \cdots + (a+2m)^2 \Rightarrow 2s_m = s_{2m} \\ \Rightarrow &2(m+1)a^2 + 2m(m+1)a + \frac{m(m+1)(2m+1)}{3} = (2m+1)a^2 + (2m)(2m+1)a + \frac{(m)(2m+1)(4m+1)}{3} \\ \Rightarrow &a^2 - 2am^2 - m^2(2m+1) = 0 \Rightarrow (a+m)\left(a - m(2m+1)\right) = 0 \Rightarrow a = m(2m+1) \end{aligned}$$

Thus $\boxed{S = \{m(2m+1), m(2m+1)+1, \ldots, m(2m+1)+2m\}.}$ □

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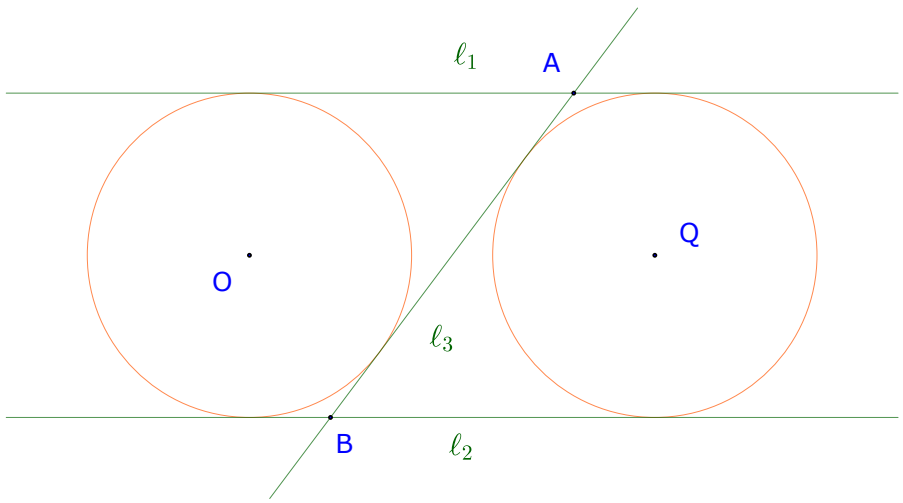
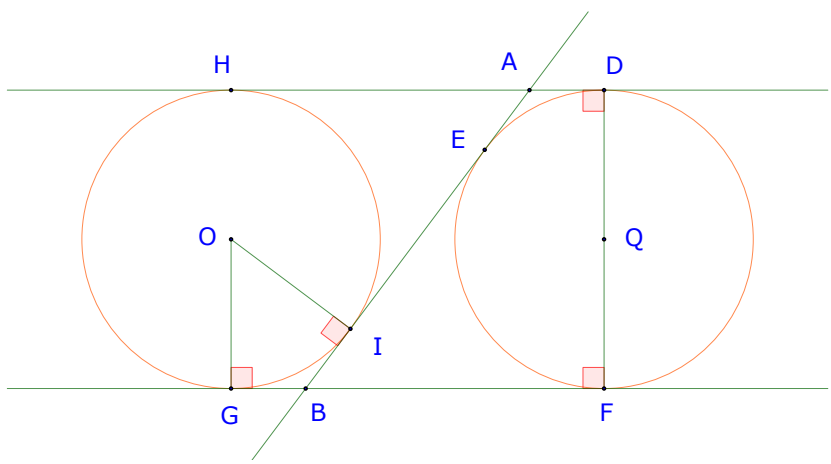
Problem 1.5.7 (Crux Mathematicorum/MA 190)**.** (*10 points*) Given two parallel lines ℓ_1 and ℓ_2, the transversal ℓ_3 intersects them at points A and B respectively. Two circles with centres at O and Q lie between the parallel lines on the left and on the right sides of the transversal such that the circles are tangent to all three lines.

Show that $OQ = AB$.



Solution. [Solution 1] Let D, E, F, G, H, I be the tangent points as shown in the diagram below.



Now,

$$\begin{aligned} &DH = DA + AH = AE + AI = 2AE + EI \\ &GF = GB + BF = BI + IE = 2BI + EI \end{aligned} \quad DH = GF \Rightarrow BI = AE$$

$$\Rightarrow AB = AE + EI + IB = 2AE + EI = DH \Rightarrow \boxed{AB = OQ.}$$

□

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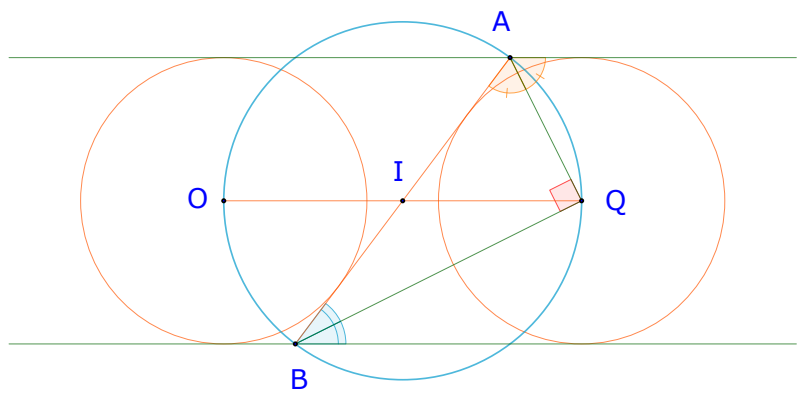
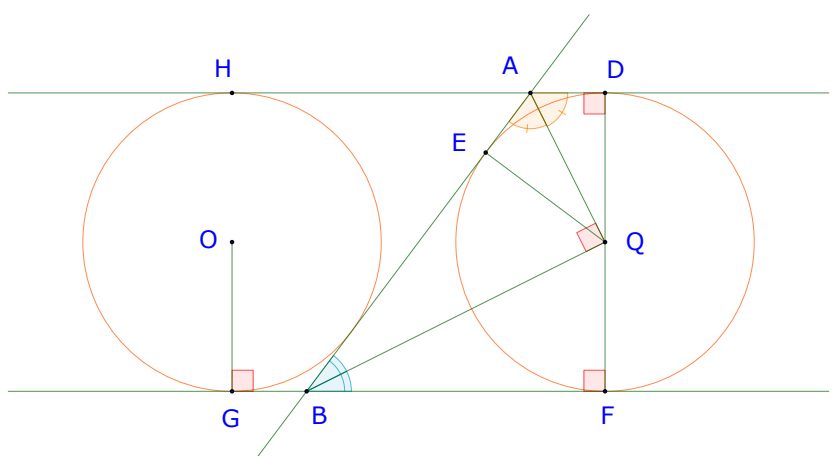
Solution. [Solution 2] Note that A is a point ot two tangents to the circle (Q), so AQ is the angle bisector. Similarly BQ is also an angle bisector as shown below.



Since the two angles are A and B sum up to $180°$, thus half of their sum is $90°$, which means that AQB is $90°$, or Q is on the circle with diameter AB. Similarly O is also on that circle.

Then $IO = IA = IB = IQ$ Note that $\angle IQA = \angle IAQ$, thus IQ parallel to ℓ_1, or I on the segment and is the midpoint of OQ. Therefore $OQ = OI + IQ = AB$. □

Solution. [Solution 3] Let D, E, F, G, H be the tangent points as shown in the diagram above. Let r be the radius of the circles.



Because of the symmetry, $AD = BG$, so $AD + BF = OQ$.

On the other hand, $\angle EAQ = \angle DAQ$, $\angle EBQ = \angle FBQ$, so $\angle EAQ + \angle EBQ = 90°$. Thus $\triangle AQB$ is a right triangle, thus $AB^2 = AQ^2 + BQ^2 = AD^2 + DQ^2 + QF^2 + BF^2$.

Since $\triangle ADQ \sim \triangle QFB$ (angle-angle-angle), so $\frac{AD}{DQ} = \frac{QF}{FB}$, or $AD \cdot BF = DQ \cdot QF = r^2$

Therefore, $AB^2 = AD^1 + BF^2 + 2AD \cdot BF = (AD + BF)^2 = OQ^2$. Hence $\boxed{OQ = AB.}$ □

Problem 1.5.8 (29th Nordic Mathematical Contest 2015/Problem 2)**.** (*10 points*) Find the primes p, q, and r, given that one of the number pqr and $p+q+r$ is 101 times the other.

Solution. [Solution 1] Since the product of two of p, q, r is at least 4, so the product pqr is at least four times the largest of p, q, and r, and therefore larger then $p+q+r$.

Thus,

$$pqr = 101(p+q+r).$$

Now, since 101 is a prime, so one of p, q, r must be 101. WLOG, let $p = 101$, then

$$qr = 101 + q + r \Rightarrow q = 1 + \frac{102}{r-1}.$$

Therefore $r-1$ is a divisor of $102 = 2 \cdot 3 \cdot 17$. Since r is prime, so $r = 2, q = 103$ or $r = 103, q = 2$. Thus the three primes p, q, and r are $\boxed{2, 101, \text{ and } 103.}$ □

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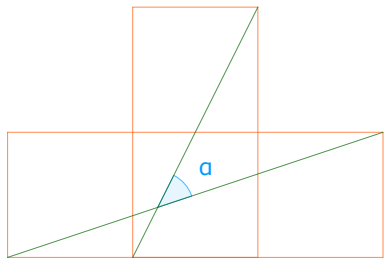
1.6 Problems for Olympiad Level

Problem 1.6.1 (23-24-ET-O-P1)**.** (*10 points*) The figure below consists of four congruent squares. Find the angle α.



Problem 1.6.2 (23-24-ET-O-P2)**.** Find the largest possible positive integer n such that $n+3$ divides $1^3+2^3+\cdots+n^3$.

Hint 1: Prove that if a prime number p divides $n+3$ and $n+3$ divides $1^3+2^3+\cdots+n^3$, then $p \in \{2,3\}$.

Hint 2: Analyze the polynomial division $1^3+2^3+\cdots+n^3$ by $n+3$.

Problem 1.6.3 (23-24-ET-O-P3)**.** (*10 points*) Show that if a 5×5 matrix is filled with zeros and ones, there must always be a 2×2 sub-matrix (that is, the intersection of the union of two rows with the union of two columns) consisting entirely of zeros or entirely of ones.

Hint: prove that if we don't have a 2×2 sub-matrix of zeroes, we must have a 2×2 sub-matrix of ones.

Problem 1.6.4 (23-24-ET-O-P4)**.** (*10 points*) The sequence (a_n) is defined by

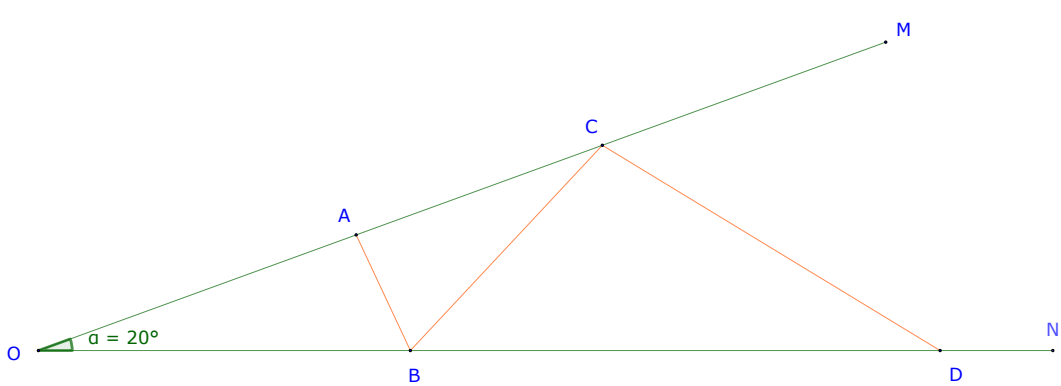
$$a_1 = 1,\ a_n = \sqrt{2a_{n-1}+a_{n-2}+\cdots+a_1},\ \text{if } n>1.$$

Find a_{2023}. *Hint: compute a_2, a_5.*

Problem 1.6.5 (23-24-ET-O-P5)**.** (*10 points*) Points A, C are on ray OM, and B, D are on ray ON. It is given that $OA = 6, OD = 16, \angle NOM = 20°$.

What is the minimum length of $AB + BC + CD$?

Hint: when is the length of a broken line minimal?



Problem 1.6.6 (23-24-ET-O-P6)**.** (*10 points*) Find all triples of non-negative integers (x, y, z) and $x \leq y$ such that:

$$x^2 + y^2 = 3 \cdot 2016^z + 77.$$

Hint: if $x^2 + y^2 \equiv 0 \pmod 7$, *then what are remainders of* x *and* y *when divided by 7?*

Problem 1.6.7 (23-24-ET-O-P7)**.** (*10 points*) There are 2023 points in the plane such that among any three of them two can be selected so that their distance is less than 1. Prove that there is a circle of radius 1 containing at least 1012 of the given points.

Problem 1.6.8 (23-24-ET-O-P8)**.** (*10 points*) Determine all polynomials $P(x)$ with real coefficients such that

1. $P(2023) = 2022$,
2. $(P(x) + 1)^2 = P(x^2 + 1)$, for all real numbers x.

Investigate the sequence (x_n) *where* $x_1 = 2023$, $x_{n+1} = x_n^2 + 1$.

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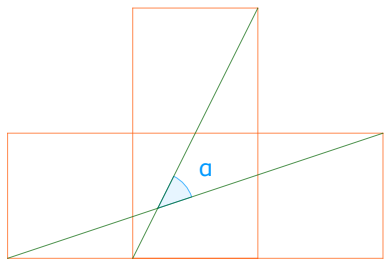
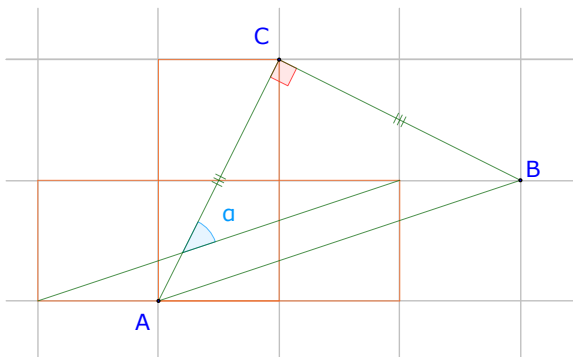
1.7 Solutions for Olympiad Level

Problem 1.7.1 (2016 Kansas MAA Undergraduate Mathematics Competition/Problem 2)**.** (*10 points*) The figure below consists of four congruent squares. Find the angle α.

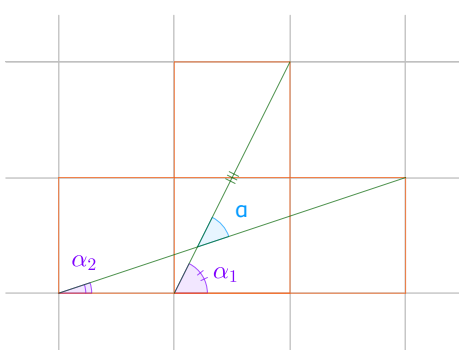


Solution. [Solution 1] It is easy to see that the segments are diagonals of a 1×2 and a 1×3 rectangles. Draw line AB parallel with the longer diagonal as show.



Since $AC^2 + CB^2 = (1^2 + 2^2) + (2^2 + 1^2) = 10 = 1^2 + 3^2 = AB^2$, so $\triangle ABC$ is an isosceles right triangle. Hence $\alpha = \angle CAB = \boxed{45°.}$ □

Solution. [Solution 2] Because the squares are congruent, the steeper line has a slope of 2, which implies that the angle of inclination, α_1, of the steeper line satisfies $\tan \alpha_1 = 2$. On the other hand, the shallower line has a slope of $\frac{1}{3}$, so the angle of inclination, α_2, of the shallower line satisfies $\tan \alpha_2 = \frac{1}{3}$.



Thus, $\alpha = \alpha_1 - \alpha_2 = \tan^{-1}(2) - \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{2-\frac{1}{3}}{1+2\cdot\frac{1}{3}}\right) = \tan^{-1}(1) = \boxed{45°.}$ □

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Problem 1.7.2 (Santa Clara University High School Mathematics 2001 Contest/Problem 4)**.** Find the largest possible positive integer n such that $n+3$ divides $1^3+2^3+\cdots+n^3$.

Hint: Prove that if a prime number p divides $n+3$ and $n+3$ divides $1^3+2^3+\cdots+n^3$, then $p \in \{2,3\}$.

Solution. [Solution 1] Note that

$$1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}n^2(n+1)^2.$$

For the first question $n+3 \mid 1^3+2^3+\cdots+n^3 \Rightarrow 4(n+3) \mid n^2(n+1)^2$. Thus if $p \mid n+3$, then $p \mid n$ or $p \mid n+1$.

Case 1: if $p \mid n$ then $p \mid (n+3)-n = 3$, thus $p = 3$.

Case 2: if $p \mid n+1$ then $p \mid (n+3)-(n+1) = 2$, thus $p = 2$.

Therefore $p \in \{2,3\}$.

Now, for the second question, from the result of the first question, since any prime factor of $n+3$ can only be 2 or 3, thus there exist k, l non-negative integers such that: $n+3 = 2^k3^\ell$.

$$4(n+3) \mid n^2(n+1)^2 \Rightarrow 2^{k+2}3^\ell \mid (2^k3^\ell-3)^2(2^k3^\ell-2)^2.$$

Let $P = (2^k3^\ell-3)^2(2^k3^\ell-2)^2$.

Case 1: if $k \geq 2$, then $2^k3^\ell-3 \equiv 1 \pmod 2$, $(2^k3^\ell-2)^2 = 2^2(2^{k-1}3^\ell-1)^2$, and $2^{k-1}3^\ell-1 \equiv 1 \pmod 2$. This means that the largest power of 2 that divides P is 2^2, but $2^4 \mid 2^{k+2}3^\ell \mid P$, which is a contradiction.

Case 2: if $\ell \geq 3$, then $(2^k3^\ell-3)^2 = 3^2(2^k3^{\ell-1}-1)^2$, where $2^k3^{\ell-1}-1 \equiv 2 \pmod 3$, and $2^k3^\ell-2 \equiv 1 \pmod 3$. Thus the largest power of 3 that divides P is 3^2, but $3^3 \mid 2^{k+2}3^\ell \mid P$, which is a contradiction.

Therefore $k \leq 1$ and $\ell \leq 2$, thus the largest possible value of n is $n = 2^13^2 = 18$, or $n = \boxed{15.}$ □

Solution. [Solution 2] Note that

$$1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}n^2(n+1)^2.$$

Thus

$$n+3 \mid \frac{1}{4}n^2(n+1)^2 \exists k \in \mathbb{Z}^+ : \ 4k = \frac{n^2(n+1)^2}{n+3} = n^3-n^2+4n-12+\frac{36}{n+3}.$$

Therefore $n+3 \mid 36$, or $n \in \{1,3,6,8,15,33\}$. It is easy to test that only $n \in \{3,6,15\}$ results in positive integer k. Thus the larges n is $\boxed{15.}$ □

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Problem 1.7.3 (2014 Science Atlantic Math Contest/Problem 3)**.** (*10 points*) Show that if a 5×5 matrix is filled with zeros and ones, there must always be a 2×2 sub-matrix (that is, the intersection of the union of two rows with the union of two columns) consisting entirely of zeros or entirely of ones.

Solution. [Solution 1] Let call a column 0-type if it has at least 3 zeros, 1-type if it has least 3 ones. Note that if a column is 0-type then it is not 1-type and vice versa. Since there are 5 columns in the given 5×5 matrix, thus at least 3 of them are 0-type or 1-type.

WLOG, lets assume that A, B, and C columns are 0-type. Let call a pair of columns a *perfect-match* if both of them having two same rows where the zeros are, in other words these four zeros form a 2×2 sub-matrix consisting entirely of zeros. Let call a pair of columns *semi-match* if both of them having at most one same row where the zeros are,

Case 1: If any of the three pairs of (A, B), (B, C), (C, A) is a perfect-match, then we have the desired answer.

Case 2: Now, each of the three pairs of (A, B), (B, C), (C, A) is a semi-match. Lets (A, B) share row 1. Let the other two rows containing zeros of A be row 2 and row 3. Then the other two rows containing zeros of B be row 3 and row 4. Two of three rows containing zeros of C must be in one of the two groups of rows $(1, 2, 3)$ or $(1, 3, 4)$. It means that (C, A) or (C, B) is a perfect-match. □

Solution. [Solution 2] Suppose that a 5×5 matrix of zeroes and ones contains no 2×2 sub-matrix of zeroes. We will show that it must contain a sub-matrix of ones.

Let A be the column with the fewest ones, or one of those columns if there are several with the same smallest number of ones, and let n be the number of ones in column A.

Case 1 : $n = 0$. In this case, no other column can contain more than one zero, since otherwise that column and column A would share a 2×2 sub-matrix of zeroes. Then any two columns other than A must each have at least 4 ones, so they share at least three rows of ones. Thus, there are 2×2 sub-matrices of ones using any pair of columns besides A.

Case 2 : $n = 1$. If column A contains only one one, then it has zeroes in 4 rows. In those four rows, no other column will contain a pair of zeroes, so each of the other columns must have at least 3 ones in those four rows. Therefore any two columns (excluding A) will have ones in two of the same rows, so again we have a 2×2 sub-matrix of ones.

Case 3 : $n = 2$. Each of the other columns must contain at least two ones in the rows in which A has zeroes. If any column has three ones in these rows, then it will share a 2×2 sub-matrix of ones with any of the others. If not, each of these four column has two ones among the three rows. But there are only $\binom{3}{2} = 3$ ways to do this, so two columns must have the same arrangement. Then these two contain a 2×2 sub-matrix of ones.

Case 4 : $n \geq 3$. In this case, each column has at least three ones. So, given any two columns, they either share two rows with ones, or between them have at least one one in each row. The first arrangement gives us the 2×2 sub-matrix of ones that we are looking for. In the second arrangement, any third column must share two rows of ones with one of the original two columns, again giving us a 2×2 sub-matrix of ones.

In every case, if we don't have a 2×2 sub-matrix of zeroes, we must have a 2×2 sub-matrix of ones. □

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Problem 1.7.4 (2017 Bulgaria Math Olympiad, Grade 10, Second Round/Problem 3)**.** (*10 points*) The sequence (a_n) is defined by

$$a_1 = 1,\ a_2 = 1,\ a_n = \left\lfloor \sqrt{2a_{n-1} + a_{n-2} + \cdots + a_1} \right\rfloor,\ \text{if } n > 1.$$

Find a_{2023}. *Hint: compute* a_2, a_5.

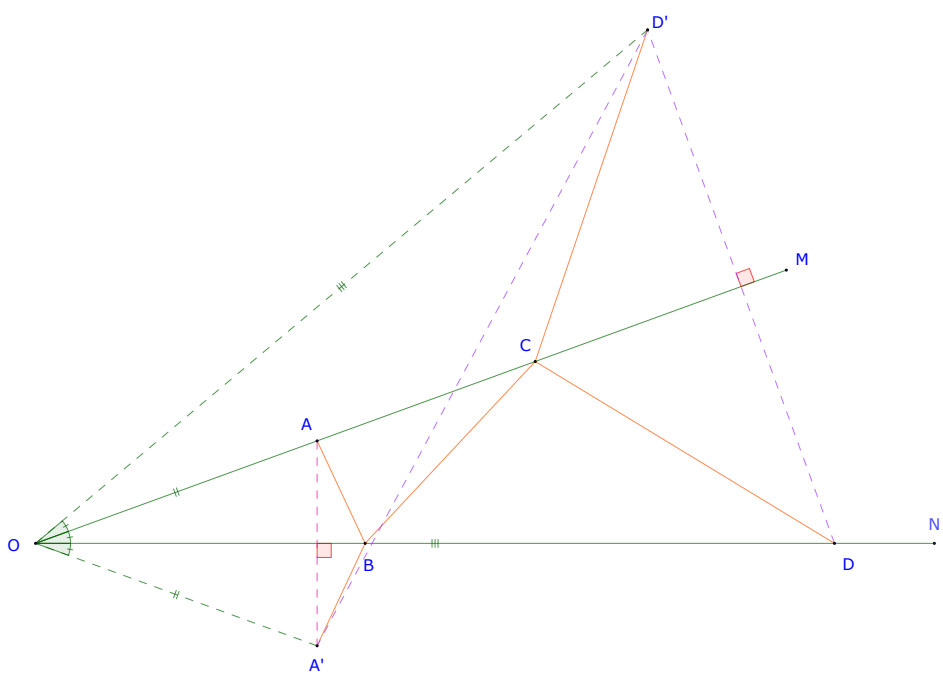
Solution. We prove that $a_{2n+1} = a_{2n} = n,\ \forall n \geq 1$. First, it is easy to verify that $a_2 = \left\lfloor \sqrt{2+1} \right\rfloor = \left\lfloor \sqrt{3} \right\rfloor = 1$. Now, let $a_{2n+1} = a_{2n} = n,\ \forall n \geq 1$.

$$\begin{aligned}
a_{2n+2} &= \left\lfloor \sqrt{2a_{2n+1} + a_{2n} + \cdots + a_1} \right\rfloor = \left\lfloor \sqrt{2n + n + 2(n-1) + 2(n-2) + \cdots + 2\cdot 1 + 1} \right\rfloor \\
&= \left\lfloor \sqrt{3n + (n-1)n + 1} \right\rfloor = \left\lfloor \sqrt{n^2 + 2n + 1} \right\rfloor = n + 1 \\
a_{2n+3} &= \left\lfloor \sqrt{2a_{2n+2} + a_{2n} + 1 + \cdots + a_1} \right\rfloor = \left\lfloor \sqrt{2(n+1) + 2n + 2(n-1) + \cdots + 2\cdot 1 + 1} \right\rfloor \\
&= \left\lfloor \sqrt{2(n+1) + n(n+1) + 1} \right\rfloor = \left\lfloor \sqrt{(n+1)(n+2) + 1} \right\rfloor = n + 1
\end{aligned}$$

Note that $(n+1)^2 = n^2 + 2n + 1 < n^2 + 3n + 3 = (n+1)(n+2) + 1 < n^2 + 4n + 4 = (n+2)^2$. Therefore $a_{2023} = \boxed{1011.}$ □

Problem 1.7.5 (2019 South Africa, Durban, Invitational World Youth Mathematics Intercity Competition, Team Contest/Problem 3)**.** (*10 points*) Points A, C are on ray OM, and B, D are on ray ON. It is given that $OA = 6, OD = 16, \angle NOM = 20°$.

What is the minimum length of $AB + BC + CD$? *Hint: when is the length of a broken line minimal?*



Solution. Let A' be the reflection of A over ON, and D' be the reflection of D over OM. It is easy to see that $AB + BC + CD = A'B + BC + CD' \geq A'D'$. Now $\angle A'OB = \angle BOA = \angle AOD' = 20° \Rightarrow \angle A'OD' = 60°$.

Thus, $(A'D')^2 = A'O^2 + D'O^2 - 2A'O \cdot D'O \cdot \cos 60° = 6^2 + 16^2 - 2\cdot 6\cdot 16\cdot \frac{1}{2} = 196$, hence $A'D' = \boxed{14.}$ □

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Problem 1.7.6 (Crux Mathematicorum/MA 136)**.** (*10 points*) Find all triples of non-negative integers (x, y, z) and $x \leq y$ such that:

$$x^2 + y^2 = 3 \cdot 2016^z + 77.$$

Hint: if $x^2 + y^2 \equiv 0 \pmod 7$, *then what are remainders of* x *and* y *when divided by 7?*

Solution. Let consider three cases.

Case 1: $z = 0$. Then $x^2 + y^2 = 80$. Since $x \leq y$, it is easy to verify that $x = 4, y = 8$.

Case 2: $z > 1$. Then $2016^z \equiv 77 \equiv 0 \pmod 7$, and $2016^z \equiv 0 \pmod{7^2}$, but $77 \not\equiv 0 \pmod{7^2}$.

Case 3: $z = 1$. It is easy to see that for any integer x, $x^2 \equiv 0, 1, 2, 4 \pmod 7$, thus if $x^2+y^2 \equiv 0 \pmod 7$ then the only possibility is $x \equiv y \equiv 0 \pmod 7$. From this claim, let $x = 7x_1, y = 7y_1$, then $x_1^2+y_1^2 = 125$. Since $x \leq y$, so $x_1 \leq y_1$, and there are two solutions $(x_1, y_1) \in \{(2, 11),\ (5, 10)\}$. Therefore $(x, y) \in \{(14, 77),\ (35, 70)\}$.

The answers are $\boxed{(4, 8, 0),\ (14, 77, 1),\ (35, 70, 1).}$ □

Problem 1.7.7 (2017 Czech-Slovakia Math Olympiad, Category B, Regional Round/Problem 2)**.** (*10 points*) There are 2023 points in the plane such that among any three of them two can be selected so that their distance is less than 1. Prove that there is a circle of radius 1 containing at least 1012 of the given points.

Solution. [Solution 1] Let A be one of the points.

Case 1: if there is no such point B among the rest of the points such that $AB \geq 1$, then all of them are within a circle radius 1 centred at A. Of these, there exist at least 1012 of the given points that are within a circle of radius 1.

Case 2: if there is a point B such that $AB \geq 1$. Then for any point C of the remaining 2021 points: $CA < 1$ or $CB < 1$.

Continue this process, by the Pigeonhole principle, there is at least $\left\lceil \frac{2021}{2} \right\rceil = 1011$ points $C_1, C_2, \ldots, C_{1011}$, all of which are within either a circle radius 1 centred at A or a circle radius 1 centred at B. In any case is a circle of radius 1 containing at least 1012 of the given points. □

Solution. [Solution 2] Let A_1 be one of the points. By the Pigeonhole Principle, one of the two situations occur.

Case 1: There exists 1011 point $A_2, A_3, \ldots, A_{1012}$ such that

$$A_1A_2 \leq 1,\ A_1A_3 \leq 1,\ \ldots, A_1A_{1012} \leq 1.$$

Then the circle centred at A_1, radius 1 contain 1012 points $A_1, A_2, \ldots, A_{1012}$.

Case 2: There exists 1012 point $B_1, B_2, \ldots, B_{1012}$ such that

$$A_1B_1 \geq 1,\ A_1B_2 \geq 1,\ \ldots, A_1B_{1012} \geq 1.$$

Now, since among points A_1, B_1, B_2 we can select two so that their distance is less than 1, thus $B_1B_2 < 1$. Similarly $B_1B_3, \ldots, B_1B_{1012} < 1$.

Then the circle centred at B_1, radius 1 contain 1012 points $B_1, B_2, \ldots, B_{1012}$. □

Problem 1.7.8 (29th Nordic Mathematical Contest 2015/Problem 2)**.** (*10 points*) Determine all polynomials $P(x)$ with real coefficients such that

1. $P(2023) = 2022$,
2. $(P(x) + 1)^2 = P(x^2 + 1)$, for all real numbers x.

Investigate the sequence (x_n) *where* $x_1 = 2023$, $x_{n+1} = x_n^2 + 1$.

Solution. We prove the following claim

Claim — For the sequence (x_n) such that $x_1 = 2023$, $x_{n+1} = x_n^2 + 1$, then $P(x_n) = x_n - 1$, for all $n \in \mathbb{Z}^+$.

Proof. By Induction Principle, $x_1 = 2023$, then $P(x_1) = 2022 = x_1 - 1$. Lets assume that the induction hypothesis is correct for n. Then $P(x_{n+1}) = P(x_n^2 + 1) = (P(x_n) + 1)^2 = (x_n)^2 = x_{n+1} - 1$. ■

From the claim it is easy to see that there is an infinite sequence (x_n) where $P(x_n) - x_n + 1 = 0$, or the polynomial $Q(x) = P(x) - x + 1$ has infinite number of distinct roots. Thus $Q(x) \equiv 0$. This means that $P(x) - x + 1 = 0$, hence $\boxed{P(x) = x - 1.}$ □

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Chapter 2

Session 1

2.1 The Pigeonhole Principle (I-Level)

Definition (Pigeonhole Principle)**.** The Pigeonhole Principle (also known as the Dirichlet box principle, Dirichlet principle or box principle) states that if $n+1$ or more holes are placed in n pigeons, then one pigeon must contain two or more holes.

Another definition could be phrased as among any $n+1$ integers, there are two with the same remainder when divided by n.

The extended version of the Pigeonhole Principle states that if k objects are placed in n boxes then at least one box must hold at least $\left\lceil \frac{k}{n} \right\rceil$ objects. Here $\lceil x \rceil$ denotes the least integer larger then or equal to x.

Example 2.1.1 (23-24-S1-I-P1)

Prove that in a group of 13 students, there are at least two students have their birthday in the same month.

Solution. Let 13 students be the pigeons and 12 months be the holes. According to the Pigeonhole Principle, there are at least 2 students have their birthday in the same month. □

Example 2.1.2 (23-24-S1-I-P2)

Prove that in a group of 32 students, there are at least two students have their birthday in the same day in some month.

Solution. Let 32 students be the pigeons and 31 days $1, 2, \ldots, 31$ be the holes. According to the Pigeonhole Principle, there are at least 2 students have their birthday the same day in some month. □

Example 2.1.3 (23-24-S1-I-P3)

The heights of 27 students in the MCC club were measured to the nearest 5cm. There was a range of heights from 150cm to 180cm. What is the least number of students with one of the heights?

Solution. For the third questions, the possible heights are $150, 155, 160, 165, 170, 175, 180$, or 7 heights. According to the Pigeonhole Principle, there are at least $\left\lceil \frac{27}{7} \right\rceil = 4$ students with one of the heights. □

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Example 2.1.4 (Mathematical Circle, Russian Experience, C4/Problem 8)

Show that in any group of five people, there are two who have the same number of friends in the group.

Proof. The number of friends a person can have in a group of five is $0, 1, \ldots, 4$. Note that it is impossible for a person to have 0 friend and another to have 4 friends at the same time. Thus there are only four possible numbers of friends among $0, 1, \ldots, 4$. Therefore by the Pigeonhole Principle, there are two out of five people having the same number of friends. □

Example 2.1.5 (LPS V2B/2.3)

10 people attended a party. Some people shake hand with others. Any person shakes hand with another only once. Nobody shakes hand with himself.

Prove that there are two people who shake hand with the same number of people.

Proof. We prove by contradiction. Lets assume that the opposite *each person has a different number of handshakes.* The numbers of handshakes should be 10 distinct numbers: $0, 1, 2, \ldots, 9$. (why there is no 10?)

Now, if there is a person with 0 handshakes (who shares hand with no one) then there would not be a a person with 9 handshakes (who shake hand with everybody) and vice versa. Thus there the numbers of handshakes can only be at most 9 distinct numbers among 10 numbers $0, 1, 2, \ldots, 9$. This is contradiction.

Hence, $\boxed{\text{there are two people who have the same number of handshakes.}}$ □

Example 2.1.6 (Mathematical Circle, Russian Experience, C4/Problem 11)

Ten students solved a total of 35 problems in a math olympiad. Each problem was solved by exactly one student. There is at least one student who solved exactly one problem, at least one student who solved exactly two problems, and at least one student who solved exactly three problem.

Prove that there is at least one student who solved exactly five problem.

Proof. By removing the three students who solved exactly one, two, and three problems, respectively, we have 7 students, who solved together $35 - 6 = 29$ problems.

By the Pigeonhole Principle, there is at least one student who solved at least $\left\lceil \frac{29}{7} \right\rceil = 5$ problems. If that student solves at least 6 problem, then together with each of the other 6 students (who solved at least 4 problems each) together they solve $6 + 6 * 4 = 30$ problems, contradiction.

Hence, there is $\boxed{\text{at least one student who solved exactly five problem.}}$ □

Example 2.1.7 (23-24-S1-I-P7)

(*10 points*) Prove that in any set of 52 integers, there exist two numbers, whose sum or whose difference is divisible by 100.

Solution. Let take a look at with the remainders when divided the numbers by 100. They are $0, 1, 2, \ldots, 99$. Lets pair them up into 51 pigeonholes: 49 remainder-pairs and two single-remainders as follow:

$$(1, 99),\ (2, 98),\ \ldots,\ (48, 52),\ (49, 51),\ (0),\ (50)$$

If we select 52 numbers from them, there would be two from the same hole. If both from a remainder-pair hole, then their difference is divisible by 100. If both from a single-remainder hole, then their sum is divisible by 100. □

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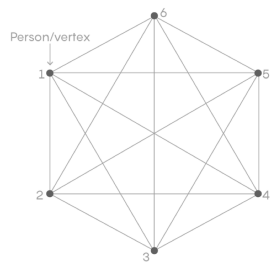
Example 2.1.8 (LPS Strategies and Methods/Exercise 2.17)

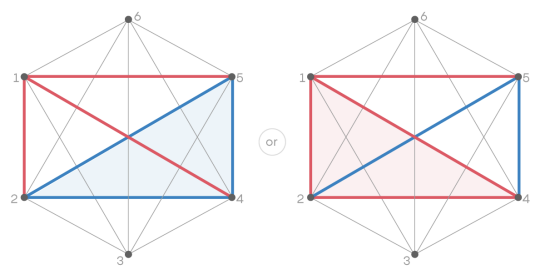
Among six persons, there are always three who know each other or three who are complete strangers.

Proof. First, let number the persons as $1, 2, \ldots, 6$, then assign them to the vertices of a hexagon shown below. We will colour the edge between two persons by red if they know each other and blue if they are strangers.



Let start with person 1. Because there are 5 other persons, by the Pigeonhole Principle, person 1 know 3 persons or don't know 3 of them.

Case 1: Lets assume that person 1 know 3 other persons. Let them be 2, 4, and 5. The left diagram below shows the three persons 2, 4, 5 connected with 1 by a red segment.



If none of the pairs $(2, 4)$,, $(2, 5)$, or $(4, 5)$ is connected by a red segment, then all of them are connected by blue segment, as show in the diagram on the left. In this case 2, 4, and 5 are complete strangers.

If one pair of $(2, 4)$,, $(2, 5)$, or $(4, 5)$ is connected by a red segment, say $(2, 4)$ then three persons who know each other are 1, 2, 4, as shown in the diagram on the right. In this case 1, 2, and 4 all know each other.

Case 2: Lets assume that person 1 don't know 3 other persons. Just follow the proof in the previous case, replace red with blue and you should have the proof. □

Example 2.1.9 (MTC 2021/Mar/C1)

Several solutions from MCC students were nominated for the Best Solution of the Month award. The evaluation committee has 10 members. Each of them secretly picked the best solution of his or her choice. It is known that out of any 4 committee members, at least 2 voted for the same solution. Prove that there exists a solution that was picked by at least 4 committee members.

Proof. First, if there were at least 4 solutions. Then since each member of the committee picked no more than one solution, thus we would have 4 committee members voting for 4 different solutions (each committee member for a distinct solution). This contradicts the fact that out of any 4 committee members, at least 2 voted for the same solution.

Therefore, we have 10 committee members and at most 3 solutions. By the Pigeonhole Principle, there is a solution voted by $\boxed{\text{at least 4 committee members.}}$ □

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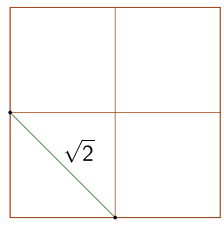
Example 2.1.10 (23-24-S1-I-P10)

Prove that for any 5 points chosen within a 2×2 square, there are two whose distance is at most $\sqrt{2}$.

Solution. First lets divide the 2×2 square into four unit squares (1×1) as shown in the diagram below. According to the Pigeonhole Principle, at least two chosen points must be in a single unit square. The largest



distance between any two points in a square is the length of the diagonal, which is $\boxed{\sqrt{2}.}$ □

Example 2.1.11 (LPS Strategies and Methods/Problem 2.31)

$3^4 = 8\underline{1}$ is a power of 3 ending in 1. $3^{20} = 3\ 486\ 784\ 4\underline{01}$ is a power of 3 ending in 01. Is there a power of 3 ending in 0001?

Solution. It is easy to see that none of 10000 numbers below is divisible by 2 or 5:

$$3^1,\ 3^2,\ \ldots, 3^{10^4}.$$

Since there are 9999 nonzero remainders $1, 2, \ldots, 9999$ when those 10000 numbers divided by 10000, by the Pigeonhole Principle, two of them 3^m and 3^n $(m > n)$ have the same remainder, thus their difference $3^m - 3^n$ is divisible by 10000.

Now $3^m - 3^n = 3^n(3^{m-n} - 1)$ and 3^n is relatively prime with 10000, so $3^{m-n} - 1$ is divisible by 10000, hence 3^{m-n} ends with 0001. □

Example 2.1.12 (LPS V2B/2.16)

A chess master has 77 days to prepare for a tournament He wants to play at least one game per day, but not more than 132 games. Prove that there is a sequence of successive days on which he plays exactly 21 games.

Proof. $a_1,\ \ldots,\ a_{77}$ are the accumulative number of games played until day 1, 2, ..., 77,

$$1 \le a_1 < a_2 < \ldots < a_{77} \le 132 \Rightarrow 22 \le a_1 + 21 < a_2 + 21 < \ldots < a_{77} + 21 \le 153$$

Now consider

$$a_1,\ a_2,\ \ldots,\ a_{77},\ a_1 + 21,\ a_2 + 21,\ \ldots,\ a_{77} + 21.$$

These are $77 + 77 = 154$ numbers between 1 and 153. By the Pigeonhole Principle, there must be two same numbers. Since all a_i are different, and all $a_i + 21$ are also different, there exist an a_i and $a_j + 21$ such that $a_i = a_j + 21$. It means the number of games on day i is 21 more than that number on day j.

Thus, $\boxed{\text{on the days } j+1, j+2 \ldots, i \text{ he plays exactly 21 games.}}$ □

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2.2 Problems (I-Level)

Submission deadline: September 29, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

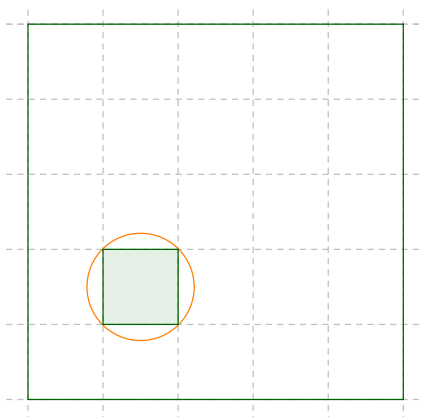
Problem 2.2.1 (23-24-S1-I-P13)**.** (*10 points*) Thirteen schools took part in an athletics competition in Super Athletic Day. There were 1514 students spectators. Show that there was one school that was cheered on by at least 117 students.

Proof. Let 1514 students be the pigeons and 13 schools be the holes. According to the Pigeonhole Principle, since $1514 = 116 \times 13 + 6$, thus there was one school that has at least $\left\lceil \frac{1514}{13} \right\rceil = \boxed{117}$ students in the event. Those are the ones cheering the team from that school. □

> **Remark.** All students correctly solved this problem.

Problem 2.2.2 (23-24-S1-I-P14)**.** (*10 points*) There are 101 points in a unit square. Prove that five of them can be covered by a circle radius $\frac{1}{7}$.

Proof. Lets divide the unit square into $5 \times 5 = 25$ smaller squares with side length $\frac{1}{5}$, as shown below.



Now, let 101 points be the pigeons and 25 smaller squares be the holes. According to the Pigeonhole Principle, there exists a smaller square where at least $\left\lceil \frac{101}{25} \right\rceil = 5$ points reside. The circle centered at the center of the square with radius $\frac{1}{7}$ cover the whole square since, since half of the diagonal of the square is $\frac{1}{2} \cdot \frac{\sqrt{2}}{5}$, which is less than $\frac{1}{7}$. □

> **Remark.** Only Minh Nhat Pham correctly solved this problem. Jeremy Nguyen has very good solutions, albeit not really cleanly presented. Khoa Anh Vu should be careful when comparing the computation result: a circle radius $\frac{1}{7}$ cannot cover a square side length $\frac{1}{4}$. This means that the decision to divide the square into 16 smaller squares is not good (enough). Giang Nguyen has a good approach, but the circles and gaps cannot be dealt with in the same way. Huy Quang Le might have had a good idea, but whatever was submitted was badly presented.

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Problem 2.2.3 (23-24-S1-I-P15)**.** (*10 points*) Show that given any set of 17 numbers it is possible to choose 5 numbers whose sum is divisible by 5.

Proof. The given 17 numbers when divided by 5 give the remainders of $0, 1, 2, 3$, and 4.

Case 1: If there are five numbers all give different remainders when divided by 5, then the sum of their remainders is $0 + 1 + 2 + 3 + 4 = 10$, which is divisible by 5.

Case 2: Now lets assume that these 17 numbers when divided by 5 give only four different remainders, Since $17 = 4 \times 4 + 1$, then there are at least five numbers when divided by 5 have the same remainder. Their sum then is divisible by 5. □

> **Remark.** Only Minh Pham, Giang Nguyen, Jérémy Nguyen, Chi-Mai Pham correctly solved this problem. Julie Cao solve one case of the problem.

Problem 2.2.4 (23-24-S1-I-P16)**.** (*10 points*) Seventeen people correspond by mail with one another-each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

Proof. Let a person among the seventeen people be P. By the Pigeonhole Principle, there is a topic, say A, among the given three topics, that at least $\left\lceil \frac{17}{3} \right\rceil = 6$ people discussed with P.

Case 1: If any two out of those 6 people also discuss A, then we are done.

Case 2: If not, let Q be a person from these 6 people. Q discussed only two other topics with the remaining 5 people, then by the Pigeonhole Principle, there is a topic, say B, among the two remaining topics, that is discussed by Q with at least $\left\lceil \frac{5}{2} \right\rceil = 3$ people.

If any two people out of those 3 people discussed B, then we are done. If not, then these 3 people only discussed C among themselves. □

> **Remark.** Only Minh Pham, Jérémy Nguyen correctly solved this problem. Julie Cao had some good ideas.

Problem 2.2.5 (23-24-S1-I-P17)**.** (*10 points*) A student has 37 days to prepare for an exam. From past experience she knows that she will require no more than 60 hours of study. She also wished to study at least one hour per day. Show that no matter how she organizes her study, there is a succession of days during which she studies exactly 13 hours. (Assume she works for a whole number of hours per day.)

Proof. Suppose that she studies a_i hours up to and including the ith day, then by the given conditions:

$$1 \leq a_1 < a_2 < \cdots < a_{37} \leq 60$$

Consider the sequence $a_1,\ a_2,\ \ldots,\ a_{37},\ a_1 + 13,\ a_2 + 13,\ \ldots,\ a_{37} + 13$.

Note that $1 \leq a_1 < a_2 < \cdots < a_{37} \leq 60$, and $14 \leq a_1 + 13 < a_2 + 13 < \ldots < a_{37} + 13 \leq 60 + 13 = 73$, thus the sequence consists of 74 terms, each term of the sequence is a positive integer from 1, 2, …, 73, thus by the Pigeonhole Principle, two terms are equal.

Since all a_i are different, and all $a_i + 13$ are also different, The two same number must be an a_i and an $a_j + 13$ such that $a_i = a_j + 13$. It means the number of hours she studies until day i is 13 more than that number on day j. □

> **Remark.** Only Minh Pham, Jérémy Nguyen correctly solved this problem. Julie Cao had some good ideas.

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Problem 2.2.6 (23-24-S1-I-P18)**.** (*10 points*) Prove that in any set of 27 odd numbers all less than 100, there is a pair of number whose sum is 102.

Proof. Let divide them into 26 holes: 24 holes containing two numbers and 2 holes containing a single number:

$$(1),\ (3,99),\ (5,97),\ \ldots,\ (49,53),\ (51).$$

Note that the sum of two numbers from any hole containing two numbers is 102. There are only 2 holes containing one number, thus while picking 27 numbers from these 26 holes, at east 25 numbers will be picked from 24 2-number hole. By the Pigeonhole Principle, at least two will be from the same 2-number hole, and their sum is 102. □

> **Remark.** Only Minh Pham, Jérémy Nguyen correctly solved this problem. Julie Cao had almost perfect approach, but she left out the number 1, that's why it was not enough for the pigeon selection.

Problem 2.2.7 (23-24-S1-I-P19)**.** (*10 points*) How many sets of 26 odd numbers all less than 100 can we choose such that no pair in any of these sets gives a sum of 102.

Solution. [Official] First, let consider the partition of the numbers 1, 2, ..., 99 in the solution of 23-24-S1-I-P18:

$$(1),\ (3,99),\ (5,97),\ \ldots,\ (49,53),\ (51),$$

To pick 26 numbers so that no pair adds up to 102, we should not pick any two numbers from the same 2-number hole. Since there are 26 holes, so we must pick the numbers 1, 51, and then one number from each of 24 holes that containing two numbers.

It means that for each 2-number hole, there are two ways to chose a number. Since there are 24 such holes, there are $\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{24} = \boxed{2^{24}}$ such choices. □

> **Remark.** Only Minh Pham, Jérémy Nguyen correctly solved this problem.

Solution (A great solution from Minh Nhat Pham with modifications for clarity) — Let $A = \{3, 5, 7, \ldots, 49\}$, $B = \{53, 55, 57, \ldots, 59\}$, and $C = \{1, 51\}$.

To select 26 numbers with no pair summing up to 102, both numbers from C must be selected and 24 numbers from both A and B. By selecting some numbers from A, for example if $3, 5 \in A$ are selected, then the numbers $97, 99$ must not be selected from B, thus $53, 57, \ldots, 93, 95 \in B$ are selected.

This means that the selection of some numbers, say k $(0 \le k \le 24)$ from A determines the selection of the rest $24 - k$ of the numbers from B. Therefore the number of possible ways is the number of ways to select k elements from A, or

$$S = \binom{24}{0} + \binom{24}{1} + \ldots + \binom{24}{24}$$

By the binomial theorem, the sum S is $\boxed{2^{24}}$.

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Problem 2.2.8 (23-24-S1-I-P20)**.** (*10 points*) Several football teams enter a tournament in which each team plays every other team exactly once. Show that at any moment during the tournament there will be two teams which have played, up to that moment, an identical number of games.

Proof. The problem looks complicated at first, in fact it is almost identical to the example LPS V2B/2.3.

Let the number of teams be n. The number of games a team can play at any moment is

$$0, 1, \ldots, (n-1).$$

Note that it is impossible for a team has played 0 game and another to have $(n-1)$ games at the same time. Thus there are only $n-1$ possible numbers of games among $0, 1, \ldots, (n-1)$. Therefore at any moment, by the Pigeonhole Principle, there are two out of n teams have played the same number of games. □

> **Remark.** Only Minh Pham, Jérémy Nguyen, and Chi-Mai Pham correctly solved this problem. Giang Nguyen should try to explain whatever you think, but don't write down your ideas in too short sentences, be as verbal as possible

Problem 2.2.9 (23-24-S1-I-P21)**.** (*10 points*) Prove that there is a set of 51 integers such that, there are no two numbers, whose sum or whose difference is divisible by 100.

Solution. The remainders when divided the numbers by 100 are $0, 1, 2, \ldots, 99$. Lets pair them up into 51 pigeonholes: 49 holes with 2 remainders and 2 holes with a single remainder as follow:

$$(1, 99),\ (2, 98),\ \ldots,\ (48, 52),\ (49, 51),\ (0),\ (50)$$

Now we select 51 numbers, one from each of the 2-remainder holes, and then the numbers 0 and 50. It is easy to see that there are no pair of numbers, from these 51 numbers, whose sum or whose difference is divisible by 100. □

Note that the problem 23-24-S1-I-p22 is a duplicate of 23-24-S1-I-p19, thus it is not considered.

> **Remark.** Only Jérémy Nguyen, Julie Cao correctly solved this problem.

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2.3 Factoring (A-Level)

Fact (Second-degree Identities)**.**

$$
\begin{aligned}
&a^2 - b^2 = (a+b)(a-b)\\
&(a+b)^2 = a^2 + 2ab + b^2\\
&(a-b)^2 = a^2 - 2ab + b^2\\
&(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca,\\
&a^2 + b^2 + c^2 - ab - bc - ca = \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2}\\
&ab + ac + bd + cd = (b+c)(a+d)
\end{aligned}
$$

Fact (Third-degree Identities)**.**

$$
\begin{aligned}
&a^3 - b^3 = (a-b)(a^2 + ab + b^2)\\
&a^3 + b^3 = (a+b)(a^2 - ab + b^2)\\
&(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\\
&(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\\
&a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)\\
&(a+b+c)^3 - (a^3 + b^3 + c^3) = 3(a+b)(b+c)(c+a)
\end{aligned}
$$

Fact.

$$
\begin{aligned}
&1 + a_1 = 1 + a_1\\
&1 + a_1 + a_2 + a_1a_2 = (1+a_1)(1+a_2)\\
&1 + a_1 + a_2 + a_3 + a_1a_2 + a_2a_3 + a_3a_1 + a_1a_2a_3 = (1+a_1)(1+a_2)(1+a_3)\\
&1 + (a_1 + a_2 + \ldots + a_n) + (a_1a_2 + \ldots + a_{n-1}a_n) + \ldots + a_1a_2\cdots a_n = (1+a_1)(1+a_2)\cdots(1+a_n)
\end{aligned}
$$

Fact (Difference of powers)**.**

$$
\begin{aligned}
&a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})\\
&a^{2n+1} + b^{2n+1} = (a+b)(a^{2n} - a^{2n-1}b + \ldots - ab^{2n-1} + b^{2n})
\end{aligned}
$$

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Example 2.3.1 (23-24-S1-A-P1)
Factor $x^4 - 3x^2y^2 + y^4$.

Solution. Let $S = x^4 - 3x^2y^2 + y^4 = (x^4 - 2x^2y^2 + 4y^2) - x^2y^2 = (x^4 - 2x^2y^2 + 4y^2) - (xy)^2$.

By the identity $(a-b)^2 = a^2 - 2ab + b^2$ for $a = x^2,\ b = y^2$:

$$x^4 - 2x^2y^2 + 4y^2 = (x^2)^2 - 2(x^2)(y^2) + (y^2)^2 = (x^2 - y^2)^2 \Rightarrow S = (x^2 - y^2)^2 - (xy)^2.$$

By the identity $a^2 - b^2 = (a+b)(a-b)$ for $a = x^2 - y^2,\ b = xy$:

$$(x^2 - y^2)^2 - (xy)^2 = (x^2 - y^2 - xy)(x^2 - y^2 + xy) \Rightarrow S = \boxed{(x^2 - y^2 - xy)(x^2 - y^2 + xy).}$$

□

Example 2.3.2 (23-24-S1-A-P2)
Factor $(x+y)(x-y) - 4(y+1)$.

Solution. Let $S = (x+y)(x-y) - 4(y+1) = x^2 - y^2 - 4y - 4 = x^2 - (y^2 + 4y + 4)$.

By the identity $(a+b)^2 = a^2 + 2ab + b^2$ for $a = y,\ b = 2$:

$$y^2 + 4y + 4 = (y+2)^2 \Rightarrow S = x^2 - (y+2)^2$$

By the identity $a^2 - b^2 = (a+b)(a-b)$ for $a = x,\ b = y+2$:

$$x^2 - (y+2)^2 = (x - y - 2)(x - y + 2) \Rightarrow S = \boxed{(x - y - 2)(x - y + 2)}$$

□

Example 2.3.3 (23-24-S1-A-P3)
Factor $4(x^2 + x - y^2) + 1$.

Solution. Let $S = 4(x^2 + x - y^2) + 1 = (4x^2 + 4x + 1) - (2y)^2$

By the identity $(a+b)^2 = a^2 + 2ab + b^2$ for $a = 2x,\ b = 1$:

$$4x^2 + 4x + 1) = (2x+1)^2 \Rightarrow S = (2x+1)^2 - (2y)^2$$

By the identity $a^2 - b^2 = (a+b)(a-b)$ for $a = 2x+1,\ b = 2y$:

$$S = (2x+1)^2 - (2y)^2 = \boxed{(2x - 2y + 1)(2x + 2y + 1).}$$

□

Example 2.3.4 (23-24-S1-A-P4)
Factor $x(x - 4y) + 4(y^2 - 1)$.

Solution.

$$x(x-4y)+4(y^2-1) = x^2-4xy+4y^2-4 = (x^2-2x(2y)+(2y)^2)-2^2 = (x-2y)^2-2 = \boxed{(x - 2y - 2)(x - 2y + 2).}$$

□

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Example 2.3.5 (23-24-S1-A-P5)
Factor $x^2 - y^2 + 2(x + 3y - 4)$.

Solution.

$$x^2 - y^2 + 2(x + 3y - 4) = (x^2 + 2x + 1) - (y^2 - 6y + 9) = (x - 1)^2 - (y - 3)^2 = \boxed{(x - y + 2)(x + y - 2).}$$

□

Example 2.3.6 (23-24-S1-A-P6)
Factor $x^3 + 9x^2 + 27x + 19$.

Solution. Let $S = x^3 + 9x^2 + 27x + 19$. Consider the identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ for $a = x,\ b = 3$:

$$x^3 + 9x^2 + 27x + 27 = (x)^3 + 3(x)^2(3) + 3(x)(3)^2 + (3)^3 \Rightarrow S = (x + 3)^3 - 8$$

Since $8 = 2^3$, consider the identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ for $a = x + 3,\ b = 2$:

$$S = (x + 3)^3 - 2^3 = (x + 3 - 2)((x + 3)^2 + (x + 3)(2) + 2^2) = \boxed{(x + 1)(x^2 + 8x + 19).}$$

□

Example 2.3.7 (23-24-S1-A-P7)
Factor $x^3 + 3x^2 + 3x - 7$.

Solution. Let $S = x^3 + 3x^2 + 3x - 7$. Consider the identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ for $a = x,\ b = 1$:

$$x^3 + 3x^2 + 3x + 1 = (x)^3 + 3(x)^2(1) + 3(x)(1)^2 + (1)^3 \Rightarrow S = (x + 1)^3 - 8$$

Since $8 = 2^3$, consider the identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ for $a = x + 3,\ b = 2$:

$$S = (x + 1)^3 - 2^3 = (x + 1 - 2)((x + 1)^2 + (x + 1)(2) + 2^2) = \boxed{(x - 1)(x^2 + 4x + 7).}$$

□

Example 2.3.8 (23-24-S1-A-P8)
Factor $(x - y)(x^2 + xy + y^2 + 3) + 3(x^2 + y^2) + 2$.

Solution. Let $S = (x - y)(x^2 + xy + y^2 + 3) + 3(x^2 + y^2) + 2$. Since $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ for $a = x, b = y$:

$$(x - y)(x^2 + xy + y^2 + 3) = (x^3 - y^3) + 3(x - y) = x^3 - y^3 + 3x - 3y$$

Thus $S = x^3 - y^3 + 3x - 3y + 3x^2 + 3y^2 + 2 = (x^3 + 3x^2 + 3x + 1) - (y^3 - 3y^2 + 3y - 1)$.

By the identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ for $a = x,\ b = 1$: $x^3 + 3x^2 + 3x + 1 = (x + 1)^3$.

By the identity $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ for $a = y,\ b = 1$: $y^3 - 3y^2 + 3y - 1 = (y - 1)^3$.

Hence $S = (x + 1)^3 - (y - 1)^3$ and by the identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ for $a = x + 1,\ b = y - 1$:

$$S = ((x + 1) - (y - 1))((x + 1)^2 + (x + 1)(y - 1) + (y - 1)^2) = \boxed{(x - y + 2)(x^2 + y^2 + xy + x - y + 1).}$$

□

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Example 2.3.9 (23-24-S1-A-P9)
Factor $a^4 + 4b^4$.

Solution. Let $S = a^4 + 4b^4$, then $S_1 = (a^2)^2 + (2b^2)^2$, thus we can add $2(a^2)(2b^2)$ to S_1 to complete the square

$$\begin{aligned} S = (a^2)^2 + (2b^2)^2 &= \left[(a^2)^2 + (2b^2)^2 + 2(a^2)(2b^2)\right] - 2(a^2)(2b^2) \\ &= (a^2 + 2b^2)^2 - (2ab)^2 = \boxed{(a^2 + 2b^2 - 2ab)(a^2 + 2b^2 - 2ab).} \end{aligned}$$

Note that this identity $a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 - 2ab)$ is called the Sophie Germain identity. □

Example 2.3.10 (23-24-S1-A-P10)
Factor $(x^2y + y^2z + z^2x) + (xy^2 + yz^2 + zx^2) + 2xyz$.

Solution. Let $S = (x^2y + y^z + z^2x) + (xy^2 + yz^2 + zx^2) + 2xyz$, note that if we group the terms x^2y, xy^2:

$$x^2y + xy^2 = (x + y)xy, \text{ so it has a term of } x + y.$$

The question is that can we group the rest of the terms to factor out $x + y$?

$$(y^2z + z^2x) + (yz^2 + zx^2) + 2xyz = z(x^2 + y^2 + 2xy) + z^2(x + y) = z(x + y)^2 + z^2(x + y) = (x + y)(z(x + y) + z^2).$$

Thus $S = (x + y)xy + (x + y)(z(x + y) + z^2) = (x + y)(xy + zx + zy + z^2) = \boxed{(x + y)(y + z)(z + x).}$ □

Example 2.3.11 (23-24-S1-A-P11)
Factor $(x - y)^3 + (y - z)^3 + (z - x)^3$.

Solution. Let $S_3 = (x-y)^3 + (y-z)^3 + (z-x)^3$. By the identity $(a+b+c)^3 - (a^3+b^3+c^3) = 3(a+b)(b+c)(c+a)$ for $a = x - y,\ b = y - z,\ c = z - x$, then

$$\begin{cases} a + b + c &= (x - y) + (y - z) + (z - x) = 0 \\ a + b &= (x - y) + (y - z) = x - z \\ b + c &= (y - z) + (z - x) = y - x \\ c + a &= (z - x) + (x - y) = z - y \end{cases}$$

Thus, $0 - S_3 = 3(x - z)(y - x)(z - y) \Rightarrow S_3 = \boxed{3(x - y)(y - z)(z - x).}$ □

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Example 2.3.12 (23-24-S1-A-P12)
Find all pairs (x, y) of integers satisfying

$$\begin{cases} x^2 + 11 = xy + y^4 \\ y^2 - 30 = xy \end{cases}$$

Solution. Note that if we add up the two equations, then we will have $x^2 + y^2$ on one side and $2xy$ on the other side, which means that we can somehow use the identity $x^2 + y^2 - 2xy = (x - y)^2$

$$x^2 + y^2 - 19 = 2xy + y^4 \Rightarrow (x - y)2 - y^4 = 18$$

But $y^4 = (y^2)^2$, so by the identity $a^2 - b^2 = (a - b)(a + b)$ for $a = x - y,\ b = y^2$:

$$19 = (x - y)2 - (y^2)^2 = ((x - y) - y^2)((x - y) + y^2) \Rightarrow (y^2 - (x - y))(y^2 - (x - y)) = -19$$

Since -19 can be factor as $(-1)\cdot 19$, $1\cdot(-19)$, $(-19)\cdot 1$, or $19\cdot(-1)$, but $y^2-(x-y)+y^2-(x-y) = 2y^2 \geq 0$ means that the sum of two factors must be positive, thus $(-1, 19)$, or $(19, -1)$ must be the pair. Hence

$$2y^2 = 19 + (-1) = 18 \Rightarrow y = \pm 3 \Rightarrow x = \mp 7 \Rightarrow (x, y) \in \boxed{\{(-7, 3),\ (7, -3)\}.}$$

□

Example 2.3.13 (23-24-S1-A-P13)
Prove that for any positive integer n, $(n + 1)^5 + n$ is not a prime.

Solution. Let $m = n + 1$, then

$$S = (n + 1)^5 + n = m^5 + m - 1$$

By the identity $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ for $a = m, b = 1$:

$$\begin{aligned} S = m^5 + m - 1 &= (m^5 + m^2) - (m^2 - m + 1) = m^2(m^3 + 1) - (m^2 - m + 1) \\ &= m^2(m + 1)(m^2 - m + 1) - (m^2 - m + 1) = (m^2 - m + 1)(m^3 + m - 1) \end{aligned}$$

It is easy to see now that both factor of S are larger then 1

$$\begin{aligned} m^2 - m + 1 &= (n + 1)^2 - (n + 1) + 1 = n^2 + n + 1 > 1 \\ m^3 + m - 1 &= n^3 + 3n^2 + 4n + 1 > 1 \end{aligned}$$

Hence , $\boxed{S \text{ cannot be a prime.}}$ □

Example 2.3.14 (23-24-S1-A-P14)
Factor $a^3 + b^3 + c^3 - 3abc$.

Solution. Here's the idea: let $P(x) = x^3 - 3xbc + b^3 + c^3$. The given expression factorable if and only if $P(x)$ has roots. Let try a few value for x such as $0, -b, -c, -(b+c)$:

$$P(0) = b^3 + c^3,\ P(-b) = c^3 + 3b^2c,\ P(-c) = b^3 + 3b^2c$$
$$P(-b-c) = -b^3 - 3b^2c - 3bc^2 - c^2 + 3(b+c)bc = 0(!)$$

Thus, $P(x)$ has a factor (divisible by) $x + b + x$ or $a + b + c$ is a factor of $a^3 + b^3 + c^3 - 3abc$. Now

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a^3 + (b+c)^3) - 3b^2c^2 - 3bc^2 + 3abc \\ &= (a+b+c)(a^2 - a(b+c) + (b+c)^2) - 3bc(a+b+c) \\ &= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \end{aligned}$$

□

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2.4 Problems (A-Level)

Submission deadline: September 29, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 2.4.1 (23-24-S1-A-P15)**.** (*10 points*) Simplify the expression

$$\frac{1}{x-1} - \frac{3}{x^3-1}.$$

Solution.

$$\begin{aligned}\frac{1}{x-1} - \frac{3}{x^3-1} &= \frac{1}{x-1} - \frac{3}{(x-1)(x^2+x+1)} = \frac{1}{x-1}\left(1 - \frac{3}{x^2+x+1}\right) \\ &= \frac{x^2+x-2}{(x-1)(x^2+x+1)} = \frac{(x-1)(x+2)}{(x-1)(x^2+x+1)} = \boxed{\frac{x+2}{x^2+x+1}.}\end{aligned}$$

□

> **Remark.** Minh Nguyen, Ryan Nguyen, Samuel Luong, Karl Le, Tran Hieu Khoa, An Nguyen, Minh Nhat (Henry) Trinh, Jason To, Dao Antoine correctly solved this problem.

Problem 2.4.2 (23-24-S1-A-P16)**.** (*10 points*) Simplify the expression

$$\frac{x^4 - 2x^3 - x^2 + 2x + 1}{x^4 - 2x^3 + x^2 - 1}.$$

Hint: factor the denominator, then see of the numerator has such a factor

Solution. Lets take a look at the denominator:

$$x^4 - 2x^3 + x^2 - 1 = x^2(x^2 - 2x + 1) - 1 = (x(x-1))^2 - 1 = (x^2 - x - 1)(x^2 - x + 1).$$

The idea is to factor the numerator to have at least one of the factors of the denominator, in this case $x^2 - x - 1$:

$$\begin{aligned}x^4 - 2x^3 - x^2 + 2x + 1 &= (x^4 - x^3 - x^2) - (x^3 - x^2 - x) + (-x^2 + x + 1) \\ &= x^2(x^2 - x - 1) - x(x^2 - x - 1) + (-x^2 + x + 1) = (x^2 - x - 1)^2 \\ \Rightarrow \frac{x^4 - 2x^3 - x^2 + 2x + 1}{x^4 - 2x^3 + x^2 - 1} &= \boxed{\frac{x^2 - x - 1}{x^2 - x + 1}.}\end{aligned}$$

□

> **Remark.** Only Samuel Luong, Karl Le, Tran Hieu Khoa, Minh Hai Vu, An Nguyen, Minh Nhat (Henry) Trinh, Jason To correctly solved this problem. Minh Nguyen did not show how to factor the numerator. Dao Antoine was unfortunate to figure out that:
>
> $(x^2-x)(x^2-x+1)-3(x^2-x+1)+4 = (x^2-x+1)(x^2-x+1)-4(x^2-x+1)+4 = (x^2-x+1-2)^2 = (x^2-x-1)^2$

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Problem 2.4.3 (23-24-S1-A-P17)**.** (*10 points*) Factoring the expression

$$2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4).$$

Solution. This is very similar to $2(ab + bc + ca) - (a^2 + b^2 + c^2)$, but that is not factorable. Lets try the technique in 23-24-S1-A-P14 to find a root of the polynomial:

$$P(X) = x^4 - 2x^2(b^2 + c^2) + (b^4 + c^4 - 2b^2c^2)$$

But this is a quadratic of x^2 (!). Thus

$$\begin{aligned}
&\Delta_P = 4(b^2 + c^2)^2 - 4(b^2 - c^2)^2 = 4^2b^2c^2 \Rightarrow x^2 = \frac{b^2 + c^2 \pm 2bc}{\Rightarrow} x = \pm(b \pm c) \\
&P(X) = (x - b - c)(x - b + c)(x + b - c)(x + b + c) \\
&\Rightarrow 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4) = \boxed{-(a - b - c)(a + b - c)(a + c - b)(a + b + c).}
\end{aligned}$$

□

> **Remark.** Only Minh Nguyen, Samuel Luong, and Tran Hieu Khoa correctly solved this problem. Karl Le was nearly able to fully factorize it. Ryan Nguyen some how stuck at the middle (should try the technique in the 23-24-S1-A-P14?)

Problem 2.4.4 (23-24-S1-A-P18)**.** (*10 points*) Factoring the expression

$$(x - a)^3(b - c)^3 + (x - b)^3(c - a)^3 + (x - c)^3(a - b)^3.$$

Hint: the given expression is very much like $a^3 + b^3 + c^3$

Solution. Note that the given expression is very much like $a^3 + b^3 + c^3$, thus we inspect the sum

$$(x - a)(b - c) + (x - b)(c - a) + (x - c)(a - b) = x(b - c + c - a + a - b) - a(b - c) - b(c - a) - c(a - b) = 0.$$

Therefore $(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3 = \boxed{3(x - a)(x - b)(x - c)(a - b)(b - c)(c - a).}$

□

> **Remark.** Only Minh Nguyen, Tran Hieu Khoa, An Nguyen, and Jason To correctly solved this problem. Karl Le was nearly able to fully factorize it. Ryan Nguyen some how stuck at the middle (should try the technique in the 23-24-S1-A-P14?)

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Problem 2.4.5 (23-24-S1-A-P19)**.** Let a, b, c be distinct real numbers such that

$$a^2(1-b+c)+b^2(1-c+a)+c^2(1-a+b)=ab+bc+ca.$$

(*5 points*) Prove that

$$(a-b)(b-c)(c-a)=(b-c)(c-a)+(c-a)(a-b)+(a-b)(b-c)$$

(*5 points*) Prove that

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}=1$$

Solution. The first question is straightforward:

$$\begin{aligned}&(a-b)(b-c)(c-a)=ab^2+bc^2+ca^2-a^2b-b^2c-c^2a\\&(b-c)(c-a)+(c-a)(a-b)+(a-b)(b-c)=-a^2-b^2-c^2+ab+bc+ca\end{aligned}$$

Since $a^2(1-b+c)+b^2(1-c+a)+c^2(1-a+b)=ab+bc+ca$, thus the desired equality follows.

For the second question, we have from the first question:

$$(a-b)(b-c)(c-a)=(b-c)(c-a)+(c-a)(a-b)+(a-b)(b-c)\Rightarrow\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}=1$$

Since

$$\frac{1}{a-b}\frac{1}{b-c}+\frac{1}{b-c}\frac{1}{c-a}+\frac{1}{c-a}\frac{1}{a-b}=\frac{(c-a)+(a-b)+(b-c)}{(a-b)(b-c)(c-a)}=0.$$

Therefore,

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}=\left(\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}\right)^2-2\left(\frac{1}{a-b}\frac{1}{b-c}+\frac{1}{b-c}\frac{1}{c-a}+\frac{1}{c-a}\frac{1}{a-b}\right)=1$$

□

> **Remark.** Only Minh Nguyen, Tran Hieu Khoa, Jason To, and Dao Antoine correctly solved this problem.

Problem 2.4.6 (23-24-S1-A-P20)**.** (*10 points*) Prove that

$$2012\cdot 503^{2011}+2013^4$$

is a composite number.

Solution. Since $2012=4\cdot 503$, thus $2012\cdot 503^{2011}+2013^4=2013^4+4(503^{503})^4$. According to the Sophie Germain identity:

$$2013^4+4(503^{503})^4=(2013^2+2(503^{503})^2-2\cdot 2013\cdot 503^{503})(2013^2+2(503^{503})^2+2\cdot 2013\cdot 503^{503}).$$

hence it is a composite number. □

> **Remark.** Only Minh Nguyen, Karl Le, Tran Hieu Khoa, and Jason To correctly solved this problem. Solutions Ryan Nguyen, Chi Khanh Pham are close, but not complete.

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Problem 2.4.7 (23-24-S1-A-P21)**.** (*10 points*) Prove that for all positive integers m, n,

$$8m^6 + 27m^3n^3 + 27n^6$$

is a composite number.

Hint: $8m^6 = (2m^2)^3$, $27n^6 = (3n^2)^3$.

Solution. Note that

$$8m^6 + 27m^3n^3 + 27n^6 = (2m^2)^3 + (-3mn)^3 + (3n^2)^3 - 3(2m^2)(3n^2)(-3mn)$$

According to the identity $^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$, for $a = 2m^2, b = -3mn, c = 3n^2$,

$$8m^6 + 27m^3n^3 + 27n^6 = (2m^2 - 3mn + 3n^2)(4m^4 + 9m^2n^2 + 3n^4 + 6m^3n + 9mn^3 - 6m^2n^2).$$

It is easy to see that the given expression is a composite number. □

Remark. Only Minh Nguyen correctly solved this problem. Tran Hieu Khoa got a good start.

Problem 2.4.8 (23-24-S1-A-P21)**.** (*10 points*) Solve the system of equations

$$\begin{cases} x + y = 2z \\ x^3 + y^3 = 2z^3 \end{cases}$$

Hint: $x + y$ is a factor of $x^3 + y^3$

Solution. [Official solution]

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) = 2z((x + y)^2 - 3xy) = 2z(4z^2 - 3xy) \Rightarrow 2z(4z^2 - 3xy) = 2z^3.$$

Case 1: $z = 0$, then $x + y = 0$, hence $(x, y, z) = (a, -a, 0)$, for a an arbitrary real number.

Case 2: $z \neq 0$, then $xy = z^2$. Therefore x, y are the roots of $t^2 - 2zt + z^2$, or $x = y = z$. Hence, $(x, y, z) = (a, a, a)$, for $a \neq 0$ an arbitrary real number. □

Remark. Only An Nguyen correctly solved this problem. Minh Nguyen, Ryan Nguyen, Samuel Luong, Karl Le, Tran Hieu Khoa, and Jason To solved one case of the problem.

Solution (A beautiful solution from An Nguyen with minor modifications) — Since $x + y = 2z$, by substituting $z = \frac{1}{2}(x + y)$,

$$4(x^3 + y^3) = (x + y)^3 \Rightarrow (x + y)(4x^2 - 4xy + 4y^2 - x^2 - 2xy - y^2) = 0$$
$$(x + y)(3x^2 - 6xy + 3y^2) = 0 \Rightarrow 3(x + y)(x - y)^2 = 0 \Rightarrow x = \pm y.$$

Hence, $\{(x, y, z)\} = \{(a, -a, 0), (a, a, a)\}$ for a an arbitrary real number.

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Problem 2.4.9 (23-24-S1-A-P22)**.** (*10 points*) If a, b, c are positive real numbers such that $a + b + c = 1$. Prove that

$$a^3 + b^3 + c^3 - 1 = 3(abc - ab - bc - ca)$$

Solution. Since

$$\begin{aligned}
&a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca),\\
&a + b + c = 1 \Rightarrow a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 1 - 2(ab + bc + ca)\\
&\Rightarrow a^3 + b^3 + c^3 - 1 = 3abc + (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) - 1\\
&= 3abc - (1 - 3(ab + bc + ca)) = 3(abc - ab - bc - ca).
\end{aligned}$$

□

Remark. All students who submitted solutions for this problem have correctly solve it.

Problem 2.4.10 (23-24-S1-A-P23)**.** a, b, c pairwise distinct real numbers.

(*5 points*) Prove that

$$\frac{bc}{(a-b)(a-c)} + \frac{ca}{(b-c)(b-a)} + \frac{ab}{(c-a)(c-b)} = 1$$

(*5 points*) Prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} \geq 2.$$

Solution. The first question is simple, all terms on the left side can be brought to the common denominator, then cross multiplying, then by simplifying we shall receive the equality.

For the second question:

$$\begin{aligned}
&\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2}\\
&= \left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right)^2 + 2\frac{bc}{(a-b)(a-c)} + \frac{ca}{(b-c)(b-a)} + \frac{ab}{(c-a)(c-b)} \geq 2.
\end{aligned}$$

□

Remark. All students who submitted solutions for this problem have correctly solve only the first question.

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2.5 Inequalities (O-Level)

We use the principle *sum-of-squares is non-negative* to prove the following inequalities.

Example 2.5.1 (23-24-S1-O-P1)
For all real numbers a, b, c, show that

$$a^2 + b^2 + c^2 + 3 \geq 2(a + b + c).$$

Solution. By grouping terms with variable a, then b, and c:

$$a^2+b^2+c^2+3-2(a+b+c) = (a^2+1-2a)+(b^2+1-2b)+(c^2+1-2c) = (a-1)^2+(b-1)^2+(c-1)^2 \geq 0.$$

□

Example 2.5.2 (23-24-S1-O-P2)
For all real numbers x, y, z, show that

$$19x^2 + 54y^2 + 16z^2 + 36xy - 24yz - 16zx \geq 0.$$

Solution. Note the presence of x^2, y^2, z^2 and then xy, yz, zz, thus it is likely that we will have quadratics involve three pairs of x, y, z.

$$\begin{aligned}
&19x^2 + 54y^2 + 16z^2 + 36xy - 24yz - 16zx \\
&= (9x^2 + 36xy + 36y^2) + (18y^2 - 24yz + 8z^2) + (8z^2 - 16zx + 8x^2) + 2x^2 \\
&= 9(x + 2y)^2 + 2(3y - 2z)^2 + 8(z - x)^2 + 2x^2 \geq 0
\end{aligned}$$

□

Remark. Is there another way to prove this?

Let $f(x) = 19x^2 + 4x(9y - 4z) + (54y^2 + 16z^2 - 24yz)$ be a quadratic for x, then

$$\Delta_f = 4^2(9y - 4z)^2 - 4(18)(2)(27y^2 + 8z^2 - 12yz) = -24(89y^2 + 7(2y - z)^2 + 33z^2) \leq 0$$

$\Delta_f \leq 0$ and the coefficient of x^2 in f is $19 > 0$ mean that $f(x) \geq 0$, for all x.

Example 2.5.3 (23-24-S1-O-P3)
For all real numbers a, b, c, show that

$$\sqrt{a^2 + ab + b^2} + \sqrt{b^2 + bc + c^2} + \sqrt{c^2 + ca + a^2} \geq \sqrt{3}(a + b + c).$$

Solution. Note that the equality stands if $a = b = c$, thus we will use $(a - b)^2 \geq 0$ as follow:

$$\begin{aligned}
a^2 + ab + b^2 &= \frac{1}{4}(a - b)^2 + \frac{3}{4}(a + b)^2 \geq \frac{3}{4}(a + b)^2 \\
\sum_{cyc} \sqrt{a^2 + ab + b^2} &\geq \frac{\sqrt{3}}{2} \sum_{cyc}(a + b) = \sqrt{3}(a + b + c)
\end{aligned}$$

Note that $\sum_{cyc}$ means *to cycle in similar way through the same set of variables*, for example $\sqrt{a^2 + ab + b^2}$ is an term with a, b, then the next one with b, c, is $\sqrt{b^2 + bc + c^2}$ and the last one is c, a is $\sqrt{c^2 + ca + a^2}$. □

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Example 2.5.4 (23-24-S1-O-P4)
For all x, y real numbers such that $xy \geq 1$, show that

$$\frac{1}{x^2+1} + \frac{1}{y^2+1} \geq \frac{2}{xy+1},$$

Solution. We investigate the difference between the two sides

$$\begin{aligned}
&\frac{1}{x^2+1} + \frac{1}{y^2+1} - \frac{2}{xy+1} = \left(\frac{1}{x^2+1} - \frac{2}{xy+1}\right) + \left(\frac{1}{y^2+1} - \frac{2}{xy+1}\right) \\
&= \frac{x(y-x)}{(x^2+1)(xy+1)} + \frac{y(x-y)}{(y^2+1)(xy+1)} = \frac{x-y}{xy+1}\left(\frac{-x}{x^2+1} + \frac{y}{y^2+1}\right) \\
&= \frac{x-y}{xy+1}\frac{x^2y+y-y^2x-x}{(x^2+1)(y^2+1)} = \frac{x-y}{xy+1}\frac{(x-y)(xy-1)}{(x^2+1)(y^2+1)} = \frac{(x-y)^2(xy-1)}{(x^2+1)(y^2+1)(xy+1)} \geq 0
\end{aligned}$$

□

We prove the following inequalities with *special identities*:

Example 2.5.5 (23-24-S1-O-P5)
For all real numbers $a \geq b \geq c$, show that

$$a^2b + b^2c + c^2a \geq ab^2 + bc^2 + ca^2.$$

Solution. Note that $a \geq b, b \geq c, c \geq a \Rightarrow a-b \geq 0, b-c \leq 0, a-c \leq 0$, thus

$$\begin{aligned}
a^2b + b^2c + c^2a - (ab^2 + bc^2 + ca^2) &= ab(a-b) - c(a^2-b^2) + c^2(a-b) = (a-b)(ab - c(a+b) + c^2) \\
&= (a-b)((ab-ca) - (bc-c^2)) = (a-b)(b-c)(a-c) \geq 0
\end{aligned}$$

□

Example 2.5.6 (23-24-S1-O-P6)
For all positive real numbers a, b, c, show that

$$(a+b)(b+c)(c+a) \geq \frac{8}{9}(a+b+c)(ab+bc+ca).$$

Solution. Note that

$$\left.\begin{aligned}
(a+b)(b+c)(c+a) &= a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2 + 2abc \\
(a+b+c)(ab+bc+ca) &= a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2 + 3abc \\
(a+b)(b+c)(c+a) &\geq (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca}) = 8abc
\end{aligned}\right\}$$

$$\Rightarrow \frac{8}{9}(a+b+c)(ab+bc+ca) = \frac{8}{9}(a+b)(b+c)(c+a) + \frac{8}{9}abc \leq (a+b)(b+c)(c+a).$$

□

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Example 2.5.7 (23-24-S1-O-P7)
For a, b, c real numbers such that $a+b \neq 0$, $b+c \neq 0$, and $c+a \neq 0$, show that

$$\frac{a^5+b^5+c^5-(a+b+c)^5}{a^3+b^3+c^3-(a+b+c)^3} \geq \frac{10}{9}(a+b+c)^2.$$

Solution. For the third, note that $a^3+b^3+c^3-(a+b+c)^3 = -3(a+b)(b+c)(c+a)$. We also have (prove it!)

$$a^5+b^5+c^5-(a+b+c)^5 = -5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+bc+ca).$$

Thus, the ratio is

$$\frac{5}{3}(a^2+b^2+c^2+ab+bc+ca) \geq \frac{10}{9}(a+b+c)^2.$$

□

We prove the following inequalities with *comparison method*.

Example 2.5.8 (23-24-S1-O-P8)
For a, b, c positive real number, show that

$$a^{2a}b^{2b}c^{2c} \geq a^{b+c}b^{c+a}c^{a+b}.$$

Solution. By dividing both sides by the right hand side, the given inequality is equivalent to:

$$a^{a-b}a^{a-c}b^{b-a}b^{b-c}c^{c-a}c^{c-b} \geq 1 \Leftrightarrow \left(\frac{a}{b}\right)^{a-b}\left(\frac{b}{c}\right)^{b-c}\left(\frac{c}{a}\right)^{c-a} \geq 1 \quad (*)$$

If both cases of $a \geq b$ and $a < b$, we have $\left(\frac{a}{b}\right)^{a-b} \geq 1$, thus by cycling through and multiplying the results together, we have (*). □

Example 2.5.9 (23-24-S1-O-P9)
For a, b, c nonnegative real numbers, show that

$$a^2+b^2+c^2+2abc+1 \geq 2(ab+bc+ca).$$

Solution. For the second, note that

$$\begin{aligned} a^2+b^2+c^2+2abc+1-2(ab+bc+ca) &= (a-1)^2+(b-c)^2+2a+2abc-2(ab+ac) \\ &= (a-1)^2+(b-c)^2+2a(1-b)(1-c) \end{aligned}$$

The quantity $(1-b)(1-c)$ is interesting because if it is nonnegative, then the whole expression on the left hand side is also nonnegative and we got the inequality proven. Can it be negative? Certainly it can. Why, because the roles of a, b, c in cycling are the same, and it is easy to see that there are two of a, b, c, say b, c, both must be larger than or not exceeding 1, thus the product $(1-b)(1-c)$ is always nonnegative. Hence, the given inequality stands. □

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Example 2.5.10 (23-24-S1-O-P10)
If a, b, c are positive real numbers and $a^2 + b^2 + c^2 = 1$, find the minimal value of

$$S = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{2(a^3 + b^3 + c^3)}{abc}$$

Solution. Our guess is that S reaches it minimum when $a = b = c = \frac{1}{\sqrt{3}}$. Let replace each of the first three numerators of 1 by $a^2 + b^2 + c^2$. Now we can see that the expression is *harmonized*, meaning the degrees of a, b, c in the terms are the same.

$$\begin{aligned}
S - 3 &= \frac{a^2 + b^2 + c^2}{a^2} + \frac{a^2 + b^2 + c^2}{b^2} + \frac{a^2 + b^2 + c^2}{c^2} - 3 - \frac{2(a^3 + b^3 + c^3)}{abc} \\
&= a^2\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + b^2\left(\frac{1}{c^2} + \frac{1}{a^2}\right) + c^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - 2\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}\right) \\
&= a^2\left(\frac{1}{b} - \frac{1}{c}\right)^2 + b^2\left(\frac{1}{c} - \frac{1}{a}\right)^2 + c^2\left(\frac{1}{a} - \frac{1}{b}\right)^2 \geq 0 \\
&\Rightarrow S \geq \boxed{3}, \text{ the equality stands when } a = b = c = \frac{1}{\sqrt{3}}.
\end{aligned}$$

□

We prove the following inequalities with the *Pigeonhole Principle*.

Example 2.5.11 (23-24-S1-O-P11)
For nonnegative real numbers x, y, z, show that

$$2 + 3\sqrt{3}xyz \geq 3(xy + yz + zx).$$

Solution. Lets take a look when the equality stands: $x = y = z = \frac{1}{\sqrt{3}}$. Thus borrowing the idea from the previous example, two of x, y must be *on the same side of* $\frac{1}{\sqrt{3}}$ *on the real number line*:

$$(1 - \sqrt{3}x)(1 - \sqrt{3}y) \geq 0 \Leftrightarrow 1 - \sqrt{3}x - \sqrt{3}y + 3xy \geq 0 \Leftrightarrow 1 + 3xy \geq \sqrt{3}x + y.$$

Now, by multiplying both sides with $\sqrt{3}z$,

$$\sqrt{3}z + 3\sqrt{3}xyz \geq 3(xz + yz) \Leftrightarrow 2 + 3\sqrt{3}xyz \geq 2 + 3(xz + yz) - \sqrt{3}z.$$

What's left is to prove that

$$2 + 3(xz + yz) - \sqrt{3}z \geq 3(xy + yz + zx) \Leftrightarrow 2 \geq \sqrt{3}z + 3xy.$$

But

$$\sqrt{3}z + 3xy \leq \sqrt{3}z + 3\frac{x^2 + y^2}{2} = \sqrt{3}z + \frac{3}{2}(1 - z^2) = 2 - \frac{(1 - \sqrt{3}z)^2}{2} \leq 2.$$

□

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Example 2.5.12 (23-24-S1-O-P12)
For positive real numbers a, b, c, let $x = a + \frac{1}{b}$, $y = b + \frac{1}{c}$, and $z = c + \frac{1}{a}$, prove that

$$xy + yz + zx \geq 2(x + y + z).$$

Solution. For the second question, again lets look for the values of a, b, c where the equality stands. It is easy to see that $a = b = c = 1$, and $x = y = z = 2$. Now, how do we use this information? From

$$xy + yz + zx \geq 2(x + y + z) \Rightarrow xy - (x + y) + yz - (y + z) = zx - (z + x) \geq 0.$$

Now with WLOG x, y both on the same side of 2, then

$$(x - 2)(y - 2) = xy - 2(x + y) + 4 \geq 0 \Leftrightarrow xy + 4 \geq 2(x + y).$$

What we need now is to prove $yz + zx \geq 2z + 4$, But this is true, by the AM-GM and AM-HM inequalities,

$$x + y = a + \frac{1}{b} + b + \frac{1}{c} = a + \left(\frac{1}{b} + b\right) + \frac{1}{c} \geq a + 2 + \frac{1}{c} = 2 + \left(a + \frac{1}{c}\right) \geq 2 + \frac{4}{\frac{1}{a} + c} = 2 + \frac{4}{z}.$$

□

We prove the following inequalities with the principle of *majorization and minorization*:

Example 2.5.13 (23-24-S1-O-P13)
For positive real numbers a, b, c, show that

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{1}{abc}$$

Solution. First, note that $a^3 + b^3 = (a + b)(a^2 - ab + b^2) \geq (a + b)ab$, thus

$$\frac{1}{a^3 + b^3 + abc} \leq \frac{1}{(a + b)ab + abc} = \frac{c}{abc(a + b + c)}$$

Similarly for the other two terms of the left hand side, and by summing up them together we shall receive an Inequalities with the right hand side as desired. □

Example 2.5.14 (23-24-S1-O-P14)
For positive real numbers a, b, c, such that $a + b + c = 3$, show that

$$1 + 8abc \geq 9 \min\{a, b, c\}$$

Solution. Let assume that $a \leq b \leq c$, then we have to prove that $1 + 8abc \geq 9a$. From $a + b + 3$, $a \leq b \leq c$, then $a \leq 1$ and $(a - b)(a - c) \geq 0$, thus

$$bc \geq a(b + c) - a^2 = a(3 - a) - a^2 = a(3 - 2a).$$

Therefore

$$\begin{aligned} 1 + 8abc - 9a &\geq 1 + 8a^2(3 - 2a) - 9a = -16a^3 + 24a^2 - 9a + 1 \\ &= -16a^2(a - 1) + 8a(a - 1) - (a - 1) = (1 - a)(1 - 4a)^2 \geq 0. \end{aligned}$$

□

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2.6 Problems (O-Level)

Submission deadline: September 29, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 2.6.1 (23-24-S1-O-P15)**.** (*10 points*) For a, b, c positive real numbers, prove that:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Official solution. Note that

$$\frac{a}{b+c} - \frac{1}{2} = \frac{(a-b)+(a-c)}{2(b+c)}.$$

Thus

$$\begin{aligned}\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} &= \frac{(a-b)+(a-c)}{2(b+c)} + \frac{(b-c)+(b-a)}{c+a} + \frac{(c-a)+(c-b)}{a+b} \\ &= \frac{a-b}{2}\left(\frac{1}{b+c} - \frac{1}{c+a}\right) + \frac{b-c}{2}\left(\frac{1}{c+a} - \frac{1}{a+b}\right) + \frac{c-a}{2}\left(\frac{1}{a+b} - \frac{1}{b+c}\right) \\ &= \frac{(a-b)^2}{2(b+c)(c+a)} + \frac{(b-c)^2}{2(c+a)(a+b)} + \frac{(c-a)^2}{2(a+b)(b+c)} \geq 0\end{aligned}$$

□

Remark. Only Henry Dinh and Albert Dinh-Le correctly solved this problem. Other students also managed to solve this, with some minor imperfections or shortcuts in simplification of complex expressions.

Solution (Solution 1 from Henry Dinh - using the substitution method) — Let $x = b+c, y = c+a, z = a+b$, then

$$a = \frac{-x+y+z}{2}, b = \frac{x-y+z}{2}, z = \frac{x+y-z}{2}$$

$$\begin{aligned}&\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{-x+y+z}{2x} + \frac{x-y+z}{2y} + \frac{x+y-z}{2z} \\ &= \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{1}{2}\left(\frac{y}{z} + \frac{z}{y}\right) + \frac{1}{2}\left(\frac{z}{x} + \frac{x}{z}\right) - \frac{3}{2} \geq \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 2 - \frac{3}{2} = \frac{3}{2}.\end{aligned}$$

Solution (Solution 2 from Henry Dinh using AM-HM) — Note that

$$\frac{a}{b+c} = \frac{a+b+c}{b+c} - 1.$$

Similarly with the other fractions, then the desired inequality is equivalent to

$$\begin{aligned}&\frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b} \geq \frac{9}{2} \iff 2(a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq 9 \\ &\iff ((a+b)+(b+c)+(c+a))\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq 9\end{aligned}$$

which is an AM-HM inequality, thus it is true.

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Definition (Convex function)**.** The function $f:\ [a,b] \to \mathbb{R}$ is *convex* on the interval $[a,b]$ if for any $x, y \in [a,b]$ and any $\alpha \in (0,1)$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y).$$

Theorem (Second derivative of a convex function)

Suppose that the function $f:\ (a,b) \to \mathbb{R}$ for every $x \in [a,b]$, there exists a second derivative $f''(x)$. f is *convex* on the interval $[a,b]$ if and only if for any $x \in [a,b] f''(x) \geq 0$.

Theorem (A linear combination of convex functions is also convex)

The functions $f_1, f_2, \ldots, f_n$ are *convex* on the interval (a,b). Then the function $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$ is *convex* on the interval $[a,b]$, for any $c_1, c_2, \ldots, c_n \in (0,\infty)$.

Theorem (Jensen's Inequality)

Let f be a convex function of one real variable. Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and let $a_1, \ldots, a_n \geq 0$ satisfy $a_1 + a_2 + \cdots + a_n = 1$. Then

$$f(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) \leq a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n).$$

Solution 3 using the normalization from Henry Dinh's idea. Lets try here a harmonization method with $s = a+b+c, x = \frac{a}{s}, y = \frac{b}{s}, z = \frac{c}{s}$, then $x+y+z = 1$. Now,

$$\begin{aligned} T &= \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{sx}{sy+sz} + \frac{sy}{sz+sx} + \frac{sz}{xy+sy} \\ &= \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = \frac{x}{1-x} + \frac{y}{1-y} + \frac{z}{1-z} \end{aligned}$$

Note that T now is a sum of three values of the function $f:\ (0,1) \to \mathbb{R}$, $f(t) = \frac{t}{1-t}$, whose derivatives

$$f(t) = \frac{(t-1)+1}{1-t} = -1 + \frac{1}{1-t} \Rightarrow f'(t) = \frac{1}{(1-t)^2} \Rightarrow f''(t) = \frac{2}{(1-t)^3} > 0$$

Thus f is a convex on $(0,1)$, therefore by Jensen's Inequality, for $a_1 = a_2 = a_3 = \frac{1}{3}$,

$$f\left(\frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z\right) \leq \frac{1}{3}f(x) + \frac{1}{3}f(y) + \frac{1}{3}f(z).$$

Now,

$$\frac{1}{3}f(x) + \frac{1}{3}f(y) + \frac{1}{3}f(z) = \frac{1}{3}T, f\left(\frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z\right) = f\left(\frac{x+y+z}{3}\right) = f\left(\frac{1}{3}\right) = \frac{1}{2} \Rightarrow T \geq \frac{3}{2}.$$

□

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Problem 2.6.2 (23-24-S1-O-P16)**.** (*10 points*) For a, b, c positive real numbers, prove that:

$$ab\left(\frac{b}{c}-1\right)+bc\left(\frac{c}{a}-1\right)+ca\left(\frac{a}{b}-1\right)\geq 0.$$

Proof.

$$ab\left(\frac{b}{c}-1\right)+bc\left(\frac{c}{a}-1\right)+ca\left(\frac{a}{b}-1\right)=\frac{ab^2}{c}+\frac{bc^2}{a}+\frac{ca^2}{b}-ab-bc-ca$$
$$=\left(\frac{ab^2}{c}+ca-2ab\right)+\left(\frac{bc^2}{c}+ab-2bc\right)+\left(\frac{ca^2}{b}+bc-2ca\right)=\frac{a(b-c)^2}{c}+\frac{b(c-a)^2}{a}+\frac{c(a-b)^2}{b}\geq 0$$

□

Remark. Only Henry Dinh, Minh Le Phuong Nguyen, Ha-Anh Le, and Anthony Pham correctly solved this problem. Albert Dinh-Le made some expensive shortcuts.

Problem 2.6.3 (23-24-S1-O-P17)**.** (*10 points*) For x, y positive real numbers such that $x^2+y^3\geq x^3+y^4$. Prove that: $x^3+y^3\leq x^2+y^2\leq x+y\leq 2$.

Proof.

$$y^2-y^3-(y^3-y^4)=y^2(1-2y+y^2)=y^2(1-y)^2\geq 0$$
$$\Rightarrow x^2-x^3+y^2-y^3\geq x^2-x^3+y^3-y^4=(x^2+y^3)-(x^3+y^4)\geq 0\Rightarrow \boxed{x^2+y^2\geq x^3+y^3}$$
$$x-x^2-(x^2-x^3)=x(1-x)^2\geq 0,\ y-y^2-(y^3-y^4)=y(1-y)^2(1+y)\geq 0$$
$$\Rightarrow (x-x^2)+(y-y^2)\geq (x^2-x^3)+(y^3-y^4)\geq 0\Rightarrow \boxed{x+y\geq x^2+y^2}$$
$$x-(1-x^2+x^3)=-(x-1)^2(x+1)\leq 0,\ y-(1-y^3+y^4)=-(y-1)^2(y^2+y+1)\leq 0$$
$$\Rightarrow x+y\leq (1-x^2+x^3)+(1-y^3+y^4)=2+(x^3+y^4)-(x^2+y^3)\Rightarrow \boxed{x+y\leq 2.}$$

□

Remark. Henry Dinh's solution is correct. Minh Le Phuong Nguyen and Ha-Anh Le made some advances.

Problem 2.6.4 (23-24-S1-O-P18)**.** (*10 points*) Prove that if $a\geq b\geq c$ such that $a+b+c\geq 0$. then

$$a^3b+b^3c+c^3a\geq ab^3+bc^3+ca^3.\text{ Does the inequality still stand if } a\leq b\leq c?$$

Proof. Note that for $a\geq b\geq c$,

$$(a^3b+b^3c+c^3a)-(ab^3+bc^3+ca^3)=ab(a^2-b^2)-c(a^3-b^3)+c^3(a-b)$$
$$=(a-b)(ab(a+b)-c(a^2+ab+b^2)+c^3)=(a-b)(c(c^2-a^2)-ab(c-a)-b^2(c-a))$$
$$=(a-b)(c-a)(c(c+a)-ab-b^2)=(a-b)(c-a)(c^2-b^2+a(c-b))$$
$$=(a-b)(b-c)(a-c)(a+b+c)\geq 0\Rightarrow a^3b+b^3c+c^3a\geq ab^3+bc^3+ca^3\geq 0$$

If $a\leq b\leq c\Rightarrow (a-b)(b-c)(a-c)(a+b+c)\leq 0$ the inequality changes direction.

□

Remark. Henry Dinh correctly solved this problem. Minh Le Phuong Nguyen and Albert Dinh-Le have near perfect solutions.

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Solution (Solution 2 from Henry Dinh using polynomial root method, with modifications for simplicity) — Let ,

$$\begin{aligned} f(a) &= a^3b + b^3c + c^3a - (ab^3 + bc^3 + ca^3) \\ &= a^3(b-c) - a(b^3 - c^3) + bc(b^2 - c^2) = (b-c)(a^3 - a(b^2 + bc + c^2) + bc(b+c)) \end{aligned}$$

Then $f(b) = f(c) = 0$. Let x be the third root of f. Since $x + b + c = 0$, so $x = -b - c$. Therefore

$$f(t) = (b-c)(t-b)(t-c)(t+b+c)$$

Thus $a^3b + b^3c + c^3a - (ab^3 + bc^3 + ca^3) = (b-c)(a-b)(a-c)(a+b+c) \geq 0$.

Problem 2.6.5 (23-24-S1-O-P19). (*10 points*) For x, y, z positive real number such that $x^2 + y^2 + z^2 = 1$, find the minimal value of

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}.$$

Proof.

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} = \frac{(xy)^2 + (yz)^2 + (zx)^2}{xyz}$$

Since $(a+b+c)^2 \geq 3(ab+bc+ca)$, and if $a = uv, b = vw, c = wu$, then $(uv+vw+wu)^2 \geq 3uvw(u+v+w)$. Now, by $u = x^2, v = y^2, w = z^2$, we have

$$\left((xy)^2 + (yz)^2 + (zx)^2\right)^2 \geq 3(xyz)^2 \Rightarrow (xy)^2 + (yz)^2 + (zx)^2 \geq \sqrt{3}(xyz).$$

Hence, the minimal value of the given expression is $\boxed{\sqrt{3}.}$ □

Remark. Onlye Henry Dinh and Albert Dinh-Le correctly solved this problem. Minh Le Phuong Nguyen make a mistake when multiplying xyz with $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$.

Problem 2.6.6 (23-24-S1-O-P20). (*10 points*) Let $0 < x < y < z < t$, such that $x + t = y + z$. Show that for any positive integer $n \geq 2$:

$$x^n + t^n > y^n + z^n.$$

Proof. Let $a = y - x = t - z$, then $x^n + t^n > y^n + z^n \Leftrightarrow (x+a)^n - x^n < (z+a)^n - z^n$. By the binomial theorem:

$$(x+a)^n - x^n = \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-1}a + \ldots + a^n < \binom{n}{1}z^{n-1} + \binom{n}{2}z^{n-1}a + \ldots + a^n = (z+a)^n - z^n.$$

□

Remark. Only Henry Dinh and Minh Le Phuong Nguyen correctly solved this problem.

Problem 2.6.7 (23-24-S1-O-P21). (*10 points*) Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2 = 3$. Prove that

$$ab^2 + bc^2 + ca^2 \leq abc + 2.$$

Proof. WLOG let b be between a, c. then $(b-a)(b-c) \leq 0$,

$$\begin{aligned} &2 + abc - ab^2 - bc^2 - ca^2 = 2 + abc - ab^2 - b(3 - a^2 - b^2) - ca^2 \\ &= (b^3 - 3b + 2) - a(b^2 - ab + ca - bc) = (b-1)^2(b+2) - a(b-a)(b-c) \geq 0 \end{aligned}$$

□

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Problem 2.6.8 (23-24-S1-O-P22)**.** (*10 points*) Let a, b, c be positive real numbers. Prove that

$$\frac{a(a-b)}{b+c} + \frac{b(b-c)}{c+a} + \frac{c(c-a)}{a+b} \geq 0.$$

Proof. Since $\frac{a(a-b)}{b+c} = \frac{(a+b+c)(a-b)}{b+c} - (a-b)$,

$$\frac{a(a-b)}{b+c} + \frac{b(b-c)}{c+a} + \frac{c(c-a)}{a+b} = (a+b+c)\sum_{cyc}\frac{a-b}{b+c} = (a+b+c)\sum_{cyc}\left(\frac{a+c}{b+c} - 1\right)$$
$$= (a+b+c)\sum_{cyc}\left(\frac{a+c}{b+c}\right) - 3(a+b+c) \geq (a+b+c)3\sqrt[3]{\prod_{cyc}\frac{a+c}{b+c}} - 3(a+b+c) = 0$$

□

> **Remark.** Only Henry Dinh and Minh Le Phuong Nguyen correctly solved this problem. Anthony Pham has nearly perfect solution, while Albert Dinh-Le made an good attempt without proper explanations.

Problem 2.6.9 (23-24-S1-O-P23)**.** (*10 points*) Let x, y, z be nonnegative real numbers such that

$$x^2 + y^2 + z^2 \leq 3.$$

Prove that

$$xy + yz + xz - xyz \leq 2.$$

Proof. WLOG let x, y be on the same side of 1, then

$$(x-1)(y-1) \geq 0 \Leftrightarrow xy - x - y + 1 \geq 0 \Leftrightarrow xyz - xz - yz + z \geq 0 \Leftrightarrow yz + zx - xyz \leq z$$

We need to prove that $z \leq 2 - xy$,

$$xy + z \leq \frac{1}{2}(x^2 + y^2) + z = \frac{1}{2}(3 - z^2) + z = 2 - \frac{(z-1)^2}{2} \leq 2.$$

□

> **Remark.** Only Henry Dinh correctly solved this problem.

Problem 2.6.10 (23-24-S1-O-P24)**.** (*10 points*) Let x, y, z be nonnegative real numbers such that $x+y+z = 3$. Prove that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq 3.$$

Proof. Let x, y be on the same side of 1, substitute $z = 3 - x - y$, we have

$$sqrtx + \sqrt{y} + \sqrt{3-x-y} \leq 3 \Leftrightarrow 3 - \sqrt{x} - \sqrt{y} \geq \sqrt{3-x-y}$$
$$\Leftrightarrow 3 + x + y - 6\sqrt{x} - 6\sqrt{y} + 2\sqrt{xy} \geq 3 - x - y$$
$$\Leftrightarrow x + y - 3\sqrt{x} - 3\sqrt{y} + \sqrt{xy} \geq 0 \Leftrightarrow (\sqrt{x} - \sqrt{y})^2 + 3(\sqrt{x} - 1)(\sqrt{y} - 1) \geq 0$$

□

> **Remark.** Only Henry Dinh and Anthony Pham correctly solved this problem.

Solution (Solution from Henry Dinh with Cauchy-Schwarz inequality) —

$$(\sqrt{x}+\sqrt{y}+\sqrt{z})^2 \le (x+y+z)(1+1+1) = 3\cdot 3 = 9 \Rightarrow \sqrt{x}+\sqrt{y}+\sqrt{z} \le 3.$$

Problem 2.6.11 (23-24-S1-O-P25)**.** (*10 points*) Let a, b, c be positive real numbers such that $a \le b \le c$, $ab+bc+ca = 3$. Prove that

$$S = \frac{1}{2a+b} + \frac{1}{2b+c} + \frac{1}{2c+a} \ge 1$$

Proof. Let $x = \frac{1}{2a+b}, y = \frac{1}{2b+c}, z = \frac{1}{2c+a}$, then since $(x+y+x)^2 \ge 3(xy+yz+zx)$:

$$S^2 \ge 3\left(\frac{1}{2a+b}\frac{1}{2b+c} + \frac{1}{2b+c}\frac{1}{2c+a} + \frac{1}{2c+a}\frac{1}{2a+b}\right) = 9\frac{a+b+c}{(2a+b)(2b+c)(2c+a)}.$$

Now $9(a+b+c) = 3(a+b+c)(ab+bc+ca)$ and

$$3(a+b+c)(ab+bc+ca)-(2a+b)(2b+c)(2c+a) = ab^2+bc^2+ca^2-(a^2b+b^2c+c^2a) = (a-b)(b-c)(c-a) \ge 0.$$

Thus $S \ge 1$. □

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Chapter 3

Session 2

3.1 Logic puzzles (I-Level)

Example 3.1.1 (23-24-S2-I-P1)

A dog and cat together weigh 27 pounds. If the dog's weight is an odd number, and if he weighs twice as much as she does, how much does each weigh?

Proof. What is he and what is she? If we *think* that the dog is heavier then the cate, then we assume *he* in the second sentence means the dog, which will lead us to the wrong answer. In fact the cat weighs twice the dog, the sum of their weights is 27. So the cat's weight is 18, the dog's 9 (and this is an odd number). □

Example 3.1.2 (23-24-S2-I-P2)

Detective James Smarties walked outside through a pouring rain for half an hour without getting a single hair on his head wet. He did not wear a hat, carry an umbrella, or hold anything over his head. His clothes got really soaked. How could this be possible?

Proof. Having no hair getting wet or having no hair? It's simple: $\boxed{\text{he is bald.}}$ □

Example 3.1.3 (23-24-S2-I-P3)

When their blind dates arrived to take them to a football game, Connie and Jennie was astonished to see that the two young men looked exactly alike. One of them explained "we are brothers," he said, "we were born on the same day of the same year from the same parents". He added "but we are not twins". How could it be?

Solution. What can be $\boxed{\text{same to twins and more than twins}}$: triplets, quadruplets, quintuplets, sextuplets, septuplets, or more! In 2009 the record was set when an octuplets were born in Bellflower, California. □

Example 3.1.4 (23-24-S2-I-P4)

How many people is "two pairs of twins twice"?

Solution. Two(2) pairs (2) of twins(1) twice(2) $= 2 \times 2 \times 2 = \boxed{8}$ people. Twin is a person, not two. □

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Example 3.1.5 (23-24-S2-I-P5)

A big fat Indian and a small thin Indian were sitting outside a teepee, each smoking a pipe. The little Indian was the son of the big Indian, but the big Indian was not the little Indian's father. How come?

Remark. Reasoning cannot be based on fact-less assumptions. A person who is a parent and smoke is not necessarily a man. Hasty conclusion is what people used to do when they draw a conclusion based on an event with high probability (but not definite). There is a chance, whatever small it is, that the conclusion is not true.

Solution. The big Indian is $\boxed{\text{the mother of the little Indian.}}$ □

Example 3.1.6 (23-24-S2-I-P6)

A visitor points to a portrait on the wall and asks who it is. "Brothers and sisters have I none," says the host, "but that man's father is my father' son." Who is in the picture?

Solution. The man is *the host's son. That man's father is my father's son* implies that the man is the grandson of the father of the host. The host does not have any brother of sister, so $\boxed{\text{that man has to be his son.}}$ □

Remark. Identify the self-contradictory sentence is a great technique.

Example 3.1.7 (23-24-S2-I-P7)

Bình was driving through Ottawa when he decided to rest. He stopped at a motel, got a room, and asked the reception girl where he can find a barbershop. "We have two barbers nearby in Ogilvie shopping malls," said the girl. "One of them is at the north entrance and the other close to the south exit."

Bình went to the one next to the north entrance. It looked like as if it had not been cleaned in months. Cut hair was all over the floor. The barber himself would have needed a shave and his haircut looked really terrible. Bình then walked to the other one. It looked neat and cheerful inside. The floor had been swept. The barber was neatly dressed, freshly shaved, and had a neat haircut.

Why did Bình walk back to the first barbershop to get his haircut?

Remark. This problem is actually simple. Let discuss a few points

1. Why the first barbershop was not well cleaned, and more importantly the barber had a terrible haircut. Who would do that to the barber?
2. Why the second barbershop was so clean? The barber was so rich to hire somebody to clean the shop? Why his hair was so good? Who did that?

Solution. The key point perhaps is that $\boxed{\text{who did the haircut for the first barber.}}$ He definitely could not do that to himself. So that should be the second barber. The haircut was terrible. And the rest just follows: the person who was busy does not have time to clean up, or the person who has to trash has no need to clean up. In addition he can does himself a favour by giving himself a shave. □

Remark. An important conclusion is that thing is not what they look like. Don't fall for some nice looks, kind words, or great promises. Look beyond the surface.

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Example 3.1.8 (23-24-S2-I-P8)

A certain small island is inhabited by rabbits of three kinds: white, grey, and spotted.

Grey mother-rabbits always have two bunnies in their litters: a white and a grey one.

The same is true for spotted mother-rabbits.

However, a white rabbit always give birth to three bunnies: a grey, a white, and a spotted one.

The first and only rabbit to ever reach the island was a rabbit who is called Great Mother by the rest of the population. Currently, 40 grey, 40 white, and 30 spotted rabbits live on the island.

If no rabbit ever died or left the island, can you tell the colour of the Great Mother?

Remark. Note that the problem assumes that there is only one mother-rabbit.

Solution. Lets identify the statement that stands out:

1. If the Great Mother was white, when she arrived to the island, the number of white rabbits is one more the number of grey rabbits.
2. Similarly, if the Great Mother was grey, the number of grey rabbits is one more the number of white rabbits.
3. However, the number of grey and white rabbits is the same.

It means that the Great Mother rabbit should be $\boxed{\text{a spotted one.}}$ □

Example 3.1.9 (23-24-S2-I-P9)

While visiting the Knights (who always tell the truth) and Liars (who always lie) Island, Huy passes a beautiful garden where three islanders, Samuel, Tony, and Zecharia are watching the sunset.

He asks Samuel, "are you a knight or a liar?". Samuel is shy, Huy cannot hear his quiet answer.

So he ask Tony, "what did Samuel say?" Tony answers, "he said that he is a liar."

"Don't trust Tony, Tony is a liar!" screams Zecharia.

What do you think, who is what?

Remark. There is an usual approach this problem, which is based on two principles:

1. Analyze possible cases and use process of elimination to eliminate impossible cases.
2. Find an insight or a logic that leads us to a few number of cases.

Proof. Because Samuel says nothing, so we start with Tony's statement: "Samuel said that he is a liar". This simply cannot be true.

Why, let use proof by contradiction: assume the opposite and prove that it is wrong. Let assume Tony is right. Samuel cannot be a truth teller because no truth teller states that he is a liar. On the other hand, Samuel cannot be a liar because no liar admits he is a liar.

Therefore Tony's statement contradicts itself: it is a lie. Hence $\boxed{\text{Tony is a liar}}$, $\boxed{\text{Zecharia is truth teller}}$ because he told Huy that Tony is a liar. What about Samuel? Tony lies about Samuel. So whatever he said about Samuel makes its opposite true. So $\boxed{\text{Samuel is a truth teller.}}$ □

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Example 3.1.10 (23-24-S2-I-P10)

The inhabitants of Truth Village always tell the truth, while those of the hamlet of Liars always lie. On the way to visit the Truth Village, you arrive at a T-intersection, where one road leads to the Truth Village, the other one to the hamlet of Liars. There is no sign, except a single person.

You have to ask but you do not know whether the man is a citizen of the hamlet of Liars or from the Truth Village. You are allowed to ask a single question, to learn how to get to the Truth Village.

What is it?

Remark. What is the relationship of a person to where that person comes from? What would that person says about where he/she's from? What would then be your question?

Solution. Can you point to the village where you come from? is the question. Both truth tellers and liars will point to the direction of the Truth Village, which is what the you want. Knowing what the person is does not help to solve the problem. In this case knowing what you want is more important. □

Example 3.1.11 (23-24-S2-I-P11)

The people from the city No-one-can-be-sure-of-any-thing are of three different types: those who always tell the truth, those who always lie, and those you alternate lie and tell the truth.

Quynh Chi meets one of them. She is allowed to ask twice so that she can know which group of people that one belongs to.

What are the questions?

Remark. We need to ask a question that the truth tellers and the liars give different answers, and each person gave the same answers twice. The one who alternate gives different answers for the same question!

Solution. The point here is how to have one question that can help to distinct three different kinds of people: the truth tellers, the liars, and the alternators.

Fortunately a single question can do so. Any question whose answer is an universal fact, for example is one plus one equal two? The truth tellers always answer with True twice, liars always say False twice, alternators answer True once and False once. □

Example 3.1.12 (23-24-S2-I-P12)

On the island of Knights and Liars, knights and liars live in separate villages. However they often visit their friends in other villages.

This week, the one and only newspaper of the island announced a contest: to come up with a single yes/no question that would allow a tourist to figure out whether he is in a knights' village or a liars' village. The question should be such that it can be addressed to anyone the tourist sees. Of course it is not known in advance that that local person is a knight or a liar, is he from the village or just a visitor. Lilian won the prize.

What was her question?

Remark. Think about a question that you want to know if a person belongs to the place.

Solution. Do you live in this village? is the question that helps. It does not matter who answer the question. Yes means this is the knight's village and No means this is a liars' village. □

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3.2 Problems (I-Level)

Submission deadline: October 6, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 3.2.1 (23-24-S2-I-P13)**.** (*10 points*) Khang plays a game of Ping-Pong in the back yard? He misses the ball. It bounces across the lawn and rolls into a small but deep hole. The hole goes down too far for him to reach the ball with his hand, and the hole bends so much to one side that he cannot get the ball by poking a stick into the hole.

How can Khang get the ball?

> **Remark.** If you never hold a Ping-pong ball in your hand, try to get one before trying to solve this. How can you get the ball coming up by itself?

Solution. The ping-pong ball floats over the water, so $\boxed{\text{pour the water into the hole.}}$ This is not a strictly math problem. It is a problem to test if you have any relationship with the reality surrounding you. The solution is sometime just from common sense, from physics, or else. It is important to know that solutions can come from very different sources □

> **Remark.** Giang Nguyen, Julie Cao, Minh Do Tue Le, and Minh Pham correctly solved this problem. Chi-Mai Pham started an attempt.

Problem 3.2.2 (23-24-S2-I-P14)**.** (*10 points*) Four students of the Math & Chess Club having a conversation about their favourite food.

1. Boy A: Jennie likes fish
2. Boy B: I likes cakes, but I am not called Anthony
3. Girl C: One of the boys - Anthony and Khang - like chicken
4. Girl D: Barbie likes salad, but I don't

Can you figure out who's who and who likes what?

> **Remark.** First, identify the names of the girls and the boys.

Solution. Since Boy B said he is not Anthony, so he is Khang and Khang likes cakes. Boy A should be Anthony and from what girl C said, Anthony likes chicken. Jenni should be girl D, and Jennie like fish. Barbie is girl C and she likes salad. □

> **Remark.** Chi-Mai Pham, Jérémy Nguyen, Julie Cao, and Minh Pham correctly solved this problem. Minh Do Tue Le and Giang Nguyen made some minor mistakes.

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Problem 3.2.3 (23-24-S2-I-P15)**.** (*10 points*) Catherine and Clemence got a nice cake from their mother. They were squabbling over who gets to cut the cake. The reason was that each of them thinks the other will cut it in a way to give herself the larger portion. Seeing this, their dad recommended a clever idea and both of them immediately agreed.

What would be the idea?

> **Remark.** The reason of the squabbling is trust, or lack of it. Lack of trust if one of them does both: cuts the cake and choose her portion.

Solution. Human tragedy is giving someone to much authority and then regretting it. In this simple case, the solution is also very simple: do not give the girl who cut the case too much authority. She can cut the cake or choose her portion but not both! Once this is enforced, trust will be reestablished.

Hence, let $\boxed{\text{the senior girl cut the cake and the younger one choose her own portion!}}$ □

> **Remark.** Chi-Mai Pham, Giang Nguyen, Jérémy Nguyen, Julie Cao, Minh Do Tue Le, and Minh Pham correctly solved this problem.

Problem 3.2.4 (23-24-S2-I-P16)**.** (*10 points*) Cindy has two treasure chests, each of them filled either with gold or silver coins. One of them has a label, "I contains gold coins, and the other one contains silver". The other is labelled with "One of us contains gold, while another contains silver". Cindy told Andy that one of the labels is correct and the other is not.

Where' the gold?

> **Remark.** Draw a table with two columns for two chests, Use what Cindy told Andy to analyze the two cases.

	Chest 1 has gold, Chest 2 has silver	Chest 1 has silver, Chest 2 has gold
Label 1		
Label 2		

Solution. The second label is always correct. It's a universal true. The first one is false.

Hence, $\boxed{\text{the gold is in the second chest.}}$ □

> **Remark.** Julie Cao, Minh Do Tue Le, and Minh Pham correctly solved this problem. Chi-Mai Pham, Giang Nguyen, and Jérémy Nguyen almost got it perfectly.

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Problem 3.2.5 (23-24-S2-I-P17)**.** (*10 points*) In the Forest of Blue fairies and Red hoodies, there are two types of people: the Blue fairies who ask only questions whose correct answer is yes; the Red hoodies who ask only questions whose correct answer is no.

For example, a Blue fairy could ask, "Does two plus two equal four?", but she could not ask whether two plus two equals five. A Red hoodie could not ask whether two plus two equals four, but could ask whether two plus two equals five.

While visiting the Forest of Blue fairies and Red hoodies, Amy met two sisters named Catherine and Clemence. Catherine asked Clemence: "Is at least one of us a Red hoodie?"

What type of people are Catherine and Clemence?

Solution. Suppose that Catherine is a Red hoodie. Then it would be true that at least one of the sisters is a Red hoodie, which means her question is answered with yes, so she is a Blue fairy. This is contradiction. So $\boxed{\text{Catherine is a Blue fairy.}}$ This means the answer to her question is yes. One of the sister is a Red hoodie, therefore $\boxed{\text{Clemence is a Red hoodie.}}$ □

> **Remark.** Chi-Mai Pham, Julie Cao, and Minh Pham correctly solved this problem. Jérémy Nguyen almost got it perfectly. Giang Nguyen made some minor mistake. Minh Do Tue Le started a good attempt.

Problem 3.2.6 (23-24-S2-I-P18)**.** (*10 points*) There are two types of people on the Island of Knights and Liars: the Knights who always tells the truth, and the Liars, who always lie. Detective Samuel came to the island to investigate a crime. He knew that:

- One of the identical twins was named Donald and he was the one who committed the crime.
- It is know that at least one of them is a Liar, but it is not known which one.

Are you Donald? Detective Samuel asked the first twin. *Yes, I am,* was the reply. *Are you Donald?* Detective Samuel asked the second twin. The second twin then answered either yes or no, and the detective knew that which one was Donald.

Was Donald the first or second twin?

Solution. If the second twin answer yes, the detective obviously cannot know which one is Donald; hence he must answered no. This means that either both twins told the truth or both lied. Since at least one of them is a Liar, so both lied. This means that $\boxed{\text{the second twin is Donald.}}$ □

> **Remark.** This is not an easy problem. The key statement was *The second twin then answered either yes or no, and the detective knew that which one was Donald,* which requires the students to make further analysis from it and come to the conclusion that the second twin answered no, which was not given in the problem text. Chi-Mai Pham, Julie Cao, Giang Nguyen, and Minh Do Tue Le had good tries.

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Problem 3.2.7 (23-24-S2-I-P19)**.** (*10 points*) Tiger always tells the truth on Monday, Tuesday, Wednesday, and Thursday. Lion always tells the truth on Monday, Friday, Saturday, and Sunday. On the rest of the days, they may tell the truth or they may lie. On one day last week, both of them said *"I lied yesterday."*.

On what day of the week did they make those statements?

Solution. Neither Tiger, nor Lion can make the statement *"I lied yesterday"* on the day which the truth must be told *if the previous day* was also one of the days on which they must tell the truth. Therefore Tiger cannot make the statement on Tuesday, Wednesday, or Thursday. Lion cannot make such statement on Saturday, Sunday, or Monday.

Thus, the day of the week when they made those statements is $\boxed{\text{Friday.}}$ □

> **Remark.** Julie Cao, and Minh Pham correctly solved this problem. Chi-Mai Pham, Jérémy Nguyen almost got it perfectly. Minh Do Tue Le, Giang Nguyen could have had to make it better.

Problem 3.2.8 (23-24-S2-I-P20)**.** (*10 points*) Hassan, the horse-trader, challenged three buyers to guess correctly the colour of his best horse. The horse's colour was either black, brown, or gray.

- Helena: The horse is not black.
- Gabriella: The horse is either gray or brown.
- Isabella: The horse is brown.

Hassan said that at least one of the buyers guessed right and at least one guessed wrong. Who were correct and who were not?

Solution. If the horse were black. Then Helena, Gabriella, and Isabella were wrong. This contradicts the given condition, since there is no one who guessed right. If the horse were brown. Then Helena, Gabriella, and Isabella were right. This again contradicts the given condition, since there is no one who guessed wrong. Therefore the horse is gray. In this case $\boxed{\text{Helena and Gabriella were right,}}$ and $\boxed{\text{Isabella were wrong.}}$ This satisfies the given condition. □

> **Remark.** Julie Cao, and Minh Pham correctly solved this problem. Chi-Mai Pham, Jérémy Nguyen almost got it perfectly. Minh Do Tue Le, Giang Nguyen could have had to make it better.

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Problem 3.2.9 (23-24-S2-I-P21)**.** (*10 points*) In Seville, a barber shaved all of and only those inhabitants who did not shave themselves. Sam met four local people Alain, Bernard, Columbus, and Daniel. Here were what they said:

- Alain: I am the only barber in this town.
- Bernard: There is only one barber in this town.
- Columbus: One of them is lying. I shaved both of them.
- Daniel: Alain is lying.

If exactly two of them telling the truth, then who were shaved?

Solution. If Alain told the truth, then Bernard did too. In this case, Columbus's statement was false, so he was not a barber. Daniel lied too. Thus, $\boxed{\text{Alain shaved Bernard, Columbus, and Daniel.}}$ Since there was no other barber in town, so Alain was not shaved.

If Alain lied, then Bernard, Columbus, and Daniel all told the truth. This is impossible. □

> **Remark.** Julie Cao, and Minh Pham correctly solved this problem. Minh Do Tue Le had a good start.

Problem 3.2.10 (23-24-S2-I-P22)**.** (*10 points*) In the Kingdom of Might and Magic, all inhabitants are either paladins or illusionists. For a *paladin*, everything he believes while he is awake is true, and everything he believes while he is asleep is false. An *illusionist* is the opposite: everything he believes while asleep is true, and everything he believes while awake is false.

Minh believes that he and his brother Khoa are both illusionists. At the same time, Khoa believes that both are paladins. As that moment, one of them was awake the other was asleep at the time.

Determine what each of them was at that moment.

Solution. Since one of them was asleep and the other was awake, and since they believed opposite things, they must be of the same type. If they were different types, their beliefs would be the opposite when they were both asleep or awake and would coincide when one was asleep and the other was awake.

If both of them were illusionists, then Minh was correct. Thus since he was an illusionist, he must be asleep at the time. Now, if both of them were paladins, then Minh was wrong. Since he was a paladin, he must also be asleep at the time.

Hence, $\boxed{\text{Minh was asleep and Khoa was awake } \textit{in every case.}}$ □

> **Remark.** Julie Cao, and Minh Pham correctly solved this problem. Chi-Mai Pham, Minh Do Tue Le had some good ideas.

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Problem 3.2.11 (23-24-S2-I-P23)**.** (*10 points*) Cuc, Dao, Lan, and Mai were hanging around in the kitchen while their mother were making some cookies. When they disappeared, their mother found out that some of cookies did also vanish. In the impromptu court held by their father, the following were discovered:

1. If both Cuc and Dao stole some cookies, then Lan were an accomplice.
2. If Cuc is guilty, then at least one of Dao and Lan was an accomplice.
3. If Lan is a cookie-thief, then Mai was an accomplice.
4. If Cuc is innocent, then Mai is indeed a cookie-stealer.

Which ones of them are definitely guilty and which ones are doubtful (nothing can be said about them)?

Solution. First, lets show that if Cuc is guilty, then so is Lan. Suppose that Cuc is guilty, then by (2), either Dao or Lan is guilty. If Dao is innocent, then it must Lan who is guilty. But, suppose Dao is guilty, then both Cuc and Dao are guilty, hence by (1) Lan is guilty too. This proves that if Cuc is guilty, then so is Lan.

Now, by (3) if Lan is guilty, so is Mai. Combining this with the previous argument, if Cuc is guilty, so is Mai. Furtheremore, by (4) if Cuc is innocent, then Mai is guilty. Therefore Mai is guilty regardless of what Cuc is.

Therefore $\boxed{\text{Mai is guilty and the rest are doubtful.}}$ □

> **Remark.** Giang Nguyen and Minh Pham correctly solved this problem.

Problem 3.2.12 (23-24-S2-I-P24)**.** (*10 points*) A school teacher in Ottawa had her purse stolen. The suspects were narrowed down to: Loo, Joo, Doo, Too, or Moo. When questioning, each child made three statements (numbering as below for ease of reference):

1. Loo: (1) I didn't take the purse; (2) I have never in my life stolen anything; (3) Too did it.
2. Joo: (4) I didn't take the purse; (5) My daddy is rich enough, and I have a purse of my own; (6) Moo knows who did it.
3. Doo: (7) I didn't take the purse; (8) I didn't know Moo before I enrolled in this school; (9) Too did it.
4. Too: (10) I am not guilty; (11) Moo did it; (12) Loo is lying when she says I stole the purse.
5. Moo: (13) I didn't take the purse; (14) Joo is guilty; (15) Doo can vouch for me because he knows me since I was born

Later, each child admitted that two of his/her statements were true and one was false.

How did Detective James Smarties solve this case?

> **Remark.** Find the one who has two sentences that both can be true, regardless whatever others said. Based on this, build a chain of reasoning, for example
>
> $$(1) \to (4) \to (8) \to \dots$$
>
> in order to clarify what known and what not.

Solution. First, it can be easily verified that both (10) and (12) are True, which means (11) is False (10) is True so (9) is False. (9) is False so (7) and (8) are True. (8) is True so (15) is False. (15) is False so (13) and (14) are True. Hence, $\boxed{\text{Joo did it}}$. □

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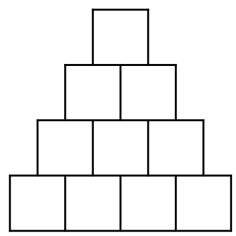
3.3 Counting revisited (A-Level)

Example 3.3.1 (23-24-S2-A-P1)

Squares with sides 1 centimeter long are arranged as below. Each row contains one more square than the row above it.

How many centimeters are in the perimeter of the figure that has 10 rows?

How many centimetres are in the perimeter of the figure is contains exactly 210 squares?



Solution. The perimeter of a figure consist of: (i) the top of 1 cm, and two sides each 1 cm; (ii) the next row having two outside half side parallel with the top, and two sides each 1 cm, and so on; (iii) the bottom having two outside half side parallel with the top, two sides each 1 cm, and as many as the number of squares in the bottom row.

So if we have the bottom having 10 squares, then there are 10 rows and the perimeter is $3\times 9+(3+10) = 40$ cm. Now let be the number of rows, note that the number of squares in the figure is $1 + 2 + \cdots + n = \frac{n(n+1)}{2} = 210$, then $n = \boxed{20.}$

By counting similarly as above, the perimeter for a $n-$row figure is $3 \times (n - 1) + (3 + n) = 4n$, or $4 \times 20 = \boxed{80.}$ □

Example 3.3.2 (23-24-S2-A-P2)

How many ways to write the number 3003 as a sum of two or more consecutive positive integers?

Solution. The sum of k consecutive integers $n, n + 1, \ldots, n + k - 1$ is:

$$n + (n + 1) + \cdots + (n + k - 1) = kn + (1 + \cdots + (k - 1)) = kn + \frac{1}{2}(k - 1)(k) = \frac{k(2n + k - 1)}{2}.$$

The two numbers k and $2n + k - 1$ (why?) have different parity: if one is odd then the other is even and vice versa. Thus this means that 3003 if can be written as a sum of two or more consecutive integers, then it has to be a product of two odd integers. Because $3003 = 3 \times 5 \times 7 \times 11$, so it has $2^4 = 16$ divisors or 16 ways to express it as a product of two odd integers. Note that $k < 2n + k - 1$, thus there are 8 such products. We have to eliminate the trivial case $k = 1$, so the number of solutions can only be $\boxed{7}$.

Note that the problem is equivalent to solve the equation $k(2n + k - 1) = 6006$. It is easy to see that

$$(k, n) \in \{(1, 3003),\ (2, 1501),\ (3, 1000),\ (6, 498),\ (7, 426),\ (11, 268),\ (13, 225),\ (14, 208)\}.$$

□

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Example 3.3.3 (23-24-S2-A-P3)

What is the number that will be listed beneath the number 121 when this triangular array is continue in the same manner? Where will the number 2023 appear?

		1		
	2	3	4	
5	6	7	8	9

Solution. It is easy to see that the last number of each row is a perfect square. In fact the last number of the nth row is n^2, which contains $2n-1$ numbers. This means that the number beneath 121 is on the 12th row, preceding the last number of that row, which is 144. Hence the number is $\boxed{143.}$

For the second question: $44^2 = 1936 < 2023 < 2025 = 45^2$. So 2023 is on the 45th row. This row starts with $44^2 + 1 = 1937$, so the position of 2023 is $2023 - 1937 + 1 = \boxed{87.}$ □

Example 3.3.4 (23-24-S2-A-P4)

The sequence shown below was formed by writing the first letter of the alphabet followed by writing the first two letters of the alphabet and continuing the pattern by writing one more letter of the alphabet each time.

$$A,\ A,\ B,\ A,\ B,\ C,\ A,\ B,\ C,\ D,\ A,\ B,\ C,\ D,\ E,\ \ldots$$

Continuing this pattern, what letter is the 280th letter in this sequence? What position is the letter Z? (the alphabet contains 26 letters).

Solution. Note that we have a series of sub-sequence with length $1, 2, 3, \ldots$:

$$\underbrace{A}_{1},\ \underbrace{A,\ B}_{2},\ \underbrace{A,\ B,\ C}_{3},\ \underbrace{A,\ B,\ C,\ D}_{4},\ \underbrace{A,\ B,\ C,\ D,\ E}_{5},\ \ldots$$

The number of letters in the $1^{\text{st}}, 2^{\text{nd}}, \ldots, 23^{\text{rd}}$ sub-sequences is:

$$1 + 2 + \cdots + 23 = \frac{23 \cdot 24}{2} = 276.$$

The number of letters in the $1^{\text{st}}, 2^{\text{nd}}, \ldots, 24^{\text{th}}$ sub-sequences is:

$$1 + 2 + \cdots + 24 = \frac{24 \cdot 25}{2} = 300.$$

Thus the 280th letter in the 24th sub-sequence, which starts at the letter $276 + 1 = 277^{\text{th}}$. Therefore the 280th letter is $\boxed{D.}$

The position of Z should be at the end of the 26th sequence, which is $1 + 2 + \cdots + 26 = \frac{26 \cdot 27}{2} = \boxed{351.}$ □

Example 3.3.5 (23-24-S2-A-P5)

There are 10 street lamps labeled $1, 2, 3, \ldots, 10$. Each is in front of a house in a row of houses. All of them are turned on. How many ways are there to turn off three lamps such that no two or three adjacent lamps will be turned off and no any lamp in each end will be turned off?

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Solution. Note that after turning of three lamps, then seven lamps are on, each of the lamps is indicated by a + sign:

$$+,\ +,\ +,\ +,\ +,\ +,\ +$$

There are 6 slots between these lamps. Thus, if we insert three − signs into these 6 slots (a slot can accept a single − sign only), then we just *reverse* the action that *turns off three lamps such that no two or three adjacent lamps will be turned off and no any lamp in each end will be turned off.*

There are altogether $\binom{6}{3} = \boxed{20}$ such ways. □

Example 3.3.6 (23-24-S2-A-P6)

A stick 100 cm long needs to be cut into 100 shorter sticks, each is 1 cm long. What is the minimum number of cuts required if Dinh Viet are allowed to cut several stick pieces at the same time?

Solution. Viet has to cut the stick into 100 pieces. With each cut, he can *at most double the number of pieces.* Since: $2^6 = 64 < 100 < 128 = 2^7$. Thus he has to make at least $\boxed{7}$ cuts. It can easily be demonstrated. □

Example 3.3.7 (23-24-S2-A-P7)

How many ways to sit seven people at a round table, if Lilian and Viet must not sit in adjacent seats?

Solution. [Solution 1] If all seven people were to sit in a row, we know that the number of permutations is

$$7 \times 6 \cdots \times 1 = 7!.$$

Since they sit in a round table, by rotating $ABCDEFG$ clockwise, we got $BCDEFGA$. For each sitting, there are 7 such rotations. So the number of ways is

$$\frac{7!}{7} = 6! = 720.$$

Now, let count the number of ways to sit seven people if Lilian and Viet sit in adjacent seats. Instead of looking at them as separate people, because they sit next to each other, we can represent them as if they were a single person. By following the above method in similar way for 6 persons, the number of ways to sit them is $5! = 120$.

If we now consider Lilian and Viet as two different persons, there are 2 ways to permute their sittings, meaning the total number of sittings is $2 \times 120 = 240$.

By subtracting this from 720 possible sittings, we obtain the number of ways to sit seven people at a round table, if Lilian and Viet must not sit in adjacent seats. Hence there are $720 - 240 = \boxed{480}$ *ways.* □

Solution. [Solution 2] There are 7 places for Lilian to sit. Since Viet cannot sit next to her, he only has 4 places to sit. Then, there are 5 places for the next person, 4 for the next, and so on. Note that because of the rotations, we need to divide the number of ways by 7. Thus, there are

$$\frac{7 \times 4 \times 5 \times 4 \times 3 \times 2 \times 1}{7} = \boxed{480}$$

ways for the seven people to sit if Lilian and Viet cannot sit next to each other. □

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Example 3.3.8 (23-24-S2-A-P8)

As a scientist at Innovative Foods Canada, Vu has almost completed the invention of the strong-like-a-Canadian egg, i.e. eggs that don't break easily. He has been asked to demonstrate for his boss, Huy, whatever he has got at the moment.

Vu wants to determine the highest floor in their 30-floor building from which an egg can fall without breaking. From previous experiments it is known that an egg still breaks: (i) if it is dropped from a certain height; or (ii) if it is dropped the times, regardless from which height.

So far, the project has only produced three eggs, so he drop some eggs to determine the floor and keeps at least one for demonstration?

What is the highest floor that Vu can **confidently** drop the last one for demonstration?

Solution. Vu can use only two eggs to determine the floor and keeps the third for demonstration.

Suppose that he drops the first egg at the kth floor.

If it breaks, then he drops the second egg at the 1st, then 2nd, then 3rd, and so on until $(k-1)^{\text{th}}$ to verify which floor from the dropping breaks the egg.

If it does not, then he can drops the first egg at the $2k^{\text{th}}$ floor. If it breaks, then he drops the second egg at $(k+1)^{\text{st}}$, then $(k+2)^{\text{nd}}$, then $(k+3)^{\text{rd}}$, and so on until $(2k-1)^{\text{th}}$ to verify which floor from the dropping breaks the egg.

He can continues to do so one more time for the first egg from the $3k^{\text{th}}$ floor and use the second egg to test from $(2k+1)^{\text{st}}$ to $(3k-1)^{\text{st}}$ floor.

Since the eggs always break at the fourth times, so only the third dropping is reliable. Hence $k-1=3$.

Here's how he tests: He drop the first egg from 4th floor, and if that breaks then use the second egg to test which floor from the 1st to the 3rd. Similarly he continues to test at the 8th, if that breaks then use the second egg to test which floor from the 5th to the 7th. Last, he tests the first egg at the 12th, floor, if that breaks then use the second egg to test which floor from the 9th to the 11th.

So the highest floor where he can drop the third egg is $\boxed{12.}$ □

Example 3.3.9 (23-24-S2-A-P9)

After collecting for the candy drive, Sally has eight \$1-bills, five \$5-bills, and three \$10-bills. She randomly selected three bills without replacement.

What is the total number of ways to select 3 bills, one of each type, from all the bills she has?

What is the probability that she chooses one of each type of bill? Express your answer as a common fraction.

Solution. The number of ways to select a \$1 bill from eight \$1-bills is $\binom{8}{1}$, a \$5 bill from five \$5-bills is $\binom{5}{1}$, and a \$10 bill from three \$10-bills is $\binom{3}{1}$.

The selections are independent, thus there are $\binom{8}{1}\binom{5}{1}\binom{3}{1} = \boxed{120}$ ways to select the three bills.

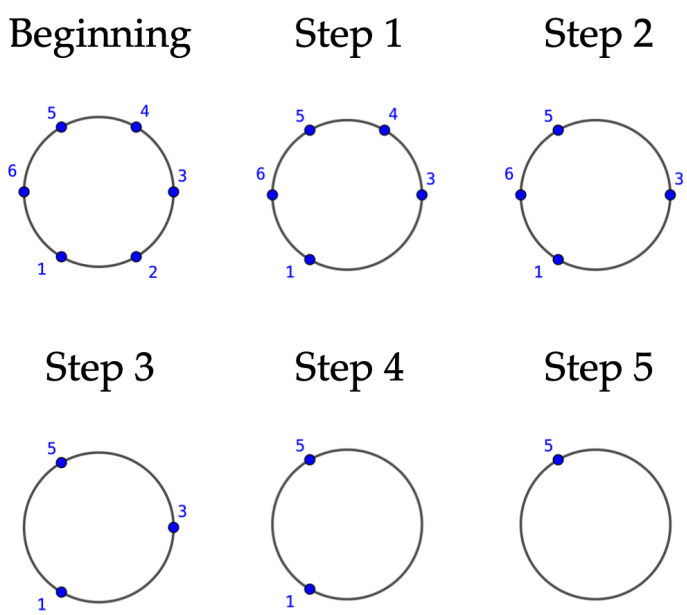
The total number of ways to choose three any bills from all the bills is $\binom{8+5+3}{3} = 560$.

Hence the probability to choose one from of each type of the bills is $\frac{120}{560} = \boxed{\frac{3}{14}.}$ □

Example 3.3.10 (23-24-S2-A-P10)

A group of n persons are standing in a circle, numbered consecutively counter-clockwise from 1 to n. Starting with the person number 2, in every step we remove every other person, proceeding counter-clockwise. For example with $n = 6$, the last person who is remaining is the person number 5 and the last person removed is the person number 1. See below.

If $n = 67$, who is the last person removed?



Solution. Easy to see that if $n = 2$, the remaining person is 1. If $n = 4$, then after removing person 2, and 4, the problem is same as $n = 2$. Similarly it can be proved that if $n = 2^k$, the remaining person is 1. In every case person $2^k - 1$ who is the last person removed.

Now $n = 67$. Let remove three person 2, 4, and 6:

$$1, \cancel{2}, 3, \cancel{4}, 5, \cancel{6}, 7, 8, 9, \ldots, 66, 67.$$

The number of people is reduced to $64 = 2^8$, and person 7 now is considered as the first one of 64 people:

$$7, 8, 9, \ldots, 66, 67, 1, 3, 5$$

Following the above reasoning, the remaining person is person 7, so the last person removed is $\boxed{5.}$ □

Remark. Who is the last person remove if $n = 2^m + k$, where m is a positive integer and $0 \leq k \leq 2^{m-1}$?

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3.4 Problems (A-Level)

Submission deadline: October 6, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 3.4.1 (23-24-S2-A-P11)**.** (*10 points*) How many ways to select in three single draws without replacement two red marbles and one blue marble from a bag containing six red marbles, eight yellow marbles, and seven blue marbles?

> **Remark.** There are some ambiguity in the original text. The correct problem text should read as follow:
>
> *How many ways to select in three single draws without replacement*, **first** *two red marbles*, **and then** *one blue marble, from a bag containing six red marbles, eight yellow marbles, and seven blue marbles?*

Solution. There are $\binom{6}{1}$ ways to select in a single draws without replacement one red marble from a bag containing six red marbles and other coloured marbles. Thus, there are $\binom{6}{1}\binom{5}{1}\binom{7}{1} = \boxed{210}$ ways to select in three single draws without replacement two red marbles and one blue marble from a bag containing six red marbles, eight yellow marbles, and seven blue marbles. □

> **Remark.** Thus, all students who arrive at the results of
>
> $$\binom{6}{1}\binom{5}{1}\binom{7}{1} = 210, \text{ or } \binom{6}{2}\binom{7}{1} = 105, \text{ or } 3 \times \binom{6}{1}\binom{5}{1}\binom{7}{1} = 630, \text{ or } 3! \times \binom{6}{1}\binom{5}{1}\binom{7}{1} = 1260.$$
>
> will earn the full point.

Problem 3.4.2 (23-24-S2-A-P12)**.** (*10 points*) What is the probability to select in three single draws without replacement two red marbles and one blue marble from a bag containing six red marbles, eight yellow marbles, and seven blue marbles?

> **Remark.** There are some ambiguity in the original text. The correct problem text should read as follow:
>
> *hat is the probability to select in three single draws without replacement*, **first** *two red marbles*, **and then** *one blue marble, from a bag containing six red marbles, eight yellow marbles, and seven blue marbles?*

Solution. From the previous problem, there are 210 ways to select in three single draws without replacement two red marbles and one blue marble from a bag containing six red marbles, eight yellow marbles, and seven blue marbles. The total number of ways to select three marbles in three single draws without replacement from a bag containing six red marbles, eight yellow marbles, and seven blue marbles is $\binom{21}{1}\binom{20}{1}\binom{19}{1}$.

Hence, the probability is $\frac{6\times5\times7}{21\times20\times19} = \boxed{\frac{3}{38}.}$ □

> **Remark.** Thus, all students who have computed the previous problem in one way and computed the total number of ways to just select three marbles (in three single draws without replacement) consistently in the same manner should arrive to the same result as this problem.

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Problem 3.4.3 (23-24-S2-A-P13)**.** (*10 points*) Catherine participated in a game of Snap. Here's how the game is played.

The house and each of fifteen players have shuffled a deck of 17 cards numbered from 1 to 17. After that everyone turned their cards over, one by one. When a player's card matched the house's one, the player shouted "Snap!" and scored a point. At the end, each player counted the number of points and that was the final score of the player.

It turned out that no two players have the same score and no player score 0 or 17 points. Catherine's score was the second highest.

What's her score?

Solution (Official solution and same as solutions from Henry Trinh, Jason To, and Minh Nguyen) — Note that there is no such score as 16. If a player has 16 cards matched, then the 17th card should also match. Therefore the player's score must be 17, which contradicts the given conditions. Therefore the only possible scores are $1, 2, \ldots, 15$, which means the second highest score is $\boxed{14.}$

Remark. Henry Trinh, Jason To, and Minh Nguyen correctly solved this problem. An Nguyen, Karl Le, Ryan Nguyen, Tran Hieu Khoa made the same mistake not considering the impossibility of the case of score 16.

Problem 3.4.4 (23-24-S2-A-P14)**.** (*10 points*) 10 distinct points are marked on a circle.

What are the maximum number of distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices?

What are the maximum possible number of intersections of diagonals formed by those vertices?

Solution (Official solution and same as solutions from An Nguyen, Karl Le, and Minh Nguyen) — The number of triangles with three points as vertices is $\binom{10}{3}$. Similarly the number of quadrilateral is $\binom{10}{4}$, and so on and the number of decagons is $\binom{10}{10}$ (the polygon itself). Therefore the total number of such polygons is:

$$\binom{10}{3} + \binom{10}{4} + \ldots + \binom{10}{10} = \boxed{968}$$

For the second question, every four points form a distinct pair of diagonals. Thus, the maximum number of intersections is $\binom{10}{4} = 210$.

Remark. An Nguyen, Karl Le, and Minh Nguyen correctly solved this problem. Tran Hieu Khoa made a calculation mistake at the end of the second question: $\binom{10}{4}$ is 210, and not 5040. Henry Trinh, Jason To, Karl Le, Ryan Nguyen, and Samuel Luong managed to solve only the first question.

Problem 3.4.5 (23-24-S2-A-P15)**.** (*10 points*) In an up-and-down counting number, the digits increase to a maximum digit and then decrease. This maximum is not the first or the last digit. For example 256850 is an up-and-dow, but 255876 is not. How many different 4-digit up-and-dow numbers have the maximum digit of 6?

Solution (Official solution and similar solutions from An Nguyen, Antoine Dao, Jason To, Minh Hai Vu, Karl Le, Samuel Luong, and Minh Nguyen) — The digit 6 can be the hundreds digit or the tens digit. It cannot be both. We count the numbers in two cases separately.

Case 1: 6 is the hundreds digit: The number is in _ 6 _ _ format. There are 5 possible values for the thousands digits: 1, 2, 3, 4, 5. There are 5 possible values for the tens digits: 5, 4, 3, 2, 1. A tens digit of 4 produces 4 possible ones digits: 3, 2, 1, 0. Similarly a tens digit of 3, 2, and 1 produces 3, 2, and 1 possible ones digits. Therefore the number of ways for the tens and ones digits is: $5+4+3+2+1=15$. Hence the total of all possible numbers in this case is $5 \times 15 = 75$.

Case 2: 6 is the tens digit: The number is in _ _ 6 _ format. There are 4 possible values for the thousands digits: 1, 2, 3, 4. A thousands digit of 1 produces 4 possible hundreds digits: 3, 2, 1, 0. Similarly to case 1, the number of ways to fill the thousands and hundreds digits is $4+3+2+1=10$. There are 6 possible values for the ones digits: 0, 1, 2, 3, 4, 5. Hence the total of numbers in this case is $10 \times 6 = 60$.

The answer is $75 + 60 = \boxed{135.}$

Remark. An Nguyen, Antoine Dao, Jason To, Minh Hai Vu, Karl Le, Samuel Luong, and Minh Nguyen correctly solved this problem. Ryan Nguyen made a mistake when determined there would be only five possibilities for the last digit.

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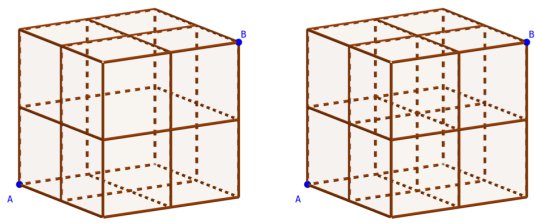
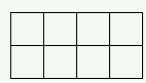
Problem 3.4.6 (23-24-S2-A-P16)**.** (*10 points*) A cube is built from 8 unit cubes as below in the left diagram. How many paths of length 6 on the surface of the cube are there from to along the edges of the component unit cubes? Now all paths through the inner of the cube are also allowed as below in the right diagram. How many paths of length 6 of the cube are there from to along the edges of the component unit cubes?



Solution (Official solution and same as Minh Nguyen's and Tran Hieu Khoa's) — Take two faces, one has A, the other has B as a vertex, share one edge. A flattened diagram with A at the lower left and B at upper right corners.



The number of path with length 6 and only two moves upwards from A to B on this is $\binom{6}{2}$. For all six pairs of faces, we have $6 \times \binom{6}{2}$. Any path along the borders as the bold line in above is counted twice by two different pairs of faces, there are $6 \times \binom{4}{2}$ such paths. Hence, the answer for the first question is:

$$6 \cdot \left(\binom{6}{2} - \binom{4}{2} \right) = 6 \cdot (15 - 6) = \boxed{54.}$$

The second question requires counting ways length 6, each formed by two moves in a dimension of three dimensions. So we can choose two moves in $z-$dimension from 6 possibilities, but only 2 moves in $y-$dimension from the remaining 4 possibilities, and 2 moves in $x-$dimension from the remaining 2 possibilities. So the total number of ways is:

$$\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{6!}{2!2!2!} = \boxed{90.}$$

Remark. Only Minh Nguyen and Tran Hieu Khoa correctly solved this problem. Antoine Dao and Ryan Nguyen managed to count a partial sum.

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Problem 3.4.7 (23-24-S2-A-P17). (*10 points*) For $n > 1$, consider an $n \times n$ chessboard and place pieces at the centres of different squares. With $2n$ chess pieces on the board, show that there are 4 pieces among them that form the vertices of a parallelogram.

Show that there is a way to place $2n - 1$ chess pieces so that no 4 of them form the vertices of a parallelogram.

Solution (Official solution and same solutions from Karl Le, Minh Nguyen, Ryan Nguyen, and Tran Hieu Khoa) — Since there can be at most n pieces that are left most in their rows (some rows may be empty), there are at least n pieces that are not the leftmost in their row.

Record the distances (the number of squares) between the leftmost piece and the other pieces on the same row. There are then at least n distances recorded. Since the distances range from 1 to $n - 1$, by the Pigeonhole Principle, at least two of these distances are the same. Thus, there are at least two rows each containing two pieces that are the same distance apart. These four pieces yield a parallelogram.

If $2n - 1$ pieces are placed, for example, $\boxed{\text{on the squares of the first column and the first row,}}$ then there is no parallelogram.

Remark. Only Karl Le, Minh Nguyen, Ryan Nguyen, and Tran Hieu Khoa correctly solved this problem. Jason To proposed some idea.

Problem 3.4.8 (23-24-S2-A-P18). (*10 points*) Binh, Anthony, and Vu participated in a beauty contest. The contest committee had a number of judges. Each judge voted for only one of the three participants.

A spectator called Minh, counted 59 judges, with 15 judges voting for Binh or Anthony, 18 voting for Anthony or Vu, and 20 voting for Vu or Binh. However Minh was notorious for poor counting, each number he counted in above differs from reality by no more than 13.

How many judges voted for Anthony?

Solution (Official solution and same solution from An Nguyen, Minh Nguyen, and Ryan Nguyen) — Let a be the number judges voted Binh or Anthony, b be the number judges voted for Anthony or Vu, and c be the number judges voted for for Vu or Binh. Let s be the total number of judges. Easy to see that:

$$a + b + c = 2s.$$

Because Minh counted poorly, so

$$\begin{aligned} &a \leq 15 + 13 = 28,\ b \leq 18 + 13 = 31,\ c \leq 20 + 13 = 33 \\ \Rightarrow &2s = a + b + c \leq 28 + 31 + 33 \Rightarrow s \leq 46 \quad (*) \end{aligned}$$

Also because Minh counted poorly, so

$$s \geq 59 - 13 = 46 \quad (**)$$

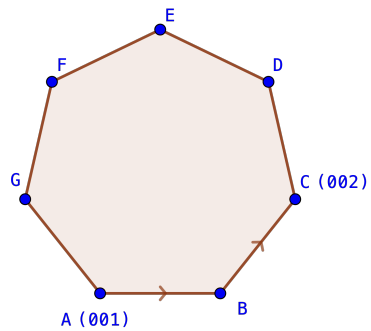
From (*) and (**), $s = 46$, thus $a = 28,\ b = 31,\ c = 33$.

Therefore the number of judges voted for Anthony is $a + b - s = 28 + 31 - 46 = \boxed{13.}$

Remark. Only An Nguyen, Minh Nguyen, and Ryan Nguyen correctly solved this problem.

Problem 3.4.9 (23-24-S2-A-P19)**.** (*10 points*) The director of a secret service has put together a list of mutual surveillance assignments for his seven agents, codename 001 through 007. Agent 001 will watch the agent who is watching 002; agent 002 will watch the agent who watches 003; and so on. Agent 007 will watch the agent watching 001. Can you deduce who watches whom?

The service recruits a new agent, and gives the agent codename 008. How could the director change the surveillance assignments to keep it working for all 8 agents?



Solution (Official solution and somewhat similar to solution from Ryan Nguyen) — First, let A, B, C, D, E, F, G be seven points that represent the agents. We will assign the points to agents a long the way as the solution is described. First, let A be agent 001. We knows that A watches B, and B watches agent 002, so let C be agent 002.

By continuing as above the points establish a series of path from 001 to 002, 002 to 003, ..., 007 to 001. All these paths are parts of a circle containing all the agents. It means that E is 003, G is 004, B is 005, D is 006, and finally F is 007.

For the second question, note that the reasoning requires an odd amount of agents. The agent 004 should watch an agent who watches the agent 001, which contradicts the given conditions.

Remark. Ryan Nguyen has the best near-perfect solution. Minh Hai Vu and Jason To had good attempt but still struggle to point out why the case does not work for 8 agents. Karl Le, Minh Nguyen, and Tran Hieu Khoa managed to solve the first question. Chi Khanh Pham had a good idea but having difficulties to express her reasoning (more reason to learn the Introductory Curriculum?)

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Problem 3.4.10 (23-24-S2-A-P20)**.** (*10 points*) Kevin has developed a taste for cookie. He has been collecting cookies to celebrate the coming Christmas. He has 15 cookie jars containing $1, 2, 3, \ldots, 15$ cookies, respectively.

On a cold dark night in December, the cookie monster came into the house. He decided to choose a subset of jars (some number of jars) and take the same number of cookies from each jar and then left the house. This hideous act was repeated for another three nights before Christmas.

Is it possible that the heartless monster ate all the cookies that Kevin has spent great effort to collect leaving him devastated on the Christmas Eve?

Solution (Official solution and somewhat similar to solutions from An Nguyen, Chi Khanh Pham, Jason To, Karl Le, Ryan Nguyen, and Samuel Luong) — The key idea is to realize that jars with the same number of cookies can be treated as if they were just one jar: for if you've got two (or more) jars with k cookies, the cookie monster might as well do the same thing with them in each subsequent step - those jars are depleted at the same rate.

With this in mind, we just need to focus on the different numbers of cookies in the jars at each night. Below is a solution:

$$\{15\ 14, \cdots,\ 1\} \xrightarrow{\text{remove 8 cookies}} \{7\ 6, \cdots,\ 1\}$$
$$\{7,\ 6, \cdots,\ 1\} \xrightarrow{\text{remove 4 cookies}} \{3,\ 2,\ 1\}$$
$$\{3,\ 2,\ 1\} \xrightarrow{\text{remove 2 cookies}} \{1\}$$
$$\{1\} \xrightarrow{\text{remove 1 cookie}} \emptyset$$

Note that $\emptyset$ means the empty set.

Remark. Only An Nguyen, Chi Khanh Pham, Jason To, Karl Le, Ryan Nguyen, and Samuel Luong correctly solved this problem.

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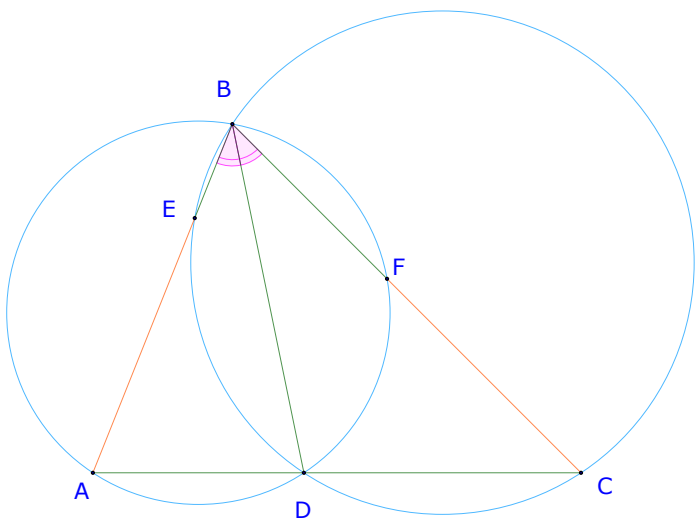
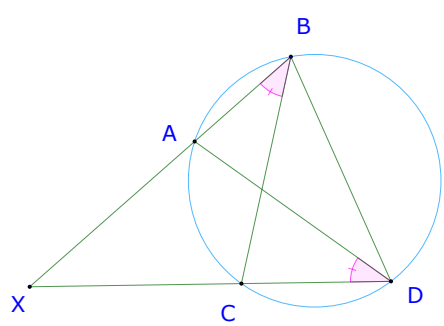
3.5 Geometry warmup (O-Level)

Example 3.5.1 (St. Petersburg City MO 1996)
Let BD be the angle bisector of $\angle ABC$ in $\triangle ABC$ with D on AC. The circumcircles of $\triangle BDC$ and $\triangle ABD$ meet AB and BC at E and F, respectively/. Prove that $AE = CF$.



Theorem (Intersecting Secants Theorem)
Let AB and CD be two lines that intersect at X, such that A and C lie between X, B and X, D, respectively. Then the quadrilateral $ABDC$ is cyclic if and only if $XA \cdot XB = XC \cdot XD$.



Intersecting Secants Theorem. Note that $\angle AXD \equiv \angle CXB$ (both angle are the same), thus

$$ABCD \text{ is cyclic } \Leftrightarrow \angle XBC = \angle XDA \Leftrightarrow \triangle XBC \sim \triangle XDA \Leftrightarrow \frac{XB}{XC} = \frac{XD}{XA} \Leftrightarrow XA \cdot XB = XC \cdot XD.$$

□

St. Petersburg City MO 1996. By Intersecting Secants Theorem for circle $(BEDC)$, secants BE, CD, $AE \cdot AB = AD \cdot AC$. Similarly for $(ADFB) : CF \cdot CB = CD \cdot CA$. Thus $\frac{AE \cdot AB}{CF \cdot CB} = \frac{AD}{AC}$. By Angle Bisector Theorem, $\frac{AB}{CB} = \frac{AD}{CD}$, hence $\boxed{AE = CF.}$ □

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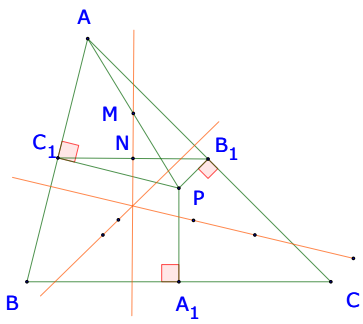
Example 3.5.2 (23-24-S2-O-P2)

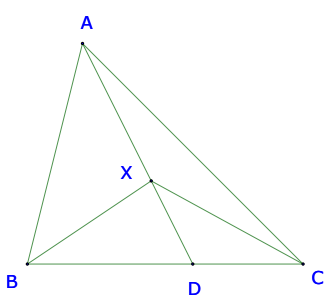
Let P be an arbitrary point inside $\triangle ABC$. Let A_1, B_1, and C_1 be the feet of the perpendiculars from P to BC, CA, and AB, respectively. Prove that the lines that pass through the midpoints of PA and B_1C_1, PB and C_1A_1, and PC and A_1B_1 are concurrent.

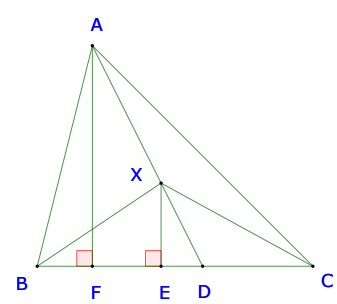


Proof. Let M and N be the midpoints of PA and B_1C_1, respectively. Since $\angle AB_1P = \angle AC_1P = 90°$, thus AC_1PB_1 is cyclic and AP is the diameter of (AC_1PB_1), hence M is its center. N is the midpoint of B_1C_1, thus MN is the perpendicular bisector of B_1C_1. Similarly the line pass through the midpoints of PB and C_1A_1 is the perpendicular bisector of C_1A_1, and the line pass through the midpoints of PC and A_1B_1 is the perpendicular bisector of A_1B_1. Thus $\boxed{\text{these lines meet at the circumcenter of } \triangle A_1B_1C_1.}$ □

Lemma (Area Lemma)

ABC is a triangle, point $D \in BC$, $X \in AD$, then $\frac{[BCX]}{[BCA]} = \frac{DX}{DA}$.





Proof. Let E, F be the feet of the perpendiculars from X and A to BC. Note that $\triangle AFD \sim \triangle XED$,:

$$\frac{[BCX]}{[BCA]} = \frac{\frac{1}{2}(XE)(BC)}{\frac{1}{2}(AF)(BC)} = \frac{XE}{AF} = \frac{DX}{DA}.$$

□

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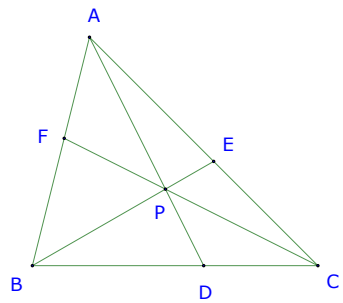
Theorem (van Aubel's Theorem)
In $\triangle ABC$, $D \in BC$, $E \in CA$, $F \in AB$, such that AD, BE, and CF are concurrent at P, prove that:

$$\frac{AP}{PD} = \frac{AE}{EC} + \frac{AF}{FB}.$$



Proof. Note that

$$\frac{AE}{EC} = \frac{[APB]}{[BPC]}, \ \frac{AF}{FC} = \frac{[APC]}{[BPC]} \Rightarrow \frac{AE}{EC} + \frac{AF}{FB} = \frac{[BAC]}{[BPC]} - 1$$

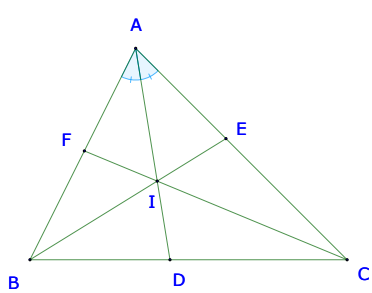
By the Area Lemma for triangle ABC and points P, D:

$$\frac{PD}{AD} = \frac{[BPC]}{[BAC]} \Rightarrow \frac{AD}{PD} = \frac{[BAC]}{[BPC]} \Rightarrow \frac{AP}{PD} = \frac{AD - PD}{PD} = \frac{[BAC]}{[BPC]} - 1 = \frac{AE}{EC} + \frac{AF}{FB}.$$

□

Example 3.5.3 (23-24-S2-O-P3)
Let a, b, c be the side lengths of $\triangle ABC$ and I be its incentre. Let point $D = AI \cap BC$. Prove that:

$$\frac{AI}{ID} = \frac{b+c}{a}, \ \frac{ID}{AD} = \frac{a}{a+b+c}$$



Proof. $E = BI \cap AC$, $F = CI \cap AB$. By the van Aubel's Theorem and the Angle Bisector Theorem:

$$\frac{AI}{ID} = \frac{b+c}{a} = \frac{AE}{EC} + \frac{AF}{FB} = \frac{AB}{BC} + \frac{AC}{BC} = \frac{b+c}{a} \Rightarrow \frac{ID}{AD} = \frac{a}{a+b+c}.$$

□

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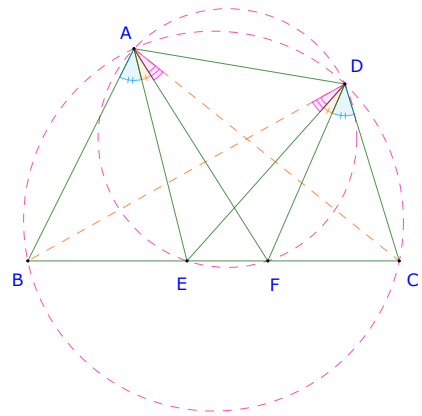
Example 3.5.4 (All-Russian Olympiad 1996, Grade 10/Problem 1)

Points E and F are given on side BC of convex quadrilateral $ABCD$ (with $BE < BF$). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.



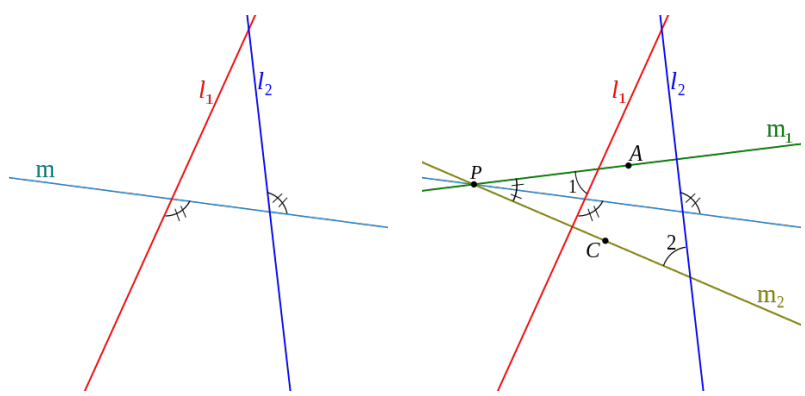
Solution. $\angle EAF = \angle FDE$ so $AEFD$ is cyclic, or $\angle ADF + \angle AEF = 180°$, now:

$\angle ADC + \angle ABC = (\angle ADF + \angle FDC) + (\angle AEF - \angle BAE) = \angle ADF + \angle AEF = 180° \Rightarrow ABCD$ is cyclic $\Rightarrow \angle BAC = \angle BDC \Rightarrow \angle FAC = \angle BDE$

□

Definition (Antiparallel lines)**.** In geometry, two lines ℓ_1 and ℓ_2 are antiparallel with respect to a given line m if they each make congruent angles with m in opposite senses (see the diagram below on the left.)

More generally, lines ℓ_1 and $\ell_2$are antiparallel with respect to another pair of lines m_1 and m_2 if they are antiparallel with respect to the angle bisector of m_1 and m_2 (see the diagram below on the right.)



Theorem (Antiparallel sides in cyclic quadrilaterals)

In any cyclic quadrilateral, any two opposite sides are antiparallel with respect to the other two sides.

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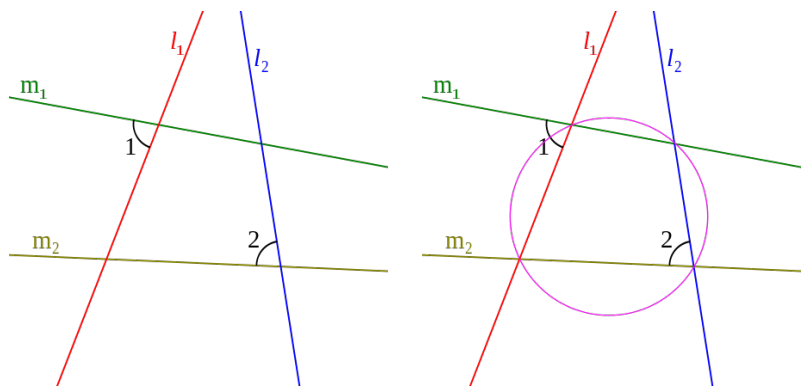
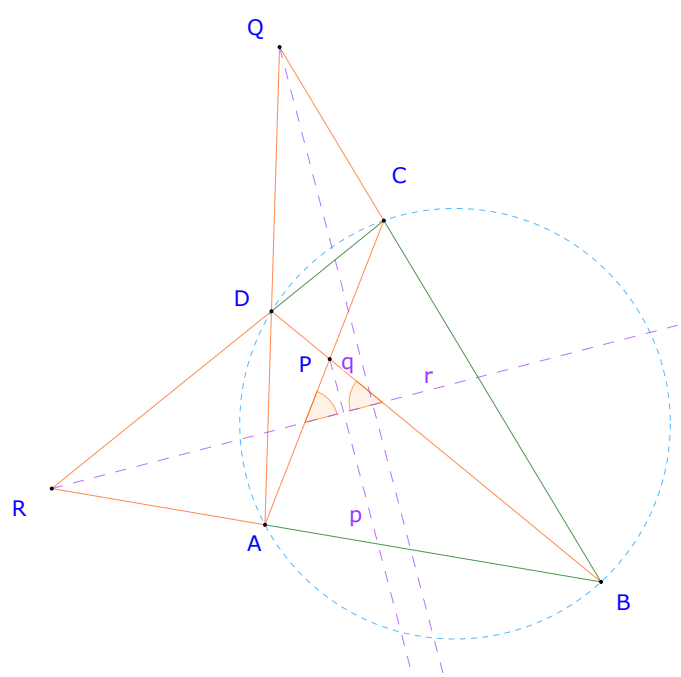
Given two lines m_1 and m_2, lines ℓ_1 and ℓ_2 are antiparallel with respect to m_1 and m_2 if $\angle 1 = \angle 2$ (see the diagram below on the left.) In any quadrilateral inscribed in a circle, any two opposite sides are antiparallel with respect to the other two sides. (see the diagram below on the right.)



Example 3.5.5 (23-24-S2-O-P5)

In a cyclic quadrilateral $ABCD$, let $P = AC \cap BD$, $Q = AD \cap BC$, and $R = AB \cap CD$. Denote p, q, r the angle bisector of $\angle APB$, $\angle AQB$, $\angle BRC$, respectively. Prove that r is perpendicular to both p, q.

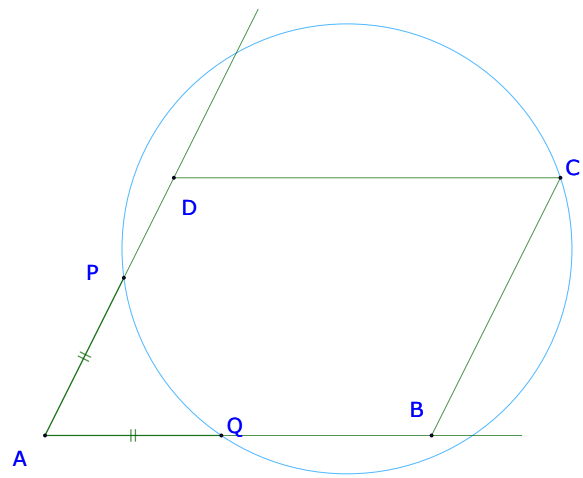


Solution. Since $ABCD$ is cyclic, lines AC and BD are antiparallel with respect to r, so they form with r an isosceles triangle with the based on r, therefore its angle bisector is perpendicular to r. Thus $r \perp p$.

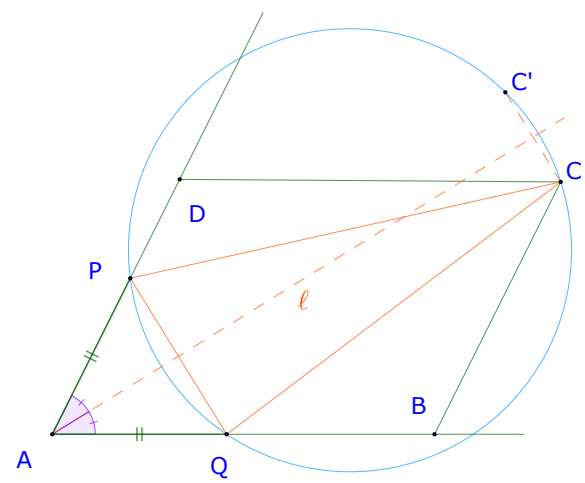
At the same time $ABCD$ formed by lines AD and BC, thus AD and BC are also antiparallel to r. Similar as above r is perpendicular to q. $\square$

Example 3.5.6 (23-24-S2-O-P6)

(*10 points*) Point P, Q are given on the sides AB, AD respectively, of a parallelogram $ABCD$ ($AB > AD$) such that $AP = AQ = x$. Prove that as x varies, the circumcircles of the triangles PQC pass through another fixed point (other than C).



Solution. Let ℓ be the angle bisector of $\angle PAQ$. Since $\triangle APQ$ is isosceles, ℓ is the perpendicular bisector of PQ.



Therefore ℓ is through the circumcenter of $\triangle APQ$. Point C', the reflection of C over ℓ is always on the circle. This is the desired fix point. $\square$

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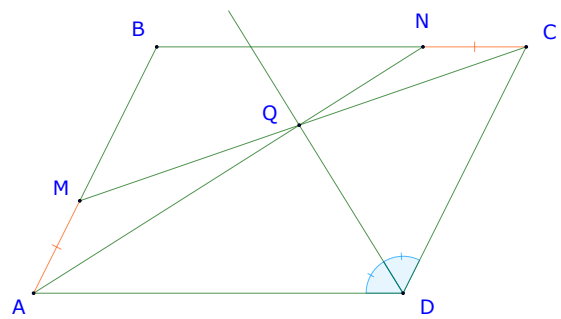
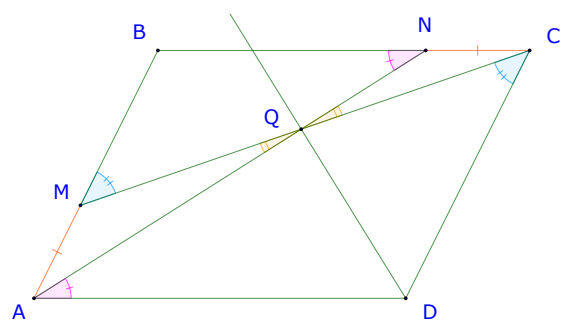
Example 3.5.7 (Bundeswettbewerb Mathematik 2003/Problem 3)

Let $ABCD$ be a parallelogram. Let M and N be points on the sides AB and BC, respectively, such that $AM = CN \neq 0$. The lines AN and CM intersect at a point Q. Prove that the point Q lies on the bisector of the angle $\measuredangle ADC$.



Remark (Solution 1)**.** In the solution below, we shows how to combine pairs of equal angles (by parallel lines) with the the Law of Sines for proving equality of angles, or a line as an angle bisector.

Solution. Since $AB \parallel DC$ and $AD \parallel BC$, thus $\angle QAD = \angle QNB$, $\angle QMB = \angle QCD$, $\angle AQM = \angle CDN$.



Now, by the Law of Sines for $\triangle AQM$ and $\triangle CQD$, note that $\angle AMQ = 180^\circ - \angle QMB$, thus:

$$\frac{\sin\angle AQM}{AM} = \frac{\sin\angle AMQ}{AQ} = \frac{\sin\angle QMB}{AQ} = \frac{\sin\angle QCD}{AQ} = \frac{QD\frac{\sin\angle QDC}{QC}}{AQ} = \frac{QD}{QC\cdot AQ}\cdot\sin\angle QDC$$

Similarly, by the Law of Sines for $\triangle CQN$ and $\triangle AQD$, note that $\angle CNQ = 180^\circ - \angle QNB$, thus:

$$\frac{\sin\angle CQN}{CN} = \frac{\sin\angle CNQ}{CQ} = \frac{\sin\angle QNB}{CQ} = \frac{\sin\angle QAD}{CQ} = \frac{QD\frac{\sin\angle QDA}{QA}}{CQ} = \frac{QD}{QA\cdot CQ}\cdot\sin\angle QDA$$

Since $\frac{\sin\angle AQM}{AM} = \frac{\sin\angle CQN}{CN}$, thus $\sin\angle QDC = \sin\angle QDA$. Hence $\boxed{QD \text{ is the angle bisector of } \angle ADC.}$ □

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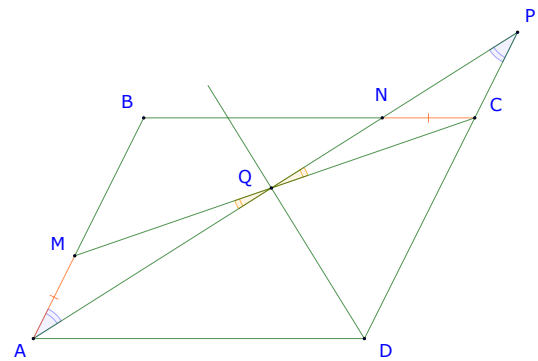
Remark (Solution 2)**.** Below, with some insights by using additional lines and intersections, we show an proof using only the Angle Bisector Theorem.

Solution. Extend AN towards N to intersect CD at P.



Note that QD is the angle bisector in $\triangle ADP$ if and only if $\frac{AQ}{QP} = \frac{AD}{DP}$. However,

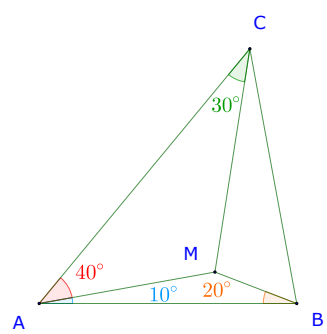
$$\triangle AQM \sim \triangle PQC \Rightarrow \frac{AQ}{QP} = \frac{AM}{CP} = \frac{CN}{CP} = \frac{AD}{DP}.$$

The last equality stands since $\triangle PCN \sim \triangle PDA$.

Note that by letting $P = MC \cap AD$ we arrive to a similar solution. □

Example 3.5.8 (USAMO 1996/Problem 5)

Let ABC be a triangle, and M an interior point such that $\angle MAB = 10°$, $\angle MBA = 20°$, $\angle MAC = 40°$ and $\angle MCA = 30°$. Prove that the triangle is isosceles.



Remark (Solution 1)**.** In this solution, we show a classic approach proving the equality of two segment by using the Law of Sines and the Law of Cosines to compute their lengths.

Solution. [Solution 1] It is easy to see that $\angle AMB = 150°$ and $\angle AMC = 110°$. Now by the Law of Sines on triangles ABM and ACM, we have

$$\frac{AB}{\sin 150°} = \frac{AM}{\sin 20°}, \quad \frac{AC}{\sin 110°} = \frac{AM}{\sin 30°} \Rightarrow \frac{AB}{AC} = \frac{\sin 150° \sin 30°}{\sin 20° \sin 110°}.$$

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WLOG, let $AB = \sin 150^\circ \sin 30^\circ = \frac{1}{4}$ and $AC = \sin 20^\circ \sin 110^\circ$ (why can we do this?). Then by the Law of Cosines, we have

$$\begin{aligned} BC^2 &= AB^2 + AC^2 - 2(AB)(BC)\cos\angle BAC \\ &= \frac{1}{16} + \sin^2 20^\circ \sin^2 110^\circ - 2\left(\frac{1}{4}\right)\sin 20^\circ \sin 110^\circ \cos 50^\circ \\ &= \frac{1}{16} + \sin^2 20^\circ \sin^2 110^\circ - \frac{1}{2}\sin 20^\circ \sin 110^\circ \sin 40^\circ \\ &= \frac{1}{16} + \sin^2 20^\circ \sin^2 110^\circ - \sin 20^\circ \sin 110^\circ \sin 20^\circ \cos 20^\circ \\ &= \frac{1}{16} \Rightarrow \boxed{AB = BC} \end{aligned}$$

□

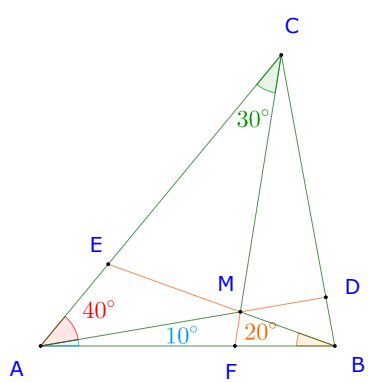
Remark (Solution 2)**.** In this solution, by extending AM, BM, and CM, to intersect the opposite sides at D, E, and F respectively,. We have three segments AD, BE, CF concurrent at M, this gives the idea to explore the Ceva Theorem in Trigonometric Form.

Theorem (Ceva Theorem in Trigonometric Form)

Cevians AD, BE, CF concur if and only if

$$\frac{\sin\angle BAD}{\sin\angle DAC} \cdot \frac{\sin\angle CBE}{\sin\angle EBA} \cdot \frac{\sin\angle ACF}{\sin\angle FCB} = 1.$$

Solution. [Solution 2] Extending AM, BM, and CM, to intersect the opposite sides at D, E, and F respectively.



Let $x = \angle CBM$, then $\angle BCM = 80^\circ - x$. We have

$$\angle BAD = 10^\circ,\ \angle DAC = 40^\circ,\ \angle CBE = x,\ \angle EBA = 20^\circ,\ \angle ACF = 30^\circ, \angle FCB = 80^\circ - x.$$

$$\sin 10^\circ \sin x \sin 30^\circ = \sin 40^\circ \sin 20^\circ \sin(80^\circ - x) \Leftrightarrow \sin 10^\circ \sin x = 2\sin 40^\circ \sin 20^\circ \cos(10^\circ + x)$$

$$\Leftrightarrow \sin 10^\circ \sin x = \left(\cos 20^\circ - \frac{1}{2}\right)\cos(10^\circ + x) \Leftrightarrow 2\sin 10^\circ \sin x = 2\cos 20^\circ \cos(10^\circ + x) - \cos(10^\circ + x)$$

$$\Leftrightarrow \cos(10^\circ - x) - \cos(10^\circ + x) = \cos(30^\circ + x) + \cos(10^\circ - x) - \cos(10^\circ + x)$$

$$\Leftrightarrow \cos(30^\circ + x) = 0 \Leftrightarrow x = 60^\circ \Leftrightarrow \boxed{\angle BAC = \angle ACB = 50^\circ.}$$

□

Remark (Solution 3)**.** Similar to solution 2, with an advanced understanding of monotonic function.

Solution. [Solution 3] Similar to the previous solution, we arrive at

$$\sin 10^\circ \sin x \sin 30^\circ = \sin 40^\circ \sin 20^\circ \sin(80^\circ - x)$$

Note that $\sin 20^\circ = 2 \sin 10^\circ \cos 10^\circ$, thus

$$\sin x = 4 \sin(80^\circ - x) \sin 40^\circ \cos 10^\circ.$$

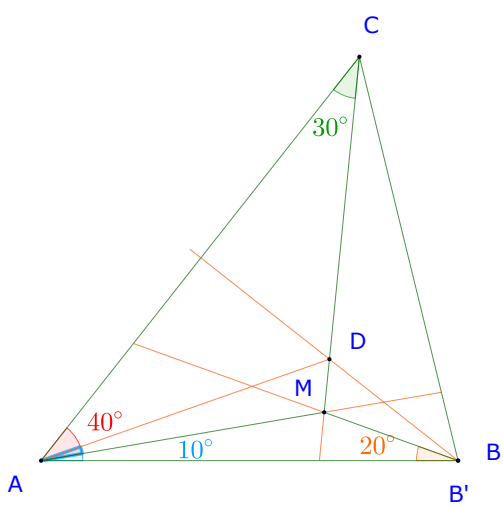
The left hand side is increasing function of x, while the right hand side is a decreasing one. For $x = 60^\circ$, then both sides are equal:

$$\begin{aligned} &4 \sin(80^\circ - 60^\circ) \sin 40^\circ \cos 10^\circ = 4 \sin 20^\circ \sin 40^\circ \cos 10^\circ = 2(\cos 20^\circ - \cos 60^\circ) \cos 10^\circ \\ &= 2 \cos 20^\circ \cos 10^\circ - \cos 10^\circ = \cos 30^\circ + \cos 10^\circ - \cos 10dg = \cos 30^\circ = \sin 60^\circ. \end{aligned}$$

Thus $x = 60^\circ$, and we have the same conclusion. □

Remark (Solution 4)**.** In this solution, we show a non-trigonometric solution using refection to create symmetry.

Solution. [Solution 4] Let the reflection of AB over AM intersecting MC at D.



First, $\angle DAC = 40^\circ - \angle MAB = 30^\circ = \angle DCA$, thus $\triangle ADC$ is isosceles at D.

Now, let the perpendicular bisector of AC, through D to intersect AB at B'.

By the symmetry created by the reflection, AM is the angle bisector of $\angle DAB'$. In addition, $\angle ADB' = 180^\circ - \frac{1}{2}\angle ADC = 120^\circ$ and $\angle ADM = \angle DAC + \angle DCA = 60^\circ$, thus DM is a bisector of $\angle ADB'$.

Therefore M is the incenter of $\triangle ADB'$. Thus MB' is the angle bisector of $\angle AB'D = 180^\circ - \angle B'AD - \angle ADB' = 40^\circ$, therefore $\angle MB'A = 20^\circ$. This means that $B \equiv B'$.

Hence, B is on the perpendicular bisector of AC, thus $\triangle ABC$ is isosceles. □

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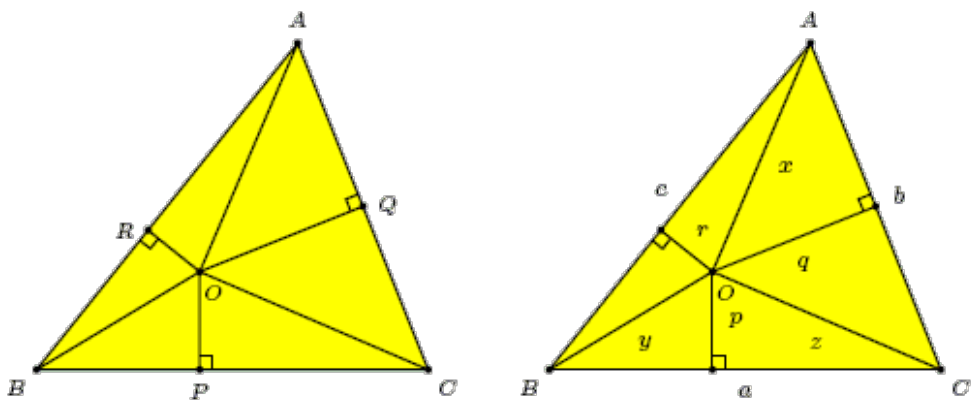
Example 3.5.9 (Erdos-Mordell's Theorem)

Let O be the point inside the triangle ABC. Denote by P, Q, R the feet of perpendiculars from O to BC, CA, and AB. Prove that

$$OA + OB + OC \geq 2(OP + OQ + OQ).$$

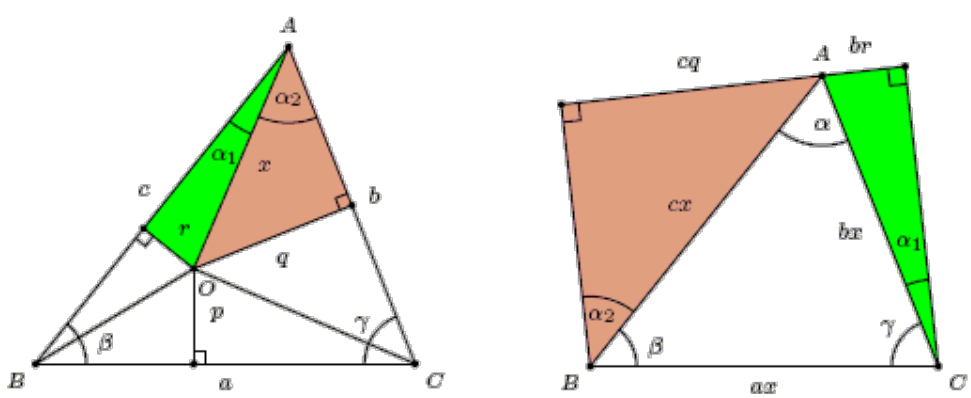


Lemma

For the quantities x, y, z, p, q, r in $\triangle ABC$, shown as above in the right diagram, we have

$$ax \geq br + cq,\ by \geq ar + cp, czgeaq + bp.$$

Proof. For the proof we construct a trapezoid as shown. The diagram makes the first inequality $ax \geq br + cq$ obvious.



The other two are shown similarly. □

Erdos-Mordell's Theorem. From the lemma we have

$$ax \geq br + cq,\ by \geq ar + cp,\ cz \geq aq + bp \Rightarrow x \geq \left(\frac{b}{a}\right) r + \left(\frac{c}{a}\right) q,\ y \geq \left(\frac{a}{b}\right) r + \left(\frac{c}{b}\right) p,\ z \geq \left(\frac{a}{c}\right) q + \left(\frac{b}{c}\right) p$$

$$\Rightarrow x + y + z \geq (\frac{b}{a} + \frac{a}{b})r + (\frac{c}{a} + \frac{a}{c})q + (\frac{c}{b} + \frac{b}{c})p \geq 2(p + q + r)$$

□

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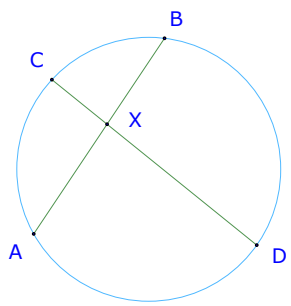
3.6 Problems (O-Level)

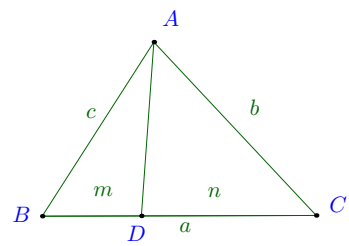
Submission deadline: October 6, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Theorem (Intersecting Chord Theorem)

Let AB and CD be two segments that intersect at X, such that A and C lie between X, B and X, D, respectively. Then the quadrilateral $ABDC$ is cyclic if and only if $XA \cdot XB = XC \cdot XD$.



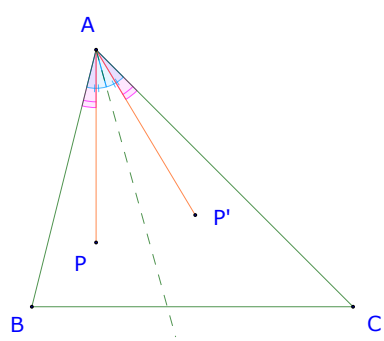


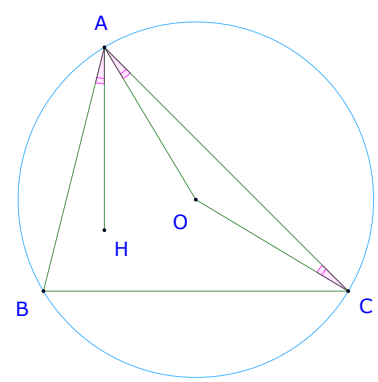
Theorem (Stewart's Theorem)

Given $\triangle ABC$, $BC = a, CA = b, AB = c$. D is a point on BC, $BD = m, DC = n, AD = d$. We have:

$$b^2m + c^2n = a(d^2 + mn).$$

Definition (Isogonal Conjugates)**.** Two points P and P' are called isogonal conjugates if line XP' is the reflection of XP across the angle bisector of the angle at the vertex X in a triangle $\triangle ABC$, where X is any of the vertices A, B, or C. In other words, the lines XP and XP' make equal angles with the sides of the triangle that contain X, e.g. $\angle PAB = \angle P'AC$.





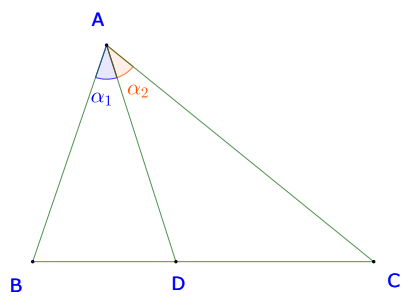
Theorem (Orthocenter-circumcircle Theorem)

The orthocenter and the circumcenter in a triangle are isogonal conjugates.

Lemma (Ratio Lemma)

In $\triangle ABC$ let D be a point on segment BC (not necessarily between B and C) and let α_1 and α_2 be the angles $\angle BAD$ and $\angle DAC$, respectively, then

$$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin \alpha_1}{\sin \alpha_2}$$



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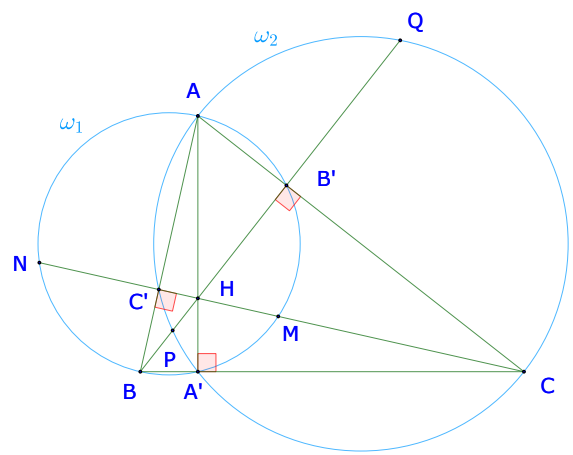
Problem 3.6.1 (23-24-S2-O-P16). (*10 points*) An acute-angled triangle ABC is given in the plane. The circle with diameter AB, intersects altitude CC', and its extension at points M and N,, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

Solution (Official solution, same as Henry Dinh's) — Let A' be the foot of the altitude from A to BC. Since $\angle AA'B = 90°$, thus $A' \in \omega_1 = (ABB')$. Similarly $A' \in \omega_2 = (ACC')$. Let H be the orthocenter of $\triangle ABC$. Now by the Intersecting Chord Theorem for ω_1 then ω_2,

$$HM \cdot HN = \text{ (in } \omega_1) \ HA \cdot HA' = \text{ (in } \omega_2) \ HP \cdot HQ \Rightarrow MPNQ \text{ is cyclic.}$$



Solution (Solution 2 from Minh Le Phuong Nguyen - with minor modifications) — Let $H = BB' \cap CC'$.

$$\left.\begin{aligned} BMB'N, CQC'P \text{ are cyclic} \Rightarrow HM \cdot HN = HB \cdot HB',\ HP \cdot HQ = HC \cdot HC' \\ \triangle BHC' \sim \triangle CHB' \Rightarrow HB \cdot HB' = HC \cdot HC' \end{aligned}\right\}$$

$$\Rightarrow HM \cdot HN = HP \cdot HQ \Rightarrow MPNQ \text{ is cyclic.}$$

Solution (Solution 2 from Albert Dinh-Le, Anthony Pham, Ha Anh Le - with minor modifications) — Let $H = BB' \cap CC'$. $BMB'N$, $CQC'P$, and $BCB'C'$ (by circle diameter BC), are cyclic.

$$HM \cdot HN = HB \cdot HB', HP \cdot HQ = HC \cdot HC', HB \cdot HB' = HC \cdot HC' \Rightarrow HM \cdot HN = HP \cdot HQ.$$

Remark. All students have submitted correct solutions. In addition, Henry Dinh and Minh Le Phuong Nguyen has proven the Intersecting Chord Theorem. Nice!

Corollary

ABC is an acute triangle. A', B', and C' are the feet of the altitudes from A, B, and C to BC, CA, and AB, respectively. Let H be the orthocenter of $\triangle ABC$, then

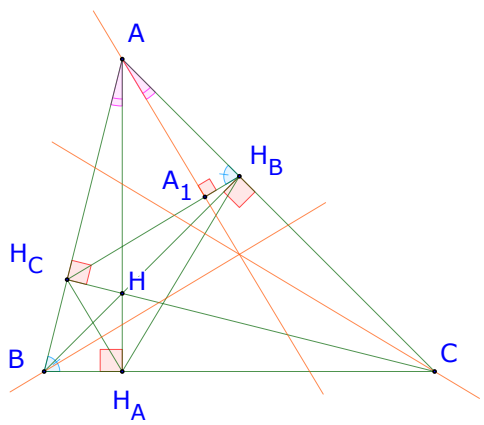
$$HA \cdot HA' = HB \cdot HB' = HC \cdot HC'.$$

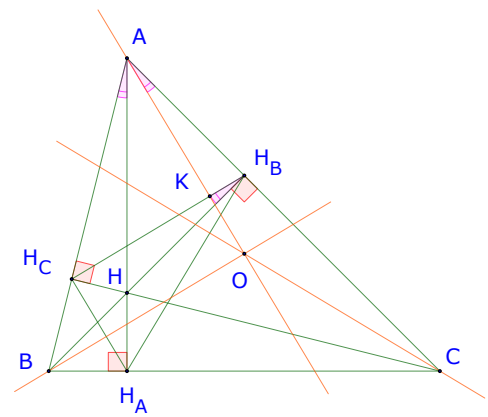
Problem 3.6.2 (23-24-S2-O-P17). (*10 points*) Let AH_A, BH_B and CH_C be altitudes in $\triangle ABC$. Let p_A, p_B, p_C be the perpendicular lines from vertices A, B, C to H_BH_C, H_CH_A, H_AH_B respectively. Prove that p_A, p_B, p_C are concurrent lines.

Solution (Official solution and almost as same as Minh Le Phuong Nguyen's) — Let $A_1 = p_A \cap H_BH_C$.

$$BCH_BH_C \text{ is cyclic} \Rightarrow \angle AH_BA_1 = \angle AH_BH_c = \angle H_CBC \equiv \angle ABC = \angle B \Rightarrow \angle CAA_1 \equiv 90^\circ - \angle B,$$
$$\angle BAH_A = 90^\circ - \angle ABH_A = 90^\circ - \angle B \Rightarrow \angle CAA_1 = \angle BAH_A$$

Thus, AH_A and $AA_1 \equiv p_A$ are symmetric with respect to the angle bisector of $\angle BAC$. The orthocenter H lies on the altitude AH_A, by the Orthocenter-circumcircle Theorem, the circumcenter O must lie on p_A. Similarly, O lies on p_B and p_C, so $\boxed{\text{the lines } p_A, p_B, \text{ and } p_C \text{ are concurrent at } O.}$





Solution (Solution 2 by Henry Dinh) — Let O be the circumcenter. Let $K = OA \cap H_BH_C$, By the Orthocenter-circumcircle Theorem,

$$\angle KAH_B = \angle HAH_C,\ AH_CHH_B \text{ is cyclic } \Rightarrow \angle HAH_C = \angle HH_BH_C \Rightarrow KAH_B = HH_BH_C$$
$$KAH_B = HH_BH_C, HH_B \perp AC \Rightarrow H_BH_C \perp KA \Rightarrow H_BH_C \perp OA \Rightarrow O \in p_A$$

Similarly, O lies on p_B and p_C, so $\boxed{\text{the lines } p_A, p_B, \text{ and } p_C \text{ are concurrent at } O.}$

Remark. Only Henry Dinh and Minh Le Phuong Nguyen submitted solutions and both correctly solved this problem. Both have proven the Orthocenter-circumcircle Theorem. Great!

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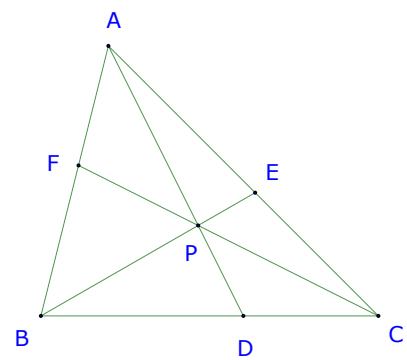
Problem 3.6.3 (23-24-S2-O-P18). (*10 points*) Point P is inside $\triangle ABC$. Line segments APD, BPE, and CPF are drawn with D on BC, E on AC, and F on AB. Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of $\triangle ABC$.

Hint: Use the Area Lemma and Stewart's Theorem.



Solution (Official solution and almost as same as Henry Dinh's and Minh Le Phuong Nguyen's) — By the Area Lemma

$$\frac{[BPC]}{[ABC]} = \frac{DP}{DA} = \frac{1}{2}, \frac{[CPA]}{[ABC]} = \frac{EP}{EB} = \frac{1}{4}.$$

Thus, $[APB] = \frac{1}{4}[ABC]$, by the Area Lemma,

$$\frac{PF}{CF} = \frac{[APB]}{[ABC]} = \frac{1}{4} \Rightarrow PF = 5, CP = 15.$$

$[ABP] = [CPA]$, thus D is mid point of BC.

By the Stewart's Theorem for $\triangle BPC$ and point D, note that,

$$\begin{aligned} &BC = 2BD = 2CD \Rightarrow PC^2 \cdot BD + PB^2 \cdot CD = BC \cdot BD \cdot CD + PD^2 \cdot BC \\ &\Rightarrow BP^2 + CP^2 = 2BD^2 + 2PD^2 \Rightarrow 9^2 + 15^2 = 2BD^2 + 2 \cdot 6^2 \Rightarrow BD^2 = 117 \end{aligned}$$

Therefore $BP^2 + PD^2 = 9^2 + 6^2 = 117 = BD^2$, thus BPD is a right triangle.

Hence $[ABC] = 2[BPC] = 4[BPD] = 2(BP)(PD) = \boxed{108.}$

Remark. Only Henry Dinh and Minh Le Phuong Nguyen submitted solutions and both correctly solved this problem. Both, instead of recognizing $\triangle BDP$ is a right triangle at P, continued to use Heron's formula to compute $[BPC]$,

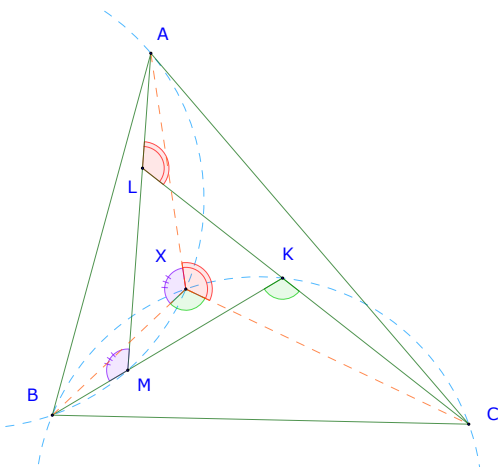
$$\begin{aligned} [BPC] &= \frac{1}{4}\sqrt{(BP+PC+BC)(BP+PC-BC)(BP-PC+BC)(-BP+PC+BC)} \\ &= \frac{1}{4}\sqrt{((BP+PC)^2 - BC^2)(BC^2 - (BP-PC)^2)} = \frac{1}{4}\sqrt{(24^2 - 4 \cdot 117)(4 \cdot 117 - 6^2)} = 54. \end{aligned}$$

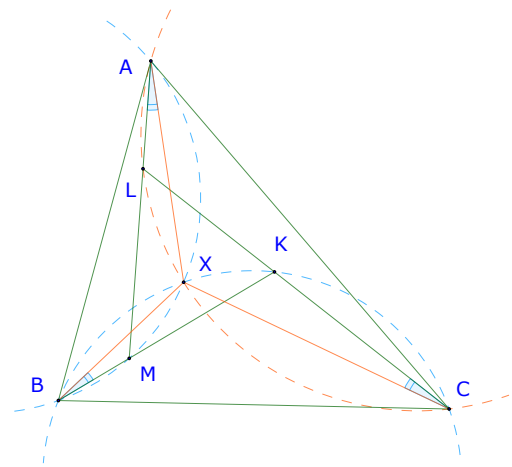
Problem 3.6.4 (23-24-S2-O-P19)**.** (*10 points*) Triangle KLM lies inside triangle ABC so that points K, L, M lie on the segments Cl, AM, BK, respectively. Prove that the circumcircles of the triangles ABM, BCK, CAL pass through a common point.

Solution (Official solution and almost as same as Anthony Pham's) — Let $X \neq B$ be the intersection of the two circumcirles of $\triangle ABM$ and $\triangle BCK$. Then $\angle AXB = \angle AMB, \angle BXC = \angle BKC$. It is easy to see that

$$\angle ALC = 180^\circ - \angle MLK = \angle LKM + \angle KML = 360^\circ - (\angle AMB + \angle BKC)$$
$$= 360^\circ - (\angle AXB + \angle BXC) = \angle CXA \Rightarrow ALXC \text{ is cyclic.}$$

Hence, the circumcircles of the triangles ABM, BCK, CAL pass through a common point X.





Solution (Solution 2 from Henry Dinh) — Let $X \neq B$ be the intersection of the two circumcirles of $\triangle ABM$ and $\triangle BCK$.

$$\left.\begin{aligned} ABMX \text{ is cyclic } &\Rightarrow \angle MBX = \angle MAX (\text{subend } \widehat{MX}) = \angle LAX \\ BXKC \text{ is cyclic } &\Rightarrow \angle XBK = \angle KCX (\text{subend } \widehat{XK}) = \angle LCX \end{aligned}\right\}$$
$$\angle MBX = \angle XBK \Rightarrow \angle LAX = \angle LCX \Rightarrow X \in (ALC)$$

Hence, the circumcircles of the triangles ABM, BCK, CAL pass through a common point X.

Remark. Only Anthony Pham and Henry Dinh and submitted solutions and both correctly solved this problem.

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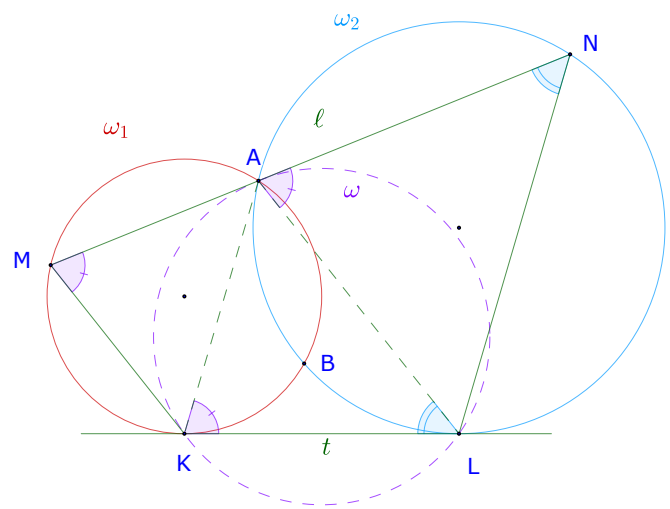
Problem 3.6.5 (23-24-S2-O-P20)**.** (*10 points*) Circles ω_1 and ω_2 intersect at A, B. Their common external tangent t is tangent to them at K, L, respectively, such that B lies inside $\triangle AKL$. Line ℓ passing through A intersects circles ω_1, ω_2 at M, N, respectively. Prove that ℓ is tangent to the circumcircle of $\triangle KLA$ if and only if $KLNM$ is cyclic.

Solution (Official solution and somewhat similar to Anthony Pham's second solution) — Denote by m the angle bisector of line ℓ and t (or axis of the stripe if they are parallel.) *Every pair of anti parallel lines will be considered with respect to m.* First, note that t is tangent to the circumcircle of $\triangle KAM$, line KA is antiparallel to KM. Similarly LA is antiparallel to LN.

For the if part, if ℓ is tangent to the circumcircle of $\triangle KLA$ the KA and AL are antiparallel. Since KM is antiparallel to KA, KA to AL, and AL to LN, together we have KM is antiparallel to LN. Hence $KLNM$ is cyclic.

On the other hand if $KLNM$ is cyclic, then AK is antiparallel to KM, KM is antiparallel to LN, and LN is antiparallel to LA. Thus KA is antiparallel to LA and ℓ is tangent to the circumcircle of $\triangle KLA$.



Solution (Solution 2 from Anthony Pham, Ha-Anh Le, and Henry Dinh - with modifications) — We prove both directions of the statement as below.

($\Rightarrow$) Let ℓ be the tangent line to the circumcircle of $\triangle KLA$.

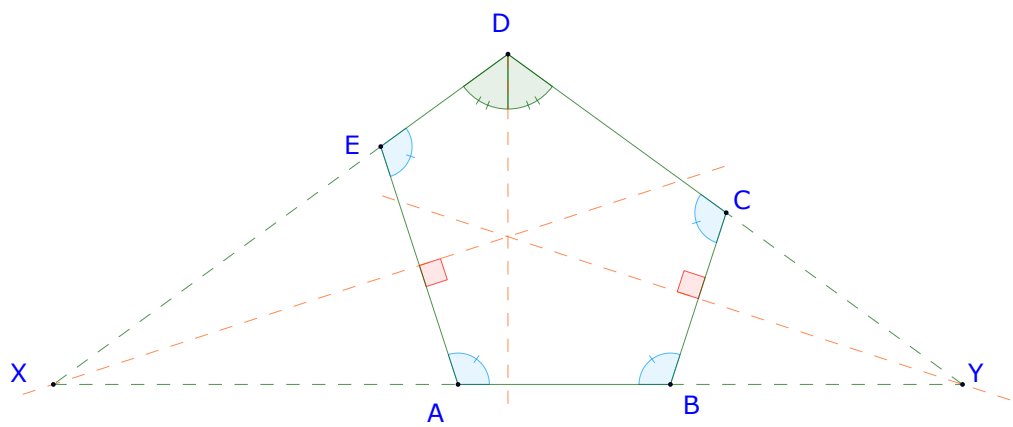
$$\angle LAN = \angle LKA \text{ (in } \omega), \angle ALK = \angle ANL \text{ (in } \omega_2), \angle KMA = \angle AKL \text{ (in } \omega_1)$$
$$\Rightarrow \angle KMA + \angle KLN = \angle LAN + \angle ANL + \angle ALN = 180^\circ \Rightarrow MKLN \text{ is cyclic.}$$

($\Leftarrow$) Let $MKLN$ be cyclic.

$$\Rightarrow \angle KMA + \angle KLN = 180^\circ,\ \angle KMA = \angle AKL, \angle ALK = \angle ANL$$
$$\Rightarrow \angle AKL + \angle ALN + \angle ANL = 180^\circ \Rightarrow \angle AKL = \angle LAN \Rightarrow \ell \text{ is tangent}$$

Remark. Only Henry Dinh, and Ha-Anh Le submitted solutions and all correctly solved this problem. Anthony Pham made so some minor mistakes, writings very difficult to read.

Problem 3.6.6 (23-24-S2-O-P21)**.** (*10 points*) In the convex pentagon $ABCDE$ all interior angles have the same measure. Prove that the perpendicular bisector of segment EA, the perpendicular bisector of segment BC, and the angle bisector of $\angle CDE$ intersect at one point.



Solution (Official solution and same as Alber Dinh-Le's and Henry Dinh's solution) — By extending AB to meet CD at X, and DE at Y then $\triangle BCX$ and $\triangle AEY$ are both isosceles. Thus their angle bisectors are the same as the angle bisectors in $\triangle DXY$. Furthermore the perpendicular bisector of segment EA is the same as the angle bisector of $\angle DYX$, and the perpendicular bisector of segment BC is the same as the angle bisector of $\angle DXY$. Thus they shall meet the angle bisector of $\angle CDE$ at one point.

Remark. Only Albert Dinh-Le and Henry Dinh submitted solutions and both correctly solved this problem.

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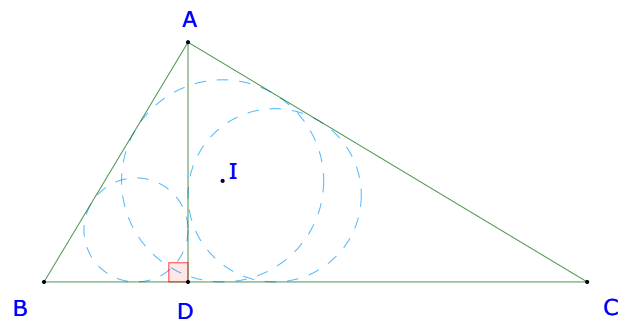
Problem 3.6.7 (23-24-S2-O-P22). (*10 points*) ABC is a right triangle with $\angle A = 90°$, and altitude AD. Let r, s, t be the inradii of the triangles ABC, ADB, and ADC, respectively. Show that:

$$r + s + t = AD.$$



Solution (Official solution and somewhat similar to Henry Dinh's and Minh Le Phuong Nguyen's solution) — First, it is easy to prove that all triangles BAC, BDA, ADC are pairwise similar. The ratio $\frac{r}{AC}$ in $\triangle ABC$ is the same as the ratio $\frac{s}{AD}$ in $\triangle BDA$,

$$\frac{r}{AC} = \frac{s}{AD}.$$

The ratio $\frac{r}{AB}$ in $\triangle ABC$ is the same as the ratio $\frac{t}{AD}$ in $\triangle ADC$,

$$\frac{r}{AB} = \frac{t}{AD}.$$

Thus, instead of $r + s + t = AD$ we need to prove that:

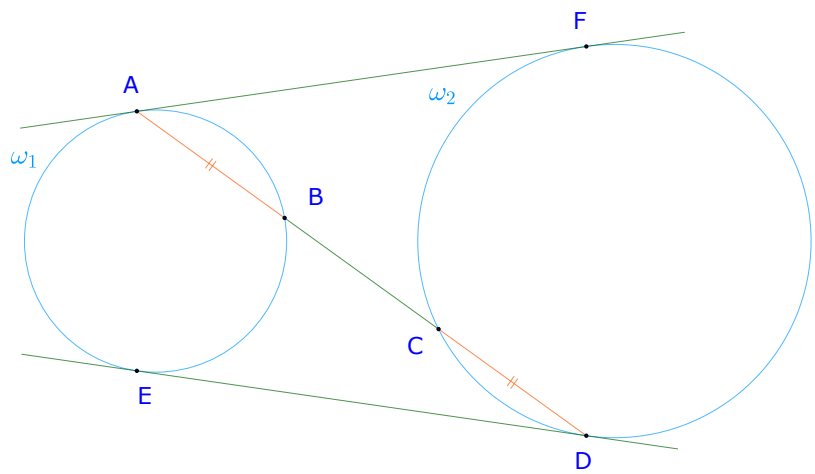
$$\frac{r}{AD} + \frac{r}{AC} + \frac{r}{AB} = 1.$$

But this is obvious, since

$$\frac{r}{AD} = \frac{[BIC]}{[ABC]}, \frac{r}{AC} = \frac{[AIB]}{[ABC]}, \frac{r}{AB} = \frac{[AIC]}{[ABC]} \text{ and } \frac{[BIC]}{[ABC]} + \frac{[AIB]}{[ABC]} + \frac{[AIC]}{[ABC]} = 1.$$

Remark. Only Henry Dinh and Minh Le Phuong Nguyen correctly solved this problem. Anthony Pham started an attempt.

Problem 3.6.8 (23-24-S2-O-P23)**.** (*10 points*) Let ω_1 and ω_2 be two circles. One of their common external tangents is tangent to ω_1 at A, the second one is tangent to ω_2 at D. Line AD intersects the circles ω_1 and ω_2 for the second time at B and C, respectively. Prove that $AB = CD$.



Solution (Official solution and similar to Albert Dinh-Le's, Anthony Pham's, and Ha-Anh Le's solution) — Let denote the points of contact of the first tangent with ω_2 as F and of the second tangent with ω_1 as E. By symmetry $AF = ED$. By Power of a Point:

$$AC \cdot AD = p(A, \omega_2) = AF^2 = ED^2 = p(D, \omega_1) = BD \cdot AD \Rightarrow \boxed{AC = BD.}$$

Remark. Only Albert Dinh-Le, Anthony Pham, and Ha-Anh Le submitted solutions and all correctly solved this problem.

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Problem 3.6.9 (23-24-S2-O-P24)**.** (*10 points*) Let $ABCD$ be a cyclic quadrilateral. Let P be the point on the ray AD such that $AP = BC$ and let Q be the point on the ray AB such that $AQ = CD$. Prove that the line AC cuts PQ at its midpoint.

Solution (Official solution and similar to Henry Dinh's) — Let X be the intersection of AC and PQ. By the Ratio Lemma,

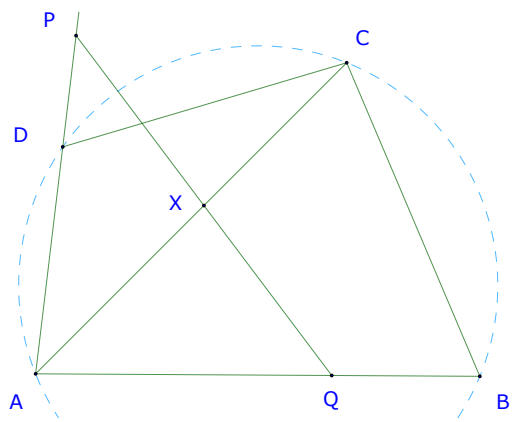
$$\frac{PX}{XQ} = \frac{AP}{AQ} \cdot \frac{\sin \angle PAX}{\sin \angle XAQ}.$$

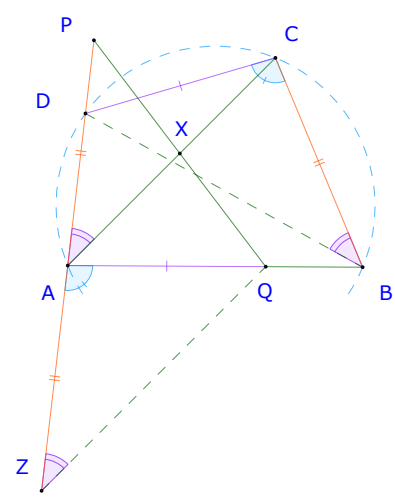
Let R be the radius of the circumcircle of $ABCD$, then, by the Extended Law of Sines:

$$\sin \angle PAX = \sin \angle DAC = \frac{CD}{2R} = \frac{AQ}{2R} \quad (*)$$

Similarly $\sin \angle XAQ = \frac{AP}{2R} \quad (**)$

From (*) and (**), it is easy to see that $\boxed{PX = XQ.}$





Solution (Solution 2 from Albert Dinh-Le) — Let Z be on ray DA towards A such that $AZ = BC$.

$$AQ = CD, AZ = CB, \angle ZAQ = \angle DCB \Rightarrow \triangle ZAQ \cong \triangle DCB \Rightarrow \angle DBC = \angle AZQ$$
$$ABCD \text{ is cyclic } \Rightarrow \angle CAD = \angle DBC \Rightarrow \angle CAD = \angle QZA \Rightarrow AC \parallel ZQ$$

A is midpoint of PZ, $AX \parallel ZQ$, hence $\boxed{PX = XQ.}$

Remark. Only Albert Dinh-Le and Henry Dinh submitted solutions and both correctly solved this problem. In addition Henry Dinh proved the Ratio Lemma. Good!

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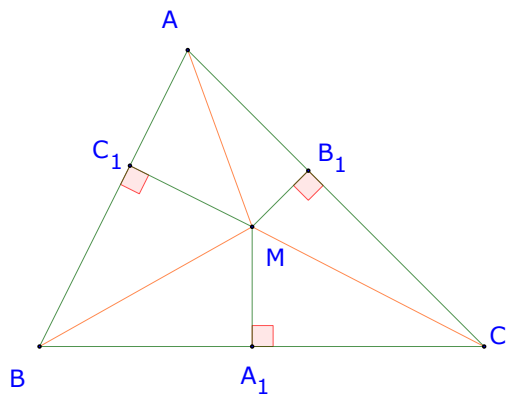
Problem 3.6.10 (23-24-S2-O-P25)**.** (*10 points*) Let M be the point inside the triangle ABC. Denote by A_1, B_1, C_1 the feet of perpendiculars from M to BC, CA, and AB. Prove that

$$MA \cdot MB \cdot MC \geq 8MA_1 \cdot MB_1 \cdot MC_1.$$



Solution (Official solution and same as Albert Dinh-Le's) — Let $BC = a, CA = b, AB = c$. Let h_a, h_b, h_c be the length of the altitudes from A, B, and C, respectively.

$$h_a \leq AM + MA_1 \Rightarrow a \cdot MA_1 + b \cdot MB_1 + c \cdot MC_1 = 2S = ah_a \leq a \cdot MA + a \cdot MA_1$$
$$\Rightarrow a \cdot MA \geq b \cdot MB_1 + c \cdot MC_1 \geq 2\sqrt{bc \cdot MB_1 \cdot MC_1}$$

Similarly $abc \cdot MA \cdot MC \cdot MC \geq 8abc \cdot MA_1 \cdot MB_1 \cdot MC_1 \Rightarrow MA \cdot MB \cdot MC \geq 8MA_1 \cdot MB_1 \cdot MC_1$.

Solution (Solution 2 from Henry Dinh) — From the lemma in the Erdos-Mordell's Theorem and follow the same notations, we have

$$\left.\begin{aligned} x &\geq \frac{b}{a}r + \frac{c}{a}q \geq 2\sqrt{\frac{bc}{a^2}rq} \\ y &\geq \frac{c}{b}p + \frac{a}{b}r \geq 2\sqrt{\frac{ca}{b^2}pr} \\ z &\geq \frac{a}{c}q + \frac{b}{c}p \geq 2\sqrt{\frac{ab}{c^2}qp} \end{aligned}\right\} \Rightarrow xyz \geq 8pqr$$

Remark. Only Albert Dinh-Le and Henry Dinh submitted solutions and both correctly solved this problem.

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Chapter 4

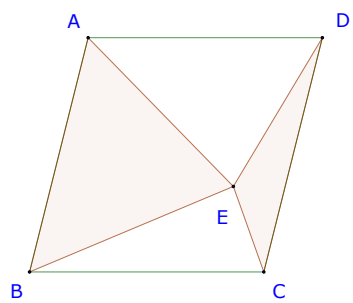
Session 3

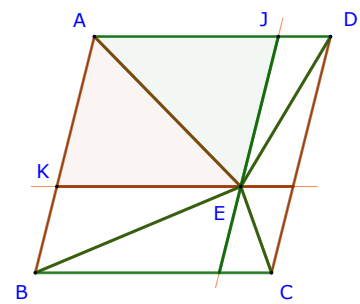
4.1 A picture is worth a thousand words (I-Level)

In this session, we will investigate a number of ways to *prove area equality without writing lengthy proof.* While it sounds simple, easy, and exciting, it is important that you need to improve your creating thinking in order to first understand the examples, and then use them as tools, guidelines, or ideas to solve the problems.

Example 4.1.1 (23-24-S3-I-P1)
E is an arbitrary point inside the parallelogram $ABCD$, prove that

$$[AEB] + [CED] = \frac{1}{2}[ABCD].$$





Solution. Draw lines through E, parallel with sides of $ABCD$, dividing the parallelogram into four smaller parallelograms. Any of the smaller parallelogram, say $AKEJ$, consists of a brown triangle from the shaded triangle and a green triangles with the same area. Thus, the area of the shaded triangles is the sum of the area of all smaller brown triangles, which is half of the sum of the area of all smaller parallelograms, of half of the $ABCD$ parallelogram. □

Remark. Here's how we use the techniques:

1. First, divide the given figure into smaller figures.
2. Deal with each of them, iif they have the same shape, then work in the same way.
3. Use all partial results to arrive at the overall result.

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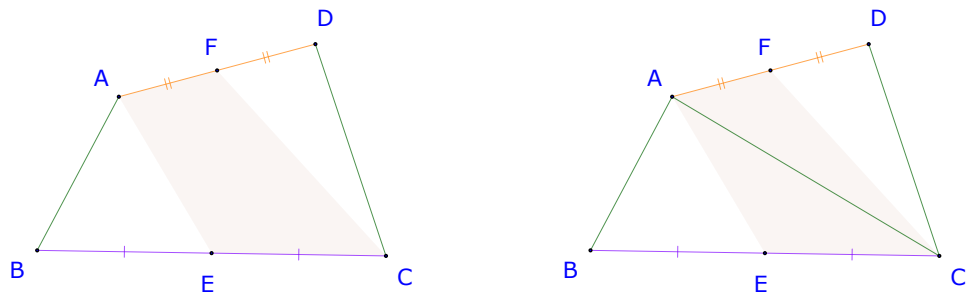
Example 4.1.2 (23-24-S3-I-P2)

E and F are midpoints of BC and DA in the convex quadrilateral $ABCD$, prove that

$$[AECF] = \frac{1}{2}[ABCD].$$

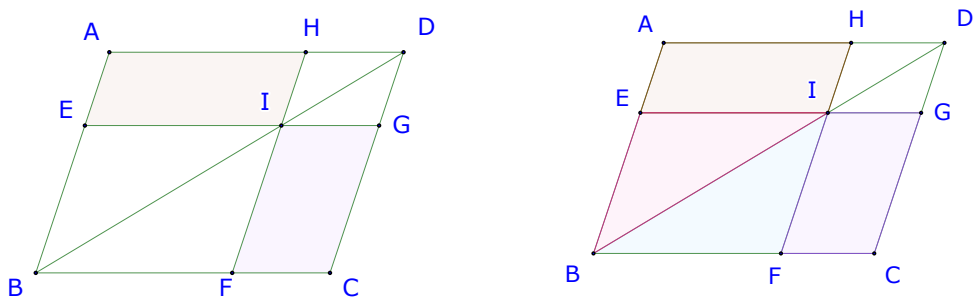


Proof. Connect AC. Since E is midpoint of BC, thus the triangles ABE and AEC have the same area. Similarly triangles CDF and CFA have the same area. Thus the area of $AECF$ is half of $ABCD$. □

Example 4.1.3 (23-24-S3-I-P3)

I is an arbitrary point on the diagonal BD in parallelogram $ABCD$. Lines through I parallel with the sides of $ABCD$ intersect AB, BC, CD, and DA at E, F, G, and H, respectively.

$$[AEIH] = [FCGI].$$



Proof. First, since BD is the diagonal in parallelogram $ABCD$, $[ABD] = [BCD]$. Now, $BEIF$ is also a parallelogram, thus $[BEI] = [BFI]$, similarly $[HID] = [IGD]$. Therefore

$$[AEIH] = [ABD] - [BEI] - [HID] = [BCD] - [BFI] - [IGD] = [FCGI].$$

□

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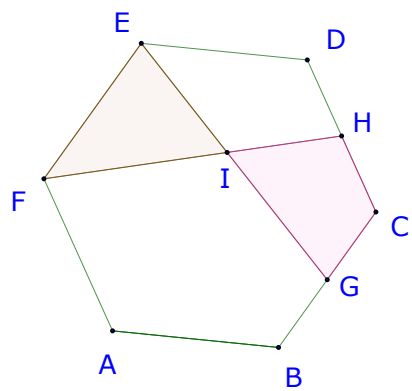
Example 4.1.4 (23-24-S3-I-P4)

G, H are midpoints of BC, CD in th regular hexagon $ABCDEF$. EG and FH intersect at I. Prove that

$$[GCHI] = [EFI].$$

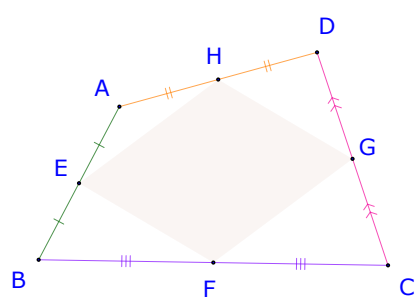


Proof. It is easy to see that the quadrilaterals $GCDE$ and $HDEF$ are congruent, thus have the same area, or $[GCDE] = [HDEF]$. Taking $HDEI$ away, we have $[GCHI] = [EFI]$. □

Example 4.1.5 (23-24-S3-I-P5)

E, F, G, and H are midpoints the sides in the convex quadrilateral $ABCD$. Prove that

$$[EFGH] = \frac{1}{2}[ABCD].$$



Proof. EH is the mid-segment (the segment connecting two midpoints) in $\triangle ABD$, therefore $[AEH] = \frac{1}{4}[ABD]$. Similarly $[BEF] = \frac{1}{4}[ABC]$, $[CFG] = \frac{1}{4}[BCD]$, and $[GDH] = \frac{1}{4}[CDA]$, therefore:

$$[AEH] + [BEF] + [CFG] + [GDH] = \frac{1}{2}[ABCD] \Rightarrow [EFGH] = \frac{1}{2}[ABCD].$$

□

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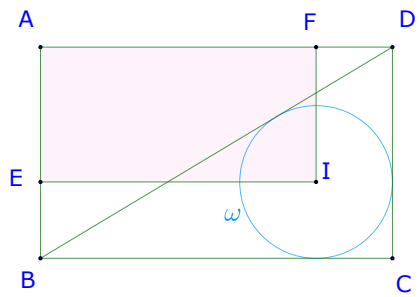
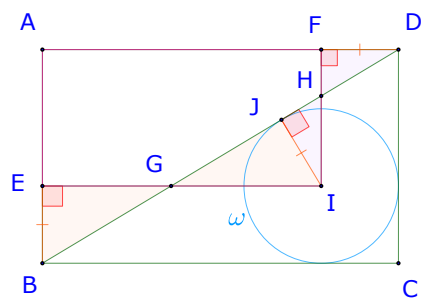
Example 4.1.6 (23-24-S3-I-P6)

$ABCD$ is a rectangle. E, F are arbitrary points on of CD, DA, respectively. Circle ω center at I is tangent to sides BC, CD, and diagonal BD. Lines through I parallel to the sides of the rectangle intersect AB at E and DA at F. Prove that

$$[AEIF] = \frac{1}{2}[ABCD].$$





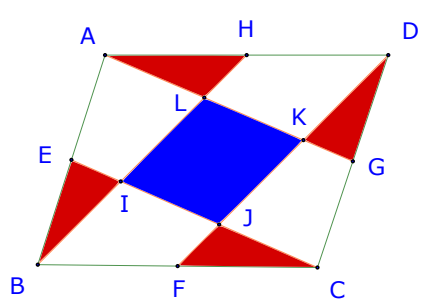
Proof. Let I be the tangent point of circle ω with BD. $\triangle BEG$ and $\triangle CJG$ are both right triangles and have the short legs with the same length $EB = IJ$ (= the radius of ω), thus they are congruent and have the same area. Similarly $\triangle IJH$ and $\triangle FDH$ have the same area.

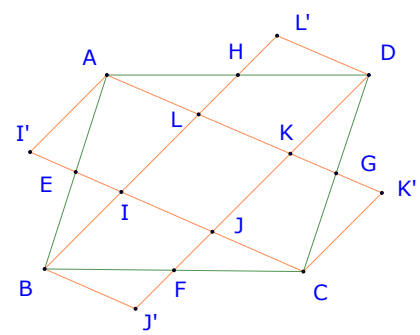
$$[AEIF] = [AEGHF] + [GIH] = [AEGHF] + [GIJ] + [IJH] = [AEGHF] + [EGB] + [HFD] = \frac{1}{2}[ABCD].$$

□

Example 4.1.7 (23-24-S3-I-P7)

$ABCD$ is a parallelogram. E, F, G, and H are midpoints of AB, BC, CD, and DA, respectively. Segments AG, BH, CE, EF intersect at I, J, K, and L, as shown below. Prove that the area of the blue region is the same as the sum of the areas of the red regions.





Proof. Let I', J', K', L' be the reflections of I, J, K, L over E, F, G, H, respectively. It is easy to prove that $ALII'$, $BIJJ'$, $CJKK'$, and $DKLL'$ as well as $IJKL$ are congruent parallelograms. Thus, the sum of the area of the red region is equivalent to the area of one of these parallelograms. □

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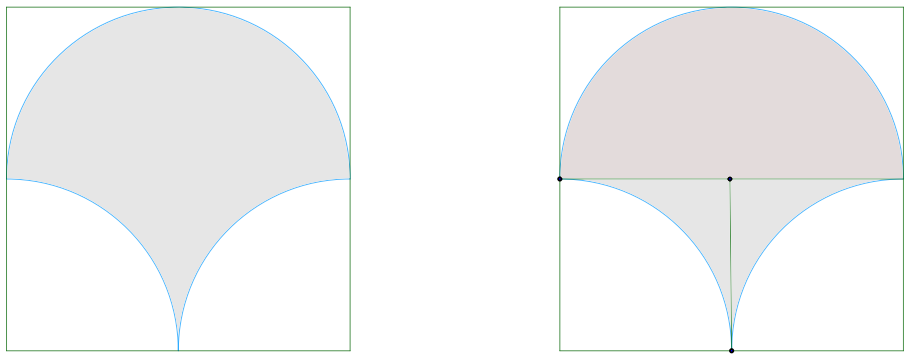
Example 4.1.8 (23-24-S3-I-P8)

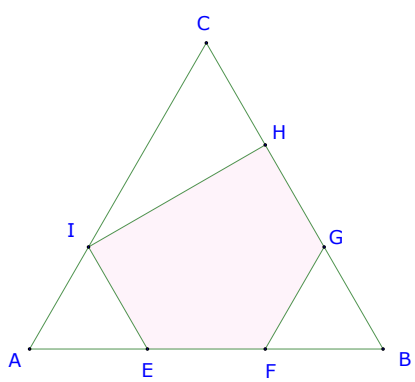
A half cirle and two quarters of circles are draw in a square as shown below. Find the ratio of the area of the shaded region to the area of the square.



Proof. The area of the shaded region contains a half circle plus twice of the difference of a quarter of the square with the area of a quarter circle. The half circle and two quarter circles cancel out, leaving twice quarter of the square, thus the ratio is $\frac{1}{2}$. □

Example 4.1.9 (23-24-S3-I-P9)

$\triangle ABC$ is equilateral. Points E, F trisect the side AB, point GF, H trisect the side BC. IE is parallel to BC. Find the ratio of the shaded area to the area of $\triangle ABC$.



Proof. Since $AE = \frac{1}{3}AB$, thus $[AIE] = \left(\frac{1}{3}\right)^2 [ABC] = \frac{1}{9}[ABC]$. $[BGF] = [AIE]$. $\triangle CHI$ is right triangle at H, $CH = \frac{1}{3}BC$, IH is two-third of the altitude from A to BC, thus $[CIH] = \frac{1}{3} \cdot \frac{2}{3}[ABC]$. Hence

$$\frac{[EFGHI]}{[ABC]} = 1 - \frac{1}{9} - \frac{1}{9} - \frac{2}{9} = \frac{5}{9}.$$

□

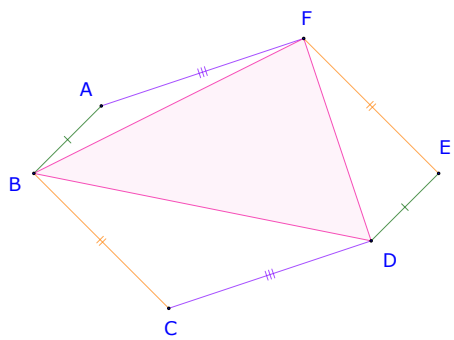
4.2 Problems (I-Level)

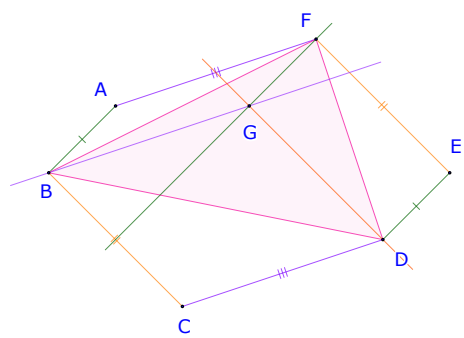
Submission deadline: October 13, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 4.2.1 (23-24-S3-I-P11). (*10 points*) In the hexagon $ABCDEF$, $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$, $AB = DE$, $BC = EF$, and $CD = FA$. Prove that

$$[BDF] = \frac{1}{2}[ABCDEF].$$





Proof. Draw lines through B parallel with AD and CD, through D parallel with CB and EF, through F parallel with AB and ED. They are concurrent at G (all meet at G, can you prove it?) and divide the hexagon into three parallelograms $ABGF$, $CDGB$, and $DEFG$. They also divide the shaded triangle BDF into three triangles, each's area is half of the parallelogram whose contains it. Therefore the area of the shaded triangle is half of the hexagon. □

> **Remark.** Giang Nguyen, Jeremy Nguyen, Julie Cao, Minh Do Tue Le, Minh Nhat Pham correctly solved this problem.

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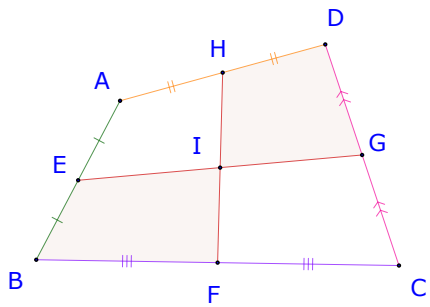
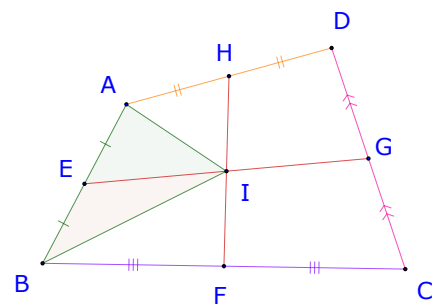
Problem 4.2.2 (23-24-S3-I-P12). (*10 points*) In the convex quadrilateral $ABCD$, E, F, G, and H are midpoints of AB, BC, CD, and DA, respectively. EG and FH intersect at I. Prove that

$$[EBFI] + [GDHI] = \frac{1}{2}[ABCD].$$





Proof. Connect AI. Since E is midpoint of AB, thus the triangles AEI and EIB have the same area. Similarly the area of BIF and CFI are the same. Thus $[EBIF] = \frac{1}{2}[ABCI]$. Similarly $[GDHI] = \frac{1}{2}[CDAI]$. Summing up, we have $[EBFI] + [GDHI] = \frac{1}{2}[ABCD]$. □

Remark. Giang Nguyen, Jeremy Nguyen, Julie Cao, Minh Do Tue Le, Minh Nhat Pham correctly solved this problem.

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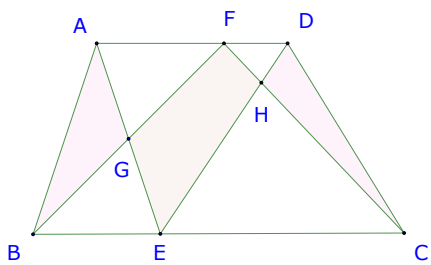
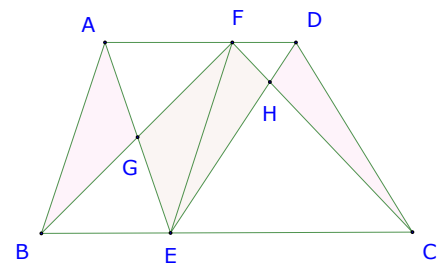
Problem 4.2.3 (23-24-S3-I-P13)**.** (*10 points*) $ABCD$ is a trapezoid. E, F are arbitrary points on BC and AD, respectively. AE, BF intersect at G and CF, DE intersect at H. Prove that:

$$[ABG] + [CDH] = [EFGH].$$





Proof. Connect EF. Since BE parallel with AF so $[ABF] = [AEF]$, thus $[ABG] = [FEG]$. Similarly $[CDH] = [FEH]$. Summing up, we have $[ABG] + [CDH] = [EFGH]$. □

Remark. Giang Nguyen, Jeremy Nguyen, Julie Cao, Minh Do Tue Le, Minh Nhat Pham correctly solved this problem.

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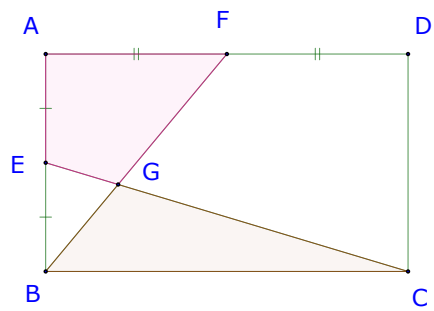
Problem 4.2.4 (23-24-S3-I-P14)**.** (*10 points*) $ABCD$ is a rectangle. E, F are midpoints of AB, AD, respectively. CE, BF intersect at G. Prove that:

$$[AEGF] = [BGC].$$



Proof. It is easy to see that the triangles ABF and EBC have the same area

$$[ABF] = \frac{AB \cdot AF}{2} = \frac{AB \cdot AD}{4},\ [EBC] = \frac{EB \cdot BC}{2} = \frac{AB \cdot AD}{4}$$
$$\Rightarrow [AEGF] = [ABF] - [EBG] = [EBC] - [EBG] = [BGC].$$

□

Remark. Giang Nguyen, Jeremy Nguyen, Julie Cao, Minh Do Tue Le, Minh Nhat Pham correctly solved this problem.

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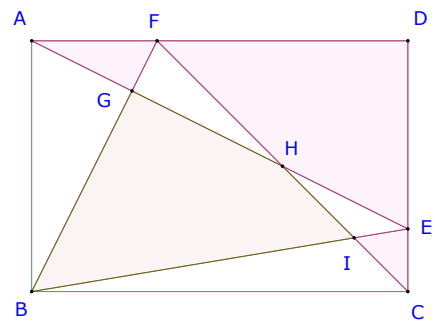
Problem 4.2.5 (23-24-S3-I-P15)**.** (*10 points*) $ABCD$ is a rectangle. E, F are arbitrary points on of CD, DA, respectively. AE intersects BF, BE at G, H, respectively. CF intersects BE at I. Prove that:

$$[AGF] + [FHED] + [CEI] = [BGHI].$$

Hint: look at the triangle BDF and CDF.



Proof. It is easy to see that the triangles BDF and CDF have the same area, similarly the triangles BDE and ADE have the same area, therefore: $[BFDE] = [CDF] + [ADE]$. But

$$[BFDE] = [BGHI] + [GFH] + [FHED] + [HIE],$$
$$[CDF] + [ADE] = [AGF] + [GFH] + 2[FHED] + [HIE] + [CEI]$$

Hence, $[BGHI] = [AGF] + [FHED] + [CEI]$. □

Remark. Giang Nguyen, Jeremy Nguyen, Minh Nhat Pham correctly solved this problem.

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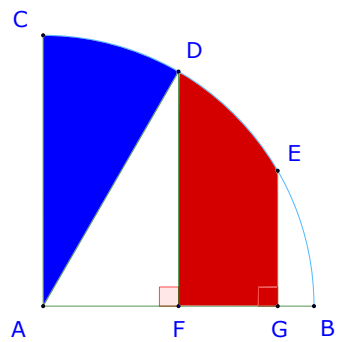
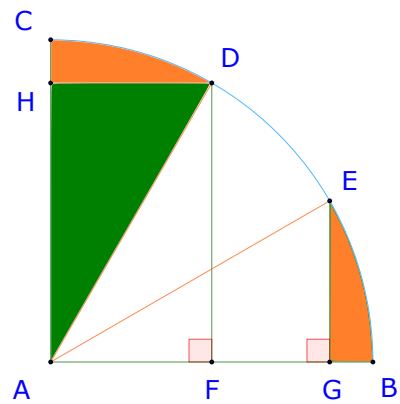
Problem 4.2.6 (23-24-S3-I-P16)**.** (*10 points*) ABC is a quarter of circle. D, E trisect arc $\widehat{BC}$: $\widehat{CD} = \widehat{DE} = \widehat{EB}$. Prove that the blue circular sector ACD and the red region $DEGF$ have the same area.





Proof. Since D, E trisect arc $\widehat{BC}$, the circula sector ACD and AEB are congruent. Thus, if $DH \perp AC$, then the orange regions CDH and EGB are the same. It is easy to see that $\triangle AHD \cong \triangle ADF$, thus the sum of the area of $\triangle ADF$ and the region EGB is the same as the blue circular sector ACD. This means that the area $DEGF$ is one-third of the quarter circle ABC, thus it is the same as the area of the blue circular sector ACD □

Remark. Jeremy Nguyen, Minh Nhat Pham correctly solved this problem.

The diagram in Giang Nguyen's solution does not show the points and segments mentioned in the text, thus it is impossible to verify her proof.

Julie Cao started an attempt but the writing is not clear to understand.

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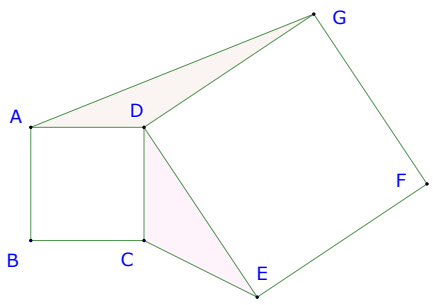
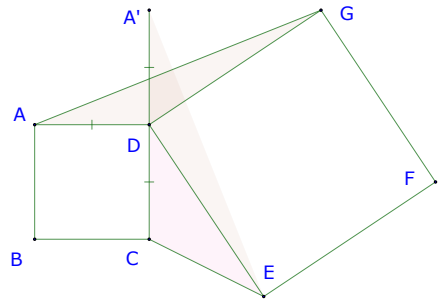
Problem 4.2.7 (23-24-S3-I-P17)**.** (*10 points*) $ABCD$ and $DEFG$ are two squares as shown below, prove that

$$[ADG] = [CDE].$$





Proof. Rotate the triangle ADG clockwise around D. Because $\angle CDA = \angle EDG = 90°$, $DG = DE$, we receive the triangle $A'DE$. A', D, C are collinear (lie on the same line). Thus $[ADG] = [A'DE] = [DCE]$. $\square$

> **Remark.** Giang Nguyen, Jeremy Nguyen, Minh Nhat Pham correctly solved this problem.
>
> It is not clear in Julie Cao's solution why the two altitudes must be equal.

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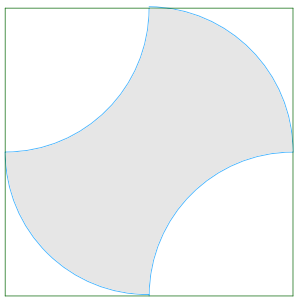
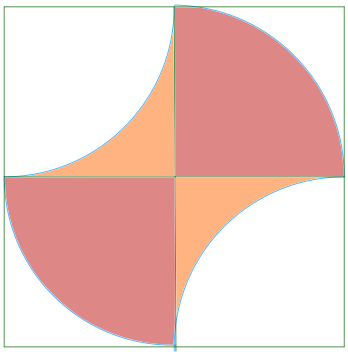
Problem 4.2.8 (23-24-S3-I-P18)**.** (*10 points*) A half cirle and two quarters of circles are draw in a square as shown below. Find the ratio of the area of the shaded region to the area of the square.





Proof. Similar to the 23-24-S3-I-P8, but splitting into four regions, it is easy to see that they all make up two quarter square, thus the ratio is $\frac{1}{2}$. □

Remark. Giang Nguyen, Jeremy Nguyen, Julie Cao, Minh Nhat Pham correctly solved this problem.

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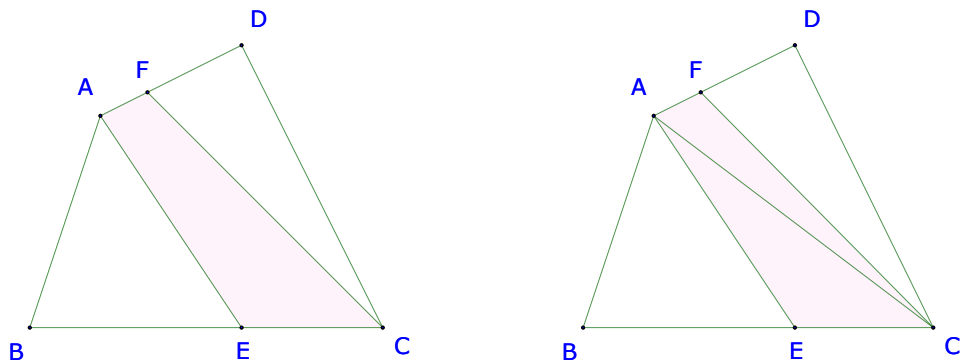
Problem 4.2.9 (23-24-S3-I-P19)**.** (*10 points*) $ABCD$ is a convex quadrilateral. E and F are points on BC and DA such that $BE = 2EC$, $DF = 2FA$. Find the ratio of the area of the shaded region to the area of the quadrilateral.



Proof. Connect AC. Since $BE = 2EC$ and $DF = 2AF$,

$$\frac{[ACE]}{[ABC]} = \frac{[CFD]}{[CAD]} = \frac{[ACE]+[CFD]}{[ABC]+[CDA]} = \frac{[ACE]+[CFD]}{[ABCD]}$$
$$\Rightarrow \frac{[AECF]}{ABCD} = 1 - \frac{[ACE]+[CFD]}{[ABCD]} = \frac{1}{3}.$$

□

Remark. Giang Nguyen, Jeremy Nguyen, Julie Cao, Minh Nhat Pham correctly solved this problem.

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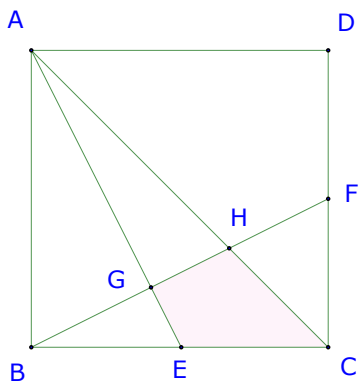
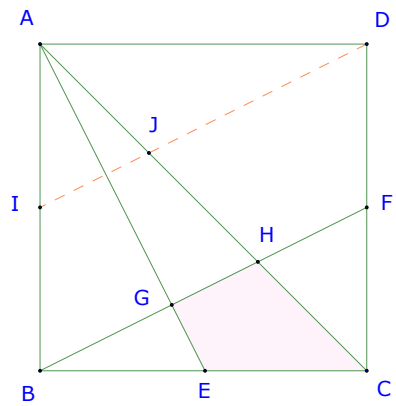
Problem 4.2.10 (23-24-S3-I-P20)**.** (*10 points*) $ABCD$ is a square. E and F are midpoints of BC and CD. BF intersect AE and AC at G and H. Find the ratio of the area of the shaded region to the area of the square.





Proof. WLOG, let the side length of the square be 1. $BE = \frac{1}{2}$, $AE = \sqrt{AB^2 + BE^2} = \frac{\sqrt{5}}{2}$.

$$\triangle BGE \sim \triangle ABE \Rightarrow \frac{[BGE]}{[ABE]} = \left(\frac{BE}{AE}\right)^2 = \left(\frac{\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right)^2 = \frac{1}{5} \Rightarrow [BGE] = \frac{1}{20}.$$

$$[CFH] = \frac{HF}{DJ}[CJD] = \frac{1}{4}[CJD] = \frac{1}{4}\frac{CJ}{AC}[ACD] = \frac{1}{6}[ACD] \Rightarrow [CFH] = \frac{1}{12}$$

$$\Rightarrow [GHCE] = [BCF] - [BGE] - [CFH] = \frac{1}{4} - \frac{1}{20} - \frac{1}{12} = \frac{7}{60}$$

Thus, the ratio is $\frac{7}{60}$. □

Remark. Giang Nguyen, Jeremy Nguyen correctly solved this problem.

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4.3 Perfect squares are everywhere (A-Level)

In this session, we will discuss properties of perfect squares through a number of interesting examples.

Example 4.3.1 (23-24-S3-A-P1)
Cancelling the exponents yields

$$\frac{37^3+13^3}{37^3+24^3}=\frac{37+13}{37+24}=\frac{50}{61}.$$

which is correct.

Find the necessary and sufficient conditions for the positive integer triple (A, B, C) to satisfy

$$\frac{A^3+B^3}{A^3+C^3}=\frac{A+B}{A+C}.$$

Solution. For A, B, and C positive real numbers,

$$\frac{A^3+B^3}{A^3+C^3}=\frac{A+B}{A+C}\Leftrightarrow\frac{A^3+B^3}{A+B}=\frac{A^3+C^3}{A+C}\Leftrightarrow A^2-AB+B^2=A^2-AC+C^2$$
$$\Leftrightarrow(B-C)(A-B-C)=0\Leftrightarrow\boxed{B=C}\text{ or }\boxed{A=B+C.}$$

□

Example 4.3.2 (23-24-S3-A-P2)
For which natural number n is $2^8+2^{11}+2^n$ a perfect square?

Solution. Let m be an integer such that $m^2=2^8+2^{11}+2^n$. Note that:

$$2^8+2^{11}=2^8(1+2^3)=2^8\cdot 3^2=48^2.$$

Thus,

$$2^n=m^2-48^2=(m-48)(m+48).$$

Therefore there exists positive integers k, ℓ, such that

$$\left.\begin{aligned} m-48&=2^k \quad (1)\\ m+48&=2^\ell \quad (2)\\ k+\ell&=n \end{aligned}\right\}(2)-(1)\Rightarrow 96=2^5\cdot 3=2^k(2^{\ell-k}-1)\Rightarrow k=5,\ell=7\Rightarrow\boxed{n=12.}$$

□

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Example 4.3.3 (23-24-S3-A-P3)
Prove that for n positive integer the following number is a perfect square:

$$m = \underbrace{99\ldots9}_{n}\underbrace{00\ldots0}_{n}25.$$

Solution. Note that $\underbrace{99\ldots9}_{n} = 10^n - 1$, thus:

$$m = \underbrace{99\ldots9}_{n}10^{n+2} + 25 = 10^{2n+2} - 10^{n+2} + 25 = \boxed{(10^{n+1} - 5)^2.}$$

□

Example 4.3.4 (23-24-S3-A-P4)
Find all prime p such that $2p^4 - p^2 + 16$ is a perfect square.

Solution. Note that a perfect square has a remainder 0 or 1 when divided by 3. Thus

$$2p^4 \equiv 2p^2 \pmod 3 \Rightarrow 2p^4 - p^2 + 16 \equiv p^2 + 1 \pmod 3.$$

Since $2p^4 - p^2 + 16$ is a perfect square, so $p^2 + 1 \equiv 0$ or $1 \pmod 3 \Rightarrow \boxed{p = 3.}$ In this case $2p^4 - p^2 + 16 = 169 = 13^2$. □

Example 4.3.5 (23-24-S3-A-P5)
Prove that the sum of the squares of 1984 consecutive positive integers cannot be the square of an integer.

Solution. Let $n \geq 0$ be an integer. Let $S(n,k) = (n+1)^2 + (n+2)^2 + \cdot + (n+k)^2$, where k is a positive integer, then:

$$S(n,k) = kn^2 + 2n(1 + 2 + \cdot + k) + (1^2 + 2^2 + \cdots + k^2) = kn^2 + nk(k+1) + \frac{k(k+1)(2k+1)}{6}$$

With $k = 1984$,

$$S(n, 1984) = 992(2n^2 + (2 \cdot 1985)n + 1985 \cdot 1323)$$

Since the second term of $S(n, 1984)$ is an odd number and $992 = 2^5 \cdot 31$, $S(n, 1984)$ is divisible by 2^5, but not by 2^6. Hence, it is not a perfect square. □

Example 4.3.6 (23-24-S3-A-P6)
a, b, c are positive integers such that ab and bc are perfect squares. Prove that ca is also a perfect square.

Solution. Let $ab = m^2, bc = n^2$, then $(ab)(bc) = (mn)^2$, so $(ac)b^2 = (mn)^2$. It is easy to see that if p is a prime factor of b, then it is also a prime factor of mn, thus by canceling these prime factors of b, then $\frac{(mn)^2}{b^2}$ is an integer and a perfect square. Hence ac must be a perfect square. □

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Example 4.3.7 (23-24-S3-A-P7)

Prove that if x and y are sum of two perfect squares, then xy is also a sum of two perfect squares. In other words, if a, b, c, d integers such that $x = a^2 + b^2$, $y = c^2 + d^2$ then $xy = (a^2 + b^2)(c^2 + d^2)$ can be written as a sum of two perfect squares.

Prove that if x and y are sum of a perfect square and twice of another perfect square, then xy is also a sum of a perfect square and twice of another perfect square. In other words, if a, b, c, d integers such that $x = a^2 + 2b^2$, $y = c^2 + 2d^2$ then $xy(a^2 + 2b^2)(c^2 + 2d^2)$ can be written as a sum of a perfect square and twice of another perfect square.

Solution. See below,

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ac - bd)^2$$
$$(a^2 + 2b^2)(c^2 + 2d^2) = (ac + 2bd)^2 + 2(ac - bd)^2$$

□

Example 4.3.8 (23-24-S3-A-P8)

x, y, z are positive integers and their greatest common divisor is 1, such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

Prove that $x + y$ is a perfect square.

Solution. It is easy to transform the given equation as follow,

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z} \Rightarrow z(x + y) = xy \Rightarrow z^2 = (x - z)(x - y).$$

Let d be a common dividor of $x - z$ and $y - z$, then d^2 is a common divisor of z^2, thus $d \mid z$. But $x = (x - z) + z,\ y = (y - z) + z$, therefore $d \mid x, d \mid y$, which means that d is a common divisor of x, y, z. Therefore $\gcd(x - z, y - z) = 1$.

Thus, there exist k, l positive integers such that $x - z = k^2$, $y - z = l^2$, or $(kl)^2 = z^2$, so $kl = z$, which means that:

$$x + y = (z + k^2) + (z + l^2) = k^2 + l^2 + 2kl = \boxed{(k + l)^2.}$$

□

Example 4.3.9 (23-24-S3-A-P9)

n is called *interesting* number if it can be written as $3x^2 + 32y^2$, where x, y are integers. Prove that if n is an *interesting* number, then $97n$ is *interesting* number too.

Solution. Note that

$$96n = 96 \cdot 3x^2 + 96 \cdot 32y^2 = 3(32)y^2 + 32(3x^2).$$

Thus, $96n$ is an *interesting* number. Now,

$$97n = n + 96n = \left[3x^2 + 32y^2\right] + \left[3(32)y^2 + 32(3x^2)\right] = 3[x^2 + (32y)^2] + 32[y^2 + (3x)^2]$$
$$= 3[x^2 + 64xy + (32y)^2] + 32(y^2 - 6yx + (3x)^2] = 3(x + 32y)^2 + 32(y - 3x)^2.$$

Thus $97n$ is an *interesting* number too. □

Fact. $n = ax^2 + by^2$, then $(ab+1)n = a(by-x)^2 + b(ax+y)^2$.

Example 4.3.10 (23-24-S3-A-P10)
Determine all perfect squares in the sequence $\{a_1, a_2, \ldots\}$, where

$$a_3 = 91,\ a_{n+1} = 10a_n + (-1)^n,\ \forall n \geq 0.$$

Solution. Note that $a_2 = 9,\ a_1 = 1,\ a_0 = 0$ are perfect square. By induction we can prove that, for all $n \geq 2$,

$$\begin{cases} a_2 n \equiv 5 \pmod 8 \\ a_{2n+1} \equiv 3 \pmod 8 \end{cases}$$

Since every odd perfect square is congruent to 1 mod 8, thus none of them can be a perfect square. The answer is $\boxed{a_2 = 9,\ a_1 = 1,\ a_0 = 0}$. □

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4.4 Problems (A-Level)

Submission deadline: October 13, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 4.4.1 (23-24-S3-A-P11)**.** (*10 points*) Prove that, for all distinct values of a, b, and c, such that

$$\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}$$

is an integer (not necessarily positive), then the expression

$$\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}$$

is a perfect square.

Solution (Official solution and similar to the ones from Jason To, Karl Le, Ryan Nguyen, Samuel Luong, Tran Hieu Khoa) — Let $x=\dfrac{1}{b-c},\ y=\dfrac{1}{c-a},\ z=\dfrac{1}{a-b}$, then

$$xy+yz+zx=\frac{(a-b)+(b-c)+(c-a)}{(a-b)(b-c)(c-a)}=0\Rightarrow x^2+y^2+z^2=(x+y+z)^2.$$

$$\Rightarrow \frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}=\boxed{\left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)^2.}$$

Solution (Solution 2 similar to the ones from An Nguyen, Antoine Dao, Minh Hai Vu) — Let $A=b-c, B=c-a, C=a-b$, then $A+B+C=0$, therefore:

$$\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)^2=\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}+\frac{2ABC(A+B+C)}{(ABC)^2}$$
$$=\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}=\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}.$$

The conclusion follows.

Remark. An Nguyen, Antoine Dao, Jason To, Karl Le, Minh Hai Vu, Ryan Nguyen, Samuel Luong, Tran Hieu Khoa correctly solved this problem.

Problem 4.4.2 (23-24-S3-A-P12)**.** (*10 points*) Find all (including negative) integer solutions of

$$n^2(n-1)^2 = 4(m^2-1).$$

Solution (Official solution) — The left hand side of the equation is a perfect square, thus the right hand side must be a perfect square, too. It means that m^2-1 is a perfect square. Since m^2 is a perfect square and the only pair of perfect square with difference 1 is $(0,1)$. Therefore $m^2=1$, or $m=\pm 1$. Now, $n^2(n-1)^2=0$, so $n^2=0$, or $(n-1)^2=0$, thus $n=1$.

Hence, the solutions are $\boxed{\{(m,n)\} = \{(0,-1),(0,1),(1,-1),(1,1).\}.}$

Solution (Solution 2 similar to the ones from An Nguyen) —

$$n^2(n^2-1) = 4(m^2-1) \Rightarrow (n(n-1))^2 - (2m)^2 = 4 \Rightarrow (n(n-1)-2m)(n(n-1)+2m) = 4.$$

Note that both $n(n-1)$ and $2m$ are even, thus their sum and difference are even, too. Therefore

$$\begin{cases} n(n-1)-2m = 2, n(n-1)+2m = -2, \text{ or} \\ n(n-1)-2m = -2, n(n-1)+2m = 2, \text{ or} \end{cases} \Rightarrow \begin{cases} n(n-1) = 0, m = -1 \text{ or} \\ n(n-1) = 0, m = 1 \end{cases}$$

Hence, the solutions are $\boxed{\{(m,n)\} = \{(0,-1),(0,1),(1,-1),(1,1).\}.}$

Solution (Solution 2 similar to the ones from Antoine Dao, Karl Le, Minh Hai Vu, Ryan Nguyen) —

$$n^2(n^2-1) = 4(m^2-1) \Rightarrow \left(\frac{n(n-1)}{2}\right)^2 = m^2-1.$$

The only pair of perfect square with difference 1 is 1 and 0, or $m^2=1, n(n-1)=0$. Hence, the solutions are $\boxed{\{(m,n)\} = \{(0,-1),(0,1),(1,-1),(1,1).\}.}$

Remark. An Nguyen, Antoine Dao, Karl Le, Minh Hai Vu, Ryan Nguyen correctly solved this problem.

Jason To's proof stated that if (a,b,c) is a Pythagorean triple $(a^2+b^2=c^2)$ and $b=2$ then $a=0$ and $c=2$. This argument without proof is not acceptable.

It is not clear how the the answers can come straight from $(n^2-n)^2 = (2m-2)(2m+2)$ in Tran Hieu Khoa's solutions.

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Problem 4.4.3 (23-24-S3-A-P13). (*10 points*) Prove that for n positive integer the following number is a perfect square:

$$m = 1\underbrace{77\ldots7}_{n}92\underbrace{88\ldots8}_{n-1}921.$$

Solution (Official solution and same as Minh Hai Vu's) — Note that,

$$\underbrace{77\ldots7}_{n} = \frac{7}{9}\underbrace{99\ldots9}_{n} = \frac{7}{9}(10^n - 1),\ \underbrace{88\ldots8}_{n-1} = \frac{8}{9}(10^{n-1} - 1)$$

$$m = 1 \cdot 10^{2n+4} + \frac{7}{9}(10^n - 1) \cdot 10^{n+4} + 92 \cdot 10^{n+2} + \frac{8}{9}(10^{n-1} - 1) \cdot 10^3 + 921$$

$$= \frac{1}{9}(16 \cdot 10^{2n+4} + 136 \cdot 10^{n+2} + 289) = \frac{1}{9}\left(4 \cdot 10^{n+2} + 17\right)^2.$$

$4 \cdot 10^{n+2} + 17$ is divisible by 3, $(4 \cdot 10^{n+2} + 17)^2$ is divisible by 9, thus $\boxed{\frac{(4 \cdot 10^{n+2}+17)^2}{9} \text{ is a perfect square.}}$

Remark. Only Minh Hai Vu correctly solved this problem.

Jason To started an attempt.

Problem 4.4.4 (23-24-S3-A-P14). (*10 points*) N is 4−digit perfect square all of whose decimal digits are less than seven. Increasing each digit by three we obtain a perfect square again. Find N.

Solution (Official solution and similar to the ones from Jason To, Karl Le, Minh Nguyen, Tran Hieu Khoa) — Let $n^2 = N = \overline{abcd} = 10^3a + 10^2b + 10c + d$, where a, b, c, d are integers and $1 \leq a \leq 6$, $0 \leq b, c, d\ le6$, then there exists integer m such that:

$$m^2 = 10^3(a+3) + 10^2(b+3) + 10(c+3) + (d+3) = n^2 + 3333 \Rightarrow (m-n)(m+n) = 3 \cdot 11 \cdot 101.$$

Since $m < 99$, $n^2 \leq 6666 \Rightarrow n \leq 61$, thus $m + n = 101, m - n = 33$, or $n = 34$, hence $\boxed{N = 1156.}$

Solution (Solution 2 similar to the ones from Minh Hai Vu, Minh Nguyen, Ryan Nguyen) — Note that the unit digit of the perfect square N must be one of $0, 1, 4, 6, 9$. Since $N + 3333$ is also a perfect square thus the unit digit of N must be 1 or 6. By testing all 4-digit perfect numbers between 1000 and 9999 where its unit digit is one of 1, 6 and none of its other digits can be larger than 6, we find $\boxed{N = 1156.}$

Remark. Jason To, Karl Le, Minh Hai Vu, Minh Nguyen, Ryan Nguyen, Tran Hieu Khoa correctly solved this problem.

An Nguyen started an attempt.

Antoine Dao misinterpreted the problem.

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Problem 4.4.5 (23-24-S3-A-P15)**.** (*10 points*) n is a non-negative integer, prove that

$$3^n + 2 \cdot 17^n$$

is never a perfect square.

Solution (Official solution and similar to the ones from Jason To, Karl Le, Ryan Nguyen, Tran Hieu Khoa) — First $3^n + 2 \cdot 17^n = 3$ if $n = 0$, so it is not a perfect square.

For $n \geq 1$, $17^n \equiv 1 \pmod 8$, $3^{2k} \equiv 1 \pmod 8$, $3^{2k+1} \equiv 3 \pmod 8$, thus

$$3^n + 2 \cdot 17^n \equiv \begin{cases} 3 \pmod 8, & \text{if } n \text{ is even} \\ 5 \pmod 8, & \text{if } n \text{ is odd.} \end{cases}$$

However, for m integer,

$$m^2 \equiv \begin{cases} 0 \pmod 8, & \text{if } 4 \mid k \\ 4 \pmod 8, & \text{if } 2 \mid k,\ 4 \nmid k \\ 1 \pmod 8, & \text{if } 2 \nmid k \end{cases}$$

Hence, $\boxed{3^n + 2 \cdot 17^n \text{ is never a perfect square.}}$

Solution (Solution 2 from Chi Khanh Pham, Minh Nguyen) — First $3^n + 2 \cdot 17^n = 3$ if $n = 0$, so it is not a perfect square.

For $n \geq 1$, $17^n \equiv 2^n \pmod 5$, thus the terms $2 \cdot 17^n$ leave a periodic sequence of remainders with period 4:

$$\underbrace{4, 3, 1, 2}_{\text{period}}, 4, 3, 1, 2, 4$$

The terms 3^n also leave a periodic sequence of remainders with period 4, when divided by 5:

$$\underbrace{3, 4, 2, 1}_{\text{period}}, 3, 4, 2, 1, 3$$

Thus the terms $3^n + 2 \cdot 17^n$ leave a periodic sequence of remainders with period 4, when divided by 5:

$$\underbrace{2, 2, 3, 3}_{\text{period}}, 2, 2, 3, 3, 2$$

However any perfect square when divided by 5 leaves a remainder of $0, 1$, or 4.

Hence, $\boxed{3^n + 2 \cdot 17^n \text{ is never a perfect square.}}$

Remark. Chi Khanh Pham , Jason To, Karl Le, Minh Nguyen, Ryan Nguyen, Tran Hieu Khoa correctly solved this problem.

An Nguyen shown an incomplete solution (without conclusion).

Problem 4.4.6 (23-24-S3-A-P16)**.** (*10 points*) n is an arbitrary positive integer. Find positive integer m such that

$$\binom{m}{2} = 3\binom{n}{4}.$$

Solution (Official solution and similar to An Nguyen's) — First we prove the claim,

Claim — The product of 4 consecutive positive integers plus one is always a perfect square.

Proof. Let n be an arbitrary positive integer

$$n(n-1)(n-2)(n-3)+1 = n(n-3)\cdot(n-1)(n-2)+1 = (n^2-3n)(n^2-3n+2)+1 = \left((n^2-3n)+1\right)^2.$$

■

Now, by the given condition, then by the claim:

$$\binom{m}{2} = 3\binom{n}{4} \Leftrightarrow \frac{m(m-1)}{2} = \frac{3n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$

$$\Leftrightarrow 4m(m-1) = n(n-1)(n-2)(n-3) = \left((n^2-3n)+1\right)^2 - 1$$

$$\Leftrightarrow (2m-1)^2 = \left((n^2-3n)+1\right)^2 \Leftrightarrow m = \frac{1}{2}(n^2-3n+2) = \frac{(n-1)(n-2)}{2} = \boxed{\binom{n-1}{2}}.$$

Remark. An Nguyen correctly solved this problem.

Jason To, Ryan Nguyen started an attempt.

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Problem 4.4.7 (23-24-S3-A-P17). (*10 points*) Prove that if the sum $x+y$ can be written as sum of a perfect square and thrice of another perfect square, then the sum x^3+y^3 can also be written as sum of a perfect square and thrice of another perfect square. In other words, if there exist a, b integers such that $x+y=a^2+3b^2$, then there exist c, d integers, such that

$$x^3+y^3=c^2+3d^2.$$

Solution (Official solution and same as Karl Le's) — First we prove the claim,

> **Claim** — Prove that if x and y are sum of a perfect square and thrice of another perfect square, then xy is also a sum of a perfect square and thrice of another perfect square. In other words, if a, b, c, d integers such that $x=a^2+3b^2$, $y=c^2+2=3d^2$ then $xy=(a^2+3b^2)(c^2+3d^2)$ can be written as a sum of a perfect square and thrice of another perfect square.

Proof. It is easy to verify that $(a^2+3b^2)(c^2+3d^2)=(ac+2=3bd)^2+3(ac-bd)^2$. ■

Now, let $x+y=a^2+3b^2$, we have $x^3+y^3=(x+y)(x^2-xy+y^2)$, where:

$$x^2-xy+y^2 \begin{cases} \left(\frac{x+y}{2}\right)^2+3\left(\frac{x-y}{2}\right)^2, & \text{if } x, y \text{ have the same parity} \\ \left(\frac{x}{2}-y\right)^2+3\left(\frac{y}{2}\right)^2, & \text{if } x, y \text{ have different parity} \end{cases}$$

Thus, x^2-xy+y^2, can be written as sum of a perfect square and thrice of another perfect square. Hence, x^3+y^3 can be written as sum of a perfect square and thrice of another perfect square.

> **Remark.** Karl Le correctly solved this problem.

Problem 4.4.8 (23-24-S3-A-P18). (*10 points*) Prove that for integers x, y such that

$$2x^2+x=3y^2+y,$$

then $2x+2y+1$ and $3x+3y+1$ are perfect squares.

Hint: investigate $(2x+2y+1)(x-y)$, *then consider* $(2x+2y+1)(3x+3y+1)(x-y)^2$.

Solution (Official solution and same as Minh Nguyen's and Tran Hieu Khoa's) — First, note that:

$$(2x+2y+1)(x-y)=2x^2-2xy+2yx-2y^2+x-y=2x^2-2y^2+3y^2-2x^2=y^2$$
$$(3x+3y+1)(x-y)=3x^2-3xy+3yx-3y^2+x-y=3x^2-3y^2+3y^2-2x^2=x^2$$

Now, $(2x+2y+1)(3x+3y+1)(x-y)^2=(xy)^2$. If $xy=0$, then one of $x=0$ or $y=0$, which means from the given equality that then the other one of $x=0$ or $y=0$ is also 0 (why?) In any case both $2x+2y+1$ and $3x+3y+1$ are 1 and thus are perfect squares.

Now, suppose that $xy \neq 0$. Let $d \mid \gcd(2x+2y+1, 3x+3y+1)$, then $d \mid (3x+3y+1)-(2x+2y+1)=x+y$. Which means that $d \mid (2x+2y+1)-2(x+y)=1$. Therefore $\gcd(2x+2y+1, 3x+3y+1)=1$, and since $(2x+2y+1)(3x+3y+1)(x-y)^2=(xy)^2$, hence each of $2x+2y+1$ and $3x+3y+1$ is a perfect square.

> **Remark.** Minh Nguyen, Tran Hieu Khoa correctly solved this problem.
>
> Ryan Nguyen, Samuel Luong comes to the conclusion that $(2x+2y+1)(3x+3y+1)(x-y)^2=(xy)^2$.

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Problem 4.4.9 (23-24-S3-A-P19). (*10 points*) Prove that the sum of squares of twelve positive consecutive integers is not divisible by the sum of them. In other words, if n non-negative integers, then

$$(n+1)+(n+2)+\cdots+(n+12) \not| \ (n+1)^2+(n+2)^2+\cdots+(n+12)^2$$

Solution (Official solution) — First, note that the sum of any three positive consecutive integers is divisible by 3, thus the sum of twelve positive consecutive integers is divisible by 3, too. On the other hand, the sum of squares of three consecutive integers when divided by 3 leaves a remainder of 2, the sum of twelve positive consecutive integers when divided by 3 has a remainder 2. The conclusion follows.

Solution (Solution 2 from Tran Hieu Khoa, Ryan Nguyen with modifications) — Note that

$$\begin{aligned}
&(n+1)+(n+2)+\cdots+(n+12)=12n+78\\
&(n+1)^2+(n+2)^2+\cdots+(n+12)^2=12n^2+156n+650\\
&\Rightarrow \frac{12n^2+156n+650}{12n+78}=\left(n+\frac{13}{2}\right)(12n+78)+143
\end{aligned}$$

The last expression shows that it cannot be an integer for any non-negative integer n.

Remark. Ryan Nguyen, Tran Hieu Khoa submitted almost complete solution for this problem.

Problem 4.4.10 (23-24-S3-A-P20). (*10 points*) Let N be a 16−digit number, where none of its digits can be $0, 1, 4$, or 9. Prove that there exist some of its consecutive digits such that their product is a perfect square.

Solution (Minh Nguyen) — Let $N=\overline{a_1a_2\ldots a_{16}}$. Consider the following sequence of 16 terms:

$$a_1,\ a_1a_2,\ ldots,\ a_1a_2\ldots a_{16} \quad (*)$$

Note that since each of $a_1, a_2, \ldots, a_{16}$ can only be $2, 3, 5, 6, 7, 8$, thus each of the term can only be factored as

$$2^{2\alpha_2}+\beta_2\cdot 3^{2\alpha_3}+\beta_3\cdot 5^{2\alpha_5}+\beta_5\cdot 7^{2\alpha_7}+\beta_7,$$

where α_i $(i=2,3,5,7)$ are non-negative integers, β_i $(i=2,3,5,7)$ are 0 or 1.

Thus, the (*) sequence has a one-on-one correspondence map to a sequence of four-digit binaries $\beta_2\beta_3\beta_5\beta_7$. If there is a 0000 binary, then it is the term in (*) which is the desired sequence of consecutive digits, whose product is a perfect square. If there is no such binary, then among 16 terms, two should have the same four-digit binaries $\beta_2\beta_3\beta_5\beta_7$. Let assume that they are $a_1a_2\ldots a_i$ and $a_1a_2\ldots a_j$, where $i<j$, then

$$\frac{a_1a_2\ldots a_j}{a_1a_2\ldots a_i}=a_{i+1}a_{i+2}\ldots a_j$$

has a 0000 binary representation of its prime factorization. This product is a perfect square.

Note that 16 is the best possible lower limit. For 15, the following number 232523272325232 is a counter example.

Remark. Only Minh Nguyen correctly solved this problem.

4.5 The Extremal Principle (O-Level)

Definition (Order relation)**.** An order on a set is a relation $\leq$ with three properties:

1. $a \leq a$,
2. if $a \leq b$ and $b \leq a$ then $a = b$.
3. if $a \leq b$ and $b \leq c$ then $a \leq c$.

Definition (Total order relation)**.** The order is called total if any two elements are comparable, that is, if for every a and b, either $a \leq b$ or $b \leq a$.

Definition (Well-ordering principle on finite set)**.** The *well-ordering principle* states that every *finite* non-empty total order set contains *a maximal* and *a minimal element.*

Definition (Well-ordering principle for positive integers)**.** The *well-ordering principle* states that every non-empty set of positive integers contains a *a least element.*

We use the term *Extremal Principle* to indicate the existence of a minimal or maximal element in a set. The extremal principle is just a clean way of writing up a proof that uses greedy algorithm to find a case by trying to optimize a well-chosen quantity.

We used to combine the Extremal Principle with the Proof by Contradiction to prove an existence or a non-existence. We used to combine the Extremal Principle with the Pigeonhole Principle to establish the condition for existence or to increase or decrease a quantity. We used to combine the Extremal Principle with the Induction Principle to simplify a proof.

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Example 4.5.1 (23-24-S3-O-P1)

Positive integers are written in the cells of a regular square grid such that every number is the average of the numbers in the adjacent cells. Prove that all numbers are equal.

Solution. Lets assume that not all the numbers are equal, thus there exist some numbers that are strictly less some of the numbers in the adjacent cells. By the Extremal Principle, the set of all these numbers must has a *minimal element, which is strictly less then some of the numbers in the adjacent cells.* However since this is the average of the numbers in the adjacent cells, thus it is not less than any of the numbers in the adjacent cells. This is a contradiction. Thus all the numbers are equal. □

Example 4.5.2 (23-24-S3-O-P2)

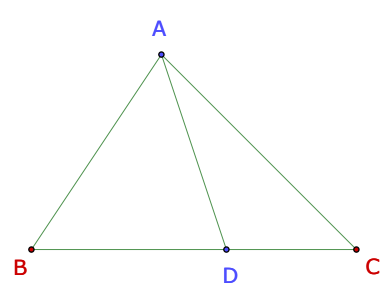
n students ($n \geq 2$) are standing in a field such that the distance between each pair is distinct. Each student is holding a ball, and when the teacher blows a whistle, each student throws their ball to the nearest student. Prove that there is a pair of students that throw their balls to each other.

Solution. Let S be the set of all distances between any pair of students. Since there are n students and $n \geq 2$, S has at least one element, thus it is not empty. S has at most $\frac{1}{2}n(n-1)$ elements, thus it is finite. By the Extremal Principle, S must have the smallest element, which is *the smallest distance of a pair of students.* There is only one such distance because the distance between each pair is distinct. Since this is the smallest distance, the closest student to each of these is the other, so these students throw their ball to each other. □

Example 4.5.3 (23-24-S3-O-P3)

Given a finite set S of red and blue points in a plane. Every line segment that joins two points of the same color contains a point of another color. Prove that all the points lie on a single straight line.

Solution. Assume that not all the points line on a single straight line. Then some of these points form non-degenerate triangles. Let $\triangle ABC$ be *the triangle with smallest area.* Since S is finite and not all points lie on the same line, this triangle exists. According to the Pigeonhole Principle, at least two vertices out of A, B, and C, must have the same colour. WLOG, let B and C be red.



Now, by the given condition, there exists a point $D \in S$ on the segment BC with different colour. It is easy to see that the area of either $\triangle ABD$ or $\triangle ACD$ is strictly less than the area of $\triangle ABC$. This is a contradiction, thus all the points line on a single straight line. □

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Example 4.5.4 (23-24-S3-O-P4)

In some country all roads between cities are one-way and such that once you leave a city you cannot return to it again. Prove that there exists a city into which all roads enter and a city from which all roads exit.

Solution. Consider the oriented graph of roads and cities. By hypothesis, the graph has no cycles. *Define a partial order of the cities, saying that $A < B$ if one can travel from A to B.* A partial order on a finite set has maximal and minimal elements. In a maximal city all roads enter, and from a minimal city all roads exit. □

Example 4.5.5 (23-24-S3-O-P5)

Given $n \geq 3$ points on the plane. Not all of them are collinear. Show that there exists a circle passing through three of the points, the interior of which does not contains any of the remaining points.

Solution. Consider A, B, two of the points with minimal distance. Then there are no additional points inside the circle with diameter AB. *Let C be one of the remaining points with maximal angle $\angle ACB$.* Then there are no points of the given set of points inside the circle through A, B, C. Although they could lie on the circle. □

Example 4.5.6 (23-24-S3-O-P6)

Every participant of a tournament plays with every other participant exactly once. No game is a draw. After the tournament, every player makes a list with the names of all players, who

1. were beaten by him, and
2. were beaten by the players beaten by him.

Prove that the list of some player contains the names of all other players.

Solution. Let A be participant who has won the maximum number of plays. If A would not have the property of the problem, then there would be another player B, who has won against A and against all players beaten by A. Thus B won more times than A. This is a contradiction. □

Example 4.5.7 (23-24-S3-O-P7)

For $n > 1$, the integers from 1 to n^2 are placed in the cells of an $n \times n$ chessboard. Show that there is a pair of horizontally, vertically, or diagonally adjacent cells whose values differ by at least $n + 1$.

Solution. Consider the cells labelled $1, 2, \ldots, n^2$. It is possible to *choose a path of $k \leq n$ cells $1 = c_1, c_2, \ldots, c_k = n$ such that c_i and c_{i+1} are adjacent, for all $i = 1, 2, \ldots, k-1$.* Now, if each pair of adjacent cells had value that differed by at most n, then the difference

$$c_k - c_1 = (c_k - c_{k-1}) + \cdots + (c_2 - c_1) \leq (n-1)n$$

But $c_k - c_1 = n^2 - 1 > n^2 - n$. This is a contradiction. Hence , there is a pair of horizontally, vertically, or diagonally adjacent cells whose values differ by at least $n + 1$. □

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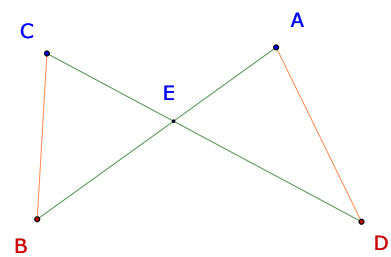
Example 4.5.8 (23-24-S3-O-P8)

There are n red points and n blue points in the plane, with no 3 points collinear. Prove that we can draw n segments joining a red point to a blue point such that no pair of segments intersect.

Solution. For a way to draw n segments joining a red point to a blue point, let calculate the sum of the lengths of these n segments. Let S be the set of all these sums, calculated from all the possible ways to draw such n segments. It is easy to see that S is not empty and is finite, thus there exists *the minimal sum of the segment lengths.* WLOG, let's assume that there exist segments AB and CD, where A, C are red and B, D are blue points, such that AB intersects CD, as shown below.



By the triangle inequality: $AD + BC < (AE + ED) + (CE + EB) = AB + CD$. Thus by disjoining AB, CD and joining CB, AD, we can reduce the sum of the joining segments. This is a contradiction. Thus there are no pair of segments that intersect. □

Example 4.5.9 (Putnam 1956/Problem B3)

Consider a set of $2n$ points in space ($n > 1$). These points are connected by $n^2 + 1$ segments. Show that there exists a triple of points that are connected by straight segments, thus forms a triangle. Show that for each n it is possible to have $2n$ points connected by n^2 segments without any triangles being formed.

Solution. We will prove the first proposition by induction. Case $n = 2$ is obviously true: 4 points are connected by 5 segments thus obviously containing a triangle. Now, let assume that case n is true, we prove case $n + 1$ is true. Let A, B be two points joined by a segment.

Case 1: If there are *less than* $2n + 1$ *other segments with* A *and* B *as vertices,* then if we remove A and B, we get a graph of $2n$ points and at least

$$(n + 1)^2 + 1 - (1 + 2n) = n^2 + 1.$$

This graph, by the induction hypothesis, must contain a triangle.

Case 2: If there are *at least than* $2n + 1$ *other segments with* A *and* B *as vertices,* because by the Pigeonhole Principle, there exists point C, among the other $2n$ points, which is connected to both A and B. Hence, (A, B, C) forms a triangle.

For the second proposition, let G and H be disjoint sets each containing n points. Join every point in G to every point in H by a segment. It is easy to see that there are n^2 segments connecting $2n$ points but there is no triangle. □

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Example 4.5.10 (Poland MO 1975)

A sequence $\{a_n\}$ satisfies the following properties:

$$\exists n \in \mathbb{Z}^+ : \ a_1 + a_2 + \cdots + a_n = 0, \text{ and } a_{n+k} = a_k, \ k = 1, 2, \ldots$$

Prove that there exists a positive integer N such that, for $k = 1, 2, \ldots,$

$$a_N + a_{N+1} + \cdots + a_{N+k} \geq 0.$$

Solution. Let $S_i = a_1 + a_2 + \cdots + a_i, \ i = 1, 2, \ldots$. Note that $S_{pn} = 0, \ p = 1, 2, \ldots$ and $S_{n+i} = S_i, i = 1, 2, \ldots$. Therefore *the set* $\{S_i\}$ *contains a finite number of distinct values.* Let S_m *be the least element in* $\{S_i\}$, then for $N = m + 1$,

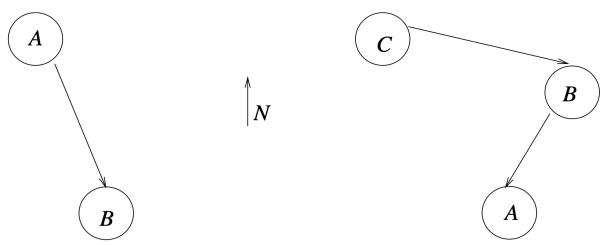
$$a_N + a_{N+1} + \cdots + a_{N+k} = a_{m+1} + a_{m+2} + \cdots + a_{m+k+1} = S_{m+k+1} - S_m \geq 0.$$

□

Example 4.5.11 (IMO SL 1981/Problem 17)

Given is a finite set of spherical planets, all of the same radius and no two intersecting. On the surface of each planet consider the set of points not visible from any other planet. Prove that the total area of these sets is equal to the surface area of one planet.

Solution. Choose a preferential direction in space, which defines the north pole of each planet. Next, *define an order on the set of planets by saying that planet A is greater than planet B if on removing all other planets from space, the north pole of B is visible from A.*



The figure above shows that for two planets A and B, either $A < B$ or $B < A$, and also that for three planets A, B, C, if $A < B$ and $B < C$ then $A < C$. The only case in which something can go wrong is that in which the preferential direction is perpendicular to the segment joining the centers of two planets. If this is not the case, then $<$ defines a total order on the planets. This order has a unique maximal element M. The north pole of M is the only north pole not visible from another planet.

Now consider a sphere of the same radius as the planets. Remove from it all north poles defined by directions that are perpendicular to the axes of two of the planets. This is a set of area zero. For every other point on this sphere, there exists a direction in space that makes it the north pole, and for that direction, there exists a unique north pole on one of the planets that is not visible from the others. As such, the surface of the newly introduced sphere is covered by patches translated from the other planets. Hence the total area of invisible points is equal to the area of this sphere, which in turn is the area of one of the planets. □

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Example 4.5.12 (23-24-S3-O-P12)

Prove that in the coordinate plane, the vertices of a regular pentagon cannot be all lattice points.

Solution. Suppose there exist such pentagons. *Among all such pentagons, consider the one with the least area.* Let p be its side length. Let $\vec{v_i} = a_i\vec{x} + b_i\vec{y}$, for $i \in \{1, 2, 3, 4, 5\}$, be the five side vectors that point from one vertex to the next. Note that

$$a_i^2 + b_i^2 = p^2,\ \forall i \in \{1, 2, 3, 4, 5\},\ \sum_{i=1}^{5} \vec{v_i} = 0 \Rightarrow \sum_{i=1}^{5} a_i = \sum_{i=1}^{5} b_i = 0$$

$$\Rightarrow \left(\sum_{i=1}^{5} a_i\right)^2 + \left(\sum_{i=1}^{5} b_i\right) = 0 \Rightarrow \sum_{i=1}^{5} a_i^2 + \sum_{i=1}^{5} b_i^2 + 2 \sum_{1\le i\le j\le 5} (a_ia_j + b_ib_j) = 0$$

Now $2\sum_{1\le i\le j\le 5}(a_ia_j + b_ib_j)$ is an even number, thus $\sum_{i=1}^{5} a_i^2 + \sum_{i=1}^{5} b_i^2$ is even too. Since $a_i^2 + b_i^2 = 1,\ \forall i \in \{1, 2, 3, 4, 5\}$, thus $5p^2 = \sum_{i=1}^{5} a_i^2 + \sum_{i=1}^{5} b_i^2$ is an even number. Thus p^2 is an even number. It means that a_i, b_i have same parity.

Case 1: If a_1, b_1 are both odd, then $4 \nmid a_1^2 + b_1^2 = p^2$, hence all other a_i and b_i are also odd. But this contradicts $\sum_{i=1}^{5} a_i = \sum_{i=1}^{5} b_i = 0$.

Case 2: If a_1, b_1 are both even, then $4 \mid a_1^2 + b_1^2 = p^2$, which means we can *scale down our pentagon* by a factor of $\frac{1}{2}$ and receiving a regular pentagon with lattice points having smaller area, a contradiction. □

Example 4.5.13 (China MO 1988)

If $a_1 = 1, a_2 = 2$, and

$$a_{n+2} = \begin{cases} 5a_{n+1} - 3a_n, & \text{when } a_na_{n+1} \text{ is even,} \\ a_{n+1} - a_n, & \text{when } a_na_{n+1} \text{ is odd,} \end{cases}$$

prove that, for any positive integer n, $a_n \neq 0$.

Solution. Let *get your hand dirty by calculating the first few values of* a_n.

$$a_3 = 5a_2 - 3a_1 = 7,\ a_4 = 5a_3 - 3a_2 = 29,\ a_5 = a_4 - a_3 = 22,\ a_6 = 5a_5 - 3a_4 = 23.$$

Thus we can notice, and then by induction, prove that the parity of the sequence follows the pattern below:

$$\text{odd, even, odd, odd, even, odd, odd}, \ldots$$

In other words, for $k = 0, 1, \ldots$, a_{3k+1} is odd, a_{3k+2} is even, and a_{3k+3} is odd.

Now, *lets assume that the set of positive integers n where $a_n = 0$ is not empty. Thus, there exists m the least value such that* $a_m = 0$. Since a_m is even, thus $m = 3k + 2$,

$$a_{3k+2} = a_{3k+1} - a_{3k} = (5a_{3k} - 3a_{3k-1}) - a_{3k} = 4a_{3k} - 3a_{3k-1} \Rightarrow 4 \mid a_{3k-1}.$$

Thus $a_{3k-1}, a_{3k-4}, \ldots$ are all multiples of 4. However $2 \nmid a_2$. This is a contradiction. Hence, for any positive integer n, $a_n \neq 0$. □

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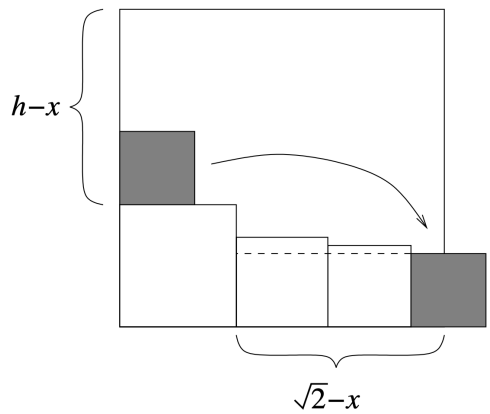
Example 4.5.14 (Kvant (Quantum))

Given finitely many squares whose areas add up to 1, show that they can be arranged without overlaps inside a square of area 2.

Solution. The guess is that a tight way of arranging the small squares inside the big square is *by placing the squares in order of decreasing side length.*

To prove that this works, denote by x the side length of the first (that is, the largest) square. Arrange the squares inside a square of side first in the lower-left corner, the next to its right, and so on, until obstructed by the right side of the big square. Then jump to height x, and start building the second horizontal layer of squares by the same rule. Keep going until the squares have been exhausted.



Let h be the total height of the layers. We are to sow that $h \leq \sqrt{2}$, 2, which in turn will imply that all the squares lie inside the square of side $\sqrt{2}$. To this end, we will find a lower bound for the total area of the squares in terms of x and h. Let us mentally transfer the first square of each layer to the right side of the previous layer. Now each layer exits the square, as shown above.

It follows that the sum of the areas of all squares but the first is greater than or equal $(\sqrt{2}-x)(h-x)$. This is because each newly obtained layer includes rectangles of base $\sqrt{2}-x$ and with the sum of heights equal to $h-x$. From the fact that the total area of the squares is 1, it follows that

$$x^2 + (\sqrt{2}-x)(h-x) \leq 1 \Rightarrow h \leq \frac{2x^2 - \sqrt{2}x - 1}{x - \sqrt{2}}$$

It is easy to prove that

$$\frac{2x^2 - \sqrt{2}x - 1}{x - \sqrt{2}} \Leftrightarrow 2x^2 - 2\sqrt{2}x + 1 \geq 0 \Leftrightarrow (x\sqrt{2}-1)^2 \geq 0.$$

Thus $h \leq \sqrt{2}$. □

Example 4.5.15 (23-24-S3-O-P24)

Rooks are placed on the $n \times n$ chessboard satisfying the following condition: If the square (i, j) is free, then at least n rooks are on the ith row and j^{th} column together. Show that there are at least $\frac{n^2}{2}$ rooks on the board.

Solution. The board has n rows and n columns. Among the rows and columns combined, consider one with the minimal number of rooks. WLOG, suppose that it is a row. Now if this minimal row has at least $\frac{n}{2}$ rooks, then the conclusion follows (why?).

Suppose not. Let, if possible, this row have k rooks where $k < \frac{n}{2}$. Consider the $n-k$ empty squares in this row. In each of their column, we must have at least $n-k$ rooks (due to the given condition). What about the remaining columns? Each of them must have at least k rooks (since k is the minimal number of rooks in any row/column). Therefore, the total number of rooks in the board is at least

$$(n-k)^2 + k^2 \geq \frac{((n-k)+k)^2}{2} = \frac{n^2}{2}.$$

□

Example 4.5.16 (23-24-S3-O-P25)

Any two of a finite number of polygons have a common point. Prove that there is a ine which has a common point with all these polygons.

Solution. Let ℓ be a line on the plane. Project all polygons onto ℓ. We get several segments any two of which have a common point. Consider the left endpoints of these segments and, of these, the one farthest to the right. We gor a point R belong to all segments. The perpendicular line to ℓ through$R$ intersects all polygon. □

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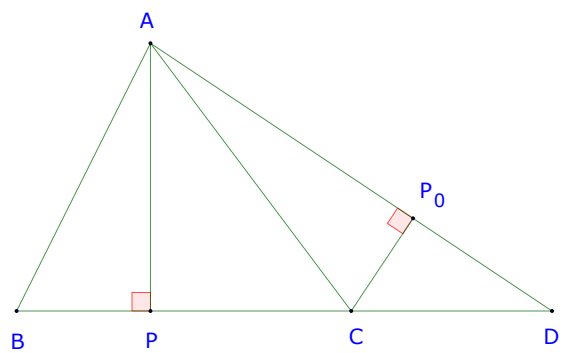
4.6 Problems (O-Level)

Submission deadline: October 13, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 4.6.1 (Sylvester-Galai)**.** Given a finite set S of points in a plane. Prove that, either they all lie on a line, or there is a line containing exactly two points of S.

Hint: Consider the triple (A, B, C) of distinct points in S that are not collinear and such that the distance from A to the line ℓ passing through B and C is minimal.

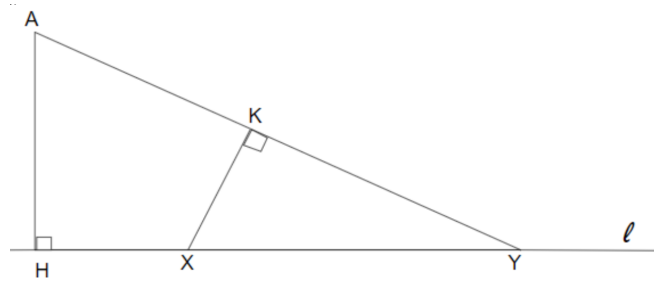


Solution (Official solution and similar to the one from Anthony Pham) — If there is a line containing all points of S, then we are done. Let's assume that not all points are collinear. We prove that there is a line containing exactly two points of S.

Assume for contradiction that the points of S do not all lie on a line and there is no line containing exactly two points of S. The cases of S containing two or three points are trivial. Let's assume that S has at least four (distinct) points.

Consider the triple (A, B, C) of distinct points in S that are not collinear and such that the distance from A to the line ℓ passing through B and C is minimal. Since S is finite and not all contained in a line, this triple exists. Since there is no line passing through exactly two points of S, there is a point $D \in S$ lie on ℓ, such that D lies outside of BC (why?) and $D \neq B$, $D \neq C$.

Let P be the projection of A onto ℓ and note that two of B, C, and D lie on the same side of P on ℓ. Without loss of generality, assume these are C and D and that C is closer to P than D. Let P_0 be the projection of C onto AD. Note that $\triangle DCP_0$ is similar to $\triangle DAP$ with ratio less than 1. This implies that $AP > CP_0$ and therefore the triple (C, A, D) contradicts the minimality of (A, B, C). This shows that there could not have been a third point $D \in S$ on ℓ, completing the proof.



Solution (Solution 2 from Henry Dinh - with minor modifications) — If there is a line containing all points of S, then we are done. Assume that all points of S do not lie on the same line. We prove that there is a line containing exactly 2 points of S. There are a finite number of lines passing through 2 points of S. Consider the positive distances from each point of S to those lines that do not go through that point. Since there are finitely many such distances, there is a distance with minimum value, say from a point A to a line ℓ. Let AH be the perpendicular from A to ℓ at H. We show that ℓ contains exactly 2 points of S.

Suppose to the contrary that ℓ contains 3 points, say X, Y, and Z. Among the 3 points X, Y, and Z, there are 2 points on the same side of H, say X and Y as in the figure below.

Now, let XK be the line perpendicular to AY at K that goes through X. Then, XK is the distance from X to the line passing through A, K and $XK < AH$, which is a contradiction to the minimality of AH. Therefore, the line ℓ contains exactly two points.

Remark. It is very important to use the Extremal Principle in the right way:

1. first, the set of S, where this principle is applied for, should not be empty.
2. second, the cardinality of S, in order words, the number of elements in S has to be finite.

If any of the two arguments above is not present in the proof, the proof is considered invalid. Why? If the first argument is missing, then there any statement on an empty set makes no sense, thus invalid. If the second argument is missing, then there is no guarantee that there exists an upper or lower bound for the elements of the set.

All students forgot there are two statements in the problem: either they all lie on a line, or there is a line containing exactly two points of S. It is fine if you prove the second statement by assuming the first one invalid. However they are two distinct cases and must be treated so.

Anthony Pham correctly solved this problem.

Minh Le Phuong Nguyen's proof does not explore the case D is between B and C (although it is trivial enough to so that would lead to the same conclusion.)

The reasoning that Albert Dinh-Le used has several issues. For example what is line ℓ, then what is point D. The construction of the proof is in the right direction, but missing these nuances made the argument baseless and weak.

Ha-Anh Le's proof does not show clearly how to define the set of distance from a point to a line going through any pair of other points of S. Note that the distances defined as such must be positive. In order to contradicts this assumption, the proof must shows that there exists another point and line with smaller but still positive distance. Thus the argument that the existence of a P point on BC invalidating the existence of the smallest distance is meaningless. The proof have contradicted itself and has not been able to bring out a contradiction.

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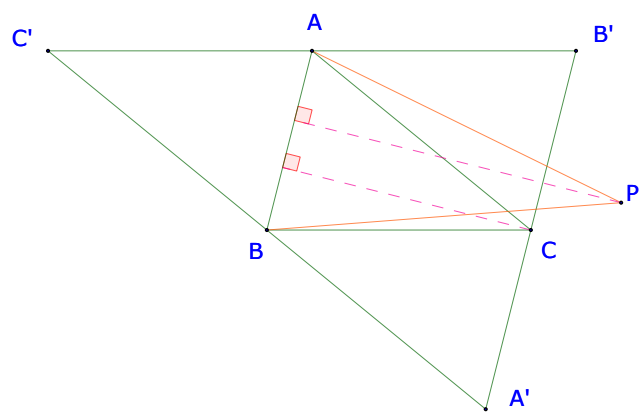
Problem 4.6.2 (Korea MO, 1995)**.** There are n points in a plane such that the area enclosed by any three of the points do not exceed 1. Prove that a triangle can be drawn with area not more than 4 that contains all n points.

Hint: Let $A'B'C'$ be it's anti-complementary triangle, i.e., the triangle formed by the parallels to the sides of $\triangle ABC$ and passing through it's vertices.

Solution (Official solution and similar to Albert Dinh-Le, Anthony Pham, and Henry Dinh) — Among all the triangles formed by the n points, consider the triangle (say, ABC) with maximum area (not exceeding 1). Let $A'B'C'$ be it's anti-complementary triangle, i.e., the triangle formed by the parallels to the sides of $\triangle ABC$ and passing through it's vertices. So $\triangle ABC$ is the medial triangle of $\triangle A'B'C'$, with area of $\triangle A'B'C'$ not exceeding 4. Now we shall prove that $\triangle A'B'C'$ contains all n points. Assume to the contrary that there exists a point (say, P) outside $\triangle A'B'C'$. WLOG, assume that the line $A'B'$ separates the point P from $\triangle A'B'C'$. It is easy to see that $[ABC] < [ABP]$ (why?), this is a contradiction. Hence, $\triangle A'B'C'$ contains all n points.



Solution (Solution 2 from Ha-Anh Le and Minh Le Phuong Nguyen) — Among all the triangles formed by the n points, consider the triangle (say, ABC) with maximum area (not exceeding 1). Let $A'B'C'$ be it's anti-complementary triangle, i.e., the triangle formed by the parallels to the sides of $\triangle ABC$ and passing through it's vertices. So $\triangle ABC$ is the medial triangle of $\triangle A'B'C'$, with area of $\triangle A'B'C'$ not exceeding 4.

Now we shall prove that $\triangle A'B'C'$ contains all n points. It is easy to see that any point P in S must be within the parallel strip formed by line through $A'B'$ and line through C' parallel to $A'B'$. Similarly that P must be within the strip formed by line through $B'C'$ and line through A' parallel to $B'C'$, and within the strip formed by line through $C'A'$ and line through B' parallel to $C'A'$. Therefor P must be inside $\triangle A'B'C'$, or $\triangle A'B'C'$ contains all n points.

Remark. Albert Dinh-Le, Anthony Pham, Henry Dinh, Minh Le Phuong Nguyen correctly solve this problem.

Ha-Anh Le almost made the perfect solution, however the statement that P must be within one of the three triangles $A'BC$, $B'CA$, and $C'AB$ is wrong (P can be inside ABC as well)

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Problem 4.6.3 (Balkan MO 2010)**.** A strip of width w is the set of all points which lie on or between two parallel lines that are at at a distance w apart. Let S be a set of n ($n \geq 3$) points on the plane such that any three different points of S can be covered by a strip of width 1. Prove that S can be covered by a strip of width 2.

Hint: Among all the triangles formed by the n points in S, consider the triangle (say, ABC) with maximum area. Consider the anti-complementary triangle of $\triangle ABC$ and other given points.

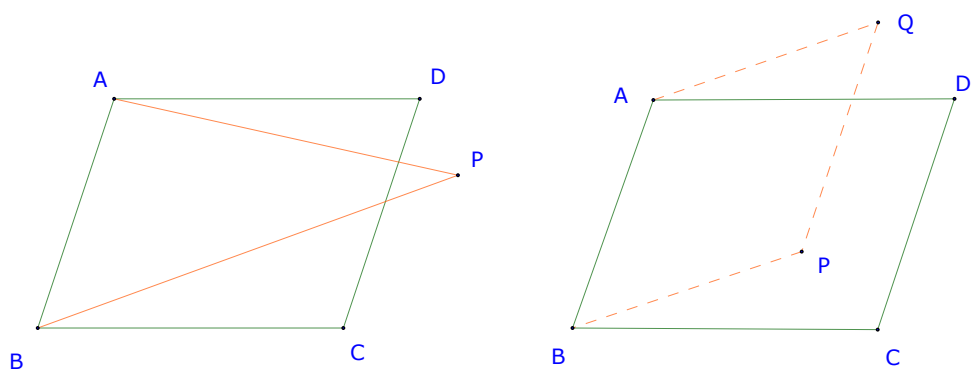
Solution (Official solution and similar to solutions from all students) — Among all the triangles formed by the n points in S, consider the triangle (say, ABC) with maximum area. Proceeding as in the problem Korea MO, 1995, we can prove that all the points in S should lie inside the anti-complementary triangle (say $\triangle A'B'C'$) of $\triangle ABC$, otherwise we have a contradiction. By the property of S, $\triangle ABC$, has altitudes of length at most 1, so $\triangle A'B'C'$ has altitudes of length at most 2 (Why?). Hence, S can be covered by a strip of length 2.

Remark. Albert Dinh-Le, Anthony Pham, Henry Dinh, Minh Le Phuong Nguyen correctly solve this problem.

Ha-Anh Le's proof stated that the altitude from $\triangle ABC$ has an altitude of 1 is wrong.

Problem 4.6.4 (USA TST 2005)**.** Find all sets S of finitely many points in the plane, no three of which are collinear and such that for any three points A, B, C in S, there is another point D in S such that A, B, C, D (in some order) are the vertices of a parallelogram.

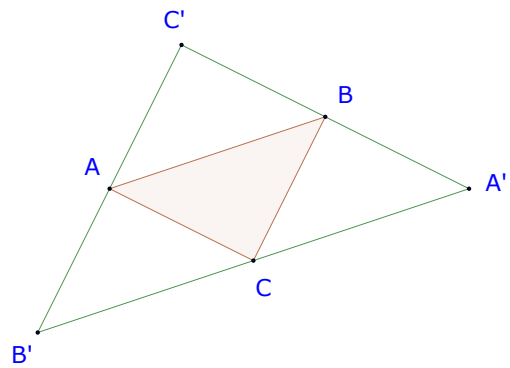
Hint: Assume, to the contrary, that S has more than 4 points. Consider points A, B, C in S such that $\triangle ABC$ has the maximum possible area over all choices of A, B, C. Let point D be such that $ABCD$ is a parallelogram. Consider $ABCD$ and other points of S.



Solution (Official solution and same as Henry Dinh's) — We claim that any such set S contains at most 4 points. Assume, to the contrary, that S has more than 4 points. Consider points A, B, C in S such that $\triangle ABC$ has the maximum possible area over all choices of A, B, C. There is a point D such that $ABCD$ is a parallelogram.

Now, any other point P in S cannot lie outside $ABCD$ because in that case, see the diagram on the left, the area of $\triangle ABP$ will be greater than that of ABD (assuming WLOG that the line CD separates P from AB). Therefore, P lies inside $ABCD$.

But then for any point Q in S such that $ABPQ$ is a parallelogram, Q lies outside $ABCD$, see the diagram on the right. Now, the area of $\triangle BQC$ is greater than that of $\triangle ABC$ (assuming WLOG that the line AD separates Q from BC), a contradiction. Therefore, any such set S contains at most 4 points.



Solution (Solution 2 and similar to the ones from Anthony Pham, Minh Le Phuong Nguyen) — We claim that any such set S contains at most 4 points. Assume, to the contrary, that S has more than 4 points. Consider points A, B, C in S such that $\triangle ABC$ has the maximum possible area over all choices of A, B, C. There is a point D such that $ABCD$ is a parallelogram.

Now, D can be only one of A', B', C'. Let assume that $D' \equiv A'$. Now for any other point $E \in S, E \not\equiv D$, $ABCE$ is a parallelogram, thus E can be only one of B', C', let's assume that $E \equiv B'$, then

$$[ADE] = [AA'B'] > [CAB], \text{ a contradiction.}$$

Therefore, any such set S contains at most 4 points.

Remark. Anthony Pham, Henry Dinh, Minh Le Phuong Nguyen correctly solve this problem.

Minh Le Phuong Nguyen's solution is a brilliant one, showing a great insight of the problem.

Problem 4.6.5 (23-24-S3-O-P20)**.** Prove that among any 50 distinct positive integers strictly less than 100 there are two that are co-prime.

Hint: Order the numbers $x_1 < x_2 < \ldots < x_{50}$. Compare x_1 and x_{50}.

Solution (Official solution and similar to solutions from all students) — Order the numbers:

$$x_1 < x_2 < \ldots < x_{50}.$$

If in this sequence there are two consecutive integers, they are co-prime and we are done. Otherwise, $x_{50} \geq x1 + 2 \cdot 49 = 99$. Equality must hold, since $x_{50} < 100$, and in this case the numbers are precisely the 50 odd integers less than 100. Among them the number 3 is co-prime to the number 7.

Remark. All students correctly solve this problem.

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Problem 4.6.6 (23-24-S3-O-P21)**.** Prove that it is impossible to dissect a cube into finitely many cubes, no two of which are the same size.

Hint: assume that such a dissection exists, and look at the bottom face. It is cut into squares. Take the smallest of these squares. It is not hard to see that this square lies in the interior of the face (why?). Look at the cube that lies right above this square!

Solution (Official solution and similar to submitted solutions) — Assume that such a dissection exists, and look at the bottom face. It is cut into squares. Take the smallest of these squares. It is not hard to see that this square lies in the interior of the face, meaning that it does not touch any side of the bottom face. Look at the cube that lies right above this square! This cube is surrounded by bigger cubes, so its upper face must again be dissected into squares by the cubes that lie on top of it. Take the smallest of the cubes and repeat the argument. This process never stops, since the cubes that lie on top of one of these little cubes cannot end up all touching the upper face of the original cube. This contradicts the finiteness of the decomposition. Hence the conclusion.

Remark. Anthony Pham, Ha-Anh Le, Henry Dinh, and Minh Le Phuong Nguyen correctly solve this problem.

Problem 4.6.7 (Romania 1999 TST/Problem 5)**.** Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of distinct positive integers. Prove that, for any positive integer n

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{2n+1}{3}(a_1 + a_2 + \cdots + a_n).$$

Hint: first, WLOG, rearrange $\{a_i\}$ as a strictly increasing sequence. Then prove that:

$$2(a_1 + a_2 + \cdots + a_n) \leq (a_{n+1} - 1)a_{n+1} \leq 3a_{n+1}^2 - (2n+3)a_{n+1}.$$

Solution (Official solution and similar to Henry Dinh's) — First, WLOG, rearrange $\{a_i\}$ as a strictly increasing sequence: $a_1 < a_2 < \cdots < a_n < \cdots$ Then since $\{a_i\}$ is a sequence of distinct positive integers, thus

$$a_n \geq n, \ \forall n \quad (*)$$

Now, we will prove by induction the desired inequality. It is easy to verify that $a_1^2 \geq \frac{2\cdot 1+1}{3}a_1$ since $a_1 \geq 1$. Let's assume that the induction hypothesis is correct and we have to prove that,

$$a_1^2 + a_2^2 + \cdots + a_{n+1}^2 \geq \frac{2(n+1)+1}{3}(a_1 + a_2 + \cdots + a_{n+1}).$$

By the induction hypothesis, we just need to prove that:

$$a_{n+1}^2 \geq \frac{2}{3}(a_1 + a_2 + \cdots + a_n) + \frac{2n+3}{3}(a_{n+1}) \Leftrightarrow 3a_{n+1}^2 - (2n+3)a_{n+1} \geq 2(a_1 + a_2 + \cdots + a_n).$$

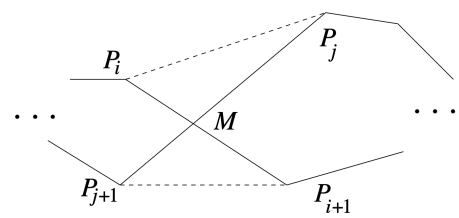
But by (*),

$$\begin{aligned} &3a_{n+1}^2 - (2n+3)a_{n+1} = a_{n+1}(3a_{n+1} - 2n - 3) \geq a_{n+1}(a_{n+1} - 1) \\ &\geq (a_n + 1)a_n = 2(1 + 2 + \ldots + a_n) \geq 2(a_1 + a_2 + \cdots + a_n). \end{aligned}$$

Remark. Henry Dinh correctly solve this problem. Ha-Anh Le managed to solve part of the problem

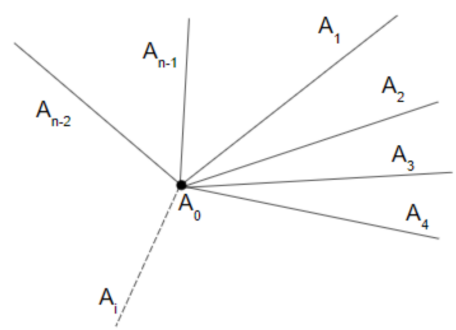
Problem 4.6.8 (23-24-S3-O-P23)**.** Given n points in the plane, no three of which are collinear, show that there exists a closed polygonal line with no self-intersections having these points as vertices.

Hint: study the example 23-24-S3-O-P4. There are only finitely many polygonal lines with these points as vertices.



Solution (Official solution and similar to Anthony Pham's and Henry Dinh's first solution) — There are only finitely many polygonal lines with these points as vertices. Choose the one of minimal length $P_1P_2\dots P_n$. If two sides, say P_iP_{i+1} and P_jP_{j+1}, intersect at some point M, then replace them by P_iP_j and $P_{i+1}P_{j+1}$ to obtain the closed polygonal line $P_1\dots P_iP_jP_{j-1}\dots P_{i+1}P_{j+1}\dots P_n$, see the figure below.

The triangle inequality in triangles MP_iP_j and $MP_{i+1}P_{j+1}$ shows that this polygonal line has shorter length, a contradiction. It follows that $P_1P_2\dots P_n$. has no self-intersections, as desired.



Solution (Solution 2 from Henry Dinh) — Fix a point A_0 and another point A_1. From these n points, there is a finite number of angles with vertex A_0 and first arm A_0A_1 and second arm is the line segment connecting A_0 with another point among the remaining points.

Note that any line A_0A_i does not contain any other point among those n points because no three points are collinear. We can label and order these points as A_2, A_3, $\dots$, A_{n-1} such that

$$\angle A_1A_0A_2 < \angle A_1A_0A_3 < \angle A_1A_0A_4 < \cdots < \angle A_1A_0A_{n-1}.$$

In other words, we label the $n-1$ points around A_0 clockwise. Clearly, $A_0A_1\dots A_{n-1}A_0$ is a closed polygonal line with no self-intersections.

Remark. Anthony Pham, Henry Dinh correctly solve this problem.

The second solution from Henry Dinh is a marvel of proof in combinatorial geometry.

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Chapter 5

Session 4

5.1 Colouring with two colours (I-Level)

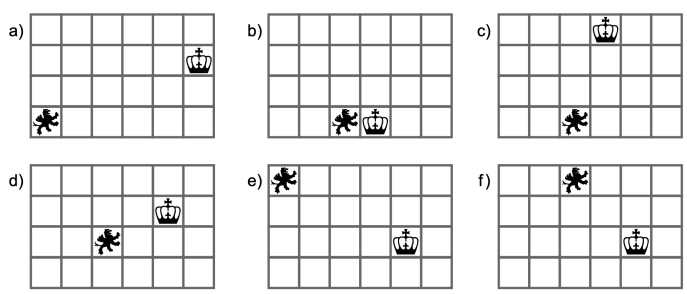
Identifying *a quantity or property that remains unchanged is a very powerful problem-solving technique*: such a property is called an **invariant.** Invariants could help us crack really tough problems.

First, we identify the invariant that uses the ideas of *alternation and parity.*

Example 5.1.1 (23-24-S4-I-P1)

The castle has 24 rooms arranged as a 6×4*grid*, in each wall between the rooms there is a door. Can you walk through the castle in such a way that you would start in the room marked with the coat of arms (a winged lion), visit each room of the castle exactly once, and end up in the room marked with a crown?

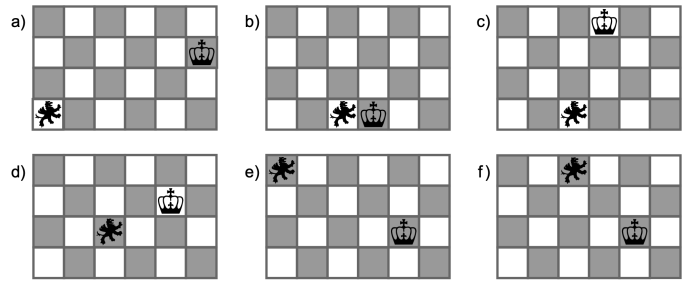
There are six different cases show as six diagrams. Solve all of them.



Remark. By *trial-and-error,* you might see that cases (a), (b), and (d) have solution, while (c), (e), and (f) are impossible to solve. Why is that? We cannot reason such as *" "If we start at this room and walk this way, we get stuck here; if we walk that way we get stuck there .... We tried all possible paths, and we failed to reach our goal..."*. We have to reason with firm mathematical arguments, which are not only to solve a particular case, but for similar cases as well.

Here is an interesting observation: if we take *the shortest route* from the first to the last room and *pass through an even number of empty rooms,* the puzzle can be solved; if we pass through an odd number of rooms, it cannot. This observation seems to work: in (a) we pass through 6 rooms, in (b)—through 0, in (c)—through 3, in (e)—through 5 rooms. However, this is not a proof yet; so far, it is just an observation that happens to be correct for several specific puzzles. Is there a way to prove that this pattern is always true? We do need an insight. How about coloring the rooms in a checkerboard pattern?

Solution. Lets colour the maps by checkerboard coloring.



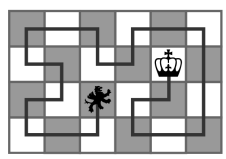
Here are our observations:

- The rooms along a route alternate in color!;
- If the first and the last rooms are of different colors, then the puzzle has a solution!
- If the first and the last rooms are of the same color, the puzzle cannot be solved!

Hence, we devised a hypothesis:

A proper route has to start and end in rooms of different colors!

Indeed, the rooms along a route always alternate in color. Therefore, all the odd-numbered rooms must be of one color, and all the even-numbered rooms—of the other. The castle has 24 rooms altogether; thus, a path should be 24 rooms long. However, 24 is an even number. Therefore, the color of the last room will always be different from the color of the first room, see the diagram below. It explains why (c), (e), and (f) are impossible.



□

Remark. For this problem, we found a property that remains unchanged: *a path of even length should always start and end in rooms of different colors.*

So, the conclusion is that it would be impossible to find a path of an even length connecting rooms of the same color!

Congratulation! You just have discovered the first invariant that uses the ideas of alternation and parity.

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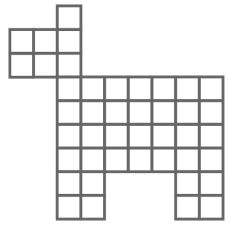
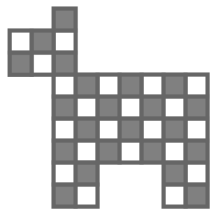
Example 5.1.2 (23-24-S4-I-P2)

Can you tile this doggy with dominoes? (A domino is a 2×1 rectangle. "To tile" means to cover completely and without overlaps.)



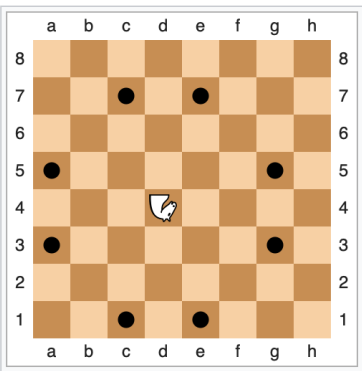
Solution. If you colored the doggy in a checkerboard pattern, you would see that the number of black squares is 2 bigger than the number of white squares, see the diagram below.



However, each tile covers 1 black and 1 white square. Therefore, such a tiling is not possible. □

Example 5.1.3 (LPS V2B/2.10)

A special chess piece called a *camel* moves along a board in almost the same way as a knight does: it moves to any adjacent square and then move three squares in any particular direction (up, down, left, right), see below. Is it possible for a *camel* to go from some square to an adjacent square?



The camel (represented as an inverted knight) may move to any of eight squares (black dots).

Solution. A move of a chess *camel* always starts and ends on squares of the same color. Thus, if a camel starts on black, it stays on black. However, the target square is white. Therefore, the camel cannot get there. □

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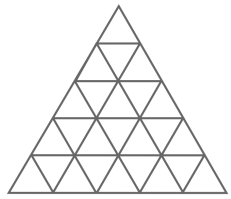
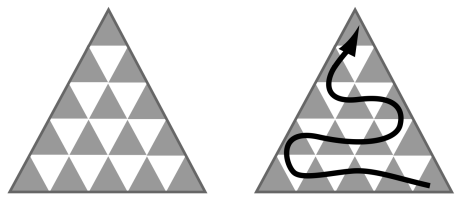
Example 5.1.4 (23-24-S4-I-P4)

The Triangular Castle, shown as below, has triangular rooms and a door in each of its interior walls. A tour guide wants to trace a route through this castle that would not visit any room more than once. What is the maximum number of rooms one can visit on such a tour? (A tour can start and end in any two rooms.)



Solution. Let's color the rooms of the castle black and white, as shown in the picture below on the left. You can see that there are 15 black rooms and only 10 white rooms. Since black and white rooms alternate along any route, one can't visit more than 11 black rooms, so 4 rooms will always be left unvisited.

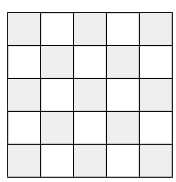


It is easy to plot a route that visits all rooms except 4, see the picture above on the right. □

Example 5.1.5 (23-24-S4-I-P5)

There are 25 beetles sitting on the 25 squares of a 5×5 chessboard. On a blow of a whistle, each beetle relocates to an adjacent square (a square above or below or to the right or to the left). Prove that at least one of the squares will end up being empty.

Solution. Color the 5×5 square in a checkerboard pattern as shown below.



Altogether, the beetles are located at 13 black and 12 white squares, 1 per square. Each beetle changes the color of his square when he relocates. Therefore, 13 beetles that were used to be on black squares are all now sitting on 12 white squares. Therefore, by the Pigeonhole Principle, there will be at least 1 square with at least 2 beetles on it. □

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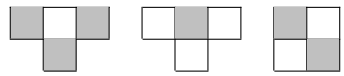
Example 5.1.6 (LPS V2B/2.2)

Prove that the 8×8 chessboard cannot be covered by 15 tetrominoes and 1 square tetromino, shown below.



Proof. Assume that it is possible. A tetromino has three black squares and one white square or three white squares and one black square. A square tetromino contains two black and two white squares. Thus, *the number of black squares in* 15 *tetrominoes and* 1 *square tetromino is an odd number.* This is not possible because the 8×8 chessboard has an even number of black squares. Hence, the 8×8 chessboard $\boxed{\text{cannot be covered}}$ by 15 tetrominoes and 1 square tetromino. □

Example 5.1.7 (23-24-S4-I-P7)

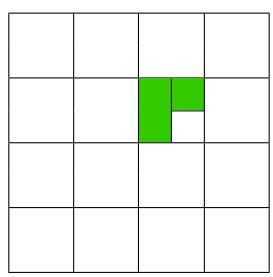
Twenty-five checkers are placed on a 25×25 checkerboard in such a way that their positions are symmetric with respect to *one of its diagonals.* Prove that at least one of the checkers is positioned on that diagonal.

Proof. If no checker occurred on the diagonal, then the checkers could be partitioned into pairs, placed symmetrically with respect to the diagonal. Therefore, there must be one (and in fact an odd number) of checkers on the diagonal. First, note that there may be not just one, but any odd number of checkers on the diagonal. We partition into pairs thus: if we form a number of pairs from a set of oddly many objects, then at least one object will remain unpaired. □

Example 5.1.8 (23-24-S4-I-P8)

What is the largest number of squares on an 8 x 8 checkerboard which can be colored green, so that in any arrangement of a *tromino* of three squares, shown as below, at least one square is not colored green? (The tromino may appear as in the figure, or it may be rotated through some multiple of 90 degrees.)

Solution. The answer is 32. Indeed, suppose that 33 or more squares are colored green. Then, after we have divided the board into sixteen 2×2 squares,



the Pigeonhole Principle guarantees that at least one of these squares contains 3 or more small green squares. These 3 green squares form the forbidden tromino in some position, and we have a contradiction. On the other hand we can color all the black fields (of the usual coloring) green, and this is an example of 32 green squares with the property needed. □

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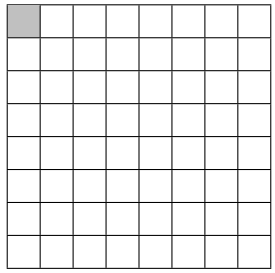
Example 5.1.9 (23-24-S4-I-P9)

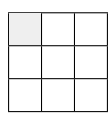
In an 8×8 table one of the boxes is colored black and all the others are white. Prove that one cannot make all the boxes white by recoloring the rows and columns. *Recoloring* is the operation of changing the color of all the boxes in a row or in a column.



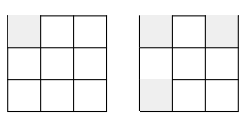
Proof. If we recolor a column (or a row) containing an odd number of black boxes, then after the operation the number of black boxes in that column (or that row) is still odd. If we recolor a column (or a row) containing an even number of black boxes, then after the operation the number of black boxes in that column (or that row) is still even. It means that the number of odd black boxes in the table is *invariant* and thus remains odd. Hence, it is impossible to make all the boxes white by recoloring the rows and columns. □

Example 5.1.10 (23-24-S4-I-P10)

In an 3×3 table one of the boxes is colored black and all the others are white. Prove that one cannot make all the boxes white by recoloring the rows and columns. *Recoloring* is the operation of changing the color of all the boxes in a row or in a column.



Proof. Let's consider the four corner boxes. Initially it is odd since there is a single black box in one of the corners. Now, let assume that the number of the black boxes in these corners is odd: 1 or 3, see below.



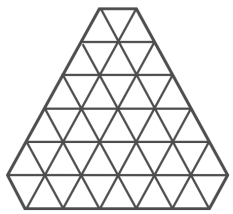
In any case an operation will leave the parity of number unchanged if it is performed on a column or a row containing a single black box (since it swap one black box for another), or two boxes of the same colour (two boxes of the same colour to two boxes of the other colour). Thus, the parity of the number of black boxes among the four corner boxes is *invariant* (read *unchanged*) under recolorings. Hence, it is impossible to make all the boxes white by recoloring the rows and columns. □

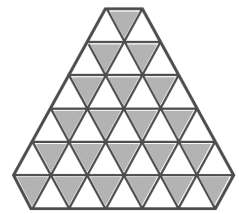
5.2 Problems (I-Level)

Submission deadline: October 20, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 5.2.1 (23-24-S4-I-P11)**.** (*10 points*) Can you cut this hexagon-shaped cake into 23 identical pieces? (All cuts should follow the grid lines.)





Solution (Official solution and similar to the ones from Chi-Mai Pham, Jérémy Nguyen, Khoa Anh Vu, Minh Pham with modifications) — The cake is made of $3+5+7+9+11+11=46$ little triangles. Therefore, each of the 23 pieces should be composed of 2 triangles that share a side.

Let's *color the triangles in alternating black and white pattern as shown above.* This way, each piece should have 1 black and 1 white triangle. However, the numbers of black and white triangles are 21 and 25. Therefore, it is not possible to cut the cake into 23 identical pieces because that requires *the equal numbers for black and white triangles.*

Remark. Only Chi-Mai Pham, Jérémy Nguyen, Khoa Anh Vu, Minh Pham correctly solved this problem.

Julie Cao understood how to use checkerboard pattern but had not recognized the difference in numbers of black and white triangles.

Giang Nguyen, Minh Do Tue Le could not use the checkerboard pattern to solve the problem.

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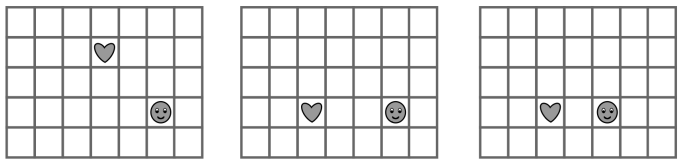
Problem 5.2.2 (23-24-S4-I-P12)**.** (*10 points*) The summer cottage has 35 rooms arranged as a 7×5 grid; each wall between the rooms has a door. Can you walk through the cottage in such a way that:

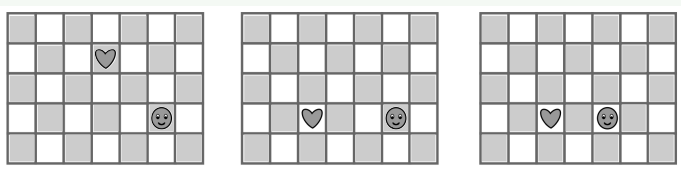
- you would start in the drawing room (marked with a smiley face);
- you would visit each room of the cottage exactly once;
- you would end up in the room marked with the heart?

For every map, either draw a path or prove that such a route is impossible.



Solution (Official solution and similar to the ones from Giang Nguyen, Jérémy Nguyen with modifications) — A solution for the first map is easy to find.

The board is 7×5; thus, a proper route should be 35 squares long. If we *color the board in a checkerboard pattern,* the squares along the route should alternate in color. Since 35 is an odd number, *the first and last rooms should be of the same color.* Therefore, the second map is not possible.



Next, suppose that the low left corner of the board is black. Then 18 squares are black, and 17 are white. If we line these squares up in an alternating pattern, then the *first and last squares must be of the dominating color—black.* However, in the third map both squares are white. Thus, it is impossible.

Remark. Only Giang Nguyen, Jérémy Nguyen correctly solved this problem.

Chi-Mai Pham, Julie Cao, Minh Do Tue Le, Minh Pham solved correctly the first two questions but cannot managed the last.

Khoa Anh Vu must mistakenly interpreted the problem text.

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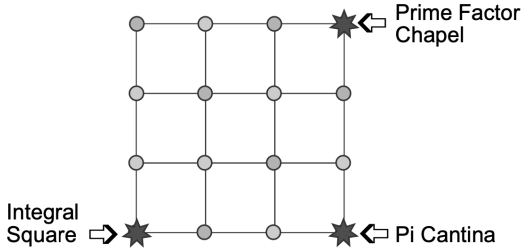
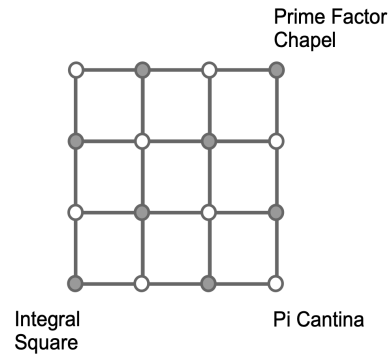
Problem 5.2.3 (23-24-S4-I-P13)**.** (*10 points*) Downtown Math-Hattan has 4 streets going east-west, and 4 avenues going north-south, with a small plaza at each intersection. All roads can be walked in both directions.

- Can you walk from the Integral Square to the Pi Cantina, visiting each plaza exactly once?
- Can you walk from the Integral Square to the Prime Factor Chapel, visiting each plaza exactly once?

For each part, either present the route or explain why it doesn't exist.





Solution (Official solution and similar to the ones from Chi-Mai Pham, Jérémy Nguyen, Khoa Anh Vu, Minh Pham with modifications) — A route from the Square to the Cantina is easy to draw to answer the first question.

For the second question: a proper route from the Square to the Chapel is not possible. Indeed, let's *color the intersections into alternating black and white colors.* Then *the colors of the intersections along a route would alternate.* Moreover, the Square-to-Chapel route that visits all intersections exactly once should pass through all 16 of them. Therefore, the first and last intersections on this route should be of different colors. However, the Square and the Chapel are of the same color. Hence. it is impossible to do so.

Remark. Only Chi-Mai Pham, Jérémy Nguyen, Khoa Anh Vu, Minh Pham correctly solved this problem.

Giang Nguyen, Julie Cao, Minh Do Tue Le could not color the intersections by alternating colors, therefore didn't know how to continue.

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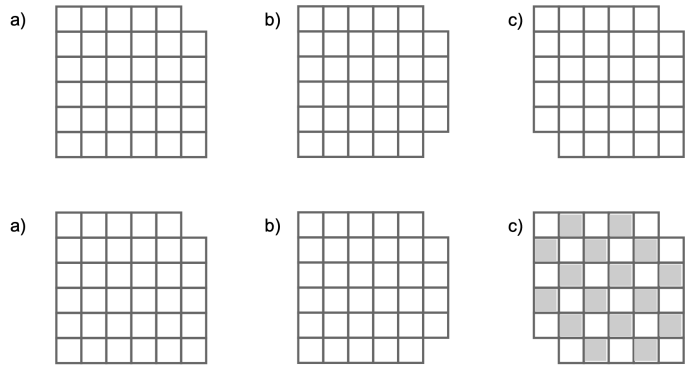
Problem 5.2.4 (23-24-S4-I-P14)**.** You start off with a 6×6 chessboard. In part (a), one corner of it is missing. In parts (b) and (c), each chessboard is missing 2 corners. Which of these boards can you tile with dominoes? (A domino is a 2×1 tile. "To tile" means to cover completely and without overlaps.)



Solution (Official solution and similar to the ones from Chi-Mai Pham, Jérémy Nguyen, Julie Cao, Khoa Anh Vu, Minh Pham with modifications) — For the first question: each tile covers exactly 2 squares (an even number). Therefore, a shape that can be tiled should have an even number of squares. Thus, since the big shape is made of 35 squares, it cannot be tiled.

For the second question: such a tiling is easy to draw.

For the last question: If we *color the board in a checkerboard pattern,* as shown above, we can observe that both missing squares are black.

Thus, the numbers of black and white squares are different. However, any 2×1 tile would always cover 1 black and 1 white square. Therefore, it is not possible to tile this shape.

Remark. Only Chi-Mai Pham, Jérémy Nguyen, Julie Cao, Khoa Anh Vu, Minh Pham correctly solved this problem.

Giang Nguyen, Minh Do Tue Le could not color the intersections by alternating colors, therefore didn't know how to continue.

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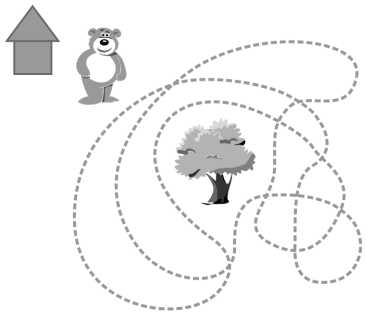
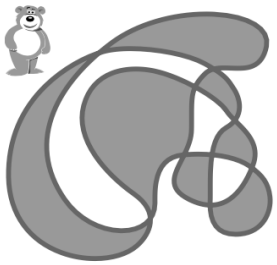
Problem 5.2.5 (23-24-S4-I-P15)**.** (*10 points*) Little Bear came out of his house and found strange footprints going around his favorite tree (see the picture below). He walked around the tree several times, studying the tracks and sometimes jumping over them to get a better look. In the end, the bear returned to the house. Could he have jumped over the mystery footprints 7 times? (Little Bear never jumps over the places where the track intersects itself.)





Solution (Official solution and similar to the ones from Jérémy Nguyen, Khoa Anh Vu with modifications) — Color the picture as shown.

Then each time Little Bear jumps over footprints he changes the color. Thus, after 7 jumps he will be on a color that is different from where he started.

Remark. Only Jérémy Nguyen, Khoa Anh Vu, correctly solved this problem.

Minh Do Tue Le was close to the key argument. Giang Nguyen did not realise the way to do alternate colouring.

Problem 5.2.6 (23-24-S4-I-P16)**.** (*10 points*) Twenty-five checkers are placed on a 25×25 checkerboard in such a way that their positions are symmetric with respect to *both of its diagonals.* Prove that one of the checkers is placed in the center square.

Hint: connect all the pairs of checkers which are located symmetrically with respect to one of the diagonals with a thread into necklaces.

Solution (Official solution and similar to the ones from Jérémy Nguyen, Julie Cao with modifications) — Suppose that no checker is placed in the center square. We connect all the pairs of checkers which are located symmetrically with respect to one of the diagonals with a thread. We then divide all the.checkers into *necklaces*: groups connected by threads. Then in each *necklaces* there will be either two or four checkers. This means that the total number of checkers will be even, which is a contradiction.

Remark. Only Jérémy Nguyen, Julie Cao, correctly solved this problem.

Chi-Mai Pham, Minh Do Tue Le made weak arguments.

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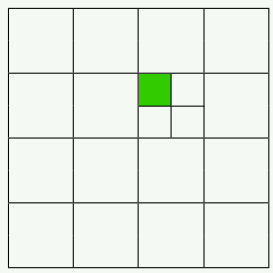
Problem 5.2.7 (23-24-S4-I-P17)**.** What is the smallest number of squares on an 8 x 8 checkerboard which can be colored green, so that in any tromino such as below, at least one square is colored green? (The tromino may appear as in the figure, or it may be rotated through some multiple of 90 degrees.)

Solution (Official solution and similar to the ones from Jérémy Nguyen, Minh Do Tue Le, Minh Pham with modifications) — The answer is 32. Indeed, if 31 or fewer squares are colored green, then, after we have divided the board into sixteen 2×2 squares, one of those sixteen 2×2 squares contains 1 (shown as below,) or 0 green squares.

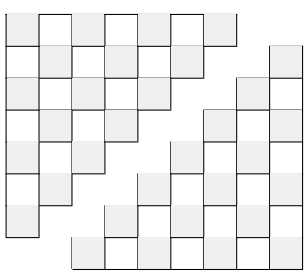


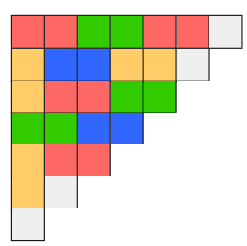
Then the other 3 or 4 squares are not colored green, and they form the tromino without green.

Remark. Only Jérémy Nguyen, Minh Do Tue Le, Minh Pham correctly solved this problem.

Chi-Mai Pham made a wrong argument. Julie Cao regretfully misinterpreted the problem text?

Problem 5.2.8 (23-24-S4-I-P18)**.** (*10 points*) One main diagonal of the 8×8 chessboard is removed, leaving two triangular parts, as shown in below on the left. What is the *largest number* of 1×2 dominoes can be used to tile the two triangular parts?





Solution (Official solution and similar to the ones from Chi-Mai Pham, Jérémy Nguyen, Julie Cao, Minh Pham with modifications) — Note that each triangular part contains 16 square of one colour and 12 of the other. A domino contains one square for each colours, so the total numbers of squares for each colour the dominoes cover are the same. Thus, at most 12 dominoes can be used to tile it. Above on the right shows an example of tiling with 12 dominoes. The maximum number of dominoes can be used for both part is $\boxed{2 \cdot 12 = 24.}$

Remark. Only Chi-Mai Pham, Jérémy Nguyen, Julie Cao, Minh Pham correctly solved this problem.

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Problem 5.2.9 (23-24-S4-I-P19)**.** (*10 points*) Someone has put a rectangular playing card *face down* on the table. You can flip the card over one of its edges as many times as you want. Can you end up with the card sitting *face up* on the same spot?

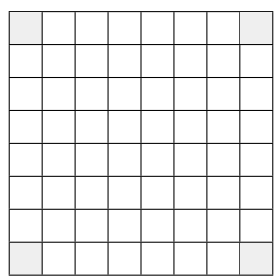
Hint: create a rectangular grid, each rectangle has the same size as the card. Let the card be on a rectangle of the grid. Colour the grid. When the card is turned over one of its sides, shown below, what happens?

Solution (Official solution and similar to the ones from Chi-Mai Pham, Giang Nguyen, Jérémy Nguyen, Julie Cao, Minh Pham with modifications) — First, let's create a rectangular grid, each rectangle has the equal size as the card. *Color the grid's rectangles alternately black and white.* Suppose the card is on white. If you flip it around the edge, it will be *face up and on black.* If you flip it one more time, it will be on *white and face down,* and so on. Thus, whenever the card is face up, it is on black, and whenever the card is face down, it is on white. Therefore, it cannot end up sitting face up on the original spot.

Remark. Only Chi-Mai Pham, Giang Nguyen, Jérémy Nguyen, Julie Cao, Minh Pham correctly solved this problem.

Problem 5.2.10 (23-24-S4-I-P20)**.** (*10 points*) In an 8×8 table four corner of the boxes is colored black and all the others are white. Prove that one cannot make all the boxes white by recoloring the rows and columns. *Recoloring* is the operation of changing the color of all the boxes in a row or in a column.

Hint: look at both 23-24-S4-I-P9 and 23-24-S4-I-P10 for ideas, can you find a 2×2 square where the number of black boxes is invariant during recolorings?



Solution (Official solution and similar to the one from Jérémy Nguyen with modifications) — This problem can be solved in just the same way as 23-24-S4-I-P9, if we consider a set of four boxes with the same property. One such set is the four boxes forming a 2×2 square in the upper left corner of the table.

Remark. Only Jérémy Nguyen correctly solved this problem.

Chi-Mai Pham, Julie Cao made a good start.

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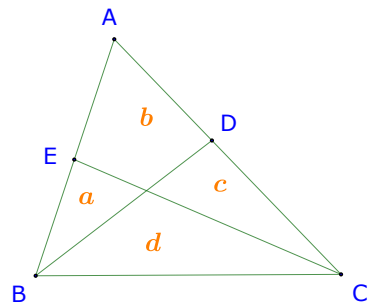
5.3 Basic Geometry at Advanced level (A-Level)

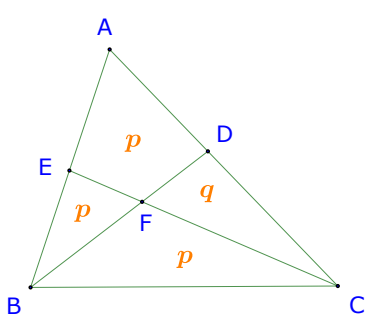
In this session, we will discuss a few basic geometry problems from different angles.

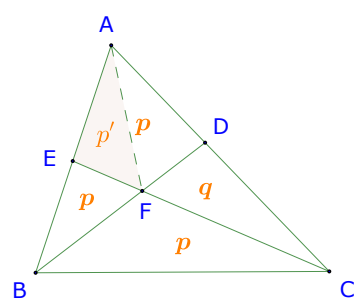
Example 5.3.1 (23-24-S4-A-P1)
It is possible for the segments BD and CE to divide the $\triangle ABC$ into four equal parts $a = b = c = d$ as shown below.



Solution. Let's assume that it is possible that $a = b = c = d$. $a + b = c + d$ means that $[ABD] = [CBD]$, since they have the equal height from B, thus the equal base $AD = DC$. But that means $[AED] = [ECD]$, which implies that $b > c$, a contradiction. Hence, it is impossible for the segments BD and CE to divide the $\triangle ABC$ into four equal parts. $\square$

Example 5.3.2 (23-24-S4-A-P2)
Segments BD and CE to divide the $\triangle ABC$ into four parts. $[ADFE] = [BFC] = [BEF] = p$, $[CFD] = q$. Compute $\dfrac{p}{q}$.





Solution. Let $x = \frac{p}{q}$, $p' = [AEF]$, then since $\triangle BFC$ and $\triangle CFD$ have the same height from C,

$$\frac{BF}{FD} = \frac{[BFC]}{[CFD]} = x \Rightarrow x = \frac{BF}{FD} = \frac{[ABF]}{[AFD]} = \frac{p + p'}{p - p'} \quad (*)$$

$$[BEF] = [BFC] \Rightarrow EF = FC \Rightarrow 1 = \frac{EF}{FC} = \frac{[AEF]}{[AFC]} = \frac{p'}{p - p' + q} \Rightarrow p - p' + \frac{p}{x} = p' \Rightarrow x = \frac{p}{2p' - p} \quad (**)$$

From (*) and (**)

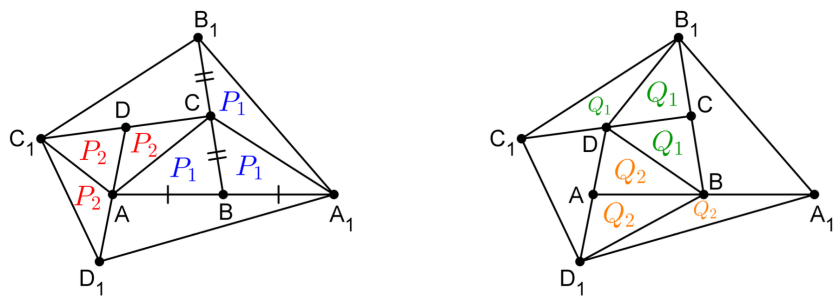
$$\frac{p+p'}{p-p'} = \frac{p}{2p'-p} \Rightarrow -p^2 + pp' + 2p'^2 = p^2 - pp' \Rightarrow p^2 - pp' - p'^2 = 0 \Rightarrow p = \left(\frac{1+\sqrt{5}}{2}\right)p'$$

$$\Rightarrow x = \frac{\frac{1+\sqrt{5}}{2}+1}{\frac{1+\sqrt{5}}{2}-1} = \frac{3+\sqrt{5}}{\sqrt{5}-1} = \frac{\left(3+\sqrt{5}\right)\left(\sqrt{5}+1\right)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \boxed{2+\sqrt{5}.}$$

□

Example 5.3.3 (23-24-S4-A-P3)

Let $ABCD$ be a convex quadrilateral with area 1. Let A_1 be a point on the ray AB beyond B, such that $AB = BA_1$. Similarly define the points B_1, C_1, and D_1. Prove that the area of $A_1B_1C_1D_1$ is 5.



Proof. Let P_1, P_2 denote the areas of $\triangle ABC$, $\triangle ADC$, respectively. The triangles $\triangle ABC$ and $\triangle BA_1C$ have base sides of equal length ($AB = BA_1$) and have a common altitude from C to the line AA_1. Therefore, we have $[BA_1C] = [ABC] = P_1$.

Similarly, the triangles $\triangle BCA_1$ and CB_1A_1 have base sides of equal length ($BC = CB_1$) and have a common altitude from A_1 to the line BB_1, so $[CB_1A_1] = [BCA_1] = P_1$.

Analogously, $[D_1AC_1] = [ADC_1] = [ADC] = P_2$. Therefore,

$$[BB_1A_1] + [DD_1C_1] = 2(P_1 + P_2) = 2[ABCD] = 2.$$

Similarly, if Q_1, Q_2 denote the area of $\triangle BCD$, $\triangle AD$, respectively, we get

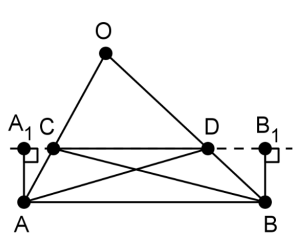
$$[CC_1B_1] + [AA_1D_1] = 2(Q_1 + Q_2) = 2[ABCD] = 2.$$

Hence, $[A_1B_1C_1D_1] = [ABCD] + ([BB_1A_1] + [DD_1C_1]) + ([CC_1B_1] + [AA_1D_1]) = \boxed{5.}$ □

Example 5.3.4 (Thales' Proportionality Theorem)

Let OAB be a triangle and let CD be a line that intersects its sides OA and OB at C and D, respectively. Prove that

$$AB \parallel CD \iff \frac{OC}{CA} = \frac{OD}{DB}.$$



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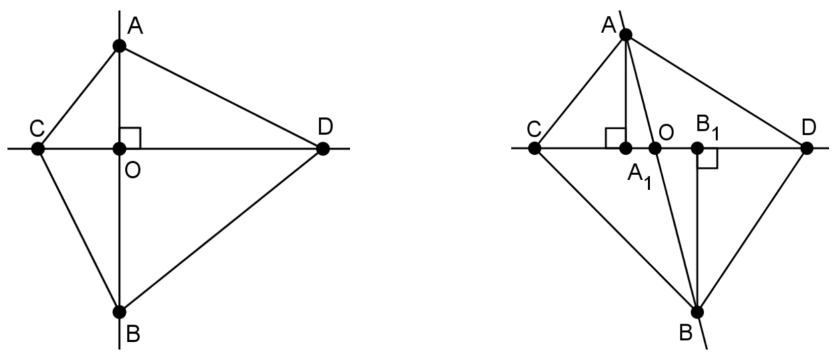
Proof. Let A_1 and B_1 be the feet of the perpendiculars from A and B, respectively, to the line CD. Then,

$$AB \parallel CD \iff AA_1 = BB_1 \iff [CDA] = [CDB] \iff \frac{[OCD]}{[CDA]} = \frac{[OCD]}{[CDB]} \iff \frac{OC}{CA} = \frac{OD}{DB}..$$

□

Example 5.3.5 (23-24-S4-A-P5)
Let AB and CD be two intersecting lines. Then,

$$AB \perp CD \iff CA^2 + CB^2 = DA^2 - DB^2.$$



Proof. Let $AB \cap CD = O$. We will firstly prove the first direction, so let $AB \perp CD$. Then, the triangles $\triangle ACO$, $\triangle BCO$, $\triangle ADO$, and $\triangle BDO$ are right triangles, so by the Pythagorean Theorem, we get

$$CA^2 + CB^2 = (OC^2 + OA^2) - (OC^2 + OB^2) = OA^2 - OB^2 = (OD^2 + OA^2) - (OD^2 + OB^2) = DA^2 - DB^2.$$

Now, let's prove the other direction. Let $CA^2 - CB^2 = DA^2 - DB^2$. We will discuss the case where O is between A and B and between C and D. Let the feet of the perpendiculars from A and B to CD be A_1 and B_1, respectively. Then, triangles $\triangle CAA_1$, $\triangle CBB_1$, $\triangle DAA_1$, and $\triangle DBB_1$ are right triangles, so by using the Pythagorean Theorem and substituting in the condition, we get

$$\begin{aligned}
&(CA_1^2 + AA_1^2) - (CB_1^2 + BB_1^2) = (DA_1^2 + AA_1^2) - (DB_1^2 + BB_1^2)\\
&\Rightarrow CA_1^2 - CB_1^2 = DA_1^2 - DB_1^2 \Rightarrow CA_1^2 - DA_1^2 = CB_1^2 - DB_1^2\\
&\Rightarrow (CA_1 - DA_1)(CD) = (CB_1 - DB_1)(CD)\\
&\Rightarrow CA_1 - DA_1 = CB_1 - DB_1 \Rightarrow CA_1 - CB_1 = DA_1 - DB_1\\
&\Rightarrow -A_1B_1 = A_1B_1 \Rightarrow A_1B_1 = 0 \Rightarrow A_1 \equiv B_1.
\end{aligned}$$

Therefore, the perpendiculars to CD from A and B pass through a common point on CD, so they must be the same line, i.e. $AB \perp CD$.

In the cases where O is not between A and B or between C and D, the proof follows exactly the same steps. There might be a different operation when dealing with the line segments (addition or subtraction) depending on the configuration, but the result will always be the same. □

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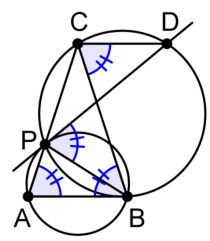
Example 5.3.6 (23-24-S4-A-P6)

Let ABC be an isosceles triangle, such that $AC = BC$. Let P be a point on the side AC. The tangent to the circle (ABP) at the point P intersects the circle (BCP) at D. Prove that $CD \parallel AB$.

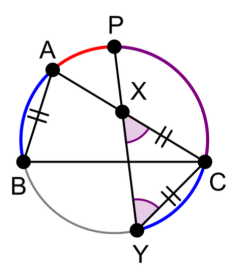
(ABC) denotes the circle through all three points A, B, and C.



Proof. We need to prove that $\angle DCB = \angle CBA$. Since the points D, C, P, B lie on the same circle, we get $\angle DCB = \angle DPB$. Since DP is tangent to (ABP), we get $\angle DPB = \angle PAB$. Finally, since $\triangle CAB$ is isosceles, we get $\angle CAB = \angle CBA$. Hence, $\angle DCB = \angle DPB = \angle PAB \equiv \angle CAB = \angle CBA$. $\square$

Example 5.3.7 (IGO 2016, Elementary)

Let ω be the circumcircle of $\triangle ABC$ with $AC > AB$. Let X be a point on AC and Y be a point on ω, such that $CX = CY = AB$. (The points A and Y lie on different sides of the line BC). The line XY intersects ω for the second time at P. Show that $PB = PC$.



Proof. Triangle $\triangle CXY$ is isosceles, so we have $\angle CXY = \angle CYX$, thus

$$\widehat{CP} = 2\angle CYP \equiv 2\angle CYX = 2\angle CXY = \widehat{CY} + \widehat{AP} = \widehat{BA} + \widehat{AP} = \widehat{BP}.$$

Therefore, $CP = BP$. $\square$

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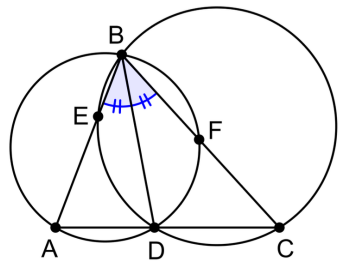
Example 5.3.8 (St. Petersburg City MO 1996)

Let BD be the angle bisector of angle ABC in $\triangle ABC$ with D on the side AC. Let the circle (BDC) meet AB at E and (ABD) meet BC at F. Prove that $AE = CF$.



Proof. By Intersecting Secants Theorem for circle $(BEDC)$ and secants BE, CD, we get $AE \cdot AB = AD \cdot AC$. Similarly, for $(ABFD)$, we get $CF \cdot CB = CD \cdot CA$.

By dividing these equations side by side, we get

$$\frac{AE}{CF} \cdot \frac{AB}{CB} = \frac{AD}{CD}.$$

From Angle Bisector Theorem,we know that

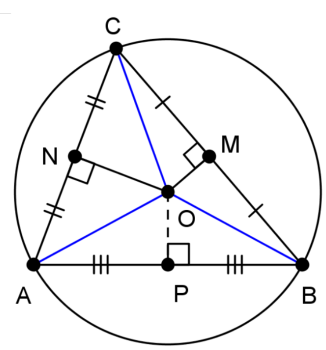
$$\frac{AB}{CB} = \frac{AD}{CD} \Rightarrow AE = CF.$$

□

Example 5.3.9 (Circumcenter)

The three side bisectors of a triangle are concurrent. The point of concurrence is the center of a circle that passes through all three vertices of the triangle.

The point of concurrence is called the circumcenter of the triangle. The circle that is circumscribed around the triangle is called the circumcircle of the triangle.



Solution. Let $M, N and P$ be the midpoints of the sides BC, CA, and AB, respectively. Let O be the intersection of the side bisectors of BC and CA. Then $OM \perp BC$ and $ON \perp CA$.

Let's take a look at the triangles $\triangle OMB$ and $\triangle OMC$. They have a common side OM, $\angle OMB = 90° = \angle OMC$, and $MB = MC$, so by S-A-S, they are congruent. Therefore, their corresponding sides are equal, i.e. $OB = OC$. Similarly, $\triangle ONC \cong \triangle ONA$, so $OC = OA$. Therefore, $OA = OB$, so $\triangle OAB$ is isosceles. Therefore, since P is the midpoint of AB, we get that OP is the side bisector of AB. Since $OA = OB = OC$, then O is the center of a circle that passes through the vertices of $\triangle ABC$. □

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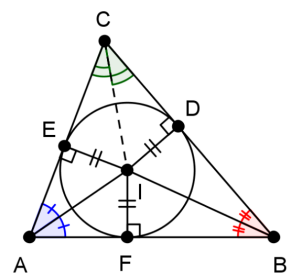
Example 5.3.10 (Incenter)

The three angle bisectors of a triangle are concurrent. The point of concurrence is the center of a circle that is tangent to all three sides of the triangle.

The point of concurrence is called the incenter of the triangle. The circle that is inscribed inside the triangle is called the incircle of the triangle.



Solution. Let I be the intersection of the angle bisectors of $\angle CAB$ and $\angle ABC$. Let D, E, and F be the feet of the perpendiculars from I to the sides BC, CA, and AB, respectively.

Let's take a look at the triangles $\triangle AIE$ and $\triangle AIF$. They are right triangles and $\angle IAE = \frac{1}{2}\alpha = \angle IAF$, so they are similar. But they have a common corresponding side AI, so their ratio of similarity is 1, i.e. they are congruent. Therefore, $IE = IF$. Similarly, $\triangle BIF \cong \triangle BID$, so $IF = ID$. Therefore, $IE = ID$.

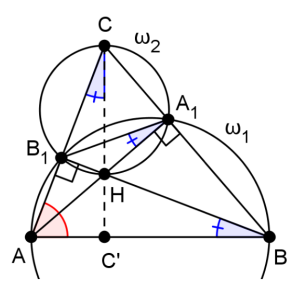
The triangles $\triangle CIE$ and $\triangle CID$ are right triangles, so by the Pythagorean Theorem, we get

$$CE^2 = IC^2 - IE^2 = IC^2 - ID^2 = CD^2 \Rightarrow CE = CD.$$

So, by S-S-S, $\triangle CIE \cong \triangle CID$ and therefore $\angle ICE = \angle ICD$, i.e. CI is the angle bisector of $\angle ECD \equiv \angle ACB$. Since $ID = IE = IF$, $ID \perp BC$, $IE \perp CA$, and $IF \perp AB$, then I is the center of a circle that is tangent to the sides of $\triangle ABC$. $\square$

Example 5.3.11 (Orthocenter)

The three altitudes of a triangle are concurrent. The point of concurrence is called the orthocenter of the triangle.



Solution. Let the altitudes AA_1 and BB_1 intersect at H.

Since $AA_1 \perp BC$ and $BB_1 \perp AC$, then $\angle AA_1B = 90^\circ = \angle AB_1B$. Therefore, ABA_1B_1 is a cyclic quadrilateral. Let (ABA_1B_1) be ω_1. Also,

$$\angle CB_1H + \angle CA_1H \equiv \angle CB_1B + \angle CA_1A = 90^\circ + 90^\circ = 180^\circ,$$

so CB_1HA_1 is a cyclic quadrilateral. Let (CB_1HA_1) be ω_2.

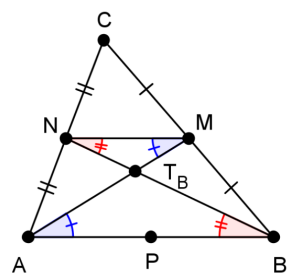
Let $CH \cap AB = C'$. Then,

$$\angle ACC \equiv \angle B_1CH = \angle B_1A_1H \equiv \angle B_1A_1A = \angle B_1BA = 90 - \angle A.$$

Finally, from sum of angles in triangle $\triangle ACC'$, we get that $\angle AC'C = 90°$, i.e. $CH \perp AB$. □

Example 5.3.12 (Centroid)

The three altitudes of a triangle are concurrent. The point of concurrence is called the orthocenter of the triangle.



Solution. Let M, N, and P be the midpoints of the sides BC, CA, and AB, respectively. Then, MN is a midsegment in $\triangle ABC$, so $MN \parallel AB$ and $AB = 2MN$.

Let the $B-$median intersect the $A-$median at a point T_B. Then,

$$\angle T_BAB = \angle T_BMN,\ \angle T_BBA = \angle T_BNM \Rightarrow \triangle TBAB \sim \triangle T_BMN \Rightarrow \frac{AT_B}{T_BM} = \frac{AB}{MN} = 2.$$

Similarly, if the $C-$median intersect the $A-$median at T_C, we can get $\frac{AT_C}{T_CM} = 2$. So $T_B \equiv T_C \equiv T$, i.e. the $B-$median and the $C-$median intersect the $A-$median at the same point T. Additionally, $\frac{AT}{TM} = \frac{BT}{TN} = \frac{CT}{TP} = 2$. □

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5.4 Problems (A-Level)

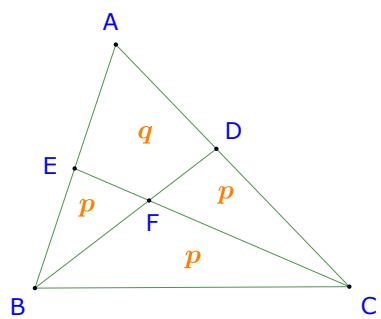
Submission deadline: October 20, 2023.

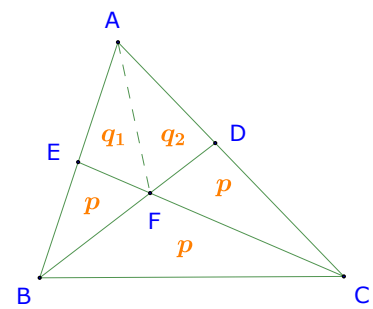
Note that for every problem you earn an additional same number of points for every different solution.

Problem 5.4.1 (23-24-S4-A-P11)**.** (*10 points*) Is it possible for the segments BD and CE to divide the $\triangle ABC$ into four parts such that $[BEF] = [BFC] = [CFD] = p$, $[ADFE] = q$.

Solution (Official solution and similar as the ones from An Nguyen, Antoine Dao, Karl Le with modifications) — $[BFC] = [CFD]$, thus $BF = DF$. Similarly $CF = EF$. Since $\angle EFB = \angle CFD$, thus $\triangle EFB \cong \triangle CFD$ by S-A-S, see the diagram below on the left.

Therefore $\angle BEC = \angle BEF = \angle DCF = \angle DCE$, which implies that $BE \parallel CD$, a contradiction to the fact that BE intersecting CD at A. Hence, it is impossible that $[BEF] = [BFC] = [CFD] = p$.





Solution (Solution 2 from Jason To, Tran Hieu Khoa with modifications) — $[BFC] = [CFD]$, thus $BF = DF$. Similarly $CF = EF$. Since $\angle EFB = \angle CFD$, thus $\triangle EFB \cong \triangle CFD$ by S-A-S, see the diagram above on the right.

Now, since $[EAF] = [EFC]$, $[AFD] = [APB]$, thus

$$q_1 = q_2 + p, q_2 = q_1 + p \Rightarrow q_1 - q_2 = p = -p \Rightarrow p = 0.$$

Hence, it is impossible that $[BEF] = [BFC] = [CFD] = p$.

Remark. Antoine Dao, Karl Le, Jason To, Tran Hieu Khoa correctly solved this problem.

An Nguyen made the assumption that $EDCB$ to be a rectangle.

Minh Hai Vu, Samuel Luong had a good start but can't find the right arguments, so ended up with a wrong conclusion.

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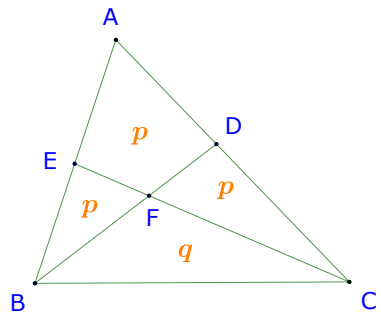
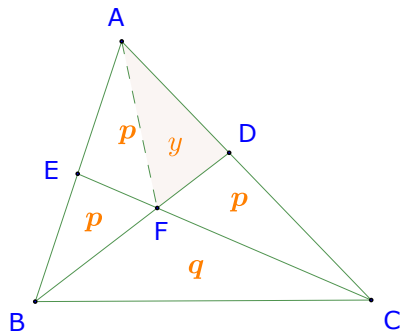
Problem 5.4.2 (23-24-S4-A-P12)**.** (*10 points*) Segments BD and CE to divide the $\triangle ABC$ into four parts such that $[ADFE] = [BEF] = [CFD] = p$, $[BFC] = q$. Compute the ratio $\frac{p}{q}$.





Solution (Official solution and similar as the ones from An Nguyen, Antoine Dao, Jason To, Karl Le, Minh Hai Vu, Tran Hieu Khoa with modifications) — Let $[ADF] = y$,

$$\frac{q}{p} = \frac{[BFC]}{[CFD]} = \frac{BF}{FD} = \frac{[ABF]}{[AFD]} = \frac{2p-y}{y}, \ \frac{q}{p} = \frac{FC}{EF} = \frac{[CFA]}{[EFA]} = \frac{p+y}{p-y}$$

$$\Rightarrow \frac{2p-y}{y} = \frac{p+y}{p-y} \Rightarrow y^2 - 3yp + 2p^2 = y^2 + yp \Rightarrow y = \frac{1}{2}p \Rightarrow \frac{p}{q} = \frac{p-y}{p+y} = \boxed{\frac{1}{3}.}$$

Remark. An Nguyen, Antoine Dao, Jason To, Karl Le, Minh Hai Vu, Tran Hieu Khoa correctly solved this problem.

Jason To made a minor mistake.

Samuel Luong had a good start but no further.

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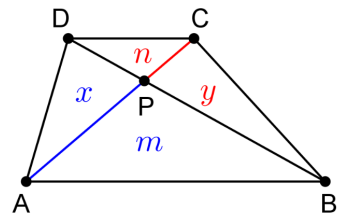
Problem 5.4.3 (23-24-S4-A-P13)**.** (*10 points*) Let $ABCD$ be a trapezoid ($AB \parallel CD$). Let its diagonals AC and BD intersect at P. Let the areas of the triangle $\triangle ABP$ and $\triangle CDP$ be m and n, respectively. Prove that

$$[ABCD] = (\sqrt{m} + \sqrt{n})^2.$$



Solution (Official solution and similar as the ones from Antoine Dao, Karl Le, Minh Hai Vu, Samuel Luong, with modifications) — Let x, y denote the areas of $\triangle ADP$, $\triangle BCP$, respectively.

The triangles $\triangle ABD$ and $\triangle ABC$ have a common base side AB and altitudes of equal length from D, C to AB since $AB \parallel CD$. Therefore, $[ABD] = [ABC]$, thus $x + m = y + m$, so $x = y$.

On the other hand, for the collinear points A, P, C and for the points D, B that do not lie on AC, we have

$$\frac{[APD]}{[PCD]} = \frac{AP}{PC} = \frac{[APB]}{[PCB]} \Rightarrow \frac{x}{n} = \frac{y}{m} \Rightarrow mn = xy = x^2.$$

Hence, $[ABCD] = m + 2x + n = m + 2\sqrt{mn} + n = \boxed{(\sqrt{m} + \sqrt{n})^2.}$

Remark. Antoine Dao, Karl Le, Minh Hai Vu, Samuel Luong correctly solved this problem.

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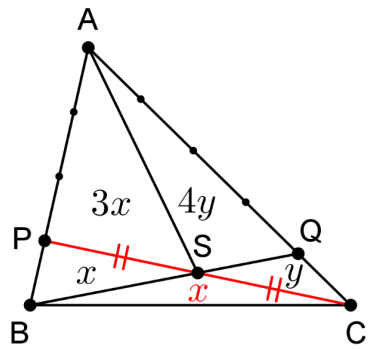
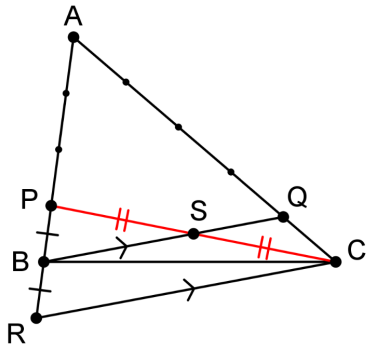
Problem 5.4.4 (23-24-S4-A-P14)**.** (*10 points*) Let P be a point on the side AB in $\triangle ABC$, such that $AP = 3PB$. Let $Q \in AC$, such that $AQ = 4QC$. Prove that BQ bisects the line segment CP.



Solution (Official solution 1 and similar as the ones from An Nguyen, Antoine Dao, Jason To, Karl Le, Tran Hieu Khoa with modifications) — Let $BQ \cap CP = S$. We need to prove that $CS = SP$, that is equivalent to proving that $[CSB] = [SPB]$.

Let $x = [SPB]$. Since $AP = 3PB$, $[SAP] = 3x$. Similarly, if $y = [SQC]$, since $AQ = 4QC$, we get $[SAQ] = 4y$. But also, since $AQ = 4QC$, we get $[BAQ] = 4[BQC]$, thus

$$x + 3x + 4y = 4([CSB] + y) \Rightarrow [CSB] = x = [SPB].$$



Solution (Official solution 2) — Let $BQ \cap CP = S$. Let R be a point on the extension of AB, such that $PB = BR$. Now,

$$\frac{AB}{BR} = 4 = \frac{AQ}{QC}.$$

by 23-24-S1-A-P4, $BQ \parallel RC$. Focusing on $\triangle PRC$, we see that B is midpoint of PR and $BS \parallel RC$, so BS is midsegment, and thus S is the midpoint of PC.

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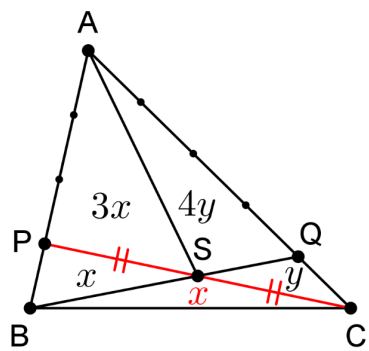
Solution (Solution 3 from Samuel Luong) — Applying the Menelaus theorem for $\triangle APC$ and line BSQ,

$$\frac{AB}{BP} \cdot \frac{PS}{SC} \cdot \frac{CQ}{QA} = 1 \Rightarrow \frac{4}{1} \cdot \frac{PS}{SC} \cdot \frac{1}{4} = 1 \Rightarrow PS = SC.$$



Solution (Solution 4 from Minh Hai Vu) — Let $\vec{b} = \overrightarrow{AB}, \vec{c} = \overrightarrow{AC}$. Let n, k be positive real numbers such that $\overrightarrow{CS} = n\overrightarrow{CP}$, $\overrightarrow{QS} = k\overrightarrow{QB}$.

$$\overrightarrow{AQ} = \frac{4}{5}\vec{c},\ \overrightarrow{QB} = \overrightarrow{QA} + \overrightarrow{AB} = -\frac{4}{5}\vec{c} + \vec{b},\ \overrightarrow{CP} = \overrightarrow{CA} + \overrightarrow{AP} = -\vec{c} + \frac{3}{4}\vec{b}$$
$$\overrightarrow{CS} = n\overrightarrow{CP} = \frac{3n}{4}\vec{b} - n\vec{c},\ \overrightarrow{CS} = \overrightarrow{CQ} + \overrightarrow{QS} = -\frac{1}{5}\vec{c} + k\left(-\frac{4}{5}\vec{c} + \vec{b}\right) = k\vec{b} - \frac{4k+1}{5}\vec{c}$$
$$\frac{3n}{4}\vec{b} - n\vec{c} = k\vec{b} - \frac{4k+1}{5}\vec{c}$$

Now since $\vec{b} \not\equiv \vec{0}, \vec{c} \not\equiv \vec{0}$, thus the coefficients of $\vec{b}$ and $\vec{c}$ on both sides should be the same

$$\begin{cases} \frac{3n}{4} = k \\ -n = -\frac{4k+1}{5} \end{cases} \Rightarrow 5n = (3n) + 1 \Rightarrow n = \frac{1}{2},\ k = \frac{3}{8}.$$

Thus $CP = 2CS$, or BQ bisects the line segment CP.

Remark. An Nguyen, Antoine Dao, Jason To, Karl Le, Minh Hai Vu, Samuel Luong, Tran Hieu Khoa correctly solved this problem.

Samuel Luong showed that he is able to apply other learned knowledge into solving problem, which is great!

Minh Hai Vu's solution is brilliant, a beautiful application of vector computation and principle of unique linear combination of base vectors.

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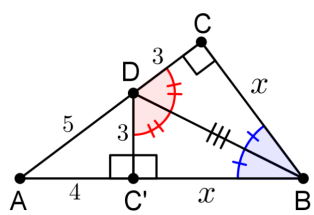
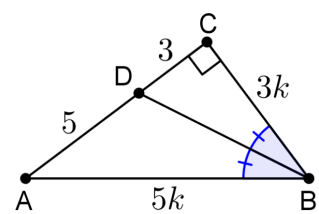
Problem 5.4.5 (23-24-S4-A-P15)**.** (*10 points*) Let $\triangle ABC$ be a right triangle. The angle bisector of $\angle ABC$ intersects AC at D. If $AD = 5$ and $CD = 3$, find AB.

Solution (Official solution 1 and similar as the ones from Karl Le, Minh Hai Vu, Tran Hieu Khoa with modifications) — Let C' be the foot of the perpendicular from D to AB. Now, we focus on $\triangle BDC$ and $\triangle BDC'$. We have $\angle CBD = \frac{1}{2}\angle B = \angle C'BD$, BD is a common side and $\angle BDC = 90^\circ - \frac{1}{2}\angle B = \angle BDC'$, so by the criterion A-S-A for congruent triangles, we get $\triangle BDC \cong \triangle BDC'$. Therefore, $DC' = DC = 3$ and $BC' = BC$.

Let $x = BC$. from the right $\triangle ADC'$, by Pythagorean Theorem, we get $AD^2 = DC'^2 + C'A^2$, thus i.e. $C'A = 4$. Finally, from the right $\triangle ABC$, we get $AB^2 = BC^2 + CA^2$, or $(4+x)^2 = x^2 + 8^2$ which has a solution $x = 6$. Therefore, $AB = AC' + C'B = 4 + 6 = \boxed{10.}$





Solution (Official solution 2) — From Angle Bisector Theorem in $\triangle ABC$, we get $\frac{BA}{BC} = \frac{DA}{DC} = \frac{5}{3}$. Let $BA = 5k$ and $BC = 3k$. Now, using Pythagorean Theorem in $\triangle ABC$, we get $(5k)^2 = (3k)^2 + 8^2$, which has a positive solution $k = 2$, hence $AB = 5k = \boxed{10.}$

Solution (Solution 3 from An Nguyen with modifications) — Let $\alpha = \angle CBD$, $x = BC$, then $\tan\alpha = \tan\angle CBD = \frac{3}{x}$, $\tan 2x = \tan\angle ABC = \frac{8}{x}$. Since

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} \Leftrightarrow \frac{8}{x} = \frac{2\cdot\frac{3}{x}}{1-\left(\frac{3}{x}\right)^2} \Leftrightarrow \frac{8}{x} = \frac{6x}{x^2-9} \Leftrightarrow 8x^2 - 72 = 6x^2 \Leftrightarrow x^2 = 36 \Leftrightarrow x = 6.$$

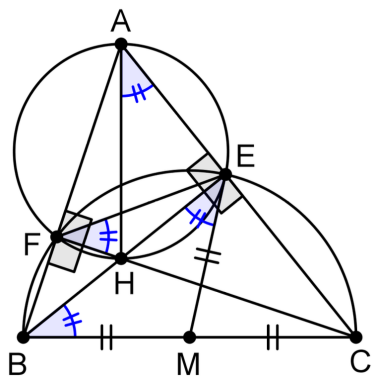
Therefore, $AB = AC' + C'B = 4 + 6 = \boxed{10.}$

Remark. An Nguyen, Karl Le, Minh Hai Vu, Tran Hieu Khoa correctly solved this problem.

The solution from An Nguyen is particular interesting because of a direct application of trigonometry plus manipulation of trigonometric identities. Lots to learn but great when applying correctly.

Problem 5.4.6 (23-24-S4-A-P16)**.** (*10 points*) Let ABC be an acute triangle. Let E and F be the feet of the altitudes in $\triangle ABC$ from B and C, respectively, and let M be the midpoint of BC. Prove that ME and MF are tangents to the circle (AEF).

Hint: let $BE \cap CF = H$, prove that $BFEC$ is cyclic, then $\angle MEH = \angle EAH$.



Solution (Official solution and similar as the ones from Karl Le, Minh Hai Vu with modifications) — Let $BE \cap CF = H$. Since $\angle HEA + \angle HFA = 90^\circ + 90^\circ = 180^\circ$, $AEHF$ is a cyclic quadrilateral. In order to prove that ME is tangent to $(AEHF)$,, we need to prove that $\angle MEH = \angle EAH$.

Since $AEHF$ is cyclic, we get $\angle EAH = \angle EFH$. Since $\angle BEC = 90^\circ = BFC$, E, F lie on the circle with diameter BC, so $BFEC$ is cyclic. Thus, we get $\angle EFC = \angle EBC$. Since M is midpoint of the diameter BC, we get that M is the center of $(BFEC)$, so $MB = ME$ as radii. Therefore, $\angle MBE = \angle MEB$.

Thus, $\angle EAH = \angle EFH \equiv \angle EFC = \angle EBC \equiv EBM = \angle MEB \equiv \angle MEH$, so ME is tangent to (AEF). Similarly, MF is tangent to (AEF).

Remark. Only Karl Le, Minh Hai Vu correctly solved this problem.

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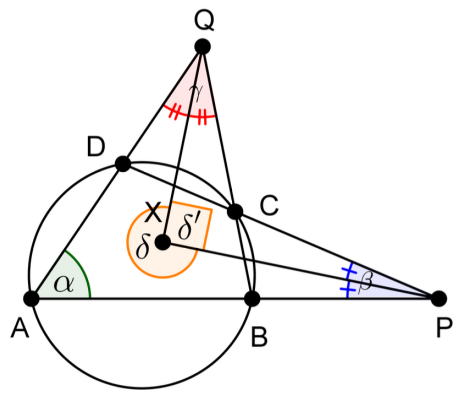
Problem 5.4.7 (23-24-S4-A-P17)**.** (*10 points*) Let $ABCD$ be a cyclic quadrilateral such that the rays AB and DC intersect at P and the rays BC and AD intersect at Q. Prove that the angle bisectors of $\angle BPC$ and $\angle CQD$ are perpendicular.

Hint: let the angle bisectors of $\angle BPC$ and $\angle CQD$ intersect at X What is the sum of all internal angles of $APXQ$?



Solution (Official solution and similar as the one from Karl Le with modifications) — Let the angle bisectors of $\angle BPC$ and $\angle CQD$ intersect at X. Let's introduce the following notations: $\alpha = \angle BAD, \beta = \angle BPC, \gamma = \angle CQD$. Then, three of the interior angles in the quadrilateral $APXQ$ are α, $\frac{1}{2}\beta$, and $\frac{1}{2}\gamma$. Let the fourth interior angle, which is a *reflex angle* (reflex angle is an angle that is greater than $180°$, but less than $360°$.), be δ, and let its *explementary angle* (two angles are explementary if their sum is $360°$) be δ'. We want to prove that $\delta' = 90°$.

From the sum of the interior angles in the quadrilateral $APXQ$, we get

$$\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma + \delta = 360°,\ \delta + \delta' = 360° \Rightarrow \delta' = \alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma.$$

Using the angle measure of arcs, we get

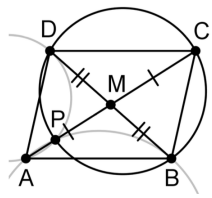
$$\alpha = \frac{1}{2}\widehat{BD} = \frac{1}{2}\left(\widehat{BC} + \widehat{CD}\right), \beta = \frac{1}{2}\left(\widehat{DA} - \widehat{BC}\right), \gamma = \frac{1}{2}\left(\widehat{AB} - \widehat{CD}\right).$$

Finally, we get

$$\delta' = \alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma = \frac{1}{4}\left(\widehat{BC} + \widehat{CD} + \widehat{DA} - \widehat{BC} + \widehat{AB} - \widehat{CD}\right) = \frac{1}{4}(\widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DA}) = \frac{360°}{4} = 90°.$$

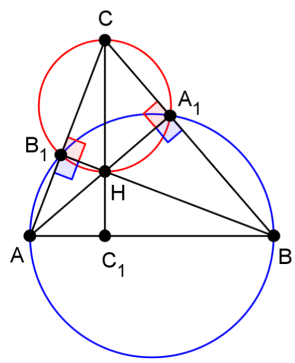
Remark. Only Karl Le correctly solved this problem.

Problem 5.4.8 (23-24-S4-A-P18)**.** (*10 points*) Let $ABCD$ be a parallelogram with $AC > BD$. The circumcircle of $\triangle BCD$ intersects AC again at P. Prove that BD is a common tangent for the circumcircles of $\triangle ABP$ and $\triangle ADP$.



Proof. Let M be the intersection of the diagonals AC and BD. We know that $MA = MC$ and $MB = MD$. Since $BCDP$ is cyclic, by Intersecting Chords Theorem, we get $MB \cdot MD = MP \cdot MC$. By substituting the equal lengths, we get $MB^2 = MP \cdot MA$, so by the converse of Secant- Tangent Theorem we get that $MB \equiv BD$ is tangent to (ABP). Similarly, BD is tangent to (ADP). □

Problem 5.4.9 (23-24-S4-A-P19)**.** (*10 points*) Let AA_1, BB_1, and CC_1 be the altitudes in a $\triangle ABC$ and let H be its orthocenter. Prove that ABA_1B_1, BCB_1C_1, and CAC_1A_1 are cyclic quadrilaterals. Prove that AB_1HC_1, BC_1HA_1, and CA_1HB_1 are also cyclic quadrilaterals.



Solution (Official solution and similar as the ones from Karl Le, Minh Hai Vu with modifications) — Since $AA_1 \perp BC$ and $BB_1 \perp AC$, then $\angle AA_1B = 90^\circ = \angle AB_1B$. Therefore, ABA_1B_1 is a cyclic quadrilateral. Similarly, BCB_1C_1 and CAC_1A_1 are cyclic quadrilaterals.

On the other hand,

$$\angle CB_1H + \angle CA_1H \equiv \angle CB_1B + \angle CA_1A = 90^\circ + 90^\circ = 180^\circ,$$

so CA_1HB_1 is a cyclic quadrilateral. Similarly, AB_1HC_1 and BC_1HA_1 are cyclic quadrilaterals.

Remark. Only Karl Le, Minh Hai Vu correctly solved this problem.

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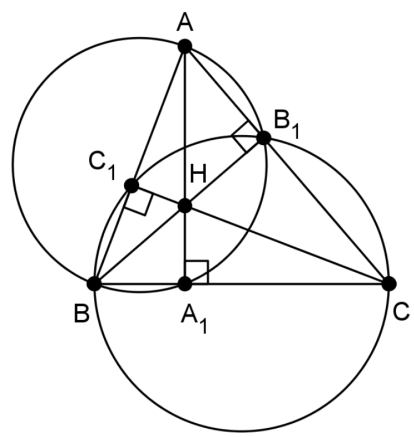
Problem 5.4.10 (23-24-S4-A-P20). (*10 points*) Let AA_1, BB_1, and CC_1 be the altitudes in a $\triangle ABC$ and let H be its orthocenter. Prove that,

$$AH \cdot HA_1 = BH \cdot HB_1 = CH \cdot HC_1.$$



Solution (Official solution and similar as the one from Karl Le with modifications) — From 23-24-S4-A-P19 we know that ABA_1B_1 is a cyclic quadrilateral. Since the altitudes AA_1 and BB_1 intersect at the orthocenter H, by the Intersecting Chords Theorem, we get $AH \cdot HA_1 = BH \cdot HB_1$. Similarly, from the cyclic quadrilateral BCB_1C_1, we get $BH \cdot HB_1 = CH \cdot HC_1$.

Remark. Only Karl Le correctly solved this problem.

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5.5 Integer Sequences (O-Level)

In this session, we explore the sequences made from integers, understand, use, and prove their properties and relations.

Example 5.5.1 (23-24-S4-O-P1)

Find the smallest positive integer n such that for the sequence $a_1, a_2, \ldots, a_n$ consisting of n distinct integers, the product of all differences $a_i - a_j$ $(i < j)$ is divisible by 1991.

Remark. Note that $1991 = 11 \cdot 181$. Therefore, n must be at least 182?

Solution. Let

$$S = \prod_{1 \le i < j \le n} (a_i - a_j)$$

Note that $1991 = 11 \cdot 181$. Therefore S is divisible by 1991 if and only if it is divisible by both 11 and 181.

If $n \le 181$ then we can *choose* the numbers $a_1, a_2, \ldots, a_n$ from distinct congruence classes modulo 181 (in other words each has a different remainder when divided by 181), so that S will not be divisible by 181.

On the other hand, if $n \ge 182$ then according to the pigeonhole principle there always exist a_i and a_j $(1 \le i < j \le n)$ such that $a_i - a_j$ is divisible by 181. It is obvious that there exist a_k and a_ℓ $(1 \le k < \ell \le n)$ such that $a_k - a_l$ is divisible by 11.

Hence, the answer is $\boxed{182.}$ □

Example 5.5.2 (23-24-S4-O-P2)

A sequence of integers $a_1, a_2, \ldots$ is such that $a_1 = 1, a_2 = 2$ and for $n \ge 1$,

$$a_{n+2} = \begin{cases} 5a_{n+1} - 3a_n, & \text{if } a_n \cdot a_{n+1} \text{ is even,} \\ a_{n+1} - a_n, & \text{if } a_n \cdot a_{n+1} \text{ is odd,} \end{cases}$$

Prove that $a_n \neq 0$ for all n.

Solution. Consider the sequence of remainders of a_n when divided by 6. The sequence of remainders is $1, 2, 1, 5, 4, 5, 1, 2, \ldots$ with a period of $1, 2, 1, 5, 4, 5$. Therefore $\boxed{a_n \neq 0,\ \forall n.}$ □

Example 5.5.3 (23-24-S4-O-P3)

Given a sequence $a_1, a_2, a_3, \ldots$ of positive integers in which every positive integer occurs exactly once. Prove that there exist integers ℓ and m, $1 < \ell < m$, such that:

$$a_1 + a_m = 2a_\ell.$$

Remark. Instead looking for both m, ℓ, we consider the difference $2a_\ell - a_1$. This is an integer, thus it should appear somewhere in the sequence, which is a term a_m. How can we make sure that $\ell < m$?

Solution. By the Extremal Principle, there exists *the least index ℓ such that $a_\ell > a_1$*. Since $2a_\ell - a_1$ is a positive integer larger than a_1, it occurs in the given sequence beyond a_ℓ. In other words, there exists an index $m > \ell$ such that $\boxed{a_m = 2a_\ell - a_1.}$ □

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Example 5.5.4 (23-24-S4-O-P4)

Does there exist a finite sequence of integers $c_1, \ldots, c_n$ such that all the numbers $a + c_1, \ldots, a + c_n$ are primes for more than one but not infinitely many different integers a?

Solution. Let $n = 5$ and consider the integers $0, 2, 8, 14, 26$. Adding $a = 3$ or $a = 5$ to all of these integers we get primes. Since the numbers $0, 2, 8, 14$ and 26 have pairwise different remainders modulo 5 then for any integer a the numbers $a + 0, a + 2, a + 8, a + 14$ and $a + 26$ have also pairwise different remainders modulo 5; therefore one of them is divisible by 5.

Hence if the numbers $a + 0, a + 2, a + 8, a + 14$ and $a + 26$ are all primes then one of them must be equal to 5, which is only true for $a = 3$ and $a = 5$. □

Example 5.5.5 (23-24-S4-O-P5)

Let $p \neq 3$ be a prime number. Show that there is a non-constant arithmetic sequence of positive integers $x_1, x_2, \ldots, x_p$ such that the product of the terms of the sequence is a cube.

Solution. Let $a_1, a_2, \ldots, a_p$ be any arithmetic sequence of positive integers and let P be the product of the terms of this sequence. For any n, the sequence

$$P^n a_1, P^n a_2, \ldots, P^n a_p$$

is also arithmetic, and the product of terms is P^{np+1}.

Now, either $p \equiv 1 \pmod 3$, or $p \equiv -1 \pmod 3$.. In the former case, $2p + 1 = 3q$ for some q and in the latter case, $1p + 1 = 3q$ for some q. So we can choose either $x_i = P^2 a_i$ or $x_i = P a_i$ to obtain the sequence we are looking for. □

Example 5.5.6 (23-24-S4-O-P6)

Consider the sequence of positive integers defined by $s_1, s_2, s_3, \ldots$ of positive integers defined by $s_1 = 2$, and for each positive integer n, s_{n+1} is equal to s_n plus the product of prime factors of s_n. The first terms of the sequence are $2, 4, 6, 12, 18, 24$. Prove that the product of the 2023 smallest primes is a term of the sequence.

Solution. Denote the n-th prime number by $p_n : p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$ and so on. Using induction, we prove that

Claim — $\forall k \in \mathbb{N}$, $p_1 p_2 \ldots p_k$ is a term of the sequence $\{s_n\}$.

Proof. It is true for $k = 1$. Assume that it is true for k, $s_n = p_1 p_2 \ldots p_k$, then it is easy to see that the terms follow s_n form an arithmetic sequence (all are multiples of $p_1 p_2 \ldots p_k$) until they reach $p_1 p_2 \ldots p_{k+1}$. ■

□

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Example 5.5.7 (23-24-S4-O-P7)

Find all real number a_0 such that the sequence defined by

$$a_{n+1} = 2^n - 3a_n, \ \forall n \geq 0$$

satisfies $a_0 < a_1 < a_2 < \dots$.

Solution. Note that

$$a_{n+1} = 2^n - 3a_n = 2^n - 3 \cdot 2^{n-1} + 3^2 a_{n-1} = \dots = 2^n - \sum_{i=1}^{n} (-3)^i 2^{n-i} + (-3)^{n+1} a_0$$
$$= \frac{2^{n+1}}{5} + \left(a_0 - \frac{1}{5}\right)(-3)^{n+1} \Rightarrow \frac{a_n}{3^n} = \frac{1}{5}\left(\frac{2}{3}\right)^n + \left(a_0 - \frac{1}{5}\right)(-1)^n$$

Now $a_0 = \frac{1}{5}$ is a desired value. If $a_0 \neq \frac{1}{5}$, then since $\left(\frac{2}{3}\right)^n \to 0$ when $n \to \infty$, so $\frac{a_n}{3^n}$ has the same sign as $\left(a_0 - \frac{1}{5}\right)(-1)^n$ when n is large enough, thus $a_n < a_{n+1}$ relation will not hold. □

Example 5.5.8 (23-24-S4-O-P8)

A sequence of integers $a_1, a_2, a_3, \dots$ is called *exact* if

$$a_n^2 - a_m^2 = a_{n-m} a_{n+m}, \ \forall n > m.$$

Prove that there exists an exact sequence with $a_1 = 1$, $a_2 = 0$, and determine a_2007.

Solution. We prove that, for any n, the terms of an exact sequence starting with $a_1 = 1$, $a_2 = 0$,

Claim — $a_{2n} = 0, \ a_{2n+1}^2 = (-1)^n$.

Proof. By the given formula, for any k,

$$a_{2n}^2 = a_{2n-2}^2 + a_{4n-2} a_2 = a_{2n-2}^2 = \dots = a_2^2 = 0.$$

For the second statement, $a_3 = a_2^2 - a_1^2 = -1$, by induction,

$$a_{4k+1} = a_{2k+1}^2 - a_{2k}^2 = 1, \ a_{4k+3} = a_{2k+2}^2 - a_{2k+1}^2 = -1, \ \forall k \geq 1.$$

■

Hence if the sequence is exact then $\boxed{a_{2007} = -1.}$

Now we prove that, the sequence defined by $a_1 = 1$, $a_2 = 0$, and

$$a_n = \begin{cases} +1, \ if \ n \equiv 1 \pmod 4 \\ -1, \ if \ n \equiv 3 \pmod 4 \end{cases} \quad \text{is exact.}$$

Case 1: If n and m have the same parity, then $n - m$ and $n + m$ are both even, then $a_n^2 - a_m^2 = 0 = a_{n-m} a_{n+m}$.

Case 2: If n odd and m even, $n - m \equiv n + m \pmod 4$, so $a_n^2 - a_m^2 = 1 = a_{n-m} a_{n+m}$, since both factors are either -1 or 1.

Case 3: Finally, if n even and m odd, $n - m \not\equiv n + m \pmod 4$, so $a_n^2 - a_m^2 = -1 = a_{n-m} a_{n+m}$, since exactly one of the factors is -1 and the other is 1. □

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Example 5.5.9 (23-24-S4-O-P9)

For an integer $n \geq 1$ let $a(n)$ denote the total number of carries which arise when adding 2017 and $n \cdot 2017$. The first few values are given by $a(1) = 1$, $a(2) = 1$, $a(3) = 0$, which can be seen from the following:

$$\begin{array}{r} 001 \\ 2017 \\ +2017 \\ = 4034 \end{array} \qquad \begin{array}{r} 001 \\ 4034 \\ +2017 \\ = 6051 \end{array} \qquad \begin{array}{r} 000 \\ 6051 \\ +2017 \\ = 8068 \end{array}$$

Prove that

$$a(1) + a(2) + \cdots + a(10^{2017} - 1) = 10 \cdot \frac{10^{2017} - 1}{9}.$$

Solution. Let $k(m)$ be the residue of m when divided by 10^k. There is a carry at the digit representing 10^k exactly when

$$k(2017) + k(n \cdot 2017) > 10^k.$$

Thus the number of $10-$, $100-$, $1000-$, and $10000-$carries are, respectively,

$$\left\lfloor \frac{7 \cdot 10^{2017}}{10} \right\rfloor, \left\lfloor \frac{7 \cdot 10^{2017}}{100} \right\rfloor, \left\lfloor \frac{7 \cdot 10^{2017}}{1000} \right\rfloor, \left\lfloor \frac{7 \cdot 10^{2017}}{10000} \right\rfloor,$$

and similarly for the rest of the carries. Thus.

$$\sum_{n=1}^{10^{2017}-1} a(n) = 7 \cdot 10^{2016} + 17 \cdot 1 -^{2015} + \cdots + 2017 + 201 + 20 + 2$$

$$= (2+0+1+7)10^{2016} + (2+0+1+7)10^{2015} + \cdots + (2+0+1+7) = \boxed{10 \cdot \frac{10^{2017} - 1}{9}.}$$

□

Example 5.5.10 (23-24-S4-O-P10)

Let a_0 be a positive integer and $a_n = 5a_{n-1} + 4$ for all $n \geq 1$. Can a_0 be chosen so that a_{54} is a multiple of 2013?

Lemma (Bézout's Lemma)

If $a, b \in \mathbb{Z},\ a, b \neq 0$ and $d = \gcd(a, b)$, then $\exists m, n \in \mathbb{Z}$ such that:

$$am + bn = d \quad (\textbf{Bézout's Identity})$$

In other words, *there exists a linear combination* of a and b equal to d. m and n are **Bézout coefficients**.

Solution.

$$a_n = 5a_{n-1} + 4 \Rightarrow a_n + 1 = 5(a_{n-1} + 1) = \ldots = 5^n(a_0 + 1) \Rightarrow a_{54} = 5^{54}(a_0 + 1) - 1.$$

Now, if a_{54} is a multiple of 2013, then there exists k integer such that

$$5^{54}(a_0 + 1) - 1 = 2013k \Rightarrow 5^{54}(a_0 + 1) - 2013k = 1$$

Since $\gcd(5^{54}, 2013) = 1$, according to Bézout's Lemma, there exists integer ℓ_1, ℓ_2, such that

$$5^{54}\ell_1 + 2013\ell_2 = 1 \Rightarrow a_0 = \ell_1 - 1, k = -\ell_1.$$

Hence, the answer is $\boxed{\text{yes.}}$ □

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5.6 Problems (O-Level)

Submission deadline: October 20, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 5.6.1 (23-24-S4-O-P11)**.** A sequence is given as

$$t_1 = 1,\ t_{n+1} = t_n + 3n^2 + 3n + 1.$$

Find t_{2023}.

Solution (Official solution and similar the one from to Minh Nguyen) —

$$t_{n+1} = t_n - n^3 + n^3 + 3n^2 + 3n + 1 = t_n - n^3 + (n+1)^3 \Rightarrow t_{n+1} - (n+1)^3 = t_n - n^3 = \ldots = t_1 - 1^3 = 0.$$

Hence, $\boxed{t_{2023} = 2023^3.}$

Solution (Solution 2 similar the ones from to all students) — Our induction hypothesis is that for all $n \geq 1$, $t_n = n^3$.

For the base case $n = 1$, $t_1 = 1^3$, thus it is true.

Now, for the inductive step, let's assume that the hypothesis is true for n, or $t_n = n^3 \quad (*)$

We shall prove that $t_{n+1} = (n+1)^3 \quad (**)$

Using the assumption (*) that the hypothesis is true for n, the left side of (**) can be written as

$$t_{n+1} = t_n + 3n^2 + 3n + 1 = n^3 + 3n^2 + 3n + 1 = (n+1)^3.$$

Thus, the hypothesis is true for $n + 1$, therefor it is true for all $n \geq 1$.

Remark. All students correctly solved this problem. There are three different ways to solve this: (1) using recurrence formula, as shown in the official solution; (2) use telescopic sum, an implicit use of recurrence formula, like Minh Nguyen shown in her solution; (3) using induction principle, as all of the students were able to used.

Note that Henry Dinh could have shortened his first proof by using $t_{n+1} = t_n - n^3 + (n+1)^3$, so proper recognition of algebra identities are useful.

Second note on using induction principle: you have to explicitly state your hypothesis. Shortcut in proof might results in losing some points. Anthony Pham needs to pay attention to this.

Last but not least: it is perfectly enough to state that $t_{2023} = 2023^3$, writing down the decimal representation of 2023^3 takes time, does not gain any extra point if correctly done, however it would result in point reduction if one or two digits are misplaced. So unless it is specifically required by the problem, don't do that.

Problem 5.6.2 (23-24-S4-O-P12)**.** Prove that none of the terms of the sequence

$$a_n = 2^{2^n} + 1,\ \forall n \geq 0 \text{ is a perfect cube.}$$

Solution (Official solution and similar the one from Albert Dinh-Le, Ha-Anh Le, Henry Dinh, and Minh Nguyen) — Let's assume the opposite: there exists m integers such that

$$2^{2^n} + 1 = m^3 \Rightarrow 2^{2^n} = m^3 - 1 = (m-1)(m^2 - +m + 1) = (m-1)(m(m+1)+1)$$

Note that $m(m+1)$ is always even for any integer m, thus $m(m+1)+1$ is odd. Therefore $m(m+1) = 0$. However, there is $\boxed{\text{no non-negative integer solution for } n}$ in those cases.

Solution (Solution 2 from Anthony Pham with modifications) — It is easy to see that the remainders modulo 7 of the consecutive cubes $1^3, 2^3, 3^3, \ldots$ can only be $0, 1$, or 6,

$$\underbrace{1, 1, 6, 1, 6, 6, 0}_{\text{period}}, 1, 1, 6, 1, 6, 6, 0, 1, \ldots$$

On the other hands, the remainders modulo 7 of the consecutive powers of $2 : 2^1, 2^2, 2^3, \ldots$ can only be $1, 2$, or 4,

$$\underbrace{2, 4, 1}_{\text{period}}, 2, 4, 1, 2, \ldots$$

Thus, the remainder modulo 7 of $2^{2^n} + 1$ can only be $2, 3$, or 5. Hence, it cannot a perfect cube, for any n.

Remark. All students correctly solved this problem.

Henry Dinh, and Minh Nguyen (almost) have some neat solution presented.

Albert Dinh-Le, Anthony Pham, Ha-Anh Le were entangled in a complex and sometime unnecessary jungle of arguments. Remember, writing a clear and clean proof is half of solving a problem. The reason you write a solution is for other people to read, not for yourself.

Students must read their proofs before submission. The struggle of Albert Dinh-Le with parity and Anthony Pham with periods of the base or exponent n cause pains while reading.

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Problem 5.6.3 (23-24-S4-O-P13)**.** Let a be any integer. Define the sequence $x_0, x_1, \ldots$ by $x_0 = a$, $x_1 = 3$, and for all $n > 1$

$$x_n = 2x_{n-1} - 4x_{n-2} + 3.$$

Determine the largest integer k_a for which there exists a prime p such that p^{k_a} divides $x_{2011} - 1$.

Solution (Official solution and similar the ones from all students) — Let $y_n = x_n - 1$, then $y_n + 1 = 2(y_{n-1} + 1) - 4(y_{n-2} + 1) + 3$, thus

$$y_n = 2y_{n-1} - 4y_{n-2} = 2(2y_{n-2} - 4y_{n-3}) - 4y_{n-2} = -8y_{n-3} = 2^6 y_{n-6}.$$

Thus $x_{2011} - 1 = y_{2011} = (2^6)^{335} y_1 = 2^{2011}$. Hence $k_a = \boxed{2011}, p = 2$ regardless of a.

Remark. All students correctly solved this problem.

There were initially typos in the problem text. I am very sorry for that. It is good to see that all of you managed to use recurrence relations to solve the problem. Note that recurrence relations in lots of cases are good enough, especially if used in finite scale, meaning for example to find a_n from a_{n-1}, a_{n-1} from a_{n-2}, and so on. On the other hand, induction principle must be used to prove any case that infinity is involved, for example a_n is equal to some formula for all n.

Problem 5.6.4 (23-24-S4-O-P14)**.** Let $a_0, a_1, a_2, \ldots, a_n$ be integers such that $a_0 = a_n = 0$,

$$a_{i+1} - 2a_i + a_{i-1} = a_i^2, \ \forall i = 1, 2, \ldots, n-1.$$

Prove that $a_i \leq 0, \forall i = 1, 2, \ldots, n-1$.

Solution (Official solution and similar the ones from Anthony Pham, Ha-Anh Le, Henry Dinh) — Assume that is it not true, thus if a_i be the maximal term among $a_0, a_1, a_2, \ldots, a_n$, then $a_i > 0$.

Now $i \neq 0, i \neq n$, and since $a_i \geq a_{i-1}, a_i \geq a_{i+1}$,

$$a_i^2 > 0 > (a_{i+1} - a_i) + (a_{i-1} - a_i), \text{ a contradiction.}$$

Remark. Only Anthony Pham, Ha-Anh Le, Henry Dinh correctly solved this problem.

Minh Nguyen submitted a very confusing, badly written, and wrong solution. It seems that Minh Nguyen need to use bigger paper sheets in order to add later comments. Her writing deteriorates this semester. I suggest her to start learning and using Latex for better solution writing, or to rewrite the solution in a different paper before submission.

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Problem 5.6.5 (23-24-S4-O-P15)**.** Let n and k be integers, $1 < k \leq n$. Find an integer b and a set A of n integers satisfying the following conditions:

- No product of $k-1$ distinct elements of A is divisible by b.
- Every product of k distinct elements of A is divisible by b.
- For all distinct a, a' in A, a does not divide a'.

Solution (Official solution and similar the ones from Anthony Pham, Ha-Anh Le, Henry Dinh) — Let $p_1, p_2, \ldots, p_n$ be the first n odd primes. Let $\boxed{A = \{2p_1, 2p_2, \ldots, 2p_n\}}$ and $\boxed{b = 2^k.}$

Any product of $k-1$ elements of A cannot be divisible by b since the highest power of 2 factor of A is 2^{k-1}. Any product of k elements of A is divisible by b since the highest power of 2 factor of A is 2^k. No element of A is divisible by the other.

Remark. Only Anthony Pham, Ha-Anh Le, Henry Dinh correctly solved this problem.

Minh Nguyen made a mistake by choosing $A = \{p_1, p_2, \ldots, p_n\}$ and $\boxed{b = p_1 p_2 \cdots p_n.}$ She did not check if the choice satisfies all three conditions (why?). It is obvious that the first condition is false because for $1 < k \leq n$ any product of $k-1$ distinct elements of A is divisible by b.

Problem 5.6.6 (23-24-S4-O-P16)**.** Define the sequence of integers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, and

$$a_{n+1} = (n + 1 - \gcd(a_n, n)) \times a_n, \ \forall n \geq 1$$

Prove that

$$\frac{a_{n+1}}{a_n} = n \iff n \text{ is prime or } n = 1.$$

Solution (Official solution and similar the ones from Anthony Pham, Henry Dinh, Minh Nguyen) — The problem is equivalent to the following claims.

Claim (Claim 1) — When n is prime then $\gcd(n, a_n) = 1$,thus $a_{p+1} = pa_p$ for all prime p.

Proof. For prime p,

$$a_p = \prod_{i=1}^{p-1} (i + 1 - \gcd(i, a_i))$$

For $1 \leq i \leq p-1$, $i + 1 - \gcd(i, a_i) < p$ thus $p \nmid a_p$. Therefore $\gcd(p, a_p) = 1$, thus $\boxed{a_{p+1} = pa_p.}$ ■

Claim (Claim 2) — When n is composite then $\gcd(a_n, n) > 1$.

Proof. Note that $a_n \mid a_{n+1}$ implies that $a_m \mid a_n, \forall m \leq n$. Let p be a prime divisor of n, or $p < n$. Then $pa_p = a_{p+1} \mid a_n$. This means that $p \mid \gcd(n, a_n) \implies 1 < p \leq \gcd(n, a_n)$. ■

Remark. Anthony Pham, Henry Dinh, Minh Nguyen correctly solved this problem.

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Problem 5.6.7 (23-24-S4-O-P17)**.** A sequence of positive integers $a_1, a_2, \ldots$ is such that for each m and n the following holds: if m is a divisor of n and $m < n$, then a_m is a divisor of a_n and $a_m < a_n$. Find the least possible value of a_{2000}.

Solution (Official solution and similar the ones from Anthony Pham, Henry Dinh) — Let d denote the least possible value of a_{2000}. We shall prove that $d = 128$.

Note that in the sequence $1 < 5 < 25 < 125 < 250 < 500 < 1000 < 2000$ each term is divisible by the previous one. As $a_1 > 1$ it follows that

$$a_{2000} \geq 2a_{1000} \geq 2^2 a_{500} \geq 2^3 a_{250} \geq \ldots \geq 2^7 a_1 \geq 2^7 = \boxed{128.}$$

Clearly the sequence defined by $a_1 = 1$ and $a_n = 2^{\alpha_1+\alpha_2+\cdots+\alpha_k}$, for $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, has the required property with the least possible values for a_i.

Since $2000 = 2^4 \cdot 5^3$, we have $a_{2000} = 2^{4+3} = 128$ and therefore $d \leq 128$.

Note that it can be proven by prime factorization that for all n,

$$a_n \geq 2^{\alpha_1+\alpha_2+\cdots+\alpha_k}, \text{ where } n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}.$$

Remark. Anthony Pham, Henry Dinh correctly solved this problem. Both solutions are very well written.

Solutions from Ha-Anh Le, Minh Nguyen contain with somewhat weak arguments.

Problem 5.6.8 (23-24-S4-O-P18)**.** Is there an infinite sequence of prime numbers p_1, p_2, …, p_n, p_{n+1}, … such that

$$|p_{n+1} - 2p_n| = 1$$

for each n positive integer?

Hint: first prove that $p_3 > 3$, then for $p_3 \equiv 1 \pmod 3$, prove that $p_{n+1} = 2^{n-2}(p_3 - 1)$, $n = 3, 4, 5, \ldots$.

Solution (Official solution and similar the ones from Anthony Pham, Ha-Anh Le with modifications) — Suppose the contrary. It is easy to see that $p_3 > 3$ (why?)

Case 1: $p_3 \equiv 1 \pmod 3$,

Case 1a: $p_4 = 2p_3 + 1$, then $p_4 \equiv 0 \pmod 3$, impossible.

Case 1b: $p_4 = 2p_3 - 1$, so $p_4 \equiv 1 \pmod 3$. Continuing in similar way,

$$p_5 = 2p_4 - 1,\ p_6 = 2p_5 - 1, \ldots \Rightarrow p_{n+1} = 2^{n-2}(p_3 - 1),\ n = 3, 4, 5, \ldots$$

For $n = p_3 + 1$, by Fermat's little theorem: $\gcd(2, p_3) = 1$, thus $p_3 \mid 2^{p_3-1} - 1$,

$$p_{p_3+2} - 1 = 2^{p_3-1}(p_3 - 1) \Rightarrow p_{p_3+2} \equiv 1 + 1(p_3 - 1) \pmod{p_3}, \text{ a contradiction.}$$

Case 2: $p_3 \equiv 2 \pmod 3$, can be treated in the same way.

Remark. Anthony Pham, Ha-Anh Le solved this problem with some solutions causing bit of confusion.

Problem 5.6.9 (23-24-S4-O-P19)**.** Consider the sequence:

$$x_1 = 19,\ x_2 = 95,\ x_{n+2} = \operatorname{lcm}(x_{n+1}, x_n) + x_n,\ \forall n > 1$$

where $\operatorname{lcm}(a, b)$ means the least common multiple of a and b. Find $\gcd(x_{1995}, x_{1996})$.

Hint: let $d = \gcd(x_{n+1}, x_n)$, *lcm*$(x_{n+1}, x_n)\gcd(x_{n+1}, x_n) = x_{n+1}x_n$, $\gcd\left(x_{n+1}, \frac{x_n}{d}\right) = 1$.

Solution (Official solution and similar the ones from Henry Dinh) — Let $d = \gcd(x_{n+1}, x_n)$, since $\operatorname{lcm}(x_{n+1}, x_n)\gcd(x_{n+1}, x_n) = x_{n+1}x_n$, thus

$$\begin{aligned}
&\operatorname{lcm}(x_{n+1}, x_n) + x_n = \frac{x_{n+1}x_n}{d} + x_n = (x_{n+1} + d)\frac{x_n}{d} \\
&\gcd(x_{n+2}, x_{n+1}) = \gcd(\operatorname{lcm}(x_{n+1}, x_n) + x_n, x_{n+1}) = \gcd\left((x_{n+1} + d)\frac{x_n}{d}, x_{n+1}\right) \\
&\gcd\left(x_{n+1}, \frac{x_n}{d}\right) = 1 \Rightarrow \gcd(x_{n+2}, xn + 1) = \gcd(x_{n+1} + d, x_{n+1}) = d.
\end{aligned}$$

Now, by induction $\gcd(x_{n+1}, x_n) = \ldots = \gcd(x_2, x_1) = 19$. Hence $\gcd(x_{1996}, x_{1995}) = \boxed{19.}$

Remark. Only Henry Dinh correctly solved this problem.

Problem 5.6.10 (23-24-S4-O-P20)**.** Let $x_1 = 1$ and

$$x_{n+1} = x_n + \left\lfloor \frac{x_n}{n} \right\rfloor + 2,\ \forall n \geq 1.$$

where x denotes the largest integer not greater than x. Determine x_{1997}.

Hint: if $x_N = AN$, *for some positive integers* A *and* N, *then for* $i = 0, 1, \ldots, N$, *prove that*

$$\begin{aligned}
&x_{N+i} = A(N + i) + 2i,\ x_{2N} = (A + 1)(2N),\ \textit{then} \\
&x_{2^{k-1}+i} = k(2^{k-1} + i) + 2i,\ \forall i = 0, 1, \ldots, 2^{k-1}.
\end{aligned}$$

Solution (Official solution and similar the ones from Henry Dinh) — It is easy to prove the claim below,

Claim — If $n_n = an + b$, for a non-negative integer, and $0 \leq b < n$, then

$$x_{n+1} = x_n + a + 2 = a(n + 1) + b + 2.$$

Thus, if $x_N = AN$, for some positive integers A and N, then for $i = 0, 1, \ldots, N$, by the claim

$$x_{N+i} = A(N + i) + 2i,\ x_{2N} = (A + 1)(2N)$$

Now, for $N = 1, x_1 = 1 \cdot 1$, so $A = 1$. $x_2 = (1+1)\cdot(2\cdot 1)$, so $x_2 = (2)2$. Thus, we form a hypothesis

Claim — $x_{2^{k-1}+i} = k(2^{k-1} + i) + 2i,\ \forall i = 0, 1, \ldots, 2^{k-1}$.

It can easily be proved from the reasoning above. Now, for $A = 11, N = 2^{10} = 1024, i = 973, x_{1997} = 11 \cdot (1024 + 993) + 2 \cdot 993 = \boxed{23913.}$

Remark. Only Henry Dinh correctly solved this problem.

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Chapter 6

October Test

6.1 Rules

- Each level test consisting of 8 problems will be delivered by automatic scheduled email at the registered time. No test paper will be sent twice.
- Students have 270 minutes to solve the problems. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET.
- After the allowed time, the student has 15 minutes to scan the solutions. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET, and has to submit the scanner solutions before 11:45 AM. Any solution submitted later than the allowed time will not be accepted.
- All submissions must be done by email to nghia71@gmail.com.
- Solutions must be written in English. Any solution looks unreadable, or is presented badly will not be considered for grading. Answer without proper solution will not be considered as solution. Answer consists of self-explanatory and easy-to-understand diagrams are acceptable but might not be considered as full solution.
- A problem can contain more than one questions, each with an indication of the maximum number of points can be awarded if a correct solution is provided. Partially correct solution is awarded with a some points depending on its quality. Multiple solutions can be submitted for a single problem. Each different solution, if correct, will be awarded the same number of points. Partially distinct solution also counts.

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6.2 Problems for Intermediate Level

Problem 6.2.1 (23-24-OT-I-P1)**.** (*10 points*) A bag contains 200 marbles: 60 red, 60 blue, 60 green, and 20 of white and yellow ones. It is not known how many white or yellow marbles are there, is there at least one white, or is there at least one yellow marble.

If the marbles are chosen from the bag without looking, at least how many marbles you have to pick in order to ensure that you will always have at least 20 marbles of the same colour?

Solution. First, there is no guarantee that picking the 20 of white and yellow ones will result in 20 marbles of the same colour. As stated, there might be at least some white and some yellow marbles, thus the number of marbles in each of the white or yellow colours can never be 20.

Second, picking 19 marbles of each red, blue, and green colours will not result in a 20 marbles of the same colour. However, if one marble of red, blue, or green colours to these $19 + 19 + 19 = 57$ marbles, then according to the Pigeonhole Principle, there is one colour with at least $\left\lceil \frac{58}{3} \right\rceil = 20$ marbles of the same colour - red, blue, or green.

Hence, picking $20 + 57 = 77$ marbles might not result in at least 20 marbles of the same colour, but picking $\boxed{78}$ of them definitely will. □

Remark. Jeremy Nguyen, Chi-Mai Pham, Giang Nguyen, Julie Cao, Khoa Vu correctly solved this problem.

Problem 6.2.2 (23-24-OT-I-P2)**.** (*10 points*) Samuel met three girls, Emily, Isabella, and Ophelia. The following were his thoughts:

1. I love at least one of the three girls.
2. If I love Emily but not Isabella, then I also love Ophelia.
3. I either love both Ophelia and Isabella or I love neither one.
4. If I love Ophelia, then I also love Emily.

Which of the girls does Samuel love? Which ones you are not sure about?

Solution. Lets examine the given statements.

By (3), Samuel either loves both Ophelia and Isabella or he loves neither. Suppose that he loves neither. Then by (1) he must love Emily. Therefore he loves Emily, he doesn't love Isabella, and he doesn't love Ophelia. This is a contradiction of (2). Hence, Samuel loves both Ophelia and Isabella.

By (4), he loves Ophelia, so he loves Emily. Hence, $\boxed{\text{he loves all the girls.}}$ □

Remark. Chi-Mai Pham, Huy Quang Le, Julie Cao, Khoa Vu, Minh Do Tue Le correctly solved this problem.

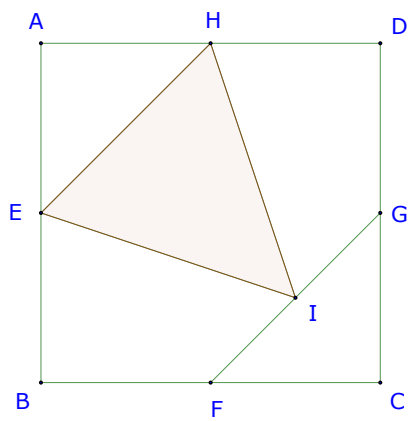
Giang Nguyen seems to get the solution, but her writing needs improvement. She has to express her ideas in more details.

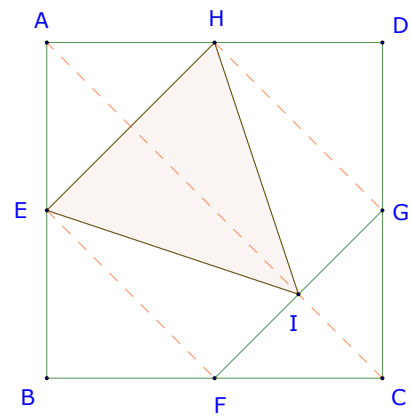
Minh Nhat Pham, Jeremy Nguyen was very close to the correct solution.

Problem 6.2.3 (23-24-OT-I-P3). (*10 points*) $ABCD$ is a convex quadrilateral. E, F, G, and H are midpoints of AB, BC, CD, and DA, respectively.

Find the ratio of the areas of $\triangle EHI$ to the square $ABCD$, in other words

$$\frac{[EHI]}{[ABCD]}.$$

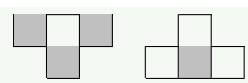




Solution. Connect EF and GH. It is easy to see that $[EFGH]$ is half of $[ABCD]$, and $[EHI]$ is half of $[EFHG]$. Thus $\frac{[EHI]}{[ABCD]} = \boxed{\frac{1}{4}.}$ □

> **Remark.** Giang Nguyen, Huy Quang Le, Jeremy Nguyen, Julie Cao, Khoa Vu, and Minh Nhat Pham correctly solved this problem.

Problem 6.2.4 (23-24-OT-I-P4). (*10 points*) Is it possible to tile a 10×10 board with only $T-$tetrominos? Two $T-$tetromino examples are show below.

> **Solution** (Official solution and same as the one from Minh Nhat Pham) —


Solution. Let's colour the board in a checkerboard pattern. Assume that it is possible. A tetromino has three black squares and one white square or three white squares and one black square. If 25 of them can cover the 10 board, then the number of black squares covered by them is an odd number, while the number of the black squares of the 10 board is 50, an even number, a contradiction. Thus it is $\boxed{\text{impossible}}$ to tile a 10×10 board with only $T-$tetrominos. □

> **Remark.** Minh Nhat Pham correctly solved this problem.
>
> Chi-Mai Pham recognized the types of the numbers of black/white squares of a tetromino, but she did not know how to apply the parity (odd/even) here to come to the conclusion that the sum of all black (or white) squares on 25 tetrominos is an odd number.
>
> Huy Quang Le saw that 25 tetrominos are needed but can't explain why they cannot tile the square.
>
> Giang Nguyen gave one of the cutest answers, unfortunately cuteness isn't among reason to awarding points.

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Problem 6.2.5 (23-24-OT-I-P5)**.** (*10 points*) Find the least numbers to be chosen from the set $\{1, 2, \ldots, 100\}$ to ensure that there are always two numbers, one is divisible by the other.

Hint: if any two numbers are chosen from the set $\{3, 6, 12, 24, 48, 96\}$, *one is divisible by the other.*

Solution. First, note that if any two numbers are chosen from the set $S_3 = \{3, 612, 24, 48, 96\}$, one number is divisible by the other. The notable fact about this set is that, *the first element is an odd number between 1 and 100, each of subsequent element in the set is twice the previous one,* shown as below:

$$3, 6 = 2 \cdot 3, 12 = 2^2 \cdot 3, 24 = 2^3 \cdot 3, 48 = 2^4 \cdot 3, 96 = 2^5 \cdot 3.$$

Second, there are exactly 25 prime numbers between 1 and 100: $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$. Thus, by partitioning the given set $\{1, 2, \ldots, 100\}$ into 24 subsets $S_2, S_3, \ldots, S_{25,}$, each *start with an odd prime,* then the next term is double of the previous one, if it is still not exceeding 100, as follow:

$$S_i = \{p_i, 2p_i, 2^2 p_i, \ldots\}$$

then if any two numbers are chosen from the same subset S_i, one number is divisible by the other, any number from subset S_i is not divisible by a number from subset S_j $(2 \leq i < j \leq 25.)$

Now, let S_1 be the set containing only the highest power of 2 not exceeding 100.

Now, by Pigeonhole Principle, if $\boxed{26}$ numbers are chosen from $\{1, 2, \ldots, 100\}$, then there are at least two chosen from the same one of the subsets, thus divisible by each other. □

> **Remark.** Jeremy Nguyen has a good solution but his method of partitioning numbers will result in same numbers in different subsets. This would require additional arguments beside the pigeonhole principle.
>
> Minh Nhat Pham has a good approach showing that because there are 25 primes not exceeding 100, by choosing 26 numbers, there are two numbers *having the same prime as common divisor*, but that is not enough for one number to be divisible by the other.

Problem 6.2.6 (23-24-OT-I-P6)**.** (*10 points*) On the Planet of the Apes, there live the *green* apes and the *red* apes. Furthermore these inhabitants were born in the northern or southern hemisphere. It is known that,

- the green northeners and the red southerners always tell the truth, and
- the red northeners and the green southerners always lie.

Melanie visited the Planet of the Apes on a bright day. At the center plaza, she met a native, who said *"I am a green northerner"*. In broad daylight, of course she saw what colour the native was, but from the answer, she did not know where the native was from.

What *colour* was he? Was he a *northerner* or a *southerner*?

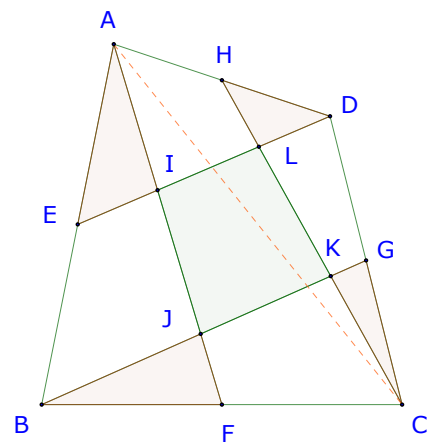
Solution. If the native was red, then he lied. Since he was red, he was a red northerner. Because Melanie would not know where the native was from, so the native was $\boxed{\text{green.}}$ Obviously she could not decide that he was a green northerner or a green southerner, since both could have given her the same statement. Thus $\boxed{\text{it is impossible to decide where he was from.}}$ □

> **Remark.** Huy Quang Le, Jeremy Nguyen, Khoa Vu correctly solved this problem.
>
> Chi-Mai Pham, Julie Cao, Minh Do Tue Le had a near perfect solution. She would had arrived at the conclusion that Melanie saw a green ape, but she could not decide whether he was a green northerner or a green southerner. Giang Nguyen seems to get the solution, but her writing needs improvement. She has to express her ideas in more details. Minh Nhat Pham made a wrong deduction.

Problem 6.2.7 (23-24-OT-I-P7). (*10 points*) $ABCD$ is a convex quadrilateral. E, F, G, and H are midpoints of AB, BC, CD, and DA, respectively. AF, BG, CH, and DE pairwise intersecting as shown below at I, J, K, and L.

Prove that the green quadrilateral $IJKL$ has the same area as the sum of the area of the brown triangles AEI, BFJ, CGK, and DHL, in other words:

$$[IJKL] = [AEI] + [BFJ] + [CGK] + [DHL]$$



Solution (Official solution and same one from Jeremy Nguyen, Julie Cao) — Connect AC, note that $\triangle ABF$ and $\triangle ACF$ has the same area, since they have the same height from A and bases BF, FC with the same length. Similarly with $\triangle CDH$ and $\triangle CAH$, therefore

$$\begin{aligned}&[ABF] = [ACF], [CDH] = [CAH] \Rightarrow [ABF] + [CDH] = [ACF] + [CDH]\\ \Rightarrow &[AEI] + [BEIJ] + [BFJ] + [CGK] + [DGKL] + [DHL] = [IJKL] + [AHLI] + [CFJK] \quad (*)\end{aligned}$$

In the same way

$$\begin{aligned}&[ADE] = [DEB], [BCG] = [BDG] \Rightarrow [ADE] + [BCG] = [DEB] + [BDG]\\ \Rightarrow &[AEI] + [AHLI] + [DHL] + [BFJ] + [CFJK] + [CGK] = [IJKL] + [BEIJ] + [DGKL] \quad (**)\end{aligned}$$

Comparing (*) and (**), the desired quality follows.

Remark. Jeremy Nguyen, Julie Cao correctly solved this problem.

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Problem 6.2.8 (23-24-OT-I-P8)**.** (*10 points*) A cook has a cube of cheese containing 27 unit cubes. The central unit cube is poisoned. A very clever mouse is found the cube and start to gnaw. When the mouse is done with a unit cube, it moves to the next unit cube that shares a face with the one it just finished.

Will the mouse finished the entire cube except the central one?

Solution (Official solution and same as the one from Minh Nhat Pham) — Let colour the unit cubes alternately similar to a checkerboard pattern: let the corner cubes be black and every two neighbours have the opposite colours. It easy to see that the number of the black ones is 14 and the number of the white ones is 13. Since the mouse wants to avoid the central one, it must eat 14 black and 12 white cubes. However, because the colours of the unit cubes that the mouse eat changes each step from black to white or from white to black, so the difference between the number of black and white cubes the mouse can eat is 0 or 1. This means that the mouse $\boxed{\text{cannot finish the entire cube, except the central one.}}$

Remark. Minh Nhat Pham correctly solved this problem.

Jeremy Nguyen's solution was near to perfect. He missed the argument that the colours of the cubes are alternate in the order the mouse eating.

Chi-Mai Pham saw that some surrounding cubes might have to be taken off first then following the ones with shared faces.

If I am grading story telling, Giang Nguyen must have one of the top positions, unfortunately this is math contest. Life is hard, for me too not award some points to these great and very enjoyable thoughts. However, the process of thought is great! You should just try to be naturally and add more math to it!

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6.3 Problems for Advanced Level

Problem 6.3.1 (23-24-OT-A-P1)**.** (*10 points*) Factorize

$$(2y-3z)^3+(3z-4x)^3+(4x-2y)^3$$

Solution (Officical solution and same as Henry Trinh, Karl Le, Minh Hai Vu) — Let $a=2y-3z, b=3z-4x, c=4x-2y$, then

$$a+b+c=0,\ a+b=2y-4x,\ b+c=3z-2y,\ c+a=4x-3z$$

By the identity $(a+b+c)^3-(a^3+b^3+c^3)=3(a+b)(b+c)(c+a)$ with $a+b+c=0$,

$$(2y-3z)^3+(3z-4x)^3+(4x-2y)^3=a^3+b^3+c^3=-3(a+b)(b+c)(c+a)$$
$$=-3(2y-4x)(3z-2y)(4x-3z)=\boxed{6(2x-y)(2y-3z)(3z-4x).}$$

Solution (Solution by An Nguyen, Jason To, Minh Nguyen, Ryan Nguyen, Samuel Luong, Tran Hieu Khoa) — Let $a=2y-3z, b=3z-4x, c=4x-2y$, then by the identity $a^3+b^3+c^3=3(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc$ with $a+b+c=0$,

$$(2y-3z)^3+(3z-4x)^3+(4x-2y)^3=3(2y-3z)(3z-4x)(4x-2y)=\boxed{6(2x-y)(2y-3z)(3z-4x).}$$

Remark. An Nguyen, Henry Trinh, Jason To, Karl Le, Minh Nguyen, Ryan Nguyen, Samuel Luong, Tran Hieu Khoa correctly solved this problem. An Nguyen, Ryan Nguyen did it without using substitutions.

Antonie Dao made a mistake by letting $S=a^3+b^3+c^3$ then $S=3(a+b)(b+c)(c+a)-(a^3+b^3+c^3)$.

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Problem 6.3.2 (23-24-OT-A-P2)**.** (*10 points*) How many 6-letter words in which one letter appears more than once, can be made from the letters F, L, I, G, H, T ? (You may use any letter as often as you like.)

Solution (Officical solution and same ones from An Nguyen, Antoine Dao, Henry Trinh, Jason To, Minh Hai Vu, Minh Nguyen) — We will use *complementary counting technique: first, we count all possible cases, then we subtract from that all undesirable cases.*

First, we need to fill the six slots below by using the six letters

_ _ _ _ _ _

There are six ways to fill one slot with one of the letters F, L, I, G, H, T. Thus, the total number to fill the six slots, or the total number of ways to create 6-letter words is

$$\underbrace{6 \times 6 \times \cdots \times 6}_{6} = 6^6.$$

Now, an undesirable case is a 6-letter word where all the letters are different, or it is a way to permute the letters F, L, I, G, H, T (because all of them are different), which is $6 \times 5 \times \cdots \times 1 = 6!$.

Hence, the number of desired cases is $6^6 - 6! = \boxed{45936.}$

Remark. An Nguyen, Antoine Dao, Henry Trinh, Jason To, Minh Hai Vu, Minh Nguyen correctly solved this problem.

Ryan Nguyen wrote a program to find the possible cases and came out with a correct answer. Unfortunately, he did not submit the program source code, that would help to understand what algorithm he relied on.

Tran Hieu Khoa did not pay attention to the required condition that *one letter appears more than once.*

Samuel Luong misinterpreted the problem.

Problem 6.3.3 (23-24-OT-A-P3)**.** (*10 points*) Find all (not necessarily positive) integers (a, b, c) such that

$$a^2 + b^2 = 8c + 6.$$

Solution (Officical solution and same as the ones from An Nguyen, Karl Le, Minh Nguyen, Ryan Nguyen, Samuel Luong, Tran Hieu Khoa) — Note that for any perfect square n^2, its remainder when divided by 8 can only be $0, 1$, or 4. Thus $a^2 + b^2$ has a remainder of $0, 1, 2, 4$, or 5 when divided by 8. Hence, there is $\boxed{\text{no integers } a, b, \text{ and } c}$ satisfying the given equation.

Remark. An Nguyen, Jason To, Karl Le, Minh Nguyen, Ryan Nguyen, Samuel Luong, Tran Hieu Khoa correctly solved this problem.

Jason To's and Minh Hai Vu's solutions are long because they investigated the equality modulo 4 in order to arrive at an intermediate result that both a and b are odd, then using modulo 8 to so that the equality is impossible. They could have had cut it short by just using modulo 8, as shown above.

Problem 6.3.4 (23-24-OT-A-P4)**.** (*10 points*) ABC is an isosceles triangle with $AB = AC$. M is midpoint of BC. P is an arbitrary point on AM. The line through C perpendicular to BC intersects ray BP at Q. $QH \perp AB$. Prove that,

$$\frac{AB}{AP} = \frac{BC}{HQ}.$$

Solution (Official solution) — Connect CP. M is midpoint of BC in the isosceles triangle ABC, thus $AM \perp BC$, so $PM \parallel CQ$, or P is midpoint of BQ. $\triangle BCQ$ is right at C, thus $BP = PQ$.

Now, BP is half of BQ, so $(AB)(HQ) = 2[ABQ] = 4[ABP]$. On the other hand, $(AP)(BC) = (AP)(2BM) = 2(AP)(BM) = 4[ABP]$. The desired equality follows.

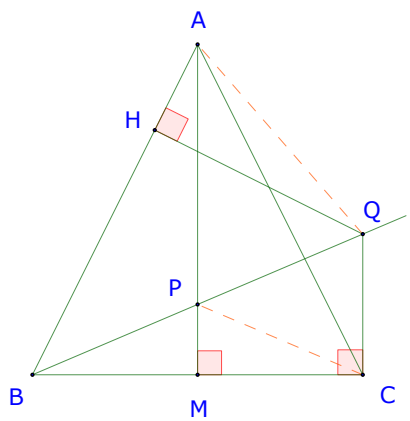

Figure 6.1: Official solution

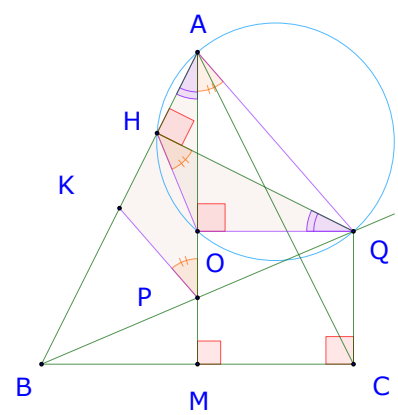

Figure 6.2: Karl Le's solution

Solution (Solution 2 from Karl Le) — Since M is midpoint of BC, $MP \parallel CQ$, so P is midpoint of BQ. See above. Let K be the mid point of AB, then PK is parallel to AQ, and $\angle KPA = \angle OAQ$.

Let O be the foot of the perpendicular from Q to AP. A, H, O, Q all are on the circle diameter AQ because $\angle AHQ = \angle AOQ = 90°$. Thus $\angle OHQ = \angle OAQ = \angle KPA \quad (*)$. On the other hand, $\angle HAQ = \angle OQH \quad (**)$.

From (*) and (**), $\triangle KPA \sim \triangle OHQ$, or

$$\frac{KA}{AP} = \frac{OQ}{HQ} \Rightarrow \frac{AB}{AP} = \frac{2KA}{AP} = \frac{2OQ}{HQ} = \frac{BC}{HQ}.$$

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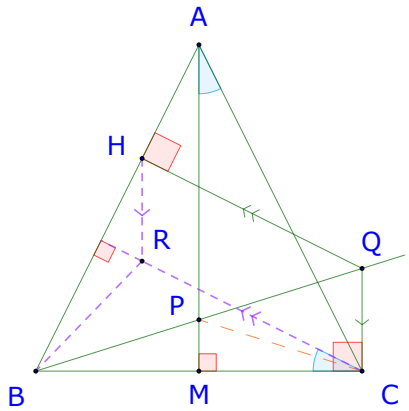

Figure 6.3: An Nguyen's attempt

Remark. Karl Le has shown an amazing solution. Very original.

An Nguyen drawn a diagram which led her into a wrong argument, see above. First, R, the intersection of lines through H, C parallel to QC, HQ, is not necessarily on CP. Second, even if R is on PC, $\triangle BRC$ and $\triangle APC$ can never be congruent since $AC = BC$ equality is not necessarily true. Third, it is indeed $\triangle BRC \sim \triangle CPA$, but that is not a trivial deduction. I would suggest to try to prove it, with perhaps an obvious application of trigonometry.

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Problem 6.3.5 (23-24-OT-A-P5). (*10 points*) Factorize

$$(3a+3b-18ab)(3a+3b-2)+(1-9ab)^2$$

Solution (Officical solution and same one from Antoine Dao, Tran Hieu Khoa, Minh Hai Vu, Minh Nguyen) — Let $u=3a+3b, v=9ab$, then by substitution,

$$\begin{aligned}&(3a+3b-18ab)(3a+3b-2)+(1-9ab)^2=(u-2v)(u-2)+(1-v)^2\\&=u^2-2uv-2u+4v+1-2v+v^2=(u^2-2uv+v^2)-2(u-v)+1\\&=(u-v)^2-2(u-v)+1=(u-v-1)^2\end{aligned}$$

Substituting back,

$$(u-v-1)^2=(3a+3b-9ab-1)^2=(3a(1-3b)-(1-3b))^2=\boxed{(3a-1)^2(3b-1)^2}$$

Solution (Solution 2 from Minh Hai Vu - with modifications) — Let $f(x)=(3x+3b-18xb)(3x+3b-2)+(1-9xb)^2$,

$$f\left(\frac{1}{3}\right)=(1+3b-6b)(1+3b-2)+(1-3b)^2=(1-3b)(3b-1)+(1-3b)^2=0$$

Thus $\frac{1}{3}$ is a root of f, or $3a-1$ is a factor of f. By symmetry, $3b-1$ is also a factor of f. Note that

$$3a+3b-18ab=(3a-1)+(3b-1)-2(9ab-1),\ 3a+3b-2=(3a-1)+(3b-1)$$

Now, let $u=3a-1, v=3b-1, w=9ab-1$, then the given expression is:

$$(u+v-2w)(u+v)+w^2=(u+v)^2-2w(u+v)+w^2=(u+v-w)^2=\boxed{(3a-1)^2(3b-1)^2.}$$

Solution (Solution 3 from Ryan Nguyen - with modifications) — Let $x=3a+3b-18ab, y=3a+3b-2$, and $z=1-9ab$ then $x-y=2z$, thus

$$\begin{aligned}&xy+\left(\frac{x-y}{2}\right)^2=\frac{4xy+x^2-2xy+y^2}{4}=\left(\frac{x+y}{2}\right)=(3a+3b-9ab-1)^2\\&(3a+3b-9ab-1)^2=(3a(1-3b)-(1-3b))^2=\boxed{(3a-1)^2(3b-1)^2}\end{aligned}$$

Remark. Antoine Dao, Minh Hai Vu (first solution), Tran Hieu Khoa correctly solved this problem.

Jason To, Ryan Nguyen, Samuel Luong had nearly perfect solutions. These are considered incomplete since they did not investigate if $3a+3b-9ab-1$ can be factorized further as shown above. Note that in factorization problem, a complete factorization is always required - until each of the obtained terms in the product are non-factorable.

Minh Hai Vu's second solution had a brilliant start using polynomial roots, but then he was stuck at using them.

Chi Khanh Pham confused factorization with expansion. Does she has enough basic knowledge in IC?

Problem 6.3.6 (23-24-OT-A-P6). (*10 points*) The MCC club sends 8 students to participate in a tennis tournament. In how many ways the organizer can pick 6 students who then can be organized into 3 pairs?

Note that a pair (A,B) is considered the same as (B,A).

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Solution (Official solution 1 and same as Jason To's) — Note that if we can select three pairs of student, then those 6 students are the 6 students choosen from 8 students. Thus, we just need to consider to choose three pairs of students from 8 students.

There are $\binom{8}{2}$ ways to choose 2 from 8 students for the first pair. There are $\binom{6}{2}$ ways to choose 2 from $8-2=6$ students for the second pair. And obviously $\binom{4}{2}$ ways to choose 2 from $6-2=4$ students for the third pair. Furthermore, the pairs can be permuted, below is an example

$$\underbrace{A\ B}_{\text{pair1}}\ \underbrace{C\ D}_{\text{pair2}}\ \underbrace{E\ F}_{\text{pair3}} \text{ is the same as } \underbrace{C\ D}_{\text{pair2}}\ \underbrace{E\ F}_{\text{pair3}}\ \underbrace{A\ B}_{\text{pair1}}$$

and there are 3! ways to do so.

Hence, the number of ways the organizer can pick 6 students into 3 pairs is:

$$\frac{\binom{8}{2}\binom{6}{2}\binom{2}{2}}{3!} = \frac{8!}{6!2!}\cdot\frac{6!}{4!2!}\cdot\frac{4!}{2!2!}\cdot\frac{1}{3!} = \frac{7\cdot 8}{2}\cdot\frac{6\cdot 5}{2}\cdot\frac{4\cdot 3}{2}\cdot\frac{1}{6} = 28\cdot 15 = \boxed{420.}$$

Solution (Official solution 2) — First, we solve a smaller problem: how can we divide 6 students into 3 pairs (considering permuting two persons will not result in a different pair.)

We will use *over-counting technique: first, we count all possible cases, then we reduce them by looking at how over-counting happens.*

Considering the six slots we can put the six persons: _ _ _ _ _ _ where all the persons are different, so it is a way to permute them which is $6\times 5\times\cdots\times 1 = 6!$.

If we then group them into pairs,

$$\underbrace{A\ B}_{\text{pair1}}\ \underbrace{C\ D}_{\text{pair2}}\ \underbrace{E\ F}_{\text{pair3}}$$

then are 2! ways to permute persons in a pair, and there are three such pairs, thus every case appears twice.

Furthermore, the pairs can be permuted, below is an example

$$\underbrace{C\ D}_{\text{pair2}}\ \underbrace{E\ F}_{\text{pair3}}\ \underbrace{A\ B}_{\text{pair1}} \text{ is the same as } \underbrace{A\ B}_{\text{pair1}}\ \underbrace{C\ D}_{\text{pair2}}\ \underbrace{E\ F}_{\text{pair3}}$$

and there are 3! ways to do so.

Therefore, the number of ways to group 6 people into 3 pairs is

$$\frac{\frac{6!}{2!2!2!}}{3!}$$

Now, there are $\binom{8}{6}$ ways to pick 6 students out of 8. Hence, the number of ways the organizer can pick 6 students can be divided into 3 pairs is:

$$\binom{8}{6}\cdot\frac{\frac{6!}{2!2!2!}}{3!} = \frac{8!}{6!2!}\cdot\frac{6!}{2!2!2!3!} = \frac{7\cdot 8}{2}\cdot\frac{4\cdot 5\cdot 6}{2^3} = 28\cdot 15 = \boxed{420.}$$

Solution (Official solution 3 and same as the ones from Karl Le, Minh Nguyen, Ryan Nguyen, Samuel Luong) — First, we solve a smaller problem: how can we divide 6 students into 3 pairs (considering permuting two persons will not result in a different pair.)

There are $\binom{6}{2}$ ways to choose 2 from 6 students for the first pair. There are $\binom{4}{2}$ ways to choose 2

from $6-2=4$ students for the second pair. And obviously $\binom{2}{2}$ ways to choose 2 from $4-2=2$ students for the third pair. Furthermore, the pairs can be permuted, below is an example

$$\underbrace{A\ B}_{\text{pair1}}\ \underbrace{C\ D}_{\text{pair2}}\ \underbrace{E\ F}_{\text{pair3}} \text{ is the same as } \underbrace{C\ D}_{\text{pair2}}\ \underbrace{E\ F}_{\text{pair3}}\ \underbrace{A\ B}_{\text{pair1}}$$

and there are 3! ways to do so.

Therefore, the number of ways to group 6 people into 3 pairs is

$$\frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!}.$$

Now, there are $\binom{8}{6}$ ways to pick 6 students out of 8. Hence, the number of ways the organizer can pick 6 students can be divided into 3 pairs is:

$$\binom{8}{6}\cdot\frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!}=\frac{8!}{6!2!}\cdot\frac{6!}{4!2!}\cdot\frac{4!}{2!2!}\cdot\frac{2!}{0!2!}\cdot\frac{1}{3!}=\frac{7\cdot 8}{2}\cdot\frac{4\cdot 5\cdot 6}{2^3}=28\cdot 15=\boxed{420.}$$

Remark. The selection of 6 students is a simple trick to make the problem more complicated.

Jason To, Karl Le, Minh Nguyen, Ryan Nguyen, Samuel Luong correctly solved this problem.

An Nguyen calculated correctly the number of ways to pick 6 students out of 8. She has also provided the correct number of ways the organizer can pick 6 students can be divided into 3 pairs, but with a wrong argument: *5 'starting' pairs times 3 more pairs that can be made from the remaining 4 students.* What are the 5 starting pairs can be made from 6 students?

Henry Trinh followed the roadmap of solution 3, however he did not eliminated the duplicates in permutation of the pairs.

Minh Hai Vu, Tran Hieu Khoa successfully followed the roadmap of solution 3, however they did not eliminated the duplicates in permutation of the pairs.

Chi Khanh Pham started by following the roadmap of solution 3, but she seems to not grasping the counting methods properly.

Antoine Dao made two mistakes. First to assume that there are $\frac{6!}{6}$ ways to choose 6 students out of 8. Second mistake was made when he did not eliminated the duplicates in permutation of the pairs.

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Problem 6.3.7 (23-24-OT-A-P7)**.** (*10 points*) Find all (not necessarily positive) integers n such that

$$\text{at least one of } n^2 - n + 2 \text{ and } n^4 - n + 2 \text{ is a perfect square.}$$

Solution. We will do casework based on n.

Case 1: for $n > 2$,

$$\left.\begin{aligned} n^2 - n + 2 = n(n-1) + 2 > (n-1)^2 \\ n^2 - n + 2 = n^2 + (2-n) < n^2 \end{aligned}\right\} \Rightarrow (n-1)^2 < n^2 - n + 2 < n^2,$$

thus $n^2 - n + 2$ is not a perfect square.

Similarly, for $n > 2$,

$$(n^2)^2 > n^4 + (2-n) = n^4 - n + 2 > n^4 - (n^2) + (1 - n^2) = (n^2 - 1)^2,$$

thus $n^4 - n + 2$ is not a perfect square, either.

Case 2: if $n = 2$, then $n^2 - n + 2 = 2^2$ and $n^4 - n + 2 = 4^2$, thus $\boxed{n = 2}$ is a solution.

Case 3: if $n = 0$ or $n = 1$ then $n^2 - n + 2 = 2$ and $n^4 - n + 2 = 2$. None of them is a perfect square.

Case 4: $n < 0$, let $m = -n \geq 1$, then $n^2 - n + 2 = m^2 + m + 2$, $n^4 - n + 2 = 2 = m^4 + m + 2$,

$$\begin{aligned} m^2 < m^2 + m + 2 \leq m^2 + 2m + 1 = (m+1)^2 \\ (m^2)^2 < m^4 + m + 2 \leq m^4 + 2m^2 + 1 = (m^2+1)^2 \end{aligned}$$

In both case the equalities on the right can only stand if $m = 1$, Thus for $n = -1$, $n^2 - n + 2 = 4$ and $n^4 - n + 2 = 4$ are both perfect square, hence $\boxed{n = -1}$ is also a solution. □

Remark. Ryan Nguyen gave the answers with some minor tests but no adequate reasoning.

Tran Hieu Khoa wrote a program to find the possible cases. However the program is capable to verify n between -1000000 and 1000000, which hardly is enough comparing to all integers. Second, he did not submit the program source code, that would help to understand what algorithm he relied on.

Chi Khanh Pham gave the two answers without proof.

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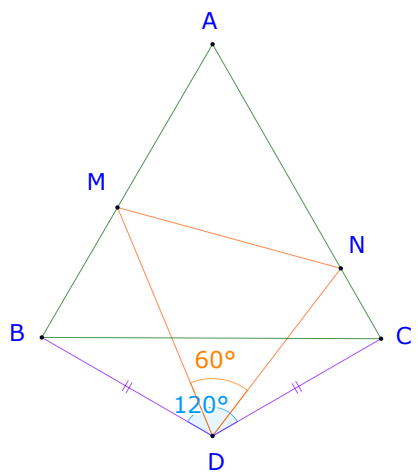
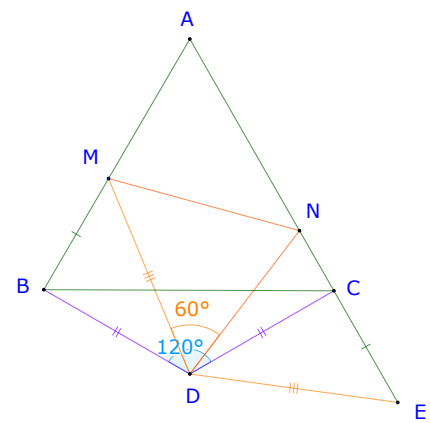
Problem 6.3.8 (23-24-OT-A-P8)**.** (*10 points*) ABC is an equilateral triangle. D is a point outside of $\triangle ABC$ such that $BD = DC$ and $\angle BDC = 120°$. M and N are points on AB and AC, respectively, such that $\angle MDN = 60°$.

Find the perimeter of $\triangle AMN$.

Hint: Let E be on the extension of AC such as $CE = BM$. Investigate $\triangle CDE$.





Solution. Let E be on the extension of AC such as $CE = BM$. Then $CD = BD$, and

$$\angle ABC = \angle ACB,\ \angle DBC = \angle DCB \Rightarrow \angle ABD = \angle ACD$$

Thus $\triangle MBD \cong \triangle ECD$, so $MD = ED$ and $\angle BDM = \angle CDE$.

$$\angle EDN = \angle EDC + \angle CDN = \angle BDM + \angle CDN = 60°.$$

Now, $\triangle MDN$ and $\triangle EDN$ with shared side DN, having $DM = DE$, $\angle MDN = \angle EDN = 60°$, thus $\triangle MDN \cong \triangle EDN$. Therefore $MN = NE = NC + CE = NC + MB$.

Hence,

$$AM + MN + NA = AM + (NC + MB) + NA = AB + AC = \boxed{2AB.}$$

□

Remark. Samuel Luong solved a minor (important) step.

Ryan Nguyen had a good observation but it is not true that if something is true for three separate values of x then it is true for all x.

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6.4 Problems for Olympiad Level

Students have **270** *minutes to solve the problems,* **15** *additional minutes to scan and submit the solutions.*

Note that for every problem you earn an additional same number of points for every different solution.

Problem 6.4.1 (23-24-OT-O-P1)**.** Any a, b, c are non-negative real number such that $a^2 + b^2 + c^2 = 3$.

Prove that

$$ab + bc + ca \leq a + b + c$$

Solution (Official Solution) — Note that $a^2 + b^2 + c^2 = 3$, thus two of a, b, c must be on the opposite side of 1 on the real number line. Lets assume that they are a and b:

$$(a-1)(b-1) \leq 0 \Rightarrow ab + 1 \leq a + b.$$

If we can prove that $bc + ca - 1 \leq c$ then we are done. Note that $(a+b)^2 \leq 2(a^2+b^2)$, thus

$$bc + ca - 1 = c(a+b) - 1 \leq c\sqrt{2(a^2+b^2)} - 1 = c\sqrt{2(3-c^2)} - 1.$$

Now,

$$c\sqrt{2(3-c^2)} - 1 \leq c \Leftrightarrow c^2(6-2c^2) \leq (c+1)^2 \Leftrightarrow 6c^2 - 2c^4 \leq c^2 + 2c + 1 \Leftrightarrow 2c^4 - 5c^2 + 2c + 1 \geq 0$$
$$2c^4 - 5c^2 + 2c + 1 = (2c^4 - 4c^2 + 2) - (c^2 - 2c + 1) = (c-1)^2(2(c+1)^2 - 1) \geq 0$$

Since $c \geq 0$, the last inequality is true.

Solution (Solution 2 from Henry Dinh - with modifications) — Since $a^2 + b^2 + c^2 = 3$,

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2) = 9 \Rightarrow a + b + c \leq 3$$
$$3(ab+bc+ca) \leq (a+b+c)^2 \Rightarrow ab + bc + ca \leq \frac{a+b+c}{3} \cdot (a+b+c) \leq a + b + c.$$

Remark. Henry Dinh correctly solved this problem. His two solutions are basically two-step solutions where the steps are interchanged, thus they counted as a single solution.

Ha-Anh Le has proven that $1 \geq abc$ and $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$, then implied that $a + b + c \geq 3$. This is incorrect and invalidates the rest of the proof.

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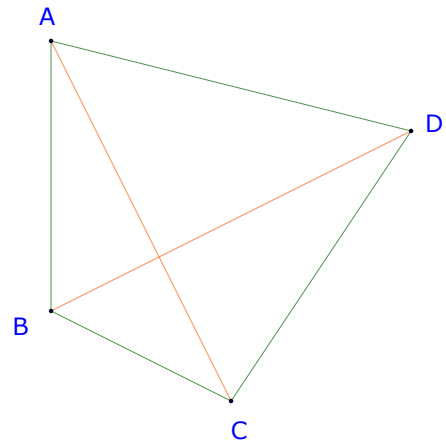
Problem 6.4.2 (23-24-OT-O-P2)**.** (*10 points*) Let $ABCD$ be a convex quadrilateral.

Prove that if $(AB + CD)(AD + BC) = 4[ABCD]$, then $AC = BD$.



Solution (Official solution and same as Henry Dinh's) — Note that $\sin x \leq 1$ for any x,

$$\begin{aligned}
&[ABCD] = [ABC] + [ADC] = \frac{1}{2}(AB \cdot BC \sin \angle ABC + AD \cdot DC \sin \angle ADC) \\
&[ABCD] = [BAD] + [BDC] = \frac{1}{2}(BA \cdot AD \sin \angle BAD + BC \cdot CD \sin \angle BCD) \\
\Rightarrow\ & 4[ABCD] = AB \cdot BC \sin \angle ABC \\
&+ AD \cdot DC \sin \angle ADC + BA \cdot AD \sin \angle BAD + BC \cdot CD \sin \angle BCD \\
&\leq AB \cdot BC + AD \cdot DC + BA \cdot AD + BD \cdot DC = (AB + CD)(BC + AD).
\end{aligned}$$

Thus, the equality can stand if and only if $\angle ABC = \angle ADC = \angle BAD = \angle BCD = 90^\circ$, or $ABCD$ is a rectangle. Hence, it is obvious then $\boxed{AC = BD.}$

Remark. Only Henry Dinh correctly solved this problem.

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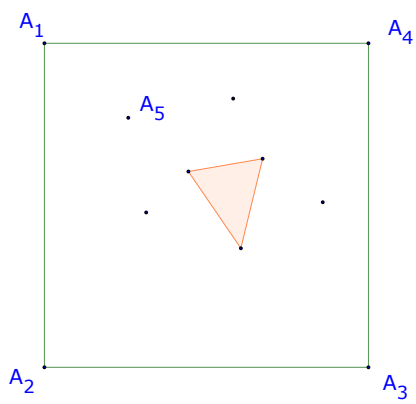
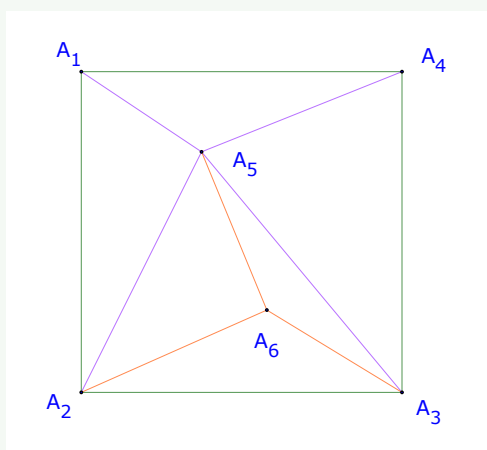
Problem 6.4.3 (23-24-OT-O-P3). (*10 points*) $A_1A_2A_3A_4$ is a unit square. n is an arbitrary positive integer. n points $A_5, A_6, \ldots, A_{n+4}$ are placed inside the square. Prove that we can choose 3 points among $A_1, A_2, \ldots, A_{n+4}$ such that the area of the triangle formed by these points does not exceed $\dfrac{1}{2(n+1)}$.



Solution (Official solution and same as Henry Dinh's) — First, since the sum of the areas of the triangles $\triangle A_1A_2A_5$, $\triangle A_2A_3A_5$, $\triangle A_2A_4A_5$, and $\triangle A_4A_1A_5$ is 1, thus the area of at least one of these triangles cannot exceed $\frac{1}{4}$.



Now, let's assume that $n > 1$. Consider point A_6. It should be with in one of the triangles $\triangle A_1A_2A_5$, $\triangle A_2A_3A_5$, $\triangle A_2A_4A_5$, and $\triangle A_4A_1A_5$. WLOG, let it be in $\triangle A_1A_2A_5$. By connecting A_6 with the vertices A_1, A_2, A_5 we gain *two* more triangles. Now if $n > 2$, then A_7 should be inside one of the $4 + 2 = 6$ triangles, by connecting it to the vertices of the triangle containing it, we gain *two* more triangles.

By continuing this process, we got in total not more than $4 + 2(n-1) = 2(n+1)$ triangles. It is obvious that at least one of them has an area not exceeding $\frac{1}{2(n+1)}$.

Remark. Only Henry Dinh correctly solved this problem.

Ha-Anh Le, Minh Le Phuong Nguyen had a good attempt but ran into a dead-end.

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Problem 6.4.4 (23-24-OT-O-P4)**.** (*10 points*) Let the sequence $\{a_n\}$ be defined as follow:

$$a_0 = a_1 = 1, a_{n+1} = a_n a_{n-1} + 1,\ \forall n \geq 1.$$

Find the remainder of a_{2023} when divided by 4.

Solution (Official solution and same one from Albert Dinh-Le) — Let's explore some of the terms modulo 4: $a_0 = a_1 = 1 \equiv 1 \pmod 4$.

$$a_2 = 1 \cdot 1 + 1 \equiv 2 \pmod 4, a_3 = 2 \cdot 1 + 1 \equiv 3 \pmod 4, a_4 = 3 \cdot 2 + 1 \equiv 3 \pmod 4$$
$$a_5 = 3 \cdot 3 + 1 \equiv 2 \pmod 4, a_6 = 2 \cdot 3 + 1 \equiv 3 \pmod 4, a_7 = 3 \cdot 2 + 1 \equiv 3 \pmod 4$$
$$a_8 = 3 \cdot 3 + 1 \equiv 2 \pmod 4, a_9 = 2 \cdot 3 + 1 \equiv 3 \pmod 4, \ldots$$

Therefore, we form a hypothesis

Claim —

$$a_n \equiv \begin{cases} 2, & \text{if } n \equiv 2 \pmod 3 \\ 3, & \text{otherwise} \end{cases}$$

Proof. The base cases already are proven by our exploration. Let assume that

$$a_{3k} \equiv a_{3k+1} \equiv 3 \pmod 4,\ a3k + 2 \equiv 2 \pmod 4$$
$$\Rightarrow \quad a_{3k+3} \equiv 2 \cdot 3 + 1 \equiv 3 \pmod 4, a_{3k+4} \equiv 3 \cdot 2 + 1 \equiv 3 \pmod 4, a_{3k+5} \equiv 3 \cdot 3 + 1 \equiv 2 \pmod 4.$$

■

Now, $2023 = 3 \cdot 674 + 1$, thus $a_{2023} \equiv \boxed{3} \pmod 4$.

Remark. Albert Dinh-Le, Anthony Pham correctly solved this problem.

Henry Dinh almost solved this problem correctly, but the careless argument $2023 = 6 \cdot 674 + 1$ took away some points.

Minh Le Phuong Nguyen wrote a computer program and correctly solved it. She has also submitted the source code. It is accepted for the test but it is recommended to solve it without code (because of what we are trying to learn.)

Ha Anh Le made a good attempt, without proving the generic case a_n to imply that the sequence is periodic modulo 4. Another unfortunate mistake she did not realize that the period length is 3 and not 4.

Problem 6.4.5 (23-24-OT-O-P5)**.** (*10 points*) a, b, c are positive integers such that $a^2 + b^2 + c^2 = 1$.

Prove that

$$S = 2(a^3 + b^3 + c^3) - (a + b + c) \geq -\frac{\sqrt{3}}{3}.$$

Solution. First, by the Rearrangement inequality

$$3(a^3 + b^3 + c^3) \geq (a + b + c)(a^2 + b^2 + c^2) \quad (*)$$

Furthermore by Cauchy-Schwarz inequality

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2 \quad (**)$$

Now, since $a^2 + b^2 + c^2 = 1$,

$$S \geq \frac{2(a + b + c)(a^2 + b^2 + c^2)}{3} - (a + b + c) = -\frac{1}{3}(a + b + c) \geq -\frac{\sqrt{3(a^2 + b^2 + c^2)}}{3} = -\frac{\sqrt{3}}{3}.$$

□

Remark. Note that (*) and (**) can be proven as below

$$\begin{aligned}
(*) \quad & 3(a^3 + b^3 + c^3) \geq (a + b + c)(a^2 + b^2 + c^2) \\
\Leftrightarrow \quad & 2(a^3 + b^3 + c^3) - (a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2) \geq 0 \\
\Leftrightarrow \quad & (a - b)^2(a + b) + (b - c)^2(b + c) + (c - a)^2(c + a) \geq 0.
\end{aligned}$$

$$\begin{aligned}
(**) \quad & 3(a^2 + b^2 + c^2) \geq (a + b + c)^2 \\
\Leftrightarrow \quad & 2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca) \\
\Leftrightarrow \quad & (a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0.
\end{aligned}$$

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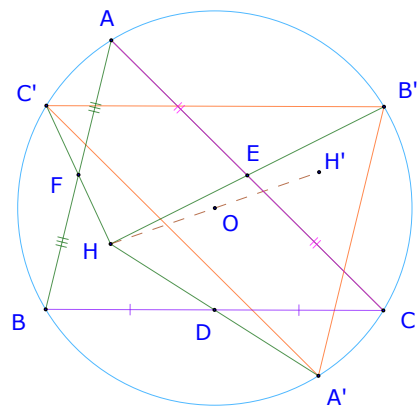
Problem 6.4.6 (23-24-OT-O-P6)**.** ABC is an acute triangle. D, E, and F are midpoints of BC, CA, and AB, respectively. Let H and O be the orthocenter and circumcenter of $\triangle ABC$. Rays HD, HE, and HF intersect the circumcircle of $\triangle ABC$ at A', B', and C', respectively. Let H' be the orthocenter of $\triangle A'B'C'$. Prove that:

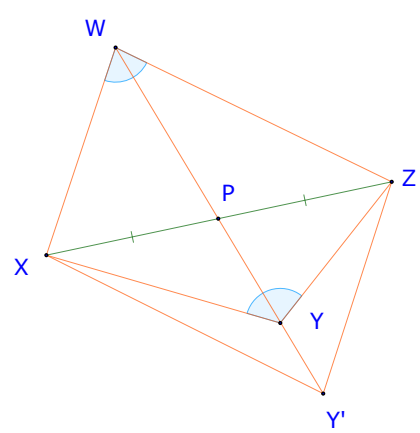
- (*2 points*) $BHCA'$ is a parallelogram.
- (*3 points*) AA', BB', and CC' are diameters of the circumcircle of $\triangle ABC$.
- (*5 points*) H, O, and H' are collinear.



Solution. First, we prove a minor claim

Claim — In quadrilateral $XYZW$, $\angle XYZ = \angle XWZ$, diagonals XZ and YW intersects at the midpoint P of XZ. Then $XYZW$ is a parallelogram.

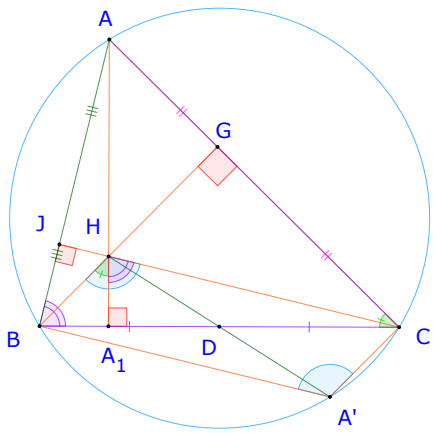
Proof. We give a synthetic proof.

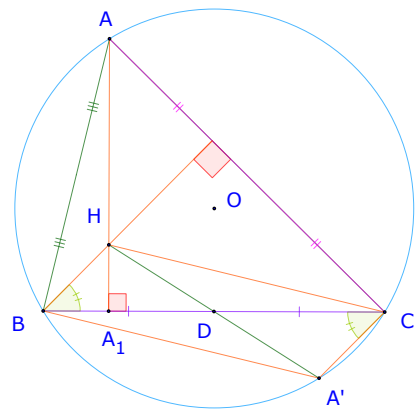


Let's assume that $WP \neq PY$. WLOG let's assume that $WP > PY$. Y' is a point on the extension of PY towards Y such that $WP = PY'$. Then $XY'ZW$ is a parallelogram and $\angle XY'Z = \angle XWZ = \angle XYZ$, impossible. Thus $WP = PY$, or $XYZW$ is a parallelogram. ■

Now, BA_1HJ and CA_1HG are cyclic quadrilaterals, thus

$$\angle BHC = \angle BHA_1 + \angle A_1HC = \angle ACB + \angle ABC = 180^\circ - \angle BAC = \angle BA'C.$$



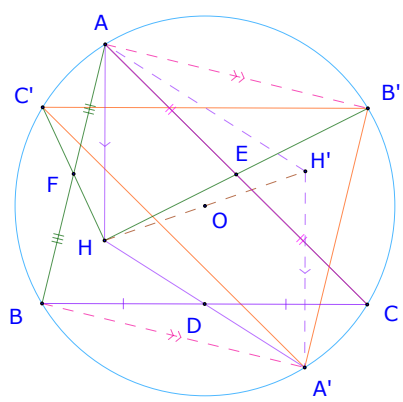


Since $BD = DC$, by the claim, $BHCA'$ is a parallelogram. This implies that

$$\angle DCA' = \angle DBH = 90^\circ - \angle C \Rightarrow \angle ACA' = 90^\circ.$$

Therefore AA', and similarly BB' and CC' are diameters of the circumcircle of $\triangle ABC$.

Since AA', BB' intersecting each other at midpoints, $ABA'B'$ is a parallelogram, so $AB = A'B'$. Similarly $BC = B'C', CA = C'A'$. Thus $\triangle ABC \cong \triangle A'B'C'$.



Now, consider $AHA'H'$, it is easy see that $AH = A'H'$ (because of corresponding segments in two congruent triangles) and $AH \parallel A'H'$ (perpendicular to a pair of parallel bases), so $AHA'H'$ is a parallelogram, thus HH' shall be bisected by the midpoint of AA', which is O. $\square$

Remark. Only Henry Dinh correctly solved this problem.

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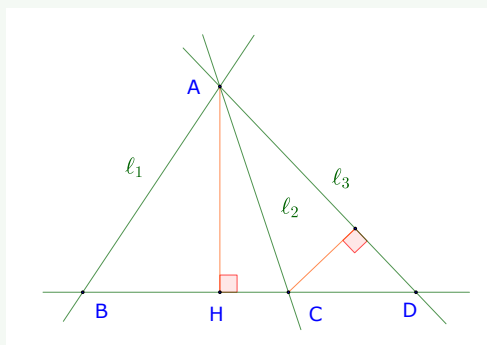
Problem 6.4.7 (23-24-OT-O-P7). (*10 points*) S is a set of n lines which are drawn on the plane. No two lines are parallel. For every pair (ℓ_1, ℓ_2), there exists a line ℓ passes through the intersection of ℓ_1 and ℓ_2 $(\ell_1, \ell_2, \ell \in S.)$

Prove that all lines are concurrent.

Solution (Official solution and same as Henry Dinh) — Let's assume that there all lines are not concurrent.

Consider the intersections of all pairs of lines. There are $\frac{n(n-1)}{2}$ such points. Now consider the distances from an intersection to all lines not passing through it. Obviously this set is not empty because of no two lines are parallel.

Thus by the Extremal Principle, there exist a point A intersection of a pair of lines (ℓ_1, ℓ_2) with the smallest distance to a line ℓ not passing through it. Let B and C be the intersection of ℓ_1 and ℓ_2 with ℓ. By the given condition, there exists line ℓ_3 passing through A and thus intersects with ℓ at a point D.



Now let H be the foot of the perpendicular from A to ℓ. By the Pigeonhole Principle, two points from B, C, D must be on the same side of H, see the diagram below for an example. Let say C, D are such points. Then the distance from C to ℓ_3 is less than AH, which supposes to be the smallest among such distances. This is a contradiction, thus all lines are concurrent.

Solution (Solution from Ha-Anh Le, Henry Dinh) — Let's prove it by induction principle.

It is easy to verify the base case when $n = 1, 2$, or 3.

Let assume that the case it true for $n = k$. Now consider lines $\ell_1, \ell_2, \ldots, \ell_k$, and ℓ_{k+1}. From the induction hypothesis, for lines $\ell_1, \ell_2, \ldots, \ell_k$, there a point P, where all lines are concurrent. Now, for the intersection Q of lines ℓ_{k+1} and line ℓ_1 there is a line among $\ell_2, \ell_3, \ldots, \ell_k$, which is through the Q. It means that $P \equiv Q$, or ℓ_{k+1} is through P, thus all lines are concurrent.

Remark. Ha Anh Le, Henry Dinh correctly solved the problem.

Albert Dinh-Le, Minh Le Phuong Nguyen tried to create a proof of contradiction by stating that it is possible to construct an infinite number of lines with a very weak argument.

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Problem 6.4.8 (23-24-OT-O-P8)**.** Let the sequence $\{a_n\}$ be defined as follow:

$$a_0 = 3, a_1 = 17, a_{n+1} = 6a_n - a_{n-1},\ \forall n \geq 1.$$

(*1 points*) Prove that

$$a_{n+1}^2 - 6a_{n+1}a_n = -6a_na_{n-1} + a_{n-1}^2,\ \forall n \geq 1$$

(*9 points*) Prove that for all $n \geq 0$, $\frac{1}{2}(a_n^2 - 1)$ is a perfect square.

Note that student shall not use without proof the results of any preceding statement in the proof of any later statement. Student can earn a full 10 pts if he/she can prove the second statement without using the result of the first statement.

Solution. [Official solution 1] Let's verify the base cases:

$$\frac{1}{2}(a_0^2 - 1) = \frac{1}{2}(3^2 - 1) = 2^2,\ \frac{1}{2}(a_1^2 - 1) = \frac{1}{2}(17^2 - 1) = 12^2.$$

Now, let assume that the hypothesis is true for all $a_0, a_1, \ldots, a_n$ $(n \geq 1)$.

First, note that:

$$\begin{aligned}
&a_{n+1} = 6a_n - a_{n-1} \Rightarrow a_{n+1} - 3a_n = 3a_n - a_{n-1} \\
&\Rightarrow (a_{n+1} - 3a_n)^2 = (3a_n - a_{n-1})^2 \Rightarrow a_{n+1}^2 - 6a_{n+1}a_n = -6a_na_{n-1} + a_{n-1}^2
\end{aligned}$$

Therefore if we want to use these in order to perform a telescopic summation:

$$\begin{aligned}
a_{n+1}^2 - 6a_{n+1}a_n &= -6a_na_{n-1} + a_{n-1}^2 \\
a_n^2 - 6a_na_{n-1} &= -6a_{n-1}a_{n-2} + a_{n-2}^2 \\
&\cdots \\
a_3^2 - 6a_3a_3 &= -6a_2a_1 + a_1^2 \\
a_2^2 - 6a_2a_1 &= -6a_1a_0 + a_0^2
\end{aligned}$$

Then

$$\begin{aligned}
&a_{n+1}^2 - 6a_{n+1}a_n + a_n^2 = a_1^2 - 6a_1a_0 + a_0^2 = -8 \\
&\Rightarrow 9a_{n+1}^2 - 6a_{n+1}a_n + a_n^2 = 8a_{n+1}^2 - 8 \Rightarrow (3a_{n+1} - a_n)^2 = 8(a_{n+1}^2 - 1) \\
&\Rightarrow 8 \mid (3a_{n+1} - a_n)^2 \Rightarrow 16 \mid (3a_{n+1} - a_n)^2 \Rightarrow \frac{1}{2}(a_{n+1}^2 - 1) = \frac{1}{16}(3a_{n+1} - a_n)^2 \text{ is a perfect square.}
\end{aligned}$$

□

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Solution. [Official solution 2] The characteristic polynomial of $a_{n+1} = 6a_n - a_{n-1}$, or $a_{n+1} - 6a_n + a_{n-1} = 0$ is $P(x) = x^2 - 6x + 1$, which has two roots $3 \pm 2\sqrt{2}$. It means that $a_n = c_1(3+2\sqrt{2})^n + c_2(3-2\sqrt{2})^n$. Since $a_0 = 3, a_1 = 17$, so

$$\begin{cases} c_1 + c_2 = 3 \\ c_1(3+2\sqrt{2}) + c_2(3-2\sqrt{2}) = 17 \end{cases} \Rightarrow 2\sqrt{2}(c_1 - c_2) = 8 \Rightarrow c_1 - c_2 = 2\sqrt{2} \Rightarrow \begin{cases} c_1 = \frac{1}{2}(3+2\sqrt{2}) \\ c_2 = \frac{1}{2}(3-2\sqrt{2}) \end{cases}$$
$$\Rightarrow a_n = \frac{1}{2}\left((3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}\right)$$

Now, $3 \pm 2\sqrt{2} = (1 \pm \sqrt{2})^2$, so let $u = 1+\sqrt{2}, v = 1-\sqrt{2}$, then $uv = 1, u - v = 2\sqrt{2}$ and

$$a_n = \frac{1}{2}(u^{2n+2} + v^{2n+2}) \Rightarrow \frac{1}{2}(a_n^2 - 1) = \frac{1}{2}\left[\left(\frac{1}{2}(u^{2n+2} + v^{2n+2})^2\right) - 1\right] = \frac{1}{8}\left((u^{2n+2} + v^{2n+2})^2 - 4\right)$$
$$= \frac{1}{8}\left(u^{2n+2} + v^{2n+2} - 2\right)\left(u^{2n+2} + v^{2n+2} + 2\right) = \frac{1}{8}(u^{n+1} - v^{n+1})^2(u^{n+1} + v^{n+1})^2$$
$$= \frac{1}{8}(u^{n+1} + v^{n+1})^2(u-v)^2(u^n + u^{n-1}v + \cdots + v^n)^2 = (u^{n+1} + v^{n+1})^2(u^n + u^{n-1}v + \cdots + v^n)^2$$

It is not difficult to prove that each of the factors of $u^{n+1} + v^{n+1}$ and $u^n + u^{n-1}v + \cdots + v^n$ is an integer, thus the given expression is a perfect square. □

Remark. Ha-Anh Le has proven the first question and has an attempt towards the generic form of a_n, similar to the second solution, but could not finish it.

Henry Dinh has proven the first question.

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6.5 IC Level Test Rules

- All test lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *at most one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

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6.6 IC Test for Level 1

Problem 6.6.1 (ICLT-2023-SM1-R1-L1-P1)**.** Evaluate

$$(10^{\frac{1}{2}})^{\frac{1}{2}} \times 5^{\frac{1}{2}} \times 2^{\frac{1}{2}}.$$

(A) $(10)^{\frac{3}{4}}$ (B) $10^{\frac{1}{4}}$ (C) $2^{\frac{3}{4}} \times 5^{\frac{3}{4}}$ (D) $2^{\frac{3}{4}} \times 5^{\frac{7}{12}}$ (E) $2^{\frac{7}{12}} \times 5^{\frac{7}{12}}$

Solution. ICLT-2023-SM1-R1-L1-P1 *Exercise 1.7.5.c, Chapter 1, Introductory to Algebra.*

$$10^{\frac{1}{2}} = (2 \times 5)^{\frac{1}{2}} = 2^{\frac{1}{2}} \times 5^{\frac{1}{2}} \Rightarrow (10^{\frac{1}{2}})^{\frac{1}{2}} = (2^{\frac{1}{2}})^{\frac{1}{2}} \times (5^{\frac{1}{2}})^{\frac{1}{2}} = 2^{\frac{1}{4}} \times 5^{\frac{1}{4}}$$

$$\Rightarrow (10^{\frac{1}{2}})^{\frac{1}{2}} \times 5^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{4}} \times 5^{\frac{1}{4}} \times 5^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\left(\frac{1}{4}+\frac{1}{2}\right)} \times 5^{\left(\frac{1}{4}+\frac{1}{2}\right)} = 2^{\frac{3}{4}} \times 5^{\frac{3}{4}} = \boxed{10^{\frac{3}{4}}.}$$

The number answer is $\boxed{A.}$ □

> **Remark.** The problem requires students to choose at most one from the five answers. However A and C are the same, just different ways of expressing the product of powers with the same exponent.
>
> Anna Dao, Liam Bao, Minh Chau Nguyen, Minh-Quang Pham gave the correct answer.

Problem 6.6.2 (ICLT-2023-SM1-R1-L1-P2)**.** My team has 10 members. In how many ways can we choose a captain, a lieutenant, a secretary, and a technician, if no member can hold more than one position?

(A) 34 (B) 40 (C) 210 (D) 1000 (E) 5040

Solution. ICLT-2023-SM1-R1-L1-P2 *Exercise 1.4.8, Chapter 1, Introductory to Counting & Probability.*

There are 10 choices for captain, 9 choices for lieutenant, 8 choices for secretary, and 7 choices for treasurer, for a total of $10 \times 9 \times 8 \times 7 = \boxed{5040}$ different choices. The answer is $\boxed{E.}$ □

Problem 6.6.3 (ICLT-2023-SM1-R1-L1-P3)**.** What is the maximum number of possible points of intersection of a circle and a square?

(A) 2 (B) 4 (C) 5 (D) 6 (E) 8

Solution. ICLT-2023-SM1-R1-L1-P3 *Exercise 1.3.3, Chapter 1, Introductory to Geometry.*

A line segment can intersect a circle in at most two points. Since a square consists of four line segments, the maximum number of intersections between a square and a circle is $4 \times 2 = \boxed{8.}$ The answer is $\boxed{E.}$ □

> **Remark.** Anna Dao, Liam Bao, Minh Chau Nguyen, Minh-Quang Pham gave the correct answer.

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Problem 6.6.4 (ICLT-2023-SM1-R1-L1-P4)**.** Billy puts 18 marbles in b boxes such that each box contains the same number of marbles. If there are at least 2 boxes and each box has more than two marbles, how many possible values of b are there?

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

Solution. ICLT-2023-SM1-R1-L1-P4 *Exercise 1.6.1, Chapter 1, Introductory to Number Theory.*

The number of marbles in each box is $\frac{18}{b}$. Thus, b is a divisor of 18. We know that b is at least 2 and the number of marbles in each box $\frac{18}{b}$ is at least 3, thus b can only be 2, 3, or 6. Hence, there are $\boxed{3}$ possible values for b. The answer is $\boxed{D.}$ □

> **Remark.** Liam Bao gave the correct answer. Some other students had not paid attention to the fact that *each box has more than two marbles*, meaning there should be at least 3 marbles in a box.

Problem 6.6.5 (ICLT-2023-SM1-R1-L1-P5)**.** Factor $18a^4 - 24a^3 + 6a^2$ completely as you can.

(A) $6a^2(3a-1)(1-a)$ (B) $6a^2(3a-1)(a-1)$ (C) $6(3a^4 - 24a^3 + 6a^2)$
(D) $a^2(18a^2 - 24a + 6)$ (E) $6a^2(3a^2 - 4a + 1)$

Solution. ICLT-2023-SM1-R1-L1-P5 *Problem 2.10.d, Chapter 2, Introductory to Algebra.*

Note that each terms $6a^4, 24a^3$, and $18a^2$ of the expression contains a factor $6a^2$, thus

$$18a^4 - 24a^3 + 6a^2 = (6a^2)(3a^2) - (6a^2)(4a) + (6a^2)(1) = (6a^2)(3a^2 - 4a + 1)$$
$$= 6a^2(3a(a-1) - (a-1)) = \boxed{6a^2(3a-1)(a-1).}$$ The answer is $\boxed{B.}$

□

> **Remark.** Anna Dao gave the correct answer. Liam Bao, Minh Chau Nguyen, Minh-Quang Pham had chosen an answer that is not a complete factorization.

Problem 6.6.6 (ICLT-2023-SM1-R1-L1-P6)**.** The class has 7 students: 4 boys and 3 girls. In how many ways can the students seated in a row such that 3 girls seat next to each other?

(A) 720 (B) 576 (C) 384 (D) 288 (E) 144

Solution. ICLT-2023-SM1-R1-L1-P6 *Problem 2.12, Chapter 2, Introductory to Counting & Probability.*

Without worrying about which specific boys and girls go in which seats, in how many ways can the girls sit together? There are 5 basic configurations of boys and girls, note that B is a boy and G is a girl,

$$GGGBBBB, BGGGBBB, BBGGGBB, BBBGGGB, BBBBGGG$$

Then, within each configuration, there are $4! = 24$ ways in which we can assign the 4 boys to seats, and $3! = 6$ ways in which we can assign the 3 girls to seats. Therefore the number of possible seatings is $5 \times 24 \times 6 = \boxed{720}$. The answer is $\boxed{A.}$ □

> **Remark.** Minh-Quang Pham gave the correct answer.

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Problem 6.6.7 (ICLT-2023-SM1-R1-L1-P7)**.** Solve

$$\frac{x}{x-1}+\frac{1}{5}=\frac{5}{x-1}.$$

Solution. ICLT-2023-SM1-R1-L1-P7 *Problem 3.18.c, Chapter 3, Introductory to Algebra.*

$$\frac{x}{x-1}+\frac{1}{5}=\frac{5}{x-1}\Rightarrow\frac{1}{5}=\frac{5}{x-1}-\frac{x}{x-1}=\frac{5-x}{x-1}\Rightarrow x-1=5(5-x)=25-5x\Rightarrow 6x=26\Rightarrow x=\boxed{\frac{13}{3}.}$$

□

Remark. Minh-Quang Pham correctly solved the problem.

Problem 6.6.8 (ICLT-2023-SM1-R1-L1-P8)**.** The first angle in a triangle is twice of one of the two other angles. The second angle is twice of of one of the two other angles. What are the possible measures of the smallest angle in the triangle?

Solution. ICLT-2023-SM1-R1-L1-P8 *Problem 2.18, Chapter 2, Introductory to Geometry.*

There are two possible cases.

Case 1: The first and the second angle are both twice the third angle. Let the measure of the smallest angle, the third angle, be x,

$$2x+2x+x=180^\circ\Rightarrow x=\frac{180^\circ}{5}.$$

Case 2: One of the first two angles is twice the other angle of the first two angles, the smaller of of the first two angles is is twice the third angle. Let the measure of the smallest angle, the third angle, be x,

$$4x+2x+x=180^\circ\Rightarrow x=\frac{180^\circ}{7}.$$

□

Remark. Minh-Quang Pham managed to solve the problem with a solution that is very difficult to read.

Problem 6.6.9 (ICLT-2023-SM1-R1-L1-P9)**.** What is the second largest two-digit prime number whose digits are different and each digit is a prime?

Solution. ICLT-2023-SM1-R1-L1-P9 *Problem 2.11, Chapter 2, Introductory to Number Theory.*

A two-digit number whose digits are prime is made up of the digits $2, 3, 5$, or 7. We check the possible numbers from the largest in this list and work backwards until we find the largest prime number satisfying the required conditions:

- 77 is divisible by 7, 75 is divisible by 3. 73 is prime, so it is the largest two-digit prime number whose digits are both prime and both are different.
- 57 is divisible by 3, 55 is divisible by 5. $\boxed{53}$ is prime, so it is the second largest two-digit prime number whose digits are both prime and both are different.

□

Remark. Minh-Quang Pham managed to solve the problem with a solution that is very difficult to read.

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Problem 6.6.10 (ICLT-2023-SM1-R1-L1-P10)**.** Each of the grades 7 and 8 has 5 clubs. The school decides to hold a competition. A pair of clubs in each grade play each other twice. A pair of clubs in different grades play each other once. What is the total number of matches?

Solution. [First solution] ICLT-2023-SM1-R1-L1-P10 *Exercise 3.3.3, Chapter 3, Introductory to Counting & Probability.*

We call a match *intra-grade* if it is played between the two clubs from the same grade, and *cross-grade* if it is played between the two clubs from different grades.

It is easy to see that the total number of intra-grade matches is the number of intra-grade matches for grade 7 plus the number of intra-grade matches for grade 8. A pairs of clubs in each grade play each other twice, so the number of intra-grade matches for grade 7 is $2 \cdot \frac{5 \cdot 4}{2} = 20$. Grade 8 has the same number of clubs, so the number of intra-grade matches for grade 8 is also 20. Therefore there are $2 \cdot 20 = 40$ *intra-grade* matches.

For a club in grade 7 there are 5 clubs in grade 8, so the number of *cross-grade* matches (played by all the clubs in grade 7 with all the clubs in grade 8) is $5 \cdot 5 = 25$.

Thus, the total number of matches is $40 + 25 = \boxed{65.}$ □

Solution. [Second solution] ICLT-2023-SM1-R1-L1-P10 The other way to count is to look at each club. Each club plays twice a club in the same grade, so each club plays $(5 - 1) \cdot 2 = 8$ matches with a club in the same grade. Each club plays once a club in different grade, so each club plays $5 \cdot 1 = 5$ matches with a club in the other grade. In total, each club plays $8 + 5 = 13$ matches. There are $5 + 5 = 10$ clubs, so there are $10 \cdot 13 = 130$ matches. However each match counted this way is counted twice (we count both $A - play - B$ and $B - play - A$), hence the total number of matches is $\frac{130}{2} = \boxed{65.}$ □

Remark. Minh-Quang Pham correctly solved the problem.

6.7 Grading

Answers for multiple-choice problems.

Problem 1: $A,\ C$

Problem 2: E

Problem 3: E

Problem 4: C

Problem 5: B

Problem 6: A

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can bring it to $\frac{1}{5} = \frac{5-x}{x-1}$.

Problem 8: 2 points if the student can see that there are two different cases of $2x, 2x, x$ and $4x, 2x, x$ angles.

Problem 9: 2 points if the student can find the largest such prime 73.

Problem 10: 2 points if the student can devise a counting method of both within and outside of the grades.

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6.8 IC Test for Level 2

Problem 6.8.1 (ICLT-2023-SM1-R1-L2-P1)**.** The first angle in a triangle is twice of one of the two other angles. The second angle is twice of of one of the two other angles. What would be the smallest value of the smallest angle in the triangle?

$(A)\quad 37.5^\circ \qquad (B)\quad \dfrac{180^\circ}{5} \qquad (C)\quad \dfrac{180^\circ}{7} \qquad (D)\quad 35^\circ \qquad (E)\quad 15^\circ$

Solution. ICLT-2023-SM1-R1-L2-P1 *Problem 2.18, Chapter 2, Introductory to Geometry.* There are two possible cases.

Case 1: The first and the second angle are both twice the third angle. Let the measure of the smallest angle, the third angle, be x, $2x + 2x + x = 180^\circ \Rightarrow x = \frac{180^\circ}{5}$.

Case 2: One of the first two angles is twice the other angle of the first two angles, the smaller of of the first two angles is is twice the third angle. Let the measure of the smallest angle, the third angle, be x, $4x + 2x + x = 180^\circ \Rightarrow x = \frac{180^\circ}{7}$.

The number answer is $\boxed{C.}$ □

Remark. Jade Pham, Martin Nguyen gave the correct answer.

Problem 6.8.2 (ICLT-2023-SM1-R1-L2-P2)**.** Each of the grades 7 and 8 has 5 clubs. The school decides to hold a competition. A pair of clubs in each grade play each other twice. A pair of clubs in different grades play each other once. What is the total number of matches?

$(A)\quad 65 \qquad (B)\quad 40 \qquad (C)\quad 38 \qquad (D)\quad 36 \qquad (E)\quad 34$

Solution. [First solution] ICLT-2023-SM1-R1-L2-P2 *Exercise 3.3.3, Chapter 3, Introductory to Counting & Probability.* We call a match *intra-grade* if it is played between the two clubs from the same grade, and *cross-grade* if it is played between the two clubs from different grades.

It is easy to see that the total number of intra-grade matches is the number of intra-grade matches for grade 7 plus the number of intra-grade matches for grade 8. A pairs of clubs in each grade play each other twice, so the number of intra-grade matches for grade 7 is $2 \cdot \frac{5 \cdot 4}{2} = 20$. Grade 8 has the same number of clubs, so the number of intra-grade matches for grade 8 is also 20. Therefore there are $2 \cdot 20 = 40$ *intra-grade* matches. For a club in grade 7 there are 5 clubs in grade 8, so the number of *cross-grade* matches (played by all the clubs in grade 7 with all the clubs in grade 8) is $5 \cdot 5 = 25$.

Thus, the total number of matches is $40 + 25 = 65$. The answer is $\boxed{A.}$ □

Solution. [Second solution] ICLT-2023-SM1-R1-L1-P10 The other way to count is to look at each club. Each club plays twice a club in the same grade, so each club plays $(5 - 1) \cdot 2 = 8$ matches with a club in the same grade. Each club plays once a club in different grade, so each club plays $5 \cdot 1 = 5$ matches with a club in the other grade. In total, each club plays $8 + 5 = 13$ matches. There are $5 + 5 = 10$ clubs, so there are $10 \cdot 13 = 130$ matches.

However each match counted this way is counted twice (we count both $A - play - B$ and $B - play - A$), hence the total number of matches is $\frac{130}{2} = 65$. The answer is $\boxed{A.}$ □

Remark. Jade Pham, Martin Nguyen gave the correct answer.

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Problem 6.8.3 (ICLT-2023-SM1-R1-L2-P3)**.** Simplify as much as possible the following product

$$\frac{2}{3a^2-12b}\cdot\frac{9a^3-36ab}{8a^2}$$

$$(A)\quad \frac{3}{5a}\qquad (B)\quad \frac{2}{3a}\qquad (C)\quad \frac{3a^2-12b}{4a}\qquad (D)\quad \frac{1}{3(3a^2-12b)}(E)\quad \frac{3}{4a}$$

Solution. ICLT-2023-SM1-R1-L2-P3 *Exercise 4.3.4, Chapter 4, Introductory to Algebra.*

$$3a^2-12b=3(a^2-4b),\ 9a^3-36ab=9a(a^2-4b)$$
$$\Rightarrow \frac{2}{3a^2-12b}\cdot\frac{9a^3-36ab}{10a^2}=\frac{2}{3(a^2-4b)}\cdot\frac{9a(a^2-4b)}{8a^2}=\boxed{\frac{3}{4a}}$$

The answer is $\boxed{E.}$ □

Problem 6.8.4 (ICLT-2023-SM1-R1-L2-P4)**.** Which pair of triangles must be congruent?

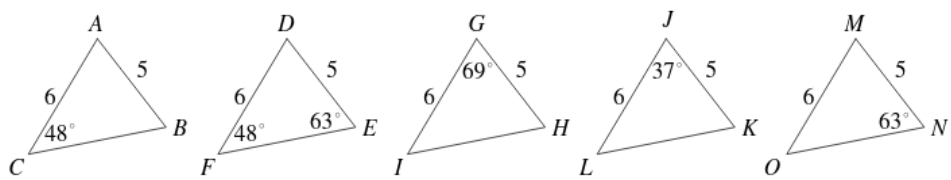

Figure 6.4: ICLT-2023-SM1-R1-L2-P4

$$(A)\quad (\triangle ABC,\triangle GHI)\qquad (B)\quad (\triangle GHI,\triangle DEF)\qquad (C)\quad (\triangle MNO,\triangle ABC)$$
$$(D)\quad (\triangle JKL,\triangle MNO)\qquad (E)\quad (\triangle DEF,\triangle JKL)$$

Solution. ICLT-2023-SM1-R1-L2-P4 *Exercise 3.3.1, Chapter 3, Introductory to Geometry.*

In $\triangle DEF$, we have $\angle D=180^\circ-\angle E-\angle F=69^\circ$.

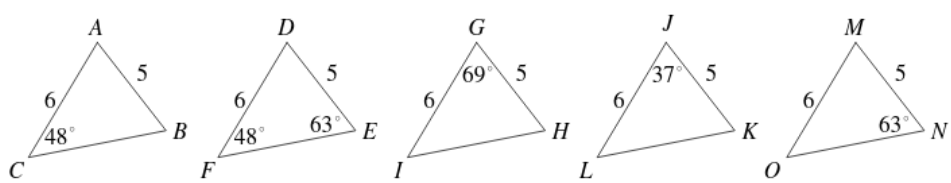

Figure 6.5: ICLT-2023-SM1-R1-L2-P4

Therefore, by SAS we have $\triangle DEF\cong\triangle GHI$. No other two triangles shown need be congruent. The answer is $\boxed{B.}$ □

Remark. Jade Pham gave the correct answer.

Problem 6.8.5 (ICLT-2023-SM1-R1-L2-P5)**.** Compute the least common multiple of $24, 18$, and 36.

(A) 504 (B) 252 (C) 168 (D) 72 (E) 36

Solution. ICLT-2023-SM1-R1-L2-P5 *Exercise 3.4.1.i, Chapter 3, Introductory to Number Theory.*

Note that

$$24 = 2^3 \cdot 3, 18 = 2 \cdot 3^2, 36 = 2^2 \cdot 3^2 \Rightarrow \text{lcm}(24, 18, 36) = 2^3 \cdot 3^2 = \boxed{72.}$$

The answer is $\boxed{D.}$ □

> **Remark.** Jade Pham, Martin Nguyen gave the correct answer.

Problem 6.8.6 (ICLT-2023-SM1-R1-L2-P6)**.** Which one is the largest?

(A) $\binom{6}{3}$ (B) $\binom{7}{4}$ (C) $\binom{9}{7}$ (D) $\binom{10}{2}$ (E) $\binom{44}{1}$

Solution. ICLT-2023-SM1-R1-L2-P6 *Problem 4.7, Chapter 4, Introductory to Counting & Probability.*

$$\binom{6}{3} = \frac{6!}{3!3!} = \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} = 20$$
$$\binom{7}{4} = \frac{7!}{4!3!} = \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3} = 35$$
$$\binom{9}{7} = \frac{8 \cdot 9}{1 \cdot 2} = 36$$
$$\binom{10}{2} = \frac{9 \cdot 10}{1 \cdot 2} = \boxed{45}$$
$$\binom{44}{1} = 44$$

The answer is $\boxed{D.}$ □

Problem 6.8.7 (ICLT-2023-SM1-R1-L2-P7)**.** There are red and blue balls in the Christmas shop. A red ball costs \$5 and a blue ball costs \$3. 6 red balls together weight 1 kg. 2 blue balls together weight 1 kg.

Marianna bought some red and blue balls. Altogether they cost \$108, and weight 10 kg. How many blue balls did Marianna buy?

Solution. ICLT-2023-SM1-R1-L2-P7 *Problem 5.11, Chapter 5, Introductory to Algebra.*

Let r be the number of red balls and b be the number of blue balls that Marianna bought.

$$\begin{cases} 5r + 3b &= 108 \\ \frac{1}{6}r + \frac{1}{2}b &= 10 \end{cases} \Rightarrow \begin{cases} 5r + 3b &= 108 \\ r + 3b &= 60 \end{cases} \Rightarrow 4r = 108 - 60 = 48 \Rightarrow r = 12 \Rightarrow b = \boxed{16.}$$

□

> **Remark.** Jade Pham, Martin Nguyen gave the correct answer without proof.

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Problem 6.8.8 (ICLT-2023-SM1-R1-L2-P8)**.** AC and BD meet at X as shown. Given $[ABX] = 24$, $[BCX] = 15$, and $[CDX] = 10$, find $[ADX]$.

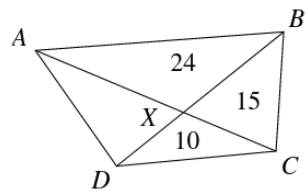

Figure 6.6: ICLT-2023-SM1-R1-L2-P8

Solution. ICLT-2023-SM1-R1-L2-P8 *Problem 4.12, Chapter 4, Introductory to Geometry.*

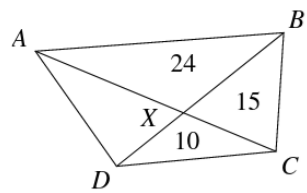

Figure 6.7: ICLT-2023-SM1-R1-L2-P8

Since $\triangle ABX$ and $\triangle CBX$ share an altitude from B, we have

$$\frac{AX}{CX} = \frac{[ABX]}{[CBX]} = \frac{8}{5}.$$

Turning to triangles $\triangle ADX$ and $\triangle CDX$, we have

$$\frac{[ADX]}{[CDX]} = \frac{AX}{CX} = \frac{8}{5}.$$

Therefore, $[ADX] = \frac{8}{5}[CDX] = \boxed{16.}$ □

Problem 6.8.9 (ICLT-2023-SM1-R1-L2-P9)**.** How many common divisors that 108, 144 and 72 have?

Solution. ICLT-2023-SM1-R1-L2-P9 *Problem 4.4.2, Chapter 4, Introductory to Number Theory.*

A common divisor of 108, 144, and 72 is a divisor of the greatest common divisor of 108, 144, and 72. Thus the number of common divisors of 108, 144, and 72 is the number of divisors of the greatest common divisor of 108, 144, and 72. Now, since

$$108 = 2^2 \cdot 3^3, 144 = 2^4 \cdot 3^2, 72 = 2^3 \cdot 3^2 \Rightarrow \gcd(108, 144, 72) = 2^2 \cdot 3^2 = 36$$

It is easy to see that 36 has $(2+1)(2+1) = \boxed{9}$ distinct divisors: $1, 2, 3, 4, 6, 9, 12, 18$, and 36. □

Remark. Jade Pham gave the almost perfect answer. She forgot that 1 is also a common divisor.

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Problem 6.8.10 (ICLT-2023-SM1-R1-L2-P10)**.** The teacher wants to divide 8 students into two groups: one group has 3 students and the other with 5 students, such that two of the 8 students, Lena and Jean, should not be in the same group. In how many ways can the teacher do that?

Solution. [First solution] ICLT-2023-SM1-R1-L2-P10 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

Let's first do the problem by complementary counting. If we have no restrictions on the groups, then we simply need to choose 3 of the 8 students to be in the smaller group, and the rest of the students will make up the larger group. There are $\binom{8}{3}$ ways to do this.

But we can't have Lena and Jean in the same group. So we have to subtract the number of ways that we can form the two groups with Lena and Jean in the same group.

Case 1: Lena and Jean are both in the smaller group. If they are both in the smaller group, then we have to choose 1 more student from the 6 remaining to complete the smaller group, and we can do this in $\binom{6}{1}$ ways.

Case 2: Lena and Jean are both in the larger group. If they are both in the larger group, then we have to choose 3 students from the 6 remaining to compose the larger group, and we can do this in $\binom{6}{3}$ ways.

So to get the number of ways to form groups such that Lena and Jean are both in the same group, we add the counts from our two cases, to get $\binom{6}{1} + \binom{6}{3}$.

But remember that these are the cases that we don't want, so to solve the problem, we subtract this count from the number of ways to form the two groups without restrictions. Thus, our answer is

$$\binom{8}{3} - \left(\binom{6}{1} + \binom{6}{3}\right) = 56 - (6 + 20) = \boxed{30.}$$

□

Solution. [Second solution] ICLT-2023-SM1-R1-L2-P10 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

We could solve this problem is by direct counting. There are two cases of possible groupings.

Case 1: Lena is in the smaller group, Jean is in the larger group To complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

Case 2: Jean is in the smaller group, Lena is in the larger group Again, to complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

So to count the total number of groupings, we add the counts from our two cases, to get $\binom{6}{2} + \binom{6}{2} = 15 + 15 = \boxed{30}$ as our final answer. □

Remark. Jade Pham gave the wrong answer without any solution.

6.9 Grading

Answers for multiple-choice problems.

Problem 1: C

Problem 2: A

Problem 3: E

Problem 4: B

Problem 5: D

Problem 6: D

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can establish the system of equations

$$\begin{cases} 5r + 3b &= 108 \\ \frac{1}{6}r + \frac{1}{2}b &= 10 \end{cases}$$

Problem 8: 2 points if the student can show that $\triangle ABX, \triangle CBX$ or $\triangle ADX, \triangle CDX$ have same height, different bases.

Problem 9: 2 points if the student can see that a common divisor of some numbers is a divisor of the greatest common divisor of them.

Problem 10: 2 points if the student can find the total number of ways to split $\binom{8}{3}$ if use complementary counting (solution 1), or can show casework if use direct counting (solution 2.)

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6.10 IC Test for Level 3

Problem 6.10.1 (ICLT-2023-SM1-R1-L3-P1)**.** There are red and blue balls in the Christmas shop. A red ball costs \$5 and a blue ball costs \$3. 6 red balls together weight 1 kg. 2 blue balls together weight 1 kg.

Marianna bought some red and blue balls. Altogether they cost \$108, and weight 10 kg. How many more blue balls than red balls did Marianna buy?

(A) 2 (B) 4 (C) 5 (D) 6 (E) 12

Solution. ICLT-2023-SM1-R1-L2-P7 *Problem 5.11, Chapter 5, Introductory to Algebra.*

Let r be the number of red balls and b be the number of blue balls that Marianna bought.

$$\begin{cases} 5r + 3b &= 108 \\ \frac{1}{6}r + \frac{1}{2}b &= 10 \end{cases} \Rightarrow \begin{cases} 5r + 3b &= 108 \\ r + 3b &= 60 \end{cases} \Rightarrow 4r = 108 - 60 = 48 \Rightarrow r = 12 \Rightarrow b = 16$$

The answer is $\boxed{B.}$ □

Remark. Luan Tran gave the correct answer.

Problem 6.10.2 (ICLT-2023-SM1-R1-L3-P2)**.** AC and BD meet at X as shown. Given $[ABX] = 24$, $[BCX] = 15$, and $[CDX] = 10$, find $[ABCD]$.

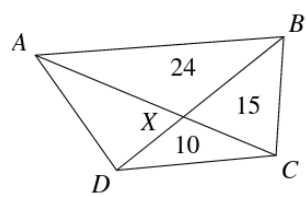

Figure 6.8: ICLT-2023-SM1-R1-L3-P2

(A) 61 (B) 63 (C) 65 (D) 69 (E) 73

Solution. ICLT-2023-SM1-R1-L3-P2 *Exercise 3.3.3, Chapter 3, Introductory to Counting & Probability.*

Problem 4.12, Chapter 4, Introductory to Geometry.

Since $\triangle ABX$ and $\triangle CBX$ share an altitude from B, we have $\frac{AX}{CX} = \frac{[ABX]}{[CBX]} = \frac{8}{5}$.

Turning to triangles $\triangle ADX$ and $\triangle CDX$, we have $\frac{[ADX]}{[CDX]} = \frac{AX}{CX} = \frac{8}{5}$.

Therefore, $[ADX] = \frac{8}{5}[CDX] = 16$, thus $[ABCD] = 24 + 15 + 10 + 16 = \boxed{65.}$ The answer is $\boxed{C.}$ □

Remark. Luan Tran gave the correct answer.

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Problem 6.10.3 (ICLT-2023-SM1-R1-L3-P3)**.** How many common divisors that 108, 144 and 72 have?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Solution. ICLT-2023-SM1-R1-L2-P9 *Problem 4.4.2, Chapter 4, Introductory to Number Theory.*

A common divisor of 108, 144, and 72 is a divisor of the greatest common divisor of 108, 144, and 72. Thus the number of common divisors of 108, 144, and 72 is the number of divisors of the greatest common divisor of 108, 144, and 72. Now, since

$$108 = 2^2 \cdot 3^3, 144 = 2^4 \cdot 3^2, 72 = 2^3 \cdot 3^2 \Rightarrow \gcd(108, 144, 72) = 2^2 \cdot 3^2 = 36$$

36 has $(2+1)(2+1) = 9$ distinct divisors: $1, 2, 3, 4, 6, 9, 12, 18$, and 36. The answer is $\boxed{E.}$ □

Remark. Luan Tran gave the correct answer.

Problem 6.10.4 (ICLT-2023-SM1-R1-L3-P4)**.** The teacher wants to divide 8 students into two groups: one group has 3 students and the other with 5 students, such that two of the 8 students, Lena and Jean, should not be in the same group. In how many ways can the teacher do that?

(A) 44 (B) 40 (C) 36 (D) 30 (E) 24

Solution. [First solution] ICLT-2023-SM1-R1-L3-P4 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

Let's first do the problem by complementary counting. If we have no restrictions on the groups, then we simply need to choose 3 of the 8 students to be in the smaller group, and the rest of the students will make up the larger group. There are $\binom{8}{3}$ ways to do this.

But we can't have Lena and Jean in the same group. So we have to subtract the number of ways that we can form the two groups with Lena and Jean in the same group.

Case 1: Lena and Jean are both in the smaller group. If they are both in the smaller group, then we have to choose 1 more student from the 6 remaining to complete the smaller group, and we can do this in $\binom{6}{1}$ ways.

Case 2: Lena and Jean are both in the larger group. If they are both in the larger group, then we have to choose 3 students from the 6 remaining to compose the larger group, and we can do this in $\binom{6}{3}$ ways.

So to get the number of ways to form groups such that Lena and Jean are both in the same group, we add the counts from our two cases, to get $\binom{6}{1} + \binom{6}{3}$.

But remember that these are the cases that we don't want, so to solve the problem, we subtract this count from the number of ways to form the two groups without restrictions. Thus, our answer is

$$\binom{8}{3} - \left(\binom{6}{1} + \binom{6}{3}\right) = 56 - (6 + 20) = \boxed{30.}$$

The answer is $\boxed{D.}$ □

Solution. [Second solution] ICLT-2023-SM1-R1-L3-P4 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

We could solve this problem is by direct counting. There are two cases of possible groupings.

Case 1: Lena is in the smaller group, Jean is in the larger group To complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

Case 2: Jean is in the smaller group, Lena is in the larger group Again, to complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

So to count the total number of groupings, we add the counts from our two cases, to get $\binom{6}{2} + \binom{6}{2} = 15 + 15 = \boxed{30}$ as our final answer. The answer is $\boxed{D.}$ □

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Problem 6.10.5 (ICLT-2023-SM1-R1-L3-P5)**.** What is the ratio of y to x if

$$\frac{7x-3y}{9x-2y}=\frac{4}{7}$$

(A) $13:18$ (B) $11:15$ (C) $9:11$ (D) $1:1$ (E) $1:2$

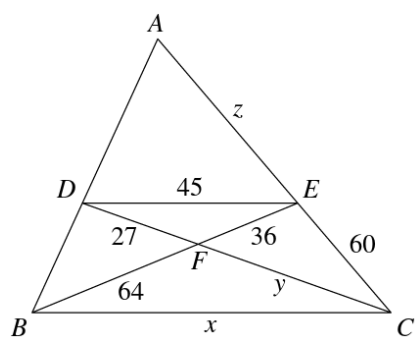
Solution. ICLT-2023-SM1-R1-L3-P5 *Problem 6.5, Chapter 6, Introductory to Algebra.*

$$\frac{7x-3y}{9x-2y}=\frac{4}{7}\Rightarrow 7(7x-3y)=4(9x-2y)\Rightarrow 49x-21y=36x-8y\Rightarrow 13x=13y\Rightarrow \frac{y}{x}=\frac{1}{1}$$

Thus, the ratio of y to x is $\boxed{1:1}$. The answer is $\boxed{D.}$ □

Remark. Luan Tran gave the correct answer.

Problem 6.10.6 (ICLT-2023-SM1-R1-L3-P6)**.** In the diagram below, $DE \parallel BC$, and the segments have the lengths as shown in the diagram, $x = BC$, $y = CF$, and $z = EA$.



Find $x+y+z$.

(A) $\frac{1430}{7}$ (B) 188 (C) $\frac{1436}{7}$ (D) 109 (E) $\frac{764}{7}$

Solution. ICLT-2023-SM1-R1-L3-P6 *Problem 5.12, Chapter 5, Introductory to Geometry.*

Since $DE \parallel BC$, we have $\triangle FBC \sim \triangle FED$ by AA Similarity. Therefore, we have

$$\frac{FC}{FD}=\frac{BC}{DE}=\frac{FB}{FE}=\frac{64}{36}=\frac{16}{9}\Rightarrow x=BC=\frac{16}{9}(DE)=80,\ y=FC=\frac{16}{9}(DF)=48.$$

Since $\triangle ADE \sim \triangle ABC$ by AA Similarity, we have

$$\frac{AE}{AC}=\frac{DE}{BC}=\frac{45}{80}=\frac{9}{16},\ AE=z,\ AC=AE+EC=z+60$$
$$\Rightarrow \frac{z}{z+60}=\frac{9}{16}\Rightarrow 16z=9z+540\Rightarrow z=\frac{540}{7}\Rightarrow x+y+z=80-48+\frac{540}{7}=\boxed{\frac{764}{7}.}$$

The answer is $\boxed{C.}$ □

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Problem 6.10.7 (ICLT-2023-SM1-R1-L3-P7)**.** What are the positive divisors of 540 having exactly 4 positive divisors?

For example 21 is a positive divisor of 42 *and 21 has exactly 4 positive divisors:* $1, 3, 7, 21$.

Solution. ICLT-2023-SM1-R1-L3-P7 *Problem 5.9, Chapter 5, Introductory to Number Theory.*

The problem requires us to find divisors of 540 with a particular property, so we begin by finding the prime factorization of 540 in order to learn more about its divisors:

$$540 = 2^2 \cdot 3^3 \cdot 5^1.$$

A divisor d of 540 has the form

$$d = 2^a \cdot 3^b \cdot 5^c, \text{ where } a = 0, 1 \text{ or } 2;\ b = 0, 1, 2 \text{ or } 3;\ b = 0, 1;\ c = 0 \text{ or } 1$$

We must now count possible combinations of a, b and c such that d has exactly 4 positive divisors. In other words, we are counting combinations of a, b and c such that

$$(a + 1)(b + 1)(c + 1) = 4.$$

The only ways to get 6 as the product of 3 positive integers are

$$6 = 1 \cdot 1 \cdot 4 = 1 \cdot 2 \cdot 2.$$

These products imply that (a, b, c) must include either two 0's and one 3 or else one 0, one 1, and one 1. Since none of a, b, or c is greater than 3, this leaves us with only four possibilities:

$$(0, 0, 3), (1, 1, 0), (1, 0, 1), (0, 1, 1) \Rightarrow 3^3 = 27, 2 \cdot 3 = 6, 2 \cdot 5 = 10, 3 \cdot 5 = 15.$$

It is easy to verify that each of $\boxed{27, 6, 10, 15}$ has exactly 4 divisors. □

Remark. Luan Tran solved this problem correctly.

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Problem 6.10.8 (ICLT-2023-SM1-R1-L3-P8)**.** Six parallel lines in a plane intersect a set of n parallel lines that go in another direction. The lines form a total of 150 parallelograms, many of which overlap each other. Find n.

Solution. ICLT-2023-SM1-R1-L3-P8 *Problem 6.4, Chapter 6, Introductory to Counting & Probability.*

The goal here is to find an expression for the number of parallelograms in terms of n. Then we'll set that expression equal to 150, and solve for n.

We start with a constructive approach, wondering how we can create a parallelogram by choosing some of our lines. We need to choose 2 lines from each set - this will give us the 4 sides of a parallelogram, as shown in the figure below.

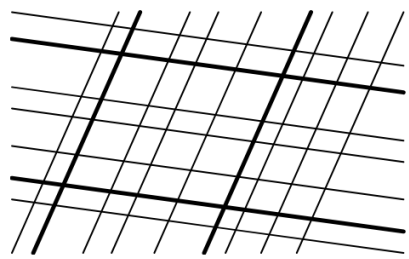

Figure 6.9: ICLT-2023-SM1-R1-L3-P8

There are 6 lines in one set of parallel lines, so the number of ways we can choose 2 lines from that set is $\binom{6}{2} = 15$. There are n lines in the other set of parallel lines, so the number of ways we can choose 2 lines from that set is $\binom{n}{2}$.

Hence there are $15\binom{n}{2}$ parallelograms. But we're told that there are 150 parallelograms, so we have

$$15\binom{n}{2} = 150 \Rightarrow \binom{n}{2} = \frac{150}{15} = 10 \Rightarrow \frac{n(n-1)}{2} = 10 \Rightarrow n = \boxed{5.}$$

□

Problem 6.10.9 (ICLT-2023-SM1-R1-L3-P9)**.** Nine people can mow a lawn in fifteen hours. How many more people are needed to mow the lawn in just five hours, assuming each person mows at the same rate?

Solution. ICLT-2023-SM1-R1-L3-P9 *Exercise 7.2.2, Chapter 7, Introductory to Algebra.*

The number of people mowing and the time required to mow are inversely proportional. Letting n be the number of people and t be the amount of time, we have $n \cdot t = (9)(15) = 135$ because 9 people can mow a lawn in 15 hours. If m people can mow the lawn in 5 hours, then we must have $m(5) = 135$, so $m = 27$. Therefore, we need to add $27 - 9 = \boxed{18}$ people to the job. □

> **Remark.** Luan Tran solved this problem correctly.

Problem 6.10.10 (ICLT-2023-SM1-R1-L3-P10)**.** Find the highest power of 43 that divides $41! + 42! + 43!$.

Solution. ICLT-2023-SM1-R1-L3-P10 *Problem 6.4, Chapter 6, Introductory to Number Theory.*

$41! + 42! + 43! = 41!(1 + 42 + 42 \cdot 43) = 41! \cdot 43^2$. The highest power of 43 that divides $41! + 42! + 43!$ is $\boxed{43^2.}$ □

6.11 Grading

Answers for multiple-choice problems.

Problem 1: B

Problem 2: C

Problem 3: E

Problem 4: D

Problem 5: D

Problem 6: C

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can factor 540 and find how to form a divisor that has 4 factors.

Problem 8: 2 points if the student can see that choosing two pairs of parallel lines will form a parallelogram.

Problem 9: 2 points if the student can find that the number of people mowing and the time required to mow are inversely proportional.

Problem 10: 2 points if the student can factor 5! out of the sum.

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6.12 IC Test for Level 6

Problem 6.12.1 (ICLT-2023-SM1-R1-L6-P1)**.** Suppose two numbers x and y are each chosen such that $0 < x < 1$ and $0 < y < 1$. What is the probability that $x + y > \frac{1}{2}$?

$(A)\ \frac{7}{8}$ $\quad (B)\ \frac{5}{8}$ $\quad (C)\ \frac{3}{8}$ $\quad (D)\ \frac{3}{4}$ $\quad (E)\ \frac{1}{2}$

Solution. ICLT-2023-SM1-R1-L6-P1 *Problem 10.5, Chapter 10, Introductory to Geometry.*

The "total outcomes" region is the region in the xy-plane with $0 < x < 1$ and $0 < y < 1$. This is the interior of the square with corners $(0, 0)$, $(0, 1)$, $(1, 1)$, and $(1, 0)$. The area of the square is 1.

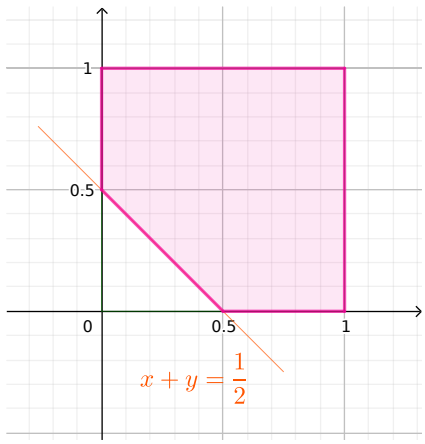

Figure 6.10: ICLT-2023-SM1-R1-L6-P1

Now we need to describe the region corresponding to successful outcomes. We need $x + y > \frac{1}{2}$, so this will be the region inside the square of which is above the line $x + y = \frac{1}{2}$. The area of the region is $1 - \frac{\frac{1}{2} \cdot \frac{1}{2}}{2} = \frac{7}{8}$. Hence, the probability is $\frac{\frac{7}{8}}{1} = \boxed{\frac{7}{8}.}$ The answer is $\boxed{A.}$ □

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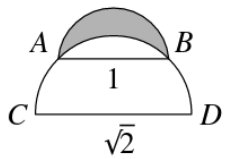
Problem 6.12.2 (ICLT-2023-SM1-R1-L6-P2)**.** The shaded portion of the figure is called a lune. Given that $AB = 1$, $CD = \sqrt{2}$, and that AB and CD are diameters of the respective semicircles shown, find the area of the lune.



$(A)\ \dfrac{3}{8}$ $\quad(B)\ \dfrac{1}{4}$ $\quad(C)\ \dfrac{\pi}{4}$ $\quad(D)\ \dfrac{\pi-1}{8}$ $\quad(E)\ \dfrac{3\pi-4}{16}$

Solution. ICLT-2023-SM1-R1-L6-P2 *Problem 8.11, Chapter 8, Introductory to Geometry.*

$$\begin{aligned}&\text{Area of lune} = \text{Area of small semicircle} - \text{Area of circular segment } AB\\ &\text{Area of circular segment } AB = \text{Area of sector } AOB - \text{Area of } \triangle AOB\\ \Rightarrow\ &\text{Area of lune} = \text{Area of small semicircle} - \text{Area of sector } AOB + \text{Area of } \triangle AOB\end{aligned}$$

Now $CD = \sqrt{2}$, so $OC = OA = OB = OD = \dfrac{\sqrt{2}}{2}$, so $AB^2 = AO^2 + BO^2 = 1$. Thus, The radius of the small semicircle is $\frac{1}{2}$, so its area is $\frac{1}{2} \cdot \pi \left(\dfrac{1}{2}\right)^2 = \dfrac{\pi}{8}$. Sector AOB is $\dfrac{90^\circ}{360^\circ} = \dfrac{1}{4}$ of a circle with radius $\dfrac{\sqrt{2}}{2}$, so its area is $\dfrac{1}{4} \cdot \pi \left(\dfrac{\sqrt{2}}{2}\right)^2 = \dfrac{\pi}{8}$. Finally, $[AOB] = \frac{1}{2} \cdot (AO)(OB) = \dfrac{1}{4}$, so we have

$$\text{Area of small semicircle} - \text{Area of sector } AOB + \text{Area of } \triangle AOB = \frac{\pi}{8} - \frac{\pi}{8} + \frac{1}{4} = \boxed{\frac{1}{4}.}$$

The answer is $\boxed{B.}$ □

Problem 6.12.3 (ICLT-2023-SM1-R1-L6-P3)**.** Write $\dfrac{5-i}{-1+3i}$ as a single complex number, where i is the imaginary unit $i = \sqrt{-1}$.

$(A)\ \dfrac{1}{5} - \dfrac{8}{5}i$ $\quad(B)\ -\dfrac{4}{5} + \dfrac{7}{5}$ $\quad(C)\ -\dfrac{4}{5} - \dfrac{7}{5}$ $\quad(D)\ \dfrac{4}{5}\dfrac{7}{5}$ $\quad(E)\ \dfrac{1}{5} + \dfrac{8}{5}$

Solution. ICLT-2023-SM1-R1-L6-P3 *Problem 12.10, Chapter 12, Introductory to Algebra.*

To remove $-1+3i$ from the denominator, we multiply it with its conjugate $-1-3i$,

$$\frac{5-i}{-1+3i} = \frac{(5-i)(-1-3i)}{(-1)^2-(3i)^2} = \frac{-5+i-15i+3i^2}{10} = \boxed{-\frac{4}{5} - \frac{7}{5}i.}$$

The answer is $\boxed{C.}$ □

Problem 6.12.4 (ICLT-2023-SM1-R1-L6-P4)**.** What is the unit digit of $3^{2022} - 2^{2022}$.

$(A)\ 9$ $\quad(B)\ 7$ $\quad(C)\ 5$ $\quad(D)\ 3$ $\quad(E)\ 1$

Solution. ICLT-2023-SM1-R1-L6-P4 *Problem 10.20, Chapter 10, Introductory to Number Theory.*

The units digit of powers of 2 and 3 both repeat in cycles four long, $2, 4, 8, 6$, and $3, 9, 7, 1$. Since $2022 \div 4$ has a remainder of 2, the units digit of $3^{2022} - 2^{2022}$ is the same as $3^2 - 2^2$, or $\boxed{5}$. The answer is $\boxed{C.}$ □

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Problem 6.12.5 (ICLT-2023-SM1-R1-L6-P5)**.** The numbers on a standard six-sided die are arranged such that numbers on opposite faces always add to 7. The die is rolled, and the product of the numbers appearing on the top and bottom faces of the die is calculated. What is the expected value of this product?

(A) 84 $\quad$ (B) 28 $\quad$ (C) 12 $\quad$ (D) $\frac{28}{3}$ $\quad$ (E) $\frac{14}{3}$

Solution. ICLT-2023-SM1-R1-L6-P5 *Problem 11.11, Chapter 11, Introductory to Counting & Probability.*

The products of opposite faces are $1 \times 6 = 6$, $2 \times 5 = 10$, and $3 \times 4 = 12$. Therefore, after rolling the die, the outcomes of $6, 10$, and 12 are equally likely. So the expected value is just the average of these outcomes, namely $\frac{6+10+12}{3} = \boxed{28}3$. The answer is $\boxed{D.}$ □

Problem 6.12.6 (ICLT-2023-SM1-R1-L6-P6)**.** x, y, and z are positive real numbers such that $2x > z > 3y > 0$ and

$$\begin{cases} (2x-z)^2 &= z^2 - 5 \\ (3y-z)^2 &= z^2 - 5 \end{cases}$$

Find the product xy.

(A) 1 $\quad$ (B) $\frac{1}{2}$ $\quad$ (C) $\frac{3}{4}$ $\quad$ (D) $\frac{4}{5}$ $\quad$ (E) $\frac{5}{6}$

Solution. ICLT-2023-SM1-R1-L6-P6 *Problem 11.2, Chapter 11, Introductory to Algebra.*

Since $2x > z > 3y > 0$, $\sqrt{(2x-z)^2} = 2x - z$, and $\sqrt{(3y-z)^2} = z - 3y$, thus

$$\begin{cases} 2x - z &= \sqrt{z^2-5} \\ z - 3y &= \sqrt{z^2-5} \end{cases} \Rightarrow \begin{cases} 2x &= z + \sqrt{z^2-5} \\ 3y &= z - \sqrt{z^2-5} \end{cases} \Rightarrow 6xy = z^2 - (z^2-5) = 5 \Rightarrow xy = \boxed{\frac{5}{6}.}$$

The answer is $\boxed{E.}$ □

Problem 6.12.7 (ICLT-2023-SM1-R1-L6-P7)**.** A bin has 6 white balls and k black balls in it, where k is an unknown positive integer. A ball is drawn at random from the bin. If a white ball is drawn, the player wins \$2, but if a black ball is drawn, the player loses \$1. If the expected loss for playing the game is $\frac{1}{2}$, then what is k?

Solution. ICLT-2023-SM1-R1-L6-P7 *Exercise 11.3.5, Chapter 11, Introductory to Counting & Probability.*

There are 6 white balls and $6 + k$ total balls, so the probability that a white ball is drawn is $\frac{6}{6+k}$. Similarly, the probability that a black ball is drawn is $\frac{k}{6+k}$. Since the expected loss is \$1, then

$$E = \frac{6}{6+k}(2) + \frac{k}{6+k}(-1) = -\frac{1}{2} \Rightarrow 2(12-k) = -(k+6) \Rightarrow k = \boxed{30.}$$

□

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Problem 6.12.8 (ICLT-2023-SM1-R1-L6-P8)**.** What is the unit digit of the sum

$$1! + 2! + 3! + \cdots + 2023!.$$

when written in base-8?

Solution. ICLT-2023-SM1-R1-L6-P8 *Exercise 10.4.4, Chapter 10, Introductory to Number Theory.*

We can ignore $n!$ when n is 8 or higher, because each is a multiple of 8 and will therefore have a base-8 units digit of 0. Let's find the remainder when divided by 8 of the sum of the terms $n!$ when $n < 8$, which will be the base-8 units digit of the original sum:

$$1! + 2! + 3! + 4! + 5! + 6! + 7! = 1 + 2 + 6 + 24 + 120 + 720 + 5040 = 5913 \equiv \boxed{1} \pmod 8.$$

□

Problem 6.12.9 (ICLT-2023-SM1-R1-L6-P9)**.** Solve the inequality

$$\frac{2}{x-3} - 3- + 2 > -1.$$

Solution. ICLT-2023-SM1-R1-L6-P9 *Problem 15.7, Chapter 15, Introductory to Algebra.*

$$\frac{2}{x-3} - 3x + 6 > -1 \Rightarrow \frac{2 + (-3x+7)(x-3)}{x-3} > 0 \Rightarrow \frac{3x^2 - 16x + 19}{x-3} < 0$$

Case 1: $x - 3 < 0$, or $x < 3$, then $3x^2 - 16x + 19 > 0$,

$$\Delta = 16^2 - 4 \cdot 3 \cdot 19 = 28 \Rightarrow x < \frac{16 - 4\sqrt{7}}{6} = \frac{8 - 2\sqrt{7}}{3} \approx 0.9 \text{ or } x > \frac{8 + 2\sqrt{7}}{3} \approx 4.4.3 \Rightarrow x < \frac{8 - 2\sqrt{7}}{3}.$$

Case 2: $x - 3 > 0$, or $x > 3$, then $3x^2 - 16x + 19 < 0$, from above,

$$\frac{8 - 2\sqrt{7}}{3} < x < \frac{8 + 2\sqrt{7}}{3} \Rightarrow 3 < x < \frac{8 + 2\sqrt{7}}{3}$$

Hence, $\boxed{x \in (-\infty, \frac{8-2\sqrt{7}}{3}) \cup (3, \frac{8+2\sqrt{7}}{3}).}$ □

Problem 6.12.10 (ICLT-2023-SM1-R1-L6-P10)**.** In isosceles triangle $\triangle ABC$, BC is the base. Given that $BC < AB$, prove that that $\angle BAC < 60°$.

Solution. ICLT-2023-SM1-R1-L6-P10 *Problem 10.27, Chapter 10, Introductory to Geometry.*

Since $BC < AB$, we have $\angle BAC = \angle A < \angle ACB = \angle C$. Since $\angle B$ and $\angle C$ are the base angles of isosceles $\triangle ABC$, we have $\angle B = \angle C$. From $\triangle ABC$, we have $\angle A + \angle B + \angle C = 180°$. We also have $\angle A < \angle C$ and $\angle B = \angle C$, so $\angle A + \angle B + \angle C > 3\angle A$. Hence, $180° > 3\angle A$, so $\boxed{\angle A < 60°.}$ □

6.13 Grading

Answers for multiple-choice problems.

Problem 1: A

Problem 2: B

Problem 3: C

Problem 4: C

Problem 5: D

Problem 6: E

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can establish the correct formula for the expected value.

Problem 8: 2 points if the student can find that $7! + \ldots + 2022!$ is divisible by 7, thus shall not influence the unit digit of the sum.

Problem 9: 2 points if the student can casework depeding on the signs of $(x - 3)$ and $3x^2 - 16x + 19$.

Problem 10: 2 points if the student can đeduce $\angle A > \angle C$ from $BC > AB$.

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Chapter 7

Session 5

7.1 One two three (I-Level)

In this session we discuss basic counting methods.

Example 7.1.1 (23-24-S5-I-P1)

There are five different shirts and three different skirts in the Girl's Stuffs store?

How many ways to buy a shirt and a skirt?

Solution. First, let choose a shirt, there are five ways to do so. To complete the set, we choose a skirt, there are three ways to do so. Hence, altogether there are $5 \times 3 = \boxed{15}$ ways. □

Let's extend the previous problem a bit more.

Example 7.1.2 (23-24-S5-I-P2)

Julie visits the Girl's Stuffs store mentioned in the 23-24-S5-I-P1 example, the store has four different pairs of shoes.

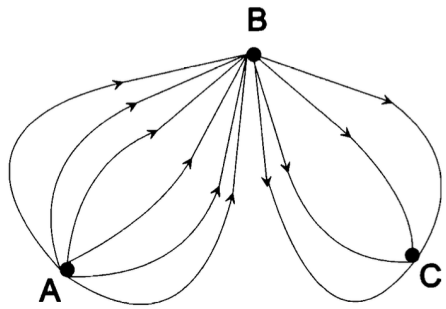
How many ways Julie can have a shirt, a skirt, and a pair of shoes?

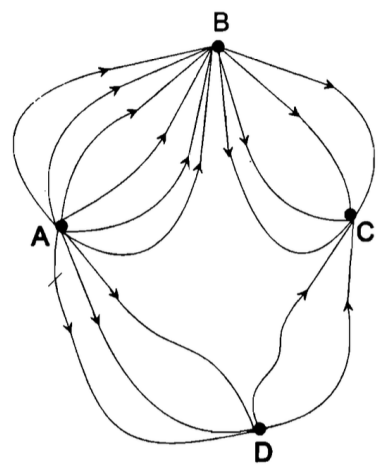
Solution. It is easy to see that there are four ways to choose a pair of shoes. Hence, by the result of the 23-24-S5-I-P1 example, she now has $15 \times 4 = \boxed{60}$ ways to choose a shirt, a skirt, and a pair of shoes. □

Example 7.1.3 (23-24-S5-I-P3)

There are three towns A, B, C in the Counting County. Six roads go from A to B, four roads go from B to C, see the figure on the left below. In how many way Jeremy can drive from A to C?

The Chief of the County announces the establishment of a new town, D with several roads connecting it to A and C, see the figure below on the right. Now, in how many way Jeremy can drive from A to C?





Solution. For the first question, there are $6 \times 4 = \boxed{24}$ ways to drive from A to C according to the left figure.

For the second question, there are two ways to drive from A to C, via B or via D, thus we consider two separate cases.

Case 1: Jeremy drive from A to B then to C, then he has 24 ways as discussed in the first question.

Case 2: Jeremy drive from A to D then to C, then similar to the first question he has $3 \times 2 = 6$ ways to do so.

Hence, altogether, he has $24 + 6 = \boxed{30}$ ways. □

Note that if you have *multiple independent choices,* like three ways to go from A to D, and two ways to drive from D to C, then you *multiply them* to get the number of choices.

If you can divide the counting into *multiple separate cases,* like going A-B-C and going A-D-C, then you *add the number of ways in each of the cases together* in order to get the total number of choices.

Now we visit other problems where counting requires a bit of understanding what choices are there.

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Example 7.1.4 (23-24-S5-I-P4)

Julie visits the Girl's Stuffs store. There are five different shirts, three different skirts, and four different hats. If Julie just want to buy two items of different types (shirt, shirt, hat), in how many ways can she do it?

Solution. What kind of choices can she make? She can buy a pair of (shirt, skirt), (skirt, hat), or (hat, shirt). Aha, so we have three separate cases to consider. It is now easy to see that she has $5 \times 3 = 15$, $3 \times 4 = 12$, and $4 \times 5 = 20$ ways to choose in each of the cases. Hence, the total number of ways is $15 + 12 + 20 = \boxed{47}$. □

Example 7.1.5 (23-24-S5-I-P5)

We call a positive integer *odd-looking* if it has only odd digits, e.g. 135, and *even-looking* if it has only even digits, e.g. 246.

Khoa counts the number of four-digit *odd-looking* numbers, what's he got? Knowing what's Khoa got, Giang can immediately tells the number of four-digit numbers, which are not *odd-looking*, how could she do that?

Now, Khoa challenges Giang to count the number of four-digit *even-looking* numbers, what can she tell Khoa?

Solution. For each digit in a four-digit number, Khoa has five choices: 1, 3, 5, 7, and 9. Thus, he got $5 \times 5 \times 5 \times 5 = 5^4 = \boxed{625}$ *odd-looking* four-digit numbers.

Giang is clever, she know that there are $9999 - 1000 + 1 = 9000$ four-digit number, so she just need to make a subtraction $9000 - 625 = \boxed{8375.}$

For Giang, the problem of counting four-digit *even-looking* numbers is a bit different from what Khoa has done. For the thousands digit, she has only four choices: 2, 4, 6, and 8. However, for any of the hundreds, tens, of unit digits, she has 5 choices: 0, 2, 4, 6, and 8. Hence, the number she's got is $4 \times 5 \times 5 \times 5 = \boxed{500.}$ □

Example 7.1.6 (23-24-S5-I-P6)

Nhat Minh toss a coin three times, how many sequence of heads (H) and tails (T) can he obtain?

Each square in the 2×2 grid can be coloured black or white. How many different ways can Minh colour them?

In how many ways can Minh fill a Super Easy Winner lotto card? In this lotto, Minh must predict the result of 13 Premier League games, indicating either a victory for one of two teams, or a draw.

Solution. For each toss, he has two different outcomes: H or T. Thus there are $2 \times 2 \times 2 = 2^3 = \boxed{8}$ different sequences. It is also easy to see what those sequences are

$$HHH, HHT, HTH, THH, HTT, THT, TTH, TTT.$$

Applying the same technique, we can easily see that there are 2 ways to colour a square, and colouring each square is an independent event from colouring another, thus, there are $2^4 = \boxed{16}$ different colourings.

For the last question, for each game between two teams, say A and B, Minh can predict one of the three outcomes: A wins, B wins, or A and B draw. Thus the number of ways to fill the lotto card is 3^{13}.

On the side note this is quite a big number, contradicting the name of Super Easy Winner (a marketing plot, perhaps?) □

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Example 7.1.7 (23-24-S5-I-P7)

Giang's younger sister learns to play with lego. She has a number of letters, A, B and C. Being very creative, she wants to invent a new language, each word has no more than four letters, choosing from those letter legos. How many words does this new language contain?

Proof. It's a casework! We need to count the words that has one, two, three, or four letters, and then add up the results. It is not difficult to find that there are

$$3^1 + 3^2 + 3^3 + 3^4 = \boxed{120.}$$

□

Example 7.1.8 (23-24-S5-I-P8)

Nhat Minh and Tue Minh with another nine MCC students represent the club football team. In how many ways the team can choose a captain and a vice captain?

In how many ways the team can sew a three-coloured flag with three horizontal strips of equal height if they have pieces of fabric of six colours? Note that they of course can make a distinction between the top and the bottom of the flag.

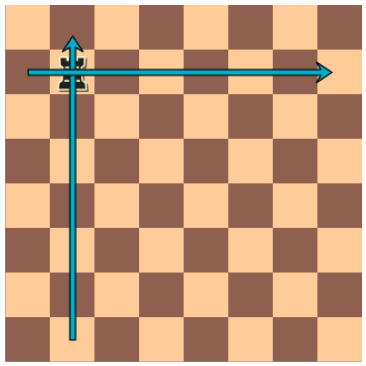
Solution. The first question is simple, there should be 11 ways to choose the captain. If the captain is already chosen, then there are 10 ways to choose the vice captain, so $11 \times 10 = \boxed{110}$ ways.

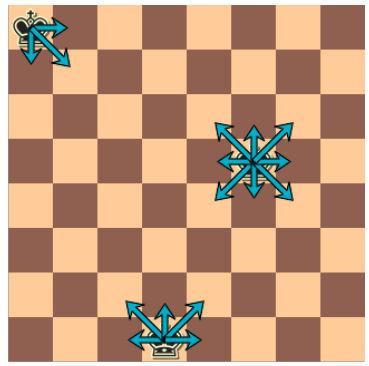
For the second question, since we have pieces of fabric of six colours, thus there are six ways to choose the first horizontal strip. Now, the colour of the first strip is chosen, thus the colour for the second strip must be different, thus there are only five ways to choose the second strip. Similarly there are four ways to choose the last strip. Altogether, there are $6 \times 5 \times 4 = \boxed{120}$ ways. □

Example 7.1.9 (23-24-S5-I-P9)

In how many ways can Huy put a white and a black rook on the chessboard such that they do not attack each other?

In how many ways can Huy put a white and a black king on the chessboard such that they do not attack each other?





Solution. Huy can put a black rook any where on the chessboard. The rook attacks any piece on the row and column of its square, including its square. There are $8 + 8 - 1 = 15$ squares under attack. The remaining $64 - 15 = \boxed{49}$ squares are where Huy can put the white rook. Thus there are $64 \times 49 = \boxed{3136}$ different ways. Note that each choice of two positions is unique because we cannot swap the rooks since they are in different colours.

For the kings, note that the number of squares under attack of a king depends on where it stand. Let's split the problem into three cases.

Case 1: The black king stands at a corner, there are 4 ways to do so. Then there are exactly four squares under attack, thus there are $4 \times 60 = 240$ ways to set them up.

Case 2: The black king stands on an edge of the board but not at a corner, there are $6 \times 4 = 24$ ways to do so. For each choice there are exactly six squares under attack, thus $24 \times 58 = 1392$ ways to put them.

Case 3: The black king does not stand on an edge of the board, there are $6 \times 6 = 36$ ways to do so. For each choice there are exactly nine squares under attack, thus $36 \times 55 = 1392$ ways to put them.

Altogether, there are $240 + 1392 + 1392 = \boxed{3612}$ ways to put both kings on the chessboard. Similar to the first question, since the kings have different colours, each of these settings is unique (we can not swap the kings), thus there is no duplicate. □

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In this last example, we discuss permutations.

Example 7.1.10 (23-24-S5-I-P10)

How many ways are there for Chi-Mai to pay four balls, coloured red, black, blue, and green in a row?

How many different words can Chi-Mai have if she rearranges the letters in the word BEAUTY?

How many different words can Chi-Mai have if she rearranges the letters in the word FOOT?

How many different words can Chi-Mai have if she rearranges the letters in the word HAVANA?

Solution. The first place in the row can be occupied by any of the given balls. The second can be occupied by any of the three remaining balls, et cetera. Finally, we have the answer $4 \times 3 \times 2 \times 1 = 4! = \boxed{24.}$

Analogously we can prove that n different objects can be laid out in a row in

$$n \cdot (n-1) \cdot (n-2) \cdot 2 \cdot 1 = n!$$

ways, that is the number of permutations of n objects is $n!$.

Similar to the first question, since all the letters are different, there are 6! words.

For the third question, This word contains two letters O, and all the other letters are different. Let us temporarily think of these letters O as two different letters O_1 and O_2. Under this assumption we have $4! = 24$ different words. However, any two words which can be obtained from each other just by transposing the letters O_1 and O_2 are, in fact, identical, for example

$$FO_1TO_2,\ FO_2TO_1$$

Thus, Chi-Mai's 24 words split into pairs of identical words. This means that the answer is $frac242 = \boxed{12.}$

Using the same technique from the third question, there are 3! ways to permute three copies of the letter A: A_1, A_2, A_3. Hence the number of different words is: $\frac{6!}{3!} = 6 \cdot 5 \cdot 4 = \boxed{120.}$ □

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7.2 Problems (I-Level)

Problem 7.2.1 (23-24-S5-I-P11)**.** (*10 points*) How many different words can Chi-Mai have if she rearranges the letters in the word MATHEMATICAL?

Solution (Official solution - similar to submissions from Chi-Mai Pham, Giang Nguyen, Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham) — We have two letters M, three letters A, two letters T, thus, similar to the problem 23-24-S5-I-P10, the total number of different words is

$$\frac{12!}{2!3!2!} = \boxed{19\ 958\ 400.}$$

Remark. All students who submitted solutions have achieved a perfect solution.

Problem 7.2.2 (23-24-S5-I-P12)**.** Each of the problems can be solved separately.

- (*5 points*) There are 19 islands in the Scatter archipelago, and every pair of them is connected by an air route. How many air routes are there?
- (*5 points*) Each of two novice collectors has 10 stamps and 5 postcards. We call an exchange fair if they exchange a stamp for a stamp or a postcard for a postcard. How many ways are there to carry out one fair exchange between these two collectors?

Solution (Official solution - similar to submissions from Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham) — Every route connects two islands. We can choose any of the 19 islands in the archipelago (say, island A) as the beginning of a route, and we have 18 remaining islands to choose the end of a route (say, island B) from. Multiplying, we have $19 \times 18 = 342$. However, this calculation counted every route AB twice: when A was chosen as the beginning of the route,and when B was chosen as the beginning. Hence, the number of routes is $\frac{1}{2}342 = \boxed{171.}$

For the second question, there are $10 \times 10 = 100$ ways carry out a fair exchange of a stamp for a stamp and $5 \times 5 = 25$ ways of a postcard for a postcard. Thus the answer is $\boxed{125.}$

Remark. Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham have submitted the correct solutions.

Chi-Mai Pham did not correctly solved the first problem, Giang Nguyen did not correctly solved the second question.

Problem 7.2.3 (23-24-S5-I-P13)**.** Each of the problems can be solved separately.

- (*2 points*) How many diagonals are there in a pentagon?
- (*8 points*) How many diagonals are there a convex $n-$gon?

Solution (Official solution - similar to submissions from Chi-Mai Pham, Jeremy Nguyen, and Khoa Anh Vu) — For the first question, by drawing a simple diagram, it is easy to count the number of diagonals in a pentagon. It is $\boxed{5.}$

Consider a $n-$gon. For any vertex, a diagonal can be constructed if it is connected with any of the other vertices, except itself and its two neighbours. Thus, the number of ways to connect a vertex with its non-neighbours is $n \times (n-3)$. Similar to the first question of the problem 23-24-S5-I-P12, a diagonal AB drawn this way is counted twice, therefore the number of diagonals is $\boxed{\frac{n(n-3)}{2}.}$

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Solution (Solution 2 from Giang Nguyen) — Consider a $n-$gon. We count the number of ways a vertex can be connected to another (different) vertex, $n(n-1)$, then divided by 2 because of the over-counting (AB is same as BA), then subtracting the number of sides (which obviously not diagonals)

$$\frac{n(n-1)}{2} - n = \boxed{\frac{n(n-3)}{2}.}$$

Solution (Solution 3 from Julie Cao) — Consider a $n-$gon.

We count the number of ways the first vertex can be connected to all other vertices, except its neighbours, which is $(n-1)-(2) = n-3$. For the second vertex, count the number of ways the first vertex can be connected to all other vertices, except its neighbours, which is then $(n-1)-(2) = n-3$.

Now, for the third vertex, count the number of ways the third vertex can be connected to all other vertices, except *the first vertex,* its neighbours, which is then $(n-1)-(1+2) = n-4$, For the fourth vertex, count the number of ways the fourth vertex can be connected to all other vertices, except *the first* and *the second vertex,* its neighbours, which is then $(n-1)-(2+2) = n-5$, For the fifth vertex, count the number of ways the fifth vertex can be connected to all other vertices, except *the first, the second*, and *the third vertex,* its neighbours, which is then $(n-1)-(3+2) = n-6$, and so on.

Then the number of diagonals is

$$(n-3)+(n-3)+(n-4)+(n-5)+\cdots+1+0+0 = (n-3)+\frac{(n-3)(n-2)}{2} = \boxed{\frac{n(n-3)}{2}.}$$

Remark. Chi-Mai Pham, Giang Nguyen, Jeremy Nguyen, Julie Cao and Khoa Anh Vu have submitted the correct solutions.

Giang Nguyen showed a beautiful solution based on another technique when eliminating over-counting.

Julie Cao showed an amazing solution of direct counting without over-counting.

Problem 7.2.4 (23-24-S5-I-P14)**.** Each of the problems can be solved separately.

- (*5 points*) A *necklace* is a circular string with several beads on it. It is allowed to rotate a necklace but not to turn it over. How many different necklaces can be made using 11 different beads?
- (*5 points*) Assume now that it is allowed to turn a necklace over. How many necklaces can be made using 11 different beads?

Solution (Official solution - similar to submissions from Giang Nguyen, Jeremy Nguyen, Minh Nhat Pham) — Let us first assume that it is prohibited to rotate the necklace. The way the beads can be selected one by one and put on a row is exactly the ways we can permute the beads. Then it is clear that we have 11! different necklaces.

However, by connecting the last bead to the first one, any bead can be the first one if we rotate the necklace by one bead (clockwise or counter-clockwise). This means that any arrangement of beads must be considered identical to those 10 that can be obtained from it by rotation. Thus, the number of different necklaces can be made using 11 different beads is $\frac{11!}{11} = \boxed{10!.}$

By turning the necklace over we divides the number of necklaces by 2, thus we have $\boxed{5 \cdot 9!}$.

Remark. Giang Nguyen, Jeremy Nguyen, Minh Nhat Pham have submitted the correct solutions.

Chi-Mai Pham did make an attempt for the first question. Julie Cao reasoning is not correct.

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Problem 7.2.5 (23-24-S5-I-P15). (*10 points*) How many six-digit numbers have at least one even digit?

Remark. The main idea in this solution was to use the method of complements; that is, counting (or, considering) the "undesired" objects instead of those "desired".

Solution (Official solution - similar to submissions from Giang Nguyen, Jeremy Nguyen, Khoa Anh Vu, and Minh Nhat Pham) — Instead of counting the numbers with at least one even digit, let us find the number of six-digit numbers that do not possess this property. Since these are exactly the numbers with all their digits odd, there are $5^6 = 15625$ of them (see 23-24-S5-I-P5). Since there are $999999 - 99999 = 900000$ six-digit numbers in all, we conclude that the number of six-digit numbers with at least one even digit is $900000 - 15625 = \boxed{884\ 375.}$

Remark. Giang Nguyen, Jeremy Nguyen, Khoa Anh Vu, and Minh Nhat Pham have submitted the correct solutions.

Chi-Mai Pham, Julie Cao made incorrect calculations.

Problem 7.2.6 (23-24-S5-I-P16). (*10 points*) How many six-digit numbers have at least two equal digits?

Note that 123455 is such a number, and 345728 is not.

Solution (Official solution - similar to submissions from Jeremy Nguyen, Khoa Anh Vu, and Minh Nhat Pham) — Let first count the number of six-digit numbers whose digits are all distinct. There are 9 ways to choose the first digit from $0, 1, 2, \ldots, 9$, note that 0 cannot be chosen. 9 ways the second digit from the remaining 9 digits from $1, 2, \ldots, 9$; 8 ways the third digit from the remaining 8 digits from $1, 2, \ldots, 9$; and so on. Thus the number of ways is $9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136080$.

Thus the number of six-digit numbers have at least two equal digits is $900000 - 136080 = \boxed{763\ 920.}$

Remark. Jeremy Nguyen, Khoa Anh Vu, and Minh Nhat Pham have submitted the correct solutions.

Jeremy Nguyen, unfortunately did not complete the calculation. Julie Cao made incorrect calculation.

Problem 7.2.7 (23-24-S5-I-P17). (*10 points*) There are five books on a shelf. How many ways are there to arrange some (or all) of them in a stack? The stack may consist of a single book.

Solution (Official solution - similar to submissions from Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham) — Consider five cases, depending on how many books the stack consists of.

If there is only one book, then it can be chosen in 5 ways.

A stack of two books can be chosen in $5 times 4$ ways, since the number of ways to choose the second book in the stack is 4.

Similarly, we can calculate the numbers of ways to arrange 3, 4, and 5 books in the stack.

The answer is

$$5 + 5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1 = \boxed{325.}$$

Remark. Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham have submitted the correct solutions.

Chi-Mai Pham, very unfortunately, made a mistake when add the numbers to get the final sum.

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Problem 7.2.8 (23-24-S5-I-P18)**.** (*10 points*) How many ways are there to choose four cards of different suits and different values from a deck of 52 cards?

Solution (Official solution - similar to submissions from Giang Nguyen, Julie Cao, Khoa Anh Vu) — We must have one card of each suit, and the spade can be chosen in 13 ways. The club cannot be of the same value as the spade, and therefore there are only 12 ways to choose it. The number of choices for the diamond is 11, and for the heart is 10. The order in which we have named the suits does not affect the analysis. The answer is $13 \times 12 \times 11 \times 10 = \boxed{17\ 160.}$

Solution (Solution 2 from Jeremy Nguyen) — There are 52 ways to choose the first card. For the second card, we cannot chose from the suit of the first one, neither the cards with the same value, thus limiting to only $3 * (13 - 1) = 36$ ways. For the third card, we can chose from two different suits after remove two cards of chosen values from them, thus $2 * (13 - 2) = 22$ ways. The last one is obvious, there are $13 - 3 = 10$ ways.

Now, since a choice of 4 cards can be permuted, thus we have to divide the result above by 4!, of

$$\frac{52 \cdot 36 \cdot 22 \cdot 10}{4!} = \boxed{17\ 160.}$$

Remark. Giang Nguyen, Jeremy Nguyen, Julie Cao, Khoa Anh Vu have submitted the correct solutions.

Chi-Mai Pham and Minh Nhat Pham did not realized that the chosen cards can be permuted.

Problem 7.2.9 (23-24-S5-I-P19)**.** Each of the problems can be solved separately.

- (*5 points*) How many ways are there to put two knights on a chessboard so that they do not attack each other?
- (*5 points*) How many ways are there to put eight rooks on a chessboard so that they do not attack each other?

Solution (Official solution for question 1) — For the first question, note that when we have two knights threatening each other, it actually forms either a 2×3 or 3×2 board. And for each of 2×3 and 3×2 boards, there are 2 ways of placing two knights so that they threaten each other. So, what we should do is to count how many 2×3 and 3×2 boards on 8×8 board.

By moving the 2×3 on the 8×8 board from left to right, from top bottom we can see that there are $(8 - 1)(8 - 2)$ ways to do so. Similarly there are $(8 - 2)(8 - 1)$ ways for the 3×2 board. Thus the number of ways to put two knights on a chessboard so that they threaten each other is $4 \times 7 \times 6$. The total number of ways to put any two knights on the chessboard is $\frac{1}{2}(64 \times 63)$. Therefore the number of ways to put two knights on a chessboard so that they do not attack each other is

$$\frac{1}{2}(64 \times 63) - 4 \times 7 \times 6 = \boxed{1848.}$$

Solution (Official solution for question 2 - similar to submissions from Giang Nguyen, Jeremy Nguyen, Minh Nhat Pham) — For the second question, exactly one of the rooks must be in each row. Whether some of the rooks attack each other depends only on the choice of the columns the rooks are in. Since the numbers of the columns belong to the set of natural numbers 1 through 8, and two rooks can attack each other if and only if they stand in the same column, we have the familiar problem about the number of ways to arrange eight objects in a row. The answer is $8! = \boxed{40320.}$

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

Solution (Solution 2 - similar to submissions from Giang Nguyen, Jeremy Nguyen, and Minh Nhat Pham) — Let's casework based on the position of a knight. The board above shows the board where each of the squares containing the number of accessible squares from that square by a knight.

So what we need to do is to add them up, divide by 2! because the two knights can swap their places, and then subtract the result from the total number of ways to put two knights on the board:

$$2 \cdot 4 + 3 \cdot 8 + 4 \cdot 20 + 6 \cdot 16 + 8 \cdot 16 = 336 \Rightarrow \frac{1}{2}(64 \times 63) - \frac{1}{2}336 = \boxed{1848.}$$

Remark. Jeremy Nguyen has submitted the correct solution for the first question.

Giang Nguyen, for the first question, very brilliantly showed the board with the number of accessible squares from that square by a knight, was so unfortunately because she did not recognized that the positions of the two knights can be swapped, thus not dividing the result by 2!.

Minh Nhat Pham, too, forgot the permutation of the knights.

Giang Nguyen, Jeremy Nguyen, Minh Nhat Pham have submitted the correct solution for the second question.

Julie Cao made a complicated but incorrect computations.

Problem 7.2.10 (23-24-S5-I-P20). (*10 points*) There are three rooms in a dormitory: one single, one double, and one for four students. How many ways are there to house seven students in these rooms?

Solution (Official solution - similar to submissions from Giang Nguyen, Jeremy Nguyen, and Minh Nhat Pham) — Note that distributing the students is equivalent to arranging them in a row, since after doing that the first student can be sent to live in a single room, the next two to the double, and the remaining four to the room for four students.

However, each of the distributions can be obtained from several arrangements. Indeed, we can permute the students within the pair and within the quadruple (we can do this with the first student too, though nobody will notice our efforts).

Since there are 2! and 4! possible permutations of the pair and the quadruple respectively, we must divide the number of these arrangements (which is equal to 7!) by the product of 2! and 4!. Thus, the answer is $\frac{7!}{1!2!4!} = \boxed{105.}$

Remark. Giang Nguyen, Jeremy Nguyen, and Minh Nhat Pham have submitted the correct solution for the problem.

Chi-Mai Pham made some good attempt. Khoa Anh Vu did not pay attention to the different settings of the rooms.

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7.3 The Induction Principle (A-Level)

In this session we discuss the Induction Principle.

Definition (Induction Principle)**.** Let a be an integer, and let $P(n)$ be a statement (or proposition) about n for each integer $n \geq a$. The **principle of induction** is a way of proving that $P(n)$ is true for all integers $n \geq a$ in two steps:

1. *The Base case*: Prove that $P(a)$ is true.
2. *The Inductive step*: Assume that $P(k)$ is true for some integer $k \geq a$, and use this to prove that $P(k+1)$ is true.

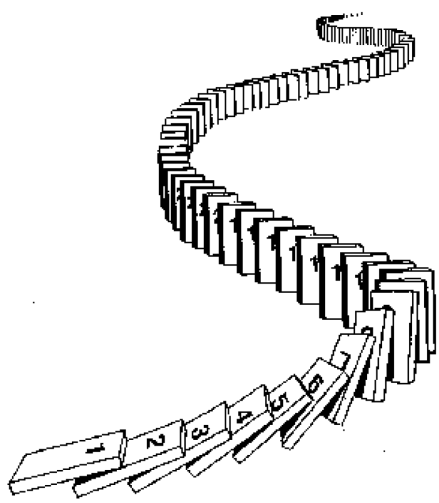
Then we may conclude that $P(n)$ is true for all integers $n \geq a$.

Our reasoning with the wave of falling dominoes shows that an inductive step is but a shortened form of the chain of theorems shown in the figure below;

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \cdots \Rightarrow P(k) \Rightarrow P(k+1) \Rightarrow \cdots$$

We will call theorems in this chain *steps*, and the process of their successive proof *the process of induction*. This process can be visually represented as a wave of proofs, running from statement to statement along a chain of theorems.

Below is how you can visualize the wave of proofs.



The domino effect is the chain reaction consisting of a row of falling dominoes. The dominoes are vertical and close enough to one another. One pushes the first domino of the row, and this falls onto the second domino, which falls onto the third domino and so on. In the end an infinitely long row has fallen!

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Example 7.3.1 (23-24-S5-A-P1)
Show that for all $n \geq 2$,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = \frac{(n-1)n(n+1)}{3}.$$

Solution. Our hypothesis is that for all $n \geq 2$,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = \frac{(n-1)n(n+1)}{3}.$$

For the base case $n = 2$, $1 \cdot 2 = 2 = \frac{1 \cdot 2 \cdot 3}{3}$, thus the hypothesis is true.

Now, for the Inductive step, let's assume that the hypothesis is true for n, or

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = \frac{(n-1)n(n+1)}{3} \quad (*)$$

We shall prove that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3} \quad (**)$$

Using the assumption (*) that the hypothesis is true for n, the left side of (**) can be written as

$$\frac{(n-1)n(n+1)}{3} + n \cdot (n+1) = n(n+1)\left(\frac{n-1}{3} + 1\right) = \frac{n(n+1)(n+2)}{3}.$$

Thus, the hypothesis is true for $n+1$, therefore it is true for all $n \geq 2$. □

Example 7.3.2 (23-24-S5-A-P2)
Show that for all $n \geq 2$,

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

Solution. Our hypothesis is that for all $n \geq 2$,

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

For the base case $n = 2$,

$$\left(1 - \frac{1}{4}\right) = \frac{3}{4} = \frac{2+1}{2 \cdot 2}, \text{ thus the hypothesis is true.}$$

Now, for the Inductive step, let's assume that the hypothesis is true for n, or

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n} \quad (*)$$

We shall prove that

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+2}{2(n+1)} \quad (**)$$

Using the assumption (*) that the hypothesis is true for n, the left side of (**) can be written as

$$\frac{n+1}{2n}\cdot\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+1}{2n}\frac{(n+1)^2-1}{(n+1)^2}=\frac{n+1}{2n}\frac{n(n+2)}{(n+1)^2}=\frac{n+2}{2(n+1)}.$$

Thus, the hypothesis is true for $n+1$, therefore it is true for all $n \geq 2$. □

Example 7.3.3 (23-24-S5-A-P3)
Show that for all $n \geq 1$,

$$n^3+(n+1)^3+(n+2)^3 \text{ is divisible by } 9.$$

Solution. Our hypothesis is that for all $n \geq 1$,

$$n^3+(n+1)^3+(n+2)^3 \text{ is divisible by } 9.$$

For the base case $n=1$,

$$1^3+2^3+3^3=9+27 \text{ is divisible by } 9.$$

Now, for the Inductive step, let's assume that the hypothesis is true for n, or

$$n^3+(n+1)^3+(n+2)^3 \text{ is divisible by } 9 \quad (*)$$

We shall prove that

$$(n+1)^3+(n+2)^3+(n+3)^3 \text{ is divisible by } 9 \quad (**)$$

Using the assumption (*) that the hypothesis is true for n, the left side of (**) can be written as

$$(n+1)^3+(n+2)^3+(n+3)^3=n^3+(n+1)^3+(n+2)^3+(n+3)^3-n^3.$$

Since

$$(n+3)^3-n^3=((n+3)-n)\left((n+3)^2+(n+3)n+n^2\right)=3(3n^2+9n+9) \text{ is divisible by } 9.$$

Thus, the hypothesis is true for $n+1$, therefore it is true for all $n \geq 1$. □

Example 7.3.4 (23-24-S5-A-P4)
Show that for all $n \geq 1$,

$$11^{n+2}+12^{2n+1} \text{ is divisible by } 133.$$

Solution. Our hypothesis is that for all $n \geq 1$,

$$11^{n+2}+12^{2n+1} \text{ is divisible by } 133.$$

For the base case $n=1$,

$$11^3+12^3=(11+12)(11^2-11\cdot 12+12^2)=23\cdot 133 \text{ is divisible by } 133.$$

Now, for the Inductive step, let's assume that the hypothesis is true for n, or

$$11^{n+2}+12^{2n+1} \text{ is divisible by } 133 \quad (*)$$

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We shall prove that

$$11^{n+3} + 12^{2n+3} \text{ is divisible by } 133 \quad (**)$$

Note that

$$11^{n+3} + 12^{2n+3} - 11(11^{n+2} + 12^{2n+1}) = 12^{2n+1}(12^2 - 11) = 133 \cdot 12^{2n+1} \text{ is divisible by } 133.$$

By (*), $11^{n+3} + 12^{2n+3}$ is divisible by 133, thus the hypothesis is true for $n+1$, therefore it is true for all $n \geq 1$. □

Example 7.3.5 (23-24-S5-A-P5)
Show that for all $n \geq 2$,

$$2^n > 1 + n\sqrt{2^{n-1}}.$$

Solution. Our hypothesis is that for all $n \geq 2$,

$$(2^n - 1)^2 > n^2 2^{n-1}.$$

For the base case $n = 2$,

$$(2^2 - 1) = 3^2 = 9 > 2^2 \cdot 2^1 = 8.$$

Now, for the Inductive step, let's assume that the hypothesis is true for n, or

$$(2^n - 1)^2 > n^2 2^{n-1} \quad (*)$$

We shall prove that

$$(2^{n+1} - 1)^2 > (n+1)^2 2^n \quad (**)$$

By the assumption (*),

$$(2^{n+1} - 1)^2 = (2(2^n) - 1)^2 = 4(2^n - 1)^2 + 4(2^n) + 3 > 4n^2 2^{n-1} + 42^n + 3$$

Thus we need to show that

$$4n^2 2^{n-1} + 42^n + 3 > (n+1)^2 2^n = n^2 2^n + (2n)2^n + 2^n \Leftrightarrow n^2 2^n + 3(2^n) + 3 > (2n)2^n$$

Since $n^2 > 2n$, the last inequality is obvious. Thus the hypothesis is true for $n+1$, therefore it is true for all $n \geq 2$. □

Example 7.3.6 (23-24-S5-A-P6)
Show that for all $n \geq 1$,

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \leq \frac{1}{\sqrt{2n+1}}.$$

Solution. Our hypothesis is that for all $n \geq 1$,

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \leq \frac{1}{\sqrt{2n+1}}.$$

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For the base case $n = 2$, it is easy to verify that

$$\frac{1}{2} < \frac{1}{\sqrt{3}}.$$

Now, for the Inductive step, let's assume that the hypothesis is true for n, or

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \leq \frac{1}{\sqrt{2n+1}}. \quad (*)$$

We shall prove that

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} \leq \frac{1}{\sqrt{2n+3}}. \quad (**)$$

By the assumption (*),

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{2n+1}} \cdot \frac{2n+1}{2n+2} = \frac{\sqrt{2n+1}}{2n+2}$$

Since

$$\frac{\sqrt{2n+1}}{2n+2} < \frac{1}{\sqrt{2n+3}} \Leftrightarrow (2n+1)(2n+3) < (2n+2)^2 \Leftrightarrow 4n^2 + 8n + 3 < 4n^2 + 8n + 4, \text{ which is true.}$$

Thus the hypothesis is true for $n+1$, therefore it is true for all $n \geq 2$. □

Example 7.3.7 (23-24-S5-A-P7)

Prove that every positive integer can be represented as a sum of several distinct powers of 2.

Solution. Our hypothesis is that every positive integer can be represented as a sum of several distinct powers of 2.

It is easy to verify the base cases $n = 1$ and $n = 2$.

Now, for the Inductive step, let's assume that the hypothesis is true for n or let's assume that every positive integer less than or equal to n can be represented as a sum of several distinct powers of 2.

We shall prove that $n+1$ can be represented as a sum of several distinct powers of 2.

Now, for $n+1 \geq 4$, there exists an positive integer m such that

$$2^m \leq (n+1) < 2^{m+1}.$$

If $n+1 = 2^m$, then we are done, if not then $n+1 = 2^m + (n+1-2^m)$, where $n+1-2^m < n$ and can be represented as a sum of several distinct powers of 2. It is easy to see that any power of 2 in the sum representing $n+1-2^m$ is less than 2^m, otherwise $n+1 > 2^{m+1}$.

Thus the hypothesis is true for $n+1$, therefore it is true for all $n \geq 2$. □

Example 7.3.8 (23-24-S5-A-P8)

There are $n \geq 1$ real numbers with non-negative sum written on a circle. Prove that one can enumerate them $a_1, a_2, \ldots, a_n$ such that they are consecutive on the circle and

$$a_1 \geq 0, a_1 + a_2 \geq 0, \ldots, a_1 + a_2 + \cdots + a_{n-1} \geq 0, a_1 + a_2 + \cdots + a_{n-1} \geq 0.$$

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Solution. Our hypothesis is based on n.

It is easy to verify the base cases when $n = 1$ or we have only one number.

For the Inductive step, let's assume that the hypothesis is true for $n - 1$. We shall prove for n.

As the sum of these numbers are non-negative, there are non-negative numbers. If all of them are non-negative, we can chose any number to be a_1 and then enumerate the rest clockwise, and we have the desired inequalities.

Now, let's assume that there exists $a_n < 0$, then by applying the hypothesis for

$$a_1, a_2, \ldots, a_{n-2}, a_{n-1} + a_n \text{ (note that the last number is a sum)}$$

we can find a_j such that

$$a_j, a_j + a_{j+1}, \ldots, a_j + \ldots a_{n-2}, a_j + \ldots a_{n-1} + a_n, \ldots \text{ are all non-negative.}$$

Since $a_n < 0$, thus $a_j + \ldots a_{n-1} > 0$, therefore this sum plus $n - 1$ of the above sums are the n desired sums with a_j as the first number in the re-enumeration. □

Example 7.3.9 (23-24-S5-A-P9)

The sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined as follow,

$$a_1 = 3, a_2 = 5, a_{n+1} = 3a_n - 2a_{n-1}, \text{ for } n \geq 2.$$

Prove that $a_n = 2^n + 1$, for all n positive integer.

Solution. Our hypothesis is that $a_n = 2^n + 1$, for all n positive integer

It is easy to verify the base cases when $n = 1$ and $n = 2$.

For the Inductive step, let's assume that the hypothesis is true for all positive integers less than or equal to n. We shall prove for $n + 1$.

It is easy to verify that $a_{n+1} = 3a_n - 2a_{n-1} = 3(2^{n-1} + 1) - 2(2^{n-2} + 1) = 2^n + 1$. Thus the hypothesis is true for $n + 1$, therefore it is true for all $n \geq 1$. □

Example 7.3.10 (23-24-S5-A-P10)

A bank has an unlimited supply of 3-peso and 5-peso notes. Prove that it can pay any number of pesos greater than 7.

Solution. Our hypothesis is that any positive integer larger than 7 can be expressed a sum of 3s and 5s.

It is easy to verify the base cases of 8, 9, and 10:

$$8 = 3 + 5, 9 = 3 + 3 + 3, 10 = 5 + 5.$$

For the Inductive step, let's assume that the hypothesis is true for $k, k + 1, k + 2$. We can easily add 3 to any of the number to prove for $k + 3, k + 4, k + 5$.

This means that this induction proof with a compound base may be split into three standard inductions using the following schemes:

$$8 \to 11 \to 14 \to \cdots, 9 \to 12 \to 15 \to \cdots, 10 \to 13 \to 16 \to \cdots$$

□

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7.4 Problems (A-Level)

Submission deadline: November 12, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 7.4.1 (23-24-S5-A-P11)**.** (*10 points*) Show that for a, b, n positive integers

$$\frac{1}{a(a+b)} + \frac{1}{(a+b)(a+2b)} + \cdots + \frac{1}{(a+(n-1)b)(a+nb)} = \frac{n}{a(a+nb)}$$

Solution (Official solution - similar to students' submissions) — We give a proof based on the Induction principle with n.

Our hypothesis is the given equality. It is easy to verify the base cases of $n = 1$:

For the Inductive step, let's assume that the hypothesis is true for n.

Then

$$\begin{aligned}
&\frac{1}{a(a+b)} + \frac{1}{(a+b)(a+2b)} + \cdots + \frac{1}{(a+(n-1)b)(a+nb)} + \frac{1}{(a+nb)(a+(n+1)b)} \\
&= \frac{n}{a(a+nb)} + \frac{1}{(a+nb)(a+(n+1)b)} = \frac{1}{a+nb}\left(\frac{n}{a} + \frac{1}{a+(n+1)b}\right) \\
&= \frac{1}{a+nb}\frac{(n+1)a+n(n+1)b}{a(a+(n+1)b)} = \frac{n+1}{a+(n+1)b}.
\end{aligned}$$

Thus the hypothesis is true for $n+1$, therefore it is true for all $n \geq 1$.

Solution (Solution 2 - from Samuel Luong) — We give write that, for $k = 0, 1, \ldots, n$:

$$\frac{1}{(a+kb)(a+(k+1)b)} = \frac{1}{b}\left(\frac{1}{a+kb} - \frac{1}{a+(k+1)b}\right)$$

Then the given sum is equivalent to the following telescopic sumL

$$\begin{aligned}
&\frac{1}{b}\left[\left(\frac{1}{a} - \frac{1}{a+b}\right) + \left(\frac{1}{a+b} - \frac{1}{a+2b}\right) + \cdots + \left(\frac{1}{a+(n-1)b} - \frac{1}{a+nb}\right)\right] \\
&= \frac{1}{b}\left(\frac{1}{a} - \frac{1}{a+nb}\right) = \frac{n}{a+nb}.
\end{aligned}$$

Remark. All students who submitted solution(s) have correctly solved this problem.

Samuel Luong showed a good insight how to turn a sum into a telescopic one so that can quickly be computed.

Chi Khanh Pham and Tran Hieu Khoa made some minor mistake.

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Problem 7.4.2 (23-24-S5-A-P12)**.** (*10 points*) Show that for m, n non-negative integers

$$\frac{m!}{0!} + \frac{(m+1)!}{1!} + \cdots + \frac{(m+n)!}{n!} = \frac{(m+n+1)!}{n!(m+1)}$$

Solution (Official solution - similar to students' submissions) — We give a proof based on the Induction principle with n.

Our hypothesis is the given equality. It is easy to verify the base cases of $n = 1$:

$$\frac{m!}{0!} + \frac{(m+1)!}{1!} + \frac{(m+2)!}{2!} = m! + (m+1)! + \frac{1}{2}(m+2)! = m!(1 + (m+1) + \frac{1}{2}(m+1)(m+2))$$
$$= m!\frac{2 + 2m + 2 + m^2 + 3m + 2}{2} = m!\frac{m^2 + 5m + 6}{2} = m!\frac{(m+2)(m+3)}{2} = \frac{(m+3)!}{2!(m+1)}.$$

For the Inductive step, let's assume that the hypothesis is true for n.

Then

$$\frac{m!}{0!} + \frac{(m+1)!}{1!} + \cdots + \frac{(m+n)!}{n!} + \frac{(m+n+1)!}{(n+1)!} = \frac{(m+n+1)!}{n!(m+1)} + \frac{(m+n+1)!}{(n+1)!}$$
$$= \frac{(m+n+1)!}{n!}\left(\frac{1}{m+1} + \frac{1}{n+1}\right) = \frac{(m+n+1)!}{n!}\frac{m+n+2}{(n+1)(m+1)} = \frac{(m+n+2)!}{(n+1)!(m+1)}.$$

Thus the hypothesis is true for $n+1$, therefore it is true for all $n \geq 1$.

Remark. Most of students who submitted solution(s) have correctly solved this problem, read below for details.

Antoine Dao and Minh Hai Vu made some minor mistake.

Tran Hieu Khoa introduced foreign terms into the calculation, thus invalidating it.

Problem 7.4.3 (23-24-S5-A-P13)**.** (*10 points*) Show that for all $n \geq 1$,

$$4^n + 15n - 1 \text{ is divisible by } 9.$$

Solution (Official solution - similar to students' submissions) — We give a proof based on the Induction principle with n.

Our hypothesis is that for all $n \geq 1$, $9 \mid 4^n + 15n - 1$.

It is easy to verify the base cases of $n = 1$: $4^1 + 15 \cdot 1 - 1 = 18$, which is divisible by 9.

For the Inductive step, let's assume that the hypothesis is true for n.

Then, it is simple to verify that the expression for $n+1$ is divisible by 9, as show below:

$$4^(n+1) + 15(n+1) - 1 = 4(4^n + 15n - 1) - 3(15n) + 18.$$

Thus the hypothesis is true for $n+1$, therefore it is true for all $n \geq 1$.

Remark. All students who submitted solution(s) have correctly solved this problem, read below for details.

Jason To made a minor mistake.

Minh Nguyen solutions are just replication with simple change in expression in use, thus count as a single solution.

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Problem 7.4.4 (23-24-S5-A-P14)**.** (*10 points*) Show that for all $n \geq 1$,

$$2^{3^n} + 1 \text{ is divisible by } 3^{n+1}.$$

Solution (Official solution - similar to students' submissions) — We give a proof based on the Induction principle with n.

Our hypothesis is that for all $n \geq 1$, $3^{n+1} \mid 2^{3^n} + 1$.

It is easy to verify the base cases of $n = 1$: $3^2 = 9 \mid 2^3 + 1 = 9$.

For the Inductive step, let's assume that the hypothesis is true for n.

Then,

$$2^{3^{n+1}} + 1 = 2^{3n \cdot 3} + 1 = \left(2^{3^n}\right)^3 + 1^3$$

By the identity for a, b real numbers,

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$\left(2^{3^n}\right)^3 + 1^3 = (2^{3^n} + 1)\left(\left(2^{3^n}\right)^2 - 2^{3^n} + 1\right) = (2^{3^n} + 1)\left(2^{3^n}(2^{3^n} - 1) + 1\right) \quad (*)$$

Since $2^{3^n} + 1$ is divisible by 3^{n+1}, thus it is divisible by 3, or 2^{3^n} has a remainder of 2 when divided by 3, which means that $2^{3^n}(2^{3^n} - 1)$ has a remainder of 2 when divided by 3. Therefore the first term of the expression (*) is divisible by 3^{n+1}, and the second term is divisible by 3. Thus $2^{3^{n+1}} + 1$ is divisible by 3^{n+2}.

Hence the hypothesis is true for $n + 1$, therefore it is true for all $n \geq 1$.

Remark. Most students who submitted solution(s) have correctly solved this problem, read below for details.

Minh Nguyen second solution is based on advanced theorem of Lifting the Exponent. That is worth to be used in the context of olympiad competition, but not likely here. For example, you don't use the Fermat's Theorem of $a^n + b^n = c^n$ having no root when $n \geq 5$, to prove smaller case such as $a^4 + b^4 = 10^4$ having no roots. On the other hand, the use of such advance theorem can be allowed if you are able to provide its proof together with the solution.

Chi Khanh Pham made lots of small mistakes.

Tran Hieu Khoa incorrectly derive the formula of $2^{3^{n+1}} = 8^{n+1}$, thus invalidating the proof.

Note that students are encouraged to use *mathematical notation*, whenever possible. They help to make the proof clearer, more precise, and short.

For example, instead of writing *a is divisible by b,* you can write $b \mid a$; instead of *a leaves remainder 1 when divided by 3,* use $a \equiv 1 \pmod 3$; and so on.

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Problem 7.4.5 (23-24-S5-A-P15)**.** (*10 points*) Show that for all $n \geq 1$,

$$2(\sqrt{n+1}-1) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

Solution (Official solution - similar to submissions from An Nguyen, Chi Khanh Pham, Karl Le, Minh Hai Vu, and Ryan Nguyen) — For the inequality on the left, we give a proof based induction by n.

Our hypothesis is that for all $n \geq 1$,

$$2(\sqrt{n+1}-1) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}.$$

It is easy to verify the base cases of $n = 1$: $2(\sqrt{2}-1) < 2(1.45-1) = 0.9 < 1$. For the Inductive step, let's assume that the hypothesis is true for n. Then,

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > 2(\sqrt{n+1}-1) + \frac{1}{\sqrt{n+1}}$$

We need to prove that

$$2(\sqrt{n+1}-1) + \frac{1}{\sqrt{n+1}} > 2(\sqrt{n+2}-1) \Leftrightarrow \frac{1}{\sqrt{n+1}} > 2\left(\sqrt{n+2}-\sqrt{n+1}\right) = \frac{2}{\sqrt{n+2}+\sqrt{n+1}}.$$

The last inequality is obvious. Hence the hypothesis is true for $n+1$, therefore it is true for all $n \geq 1$.

For the inequality on the right, we give a proof based on the Induction principle with n.

Our hypothesis is that for all $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

It is easy to verify the base cases of $n = 1$: $1 < 2\sqrt{1} = 2$. For the Inductive step, let's assume that the hypothesis is true for n. Then,

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$$

We need to prove that

$$2\sqrt{n} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1} \Leftrightarrow \frac{1}{\sqrt{n+1}} < 2\left(\sqrt{n+1}-\sqrt{n}\right) = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

The last inequality is obvious. Hence the hypothesis is true for $n+1$, therefore it is true for all $n \geq 1$.

Remark. An Nguyem Chi Khanh Pham, Karl Le, Minh Hai Vu, and Ryan Nguyen have correctly solved this problem, read below for details.

Antoine Dao made several mistakes.

Tran Hieu Khoa proved only the left hand side inequality.

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Problem 7.4.6 (23-24-S5-A-P16). (*10 points*) Let $a_1, a_2, \ldots, a_n$ be integers, not all zeros, such as

$$a_1 + a_2 + \cdots + a_n = 0.$$

Prove that, for some all $k \in \{1, 2, \ldots, n\}$,

$$|a_1 + 2a_2 + \cdots + 2^{k-1}a_k| > \frac{2^k}{3}.$$

Solution (Official solution) — Let's use Proof by Contradiction to solve this problem. Let's reformulate the problem. For all $n \geq 1$, where $a_1, a_2, \ldots, a_n$ be integers, not all zeros, and $a_1 + a_2 + \cdots + a_n = 0$, if for all $k \in \{1, 2, \ldots, n\}$,

$$|a_1 + 2a_2 + \cdots + 2^{k-1}a_k| \leq \frac{2^k}{3}.$$

then $a_1 = a_2 = \ldots = a_n = 0$.

We give a proof for this hypothesis based the Induction principle with k. If $k = 1$,

$$|a_1| \leq \frac{2}{3}, \text{ since } a_1 \text{ is integer } \Rightarrow a_1 = 0.$$

For the Inductive step, let's assume that the hypothesis is true for $1, 2, \ldots, k-1$. Then, since $a_1 = a_2 = \ldots a_{k-1} = 0$,

$$\frac{2^k}{3} \geq |a_1 + 2a_2 + \cdots + 2^{k-1}a_k| = 2^{k-1}|a_k| \Rightarrow a_k = 0.$$

Hence the hypothesis is true for k, therefore it is true for all $k = 1, 2, \ldots, n$.

Remark. No student submitted solution for this problem.

Problem 7.4.7 (23-24-S5-A-P17). (*10 points*) Prove that any polygon (not necessarily convex) can be dissected into triangles by disjoint diagonals (they are allowed to meet only at vertices of the polygon).

Solution (Official solution) — Let use the following lemma without proof.

Claim — Any polygon (except a triangle) has at least one diagonal which lies completely within it.

We give a proof based on the Induction principle with n.

It is easy to verify the base case of $n = 3$. For the Inductive step, let's assume that the hypothesis is true for $1, 2, \ldots, n-1$.

By the lemma there exists such a diagonal which lies completely within the polygon. This diagonal dissects the polygon into two polygons with fewer sides. We just need to deal with a polygon (if there is) that is not a triangle. By the hypothesis, this polygon has fewer sides, this can be dissected into triangles by disjoint diagonals. Thus, all we have now are triangles dissected from the polygon by disjoint diagonals.

Hence the hypothesis is true for n, therefore it is true for all n.

Remark. Antoine Dao is the only one who submitted a solution, which is based on the assumption that the existence of an interior angle less than $180°$. This is not obvious, nor trivial.

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Problem 7.4.8 (23-24-S5-A-P18). (*10 points*) For a is a non-zero number, it is known that $a + \frac{1}{a}$ is an integer. Prove that for any $n \geq 1$ positive integer,

$$a^n + \frac{1}{a^n} \text{ is an integer.}$$

Solution (Official solution - similar to submissions from An Nguyen, Chi Khanh Pham, Minh Hai Vu, and Samuel Luong) — We give a proof based on the Induction principle with n.

It is easy to verify the base case of $n = 1$. For the Inductive step, let's assume that the hypothesis is true for $1, 2, \ldots, n$. Then,

$$a^{n+1} + \frac{1}{a^{n+1}} = \left(a^n + \frac{1}{a^n}\right)\left(a + \frac{1}{a}\right) - \left(a^{n-1} + \frac{1}{a^{n-1}}\right).$$

By the hypothesis $a + \frac{1}{a}$, $a^n + \frac{1}{a^n}$, and $a^{n-1} + \frac{1}{a^{n-1}}$ are all integers, thus $a^{n+1} + \frac{1}{a^{n+1}}$ is

Hence the hypothesis is true for n, therefore it is true for all n.

Remark. An Nguyen, Chi Khanh Pham, Minh Hai Vu, and Samuel Luong have correctly solved this problem, read below for details.

Ryan Nguyen solved the case $n \to 5n$, but did not deal with $n \to n + 1$.

Tran Hieu Khoa made an attempt.

Problem 7.4.9 (23-24-S5-A-P19). (*10 points*) The sequence of Fibonacci number is defined by

$$f_1 = f_2 = 1, f_{n+1} = f_n + f_{n-1}, \text{ for } n \geq 2.$$

Prove that the nth Fibonacci number is divisible by 3 if and only if n is divisible by 4.

Solution (Official solution - similar to submissions from Minh Nguyen and Ryan Nguyen) — Let consider the remainders of the first few terms of the Fibonacci when divided by 3:

$$\underbrace{1, 1, 2, 0, 2, 2, 1, 0,}_{\text{period8}} 1, 1, 2, 0, 2, 2, 1, 0, 1$$

Now, by Induction principle using $f_{n+1} = f_n + f_{n-1}$, for $n \geq 2$, with the base cases $n = 1, 2, \ldots, 8$,

$$f_n \equiv \begin{cases} 0 \pmod 3, & \text{if } n \equiv 0, 4 \pmod 8 \\ 1 \pmod 3, & \text{if } n \equiv 1, 2, 7 \pmod 8 \\ 2 \pmod 3, & \text{if } n \equiv 3, 5, 6 \pmod 8 \end{cases}$$

Thus, the nth Fibonacci number is divisible by 3 if and only if n is divisible by 4.

Remark. Minh Nguyen and Ryan Nguyen have correctly solved this problem, read below for details.

Minh Hai Vu, Karl Le made proofs that containing partial reasoning. Note that the problem's statement is an *A if and only if B* logical relation. Thus you have to prove both cases *if A then B* and *if B then A*.

Jason To showed his guess that the sequence of remainder is periodic. He should use the formula to turn the guess into a formal proof.

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Problem 7.4.10 (23-24-S5-A-P20)**.** (*10 points*) A set of n points is taken on a circle and each pair is connected by a segment. It happens that no three of these segments meet at the same point. Into how many parts do they divide the interior of the circle?

Remark. The idea is simple. Here it is.

The number of line segments created by connecting n points is $\binom{n}{2}$ (one for every pair of points) and the number of intersection points is $\binom{n}{4}$ (one for every four points). With no points, we have one region. Each time we add a line segment with k intersections, we add $k+1$ regions (by splitting $k+1$ regions); think of this as one region for each line segment and one region for each intersection. Thus, the number of regions is

$$\binom{n}{4} + \binom{n}{2} + 1.$$

However, this higher level of abstraction is explained in more details in the below proof.

Solution (Official solution) — Our hypothesis is that the number of regions is

$$\binom{n}{4} + \binom{n}{2} + 1$$

The base cases for $n = 1$ or $n = 2$ can easily be verified.

To prove the inductive step let us take $k+1$ points on a circle. The segments connecting all of these points but the $(k+1)^{\text{th}}$ divide the interior of the circle into

$$\frac{k(k-1)(k-2)(k-3)}{24} + \frac{k(k-1)}{2} + 1$$

parts by the inductive assumption.

The segment connecting the $(k+1)^{\text{th}}$ point with the ith one (where i is a positive integer not greater than k) intersects $(i-1)(k-i)$ other segments.

Thus, adding this segment would increase the number of the parts by $(i-l)(k-i)+1$. Adding all the segments connecting the $(k+1)^{\text{th}}$ point with the other k points would increase the number of parts by

$$\begin{aligned}
&1 + ((1)(k-2)+1) + \cdots + ((i-1)(k-i)+1) + \cdots + ((k-2)(1)+1) + 1 \\
&= (k+1)(1+2+\cdots+k) - (1^2+2^2+\cdots+k^2) - k^2 + k = \frac{k(k+1)^2}{2} - \frac{k(k+1)(k+2)}{6} - k^2 + k \\
&= \frac{(k+1)k(k-1)(k-2)}{24} + \frac{k(k+1)}{2} + 1.
\end{aligned}$$

Hence the hypothesis is true for $n = k+1$, therefore it is true for all n.

Remark. No student submitted solution for this problem.

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7.5 Solving Problems with Induction Principle (O-Level) - Part 1

Submission deadline: November 12, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 7.5.1 (Expression as sum of dictinct divisors)**.** (*10 points*) $n \geq 3$ is a positive integer. Prove that $n!$ can be written as a sum of n distinct divisors of itself.

Remark. First, it is easy to see that $3! = 6 = 1 + 2 + 3$ and each of $1, 2$, and 3 is a divisor of $3!$.

Normally, to proceed with the inductive step, if we assume that $n! = d_1 + d_2 + \cdots + d_n$, where $d_1, d_2, \cdots, d_n$ are all distinct, then $(n+1)! = (n+1)d_1 + (n+1)d_2 + \cdots + (n+1)d_n$ is a sum of n distinct divisors of $(n+1)!$. We can attempt to change it into a sum of $n+1$ terms by for example breaking up the first term $(n+1)d_1 = d_1 + nd_1$,

$$(n+1)! = d_1 + nd_1 + (n+1)d_2 + \cdots + (n+1)d_n$$

Now, it is easy to see that $d_1, (n+1)d_2, \cdots, (n+1)d_n$ are n distinct divisors of $(n+1)!$. However nd_1 might not be a divisor of $(n+1)!$. But can we choose such d_1 so that nd_1 will always be a divisor of $(n+1)!$?

The answer is surprisingly simple: *if initially* $d_1 = 1 < d_2 < \cdots < d_n$, *then* $nd_1 = n \mid (n+1)!$, *but it means to change the original statement into a stronger one (why?)*

Can you rewrite the original statement and then prove it by the Induction Principle?

If you can, then you just make an advancement in using the principle. Sometimes going straight to prove it is more harder to look at the statement and see how you can make the approach.

Solution (Official solution - similar to submissions from Anthony Pham, Ha-Anh Le, Henry Dinh, anh Minh Le Phuong Nguyen) — Our hypothesis is a *stronger* version of the problem's statement: for all $n \geq 3$, $n!$ can be written as a sum of n distinct divisors of itself with the smallest divisor is 1, in other words there exists positive integer k such that

$$n! = d_1 + d_2 + \cdots + d_k, \text{ where } d_i \mid n!,\ i = 1, 2, \ldots, k, \text{ and } d_1 = 1 < d_2 = n - 1 < \ldots < d_k \quad (*)$$

For the base case $n = 3$,

$$3! = 6 = 1 + 2 + 3,\ 1 \mid 3!, 2 \mid 3!, 3 \mid 3!.$$

Now, let's assume that the hypothesis is true for n, or (*) stands, then $\exists d_1 = 1 < d_2 = n-1 < \ldots < d_k$.

$$(n+1)! = (n+1)(d_1 + d_2 + \cdots + d_k) = (n+1)(1 + d_2 + \cdots + d_k) = 1 + n + d_2 + \cdots + d_k$$

Thus there exists $\ell = k + 1$, and $e_1 = 1, e_2 = n, e_3 = (n+1)d_2, \ldots, e_{k+1} = (n+1)d_k$, such that

$$(n+1)! = e_1 + e_2 + \cdots + e_\ell, \text{ where } e_i \mid (n+1)!,\ i = 1, 2, \ldots, \ell, \text{ and } 1 = e_1 < n = e_2 < \ldots < e_\ell$$

Thus the hypothesis (*) is true for $n + 1$, therefore it is true for all $n \geq 3$. Hence, the weaker hypothesis of the problem follows.

Remark. Anthony Pham, Ha-Anh Le, Henry Dinh, anh Minh Le Phuong Nguyen have correctly solved this problem, read below for details.

The proof shows the validity of the summation:

$$1 + (n-1) + (n-1)(n-2) + \cdots + (n-1)(n-2) \cdot 1 = n!.$$

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Problem 7.5.2 (Upper bound of the sum of reciprocals)**.** $n \geq 1$ is a positive integer.

(*5 points*) Prove that

$$x_n = \frac{1}{1} + \frac{1}{2^1} + \cdots + \frac{1}{2^n} < 2.$$

(*5 points*) $n \geq 1$ is a positive integer. Prove that

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \frac{7}{10}.$$

Remark. For both questions, applying the Induction Principle on the given statements will not help. Why? For the first question, the sequence (x_n) is strictly increasing:

$$x_{n+1} = x_n + \frac{1}{2^n} > x_n, \ \forall n \geq 1.$$

The upper bound 2 is unchanged regardless of n, thus using the hypothesis for n will not help to prove for $n+1$. Luckily you can easily find *a generic formula for x_n, and use that to prove the statement, perhaps without using the Induction Principle.*

The second question is similar in the sense that (y_n) is also a is strictly increasing sequence,

$$y_{n+1} - y_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{(2n+1)(2n+2)} > 0$$

However, it is not trivial to determine a generic formula for y_n. In this case we will try to *change the original statement by experimenting adding the value of a function f at n to y_n in order to make the sequence $(y_n + f(n))$ to be strictly decreasing from a certain value of $n = k$,* in other words

$$y_{n+1} + f(n+1) \leq y_n + f(n), \ \forall n \geq k \Leftrightarrow \frac{1}{(2n+1)(2n+2)} \leq f(n) - f(n+1), \ \forall n \geq k \quad (*)$$

Since a generic term of y_n is in the format $\frac{1}{n}$, let experiment with $f(n) = \frac{\ell}{n}$, where ℓ is a positive constant. Now, try to determine k, ℓ such that for all $n \geq k$,

$$f(n) = \frac{\ell}{n} \Rightarrow \frac{1}{(2n+1)(2n+2)} \leq f(n) - f(n+1)$$

Once you are done, rewrite the original statement to a stronger one, perhaps not covering all $n \geq 1$ cases (certain cases to be dealt with separately, there are fortunately only a few). Then use the Induction Principle to prove the changed statement. Don't forget to make sure to show that the original proposition is true.

Remark. Let's continue the thought previously, we want to find a function $f: \mathbb{Z}^+ \to \mathbb{R}^+$ such that, from a positive integer k, if $n \geq k$, then

$$y_{n+1} + f(n+1) \leq y_n + f(n) \leq \frac{7}{10} \Rightarrow f(n) - f(n+1) \geq y_{n+1} - y_n = \frac{1}{(2n+1)(2n+2)}$$

Now, since a generic term of y_n is in the format $\frac{1}{n}$, let experiment with $f(n) = \frac{\ell}{n}$, where ℓ is a constant,

$$\frac{1}{(2n+1)(2n+2)} \leq f(n) - f(n+1) = \frac{\ell}{n} - \frac{\ell}{n+1} = \frac{\ell}{n(n+1)} \Leftrightarrow \ell \leq \frac{n}{2(2n+1)} = \frac{1}{4 + \frac{1}{n}}$$

Thus, the selection of $\ell = \frac{1}{4}$ should be sufficient. By verifying the first few terms of $y_n + f(n)$ we find $k = 4$, so for all $n \geq k$,

$$y_n + \frac{1}{4n} < \frac{7}{10}.$$

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Solution (Official solution - similar to submissions from Anthony Pham, Ha-Anh Le, Henry Dinh) — For the first question, by the formula for a geometric series

$$x_n = \frac{1}{1} + \frac{1}{2^1} + \cdots + \frac{1}{2^n} = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^n} < 2.$$

For the second question, we prove a *stronger* inequality, for $n \geq 4$,

$$y_n + \frac{1}{4n} < \frac{7}{10}$$

It is easy to verify that given inequality for $n \leq 3$, or $y_n < \frac{7}{10}$, $\forall n \leq 3$, and for $n = 4$,

$$y_4 + \frac{1}{4 \cdot 4} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{16} = \frac{1171}{1680} \approx 0.697 < 0.7 = \frac{7}{10}.$$

For the inductive step, if for $n \geq 4$,

$$y_n + \frac{1}{4n} < \frac{7}{10} \Rightarrow y_{n+1} + \frac{1}{4(n+1)} - y_n - \frac{1}{4n} = \left(\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}\right) + \left(\frac{1}{4(n+1)} - \frac{1}{4n}\right)$$
$$= \frac{1}{(2n+1)(2n+2)} - \frac{1}{4n(n+1)} = -\frac{1}{4n(n+1)(2n+1)} < 0 \Rightarrow y_{n+1} + \frac{1}{4(n+1)} < \frac{7}{10}$$

Thus the hypothesis (*) is true for $n+1$, therefore it is true for all $n \geq 1$. Hence, the weaker hypothesis of the problem follows.

Remark. Ha-Anh Le, Henry Dinh have correctly solve the problem.

Anthony Pham made some mistake when skipping verification of the base cases.

Albert Dinh-Le, Minh Le Phuong Nguyen solved only the first question.

Problem 7.5.3 (Fibonacci sequence)**.** (*10 points*) $n \geq 1$ is a positive integer. (f_n) is the Fibonacci sequence: $f_0 = f_1 = 1, f_{n+1} = f_n + f_{n-1}$. Prove that

$$f_{2n+1} = f_{n+1}^2 + f_n^2 \quad (*)$$

Remark. For the first solution, using your knowledge of characteristic polynomial, find the generic formula for f_n, then prove (*). Beside a few simple algebraic manipulations, you would not face substantial problems.

For the second one, *instead of making the statement stronger, we shall generalize it in a way to make it easier to prove by the Induction Principle.* Note that $2n + 1 = (n) + (n) + 1$, so (*) can be rewritten as:

$$f_{(n)+(n)+1} = f_{(n)+1} \cdot f_{(n)+1} + f_{(n)} \cdot f_{(n)} \text{ or } f_{m+n+1} = f_{m+1}f_{n+1} + f_m fn, \quad \text{where } m = n.$$

Now, can you prove that for $m, n \geq 1$,

$$f_{m+n+1} = f_{m+1}f_{n+1} + f_m f_n \quad (**)$$

by a modified Induction Principle: fix a value of m positive integer and prove (**) is true for all n. I let you to conclude the proof for the original statement.

Remark. There was *typo error* in the problem text: the initial condition must be stated as $f_0 = 0, f_1 = 1$.

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Solution (Official solution - similar to submissions from all students) — We consider a *stronger* hypothesis, for m, n non-negative integers:

$$f_{m+n+1} = f_{m+1}f_{n+1} + f_m f_n \quad (**)$$

We prove it by induction based on $n \geq 0$.

The base case is trivial for $n = 0$, which means $f_{m+1} = f_{m+1}$, and for $n = 1$, which becomes $f_{m+2} = f_{m+1} + f_m$ that is the formula for the Fibonacci sequence.

For the inductive step, with the assumption that (**) is true for all $k = 0, 1, \ldots, n$,

$$\begin{aligned} f_{m+n+2} &= f_{m+n+1} + f_{m+n} = (f_{m+1}f_{n+1} + f_m f_n) + (f_{m+1}f_n + f_m f_{n-1}) \\ &= f_{m+1}(f_{n+1} + f_n) + f_m(f_n + f_{n-1}) = f_{m+1}f_{n+2} + f_m f_{n+1} \end{aligned}$$

Thus the hypothesis (*) is true for $n + 1$, therefore it is true for all $n \geq 0$.

The desired identity is the case of (**) when $m = n$

$$f_{2n+1} = f_{(n)+(n)+1} = f_{(n)+1}f_{(n)+1} + f_{(n)}f_{(n)} = f_{n+1}^2 + f_n^2.$$

Solution (Solution 2 - similar to submissions from Albert Dinh-Le, Anthony Pham, Ha-Anh Le, Minh Le Phuong Nguyen) — Let $\alpha_1 = \frac{1+\sqrt{5}}{2}$, $\alpha_2 = \frac{1+\sqrt{5}}{2}$, the generic formula (closed-form) of a Fibonacci number is

$$f_n = \frac{1}{\sqrt{5}}(\alpha_1^n - \alpha_2^n)$$

Note that

$$\alpha_1\alpha_2 = -1 \Rightarrow \alpha_1 + \frac{1}{\alpha_1} = \alpha_1 - \alpha_2 = \sqrt{5}, \alpha_2 + \frac{1}{\alpha_2} = \alpha_2 - \alpha_1 = -\sqrt{5}$$

Therefore

$$\begin{aligned} f_{n+1}^2 + f_n^2 &= \frac{1}{5}\left[(\alpha_1^{2n+2} - 2(\alpha_1\alpha_2)^{n+1} + \alpha_2^{2n+2}) + (\alpha_1^{2n} - 2(\alpha_1\alpha_2)^n + \alpha_2^{2n})\right] \\ &= \frac{1}{5}\left[\alpha_1^{2n+1}\left(\alpha_1 + \frac{1}{\alpha_1}\right) + \alpha_2^{2n+1}\left(\alpha_2 + \frac{1}{\alpha_2}\right)\right] = \frac{1}{5}\left(\alpha_1^{2n+1}\sqrt{5} + \alpha_2^{2n+1}(-\sqrt{5})\right) \\ &= \frac{1}{\sqrt{5}}(\alpha_1^{2n+1} - \alpha_2^{2n+1}) = f_{2n+1} \end{aligned}$$

Remark. All students have correctly solved the problem.

Students with the second solution demonstrated the use of the closed-form of Fibonacci terms, which is very helpful in many cases.

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Problem 7.5.4 (Sum of reciprocals of binomial coefficients)**.** (*10 points*) $n \geq 1$ is a positive integer. Prove that

$$\binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1} = \frac{n+1}{2^{n+1}}\left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1}\right)$$

Remark. It would be hard to use the Induction Principle to prove it by directly manipulating the left side for $n+1$ to be the right side. Hence, we demonstrate the use of *bijection mapping* in an unusual way.

Let the (a_n) be the sequence of the terms in the left side, (b_n) be the sequence of the term on the right side. *We prove that $a_n = b_n$ leads to $a_{n+1} = b_{n+1}$ by finding the same recurrence relations for both sequences*, as shown below:

$$\frac{2^{n+2}}{n+2}b_{n+1} = \frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1} + \frac{2^{n+2}}{n+2} = \frac{2^{n+1}}{n+1}b_n + \frac{2^{n+2}}{n+2} \Rightarrow b_{n+1} = \frac{n+2}{2(n+1)}b_n + 1.$$

Solution (Official solution - similar to submissions from Albert Dinh-Le, Henry Dinh) — Let

$$a_n = \binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1}, \; b_n = \frac{n+1}{2^{n+1}}\left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1}\right)$$

We prove that for all $n \geq 1$,

$$\begin{cases} a_{n+1} = \frac{n+2}{2(n+1)}a_n + 1 & (*) \\ b_{n+1} = \frac{n+2}{2(n+1)}b_n + 1 & (**) \end{cases}$$

For the first identity (*), consider a term of a_n, for $i = 0, 1, \ldots n$,

$$\binom{n}{i}^{-1} = \frac{i!(n-i)!}{n!} = \frac{n+1}{n+2} \cdot \frac{(n+2)i!(n-i)!}{(n+1)!} = \frac{n+1}{n+2} \cdot \frac{((i+1)+(n+1-i))i!(n-i)!}{(n+1)!}$$
$$= \frac{n+1}{n+2} \cdot \left(\frac{(i+1)!(n-i)!}{(n+1)!} + \frac{i!(n+1-i)!}{(n+1)!}\right) = \frac{n+1}{n+2} \cdot \left(\binom{n+1}{i+1}^{-1} + \binom{n+1}{i}^{-1}\right)$$

By summing up

$$a_n = \frac{n+1}{n+2} \cdot \left(2a_{n+1} - \binom{n+1}{n+1}^{-1} - \binom{n+1}{0}^{-1}\right) = \frac{n+1}{n+2} \cdot (2a_{n+1} - 2) \Rightarrow a_{n+1} = \frac{n+2}{2(n+1)}a_n + 1.$$

For (**)

$$\frac{2^{n+2}}{n+2}b_{n+1} = \frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1} + \frac{2^{n+2}}{n+2} = \frac{2^{n+1}}{n+1}b_n + \frac{2^{n+2}}{n+2} \Rightarrow b_{n+1} = \frac{n+2}{2(n+1)}b_n + 1.$$

Starting with $a_1 = b_1$, applying the Induction Principle on both sequences (a_n) and (b_n) at the same time, it is easy arrive at the conclusion that $a_n = b_n$ for all n.

Remark. Albert Dinh-Le, Henry Dinh have correctly solved the problem.

Although it is an unusual use of the Induction Principle, but similar to counting-in-two-ways or bijection (one-on-one correspondence), there are many situation when proving an equivalent of the two ways of computing often staring from the same point and follow the same rules.

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Problem 7.5.5 (An identity fuction)**.** (*5 points*) Let $f: \mathbb{Z}^+ \to \mathbb{Z}$ be a function with the following properties:

1. $f(2) = 2$;
2. $f(mn) = f(m)f(n),\ \forall m, n$;
3. $f(m) > f(n),\ \forall m > n$.

Prove that $f(n) = n,\ \forall n \in \mathbb{Z}^+$.

(*5 points*) Let $f: \mathbb{Z}^+ \to \mathbb{Z}$ be a function such that:

$$f(f(n)) + f(n) = 2n.$$

Prove that $f(n) = n,\ \forall n \in \mathbb{Z}^+$.

Remark. For the first question, what do we know about this function f? Is has *a fix point at 2:* $f(2) = 2$. It is a *multiplicative function.* It is also a *strictly increasing or a strict monotonic function.* We have to prove that every n is its fix point, or f *is an identity function.*

The difficult to use the Induction Principle as-is is the disconnect between $n+1$ and n while looking at the given conditions. But by looking carefully when considering $m = 2$,

$$f(2n) = f(2 \cdot n) = f(2)f(n) = 2f(n).$$

Thus, what is true for n would be true for $2n$? Similarly, what is true for $n+1$ would be true for $2n+2$?

$$f(2n+2) = f(2(n+1)) = 2f(n+1).$$

How about $2n+1$? What do we know about $f(2n+1)$ in comparison with $f(2n)$ and $f(2n+2)$? Thus, it is worth to rewrite the proposition: *instead of proving statement* $P(n) \to P(n+1)$, *let prove that if* $P(1), P(2), \ldots, P(2n-1), P(2n)$ *are true, then* $P(2n+1), P(2n+2)$ *are true?*

The second question is just to see whether you have mastered some of the techniques learned by solving the first one.

Remark. *Typo error*: For the second question f function must have positive integer value, in other words $f: \mathbb{Z}^+ \to \mathbb{Z}^+$.

Solution (Official solution for the first question - similar to submissions from most students) — For the first question, by (3):

$$f(2) = f(2 \cdot 1) = f(2)f(1) \Rightarrow f(1) = 1$$

Thus the base cases of $n = 1, 2$ are proven.

For the inductive step, let's assume that for all $k \leq 2n$, $f(k) = k$. We shall prove that

$$f(2n+1) = 2n+1, f(2n+2) = 2n+1.$$

Note that

$$f(2n+2) = f(2)f(n+1) = 2(n+1) \Rightarrow 2n = f(2n) < f(2n+1) < f(2n+2) = 2n+2 \Rightarrow f(2n+1) = 2n+1.$$

Thus for all n, $f(n) = n$.

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Solution (Solution 2 for question 1 - Anthony Pham) — For the first question, by (3):

$$f(2) = f(2 \cdot 1) = f(2)f(1) \Rightarrow f(1) = 1$$

Thus the base cases of $n = 1, 2$ are proven.

For the inductive step, let's assume that for all $k < n$, $f(k) = k$.

Case 1: n is composite, then

$$\exists a, b \in \mathbb{Z}^+ : \ 1 < a, b < n, n = ab \Rightarrow f(n) = f(ab) = f(a)f(b) = ab = n$$

Case 2: n is prime, then $n - 1$ and $n + 1$ are composite. Following the same reasoning as in the previous case

$$f(n-1) = n-1, f(n+1) = n+1 \Rightarrow n-1 = f(n-1) < f(n) < f(n+1) = n+1 \Rightarrow f(n) = n.$$

The hypothesis follows.

Solution (Official solution for the second question - similar to submissions from Henry Dinh, Minh Le Phuong Nguyen) — For the inductive step, let $f(k) = k, \forall k < n$. Then

$$f(f(n)) + f(n) = 2n$$

Case 1: if $f(n) = k < n$, then

$$f(f(n)) = f(k) = k \Rightarrow f(f(n)) + f(n) = f(k) + k = 2k < 2n.$$

Case 2: if $f(n) = k > n$, then $f(f(n)) = 2n - k < n$. By the hypothesis

$$f(f(f(n))) + f(f(n)) = 2f(n).$$

However, $f(f(n)) < n$, so $f(f(f(n))) = f(n)$, thus $f(f(f(n))) + f(n) < 2f(n)$, a contradiction.

Therefore $f(n) = n$. The hypothesis follows.

Remark. Henry Dinh has correctly solved the problem.

Minh Le Phuong Nguyen needs to make the proof more precise. Such as the base case $f(1) = 1$, which was used later on during the inductive step.

Anthony Pham and Ha-Anh Le solved only the first question.

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Problem 7.5.6 (Classical inequalities)**.** (*5 points*) Let $a_1, a_2, \ldots, a_n$ be any real numbers and $b_1, b_2, \ldots, b_n$ be any positive real numbers, then use the Induction Principle to prove that

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

(*5 points*) Let $a_1, a_2, \ldots, a_n$ be any positive real numbers, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Solution (Official solution for the first question - similar to submissions from Albert Dinh-Le, Ha-Anh Le, Henry Dinh) — For the first question, the base case of $n = 1$ is clear. Let prove the case $n = 2$ in order to reuse it in the inductive step.

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} \geq \frac{(a_1 + a_2)^2}{b_1 + b_2} \Leftrightarrow (a_1^2 b_2 + a_2^2 b_1)(b_1 + b_2) \geq (a_1 + a_2)^2 b_1 b_2 \Leftrightarrow a_1^2 b_2^2 + a_2^2 b_1^2 \geq 2a_1 a_2 b_1 b_2.$$

It is easy to verify the last inequality. Now, for the inductive step,

$$\begin{aligned}\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} + \frac{a_{n+1}^2}{b_{n+1}} &\geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n} + \frac{a_{n+1}^2}{b_{n+1}} \\ &\geq \frac{((a_1 + a_2 + \cdots + a_n) + a_{n+1})^2}{((b_1 + b_2 + \cdots + b_n) + b_{n+1})}\end{aligned}$$

The last inequality is an application of the case $n = 2$ for two pairs of numbers $(a_1 + a_2 + \cdots + a_n), a_{n+1}$ and $(b_1 + b_2 + \cdots + b_n), b_{n+1}$. Hence the hypothesis follows.

Solution (Official solution for the second question - similar to submissions from Albert Dinh-Le, Ha-Anh Le) — For the second question, for the case $n \to 2n$,

$$\begin{aligned}&\frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = \frac{1}{2}\left(\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + a_{2+1} + \cdots + a_{2n}}{n}\right) \\ &\geq \sqrt{\sqrt[n]{a_1 a_2 \cdots a_n} \cdot \sqrt[n]{a_{n+1} a_{n+2} \cdots a_{2n}}} = \sqrt[2n]{a_1 a_2 \cdots a_{2n}}.\end{aligned}$$

The case $n \to n - 1$ is as follow,

$$\begin{aligned}&\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} = \frac{a_1 + a_2 + \cdots + a_{n-1} + \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}}{n} \\ &\geq \sqrt[n]{a_1 a_2 \cdots a_{n-1}\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)} \\ &\Rightarrow \left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^n \geq a_1 a_2 \cdots a_{n-1}\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right) \\ &\Rightarrow \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \geq \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}\end{aligned}$$

Hence, by the base case and following $n \to 2n \to 2n - 1 \to \cdots \to n + 1$ chain of applications, the hypothesis follows.

Remark. For the first question, it would surprises you that after proving the case for $n = 2$, you will find the way to reuse that to prove the inductive step.

For the second question, you would learn another technique for the inductive step by proving that $P(n) \Rightarrow P(2n)$, and $P(n) \Rightarrow P(n-1)$. It would means that

$$\text{instead of going } n \to n+1 \text{ we rather go } n \to 2n \to 2n-1 \to \cdots \to n+1$$

It's just another way to top the sequence of infinitely many dominos.

Solution (Solution 2 for the second question from Henry Dinh) — We prove the following claim

Claim — For $0 < x_1, x_2, \ldots, x_n$, where $x_1 x_2 \cdots x_n = 1$,

$$x_1 + x_2 + \cdots + x_n \geq n \quad (*)$$

Proof. For $0 < x_1 \leq 1 \leq x_2$, then

$$x_2 - x_1 x_2 \geq 1 - x_1 \Rightarrow x_1 + x_2 \geq x_1 x_2 + 1.$$

Let's assume that (*) stand for $n-1$. Then $x_1 x_2 \cdots x_n = 1$. The trivial case when all x_i is 1 is obvious. WLOG, since the roles of x_i are interchangeable, let's assume that $x_{n-1} \leq 1 \leq x_n$.

From the case $n = 2$

$$x_1 + x_2 + \cdots + x_n = (x_1 + x_2 + \cdots + x_{n-2}) + (x_{n-1} + x_n) \geq x_1 + x_2 + \cdots + x_{n-2} + (x_{n-1} x_n) + 1$$

By the hypothesis,

$$x_1 + x_2 + \cdots + x_{n-2} + (x_{n-1} x_n) \geq n - 1 \Rightarrow x_1 + x_2 + \cdots + x_n \geq n.$$

■

By applying $x_k = \frac{a_k}{\sqrt{\prod_{i=1}^n a_i}}$, $k = 1, 2, \ldots, n$, Then $x_1 x_2 \cdots x_n = 1$, thus

$$\sum_{k=1}^n \frac{a_k}{\sqrt{\prod_{i=1}^n a_i}} \geq n \Rightarrow \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Remark. Albert Dinh-Le, Ha-Anh Le, Henry Dinh has correctly solved the problem.

Anthony Pham solved only the first question.

Minh Le Phuong Nguyen make an attempt.

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Problem 7.5.7 (Algebraic manipulation)**.** (*10 points*) Let $a_1, a_2, \ldots, a_n$ be any real numbers satisfying $a_{i+j} \leq a_i + a_j,\ \forall i, j \in \mathbb{Z}^+$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n,\ \forall n \geq 1.$$

Remark. Straight Induction Principle plus algebraic manipulation.

Solution (Official solution - similar to submission from Henry Dinh) — The base case $n = 1$ is trivial. Let's now list all the inequalities for $1 \leq k \leq n$.

$$\begin{cases} a_1 & \geq a_1 \\ a_1 + \frac{a_2}{2} & \geq a_2 \\ a_1 + \frac{a_2}{2} + \frac{a_3}{3} & \geq a_3 \\ & \ldots \\ a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_{n-1}}{n-1} & \geq a_{n-1} \\ a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_{n-1}}{n-1} + \frac{a_n}{n} & \geq a_n \end{cases}$$

By summing up

$$n \cdot a_1 + (n-1) \cdot \frac{a_2}{2} + \cdots + 1 \cdot \frac{a_n}{n} \geq a_1 + a_2 + \cdots + a_n.$$

By adding $a_1 + a_2 + \cdots + a_n$ to both sides, and note that $a_i + a_{n+1-i} \geq a_{n+1}$,

$$(n+1)\left(a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n}\right) \geq (a_1 + a_n) + (a_2 + a_{n-1}) + \cdots + (a_n + a_1) \geq n a_{n+1}$$
$$\Rightarrow a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n} \geq \frac{n}{n+1} a_{n+1} \Rightarrow a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n} + \frac{a_{n+1}}{n+1} \geq a_{n+1}$$

The hypothesis follows.

Remark. Only Henry Dinh has correctly solved the problem.

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Problem 7.5.8 (Recurrence relation for solutions to equations in similar form)**.** (*10 points*) Let $k \geq 3$ be an integer. Show that there exist odd integer x and y such that

$$2^k = 7x^2 + y^2.$$

Remark. The key to the solution based on Induction Principle perhaps is how to produce the solution (x_{k+1}, y_{k+1}) to the equation $2^{k+1} = 7x^2 + y^2$ based on a solution (x_k, y_k) to the equation $2^k = 7x^2 + y^2$, in other words to find (x_{k+1}, y_{k+1}), if

$$7x_{k+1}^2 + y_{k+1}^2 = 2(7x_k^2 + y_k^2) \quad (*)$$

One of the usual ideas is to find some linear combinations of x_k, y_k to represent x_{k+1}, y_{k+1}, in order words to find constant α, β, γ, δ, so that

$$x_{k+1} = \alpha x_k + \beta y_k,\ y_{k+1} = \gamma x_k + \delta y_k \text{ to satisfy } (*)$$

Solution (Official solution - similar to submission from Henry Dinh) — For $k = 3$, $2^3 = 7 \cdot 1^2 + 1^2$. So $(1, 1)$ is a solution for $k = 3$.

Let assume that (x_k, y_k) is a solution for $2^k = 7x^2 + y^2$.

Now,

$$2^{k+1} = 2(7x_k^2 + y_k^2) = 7\left(\frac{x_k \pm y_k}{2}\right)^2 + \left(\frac{7x_k \mp y_k}{2}\right)^2.$$

It is easy to see that if x_k, y_k both odd, then $\dfrac{x_k + y_k}{2}$, or $\left|\frac{x_k - y_k}{2}\right|$ is odd (their sum is x_k or y_k).

If $\dfrac{x_k + y_k}{2}$ is odd, then we can choose $x_{k+1} = \dfrac{x_k + y_k}{2}$, $y_{k+1} = \left|\dfrac{7x_k - y_k}{2}\right|$, and both are odd.

If $\dfrac{x_k + y_k}{2}$ is even, then we can choose $x_{k+1} = \left|\dfrac{x_k - y_k}{2}\right|$, $y_{k+1} = \dfrac{7x_k + y_k}{2}$, and both are odd.

Thus, there exists a pair of odd integers (x_{k+1}, y_{k+1}), which is a solution for $2^{k+1} = 7x^2 + y^2$.

Therefore, the hypothesis is true for $k + 1$. Hence, it is true for all k.

Remark. Only Henry Dinh has correctly solved the problem.

Albert Dinh-Le recognized the recurrence relations of the roots but have not proving his arguments adequately.

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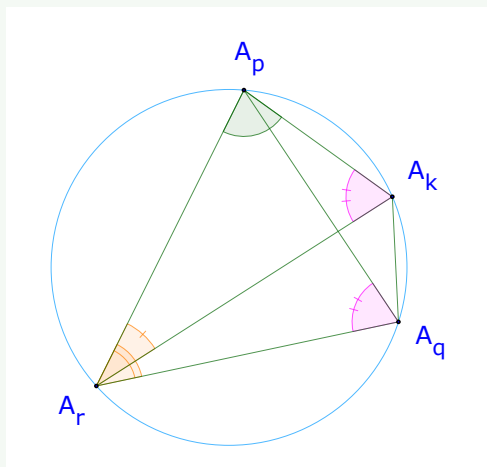
Problem 7.5.9 (Induction principle in combinatorial geometry)**.** (*10 points*) Let $A_1A_2\dots A_n$ be a convex polygon inscribed in a circle such that among its vertices, there are no two which form a diameter. Prove that if among the triangles $A_pA_qA_r$ as distinct p, q, r range over $1, 2, \dots, n$, there is at least one acute triangle, then there are at least $n-2$ such acute triangle.

Remark. The key to the solution perhaps in the inductive step with an acute $A_pA_qA_r$ identify then an A_k ($k \notin \{p, q, r\}$) vertex removed to obtain the $n-1$ polygon.

Solution (Official solution - similar to submission from Henry Dinh and Minh Le Phuong Nguyen) — Let's prove it by induction based on n. The base case $n = 3$ is simple to verify. Let $n \geq 4$. Let $\triangle A_pA_qA_r$ be an acute triangle and let's remove an vertex A_k distinct from A_p, A_q, A_r ($1 \leq k \leq n$, $k \notin \{p, q, r\}$) The induction hypothesis now should be true for the newly obtained $(n-1)-$gon, there are $n-3$ acute triangles with vertexes among $\{A_i, i = 1, 2, \dots, n\} \setminus \{A_k\}$.

Now, WLOG, let's assume that $A_k \in \widehat{A_pA_q}$. Consider $\angle A_kA_pA_r$ and $\angle A_kA_qA_r$. They can not be equal, which then implies that A_rA_k is a diameter. Thus one is strictly less than the other. Let assume that $\angle A_kA_pA_r < \angle A_kA_qA_r$. then $\angle A_kA_pA_r < 90°$.



In this $\triangle A_kA_pA_r$, from above $\angle A_kA_pA_r < 90°$; $\angle A_pA_kA_r = \angle A_pA_qA_r < 90°$ (because $\triangle A_pA_qA_r$ is acute); and $\angle A_pA_rA_k < \angle A_pA_rA_q < 90°$. Therefore $\triangle A_kA_pA_r$ is acute.

This $\triangle A_kA_pA_r$ acute triangle is distinct from the other $n-3$ acute triangles in the $(n-1)-$gon, and together with them make $n-2$ acute triangles in the $n-$gon.

The hypothesis is now proven for n and thus should be true for all $n \geq 3$.

Remark. Only Henry Dinh and Minh Le Phuong Nguyen have correctly solved the problem.

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Problem 7.5.10 (Divisibility by 9 as invariant in games)**.** (*5 points*) There are 56 candies on the table, Players play by turns (one after another). In each turn it is allowed to take either one, three, or five candies. The winner is the person who takes the last set of candies. Given the perfect play on both sides, who will win?

(*5 points*) Three players play the following game. There are 54 candies on the table. Players play by turns (one after the other). In each turn it is allow to take either one, three, or five candies such that the same number of candies cannot be taken in two consecutive turns. The winner is the person who takes the last set of candies. Given the perfect play of all players, who will win?

Remark. For the first question, let prove that if the number of candies in $6n - 4$ format then the second player wins. Prove the base case $n = 1$. Reiterate that case in the inductive step, when there are $6(n + 1) - 4$ candies.

For the second question, let $9n$ be the number of the candies, find out who wins for the base case $n = 1$. Reiterate that case in the inductive step, when there are $9(n + 1)$ candies.

Solution (Official solution - similar to submission from all students) — For the first question, we prove that if the number of candies is in $6n - 4$ format then the second player wins. It is easy to see that the base case is true. For $6(n + 1) - 4$ candies ($n \geq 1$), the second player always take $6 - k$ candies if the first player takes k candies in the first round, thus leading the game back to a *known winning position.*

For the second game, we prove by induction based on $9n$, the number of candies. For $n = 1$, the first two players can take altogether 4, 6, or 8, candies which means that the third player will always win. Let that be the assumption for $9n$. For $9(n + 1)$ candies, the third player can always finish the first round such that there remains $9n$ candies, thus leading the game back to a *known winning position.*

Remark. All students have correctly solved the problem.

Minh Le Phuong Nguyen uses analysis for the base cases then considering the winning positions, which is a great approach.

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Chapter 8

Session 6

8.1 Word problems (I-Level)

In this session we discuss solving words problem, that is, how to establish equations or systems of equations, and solve them.

Example 8.1.1 (23-24-S6-I-P1)
In eight years Henry will be three times the age that Sally was last year. Twenty five years ago their ages added to 83. How old is Henry now?

Solution. Let h represent Henry's current age, the s represent Sally's current age. In 8 years, Henry will have age $h+8$. Last year Sally had age $s-1$. Thus

$$h+8=3(s-1) \Rightarrow h=3s-11 \quad (*)$$

25 years ago their ages were $h-25$ and $s-25$. Thus

$$(h-25)+(s-25)=83 \Rightarrow h+s=133 \Rightarrow h=133-s \quad (**)$$

Therefore, from (*) and (**)

$$3s-11=133-s \Rightarrow 4s=144 \Rightarrow s=36,\ h=133-36=\boxed{97.}$$

□

Remark. It is always important to test your answers. Here's how can you do it for the solution above.

Let's test if these are the correct answers. In eight years, Henry will be $97+9=105$, which is $105=3\times 35$, or 3 times $36-1=35$, which is Sally's age last year. 25 years agon Henry was $97-25=72$ and Sally was $36-15=11$, thus their sum of ages was $72+11=83$. All given conditions are satisfied.

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Example 8.1.2 (23-24-S6-I-P2)

The lengths of the diagonals of a rhombus are, in inches, two consecutive integers. The area of the rhombus is 210 square inches. Find its perimeter, in inches.

Solution. Let x and $x+1$ be the lengths of the two diagonals. Then, from the formula of the area for rhombus,

$$\frac{1}{2}x(x+1) = 210 \Rightarrow x(x+1) = 420 = 20 \times 21 \Rightarrow x = 20.$$

The side length s of the rhombus is the hypotenuse of a right triangle with legs 10 and $\frac{21}{2}$, or

$$\sqrt{10^2 + \left(\frac{21}{2}\right)^2} = \sqrt{\frac{400+441}{4}} = \frac{29}{2}.$$

Therefore the perimeter is $4 \times \frac{29}{2} = \boxed{58}$. □

Example 8.1.3 (23-24-S6-I-P3)

How many gallons of a solution which is 15% alcohol do we have to mix with a solution that is 35% alcohol to make 250 gallons of a solution that is 21% alcohol?

Solution. Let x represent the number of gallons of 15% alcohol solution that we need to mix. To make 250 gallons, we need to mix the x gallons of 15% alcohol solution with $250 - x$ gallons of the 35% alcohol solution.

The mix will then contain $(0.15)(x) + (0.35)(250 - x)$ gallons of alcohol solution which should be 21% of the total 250 gallons, in other words $(0.21)(250)$, therefore

$$(0.15)(x) + (0.35)(250 - x) = (0.21)(250) \Rightarrow 15x + 35(250 - x) = 21(250)$$
$$\Rightarrow 35(250) - 20x = 21(250) \Rightarrow 20x = 14(250) \Rightarrow x = \boxed{175}.$$

□

Example 8.1.4 (23-24-S6-I-P4)

A rectangle has area of 1100. If the length is increased by ten percent and the width is decreased by ten percent, what is the area of the rectangle?

Solution. Let l and w be the length and width of the rectangle, then $lw = 1100$. Now, if the length is increased by ten percent and the width is decreased by ten percent, then the area of the modified rectangle is,

$$1.1l \cdot 0.9w = (1.1 \cdot 0.9)lw = 0.99lw = 0.99 \cdot 1100 = \boxed{1089.}$$

□

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Example 8.1.5 (23-24-S6-I-P5)

A week ago, Sandy's seasonal Little League batting average was 360. After five more at bats this week, Sandy's batting average is up to 400. What is the smallest number of hits that Sandy could have had this season?

Remark. *Batting average* is a statistic in cricket, baseball, and softball that measures the performance of batters. In baseball, the batting average is defined by the number of hits divided by at bats. It is usually reported to three decimal places and read without the decimal: A player with a batting average of .300 is *batting three-hundred.*

Solution. Suppose that a week ago Sandy had been at bat x times. Then Sandy would have made $(0.360)x$ hits. Suppose that Sandy made y more hits in the next five times at bat. That makes a total of $(0.360)x + y$ hits after $x + 5$ times at bat, which gives a batting average of 400, thus

$$\frac{(0.360)x + y}{x + 5} = 0.400 \Rightarrow (0.360)x + y = (0.4)(x + 5) \Rightarrow y = 0.04x + 2$$

We want to find the smallest number of hits, which is

$$0.36x + 0.04x + 2 = 0.4x + 2.$$

Thus we want to find the smallest possible integer x such that both $0.36x$ and $0.04x + 2$ are integers. Obviously the least possible integer value for the second expression is 3, which means $x = 25$. Then the least number of hits is $(0.4)25 + 2 = \boxed{12.}$ □

Example 8.1.6 (23-24-S6-I-P6)

Fill in numbers in the boxes below so that the sum of the entries in each three consecutive boxes is 2005. What is the number that goes into the leftmost box?

		999					888	

Solution. Suppose that four adjacent boxes contain the numbers a, b, c, and d as read from left to right, see the diagram below.

		999	a	b	c	d	888	

Then

$$\begin{cases} a + b + c = 2005 \\ b + c + d = 2005 \end{cases} \Rightarrow a = d$$

In order words, the number in the boxes repeat every three boxes as shown below

a	b	999	a	b	c	a	888	c

Therefore $b = 888$, $c = 999$, and $a = 2005 - (888 + 999) = \boxed{118.}$ □

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Example 8.1.7 (23-24-S6-I-P7)

Bill's age is one third larger than Tracy's age. In 30 years Bill's age will be one eighth larger than Tracy's age. How many years old is Bill?

Proof. Let Bill's current age be b and Tracy's current age be t. Then

$$\begin{cases} b = t\left(1+\frac{1}{3}\right) \\ (b+30) = (t+30)\left(1+\frac{1}{8}\right) \end{cases} \Rightarrow \begin{cases} 3b = 4t \\ 8b+240 = 9t+270 \end{cases} \Rightarrow 8b = 9\left(\frac{3}{4}\right)b + 30 \Rightarrow \frac{5}{4}b = 30 \Rightarrow b = \boxed{24.}$$

□

Example 8.1.8 (23-24-S6-I-P8)

At a movie theater tickets for adults cost 4 dollars more than tickets for children. One afternoon the theater sold 100 more child tickets than adult tickets for a total sales amount of 1475 dollars. How much money would the theater have taken in if the same tickets were sold, but the costs of the child tickets and adult tickets were reversed?

Solution. Let x be the cost of a child ticket, and let a be the number of adult tickets sold. Then the cost of an adult ticket is $x+4$ and the number of child tickets sold is $a+100$. This means that

$$(x+4)a + x(a+100) = 1475 \Rightarrow 2xa + 4a + 100x = 1475.$$

If the cost of the tickets were reversed, then the theater would have taken in

$$xa + (x+4)(a+100) = 2xa + 4a + 100x + 400 = 1475 + 400 = \boxed{1875.}$$

□

Example 8.1.9 (23-24-S6-I-P9)

A rogue spaceship escapes. 54 minutes later the police leave in a spaceship in hot pursuit. If the police spaceship travels 12% faster than the rogue spaceship along the same route, how many minutes will it take for the police to catch up with the rogues?

Solution. Let x be the number of minutes it will take the police to catch the rogues, and let r be the rate at which the rouges are traveling. Then the police are traveling at a rate of $1.12r$, and the rogues will have traveled for $54+x$ minutes. Since distance = time × rate, thus

$$(54+x)r = x(1.12r) \Rightarrow 54r = (.12)xr \Rightarrow x = \frac{54}{.12} = \boxed{450.}$$

□

Example 8.1.10 (23-24-S6-I-P10)

The sizes of the freshmen class and the sophomore class are in the ratio 5:4. The sizes of the sophomore class and the junior class are in the ratio 7:8. The sizes of the junior class and the senior class are in the ratio 9:7. If these four classes together have a total of 2158 students, how many of the students are freshmen?

Solution. Let the sizes of the freshman, sophomore, junior, and senior classes be w, x, y, and z, respectively. Then

$$4w = 5x,\ 8x = 7y,\ 7y = 9z,\ w + x + y + z = 2158$$

By computing x, y, z based on w using the first three equations and substitute those into the last one:

$$x = \frac{4}{5}w,\ y = \frac{8}{7}x = \frac{8}{7} \cdot \frac{4}{5}w, z = \frac{7}{9}\frac{8}{7} \cdot \frac{4}{5}w$$
$$\Rightarrow 2158 = w + \frac{4}{5}w + \frac{32}{35}w + \frac{32}{45}w = \frac{1079}{315}w \Rightarrow w = \frac{315 \cdot 2158}{1079} = \boxed{630.}$$

□

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8.2 Problems (I-Level)

Submission deadline: November 19, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 8.2.1 (23-24-S6-I-P11)**.** (*10 points*) The product of two positive numbers is equal to 50 times their sum and 75 times their difference. Find their sum.

Solution (Official solution - similar to submissions from Giang Nguyen, Huy Quang Le, Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham) — Let the two numbers be x and y. WLOG, let's assume that $x \geq y$. Then

$$xy = 50(x+y) = 75(x-y) \Rightarrow 25x = 125y \Rightarrow x = 5y \Rightarrow 5y^2 = 300y \Rightarrow y = 60, x = 300 \Rightarrow x+y = \boxed{360.}$$

Remark. Giang Nguyen, Huy Quang Le, Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham have correctly solved the problem.

Problem 8.2.2 (23-24-S6-I-P12)**.** (*10 points*) There were 891 people voting at precinct 91. There were 20 percent more female voters than male voters. How many female voters were there?

Solution (Official solution - similar to submissions from Giang Nguyen, Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham) — Let m and n be the numbers of male and female voters, respectively. Then

$$\begin{cases} m + f = 891 \\ f = (1.2)m \end{cases} \Rightarrow (2.2)m = 891 \Rightarrow m = 405 \Rightarrow f = (1.2)(405) = \boxed{486.}$$

Remark. Giang Nguyen, Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham have correctly solved the problem.

Huy Quang Le incorrectly interpreted the number of female voters, thus provided the number of male voters as the answer.

Problem 8.2.3 (23-24-S6-I-P13)**.** (*10 points*) Find a positive rational number that the number is equal to the ratio of 7 plus the number's reciprocal and 65 minus the number's reciprocal.

Solution (Official solution - similar to submissions from Giang Nguyen, Huy Quang Le, Jeremy Nguyen, and Khoa Anh Vu) — Let x be the positive rational number. Then

$$x = \frac{7 + \frac{1}{x}}{65 - \frac{1}{x}} = \frac{7x+1}{65x-1} \Rightarrow 65x^2 - x = 7x + 1 \Rightarrow 65x^2 - 8x - 1 = 0 \Rightarrow (13x+1)(5x-1) = 0.$$

Since $x > 0$, so $13x + 1 > 0$, thus $5x - 1 = 0$, hence $x = \boxed{\frac{1}{5}.}$

Remark. Giang Nguyen, Huy Quang Le, Jeremy Nguyen, and Khoa Anh Vu have correctly solved the problem.

Giang Nguyen, Jeremy Nguyen, and Khoa Anh Vu, showed clearly how did they solved the quadratic equation $65x^2 - 8x - 1 = 0$, great!

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Problem 8.2.4 (23-24-S6-I-P14). (*10 points*) While driving his car, Ken pulled off the road to get gasoline when he was $\frac{7}{12}$ the way through his trip. After driving another eleven miles, he noticed that he was $\frac{13}{20}$ of the way through his trip. How many miles long was his entire trip?

Solution (Official solution - similar to submissions from Giang Nguyen, Huy Quang Le, Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham) — Let the length (in miles) of the entire trip be ℓ, then

$$\frac{7}{12}x + 11 = \frac{13}{20}x \Rightarrow 11 = x\left(\frac{13}{20} - \frac{7}{12}\right) = x\frac{156-140}{(12)(20)} = x\frac{16}{(12)(20)} = \frac{x}{15} \Rightarrow x = \boxed{165.}$$

Remark. Giang Nguyen, Huy Quang Le, Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham have correctly solved the problem.

Problem 8.2.5 (23-24-S6-I-P15). (*10 points*) A bowl contained 10% blue candies and 25% red candies. A bag containing three quarters red candies and one quarter blue candies was added to the bowl. Now the bowl is 16% blue candies. What percentage of the candies in the bowl are now red?

Solution (Official solution - similar to submissions from Jeremy Nguyen and Julie Cao) — Suppose that the bow originally had x candies. Then $0.1x$ and $0.25x$ are the numbers of blue and red candies, respectively. Note that $0.1 + 0.25 = 0.35 < 1$, thus there are other candies in the bowl that are neither blue, nor red.

By adding y candies to the bowl, where the numbers of blue and red candies are $\left(\frac{1}{4} = 0.25\right)$ $0.25y$ and $\left(\frac{3}{4} = 0.25\right)$ $0.75y$, respectively. Then

$$\frac{0.1x + 0.25y}{x+y} = 0.16 \Rightarrow 0.1x + 0.25y = 0.16x + 0.16y \Rightarrow 0.09y = 0.06x \Rightarrow 2x = 3y.$$

We need to find the percentage

$$\frac{0.25x + 0.75y}{x+y} = 0.25 \cdot \frac{x+3y}{x+y} = 0.25\frac{3x+9y}{3x+3y} = 0.25\frac{3x+6x}{3x+2x} = 0.25\frac{9}{5} = 0.45 \text{ or } \boxed{45\%.}$$

Remark. Jeremy Nguyen and Julie Cao have correctly solved the problem.

Giang Nguyen was very close to the solution but made some unfortunate wrong reasoning.

Problem 8.2.6 (23-24-S6-I-P16). (*10 points*) One plant is now 44 centimeters tall and will grow at a rate of 3 centimeters every 2 years. A second plant is now 80 centimeters tall and will grow at a rate of 5 centimeters every 6 years. In how many years will the plants be the same height?

Solution (Official solution - similar to submissions from Giang Nguyen, Huy Quang Le, Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham) — Let n be the number of years after which the plants will be the same height. The first plan will reach a height at $44 + \frac{3}{2}n$, the second $80 + \frac{5}{6}n$, then

$$44 + \frac{3}{2}n = 80 + \frac{5}{6}n \Rightarrow n\left(\frac{3}{2} - \frac{5}{6}\right) = 36 \Rightarrow n\left(\frac{4}{6}\right) = 36 \Rightarrow n = \boxed{54.}$$

Remark. Giang Nguyen, Huy Quang Le, Jeremy Nguyen, Julie Cao, Khoa Anh Vu, and Minh Nhat Pham have correctly solved the problem.

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Problem 8.2.7 (23-24-S6-I-P17)**.** (*10 points*) Tom can run to Beth's house in 63 minutes. Beth can run to Tom's house in 84 minutes. At noon Tom starts running from his house toward Beth's house while at the same time Beth starts running from her house toward Tom's house. When they meet, they both run at Beth's speed back to Beth's house. At how many minutes after noon will they arrive at Beth's house?

Solution (Official solution - similar to submissions from Jeremy Nguyen, Julie Cao and Minh Nhat Pham) — For easy counting, let's assume that the distance from Tom's to Beth's house is d. Then Tom runs a distance $\frac{d}{63}$ every minute, Beth runs a distance $\frac{d}{84}$ every minute. When they run *towards*, together they cover a the distance of

$$\frac{d}{63}+\frac{d}{84}=\frac{d}{7\cdot 3\cdot 3}+\frac{d}{7\cdot 3\cdot 4}=\frac{7d}{7\cdot 3\cdot 3\cdot 4}=\frac{d}{36}.$$

Hence it will take them 36 minutes to meet each other. Now, this is exactly the time Beth needs to run back to her house at her speed. Thus the time when they both arrive at Beth's house is $36+36=\boxed{72}$ minutes after noon.

Remark. Jeremy Nguyen, Julie Cao and Minh Nhat Pham have correctly solved the problem.

Huy Quang Le successfully computed Tom's speed.

Problem 8.2.8 (23-24-S6-I-P18)**.** (*10 points*) Bill bought 13 notebooks, 26 pens, and 19 markers for 25 dollars. Paula bought 27 notebooks, 18 pens, and 31 markers for 31 dollars. How many dollars would it cost Greg to buy 24 notebooks, 120 pens, and 52 markers?

Solution (Official solution) — Let x,, y, and z be the cost in dollars of a notebook, a pen, and a marker, then,

$$\begin{cases} 13x+26y+19z=25 \\ 27x+18y+31z=31 \end{cases}$$

Obviously we cannot solve a three-variable system of equations containing only two equation. But, do we really need to solve it? All what we need to do is to find out the value of

$$24x+120y+52z=2(12x+60y+26z)$$

Note that $12=13\cdot 3-27, 60=3\cdot 26-18, 26=3\cdot 19-31$, thus

$$24x+120y+52z=2\left(3(13x+26y+19z)-(27x+18y+31z)\right)=2(3\cdot 25-31)=\boxed{88}.$$

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Problem 8.2.9 (23-24-S6-I-P19)**.** (*10 points*) Purple College keeps a careful count of its students as they progress each year from the freshman class to the sophomore class to the junior class and, finally, to the senior class. Each year at the college one third of the freshman class drops out of school, 40 students in the sophomore class drop out of school, and one tenth of the junior class drops out of school.

Given that the college only admits new freshman students, and that it wants to begin each school year with 3400 students enrolled, how many students does it need to admit into the freshman class each year?

Solution (Official solution - similar to submissions from Huy Quang Le, Jeremy Nguyen, and Minh Nhat Pham) — Let the number of students in each new freshman class be n. Since one third of the freshman class drops out of school, the number of students in each sophomore class is $\frac{2}{3}n$; 40 students in the sophomore class drop out of school, so the number of students in each junior class is $\frac{2}{3}n - 40$; one tenth of the junior class drops out of school, thus the number of students in each senior class is $\frac{9}{10}\left(\frac{2}{3}n - 40\right)$.

Now,

$$\begin{aligned} &n + \left(\frac{2}{3}n\right) + \left(\frac{2}{3}n - 40\right) + \frac{9}{10}\left(\frac{2}{3}n - 40\right) = 3400 \\ &\Rightarrow n\left(1 + \frac{2}{3} + \frac{2}{3} + \frac{3}{5}\right) = 3400 + 40 + 36 \\ &\Rightarrow n\frac{44}{15} = 3476 \Rightarrow n = \boxed{1185.} \end{aligned}$$

Remark. Huy Quang Le, Jeremy Nguyen, and Minh Nhat Pham have correctly solved the problem.

Problem 8.2.10 (23-24-S6-I-P20)**.** (*10 points*) Sally's salary in 2006 was \$37,500. For 2007 she got a salary increase of x percent. For 2008 she got another salary increase of x percent. For 2009 she got a salary decrease of $2x$ percent. Her 2009 salary is \$34,825.

Suppose instead, Sally had gotten a $2x$ percent salary decrease for 2007, an x percent salary increase for 2008, and an x percent salary increase for 2009. What would her 2009 salary be then?

Solution (Official solution - similar to submissions from Jeremy Nguyen, Julie Cao, and Minh Nhat Pham) — In order to make it easier to read, let $y\frac{x}{100}$, then Sally's salary in 2009 is

$$37500(1+y)(1+y)(1-2y) = 34825.$$

In the second case, her salary would be

$$37500(1-2y)(1+y)(1+y)$$

which is still equal to $\boxed{34825.}$

Note: The value of y when $37500(1+y)(1+y)(1-2y) = 34825$ is $y \approx 14.715$

Remark. Jeremy Nguyen, Julie Cao, and Minh Nhat Pham have correctly solved the problem.

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8.3 The Locus (A-Level)

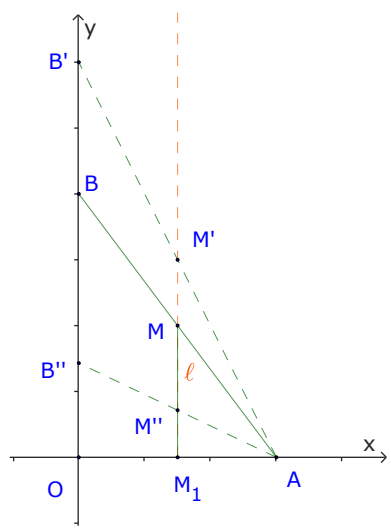
In geometry, a *locus* (plural: *loci*) (Latin word for "place", "location") is a set of all points (commonly, a line, a line segment, a curve or a surface), whose location satisfies or is determined by one or more specified conditions.

A proof for the hypothesis about the existence of a locus - a set of of points S,, must contains three parts:

- if point P satisfies certain condition C then P must be an element of the set S, and
- if point P is an element of S then P must satisfies the condition C.

Example 8.3.1 (23-24-S6-A-P1)

Let A be a fixed point on ray $\vec{x}$. B is an arbitrary point on ray $\vec{y}$, as shown below. Find the locus of the midpoints M of segments AB.



Solution. By choosing a a few locations for B, we establish a hypothesis that

The locus of the midpoints M of the segment AB is the ray ℓ through the midpoint M_1 of OA and perpendicular to the ray $\vec{x}$ (*)

We have to prove the hypothesis in two steps:

Claim — If M is the midpoint of AB then M is on line ℓ.

Proof. $\triangle AOB$ is a right triangle, $\angle AOB = 90°$, M is midpoint of AB, thus MM_1 is the median segment of $\triangle AOB$, thus $MM_1 \parallel OB$.

$\ell \perp OA$, so $\ell \parallel OB$. Since ℓ is through M_1, therefore M_1 must be on ℓ. ■

Claim — If M is the intersection of AB with ℓ, then M must be the midpoint of AB

Proof. $\triangle AOB$ is a right triangle, $\angle AOB = 90°$, M_1 is midpoint of OA, $\ell \perp OA$, so $\ell \parallel OB$, since ℓ is through M_1, therefore MM_1 is the median segment of $\triangle AOB$, hence M is the midpoint of AB. ■

□

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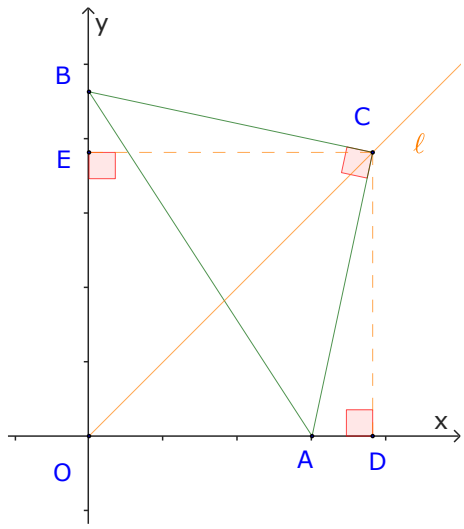
Example 8.3.2 (23-24-S6-A-P2)

Let A be a fixed point on ray $\vec{x}$. B is an arbitrary point on ray $\vec{y}$, as shown below. Find the locus of points C such that $\triangle ABC$ is an isosceles right triangle, $\angle ACB = 90°$, $CA = CB$.



Solution. By choosing a a few locations for C, we establish a hypothesis that

The locus of points C such that $\triangle ABC$ is an isosceles right triangle, $\angle ACB = 90°$, $CA = CB$; is the ray ℓ through O and bisects the angle $\angle xOy$ (*)

We have to prove the hypothesis in two steps:

Claim — If C is a point such that $\triangle ABC$ is an isosceles right triangle, $\angle ACB = 90°$, $CA = CB$, then C is on ℓ.

Proof. Let D, E be the feet of the perpendiculars from C to OA an OB. $\angle CAD = \angle CBE$ ($CA \perp CB$, $AD \perp BE$), both $\triangle CAD$, $\triangle CBE$ are right triangles so $\angle DCA = \angle ECB$, and $CA = CB$, thus $\triangle CAD \cong \triangle CBE$. Therefore $CD = CE$, or C is on ℓ, the angle bisector of $\angle xOy$. ■

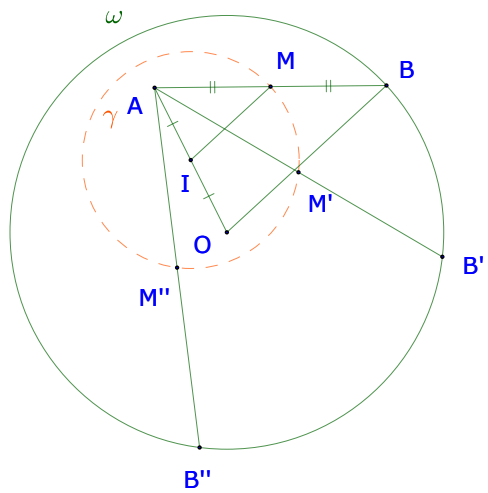
Claim — If C is on ℓ, B is on $\vec{y}$ such that $\angle ACB = 90°$, then $CA = CB$,

Proof. Let D, E be the feet of the perpendiculars from C to OA an OB. Since $CD = CE$, $CA \perp CB$, similarly as $\triangle CAD \cong \triangle CBE$, hence $CA = CB$. ■

□

Example 8.3.3 (23-24-S6-A-P3)

Given the circle ω centred at O. A is a fixed point inside the circle, B is a point moving on the perimeter of the circle. Find the locus of the midpoints M of the segment AB.



Solution. By choosing a a few locations for B, we establish a hypothesis that

The locus of the midpoints M of the segment AB is the circle γ centred at I, the midpoint of OA, with the radius equal to half of the radius of ω.

We have to prove the hypothesis in two steps:

Claim — If M is a midpoint of a segment AB, then M is on γ.

Proof. Let R be the radius of ω. Since M, I are the midpoints of AB, AO, thus MI is the median segment, $MI = \frac{1}{2}OB = \frac{1}{2}R$. Thus M is on γ. ■

Claim — If M is on γ, segment AB contains M such that $AM = MB$, then B is on the circle ω.

Proof. Since M, I are the midpoints of AB, AO, thus MI is the median segment, $MI = \frac{1}{2}OB$. But M is on γ, so $OB = 2MI = R$, hence B is on ω. ■

□

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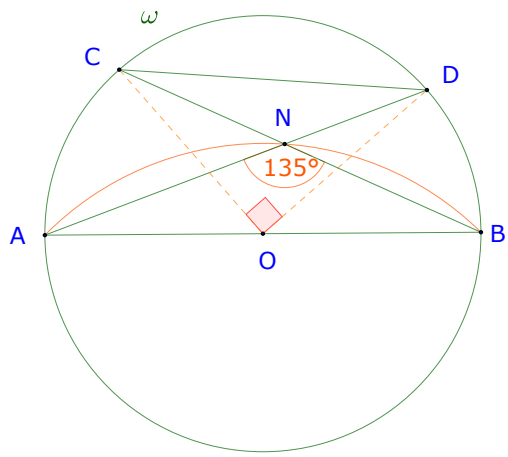
Example 8.3.4 (23-24-S6-A-P4)

ω is a semi-circle centred at O with diameter AB, radius R. C, D are arbitrary points on ω such that $CD = R\sqrt{2}$. AD intersects BC at N. Find the locus of such points N.



Solution. By choosing a a few locations for C, we establish a hypothesis that

The locus of the intersection N of the segments AD and BC is the set of points from an arc through A, B such that for each point N on the arc $\angle ANB = 135°$.

We have to prove the hypothesis in two steps:

Claim — If N is the intersection of the segments AD and BC then $\angle ANB = 135°$.

Proof. Since $CD = R\sqrt{2}$, so CD is a side of a square inscribed inside the circle centred at O radius R.

$$\widehat{CD} = 90° \Rightarrow \angle ANB = \frac{1}{2}\left(\widehat{AB} + \widehat{CD}\right) = \frac{1}{2}180° + 90° = 135°.$$

■

Claim — Now, if $\angle ANB = 135°$, AN and BN intersect γ at D and C, respectively. Then $CD = R\sqrt{2}$.

Proof. Since $\angle ANB = 135°$, $\angle ANB = \frac{1}{2}\left(\widehat{AB} + \widehat{CD}\right)$, so $\widehat{CD} = 90°$, CD is a side of a square inscribed inside the circle centred at O radius R, hence $CD = R\sqrt{2}$. ■

□

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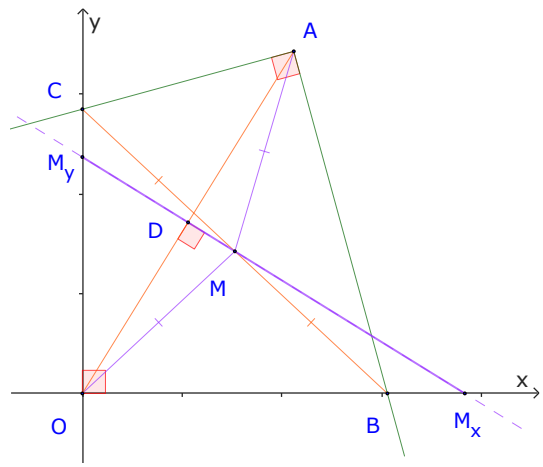
8.4 Problems (A-Level)

Problem 8.4.1 (23-24-S6-A-P5)**.** (*15 points*) A is a point in the quadrant determined by rays $\vec{x}$ and $\vec{y}$, as shown below. A right angle with A as its vertex rotates around A. The rays of the right angle intersect $\vec{x}$ at B and $\vec{y}$ at C, respectively. Find the locus of the midpoints M of the segment BC.



Solution (Official solution - similar to submissions from An Nguyen, Antoine Dao, Chi Khanh Pham, Karl Le, Minh Nhat Trinh, Samuel Luong) — First, M is the midpoint of the hypotenuse BC of both right triangles ABC and OBC, thus $MA = \frac{1}{2}BC = MO$. This means that M is on the perpendicular bisector of OA. Our hypothesis is that:

The locus of the midpoints M of the segment BC is the set of points constituting the segment M_xM_y.

We prove the hypothesis in two steps:

Claim — If M is the midpoint of BC then M is on the segment M_xM_y.

Proof. Since we have proved that M is on the perpendicular bisector of OA and since M is between B and C so M must be in the quadrant determine by the rays $\vec{x}$ and $\vec{y}$. The intersection of the perpendicular bisector of OA with the quadrant determine by the rays $\vec{x}$ and $\vec{y}$ is exactly the M_xM_y segment, thus M must be on the segment M_xM_y. ■

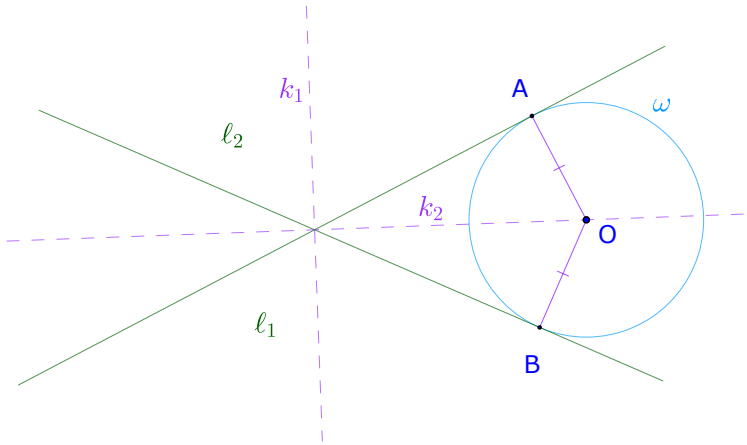
Claim — Now, if M is on the segment M_xM_y, then a segment BC with M is midpoint must form a right angle BAC.

Proof. Since M is the midpoint of BC so M is the midpoint of the hypotenuse BC of the right triangle OBC, thus $MO = \frac{1}{2}BC$. Now M_xM_y is the perpendicular bisector of OA, thus $MA = MO = \frac{1}{2}BC$. This means that A is on the circle centered at M with diameter BC. Hence $\angle BAC = 90°$. ■

Remark. An Nguyen, Antoine Dao, Chi Khanh Pham, Karl Le, Minh Nhat Trinh, Samuel Luong have successfully solved the problem.

Some students proved only one part of the hypothesis. Some students successfully establish a hypothesis, but did not proceed to prove them.

Problem 8.4.2 (23-24-S6-A-P6). (*15 points*) Lines ℓ_1, ℓ_2 intersect at O. Find the locus of the centres of the circle ω that is tangent to both lines.



Solution (Official solution - similar to submissions from An Nguyen, Antoine Dao, Chi Khanh Pham, Jason To, Karl Le, and Minh Hai Vu) — First, let A, B be the tangent points of the circle ω with the lines ℓ_1 and ℓ_2. Then $OA = OB$, which implies that O is on one of the angle bisectors k_1 and k_2 of the four angles form by the lines ℓ_1 and ℓ_2. Our hypothesis is that:

The locus of the centres of the circle ω that is tangent to both lines ℓ_1 and ℓ_2 is the set of points on the angle bisectors k_1 and k_2 of the four angles form by the lines ℓ_1 and ℓ_2.

We prove the hypothesis in two steps:

Claim — If O the center circle ω that is tangent to both lines ℓ_1 and ℓ_2, then O is on one of the angle bisectors k_1 and k_2.

Proof. The proof is obvious based on our exploration above. ■

Claim — O is on one of the angle bisectors k_1 and k_2, then there exists a circle ω centred at O that is tangent to both lines ℓ_1 and ℓ_2,

Proof. This, again, is obvious based on the property of the angle bisector. ■

Remark. An Nguyen, Antoine Dao, Chi Khanh Pham, Jason To, Karl Le, and Minh Hai Vu have successfully solved the problem.

Some students made minor mistakes. Ryan Nguyen could not complete the solution.

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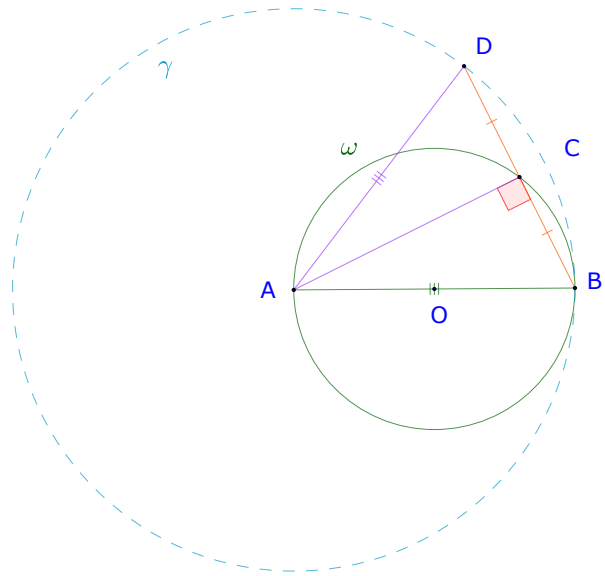
Problem 8.4.3 (23-24-S6-A-P7). (*15 points*) Given the circle ω centred at O with diameter AB. C is a point moving on the perimeter of the circle ω. Point D is on ray $\overrightarrow{BC}$ towards C such that $BC = CD$, as shown below. Find the locus of such points D.



Solution (Official solution - similar to submissions from most of the students) — First, since AB is the diameter and C is on the perimeter of ω, thus $\triangle ABC$ is right at C. $CD = CB$, so AC is both altitude and median of the triangle DAB thus $\triangle DAB$ is isosceles, so $DA = AB$. Then D is on the circle γ centered at A, radius AB. Our hypothesis is that:

The locus of points D, such that the circle ω bisects the segment BD, is the circle γ centered at A, radius AB.

We prove the hypothesis in two steps:

Claim — If D is a point such that the circle ω bisects the segment BD, then D is on the perimeter of the circle γ centered at A, radius AB.

Proof. This is what we just proved above. ■

Claim — If D is on the perimeter of the circle γ centered at A, radius AB, then the circle ω bisects the segment BD.

Proof. if D is on the perimeter of the circle γ centered at A, radius AB, then C, the inter section of the circle ω with BD is the foot of the altitude from A to BD. Since $AD = AB$ so $\triangle DAB$ is isosceles, thus C is also the midpoint of BD. Hence the circle ω bisects the segment BD. ■

Remark. Most of the students have successfully solved the problem.

Ryan Nguyen could not provide sufficient arguments for a complete solution.

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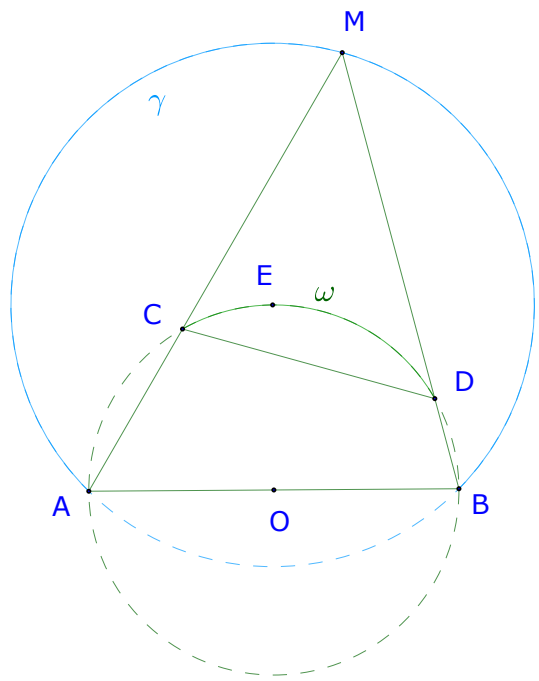
Problem 8.4.4 (23-24-S6-A-P8)**.** (*15 points*) ω is a semi-circle centred at O with diameter AB, radius R. C, D are arbitrary points on ω such that $CD = R\sqrt{2}$. AC intersects BD at M. Find the locus of such points M.



Solution (Official solution - similar to submissions from An Nguyen, Chi Khanh Pham, and Karl Le) — The measure of the $\angle CMD = \frac{1}{2}\left(\widehat{AB} - \widehat{CD}\right)$. Now, since $\widehat{CD} = 90°$, since it is the side of a square inscribed in the circle ω ($\angle COD = 90°$), and since $\widehat{AB} = 180°$, thus $\angle CMD = 45°$. Our hypothesis is as follows,

The locus of the intersection M of the segments AC and BD is the set of points from an arc through A, B such that for each point N on the arc $\angle AMB = 45°$.

We have to prove the hypothesis in two steps. The first one we have already proved as above.

Claim — If M is the intersection of the segments AC and BD then $\angle AMB = 45°$.

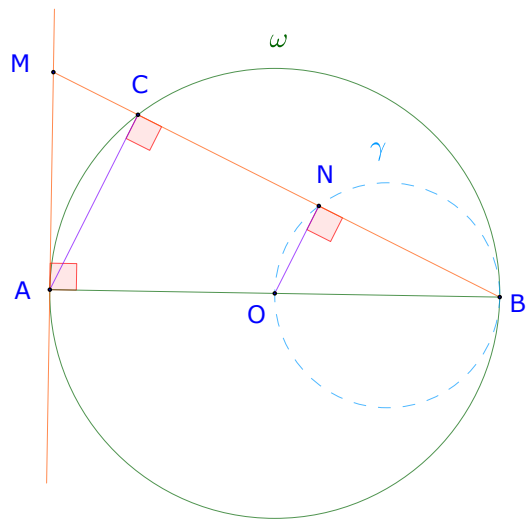
Claim — Now, if $\angle AMB = 45°$, AM and BM intersect ω at C and D, respectively. Then $CD = R\sqrt{2}$.

Proof. Since $\angle AMB = 45°$, $\angle AMB = \frac{1}{2}\left(\widehat{AB} - \widehat{CD}\right)$, so $\widehat{CD} = 90°$, CD is a side of a square inscribed inside the circle centred at O radius R, hence $CD = R\sqrt{2}$. ■

Remark. An Nguyen, Chi Khanh Pham, and Karl Le have successfully solved the problem.

Ryan Nguyen should have followed the two-part arguments for a complete solution.

Problem 8.4.5 (23-24-S6-A-P9)**.** (*20 points*) Given the circle ω centred at O with diameter AB. ℓ is the tangent line of ω at A. M is an arbitrary point moving ℓ. BM intersects ω at C. N is such a point on BC that $BN = \frac{1}{2}BC$. Find the locus of such points N.



Solution (Official solution - similar to submissions from An Nguyen, Antoine Dao, Chi Khanh Pham, Jason To, Karl Le, and Samuel Luong) — We show how the hypothesis is formed and leave the proof (which is simple) for the readers.

Note that N is the midpoint of CB and O is the midpoint of AB, thus NO is a mid-segment in $\triangle ABC$, thus $NO \parallel AC$, then $NO \perp BC$, thus $\triangle ONB$ is a right triangle at N.

Therefore, *the locus of points N where $BN = \frac{1}{2}BC$, is the circle γ diameter OB.*

Remark. An Nguyen, Antoine Dao, Chi Khanh Pham, Jason To, Karl Le, and Samuel Luong have successfully solved the problem.

Tran Hieu Khoa, Ryan Nguyen managed some attempts.

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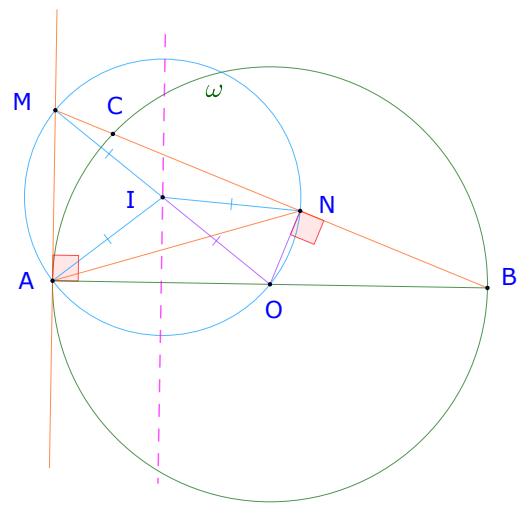
Problem 8.4.6 (23-24-S6-A-P10)**.** (*20 points*) Given the circle ω centred at O with diameter AB. ℓ is the tangent line of ω at A. M is an arbitrary point moving ℓ. BM intersects ω at C. N is the midpoint of BC. I is the circumcentre of $\triangle AMN$. Find the locus of such points I.



Solution (Official solution - similar to submissions from Antoine Dao, Chi Khanh Pham, and Karl Le) — We show how the hypothesis is formed and leave the proof (which is simple) for the readers.

From the previous problem, $\triangle ONB$ is a right triangle at N, thus N is on the circle diameter MO. Since A is also on the circle diameter MO, so A, M, N, O on the circle centred at I, or $IA = IO$.

Therefore, *the locus of points I circumcentre of $\triangle AMN$ is the perpendicular bisector of AO.*

Remark. Antoine Dao, Chi Khanh Pham, and Karl Le have successfully solved the problem.

An Nguyen, Ryan Nguyen managed some attempts.

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8.5 Solving Problems with Induction Principle (O-Level) - Part 2

Submission deadline: November 19, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 8.5.1 (Fundamental Theorem of Arithmetic)**.** (*10 points*) Prove that every positive integer $n \geq 2$ is expressible as a product of primes in exactly one way. In particular, if

$$p_1 p_2 \cdots p_k = q_1 q_2 \cdot q_\ell, \text{ where } p_1, p_2, \ldots p_k, q_1, q_2, \ldots q_l \text{ are primes,}$$

then $k = l$ and $q_1, q_2, \ldots q_l$ are $p_1, p_2, \ldots p_k$ in some order.

Remark. Here we take an unusual application of the Induction Principle, we prove $P(n)$ statement is true by induction on the number of (prime) (and not necessarily distinct) factors of n. The problem consists two distinct statements: (i) every positive integer $n \geq 2$ is expressible as a product of primes; (ii) the expression is unique.

For the first one perhaps you need to consider a proof by contradiction, equipped with the Extremal Principle on the minimal n *not* satisfying, Proving the lemma on the existence of infinitely many primes will help.

For the second, start with the base case when n has a single prime factor. For the inductive step, use a corollary from the Bezout's theorem: if p prime divides a product ab then $p \mid a$ or $p \mid b$.

Solution (Official solution - similar to submissions from Albert Dinh-Le, Anthony Pham, Henry Dinh, and Minh Le Phuong Nguyen) — We prove the statement in two parts.

Claim — Every positive integer $n \geq 2$ is expressible as a product of primes.

Proof. Assume the contrary, then by the Extremal principle, there exists a minimal n that cannot be written as product of primes. If n is a prime, then it is a product of one prime, thus n is not a prime. Then there exist $1 < a, b < n$ integers such that $n = a \cdot b$. By the Extremal principle, a and b must be product of primes, thus n is. This is a contradiction, thus n is expressible as a product of primes. ■

Claim — There is a single way to express n as a product of primes.

Proof. We prove by induction on $k + \ell$, in other words

$$n = p_1 p_2 \cdots p_k = q_1 q_2 \cdot q_\ell, \text{ where } p_1, p_2, \ldots p_k, q_1, q_2, \ldots q_\ell \text{ are primes,}$$

then $k = \ell$ and $q_1, q_2, \ldots q_\ell$ are $p_1, p_2, \ldots p_k$ in some order.

It is easy to see that if $k + \ell = 2$, then $k = \ell = 1$, otherwise we have 1 as a product of two primes, which is impossible.

Now consider the case $l + k$,

$$p_1 p_2 \cdots p_k = q_1 q_2 \cdot q_\ell \Rightarrow p_1 \mid q_1 q_2 \cdot q_\ell \Rightarrow \exists i \in \{1, \ldots, \ell\} p_1 \mid q_i \Rightarrow p_1 = q_i \Rightarrow p_2 \cdots p_k = q_1 \cdots q_{i-1} q_{i+1} \cdots q_\ell$$

By the induction hypothesis for $k + \ell - 2$, $k = 1 = \ell - 1$, $q_1, \ldots, q_{i-1}, q_{i+1} \ldots q_\ell$ are $p_2, \ldots p_k$. Thus $k = \ell$, and $q_1, q_2, \ldots q_\ell$ are $p_1, p_2, \ldots p_k$ in some order.

Therefore the statement stands. ■

Remark. Albert Dinh-Le, Anthony Pham, Henry Dinh, and Minh Le Phuong Nguyen have successfully solved the problem.

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Problem 8.5.2 (Another p and $\left\lfloor \frac{n}{p} \right\rfloor$ technique)**.** (*10 points*) Let n be a positive integer. Does n^2 have more positive divisors of the form $4k+1$ or of the form $4k-1$?

Remark. Take a few small n and test them. By doing so you form your own hypothesis. Remember to take some other n *after forming* the hypothesis to test the hypothesis!

Now, let $M(n)$ be the number of positive divisors of n in the form $4k-1$, $P(n)$ be the number of positive divisors of n in the form $4k+1$.

1. How $M(n)$ is compared to $P(n)$ if $n = 2^k$.
2. For the inductive step, consider np^α ($p \nmid n$) in two cases $p \equiv 1 \pmod 4$, and $p \equiv -1 \pmod 4$.

Makes sure the base case and the inductive step cover all possible $n \geq 2$.

Solution (Official solution - similar to submissions from Anthony Pham and Henry Dinh) — Let $M(n)$ be the number of positive divisors of n^2 in the form $4k-1$, $P(n)$ be the number of positive divisors of n^2 in the form $4k+1$.

It is easy to see that $M(2^k) = 0 < 1 = P(2^k)$.

We prove by strong induction based on the number of prime powers in prime factorization of n,

Claim — $M(n) < P(n)$.

Proof. Let p be a prime such that $p \nmid n,\ p \equiv 1 \pmod 4$. Then every divisor of $(p^\alpha n)^2 = p^{2\alpha}n^2$ of the form $4k-1$ is a power of p times a divisor of n^2 of the form $4k-1$.

Similarly every divisor of $(p^\alpha n)^2 = p^{2\alpha}n^2$ of the form $4k+1$ is a power of p times a divisor of n^2 of the form $4k+1$. Hence

$$M(p^\alpha n) = (2\alpha_1)M(n),\ P(p^\alpha n) = (2\alpha_1)P(n)$$
$$M(n) < P(n) \Rightarrow M(p^\alpha n) < P(p^\alpha n) \quad (*)$$

Now, consider p be a prime such that $p \nmid n,\ p \equiv -1 \pmod 4$. Then every divisor of $(p^\alpha n)^2 = p^{2\alpha}n^2$ of the form $4k-1$ is either p^{2r} times a divisor of n^2 of the form $4k-1$ $(0 \leq r \leq \alpha)$ or p^{2r-1} times a divisor of n^2 of the form $4k+1$ $(1 \leq r \leq \alpha)$.

Similarly every divisor of $(p^\alpha n)^2 = p^{2\alpha}n^2$ of the form $4k+1$ is either p^{2r} times a divisor of n^2 of the form $4k+1$ $(0 \leq r \leq \alpha)$ or p^{2r-1} times a divisor of n^2 of the form $4k-1$ $(1 \leq r \leq \alpha)$. Hence

$$M(p^\alpha n) = (\alpha_1+1)M(n) + \alpha P(n),\ P(p^\alpha n) = (\alpha_1+1)P(n) + \alpha M(n)$$
$$M(n) < P(n) \Rightarrow M(p^\alpha n) < P(p^\alpha n) \quad (**)$$

(*) and (**) imply that the hypothesis stands for $p^\alpha n$. ■

Remark. Anthony Pham and Henry Dinh have successfully solved the problem.

Albert Dinh-Le, Minh Le Phuong Nguyen provided some correct partial arguments.

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Problem 8.5.3 (Algebraic manipulations on divisibility and Fermat's Little Theorem)**.** Prove that for all $n \in \mathbb{Z}^+$.

(*5 points*) $17 \mid 2^{5n+3} + 5^n \cdot 3^{n+2}$.

(*5 points*) $p \mid n^p - n$, for any p prime.

Remark. The first one is just an warmup exercise. Pay attention to the ways the powers are manipulated

$$2^{5n+8} = 2^{5n+3} \cdot 34 - 2^{5n+3} \cdot 2,\ 5^{n+1} \cdot 3^{n+3} = 5^n \cdot 3^{n+2} \cdot 17 - 5^n \cdot 3^{n+2} \cdot 2.$$

For the Fermat's Little Theorem, in the inductive step,

$$(n+1)^p - (n+1) = \left(n^p + \binom{p}{1} n^{p-1} + \cdots + \binom{p}{p-1} n + 1\right) - (n+1)$$

For what $k:\ 1 \le k \le p-1$, $p \mid \binom{p}{k}$?

Solution (Official solution - similar to submissions from Albert Dinh-Le, Anthony Pham, Henry Dinh, and Minh Le Phuong Nguyen) — Let $P(n):\ 17 \mid 2^{5n+3} + 5^n \cdot 3^{n+2}$. Then $P(0)$ is true because $17 \mid 2^3 + 5^0 \cdot 3^2 = 17$. Assume $P(k)$ is true, for all $k \le n$, then

$$\begin{aligned}
&2^{5(n+1)+3} + 5^{n+1} \cdot 3^{(n+1)+2} = 2^{5n+8} + 5^{n+1} \cdot 3^{n+3} \\
&= (2^{5n+3} \cdot 34 - 2^{5n+3} \cdot 2) + (5^n \cdot 3^{n+2} \cdot 17 - 5^n \cdot 3^{n+2} \cdot 2) \\
&= 2^{5n+3} \cdot 17 + 17(2^{5n+3} + 5^n \cdot 3^{n+2}) - 2(2^{5n+3} + 5^n \cdot 3^{n+2}) \quad (*)
\end{aligned}$$

The expression in (*) is a multiple of 17, thus $P(n+1)$ is true. Therefore $P(n)$ is true for all n.

For the second question, let analyze the inductive step. Note that

$$(n+1)^p - (n+1) = \left(n^p + \binom{p}{1} n^{p-1} + \cdots + \binom{p}{p-1} n + 1\right) - (n+1)$$

Since $\binom{p}{i} = p \cdot \frac{(p-1)!}{i!(p-i)}$, thus it is a multiple of p, therefore the inductive step is obvious base on $p \mid n^p - n$.

Remark. Albert Dinh-Le, Anthony Pham, Henry Dinh, and Minh Le Phuong Nguyen have successfully solved the problem.

Problem 8.5.4 (Another sum of divisors)**.** (*10 points*) Prove that for $n \geq 1$, the number $n!$ has the following property: to any divisor of $n!$, except $n!$, we can add another divisor of $n!$ so that their sum is a divisor of $n!$.

Remark. The Induction Principle by n looks obvious. There are more nuances with the inductive step.

If $k \mid (n+1)!$, then there exist a, b so that $k = ab$, where $a \mid n!, b \mid n+1$ (why?)

Explore the cases $a < n!$ and $a = n!$, you might have to pay attention to whether $n+1$ is a prime.

Solution (Official solution - similar to submissions from Anthony Pham and Henry Dinh) — Consider an induction based on $n \geq 2$ The case $n = 2$ is obvious $1 + 1 = 2!$.

Now, for the inductive step.

Let $k \mid (n+1)!$, then there exist a, b so that $k = ab$, where $a \mid n!, b \mid n+1$.

Case 1: $a < n!$, then by induction hypothesis,

$$\exists c \mid n!,\ a + c \mid n! \Rightarrow k + cb = ab + cb = (a+c)b \mid n!(n+1) = (n+1)!.$$

Case 2: $a = n!$, then $b < n+1$.

Case 2a: $n+1$ is composite, then

$$\exists c > 1,\ n+1 = bc \Rightarrow c < n \Rightarrow \frac{n!}{c} \mid n! \Rightarrow \exists d,\ \frac{n!}{c} + d \mid n!$$
$$\Rightarrow k + (n+1)d = n!b + (n+1)d = \frac{(n+1)!}{c} + (n+1)d = \left(\frac{n!}{c} + d\right)(n+1) \mid n!(n+1) = (n+1)!$$

Remark. Anthony Pham and Henry Dinh have successfully solved the problem.

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Problem 8.5.5 (Circular route with n cars)**.** (*10 points*) On a circular route, there are n identical cars. Together they have enough gas for one car to make a complete tour. Prove that there is a car that can make a complete tour by taking gas from all the cars that is encounters.

Remark. In the inductive step, let prove that:

1. There exist cars A and B such that A has enough gas to get to B.
2. Give the gas of B to A and remove B to obtain a starting point of the hypothesis for n.

Make sure to correctly conclude the proof.

Solution (Official solution - similar to one submission from Henry Dinh) — We will prove this by induction on n. It is clearly true for $n = 1$.

Consider $n + 1$ cars. Clearly there is a car, say A has enough gas to reach another car, say B. (Why there is such A? Prove by contradiction can help.) The consider a configuration without B. A now has the total amount of the gas of A and B. It is clear that this configuration satisfies the induction hypothesis. Thus, there is a car C which can make a complete tour by taking gas from all the cars that is encounters. Now, if C is A then it is equivalent to the case A goes to B and takes B gas and then continue. If C is not A, then C will be able to reach A, which then has enough gas to reach B, and then the next car, which is not A.

Solution (Solution 2 from Henry Dinh) — Let the cars be $C_1, C_2, \ldots, C_n$. Let's imagine that we have a similar car Z with a huge tank that has more than enough gas to complete a tour. Let this car Z start from the location of some car C_i on the circular route, and takes all the gas from all cars $C_1, \ldots, C_n$ it encounters, and complete the tour.

WLOG, let C_1 be a car where the level of gas in the car Z is the lowest when Z encounters C_1. We claim that if the car C_1 is the starting point and travel in the same direction as the car Z, then it will be able to complete a tour by taking gas from all the cars that it encounters on its way.

Suppose not, that C_1 runs out of gas when getting from car C_j to car C_{j+1} for some $1 \leq j \leq n - 1$. Then the amount of gas required to travel from car C_1 to car C_{j+1} is more than the total amount of gas from cars $C_1, C_2, \ldots, C_{j+1}$. That implies that the level of gas in the car Z when meeting car C_j is strictly lower than its level of gas when Z encountered C_1, which is a contradiction to the minimality of the level of gas when Z encounters C_1. Therefore, C_1 can make a complete tour.

Remark. Henry Dinh has successfully solved the problem. His second solution is very subtle use of the Extremal Principle.

Albert Dinh-Le, Anthony Pham, and Minh Le Phuong Nguyen submitted good solutions with some minor weaknesses.

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Problem 8.5.6 (Santa Claus coming to town)**.** (*10 points*) Santa Claus has $n \geq 2$ different presents and several identical bags. Each bag contains exactly two objects: either two bags, or a bag and a presents, or two presents. In particular, the big bag of presents Santa Claus has contains two objects. How many ways are there to split the presents into bags?

Remark. Test out the first few cases to obtain an hypothesis on the formula of the number of ways. You will see some similarity to the permutation $n!$. Now, in the inductive step, what happens if we remove the $n+1$ present? Or what happen if we add another present? Pay attention to the number of bags.

Solution — We show by induction that the number of ways is $(2n-3)!! = 1 \cdot 3 \cdot 5 \cdots (2n-3)$, and the number of needed bags is $n-1$. For $n = 2$ we put two presents into a bag. There is no other way.

Lets assume we have $n+1$ present, and we find a ways to put them in bags as required. Then the $(n=1)^{1^{\text{st}}}$ present is in a bag with another a bag, or another present. Let's remove the $(n=1)^{1^{\text{st}}}$ present, the bag that contains it, but not the content of the bag. Then the remaining configuration is a valid way. Conversely, given a valid way to distribute n presents, we just need to add the $(n=1)^{1^{\text{st}}}$ present to any of the existing objects - a bag or a present into a new bag.

Now, by induction, for n present we use $2n-1$ objects: n present and $n-1$ bags, thus $2n-1$ objects. To add a new object we have $2n-1$ choices for any of these $2n-1$ objects and perform the procedure as described above. Thus the number of valid ways now is $(2n-3)!! \cdot (2n-1) = \boxed{(2n-1)!!.}$

Problem 8.5.7 (The number of edges in a tree)**.** (*10 points*) Prove that if the graph G is a tree with n vertices then it has exactly $n-1$ edges.

Read the Graph Theory chapter in the LPS V3 book for concepts and definitions.

Remark. The key observation is the following. Suppose we select a vertex of G and start to *walk* on the edges from that vertex, forming a trail $v_1, v_2, \ldots,$ By the Extremal Principle, the trail has to stop at some vertex (why?). The trail cannot self-intersect (why?). The trail will stop at a vertex of degree 1 (only one edge coming in/out of that vertex) (why?)

Solution (Official solution - similar to submissions from Albert Dinh-Le, Henry Dinh, and Minh Le Phuong Nguyen) **—** Suppose we select a vertex of G and start to walk on the edges from that vertex, forming a trail $v_1, v_2, \ldots$ The trail cannot self-intersect, as then we would have a circle, which contradicts the existence of the tree. Thus the walk should eventually stop ar some vertex of degree 1, otherwise it is still possible to continue the walk.

Now we prove that the tree contains $n-1$ edges. The case is clear if $n = 1$. Suppose we have a tree with $n+1$ vertices. As above, we shall find a vertex with degree 1. By removing that vertex together with the edge connecting it, we obtain again a graph with n vertices. The graph is still connected, and contains no cycle, thus it is a tree. Therefore the number of edges before removing the vertex is $n-1+1 = n$.

Remark. Albert Dinh-Le, Henry Dinh, and Minh Le Phuong Nguyen have successfully solved the problem.

Problem 8.5.8 (An application of Sylvester-Gallai theorem)**.** (*10 points*) There are $n \geq 3$ distinct points on the plane, not all of them collinear. Prove that these point determine at least n lines (each line through at least two points.)

Remark. The application of Sylvester-Gallai theorem (first problem, Session 3) is crucial. Removing P_{n+1} to obtain a configuration with the case of n.

Solution (Official solution - similar to submissions from Albert Dinh-Le, Henry Dinh, and Minh Le Phuong Nguyen) — We prove the given statement by induction. For $n = 3$, the statement is obvious. For $n > 3$, by Sylvester-Gallai theorem, there is a ;ine passing through only two out of $P_1, P_2, \ldots, P_{n+1}$ points. WLOG, let that be the line through P_nP_{n+1}, If all $P_1, P_2, \ldots, P_n$ are collinear, then we have n line through P_{n+1} and each of the points $P_1, P_2, \ldots, P_n$, thus altogether $n + 1$ lines. Otherwise we have n distinct lines through pairs of $P_1, P_2, \ldots, P_n$. These, together with the line P_nP_{n+1}, make $n + 1$ lines.

Remark. Albert Dinh-Le, Henry Dinh, and Minh Le Phuong Nguyen have successfully solved the problem.

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Problem 8.5.9 (Another example in combinatorial geometry)**.** Let $A_1, A_2, \ldots, A_n$ $(n \geq 4)$ be a convex polygon. Denote by R_i the radius of the circumcircle of the triangle $A_{i-1}A_iA_{i+1}$ $(i = 2, 3, \ldots, n$ and $A_{n+1} = A_1.)$ Prove that if $R_2 = R_3 = \ldots = R_n$ then one can draw a circumcircle for the polygon $A_1A_2 \ldots A_n$.

Remark. Prove the following lemmas, then use them with an induction on n.

Lemma (Lemma 1)

If the convex quadrilateral $MNPK$ cannot be inscribed in a circle, but the radii of the circumcircles of the triangles MNP and MNK are equal, then $\angle MPN + \angle MKN = 180°$.

Lemma (Lemma 2)

Let $ABCDE$ be a convex pentagon such that $R_B = R_C = R_D$, where R_B, R_C, and R_D denote the radii of the circumcircles of the triangles ABC, BCD, and CDE, respectively. Then at least one of the quadrilaterals $ABCD$ or $BCDE$ can be inscribed in a circle.

Lemma (Lemma 3)

Let $ABCDE$ be a convex pentagon such that $R_B = R_C = R_D = R_E$ where R_B, R_C, R_D, and R_E denote the radii of the circumcircles of the triangles ABC, BCD, CDE, and DEA, respectively. Then at least one of the quadrilaterals $ABCD$ or $CDEF$ can be inscribed in a circle.

Solution (Official solution - similar to one submission from Henry Dinh) — We leave the proofs of the lemmas for the students. We prove the given statement by induction. The case $n =$ is handled by the lemmas. For $n = 5$, applying Lemma 2 and Lemma 3 $(A = F)$ we see that at least two of the quadrilaterals $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $A_3A_4A_5A_1$ can be inscribed in a circle. From this it follows that it is possible to draw a circumcircle for the pentagon $A_1A_2A_3A_4A_5$.

For the induction step, we show that for $n \geq 5$, then the case $n+1$ stands. Consider the quadrilaterals $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $A_3A_4A_5A_6$, and $A_4A_5A_6A_7$, (with $n = 6$, consider $A_7 = A_1$.) Using Lemma 2 and Lemma 3 we have that a pair of consecutive quadrilaterals in the list can be inscribed in a circle.

Assume that these are $A_1A_2A_3A_4$, $A_2A_3A_4A_5$ (the argument is basically same for other cases). Then it can be verified that the convex $n-$gon $A_1A_2A_4A_5 \ldots A_nA_{n+1}$ satisfy the conditions, thus can be inscribed in a circle. Moreover A_3 is also on that circle. This completes our proof.

Remark. Henry Dinh has successfully solved the problem.

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Problem 8.5.10 (Induction principle in Calculus)**.** (*10 points*) Let I be an interval, $f, g: I \to \mathbb{R}$ be two n times differentiable functions. Prove that

$$(fg)^{(n)} = \binom{n}{0} f^{(n)} g + \binom{n}{1} f^{(n-1)} g^{(1)} + \cdots + \binom{n}{n-1} f^{(1)} g^{(n-1)} + \binom{n}{n} f g^{(n)}$$

where $f^{(k)}$ denotes the kth derivative of f.

Remark. Quite straightforward application of the Induction Principle with the derivate formula of a product. A few manipulations of binomial coefficients will help.

Solution (Official solution - similar to submissions from Henry Dinh and Minh Le Phuong Nguyen) — The key is the rule for derivative of a product

$$(fg)' = f'g + fg'.$$

The rest is simple with the Pascal Identity

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Remark. Henry Dinh and Minh Le Phuong Nguyen have successfully solved the problem.

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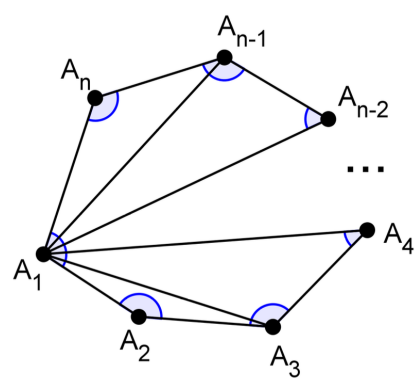
Chapter 9

Session 7

9.1 Angle chasing (I-Level)

In this session we discuss how to determine measure of an angle via multiple different techniques.

Example 9.1.1 (23-24-S7-I-P1)
Find the sum of the interior angles in an $n-$gon.



Solution. Let $A_1A_2A_3\dots A_n$ be a polygon with n sides. If we draw the diagonals from A_1 to all the other $(n-3)$ vertices, we get $(n-2)$ distinct triangles.

Since the sum of all the interior angles in a triangle is $180°$, thus the sum of all the interior angles in these triangles is $(n-2)\cdot 180°$. Note that these angles actually form all the interior angles in the $n-$gon. So, the sum of the interior angles in an $n-$gon is $\boxed{(n-2)\cdot 180°.}$ □

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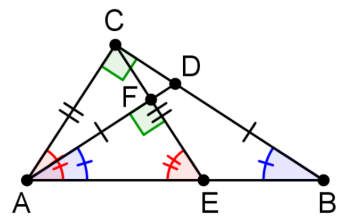
Example 9.1.2 (23-24-S7-I-P2)

Let ABC be a right triangle with $\angle BCA = 90°$ and $CA < CB$. Let $D \in BC$ (D on BC) such that $DA = DB$ and $E \in AB$ such that $CA = CE$. Prove that $AD \perp CE$.



Proof. Since $CA = CE$,, the triangle ACE is isosceles at C, we have $\angle CEA = \angle CAE$. Similarly, since $DA = DB$, we have $\angle DAB = \angle DBA$.

From the sum of angles in a triangle, we have $\angle CAE + \angle DBA = 180° - \angle ACB = 90°$.

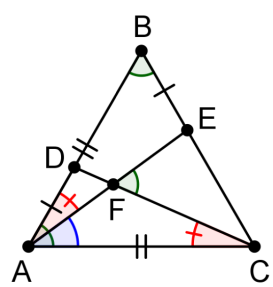
Now,let F be the intersection of AD and CE. From $\triangle AFE$ we get

$$\angle AFE = 180° - (\angle FEA + \angle EAF) = 180° - (\angle CAE + \angle DBA) = \boxed{90°.}$$

□

Example 9.1.3 (23-24-S7-I-P3)

Let ABC be an equilateral triangle. Let $D \in AB$ (D on AB) and $E \in BC$, such that $AD = BE$. let F be the intersection of AE and CD. Find $\angle CFE$.



Solution. We focus on the triangles0$\triangle ABE$ and $\triangle CAD$. Since $\triangle ABC$ is equilateral triangle, we get $AB = CA$ and $\angle ABE = 60° = \angle CAD$.

Using $BE = AD$, by the criterion side-angle-side, we get $\triangle ABE \cong \triangle CAD$. Therefore, $\angle EAB = \angle DCA$.

Finally, as an exterior angle in triangle $\triangle FAC$, we get:

$$\angle CFE = \angle CAF + \angle FCA \equiv \angle CAE + \angle DCA = \angle CAE + \angle EAB = \angle CAB = \boxed{60°.}$$

□

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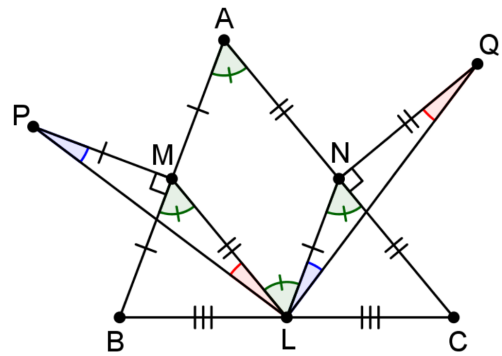
Example 9.1.4 (23-24-S7-I-P4)

Let M and N be midpoints of the sides AB and AC, respectively, in a triangle ABC. Let P and Q be points outside the triangle, such that $PM \perp AB$, $PM = \frac{1}{2}AB$ and $QN \perp AC$, $QN = \frac{1}{2}AC$. If L is the midpoint of BC, prove that $LP = LQ$ and $\angle PLQ = 90°$.



Solution. Since LM is a midsegment in $\triangle ABC$, we get $LM = \frac{1}{2}CA$ and $LM \parallel CA$. Therefore, $LM = CN = AN = QN$ and $\angle LMB = \angle CAB$, thus $\angle LMP = 90° + \angle A$. Similarly, $NL = MP$ and $\angle QNL = 90° + \angle A$. Therefore, by side-angle-side, $\triangle LMP \cong \triangle QNL$, so $LP = QL$.

Let $x = \angle MPL = \angle NLQ$, $y = \angle PLM = \angle LQN$. Since $LM \parallel NA$ and $LN \parallel MA$, we get that $LMAN$ is a parallelogram, so $\angle MLN = \angle A$. From the sum of angles in triangle $\triangle PML$, $(90° + \angle A) + x + y = 180°$.

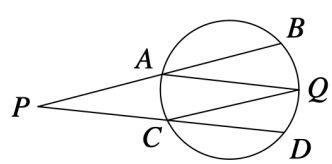
Finally,

$$\angle PLQ = \angle PLM + \angle MLN + \angle NLQ = y + \angle A + x = 180° - 90° = \boxed{90°.}$$

□

Example 9.1.5 (23-24-S7-I-P5)

Points A, B, Q, D, and C lie on the circle as shown and the measures of arcs $\widehat{BQ}$ and $\widehat{QD}$ are $42°$ and $38°$ respectively. What is the sum of angles P and Q?



Solution. As an angle formed by two secants, $\angle P = \frac{1}{2}(\widehat{BD} - \widehat{AC})$. As an inscribed angle, $\angle Q = \frac{1}{2}\widehat{AC}$. Thus

$$\angle P + \angle Q = \frac{1}{2}\widehat{BD} = \frac{1}{2}(\widehat{BQ} + \widehat{QD}) = \frac{1}{2}(42° + 38°) = \boxed{40°.}$$

□

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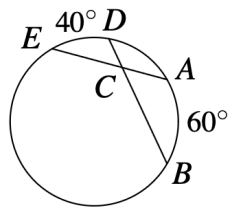
Example 9.1.6 (23-24-S7-I-P6)

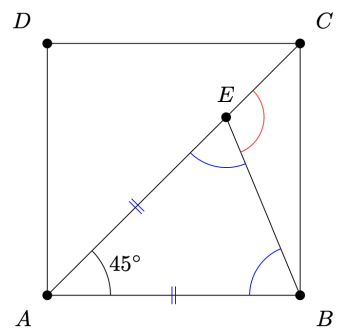
In the figure, if $\widehat{AB} = 60°$ and $\widehat{DE} = 40°$, then what is $\angle ACD$?



Solution. Since $\angle ACB$ is one-half the sum of $\widehat{AB}$ and $\widehat{DE}$, we have $\angle ACB = 50°$. Since $\angle ACD + \angle ACB = 180°$, we find $\angle ACD = \boxed{130°.}$ □

Example 9.1.7 (Fall Animath Cup/Problem 1)

Let $ABCD$ be a square with sides of length 1. Let E be the point on the segment AC such that $AE = 1$. Calculate the value, in degrees, of the angle $\angle BEC$.



Remark. The problem translation from French to English is courtesy of Albert Dinh-Le.

Solution. It is easy to see that $\angle BAC = 45°$. Since $AE = AB = 1$, triangle ABE is isosceles at A. Consequently, angles $\angle BEA$ and $\angle EBA$ are equal. Given that the sum of angles in triangle AEB is $180°$, we can deduce that:

$$\angle AEB = \frac{1}{2}(180° - \angle EAB) = 67.5° \Rightarrow \angle BEC = 180° - \angle AEB = 112.5°.$$

□

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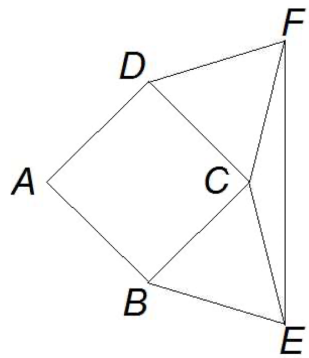
Example 9.1.8 (Purple Comet Math Meet 2014/Problem 1)

In the diagram below $ABCD$ is a square and both $\triangle CFD$ and $\triangle CBE$ are equilateral. Find the degree measure of $\angle CEF$.



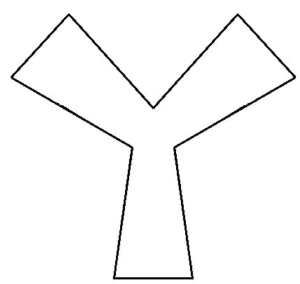
Solution. Since the sum of all angles at C is $360°$, or

$$\angle BCD + \angle DCF + \angle FCE + \angle ECB = 360° \Rightarrow \angle FCE = 360° - 90° - 2(60°) = 150°.$$

$\triangle FCE$ is isosceles at C, thus $\angle CEF = \frac{1}{2}(180° - \angle FCE) = \boxed{15°.}$ □

Example 9.1.9 (23-24-S7-I-P9)

The figure below has only two sizes for its internal angles. The larger angles are three times the size of the smaller angles. Find the degree measure of one of the larger angles.



Solution. By connecting the three points in the middle of the figure, we divide the figure into three convex quadrilaterals and one triangle. The sum of angles in a convex quadrilateral is $360°$ (because one of its diagonal divides the quadrilateral into two distinct triangle.) Thus the sum of all internal angle is $3 \cdot 360° + 1 \cdot 180° = 1260°$. Let the measure of the larger angle be x, then

$$3 \cdot x + 6 \cdot \frac{x}{3} = 1260° \Rightarrow x = \frac{1260°}{5} = \boxed{252°.}$$

□

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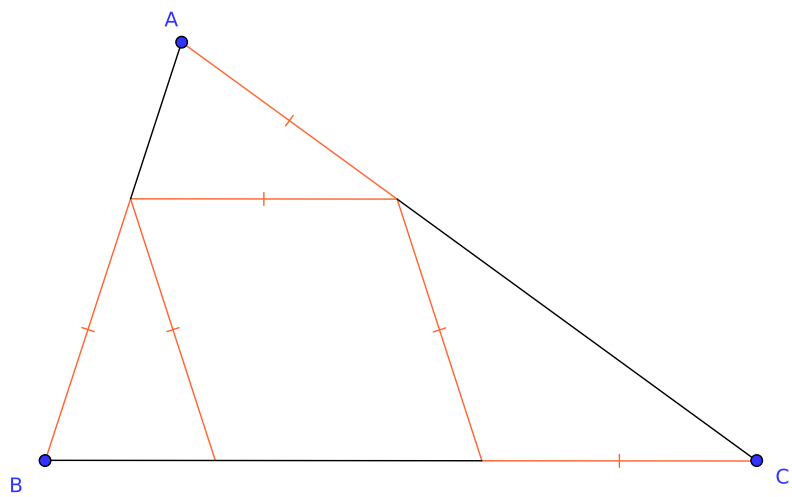
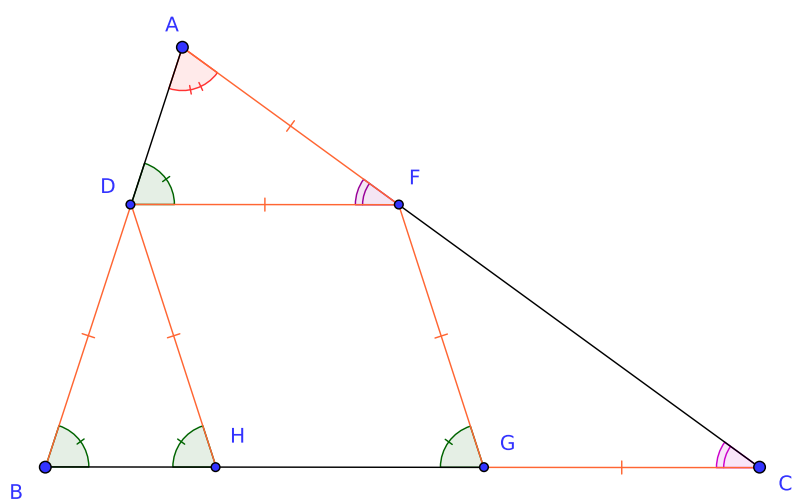
Example 9.1.10 (ET-2022-SM2-R1-J2)
Find the value of
$$\frac{\angle A}{2}+\frac{\angle B}{3}-\frac{\angle C}{4}.$$



Solution. First, lets denote the angles $\angle A, \angle B, \angle C$. as shown in the diagram below.



It is easy to see that
$$\angle B = \angle BHD = \angle HGF = 2\angle C$$
$$\angle B = \angle ADF = \angle A,\ \angle C = \angle DFA$$

In $\triangle ADF$,
$$\angle A + \angle ADF + \angle DFA = 180^\circ \Rightarrow 2\angle B + \angle C = 180^\circ \Rightarrow 5\angle C = 180^\circ$$
$$\Rightarrow \angle C = 36^\circ \Rightarrow \angle A = \angle B = 72^\circ$$
$$\Rightarrow \frac{\angle A}{2}+\frac{\angle B}{3}-\frac{\angle C}{4} = \frac{72^\circ}{2}+\frac{72^\circ}{3}-\frac{36^\circ}{4} = \boxed{51^\circ.}$$

□

9.2 Problems (I-Level)

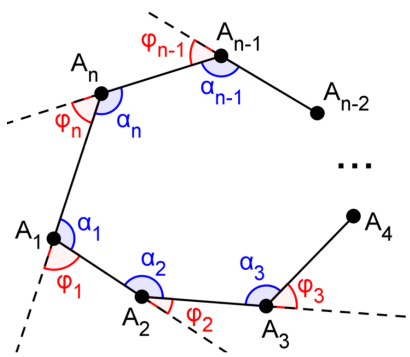
Submission deadline: November 26, 2023.

Important notices to all students:

- **all proofs must be accompanied with clean, clear, and precisely drawn diagrams** - by hand with compass and straightedge or by computer software such as GeoGebra. Submitted **solutions without diagrams will not be graded at all.**
- A proof for the hypothesis about the existence of a locus must contains two parts. Each part is worth 50% of the awardable points.
- for every problem you earn an additional same number of points for every different solution

Problem 9.2.1 (23-24-S7-I-P11)**.** (*10 points*) Find the sum of $\varphi_1, \varphi_2, \ldots, \varphi_n$, the exterior angles on an $n-$gon, as show in the diagram below. In other words, compute

$$\varphi_1 + \varphi_2 + \ldots + \varphi_n.$$



Solution. Since each exterior and its corresponding interior angle form a linear pair, we have $\alpha_i + \beta_i = 180°, i = 1, 2, \ldots, n$. If we sum these equations, we get

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n + \beta_1 + \beta_2 + \cdots + \beta_n = n(180°)$$

From 23-24-S7-I-P1, we know that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = (n-2) \cdot 180° \Rightarrow \beta_1 + \beta_2 + \cdots + \beta_n = 2 \cdot 180° = 360°$$

Hence, the sum of the exterior angles in any polygon is always $\boxed{360°.}$ □

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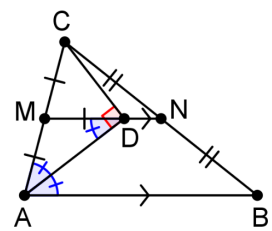
Problem 9.2.2 (23-24-S7-I-P12)**.** (*10 points*) Let M and N be midpoints of the sides CA and CB, respectively, in a triangle ABC. The angle bisector of $\angle BAC$ intersects the line MN at D. Prove that $\angle ADC = 90°$.



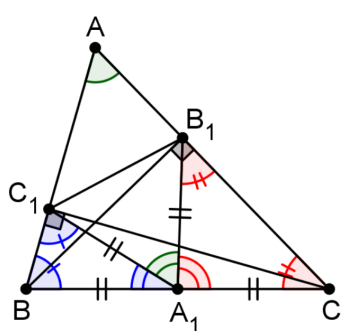
Solution. We have that MN is a midsegment in $\triangle ABC$, so, $MN \parallel AB$. This means that $\angle MDA = \angle DAB$. But, since AD is angle bisector of $\angle CAB$, we have $\angle DAB = \angle DAC$.

$$\angle MDA = \angle DAB = \angle DAC \equiv \angle DAM$$

Thus, $\triangle MAD$ is isosceles, and therefore $MD = MA$. Since $MD = MA = MC$, M is the center of the circumcircle of $\triangle ACD$ with diameter AC, hence $\angle ADC = \boxed{90°.}$ □

Problem 9.2.3 (23-24-S7-I-P13)**.** (*10 points*) In the triangle ABC, let A_1 be the midpoint of BC and let B_1 and C_1 be the feet of the altitudes from the vertices B and C, respectively. Prove that the triangle $A_1B_1C_1$ is equilateral if and only if $\angle BAC = 60°$.

> **Remark.** Note that the statement of the problem is *if and only if*, thus the proof must consists of two parts.
>
> Part one: If $\angle BAC = 60°$ then $A_1B_1C_1$ is equilateral triangle.
>
> Part two: If $A_1B_1C_1$ is equilateral triangle then $\angle BAC = 60°$.



Solution. Since A_1 is midpoint of the hypotenuse BC in the right triangle $\triangle B_1BC$, we have $A_1B_1 = A_1B = A_1C$. Similarly, from $\triangle C_1BCwegetA_1C_1 = A_1B = A_1C$. Therefore, triangles $\triangle A_1B_1C_1$, $\triangle A_1BC_1$, and A_1CB_1 are isosceles. Therefore, $\angle BA_1C_1 = 180° - 2\angle B$, and $\angle CA_1B_1 = 180° - 2\angle C$. So,

$$\angle B_1A_1C_1 = 180° - (\angle BA_1C_1 \angle CA_1B_1) = 180° - 2(180° - \angle B - \angle C) = 180 - 2\angle A.$$

Finally, we see that the isosceles $\triangle A_1B_1C_1$ is equilateral if and only if $\angle B_1A_1C_1 = 60°$, which is equivalent to $180° - 2\angle A = 60°$, which is true if and only if $\angle A = 60°$. □

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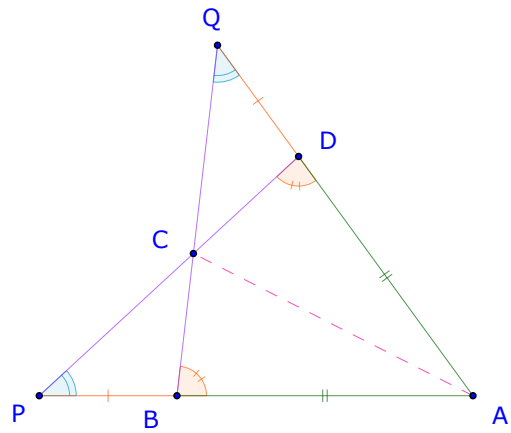
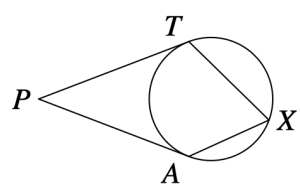
Problem 9.2.4 (23-24-S7-I-P14)**.** (*10 points*) Let $ABCD$ be a convex quadrilateral ($AB > CD$, $AD > BC$). Let P and Q be the intersection of AB with DC and AD with BC, respectively. If $AP = AQ$ and $AB = AD$, prove that AC is the angle bisector of $\angle BAD$.



Solution. First, $AB = AD, AQ = AP, \angle QAB = \angle PAD$, thus $\triangle ABQ \cong \triangle ADP$, so $\angle AQB = \angle APD$, $\angle ABQ = \angle ADP$, thus $\angle CBP = \angle CDQ$. Therefore $\triangle CBP \cong \triangle CDQ$ by angle-side-angle. This means that $CB = CD$, which implies that $\triangle CBA \cong \triangle CDA$. Hence, $\boxed{\angle BAC = \angle DAC.}$ □

Problem 9.2.5 (23-24-S7-I-P15)**.** (*10 points*) Segments PA and PT are tangent to the circle. Find the measure of $\angle TXA$ if $\angle P = 42°$.



Solution. Note that the measure of $\angle P = 42°$ is equal to half the difference of major arc $\widehat{AXT}$ (containing X) and minor arc $\widehat{AT}$ (no containing X), thus the difference of the two arcs is $84°$. Now, the sum of the two arcs is $360°$, thus the measure of the minor arc is:

$$\frac{1}{2}(360° - 84°) = 138° \Rightarrow \angle TXA = \frac{1}{2}138° = \boxed{69°.}$$

□

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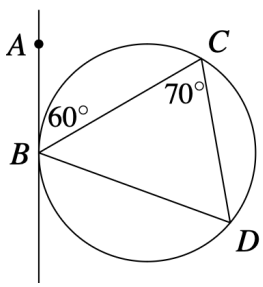
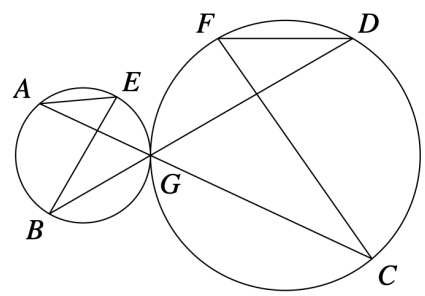
Problem 9.2.6 (23-24-S7-I-P16)**.** Consider the two questions separately.

(*5 points*) In the left figure, given that $\angle ABC = 60°$ and $\angle BCD = 70°$, find $\angle CBD$.

(*5 points*) In the right figure, two circles are tangent at G. Prove that $\angle AEB = \angle CFD$.





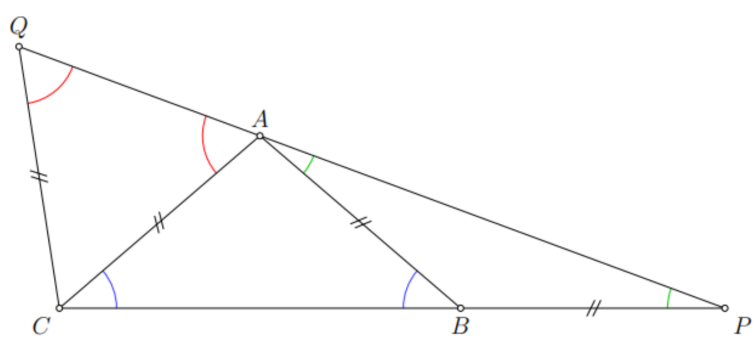
Solution. For the first question, $\angle BDC = \frac{1}{2}\widehat{BC} = \angle ABC = 60°$. Thus $\angle CBD = 180° - 60° - 70° = \boxed{50°.}$

For the second question, $\angle AEB = \frac{1}{2}\widehat{AB}\angle AGB = \angle CGD = \frac{1}{2}\widehat{CD} = \angle CFD$. □

Problem 9.2.7 (23-24-S7-I-P17)**.** (*10 points*) Let ABC be an isosceles triangle at A ($AB = AC$) such that $AB < BC$. Let P be the point on the line (BC) located outside the segment BC such that $BP = BA$. Let Q be the point on line AP, different from A, such that $CQ = CA$.

If the angle $\angle QPC$ is denoted as x, what is the measure of angle $\angle PCQ$?

> **Remark.** The problem translation from French to English is courtesy of Albert Dinh-Le.



Solution. Let $x = \angle QPC$. Note that $CQ = CA = BA = BP$, thus

$$\angle ACB = \angle ABC = \angle APB + \angle BAP = 2\angle APC = 2x,$$
$$\angle AQC = \angle QAC = 180° - \angle CAB - \angle BAP = 180° - (180° - 2\angle ACB) - x = 3x,$$
$$\Rightarrow \angle QCP = 180° - \angle AQC - \angle APC = \boxed{180° - 4x.}$$

□

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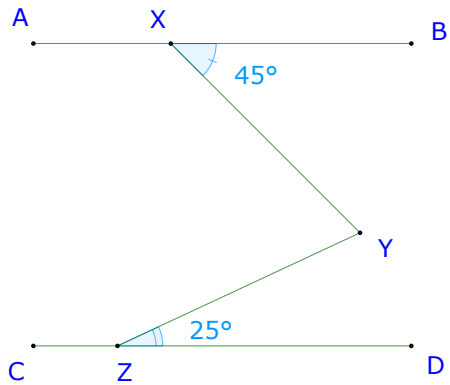
Problem 9.2.8 (23-24-S7-I-P18)**.** (*10 points*) In the diagram below $AB \parallel CD$, $\angle BXY = 45°$, $\angle DZY = 25°$, and $XY = YZ$. Find the degree measure of $\angle YXZ$.

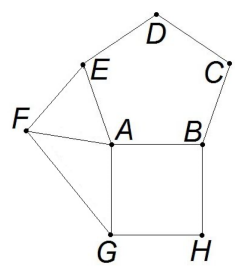


Solution. First, by drawing a line through Y, parallel with AB and CD, dividing $\angle XYZ$ into two angles, one is equal to $\angle YXB$, and the other is equal to $\angle YZD$, then

$$\angle XYZ = \angle YXB + \angle YZD = 70°.$$

Now, $XY = YZ$. so $\triangle XYZ$ is isosceles at Y, or $\angle YXZ = \frac{1}{2}(180° - \angle XYZ) = \boxed{55°.}$ □

Problem 9.2.9 (23-24-S7-I-P19)**.** (*10 points*) On side AE of regular pentagon $ABCDE$ there is an equilateral triangle AEF, and on side AB of the pentagon there is a square $ABHG$ as shown. Find the degree measure of angle AFG.



Solution. First $AF = AG$, so $\angle AFG = \frac{1}{2}(180° - \angle FAG)$.

$$\angle FAG = 360° - (60° + 90° + \frac{(5-2)180°}{5}) = 102° \Rightarrow \angle AFG = \boxed{39°.}$$

□

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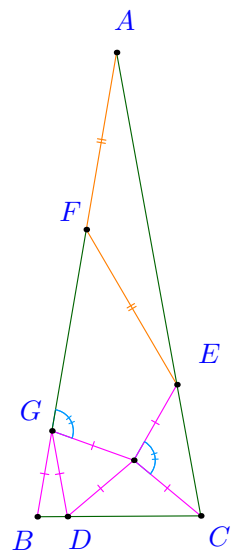
Problem 9.2.10 (23-24-S7-I-P20)**.** (*10 points*) In $\triangle ABC$, $AB = AC$, $AF = EF$, $\angle CHE = \angle FGH$, and $EH = CH = DH = GH = DG = BG$. Find $\angle BAC$.



Solution. Let $x = \angle ABC$,

$$\triangle BAC \sim \triangle BGD \Rightarrow \angle BDG = \angle DBG = x \angle BGD + \angle GDH + \angle HDC = 180^\circ \Rightarrow \angle HDC = 120^\circ - x$$
$$\angle HCD = \angle HDC,\ \angle ECH = \angle ACB - \angle HCD \Rightarrow \angle ECH = x - (120^\circ - x) = 2x - 120^\circ$$
$$\angle EHC = 180^\circ - 2(2x - 120^\circ) = 420^\circ - 4x,\ \angle FGH = 180^\circ - \angle HGD - \angle DGB$$
$$\Rightarrow \angle HGF = 180^\circ - 60^\circ - (180^\circ - 2x) = 2x - 60^\circ$$
$$\angle CHE = \angle FGH \Rightarrow 420^\circ - 4x = 2x - 60^\circ \Rightarrow x = 80^\circ$$

Therefore, $\boxed{\angle BAC = 180^\circ - 2 \cdot 80^\circ = 20^\circ.}$ $\square$

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9.3 Advanced counting techniques (A-Level)

Definition (Set)**.** A set is *a collection of objects.*

There might only be a *finite* number of objects in the set, so we could count them if we like, in which case the set is called a finite set. Otherwise we call it an *infinite* set.

The objects in the set are called the *elements or members* of the set.

The set with no elements at all is called the *empty set,* denoted by $\emptyset$.

Definition (Supersets & Subsets)**.** A set A is called a *subset* of a set B if every element of A is also an element of B, denoted by

$$A \subseteq B$$

If A is a subset of B, we also say that B is a *superset* of A.

Definition (Intersections & Unions)**.** The *union* $A \cup B$ of two sets A and B is the set of all objects that are elements of A or of B,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

The *intersection* $A \cap B$ of two sets A and B is the set of all objects that are elements of both A and B,

$$A \cap B = \{x \mid x \in A \text{ or } x \in B\}.$$

Definition (Disjoint)**.** If $A \cap B = \emptyset$, then we say that A and B are *disjoint.*

Definition (Difference)**.** The set difference of two sets A and B is

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

Definition (Cardinality)**.** For a set S, $|S|$ denotes the **cardinality** of S, in other words the number of elements in S, *for example* $|\{1, 3, 5, 7\}| = 4$. The **empty** set is denoted by $\emptyset$.

Theorem (The Principle Inclusion-Exclusion for two/three sets)

If A and B are two sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

If A, B, and C are three sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

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Example 9.3.1 (23-24-S7-A-P1)

In how many rearrangements of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ do the numbers form a *hill,* that is, the numbers form an increasing sequence at the beginning up to a peak, and then form a decreasing sequence to the end such as in 129876543 or 258976431?

Solution. Since in any such arrangement the number 9 must be the number of the peak, the numbers in the arrangement must increase to the position of the 9, and then decrease after that position.

The sequence is then *completely determine by the numbers that come before the 9.* There are 2^8 subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that could make up the numbers that come before the 9, except that neither the empty set or the entire set result in an arrangement that both increase up to 9 and decrease after 9.

Therefore, there are $2^8 - 2 = \boxed{254}$ possible sequences that form a *hill.* □

Example 9.3.2 (23-24-S7-A-P2)

Find the least six-digit palindrome that is a multiple of 45. Note that a palindrome is a number that reads the same forward and backwards such as 1441 or 35253.

Solution. A number is a multiple of 45 exactly when it is a multiple of both 5 and 9.

The decimal representation of a number divisible by 5 must end in the digit 0 or 5. A six-digit palindrome cannot end in 0.

A number is a multiple of 9 if the sum of its digits is a multiple of 9. A six-digit palindrome must have the sum of its digits as a multiple of 18 since each of its digits appears twice.

The least such six-digit palindrome must then start with 5, follow by a pair of two digits adding up to 4, then another 5. The least such number is $\boxed{504405.}$ □

Example 9.3.3 (23-24-S7-A-P3)

How many four-digit positive integers have no adjacent equal even digits? For example, count numbers such as 1164 and 2035 but not 6447 or 5866.

Solution. There are $9999 - 999 = 9000$ four-digit positive integers.

Let A be the set of four-digit positive integers where the first two digits are the same even digits, B be the set of four-digit positive integers where the second and third digits are the same even digits, C be the set of four-digit positive integers where the last two digits are the same even digits.

Thus we count four-digit positive integers have no adjacent equal even digits, or

$$9000 - |A \cup B \cup C|.$$

By the The Principle Inclusion-Exclusion for two/three sets,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

Now it is easy to find that for any number in A, there are 4 choices for the first (and second) digits: $2, 4, 6, 8$ and 10 choices for the third or the fourth choices, thus $|A| = 4 \cdot 100$. Similarly,

$$|B| = 5 \cdot 90, |C| = 5 \cdot 90, |A \cap B| = 4 \cdot 10, |B \cap C| = 5 \cdot 9, |A \cap C| = 4 \cdot 5, |A \cap B \cap C| = 4.$$

Then $|A \cup B \cup C| = (400 + 450 + 450) - (40 + 45 + 20) + 4 = \boxed{1199.}$ □

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Example 9.3.4 (23-24-S7-A-P4)

In how many ways can the letters in the word MEMENTO be arranged if no two of the same letter may appear consecutively in an arrangement?

Solution. Let use the The Principle Inclusion-Exclusion for two/three sets to solve this problem.

The total number of arrangement is $A = \frac{7!}{2!}2!$.

The number of ways for the two Ms to appear consecutively is $B = \frac{6!}{2!}$ because the MM block can be seen as a single letter. Similarly there are $B = \frac{6!}{2!}$ ways for two Es to be together.

Furthermore there are $C = 5!$ ways to arrange if both letter Ms and both letter Es appear together.

The total number of ways no two Ms or two Es appear together is *the total number of arrangement subtracts the number of ways for the two Ms to appear consecutively, then subtracts the number ways for two Es to be together, then adds back the number of ways to arrange if both letter Ms and both letter Es appear together,* or

$$A - 2B + C = \frac{7!}{2!}2! - 2\frac{6!}{2!} + 5! = \boxed{660.}$$

□

Example 9.3.5 (23-24-S7-A-P5)

How many 6-digit numbers, written in decimal notation, have at least one 1, one 2, and one 3 among their digits?

Solution. Let S_1, S_2, and S_3 be the sets of 6-digit numbers that do not contain the digits 1, 2, and 3, respectively. Then by The Principle Inclusion-Exclusion for two/three sets, the number of 6-digit numbers that are missing at least one of the digits 1, 2, or 3 is

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|$$

First we count $|S_1|$, the number of 6-digit numbers that do not contain the digit 1. The first digit has 8 possible values (2 through 9), and each of the remaining 5 digits has 9 possible values, so $|S_1 = 8 \cdot 9^5$. Similarly, $|S_2| = |S_3| = 8 \cdot 9^5$.

Next we count $|S_1 \cap S_2|$, the number of 6-digit numbers that do not contain the digit 1 or 2. The first digit has 7 possible values (3 through 9), and each of the remaining 5 digits has 8 possible values, so $|S_1 \cap S_2| = 7 \cdot 8^5$. Similarly,

$$|S_1 \cap S_3| = |S_2 \cap S_3| = 7 \cdot 8^5.$$

Finally, we count $|S_1 \cap S_2 \cap S_3|$, the number of 6-digit numbers that do not contain the digits 1, 2, or 3. The first digit has 6 possible values (4 through 9), and each of the remaining 5 digits has 7 possible values, so $|S_1 \cap S_2 \cap S_3| = 6 \cdot 7^5$.

Therefore, $|S_1 \cup S_2 \cup S_3| = 3 \cdot 8 \cdot 9^5 - 3 \cdot 7 \cdot 8^5 + 6 \cdot 7^5 = 829\ 890$.

There are a total of $9 \cdot 10^5 = 900\ 000$ 6-digit numbers, so there are $900\ 000 - 829\ 890 = \boxed{70\ 110}$ 6-digit numbers that have at least one 1, one 2, and one 3. □

Example 9.3.6 (23-24-S7-A-P6)

How many subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ have the property that no two of its elements differ by more than 5? For example, count the subsets $\{3\}$, $\{2, 5, 7\}$, and $\{5, 6, 7, 8, 9\}$ but not the subset $\{1, 3, 5, 7\}$.

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Solution. Let S be a subset such that no two of its elements differ by more than 5. Obviously the empty set is such a subset. Let consider non-empty subset S. Because S has a finite number of element, so there exists the smallest element $e \in S$.

Case 1: $1 \leq e \leq 7$, any element from $e+1,\ e+2,\ \ldots,\ e+5$ can be element of S. So there are $7 \cdot 2^5 = 224$ ways to choose from those number to form S (together with e).

Case 2: $8 \leq e \leq 12$, for each e, there are $12 - e$ elements larger than e. From the previous case, there are 2^{12-e} ways to choose from those number to form S (together with e). It means the number of ways for this case is $2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 31$.

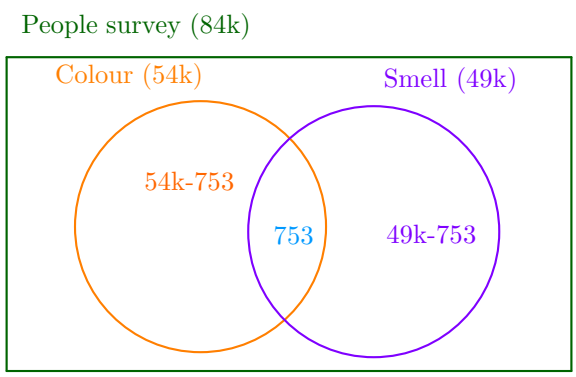
Thus, the number of such subsets is $1 + 224 + 31 = \boxed{256.}$ □

Example 9.3.7 (Gauss G8 2014/25)

Residents were surveyed in order to determine which flowers to plant in the new Public Garden. A total of N people participated in the survey. Exactly $\frac{9}{14}$ of those surveyed said that the colour of the flower was important. Exactly $\frac{7}{12}$ of those surveyed said that the smell of the flower was important. In total, 753 people said that both the colour and smell were important. How many possible values are there for N?

Solution. The difficulty of the problem is that the quantities are given as fractions of unknown value N. Our approach is to *turn N into a multiple of the least common multiple of the denominators* of the fractions. This ways the fractions can be transformed into integer variables. The rest is based *using a Venn-diagram to estimate*,

1. the lowest value of k based on the number of common people in both sets, and
2. the highest value of k based on the number of people not in any of the sets.



Since the least common multiple of the denominator $\text{lcm}(12, 14) = 84$, thus $84 \mid N$. Let $N = 84k$, where k is a positive integer.

The minimum value of k, since 753 is the number of people in both sets, so

$$\begin{cases} 54k - 753 \geq 0 \\ 49k - 753 \geq 0 \end{cases} \Rightarrow 49k - 753 \geq 0 \Rightarrow k \geq 16$$

The maximum value of k, the number of people outside of both sets is

$$84k - (54k - 753) - (49k - 753) - 753 = 753 - 19k \geq 0 \Rightarrow k \leq 39$$

It is easy to see that all values of k between 16 and 39, including both 16 and 39, satisfy the given conditions. Thus there are $\boxed{39 - 16 + 1 = 24}$ such values. □

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9.4 Problems (A-Level)

Submission deadline: November 26, 2023.

For every problem you earn an additional same number of points for every different solution.

Problem 9.4.1 (23-24-S7-A-P8)**.** (*10 points*) How many four-digit positive integers have exactly one digit equal to 1 and exactly one digit equal to 3?

Solution. A desired four-digit number could start with 1 or 3: there are two ways to chose one of them. There are three ways to choose the position for the other digit. Then there are $8 \cdot 8 = 64$ ways to choose the remaining two digits. All together $2 \cdot 3 \cdot 64 = 384$ ways.

A desired four-digit number could start with a digit other than 1 or 3. There are 7 ways to do so. Then there are three ways to choose the position for the 1 digit, two ways for the 3 digit. Last, there are 8 ways to choose the last digit. All together $7 \cdot 3 \cdot 2 \cdot 8 = 336$ ways.

Hence. the total of such numbers is $384 + 336 = \boxed{720.}$ □

Problem 9.4.2 (23-24-S7-A-P9)**.** (*10 points*) How many subsets of $\{1, 2, 3, 4, \ldots, 12\}$ contain exactly one prime number?

Solution. The set $\{1, 2, 3, \ldots, 12\}$ contains five prime numbers $2, 3, 5, 7, 11$, thus it contains 7 numbers which are not prime.

To make a subset containing exactly one prime number we have 5 ways to choose a prime number and 2^7 ways to choose a subset of 7 non-prime numbers.

Thus, the number of ways is $5 \cdot 2^7 = \boxed{640.}$ □

Problem 9.4.3 (23-24-S7-A-P10)**.** (*10 points*) At Mallard High School there are three intermural sports leagues: football, basketball, and baseball. There are 427 students participating in these sports: 128 play on football teams, 291 play on basketball teams, and 318 play on baseball teams.

If exactly 36 students participate in all three of the sports, how many students participate in exactly two of the sports?

Solution. Let A, B, and C be the set of students who play football, basketball, and baseball, respectively. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$
$$427 = 128 + 291 + 318 - |A \cap B| - |B \cap C| - |C \cap A| + 36 \Rightarrow |A \cap B| - |B \cap C| - |C \cap A| = 346$$

Now, the number of students playing exactly two is each of

$$|A \cap B| - |A \cap B \cap C|,\ |B \cap C| - |A \cap B \cap C|,\ |C \cap A| - |A \cap B \cap C|,\ \text{or } 346 - 3 \cdot 36 = \boxed{238.}$$

, □

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Problem 9.4.4 (23-24-S7-A-P11)**.** (*15 points*) How many 6-digit binary numbers (numbers with 0's and 1's as digits) have a string of three consecutive 1's appearing in them? *For example, 101110 and 111100 both have a string of three consecutive 1's, but 100110 doesn't.*

Hint: define four sets A_1, A_2, A_3, A_4 *as follow* $A_1 = \{111???\},\ A_2 = \{?111??\},\ A_3 = \{??111?\},\ A_4 = \{???111\}$, *where ? is a place holder for 0 or 1 digit. Try the Inclusion-Exclusion Principle.*

Solution. There are four possible positions for a run of three ones: 111???, ?111??, ??111?, and ???111. Let define a set for each of the types:

$$A_1 = \{111???\},\ A_2 = \{?111??\},\ A_3 = \{??111?\},\ A_4 = \{???111\}.$$

Note that every 6-digit number begins with a 1, so actually

$$A_1 = \{111???\},\ A_2 = \{1111??\},\ A_3 = \{1?111?\},\ A_4 = \{1??111\}.$$

Now

$$\begin{aligned}
&A_1 \cap A_2 = \{1111??\},\ A_1 \cap A_3 = \{11111?\},\ A_1 \cap A_4 = \{111111\}\\
&A_2 \cap A_3 = \{11111?\},\ A_2 \cap A_4 = \{111111\}, A_3 \cap A_4 = \{1?1111\},\\
&A_1 \cap A_2 \cap A_3 = \{11111?\},\ A_1 \cap A_2 \cap A_4 = \{111111\},\\
&A_1 \cap A_3 \cap A_4 = \{111111\},\ A_2 \cap A_3 \cap A_4 = \{111111\},\\
&A_1 \cap \cap A_2 \cap A_3 \cap A_4 = \{111111\}
\end{aligned}$$

By The Principle Inclusion-Exclusion for two/three sets,

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = (8+4+4+4) - (4+2+1+2+1+2) + (2+1+1+1) - 1 = \boxed{12}.$$

It is also pretty easy to list them: $100111, 101110, 101111, 110111$, and all 8 numbers of the form 111???. □

Problem 9.4.5 (23-24-S7-A-P12)**.** (*15 points*) Two biologists, two chemists, and two physicists go out to dinner and sit at a round table with 6 equally spaced chairs. In how many ways can they sit so that no two scientists of the same type (for example, two biologists) are seated next to each other? (Two seatings that are merely rotations of each other are not considered distinguishably different.)

Hint for Approach 1: Start by arbitrarily seating one of the biologists Considering following casework: (i) The other biologist sits directly opposite the first; (ii) The other biologist sits 2 seats away from the first.

Hint for Approach 2: what we need to count is the number of seatings with no pair adjacent. The The Principle Inclusion-Exclusion for two/three sets can help if we can count number of seatings with no restriction, with 1 pair adjacent, with 2 pairs adjacent, and with all three pairs adjacent.

Solution. [Solution 1] We start by arbitrarily seating one of the biologists in any seat. Since we're dealing with a round table, this isn't really a *choice*, since we can always rotate the table. Now there are two cases, depending on where we seat the other biologist.

Case 1: The other biologist sits directly opposite the first. We have 4 choices for where the first chemist sits (since there are 4 empty seats), then 2 choices for where the other chemist sits (since she cannot sit next to the first chemist). To finish, we have 2 choices for how to seat the two physicists. So there are a total of $4 \times 2 \times 2 = 16$ seatings in this case.

Case 2: The other biologist sits 2 seats away from the first. There are 2 choices here (2 seats to the left or 2 seats to the right). We then have 4 choices for the person to sit between the biologists. The other person of the same science must then sit directly opposite, so that the remaining two people are not next to each other. Finally, there are 2 choices for seating the remaining two people. So there are $4 \times 2 \times 2 = 16$ seatings in this case.

This gives a total of $16 + 16 = 32$ seatings. □

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Solution. [Solution 2] What we need to count is the number of seatings with no pair adjacent using The Principle Inclusion-Exclusion for two/three sets,

$$\begin{aligned}&\#\text{of seatings with no pair adjacent}\\&= \#\text{ of seatings (with no restriction)} - \#\text{ of seatings with 1 pair adjacent}\\&+ \#\text{ of seatings with 2 pairs adjacent} - \#\text{ of seatings with all 3 pairs adjacent.}\end{aligned}$$

Because this is a round table, there are $\frac{6!}{6} = 5! = 120$ ways to seat the six people - there are 6! ways to arrange 6 people, but we have to divide by 6 due to the symmetries of rotation.

Suppose we want to sit one specified pair together. We can seat them anywhere at the table (it doesn't matter where because of the rotational symmetry), and then there are 4! ways to seat the remaining 4 people. There are also 2 ways to seat the two people within the pair, so there are a total of $2 \cdot 4! = 48$ ways to seat all the people with one specified pair together. There are 3 different pairs, so we have to subtract 3(48) from our initial count of 120.

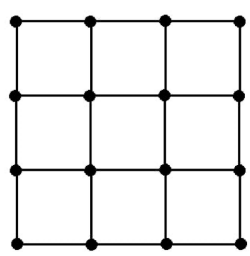
Now suppose we want to seat two specified pairs together. We can seat the first pair anywhere. We then consider the second pair as a unit, so there are 3! ways to seat the second pair and the other two people. There are also 2 ways to seat the people within each pair, so there are a total of $2^2 \cdot 3! = 24$ ways to seat all the people with two specified pairs together. There are 3 choices of two pairs, so we have to add 3(24) back to our running count.

Finally, to seat all three pairs together, there are 2 ways to arrange the pairs, and 2 ways to arrange the people within each pair, for a total of $2^3 \cdot 2 = 16$ ways to seat all the people with all three pairs together.

Therefore, the final answer is: $120 - 3(48) + 3(24) - 16 = \boxed{32.}$ □

Problem 9.4.6 (23-24-S7-A-P13)**.** (*20 points*) Sixteen dots are arranged in a four by four grid as shown. The distance between any two dots in the grid is the minimum number of horizontal and vertical steps along the grid lines it takes to get from one dot to the other. For example, two adjacent dots are a distance 1 apart, and two dots at opposite corners of the grid are a distance 6 apart.

Find the mean value of all distances between any two distinct dots.



Solution. There are $\binom{16}{2} = 120$ pairs of points in the grid.

For $4 \cdot \binom{4}{2} = 24$ of the pairs (the ones on the same row), the horizontal distance between the points is 0. For $3 \cdot 4 \cdot 4 = 48$ of the pairs, the horizontal distance between the points is 1. For $2 \cdot 4 \cdot 4 = 32$ of the pairs, the horizontal distance between the points is 2. For $4 \cdot 4 = 16$ of the pairs, the horizontal distance between the points is 3.

Thus, the sum of all horizontal distances between pairs of points for all the pairs is

$$0 \cdot 24 + 1 \cdot 48 + 2 \cdot 32 + 3 \cdot 16 = 160.$$

Similarly the sum of all vertices distances between pairs of points for all the pairs is also 160.

Thus, the mean of the distances is $\frac{2 \cdot 160}{120} = \boxed{\frac{8}{3}.}$ □

Problem 9.4.7 (23-24-S7-A-P14)**.** (*20 points*) Omar has four fair standard six-sided dice. Omar invented a game where he rolls his four dice over and over again until the number 1 does not appear on the top face of any of the dice. Omar wins the game if on that roll the top faces of his dice show at least one 2 and at least one 5. Find the probability that Omar wins the game.

Hint: Consider two events (A) when no 2 appears, and (B) when no 5 appears.

Solution. Let a, b, c, d be the numbers that show up on Omar's dice. Given that Omar does not roll a 1, the set of all possible (a, b, c, d) 4-tuples corresponding to Omar's dice when he is done rolling is the set S containing 5^4 equally likely possible (a, b, c, d) 4-tuples.

Let A be the set of (a, b, c, d) 4-tuples in S where 2 does not appear, and B be the set of (a, b, c, d) 4-tuples in S where 5 does not appear. It is easy to see that $|A|$, the number of 4-tuples in A is 4^4 (because of no 1 or 2 can appear), similarly $|B| = 4^4$, and $A \cap B$, the number of 4-tuples in A and B is 3^4 (because of no 1 or 2 or 5 can appear), thus Then

$$|A \cup B| = |A| + |B| - |A \cup B| = 2 \cdot 4^4 - 3^4 = 431.$$

Therefore the probability is $1 - \dfrac{431}{5^4} = \boxed{\dfrac{194}{625}}.$ $\square$

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9.5 Solving Problems with Induction Principle (O-Level) - Part 3

Submission deadline: November 26, 2023.

Note that for every problem you earn an additional same number of points for every different solution.

Problem 9.5.1 (Sharpen the inequality)**.** (*10 points*) The sequence (a_n) is defined by,

$$a_1 = \frac{1}{2}, a_{n+1} = \frac{2n-1}{2n+2}a_n$$

Prove that $a_1 + a_2 + \cdots + a_n < 1$ for $n \geq 1$.

Remark. Let's sharpen the inequality to $a_1 + a_2 + \cdots + a_{n-1} + f(n)a_n < 1$ for $n \geq 1$, where $f(n) \geq 1$. Try to explore the base case to see that $f(1) \leq 2$, then the inductive step $(n-1 \to n)$ to see that

$$f(n-1) \geq \frac{2n-3}{2n}f(n).$$

Now, determine $f(n)$, perhaps as a linear function of n?

Remark. We determine the function $f(n) = cn$, where $c > 0$ is a constant such that $f(n) \geq 1$, but

$$s_{n+1} = a_1 + a_2 + \cdots + a_n + c(n+1)a_{n+1} \leq s_n + a_1 + a_2 + \cdots + a_{n-1} + c(n)a_n \quad (*)$$

In other word (s_n) is decreasing.

Now (*) is equivalent to

$$c(n+1)a_{n+1} \leq a_n\,(cn-1) = \frac{2n+2}{2n-1}a_{n+1}(cn-1) \Leftrightarrow \frac{(c-2)(n+1)}{2n-1} \geq 0$$

Thus $c = 2$ is a good choice.

Solution. Consider $s_n = a_1 + a_2 + \cdots + a_{n-1} + 2na_n,\ \forall n \geq 2$, then , so

$$a_1 = \frac{1}{2},\ a_2 = \frac{1}{4}\frac{1}{2},\ \Rightarrow s_2 = a_1 + 2(2)a_2 = \frac{1}{2} + \frac{1}{2} = 1.$$

We prove by induction $s_n = 1, \forall n \geq 2$, with the base case verified.

$$s_{n+1} = a_1 + a_2 + \cdots + a_n + 2(n+1)a_{n+1} = s_n + (1-2n)a_n + 2(n+1)a_{n+1} = s_n + (1-2n)a_n + (2n-1)a_n = s_n = 1.$$

Therefore the hypothesis is true, thus

$$a_1 + a_2 + \cdots + a_n = s_n - (2n-1)a_n < 1.$$

□

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Problem 9.5.2 (An application of Pascal's identity)**.** (*10 points*) Show that for any positive integer n

$$\sum_{r=1}^{n} \frac{1}{r}\binom{n}{r} = \sum_{r=1}^{n} \frac{2^r - 1}{r} \quad (*)$$

Remark. Remember the Pascal's identity?

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \ 1 \le k \le n.$$

Solution. For $n = 1$, both sides become 1, thus true.

Assuming that it is true for $n \ge 1$, then

$$\begin{aligned}
\sum_{r=1}^{n+1} \frac{1}{r}\binom{n+1}{r} - \sum_{r=1}^{n} \frac{1}{r}\binom{n}{r} &= \sum_{r=1}^{n} \frac{1}{r}\left(\binom{n+1}{r} - \binom{n}{r}\right) + \frac{1}{n+1}\binom{n+1}{n+1} \\
&= \sum_{r=1}^{n} \frac{1}{r}\left(\binom{n}{r-1}\right) + \frac{1}{n+1} = \frac{1}{n+1}\left(\sum_{r=1}^{n}\binom{n+1}{r}\right) + \frac{1}{n+1} \\
&= \frac{1}{n+1}\left(\sum_{r=0}^{n}\binom{n+1}{r} - 2 + 1\right) = \frac{2^{n+1} - 1}{n+1}
\end{aligned}$$

Thus

$$\sum_{r=1}^{n+1} \frac{1}{r}\binom{n+1}{r} = \sum_{r=1}^{n} \frac{1}{r}\binom{n}{r} + \frac{2^{n+1} - 1}{n+1} = \sum_{r=1}^{n+1} \frac{2^r - 1}{r}$$

□

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Problem 9.5.3 (Another functional equation)**.** (*10 points*) Find all functions $f: \mathbb{Z} \to \mathbb{R}$ such that:

$$f(1) = \frac{5}{2}, f(m)f(n) = f(m+n) + f(m-n), \ \forall m, n \in \mathbb{Z}.$$

> **Remark.** Compute a first few values for f at $0, 1, 2$, and 3 to arrive at a hypothesis that $f(n) = 2^n + 2^{-n}$.

Solution. We prove that $f(n) = 2^n + 2^{-n}, \ \forall n \geq 0 \quad (*)$ First

$$f(0)f(1) = 2f(1) \Rightarrow f(0) = 2$$

Assuming that (*) is true for all $k \leq n$, then

$$f(n)f(1) = f(n+1) + f(n-1) \Rightarrow f(n) = f(n)f(1) - f(n-1) = 2^{n+1} + 2^{-(n+1)}$$

For n negative, note that $f(0)f(n) = f(n) + f(-n) \Rightarrow f(-n) = f(n)$, in other words f is an even function, thus $f(-n) = f(n) = 2^n + 2^{-n}, \ \forall n \in \mathbb{Z}$. $\square$

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Problem 9.5.4 (Inequality of alternate sums)**.** (*10 points*) Let $n \geq 1$ be a positive integer and let $x_1, x_2, \ldots, x_n$ be real numbers such that $0 \leq x_n \leq x_{n-1} \leq x_2 \leq x_1$. Let

$$s_n = x_1 - x_2 + \cdots + (-1)^n x_{n-1} + (-1)^{n+1} x_n$$
$$S_n = x_1^2 - x_2^2 + \cdots + (-1)^n x_{n-1}^2 + (-1)^{n+1} x_n^2$$

Prove that $s_n^2 \leq S_n$.

Remark. First, prove that $s_n \geq 0,\ \forall n \geq 1$ by the Induction Principle.

Second, when proving the given inequality by induction, you might need to consider casework based on the parity of n.

Solution. First we prove that

Claim — $s_n \geq 0$, for all $n \geq 1$.

Proof. For the base case $s_1 = x_1 \geq 0$. Assume that $s_n \geq 0$.

Case 1: n is even

$$s_n \geq 0, x_{n+1} \geq 0 \Rightarrow s_{n+1} = s_n + (-1)^{n+2} x_{n+1} = s_n + x_{n+1} \geq 0.$$

Case 2: n is odd

$$s_n \geq 0, x_n \geq x_{n+1} \Rightarrow s_{n+1} = s_{n-1} + (-1)^{n+1} x_{n-1} + (-1)^{n+2} x_{n+1} = s_n + x_n - x_{n+1} \geq 0.$$

Thus $s_n \geq 0,\ \forall n \geq 1$. ■

Now we proof the problem statement by induction.

For the base case $n = 1$, $S_1 = s_1^2 \geq s_1^2$. For $n = 2$

$$S_2 = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) \geq (x_1 - x_2)^2 = s_2^2.$$

Let assume that the hypothesis is true for n, then

Case 1: $n = 2k$

$$\begin{aligned} s_{2k+2}^2 &= (s_{2k+1} - x_{2k+2})^2 = s_{2k+1}^2 - 2s_{2k+1}x_{2k+2} + x_{2k+2}^2 \\ &\leq S_{2k+1} - x_{2k+2}^2 + 2x_{2k+2}(s_{2k+1} - x_{2k+2}) = S_{2k+2} - 2s_{2k+2}x_{2k+2} \\ &\leq S_{2k+2}. \end{aligned}$$

Case 2: $n = 2k + 1$

$$\begin{aligned} s_{2k+3}^2 &= (s_{2k+1} - x_{2k+2} + x_{2k+3})^2 = s_{2k+1}^2 - 2s_{2k+1}(x_{2k+2} - x2k + 3) + (x_{2k+2} - x_{2k+3})^2 \\ &\leq S_{2k+1} - x_{2k+2}^2 + x_{2k+3}^2 - 2s_{2k+1}(x_{2k+2} - x2k + 3) + 2x_{2k+2}^2 - 2x_{2k+2}x_{2k+3} \\ &= S_{2k+1} - x_{2k+2}^2 + x_{2k+3}^2 - 2(s_{2k+1} - x_{2k+2})(x_{2k+2} - x2k + 3) \\ &= S_{2k+3} - 2s_{2k+2}(x_{2k+2} - x2k + 3) \leq S_{2k+3} \end{aligned}$$

□

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Problem 9.5.5 (Inequality of absolute values of sines)**.** (*10 points*) Let $n \geq 1$ be a non-negative integer. Prove that for all real number x,

$$\sum_{k=0}^{n} |\sin(2^k x)| \leq 1 + \frac{\sqrt{3}}{2} n.$$

Remark. Explore the base case when $n = 0$ and $n = 1$, prove them.

Now, prove the following by induction, not that the induction step can be completed using the base case ($n = 1$ with x replaced by $2^n x$),

$$\frac{2}{3}|\sin x| + \sum_{k=1}^{n-1} |\sin(2^k x)| + \frac{1}{3}|\sin(2^n x)| \leq \frac{\sqrt{3}}{2} n$$

Comparing what you have with the original inequality, the conclusion is not far anymore.

Solution. We prove that

Claim —

$$2|\sin x| + |\sin 2x| \leq \frac{3\sqrt{3}}{2}.$$

Proof.

$$\begin{aligned}
2|\sin x| + |\sin 2x| &= 2|\sin x|(1 + |\cos x|) \\
&= 2\sqrt{(1 - (|\cos x|)^2)\,(1 + |\cos x|)^2} = 2\sqrt{(1 - |\cos x|)\,(1 + |\cos x|)^3} \\
&= \frac{2}{\sqrt{3}}\sqrt{3(1 - |\cos x|)(1 + |\cos x|)(1 + |\cos x|)(1 + |\cos x|)} \\
&\leq \frac{2}{\sqrt{3}}\sqrt{\left(\frac{3(1 - |\cos x|) + (1 + |\cos x|) + (1 + |\cos x|) + (1 + |\cos x|)}{4}\right)^4} \\
&= \frac{2}{\sqrt{3}}\left(\frac{3}{2}\right)^2 = \frac{3\sqrt{3}}{2}
\end{aligned}$$

■

For the base case for $n = 1$,

$$\sum_{k=0}^{n} |\sin(2^k x)| = \left(\frac{2}{3}|\sin x| + \frac{1}{3}|\sin 2x|\right) + \left(\frac{1}{3}|\sin x| + \frac{2}{3}|\sin 2x|\right) \leq \frac{\sqrt{3}}{2} + \left(\frac{1}{3} + \frac{2}{3}\right) = 1 + \frac{\sqrt{3}}{2} n.$$

Furthermore, from the base case, by replacing x with $2^n x$,

$$\frac{2}{3}|\sin 2^n x| + \frac{1}{3}|\sin 2^{n+1} x| \leq \frac{\sqrt{3}}{2}.$$

Now, let's assume that the inequality stands for $n - 1$,

$$\begin{aligned}
&\sum_{k=0}^{n-1} |\sin(2^k x)| + \frac{2}{3}|\sin 2^n x| + \frac{1}{3}|\sin 2^{n+1} x| \leq \frac{\sqrt{3}}{2}(n-1) + \frac{\sqrt{3}}{2} \\
&\Rightarrow \sum_{k=0}^{n+1} |\sin(2^k x)| \leq \frac{\sqrt{3}}{2}(n) + \frac{1}{3}|\sin 2^n x| + \frac{2}{3}|\sin 2^{n+1} x| \leq \frac{\sqrt{3}}{2}(n+1)
\end{aligned}$$

□

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Problem 9.5.6 (Term of a sequence)**.** (*10 points*) Define a sequence (a_n) by

$$a_1 = 1, a_2 = 2, a_{n+2} = 2a_{n+1} - a_n + 2, \ \forall n \geq 1.$$

Prove that $\forall m \geq 1, \ a_m a_{m+1}$ is also a term of this sequence.

> **Remark.** Explore a first few examples to see if you can obtain the generic formula for the terms of this sequence. Prove your hypothesis. Then prove the problem statement.

Solution. First we prove by induction that

> **Claim —** $a_n = (n-1)^2 + 1$.

Proof. It is easy to verify the base case. For the inductive step:

$$2(n^2+1) - ((n-1)^2+1) + 2 = n^2 + 2n + 2 = (n+1)^2 + 1.$$

■

Now, $\forall m \geq 1$,

$$a_m a_{m+1} = ((m-1)^2+1)(m^2+1) = (m^2 - 2m + 2)(m^2+1) = (m^2 - m + 1)^2 + 1 = a_{m^2-m+2}.$$

□

Problem 9.5.7 (A remarkable sequence)**.** (*10 points*) Let $a_1, a_2, \ldots$ be a sequence with

$$a_1 = 1, \ a_{n+1} = \begin{cases} a_n - 2 \text{ if } a_n - 2 \notin \{a_1, a_2, \ldots, a_n\} \text{ and } a_n - 2 > 0, \\ a_n + 3 \text{ otherwise} \end{cases}$$

Prove that for every positive integer $k \geq 1$, there exist n such that

$$a_n = a_{n-1} + 3 = k^2.$$

> **Remark.** This remarkable sequence is worth to explore. Try to figure out for a few values of n
>
> $$a_{5n+1}, a_{5n+2}, a_{5n+3}, a_{5n+4}, a_{5n+5}.$$
>
> Prove your hypothesis by induction on n.

Solution. It is easy to verify that by induction for a set of five numbers $5n+1, 5n+2, 5n+3, 5n+4, 5n+5$,

$$a_{5n+1} = 5n+1, \ a_{5n+2} = 5n+4, \ a_{5n+3} = 5n+2, \ a_{5n+4} = 5n+4, \ a_{5n+5} = 5n+3.$$

The required statement follows. □

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Problem 9.5.8 (Invariant property of sum of powers)**.** (*10 points*) Let x, y be real numbers such that the number $x+y$, x^2+y^2, x^3+y^3, x^4+y^4, are all integers. Prove that, for all $n \geq 5$, x^n+y^n is an integer.

> **Remark.** Can you express $x^{n+1}+y^{n+1}$ based on $x+y, xy$, $x^{n-1}+y^{n-1}$, and x^n+y^n? If you see that then could it be true that xy is an integer? Once you have it, continue with a strong induction.

Solution. First note that $2xy = (x+y)^2 - (x^2+y^2)$ so this is an integer. Furthermore

$$2(x+y)^4 = 2(x^4+y^4) + 4(2xy)(x^2+y^2) + 3(2xy)^2 \Rightarrow 2 \mid 3(2xy)^2 \Rightarrow 2 \mid 2xy \Rightarrow xy \in \mathbb{Z} \quad (*)$$

Now, with (*) and the identity

$$x^{n+1}+y^{n+1} = (x+y)(x^n+y^n) - xy(x^{n-1}+y^{n-1})$$

it is easy to prove by induction that $n \geq 5$, x^n+y^n. □

Problem 9.5.9 (Complete residue class)**.** (*10 points*) Let a and n be two positive integers such that a^n-1 is divisible by n. Prove that the number $a+1, a^2+2, \ldots, a^n+n$ are all distinct modulo n.

Theorem (Euler's Theorem)

$a, n \in \mathbb{Z},\ \gcd(a,n) = 1 \Rightarrow a^{\varphi(n)} \equiv 1 \pmod{n}$, where $\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$.

> **Remark.** Euler theorem is useful to determine the existence of k, where $a^k \equiv 1 \pmod{n}$. By the given condition, such $k \mid n$. Hence $k < n$. This is where the assumption for $k < n$ in a strong inductive step can come in.
>
> The remaining perhaps to prove that
>
> $$1 \leq \ell_1 \neq \ell_2 \leq n \Rightarrow a^{\ell_1} + \ell_1 \not\equiv a^{\ell_2} + \ell_2 \pmod{n}.$$

Solution. We prove by induction. For $n = 1$ the case case is clear with single term. Assume now that the statement is true for all integers less than $n \geq 2$. Let k, the order of a modulo n, which is the least value such that $a^k \equiv 1 \pmod{n}$.

From Euler's Theorem, $k < n$, and from the condition given by the problem $n \mid a^n - 1$, we have $k \mid n$. By induction hypothesis,

$$a+1, a^2+2, \ldots, a^k+k$$

are distinct modulo k. We prove now the statement:

$$1 \leq x \neq y \leq n,\ a^x + x \not\equiv a^y + y \pmod{n}.$$

Let $x = kz + t, y = ku + v$, $(1 \leq t, uv \leq k, 0 \leq z, u < \frac{n}{k}$. Then

$$a^x \equiv a^t \pmod{n},\ a^y \equiv a^v \pmod{n}.$$

Case 1: $t \neq v$, then

$$a^x + x \equiv a^t + t \not\equiv a^v + v \equiv a^y + y \pmod{n}.$$

Case 2: $t = v$, then $z \neq u$,

$$a^x + x \equiv a^t + kz + t = a^t + ku + v + k(z-u) \equiv a^y + y + k(z-u) \not\equiv a^y + y \pmod{n}.$$

□

Problem 9.5.10 (An immitation of Fermat's Theorem)**.** (*10 points*) Let $n \geq 1$, which is not divisible by 3. Show that

$$x^3 + y^3 = z^n$$

has at least one solution (x, y, z) where x, y, z are positive integers.

> **Remark.** Prove the first two base case $n = 1, n = 2$. Then prove that based on a solution (x_n, y_n, z_n) for the case n, you can produce a solution $(x_{n+3}, y_{n+3}, z_{n+3})$ for the case $n + 3$. Here's the idea:
>
> $$(a^3)(x^3 + y^3) = (a^3)(z^n) = z^{n+1}.$$
>
> Now, make sure that your induction proof is correctly written.
>
> Side note, actually from $z = a^3 + b^3$, $n = 3r + 1$ you can try to figure out what would be the value of x, y. If you can do that then the case $n = 3r + 2$ is also simple with $z = (a^3 + b^3)^2$.

Solution. The base case $n = 1$ is obvious. For $n = 2$, we have $1^3 + 2^3 = 3^2$.

Now, let assume that (x_n, y_n, z_n) is a solution for the case n, then

$$(z_n)^{n+3} = (z_n^3)z_n^n = (z_n^3)(x_n^3 + y_n^3) = (z_n x_n)^3 + (z_n y_n)^3$$

Thus $(x_{n+3}, y_{n+3}, z_{n+3}) = (z_n x_n, z_n y_n, z_n)$ is a solution for $x^3 + y^3 = z^{n+3}$ case. $\square$

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Chapter 10

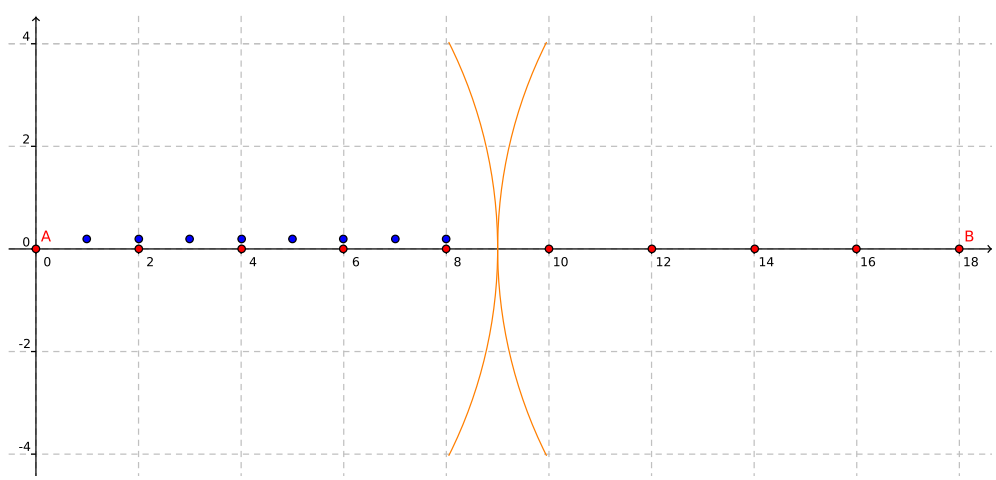
Session 8

10.1 Unusual Problems (I-Level)

In this session we discuss problems that require a thorough understanding of the problem text in order to devise some approach.

Example 10.1.1 (23-24-S8-I-P1)

On a straight line there are 10 distinct red points as show below Anna selects every two points of these 10 points, she then *marks* the midpoint of the segment connected them. What is the minimal number of distinct *marked* points?



Solution. Let A and B be the points with the greatest distance between them. Connect point A by segments with all other points except B. The midpoints of the obtained $10 - 2 = 8$ segments do not coincide, otherwise the second endpoints of the segments would coincide too, and are inside a circle centred at A with radius $\frac{AB}{2}$.

Similarly there are 8 distinct midpoints situated in a circle centred at B of radius with radius $\frac{AB}{2}$. The two circles have exactly one common point, which is the midpoint of the segment AB. Thus, together with the midpoint of AB, we have implicitly constructed $8 + 8 + 1 = \boxed{17}$ midpoints of segments.

The example above is to ensure that the minimal number is reached. All given points lie on the same straight line at $0, 2, 4, \ldots, 18$, so with a constant step between them. It is easy to see that there are exactly $\boxed{17}$ distinct midpoints, and they are at $1, 2, 3, \ldots, 17$. $\square$

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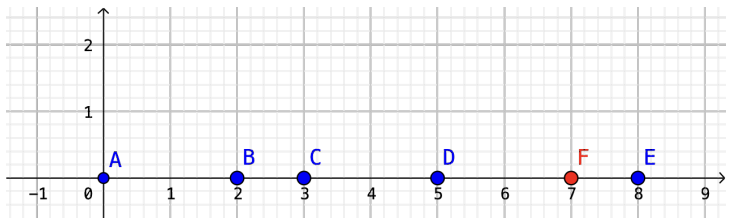
Example 10.1.2 (23-24-S8-I-P2)

Five friends live on the same street at A, B, C, D, and E, see below. Place F is not a ideal place to meet because the sum of distances from A, B, C, D, and E to F is $7 + 5 + 4 + 2 + 1 = 19$. Try to figure out *where* should they meet so that the *sum of distances from that place to their houses is minimal.*

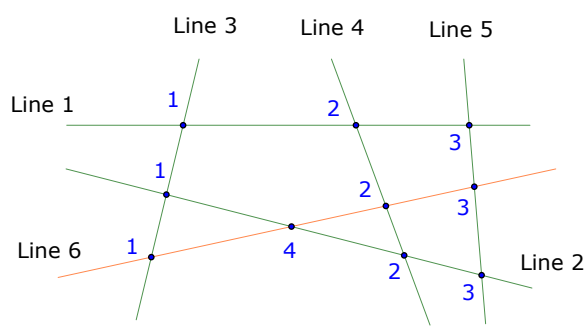


Solution. Let P be the point where they meet so that the *sum of distances from that place to their houses is minimal.* First, let's take A and B, P must be in between them in order to make sum of distances from P to that two points to be minimal. Similarly, P must be in between B, C and C, D. Thus $P \equiv \boxed{C.}$ □

Example 10.1.3 (23-24-S8-I-P3)

On the blackboard six segments are drew. Every two segments intersect one another at a point. The 1st segment contains three of the intersections, the 2nd segment contain 4 of the intersections, the 3rd, 4th, and 5th segments each contains five intersections. What segments does the 6th segment intersect with?

Solution. Each of the 3rd, 4th, and 5th are to intersect all the remaining segments. Therefore the 1st segment intersects with the 3rd, 4th, and 5th segments only. Thus, the 2nd segment intersect with the 3rd, 4th, 5th, and 6th segments.

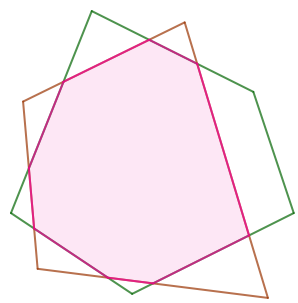


Thus, the 6th intersects with the $\boxed{2^{\text{nd}}, 3^{\text{rd}}, 4^{\text{th}}, \text{ and } 5^{\text{th}}}$ segments. □

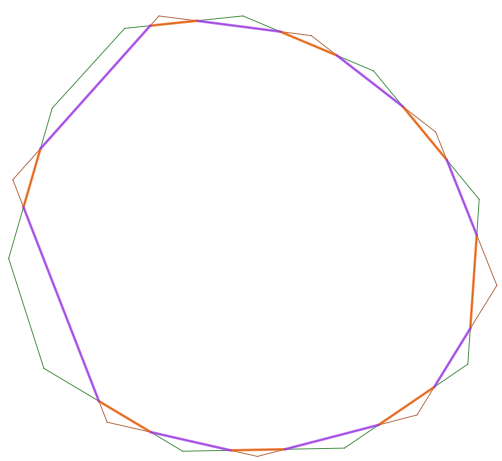
Example 10.1.4 (23-24-S8-I-P4)

In a convex polygon, any line extending one of its sides does not go through the inside of the polygon. The figure below shows the intersection of a convex pentagon and a convex quadrilateral, which is an octogon (a polygon with 8 sides.)

At most how many sides can the intersection of a convex octogon with a convex decagon have? *A decagon is a polygon with* 10 *sides.*



Solution. Consider the 8−sided polygon. Each of its sides can be intersected by at most 2 sides of the 10−sided polygon. It means that the intersection polygon would have at most *the number of sides of the* 8−*sided polygon* plus *the number of sides same as the number of vertices of the* 8−*sided polygon.* See the figure below for an example.



The number of sides is $\boxed{16.}$. □

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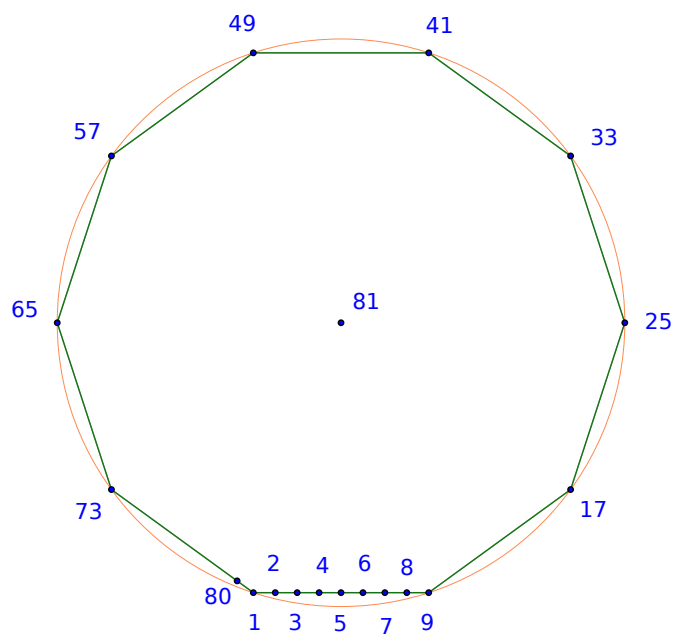
Example 10.1.5 (23-24-S8-I-P5)

In a ceremony of the guild, the master asked the members to form 10 lines, each line consists of 9 persons, so that he can stand in *a place that has the same distance to every row.*

The member were confused because all together, including the masters, the guild has 81 members. Chi Ton, an invited guest to the ceremony, pointed out how they can do that. Of course, he did neither stand into those lines, nor posed as the master. What was his solution?



Solution. The key here is that *the master is equidistant from each row*, not each member. Thus, there should be 10 rows at the same distant from a point. It reminds us of the circle where the center is at the same distant from 10 chords, if those chords have the same length. To arrange 80 persons into 10 chords, each has 9 persons, we create a regular polygon with 9 persons on each side as show below in the diagram. □

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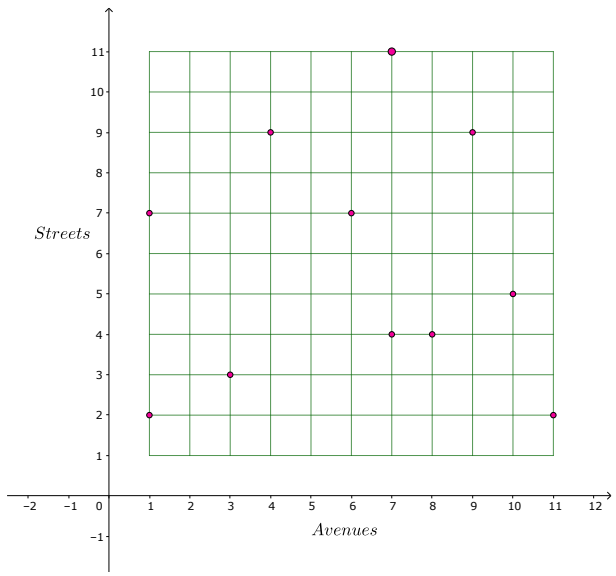
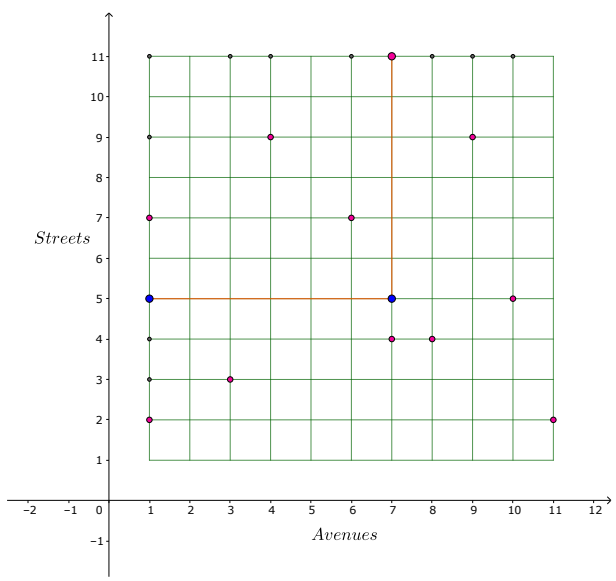
10.2 Problems (I-Level)

Submission deadline: December 3, 2023.

Problem 10.2.1 (23-24-S8-I-P6)**.** (*20 points*) In the diagram below, the red dots indicate houses of friends, who can only walk along streets and avenues. Where is the place that has the minimal sum of distances to their houses?



Solution. By applying the result of the one-dimensional problem 23-24-S8-I-P1, it is easy to see that the intersection of Avenue 1 and Street 5 at $(1, 5)$ is ideal if all the friends would have been living on the Avenue 1. Similarly, the intersection of Avenue 7 and Street 11 at $(7, 11)$ is ideal if the friends would have been living on the Street 11.



Since *the sum of distances from the friend houses consist of the two sums of distances along each axis* from the friend houses to the meeting place. The intersection $\boxed{(7, 5)}$ minimizes both sums. □

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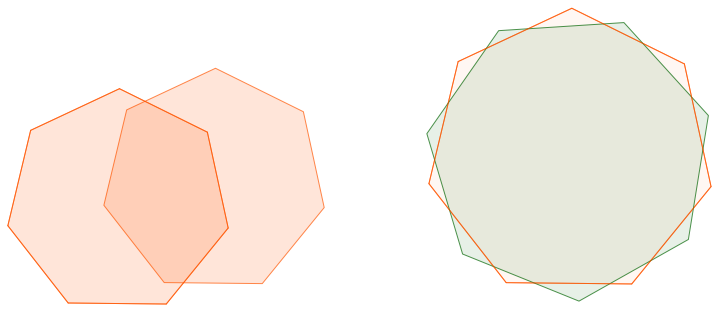
Problem 10.2.2 (23-24-S8-I-P7)**.** (*20 points*) Let ABC be a triangle. Points D, E, and F be the midpoints of sides BC, CA, and AB respectively. What is the maximum number of circles which pass through at least 3 of these 6 points?

Solution. All $\binom{6}{3}$ triples of points can produce distinct circles aside from the case where the three points are collinear. BDC, CEA, AFB. Hence, the answer is $\boxed{20 - 3 = 17.}$ □

Problem 10.2.3 (23-24-S8-I-P8)**.** Two same-size regular heptagon (regular 7−gon, or a regular polygon with 7 sides) overlap each other. The overlapping region is a convex polygon. *For example, the figure below shows an overlapping hexagon (darker shaded).*

(*10 points*) What is the *largest* number of sides can the overlapping polygon have?

(*10 points*) What is the *largest odd* number of sides can the overlapping polygon have?



Solution. The sides of the overlapping polygon are the sides of the two regular heptagons. In other words, its sides are (part of) the sides of the two polygons. So the largest number of sides it can have is the sum of the numbers of sides of both polygons. See above right for an example.

Thus, the largest number of sides is $\boxed{7 + 7 = 14.}$ □

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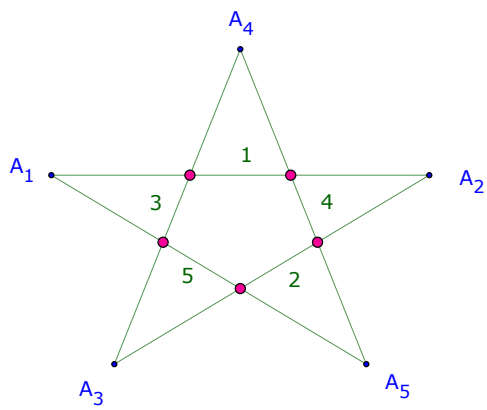
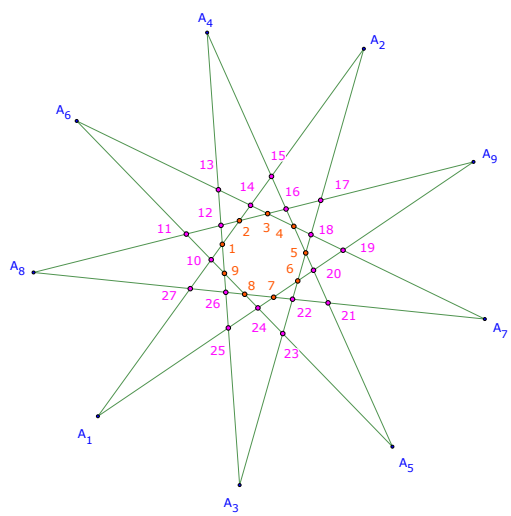
Problem 10.2.4 (23-24-S8-I-P9)**.** A *close path* $A_1A_2\ldots A_5$ contains 5 segments A_1A_2, A_2A_3, $\ldots$, A_5A_1 that has 5 *self-intersection* points . This close path has the *maximum* number of self-intersection points for any $5-$segment close path.

(*10 points*) Your first task is to find the *maximum* number of self-intersection points for a $7-$segment close path $A_1A_2\ldots A_7$.

(*10 points*) Your second task is to find the *maximum* number of self-intersection points for a $9-$segment close path $A_1A_2\ldots A_9$.





Solution. If we take two points, says A_1 and A_2, then except A_nA_1, A_1A_2, and A_2A_3 the remaining $n-3$ segments of the path $A_1A_2\ldots A_nA_1$ can intersect with A_1A_2. There are n such consecutive pairs like (A_1, A_2), and since each intersection of two segments is counted twice, so the number of intersections is atmost $\frac{n(n-3)}{2}$.

Note that if we ***walk*** *along the points $A_1, A_2, \ldots, A_n, A_1$, then for $i = 1, 2, \ldots, n-1$, the $i^{th}-$segment can intersect at most $i-2$ previous segments on the path. Finally the $n^{th}-$segment intersects at most $n-3$ previous segments. Therefore the number of intersection points cannot exceed,*

$$1 + 2 + \ldots + (n-3) + (n-3) = \frac{(n-3)(n-2)}{2} + (n-3) = \frac{n(n-3)}{2}.$$

For a $7-$segment close path, the maximum number of intersection points is $\frac{7\cdot 6}{2} = \boxed{21.}$

For a $9-$segment close path, the maximum number of intersection points is $\frac{9\cdot 6}{2} = \boxed{27.}$ $\square$

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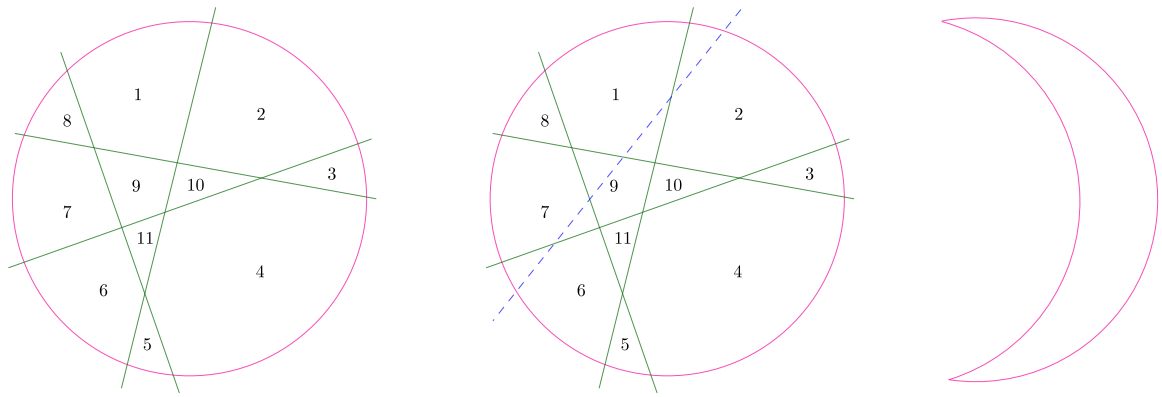
Problem 10.2.5 (23-24-S8-I-P10)**.** Four (green) cuts can dissect a circle into 11 parts as shown below in figure on the left. A (blue) cut is *divided* by four existing lines at 4 points into 3 segments and 2 half-lines. These divide 5 existing regions of the circle into 10 regions, thus increasing the number of regions by 5 as shown in the middle figure. The maximal number of regions of a circle that n cuts can produce is $\frac{n(n+1)}{2}+1$.

(*10 points*) What is the maximal number of regions of *a crescent moon* at right that 2 cuts can produce?

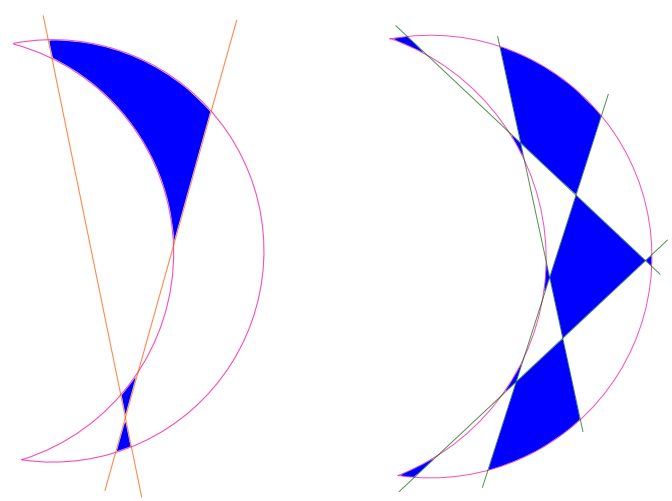
(*10 points*) What is the maximal number of regions of *a crescent moon* at right that 4 cuts can produce?



Solution. The number of regions n cuts will produce for a circle is $2+2+3+\ldots+(n-1)+n=\frac{n(n+1)}{2}+1$.

For the crescent, in addition of n newly added of regions, the nth cut can cut the crescent again, adding one more region, resulting in $3+3+\ldots+n+(n+1)=\frac{(n+1)(n+2)}{2}$ regions.

With $n=2$, we have $\frac{3\cdot 4}{2}=\boxed{6.}$ With $n=4$, we have $\frac{5\cdot 6}{2}=\boxed{15.}$ The diagrams below shows an example with alternate colouring. $\square$



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10.3 Square is never negative (A-Level)

In this session we discuss problems that cleverly uses the fact that any square a^2 is a non-negative number.

Example 10.3.1 (23-24-S8-A-P1)
a, b, c, d, and e are real numbers. Prove that

$$2a^2 + 3b^2 + 2c^2 + 3d^2 + 2e^2 \geq 2(a + c + e)(b + d).$$

Solution. Note that the right side, when expanded, contains the pair-wise product of terms $2ab$, $2ad$, $2cb$, $2cd$, $2eb$, and $2ed$, thus we investigate the square of their differences,

$$\begin{cases} (a-b)^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2ab, \\ (a-d)^2 \geq 0 \Rightarrow a^2 + d^2 \geq 2ad, \\ (c-b)^2 \geq 0 \Rightarrow c^2 + b^2 \geq 2cb, \\ (c-d)^2 \geq 0 \Rightarrow c^2 + d^2 \geq 2cd, \\ (e-b)^2 \geq 0 \Rightarrow e^2 + b^2 \geq 2eb, \\ (e-f)^2 \geq 0 \Rightarrow e^2 + d^2 \geq 2ed, \end{cases}$$

By summing up these, we receive the desired inequality. □

Example 10.3.2 (23-24-S8-A-P2)
Solve in real numbers the system of equations:

$$\begin{cases} \frac{1}{2}(x + y + 1) = 1 \\ \sqrt{x} - \sqrt{xy} + \sqrt{y} = 1 \end{cases}$$

Solution. From the existence of $\sqrt{x}$ and $\sqrt{y}$, we have $x, y \geq 0$. Now, note that

$$\begin{cases} \frac{1}{2}(x + y + 1) = 1 \\ \sqrt{x} - \sqrt{xy} + \sqrt{y} = 1 \end{cases} \Rightarrow \begin{cases} x + y = 1 \\ \sqrt{xy} = (\sqrt{x} + \sqrt{y}) - 1 \end{cases}$$

$$(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy} = 1 + 2(\sqrt{x}\sqrt{y}) \Rightarrow (\sqrt{x} + \sqrt{y} - 1)^2 = 0$$

$$\Rightarrow \sqrt{x} + \sqrt{y} = 1 \Rightarrow \sqrt{xy} = 0 \Rightarrow (x, y) \in \boxed{\{(0, 1), (1, 0)\}.}$$

□

Example 10.3.3 (23-24-S8-A-P3)
Prove that for any non-negative real numbers x, y, z, w,

$$\sqrt{(x + y)(z + w)} \geq \sqrt{xz} + \sqrt{yw}.$$

Solution. By bringing both sides (with positive values!) to the square, we obtain an equivalent inequality,

$$(x + y)(z + w) \Leftrightarrow (\sqrt{xz} + \sqrt{yw}) = xz + yw + 2\sqrt{xyzw} \Leftrightarrow xw + yz \geq 2\sqrt{xyzw} \Leftrightarrow (\sqrt{xw} - \sqrt{yz})^2 \geq 0$$

□

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Example 10.3.4 (23-24-S8-A-P4)

For $n \geq 2$, let $x_1, x_2, \ldots, x_n$ be real numbers such that,

$$\begin{cases} x_1 + x_2 + \cdots + x_n = n^2 \\ x_1^2 + x_2^2 + \cdots + x_n^2 = n(n^2+1) \end{cases}$$

Prove that $x_k \in [n - \sqrt{n}, n + \sqrt{n}]$, for all $k = 1, 2, \ldots, n$.

Solution. What we need to prove is

$$n - \sqrt{n} \leq x_k \leq n + \sqrt{n},\ k = 1, 2, \ldots, n \Leftrightarrow -\sqrt{n} \leq (x_k - n) \leq \sqrt{n} \Leftrightarrow (x_k - n)^2 \leq n.$$

Now, from the two given equalities, we try to combine them so that we can receive $(x_k - n)^2$, for all $k = 1, 2, \ldots, n$.

$$(x_1^2 + x_2^2 + \cdots + x_n^2) - 2(n)(x_1 + x_2 + \cdots + x_n) + n(n^2) = n(n^2+1) - 2n(n^2) + n(n^2) = n$$
$$\Rightarrow (x_1 - n)^2 + (x_2 - n)^2 + \cdots + (x_n - n)^2 = n.$$

Since the sum of all $(x_k - n)^2$ $(k = 1, 2, \ldots, n)$ is equal to n, thus each and everyone of them should not exceed n. □

Example 10.3.5 (23-24-S8-A-P5)

Solve in positive real numbers the equations

$$\frac{x^2 + y^2}{1 + xy} = \sqrt{2 - \frac{1}{xy}}$$

Solution. Here, we introduce a *substitution technique when you can obtain the sum $x + y$ and the product xy* such that the substitution shall not leave any x or y. Let $s = x + y, p = xy$, then $x^2 + y^2 = s^2 - 2p$, thus the given equation is equivalent to,

$$\frac{s^2 - 2p}{1 + p} = \sqrt{2 - \frac{1}{p}}.$$

Since $(x + y)^2 \geq 4xy \Rightarrow s^2 - 2p \geq 2p$, so by using this we can *remove s from the inequality*, thus leaving only one unknown (p):

$$\frac{s^2 - 2p}{1 + p} \geq \frac{2p}{1 + p}.$$

Now,

$$\frac{2p}{1+p} \geq \sqrt{2 - \frac{1}{p}} \Leftrightarrow 4p^3 \geq (p+1)^2(2p - 1 \Leftrightarrow 2p^3 - 3p^2 + 1 \geq 0 \Leftrightarrow (p-1)^2(2p+1) \geq 0.$$

Since the equality stands, thus $p = 1$ and $s^2 - 2p = 2p$, thus $\boxed{x = y = 1.}$ □

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10.4 Problems (A-Level)

Submission deadline: December 3, 2023.

Problem 10.4.1 (23-24-S8-A-P6)**.** (*10 points*) Prove that for any real numbers a and b

$$a^2 - ab + b^2 \geq \frac{1}{2}(a^2 + b^2) \geq \frac{1}{3}(a^2 + ab + b^2) \geq \frac{1}{4}(a + b)^2.$$

Solution.

$$a^2 - ab + b^2 \geq \frac{1}{2}(a^2 + b^2) \Leftrightarrow 2a^2 - 2ab + 2b^2 \geq a^2 + b^2 \Leftrightarrow (a - b)^2 \geq 0.$$
$$\frac{1}{2}(a^2 + b^2) \geq \frac{1}{3}(a^2 + ab + b^2) \Leftrightarrow 3a^2 + 3b^2 \geq 2a^2 - 2ab + 2b^2 \Leftrightarrow (a - b)^2 \geq 0.$$
$$\frac{1}{3}(a^2 + ab + b^2) \geq \frac{1}{4}(a + b)^2 \Leftrightarrow 4a^2 + 4ab + 4b^2 \geq 3a^2 + 6ab + 3b^2 \Leftrightarrow (a - b)^2 \geq 0.$$

□

Problem 10.4.2 (23-24-S8-A-P7)**.** (*10 points*) Prove that for any real numbers a,

$$(a^2 + 1)(5a^2 + 1) \geq 4a(a^3 + a^2 + 1).$$

Solution.

$$(a^2 + 1)(5a^2 + 1) - 4a(a^3 + a^2 + 1) = 5a^4 + 6a^2 + 1 - 4a^4 - 4a^3 - 4a$$
$$= a^4 - 4a^3 + 6a^2 - 4a + 1 = (a - 1)^4) \geq 0$$

□

Problem 10.4.3 (23-24-S8-A-P8)**.** (*10 points*) Find all x real numbers such that

$$x^3 - x \leq \frac{\sqrt{2}}{4}(x^4 + 1).$$

Solution. First, if $x = 0$ then the inequality is obvious.

Now, with the assumption that $x \neq 0$, dividing both side by x^2,

$$x - \frac{1}{x} \leq \frac{\sqrt{2}}{4}\left(x^2 + \frac{1}{x^2}\right).$$

Let

$$y = x - \frac{1}{x} \Rightarrow y^2 = x^2 + \frac{1}{x^2} - 2.$$

The inequality becomes

$$y \leq \frac{\sqrt{2}}{4}(y^2 + 2) \Leftrightarrow y^2 - 2\sqrt{2}y + 2 \geq 0 \Leftrightarrow (y - \sqrt{2})^2 \geq 0.$$

□

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Problem 10.4.4 (23-24-S8-A-P9)**.** (*15 points*) Solve in real numbers the equation

$$\sqrt{x_1-1}+2\sqrt{x_2-2^2}+\cdots+n\sqrt{x_n-n^2}=\frac{1}{2}(x_1+x_2+\cdots+x_n).$$

Solution. First, from the existence of the square roots: $x_1-1\geq 0, x_2-2^2\geq 0,\ldots,x_n-n^2\geq 0$. Then,

$$\begin{aligned}
&\sqrt{x_1-1}+2\sqrt{x_2-2^2}+\cdots+n\sqrt{x_n-n^2}=\frac{1}{2}(x_1+x_2+\cdots+x_n)\\
\Leftrightarrow\ &(x_1-2(1)\sqrt{x_1-1})+(x_2-2(2)\sqrt{x_2-2^2})+\cdots+(x_n-2(n)\sqrt{x_n-n^2})=0\\
\Leftrightarrow\ &((\sqrt{x_1-1^2})^2-2(1)\sqrt{x_1-1^2}+1^2)+((\sqrt{x_2-2^2})^2-2(2)\sqrt{x_2-2^2}+2^2)+\cdots\\
&+((\sqrt{x_1-n^2})^2-2(2)\sqrt{x_n-n^2}+n^2)=0\\
\Leftrightarrow\ &(\sqrt{x_1-1^2}-1)^2+(\sqrt{x_2-2^2}-2)^2+\cdots++(\sqrt{x_2-2^2}-2)^2=0\\
\Leftrightarrow\ &x_1-1^2=1,\ x_2-2^2=2,\ldots,x_n-n^2=n\\
\Leftrightarrow\ &\boxed{x_1=1^2+1,\ x_2=2^2+2,\ldots,x_n=n^2+n.}
\end{aligned}$$

□

Problem 10.4.5 (23-24-S8-A-P10)**.** (*15 points*) Prove that for any a,b real numbers

$$(a^2+1)(b^2+1)+4ab\geq 2(ab+1)(a+b).$$

Solution. We have $(ab+1)(a+b)=(a^2+1)b+(b^2+1)a$. Thus the inequality can be written as:

$$(a^2+1)(b^2+1)+4ab-2(a^2+1)b-2(b^2+1)a=(a^2+1)(b^2+1-2b)+2a(2b-b^2-1)=(b-1)^2(a-1)^2\geq 0$$

□

Problem 10.4.6 (23-24-S8-A-P11)**.** (*20 points*) Find all real numbers x,y $0<x,y,z<2$ such that:

$$\frac{1}{x}+\frac{1}{2-y}=\frac{1}{y}+\frac{1}{2-z}=\frac{1}{z}+\frac{1}{2-x}=2.$$

Solution. Adding all equations up

$$\left(\frac{1}{x}+\frac{1}{2-x}\right)+\left(\frac{1}{y}+\frac{1}{2-y}\right)+\left(\frac{1}{z}+\frac{1}{2-z}\right)=6.$$
$$\frac{1}{x(2-x)}+\frac{1}{y(2-y)}+\frac{1}{z(2-z)}=3\quad(*)$$

Since $(\sqrt{x}-\sqrt{(2-x)})^2\geq 0\Rightarrow 2\geq 2\sqrt{x(2-x)}\Rightarrow x(2-x)\geq 1$. The equality stands when $x=1$. Similar with y and z. Thus (*) stands if and only if $\boxed{x=y=z=1.}$ □

Problem 10.4.7 (23-24-S8-A-P12)**.** (*20 points*) Solve in integers the equation

$$u^2+v^2+x^2+y^2+z^2=uv+vx-xy+yz+zu+3.$$

Remark. There is a typo in the problem text. It requires to find solution in positive integers only.

Solution. The equality can be rewritten as

$$(u-v)^2+(v-x)^2+(x+y)^2+(y-z)^2+(z-u)^2=6$$

Furthermore since x, y are positive integers, thus $x+y \geq 2$, or $(x+y)^2 \geq 4$.

Therefore $x+y=2$, and $x=y=1$,

$$(u-v)^2+(v-1)^2+(1-z)^2+(z-u)^2=2 \quad (*)$$

Now, since x, y, z, u, v are integers, there are exactly two squares having value 1 and the rest 0 in (*). Consider the sequence (x, v, u, z, y). It starts with 1 and ends with 1, it should jump to 2 at some point and sometime later back to 1. Thus the solutions are

$$(x, v, u, z, y) \in \{(1,1,1,2,1),(1,1,2,1,1),(1,2,1,1,1),(1,1,2,2,1),(1,2,2,1,1),(1,2,2,2,1)\}.$$

□

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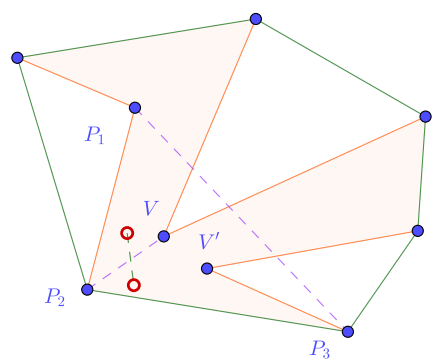
10.5 Solving Problems with Induction Principle (O-Level) - Part 4

Problem 10.5.1 (Triangulation)**.** n is a positive integer, $n \geq 3$.

(*5 points*) Prove that any $n-$gon (not necessarily convex) can be cut into triangles by non-intersecting diagonals.

(*5 points*) Prove that the sum of the inner angles of any $n-$gon (not necessarily convex) is equal to $(n-2)180°$. Hence prove that the number of triangles into which an $n-$gon is cut by non-intersecting diagonals is $n-2$.

> **Remark.** First, prove that any n-gon ($n \geq 4$) (not necessarily convex) has at least one diagonal that completely lies inside it. Then straightforward induction based on this lemma.



Solution. First, we use the problem Existence of a convex hull for existence of a convex hull. Then

> **Claim —** Any $n-$gon $\mathcal{P}$ ($n \geq 4$) has at least one diagonal that completely lies inside it.

Proof. Let us index the vertices $P_1P_2 \dots P_n$. Consider the convex hull $\mathcal{H}$. $\mathcal{H}$ is a convex polygon, so it has a vertex, which will be one of the original vertices, WLOG, A_2. This ensures that the angle $P_1P_2P_3$ is less than $180°$, i.e. if we take triangle $\triangle P_1P_2P_3$ then part of the polygon $\mathcal{P}$ near P_2 is in the interior of $\mathcal{H}$.

Now if P_1P_3 is an interior diagonal, then we are done. Otherwise, we know that there exists exterior part of the polygon inside $\triangle P_1P_2P_3$. In particular, there should be some vertices of $\mathcal{P}$ in there (see V, V'). Let V be the vertex other than P_2 inside $\triangle P_1P_2P_3$, which is farthest away from the line P_1P_3,.

Then VP_2 lies in the interior of $\mathcal{P}$. Indeed, if there is a side of $\mathcal{P}$ that intersects VP_2, then one of the ends of is will be further away from P_1P_3 than V, but will still lie inside the triangle (see the points as hollow red circles) This would contradict the choice of V. Since the points inside the triangle near P_2 are in the interior of $\mathcal{P}$, then the whole segment is, which makes it an interior diagonal. ■

For the first question, we prove by induction. The base case of $n = 3$ is clear. Now, for the inductive step, an interior diagonal divide it into two non-overlapping polygons, each with the number of sides less than n. By the hypothesis, both polygons can be cut into triangles by non-intersecting diagonals, thus the union set of triangles are the ones that the $n-$gon is cut into by non-intersecting diagonals.

For the second question, the base case for the given hypothesis is clear. Now, for the inductive step, an interior diagonal divide it into two non-overlapping polygons, one with $k+1$ sides (k of the $n-$gon plus the diagonal), the other with $n-k-1$ sides ($n-k$ sides from the $n-$gon plus the diagonals). The sum of the inner angles of any $n-$gon now is equal to $(k-1)180° + (n-k-1)180° = (n-2)180°$.

By the total measure of the sum of all angles, it implies that the number of such triangles is $n-2$. □

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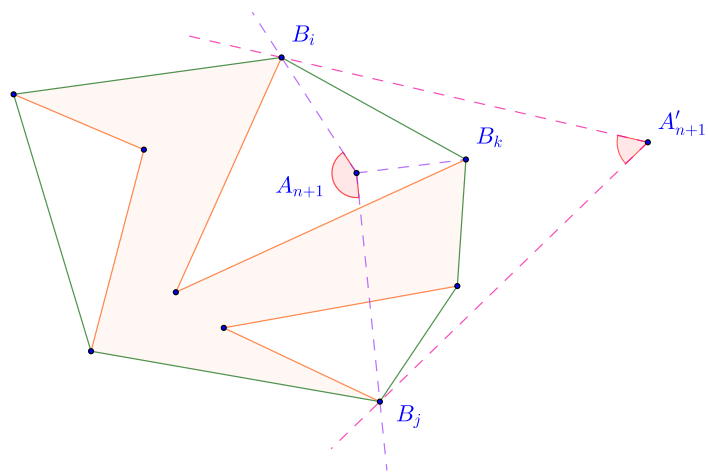
Problem 10.5.2 (Existence of a convex hull)**.** If $A_1, A_2, \ldots, A_n$ are any points in the plane, with any three not collinear, then there is a convex polygon P such that some of A_i are vertices of P and the rest of the point are inside P.

Note that P is called the convex hull of $A_1, A_2, \ldots, A_n$.

> **Remark.** In inductive step, of A_{n+1} is added to the set of the points and $B_1B_2\ldots B_m$ $(m \leq n)$ is the convex hull of $A_1, A_2, \ldots, A_n$, then investigate the angles $\angle B_iA_{n+1}B_j$ $(1 \leq i < j \leq m)$. What happen if some B_k lie outside of the greatest such angle? What happen if all of them lie inside?



Solution. We prove by induction. The base case $n = 3$ is trivial.

Let's assume the hypothesis is true for n. Now, Let $B_1B_2\ldots B_m$ $(m \leq n)$ be the convex hull of the polygon $A_1A_2\ldots A_n$, where $B_1, B_2, \ldots, B_m$ are some points of $A_1, A_2, \ldots, A_n$.

Consider point A_{n+1}. By the Extremal Principle there exists B_i, B_j such that the measure of angle $\angle B_iAB_j$ (less than $180°$) is maximal.

Case 1: There exists point B_k lies outside of this $\angle B_iA_{n+1}B_j$. B_i, B_j, B_k cannot lie the same half-plane, otherwise one of $\angle B_iA_{n+1}B_k$ or $\angle B_kAB_j$ is larger than $\angle B_iA_{n+1}B_j$, which contradicting the choice of $\angle B_iA_{n+1}B_j$. Thus B_k like outside of the same half-plane formed by the angle $\angle B_iA_{n+1}B_j$, which means that A_{n+1} is inside the triangle $\angle B_iB_kB_j$, or inside the convex hull $B_1, B_2, \ldots, B_m$. Thus, $B_1, B_2, \ldots, B_m$ is the convex hull of $A_1A_2\ldots A_nA_{n+1}$.

Case 2: All B_k other than B_i, B_k lie inside of the $\angle B_iA'_{n+1}B_j$. Then, $B_1B_2B_iA_{n+1}B_j\ldots B_m$ is the convex hull of $A_1A_2\ldots A_nA_{n+1}$. $\square$

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Problem 10.5.3 (A point inside a convex polygon)**.** Point O is inside (or on the boundary of) a convex $n-$gon $A_1A_2\dots A_n$. Prove that among the angles A_iOA_j $(1 \le i \ne j \le n)$ there are not fewer than $n-1$ non-acute ones.

> **Remark.** Similar approach as in Induction principle in combinatorial geometry.

Solution. We prove by induction based on n. For $n = 3$ there is nothing to prove.

Let consider $P_1P_2\dots P_n$ polygon, $n \ge 4$. Let p, q, r be such indexes that O is inside $\triangle A_pA_qA_r$. Let $A_k \notin \{A_p, A_q, A_r\}$. By removing A_k we receive a $(n-1)-$gon that we can apply the induction hypothesis to receive $n-2$ non-acute angles. Note that none of these triangles has A_k as a vertex.

Point O now must be inside one of the triangles with vertices A_k and two of A_p, A_q, A_r. WLOG let $O \in \triangle A_kA_qA_r$. Then

$$\angle A_kOA_q + \angle A_kOA_p = 360^\circ - \angle AqOA_r \ge 180^\circ.$$

Hence, at least of of these two $\angle A_kOA_q, \angle A_kOA_p$ is a non-acute angles. Together with $n-2$ previously determined ones, they make $n-1$ non-acute angles. □

Problem 10.5.4 (Sum of vectors)**.** On the cirle of radius 1 with the center O there are given $2n+1$ points $P_1P_2\dots P_{2n+1}$, which lie on one side of a diameter. Prove that

$$|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \dots + \overrightarrow{OP_n}| \ge 1.$$

You might want to read the Vector chapters, Geometry section of LPS V3 for refreshing some concepts, definitions.

> **Remark.** Prove the case for $n = 0, 1$. Then for the inductive step, find the two vectors with the maximal angle between them. The sum of the rest $2n+1$ vectors is another vector obeying the hypothesis. Now, work with these three vectors.

Solution. For $n = 0$ we have $|\overrightarrow{OP_1}| = 1$, thus the hypothesis stands.

Assume that it is true for $2n+1$ vectors. By the Extremal Principle, there exists the maximal angle any two vectors, WLOG, let there be $\overrightarrow{OP_1}$ and $\overrightarrow{OP_{2n+3}}$. Now, by induction hypothesis,

$$|\overrightarrow{OA}| = |\overrightarrow{OP_2} + \overrightarrow{OP_2} + \dots + \overrightarrow{OP_{2n+2}}| \ge 1$$

Note that $\overrightarrow{OA}$ is inside the angle $\angle P_1OP_{2n+3}$, therefore it forms an acute angle with the vector

$$\overrightarrow{OB} = \overrightarrow{OP_1} + \overrightarrow{OP_{2n+3}}$$

which bisects the angle $\angle P_1OP_{2n+3}$. Thus

$$|\overrightarrow{OA} + \overrightarrow{OB}| \ge |\overrightarrow{OA}| = 1.$$

□

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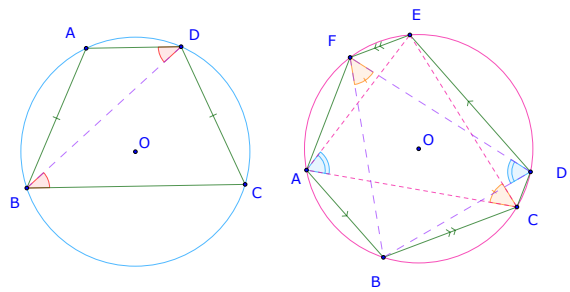
Problem 10.5.5 (No exception for the last pair)**.** $A_1A_2\dots A_{2n}$ is a polygon inscribed in a circle. It is known that all the pairs of its opposite sides except one are parallel. Prove that for any odd n, the remaining pair of sides is also parallel. Prove that for n even, the length of the exceptional sides are equal.

> **Remark.** Prove the case for $n=2, n=4$. Then consider the case when $n=3$. The induction would be a mix of two propositions.



Solution. We first prove case $n=2$, note that in the left diagram $AD \parallel BC$, then $\angle ADB = \angle CBD$, thus $AB = CD$.

For the case $n=3$, let $AB \parallel DE$, so $\angle ACE = \angle BFD$, and $BC \parallel EF$, so $\angle CAE = \angle BDF$, from here $\triangle CAE \sim \triangle FDB$, or $\angle AEC = \angle DBF$, thus $DC \parallel AF$.

Let assume that the statement is true for $(2n-2)-$gon $A_1A_2\dots A_{2n}$, where $A_1A_2 \parallel A_{n+1}A_{n+2}$, $\dots$, $A_{n-1}A_n \parallel A_{2n-1}A_{2n}$. Then considering the $(2n-2)-$gon $A_1A_2 \parallel A_{n-1}A_{n+1}\dots A_{2n-1}$, by the hypothesis, for n odd: $A_{n-1}A_{n+1} = A_{2n+1}A_1$ and for n even $A_{n-1}A_{n+1} \parallel A_{2n-1}A_1$.

Consider now the triangles $A_{n-1}A_nA_{n+1}$ and $A_{2n-1}A_{2n}A_1$.

Case 1: if n is even. then $\overrightarrow{A_{n-1}A_n}$ and $\overrightarrow{A_{2n-1}A_{2n}}$ as well as $\overrightarrow{A_{n-1}A_{n+1}}$ and $\overrightarrow{A_{2n-1}A_{2n}}$ are parallel and oppositely directed. Hence $\angle A_nA_{n-1}A_{n+1} = \angle A_1A_{2n-1}A_{2n}$ and $A_nA_{n+1} = A_{2n}A_1$ since they are chords that cut equal arcs.

Case 2: if n is odd. then $A_{n-1}A_{n+1} = A_{2n-1}A_2$, i.e. $A_1A_{n-1} \parallel A_{n+1}A_{2n-1}$. In the hexagon $A_{n-1}A_nA_{n+1}A_{2n-1}A_{2n}A_1$ we have $A_1A_{n-1} \parallel A_{n+1}A_{2n-1}$, $A_{n-1}A_n \parallel A_{2n-1}A_{2n}$, hence from the base case $n=3$, $A_nA_{n+1} \parallel A_{2n-1}A_1$. □

Problem 10.5.6 (23-24-S5-O-P36)**.** Let

$$a_1 = a_2 = 1,\ a_{n+2} = a_{n+1} + \frac{a_n}{3^n},\ \forall n \geq 1.$$

Prove that $a_n \leq 2,\ \forall n \geq 1$.

> **Remark.** Stronger induction on $a_n \leq 2 - \frac{1}{3^{n-2}}$.

Solution. Let's prove that $a_n \leq 2 - \frac{1}{3^{n-2}},\ \forall n \geq 2 \quad (*)$

For $n=2$, $a_2 = 1 \leq 2 - \frac{1}{3^0} = 1$. For $n=3$, $a_3 = a_2 + \frac{a_1}{3^1} = 1 + \frac{1}{3} < 2 - \frac{1}{3}$.

Let's assume that (*) stand for all $k \leq n$, $n \geq 3$, then

$$a_{n+1} = a_n + \frac{a_{n-1}}{3^{n-1}} < 2 - \frac{1}{3^{n-2}} + \frac{2}{3^{n-1}} - \frac{1}{3^{2n-4}} = 2 - \frac{1}{3^{n-1}} - \frac{1}{3^{2n-4}} < 2 - \frac{1}{3^{n-1}}.$$

□

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Problem 10.5.7 (23-24-S5-O-P37)**.** Let $x_0, x_1, \ldots, x_{1995}$ be positive real numbers,

$$x_0 = x_{1995} = 1,\ x_{n-1} + \frac{2}{x_{n-1}} = 2x_n + \frac{1}{x_n},\ \forall n = 1, 2, \ldots, 1995$$

Find the maximal value that x_0 can have.

> **Remark.** First, from the given equation, find the relation between x_{n-1} and x_n, then a hypothesis on the general form of x_n based on x_0. Prove that.
>
> Once done, proceed with the given condition $x_0 = x_{1995}$ to find the maximal value that x_0 can have.

Solution. First

$$x_{n-1} + \frac{2}{x_{n-1}} = 2x_n + \frac{1}{x_n} \Leftrightarrow (2x_n - x_{n-1})(x_n x_{n-1} - 1) = 0 \Rightarrow x_n = \frac{1}{2}x_{n-1},\ \text{or } x_n = \frac{1}{x_{n-1}}.$$

We prove by induction that

$$x_n = 2^{k_n} x_0^{e_n},\ k_n \in \mathbb{Z},\ |k_n| \le n,\ e_n = (-1)^{n-k_n}.$$

This is true for $n = 0$. Assume that it is true for some n, then

$$x_{n+1} = \frac{1}{2}x_n = 2^{k_n - 1} x_0^{e_n} = 2^{k_{n+1}} x_0^{e_{n+1}},\ \text{where } k_{n+1} = k_n - 1, e_{n+1} = (-1)^{n-k_n} = (-1)^{(n+1)-(k_n-1)},$$
$$x_{n+1} = \frac{1}{x_n} = 2^{-k_n} x_0^{-e_n} = 2^{k_{n+1}} x_0^{e_{n+1}} \text{ where } k_{n+1} = -k_n, e_{n+1} = (-1)^{n-k_n+1} = (-1)^{(n+1)-(k_n)}.$$

Now $x_0 = x_{1995} = 2^{k_{1995}} x_0^{e_{1995}}$, so, $e_{1995} = (-1)^{1995-k_{1995}}$. e_{1995} cannot be 1 because then k_{1995} would be odd, contradicting that $2^{k_{1995}} = 1$. So $e_{1995} = -1$, thus $x_0^2 = 2^{k_{1995}} \le 2^{1994}$.

Thus the maximal value of x_0 is $\boxed{2^{997}.}$

When this can happen? $x_k = 2^{997-k}$ for $k = 0, 1, \ldots, 1994$, and $x_{1995} = (x_{1994})^{-1} = 2^{997}$. □

Problem 10.5.8 (23-24-S5-O-P38)**.** Let $a_0 > 5$ be an odd integer,

$$a_{n+1} = \begin{cases} a_n^2 - 5 \text{ if } a_n \text{ is odd,} \\ \frac{1}{2}a_n \text{ otherwise} \end{cases} \forall n \ge 0$$

Prove that this sequence is not bounded.

> **Remark.** Explore the first few terms to establish a hypothesis on the sequence (a_{3n}). Consider remainder modulo 8 of perfect squares.

Solution. We prove that

> **Claim —** a_{3n} is odd, $a_{3n} > a_{3n-3} > \ldots > a_0 > 5$.

Proof. It is easy to verify the case $a_0 > 5$, odd integer. Let assume a_{3n} is odd, so $a_{3n+1} = a_{3n}^2 - 5 \equiv 4 \pmod 8$. This means that $a_{3n+2} = \frac{1}{2}a_{3n+1}$, $a_{3n+3} = \frac{1}{2}a_{3n+2}$, and a_{3n+3} is odd.

In addition $a_{3n+3}\frac{1}{4}(a_{3n}^2 - 5) > a_{3n}$, $(a_{3n} > 5)$ thus $a_{3n+3} > a_{3n}$. ■

□

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Problem 10.5.9 (23-24-S5-O-P39)**.** Let x be a real number and $n \geq 1$ positive integer. Prove that

$$|\sin(nx)| \leq n|\sin x|.$$

Solution. The case $n = 1$ is clear.

For the inductive step, consider

$$|\sin(n+1)x = |\sin(nx)\cos(x) + \cos(nx)\sin(x)| \leq |\sin(nx)| + |\sin(x)|.$$

□

Problem 10.5.10 (23-24-S5-O-P40)**.** Prove that for $a > 0$ and n positive integer:

$$2(1+a^{n+1})^3 \geq (1+a^3)(1+a^n)^3$$

Prove that for a, b, c positive real numbers,

$$2(a^{2023}+1)(b^{2023}+1)(c^{2023}+1) \geq (1+abc)(a^{2022}+1)(b^{2022}+1)(c^{2022}+1).$$

Solution. For $a > 0$ and n positive integer, we prove by induction that:

Claim — $2(1+a^{n+1})^3 \geq (1+a^3)(1+a^n)^3$.

Proof. For $n = 1$ then

$$2(1+a^2)^3 \geq (1+a^3)(1+a)^3 \Leftrightarrow (a-1)^4(a^2+a+1)$$

Assume that it is true for n, or

$$2(1+a^{n+1})^3 \geq (1+a^3)(1+a^n)^3.$$

Since

$$\begin{aligned}
&(1+a^{n+2})(1+a^n) \geq (1+a^{n+1})^2 \Rightarrow \frac{1+a^{n+2}}{1+a^{n+1}} \geq \frac{1+a^{n+1}}{1+a^n} \\
&\Rightarrow 2(1+a^{n+2})^3 = 2(1+a^{n+1})^3\left(\frac{1+a^{n+2}}{1+a^{n+1}}\right)^3 \\
&\geq (1+a^3)(1+a^n)^3\left(\frac{1+a^{n+1}}{1+a^n}\right)^3 = (1+a^3)(1+a^{n+1})^3
\end{aligned}$$

■

Now it is easy to verify that

$$(1+a^3)(1+b^3(1+c^3) \geq (1+abc)^3.$$

Then

$$\begin{aligned}
&(2(a^{2023}+1)(b^{2023}+1)(c^{2023}+1))^3 = 2(a^{2023}+1)^3 \cdot 2(b^{2023}+1)^3 \cdot 2(c^{2023}+1)^3 \\
&\geq (1+a^3)(1+a^{2022}+1)^3 \cdot (1+b^3)(1+b^{2022}+1)^3 \cdot (1+c^3)(1+c^{2022}+1)^3 \\
&\geq (1+abc)^3((a^{2022}+1)(b^{2022}+1)(c^{2022}+1))^3
\end{aligned}$$

By taking cubic root, the result follows. □

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Chapter 11

Test 2

11.1 Rules

- Each level test consisting of 8 problems will be delivered by automatic scheduled email at the registered time. No test paper will be sent twice.
- Students have **270 minutes to solve the problems.** For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET.
- After the allowed time, the student has 15 minutes to scan the solutions. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET, and has to submit the scanner solutions before 11:45 AM. Any solution submitted later than the allowed time will not be accepted.
- **All submissions must be sent by email to mcc.youth.global@gmail.com and dinhxuanquyen@gmail.com.**
- Solutions must be written in English. Any solution looks unreadable, or is presented badly will not be considered for grading. **Answer without proper solution will not be considered as solution and will not be accepted.** Answer consists of self-explanatory and easy-to-understand diagrams are acceptable but might not be considered as full solution.
- A problem can contain more than one questions, each with an indication of the maximum number of points can be awarded if a correct solution is provided. Partially correct solution is awarded with a some points depending on its quality. Multiple solutions can be submitted for a single problem. Each different solution, if correct, will be awarded the same number of points. Partially distinct solution also counts.
- **For geometry problems, proofs must be accompanied with clean, clear, and precisely drawn diagrams** - by hand with compass and straightedge or by computer software such as GeoGebra. Submitted **solutions without any diagram will not be graded at all.**
- A proof for the hypothesis about **the existence of a locus must contains two parts.** Each part is worth 50% of the awardable points.

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11.2 Problems for Intermediate Level

Read the rules before you start!

Problem 11.2.1 (23-24-T2-I-P1)**.** (*10 points*) In how many ways can a five letter word be written using only the first half of the alphabet with no repetitions such that the third and fifth letters are vowels and the first a consonant?

The word vowel comes from the Latin word vocalis, meaning "vocal" (i.e. relating to the voice). In English, the word vowel is commonly used to refer both to vowel sounds and to the written symbols that represent them (⟨a⟩, ⟨e⟩, ⟨i⟩, ⟨o⟩, ⟨u⟩, and sometimes ⟨w⟩ and ⟨y⟩). Consonant is a letter that is not a vowel.

Solution. In the first half of the alphabet are the vowels a, e and i, so we can choose the two vowels in $3 \cdot 2$ ways (3 for the first, then 2 for the second). We can choose the consonant in 10 ways (13 letters in the first half minus 3 vowels), and then the two unspecified letters can be chosen in $10 \cdot 9$ ways, since three letters of the 13 are taken. The total is thus $3 \cdot 2 \cdot 10 \cdot 10 \cdot 9 = \boxed{5400}$ such words. □

Problem 11.2.2 (23-24-T2-I-P2)**.** (*10 points*) Johan is swimming laps in the pool. At 12:17 he realized that he had just finished one-third of his workout. By 12:22 he had completed eight more laps, and he realized that he had just finished five-elevenths of his workout. After 12:22 how many more laps must Johan swim to complete his workout?

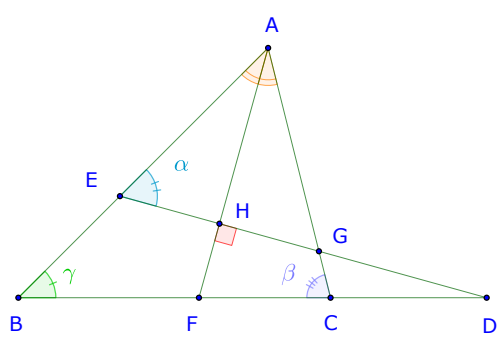
Solution. Let ℓ be the total number of laps in Johan's workout, then

$$w\left(\frac{5}{11} - \frac{1}{3}\right) = 8 \Rightarrow w\frac{4}{33} = 8 \Rightarrow w = 66.$$

Thus, after 12:22 Johan must swim $66 - \frac{5}{11}(66) = \boxed{36}$ more laps. □

Problem 11.2.3 (23-24-T2-I-P3)**.** (*10 points*) In $\triangle ABC$, $AC < AB$. AF is the angle bisector of $\angle BAC$, where F is on segment BC. D is an arbitrary point on line BC such that C is between B and D. Line through D perpendicular to AF intersects AB at E and AC at G, respectively. Prove that

$$\angle AEG = \frac{1}{2}(\angle ABC + \angle ACB).$$



Solution. Let $\alpha = \angle AEG, \angle\beta = \angle ACB$, and $\angle\gamma = \angle ABC$. AF is the angle bisector, so $\angle HAE = \angle HAG$, thus $\triangle AHC \cong AHG$, so $\angle DGC = \angle AGH = \angle AEH = \angle\alpha$.

Thus, by $\triangle GDC$, $\angle\beta = \angle GDC + \angle DGC = \angle GDC + \angle\alpha \quad (*)$

Then, in $\triangle EDB$, $\angle\alpha = \angle EDB + \angle\gamma \quad (**)$

From (*) and (**)

$$\angle\beta - \angle\alpha = \angle GDC = \angle EDB = \angle\alpha - \angle\gamma \Rightarrow \boxed{\angle\alpha = \frac{1}{2}(\angle\beta + \angle\gamma).}$$

□

Problem 11.2.4 (23-24-T2-I-P4)**.** (*10 points*) Find all four-digit perfect squares whose thousands digit and hundreds digit are the same; and whose tens and the unit digit are the same?

In other word, the perfect square would be in the form aabb.

Solution. Let $n^2 = aabb$, then $n^2 = 1000a + 100a + 10b + b = 11(100a + b)$, so $11 \mid n^2$. Since n^2 is a perfect square, thus $11^2 \mid n^2$. Note that $\frac{n^2}{11^2}$ should also be a perfect square, and

$$\frac{1000}{121} \leq \frac{n^2}{11^2} \leq \frac{9999}{121} \Rightarrow 9 \leq \frac{n^2}{11^2} \leq 82 \Rightarrow \frac{n^2}{11^2} \in \{9, 16, 25, 36, 49, 64, 81\}$$

Let's test them

$$9 \cdot 121 = \cancel{1089},\ 16 \cdot 121 = \cancel{1936},\ 25 \cdot 121 = \cancel{3023},\ 36 \cdot 121 = \cancel{4356},$$
$$49 \cdot 121 = \cancel{5929},\ 64 \cdot 121 = 7744,\ 81 \cdot 121 = \cancel{9801},$$

Thus the answer is $\boxed{7744}$. □

Problem 11.2.5 (23-24-T2-I-P5)**.** (*10 points*) In how many ways can four identical red chips and two identical white chips be arranged in a circle?

Remember, two setups which can be rotated into one another are considered the same in circular problems.

Solution. The white chips can either be adjacent, have one red between them, or have two reds between them. These $\boxed{\text{three}}$ are the only ways it can be done! □

Problem 11.2.6 (23-24-T2-I-P6)**.** (*10 points*) The Purple Plant Garden Store sells grass seed in ten-pound bags and fifteen-pound bags. Yesterday half of the grass seed they had was in ten-pound bags. This morning the store received a shipment of 27 more ten-pound bags, and now they have twice as many ten-pound bags as fifteen-pound bags. Find the total weight in pounds of grass seed the store now has.

Solution. Let x be the number of ten-pound bags, y be the number of fifteen-pound bags that the store had yesterday. Since half of the grass seed they had was in ten-pound bags, so

$$10x = 15y \Rightarrow 2x = 3y.$$

After receiving a shipment of 27 more ten-pound bags, they have twice as many ten-pound bags as fifteen-pound bags, thus

$$x + 27 = 2y \Rightarrow 4y = 2x + 54 = 3y + 54 \Rightarrow y = 54,\ x = \frac{3 \cdot 54}{2} = 81.$$

Hence, the total weight in pounds of grass seed the store now has is $(81 + 27) \cdot 10 + 54 \cdot 15 = \boxed{1890.}$ □

Problem 11.2.7 (23-24-T2-I-P7)**.** (*10 points*) P is a point outside of the circle ω centred at O. Lines through P are tangent with the circle ω at R and S. Points A and B on segments PR and PS such that segment AB is tangent with the circle ω at T. If the $\angle RPS = 40°$ find the $\angle AOB$.

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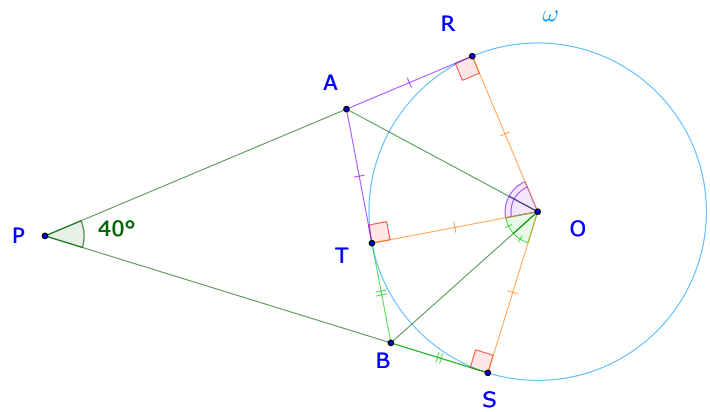


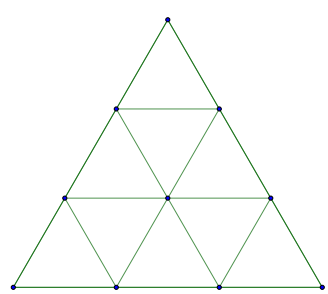
Solution. First, $\angle RPS = 40°$, so half of the difference of the major arc $\widehat{RS}$ (not containing T) and the minor arc $\widehat{RS}$ (containing T) is $40°$, so the difference is $80°$. On the other hand the sum of of the major arc $\widehat{RS}$ (not containing T) and the minor arc $\widehat{RS}$ (containing T) is the full circle, or $360°$. Therefore the minor arc $\widehat{RS}$ is $\frac{1}{2}(360° - 80°) = 140°$.

Now, note that AR and AS are tangents from the same point A to the circle ω, thus $AR = AT$, then $\triangle ARO \cong \triangle ATO$ by side-angle-side. Thus $\angle AOT = \angle AOR$, or $\angle AOT = \frac{1}{2}\angle ROT$. Similarly $\angle BOT = \frac{1}{2}\angle SOT$. Therefore

$$\angle AOB = \angle AOT + \angle BOT = \frac{1}{2}(\angle ROT + \angle SOT) = \frac{1}{2}\angle ROS = \frac{1}{2}\widehat{RS} = \boxed{70°.}$$

□

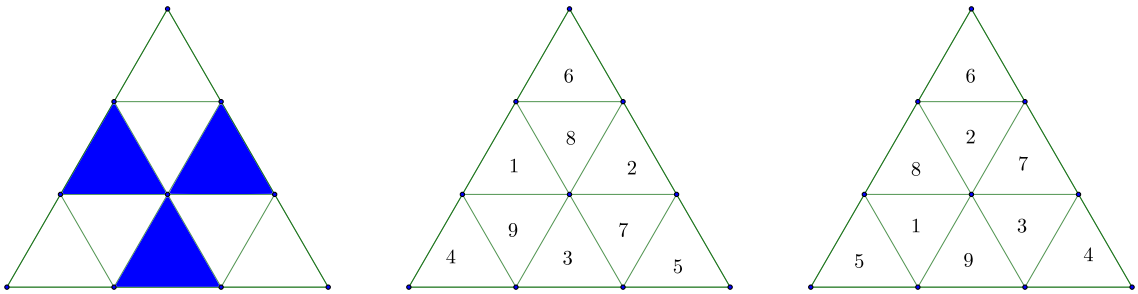
Problem 11.2.8 (23-24-T2-I-P8)**.** (*10 points*) The diagram below shows an equilateral triangle with side length 3 cut into nine equilateral triangles with side length 1. Place the numbers 1 to 9 into the nine smaller triangles (one number into one triangle) such that the sum of the numbers inside any equilateral triangle with side length 2 is the same. What are the smallest and largest values of this sum?



Solution. There are only three equilateral triangle with side length 2. Let s be the sum of the numbers inside any equilateral triangle with side length 2. Let s_1 be the sum of the number in three blue-shaded triangles.

Then by adding all the numbers in all equilateral triangle with side length 2 we have $3s$, which is equal to the sum of the numbers from 1 to 9 plus s_1, or

$$3s = (1 + 2 + \cdots + 9) + s_1 \Rightarrow s_1 = 3s - 45$$



Thus, the smallest value of s_1 can be $1+2+3$ and the largest value is $7+8+9$, which means that the smallest value of s is $\frac{45+6}{3} = \boxed{17}$ and the largest value of s is $\frac{45+24}{3} = \boxed{23.}$ The two diagrams on the right above show how to achieve such solutions. $\square$

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11.3 Problems for Advanced Level

Read the rules before you start!

Problem 11.3.1 (23-24-T2-A-P1)**.** (*10 points*) Prove that for all $2 \leq n \leq 2023$,

$$\sqrt{(n)\sqrt{(n+1)\sqrt{\cdots\sqrt{(2023)}}}} < n+1.$$

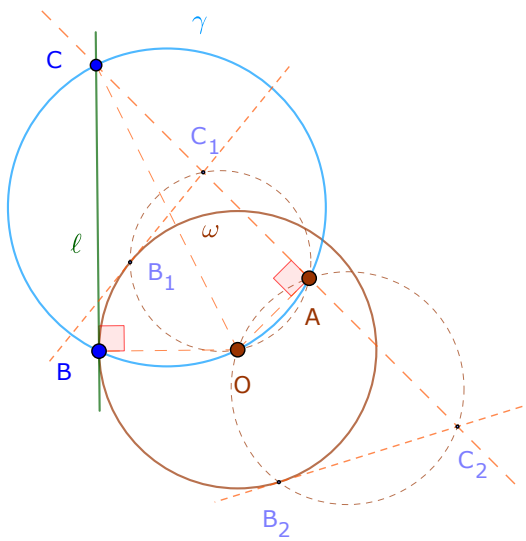
Sometimes you can top the domino in reverse direction.

Solution. For $n = 2023$, we have $\sqrt{2023} < 2024$, which is obvious. Assume that for some $n < 2023$,

$$\sqrt{(n)\sqrt{(n+1)\sqrt{\cdots\sqrt{(2023)}}}} < n+1. \Rightarrow \sqrt{(n-1)\sqrt{(n)\sqrt{\cdots\sqrt{(2023)}}}} < \sqrt{(n-1)(n+1)} = \sqrt{n^2} < n.$$

□

Problem 11.3.2 (23-24-T2-A-P2)**.** (*10 points*) Let A be a fixed point inside the circle ω. Let B be a point moving on the perimeter of O. Let ℓ be the line tangent to ω at B. The circumcircle γ of $\triangle ABO$ intersects ℓ at C. Find the locus of such C point when B moves around the circle ω.



Solution. By choosing a a few locations for C, we establish a hypothesis that:

The locus of points C is the set of points on the line δ perpendicular to AO at A.

We have to prove the hypothesis in two steps:

Claim — If C is a intersection of the circumcircle γ of $\triangle ABO$ with line ℓ, then $C \in \delta$.

Proof. It is easy to see that $OB \perp BC$, thus OC is the diameter of γ, then $\angle CAO = 90$, thus $C \in \delta$. ■

Claim — If $C \in \delta$, let B be the intersection of the circumcircle of $\triangle AOC$ with the circle ω, then CB is tangent to ω.

Proof. CO is the diameter so $CR \perp OB$. ■

□

Problem 11.3.3 (23-24-T2-A-P3)**.** (*10 points*) Mai flips a fair coin 6 times. Find the probability that she flips 3 heads in a row but not 4 heads in a row.

For example HHHTTT is a valid one, and THHHHT is not.

Solution. There are in total $2^6 = 64$ ways to flips a fair coin 6 times. There are several cases when Mai can flip 3 heads in a row but not 4 heads in a row.

Case 1: She flips in total 3 tails. There are four such ways (note the symmetry of each pairs):

$$HHHTTT,\ HHHTTT,\ TTHHHT,\ THHHTT$$

Case 2: She flips in total 2 tails. There are six such ways (note the symmetry of each pairs):

$$HHHTTH,\ HTTHHH,\ THTHHH,\ HHHTHT,\ HTHHHT,\ THHHTH$$

Case 3: She flips in total 1 tail. There are two such ways (note the symmetry of each pairs):

$$HHHTHH,\ HHTHHH,$$

Thus, the probability is

$$\frac{4+6+2}{64} = \boxed{\frac{3}{16}.}$$

□

Problem 11.3.4 (23-24-T2-A-P4)**.** (*10 points*) Solve in positive real numbers the system of equations

$$\begin{cases} x^2 + \frac{1}{y} = \sqrt{y} + \sqrt{z} \\ y^2 + \frac{1}{z} = \sqrt{z} + \sqrt{x} \\ z^2 + \frac{1}{x} = \sqrt{x} + \sqrt{y} \end{cases}$$

Solution. By summing up the equations and group the terms by x, y, then z,

$$\left(x^2 - 2\sqrt{x} + \frac{1}{x}\right) + \left(y^2 - 2\sqrt{y} + \frac{1}{y}\right) + \left(z^2 - 2\sqrt{z} + \frac{1}{z}\right) = 0$$

$$\Leftrightarrow \left(x - \frac{1}{\sqrt{x}}\right)^2 + \left(y - \frac{1}{\sqrt{y}}\right)^2 + \left(z - \frac{1}{\sqrt{z}}\right)^2 = 0 \Leftrightarrow \boxed{x = y = z = 1.}$$

□

Problem 11.3.5 (23-24-T2-A-P5)**.** (*10 points*) Let the sequence (a_n) is defined as follow,

$$a_1 = 1,\ a_{n+1} = \frac{1}{16}\left(1 + 4a_n + \sqrt{1 + 24a_n}\right).$$

Prove that

$$a_{2023} = \frac{1}{3} + \frac{1}{2^{2023}} + \frac{2}{3 \cdot 4^{2023}}.$$

Solution. We prove by induction that,

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Claim — $a_n = \frac{1}{3} + \frac{1}{2^n} + \frac{2}{3 \cdot 4^n}$, $\forall n \geq 1$.

Proof. For the base case $n = 1$,

$$\frac{1}{3} + \frac{1}{2^1} + \frac{2}{3 \cdot 4^1} = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1 = a_1.$$

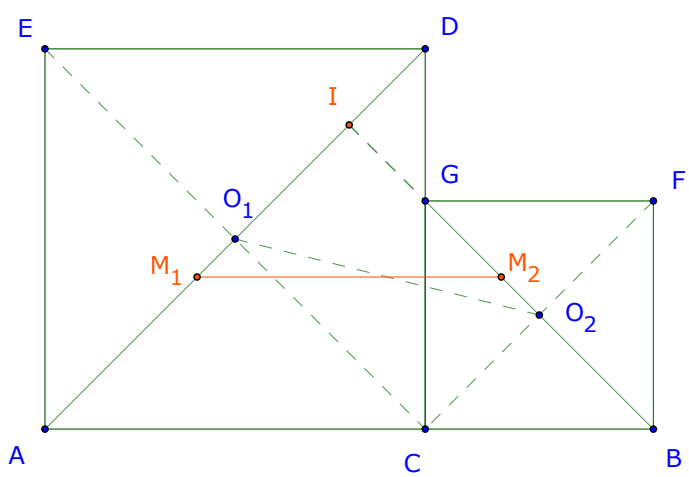
For the inductive step,

$$\begin{aligned}
&1 + 24a_n = 1 + 8 + \frac{3}{2^{n-3}} + \frac{1}{.4^{n-2}} = 3^2 + 2 \cdot 3 \cdot \frac{1}{2^{n-2}} + \left(\frac{1}{2^{n-2}}\right)^2 = \left(3 + \frac{1}{2^{n-2}}\right)^2 \\
&\Rightarrow 1 + 4a_n + \sqrt{1 + 24a_n} = 1 + \frac{4}{3} + \frac{4}{2^n} + \frac{4 \cdot 2}{3 \cdot 4^n} + 3 + \frac{1}{2^{n-2}} = \frac{16}{3} + \frac{16}{2^{n+1}} + \frac{16 \cdot 2}{3 \cdot 4^{n+1}} \\
&\Rightarrow a_{n+1} = \frac{1}{3} + \frac{1}{2^{n+1}} + \frac{2}{3 \cdot 4^{n+1}}
\end{aligned}$$

■

The desired equality follows when applying $n = 2023$ for the claim. □

Problem 11.3.6 (23-24-T2-A-P6)**.** (*10 points*) C is an arbitrary point on segment AB. On the same side of AB we construct two squares $ACDE$ and $BCGF$ (C, G, D are collinear.) Let O_1 and O_2 be the centers of the squares $ACDE$ and $BCGF$, respectively. Find the locus of midpoints M of O_1O_2.



Solution. By choosing a a few locations for M, we establish a hypothesis that:

The locus of points M is the midsegment M_1M_2 of the right isosceles triangle $\triangle AIB$, where $\angle AIB = 90°$.

We have to prove the hypothesis in two steps:

Claim — If M is the midpoint of O_1O_2, then $M \in M_1M_2$.

Proof. It is easy to see that A, I, D are collinear because $\angle BAI = \angle CAD = 45°$. Similarly B, G, I are also collinear. Then $O_1 \in AD, O_2 \in BG$, so $O_1 \in AI$, $O_2 \in BI$. Thus $\angle O_1IO_2 = 90°$. Triangle $\triangle O_1IO_2$ is right. Furthermore $CO_1 \perp O_2I$, $CO_2 \perp O_1I$ thus O_1IO_2C is a rectangle and M is the mid point of IC. Now, in triangle AIB, the midpoint of IC must be on the midsegment M_1M_2. ■

Claim — If $M \in M_1M_2$, let C be the intersection of IM with AB, on the same side of AB we construct two squares $ACDE$ and $BCGF$. Let O_1 and O_2 be the centers of the squares $ACDE$ and $BCGF$, then M is the midpoint of O_1O_2.

Proof. It is not difficult to see that M is the midpoint of IC. O_1, O_2 are on AI, BI, respectively, thus IO_1CO_2 is a rectangle, thus M must also be the midpoint of O_1O_2. ■

□

Problem 11.3.7 (23-24-T2-A-P7)**.** (*10 points*) How many positive integers not exceeding 2023 contain both even and odd digits?

Solution. We do casework.

Case 1: We find the number of two-digit numbers that contain both even and odd digits. There are in total $99 - 9 = 90$ two-digit numbers. There are 5 odd digits, so $5 \cdot 5 = 25$ two-digit numbers with both digits odd. Since the first digit cannot be 0, so there are $4 \cdot 5 = 20$ two-digit numbers with both digits even. Thus, there are $90 - 25 - 20 = 45$ two-digit numbers that contain both even and odd digits.

Case 2: We find the number of three-digit numbers that contain both even and odd digits. There are in total $999 - 99 = 900$ two-digit numbers. There are 5 odd digits, so $5 \cdot 5 \cdot 5 = 125$ three-digit numbers with all digits odd. Since the first digit cannot be 0, so there are $4 \cdot 5 \cdot 5 = 100$ three-digit numbers with all digits even. Thus, there are $900 - 125 - 100 = 675$ three-digit numbers that contain both even and odd digits.

Case 3a: If the first digit of a four-digit number is 1, then we need to count (i) three-digit numbers not containing any odd digit, which is $900 - 125 = 775$; (ii) and zero-preceding two- and one-digit numbers (to make three digits), which is 100. So the total number in this case is 885.

Case 3b: If the first digit of a four-digit number is 2, there are 2023 numbers from 2000 to 2023. Five of them between 2000 and 2009, ten of them between 2010 and 2019, two of them between 2020 and 2023. All together 17 such numbers.

Hence, the answer is $45 + 675 + 875 + 17 = \boxed{1612.}$ □

Problem 11.3.8 (23-24-T2-A-P8)**.** Let (a_n) be a sequence of real numbers such that

$$a_0 = 2, a_{n+1} \geq a_n^2 + \frac{1}{5}, \ \forall n \geq 0.$$

(*5 points*) Prove that for all $n \geq 0$,

$$a_{n+5} > (a_n)^2 + \left(a_{n+1} - \frac{1}{2}\right)^2 + \left(a_{n+2} - \frac{1}{2}\right)^2 + \left(a_{n+3} - \frac{1}{2}\right)^2 + \left(a_{n+4} - \frac{1}{2}\right)^2$$

(*5 points*) Prove that $a_{2025} > 2^{810}$.

Solution. For the first question

$$a_{n+5} \geq a_{n+4}^2 + \frac{1}{5} \Rightarrow \ a_{n+5} \geq a_{n+4} + \left(a_{n+4}^2 - a_{n+4} + \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) \geq a_{n+4} + \left(a_{n+4} - \frac{1}{2}\right)^2 + \frac{1}{20}$$

So

$$a_{n+5} > a_{n+4} + \left(a_{n+4} - \frac{1}{2}\right)^2 > a_{n+3} + \left(a_{n+3} - \frac{1}{2}\right)^2 + \left(a_{n+4} - \frac{1}{2}\right)^2 > \cdots$$
$$a_{n+1} + \left(a_{n+1} - \frac{1}{2}\right)^2 + + \left(a_{n+2} - \frac{1}{2}\right)^2 + \left(a_{n+3} - \frac{1}{2}\right)^2 + \left(a_{n+4} - \frac{1}{2}\right)^2$$

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Since $a_{n+1} \geq a_n^2 + \frac{1}{5}$, thus

$$a_{n+5} > (a_n)^2 + \left(a_{n+1} - \frac{1}{2}\right)^2 + \left(a_{n+2} - \frac{1}{2}\right)^2 + \left(a_{n+3} - \frac{1}{2}\right)^2 + \left(a_{n+4} - \frac{1}{2}\right)^2$$

For the second question, for all $n \geq 0$,

$$a_{n+5} > (a_n)^2 \Rightarrow a_{2025} > (a_{2020})^{2\cdot 1} > (a_{2015})^{2\cdot 2} > \cdots > (a_0)^{2\cdot 405} = 2^{810}.$$

$\square$

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11.4 Problems for Olympiad Level

Read the rules before you start!

Problem 11.4.1 (23-24-T2-O-P1)**.** (*10 points*) Prove that for n positive integer

$$2!4!\cdots(2n)! \geq ((n+1)!)^n$$

Solution. For $n = 1$, we have $2! = (2!)^1$, thus the base case stands.

Assume that the hypothesis is true for n, then

$$2!4!\cdots(2n)!((2n+2)!) \geq ((n+1)!)^n ((2n+2)!) \quad (*)$$

Note that

$$((n+2)!)^{n+1} = ((n+1)!)^n (n+1)!(n+2)^{n+1} \quad (**)$$

So, from (*) and (**), what we need to prove is

$$(2n+2)! \geq (n+1)!(n+2)^{n+1} \Leftrightarrow (n+2)(n+3)\cdots(2n+2) \geq (n+2)^{n+1}, \text{ which is obvious.}$$

□

Problem 11.4.2 (23-24-T2-O-P2)**.** (*10 points*) Show that in a round-robin tennis tournament with an odd number of players, it is possible for every player to win exactly half the matches.

Solution. For a tournament of 3 teams, it is easy to see that if P_1 wins against P_2, P_2 wins against P_3, and P_3 wins against P_1, then every player to win exactly half the matches. Thus the base case for the induction proof stands. *On a side note that if you create a directed graph with the teams as vertices and the match result as a directed edge from the winner to the loser then you have cycle with an odd length.*

Let assume we have a round-robin tournament with $2n-1$ players $P_1, P_2, \ldots, P_{2n-1}$ where every player to win exactly half the matches. Let A and B are two new players. A wins against with players $P_1, P_2, \ldots, P_n$ and loses against $P_{n+1}, P_{n+2}, \ldots, P_{2n-1}$, and B. B loses against with players $P_1, P_2, \ldots, P_n$ and wins against $P_{n+1}, P_{n+2}, \ldots, P_{2n-1}$, and A. This ways every (old and new) player to win exactly half the matches. □

Problem 11.4.3 (23-24-T2-O-P3)**.** (*10 points*) Find a recursive formula (recurrent relation) to count the number of ways to put n non-attacking rooks on a $n \times n$ chessboard such that the arrangement is symmetric about the diagonal from the lower left corner to the upper right corner.

Hint: take a look at the rook on the first column, it is at a corner or somewhere in the column.

Solution. Let R_n be the number of ways to put n non-attacking rooks on a $n \times n$ chessboard such that the arrangement is symmetric about the diagonal from the lower left corner to the upper right corner. It is easy to see that $R_1 = 1, R_2 = 2$.

Note that a rook in the first column may or many not occupy the square in the lower left corner of the board.

Case 1: If it does, then there are R_{n-1} ways to put the other $n-1$ rooks.

Case 2: If it doesn't, then it can occupy one of the $n-1$ squares in the first column. Once it is placed, it uniquely determines the location of a symmetrically placed rook (with respect to the given diagonal) in the first row. Removing the first column and the first row, the remaining $n-2$ rooks can be placed in R_{n-2} ways.

Thus

$$R_n = R_{n-1} + (n-1)R_{n-2}.$$

□

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Problem 11.4.4 (23-24-T2-O-P4)**.** (*10 points*) Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Show that

$$\frac{a_1}{(1+a_1)^2} + \frac{a_2}{(1+a_1+a_2)^2} + \cdots + \frac{a_n}{(1+a_1+a_2+\cdots+a_n)^2} < \frac{a_1+a_2+\cdots+a_n}{1+a_1+a_2+\cdots+a_n}.$$

Proof. We proceed with induction based on n. For $n = 1$

$$\frac{a_1}{(1+a_1)^2} < \frac{a_1}{1+a_1} \Leftrightarrow 1 < 1 + a_1, \text{ which is obvious.}$$

Let's assume that the inequality stands for $n-1$. Let $s_n = a_1 + a_2 + \ldots + a_n$, it is sufficient to show that

$$\frac{s_n - a_n}{1+s_n-a_n} + \frac{a_n}{(1+s_n)^2} < \frac{s_n}{1+s_n} \Leftrightarrow (s_n - a_n)(1+s_n)^2 \leq (s_n + s_n^2 - a_n)(1+s_n-a_n) \Leftrightarrow -a_n^2 \leq 0.$$

□

Problem 11.4.5 (23-24-T2-O-P5)**.** (*10 points*) $a > 0, b > 0$ are real numbers. n is a positive integer. Prove that

$$(n-1)a^n + b^n \geq na^{n-1}b.$$

When does the equality stand?

Proof. We prove it by induction on n.

For the base case $n = 1$, the inequality is a trivial equality.

For the inductive step,

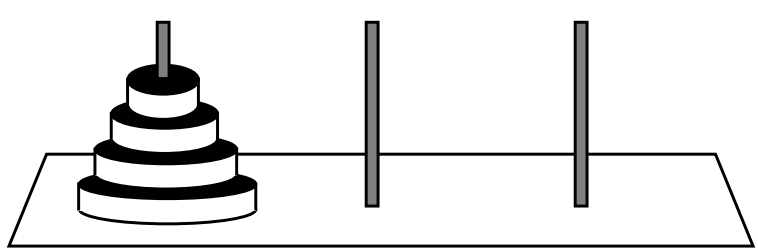
$$(n-1)a^n + b^n \geq na^{n-1}b \Rightarrow (n-1)a^{n+1} + ab^n \geq na^n b \Rightarrow na^{n+1} + ab^n \geq na^n b + a^{n+1}$$
$$\Rightarrow na^{n+1} + \geq na^n b + a^{n+1} - ab^n \Rightarrow na^{n+1} + b^{n+1} \geq na^n b + a^{n+1} - ab^n + b^{n+1}$$

Now, we need

$$na^n b + a^{n+1} - ab^n + b^{n+1} \geq (n+1)a^n b \Leftrightarrow a^{n+1} - ab^n + b^{n+1} - a^n b \geq 0 \Leftrightarrow (a^n - b^n)(a-b) \geq 0$$

The last inequality is obvious. Thus, the hypothesis is true for $n+1$. Hence it is true for all n. □

Problem 11.4.6 (23-24-T2-O-P6)**.** (*10 points*) Suppose n rings, with different outside diameters, are slipped onto an upright peg, the largest on the bottom, to form a pyramid. Two other upright pegs are placed sufficiently far apart. We wish to transfer all the rings, one at a time, to the second peg to form an identical pyramid. During the transfer, we are not permitted to place a larger ring on a smaller one (this necessitates using the third peg). Prove that the smallest number of moves necessary to complete the transfer is $2^{n+1} - 1$.



Solution. Now, let $f(n)$ be the smallest number of moves necessary to complete the transfer for n rings. In order to get the top $n-1$ rings to the third peg, we need $f(n-1)$ moves. Then we move the largest ring from the first peg to the second peg, which costs us a move. Finally we move all the $n-1$ rings from the third peg to the second peg, which against cost us $f(n-1)$ moves. Therefore

$$f(n) = 2f(n-1) + 1,\ f(1) = 1.$$

The induction to prove $f(n) = 2^{n+1} - 1$ is simple. □

Remark. *Note that in this proof, we make an assumption that the way we move the peg is optimal in respect to the number of moves. Is that true? Note that in order to get a two rings $k > l$ from peg 1 to peg 2, you shall need to reverse their order and then again reverse that, which basically cost us two moves. To move the largest one, of course, you need one move. That's roughly - but not precisely - how you can reason that this process is the optimal way.*

Problem 11.4.7 (23-24-T2-O-P7)**.** (*10 points*) $n \geq 1$ is a positive integer. Let $0 < a_1 < a_2 < \cdots < a_n$. Let $e_i = \pm 1, i = 1, 2, \ldots, n$. Prove that the sums $\sum_{i=1}^{n} e_i a_i$ have at least $\binom{n+1}{2}$ distinct values as $\{e_i\}$ range over the 2^n possible combination of signs.

Note that: for $n = 1$, there are exactly $\binom{1+1}{2} = 2$ distinct values for two sums $-a_1$ and a_1.

Hint: by subtracting a_{n+1} from a sum $\sum_{i=1}^{n} e_i a_i$, we obtain $\binom{n+1}{2}$ distinct sums.

Solution. When $n = 1$, there are exactly $\binom{1+1}{2} = 2$ distinct values for two sums $-a_1$ and a_1.

Let's assume that the hypothesis is true for n. Then for $0 < a_1 < a_2 < \cdots < a_n$ we have at least $\binom{n+1}{2}$ distinct values for the sums $\sum_{i=1}^{n} e_i a_i$.

First, by subtracting a_{n+1} from these sums, we obtain $\binom{n+1}{2}$ distinct values (*)

Second, consider the sum $S = \sum_{i=1}^{n} a_i$, which is one of the sums (*), then let

$$s_0 = S + a_{n+1},\ s_1 = -2a_1 + S + a_{n+1},\ s_2 = -2a_2 + S + a_{n+1}, \ldots, s_n = -2a_n + S + a_{n+1} \quad (**)$$

It is easy to see that all the sums in (**) are distinct, furthermore for $1 \leq i \leq n$,

$$a_i < a_{n+1} \Rightarrow S - a_{n+1} < -2a_i + S + a_{n+1}.$$

Thus none of the sums in (**) can have the same value as the one in (*).

Therefore we obtain

$$\binom{n+1}{2} + (n+1) = \binom{n+2}{2} \text{ sums of distinct values.}$$

□

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Problem 11.4.8 (23-24-T2-O-P8)**.** Arrange the number $1, 2, \ldots, n$ consecutively clockwise about the circumference of a circle. Now, remove the number 2 and proceed clockwise by removing every other number, among those that remain, until only one number is left. *For example for $n = 5$, numbers are removed in the order 2, 4, 1, 5, and 3 remains alone.*

Let $f(n)$ denote the final number which remain.

(*5 points*) Prove that

$$\begin{cases} f(2n) & = 2f(n) - 1 \\ f(2n+1) & = 2f(n) + 1 \end{cases}$$

(*5 points*) Prove that if the binary representation of $n = \overline{a_k a_{k-1} \ldots a_1 a_0}_2$, then

$$f(n) = b_k 2^k + b_{k-1} 2^{k-1} + \ldots + 2b_1 + b_0, \quad \text{where } b_i = \begin{cases} +1, & a_i = 1 \\ -1, & a_i = 0 \end{cases} \forall i = 0, 1, \ldots, k$$

Proof. For the first question, note that for $2n$ numbers, then first we remove the number $2, 4, \ldots, 2n$, thus the remaining sequence contains n numbers:

$$1, 3, 5, \ldots, 2n - 1 \quad (*)$$

If the number $f(n) = k$ remains alone from the sequence $1, 2, \ldots, n$, then the number at the position k^{th} in the sequence (*) shall remain alone. It is obvious that that number is $2k - 1$. Thus $f(2n) = 2f(n) - 1$.

Now for $2n + 1$ numbers, after we remove the number $2, 4, \ldots, 2n$, and continue to remove the number 1, the remaining sequence contains n numbers:

$$3, 5, \ldots, 2n + 1 \quad (*)$$

Reasoning similar to the above, we shall receive $f(2n + 1) = 2f(n) + 1$.

Now, we prove the second statement by induction base on k, the number of digits of n. It is easy to see for $n = a_0 = 1$. Now consider $n = \overline{a_k a_{k-1} \ldots a_1 a_0}_2$ with $k + 1$ digits.

Case 1: If $a_0 = 0$, then $n = 2\overline{a_k a_{k-1} \ldots a_1}_2$,

$$f(n) = 2f(a_k a_{k-1} \ldots a_1) - 1 = 2\left(b_k 2^{k-1} + b_{k-1} 2^{k-2} + \ldots + 2b_2 + b_1\right) - 1 = b_k 2^k + b_{k-1} 2^{k-1} + \ldots + 2b_1 + b_0$$

Case 2: If $a_0 = 1$, then $n = 2\overline{a_k a_{k-1} \ldots a_1}_2 + 1$,

$$f(n) = 2f(a_k a_{k-1} \ldots a_1) + 1 = 2\left(b_k 2^{k-1} + b_{k-1} 2^{k-2} + \ldots + 2b_2 + b_1\right) + 1 = b_k 2^k + b_{k-1} 2^{k-1} + \ldots + 2b_1 + b_0$$

□

Remark. This is a great result and simplify computation. For example

$$f(2023) = f(11111100111_2) = 11111111111_2 - 2 * 11000_2 = (2^1 1 - 1) - 2(2^4 + 2^3) = 1999.$$

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Chapter 12

Test 3

12.1 Rules

- Each level test consisting of 8 problems will be delivered by automatic scheduled email at the registered time. No test paper will be sent twice.
- Students have **270 minutes to solve the problems.** For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET.
- After the allowed time, the student has 15 minutes to scan the solutions. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET, and has to submit the scanner solutions before 11:45 AM. Any solution submitted later than the allowed time will not be accepted.
- **All submissions must be sent by email to mcc.youth.global@gmail.com and dinhxuanquyen@gmail.com.**
- Solutions must be written in English. Any solution looks unreadable, or is presented badly will not be considered for grading. **Answer without proper solution will not be considered as solution and will not be accepted.** Answer consists of self-explanatory and easy-to-understand diagrams are acceptable but might not be considered as full solution.
- A problem can contain more than one questions, each with an indication of the maximum number of points can be awarded if a correct solution is provided. Partially correct solution is awarded with a some points depending on its quality. Multiple solutions can be submitted for a single problem. Each different solution, if correct, will be awarded the same number of points. Partially distinct solution also counts.
- **For geometry problems, proofs must be accompanied with clean, clear, and precisely drawn diagrams** - by hand with compass and straightedge or by computer software such as GeoGebra. Submitted **solutions without any diagram will not be graded at all.**
- A proof for the hypothesis about **the existence of a locus must contains two parts.** Each part is worth 50% of the awardable points.

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12.2 Problems for Intermediate Level

Read the rules before you start!

Problem 12.2.1 (23-24-T3-I-P1)**.** (*10 points*) Mr. Fox, the shopkeeper, phoned Inspector Julie that his shop had been robbed. Three suspects Alligator, Bear, and Cheetah were rounded up for questioning. The following facts were established:

1. Each of Alligator, Bear, and Cheetah had been in the shop on the day of the robbery, and no one else had been in the shop that day.
2. If Alligator is guilty, then he had exactly one accomplice.
3. If Bear is innocent, so is Cheetah
4. If exactly two are guilty, then Alligator is one of them.
5. If Cheetah is innocent, so is Bear.

Whom did Inspector Julie indict?

Solution. We use casework.

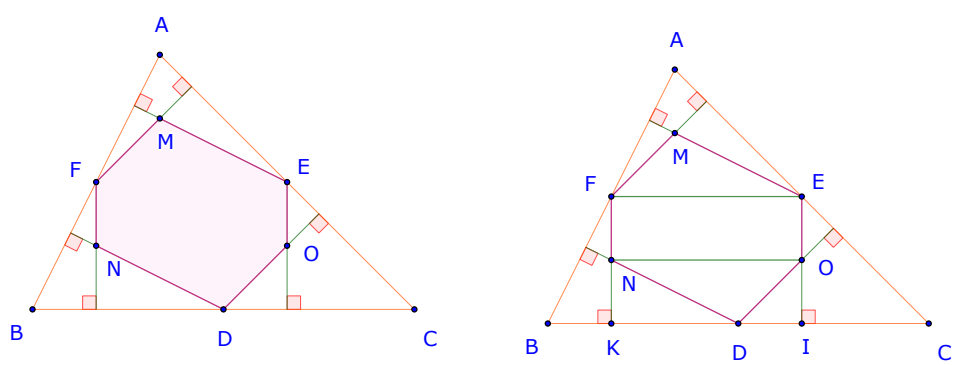
Case 1: If Alligator is guilty, then by (2) he had exactly one accomplice, who would be Bear or Cheetah. Since Alligator has exactly one accomplice, so one of Bear or Cheetah is innocent. By (3) and (5), both Bear and Cheetah are innocent. This is a contradiction. So Alligator is innocent.

Case 2: Now, Alligator is innocent. If one of Bear and Cheetah guilty then the other must also be guilty, other wise by (3) and (5), both Bear and Cheetah are innocent. Thus both of them are guilty, or exactly two are guilty. By (4), one of them must be Alligator. This again is a contradiction. Thus Bear and Cheetah are innocent.

What does this mean? None of Alligator, Bear, and Cheetah robbed the shop. But the shop was robbed. By (1), nobody else were in the shop that day. Hence, Mr. Fox was lying and he framed Alligator, Bear, and Cheetah for the robbery. That's why Inspector Julie indicted $\boxed{\text{Mr. Fox.}}$ □

Problem 12.2.2 (23-24-T3-I-P2)**.** D, E, and F are midpoints of BC, CA, and AB in $\triangle ABC$. Perpendicular lines from F to AC and E to AB intersect at M. Perpendicular lines from D to AB and F to BC intersect at N. Perpendicular lines from E to BC and D to AC intersect at O.

Find the ratio of the area of the hexagon $DNFMEO$ to the area of the triangle ABC.



Solution. Note that $\triangle AFE \cong \triangle EDC$, so $FM \parallel DO, FM = DO$, similarly $ME \parallel ND$, $ME = ND$. Then $\triangle FME \cong \triangle ODN$, furthermore $ON \parallel EF$, so $ONKI$ is a rectangle. Thus $[MFE] = [ODN] = [NKD] + [DOI]$, or $DNFMEO = [FEIK]$. It is easy to see that $FE = \frac{1}{2}BC$, and FK is half of the altitude from A in $\triangle ABC$. So $[FEIK] = \frac{1}{2}[ABC]$, hence $\frac{[DNFMEO]}{[ABC]} = \boxed{\frac{1}{2}.}$ □

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Problem 12.2.3 (23-24-T3-I-P3)**.** Mai has a round necklace containing 15 beads of two different colours: red and blue. (The necklance has no lock.)

(*5 points*) Prove that it is possible to find two beads of the same colour next to each other.

(*5 points*) Prove that it is possible to find two beads of the same colour separated by exactly two beads.

Solution. For the first question, let's assume the contrary that there are no two beads of the same colour next to each other, then the consecutive of beads can be grouped into pairs of different colours. That would mean the total number of beads is even. This contradicts that the total number of beads is 15.

For the second question, lets assume the contrary that there are no two beads of the same colour separated by exactly two other beads. WLOG, let the bead 1 be R (red), then bead 4 should be B (blue), bead 7 be R, bead 10 be B, bead 13 be R, bead 1 (again) be B! This is a contradiction. □

> **Remark.** Note that all the beads $3k+1$ should be in alternate colour going from R to B back to R and so on. However, we have only 5 such beads, so so there are two consecutive ones that should be in same colour. See the second solution.

Solution. [Solution 2 for the second question] From the bead 1, we keep the first bead, then remove the next two beads, keep the next one, remove the next two beads, and so on. We stop after bead 15. Easy to see that 5 beads remain: 1, 4, 7, 10, and 13 (the 16 is same as 1.) These 5 beads are in two colours, by the result of the first question, two of them must be in the same colour. □

Problem 12.2.4 (23-24-T3-I-P4)**.** One evening a theater sold 300 tickets for a concert. Each ticket sold for \$40, and all tickets were purchased using \$5, \$10, and \$20 bills. At the end of the evening the theater had received twice as many \$10 bills as \$20 bills, and 20 more \$5 bills than \$10 bills. How many bills did the theater receive altogether?

Solution. Let x, y, z be the numbers of \$5, \$10, and \$20 bills, then

$$\begin{cases} y = 2z \\ x = y + 20 \\ 5x + 10y + 20z = 300(40) \end{cases} \Rightarrow 5(2z+20) + 10(2z) + 20z = 12000 \Rightarrow z = 238$$

Thus $x + y + z = 2z + 20 + 2z + z = 5z + 20 = \boxed{1200.}$ □

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Problem 12.2.5 (23-24-T3-I-P5)**.** A 5×9 rectangle is cut into 10 smaller rectangles. All cuts follow the grid lines.

(*5 points*) Prove that there are at least two rectangles that have the same area.

(*5 points*) Prove that there are at least two rectangles that are congruent (same width and same length).

Hint: let's assume the contrary, what are the 10 rectangles in different shapes with smallest area?

Solution. Consider the area of the rectangles, they are positive integers. If they are all different, then their total area is at least $1 + 2 + \cdots + 10 = \frac{10 \cdot 11}{2} = 55 > 45$. Therefore there are at least two rectangles with the same area.

Now, let's assume the contrary. The 10 rectangles with smallest area and in different shapes are:

$$1 \times 1,\ 1 \times 2,\ 1 \times 3,\ 1 \times 4,\ 2 \times 2,\ 1 \times 5,\ 1 \times 6,\ 2 \times 3,\ 1 \times 7, 1 \times 8$$

Their total sum of area is $1 + 2 + 3 + 4 + 4 + 5 + 6 + 6 + 7 + 8 = 46 > 45$, thus there are at least two rectangles that are congruent. □

Problem 12.2.6 (23-24-T3-I-P6)**.** *The set $S = \{2, 3, 5\}$ is called* ***a good set*** *if for each of its subsets:*

$$\{2\},\ \{3\},\ \{5\},\ \{2,3\},\ \{3,5\},\ \{5,2\},\ \{2,3,5\},\ \textit{none of the sum of its numbers}$$
$$2 = 2,\ 3 = 3,\ 5 = 2 + 3 = 5,\ 2 + 5 = 7,\ 3 + 5 = 8,\ 2 + 3 + 5 = 10\ \textit{is a perfect square.}$$

(*5 points*) At most how many numbers from 3, 6, 10, 12, 13 can make a good set?

(*5 points*) At most how many numbers from 2, 3, 5, 6, 7, 8, 10, 11, 12, 13 can make a good set?

Solution. For the first question, for each of the pairs $(6, 10)$ and $(12, 13)$ only one number can be chosen. Since the sum of the numbers in each the groups $(3, 6)$, $(3, 13)$, and $(3, 10, 12)$ is a perfect square, so at most we can chose only $\boxed{2}$ numbers to form a good set. Here is an example $\boxed{3, 12}$.

For the second question, the numbers can be grouped into four groups:

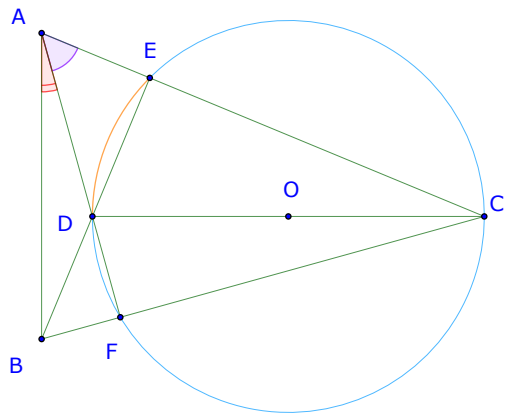
$$(8),\ (2, 7),\ (5, 11),\ (3, 6, 10, 12, 13).$$

Now we can pick one number from each of the first three groups, and two from the last group to form a good set of $\boxed{5}$ numbers. Here is one of the examples $\boxed{3, 7, 8, 11, 12.}$ □

Problem 12.2.7 (23-24-T3-I-P7)**.** (*10 points*) How many 7-digit numbers have no two adjacent digits equal?

Solution. We construct these numbers from left-to-right. We have 9 choices for the first digit (since it cannot be 0). Then, no matter what we choose for the first digit, we have 9 choices for the second digit - any digit except the one that we chose for the first digit. Similarly, for each subsequent digit, we have 9 choices, since each digit can be any of 0 through 9 as long as it does not match the previous digit. Therefore, there are 9 choices for every digit, and hence there are $9^7 = \boxed{4\ 782\ 969}$ such numbers. □

Problem 12.2.8 (23-24-T3-I-P8)**.** (*10 points*) In the figure, O is the center of the circle, $\angle EAD = 40°$ and $\widehat{ED} = 40°$. Find $\angle DAB$.



Solution. Since angles DEC and DFC are inscribed in semicircles, they are both right angles. Since two of the three lines from the vertices of $\triangle ABC$ through D are altitudes ($BE \perp AC$ and $AF \perp BC$), the third, CG, must also be an altitude. Thus, $\angle DGA = 90°$. Since $\angle EAD = 40°$ and is half the difference between arcs $\widehat{ED}$ and $\widehat{FC}$, we have $\widehat{FC} - \widehat{ED} = 2\angle EAD$, so $\widehat{FC} = 120°$. As an inscribed angle, we know $\angle CDF = \widehat{FC}/2 = 60°$, so $\angle ADG = 60°$ and $\angle DAB = 90 - \angle ADG = \boxed{30°.}$ □

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12.3 Problems for Advanced Level

Read the rules before you start!

Problem 12.3.1 (23-24-T3-A-P1)**.** (*10 points*) Nine cards are numbered $1, 2, 3, \ldots, 9$. Each of three players randomly selects and keeps three of the cards, and sums those three values. *For example if a player receives the cards* $2, 3, 5$, *then the sum she obtains is* $2 + 3 + 5 = 10$.

Find the probability that all three players obtain an odd sum.

Solution. [Solution 1] There are three ways to chose a player receives three odd cards, who can chose these three odd cards from five odd cards in $\binom{5}{3}$ ways, and the remaining two odd cards can be give to the two other players, one for each, in 2! ways. Then the four remaining even cards can be distributed to these two players, two for each, in $\binom{4}{2}$ ways. Altogether $3 \cdot \binom{5}{3} \cdot 2! \cdot \binom{4}{2} = 360$.

The total number of ways is $\binom{9}{3}\binom{6}{3} = 1680$, thus the probability is $\frac{360}{1680} = \boxed{\frac{3}{14}.}$ □

Solution. [Solution 2] The probability that the first player receives 3 odd cards is $\frac{\binom{5}{3}}{\binom{9}{3}} = \frac{5}{42}$. For the second player, two ways to chose an odd card, and $\binom{4}{2}$ ways for a pair of even cards. thus $\frac{2 \cdot \binom{4}{2}}{\binom{6}{3}} = \frac{3}{5}$.

Finally, by symmetry, there are three ways to arrange the players, so the probability is $3 \cdot \frac{5}{42} \cdot \frac{3}{5} = \boxed{\frac{3}{14}.}$ □

Problem 12.3.2 (23-24-T3-A-P2)**.** (*10 points*) Let a, b, c be positive real numbers such that

$$\frac{a(b-c)}{b+c} + \frac{b(c-a)}{c+a} + \frac{c(a-b)}{a+b} = 0.$$

Prove that $(a-b)(b-c)(c-a) = 0$.

Solution. First,

$$\begin{aligned}
\frac{b(c-a)}{c+a} &= \frac{b((c-b)+(b-a))}{c+a} = -\frac{b}{c+a} \cdot (b-c) - \frac{b}{c+a} \cdot (a-b) \\
\frac{a}{b+c} - \frac{b}{c+a} &= \frac{a^2 + ac - b^2 - bc}{(c+a)(b+c)} = \frac{(a-b)(a+b+c)}{(b+c)(c+a)} \\
\frac{c}{a+b} - \frac{b}{c+a} &= \frac{c^2 + ca - b^2 - ba}{(c+a)(b+c)} = \frac{(c-b)(a+b+c)}{(a+b)(c+a)}
\end{aligned}$$

Thus,

$$0 = \frac{a(b-c)}{b+c} + \frac{b(c-a)}{c+a} + \frac{c(a-b)}{a+b} = \frac{(a-b)(a+b+c)(b-c)}{(b+c)(c+a)} + \frac{(c-b)(a+b+c)(a-b)}{(a+b)(c+a)} \quad (*)$$

Now, if $(a-b)(b-c)(c-a) \neq 0$, then by dividing both sides of (*) by $(a-b)(b-c)(c-a)(a+b+c) \neq 0$,

$$0 = \frac{1}{(b+c)(c+a)(c-a)} - \frac{1}{(a+b)(c+a)(c-a)} \Rightarrow a+b = b+c \Rightarrow a = c, \text{ a contradiction.}$$

Thus $(a-b)(b-c)(c-a) = 0$. □

Problem 12.3.3 (23-24-T3-A-P3)**.** (*10 points*) Triangle ABC is such that $AB = 4$ and $AC = 8$. If M is the midpoint of BC and $AM = 3$, what is the length of BC?

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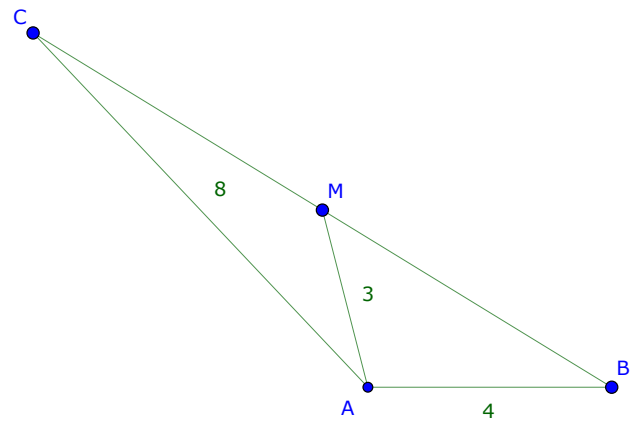

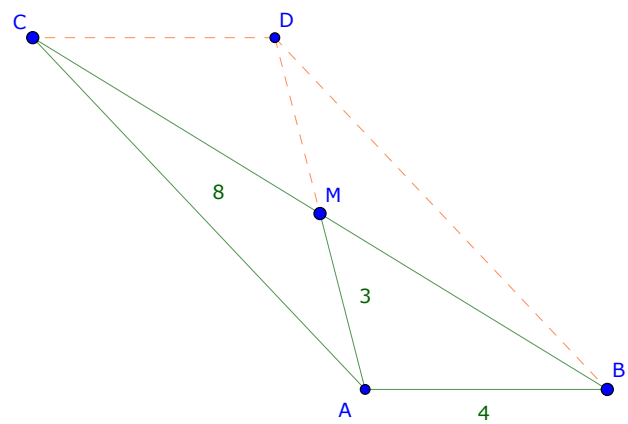

Solution. Extend median AM past M to D as shown so that $ABDC$ is a parallelogram. Note that in a parallelogram, the sum of the squares of the diagonals of a parallelogram is equal to the sum of squares of its sides. Since $AB = CD$, $AC = BD$, and $AD = 2AM$, we have

$$AD^2 + BC^2 = AC^2 + CD^2 + DB^2 + AB^2 \Rightarrow 36 + BC^2 = 2(64 + 16) \Rightarrow BC = \boxed{2\sqrt{31}}$$

□

Problem 12.3.4 (23-24-T3-A-P4)**.** (*10 points*) Let a, b, c be real number such that $a \geq b \geq c$, Prove that

$$a^2 + ac + c^2 \geq 3b(a - b + c)$$

Solution. Let

$$\begin{aligned} &f(b) = a^2 + ac + c^2 - 3b(a - b + c) = 3b^2 - 3b(a + c) + (a^2 + ac + c^2) \\ &\Rightarrow \Delta_f = 9(a + c)^2 - 12(a^2 + ac + c^2) = -3(c - a)^2 \leq 0 \end{aligned}$$

Thus the quadratic $f(b)$ is always non-negative because its leading coefficient (of b^2) is 3 ¿ 0. □

Problem 12.3.5 (23-24-T3-A-P5)**.** (*10 points*) Solve in positive real numbers the system of equations

$$\begin{cases} x^2y^2 + 1 = x^2 + xy \\ y^2z^2 + 1 = y^2 + yz \\ z^2x^2 + 1 = z^2 + xy \end{cases}$$

Solution. Note that $x^2y^2 + 1 \geq 2xy$, so $x^2 + xy \geq 2xy$, so $x \geq y$, similarly $y \geq z$, and $z \geq x$. Therefore $x = y = z$. Since the equality $x^2y^2 + 1 = 2xy$ stands so $xy = 1$, or $x = y = 1$.

Hence $\boxed{x = y = z = 1}$ is the only solution. $\square$

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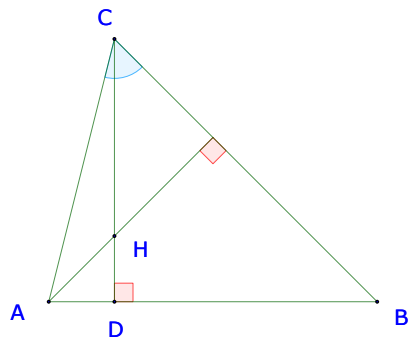
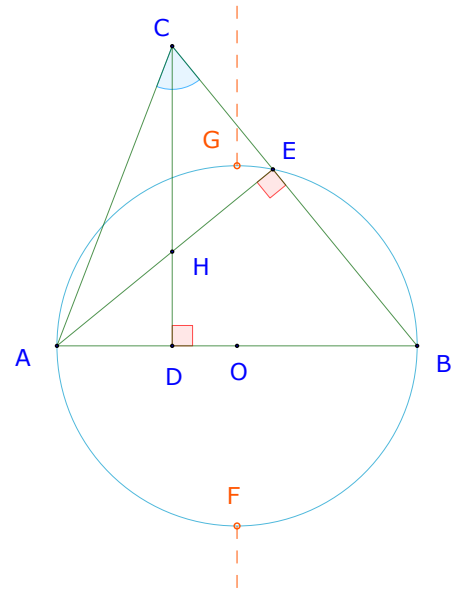
Problem 12.3.6 (23-24-T3-A-P6)**.** (*10 points*) A and B are fix points. C an arbitrary point such that $\angle ACB > 90°$. H is the orthocenter of $\triangle ABC$, D is the foot of the perpendicular from C to AB. Find the locus of points C such that $DH \cdot DC$ reaches it possible maximal value.

Hint: Consider similar triangles.





Solution. Below we show how to establish the hypothesis. Students must prove both necessary and sufficiently condition of the locus.

Let AE be the other altitude. Then $\triangle HDA \sim \triangle BDC$, so

$$\frac{DA}{DH} = \frac{DC}{DB} \Rightarrow DC \cdot DH = DA \cdot DB \leq \frac{1}{4}(DA + DB)^2 = \frac{AB^2}{4}.$$

$DC \cdot DH$ reaches the maximal if $DA = DB$, so D is on the perpendicular bisector of AB. Since $\angle ACB > 90°$, so C is on the two ray from F and G going out of the circle. □

Problem 12.3.7 (23-24-T3-A-P7)**.** (*10 points*) Prove by induction that for a, b non-negative integers, $n \geq 1$,

$$(a^2 + b^2)^n \text{ can be expressed as sum of two perfect squares}$$

Solution. For $n = 1$, the base case is obvious.

Let assume that it is true for n, or there exists u, v non-negative integers such that

$$(a^2 + b^2)^n = u^2 + v^2.$$

By the identity $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ac - bd)^2$ for all a, b, c, d,

$$(a^2 + b^2)^{n+1} = (a^2 + b^2)(a^2 + b^2)^n = (a^2 + b^2)(u^2 + v^2) = (au + bv)^2 + (au - bv)^2$$

where $|au + bv|$, and $|au - bv|$ are non-negative integers. □

Problem 12.3.8 (23-24-T3-A-P8)**.** (*10 points*) Find the greatest n positive integer such that

$$5^n \mid 2019! - 2018! + 2017!$$

Solution. Note that $2019! - 2018! + 2017! = 2017!(2019 \cdot 2018 - 2018 + 1) = 2017!(2018^2 + 1)$. Now the number of factors of 5 in 2017! is

$$\left\lfloor \frac{2017}{5} \right\rfloor + \left\lfloor \frac{2017}{25} \right\rfloor + \left\lfloor \frac{2017}{125} \right\rfloor + \left\lfloor \frac{2017}{625} \right\rfloor = 403 + 80 + 16 + 3 = 502.$$

Furthermore $2018^2 + 1 \equiv 18^2 + 1 = 325 \pmod{1000}$, 325 is divisible by 25 but not 125. Thus $2018^2 + 1$ is divisible by 25 and not 125.

Therefore the greatest n positive integer such that $5^n \mid 2019! - 2018! + 2017!$ is $502 + 2 = \boxed{504.}$ □

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12.4 Problems for Olympiad Level

Read the rules before you start!

Problem 12.4.1 (23-24-T3-O-P1)**.** (*10 points*) n is a positive integer. $a_k = 2^{2^{k-n}} + k$, $k = 0, 1, \ldots, n$. Prove that

$$(a_1 - a_0) \cdots (a_n - a_{n-1}) = \frac{7}{a_1 + a_0}.$$

Hint: let $a = 2^{2^{-n}}$, express $a_k - a_{k-1}$ base on a in order to turn the left hand side into a telescopic product.

Solution. First, note that $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$.

Let $a = 2^{2^{-n}}$, then $\forall k = 0, 1, \ldots, n$, $a^{2^k} = \left(2^{2^{-n}}\right)^{2^k} = 2^{2^{k-n}}$, and

$$a_k - a_{k-1} = 2^{2^{k-n}} + k - \left(2^{2^{k-1-n}} + (k-1)\right) = 2^{2^{k-n}} - 2^{2^{k-1-n}} + 1 = a^{2^k} - a^{2^{k-1}} + 1 = \frac{a^{2^{k+1}} - a^{2^k} + 1}{a^{2^k} + a^{2^{k-1}} + 1}$$

Thus

$$(a_1 - a_0) \cdots (a_n - a_{n-1}) = \prod_{k=1}^{n} \frac{a^{2^{k+1}} - a^{2^k} + 1}{a^{2^k} + a^{2^{k-1}} + 1} = \frac{a^{2^{n+1}} - a^{2^n} + 1}{a^{2^1} + a^{2^0+1}} = \frac{2^{2^{n+1-n}} - 2^{2^{n-n}} + 1}{2^{2^{1-n}} + 2^{2^{0-n}+1}} = \frac{7}{a_1 + a_0}$$

□

Problem 12.4.2 (23-24-T3-O-P2)**.** Let n be a positive integer. Prove that

(*5 points*) $\log_n(n+1) > \log_{n+1}(n+2)$.

(*5 points*) $\log_4(5) \cdot \log_6(7) \cdots \log_{80}(81) < 2$.

Solution. For the first question,

$$\frac{n+1}{n} > \frac{n+2}{n+1} \Rightarrow \log_n(\frac{n+1}{n}) > \log_n(\frac{n+2}{n+1}) > \log_{n+1}(\frac{n+2}{n+1})$$
$$\Rightarrow \log_n((n+1)) - \log_n(n) > \log_{n+1}((n+2)) - \log_{n+1}(n+1) \Rightarrow \log_n(n+1) > \log_{n+1}(n+2).$$

For the second question, bring the left hand side to the square, and using the result of the first question

$$\left(\log_4(5) \cdot \log_6(7) \cdots \log_{80}(81)\right)^2 = \log_4(5) \cdot \log_4(5) \cdot \log_6(7) \cdot \log_6(7) \cdots \log_{80}(81) \cdot \log_{80}(81)$$
$$< \log_3(4) \cdot \log_4(5) \cdot \log_5(6) \cdot \log_6(7) \cdots \log_{79}(80) \cdot \log_{80}(81) = \log 3(81) = 4.$$

Thus $\log_4(5) \cdot \log_6(7) \cdots \log_{80}(81) < \sqrt{4} = 2$. □

Problem 12.4.3 (23-24-T3-O-P3)**.** (*10 points*) Let $d \geq 1$ be integer, $a_1, a_2, \ldots, a_d$ be real numbers. Let $P(x) = x^d - a_1 x^{d-1} + \cdots + (-1)^d a_d$ be a polynomial with d real roots, all of which lie in the interval $[\alpha, \beta]$, where $-1 \leq \alpha \leq \beta$. Prove that

$$(1+\alpha)^d \leq (-1)^d P(-1) \leq (1+\beta)^d$$

Solution. Let $r_1, r_2, \cdots, r_d$ be the roots of $P(x)$. Then

$$P(x) = \prod_{i=1}^{d}(x - r_i) \Rightarrow (1+\alpha)^d \leq (-1)^d P(-1) = \prod_{i=1}^{d}(1 + r_i) \leq (1+\beta)^d$$

□

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Problem 12.4.4 (23-24-T3-O-P4)**.** (*10 points*) Let $a_1 = 1 + \sqrt{2}$ and for $n \geq 1$,

$$a_{n+1} = 2 - \frac{1}{a_n}.$$

Find the greatest integer less than or equal to $a_1 a_2 a_3 \cdots a_{2023}$.

Proof. Let $p_n = \prod_{k=1}^{n} a_k$. Then

$$b_1 = a_1,\ b_2 = a_1 a_2 = a_1 \left(2 - \frac{1}{a_n}\right) = 2a_1 - 1 = 1 + 2\sqrt{2}.$$

For $n \geq 2$,

$$b_{n+1} = b_n \cdot a_{n+1} = b_n \left(2 - \frac{1}{a_n}\right) = 2b_n - \frac{b_n}{a_n} = 2b_n - b_{n-1}.$$

Thus $b_3 = 1 + 3\sqrt{2}$. We prove now by induction, $b_n = 1 + n\sqrt{2} \quad (*)$

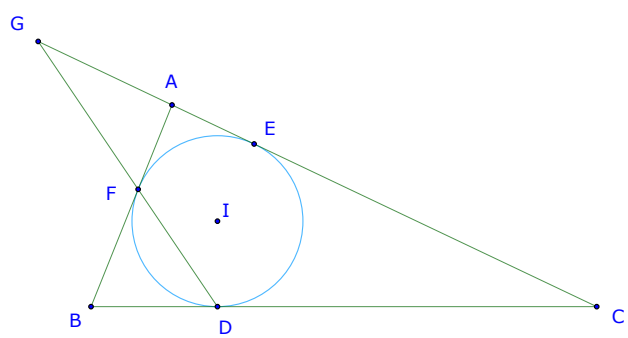
The base cases for $n = 1, 2, 3$ are clear.

$$b_{n+1} = 2b_n - b_{n-1} = 2(1 + n\sqrt{2}) - (1 + (n-1)\sqrt{2}) = 1 + (n+1)\sqrt{2}.$$

Thus (*) stands for $n + 1$, therefore for all n.

Now $b_{2023} = 1 + 2023\sqrt{2} \approx 2861.95$, thus $\lfloor b_{2023} \rfloor = \boxed{2861.}$ □

Problem 12.4.5 (23-24-T3-O-P5)**.** (*10 points*) The incircle of $\triangle ABC$ is tangent to sides BC, AC, and AB at D, E, and F, respectively. Point G is the intersection of lines AC and DF as shown. The sides of $\triangle ABC$ have lengths $AB = 73$, $BC = 123$, and $AC = 120$. Find the length EG.



Proof. Two tangents from a point to a circle are equal in length, so let $x = AE = AF, y = BF = BD, z = CD = CE$. Then

$$\begin{cases} x + y &= 73 \\ y + z &= 123 \\ z + x &= 120 \end{cases}$$

By solving the system of equations above, we obtain that $\Rightarrow x = 35, y = 38, z = 85$. By the Menelaus theorem

$$1 = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CG}{GA} = \frac{x}{y}\frac{y}{z}\frac{120 + GA}{GA} = \frac{35}{85}\frac{120 + GA}{GA}$$

Therefore $GA = 84 \Rightarrow EG = x + GA = \boxed{119}$. □

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Problem 12.4.6 (23-24-T3-O-P6)**.** (*10 points*) Find the number of three-element subsets of $\{1, 2, 3, \cdots, 13\}$ that contain at least one element that is a multiple of 2, at least one element that is a multiple of 3, and at least one element that is a multiple of 5 such as $\{2, 3, 5\}$ or $\{6, 10, 13\}$.

Hint: the set $\{1, 2, 3, \cdots, 13\}$ contains 7 elements that are not multiples of 2, thus the number of three-element subsets of S that are missing either a multiple of 2 is $\binom{7}{3}$.

Solution. Let $S = \{1, 2, 3, \cdots, 13\}$. The set S contains 7 elements that are not multiples of 2, 9 elements that are not multiples of 3, and 11 elements that are not multiples of 5.

The set S contains 5 elements that are neither multiples of 2 nor multiples of 3, 6 elements that are neither multiples of 2 nor multiples of 5, and 7 elements that are neither multiples of 3 nor multiples of 5.

The set S contains 4 elements that are neither multiples of 2, 3 or 5.

By the Inclusion-Exclusion Principle, the number of three-element subsets of S that are missing either a multiple of 2, a multiple of 3, or a multiple of 5 Is

$$\left[\binom{7}{3} + \binom{9}{3} + \binom{11}{3}\right] - \left[\binom{5}{3} + \binom{6}{3} + \binom{7}{3}\right] = 223.$$

The set S contains $\binom{13}{3} = 286$ subsets with 3 elements. Hence, the number of three-element subsets of $\{1, 2, 3, \cdots, 13\}$ that contain at least one element that is a multiple of 2, at least one element that is a multiple of 3, and at least one element that is a multiple of 5 is $286 - 223 = \boxed{63}$. □

Problem 12.4.7 (23-24-T3-O-P7)**.** (*10 points*) Find $p < q < r < s$ are prime numbers such that

$$pqrs + 1 = 4^{p+q}.$$

Solution. It is easy to see that all p, q, r, s are odd primes. thus, $p + q$ is even, so Therefore

$$15 = 4^2 - 1 \mid 4^{p+q} - 1 \Rightarrow p = 3, q = 5, rs = \frac{4^8 - 1}{15} = 4369 = 17 \cdot 257 \Rightarrow r = 17, s = 257.$$

□

Problem 12.4.8 (23-24-T3-O-P8)**.** (*10 points*) A convex polygon is given on the plane. Prove that there exists a triangle formed by three of its vertices such that the sum of squares of its side lengths is at least the sum of squares of the side lengths of the polygon.

Hint if the polygon $A_1A_2 \ldots A_n$ has a non-acute interior angle, then by removing its associated vertex, what can you say about the sum of squares of the side lengths of the remaining $(n-1)-$gon.

Proof. We prove by induction based on n the number of vertices.

The base case $n = 3$ is trivial.

Let's assume the hypothesis is true for any polygon with the number of vertices less than $n > 3$. The sum of interior angles in the polygon $A_1A_2 \ldots A_n$ is $(n-2) \cdot 180^\circ > n \cdot 90^\circ$, thus there is at least one non-acute triangle. Let that be $A_{n-1}A_nA_1$. Then

$$(A_{n-1}A_1)^2 \geq (A_{n-1}A_n)^2 + (A_nA_1)^2$$

Therefore the sum of the square of the side lengths of the polygon $A_1A_2 \ldots A_n$ does not exceed the sum of the square of the side lengths of the polygon $A_1A_2 \ldots A_{n-1}$. Using the induction hypothesis, we obtain the result. □

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Chapter 13

ICLT 2

13.1 IC Level Test Rules

- All test lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *at most one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

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13.2 IC Test for Level 1

Problem 13.2.1 (ICLT-2023-SM1-R2-L1-P1)**.** Simplify

$$\sqrt[3]{27p^4} - \frac{2\sqrt[3]{p^7}}{p}.$$

(A) $\sqrt[3]{p^5}$ (B) $25p\sqrt[3]{p}$ (C) $p\sqrt[3]{p}$ (D) $p^{\frac{5}{3}}$ (E) $2p^{\frac{5}{3}}$

Solution. Exercise 2.2.8c, Chapter 2, Introductory to Algebra.

$$\sqrt[3]{27p^4} = \sqrt[3]{3^3p^3p} = 3p\sqrt[3]{p},\ \frac{2\sqrt[3]{p^7}}{p} = \frac{2\sqrt[3]{p^6p}}{p} = \frac{2p^2\sqrt[3]{p}}{p} = 2p\sqrt[3]{p}$$

$$\Rightarrow\ \sqrt[3]{27p^4} - \frac{2\sqrt[3]{p^7}}{p} = 3p\sqrt[3]{p} - 2p\sqrt[3]{p} = p\sqrt[3]{p}$$

The number answer is $\boxed{C.}$ □

Problem 13.2.2 (ICLT-2023-SM1-R2-L1-P2)**.** Find the measure of an angle that is complementary the angle $101°$

(A) $79°$ (B) $11°$ (C) $31°$ (D) $259°$ (E) there is no such angle.

Solution. Problem 2.7, Chapter 2, Introductory to Geometry.

The angles are complementary if the sum of their measures is $90°$. Since the given angle is larger than $90°$ there is such complementary angle. The answer is $\boxed{E.}$ □

Problem 13.2.3 (ICLT-2023-SM1-R2-L1-P3)**.** How many three-digit numbers are not multiples of 5?

(A) 180 (B) 720 (C) 820 (D) 700 (E) 800

Solution. Problem 2.6, Chapter 2, Introductory to Counting & Probability.

It's easy to count the number of three-digit numbers which are multiples of 5: the smallest multiple of 5 which is a three-digit number is $100 = 5 \cdot 20$, and the largest multiple of 5 which is a three-digit number is $995 = 5 \cdot 199$. Therefore, there are $199 - 20 + 1 = 180$ three-digit numbers which are multiples of 5. There are 900 three-digit numbers in total (from 100 to 999), so there are $900 - 180 = \boxed{720}$ three-digit numbers which are not multiples of 5. The answer is $\boxed{B.}$ □

Problem 13.2.4 (ICLT-2023-SM1-R2-L1-P4)**.** Find the smallest composite number that has no prime divisors less than 23.

(A) 24 (B) 46 (C) 361 (D) 529 (E) 841

Solution. Problem 2.4, Chapter 2, Introductory to Number Theory.

Let n be the composite number that has no prime divisors less than 23. Then the least prime number that can divide such n is 23.

Since n is composite number, n must have another divisor other than 23. Since this divisor must also have no prime divisors less than 23, thus the smallest prime divisor it can have is again 23.

Hence, n is $23 \times 23 = 529$. The answer is $\boxed{D.}$ □

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Problem 13.2.5 (ICLT-2023-SM1-R2-L1-P5)**.** In how many ways can we pick a group of 3 different numbers from the group $1, 2, 3, \ldots, 10$ such that one number is the average of the other two?

(A) 10 (B) 20 (C) 25 (D) 30 (E) 35

Solution. ICLT-2023-SM1-R2-L1-P5 *Problem 2.10, Chapter 2, Introductory to Counting & Probability.*

If the three numbers are $a < b < c$ then $b = \frac{1}{2}(a + c)$.

If we choose $a = 1$, then c must be odd number $3, 5, 7, 9$, and $b = 2, 3, 4, 5$.

If we choose $a = 2$, then c must be even number $4, 6, 8, 10$, and $b = 3, 4, 5, 6$.

If we choose $a = 3$, then c must be odd number $5, 7, 9$, and $b = 4, 5, 6$.

If we choose $a = 4$, then c must be even number $6, 8, 10$, and $b = 5, 6, 7$

If we choose $a = 5$, then c must be odd number $7, 9$, and $b = 6, 7$.

If we choose $a = 6$, then c must be even number $8, 10$, and $b = 7, 8$.

If we choose $a = 7$, then c must be odd number 9, and $b = 8$.

If we choose $a = 8$, then c must be even number 10, and $b = 9$.

Altogether $4 + 4 + 3 + 3 + 2 + 2 + 1 + 1 = \boxed{20}$ ways. The answer is $\boxed{B.}$ □

Problem 13.2.6 (ICLT-2023-SM1-R2-L1-P6)**.** How many two-digit prime numbers ab are there such that ba is also a two-digit prime numbers?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Solution. ICLT-2023-SM1-R2-L1-P6 *Problem 2.11, Chapter 2, Introductory to Number Theory.*

First, if ab and ba are two-digit prime numbers then a and b can only be $1, 3, 7, 9$. Here are the pairs

$$(11, 11),\ (13, 31,),\ (17, 71),\ (37, 73),\ (79, 97).$$

Thus, there are $\boxed{9}$ such numbers. The answer is $\boxed{E.}$ □

Problem 13.2.7 (ICLT-2023-SM1-R2-L1-P7)**.** The angles in a triangle are in the ratio $5 : 6 : 7$. What are the measures of the angles?

Solution. [Solution 1] *Exercise 2.5.2, Chapter 2, Introductory to Geometry.*

The angles in a triangle are in the ratio $5 : 6 : 7$, then the angles can be written as $5x$, $6x$, $7x$. Since the three angles in a triangle add up to $180°$,

$$180° = 5x + 6x + 7x = 18x \Rightarrow x = 10° \Rightarrow (5x, 6x, 7x) = \boxed{50°, 60°, 70°.}$$

□

Solution. [Solution 2] Let the smallest angle be x, then the second largest is $\frac{6}{5}x$, and the largest is $\frac{7}{5}x$. Since the three angles in a triangle add up to $180°$,

$$180° = x + \frac{6}{5}x + \frac{7}{5}x = \frac{18x}{5} \Rightarrow x = 50° \Rightarrow \boxed{50°, 60°, 70°.}$$

□

Problem 13.2.8 (ICLT-2023-SM1-R2-L1-P8)**.** In how many ways can 4 girls and 4 boys be seated in a row of 8 chairs such that all 4 girls sit next to each other?

Solution. Problem 2.12, Chapter 2, Introductory to Counting & Probability.

If we don't distinct the boys or the girls, then there are five different configurations for 4 girls sit next to each other:

$$GGGGBBBB, BGGGGBBB, BBGGGGBB, BBBGGGGB, BBBBGGGG$$

Now, there are 4! ways to sit the 4 boys to each of their seats, and 4! ways for the girls to choose the seats Thus, the answer is $5 \cdot 4! \cdot 4! = \boxed{2880}$ ways. □

Problem 13.2.9 (ICLT-2023-SM1-R2-L1-P9)**.** Martin wanted to solve the equation $3x - 5 = 172$, but instead he solved $5x - 3 = 172$. How larger should the answer have been if he would have solved the correct equation?

Solution. Problem 3.2.4, Chapter 3, Introductory to Algebra.

Martin solved $5x - 3 = 172$, so he received $x = \frac{172+3}{5} = 35$. If he would have solved $3x - 5 = 172$, he would have gotten $x = \frac{172+5}{3} = 59$. Thus the answer he would have received is $59 - 35 = \boxed{24}$ more than he received. □

Problem 13.2.10 (ICLT-2023-SM1-R2-L1-P10)**.** Solve the equation $3t - 7 - 5 + 2t = 4t + 7 - 6t$.

Solution. Problem 3.3, Chapter 3, Introductory to Algebra.

$$3t - 7 - 5 + 2t = 4t + 7 - 6t \Rightarrow 5t - 12 = -2t + 7 \Rightarrow 7t = 19 \Rightarrow t = \boxed{\frac{19}{7}.}$$

□

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13.3 Grading for Level 1

Answers for multiple-choice problems.

Problem 1: C

Problem 2: E

Problem 3: B

Problem 4: D

Problem 5: B

Problem 6: E

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can recognize that the digits can only be $1, 3, 7, 9$.

Problem 8: 2 points if the student can write the angle as $5x$, $6x$, $7x$ or $x : y : z$ as x, $\frac{6x}{5}$, $\frac{7x}{5}$.

Problem 9: 2 points if the student can solve at least one of the two equation correctly.

Problem 10: 2 points if the student can simplify both sides correctly.

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13.4 IC Test for Level 2

Problem 13.4.1 (ICLT-2023-SM1-2-L2-P1)**.** Our math club has 18 members and 3 officers: President, Vice President, and Treasurer. Ali has a huge crush on Brenda, and won't be an officer unless she is one too. Brenda is unaware of Ali's affections, and doesn't care if he is an officer or not; she's perfectly happy to be an officer even if Ali isn't one. In how many ways can the club choose its officers?

(A) 7680 (B) 5922 (C) 5050 (D) 4176 (E) 3212

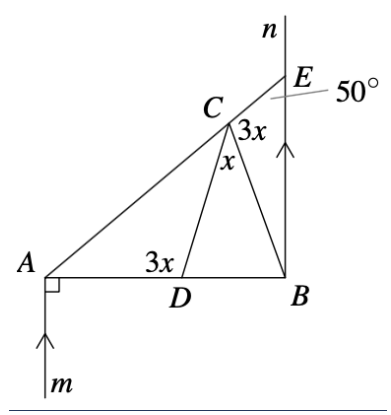
Solution. We use complementary counting approach: first we count all the permutations (without restrictions), and then subtract the ones that are not allowed.

Since there are 4 different officers, so there are $18 \cdot 17 \cdot 16$ ways to choose the officers without restrictions. The cases that we have to exclude are those where Ali is an officer but Brenda is not. So let's count the number of these cases by constructive counting. There are 3 choices for the office that Ali will hold. There are $18 - 2 = 16$ choices for the first remaining office (since Ali is already chosen and we're not allowed to choose Brenda). There are 15 choices for the last remaining office.

In total there are $\boxed{18 \cdot 17 \cdot 16 - 3 \cdot 16 \cdot 15 = 4176}$. The answer is $\boxed{D.}$ □

Problem 13.4.2 (ICLT-2023-SM1-2-L2-P2)**.** In the figure below $m \parallel n$, $AB \perp m$, $\angle ADC = \angle BCE = 3x$, $\angle CEB = 50°$, and $\angle BCD = x$. Find $\angle ACD$.

(A) 35 (B) 30 (C) 25 (D) 20 (E) 15



Solution. First, $\angle ABE = 90°$ because $m \parallel n$ and $m \perp AB$. Then, from $\triangle ABE$ we have:

$$\angle EAB + \angle ABE + \angle AEB = 180° \Rightarrow \angle EAB = 180° - \angle ABE - \angle AEB = 40°.$$

We can then use $\triangle ACD$

$$\angle ACD = 180° - \angle DAC - \angle ADC = 180° - 40° - 3x = 140° - 3x.$$

We have three angles with vertices at C that together make a straight line, so we have

$$\angle ACD + \angle DCB + \angle BCE = 180° \Rightarrow 140° - 3x + x + 3x = 180° \Rightarrow x = 40°.$$

Thus, $x = 40°$. It is easy to see that $\boxed{\angle ACD = 140° - 3x = 20°.}$ The answer is $\boxed{D.}$ □

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Problem 13.4.3 (ICLT-2023-SM1-2-L2-P3)**.** How many triples of prime numbers (p, q, r) such that:

$$p \leq q \leq r, \ p + q + r = 21.$$

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Solution. $p + q + r = 21$ an odd number, so all p, q, r should be odd or exactly two of them are odd.

First, because $p < q < r$, thus all p, q, r are odd primes. If all p, q, r are larger then 3, then $p + q + r \geq 5 + 7 + 11 = 23 > 21$, therefore $p = 3$ and $q + r = 21 - 3 = 18$. There are only two possible solutions $5 + 13 = 7 + 11 = 18$. So there are two triples $(3, 5, 13), (3, 7, 11)$.

Second, we need to investigate the case when two of them are even. This means that $p = q = 2$. In this case we have an additional triple $(2, 2, 17)$.

Thus, there are $\boxed{3}$ solutions $\boxed{(2, 2, 17), (3, 5, 13), (3, 7, 11).}$ The answer is $\boxed{B.}$ □

Problem 13.4.4 (ICLT-2023-SM1-2-L2-P4)**.** Evaluate

$$3(2y^2 + 4x - 2) - 6(x^2 - 2y - 7)$$

when $x = -y$.

(A) 36 (B) 24 (C) $2y^2$ (D) 12 (E) $2y + 1$

Solution. When $x = -y$, the given expression becomes

$$3(2y^2 + 4x - 2) - 6(x^2 - 2y - 7) = 3(2y^2 - 4y - 2) - 6(y^2 - 2y - 7) = 36$$

Thus, it is $\boxed{36.}$ The answer is $\boxed{A.}$ □

Problem 13.4.5 (ICLT-2023-SM1-2-L2-P5)**.** My debate club has 10 students: 5 girls and 5 boys. For the upcoming competition I must choose 5 students to form a team. One of the five chosen students is the team captain. Among the remaining four chosen students, two must be girls and two must be boys. How many different team can I choose?

(A) 5400 (B) 1000 (C) 600 (D) 300 (E) 200

Solution. First, for a boy captain, there is $\binom{5}{1} = 5$ ways to choose. Then, for 2 boy members, there are $\binom{4}{2}$ ways to choose them. For the 2 girl members, there are $\binom{5}{2}$ ways to choose them. So there are $5 \cdot \binom{4}{2} \cdot \binom{5}{2} = 300$ ways to choose.

It is the same numbers of ways if the team has a girl as a captain.

Thus, the total number of ways is $\boxed{2 \cdot 300 = 600.}$ The answer is $\boxed{C.}$ □

Problem 13.4.6 (ICLT-2023-SM1-2-L2-P6)**.** Two different rectangles have the same width. All four sides of both rectangles have integer lengths. The areas of the rectangles are 1818 and 1746. Find the largest possible value of the common width of the rectangles.

(A) 2 (B) 3 (C) 6 (D) 12 (E) 18

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Solution. Let ℓ_1 be the length of the larger rectangle and ℓ_2 be the length of the smaller rectangle and let w be the width of both rectangles, then

$$\begin{cases} \ell_1 \cdot w = 1818 \\ \ell_2 \cdot w = 1746 \end{cases}$$

Our goal is to find the largest possible value of w. We see that ℓ_1 and n are divisors of 1818 while ℓ_2 and n are divisors of 1746. This means that w is common divisor of 1818 and 1746. Since we are trying to find the largest possible value of w, our goal is actually to find the greatest common divisor of 1818 and 1746.

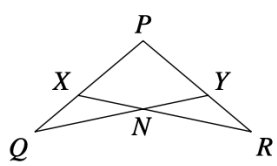
It is easy to compute that

$$\gcd(1818, 1746) = \gcd(2 \cdot 3^2 \cdot 101, 2 \cdot 3^2 \cdot 97) = 2 \cdot 3^2 = 18.$$

Therefore the largest value of w is $\boxed{18.}$ The answer is $\boxed{E.}$ □

Problem 13.4.7 (ICLT-2023-SM1-2-L2-P7)**.** In the figure below, $PQ = PR$ and $\angle PQY = \angle PRX$.

1. (*5 points*) Prove that $QY = RX$.
2. (*5 points*) Prove that $XN = YN$.



Solution. Since $PQ = PR$ and $\angle PQY = \angle PRX$. Triangles $\triangle YQP$ and $\triangle XRP$ also have angle $\angle QPR$ in common, so by ASA, they are congruent. Since QY and RX are corresponding parts of triangles $\triangle YQP$ and $\triangle XRP$, we have $QY = RX$.

Since $\triangle YQP \cong \triangle XRP$, we have $PX = PY$ because they are corresponding sides of the two triangles. We also have $PQ = PR$, and we can combine these two side length equalities to show $XQ = YR$:

$$XQ = PQ - PX = PR - PY = YR.$$

Our congruent triangles also give us $\angle Q = \angle R$. Also, $\angle XNQ = \angle YNR$ since they are vertical angles. $XQ = YR$, $\angle Q = \angle R$, and $\angle XNQ = \angle YNR$ give us $\triangle XNQ \cong \triangle YNR$ by AAS Congruence, so $\boxed{XN = YN.}$ □

Problem 13.4.8 (ICLT-2023-SM1-2-L2-P8)**.** Tim has four balls of different colours: blue, green, red, and white.

1. (*5 points*) He has two wooden boxes, both look exactly the same. In how many ways can he put the four balls into the two boxes?
2. (*5 points*) If he paints the boxes so that they look completely different, then in how many ways can he put the four balls into these two boxes?

Solution. If the two boxes are the same, then we just need to find the number of ways to choose from the 4 balls to put some into one box. He can chose all 4 balls in 1 way, or 3 balls in $\binom{4}{3} = 4$ ways, or 2 balls in $\binom{4}{2} = 6$ ways, or 1 ball in $\binom{4}{1} = 4$ ways, or no ball in 1 way. Note that the two boxes are the same, choosing 3 balls into one box is the same as choosing 1 ball into the other box.

Thus, the number of ways should be $\boxed{\frac{1}{2}(1 + 4 + 6 + 4 + 1) = 8.}$

For the second question, if the two box are different, then we don't have to divide the choices, thus the total number of ways is $\boxed{16.}$ □

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Problem 13.4.9 (ICLT-2023-SM1-2-L2-P9)**.** x and y are positive real numbers such that,

$$\begin{cases} \sqrt{x} + 2\sqrt{y} = \frac{5}{6} \\ 3\sqrt{x} - \sqrt{y} = \frac{1}{6} \end{cases}$$

1. (*5 points*) What is the value of $\sqrt{x}$?
2. (*5 points*) Solve the equation.

Solution.

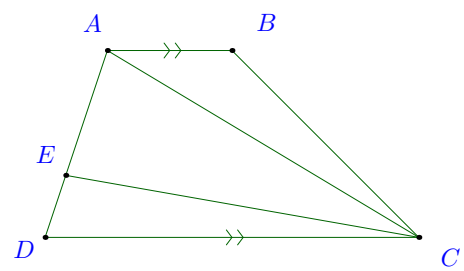
$$\begin{cases} \sqrt{x} + 2\sqrt{y} = \frac{5}{6} \\ 3\sqrt{x} - \sqrt{y} = \frac{1}{6} \end{cases} \Rightarrow (\sqrt{x} + 2\sqrt{y}) + 2(3\sqrt{x} - \sqrt{y}) = \frac{5}{6} + 2\frac{1}{6}$$

$$\Rightarrow 7\sqrt{x} = \frac{7}{6}$$

Thus $\boxed{\sqrt{x} = \frac{1}{6}.}$ Therefore $x = \frac{1}{36}$. By substitution $\sqrt{y} = 3\sqrt{x} - \frac{1}{6} = \frac{1}{3} \Rightarrow y = \frac{1}{9}$. So $\boxed{(x, y) = (\frac{1}{36}, \frac{1}{9}).}$ □

Problem 13.4.10 (ICLT-2023-SM1-2-L2-P10)**.** $ABCD$ is a trapezoid such that $AB \parallel CD$, $AB = 2$, $CD = 6$. E is on AD such that $DE = 2EA$.

1. (*4 points*) Prove that $[ADC] = 3[ABC]$.
2. (*5 points*) Prove that $[DEC] = [ABC]$.



Solution. First, since $AB \parallel CD$, so the distance from C to AB is the same as the distance from A to CD, therefore the altitudes of $\triangle ABC$ and $\triangle ACD$ from C and A, respectively, are the same. Because $CD = 3AB$, so $\boxed{[ACD] = 3[ABC].}$

For the second question, $EA = 2ED$, thus $AD = 3ED$, then $[ADC] = 3[EDC]$, and $\boxed{[ECD] = [ABC].}$ □

13.5 Grading for Level 2

Answers for multiple-choice problems.

Problem 1:	D	4176
Problem 2:	D	20
Problem 3:	B	3
Problem 4:	A	36
Problem 5:	C	600
Problem 6:	E	18

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 2 point if can find $\triangle YQP \cong \triangle XRP$.

(b) 2 point if can find $XQ = PQ - PX = PR - PY = YR$.

Problem 8: Separately for each part,

(a) 2 points if can casework the number of balls into a box.

(b) 2 points if can reuse the first case.

Problem 9: Separately for each part,

(a) 2 points if can use elimination to find the value of x.

(b) 2 points if can use substitution.

Problem 10: Separately grading for each part,

(a) 2 if found that the distance from C to AB and A to CD is the same.

(b) 3 if determined the ratio of AD and ED.

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13.6 IC Test for Level 3

Problem 13.6.1 (ICLT-2023-SM1-2-L3-P1)**.** My debate club has 10 students: 5 girls and 5 boys. For the upcoming competition I must choose 5 students to form a team. One of the five chosen students is the team captain. Among the remaining four chosen students, two must be girls and two must be boys. How many different team can I choose?

(A) 200 (B) 300 (C) 600 (D) 1000 (E) 5400

Solution. First, for a boy captain, there is $\binom{5}{1} = 5$ ways to choose. Then, for 2 boy members, there are $\binom{4}{2}$ ways to choose them. For the 2 girl members, there are $\binom{5}{2}$ ways to choose them. So there are $5 \cdot \binom{4}{2} \cdot \binom{5}{2} = 300$ ways to choose.

It is the same numbers of ways if the team has a girl as a captain.

Thus, the total number of ways is $\boxed{2 \cdot 300 = 600.}$ The answer is $\boxed{C.}$ □

Problem 13.6.2 (ICLT-2023-SM1-2-L3-P2)**.** Two different rectangles have the same width. All four sides of both rectangles have integer lengths. The areas of the rectangles are 1818 and 1746. Find the largest possible value of the common width of the rectangles.

(A) 18 (B) 12 (C) 6 (D) 3 (E) 2

Solution. Let ℓ_1 be the length of the larger rectangle and ℓ_2 be the length of the smaller rectangle and let w be the width of both rectangles, then

$$\begin{cases} \ell_1 \cdot w = 1818 \\ \ell_2 \cdot w = 1746 \end{cases}$$

Our goal is to find the largest possible value of w. We see that ℓ_1 and n are divisors of 1818 while ℓ_2 and n are divisors of 1746. This means that w is common divisor of 1818 and 1746. Since we are trying to find the largest possible value of w, our goal is actually to find the greatest common divisor of 1818 and 1746.

It is easy to compute that

$$\gcd(1818, 1746) = \gcd(2 \cdot 3^2 \cdot 101, 2 \cdot 3^2 \cdot 97) = 2 \cdot 3^2 = 18.$$

Therefore the largest value of w is $\boxed{18.}$ The answer is $\boxed{A.}$ □

Problem 13.6.3 (ICLT-2023-SM1-2-L3-P3)**.** x and y are positive real numbers such that,

$$\begin{cases} \sqrt{x} + 2\sqrt{y} = \frac{5}{6} \\ 3\sqrt{x} - \sqrt{y} = \frac{1}{6} \end{cases}$$

Find the ratio $\frac{y}{x}$.

(A) 3 (B) 4 (C) 6 (D) 8 (E) 9

Solution.

$$\begin{cases} \sqrt{x} + 2\sqrt{y} = \frac{5}{6} \\ 3\sqrt{x} - \sqrt{y} = \frac{1}{6} \end{cases} \Rightarrow (\sqrt{x} + 2\sqrt{y}) + 2(3\sqrt{x} - \sqrt{y}) = \frac{5}{6} + 2\frac{1}{6}$$

$$\Rightarrow 7\sqrt{x} = \frac{7}{6}$$

Thus $\boxed{\sqrt{x} = \frac{1}{6}.}$ Therefore $x = \frac{1}{36}$. By substitution $\sqrt{y} = 3\sqrt{x} - \frac{1}{6} = \frac{1}{3} \Rightarrow y = \frac{1}{9}$. So $\boxed{\frac{y}{x} = \frac{\frac{1}{9}}{\frac{1}{36}} = 4.}$ The answer is $\boxed{B.}$ □

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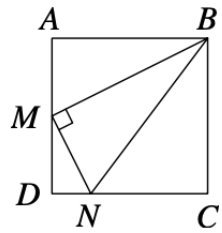
Problem 13.6.4 (ICLT-2023-SM1-2-L3-P4)**.** In the diagram below, $ABCD$ is a square. $AB = 4$. M is the midpoint of AD and $BM \perp MN$. Find BN.

(A) $2+\sqrt{2}$ (B) 4 (C) $3\sqrt{2}$ (D) 5 (E) $4\sqrt{2}$



Solution. First,

$$\angle NMD = 180^\circ - \angle NMB - \angle BMA = 180^\circ - 90^\circ - \angle BMA = 90^\circ - \angle BMA = \angle ABM.$$

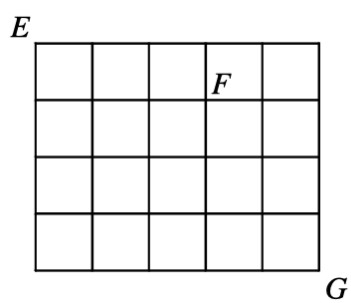
and $\angle MDN = \angle A$, we have $\triangle MDN \sim \triangle BAM$. Thus

$$\frac{MN}{BM} = \frac{MD}{AB} = \frac{1}{2} = \frac{AM}{AB} \Rightarrow \triangle BAM \sim \triangle BMN \Rightarrow \frac{BN}{BM} = \frac{BM}{BA}$$

Therefore $BN = \frac{BM^2}{BA} = \frac{BA^2+AM^2}{BA} = \frac{4^2+2^2}{4} = 5$. The answer is $\boxed{D.}$ □

Problem 13.6.5 (ICLT-2023-SM1-2-L3-P5)**.** The diagram below shows a grid. How many 9-step paths are there going from G to E through F?

(A) 12 (B) 24 (C) 30 (D) 36 (E) 40



Solution. From G to F, it is 2 steps to the left and 3 steps up, for a total of $\binom{5}{2} = 10$ different paths. From F to E, it is 3 steps to the left and 1 step up, for a total of $\binom{4}{1} = 4$ different paths.

So there are $\boxed{4 \cdot 10 = 40}$ paths from G to F. The answer is $\boxed{E.}$ □

Problem 13.6.6 (ICLT-2023-SM1-2-L3-P6)**.** Let $n = 2^a \cdot 3^b$, where a and b are positive integers. If n has exactly 22 divisors, including 1 and itself, how many divisors does n^2 have?

(A) 22 (B) 44 (C) 63 (D) 66 (E) 84

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Solution. Since n is in the format of $2^a 3^b$, the number of divisors of n is

$$22 = (a+1)(b+1)$$

Thus the pair (a, b) can only be $(1, 10)$ or $(10, 1)$, since n^2 is

$$n^2 = \left(2^a 3^b\right)^2 = (2^{2a} 3^{2b})$$

Thus, the number of divisors of n^2 is $\boxed{(2a+1)(2b+1) = 3 \cdot 21 = 63.}$ The answer is $\boxed{C.}$ □

Problem 13.6.7 (ICLT-2023-SM1-2-L3-P7)**.** There are 805 animals in the store. They are dogs, cats, and mice. The ratio of dogs to cats to mice is $2 : 3 : 30$.

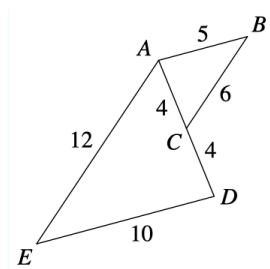
1. (*5 points*) How many cats are in the store?
2. (*5 points*) If 23 cats are added to the store, what is then the ratio of cats to dogs?

Solution. For the first question, for every 2 dogs, there are 3 cats and 30 mice. Therefore, we group the animals such that each group contains 2 dogs, 3 cats, and 30 mice. Each such group has $2 + 3 + 30 = 35$ animals, and there are 805 total animals, so there must be 23 groups. There are 3 cats in each of these groups, so there are $\boxed{23 \cdot 3 = 69}$ cats.

The number of dogs is $23 \cdot 2 = 46$. If we increase the number of cats by 23 to a total of 92, then the ratio of cats to dogs now is $\boxed{92 : 46 = 2 : 1.}$ □

Problem 13.6.8 (ICLT-2023-SM1-2-L3-P8)**.** Given the side lengths shown in the diagram below.

1. (*5 points*) Prove that $AE \parallel BC$ and $AB \parallel DE$.
2. (*5 points*) What is the ratio of areas $\frac{[\triangle ABC]}{[\triangle ADE]}$?



Solution. Since

$$\frac{AB}{DE} = \frac{AC}{AD} = \frac{BC}{AE} = \frac{1}{2},$$

so $\triangle ABC \sim \triangle DEA$ by SSS Similarity. Therefore, $\angle BAC = \angle EDA$, so $AB \parallel DE$. Also, $\angle DAE = \angle ACB$, so $AE \parallel BC$.

It is easy to see that that ratio of heights between the two triangles $\triangle ABC$ and $\triangle DEA$ is the same as their side ratio. Therefore the ratio of their areas is the square of their ratio, thus $\boxed{\frac{[\triangle ABC]}{[\triangle ADE]} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.}$ □

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Problem 13.6.9 (ICLT-2023-SM1-2-L3-P9)**.** Let $n = (30)^2 = 900$.

1. (*5 points*) Find the number of distinct positive divisors of n excluding 1 and n.
2. (*5 points*) Find the number of distinct positive divisors of n^2 excluding 1 and n^2.

Solution. Note that $n = 30^2 = (2 \cdot 3 \cdot 5)^2 = 2^2 \cdot 3^2 \cdot 5^2$. Thus, the number of divisors of n, including 1 and itself, is,

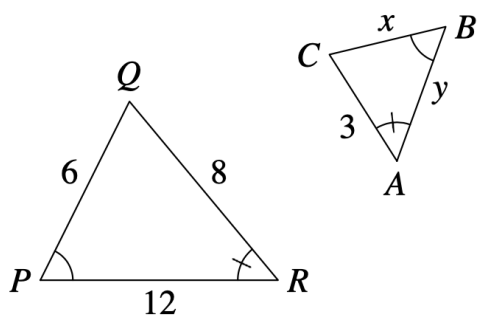
$$(2+1)(2+1)(2+1) = 27.$$

So there are $\boxed{27 - 2 = 25}$ divisors of n not including 1 and n.

Similarly, $n^2 = 30^4 = 2^4 \cdot 3^4 \cdot 5^4$ has $\boxed{(4+1)(4+1)(4+1) - 2 = 123}$ divisors, not including 1 and n^2. □

Problem 13.6.10 (ICLT-2023-SM1-4-L3-P10)**.** In the diagram at below, the angle equalities and side lengths of $\triangle PQR$ and $\triangle ABC$ are given.

1. (*5 points*) What is the value of $y + 2x$?
2. (*5 points*) What is the ratio of the areas $\frac{[PQR]}{[ABC]}$?



Solution. For the first question, Since $\angle A = \angle R$ and $\angle B = \angle P$, we have $\triangle ABC \sim \triangle RPQ$.

Therefore, we have $\frac{AB}{RP} = \frac{BC}{PQ} = \frac{AC}{RQ}$. Substitution gives $\frac{y}{12} = \frac{x}{6} = \frac{3}{8}$, from which we find $y = (12)(\frac{3}{8}) = \frac{9}{2}$ and $x = (6)(\frac{3}{8}) = \frac{9}{4}$. Thus $y + 2x = \boxed{9}$.

For the second question, $\frac{[PQR]}{[ABC]} = \left(\frac{RQ}{AC}\right)^2 = \left(\frac{8}{3}\right)^2 = \boxed{\frac{64}{9}.}$ □

13.7 Grading for Level 3

Answers for multiple-choice problems.

Problem 1:	C	600
Problem 2:	A	18
Problem 3:	B	4
Problem 4:	D	5
Problem 5:	E	40
Problem 6:	C	63

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 2 point if find 35 as the number of animals in a group, or a common denominator.

(b) 2 point if find the total number of dogs.

Problem 8: Separately for each part,

(a) 2 points if can use SSS for similarity.

(b) 2 points if can deduce the ratio of the heights.

Problem 9: Separately for each part,

(a) 1 points if correctly factor n

(b) 1 points if correctly factor n^2

Problem 10: Separately for each part,

(a) 2 points if can find $\triangle ABC \sim \triangle RPQ$.

(b) 2 points if can show ratio of areas is square of ratio of similarity.

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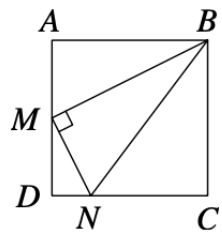
13.8 IC Test for Level 4

Problem 13.8.1 (ICLT-2023-SM1-2-L4-P1). (*6 points*)

In the figure below, $ABCD$ is a square. $AB = 4$. M is the midpoint of AD and $BM \perp MN$. Find BN.

(A) $2+\sqrt{2}$ (B) 4 (C) $3\sqrt{2}$ (D) 5 (E) $4\sqrt{2}$



Solution. First,

$$\angle NMD = 180^\circ - \angle NMB - \angle BMA = 180^\circ - 90^\circ - \angle BMA = 90^\circ - \angle BMA = \angle ABM.$$

and $\angle MDN = \angle A$, we have $\triangle MDN \sim \triangle BAM$.

Thus

$$\frac{MN}{BM} = \frac{MD}{AB} = \frac{1}{2} = \frac{AM}{AB} \Rightarrow \triangle BAM \sim \triangle BMN \Rightarrow \frac{BN}{BM} = \frac{BM}{BA}$$

Therefore $BN = \frac{BM^2}{BA} = \frac{BA^2+AM^2}{BA} = \frac{4^2+2^2}{4} = 5$. The answer is $\boxed{D.}$ □

Problem 13.8.2 (ICLT-2023-SM1-2-L4-P2). Let $n = 2^a \cdot 3^b$, where a and b are positive integers. If n has exactly 22 divisors, including 1 and itself, how many divisors does n^2 have?

(A) 22 (B) 44 (C) 63 (D) 66 (E) 84

Solution. Since n is in the format of $2^a 3^b$, the number of divisors of n is

$$22 = (a+1)(b+1)$$

Thus the pair (a, b) can only be $(1, 10)$ or $(10, 1)$, since n^2 is

$$n^2 = \left(2^a 3^b\right)^2 = (2^{2a} 3^{2b})$$

Thus, the number of divisors of n^2 is $\boxed{(2a+1)(2b+1) = 3 \cdot 21 = 63.}$ The answer is $\boxed{C.}$ □

Problem 13.8.3 (ICLT-2023-SM1-2-L4-P3). There are 805 animals in the store. They are dogs, cats, and mice. The ratio of dogs to cats to mice is $2:3:30$. How many cats are in the store?

(A) 23 (B) 46 (C) 69 (D) 92 (E) 115

Solution. For the first question, for every 2 dogs, there are 3 cats and 30 mice. Therefore, we group the animals such that each group contains 2 dogs, 3 cats, and 30 mice. Each such group has $2+3+30 = 35$ animals, and there are 805 total animals, so there must be 23 groups. There are 3 cats in each of these groups, so there are $\boxed{23cdot3 = 69}$ cats. The answer is $\boxed{C.}$ □

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Problem 13.8.4 (ICLT-2023-SM1-2-L4-P4)**.** Suppose 2 distinct integers are chosen from between 0 and 9, inclusive. What is the probability that they are both odd?

$$(A)\ \frac{2}{9} \qquad (B)\ \frac{5}{18} \qquad (C)\ \frac{1}{3} \qquad (D)\ \frac{1}{2} \qquad (E)\ \frac{5}{9}$$

Solution. Let's consider an outcome to be a choice of 2 numbers without regard to order. There are $\binom{10}{2} = 45$ ways to choose 2 numbers from our set of 10 numbers if we don't care about order. There are $\binom{5}{2} = 10$ ways to choose 2 numbers from the 5 odd numbers without regard to order. Therefore, the probability is $\boxed{\frac{10}{45} = \frac{2}{9}.}$ The answer is $\boxed{A.}$ □

Problem 13.8.5 (ICLT-2023-SM1-2-L4-P5)**.** Find the standard form of the equation of the line through $(6, 1)$ that is perpendicular to the line $2x = -3y + 7$.

$(A)\ \ 3x-2y = 10 \qquad (B)\ \ 3x-2y = 20 \qquad (C)\ \ 2x-3y = 10 \qquad (D)\ \ 2x-3y = 20 \qquad (E)\ \ 3x-4y = 20$

Solution. We find the slope of the given line by writing it in slope-intercept form. Subtracting 7 from both sides and dividing both sides by -3 gives

$$y = -\frac{2}{3}x + \frac{7}{3}.$$

So, the slope of the given line is $-2/3$. Therefore, the slope of a line perpendicular to the given line is $-1/(-2/3) = 3/2$. We find a point-slope equation of the line through $(6, 1)$ with slope $3/2$, and then convert it to standard form:

$$y - 1 = \frac{3}{2}(x - 6) \Rightarrow 2(y - 1) = 3(x - 6) \Rightarrow\ 2y - 2 = 3x - 18 \Rightarrow\ 3x - 2y = 20$$

Thus, the standard form of the line is $\boxed{3x - 2y = 20.}$ The answer is $\boxed{B.}$ □

Problem 13.8.6 (ICLT-2023-SM1-2-L4-P6)**.** In triangle $\triangle PQR$, $PQ = 3$, $\angle Q = 90°$, and $QR = 4$. Find the inradius of $\triangle PQR$.

$$(A)\ \ 2 \qquad (B)\ \ 2 - \sqrt{2} \qquad (C)\ \ \frac{3}{2} \qquad (D)\ \ \frac{1 + \sqrt{2}}{2} \qquad (E)\ \ 1$$

Solution. Since $\triangle PQR$ is a right triangle, we have $[PQR] = (PQ)(QR)/2 = 6$. $PR = \sqrt{PQ^2 + QR^2} = 5$. The semiperimeter of $\triangle PQR$ is $(3 + 4 + 5)/2 = 6$. Thus, the inradius is $\boxed{6/6 = 1.}$ The answer is $\boxed{E.}$ □

Problem 13.8.7 (ICLT-2023-SM1-2-L4-P7)**.** The Pirates play 4 games with the Mathematicians. The Pirates are 70% likely to win any given game.

1. (*5 points*) What is the probability that The Pirates win the first two games?
2. (*5 points*) What is the probability that the Mathematicians win exactly one game?

Solution. The Pirates wins the first two games if they wins both the first and the second games,

$$P(\text{Pirates win the 1st game}) \times P(\text{Pirates win the 2nd game}) = 0.7^2 = 0.49$$

Thus the probability that the Pirates wins the first two games is $\boxed{0.49.}$

The Mathematicians win exactly one game if the Pirates wins all three other games, for example for the Mathematicians to win the first game, then lose all other, the probability is

$$P(\text{Pirates lose the 1st game}) \times P(\text{Pirates win a game})^3 = 0.3 \cdot 0.7^3 = 0.1029$$

There are 4 such cases, so the probability is $\boxed{4 \cdot 0.1029 = 0.4116.}$ □

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Problem 13.8.8 (ICLT-2023-SM1-2-L4-P8)**.** In $\triangle XYZ$, $XZ = 8$. Point D is on side YZ such that $\angle YXD = \angle ZXD$, $YD = 3$, and $DZ = 4$.

1. (*5 points*) Find XY.
2. (*5 points*) Find the aread of $\triangle XYZ$.

Solution. Since $\angle YXD = \angle ZXD$, we know that XD bisects $\angle YXZ$. From the Angle Bisector Theorem, we have $\frac{XY}{YD} = \frac{XZ}{DZ} = 2$. Therefore, $\boxed{XY = 2YD = 6.}$

The semi-perimeter of the $\triangle XYZ$ with sides $XY = 6, YZ = 7, ZX = 8$ is $\frac{21}{2}$. Thus, its area is $\sqrt{\frac{21}{2}\left(\frac{21}{2}-8\right)\left(\frac{21}{2}-7\right)\left(\frac{21}{2}-6\right)}$, or $\boxed{\frac{21\sqrt{15}}{4}.}$ □

Problem 13.8.9 (ICLT-2023-SM1-2-L4-P9)**.** Find all two-digit number $\overline{ab}$, if

1. (*5 points*) $a^2 + b^2 = 4(a + b) - 8$.
2. (*5 points*) $a^2 + b^2 = 4(a + b) - 3$.

Solution.

$$a^2 + b^2 = 4(a + b) - 8 \Rightarrow (a^2 - 4a + 4) + (b^2 - 4b + 4) = 0 \Rightarrow (a - 2)^2 + (b - 2)^2 = 0$$

The left side is a sum of two perfect square. It can be 0 if and only if both perfect square is 0. Hence $\boxed{(a, b) = (2, 2).}$

$$a^2 + b^2 = 4(a + b) - 3 \Rightarrow (a^2 - 4a + 4) + (b^2 - 4b + 4) = 5 \Rightarrow (a - 2)^2 + (b - 2)^2 = 5$$

The left side is a sum of two perfect square. It can be 5 if and only if one of them is 1 and the other is 4.

Case 1: $|a - 2| = 1, |b - 2| = 2$.

$$\begin{cases} a - 2 = 1, b - 2 = 2 \\ a - 2 = 1, b - 2 = -2 \\ a - 2 = -1, b - 2 = 2 \\ a - 2 = -1, b - 2 = -2 \end{cases} \Rightarrow \overline{ab} \in \{34, 30, 14, 10\}$$

Case 2: $|a - 2| = 4, |b - 2| = 1$.

$$\begin{cases} a - 2 = 2, b - 2 = 1 \\ a - 2 = 2, b - 2 = -1 \end{cases} \Rightarrow \overline{ab} \in \{43, 41\}$$

Therefore the possible solutions are $\boxed{\{10, 14, 30, 34, 41, 43\}.}$ □

Problem 13.8.10 (ICLT-2023-SM1-2-L4-P10)**.** Three cards are chosen at random from a standard 52-card deck.

1. (*5 points*) What is the probability that they are all the same color?
2. (*5 points*) What is the probability that they are not all the same color?

Solution. For the first question, there are 26 cards of each color, so 3 of them can be selected in $\binom{26}{3}$ ways, and of course there are 2 colors. So the answer is

$$2 \cdot \frac{\binom{26}{3}}{\binom{52}{3}} = \boxed{\frac{4}{17}.}$$

For the second question, we know the probability they are all same color, so we just need to subtract that from 1. Thus, the probability is $1\frac{4}{17} = \boxed{\frac{13}{17}.}$ □

13.9 Grading for Level 4

Answers for multiple-choice problems.

Problem 1:	D	5
Problem 2:	C	63
Problem 3:	C	69
Problem 4:	A	$\frac{2}{9}$
Problem 5:	B	$3x - 2y = 20$
Problem 6:	E	1

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the below.

Problem 7: For the first part,

(a) 1 point if can find the probability is a product of probabilities to win both the first and the second games,

(b) 1 point if can casework which game the Mathematicians wins.

Problem 8: Separately for each part,

(a) 2 points if use the Angle Bisector Theorem.

(b) 2 points if use the Heron's Formula

Problem 9: Separately for each part,

(a) 2 points if can reduce to sum of squares

(b) 2 points if can argue $5 = 1^2 + 2^2$ is the only possible sum of perfect squares.

Problem 10: Separately for each part,

(a) 2 points if can find the number of favorable outcomes $2 \times \binom{26}{3}$.

(b) 2 points if can use complementary probability (result from the first question).

Ongoing school year 2023 - 2024. Last update on January 2, 2024