Ongoing school year 2023 - 2024. Last update on

School year 2023 - 2024 (Problems Only)

Nghia Doan

May 26, 2024

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Introduction

Welcome to the MCC School year 2023 - 2024!

Levels: - Introductory (I): for anyone who start middle schools, are beginners, not confident of being good at problem solving, or just simply don't have enough time for math. They are encouraged to start, or continue, or finish the Introductory Curriculum. Expected workload: 2 hours per week - Advanced (A): for returning students, who already have a good grasp of problem solving, having done a few semesters with the club, wish to learn more, and ready to achieve higher knowledge and skill sets. Expected workload: 4 hours per week - Olympiad (O): for excellent students who are already or about to take challenges at national and international competitions. Expected workload: 8 hours per week.

Have a great year!

Your teacher,

Nghia Doan

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Part I

2023

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Chapter 1

Entrance tests

1.1 Rules

Choice of level:

- There are three test papers, each for one level.
- Each student can chose multiple test levels. If the students choose more than one level, the student must choose multiple different test times.

Registration:

- At registration, student and their parents are responsible for the correct email address and the time, where and when they wish to receive the test problems.
- The club will not verify whether the registered email address is correct.

Entrance test:

- Each level test consisting of 8 problems will be delivered by automatic scheduled email at the registered time. No test paper will be sent twice.
- Students have 270 minutes to solve the problems. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET.
- After the allowed time, the student has 15 minutes to scan the solutions. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET, and has to submit the scanner solutions before 11:45 AM. Any solution submitted later than the allowed time will not be accepted.
- All submissions must be done by email to nghia71@gmail.com.
- Solutions must be written in English. Any solution looks unreadable, or is presented badly will not be considered for grading. Answer without proper solution will not be considered as solution. Answer consists of self-explanatory and easy-to-understand diagrams are acceptable but might not be considered as full solution.
- A problem can contain more than one questions, each with an indication of the maximum number of points can be awarded if a correct solution is provided. Partially correct solution is awarded with a some points depending on its quality. Multiple solutions can be submitted for a single problem. Each different solution, if correct, will be awarded the same number of points. Partially distinct solution also counts.
- Student who performs adequately will be accepted to the registered level. Student who are accepted to different levels can choose any of those levels to attend.

1.2 Problems for Intermediate Level

Problem 1.2.1 (23-24-ET-I-P1)**.** (*10 points*) Place algebraic operations $+; -; \times; \div$ between the digits 1 to 9, in that order, so that the total equals 100. You may freely use brackets before or after any of the digits in the expression. No digits may be placed together, such as 123 and 67 are not allowed.

Here is an example for a correct answer: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9 = 100$.

Here is an example for an incorrect answer: $12 + 3 + 4 + 5 - 6 - 7 + 89 = 100$.

Do not submit any of these examples as your solution.

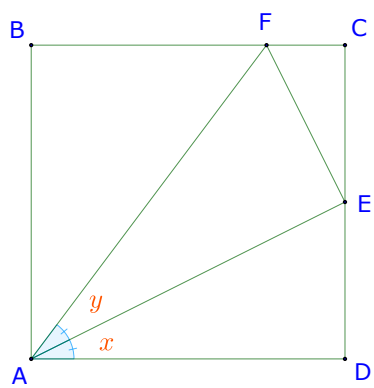
Problem 1.2.2 (23-24-ET-I-P2)**.** (*10 points*) Linh, Minh and Vinh are identical triplet sisters. Linh always tells the truth, Minh always lies and Vinh sometimes lies and sometimes tells the truth.

They are invited to a party. One of them arrives late. The party host asks this sister, who just arrived late, who she is. She answers "I am Vinh". The party host obviously cannot tell the girls apart so she asks the other two sisters the name of the sister who was late. One of them says: "Linh was late", and the other says: "Minh was late".

Which sister was late?

Problem 1.2.3 (23-24-ET-I-P3)**.** (*10 points*) $ABCD$ is a square with side length 4. E is midpoint of CD. $AE \perp EF$. Let $x = \angle EAD, y = \angle FAE$.

Prove that $x = y$.



Problem 1.2.4 (23-24-ET-I-P4)**.** (*10 points*) A pair of two-digit numbers has the following properties:

1. The sum of the four digits is 25.
2. The sum of the two numbers is 97.
3. The product of the four digits is 864.
4. The product of the two numbers is 1972.

Determine the two numbers.

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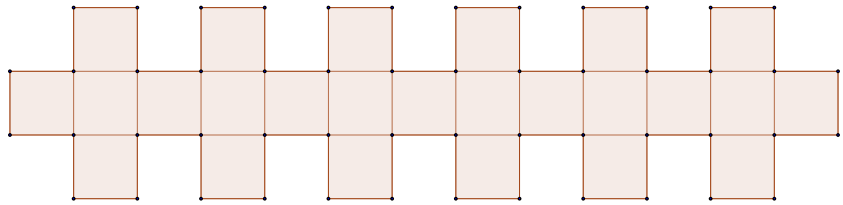
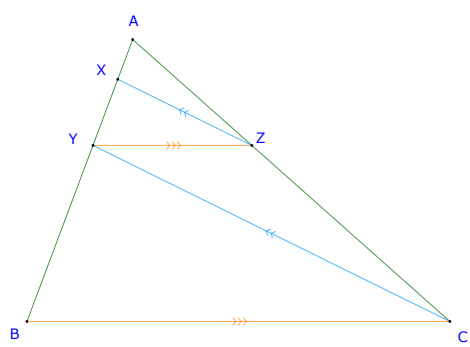
Problem 1.2.5 (23-24-ET-I-P5)**.** (*10 points*) The shape in the diagram below was created by putting together 25 unit squares. When a similar shape is created with n squares, its perimeter is 100 units. Determine n.



Problem 1.2.6 (23-24-ET-I-P6)**.** (*10 points*) N is the sum of all integers from 1 to n and M is the sum of all integers from 1 to m, where $n > m + 1$. If the difference of the two sum $N - M = 2012$, then what is the value of $n + m$?

Problem 1.2.7 (23-24-ET-I-P7)**.** (*10 points*) The area of $\triangle ABC$ is 1. X, Y are points on AB, Z is a point on AC such that $XY = 2AX$. Furthermore $XZ \parallel YC$, $YZ \parallel BC$. Find the area of $\triangle XYZ$.



Problem 1.2.8 (23-24-ET-I-P8)**.** (*10 points*) Consider a sequence of integers

$$1, 3, 2, -1, \ldots$$

where each term is equal to the term preceding it minus the term before that.

What is the sum of the first 2023 terms?

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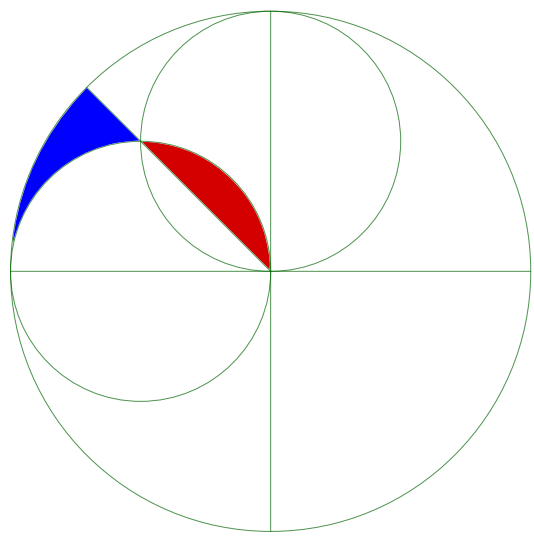
1.3 Problems for Advanced Level

Problem 1.3.1 (23-24-ET-A-P1)**.** (*10 points*) Two friends agreed to meet at a club between 3:00 PM and 5:00 PM. What is the probability that they meet at the club within 15 minutes of arrival?

Problem 1.3.2 (23-24-ET-A-P2)**.** (*10 points*) A number n written in base b reads 211, but it becomes 110 when written in base $b+2$. Find n and b in base 10.

Problem 1.3.3 (23-24-ET-A-P3)**.** (*10 points*) In the diagram below two perpendicular diameters divide a circle into four parts. On each of these diameters a circle of half the diameter is drawn, tangent to the original circle and meeting at its centre. A radius to the large circle is drawn through the intersection points of these smaller circles. Show that the red and blue shaded regions are of the same area.



Problem 1.3.4 (23-24-ET-A-P4)**.** (*10 points*) Determine all pairs of integers (not necessarily positive) (x, y) such that

$$(x-8)(x-10) = 2^y.$$

Problem 1.3.5 (23-24-ET-A-P5)**.** (*10 points*) A game is played on a 7×7 board, initially blank. Chi and Mai make alternate moves, with Chi going first. In each of her moves, Chi chooses any four blank squares which form a 2×2 block, and paints these squares brown. In each of her moves, Mai chooses any blank square and paints it green. They take alternate turns until no more moves can be made by Chi. Then Mai paints the remaining blank squares green.

Which player, if either, can guarantee to be able to paint 25 or more squares in her colour, regardless of how her opponent plays?

Problem 1.3.6 (23-24-ET-A-P6)**.** (*10 points*) Determine all sets consisting of an odd number $2m+1$ of consecutive positive integers, for some integer $m \geq 1$ such that the sum of the squares of the smallest $m+1$ integers is equal to the sum of the squares of the largest m integers.

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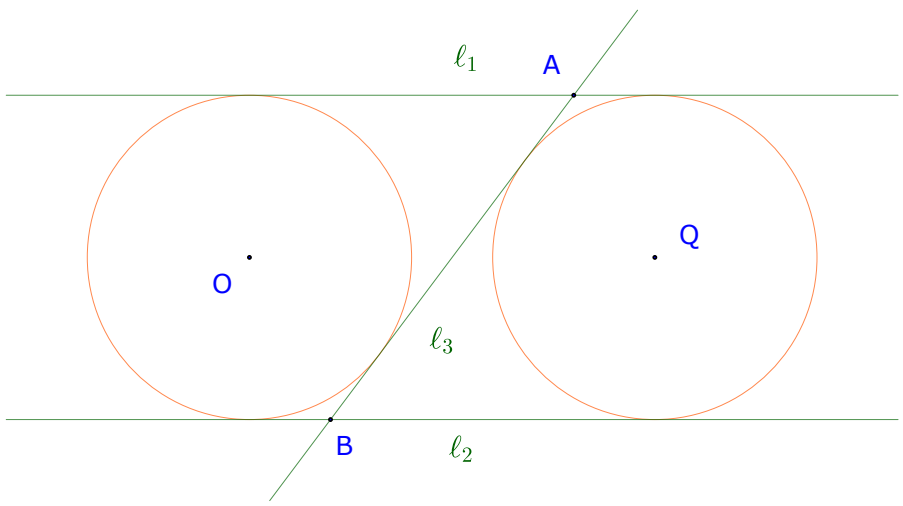
Problem 1.3.7 (23-24-ET-A-P7)**.** (*10 points*) Given two parallel lines ℓ_1 and ℓ_2, the transversal ℓ_3 intersects them at points A and B respectively. Two circles with centres at O and Q lie between the parallel lines on the left and on the right sides of the transversal such that the circles are tangent to all three lines.

Show that $OQ = AB$.



Problem 1.3.8 (23-24-ET-A-P8)**.** (*10 points*) Find the primes p, q, and r, given that one of the number pqr and $p + q + r$ is 101 times the other.

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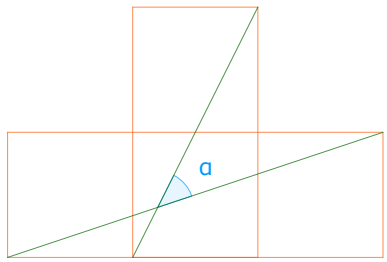
1.4 Problems for Olympiad Level

Problem 1.4.1 (23-24-ET-O-P1)**.** (*10 points*) The figure below consists of four congruent squares. Find the angle α.



Problem 1.4.2 (23-24-ET-O-P2)**.** Find the largest possible positive integer n such that $n+3$ divides $1^3+2^3+\cdots+n^3$.

Hint 1: Prove that if a prime number p divides $n+3$ and $n+3$ divides $1^3+2^3+\cdots+n^3$, then $p \in \{2,3\}$.

Hint 2: Analyze the polynomial division $1^3+2^3+\cdots+n^3$ by $n+3$.

Problem 1.4.3 (23-24-ET-O-P3)**.** (*10 points*) Show that if a 5×5 matrix is filled with zeros and ones, there must always be a 2×2 sub-matrix (that is, the intersection of the union of two rows with the union of two columns) consisting entirely of zeros or entirely of ones.

Hint: prove that if we don't have a 2×2 sub-matrix of zeroes, we must have a 2×2 sub-matrix of ones.

Problem 1.4.4 (23-24-ET-O-P4)**.** (*10 points*) The sequence (a_n) is defined by

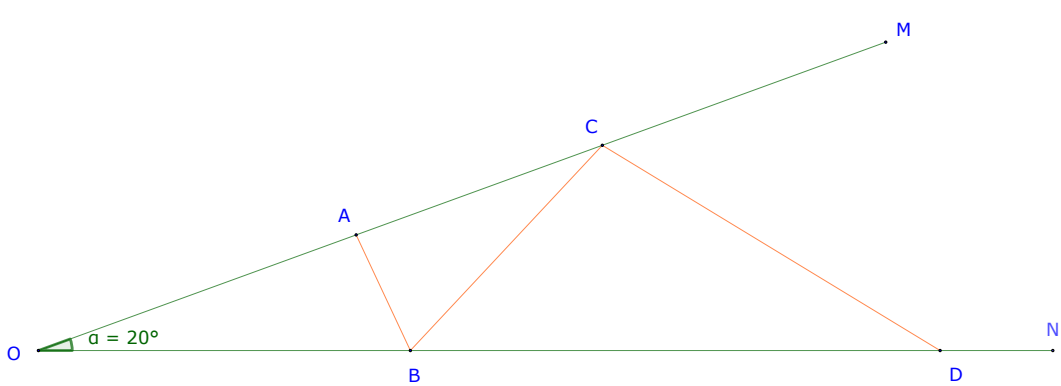
$$a_1 = 1,\ a_n = \sqrt{2a_{n-1}+a_{n-2}+\cdots+a_1},\ \text{if } n>1.$$

Find a_{2023}. *Hint: compute a_2, a_5.*

Problem 1.4.5 (23-24-ET-O-P5)**.** (*10 points*) Points A, C are on ray OM, and B, D are on ray ON. It is given that $OA = 6, OD = 16, \angle NOM = 20°$.

What is the minimum length of $AB + BC + CD$?

Hint: when is the length of a broken line minimal?



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Problem 1.4.6 (23-24-ET-O-P6)**.** (*10 points*) Find all triples of non-negative integers (x, y, z) and $x \leq y$ such that:

$$x^2 + y^2 = 3 \cdot 2016^z + 77.$$

Hint: if $x^2 + y^2 \equiv 0 \pmod 7$, *then what are remainders of* x *and* y *when divided by 7?*

Problem 1.4.7 (23-24-ET-O-P7)**.** (*10 points*) There are 2023 points in the plane such that among any three of them two can be selected so that their distance is less than 1. Prove that there is a circle of radius 1 containing at least 1012 of the given points.

Problem 1.4.8 (23-24-ET-O-P8)**.** (*10 points*) Determine all polynomials $P(x)$ with real coefficients such that

1. $P(2023) = 2022$,
2. $(P(x) + 1)^2 = P(x^2 + 1)$, for all real numbers x.

Investigate the sequence (x_n) *where* $x_1 = 2023$, $x_{n+1} = x_n^2 + 1$.

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Chapter 2

Session 1

2.1 The Pigeonhole Principle (I-Level)

Definition (Pigeonhole Principle)**.** The Pigeonhole Principle (also known as the Dirichlet box principle, Dirichlet principle or box principle) states that if $n+1$ or more holes are placed in n pigeons, then one pigeon must contain two or more holes.

Another definition could be phrased as among any $n+1$ integers, there are two with the same remainder when divided by n.

The extended version of the Pigeonhole Principle states that if k objects are placed in n boxes then at least one box must hold at least $\left\lceil \frac{k}{n} \right\rceil$ objects. Here $\lceil x \rceil$ denotes the least integer larger then or equal to x.

Example 2.1.1 (23-24-S1-I-P1)

Prove that in a group of 13 students, there are at least two students have their birthday in the same month.

Example 2.1.2 (23-24-S1-I-P2)

Prove that in a group of 32 students, there are at least two students have their birthday in the same day in some month.

Example 2.1.3 (23-24-S1-I-P3)

The heights of 27 students in the MCC club were measured to the nearest 5cm. There was a range of heights from 150cm to 180cm. What is the least number of students with one of the heights?

Example 2.1.4 (Mathematical Circle, Russian Experience, C4/Problem 8)

Show that in any group of five people, there are two who have the same number of friends in the group.

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Example 2.1.5 (LPS V2B/2.3)

10 people attended a party. Some people shake hand with others. Any person shakes hand with another only once. Nobody shakes hand with himself.

Prove that there are two people who shake hand with the same number of people.

Example 2.1.6 (Mathematical Circle, Russian Experience, C4/Problem 11)

Ten students solved a total of 35 problems in a math olympiad. Each problem was solved by exactly one student. There is at least one student who solved exactly one problem, at least one student who solved exactly two problems, and at least one student who solved exactly three problem.

Prove that there is at least one student who solved exactly five problem.

Example 2.1.7 (23-24-S1-I-P7)

(*10 points*) Prove that in any set of 52 integers, there exist two numbers, whose sum or whose difference is divisible by 100.

Example 2.1.8 (LPS Strategies and Methods/Exercise 2.17)

Among six persons, there are always three who know each other or three who are complete strangers.

Example 2.1.9 (MTC 2021/Mar/C1)

Several solutions from MCC students were nominated for the Best Solution of the Month award. The evaluation committee has 10 members. Each of them secretly picked the best solution of his or her choice. It is known that out of any 4 committee members, at least 2 voted for the same solution. Prove that there exists a solution that was picked by at least 4 committee members.

Example 2.1.10 (23-24-S1-I-P10)

Prove that for any 5 points chosen within a 2×2 square, there are two whose distance is at most $\sqrt{2}$.

Example 2.1.11 (LPS Strategies and Methods/Problem 2.31)

$3^4 = 8\underline{1}$ is a power of 3 ending in 1. $3^{20} = 3\ 486\ 784\ 4\underline{01}$ is a power of 3 ending in 01. Is there a power of 3 ending in 0001?

Example 2.1.12 (LPS V2B/2.16)

A chess master has 77 days to prepare for a tournament He wants to play at least one game per day, but not more than 132 games. Prove that there is a sequence of successive days on which he plays exactly 21 games.

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2.2 Problems (I-Level)

Problem 2.2.1 (23-24-S1-I-P13). (*10 points*) Thirteen schools took part in an athletics competition in Super Athletic Day. There were 1514 students spectators. Show that there was one school that was cheered on by at least 117 students.

Problem 2.2.2 (23-24-S1-I-P14). (*10 points*) There are 101 points in a unit square. Prove that five of them can be covered by a circle radius $\frac{1}{7}$.

Hint: use 23-24-S1-I-P10 as starting point.

Problem 2.2.3 (23-24-S1-I-P15). (*10 points*) Show that given any set of 17 numbers it is possible to choose 5 numbers whose sum is divisible by 5.

Hint: what are the remainders of those 17 numbers when divided by 5? What happen if among those given numbers there always is a number for each value of the remainders? What happen if there is no number for some remainders, i.e. no given number has remainder 1 when divided by 5?

Problem 2.2.4 (23-24-S1-I-P16). (*10 points*) Seventeen people correspond by mail with one another-each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

Hint: this is an extended version of the example LPS Strategies and Methods/Exercise 2.17.

Problem 2.2.5 (23-24-S1-I-P17). (*10 points*) A student has 37 days to prepare for an exam. From past experience she knows that she will require no more than 60 hours of study. She also wished to study at least one hour per day. Show that no matter how she organizes her study, there is a succession of days during which she studies exactly 13 hours. (Assume she works for a whole number of hours per day.)

Hint: this is an different version of the example LPS V2B/2.16.

Problem 2.2.6 (23-24-S1-I-P18). (*10 points*) Prove that in any set of 27 odd numbers all less than 100, there is a pair of number whose sum is 102.

Problem 2.2.7 (23-24-S1-I-P19). (*10 points*) How many sets of 26 odd numbers all less than 100 can we choose such that no pair in any of these sets gives a sum of 102.

Problem 2.2.8 (23-24-S1-I-P20). (*10 points*) Several football teams enter a tournament in which each team plays every other team exactly once. Show that at any moment during the tournament there will be two teams which have played, up to that moment, an identical number of games.

Problem 2.2.9 (23-24-S1-I-P21). (*10 points*) Prove that there is a set of 51 integers such that, there are no two numbers, whose sum or whose difference is divisible by 100.

2.3 Factoring (A-Level)

Fact (Second-degree Identities)**.**

$$
\begin{aligned}
&a^2 - b^2 = (a+b)(a-b)\\
&(a+b)^2 = a^2 + 2ab + b^2\\
&(a-b)^2 = a^2 - 2ab + b^2\\
&(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca,\\
&a^2 + b^2 + c^2 - ab - bc - ca = \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2}\\
&ab + ac + bd + cd = (b+c)(a+d)
\end{aligned}
$$

Fact (Third-degree Identities)**.**

$$
\begin{aligned}
&a^3 - b^3 = (a-b)(a^2 + ab + b^2)\\
&a^3 + b^3 = (a+b)(a^2 - ab + b^2)\\
&(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\\
&(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\\
&a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)\\
&(a+b+c)^3 - (a^3 + b^3 + c^3) = 3(a+b)(b+c)(c+a)
\end{aligned}
$$

Fact.

$$
\begin{aligned}
&1 + a_1 = 1 + a_1\\
&1 + a_1 + a_2 + a_1a_2 = (1+a_1)(1+a_2)\\
&1 + a_1 + a_2 + a_3 + a_1a_2 + a_2a_3 + a_3a_1 + a_1a_2a_3 = (1+a_1)(1+a_2)(1+a_3)\\
&1 + (a_1 + a_2 + \ldots + a_n) + (a_1a_2 + \ldots + a_{n-1}a_n) + \ldots + a_1a_2\cdots a_n = (1+a_1)(1+a_2)\cdots(1+a_n)
\end{aligned}
$$

Fact (Difference of powers)**.**

$$
\begin{aligned}
&a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})\\
&a^{2n+1} + b^{2n+1} = (a+b)(a^{2n} - a^{2n-1}b + \ldots - ab^{2n-1} + b^{2n})
\end{aligned}
$$

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Example 2.3.1 (23-24-S1-A-P1)
Factor $x^4 - 3x^2y^2 + y^4$.

Example 2.3.2 (23-24-S1-A-P2)
Factor $(x + y)(x - y) - 4(y + 1)$.

Example 2.3.3 (23-24-S1-A-P3)
Factor $4(x^2 + x - y^2) + 1$.

Example 2.3.4 (23-24-S1-A-P4)
Factor $x(x - 4y) + 4(y^2 - 1)$.

Example 2.3.5 (23-24-S1-A-P5)
Factor $x^2 - y^2 + 2(x + 3y - 4)$.

Example 2.3.6 (23-24-S1-A-P6)
Factor $x^3 + 9x^2 + 27x + 19$.

Example 2.3.7 (23-24-S1-A-P7)
Factor $x^3 + 3x^2 + 3x - 7$.

Example 2.3.8 (23-24-S1-A-P8)
Factor $(x - y)(x^2 + xy + y^2 + 3) + 3(x^2 + y^2) + 2$.

Example 2.3.9 (23-24-S1-A-P9)
Factor $a^4 + 4b^4$.

Example 2.3.10 (23-24-S1-A-P10)
Factor $(x^2y + y^2z + z^2x) + (xy^2 + yz^2 + zx^2) + 2xyz$.

Example 2.3.11 (23-24-S1-A-P11)
Factor $(x - y)^3 + (y - z)^3 + (z - x)^3$.

Example 2.3.12 (23-24-S1-A-P12)

Find all pairs (x, y) of integers satisfying

$$\begin{cases} x^2 + 11 = xy + y^4 \\ y^2 - 30 = xy \end{cases}$$

Example 2.3.13 (23-24-S1-A-P13)

Prove that for any positive integer n, $(n+1)^5 + n$ is not a prime.

Example 2.3.14 (23-24-S1-A-P14)

Factor $a^3 + b^3 + c^3 - 3abc$.

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2.4 Problems (A-Level)

Problem 2.4.1 (23-24-S1-A-P15)**.** (*10 points*) Simplify the expression

$$\frac{1}{x-1} - \frac{3}{x^3-1}.$$

Problem 2.4.2 (23-24-S1-A-P16)**.** (*10 points*) Simplify the expression

$$\frac{x^4 - 2x^3 - x^2 + 2x + 1}{x^4 - 2x^3 + x^2 - 1}.$$

Hint: factor the denominator, then see of the numerator has such a factor

Problem 2.4.3 (23-24-S1-A-P17)**.** (*10 points*) Factoring the expression

$$2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4).$$

Problem 2.4.4 (23-24-S1-A-P18)**.** (*10 points*) Factoring the expression

$$(x-a)^3(b-c)^3 + (x-b)^3(c-a)^3 + (x-c)^3(a-b)^3.$$

Hint: the given expression is very much like $a^3 + b^3 + c^3$

Problem 2.4.5 (23-24-S1-A-P19)**.** Let a, b, c be distinct real numbers such that

$$a^2(1-b+c) + b^2(1-c+a) + c^2(1-a+b) = ab + bc + ca.$$

(*5 points*) Prove that

$$(a-b)(b-c)(c-a) = (b-c)(c-a) + (c-a)(a-b) + (a-b)(b-c)$$

(*5 points*) Prove that

$$\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} = 1$$

Problem 2.4.6 (23-24-S1-A-P20)**.** (*10 points*) Prove that

$$2012 \cdot 503^{2011} + 2013^4$$

is a composite number.

Problem 2.4.7 (23-24-S1-A-P21)**.** (*10 points*) Prove that for all positive integers m, n,

$$8m^6 + 27m^3n^3 + 27n^6$$

is a composite number.

Hint: $8m^6 = (2m^2)^3$, $27n^6 = (3n^2)^3$.

Problem 2.4.8 (23-24-S1-A-P21)**.** (*10 points*) Solve the system of equations

$$\begin{cases} x + y = 2z \\ x^3 + y^3 = 2z^3 \end{cases}$$

Hint: $x + y$ is a factor of $x^3 + y^3$

Problem 2.4.9 (23-24-S1-A-P22)**.** (*10 points*) If a, b, c are positive real numbers such that $a + b + c = 1$. Prove that

$$a^3 + b^3 + c^3 - 1 = 3(abc - ab - bc - ca)$$

Problem 2.4.10 (23-24-S1-A-P23)**.** a, b, c pairwise distinct real numbers.

(*5 points*) Prove that

$$\frac{bc}{(a-b)(a-c)} + \frac{ca}{(b-c)(b-a)} + \frac{ab}{(c-a)(c-b)} = 1$$

(*5 points*) Prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} \geq 2.$$

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2.5 Inequalities (O-Level)

We use the principle *sum-of-squares is non-negative* to prove the following inequalities.

Example 2.5.1 (23-24-S1-O-P1)
For all real numbers a, b, c, show that

$$a^2 + b^2 + c^2 + 3 \geq 2(a + b + c).$$

Example 2.5.2 (23-24-S1-O-P2)
For all real numbers x, y, z, show that

$$19x^2 + 54y^2 + 16z^2 + 36xy - 24yz - 16zx \geq 0.$$

Example 2.5.3 (23-24-S1-O-P3)
For all real numbers a, b, c, show that

$$\sqrt{a^2 + ab + b^2} + \sqrt{b^2 + bc + c^2} + \sqrt{c^2 + ca + a^2} \geq \sqrt{3}(a + b + c).$$

Example 2.5.4 (23-24-S1-O-P4)
For all x, y real numbers such that $xy \geq 1$, show that

$$\frac{1}{x^2 + 1} + \frac{1}{y^2 + 1} \geq \frac{2}{xy + 1},$$

We prove the following inequalities with *special identities*:

Example 2.5.5 (23-24-S1-O-P5)
For all real numbers $a \geq b \geq c$, show that

$$a^2b + b^2c + c^2a \geq ab^2 + bc^2 + ca^2.$$

Example 2.5.6 (23-24-S1-O-P6)
For all positive real numbers a, b, c, show that

$$(a + b)(b + c)(c + a) \geq \frac{8}{9}(a + b + c)(ab + bc + ca).$$

Example 2.5.7 (23-24-S1-O-P7)
For a, b, c real numbers such that $a + b \neq 0$, $b + c \neq 0$, and $c + a \neq 0$, show that

$$\frac{a^5 + b^5 + c^5 - (a + b + c)^5}{a^3 + b^3 + c^3 - (a + b + c)^3} \geq \frac{10}{9}(a + b + c)^2.$$

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We prove the following inequalities with *comparison method*.

Example 2.5.8 (23-24-S1-O-P8)
For a, b, c positive real number, show that

$$a^{2a}b^{2b}c^{2c} \geq a^{b+c}b^{c+a}c^{a+b}.$$

Example 2.5.9 (23-24-S1-O-P9)
For a, b, c nonnegative real numbers, show that

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca).$$

Example 2.5.10 (23-24-S1-O-P10)
If a, b, c are positive real numbers and $a^2 + b^2 + c^2 = 1$, find the minimal value of

$$S = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{2(a^3 + b^3 + c^3)}{abc}$$

We prove the following inequalities with the *Pigeonhole Principle*.

Example 2.5.11 (23-24-S1-O-P11)
For nonnegative real numbers x, y, z, show that

$$2 + 3\sqrt{3}xyz \geq 3(xy + yz + zx).$$

Example 2.5.12 (23-24-S1-O-P12)
For positive real numbers a, b, c, let $x = a + \frac{1}{b}$, $y = b + \frac{1}{c}$, and $z = c + \frac{1}{a}$, prove that

$$xy + yz + zx \geq 2(x + y + z).$$

We prove the following inequalities with the principle of *majorization and minorization*:

Example 2.5.13 (23-24-S1-O-P13)
For positive real numbers a, b, c, show that

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{1}{abc}$$

Example 2.5.14 (23-24-S1-O-P14)
For positive real numbers a, b, c, such that $a + b + c = 3$, show that

$$1 + 8abc \geq 9 \min\{a, b, c\}$$

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2.6 Problems (O-Level)

Problem 2.6.1 (23-24-S1-O-P15)**.** (*10 points*) For a, b, c positive real numbers, prove that:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Hint: note that

$$\frac{a}{b+c} - \frac{1}{2} = \frac{(a-b)+(a-c)}{2(b+c)}$$

Definition (Convex function)**.** The function $f: [a,b] \to \mathbb{R}$ is *convex* on the interval $[a,b]$ if for any $x, y \in [a,b]$ and any $\alpha \in (0,1)$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y).$$

Theorem (Second derivative of a convex function)

Suppose that the function $f: (a,b) \to \mathbb{R}$ for every $x \in [a,b]$, there exists a second derivative $f''(x)$. f is *convex* on the interval $[a,b]$ if and only if for any $x \in [a,b] f''(x) \geq 0$.

Theorem (A linear combination of convex functions is also convex)

The functions $f_1, f_2, \ldots, f_n$ are *convex* on the interval (a,b). Then the function $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$ is *convex* on the interval $[a,b]$, for any $c_1, c_2, \ldots, c_n \in (0, \infty)$.

Theorem (Jensen's Inequality)

Let f be a convex function of one real variable. Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and let $a_1, \ldots, a_n \geq 0$ satisfy $a_1 + a_2 + \cdots + a_n = 1$. Then

$$f(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) \leq a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n).$$

Problem 2.6.2 (23-24-S1-O-P16)**.** (*10 points*) For a, b, c positive real numbers, prove that:

$$ab\left(\frac{b}{c} - 1\right) + bc\left(\frac{c}{a} - 1\right) + ca\left(\frac{a}{b} - 1\right) \geq 0.$$

Hint: investigate

$$\frac{ab^2}{c} + ca - 2ab$$

Problem 2.6.3 (23-24-S1-O-P17)**.** (*10 points*) For x, y positive real numbers such that $x^2 + y^3 \geq x^3 + y^4$. Prove that: $x^3 + y^3 \leq x^2 + y^2 \leq x + y \leq 2$.

Problem 2.6.4 (23-24-S1-O-P18)**.** (*10 points*) Prove that if $a \geq b \geq c$ such that $a + b + c \geq 0$. then

$$a^3 b + b^3 c + c^3 a \geq ab^3 + bc^3 + ca^3. \text{ Does the inequality still stand if } a \leq b \leq c?$$

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Problem 2.6.5 (23-24-S1-O-P19)**.** (*10 points*) For x, y, z positive real number such that $x^2 + y^2 + z^2 = 1$, find the minimal value of

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}.$$

Problem 2.6.6 (23-24-S1-O-P20)**.** (*10 points*) Let $0 < x < y < z < t$, such that $x + t = y + z$. Show that for any positive integer $n \geq 2$:

$$x^n + t^n > y^n + z^n.$$

Problem 2.6.7 (23-24-S1-O-P21)**.** (*10 points*) Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2 = 3$. Prove that

$$ab^2 + bc^2 + ca^2 \leq abc + 2.$$

Problem 2.6.8 (23-24-S1-O-P22)**.** (*10 points*) Let a, b, c be positive real numbers. Prove that

$$\frac{a(a-b)}{b+c} + \frac{b(b-c)}{c+a} + \frac{c(c-a)}{a+b} \geq 0.$$

Problem 2.6.9 (23-24-S1-O-P23)**.** (*10 points*) Let x, y, z be nonnegative real numbers such that

$$x^2 + y^2 + z^2 \leq 3.$$

Prove that

$$xy + yz + xz - xyz \leq 2.$$

Problem 2.6.10 (23-24-S1-O-P24)**.** (*10 points*) Let x, y, z be nonnegative real numbers such that $x+y+z = 3$. Prove that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq 3.$$

Problem 2.6.11 (23-24-S1-O-P25)**.** (*10 points*) Let a, b, c be positive real numbers such that $a \leq b \leq c$, $ab + bc + ca = 3$. Prove that

$$S = \frac{1}{2a+b} + \frac{1}{2b+c} + \frac{1}{2c+a} \geq 1$$

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Chapter 3

Session 2

3.1 Logic puzzles (I-Level)

Example 3.1.1 (23-24-S2-I-P1)

A dog and cat together weigh 27 pounds. If the dog's weight is an odd number, and if he weighs twice as much as she does, how much does each weigh?

Example 3.1.2 (23-24-S2-I-P2)

Detective James Smarties walked outside through a pouring rain for half an hour without getting a single hair on his head wet. He did not wear a hat, carry an umbrella, or hold anything over his head. His clothes got really soaked. How could this be possible?

Example 3.1.3 (23-24-S2-I-P3)

When their blind dates arrived to take them to a football game, Connie and Jennie was astonished to see that the two young men looked exactly alike. One of them explained "we are brothers," he said, "we were born on the same day of the same year from the same parents". He added "but we are not twins". How could it be?

Example 3.1.4 (23-24-S2-I-P4)

How many people is "two pairs of twins twice"?

Example 3.1.5 (23-24-S2-I-P5)

A big fat Indian and a small thin Indian were sitting outside a teepee, each smoking a pipe. The little Indian was the son of the big Indian, but the big Indian was not the little Indian's father. How come?

Example 3.1.6 (23-24-S2-I-P6)

A visitor points to a portrait on the wall and asks who it is. "Brothers and sisters have I none," says the host, "but that man's father is my father' son." Who is in the picture?

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Example 3.1.7 (23-24-S2-I-P7)

Bình was driving through Ottawa when he decided to rest. He stopped at a motel, got a room, and asked the reception girl where he can find a barbershop. "We have two barbers nearby in Ogilvie shopping malls," said the girl. "One of them is at the north entrance and the other close to the south exit."

Bình went to the one next to the north entrance. It looked like as if it had not been cleaned in months. Cut hair was all over the floor. The barber himself would have needed a shave and his haircut looked really terrible. Bình then walked to the other one. It looked neat and cheerful inside. The floor had been swept. The barber was neatly dressed, freshly shaved, and had a neat haircut.

Why did Bình walk back to the first barbershop to get his haircut?

Example 3.1.8 (23-24-S2-I-P8)

A certain small island is inhabited by rabbits of three kinds: white, grey, and spotted.

Grey mother-rabbits always have two bunnies in their litters: a white and a grey one.

The same is true for spotted mother-rabbits.

However, a white rabbit always give birth to three bunnies: a grey, a white, and a spotted one.

The first and only rabbit to ever reach the island was a rabbit who is called Great Mother by the rest of the population. Currently, 40 grey, 40 white, and 30 spotted rabbits live on the island.

If no rabbit ever died or left the island, can you tell the colour of the Great Mother?

Example 3.1.9 (23-24-S2-I-P9)

While visiting the Knights (who always tell the truth) and Liars (who always lie) Island, Huy passes a beautiful garden where three islanders, Samuel, Tony, and Zecharia are watching the sunset.

He asks Samuel, "are you a knight or a liar?". Samuel is shy, Huy cannot hear his quiet answer.

So he ask Tony, "what did Samuel say?" Tony answers, "he said that he is a liar."

"Don't trust Tony, Tony is a liar!" screams Zecharia.

What do you think, who is what?

Example 3.1.10 (23-24-S2-I-P10)

The inhabitants of Truth Village always tell the truth, while those of the hamlet of Liars always lie. On the way to visit the Truth Village, you arrive at a T-intersection, where one road leads to the Truth Village, the other one to the hamlet of Liars. There is no sign, except a single person.

You have to ask but you do not know whether the man is a citizen of the hamlet of Liars or from the Truth Village. You are allowed to ask a single question, to learn how to get to the Truth Village.

What is it?

Example 3.1.11 (23-24-S2-I-P11)

The people from the city No-one-can-be-sure-of-any-thing are of three different types: those who always tell the truth, those who always lie, and those you alternate lie and tell the truth.

Quynh Chi meets one of them. She is allowed to ask twice so that she can know which group of people that one belongs to.

What are the questions?

Example 3.1.12 (23-24-S2-I-P12)

On the island of Knights and Liars, knights and liars live in separate villages. However they often visit their friends in other villages.

This week, the one and only newspaper of the island announced a contest: to come up with a single yes/no question that would allow a tourist to figure out whether he is in a knights' village or a liars' village. The question should be such that it can be addressed to anyone the tourist sees. Of course it is not known in advance that that local person is a knight or a liar, is he from the village or just a visitor. Lilian won the prize.

What was her question?

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3.2 Problems (I-Level)

Problem 3.2.1 (23-24-S2-I-P13)**.** (*10 points*) Khang plays a game of Ping-Pong in the back yard? He misses the ball. It bounces across the lawn and rolls into a small but deep hole. The hole goes down too far for him to reach the ball with his hand, and the hole bends so much to one side that he cannot get the ball by poking a stick into the hole.

How can Khang get the ball?

Problem 3.2.2 (23-24-S2-I-P14)**.** (*10 points*) Four students of the Math & Chess Club having a conversation about their favourite food.

1. Boy A: Jennie likes fish
2. Boy B: I likes cakes, but I am not called Anthony
3. Girl C: One of the boys - Anthony and Khang - like chicken
4. Girl D: Barbie likes salad, but I don't

Can you figure out who's who and who likes what?

Problem 3.2.3 (23-24-S2-I-P15)**.** (*10 points*) Catherine and Clemence got a nice cake from their mother. They were squabbling over who gets to cut the cake. The reason was that each of them thinks the other will cut it in a way to give herself the larger portion. Seeing this, their dad recommended a clever idea and both of them immediately agreed.

What would be the idea?

Problem 3.2.4 (23-24-S2-I-P16)**.** (*10 points*) Cindy has two treasure chests, each of them filled either with gold or silver coins. One of them has a label, "I contains gold coins, and the other one contains silver". The other is labelled with "One of us contains gold, while another contains silver". Cindy told Andy that one of the labels is correct and the other is not.

Where' the gold?

Problem 3.2.5 (23-24-S2-I-P17)**.** (*10 points*) In the Forest of Blue fairies and Red hoodies, there are two types of people: the Blue fairies who ask only questions whose correct answer is yes; the Red hoodies who ask only questions whose correct answer is no.

For example, a Blue fairy could ask, "Does two plus two equal four?", but she could not ask whether two plus two equals five. A Red hoodie could not ask whether two plus two equals four, but could ask whether two plus two equals five.

While visiting the Forest of Blue fairies and Red hoodies, Amy met two sisters named Catherine and Clemence. Catherine asked Clemence: "Is at least one of us a Red hoodie?"

What type of people are Catherine and Clemence?

Problem 3.2.6 (23-24-S2-I-P18)**.** (*10 points*) There are two types of people on the Island of Knights and Liars: the Knights who always tells the truth, and the Liars, who always lie. Detective Samuel came to the island to investigate a crime. He knew that:

- One of the identical twins was named Donald and he was the one who committed the crime.
- It is know that at least one of them is a Liar, but it is not known which one.

Are you Donald? Detective Samuel asked the first twin. *Yes, I am,* was the reply. *Are you Donald?* Detective Samuel asked the second twin. The second twin then answered either yes or no, and the detective knew that which one was Donald.

Was Donald the first or second twin?

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Problem 3.2.7 (23-24-S2-I-P19)**.** (*10 points*) Tiger always tells the truth on Monday, Tuesday, Wednesday, and Thursday. Lion always tells the truth on Monday, Friday, Saturday, and Sunday. On the rest of the days, they may tell the truth or they may lie. On one day last week, both of them said *"I lied yesterday."*.

On what day of the week did they make those statements?

Problem 3.2.8 (23-24-S2-I-P20)**.** (*10 points*) Hassan, the horse-trader, challenged three buyers to guess correctly the colour of his best horse. The horse's colour was either black, brown, or gray.

- Helena: The horse is not black.
- Gabriella: The horse is either gray or brown.
- Isabella: The horse is brown.

Hassan said that at least one of the buyers guessed right and at least one guessed wrong. Who were correct and who were not?

Problem 3.2.9 (23-24-S2-I-P21)**.** (*10 points*) In Seville, a barber shaved all of and only those inhabitants who did not shave themselves. Sam met four local people Alain, Bernard, Columbus, and Daniel. Here were what they said:

- Alain: I am the only barber in this town.
- Bernard: There is only one barber in this town.
- Columbus: One of them is lying. I shaved both of them.
- Daniel: Alain is lying.

If exactly two of them telling the truth, then who were shaved?

Problem 3.2.10 (23-24-S2-I-P22)**.** (*10 points*) In the Kingdom of Might and Magic, all inhabitants are either paladins or illusionists. For a *paladin*, everything he believes while he is awake is true, and everything he believes while he is asleep is false. An *illusionist* is the opposite: everything he believes while asleep is true, and everything he believes while awake is false.

Minh believes that he and his brother Khoa are both illusionists. At the same time, Khoa believes that both are paladins. As that moment, one of them was awake the other was asleep at the time.

Determine what each of them was at that moment.

Problem 3.2.11 (23-24-S2-I-P23)**.** (*10 points*) Cuc, Dao, Lan, and Mai were hanging around in the kitchen while their mother were making some cookies. When they disappeared, their mother found out that some of cookies did also vanish. In the impromptu court held by their father, the following were discovered:

1. If both Cuc and Dao stole some cookies, then Lan were an accomplice.
2. If Cuc is guilty, then at least one of Dao and Lan was an accomplice.
3. If Lan is a cookie-thief, then Mai was an accomplice.
4. If Cuc is innocent, then Mai is indeed a cookie-stealer.

Which ones of them are definitely guilty and which ones are doubtful (nothing can be said about them)?

Problem 3.2.12 (23-24-S2-I-P24)**.** (*10 points*) A school teacher in Ottawa had her purse stolen. The suspects were narrowed down to: Loo, Joo, Doo, Too, or Moo. When questioning, each child made three statements (numbering as below for ease of reference):

1. Loo: (1) I didn't take the purse; (2) I have never in my life stolen anything; (3) Too did it.

2. Joo: (4) I didn't take the purse; (5) My daddy is rich enough, and I have a purse of my own; (6) Moo knows who did it.
3. Doo: (7) I didn't take the purse; (8) I didn't know Moo before I enrolled in this school; (9) Too did it.
4. Too: (10) I am not guilty; (11) Moo did it; (12) Loo is lying when she says I stole the purse.
5. Moo: (13) I didn't take the purse; (14) Joo is guilty; (15) Doo can vouch for me because he knows me since I was born

Later, each child admitted that two of his/her statements were true and one was false.

How did Detective James Smarties solve this case?

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3.3 Counting revisited (A-Level)

Example 3.3.1 (23-24-S2-A-P1)

Squares with sides 1 centimeter long are arranged as below. Each row contains one more square than the row above it.

How many centimeters are in the perimeter of the figure that has 10 rows?

How many centimetres are in the perimeter of the figure is contains exactly 210 squares?

Example 3.3.2 (23-24-S2-A-P2)

How many ways to write the number 3003 as a sum of two or more consecutive positive integers?

Example 3.3.3 (23-24-S2-A-P3)

What is the number that will be listed beneath the number 121 when this triangular array is continue in the same manner? Where will the number 2023 appear?

		1		
	2	3	4	
5	6	7	8	9

Example 3.3.4 (23-24-S2-A-P4)

The sequence shown below was formed by writing the first letter of the alphabet followed by writing the first two letters of the alphabet and continuing the pattern by writing one more letter of the alphabet each time.

$$A,\ A,\ B,\ A,\ B,\ C,\ A,\ B,\ C,\ D,\ A,\ B,\ C,\ D,\ E,\ \ldots$$

Continuing this pattern, what letter is the 280th letter in this sequence? What position is the letter Z? (the alphabet contains 26 letters).

Example 3.3.5 (23-24-S2-A-P5)

There are 10 street lamps labeled $1, 2, 3, \ldots, 10$. Each is in front of a house in a row of houses. All of them are turned on. How many ways are there to turn off three lamps such that no two or three adjacent lamps will be turned off and no any lamp in each end will be turned off?

Example 3.3.6 (23-24-S2-A-P6)

A stick 100 cm long needs to be cut into 100 shorter sticks, each is 1 cm long. What is the minimum number of cuts required if Dinh Viet are allowed to cut several stick pieces at the same time?

Example 3.3.7 (23-24-S2-A-P7)

How many ways to sit seven people at a round table, if Lilian and Viet must not sit in adjacent seats?

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Example 3.3.8 (23-24-S2-A-P8)

As a scientist at Innovative Foods Canada, Vu has almost completed the invention of the strong-like-a-Canadian egg, i.e. eggs that don't break easily. He has been asked to demonstrate for his boss, Huy, whatever he has got at the moment.

Vu wants to determine the highest floor in their 30-floor building from which an egg can fall without breaking. From previous experiments it is known that an egg still breaks: (i) if it is dropped from a certain height; or (ii) if it is dropped the times, regardless from which height.

So far, the project has only produced three eggs, so he drop some eggs to determine the floor and keeps at least one for demonstration?

What is the highest floor that Vu can **confidently** drop the last one for demonstration?

Example 3.3.9 (23-24-S2-A-P9)

After collecting for the candy drive, Sally has eight \$1-bills, five \$5-bills, and three \$10-bills. She randomly selected three bills without replacement.

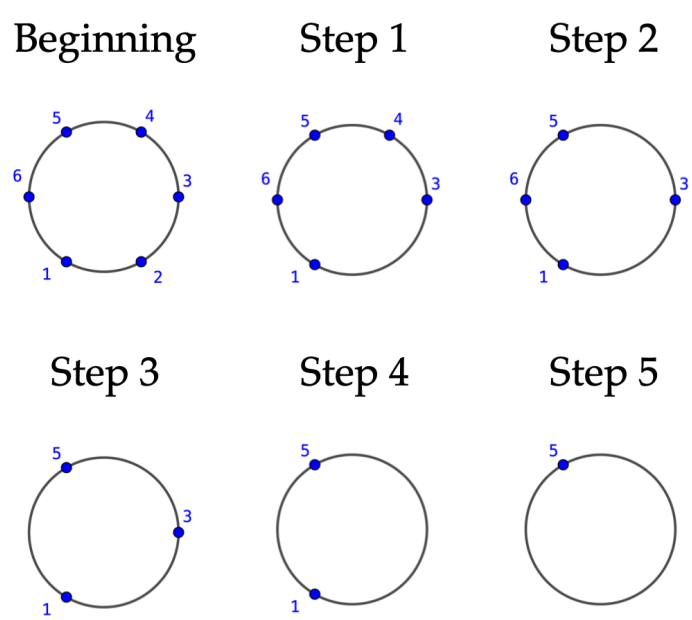
What is the total number of ways to select 3 bills, one of each type, from all the bills she has?

What is the probability that she chooses one of each type of bill? Express your answer as a common fraction.

Example 3.3.10 (23-24-S2-A-P10)

A group of n persons are standing in a circle, numbered consecutively counter-clockwise from 1 to n. Starting with the person number 2, in every step we remove every other person, proceeding counter-clockwise. For example with $n = 6$, the last person who is remaining is the person number 5 and the last person removed is the person number 1. See below.

If $n = 67$, who is the last person removed?



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3.4 Problems (A-Level)

Problem 3.4.1 (23-24-S2-A-P11)**.** (*10 points*) How many ways to select in three single draws without replacement two red marbles and one blue marble from a bag containing six red marbles, eight yellow marbles, and seven blue marbles?

Problem 3.4.2 (23-24-S2-A-P12)**.** (*10 points*) What is the probability to select in three single draws without replacement two red marbles and one blue marble from a bag containing six red marbles, eight yellow marbles, and seven blue marbles?

Problem 3.4.3 (23-24-S2-A-P13)**.** (*10 points*) Catherine participated in a game of Snap. Here's how the game is played.

The house and each of fifteen players have shuffled a deck of 17 cards numbered from 1 to 17. After that everyone turned their cards over, one by one. When a player's card matched the house's one, the player shouted "Snap!" and scored a point. At the end, each player counted the number of points and that was the final score of the player.

It turned out that no two players have the same score and no player score 0 or 17 points. Catherine's score was the second highest.

What's her score?

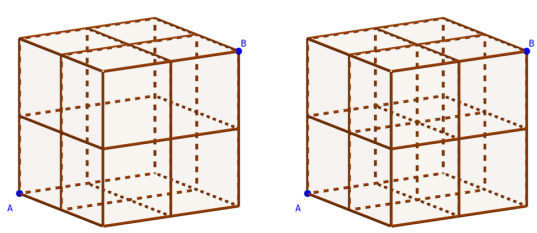
Problem 3.4.4 (23-24-S2-A-P14)**.** (*10 points*) 10 distinct points are marked on a circle.

What are the maximum number of distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices?

What are the maximum possible number of intersections of diagonals formed by those vertices?

Problem 3.4.5 (23-24-S2-A-P15)**.** (*10 points*) In an up-and-down counting number, the digits increase to a maximum digit and then decrease. This maximum is not the first or the last digit. For example 256850 is an up-and-dow, but 255876 is not. How many different 4-digit up-and-dow numbers have the maximum digit of 6?

Problem 3.4.6 (23-24-S2-A-P16)**.** (*10 points*) A cube is built from 8 unit cubes as below in the left diagram. How many paths of length 6 on the surface of the cube are there from to along the edges of the component unit cubes? Now all paths through the inner of the cube are also allowed as below in the right diagram. How many paths of length 6 of the cube are there from to along the edges of the component unit cubes?



Problem 3.4.7 (23-24-S2-A-P17)**.** (*10 points*) For $n > 1$, consider an $n \times n$ chessboard and place pieces at the centres of different squares. With $2n$ chess pieces on the board, show that there are 4 pieces among them that form the vertices of a parallelogram.

Show that there is a way to place $2n - 1$ chess pieces so that no 4 of them form the vertices of a parallelogram.

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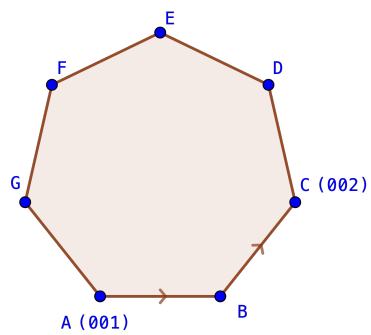
Problem 3.4.8 (23-24-S2-A-P18)**.** (*10 points*) Binh, Anthony, and Vu participated in a beauty contest. The contest committee had a number of judges. Each judge voted for only one of the three participants.

A spectator called Minh, counted 59 judges, with 15 judges voting for Binh or Anthony, 18 voting for Anthony or Vu, and 20 voting for Vu or Binh. However Minh was notorious for poor counting, each number he counted in above differs from reality by no more than 13.

How many judges voted for Anthony?

Problem 3.4.9 (23-24-S2-A-P19)**.** (*10 points*) The director of a secret service has put together a list of mutual surveillance assignments for his seven agents, codename 001 through 007. Agent 001 will watch the agent who is watching 002; agent 002 will watch the agent who watches 003; and so on. Agent 007 will watch the agent watching 001. Can you deduce who watches whom?

The service recruits a new agent, and gives the agent codename 008. How could the director change the surveillance assignments to keep it working for all 8 agents?



Problem 3.4.10 (23-24-S2-A-P20)**.** (*10 points*) Kevin has developed a taste for cookie. He has been collecting cookies to celebrate the coming Christmas. He has 15 cookie jars containing $1, 2, 3, \ldots, 15$ cookies, respectively.

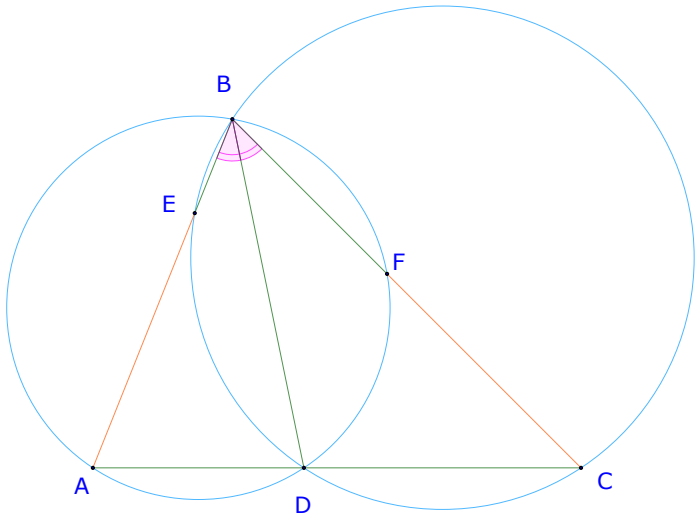
On a cold dark night in December, the cookie monster came into the house. He decided to choose a subset of jars (some number of jars) and take the same number of cookies from each jar and then left the house. This hideous act was repeated for another three nights before Christmas.

Is it possible that the heartless monster ate all the cookies that Kevin has spent great effort to collect leaving him devastated on the Christmas Eve?

3.5 Geometry warmup (O-Level)

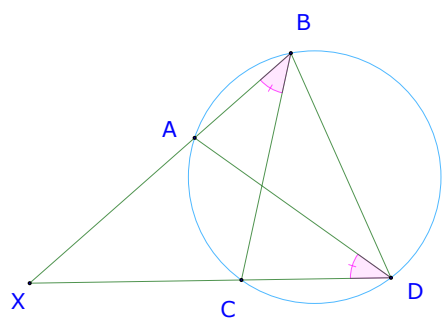
Example 3.5.1 (St. Petersburg City MO 1996)

Let BD be the angle bisector of $\angle ABC$ in $\triangle ABC$ with D on AC. The circumcircles of $\triangle BDC$ and $\triangle ABD$ meet AB and BC at E and F, respectively/. Prove that $AE = CF$.



Theorem (Intersecting Secants Theorem)

Let AB and CD be two lines that intersect at X, such that A and C lie between X, B and X, D, respectively. Then the quadrilateral $ABDC$ is cyclic if and only if $XA \cdot XB = XC \cdot XD$.



Intersecting Secants Theorem. Note that $\angle AXD \equiv \angle CXB$ (both angle are the same), thus

$$ABCD \text{ is cyclic } \Leftrightarrow \angle XBC = \angle XDA \Leftrightarrow \triangle XBC \sim \triangle XDA \Leftrightarrow \frac{XB}{XC} = \frac{XD}{XA} \Leftrightarrow XA \cdot XB = XC \cdot XD.$$

□

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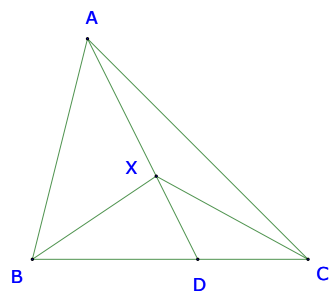
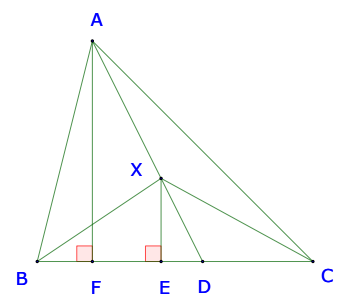
Example 3.5.2 (23-24-S2-O-P2)

Let P be an arbitrary point inside $\triangle ABC$. Let A_1, B_1, and C_1 be the feet of the perpendiculars from P to BC, CA, and AB, respectively. Prove that the lines that pass through the midpoints of PA and B_1C_1, PB and C_1A_1, and PC and A_1B_1 are concurrent.

Lemma (Area Lemma)

ABC is a triangle, point $D \in BC$, $X \in AD$, then $\frac{[BCX]}{[BCA]} = \frac{DX}{DA}$.

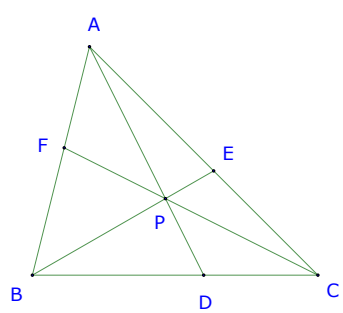




Theorem (van Aubel's Theorem)

In $\triangle ABC$, $D \in BC$, $E \in CA$, $F \in AB$, such that AD, BE, and CF are concurrent at P, prove that:

$$\frac{AP}{PD} = \frac{AE}{EC} + \frac{AF}{FB}.$$



Example 3.5.3 (23-24-S2-O-P3)

Let a, b, c be the side lengths of $\triangle ABC$ and I be its incentre. Let point $D = AI \cap BC$. Prove that:

$$\frac{AI}{ID} = \frac{b+c}{a}, \quad \frac{ID}{AD} = \frac{a}{a+b+c}$$

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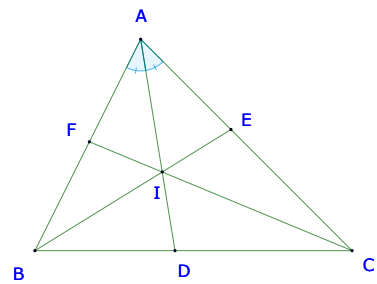
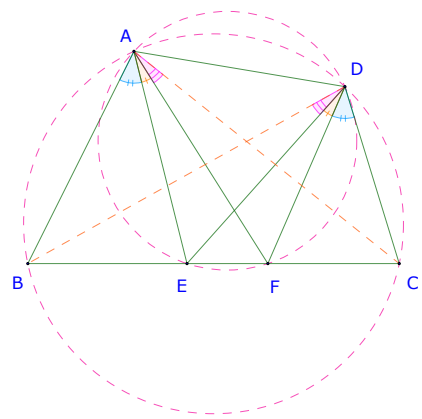


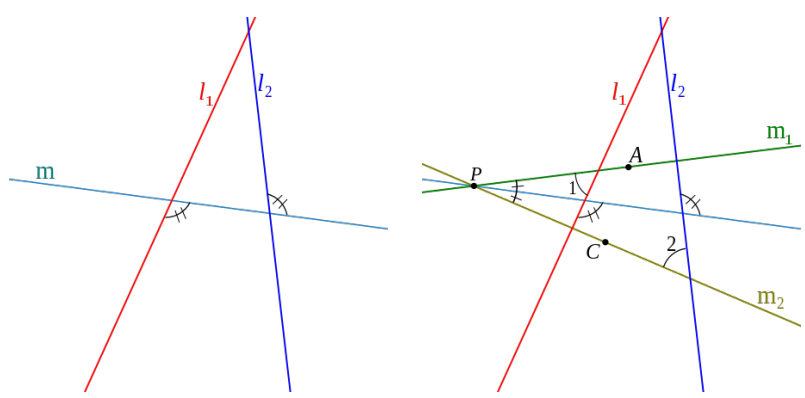
Example 3.5.4 (All-Russian Olympiad 1996, Grade 10/Problem 1)

Points E and F are given on side BC of convex quadrilateral $ABCD$ (with $BE < BF$). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.



Definition (Antiparallel lines)**.** In geometry, two lines ℓ_1 and ℓ_2 are antiparallel with respect to a given line m if they each make congruent angles with m in opposite senses (see the diagram below on the left.)

More generally, lines ℓ_1 and $\ell_2$are antiparallel with respect to another pair of lines m_1 and m_2 if they are antiparallel with respect to the angle bisector of m_1 and m_2 (see the diagram below on the right.)



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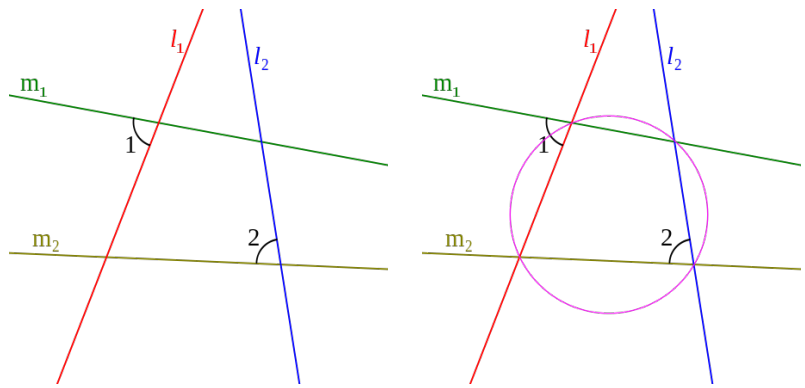
Theorem (Antiparallel sides in cyclic quadrilaterals)

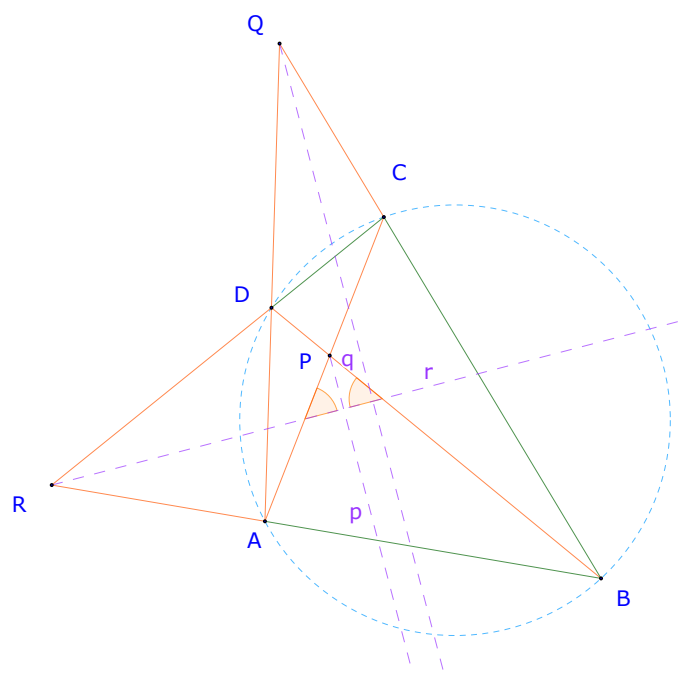
In any cyclic quadrilateral, any two opposite sides are antiparallel with respect to the other two sides.

Given two lines m_1 and m_2, lines ℓ_1 and ℓ_2 are antiparallel with respect to m_1 and m_2 if $\angle 1 = \angle 2$ (see the diagram below on the left.) In any quadrilateral inscribed in a circle, any two opposite sides are antiparallel with respect to the other two sides. (see the diagram below on the right.)



Example 3.5.5 (23-24-S2-O-P5)

In a cyclic quadrilateral $ABCD$, let $P = AC \cap BD$, $Q = AD \cap BC$, and $R = AB \cap CD$. Denote p, q, r the angle bisector of $\angle APB$, $\angle AQB$, $\angle BRC$, respectively. Prove that r is perpendicular to both p, q.



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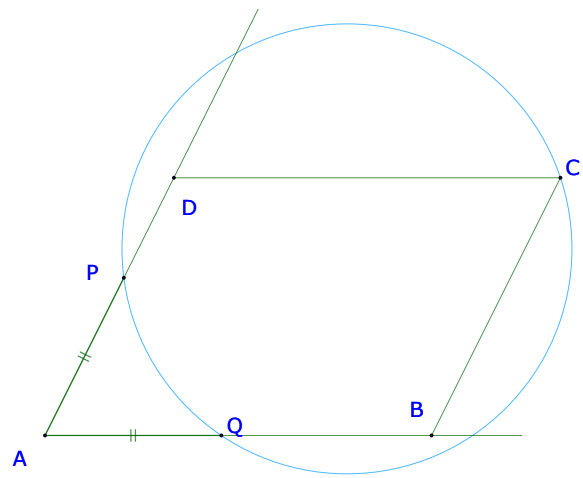
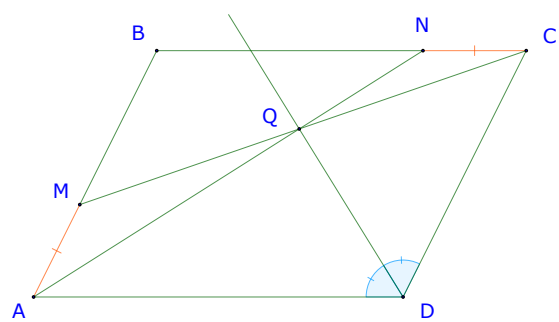
Example 3.5.6 (23-24-S2-O-P6)

(*10 points*) Point P, Q are given on the sides AB, AD respectively, of a parallelogram $ABCD$ ($AB > AD$) such that $AP = AQ = x$. Prove that as x varies, the circumcircles of the triangles PQC pass through another fixed point (other than C).



Example 3.5.7 (Bundeswettbewerb Mathematik 2003/Problem 3)

Let $ABCD$ be a parallelogram. Let M and N be points on the sides AB and BC, respectively, such that $AM = CN \neq 0$. The lines AN and CM intersect at a point Q. Prove that the point Q lies on the bisector of the angle $\measuredangle ADC$.



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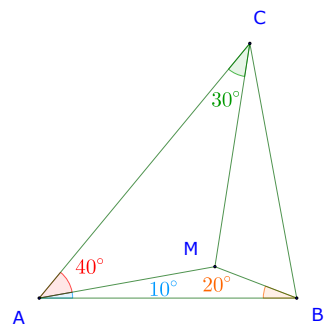
Example 3.5.8 (USAMO 1996/Problem 5)

Let ABC be a triangle, and M an interior point such that $\angle MAB = 10°$, $\angle MBA = 20°$, $\angle MAC = 40°$ and $\angle MCA = 30°$. Prove that the triangle is isosceles.



Theorem (Ceva Theorem in Trigonometric Form)

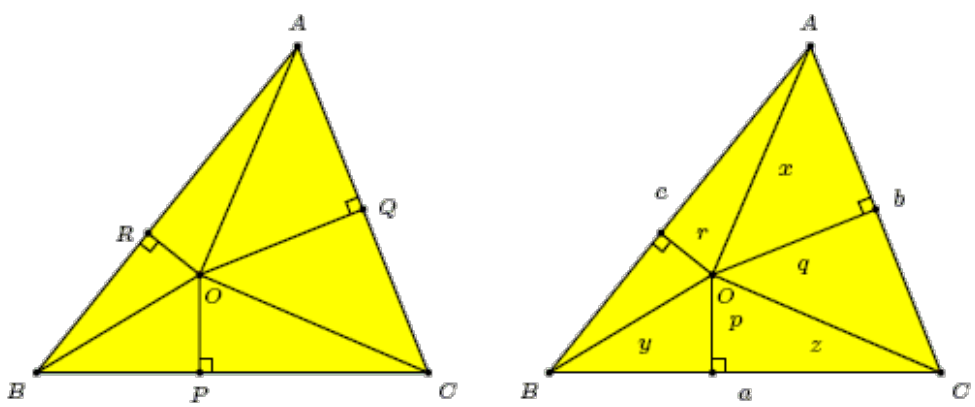
Cevians AD, BE, CF concur if and only if

$$\frac{\sin\angle BAD}{\sin\angle DAC} \cdot \frac{\sin\angle CBE}{\sin\angle EBA} \cdot \frac{\sin\angle ACF}{\sin\angle FCB} = 1.$$

Example 3.5.9 (Erdos-Mordell's Theorem)

Let O be the point inside the triangle ABC. Denote by P, Q, R the feet of perpendiculars from O to BC, CA, and AB. Prove that

$$OA + OB + OC \geq 2(OP + OQ + OQ).$$



Lemma

For the quantities x, y, z, p, q, r in $\triangle ABC$, shown as above in the right diagram, we have

$$ax \geq br + cq,\ \ by \geq ar + cp,\ \ cz \geq aq + bp.$$

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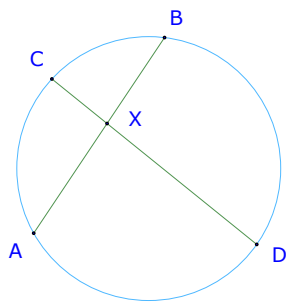
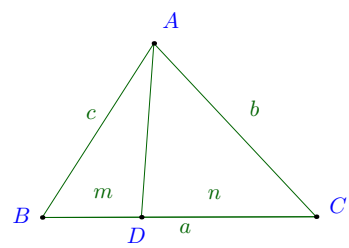
Theorem (Intersecting Chord Theorem)

Let AB and CD be two segments that intersect at X, such that A and C lie between X, B and X, D, respectively. Then the quadrilateral $ABDC$ is cyclic if and only if $XA \cdot XB = XC \cdot XD$.



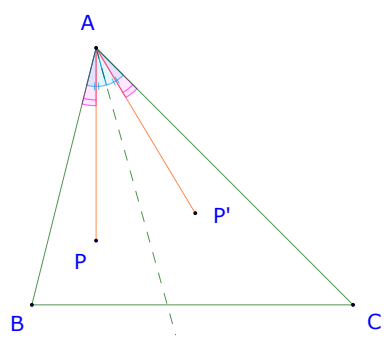


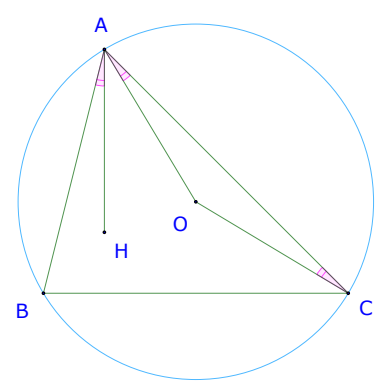
Theorem (Stewart's Theorem)

Given $\triangle ABC$, $BC = a, CA = b, AB = c$. D is a point on BC, $BD = m, DC = n, AD = d$. We have:

$$b^2m + c^2n = a(d^2 + mn).$$

Definition (Isogonal Conjugates)**.** Two points P and P' are called isogonal conjugates if line XP' is the reflection of XP across the angle bisector of the angle at the vertex X in a triangle $\triangle ABC$, where X is any of the vertices A, B, or C. In other words, the lines XP and XP' make equal angles with the sides of the triangle that contain X, e.g. $\angle PAB = \angle P'AC$.





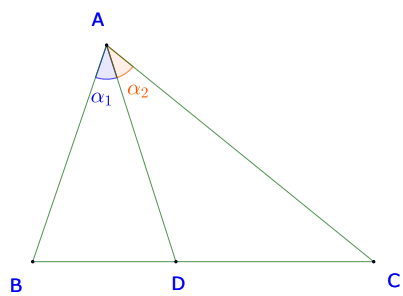
Theorem (Orthocenter-circumcircle Theorem)

The orthocenter and the circumcenter in a triangle are isogonal conjugates.

Lemma (Ratio Lemma)

In $\triangle ABC$ let D be a point on segment BC (not necessarily between B and C) and let α_1 and α_2 be the angles $\angle BAD$ and $\angle DAC$, respectively, then

$$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin \alpha_1}{\sin \alpha_2}$$



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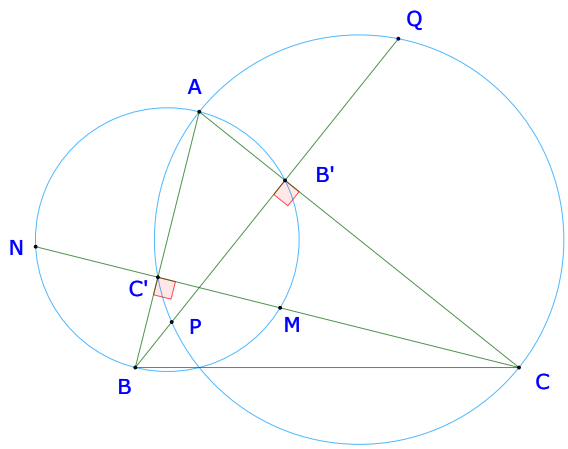
3.6 Problems (O-Level)

Problem 3.6.1 (23-24-S2-O-P16)**.** (*10 points*) An acute-angled triangle ABC is given in the plane. The circle with diameter AB, intersects altitude CC', and its extension at points M and N,, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

Hint: Investigate the intersection of the two circles. Prove and use the Intersecting Chord Theorem.



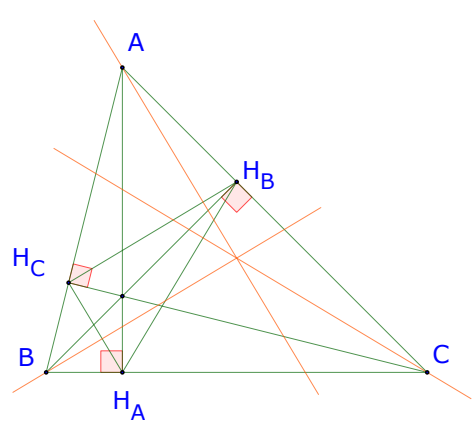
Corollary

ABC is an acute triangle. A', B', and C' are the feet of the altitudes from A, B, and C to BC, CA, and AB, respectively. Let H be the orthocenter of $\triangle ABC$, then

$$HA \cdot HA' = HB \cdot HB' = HC \cdot HC'.$$

Problem 3.6.2 (23-24-S2-O-P17)**.** (*10 points*) Let AH_A, BH_B and CH_C be altitudes in $\triangle ABC$. Let p_A, p_B, p_C be the perpendicular lines from vertices A, B, C to H_BH_C, H_CH_A, H_AH_B respectively. Prove that p_A, p_B, p_C are concurrent lines.

Hint: Use the Orthocenter-circumcircle Theorem.



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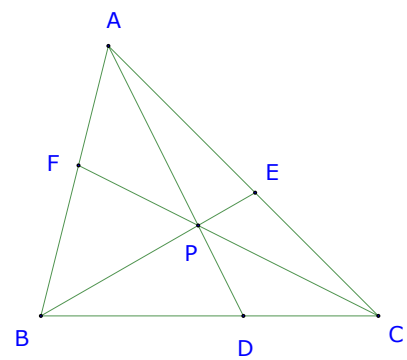
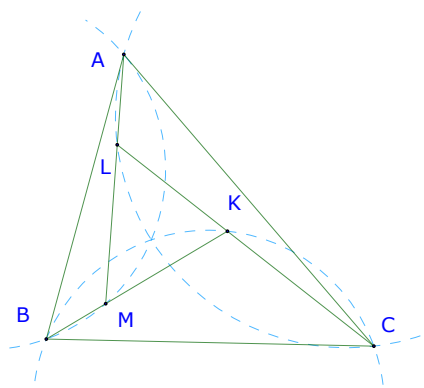
Problem 3.6.3 (23-24-S2-O-P18)**.** (*10 points*) Point P is inside $\triangle ABC$. Line segments APD, BPE, and CPF are drawn with D on BC, E on AC, and F on AB. Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of $\triangle ABC$.

Hint: Use the Area Lemma and Stewart's Theorem.



Problem 3.6.4 (23-24-S2-O-P19)**.** (*10 points*) Triangle KLM lies inside triangle ABC so that points K, L, M lie on the segments Cl, AM, BK, respectively. Prove that the circumcircles of the triangles ABM, BCK, CAL pass through a common point.

Hint: Cyclic quadrilaterals?



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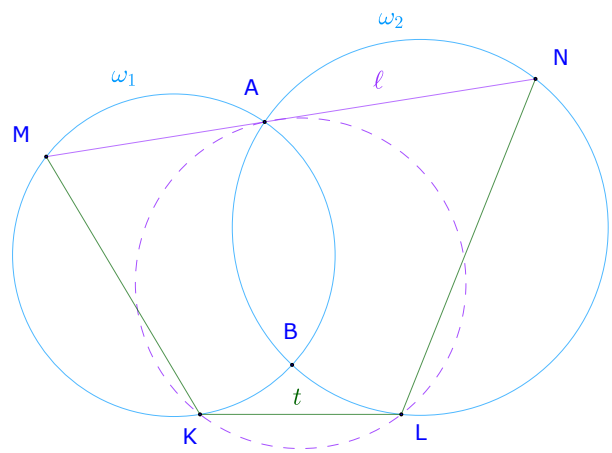
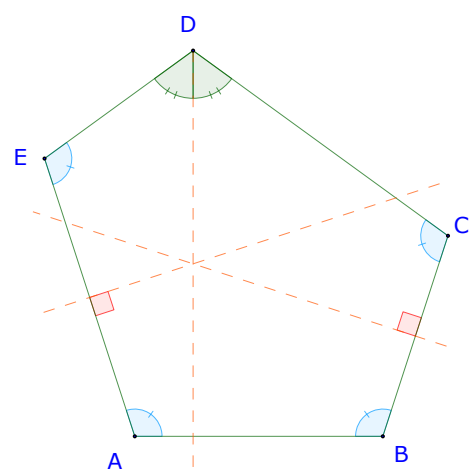
Problem 3.6.5 (23-24-S2-O-P20)**.** (*10 points*) Circles ω_1 and ω_2 intersect at A, B. Their common external tangent t is tangent to them at K, L, respectively, such that B lies inside $\triangle AKL$. Line ℓ passing through A intersects circles ω_1, ω_2 at M, N, respectively. Prove that ℓ is tangent to the circumcircle of $\triangle KLA$ if and only if $KLNM$ is cyclic.

Hint: let m be the angle bisector of line ℓ and t. Note that t is tangent to the circumcircles of $\triangle KAM$ and $\triangle LAN$. Find antiparallel pairs.



Problem 3.6.6 (23-24-S2-O-P21)**.** (*10 points*) In the convex pentagon $ABCDE$ all interior angles have the same measure. Prove that the perpendicular bisector of segment EA, the perpendicular bisector of segment BC, and the angle bisector of $\angle CDE$ intersect at one point.

Hint: three concurrent lines and one of them is already an angle bisector.



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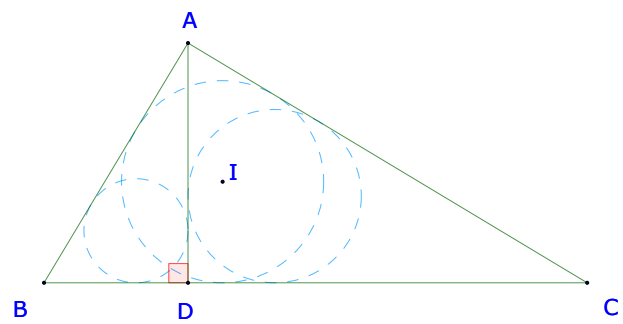
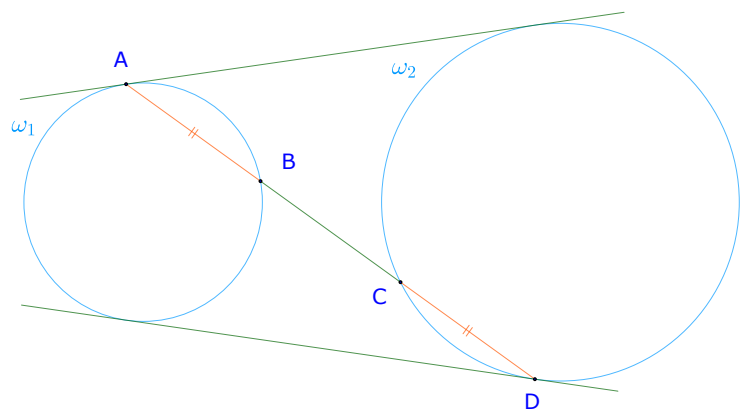
Problem 3.6.7 (23-24-S2-O-P22)**.** (*10 points*) ABC is a right triangle with $\angle A = 90°$, and altitude AD. Let r, s, t be the inradii of the triangles ABC, ADB, and ADC, respectively. Show that:

$$r + s + t = AD.$$

Hint: Find pairs of similar triangles.



Problem 3.6.8 (23-24-S2-O-P23)**.** (*10 points*) Let ω_1 and ω_2 be two circles. One of their common external tangents is tangent to ω_1 at A, the second one is tangent to ω_2 at D. Line AD intersects the circles ω_1 and ω_2 for the second time at B and C, respectively. Prove that $AB = CD$.



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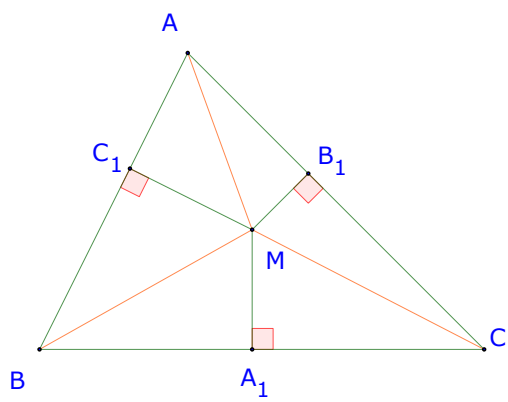
Problem 3.6.9 (23-24-S2-O-P24)**.** (*10 points*) Let $ABCD$ be a cyclic quadrilateral. Let P be the point on the ray AD such that $AP = BC$ and let Q be the point on the ray AB such that $AQ = CD$. Prove that the line AC cuts PQ at its midpoint.

Problem 3.6.10 (23-24-S2-O-P25)**.** (*10 points*) Let M be the point inside the triangle ABC. Denote by A_1, B_1, C_1 the feet of perpendiculars from M to BC, CA, and AB. Prove that

$$MA \cdot MB \cdot MC \geq 8MA_1 \cdot MB_1 \cdot MC_1.$$



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Chapter 4

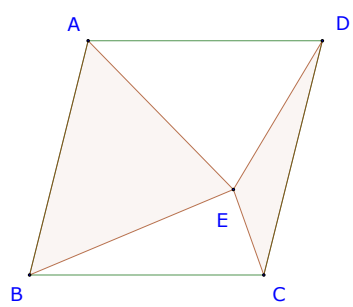
Session 3

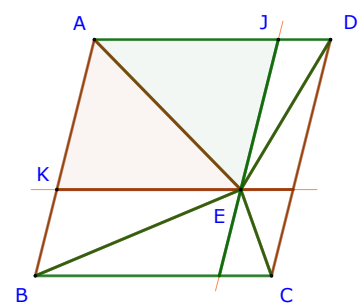
4.1 A picture is worth a thousand words (I-Level)

In this session, we will investigate a number of ways to *prove area equality without writing lengthy proof.* While it sounds simple, easy, and exciting, it is important that you need to improve your creating thinking in order to first understand the examples, and then use them as tools, guidelines, or ideas to solve the problems.

Example 4.1.1 (23-24-S3-I-P1)
E is an arbitrary point inside the parallelogram $ABCD$, prove that

$$[AEB] + [CED] = \frac{1}{2}[ABCD].$$





Example 4.1.2 (23-24-S3-I-P2)
E and F are midpoints of BC and DA in the convex quadrilateral $ABCD$, prove that

$$[AECF] = \frac{1}{2}[ABCD].$$

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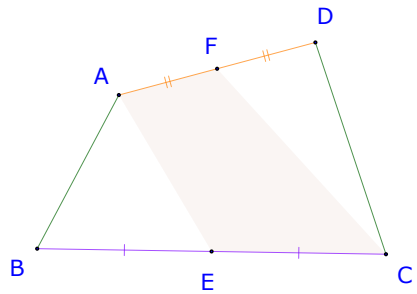
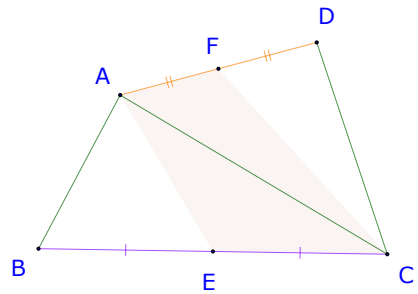
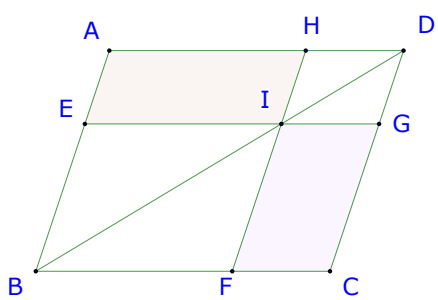


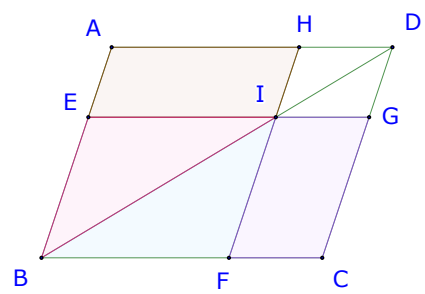


Example 4.1.3 (23-24-S3-I-P3)

I is an arbitrary point on the diagonal BD in parallelogram $ABCD$. Lines through I parallel with the sides of $ABCD$ intersect AB, BC, CD, and DA at E, F, G, and H, respectively.

$$[AEIH] = [FCGI].$$

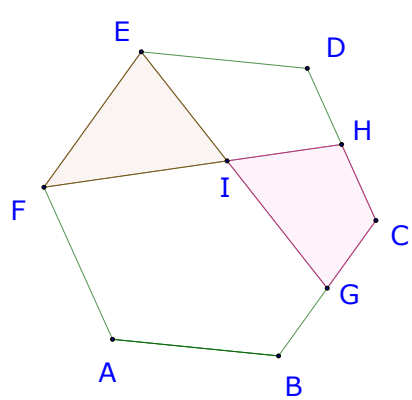




Example 4.1.4 (23-24-S3-I-P4)

G, H are midpoints of BC, CD in th regular hexagon $ABCDEF$. EG and FH intersect at I. Prove that

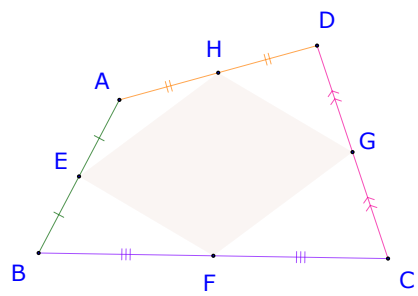
$$[GCHI] = [EFI].$$



Example 4.1.5 (23-24-S3-I-P5)

E, F, G, and H are midpoints the sides in the convex quadrilateral $ABCD$. Prove that

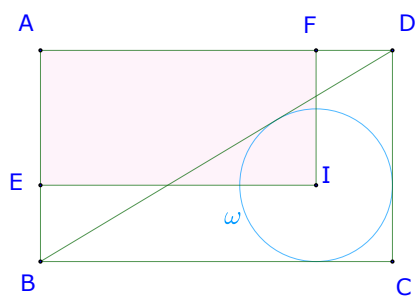
$$[EFGH] = \frac{1}{2}[ABCD].$$

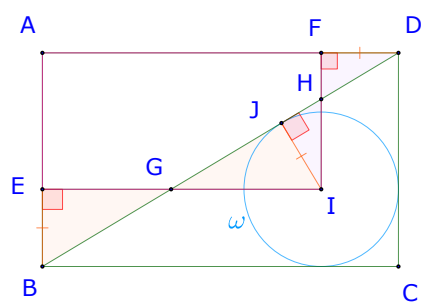


Example 4.1.6 (23-24-S3-I-P6)

$ABCD$ is a rectangle. E, F are arbitrary points on of CD, DA, respectively. Circle ω center at I is tangent to sides BC, CD, and diagonal BD. Lines through I parallel to the sides of the rectangle intersect AB at E and DA at F. Prove that

$$[AEIF] = \frac{1}{2}[ABCD].$$





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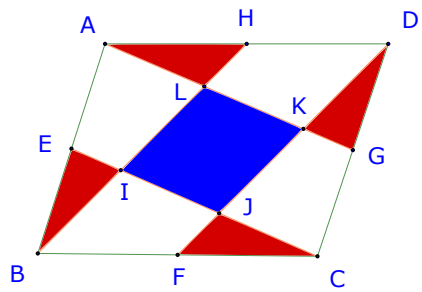
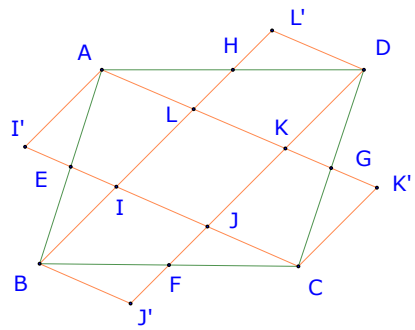
Example 4.1.7 (23-24-S3-I-P7)

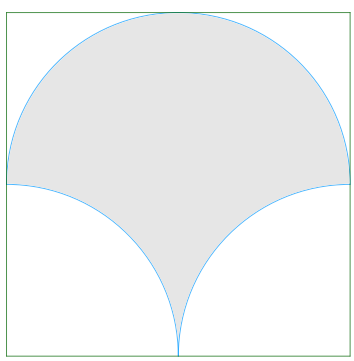
$ABCD$ is a parallelogram. E, F, G, and H are midpoints of AB, BC, CD, and DA, respectively. Segments AG, BH, CE, EF intersect at I, J, K, and L, as shown below. Prove that the area of the blue region is the same as the sum of the areas of the red regions.

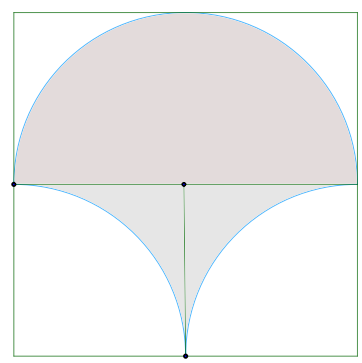




Example 4.1.8 (23-24-S3-I-P8)

A half cirle and two quarters of circles are draw in a square as shown below. Find the ratio of the area of the shaded region to the area of the square.





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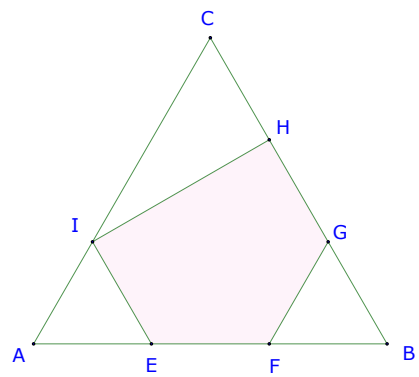
Example 4.1.9 (23-24-S3-I-P9)

$\triangle ABC$ is equilateral. Points E, F trisect the side AB, point G, H trisect the side BC. IE is parallel to BC. Find the ratio of the shaded area to the area of $\triangle ABC$.



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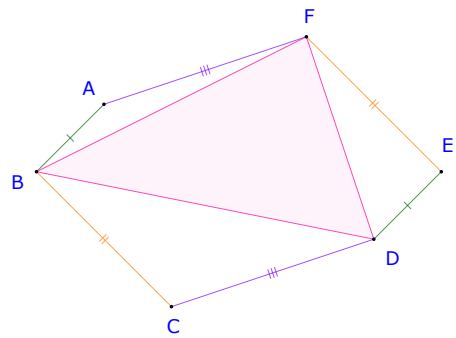
4.2 Problems (I-Level)

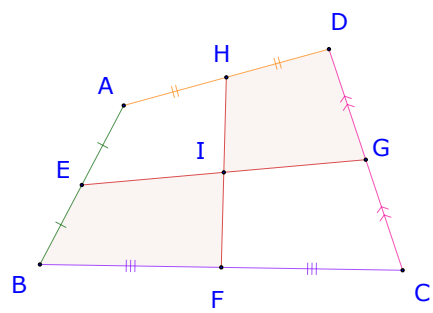
Problem 4.2.1 (23-24-S3-I-P11)**.** (*10 points*) In the hexagon $ABCDEF$, $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$, $AB = DE$, $BC = EF$, and $CD = FA$. Prove that

$$[BDF] = \frac{1}{2}[ABCDEF].$$



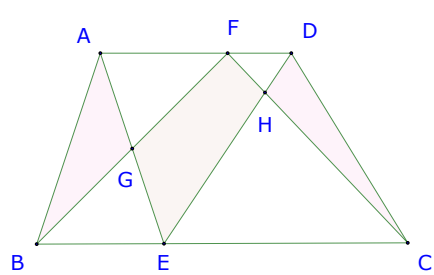
Problem 4.2.2 (23-24-S3-I-P12)**.** (*10 points*) In the convex quadrilateral $ABCD$, E, F, G, and H are midpoints of AB, BC, CD, and DA, respectively. EG and FH intersect at I. Prove that

$$[EBFI] + [GDHI] = \frac{1}{2}[ABCD].$$



Problem 4.2.3 (23-24-S3-I-P13)**.** (*10 points*) $ABCD$ is a trapezoid. E, F are arbitrary points on BC and AD, respectively. AE, BF intersect at G and CF, DE intersect at H. Prove that:

$$[ABG] + [CDH] = [EFGH].$$



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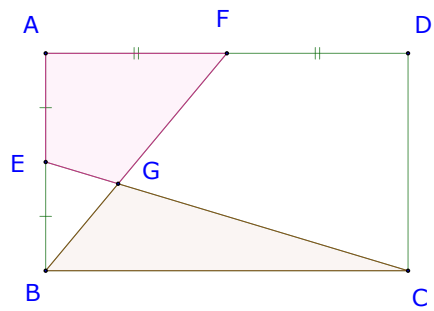
Problem 4.2.4 (23-24-S3-I-P14)**.** (*10 points*) $ABCD$ is a rectangle. E, F are midpoints of AB, AD, respectively. CE, BF intersect at G. Prove that:

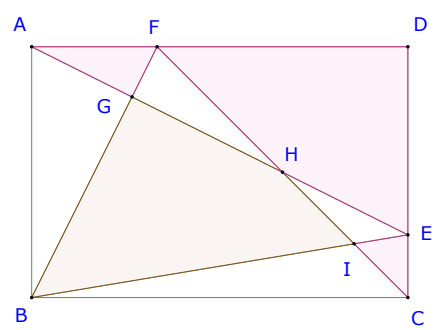
$$[AEGF] = [BGC].$$



Problem 4.2.5 (23-24-S3-I-P15)**.** (*10 points*) $ABCD$ is a rectangle. E, F are arbitrary points on of CD, DA, respectively. AE intersects BF, BE at G, H, respectively. CF intersects BE at I. Prove that:

$$[AGF] + [FHED] + [CEI] = [BGHI].$$

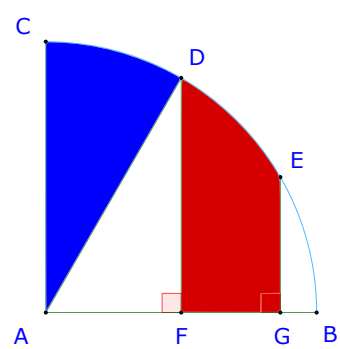
Hint: look at the triangle BDF and CDF.



Problem 4.2.6 (23-24-S3-I-P16)**.** (*10 points*) ABC is a quarter of circle. D, E trisect the arc $\widehat{BC}$

$$\widehat{CD} = \widehat{DE} = \widehat{EB}.$$

Prove that the blue circular sector ACD and the red region $DEGF$ have the same area.



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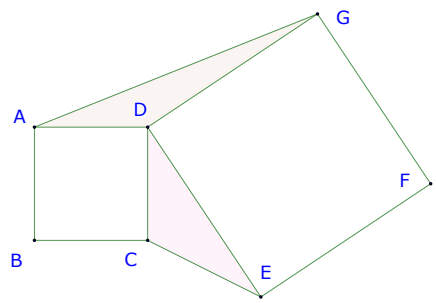
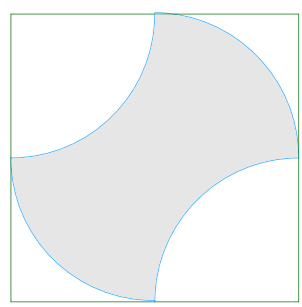
Problem 4.2.7 (23-24-S3-I-P17)**.** (*10 points*) $ABCD$ and $DEFG$ are two squares as shown below, prove that

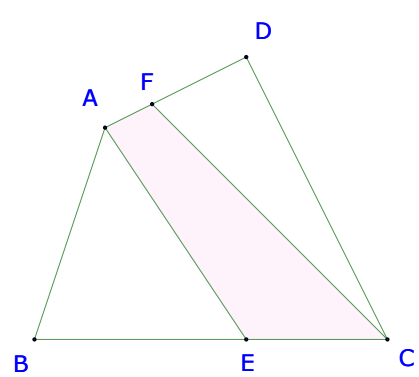
$$[ADG] = [CDE].$$



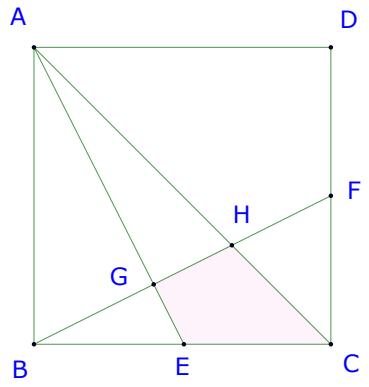
Problem 4.2.8 (23-24-S3-I-P18)**.** (*10 points*) A half cirle and two quarters of circles are draw in a square as shown below. Find the ratio of the area of the shaded region to the area of the square.



Problem 4.2.9 (23-24-S3-I-P19)**.** (*10 points*) $ABCD$ is a convex quadrilateral. E and F are points on BC and DA such that $BE = 2EC$, $DF = 2FA$. Find the ratio of the area of the shaded region to the area of the quadrilateral.



Problem 4.2.10 (23-24-S3-I-P20)**.** (*10 points*) $ABCD$ is a square. E and F are midpoints of BC and CD. BF intersect AE and AC at G and H. Find the ratio of the area of the shaded region to the area of the square.



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4.3 Perfect squares are everywhere (A-Level)

In this session, we will discuss properties of perfect squares through a number of interesting examples.

Example 4.3.1 (23-24-S3-A-P1)

Cancelling the exponents yields

$$\frac{37^3+13^3}{37^3+24^3}=\frac{37+13}{37+24}=\frac{50}{61}.$$

which is correct.

Find the necessary and sufficient conditions for the positive integer triple (A, B, C) to satisfy

$$\frac{A^3+B^3}{A^3+C^3}=\frac{A+B}{A+C}.$$

Example 4.3.2 (23-24-S3-A-P2)

For which natural number n is $2^8+2^{11}+2^n$ a perfect square?

Example 4.3.3 (23-24-S3-A-P3)

Prove that for n positive integer the following number is a perfect square:

$$m=\underbrace{99\ldots9}_{n}\underbrace{00\ldots0}_{n}25.$$

Example 4.3.4 (23-24-S3-A-P4)

Find all prime p such that $2p^4-p^2+16$ is a perfect square.

Example 4.3.5 (23-24-S3-A-P5)

Prove that the sum of the squares of 1984 consecutive positive integers cannot be the square of an integer.

Example 4.3.6 (23-24-S3-A-P6)

a, b, c are positive integers such that ab and bc are perfect squares. Prove that ca is also a perfect square.

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Example 4.3.7 (23-24-S3-A-P7)

Prove that if x and y are sum of two perfect squares, then xy is also a sum of two perfect squares. In other words, if a, b, c, d integers such that $x = a^2 + b^2$, $y = c^2 + d^2$ then $xy = (a^2 + b^2)(c^2 + d^2)$ can be written as a sum of two perfect squares.

Prove that if x and y are sum of a perfect square and twice of another perfect square, then xy is also a sum of a perfect square and twice of another perfect square. In other words, if a, b, c, d integers such that $x = a^2 + 2b^2$, $y = c^2 + 2d^2$ then $xy(a^2 + 2b^2)(c^2 + 2d^2)$ can be written as a sum of a perfect square and twice of another perfect square.

Example 4.3.8 (23-24-S3-A-P8)

x, y, z are positive integers and their greatest common divisor is 1, such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

Prove that $x + y$ is a perfect square.

Example 4.3.9 (23-24-S3-A-P9)

n is called *interesting* number if it can be written as $3x^2 + 32y^2$, where x, y are integers. Prove that if n is an *interesting* number, then $97n$ is *interesting* number too.

Fact. $n = ax^2 + by^2$, then $(ab + 1)n = a(by - x)^2 + b(ax + y)^2$.

Example 4.3.10 (23-24-S3-A-P10)

Determine all perfect squares in the sequence $\{a_1, a_2, \ldots\}$, where

$$a_3 = 91,\ a_{n+1} = 10a_n + (-1)^n,\ \forall n \geq 0.$$

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4.4 Problems (A-Level)

Problem 4.4.1 (23-24-S3-A-P11)**.** (*10 points*) Prove that, for all distinct values of a, b, and c, such that

$$\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}$$

is an integer (not necessarily positive), then the expression

$$\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}$$

is a perfect square.

Problem 4.4.2 (23-24-S3-A-P12)**.** (*10 points*) Find all (including negative) integer solutions of

$$n^2(n-1)^2 = 4(m^2-1).$$

Problem 4.4.3 (23-24-S3-A-P13)**.** (*10 points*) Prove that for n positive integer the following number is a perfect square:

$$m = 1\underbrace{77\ldots7}_{n}92\underbrace{88\ldots8}_{n-1}921.$$

Problem 4.4.4 (23-24-S3-A-P14)**.** (*10 points*) N is 4−digit perfect square all of whose decimal digits are less than seven. Increasing each digit by three we obtain a perfect square again. Find N.

Problem 4.4.5 (23-24-S3-A-P15)**.** (*10 points*) n is a non-negative integer, prove that

$$3^n + 2\cdot 17^n$$

is never a perfect square.

Problem 4.4.6 (23-24-S3-A-P16)**.** (*10 points*) n is an arbitrary positive integer. Find positive integer m such that

$$\binom{m}{2} = 3\binom{n}{4}.$$

Problem 4.4.7 (23-24-S3-A-P17)**.** (*10 points*) Prove that if the sum $x+y$ can be written as sum of a perfect square and thrice of another perfect square, then the sum x^3+y^3 can also be written as sum of a perfect square and thrice of another perfect square. In other words, if there exist a, b integers such that $x+y = a^2+3b^2$, then there exist c, d integers, such that

$$x^3+y^3 = c^2+3d^2.$$

Hint: investigate x^2-xy+y^2, depending on the parity of x, y whether it can be written as sum of a perfect square and thrice of another perfect square.

Problem 4.4.8 (23-24-S3-A-P18)**.** (*10 points*) Prove that for integers x, y such that

$$2x^2+x = 3y^2+y,$$

then $2x+2y+1$ and $3x+3y+1$ are perfect squares.

Hint: investigate $(2x+2y+1)(x-y)$, then consider $(2x+2y+1)(3x+3y+1)(x-y)^2$.

Problem 4.4.9 (23-24-S3-A-P19)**.** (*10 points*) Prove that the sum of squares of twelve positive consecutive integers is not divisible by the sum of them. In other words, if n non-negative integers, then

$$(n+1)+(n+2)+\cdots+(n+12) \not| \ (n+1)^2+(n+2)^2+\cdots+(n+12)^2$$

Problem 4.4.10 (23-24-S3-A-P20)**.** (*10 points*) Let N be a 16−digit number, where none of its digits can be $0, 1, 4$, or 9. Prove that there exist some of its consecutive digits such that their product is a perfect square.

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4.5 The Extremal Principle (O-Level)

Definition (Order relation)**.** An order on a set is a relation $\leq$ with three properties:

1. $a \leq a$,
2. if $a \leq b$ and $b \leq a$ then $a = b$.
3. if $a \leq b$ and $b \leq c$ then $a \leq c$.

Definition (Total order relation)**.** The order is called total if any two elements are comparable, that is, if for every a and b, either $a \leq b$ or $b \leq a$.

Definition (Well-ordering principle on finite set)**.** The *well-ordering principle* states that every *finite* non-empty total order set contains *a maximal* and *a minimal element.*

Definition (Well-ordering principle for positive integers)**.** The *well-ordering principle* states that every non-empty set of positive integers contains a *a least element.*

We use the term *Extremal Principle* to indicate the existence of a minimal or maximal element in a set. The extremal principle is just a clean way of writing up a proof that uses greedy algorithm to find a case by trying to optimize a well-chosen quantity.

We used to combine the Extremal Principle with the Proof by Contradiction to prove an existence or a non-existence. We used to combine the Extremal Principle with the Pigeonhole Principle to establish the condition for existence or to increase or decrease a quantity. We used to combine the Extremal Principle with the Induction Principle to simplify a proof.

Example 4.5.1 (23-24-S3-O-P1)

Positive integers are written in the cells of a regular square grid such that every number is the average of the numbers in the adjacent cells. Prove that all numbers are equal.

Example 4.5.2 (23-24-S3-O-P2)

n students ($n \geq 2$) are standing in a field such that the distance between each pair is distinct. Each student is holding a ball, and when the teacher blows a whistle, each student throws their ball to the nearest student. Prove that there is a pair of students that throw their balls to each other.

Example 4.5.3 (23-24-S3-O-P3)

Given a finite set S of red and blue points in a plane. Every line segment that joins two points of the same color contains a point of another color. Prove that all the points lie on a single straight line.

Example 4.5.4 (23-24-S3-O-P4)

In some country all roads between cities are one-way and such that once you leave a city you cannot return to it again. Prove that there exists a city into which all roads enter and a city from which all roads exit.

Example 4.5.5 (23-24-S3-O-P5)

Given $n \geq 3$ points on the plane. Not all of them are collinear. Show that there exists a circle passing through three of the points, the interior of which does not contains any of the remaining points.

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Example 4.5.6 (23-24-S3-O-P6)

Every participant of a tournament plays with every other participant exactly once. No game is a draw. After the tournament, every player makes a list with the names of all players, who

1. were beaten by him, and
2. were beaten by the players beaten by him.

Prove that the list of some player contains the names of all other players.

Example 4.5.7 (23-24-S3-O-P7)

For $n > 1$, the integers from 1 to n^2 are placed in the cells of an $n \times n$ chessboard. Show that there is a pair of horizontally, vertically, or diagonally adjacent cells whose values differ by at least $n + 1$.

Example 4.5.8 (23-24-S3-O-P8)

There are n red points and n blue points in the plane, with no 3 points collinear. Prove that we can draw n segments joining a red point to a blue point such that no pair of segments intersect.

Example 4.5.9 (Putnam 1956/Problem B3)

Consider a set of $2n$ points in space ($n > 1$). These points are connected by $n^2 + 1$ segments. Show that there exists a triple of points that are connected by straight segments, thus forms a triangle. Show that for each n it is possible to have $2n$ points connected by n^2 segments without any triangles being formed.

Example 4.5.10 (Poland MO 1975)

A sequence $\{a_n\}$ satisfies the following properties:

$$\exists n \in \mathbb{Z}^+ : \ a_1 + a_2 + \cdots + a_n = 0, \text{ and } a_{n+k} = a_k, \ k = 1, 2, \ldots$$

Prove that there exists a positive integer N such that, for $k = 1, 2, \ldots,$

$$a_N + a_{N+1} + \cdots + a_{N+k} \geq 0.$$

Example 4.5.11 (IMO SL 1981/Problem 17)

Given is a finite set of spherical planets, all of the same radius and no two intersecting. On the surface of each planet consider the set of points not visible from any other planet. Prove that the total area of these sets is equal to the surface area of one planet.

Example 4.5.12 (23-24-S3-O-P12)

Prove that in the coordinate plane, the vertices of a regular pentagon cannot be all lattice points.

Example 4.5.13 (China MO 1988)

If $a_1 = 1, a_2 = 2$, and

$$a_{n+2} = \begin{cases} 5a_{n+1} - 3a_n, & \text{when } a_n a_{n+1} \text{ is even,} \\ a_{n+1} - a_n, & \text{when } a_n a_{n+1} \text{ is odd,} \end{cases}$$

prove that, for any positive integer n, $a_n \neq 0$.

Example 4.5.14 (Kvant (Quantum))

Given finitely many squares whose areas add up to 1, show that they can be arranged without overlaps inside a square of area 2.

Example 4.5.15 (23-24-S3-O-P24)

Rooks are placed on the $n \times n$ chessboard satisfying the following condition: If the square (i, j) is free, then at least n rooks are on the ith row and j^{th} column together. Show that there are at least $\frac{n^2}{2}$ rooks on the board.

Example 4.5.16 (23-24-S3-O-P25)

Any two of a finite number of polygons have a common point. Prove that there is a ine which has a common point with all these polygons.

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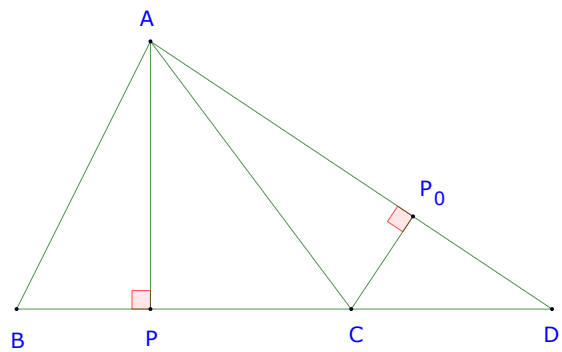
4.6 Problems (O-Level)

Problem 4.6.1 (Sylvester-Galai)**.** Given a finite set S of points in a plane. Prove that, either they all lie on a line, or there is a line containing exactly two points of S.

Hint: Consider the triple (A, B, C) of distinct points in S that are not collinear and such that the distance from A to the line ℓ passing through B and C is minimal.



Problem 4.6.2 (Korea MO, 1995)**.** There are n points in a plane such that the area enclosed by any three of the points do not exceed 1. Prove that a triangle can be drawn with area not more than 4 that contains all n points.

Hint: Let $A'B'C'$ be it's anti-complementary triangle, i.e., the triangle formed by the parallels to the sides of $\triangle ABC$ and passing through it's vertices.

Problem 4.6.3 (Balkan MO 2010)**.** A strip of width w is the set of all points which lie on or between two parallel lines that are at at a distance w apart. Let S be a set of n ($n \geq 3$) points on the plane such that any three different points of S can be covered by a strip of width 1. Prove that S can be covered by a strip of width 2.

Hint: Among all the triangles formed by the n points in S, consider the triangle (say, ABC) with maximum area. Consider the anti-complementary triangle of $\triangle ABC$ and other given points.

Problem 4.6.4 (USA TST 2005)**.** Find all sets S of finitely many points in the plane, no three of which are collinear and such that for any three points A, B, C in S, there is another point D in S such that A, B, C, D (in some order) are the vertices of a parallelogram.

Hint: Assume, to the contrary, that S has more than 4 points. Consider points A, B, C in S such that $\triangle ABC$ has the maximum possible area over all choices of A, B, C. Let point D be such that $ABCD$ is a parallelogram. Consider $ABCD$ and other points of S.

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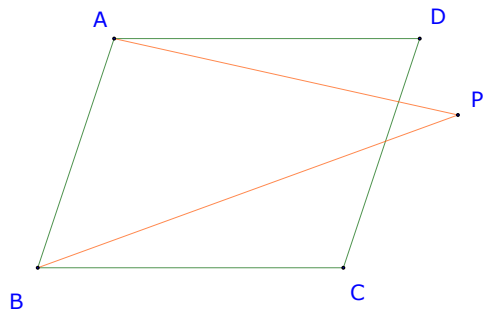
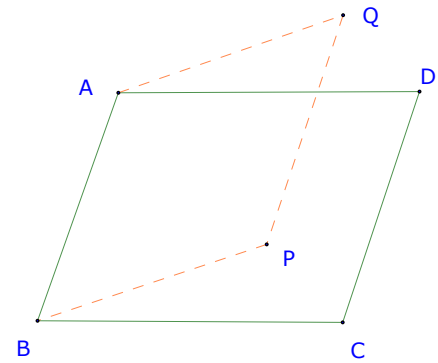




Problem 4.6.5 (23-24-S3-O-P20)**.** Prove that among any 50 distinct positive integers strictly less than 100 there are two that are co-prime.

Hint: Order the numbers $x_1 < x_2 < \ldots < x_{50}$. Compare x_1 and x_{50}.

Problem 4.6.6 (23-24-S3-O-P21)**.** Prove that it is impossible to dissect a cube into finitely many cubes, no two of which are the same size.

Hint: assume that such a dissection exists, and look at the bottom face. It is cut into squares. Take the smallest of these squares. It is not hard to see that this square lies in the interior of the face (why?). Look at the cube that lies right above this square!

Problem 4.6.7 (Romania 1999 TST/Problem 5)**.** Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of distinct positive integers. Prove that, for any positive integer n

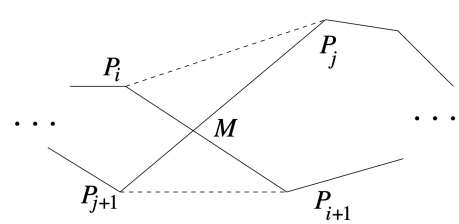
$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{2n+1}{3}(a_1 + a_2 + \cdots + a_n).$$

Hint: first, WLOG, rearrange $\{a_i\}$ as a strictly increasing sequence. Then prove that:

$$2(a_1 + a_2 + \cdots + a_n) \leq (a_{n+1} - 1)a_{n+1} \leq 3a_{n+1}^2 - (2n+3)a_{n+1}.$$

Problem 4.6.8 (23-24-S3-O-P23)**.** Given n points in the plane, no three of which are collinear, show that there exists a closed polygonal line with no self-intersections having these points as vertices.

Hint: study the example 23-24-S3-O-P4. There are only finitely many polygonal lines with these points as vertices.



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Chapter 5

Session 4

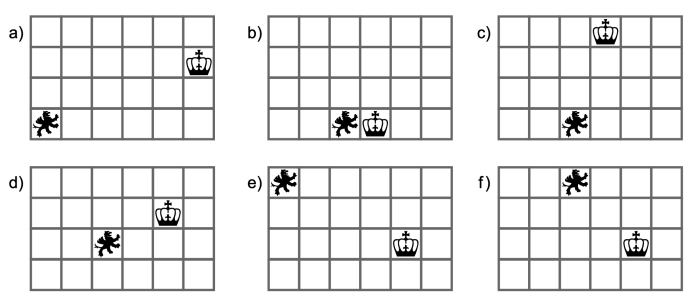
5.1 Colouring with two colours (I-Level)

Identifying *a quantity or property that remains unchanged is a very powerful problem-solving technique*: such a property is called an **invariant.** Invariants could help us crack really tough problems.

Example 5.1.1 (23-24-S4-I-P1)

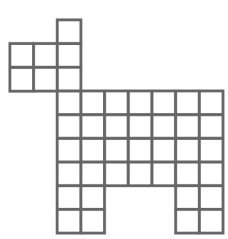
The castle has 24 rooms arranged as a 6×4*grid*, in each wall between the rooms there is a door. Can you walk through the castle in such a way that you would start in the room marked with the coat of arms (a winged lion), visit each room of the castle exactly once, and end up in the room marked with a crown?

There are six different cases show as six diagrams. Solve all of them.



Example 5.1.2 (23-24-S4-I-P2)

Can you tile this doggy with dominoes? (A domino is a 2×1 rectangle. "To tile" means to cover completely and without overlaps.)



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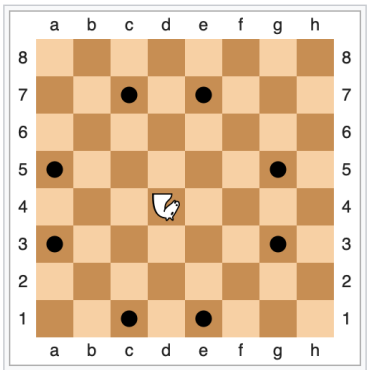
Example 5.1.3 (LPS V2B/2.10)

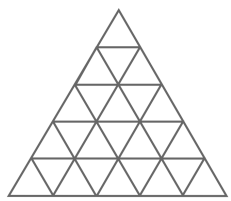
A special chess piece called a *camel* moves along a board in almost the same way as a knight does: it moves to any adjacent square and then move three squares in any particular direction (up, down, left, right), see below. Is it possible for a *camel* to go from some square to an adjacent square?



The camel (represented as an inverted knight) may move to any of eight squares (black dots).

Example 5.1.4 (23-24-S4-I-P4)

The Triangular Castle, shown as below, has triangular rooms and a door in each of its interior walls. A tour guide wants to trace a route through this castle that would not visit any room more than once. What is the maximum number of rooms one can visit on such a tour? (A tour can start and end in any two rooms.)

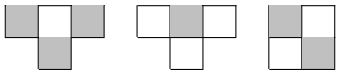


Example 5.1.5 (23-24-S4-I-P5)

There are 25 beetles sitting on the 25 squares of a 5×5 chessboard. On a blow of a whistle, each beetle relocates to an adjacent square (a square above or below or to the right or to the left). Prove that at least one of the squares will end up being empty.

Example 5.1.6 (LPS V2B/2.2)

Prove that the 8×8 chessboard cannot be covered by 15 tetrominoes and 1 square tetromino, shown below.



Example 5.1.7 (23-24-S4-I-P7)

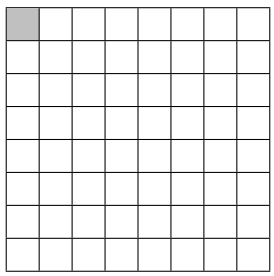
Twenty-five checkers are placed on a 25×25 checkerboard in such a way that their positions are symmetric with respect to *one of its diagonals.* Prove that at least one of the checkers is positioned on that diagonal.

Example 5.1.8 (23-24-S4-I-P8)

What is the largest number of squares on an 8 x 8 checkerboard which can be colored green, so that in any arrangement of a *tromino* of three squares, shown as below, at least one square is not colored green? (The tromino may appear as in the figure, or it may be rotated through some multiple of 90 degrees.)

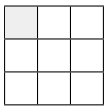
Example 5.1.9 (23-24-S4-I-P9)

In an 8×8 table one of the boxes is colored black and all the others are white. Prove that one cannot make all the boxes white by recoloring the rows and columns. *Recoloring* is the operation of changing the color of all the boxes in a row or in a column.



Example 5.1.10 (23-24-S4-I-P10)

In an 3×3 table one of the boxes is colored black and all the others are white. Prove that one cannot make all the boxes white by recoloring the rows and columns. *Recoloring* is the operation of changing the color of all the boxes in a row or in a column.



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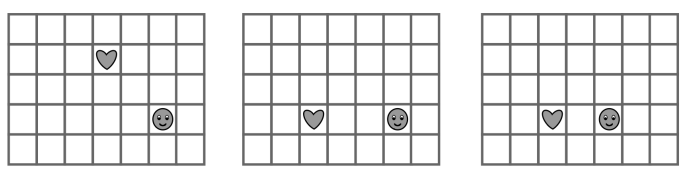
5.2 Problems (I-Level)

Problem 5.2.1 (23-24-S4-I-P11)**.** (*10 points*) Can you cut this hexagon-shaped cake into 23 identical pieces? (All cuts should follow the grid lines.)

Problem 5.2.2 (23-24-S4-I-P12)**.** (*10 points*) The summer cottage has 35 rooms arranged as a 7×5 grid; each wall between the rooms has a door. Can you walk through the cottage in such a way that:

- you would start in the drawing room (marked with a smiley face);
- you would visit each room of the cottage exactly once;
- you would end up in the room marked with the heart?

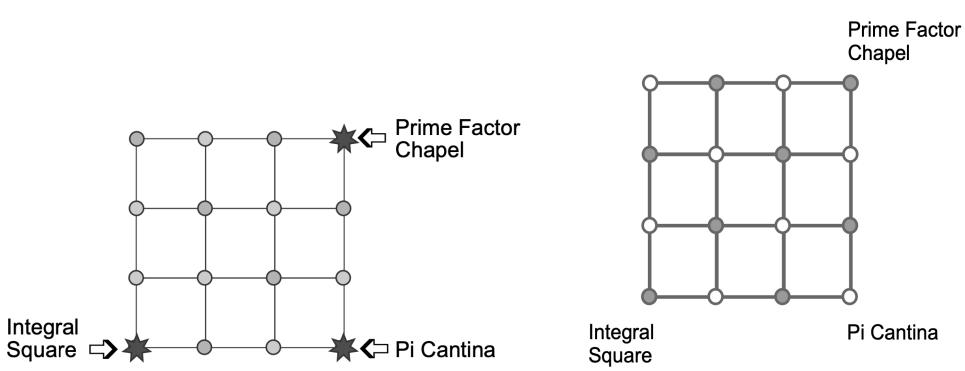
For every map, either draw a path or prove that such a route is impossible.



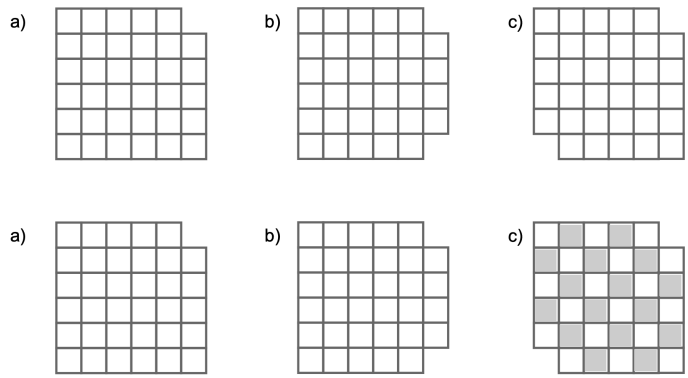
Problem 5.2.3 (23-24-S4-I-P13)**.** (*10 points*) Downtown Math-Hattan has 4 streets going east-west, and 4 avenues going north-south, with a small plaza at each intersection. All roads can be walked in both directions.

- Can you walk from the Integral Square to the Pi Cantina, visiting each plaza exactly once?
- Can you walk from the Integral Square to the Prime Factor Chapel, visiting each plaza exactly once?

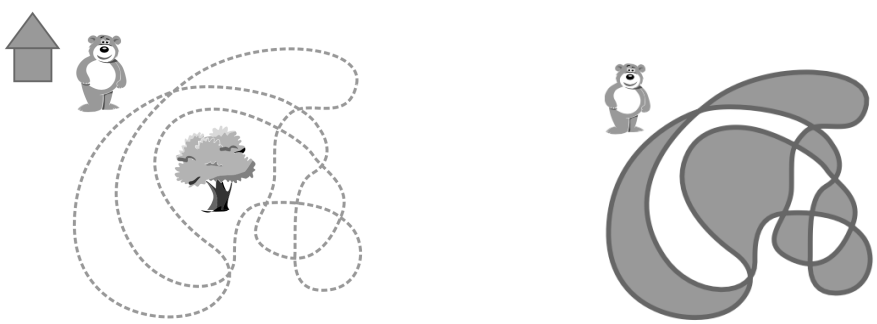
For each part, either present the route or explain why it doesn't exist.



Problem 5.2.4 (23-24-S4-I-P14)**.** You start off with a 6×6 chessboard. In part (a), one corner of it is missing. In parts (b) and (c), each chessboard is missing 2 corners. Which of these boards can you tile with dominoes? (A domino is a 2×1 tile. "To tile" means to cover completely and without overlaps.)



Problem 5.2.5 (23-24-S4-I-P15)**.** (*10 points*) Little Bear came out of his house and found strange footprints going around his favorite tree (see the picture below). He walked around the tree several times, studying the tracks and sometimes jumping over them to get a better look. In the end, the bear returned to the house. Could he have jumped over the mystery footprints 7 times? (Little Bear never jumps over the places where the track intersects itself.)



Problem 5.2.6 (23-24-S4-I-P16)**.** (*10 points*) Twenty-five checkers are placed on a 25×25 checkerboard in such a way that their positions are symmetric with respect to *both of its diagonals.* Prove that one of the checkers is placed in the center square.

Hint: connect all the pairs of checkers which are located symmetrically with respect to one of the diagonals with a thread into necklaces.

Problem 5.2.7 (23-24-S4-I-P17)**.** What is the smallest number of squares on an 8 x 8 checkerboard which can be colored green, so that in any tromino such as below, at least one square is colored green? (The tromino may appear as in the figure, or it may be rotated through some multiple of 90 degrees.)

Problem 5.2.8 (23-24-S4-I-P18)**.** (*10 points*) One main diagonal of the 8×8 chessboard is removed, leaving two triangular parts, as shown in below. What is the *largest number* of 1×2 dominoes can be used to tile the two triangular parts?

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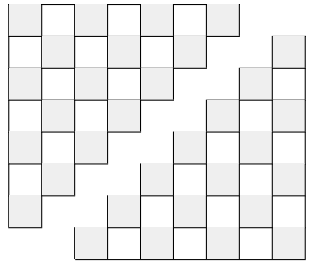
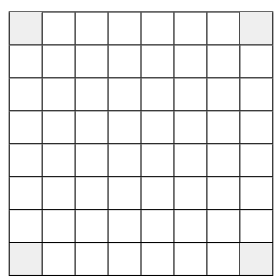


Problem 5.2.9 (23-24-S4-I-P19)**.** (*10 points*) Someone has put a rectangular playing card *face down* on the table. You can flip the card over one of its edges as many times as you want. Can you end up with the card sitting *face up* on the same spot?

Hint: create a rectangular grid, each rectangle has the same size as the card. Let the card be on a rectangle of the grid. Colour the grid. When the card is turned over one of its sides, shown below, what happens?

Problem 5.2.10 (23-24-S4-I-P20)**.** (*10 points*) In an 8×8 table four corner of the boxes is colored black and all the others are white. Prove that one cannot make all the boxes white by recoloring the rows and columns. *Recoloring* is the operation of changing the color of all the boxes in a row or in a column.

Hint: look at both 23-24-S4-I-P9 and 23-24-S4-I-P10 for ideas, can you find a 2×2 square where the number of black boxes is invariant during recolorings?



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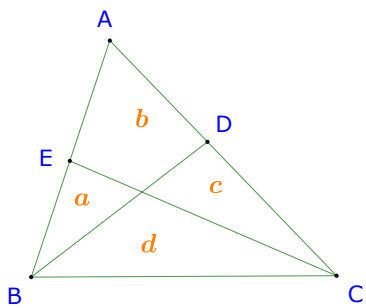
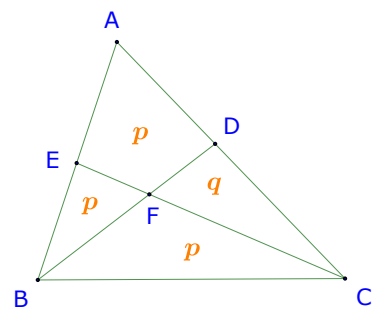
5.3 Basic Geometry at Advanced level (A-Level)

In this session, we will discuss a few basic geometry problems with areas.

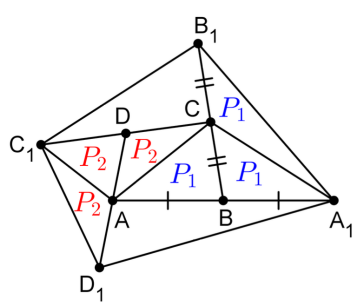
Example 5.3.1 (23-24-S4-A-P1)
It is possible for the segments BD and CE to divide the $\triangle ABC$ into four equal parts $a = b = c = d$ as shown below on the left.

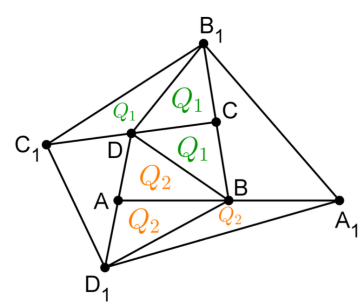




Example 5.3.2 (23-24-S4-A-P2)
Segments BD and CE to divide the $\triangle ABC$ into four parts. $[ADFE] = [BFC] = [BEF] = p$, $[CFD] = q$. See above on the right. Compute $\dfrac{p}{q}$.

Example 5.3.3 (23-24-S4-A-P3)
Let $ABCD$ be a convex quadrilateral with area 1. Let A_1 be a point on the ray AB beyond B, such that $AB = BA_1$. Similarly define the points B_1, C_1, and D_1. Prove that the area of $A_1B_1C_1D_1$ is 5.





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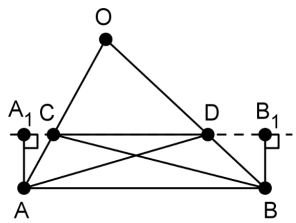
Example 5.3.4 (Thales' Proportionality Theorem)

Let OAB be a triangle and let CD be a line that intersects its sides OA and OB at C and D, respectively. Prove that

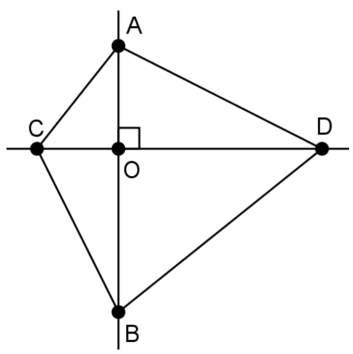
$$AB \parallel CD \iff \frac{OC}{CA} = \frac{OD}{DB}.$$

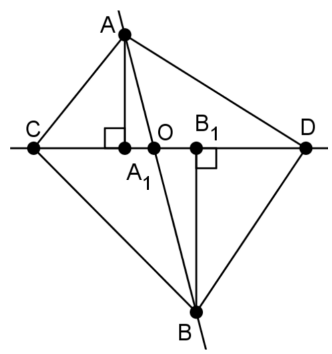


Example 5.3.5 (23-24-S4-A-P5)

Let AB and CD be two intersecting lines. Then,

$$AB \perp CD \iff CA^2 + CB^2 = DA^2 - DB^2.$$

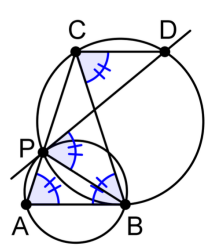




Example 5.3.6 (23-24-S4-A-P6)

Let ABC be an isosceles triangle, such that $AC = BC$. Let P be a point on the side AC. The tangent to the circle (ABP) at the point P intersects the circle (BCP) at D. Prove that $CD \parallel AB$.

(ABC) denotes the circle through all three points A, B, and C.



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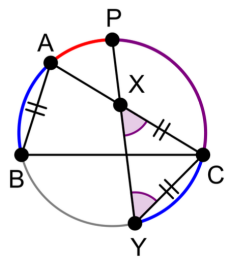
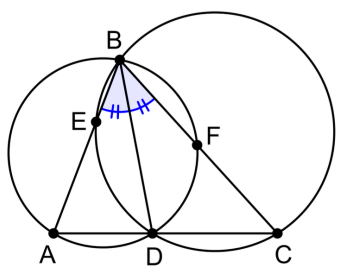
Example 5.3.7 (IGO 2016, Elementary)

Let ω be the circumcircle of $\triangle ABC$ with $AC > AB$. Let X be a point on AC and Y be a point on ω, such that $CX = CY = AB$. (The points A and Y lie on different sides of the line BC). The line XY intersects ω for the second time at P. Show that $PB = PC$.



Example 5.3.8 (St. Petersburg City MO 1996)

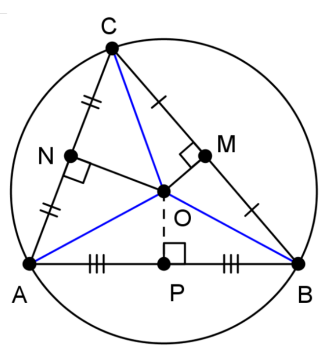
Let BD be the angle bisector of angle ABC in $\triangle ABC$ with D on the side AC. Let the circle (BDC) meet AB at E and (ABD) meet BC at F. Prove that $AE = CF$.



Example 5.3.9 (Circumcenter)

The three side bisectors of a triangle are concurrent. The point of concurrence is the center of a circle that passes through all three vertices of the triangle.

The point of concurrence is called the circumcenter of the triangle. The circle that is circumscribed around the triangle is called the circumcircle of the triangle.



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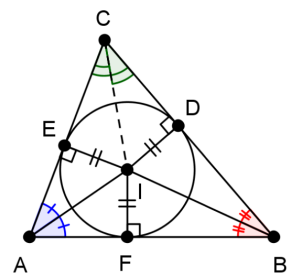
Example 5.3.10 (Incenter)

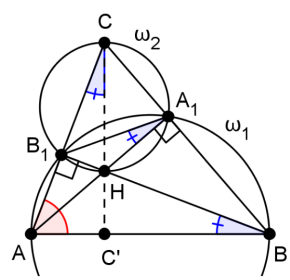
The three angle bisectors of a triangle are concurrent. The point of concurrence is the center of a circle that is tangent to all three sides of the triangle.

The point of concurrence is called the incenter of the triangle. The circle that is inscribed inside the triangle is called the incircle of the triangle.



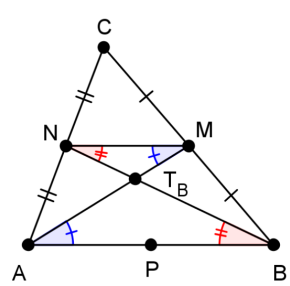
Example 5.3.11 (Orthocenter)

The three altitudes of a triangle are concurrent. The point of concurrence is called the orthocenter of the triangle.



Example 5.3.12 (Centroid)

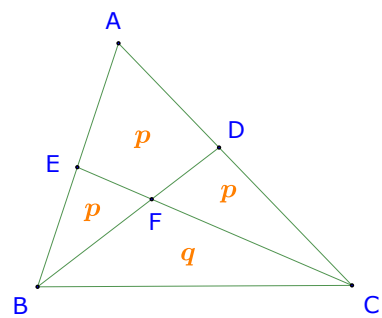
The three altitudes of a triangle are concurrent. The point of concurrence is called the orthocenter of the triangle.



5.4 Problems (A-Level)

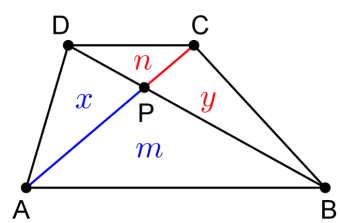
Problem 5.4.1 (23-24-S4-A-P11)**.** (*10 points*) Is it possible for the segments BD and CE to divide the $\triangle ABC$ into four parts such that $[BEF] = [BFC] = [CFD] = p$, $[ADFE] = q$.

Problem 5.4.2 (23-24-S4-A-P12)**.** (*10 points*) Segments BD and CE to divide the $\triangle ABC$ into four parts such that $[ADFE] = [BEF] = [CFD] = p$, $[BFC] = q$. Compute the ratio $\frac{p}{q}$.

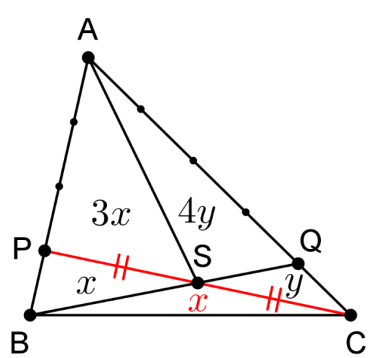


Problem 5.4.3 (23-24-S4-A-P13)**.** (*10 points*) Let $ABCD$ be a trapezoid ($AB \parallel CD$). Let its diagonals AC and BD intersect at P. Let the areas of the triangle $\triangle ABP$ and $\triangle CDP$ be m and n, respectively. Prove that

$$[ABCD] = (\sqrt{m} + \sqrt{n})^2.$$



Problem 5.4.4 (23-24-S4-A-P14)**.** (*10 points*) Let P be a point on the side AB in $\triangle ABC$, such that $AP = 3PB$. Let $Q \in AC$, such that $AQ = 4QC$. Prove that BQ bisects the line segment CP.



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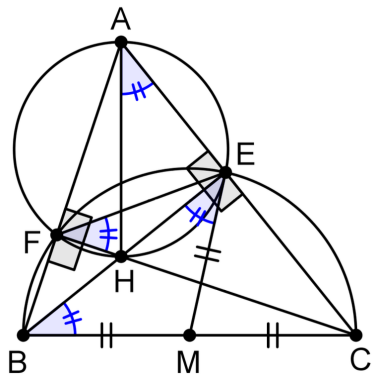
Problem 5.4.5 (23-24-S4-A-P15)**.** (*10 points*) Let $\triangle ABC$ be a right triangle. The angle bisector of $\angle ABC$ intersects AC at D. If $AD = 5$ and $CD = 3$, find AB.

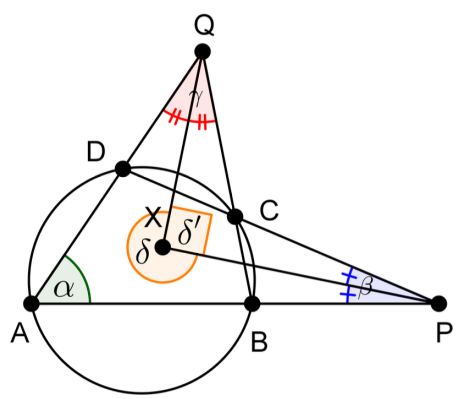
Problem 5.4.6 (23-24-S4-A-P16)**.** (*10 points*) Let ABC be an acute triangle. Let E and F be the feet of the altitudes in $\triangle ABC$ from B and C, respectively, and let M be the midpoint of BC. Prove that ME and MF are tangents to the circle (AEF).

Hint: let $BE \cap CF = H$, prove that $BFEC$ is cyclic, then $\angle MEH = \angle EAH$.

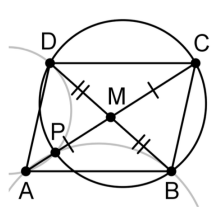


Problem 5.4.7 (23-24-S4-A-P17)**.** (*10 points*) Let $ABCD$ be a cyclic quadrilateral such that the rays AB and DC intersect at P and the rays BC and AD intersect at Q. Prove that the angle bisectors of $\angle BPC$ and $\angle CQD$ are perpendicular.

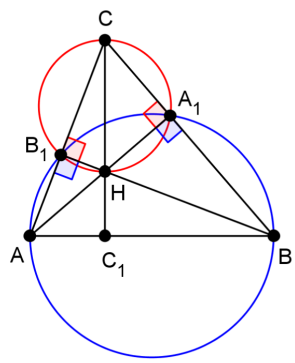
Hint: let the angle bisectors of $\angle BPC$ and $\angle CQD$ intersect at X What is the sum of all internal angles of $APXQ$?



Problem 5.4.8 (23-24-S4-A-P18)**.** (*10 points*) Let $ABCD$ be a parallelogram with $AC > BD$. The circumcircle of $\triangle BCD$ intersects AC again at P. Prove that BD is a common tangent for the circumcircles of $\triangle ABP$ and $\triangle ADP$.

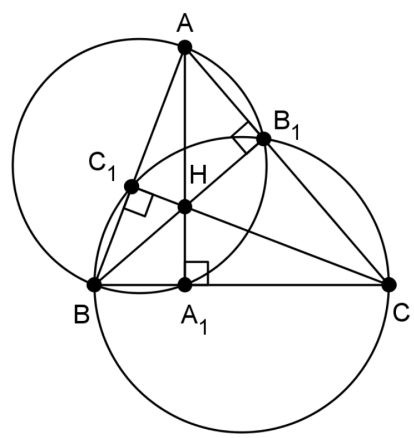


Problem 5.4.9 (23-24-S4-A-P19)**.** (*10 points*) Let AA_1, BB_1, and CC_1 be the altitudes in a $\triangle ABC$ and let H be its orthocenter. Prove that ABA_1B_1, BCB_1C_1, and CAC_1A_1 are cyclic quadrilaterals. Prove that AB_1HC_1, BC_1HA_1, and CA_1HB_1 are also cyclic quadrilaterals.



Problem 5.4.10 (23-24-S4-A-P20)**.** (*10 points*) Let AA_1, BB_1, and CC_1 be the altitudes in a $\triangle ABC$ and let H be its orthocenter. Prove that,

$$AH \cdot HA_1 = BH \cdot HB_1 = CH \cdot HC_1.$$



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5.5 Integer Sequences (O-Level)

In this session, we explore the sequences made from integers, understand, use, and prove their properties and relations.

Example 5.5.1 (23-24-S4-O-P1)

Find the smallest positive integer n such that for the sequence $a_1, a_2, \ldots, a_n$ consisting of n distinct integers, the product of all differences $a_i - a_j$ $(i < j)$ is divisible by 1991.

Example 5.5.2 (23-24-S4-O-P2)

A sequence of integers $a_1, a_2, \ldots$ is such that $a_1 = 1, a_2 = 2$ and for $n \geq 1$,

$$a_{n+2} = \begin{cases} 5a_{n+1} - 3a_n, & \text{if } a_n \cdot a_{n+1} \text{ is even,} \\ a_{n+1} - a_n, & \text{if } a_n \cdot a_{n+1} \text{ is odd,} \end{cases}$$

Prove that $a_n \neq 0$ for all n.

Example 5.5.3 (23-24-S4-O-P3)

Given a sequence $a_1, a_2, a_3, \ldots$ of positive integers in which every positive integer occurs exactly once. Prove that there exist integers ℓ and m, $1 < \ell < m$, such that:

$$a_1 + a_m = 2a_\ell.$$

Example 5.5.4 (23-24-S4-O-P4)

Does there exist a finite sequence of integers $c_1, \ldots, c_n$ such that all the numbers $a + c_1, \ldots, a + c_n$ are primes for more than one but not infinitely many different integers a?

Example 5.5.5 (23-24-S4-O-P5)

Let $p \neq 3$ be a prime number. Show that there is a non-constant arithmetic sequence of positive integers $x_1, x_2, \ldots, x_p$ such that the product of the terms of the sequence is a cube.

Example 5.5.6 (23-24-S4-O-P6)

Consider the sequence of positive integers defined by $s_1, s_2, s_3, \ldots$ of positive integers defined by $s_1 = 2$, and for each positive integer n, s_{n+1} is equal to s_n plus the product of prime factors of s_n. The first terms of the sequence are $2, 4, 6, 12, 18, 24$. Prove that the product of the 2023 smallest primes is a term of the sequence.

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Example 5.5.7 (23-24-S4-O-P7)

Find all real number a_0 such that the sequence defined by

$$a_{n+1} = 2^n - 3a_n, \ \forall n \geq 0$$

satisfies $a_0 < a_1 < a_2 < \dots$.

Example 5.5.8 (23-24-S4-O-P8)

A sequence of integers $a_1, a_2, a_3, \dots$ is called *exact* if

$$a_n^2 - a_m^2 = a_{n-m}a_{n+m}, \ \forall n > m.$$

Prove that there exists an exact sequence with $a_1 = 1$, $a_2 = 0$, and determine a_2007.

Example 5.5.9 (23-24-S4-O-P9)

For an integer $n \geq 1$ let $a(n)$ denote the total number of carries which arise when adding 2017 and $n \cdot 2017$. The first few values are given by $a(1) = 1$, $a(2) = 1$, $a(3) = 0$, which can be seen from the following:

$$\begin{array}{r} 001 \\ 2017 \\ +2017 \\ = 4034 \end{array} \qquad \begin{array}{r} 001 \\ 4034 \\ +2017 \\ = 6051 \end{array} \qquad \begin{array}{r} 000 \\ 6051 \\ +2017 \\ = 8068 \end{array}$$

Prove that

$$a(1) + a(2) + \cdots + a(10^{2017} - 1) = 10 \cdot \frac{10^{2017} - 1}{9}.$$

Example 5.5.10 (23-24-S4-O-P10)

Let a_0 be a positive integer and $a_n = 5a_{n-1} + 4$ for all $n \geq 1$. Can a_0 be chosen so that a_{54} is a multiple of 2013?

Lemma (Bézout's Lemma)

If $a, b \in \mathbb{Z}$, $a, b \neq 0$ and $d = \gcd(a, b)$, then $\exists m, n \in \mathbb{Z}$ such that:

$$am + bn = d \quad (\textbf{Bézout's Identity})$$

In other words, *there exists a linear combination* of a and b equal to d. m and n are **Bézout coefficients**.

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5.6 Problems (O-Level)

Problem 5.6.1 (23-24-S4-O-P11)**.** A sequence is given as

$$t_1 = 1,\ t_{n+1} = t_n + 3n^2 + 3n + 1.$$

Find t_{2023}.

Problem 5.6.2 (23-24-S4-O-P12)**.** Prove that none of the terms of the sequence

$$a_n = 2^{2^n} + 1,\ \forall n \geq 0 \text{ is a perfect cube.}$$

Problem 5.6.3 (23-24-S4-O-P13)**.** Let a be any integer. Define the sequence $x_0, x_1, \ldots$ by $x_0 = a$, $x_1 = 3$, and for all $n > 1$

$$x_n = 2x_{n-1} - 4x_{n-2} + 3.$$

Determine the largest integer k_a for which there exists a prime p such that p^{k_a} divides $x_{2011} - 1$.

Problem 5.6.4 (23-24-S4-O-P14)**.** Let $a_0, a_1, a_2, \ldots, a_n$ be integers such that $a_0 = a_n = 0$,

$$a_{i+1} - 2a_i + a_{i-1} = a_i^2,\ \forall i = 1, 2, \ldots, n-1.$$

Prove that $a_i \leq 0, \forall i = 1, 2, \ldots, n-1$.

Hint: assume the opposite and consider application of the Extremal Principle on $\{a_i\}$.

Problem 5.6.5 (23-24-S4-O-P15)**.** Let n and k be integers, $1 < k \leq n$. Find an integer b and a set A of n integers satisfying the following conditions:

- No product of $k-1$ distinct elements of A is divisible by b.
- Every product of k distinct elements of A is divisible by b.
- For all distinct a, a' in A, a does not divide a'.

Hint: consider $p_1, p_2, \ldots, p_n$ *the first* n *odd primes.*

Problem 5.6.6 (23-24-S4-O-P16)**.** Define the sequence of integers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, and

$$a_{n+1} = (n + 1 - \gcd(a_n, n)) \times a_n,\ \forall n \geq 1$$

Prove that

$$\frac{a_{n+1}}{a_n} = n \Longleftrightarrow n \text{ is prime or } n = 1.$$

Hint: the problem is equivalent to the following claims.

- *When* n *is prime then* $\gcd(n, a_n) = 1$*, thus* $a_{p+1} = pa_p$ *for all prime* p.
- *When* n *is composite then* $\gcd(a_n, n) > 1$.

Problem 5.6.7 (23-24-S4-O-P17)**.** A sequence of positive integers $a_1, a_2, \ldots$ is such that for each m and n the following holds: if m is a divisor of n and $m < n$, then a_m is a divisor of a_n and $a_m < a_n$. Find the least possible value of a_{2000}.

Hint: for $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$*, consider* $a_n = 2^{\alpha_1 + \alpha_2 + \cdots + \alpha_k}$.

Problem 5.6.8 (23-24-S4-O-P18)**.** Is there an infinite sequence of prime numbers p_1, p_2, ..., p_n, p_{n+1}, ... such that

$$|p_{n+1} - 2p_n| = 1$$

for each n positive integer?

Hint: first prove that $p_3 > 3$, *then for* $p_3 \equiv 1 \pmod 3$, *prove that* $p_{n+1} = 2^{n-2}(p_3 - 1),\ n = 3, 4, 5, \ldots$.

Problem 5.6.9 (23-24-S4-O-P19)**.** Consider the sequence:

$$x_1 = 19,\ x_2 = 95,\ x_{n+2} = \operatorname{lcm}(x_{n+1}, x_n) + x_n,\ \forall n > 1$$

where $\operatorname{lcm}(a, b)$ means the least common multiple of a and b. Find $\gcd(x_{1995}, x_{1996})$.

Hint: let $d = \gcd(x_{n+1}, x_n)$, *lcm*$(x_{n+1}, x_n) \gcd(x_{n+1}, x_n) = x_{n+1}x_n$, $\gcd\left(x_{n+1}, \frac{x_n}{d}\right) = 1$.

Problem 5.6.10 (23-24-S4-O-P20)**.** Let $x_1 = 1$ and

$$x_{n+1} = x_n + \left\lfloor \frac{x_n}{n} \right\rfloor + 2,\ \forall n \geq 1.$$

where x denotes the largest integer not greater than x. Determine x_{1997}.

Hint: if $x_N = AN$, *for some positive integers* A *and* N, *then for* $i = 0, 1, \ldots, N$, *prove that*

$$x_{N+i} = A(N+i) + 2i,\ x_{2N} = (A+1)(2N),\ \textit{then}$$
$$x_{2^{k-1}+i} = k(2^{k-1} + i) + 2i,\ \forall i = 0, 1, \ldots, 2^{k-1}.$$

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Chapter 6

October Test

6.1 Rules

- Each level test consisting of 8 problems will be delivered by automatic scheduled email at the registered time. No test paper will be sent twice.
- Students have 270 minutes to solve the problems. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET.
- After the allowed time, the student has 15 minutes to scan the solutions. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET, and has to submit the scanner solutions before 11:45 AM. Any solution submitted later than the allowed time will not be accepted.
- All submissions must be done by email to nghia71@gmail.com.
- Solutions must be written in English. Any solution looks unreadable, or is presented badly will not be considered for grading. Answer without proper solution will not be considered as solution. Answer consists of self-explanatory and easy-to-understand diagrams are acceptable but might not be considered as full solution.
- A problem can contain more than one questions, each with an indication of the maximum number of points can be awarded if a correct solution is provided. Partially correct solution is awarded with a some points depending on its quality. Multiple solutions can be submitted for a single problem. Each different solution, if correct, will be awarded the same number of points. Partially distinct solution also counts.

6.2 Problems for Intermediate Level

Problem 6.2.1 (23-24-OT-I-P1). (*10 points*) A bag contains 200 marbles: 60 red, 60 blue, 60 green, and 20 of white and yellow ones. It is not known how many white or yellow marbles are there, is there at least one white, or is there at least one yellow marble.

If the marbles are chosen from the bag without looking, at least how many marbles you have to pick in order to ensure that you will always have at least 20 marbles of the same colour?

Problem 6.2.2 (23-24-OT-I-P2). (*10 points*) Samuel met three girls, Emily, Isabella, and Ophelia. The following were his thoughts:

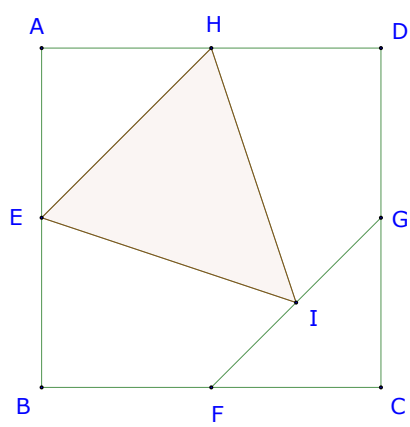
1. I love at least one of the three girls.
2. If I love Emily but not Isabella, then I also love Ophelia.
3. I either love both Ophelia and Isabella or I love neither one.
4. If I love Ophelia, then I also love Emily.

Which of the girls does Samuel love? Which ones you are not sure about?

Problem 6.2.3 (23-24-OT-I-P3). (*10 points*) $ABCD$ is a convex quadrilateral. E, F, G, and H are midpoints of AB, BC, CD, and DA, respectively.

Find the ratio of the areas of $\triangle EHI$ to the square $ABCD$, in other words

$$\frac{[EHI]}{[ABCD]}.$$



Problem 6.2.4 (23-24-OT-I-P4). (*10 points*) Is it possible to tile a 10×10 board with only $T-$tetrominos? Two $T-$tetromino examples are show below.

Problem 6.2.5 (23-24-OT-I-P5). (*10 points*) Find the least numbers to be chosen from the set $\{1, 2, \ldots, 100\}$ to ensure that there are always two numbers, one is divisible by the other.

Hint: if any two numbers are chosen from the set $\{3, 6, 12, 24, 48, 96\}$, *one is divisible by the other.*

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Problem 6.2.6 (23-24-OT-I-P6)**.** (*10 points*) On the Planet of the Apes, there live the *green* apes and the *red* apes. Furthermore these inhabitants were born in the northern or southern hemisphere. It is known that,

- the green northeners and the red southerners always tell the truth, and
- the red northeners and the green southerners always lie.

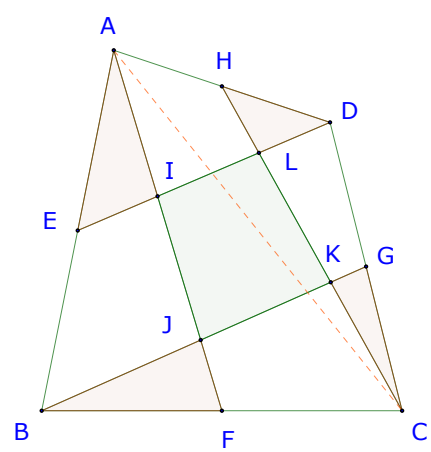
Melanie visited the Planet of the Apes on a bright day. At the center plaza, she met a native, who said *"I am a green northerner"*. In broad daylight, of course she saw what colour the native was, but from the answer, she did not know where the native was from.

What *colour* was he? Was he a *northerner* or a *southerner*?

Problem 6.2.7 (23-24-OT-I-P7)**.** (*10 points*) $ABCD$ is a convex quadrilateral. E, F, G, and H are midpoints of AB, BC, CD, and DA, respectively. AF, BG, CH, and DE pairwise intersecting as shown below at I, J, K, and L.

Prove that the green quadrilateral $IJKL$ has the same area as the sum of the area of the brown triangles AEI, BFJ, CGK, and DHL, in other words:

$$[IJKL] = [AEI] + [BFJ] + [CGK] + [DHL]$$



Problem 6.2.8 (23-24-OT-I-P8)**.** (*10 points*) A cook has a cube of cheese containing 27 unit cubes. The central unit cube is poisoned. A very clever mouse is found the cube and start to gnaw. When the mouse is done with a unit cube, it moves to the next unit cube that shares a face with the one it just finished.

Will the mouse finished the entire cube except the central one?

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6.3 Problems for Advanced Level

Problem 6.3.1 (23-24-OT-A-P1). (*10 points*) Factorize

$$(2y-3z)^3+(3z-4x)^3+(4x-2y)^3$$

Problem 6.3.2 (23-24-OT-A-P2). (*10 points*) How many 6-letter words in which one letter appears more than once, can be made from the letters F, L, I, G, H, T ? (You may use any letter as often as you like.)

Problem 6.3.3 (23-24-OT-A-P3). (*10 points*) Find all (not necessarily positive) integers (a, b, c) such that

$$a^2+b^2=8c+6.$$

Problem 6.3.4 (23-24-OT-A-P4). (*10 points*) ABC is an isosceles triangle with $AB = AC$. M is midpoint of BC. P is an arbitrary point on AM. The line through C perpendicular to BC intersects ray BP at Q. $QH \perp AB$. Prove that,

$$\frac{AB}{AP}=\frac{BC}{HQ.}$$

Problem 6.3.5 (23-24-OT-A-P5). (*10 points*) Factorize

$$(3a+3b-18ab)(3a+3b-2)+(1-9ab)^2$$

Problem 6.3.6 (23-24-OT-A-P6). (*10 points*) The MCC club sends 8 students to participate in a tennis tournament. In how many ways the organizer can pick 6 students who then can be organized into 3 pairs?

Note that a pair (A, B) is considered the same as (B, A).

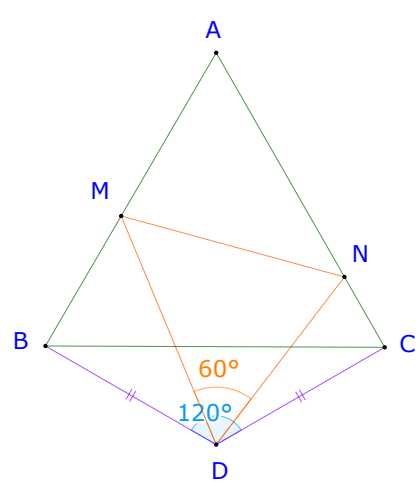
Problem 6.3.7 (23-24-OT-A-P7). (*10 points*) Find all (not necessarily positive) integers n such that

at least one of n^2-n+2 and n^4-n+2 is a perfect square.

Problem 6.3.8 (23-24-OT-A-P8). (*10 points*) ABC is an equilateral triangle. D is a point outside of $\triangle ABC$ such that $BD = DC$ and $\angle BDC = 120°$. M and N are points on AB and AC, respectively, such that $\angle MDN = 60°$.

Find the perimeter of $\triangle AMN$.

Hint: Let E be on the extension of AC such as $CE = BM$. Investigate $\triangle CDE$.



6.4 Problems for Olympiad Level

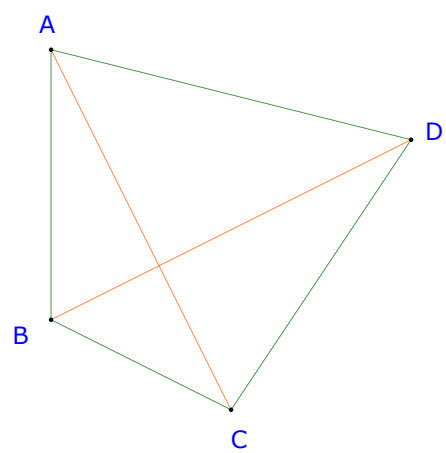
Problem 6.4.1 (23-24-OT-O-P1)**.** Any a, b, c are non-negative real number such that $a^2 + b^2 + c^2 = 3$.

Prove that

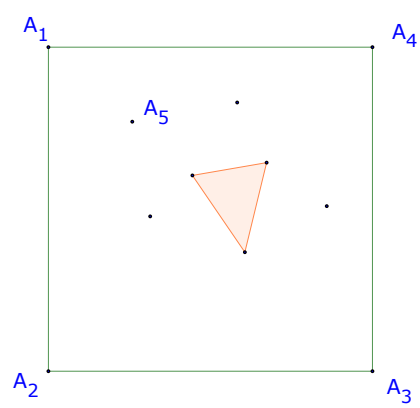
$$ab + bc + ca \leq a + b + c$$

Problem 6.4.2 (23-24-OT-O-P2)**.** (*10 points*) Let $ABCD$ be a convex quadrilateral.

Prove that if $(AB + CD)(AD + BC) = 4[ABCD]$, then $AC = BD$.



Problem 6.4.3 (23-24-OT-O-P3)**.** (*10 points*) $A_1A_2A_3A_4$ is a unit square. n is an arbitrary positive integer. n points $A_5, A_6, \ldots, A_{n+4}$ are placed inside the square. Prove that we can choose 3 points among $A_1, A_2, \ldots, A_{n+4}$ such that the area of the triangle formed by these points does not exceed $\dfrac{1}{2(n+1)}$.



Problem 6.4.4 (23-24-OT-O-P4)**.** (*10 points*) Let the sequence $\{a_n\}$ be defined as follow:

$$a_0 = a_1 = 1, a_{n+1} = a_n a_{n-1} + 1, \ \forall n \geq 1.$$

Find the remainder of a_{2023} when divided by 4.

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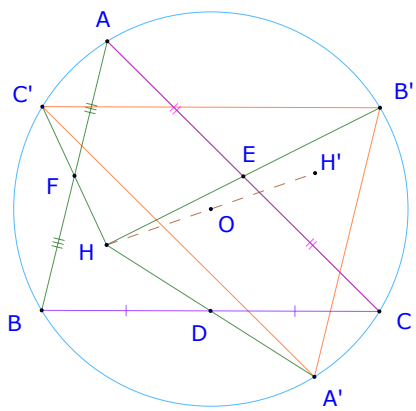
Problem 6.4.5 (23-24-OT-O-P5)**.** (*10 points*) a, b, c are positive integers such that $a^2 + b^2 + c^2 = 1$.

Prove that

$$S = 2(a^3 + b^3 + c^3) - (a + b + c) \geq -\frac{\sqrt{3}}{3}.$$

Problem 6.4.6 (23-24-OT-O-P6)**.** ABC is an acute triangle. D, E, and F are midpoints of BC, CA, and AB, respectively. Let H and O be the orthocenter and circumcenter of $\triangle ABC$. Rays HD, HE, and HF intersect the circumcircle of $\triangle ABC$ at A', B', and C', respectively. Let H' be the orthocenter of $\triangle A'B'C'$. Prove that:

- (*2 points*) $BHCA'$ is a parallelogram.
- (*3 points*) AA', BB', and CC' are diameters of the circumcircle of $\triangle ABC$.
- (*5 points*) H, O, and H' are collinear.



Problem 6.4.7 (23-24-OT-O-P7)**.** (*10 points*) S is a set of n lines which are drawn on the plane. No two lines are parallel. For every pair (ℓ_1, ℓ_2), there exists a line ℓ passes through the intersection of ℓ_1 and ℓ_2 $(\ell_1, \ell_2, \ell \in S.)$

Prove that all lines are concurrent.

Problem 6.4.8 (23-24-OT-O-P8)**.** Let the sequence $\{a_n\}$ be defined as follow:

$$a_0 = 3, a_1 = 17, a_{n+1} = 6a_n - a_{n-1},\ \forall n \geq 1.$$

(*1 points*) Prove that

$$a_{n+1}^2 - 6a_{n+1}a_n = -6a_na_{n-1} + a_{n-1}^2,\ \forall n \geq 1$$

(*9 points*) Prove that for all $n \geq 0$, $\frac{1}{2}(a_n^2 - 1)$ is a perfect square.

Note that student shall not use without proof the results of any preceding statement in the proof of any later statement. Student can earn a full 10 pts if he/she can prove the second statement without using the result of the first statement.

6.5 IC Level Test Rules

- All test lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *at most one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

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6.6 IC Test for Level 1

Problem 6.6.1 (ICLT-2023-SM1-R1-L1-P1)**.** Evaluate

$$(10^{\frac{1}{2}})^{\frac{1}{2}} \times 5^{\frac{1}{2}} \times 2^{\frac{1}{2}}.$$

(A) $(10)^{\frac{3}{4}}$ (B) $10^{\frac{1}{4}}$ (C) $2^{\frac{3}{4}} \times 5^{\frac{3}{4}}$ (D) $2^{\frac{3}{4}} \times 5^{\frac{7}{12}}$ (E) $2^{\frac{7}{12}} \times 5^{\frac{7}{12}}$

Problem 6.6.2 (ICLT-2023-SM1-R1-L1-P2)**.** My team has 10 members. In how many ways can we choose a captain, a lieutenant, a secretary, and a technician, if no member can hold more than one position?

(A) 34 (B) 40 (C) 210 (D) 1000 (E) 5040

Problem 6.6.3 (ICLT-2023-SM1-R1-L1-P3)**.** What is the maximum number of possible points of intersection of a circle and a square?

(A) 2 (B) 4 (C) 5 (D) 6 (E) 8

Problem 6.6.4 (ICLT-2023-SM1-R1-L1-P4)**.** Billy puts 18 marbles in b boxes such that each box contains the same number of marbles. If there are at least 2 boxes and each box has more than two marbles, how many possible values of b are there?

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

Problem 6.6.5 (ICLT-2023-SM1-R1-L1-P5)**.** Factor $18a^4 - 24a^3 + 6a^2$ completely as you can.

(A) $6a^2(3a-1)(1-a)$ (B) $6a^2(3a-1)(a-1)$ (C) $6(3a^4 - 24a^3 + 6a^2)$
(D) $a^2(18a^2 - 24a + 6)$ (E) $6a^2(3a^2 - 4a + 1)$

Problem 6.6.6 (ICLT-2023-SM1-R1-L1-P6)**.** The class has 7 students: 4 boys and 3 girls. In how many ways can the students seated in a row such that 3 girls seat next to each other?

(A) 720 (B) 576 (C) 384 (D) 288 (E) 144

Problem 6.6.7 (ICLT-2023-SM1-R1-L1-P7)**.** Solve

$$\frac{x}{x-1} + \frac{1}{5} = \frac{5}{x-1}.$$

Problem 6.6.8 (ICLT-2023-SM1-R1-L1-P8)**.** The first angle in a triangle is twice of one of the two other angles. The second angle is twice of of one of the two other angles. What are the possible measures of the smallest angle in the triangle?

Problem 6.6.9 (ICLT-2023-SM1-R1-L1-P9)**.** What is the second largest two-digit prime number whose digits are different and each digit is a prime?

Solution. ICLT-2023-SM1-R1-L1-P9 *Problem 2.11, Chapter 2, Introductory to Number Theory.*

A two-digit number whose digits are prime is made up of the digits $2, 3, 5$, or 7. We check the possible numbers from the largest in this list and work backwards until we find the largest prime number satisfying the required conditions:

- 77 is divisible by 7, 75 is divisible by 3. 73 is prime, so it is the largest two-digit prime number whose digits are both prime and both are different.
- 57 is divisible by 3, 55 is divisible by 5. $\boxed{53}$ is prime, so it is the second largest two-digit prime number whose digits are both prime and both are different.

□

Problem 6.6.10 (ICLT-2023-SM1-R1-L1-P10)**.** Each of the grades 7 and 8 has 5 clubs. The school decides to hold a competition. A pair of clubs in each grade play each other twice. A pair of clubs in different grades play each other once. What is the total number of matches?

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6.7 IC Test for Level 2

Problem 6.7.1 (ICLT-2023-SM1-R1-L2-P1)**.** The first angle in a triangle is twice of one of the two other angles. The second angle is twice of of one of the two other angles. What would be the smallest value of the smallest angle in the triangle?

$(A)\quad 37.5^\circ \qquad (B)\quad \dfrac{180^\circ}{5} \qquad (C)\quad \dfrac{180^\circ}{7} \qquad (D)\quad 35^\circ \qquad (E)\quad 15^\circ$

Problem 6.7.2 (ICLT-2023-SM1-R1-L2-P2)**.** Each of the grades 7 and 8 has 5 clubs. The school decides to hold a competition. A pair of clubs in each grade play each other twice. A pair of clubs in different grades play each other once. What is the total number of matches?

$(A)\quad 65 \qquad (B)\quad 40 \qquad (C)\quad 38 \qquad (D)\quad 36 \qquad (E)\quad 34$

Problem 6.7.3 (ICLT-2023-SM1-R1-L2-P3)**.** Simplify as much as possible the following product

$$\frac{2}{3a^2-12b}\cdot\frac{9a^3-36ab}{8a^2}$$

$(A)\quad \dfrac{3}{5a} \qquad (B)\quad \dfrac{2}{3a} \qquad (C)\quad \dfrac{3a^2-12b}{4a} \qquad (D)\quad \dfrac{1}{3(3a^2-12b)} \qquad (E)\quad \dfrac{3}{4a}$

Problem 6.7.4 (ICLT-2023-SM1-R1-L2-P4)**.** Which pair of triangles must be congruent?

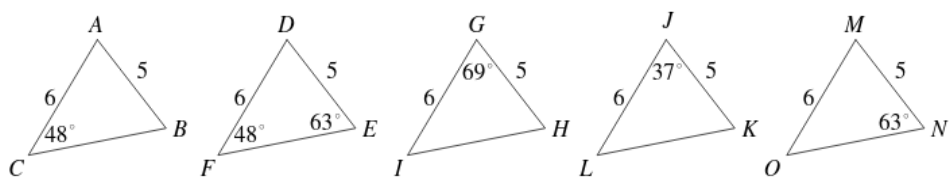

Figure 6.1: ICLT-2023-SM1-R1-L2-P4

$(A)\quad (\triangle ABC, \triangle GHI) \qquad (B)\quad (\triangle GHI, \triangle DEF) \qquad (C)\quad (\triangle MNO, \triangle ABC)$

$(D)\quad (\triangle JKL, \triangle MNO) \qquad (E)\quad (\triangle DEF, \triangle JKL)$

Problem 6.7.5 (ICLT-2023-SM1-R1-L2-P5)**.** Compute the least common multiple of $24, 18$, and 36.

$(A)\quad 504 \qquad (B)\quad 252 \qquad (C)\quad 168 \qquad (D)\quad 72 \qquad (E)\quad 36$

Problem 6.7.6 (ICLT-2023-SM1-R1-L2-P6)**.** Which one is the largest?

$(A)\quad \dbinom{6}{3} \qquad (B)\quad \dbinom{7}{4} \qquad (C)\quad \dbinom{9}{7} \qquad (D)\quad \dbinom{10}{2} \qquad (E)\quad \dbinom{44}{1}$

Problem 6.7.7 (ICLT-2023-SM1-R1-L2-P7)**.** There are red and blue balls in the Christmas shop. A red ball costs \$5 and a blue ball costs \$3. 6 red balls together weight 1 kg. 2 blue balls together weight 1 kg.

Marianna bought some red and blue balls. Altogether they cost \$108, and weight 10 kg. How many blue balls did Marianna buy?

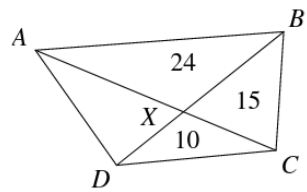

Figure 6.2: ICLT-2023-SM1-R1-L2-P8

Problem 6.7.8 (ICLT-2023-SM1-R1-L2-P8)**.** AC and BD meet at X as shown. Given $[ABX] = 24$, $[BCX] = 15$, and $[CDX] = 10$, find $[ADX]$.

Problem 6.7.9 (ICLT-2023-SM1-R1-L2-P9)**.** How many common divisors that 108, 144 and 72 have?

Problem 6.7.10 (ICLT-2023-SM1-R1-L2-P10)**.** The teacher wants to divide 8 students into two groups: one group has 3 students and the other with 5 students, such that two of the 8 students, Lena and Jean, should not be in the same group. In how many ways can the teacher do that?

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6.8 IC Test for Level 3

Problem 6.8.1 (ICLT-2023-SM1-R1-L3-P1)**.** There are red and blue balls in the Christmas shop. A red ball costs \$5 and a blue ball costs \$3. 6 red balls together weight 1 kg. 2 blue balls together weight 1 kg.

Marianna bought some red and blue balls. Altogether they cost \$108, and weight 10 kg. How many more blue balls than red balls did Marianna buy?

(A) 2 (B) 4 (C) 5 (D) 6 (E) 12

Problem 6.8.2 (ICLT-2023-SM1-R1-L3-P2)**.** AC and BD meet at X as shown. Given $[ABX] = 24$, $[BCX] = 15$, and $[CDX] = 10$, find $[ABCD]$.

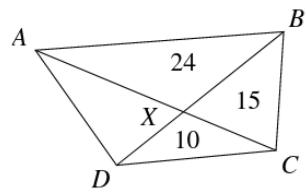


Figure 6.3: ICLT-2023-SM1-R1-L3-P2

(A) 61 (B) 63 (C) 65 (D) 69 (E) 73

Problem 6.8.3 (ICLT-2023-SM1-R1-L3-P3)**.** How many common divisors that 108, 144 and 72 have?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Problem 6.8.4 (ICLT-2023-SM1-R1-L3-P4)**.** The teacher wants to divide 8 students into two groups: one group has 3 students and the other with 5 students, such that two of the 8 students, Lena and Jean, should not be in the same group. In how many ways can the teacher do that?

(A) 44 (B) 40 (C) 36 (D) 30 (E) 24

Problem 6.8.5 (ICLT-2023-SM1-R1-L3-P5)**.** What is the ratio of y to x if

$$\frac{7x - 3y}{9x - 2y} = \frac{4}{7}$$

(A) $13 : 18$ (B) $11 : 15$ (C) $9 : 11$ (D) $1 : 1$ (E) $1 : 2$

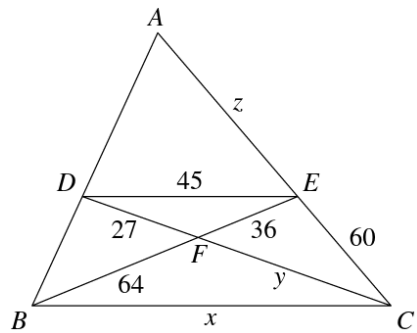
Problem 6.8.6 (ICLT-2023-SM1-R1-L3-P6)**.** In the diagram below, $DE \parallel BC$, and the segments have the lengths as shown in the diagram, $x = BC$, $y = CF$, and $z = EA$.

Find $x + y + z$.

(A) $\frac{1430}{7}$ (B) 188 (C) $\frac{1436}{7}$ (D) 109 (E) $\frac{764}{7}$

Problem 6.8.7 (ICLT-2023-SM1-R1-L3-P7)**.** What are the positive divisors of 540 having exactly 4 positive divisors?

For example 21 is a positive divisor of 42 *and 21 has exactly 4 positive divisors:* $1, 3, 7, 21$.



Problem 6.8.8 (ICLT-2023-SM1-R1-L3-P8)**.** Six parallel lines in a plane intersect a set of n parallel lines that go in another direction. The lines form a total of 150 parallelograms, many of which overlap each other. Find n.

Problem 6.8.9 (ICLT-2023-SM1-R1-L3-P9)**.** Nine people can mow a lawn in fifteen hours. How many more people are needed to mow the lawn in just five hours, assuming each person mows at the same rate?

Problem 6.8.10 (ICLT-2023-SM1-R1-L3-P10)**.** Find the highest power of 43 that divides $41! + 42! + 43!$.

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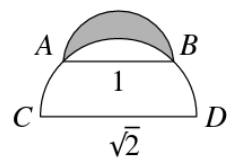
6.9 IC Test for Level 6

Problem 6.9.1 (ICLT-2023-SM1-R1-L6-P1)**.** Suppose two numbers x and y are each chosen such that $0 < x < 1$ and $0 < y < 1$. What is the probability that $x + y > \frac{1}{2}$?

$(A)\ \frac{7}{8}$ $(B)\ \frac{5}{8}$ $(C)\ \frac{3}{8}$ $(D)\ \frac{3}{4}$ $(E)\ \frac{1}{2}$

Problem 6.9.2 (ICLT-2023-SM1-R1-L6-P2)**.** The shaded portion of the figure is called a lune. Given that $AB = 1$, $CD = \sqrt{2}$, and that AB and CD are diameters of the respective semicircles shown, find the area of the lune.



$(A)\ \frac{3}{8}$ $(B)\ \frac{1}{4}$ $(C)\ \frac{\pi}{4}$ $(D)\ \frac{\pi - 1}{8}$ $(E)\ \frac{3\pi - 4}{16}$

Problem 6.9.3 (ICLT-2023-SM1-R1-L6-P3)**.** Write $\frac{5 - i}{-1 + 3i}$ as a single complex number, where i is the imaginary unit $i = \sqrt{-1}$.

$(A)\ \frac{1}{5} - \frac{8}{5}i$ $(B)\ -\frac{4}{5} + \frac{7}{5}$ $(C)\ -\frac{4}{5} - \frac{7}{5}$ $(D)\ \frac{4}{5}\frac{7}{5}$ $(E)\ \frac{1}{5} + \frac{8}{5}$

Problem 6.9.4 (ICLT-2023-SM1-R1-L6-P4)**.** What is the unit digit of $3^{2022} - 2^{2022}$.

$(A)\ 9$ $(B)\ 7$ $(C)\ 5$ $(D)\ 3$ $(E)\ 1$

Problem 6.9.5 (ICLT-2023-SM1-R1-L6-P5)**.** The numbers on a standard six-sided die are arranged such that numbers on opposite faces always add to 7. The die is rolled, and the product of the numbers appearing on the top and bottom faces of the die is calculated. What is the expected value of this product?

$(A)\ 84$ $(B)\ 28$ $(C)\ 12$ $(D)\ \frac{28}{3}$ $(E)\ \frac{14}{3}$

Problem 6.9.6 (ICLT-2023-SM1-R1-L6-P6)**.** x, y, and z are positive real numbers such that $2x > z > 3y > 0$ and

$$\begin{cases} (2x - z)^2 &= z^2 - 5 \\ (3y - z)^2 &= z^2 - 5 \end{cases}$$

Find the product xy.

$(A)\ 1$ $(B)\ \frac{1}{2}$ $(C)\ \frac{3}{4}$ $(D)\ \frac{4}{5}$ $(E)\ \frac{5}{6}$

Problem 6.9.7 (ICLT-2023-SM1-R1-L6-P7)**.** A bin has 6 white balls and k black balls in it, where k is an unknown positive integer. A ball is drawn at random from the bin. If a white ball is drawn, the player wins \$2, but if a black ball is drawn, the player loses \$1. If the expected loss for playing the game is $\frac{1}{2}$, then what is k?

Problem 6.9.8 (ICLT-2023-SM1-R1-L6-P8)**.** What is the unit digit of the sum

$$1! + 2! + 3! + \cdots + 2023!.$$

when written in base-8?

Problem 6.9.9 (ICLT-2023-SM1-R1-L6-P9)**.** Solve the inequality

$$\frac{2}{x-3} - 3- + 2 > -1.$$

Problem 6.9.10 (ICLT-2023-SM1-R1-L6-P10)**.** In isosceles triangle $\triangle ABC$, BC is the base. Given that $BC < AB$, prove that that $\angle BAC < 60°$.

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Chapter 7

Session 5

7.1 One two three (I-Level)

In this session we discuss basic counting methods.

Example 7.1.1 (23-24-S5-I-P1)

There are five different shirts and three different skirts in the Girl's Stuffs store?

How many ways to buy a shirt and a skirt?

Let's extend the previous problem a bit more.

Example 7.1.2 (23-24-S5-I-P2)

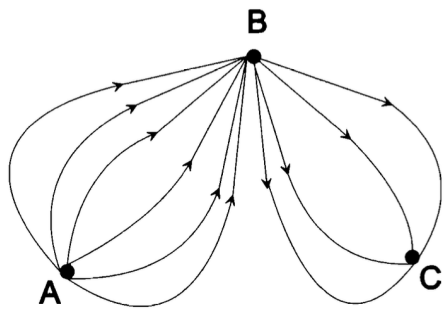
Julie visits the Girl's Stuffs store mentioned in the 23-24-S5-I-P1 example, the store has four different pairs of shoes.

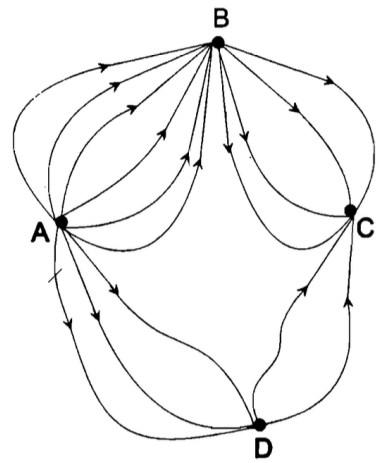
How many ways Julie can have a shirt, a skirt, and a pair of shoes?

Example 7.1.3 (23-24-S5-I-P3)

There are three towns A, B, C in the Counting County. Six roads go from A to B, four roads go from B to C, see the figure on the left below. In how many way Jeremy can drive from A to C?

The Chief of the County announces the establishment of a new town, D with several roads connecting it to A and C, see the figure below on the right. Now, in how many way Jeremy can drive from A to C?





Example 7.1.4 (23-24-S5-I-P4)

Julie visits the Girl's Stuffs store. There are five different shirts, three different skirts, and four different hats. If Julie just want to buy two items of different types (shirt, shirt, hat), in how many ways can she do it?

Example 7.1.5 (23-24-S5-I-P5)

We call a positive integer *odd-looking* if it has only odd digits, e.g. 135, and *even-looking* if it has only even digits, e.g. 246.

Khoa counts the number of four-digit *odd-looking* numbers, what's he got? Knowing what's Khoa got, Giang can immediately tells the number of four-digit numbers, which are not *odd-looking*, how could she do that?

Now, Khoa challenges Giang to count the number of four-digit *even-looking* numbers, what can she tell Khoa?

Example 7.1.6 (23-24-S5-I-P6)

Nhat Minh toss a coin three times, how many sequence of heads (H) and tails (T) can he obtain?

Each square in the 2×2 grid can be coloured black or white. How many different ways can Minh colour them?

In how many ways can Minh fill a Super Easy Winner lotto card? In this lotto, Minh must predict the result of 13 Premier League games, indicating either a victory for one of two teams, or a draw.

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Example 7.1.7 (23-24-S5-I-P7)

Giang's younger sister learns to play with lego. She has a number of letters, A, B and C. Being very creative, she wants to invent a new language, each word has no more than four letters, choosing from those letter legos. How many words does this new language contain?

Example 7.1.8 (23-24-S5-I-P8)

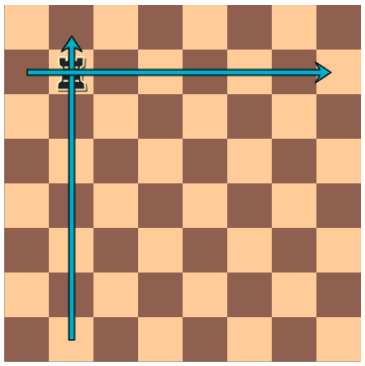
Nhat Minh and Tue Minh with another nine MCC students represent the club football team. In how many ways the team can choose a captain and a vice captain?

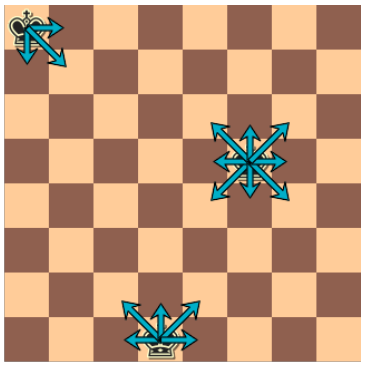
In how many ways the team can sew a three-coloured flag with three horizontal strips of equal height if they have pieces of fabric of six colours? Note that they of course can make a distinction between the top and the bottom of the flag.

Example 7.1.9 (23-24-S5-I-P9)

In how many ways can Huy put a white and a black rook on the chessboard such that they do not attack each other?

In how many ways can Huy put a white and a black king on the chessboard such that they do not attack each other?





In this last example, we discuss permutations.

Example 7.1.10 (23-24-S5-I-P10)

How many ways are there for Chi-Mai to pay four balls, coloured red, black, blue, and green in a row?

How many different words can Chi-Mai have if she rearranges the letters in the word BEAUTY?

How many different words can Chi-Mai have if she rearranges the letters in the word FOOT?

How many different words can Chi-Mai have if she rearranges the letters in the word HAVANA?

7.2 Problems (I-Level)

Problem 7.2.1 (23-24-S5-I-P11)**.** (*10 points*) How many different words can Chi-Mai have if she rearranges the letters in the word MATHEMATICAL?

Problem 7.2.2 (23-24-S5-I-P12)**.** Each of the problems can be solved separately.

- (*5 points*) There are 19 islands in the Scatter archipelago, and every pair of them is connected by an air route. How many air routes are there?
- (*5 points*) Each of two novice collectors has 10 stamps and 5 postcards. We call an exchange fair if they exchange a stamp for a stamp or a postcard for a postcard. How many ways are there to carry out one fair exchange between these two collectors?

Problem 7.2.3 (23-24-S5-I-P13)**.** Each of the problems can be solved separately.

- (*2 points*) How many diagonals are there in a pentagon?
- (*8 points*) How many diagonals are there a convex $n-$gon?

Problem 7.2.4 (23-24-S5-I-P14)**.** Each of the problems can be solved separately.

- (*5 points*) A *necklace* is a circular string with several beads on it. It is allowed to rotate a necklace but not to turn it over. How many different necklaces can be made using 11 different beads?
- (*5 points*) Assume now that it is allowed to turn a necklace over. How many necklaces can be made using 11 different beads?

Problem 7.2.5 (23-24-S5-I-P15)**.** (*10 points*) How many six-digit numbers have at least one even digit?

Problem 7.2.6 (23-24-S5-I-P16)**.** (*10 points*) How many six-digit numbers have at least two equal digits?

Note that 123455 is such a number, and 345728 is not.

Problem 7.2.7 (23-24-S5-I-P17)**.** (*10 points*) There are five books on a shelf. How many ways are there to arrange some (or all) of them in a stack? The stack may consist of a single book.

Problem 7.2.8 (23-24-S5-I-P18)**.** (*10 points*) How many ways are there to choose four cards of different suits and different values from a deck of 52 cards?

Problem 7.2.9 (23-24-S5-I-P19)**.** Each of the problems can be solved separately.

- (*5 points*) How many ways are there to put two knights on a chessboard so that they do not attack each other?
- (*5 points*) How many ways are there to put eight rooks on a chessboard so that they do not attack each other?

Problem 7.2.10 (23-24-S5-I-P20)**.** (*10 points*) There are three rooms in a dormitory: one single, one double, and one for four students. How many ways are there to house seven students in these rooms?

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7.3 The Induction Principle (A-Level)

In this session we discuss the Induction Principle.

Definition (Induction Principle)**.** Let a be an integer, and let $P(n)$ be a statement (or proposition) about n for each integer $n \geq a$. The **principle of induction** is a way of proving that $P(n)$ is true for all integers $n \geq a$ in two steps:

1. *The Base case*: Prove that $P(a)$ is true.
2. *The Inductive step*: Assume that $P(k)$ is true for some integer $k \geq a$, and use this to prove that $P(k+1)$ is true.

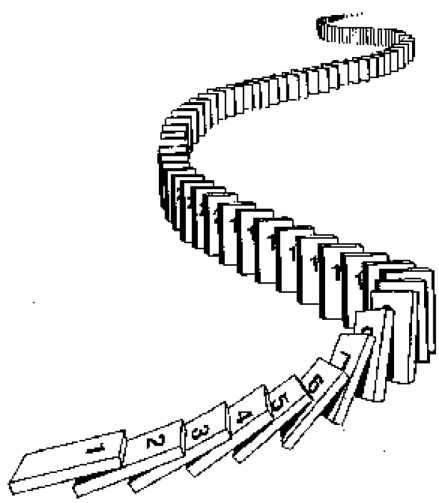
Then we may conclude that $P(n)$ is true for all integers $n \geq a$.

Our reasoning with the wave of falling dominoes shows that an inductive step is but a shortened form of the chain of theorems shown in the figure below;

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \cdots \Rightarrow P(k) \Rightarrow P(k+1) \Rightarrow \cdots$$

We will call theorems in this chain *steps*, and the process of their successive proof *the process of induction*. This process can be visually represented as a wave of proofs, running from statement to statement along a chain of theorems.

Below is how you can visualize the wave of proofs.



The domino effect is the chain reaction consisting of a row of falling dominoes. The dominoes are vertical and close enough to one another. One pushes the first domino of the row, and this falls onto the second domino, which falls onto the third domino and so on. In the end an infinitely long row has fallen!

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Example 7.3.1 (23-24-S5-A-P1)
Show that for all $n \geq 2$,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = \frac{(n-1)n(n+1)}{3}.$$

Example 7.3.2 (23-24-S5-A-P2)
Show that for all $n \geq 2$,

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

Example 7.3.3 (23-24-S5-A-P3)
Show that for all $n \geq 1$,

$$n^3 + (n+1)^3 + (n+2)^3 \text{ is divisible by } 9.$$

Example 7.3.4 (23-24-S5-A-P4)
Show that for all $n \geq 1$,

$$11^{n+2} + 12^{2n+1} \text{ is divisible by } 133.$$

Example 7.3.5 (23-24-S5-A-P5)
Show that for all $n \geq 2$,

$$2^n > 1 + n\sqrt{2^{n-1}}.$$

Example 7.3.6 (23-24-S5-A-P6)
Show that for all $n \geq 1$,

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \leq \frac{1}{\sqrt{2n+1}}.$$

Example 7.3.7 (23-24-S5-A-P7)
Prove that every positive integer can be represented as a sum of several distinct powers of 2.

Example 7.3.8 (23-24-S5-A-P8)
There are $n \geq 1$ real numbers with non-negative sum written on a circle. Prove that one can enumerate them $a_1, a_2, \ldots, a_n$ such that they are consecutive on the circle and

$$a_1 \geq 0, a_1 + a_2 \geq 0, \ldots, a_1 + a_2 + \cdots + a_{n-1} \geq 0, a_1 + a_2 + \cdots + a_{n-1} \geq 0.$$

Example 7.3.9 (23-24-S5-A-P9)

The sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined as follow,

$$a_1 = 3, a_2 = 5, a_{n+1} = 3a_n - 2a_{n-1}, \text{ for } n \geq 2.$$

Prove that $a_n = 2^n + 1$, for all n positive integer.

Example 7.3.10 (23-24-S5-A-P10)

A bank has an unlimited supply of 3-peso and 5-peso notes. Prove that it can pay any number of pesos greater than 7.

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7.4 Problems (A-Level)

Problem 7.4.1 (23-24-S5-A-P11). (*10 points*) Show that for a, b, n positive integers

$$\frac{1}{a(a+b)} + \frac{1}{(a+b)(a+2b)} + \cdots + \frac{1}{(a+(n-1)b)(a+nb)} = \frac{n}{a(a+nb)}$$

Problem 7.4.2 (23-24-S5-A-P12). (*10 points*) Show that for m, n non-negative integers

$$\frac{m!}{0!} + \frac{(m+1)!}{1!} + \cdots + \frac{(m+n)!}{n!} = \frac{(m+n+1)!}{n!(m+1)}$$

Problem 7.4.3 (23-24-S5-A-P13). (*10 points*) Show that for all $n \geq 1$,

$$4^n + 15n - 1 \text{ is divisible by } 9.$$

Problem 7.4.4 (23-24-S5-A-P14). (*10 points*) Show that for all $n \geq 1$,

$$2^{3^n} + 1 \text{ is divisible by } 3^{n+1}.$$

Problem 7.4.5 (23-24-S5-A-P15). (*10 points*) Show that for all $n \geq 1$,

$$2(\sqrt{n+1} - 1) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

Problem 7.4.6 (23-24-S5-A-P16). (*10 points*) Let $a_1, a_2, \ldots, a_n$ be integers, not all zeros, such as

$$a_1 + a_2 + \cdots + a_n = 0.$$

Prove that, for some all $k \in \{1, 2, \ldots, n\}$,

$$|a_1 + 2a_2 + \cdots + 2^{k-1}a_k| > \frac{2^k}{3}.$$

Problem 7.4.7 (23-24-S5-A-P17). (*10 points*) Prove that any polygon (not necessarily convex) can be dissected into triangles by disjoint diagonals (they are allowed to meet only at vertices of the polygon).

Hint: Use an induction on the number of sides. The inductive step is based on a lemma stating that each polygon (except a triangle) has at least one diagonal which lies completely within the polygon. Note that not necessarily convex. Such a diagonal dissects the polygon into two polygons with fewer sides. Of course you have to prove the lemma if you wish to use it.

Problem 7.4.8 (23-24-S5-A-P18). (*10 points*) For a is a non-zero number, it is known that $a + \frac{1}{a}$ is an integer. Prove that for any $n \geq 1$ positive integer,

$$a^n + \frac{1}{a^n} \text{ is an integer.}$$

Problem 7.4.9 (23-24-S5-A-P19). (*10 points*) The sequence of Fibonacci number is defined by

$$f_1 = f_2 = 1, f_{n+1} = f_n + f_{n-1}, \text{ for } n \geq 2.$$

Prove that the nth Fibonacci number is divisible by 3 if and only if n is divisible by 4.

Hint: Prove a more general statement about the repetition of remainders of Fibonacci numbers modulo 3 (with period 8).

Problem 7.4.10 (23-24-S5-A-P20). (*10 points*) A set of n points is taken on a circle and each pair is connected by a segment. It happens that no three of these segments meet at the same point. Into how many parts do they divide the interior of the circle?

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7.5 Solving Problems with Induction Principle (O-Level) - Part 1

Problem 7.5.1 (Expression as sum of dictinct divisors)**.** (*10 points*) $n \geq 3$ is a positive integer. Prove that $n!$ can be written as a sum of n distinct divisors of itself.

Problem 7.5.2 (Upper bound of the sum of reciprocals)**.** $n \geq 1$ is a positive integer.

(*5 points*) Prove that

$$x_n = \frac{1}{1} + \frac{1}{2^1} + \cdots + \frac{1}{2^n} < 2.$$

(*5 points*) $n \geq 1$ is a positive integer. Prove that

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \frac{7}{10}.$$

Problem 7.5.3 (Fibonacci sequence)**.** (*10 points*) $n \geq 1$ is a positive integer. (f_n) is the Fibonacci sequence: $f_0 = f_1 = 1, f_{n+1} = f_n + f_{n-1}$. Prove that

$$f_{2n+1} = f_{n+1}^2 + f_n^2 \quad (*)$$

Problem 7.5.4 (Sum of reciprocals of binomial coefficients)**.** (*10 points*) $n \geq 1$ is a positive integer. Prove that

$$\binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1} = \frac{n+1}{2^{n+1}}\left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1}\right)$$

Problem 7.5.5 (An identity fuction)**.** (*5 points*) Let $f: \mathbb{Z}^+ \to \mathbb{Z}$ be a function with the following properties:

1. $f(2) = 2$;
2. $f(mn) = f(m)f(n), \ \forall m, n$;
3. $f(m) > f(n), \ \forall m > n$.

Prove that $f(n) = n, \ \forall n \in \mathbb{Z}^+$.

(*5 points*) Let $f: \mathbb{Z}^+ \to \mathbb{Z}$ be a function such that:

$$f(f(n)) + f(n) = 2n.$$

Prove that $f(n) = n, \ \forall n \in \mathbb{Z}^+$.

Problem 7.5.6 (Classical inequalities)**.** (*5 points*) Let $a_1, a_2, \ldots, a_n$ be any real numbers and $b_1, b_2, \ldots, b_n$ be any positive real numbers, then use the Induction Principle to prove that

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

(*5 points*) Let $a_1, a_2, \ldots, a_n$ be any positive real numbers, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Problem 7.5.7 (Algebraic manipulation)**.** (*10 points*) Let $a_1, a_2, \ldots, a_n$ be any real numbers satisfying $a_{i+j} \leq a_i + a_j,\ \forall i, j \in \mathbb{Z}^+$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n,\ \forall n \geq 1.$$

Problem 7.5.8 (Recurrence relation for solutions to equations in similar form)**.** (*10 points*) Let $k \geq 3$ be an integer. Show that there exist odd integer x and y such that

$$2^k = 7x^2 + y^2.$$

Problem 7.5.9 (Induction principle in combinatorial geometry)**.** (*10 points*) Let $A_1A_2 \ldots A_n$ be a convex polygon inscribed in a circle such that among its vertices, there are no two which form a diameter. Prove that if among the triangles $A_pA_qA_r$ as distinct p, q, r range over $1, 2, \ldots, n$, there is at least one acute triangle, then there are at least $n - 2$ such acute triangle.

Problem 7.5.10 (Divisibility by 9 as invariant in games)**.** (*5 points*) There are 56 candies on the table, Players play by turns (one after another). In each turn it is allowed to take either one, three, or five candies. The winner is the person who takes the last set of candies. Given the perfect play on both sides, who will win?

(*5 points*) Three players play the following game. There are 54 candies on the table. Players play by turns (one after the other). In each turn it is allow to take either one, three, or five candies such that the same number of candies cannot be taken in two consecutive turns. The winner is the person who takes the last set of candies. Given the perfect play of all players, who will win?

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Chapter 8

Session 6

8.1 Word problems (I-Level)

In this session we discuss solving words problem, that is, how to establish equations or systems of equations, and solve them.

Example 8.1.1 (23-24-S6-I-P1)

In eight years Henry will be three times the age that Sally was last year. Twenty five years ago their ages added to 83. How old is Henry now?

Example 8.1.2 (23-24-S6-I-P2)

The lengths of the diagonals of a rhombus are, in inches, two consecutive integers. The area of the rhombus is 210 square inches. Find its perimeter, in inches.

Example 8.1.3 (23-24-S6-I-P3)

How many gallons of a solution which is 15% alcohol do we have to mix with a solution that is 35% alcohol to make 250 gallons of a solution that is 21% alcohol?

Example 8.1.4 (23-24-S6-I-P4)

A rectangle has area of 1100. If the length is increased by ten percent and the width is decreased by ten percent, what is the area of the rectangle?

Example 8.1.5 (23-24-S6-I-P5)

A week ago, Sandy's seasonal Little League batting average was 360. After five more at bats this week, Sandy's batting average is up to 400. What is the smallest number of hits that Sandy could have had this season?

Example 8.1.6 (23-24-S6-I-P6)

Fill in numbers in the boxes below so that the sum of the entries in each three consecutive boxes is 2005. What is the number that goes into the leftmost box?

		999					888	

Example 8.1.7 (23-24-S6-I-P7)

Bill's age is one third larger than Tracy's age. In 30 years Bill's age will be one eighth larger than Tracy's age. How many years old is Bill?

Example 8.1.8 (23-24-S6-I-P8)

At a movie theater tickets for adults cost 4 dollars more than tickets for children. One afternoon the theater sold 100 more child tickets than adult tickets for a total sales amount of 1475 dollars. How much money would the theater have taken in if the same tickets were sold, but the costs of the child tickets and adult tickets were reversed?

Example 8.1.9 (23-24-S6-I-P9)

A rogue spaceship escapes. 54 minutes later the police leave in a spaceship in hot pursuit. If the police spaceship travels 12% faster than the rogue spaceship along the same route, how many minutes will it take for the police to catch up with the rogues?

Example 8.1.10 (23-24-S6-I-P10)

The sizes of the freshmen class and the sophomore class are in the ratio 5:4. The sizes of the sophomore class and the junior class are in the ratio 7:8. The sizes of the junior class and the senior class are in the ratio 9:7. If these four classes together have a total of 2158 students, how many of the students are freshmen?

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8.2 Problems (I-Level)

Problem 8.2.1 (23-24-S6-I-P11)**.** (*10 points*) The product of two positive numbers is equal to 50 times their sum and 75 times their difference. Find their sum.

Problem 8.2.2 (23-24-S6-I-P12)**.** (*10 points*) There were 891 people voting at precinct 91. There were 20 percent more female voters than male voters. How many female voters were there?

Problem 8.2.3 (23-24-S6-I-P13)**.** (*10 points*) Find a positive rational number that the number is equal to the ratio of 7 plus the number's reciprocal and 65 minus the number's reciprocal.

Problem 8.2.4 (23-24-S6-I-P14)**.** (*10 points*) While driving his car, Ken pulled off the road to get gasoline when he was $\frac{7}{12}$ the way through his trip. After driving another eleven miles, he noticed that he was $\frac{13}{20}$ of the way through his trip. How many miles long was his entire trip?

Problem 8.2.5 (23-24-S6-I-P15)**.** (*10 points*) A bowl contained 10% blue candies and 25% red candies. A bag containing three quarters red candies and one quarter blue candies was added to the bowl. Now the bowl is 16% blue candies. What percentage of the candies in the bowl are now red?

Problem 8.2.6 (23-24-S6-I-P16)**.** (*10 points*) One plant is now 44 centimeters tall and will grow at a rate of 3 centimeters every 2 years. A second plant is now 80 centimeters tall and will grow at a rate of 5 centimeters every 6 years. In how many years will the plants be the same height?

Problem 8.2.7 (23-24-S6-I-P17)**.** (*10 points*) Tom can run to Beth's house in 63 minutes. Beth can run to Tom's house in 84 minutes. At noon Tom starts running from his house toward Beth's house while at the same time Beth starts running from her house toward Tom's house. When they meet, they both run at Beth's speed back to Beth's house. At how many minutes after noon will they arrive at Beth's house?

Problem 8.2.8 (23-24-S6-I-P18)**.** (*10 points*) Bill bought 13 notebooks, 26 pens, and 19 markers for 25 dollars. Paula bought 27 notebooks, 18 pens, and 31 markers for 31 dollars. How many dollars would it cost Greg to buy 24 notebooks, 120 pens, and 52 markers?

Problem 8.2.9 (23-24-S6-I-P19)**.** (*10 points*) Purple College keeps a careful count of its students as they progress each year from the freshman class to the sophomore class to the junior class and, finally, to the senior class. Each year at the college one third of the freshman class drops out of school, 40 students in the sophomore class drop out of school, and one tenth of the junior class drops out of school.

Given that the college only admits new freshman students, and that it wants to begin each school year with 3400 students enrolled, how many students does it need to admit into the freshman class each year?

Problem 8.2.10 (23-24-S6-I-P20)**.** (*10 points*) Sally's salary in 2006 was \$37,500. For 2007 she got a salary increase of x percent. For 2008 she got another salary increase of x percent. For 2009 she got a salary decrease of $2x$ percent. Her 2009 salary is \$34,825.

Suppose instead, Sally had gotten a $2x$ percent salary decrease for 2007, an x percent salary increase for 2008, and an x percent salary increase for 2009. What would her 2009 salary be then?

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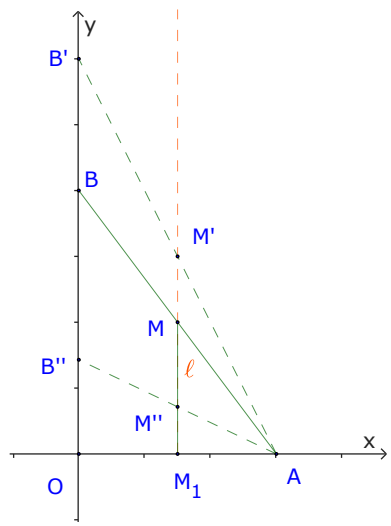
8.3 The Locus (A-Level)

In geometry, a *locus* (plural: *loci*) (Latin word for "place", "location") is a set of all points (commonly, a line, a line segment, a curve or a surface), whose location satisfies or is determined by one or more specified conditions. *A proof for the hypothesis about the existence of a locus - a set of of points S, must contains two parts*: (i) if point P satisfies certain condition C then P must be an element of the set S, and (ii) if point P is an element of S then P must satisfies the condition C.

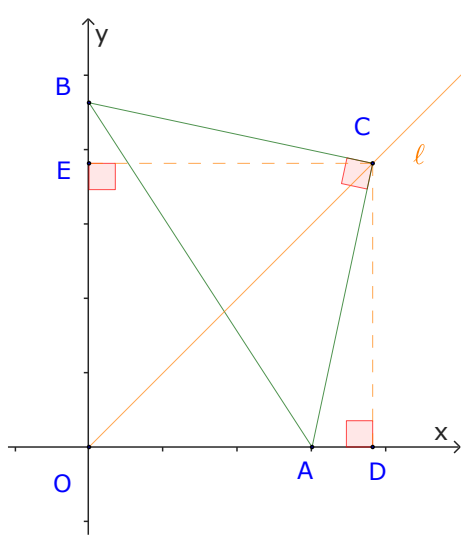
Example 8.3.1 (23-24-S6-A-P1)

Let A be a fixed point on ray $\vec{x}$. B is an arbitrary point on ray $\vec{y}$, as shown below. Find the locus of the midpoints M of segments AB.



Example 8.3.2 (23-24-S6-A-P2)

Let A be a fixed point on ray $\vec{x}$. B is an arbitrary point on ray $\vec{y}$, as shown below. Find the locus of points C such that $\triangle ABC$ is an isosceles right triangle, $\angle ACB = 90°$, $CA = CB$.



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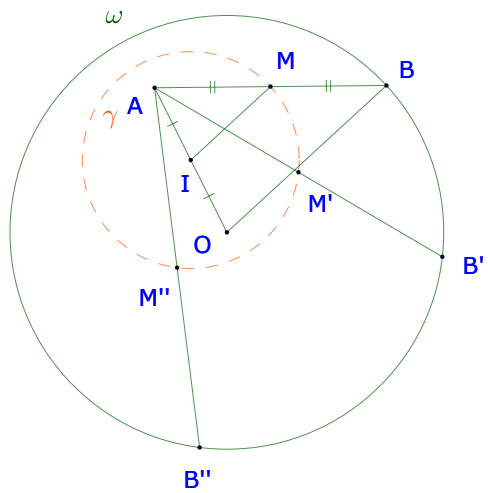
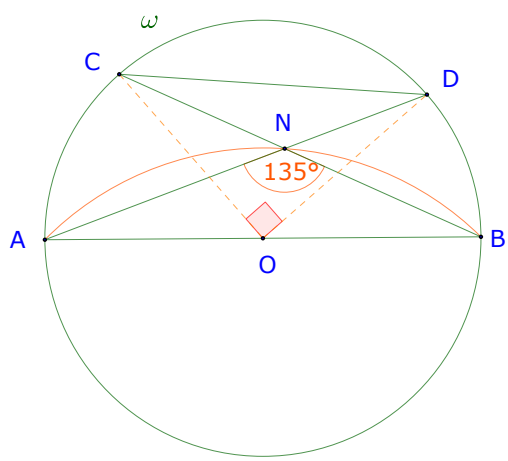
Example 8.3.3 (23-24-S6-A-P3)

Given the circle ω centred at O. A is a fixed point inside the circle, B is a point moving on the perimeter of the circle. Find the locus of the midpoints M of the segment AB.



Example 8.3.4 (23-24-S6-A-P4)

ω is a semi-circle centred at O with diameter AB, radius R. C, D are arbitrary points on ω such that $CD = R\sqrt{2}$. AD intersects BC at N. Find the locus of such points N.



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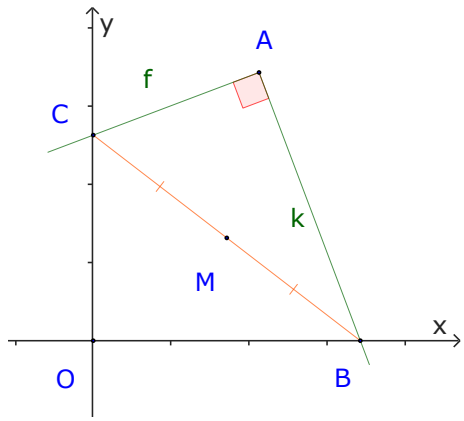
8.4 Problems (A-Level)

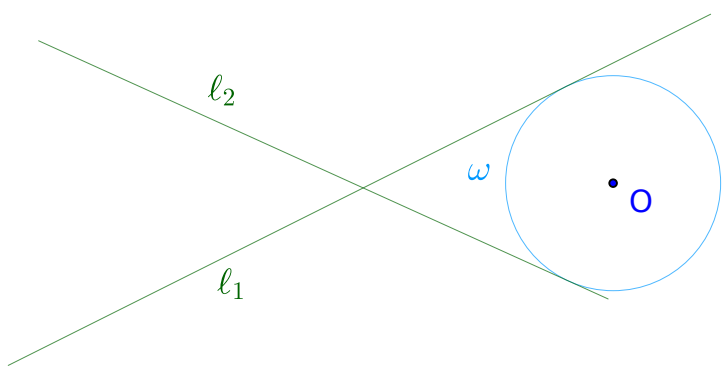
Problem 8.4.1 (23-24-S6-A-P5)**.** (*15 points*) A is a point in the quadrant determined by rays $\vec{x}$ and $\vec{y}$, as shown below. A right angle with A as its vertex rotates around A. The rays of the right angle intersect $\vec{x}$ at B and $\vec{y}$ at C, respectively. Find the locus of the midpoints M of the segment BC.

Note that the diagram below shows only one example configuration and should not be used for generalization.



Problem 8.4.2 (23-24-S6-A-P6)**.** (*15 points*) Lines ℓ_1, ℓ_2 intersect at O. Find the locus of the centres of the circle ω that is tangent to both lines.

Note that the diagram below shows only one example configuration and should not be used for generalization.



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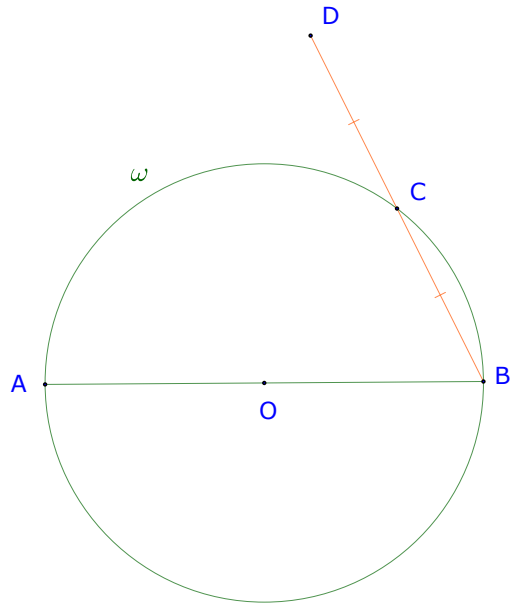
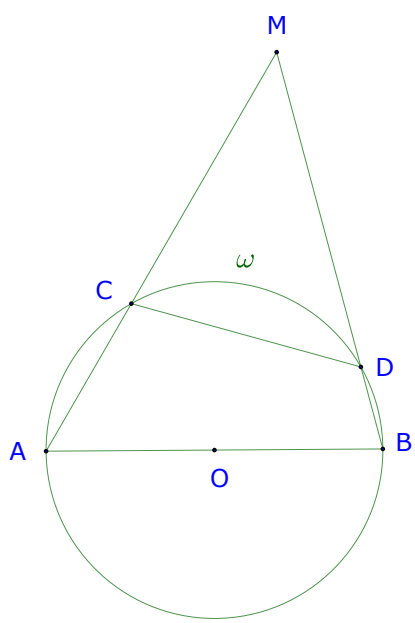
Problem 8.4.3 (23-24-S6-A-P7)**.** (*15 points*) Given the circle ω centred at O with diameter AB. C is a point moving on the perimeter of the circle ω. Point D is on ray $\overrightarrow{BC}$ towards C such that $BC = CD$, as shown below. Find the locus of such points D.

Note that the diagram below shows only one example configuration and should not be used for generalization.



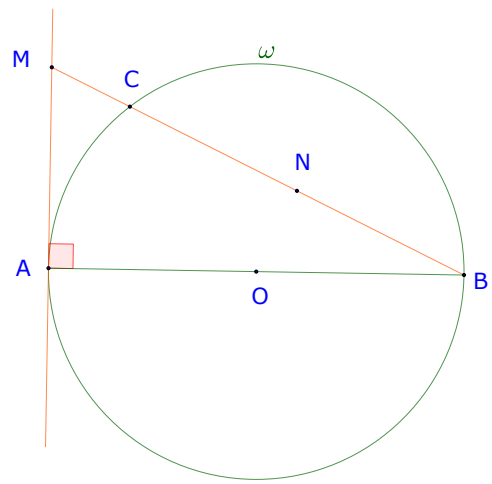
Problem 8.4.4 (23-24-S6-A-P8)**.** (*15 points*) ω is a semi-circle centred at O with diameter AB, radius R. C, D are arbitrary points on ω such that $CD = R\sqrt{2}$. AC intersects BD at M. Find the locus of such points M.

Note that the diagram below shows only one example configuration and should not be used for generalization.



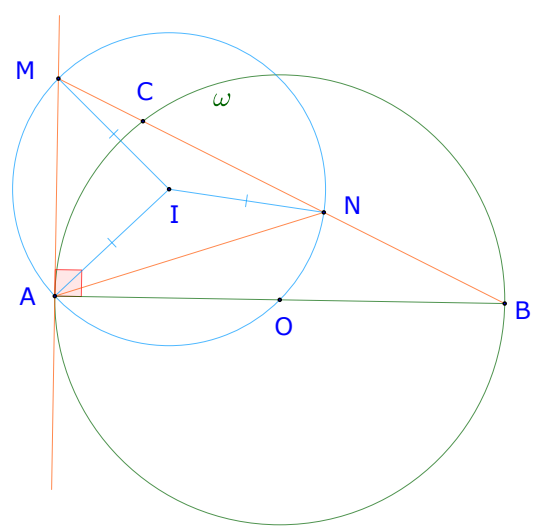
Problem 8.4.5 (23-24-S6-A-P9)**.** (*20 points*) Given the circle ω centred at O with diameter AB. ℓ is the tangent line of ω at A. M is an arbitrary point moving ℓ. BM intersects ω at C. N is such a point on BC that $BN = \frac{1}{2}BC$. Find the locus of such points N.

the diagram below shows only one example configuration and should not be used for generalization.



Problem 8.4.6 (23-24-S6-A-P10)**.** (*20 points*) Given the circle ω centred at O with diameter AB. ℓ is the tangent line of ω at A. M is an arbitrary point moving ℓ. BM intersects ω at C. N is the midpoint of BC. I is the circumcentre of $\triangle AMN$. Find the locus of such points I.

Note that the diagram below shows only one example configuration and should not be used for generalization.



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8.5 Solving Problems with Induction Principle (O-Level) - Part 2

Problem 8.5.1 (Fundamental Theorem of Arithmetic)**.** (*10 points*) Prove that every positive integer $n \geq 2$ is expressible as a product of primes in exactly one way. In particular, if

$$p_1 p_2 \cdots p_k = q_1 q_2 \cdot q_\ell, \text{ where } p_1, p_2, \ldots p_k, q_1, q_2, \ldots q_l \text{ are primes,}$$

then $k = l$ and $q_1, q_2, \ldots q_l$ are $p_1, p_2, \ldots p_k$ in some order.

Problem 8.5.2 (Another p and $\left\lfloor \frac{n}{p} \right\rfloor$ technique)**.** (*10 points*) Let n be a positive integer. Does n^2 have more positive divisors of the form $4k+1$ or of the form $4k-1$?

Problem 8.5.3 (Algebraic manipulations on divisibility and Fermat's Little Theorem)**.** Prove that for all $n \in \mathbb{Z}^+$.

(*5 points*) $17 \mid 2^{5n+3} + 5^n \cdot 3^{n+2}$.

(*5 points*) $p \mid n^p - n$, for any p prime.

Problem 8.5.4 (Another sum of divisors)**.** (*10 points*) Prove that for $n \geq 1$, the number $n!$ has the following property: to any divisor of $n!$, except $n!$, we can add another divisor of $n!$ so that their sum is a divisor of $n!$.

Problem 8.5.5 (Circular route with n cars)**.** (*10 points*) On a circular route, there are n identical cars. Together they have enough gas for one car to make a complete tour. Prove that there is a car that can make a complete tour by taking gas from all the cars that is encounters.

Problem 8.5.6 (Santa Claus coming to town)**.** (*10 points*) Santa Claus has $n \geq 2$ different presents and several identical bags. Each bag contains exactly two objects: either two bags, or a bag and a presents, or two presents. In particular, the big bag of presents Santa Claus has contains two objects. How many ways are there to split the presents into bags?

Problem 8.5.7 (The number of edges in a tree)**.** (*10 points*) Prove that if the graph G is a tree with n vertices then it has exactly $n-1$ edges.

Read the Graph Theory chapter in the LPS V3 book for concepts and definitions.

Problem 8.5.8 (An application of Sylvester-Gallai theorem)**.** (*10 points*) There are $n \geq 3$ distinct points on the plane, not all of them collinear. Prove that these point determine at least n lines (each line through at least two points.)

Problem 8.5.9 (Another example in combinatorial geometry)**.** Let $A_1, A_2, \ldots, A_n$ ($n \geq 4$) be a convex polygon. Denote by R_i the radius of the circumcircle of the triangle $A_{i-1}A_iA_{i+1}$ ($i = 2, 3, \ldots, n$ and $A_{n+1} = A_1$.) Prove that if $R_2 = R_3 = \ldots = R_n$ then one can draw a circumcircle for the polygon $A_1A_2 \ldots A_n$.

Lemma (Lemma 1)

If the convex quadrilateral $MNPK$ cannot be inscribed in a circle, but the radii of the circumcircles of the triangles MNP and MNK are equal, then $\angle MPN + \angle MKN = 180°$.

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Lemma (Lemma 2)

Let $ABCDE$ be a convex pentagon such that $R_B = R_C = R_D$, where R_B, R_C, and R_D denote the radii of the circumcircles of the triangles ABC, BCD, and CDE, respectively. Then at least one of the quadrilaterals $ABCD$ or $BCDE$ can be inscribed in a circle.

Lemma (Lemma 3)

Let $ABCDE$ be a convex pentagon such that $R_B = R_C = R_D = R_E$ where R_B, R_C, R_D, and R_E denote the radii of the circumcircles of the triangles ABC, BCD, CDE, and DEA, respectively. Then at least one of the quadrilaterals $ABCD$ or $CDEF$ can be inscribed in a circle.

Problem 8.5.10 (Induction principle in Calculus)**.** (*10 points*) Let I be an interval, $f, g : I \to \mathbb{R}$ be two n times differentiable functions. Prove that

$$(fg)^{(n)} = \binom{n}{0} f^{(n)} g + \binom{n}{1} f^{(n-1)} g^{(1)} + \cdots + \binom{n}{n-1} f^{(1)} g^{(n-1)} + \binom{n}{n} f g^{(n)}$$

where $f^{(k)}$ denotes the kth derivative of f.

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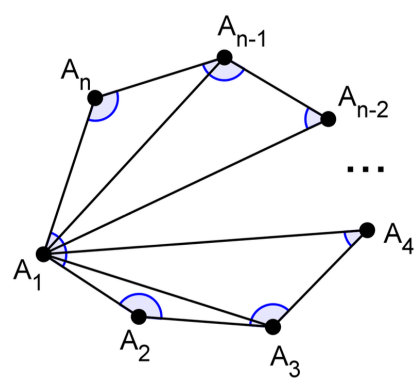
Chapter 9

Session 7

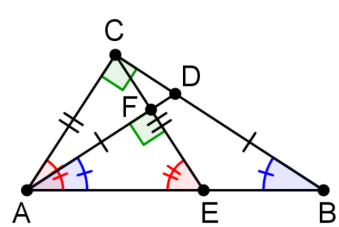
9.1 Angle chasing (I-Level)

In this session we discuss how to determine measure of an angle via multiple different techniques.

Example 9.1.1 (23-24-S7-I-P1)
Find the sum of the interior angles in an $n-$gon.



Example 9.1.2 (23-24-S7-I-P2)
Let ABC be a right triangle with $\angle BCA = 90°$ and $CA < CB$. Let $D \in BC$ (D on BC) such that $DA = DB$ and $E \in AB$ such that $CA = CE$. Prove that $AD \perp CE$.



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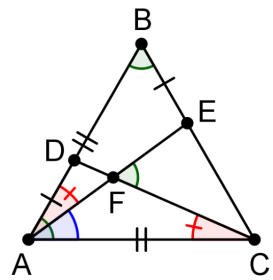
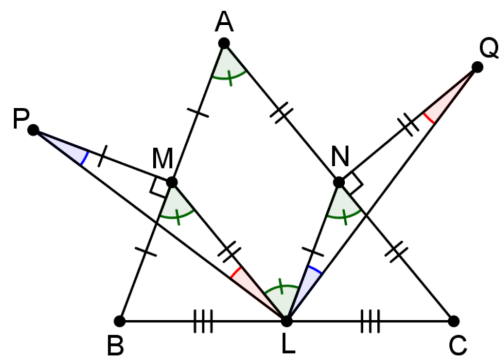
Example 9.1.3 (23-24-S7-I-P3)

Let ABC be an equilateral triangle. Let $D \in AB$ (D on AB) and $E \in BC$, such that $AD = BE$. let F be the intersection of AE and CD. Find $\angle CFE$.



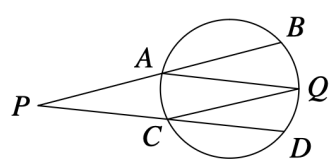
Example 9.1.4 (23-24-S7-I-P4)

Let M and N be midpoints of the sides AB and AC, respectively, in a triangle ABC. Let P and Q be points outside the triangle, such that $PM \perp AB$, $PM = \frac{1}{2}AB$ and $QN \perp AC$, $QN = \frac{1}{2}AC$. If L is the midpoint of BC, prove that $LP = LQ$ and $\angle PLQ = 90°$.



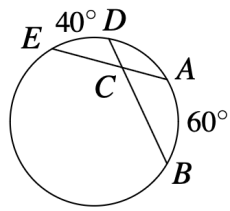
Example 9.1.5 (23-24-S7-I-P5)

Points A, B, Q, D, and C lie on the circle as shown and the measures of arcs $\widehat{BQ}$ and $\widehat{QD}$ are $42°$ and $38°$ respectively. What is the sum of angles P and Q?



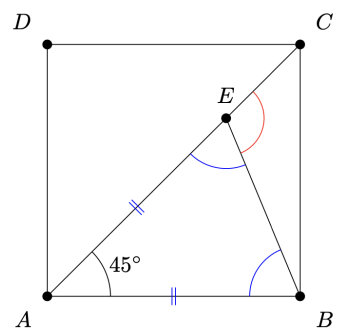
Example 9.1.6 (23-24-S7-I-P6)

In the figure, if $\widehat{AB} = 60°$ and $\widehat{DE} = 40°$, then what is $\angle ACD$?



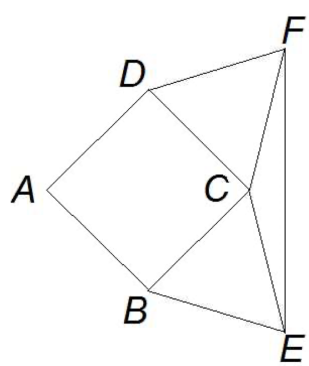
Example 9.1.7 (Fall Animath Cup/Problem 1)

Let $ABCD$ be a square with sides of length 1. Let E be the point on the segment AC such that $AE = 1$. Calculate the value, in degrees, of the angle $\angle BEC$.



Example 9.1.8 (Purple Comet Math Meet 2014/Problem 1)

In the diagram below $ABCD$ is a square and both $\triangle CFD$ and $\triangle CBE$ are equilateral. Find the degree measure of $\angle CEF$.



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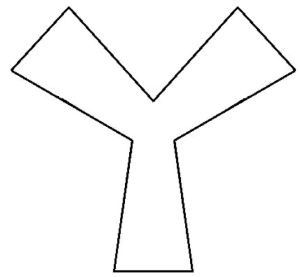
Example 9.1.9 (23-24-S7-I-P9)

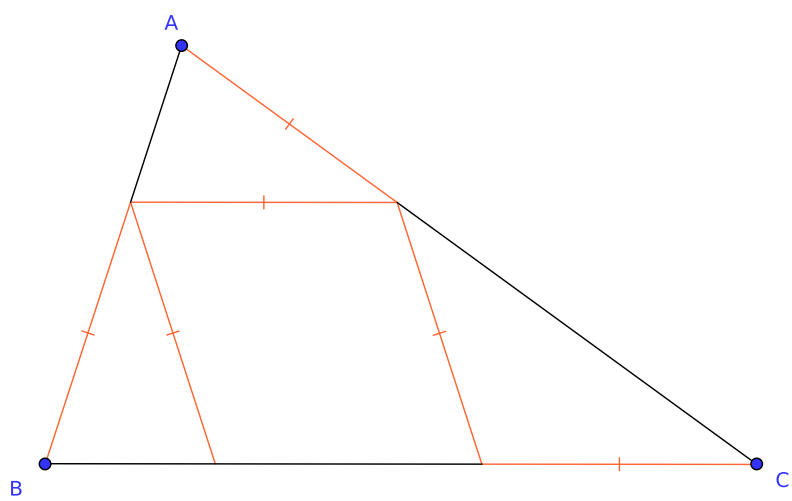
The figure below has only two sizes for its internal angles. The larger angles are three times the size of the smaller angles. Find the degree measure of one of the larger angles.



Example 9.1.10 (ET-2022-SM2-R1-J2)

Find the value of

$$\frac{\angle A}{2} + \frac{\angle B}{3} - \frac{\angle C}{4}.$$



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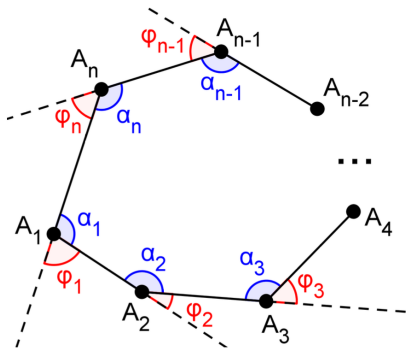
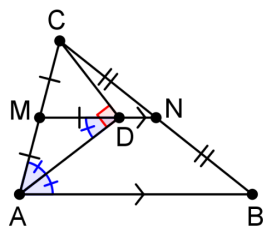
9.2 Problems (I-Level)

Problem 9.2.1 (23-24-S7-I-P11)**.** (*10 points*) Find the sum of $\varphi_1, \varphi_2, \ldots, \varphi_n$, the exterior angles on an $n-$gon, as show in the diagram below. In other words, compute

$$\varphi_1 + \varphi_2 + \ldots + \varphi_n.$$

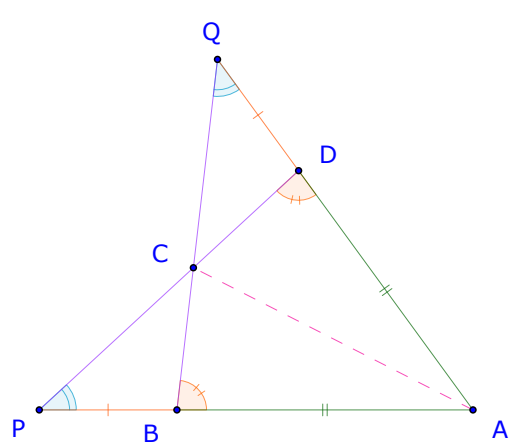


Problem 9.2.2 (23-24-S7-I-P12)**.** (*10 points*) Let M and N be midpoints of the sides CA and CB, respectively, in a triangle ABC. The angle bisector of $\angle BAC$ intersects the line MN at D. Prove that $\angle ADC = 90°$.



Problem 9.2.3 (23-24-S7-I-P13)**.** (*10 points*) In the triangle ABC, let A_1 be the midpoint of BC and let B_1 and C_1 be the feet of the altitudes from the vertices B and C, respectively. Prove that the triangle $A_1B_1C_1$ is equilateral if and only if $\angle BAC = 60°$.

Problem 9.2.4 (23-24-S7-I-P14)**.** (*10 points*) Let $ABCD$ be a convex quadrilateral ($AB > CD$, $AD > BC$). Let P and Q be the intersection of AB with DC and AD with BC, respectively. If $AP = AQ$ and $AB = AD$, prove that AC is the angle bisector of $\angle BAD$.



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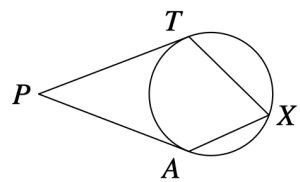
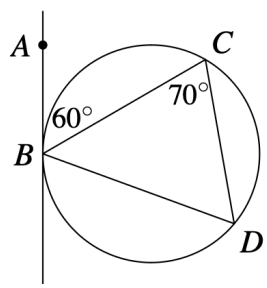
Problem 9.2.5 (23-24-S7-I-P15)**.** (*10 points*) Segments PA and PT are tangent to the circle. Find the measure of $\angle TXA$ if $\angle P = 42°$.

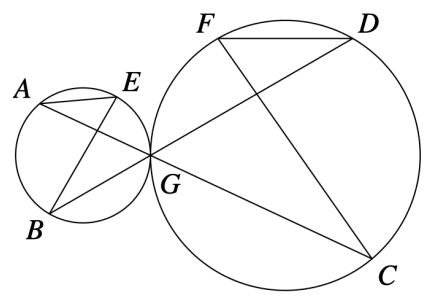


Problem 9.2.6 (23-24-S7-I-P16)**.** Consider the two questions separately.

(*5 points*) In the left figure, given that $\angle ABC = 60°$ and $\angle BCD = 70°$, find $\angle CBD$.

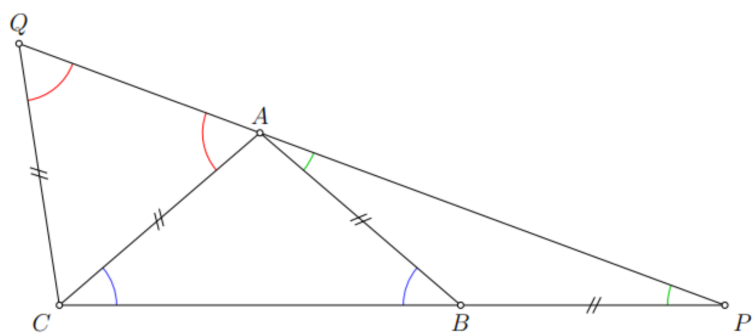
(*5 points*) In the right figure, two circles are tangent at G. Prove that $\angle AEB = \angle CFD$.





Problem 9.2.7 (23-24-S7-I-P17)**.** (*10 points*) Let ABC be an isosceles triangle at A ($AB = AC$) such that $AB < BC$. Let P be the point on the line (BC) located outside the segment BC such that $BP = BA$. Let Q be the point on line AP, different from A, such that $CQ = CA$.

If the angle $\angle QPC$ is denoted as x, what is the measure of angle $\angle PCQ$?



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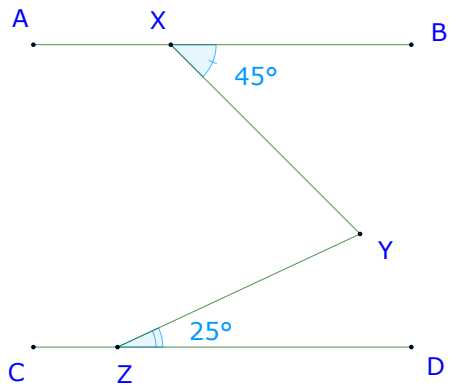
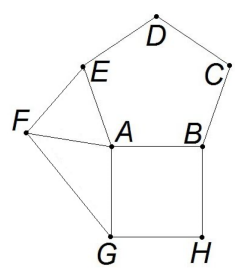
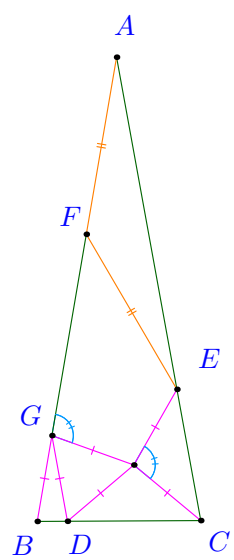
Problem 9.2.8 (23-24-S7-I-P18)**.** (*10 points*) In the diagram below $AB \parallel CD$, $\angle BXY = 45°$, $\angle DZY = 25°$, and $XY = YZ$. Find the degree measure of $\angle YXZ$.



Problem 9.2.9 (23-24-S7-I-P19)**.** (*10 points*) On side AE of regular pentagon $ABCDE$ there is an equilateral triangle AEF, and on side AB of the pentagon there is a square $ABHG$ as shown. Find the degree measure of angle AFG.



Problem 9.2.10 (23-24-S7-I-P20)**.** (*10 points*) In $\triangle ABC$, $AB = AC$, $AF = EF$, $\angle CHE = \angle FGH$, and $EH = CH = DH = GH = DG = BG$. Find $\angle BAC$.



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9.3 Advanced counting techniques (A-Level)

Definition (Set)**.** A set is *a collection of objects.*

There might only be a *finite* number of objects in the set, so we could count them if we like, in which case the set is called a finite set. Otherwise we call it an *infinite* set.

The objects in the set are called the *elements or members* of the set.

The set with no elements at all is called the *empty set,* denoted by $\emptyset$.

Definition (Supersets & Subsets)**.** A set A is called a *subset* of a set B if every element of A is also an element of B, denoted by

$$A \subseteq B$$

If A is a subset of B, we also say that B is a *superset* of A.

Definition (Intersections & Unions)**.** The *union* $A \cup B$ of two sets A and B is the set of all objects that are elements of A or of B,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

The *intersection* $A \cap B$ of two sets A and B is the set of all objects that are elements of both A and B,

$$A \cap B = \{x \mid x \in A \text{ or } x \in B\}.$$

Definition (Disjoint)**.** If $A \cap B = \emptyset$, then we say that A and B are *disjoint.*

Definition (Difference)**.** The set difference of two sets A and B is

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

Definition (Cardinality)**.** For a set S, $|S|$ denotes the **cardinality** of S, in other words the number of elements in S, *for example* $|\{1, 3, 5, 7\}| = 4$. The **empty** set is denoted by $\emptyset$.

Theorem (The Principle Inclusion-Exclusion for two/three sets)

If A and B are two sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

If A, B, and C are three sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

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Example 9.3.1 (23-24-S7-A-P1)

In how many rearrangements of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ do the numbers form a *hill,* that is, the numbers form an increasing sequence at the beginning up to a peak, and then form a decreasing sequence to the end such as in 129876543 or 258976431?

Example 9.3.2 (23-24-S7-A-P2)

Find the least six-digit palindrome that is a multiple of 45. Note that a palindrome is a number that reads the same forward and backwards such as 1441 or 35253.

Example 9.3.3 (23-24-S7-A-P3)

How many four-digit positive integers have no adjacent equal even digits? For example, count numbers such as 1164 and 2035 but not 6447 or 5866.

Example 9.3.4 (23-24-S7-A-P4)

In how many ways can the letters in the word MEMENTO be arranged if no two of the same letter may appear consecutively in an arrangement?

Example 9.3.5 (23-24-S7-A-P5)

How many 6-digit numbers, written in decimal notation, have at least one 1, one 2, and one 3 among their digits?

Example 9.3.6 (23-24-S7-A-P6)

How many subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ have the property that no two of its elements differ by more than 5? For example, count the subsets $\{3\}$, $\{2, 5, 7\}$, and $\{5, 6, 7, 8, 9\}$ but not the subset $\{1, 3, 5, 7\}$.

Example 9.3.7 (Gauss G8 2014/25)

Residents were surveyed in order to determine which flowers to plant in the new Public Garden. A total of N people participated in the survey. Exactly $\frac{9}{14}$ of those surveyed said that the colour of the flower was important. Exactly $\frac{7}{12}$ of those surveyed said that the smell of the flower was important. In total, 753 people said that both the colour and smell were important. How many possible values are there for N?

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9.4 Problems (A-Level)

Problem 9.4.1 (23-24-S7-A-P8)**.** (*10 points*) How many four-digit positive integers have exactly one digit equal to 1 and exactly one digit equal to 3?

Problem 9.4.2 (23-24-S7-A-P9)**.** (*10 points*) How many subsets of $\{1, 2, 3, 4, \ldots, 12\}$ contain exactly one prime number?

Problem 9.4.3 (23-24-S7-A-P10)**.** (*10 points*) At Mallard High School there are three intermural sports leagues: football, basketball, and baseball. There are 427 students participating in these sports: 128 play on football teams, 291 play on basketball teams, and 318 play on baseball teams.

If exactly 36 students participate in all three of the sports, how many students participate in exactly two of the sports?

Problem 9.4.4 (23-24-S7-A-P11)**.** (*15 points*) How many 6-digit binary numbers (numbers with 0's and 1's as digits) have a string of three consecutive 1's appearing in them? *For example, 101110 and 111100 both have a string of three consecutive 1's, but 100110 doesn't.*

Hint: define four sets A_1, A_2, A_3, A_4 *as follow* $A_1 = \{111???\}$, $A_2 = \{?111??\}$, $A_3 = \{??111?\}$, $A_4 = \{???111\}$, *where ? is a place holder for 0 or 1 digit. Try the Inclusion-Exclusion Principle.*

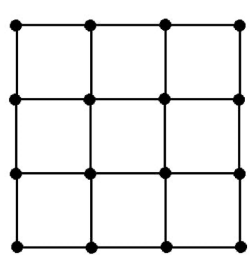
Problem 9.4.5 (23-24-S7-A-P12)**.** (*15 points*) Two biologists, two chemists, and two physicists go out to dinner and sit at a round table with 6 equally spaced chairs. In how many ways can they sit so that no two scientists of the same type (for example, two biologists) are seated next to each other? (Two seatings that are merely rotations of each other are not considered distinguishably different.)

Hint for Approach 1: Start by arbitrarily seating one of the biologists Considering following casework: (i) The other biologist sits directly opposite the first; (ii) The other biologist sits 2 seats away from the first.

Hint for Approach 2: what we need to count is the number of seatings with no pair adjacent. The The Principle Inclusion-Exclusion for two/three sets can help if we can count number of seatings with no restriction, with 1 pair adjacent, with 2 pairs adjacent, and with all three pairs adjacent.

Problem 9.4.6 (23-24-S7-A-P13)**.** (*20 points*) Sixteen dots are arranged in a four by four grid as shown. The distance between any two dots in the grid is the minimum number of horizontal and vertical steps along the grid lines it takes to get from one dot to the other. For example, two adjacent dots are a distance 1 apart, and two dots at opposite corners of the grid are a distance 6 apart.

Find the mean value of all distances between any two distinct dots.



Problem 9.4.7 (23-24-S7-A-P14)**.** (*20 points*) Omar has four fair standard six-sided dice. Omar invented a game where he rolls his four dice over and over again until the number 1 does not appear on the top face of any of the dice. Omar wins the game if on that roll the top faces of his dice show at least one 2 and at least one 5. Find the probability that Omar wins the game.

Hint: Consider two events (A) when no 2 appears, and (B) when no 5 appears.

9.5 Solving Problems with Induction Principle (O-Level) - Part 3

Problem 9.5.1 (Sharpen the inequality)**.** (*10 points*) The sequence (a_n) is defined by,

$$a_1 = \frac{1}{2}, a_{n+1} = \frac{2n-1}{2n+2}a_n$$

Prove that $a_1 + a_2 + \cdots + a_n < 1$ for $n \geq 1$.

Problem 9.5.2 (An application of Pascal's identity)**.** (*10 points*) Show that for any positive integer n

$$\sum_{r=1}^{n} \frac{1}{r}\binom{n}{r} = \sum_{r=1}^{n} \frac{2^r - 1}{r} \quad (*)$$

Problem 9.5.3 (Another functional equation)**.** (*10 points*) Find all functions $f: \mathbb{Z} \to \mathbb{R}$ such that:

$$f(1) = \frac{5}{2}, f(m)f(n) = f(m+n) + f(m-n), \ \forall m, n \in \mathbb{Z}.$$

Problem 9.5.4 (Inequality of alternate sums)**.** (*10 points*) Let $n \geq 1$ be a positive integer and let $x_1, x_2, \ldots, x_n$ be real numbers such that $0 \leq x_n \leq x_{n-1} \leq x_2 \leq x_1$. Let

$$s_n = x_1 - x_2 + \cdots + (-1)^n x_{n-1} + (-1)^{n+1} x_n$$
$$S_n = x_1^2 - x_2^2 + \cdots + (-1)^n x_{n-1}^2 + (-1)^{n+1} x_n^2$$

Prove that $s_n^2 \leq S_n$.

Problem 9.5.5 (Inequality of absolute values of sines)**.** (*10 points*) Let $n \geq 1$ be a non-negative integer. Prove that for all real number x,

$$\sum_{k=0}^{n} |\sin(2^k x)| \leq 1 + \frac{\sqrt{3}}{2}n.$$

Problem 9.5.6 (Term of a sequence)**.** (*10 points*) Define a sequence (a_n) by

$$a_1 = 1, a_2 = 2, a_{n+2} = 2a_{n+1} - a_n + 2, \ \forall n \geq 1.$$

Prove that $\forall m \geq 1, \ a_m a_{m+1}$ is also a term of this sequence.

Problem 9.5.7 (A remarkable sequence)**.** (*10 points*) Let $a_1, a_2, \ldots$ be a sequence with

$$a_1 = 1, \ a_{n+1} = \begin{cases} a_n - 2 \text{ if } a_n - 2 \notin \{a_1, a_2, \ldots, a_n\} \text{ and } a_n - 2 > 0, \\ a_n + 3 \text{ otherwise} \end{cases}$$

Prove that for every positive integer $k \geq 1$, there exist n such that

$$a_n = a_{n-1} + 3 = k^2.$$

Problem 9.5.8 (Invariant property of sum of powers)**.** (*10 points*) Let x, y be real numbers such that the number $x + y$, $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$, are all integers. Prove that, for all $n \geq 5$, $x^n + y^n$ is an integer.

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Problem 9.5.9 (Complete residue class)**.** (*10 points*) Let a and n be two positive integers such that $a^n - 1$ is divisible by n. Prove that the number $a+1, a^2+2, \ldots, a^n+n$ are all distinct modulo n.

Theorem (Euler's Theorem)

$a, n \in \mathbb{Z},\ \gcd(a,n) = 1 \Rightarrow a^{\varphi(n)} \equiv 1 \pmod{n}$, where $\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$.

Problem 9.5.10 (An immitation of Fermat's Theorem)**.** (*10 points*) Let $n \geq 1$, which is not divisible by 3. Show that

$$x^3 + y^3 = z^n$$

has at least one solution (x, y, z) where x, y, z are positive integers.

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Chapter 10

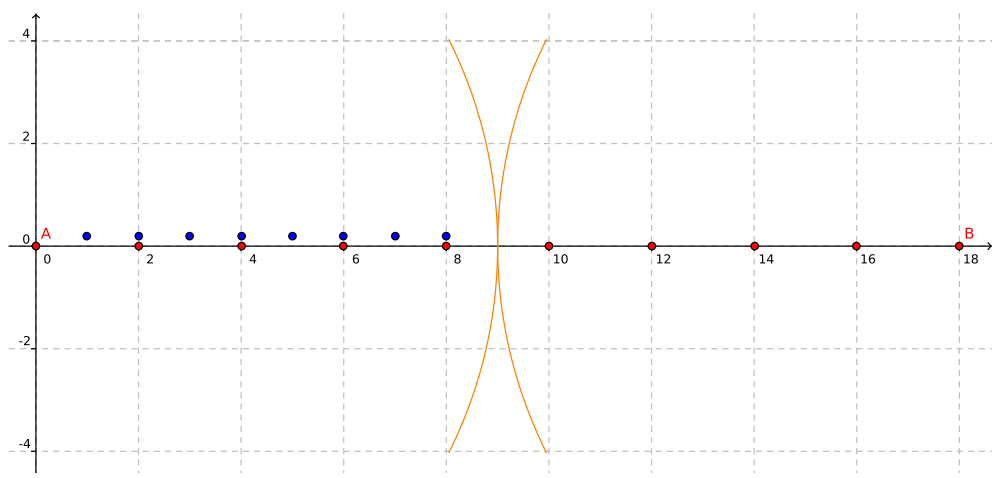
Session 8

10.1 Unusual Problems (I-Level)

In this session we discuss problems that require a thorough understanding of the problem text in order to devise some approach.

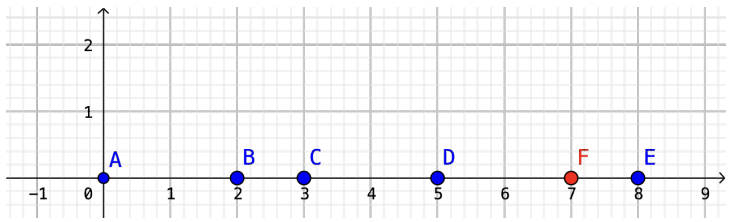
Example 10.1.1 (23-24-S8-I-P1)

On a straight line there are 10 distinct red points as show below Anna selects every two points of these 10 points, she then *marks* the midpoint of the segment connected them. What is the minimal number of distinct *marked* points?



Example 10.1.2 (23-24-S8-I-P2)

Five friends live on the same street at A, B, C, D, and E, see below. Place F is not a ideal place to meet because the sum of distances from A, B, C, D, and E to F is $7 + 5 + 4 + 2 + 1 = 19$. Try to figure out *where* should they meet so that the *sum of distances from that place to their houses is minimal.*



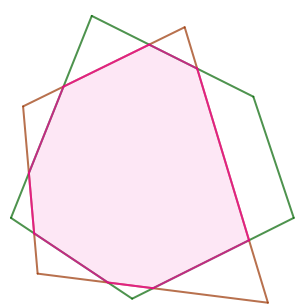
Example 10.1.3 (23-24-S8-I-P3)

On the blackboard six segments are drew. Every two segments intersect one another at a point. The 1st segment contains three of the intersections, the 2nd segment contain 4 of the intersections, the 3rd, 4th, and 5th segments each contains five intersections. What segments does the 6th segment intersect with?

Example 10.1.4 (23-24-S8-I-P4)

In a convex polygon, any line extending one of its sides does not go through the inside of the polygon. The figure below shows the intersection of a convex pentagon and a convex quadrilateral, which is an octogon (a polygon with 8 sides.)

At most how many sides can the intersection of a convex octogon with a convex decagon have? *A decagon is a polygon with* 10 *sides.*



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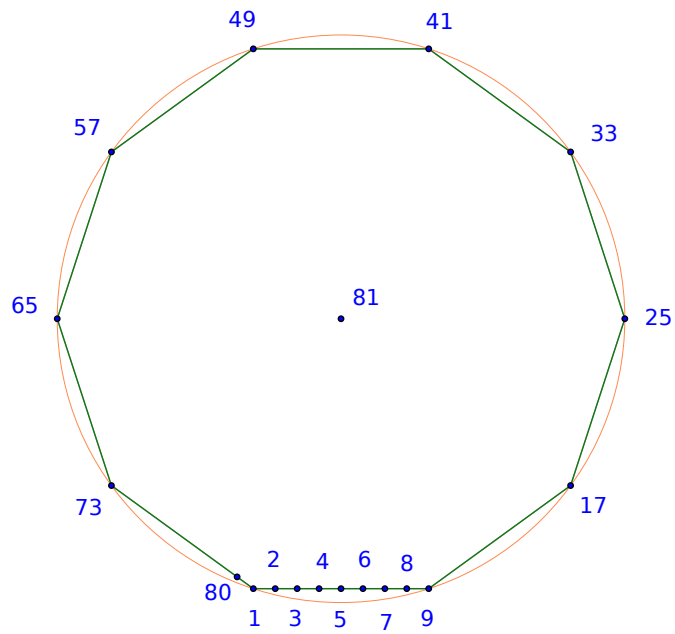
Example 10.1.5 (23-24-S8-I-P5)

In a ceremony of the guild, the master asked the members to form 10 lines, each line consists of 9 persons, so that he can stand in *a place that has the same distance to every row.*

The member were confused because all together, including the masters, the guild has 81 members. Chi Ton, an invited guest to the ceremony, pointed out how they can do that. Of course, he did neither stand into those lines, nor posed as the master. What was his solution?



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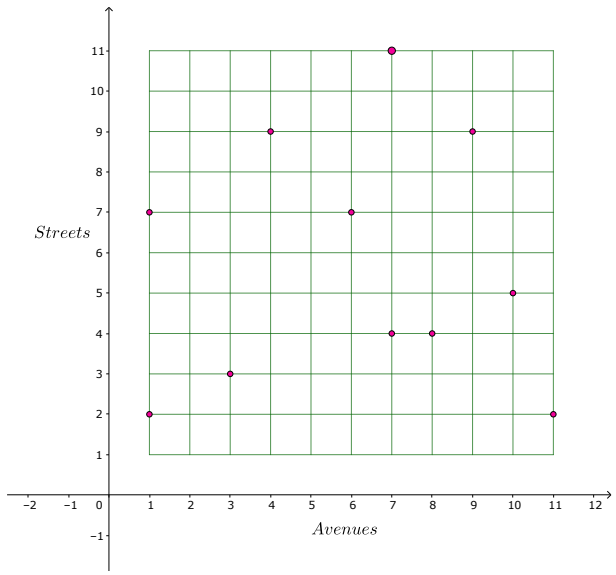
10.2 Problems (I-Level)

Problem 10.2.1 (23-24-S8-I-P6)**.** (*20 points*) In the diagram below, the red dots indicate houses of friends, who can only walk along streets and avenues. Where is the place that has the minimal sum of distances to their houses?

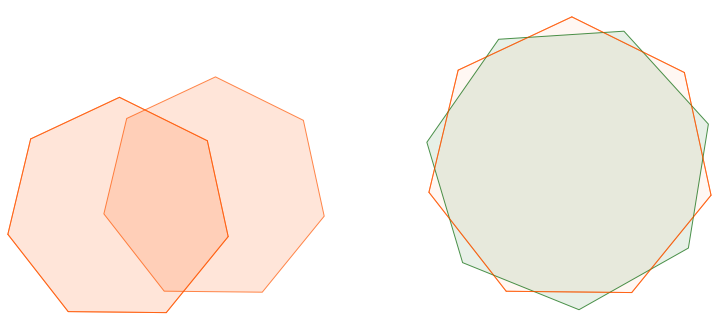


Problem 10.2.2 (23-24-S8-I-P7)**.** (*20 points*) Let ABC be a triangle. Points D, E, and F be the midpoints of sides BC, CA, and AB respectively. What is the maximum number of circles which pass through at least 3 of these 6 points?

Problem 10.2.3 (23-24-S8-I-P8)**.** Two same-size regular heptagon (regular $7-$gon, or a regular polygon with 7 sides) overlap each other. The overlapping region is a convex polygon. *For example, the figure below shows an overlapping hexagon (darker shaded).*

(*10 points*) What is the *largest* number of sides can the overlapping polygon have?

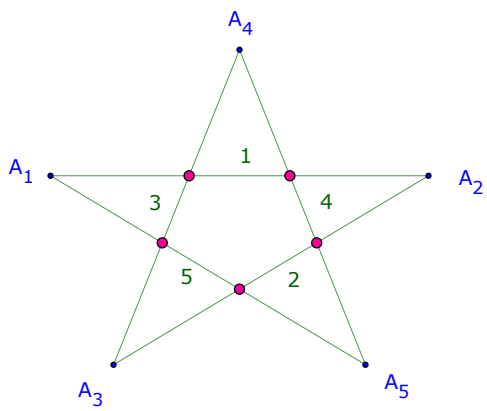
(*10 points*) What is the *largest odd* number of sides can the overlapping polygon have?



Problem 10.2.4 (23-24-S8-I-P9)**.** A *close path* $A_1A_2\ldots A_5$ contains 5 segments A_1A_2, A_2A_3, $\ldots$, A_5A_1 that has 5 *self-intersection* points . This close path has the *maximum* number of self-intersection points for any $5-$segment close path.

(*10 points*) Your first task is to find the *maximum* number of self-intersection points for a 7−segment close path $A_1A_2\ldots A_7$.

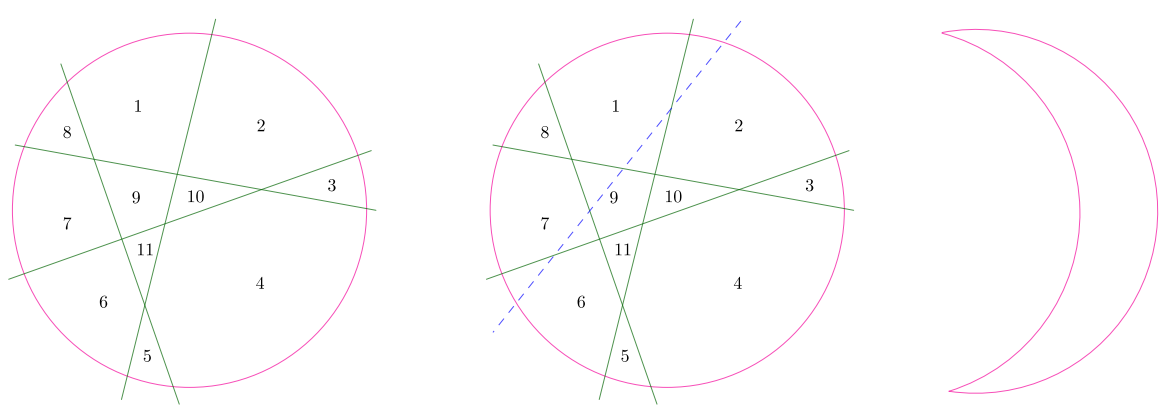
(*10 points*) Your second task is to find the *maximum* number of self-intersection points for a 9−segment close path $A_1A_2\ldots A_9$.



Problem 10.2.5 (23-24-S8-I-P10)**.** Four (green) cuts can dissect a circle into 11 parts as shown below in figure on the left. A (blue) cut in the middle diagram intersects four existing lines at 4 points, thus itself is divided into 3 segments and 2 half-lines. These divide 5 existing regions of the circle into 10 regions, thus increasing the number of regions by 5 as shown in the middle figure. The maximal number of regions of a circle that n cuts can produce is $\frac{n(n+1)}{2}+1$.

(*10 points*) What is the maximal number of regions of *a crescent moon* at right that 2 cuts can produce?

(*10 points*) What is the maximal number of regions of *a crescent moon* at right that 4 cuts can produce?



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10.3 Square is never negative (A-Level)

In this session we discuss problems that cleverly uses the fact that any square a^2 is a non-negative number.

Example 10.3.1 (23-24-S8-A-P1)
a, b, c, d, and e are real numbers. Prove that

$$2a^2 + 3b^2 + 2c^2 + 3d^2 + 2e^2 \geq 2(a + c + e)(b + d).$$

Example 10.3.2 (23-24-S8-A-P2)
Solve in real numbers the system of equations:

$$\begin{cases} \frac{1}{2}(x + y + 1) = 1 \\ \sqrt{x} - \sqrt{xy} + \sqrt{y} = 1 \end{cases}$$

Example 10.3.3 (23-24-S8-A-P3)
Prove that for any non-negative real numbers x, y, z, w,

$$\sqrt{(x + y)(z + w)} \geq \sqrt{xz} + \sqrt{yw}.$$

Example 10.3.4 (23-24-S8-A-P4)
For $n \geq 2$, let $x_1, x_2, \ldots, x_n$ be real numbers such that,

$$\begin{cases} x_1 + x_2 + \cdots + x_n = n^2 \\ x_1^2 + x_2^2 + \cdots + x_n^2 = n(n^2 + 1) \end{cases}$$

Prove that $x_k \in [n - \sqrt{n}, n + \sqrt{n}]$, for all $k = 1, 2, \ldots, n$.

Example 10.3.5 (23-24-S8-A-P5)
Solve in positive real numbers the equations

$$\frac{x^2 + y^2}{1 + xy} = \sqrt{2 - \frac{1}{xy}}$$

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10.4 Problems (A-Level)

Problem 10.4.1 (23-24-S8-A-P6)**.** (*10 points*) Prove that for any real numbers a and b

$$a^2 - ab + b^2 \geq \frac{1}{2}(a^2 + b^2) \geq \frac{1}{3}(a^2 + ab + b^2) \geq \frac{1}{4}(a + b)^2.$$

Problem 10.4.2 (23-24-S8-A-P7)**.** (*10 points*) Prove that for any real numbers a,

$$(a^2 + 1)(5a^2 + 1) \geq 4a(a^3 + a^2 + 1).$$

Problem 10.4.3 (23-24-S8-A-P8)**.** (*10 points*) Find all x real numbers such that

$$x^3 - x \leq \frac{\sqrt{2}}{4}(x^4 + 1).$$

Problem 10.4.4 (23-24-S8-A-P9)**.** (*15 points*) Solve in real numbers the equation

$$\sqrt{x_1 - 1} + 2\sqrt{x_2 - 2^2} + \cdots + n\sqrt{x_n - n^2} = \frac{1}{2}(x_1 + x_2 + \cdots + x_n).$$

Problem 10.4.5 (23-24-S8-A-P10)**.** (*15 points*) Prove that for any a, b real numbers

$$(a^2 + 1)(b^2 + 1) + 4ab \geq 2(ab + 1)(a + b).$$

Problem 10.4.6 (23-24-S8-A-P11)**.** (*20 points*) Find all real numbers x, y $0 < x, y, z < 2$ such that:

$$\frac{1}{x} + \frac{1}{2 - y} = \frac{1}{y} + \frac{1}{2 - z} = \frac{1}{z} + \frac{1}{2 - x} = 2.$$

Problem 10.4.7 (23-24-S8-A-P12)**.** (*20 points*) Solve in integers the equation

$$u^2 + v^2 + x^2 + y^2 + z^2 = uv + vx - xy + yz + zu + 3.$$

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10.5 Solving Problems with Induction Principle (O-Level) - Part 4

Problem 10.5.1 (Triangulation)**.** n is a positive integer, $n \geq 3$.

(*5 points*) Prove that any $n-$gon (not necessarily convex) can be cut into triangles by non-intersecting diagonals.

(*5 points*) Prove that the sum of the inner angles of any $n-$gon (not necessarily convex) is equal to $(n-2)180°$. Hence prove that the number of triangles into which an $n-$gon is cut by non-intersecting diagonals is $n-2$.

Problem 10.5.2 (Existence of a convex hull)**.** If $A_1, A_2, \ldots, A_n$ are any points in the plane, with any three not collinear, then there is a convex polygon P such that some of A_i are vertices of P and the rest of the point are inside P.

Note that P is called the convex hull of $A_1, A_2, \ldots, A_n$.

Problem 10.5.3 (A point inside a convex polygon)**.** Point O is inside (or on the boundary of) a convex $n-$gon $A_1A_2\ldots A_n$. Prove that among the angles A_iOA_j $(1 \leq i \neq j \leq n)$ there are not fewer than $n-1$ non-acute ones.

Problem 10.5.4 (Sum of vectors)**.** On the cirle of radius 1 with the center O there are given $2n+1$ points $P_1P_2\ldots P_{2n+1}$, which lie on one side of a diameter. Prove that

$$|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \ldots + \overrightarrow{OP_n}| \geq 1.$$

You might want to read the Vector chapters, Geometry section of LPS V3 for refreshing some concepts, definitions.

Problem 10.5.5 (No exception for the last pair)**.** $A_1A_2\ldots A_{2n}$ is a polygon inscribed in a circle. It is known that all the pairs of its opposite sides except one are parallel. Prove that for any odd n, the remaining pair of sides is also parallel. Prove that for n even, the length of the exceptional sides are equal.

Problem 10.5.6 (23-24-S5-O-P36)**.** Let

$$a_1 = a_2 = 1,\ a_{n+2} = a_{n+1} + \frac{a_n}{3^n},\ \forall n \geq 1.$$

Prove that $a_n \leq 2,\ \forall n \geq 1$.

Problem 10.5.7 (23-24-S5-O-P37)**.** Let $x_0, x_1, \ldots, x_{1995}$ be positive real numbers,

$$x_0 = x_{1995} = 1,\ x_{n-1} + \frac{2}{x_{n-1}} = 2x_n + \frac{1}{x_n},\ \forall n = 1, 2, \ldots, 1995$$

Find the maximal value that x_0 can have.

Problem 10.5.8 (23-24-S5-O-P38)**.** Let $a_0 > 5$ be an odd integer,

$$a_{n+1} = \begin{cases} a_n^2 - 5 \text{ if } a_n \text{ is odd,} \\ \frac{1}{2}a_n \text{ otherwise} \end{cases} \quad \forall n \geq 0$$

Prove that this sequence is not bounded.

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Problem 10.5.9 (23-24-S5-O-P39)**.** Let x be a real number and $n \geq 1$ positive integer. Prove that

$$|\sin(nx)| \leq n|\sin x|.$$

Problem 10.5.10 (23-24-S5-O-P40)**.** Prove that for $a > 0$ and n positive integer:

$$2(1 + a^{n+1})^3 \geq (1 + a^3)(1 + a^n)^3$$

Prove that for a, b, c positive real numbers,

$$2(a^{2023} + 1)(b^{2023} + 1)(c^{2023} + 1) \geq (1 + abc)(a^{2022} + 1)(b^{2022} + 1)(c^{2022} + 1).$$

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Chapter 11

Test 2

11.1 Rules

- Each level test consisting of 8 problems will be delivered by automatic scheduled email at the registered time. No test paper will be sent twice.
- Students have **270 minutes to solve the problems.** For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET.
- After the allowed time, the student has 15 minutes to scan the solutions. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET, and has to submit the scanner solutions before 11:45 AM. Any solution submitted later than the allowed time will not be accepted.
- **All submissions must be sent by email to mcc.youth.global@gmail.com and dinhxuanquyen@gmail.com.**
- Solutions must be written in English. Any solution looks unreadable, or is presented badly will not be considered for grading. **Answer without proper solution will not be considered as solution and will not be accepted.** Answer consists of self-explanatory and easy-to-understand diagrams are acceptable but might not be considered as full solution.
- A problem can contain more than one questions, each with an indication of the maximum number of points can be awarded if a correct solution is provided. Partially correct solution is awarded with a some points depending on its quality. Multiple solutions can be submitted for a single problem. Each different solution, if correct, will be awarded the same number of points. Partially distinct solution also counts.
- **For geometry problems, proofs must be accompanied with clean, clear, and precisely drawn diagrams** - by hand with compass and straightedge or by computer software such as GeoGebra. Submitted **solutions without any diagram will not be graded at all.**
- A proof for the hypothesis about **the existence of a locus must contains two parts.** Each part is worth 50% of the awardable points.

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11.2 Problems for Intermediate Level

Read the rules before you start!

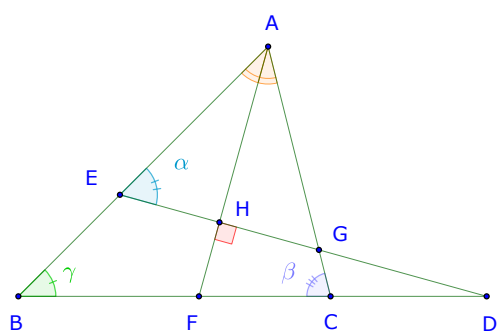
Problem 11.2.1 (23-24-T2-I-P1)**.** (*10 points*) In how many ways can a five letter word be written using only the first half of the alphabet with no repetitions such that the third and fifth letters are vowels and the first a consonant?

The word vowel comes from the Latin word vocalis, meaning "vocal" (i.e. relating to the voice). In English, the word vowel is commonly used to refer both to vowel sounds and to the written symbols that represent them (⟨a⟩, ⟨e⟩, ⟨i⟩, ⟨o⟩, ⟨u⟩, and sometimes ⟨w⟩ and ⟨y⟩). Consonant is a letter that is not a vowel.

Problem 11.2.2 (23-24-T2-I-P2)**.** (*10 points*) Johan is swimming laps in the pool. At 12:17 he realized that he had just finished one-third of his workout. By 12:22 he had completed eight more laps, and he realized that he had just finished five-elevenths of his workout. After 12:22 how many more laps must Johan swim to complete his workout?

Problem 11.2.3 (23-24-T2-I-P3)**.** (*10 points*) In $\triangle ABC$, $AC < AB$. AF is the angle bisector of $\angle BAC$, where F is on segment BC. D is an arbitrary point on line BC such that C is between B and D. Line through D perpendicular to AF intersects AB at E and AC at G, respectively. Prove that

$$\angle AEG = \frac{1}{2}(\angle ABC + \angle ACB).$$



Problem 11.2.4 (23-24-T2-I-P4)**.** (*10 points*) Find all four-digit perfect squares whose thousands digit and hundreds digit are the same; and whose tens and the unit digit are the same?

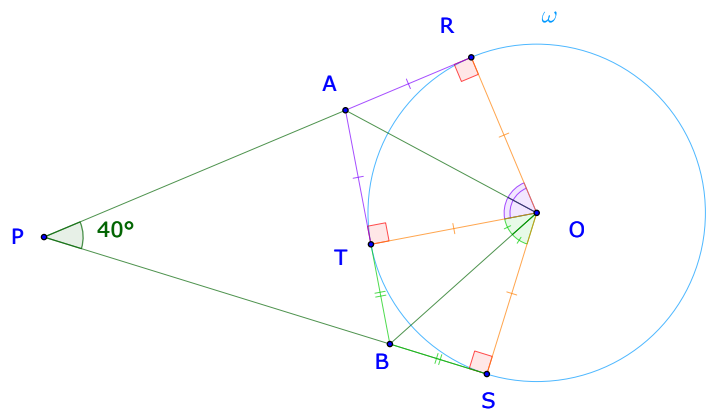
In other word, the perfect square would be in the form aabb.

Problem 11.2.5 (23-24-T2-I-P5)**.** (*10 points*) In how many ways can four identical red chips and two identical white chips be arranged in a circle?

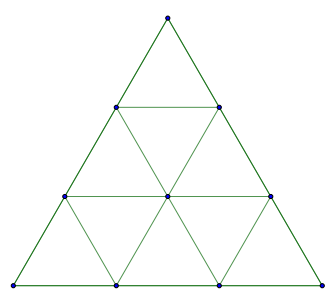
Remember, two setups which can be rotated into one another are considered the same in circular problems.

Problem 11.2.6 (23-24-T2-I-P6)**.** (*10 points*) The Purple Plant Garden Store sells grass seed in ten-pound bags and fifteen-pound bags. Yesterday half of the grass seed they had was in ten-pound bags. This morning the store received a shipment of 27 more ten-pound bags, and now they have twice as many ten-pound bags as fifteen-pound bags. Find the total weight in pounds of grass seed the store now has.

Problem 11.2.7 (23-24-T2-I-P7)**.** (*10 points*) P is a point outside of the circle ω centred at O. Lines through P are tangent with the circle ω at R and S. Points A and B on segments PR and PS such that segment AB is tangent with the circle ω at T. If the $\angle RPS = 40°$ find the $\angle AOB$.



Problem 11.2.8 (23-24-T2-I-P8)**.** (*10 points*) The diagram below shows an equilateral triangle with side length 3 cut into nine equilateral triangles with side length 1. Place the numbers 1 to 9 into the nine smaller triangles (one number into one triangle) such that the sum of the numbers inside any equilateral triangle with side length 2 is the same. What are the smallest and largest values of this sum?



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11.3 Problems for Advanced Level

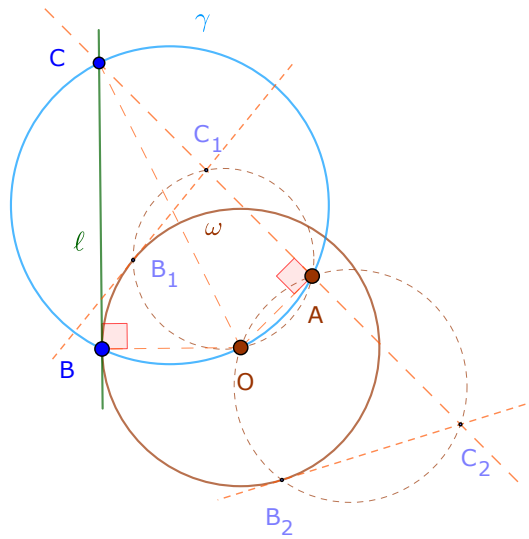
Read the rules before you start!

Problem 11.3.1 (23-24-T2-A-P1)**.** (*10 points*) Prove that for all $2 \le n \le 2023$,

$$\sqrt{(n)\sqrt{(n+1)\sqrt{\cdots\sqrt{(2023)}}}} < n+1.$$

Sometimes you can top the domino in reverse direction.

Problem 11.3.2 (23-24-T2-A-P2)**.** (*10 points*) Let A be a fixed point inside the circle ω. Let B be a point moving on the perimeter of O. Let ℓ be the line tangent to ω at B. The circumcircle γ of $\triangle ABO$ intersects ℓ at C. Find the locus of such C point when B moves around the circle ω.



Problem 11.3.3 (23-24-T2-A-P3)**.** (*10 points*) Mai flips a fair coin 6 times. Find the probability that she flips 3 heads in a row but not 4 heads in a row.

For example HHHTTT is a valid one, and THHHHT is not.

Problem 11.3.4 (23-24-T2-A-P4)**.** (*10 points*) Solve in positive real numbers the system of equations

$$\begin{cases} x^2 + \frac{1}{y} = \sqrt{y} + \sqrt{z} \\ y^2 + \frac{1}{z} = \sqrt{z} + \sqrt{x} \\ z^2 + \frac{1}{x} = \sqrt{x} + \sqrt{y} \end{cases}$$

Problem 11.3.5 (23-24-T2-A-P5)**.** (*10 points*) Let the sequence (a_n) is defined as follow,

$$a_1 = 1,\ a_{n+1} = \frac{1}{16}\left(1 + 4a_n + \sqrt{1 + 24a_n}\right).$$

Prove that

$$a_{2023} = \frac{1}{3} + \frac{1}{2^{2023}} + \frac{2}{3 \cdot 4^{2023}}.$$

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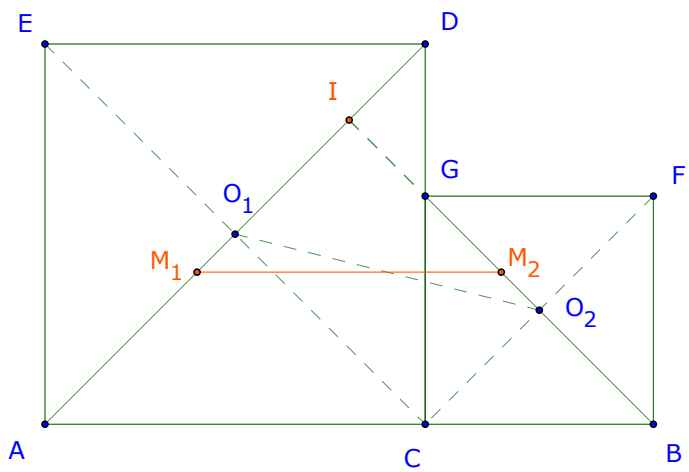
Problem 11.3.6 (23-24-T2-A-P6)**.** (*10 points*) C is an arbitrary point on segment AB. On the same side of AB we construct two squares $ACDE$ and $BCGF$ (C, G, D are collinear.) Let O_1 and O_2 be the centers of the squares $ACDE$ and $BCGF$, respectively. Find the locus of midpoints M of O_1O_2.



Problem 11.3.7 (23-24-T2-A-P7)**.** (*10 points*) How many positive integers not exceeding 2023 contain both even and odd digits?

Problem 11.3.8 (23-24-T2-A-P8)**.** Let (a_n) be a sequence of real numbers such that

$$a_0 = 2, a_{n+1} \geq a_n^2 + \frac{1}{5}, \ \forall n \geq 0.$$

(*5 points*) Prove that for all $n \geq 0$,

$$a_{n+5} > (a_n)^2 + \left(a_{n+1} - \frac{1}{2}\right)^2 + \left(a_{n+2} - \frac{1}{2}\right)^2 + \left(a_{n+3} - \frac{1}{2}\right)^2 + \left(a_{n+4} - \frac{1}{2}\right)^2$$

(*5 points*) Prove that $a_{2025} > 2^{810}$.

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11.4 Problems for Olympiad Level

Read the rules before you start!

Problem 11.4.1 (23-24-T2-O-P1)**.** (*10 points*) Prove that for n positive integer

$$2!4!\cdots(2n)! \geq ((n+1)!)^n$$

Problem 11.4.2 (23-24-T2-O-P2)**.** (*10 points*) Show that in a round-robin tennis tournament with an odd number of players, it is possible for every player to win exactly half the matches.

Problem 11.4.3 (23-24-T2-O-P3)**.** (*10 points*) Find a recursive formula (recurrent relation) to count the number of ways to put n non-attacking rooks on a $n \times n$ chessboard such that the arrangement is symmetric about the diagonal from the lower left corner to the upper right corner.

Hint: take a look at the rook on the first column, it is at a corner or somewhere in the column.

Problem 11.4.4 (23-24-T2-O-P4)**.** (*10 points*) Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Show that

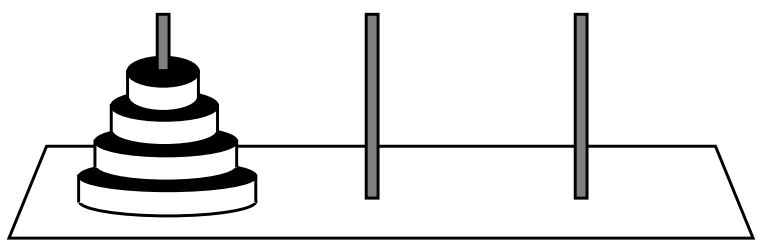
$$\frac{a_1}{(1+a_1)^2} + \frac{a_2}{(1+a_1+a_2)^2} + \cdots + \frac{a_n}{(1+a_1+a_2+\cdots+a_n)^2} < \frac{a_1+a_2+\cdots+a_n}{1+a_1+a_2+\cdots+a_n}.$$

Problem 11.4.5 (23-24-T2-O-P5)**.** (*10 points*) $a > 0, b > 0$ are real numbers. n is a positive integer. Prove that

$$(n-1)a^n + b^n \geq na^{n-1}b.$$

When does the equality stand?

Problem 11.4.6 (23-24-T2-O-P6)**.** (*10 points*) Suppose n rings, with different outside diameters, are slipped onto an upright peg, the largest on the bottom, to form a pyramid. Two other upright pegs are placed sufficiently far apart. We wish to transfer all the rings, one at a time, to the second peg to form an identical pyramid. During the transfer, we are not permitted to place a larger ring on a smaller one (this necessitates using the third peg). Prove that the smallest number of moves necessary to complete the transfer is $2^{n+1} - 1$.



Problem 11.4.7 (23-24-T2-O-P7)**.** (*10 points*) $n \geq 1$ is a positive integer. Let $0 < a_1 < a_2 < \cdots < a_n$. Let $e_i = \pm 1, i = 1, 2, \ldots, n$. Prove that the sums $\sum_{i=1}^{n} e_i a_i$ have at least $\binom{n+1}{2}$ distinct values as $\{e_i\}$ range over the 2^n possible combination of signs.

Note that: for $n = 1$, there are exactly $\binom{1+1}{2} = 2$ distinct values for two sums $-a_1$ and a_1.

Hint: by subtracting a_{n+1} from a sum $\sum_{i=1}^{n} e_i a_i$, we obtain $\binom{n+1}{2}$ distinct sums.

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Problem 11.4.8 (23-24-T2-O-P8)**.** Arrange the number $1, 2, \ldots, n$ consecutively clockwise about the circumference of a circle. Now, remove the number 2 and proceed clockwise by removing every other number, among those that remain, until only one number is left. *For example for $n = 5$, numbers are removed in the order 2, 4, 1, 5, and 3 remains alone.*

Let $f(n)$ denote the final number which remain.

(*5 points*) Prove that

$$\begin{cases} f(2n) & = 2f(n) - 1 \\ f(2n+1) & = 2f(n) + 1 \end{cases}$$

(*5 points*) Prove that if the binary representation of $n = \overline{a_k a_{k-1} \ldots a_1 a_0}_2$, then

$$f(n) = b_k 2^k + b_{k-1} 2^{k-1} + \ldots + 2b_1 + b_0, \quad \text{where } b_i = \begin{cases} +1, & a_i = 1 \\ -1, & a_i = 0 \end{cases} \forall i = 0, 1, \ldots, k$$

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Chapter 12

Test 3

12.1 Rules

- Each level test consisting of 8 problems will be delivered by automatic scheduled email at the registered time. No test paper will be sent twice.
- Students have **270 minutes to solve the problems.** For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET.
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- Solutions must be written in English. Any solution looks unreadable, or is presented badly will not be considered for grading. **Answer without proper solution will not be considered as solution and will not be accepted.** Answer consists of self-explanatory and easy-to-understand diagrams are acceptable but might not be considered as full solution.
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- A proof for the hypothesis about **the existence of a locus must contains two parts.** Each part is worth 50% of the awardable points.

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12.2 Problems for Intermediate Level

Read the rules before you start!

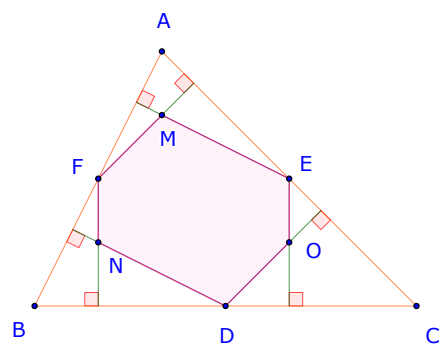
Problem 12.2.1 (23-24-T3-I-P1)**.** (*10 points*) Mr. Fox, the shopkeeper, phoned Inspector Julie that his shop had been robbed. Three suspects Alligator, Bear, and Cheetah were rounded up for questioning. The following facts were established:

1. Each of Alligator, Bear, and Cheetah had been in the shop on the day of the robbery, and no one else had been in the shop that day.
2. If Alligator is guilty, then he had exactly one accomplice.
3. If Bear is innocent, so is Cheetah
4. If exactly two are guilty, then Alligator is one of them.
5. If Cheetah is innocent, so is Bear.

Whom did Inspector Julie indict?

Problem 12.2.2 (23-24-T3-I-P2)**.** D, E, and F are midpoints of BC, CA, and AB in $\triangle ABC$. Perpendicular lines from F to AC and E to AB intersect at M. Perpendicular lines from D to AB and F to BC intersect at N. Perpendicular lines from E to BC and D to AC intersect at O.

Find the ratio of the area of the hexagon $DNFMEO$ to the area of the triangle ABC.



Problem 12.2.3 (23-24-T3-I-P3)**.** Mai has a round necklace containing 15 beads of two different colours: red and blue. (The necklance has no lock.)

(*5 points*) Prove that it is possible to find two beads of the same colour next to each other.

(*5 points*) Prove that it is possible to find two beads of the same colour separated by exactly two beads.

Problem 12.2.4 (23-24-T3-I-P4)**.** One evening a theater sold 300 tickets for a concert. Each ticket sold for \$40, and all tickets were purchased using \$5, \$10, and \$20 bills. At the end of the evening the theater had received twice as many \$10 bills as \$20 bills, and 20 more \$5 bills than \$10 bills. How many bills did the theater receive altogether?

Problem 12.2.5 (23-24-T3-I-P5)**.** A 5×9 rectangle is cut into 10 smaller rectangles. All cuts follow the grid lines.

(*5 points*) Prove that there are at least two rectangles that have the same area.

(*5 points*) Prove that there are at least two rectangles that are congruent (same width and same length).

Hint: let's assume the contrary, what are the 10 rectangles in different shapes with smallest area?

Problem 12.2.6 (23-24-T3-I-P6)**.** *The set* $S = \{2, 3, 5\}$ *is called* ***a good set*** *if for each of its subsets:*

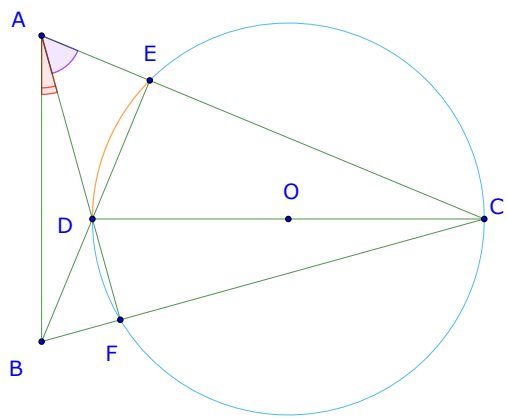
$\{2\},\ \{3\},\ \{5\},\ \{2,3\},\ \{3,5\},\ \{5,2\},\ \{2,3,5\},$ *none of the sum of its numbers*
$2 = 2,\ 3 = 3,\ 5 = 2 + 3 = 5,\ 2 + 5 = 7,\ 3 + 5 = 8,\ 2 + 3 + 5 = 10$ *is a perfect square.*

(*5 points*) At most how many numbers from $3,\ 6,\ 10,\ 12,\ 13$ can make a good set?

(*5 points*) At most how many numbers from $2,\ 3,\ 5,\ 6,\ 7,\ 8,\ 10,\ 11,\ 12,\ 13$ can make a good set?

Problem 12.2.7 (23-24-T3-I-P7)**.** (*10 points*) How many 7-digit numbers have no two adjacent digits equal?

Problem 12.2.8 (23-24-T3-I-P8)**.** (*10 points*) In the figure, O is the center of the circle, $\angle EAD = 40^\circ$ and $\widehat{ED} = 40^\circ$. Find $\angle DAB$.



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12.3 Problems for Advanced Level

Read the rules before you start!

Problem 12.3.1 (23-24-T3-A-P1)**.** (*10 points*) Nine cards are numbered $1, 2, 3, \ldots, 9$. Each of three players randomly selects and keeps three of the cards, and sums those three values. *For example if a player receives the cards* $2, 3, 5$, *then the sum she obtains is* $2 + 3 + 5 = 10$.

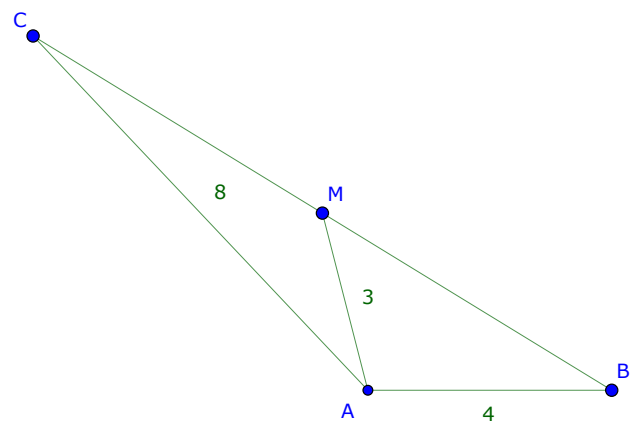
Find the probability that all three players obtain an odd sum.

Problem 12.3.2 (23-24-T3-A-P2)**.** (*10 points*) Let a, b, c be positive real numbers such that

$$\frac{a(b-c)}{b+c} + \frac{b(c-a)}{c+a} + \frac{c(a-b)}{a+b} = 0.$$

Prove that $(a-b)(b-c)(c-a) = 0$.

Problem 12.3.3 (23-24-T3-A-P3)**.** (*10 points*) Triangle ABC is such that $AB = 4$ and $AC = 8$. If M is the midpoint of BC and $AM = 3$, what is the length of BC?



Problem 12.3.4 (23-24-T3-A-P4)**.** (*10 points*) Let a, b, c be real number such that $a \geq b \geq c$, Prove that

$$a^2 + ac + c^2 \geq 3b(a - b + c)$$

Problem 12.3.5 (23-24-T3-A-P5)**.** (*10 points*) Solve in positive real numbers the system of equations

$$\begin{cases} x^2y^2 + 1 = x^2 + xy \\ y^2z^2 + 1 = y^2 + yz \\ z^2x^2 + 1 = z^2 + xy \end{cases}$$

Problem 12.3.6 (23-24-T3-A-P6)**.** (*10 points*) A and B are fix points. C an arbitrary point such that $\angle ACB > 90°$. H is the orthocenter of $\triangle ABC$, D is the foot of the perpendicular from C to AB. Find the locus of points C such that $DH \cdot DC$ reaches it possible maximal value.

Hint: Consider similar triangles.

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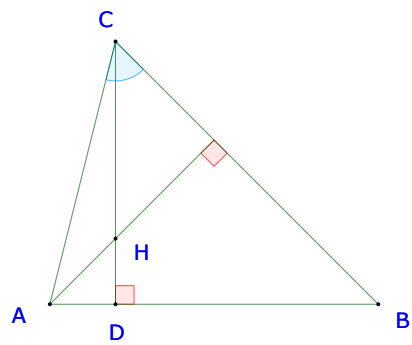

Problem 12.3.7 (23-24-T3-A-P7)**.** (*10 points*) Prove by induction that for a, b non-negative integers, $n \geq 1$,

$$(a^2 + b^2)^n \text{ can be expressed as sum of two perfect squares}$$

Problem 12.3.8 (23-24-T3-A-P8)**.** (*10 points*) Find the greatest n positive integer such that

$$5^n \mid 2019! - 2018! + 2017!$$

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12.4 Problems for Olympiad Level

Read the rules before you start!

Problem 12.4.1 (23-24-T3-O-P1). (*10 points*) n is a positive integer. $a_k = 2^{2^{k-n}} + k$, $k = 0, 1, \ldots, n$. Prove that

$$(a_1 - a_0) \cdots (a_n - a_{n-1}) = \frac{7}{a_1 + a_0}.$$

Hint: let $a = 2^{2^{-n}}$, express $a_k - a_{k-1}$ base on a in order to turn the left hand side into a telescopic product.

Problem 12.4.2 (23-24-T3-O-P2). Let n be a positive integer. Prove that

(*5 points*) $\log_n(n+1) > \log_{n+1}(n+2)$.

(*5 points*) $\log_4(5) \cdot \log_6(7) \cdots \log_{80}(81) < 2$.

Problem 12.4.3 (23-24-T3-O-P3). (*10 points*) Let $d \geq 1$ be integer, $a_1, a_2, \ldots, a_d$ be real numbers. Let $P(x) = x^d - a_1 x^{d-1} + \cdots + (-1)^d a_d$ be a polynomial with d real roots, all of which lie in the interval $[\alpha, \beta]$, where $-1 \leq \alpha \leq \beta$. Prove that

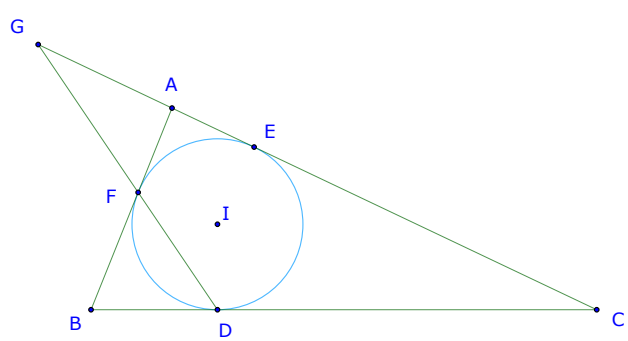
$$(1+\alpha)^d \leq (-1)^d P(-1) \leq (1+\beta)^d$$

Problem 12.4.4 (23-24-T3-O-P4). (*10 points*) Let $a_1 = 1 + \sqrt{2}$ and for $n \geq 1$,

$$a_{n+1} = 2 - \frac{1}{a_n}.$$

Find the greatest integer less than or equal to $a_1 a_2 a_3 \cdots a_{2023}$.

Problem 12.4.5 (23-24-T3-O-P5). (*10 points*) The incircle of $\triangle ABC$ is tangent to sides BC, AC, and AB at D, E, and F, respectively. Point G is the intersection of lines AC and DF as shown. The sides of $\triangle ABC$ have lengths $AB = 73$, $BC = 123$, and $AC = 120$. Find the length EG.



Problem 12.4.6 (23-24-T3-O-P6). (*10 points*) Find the number of three-element subsets of $\{1, 2, 3, \cdots, 13\}$ that contain at least one element that is a multiple of 2, at least one element that is a multiple of 3, and at least one element that is a multiple of 5 such as $\{2, 3, 5\}$ or $\{6, 10, 13\}$.

Hint: the set $\{1, 2, 3, \cdots, 13\}$ contains 7 elements that are not multiples of 2, thus the number of three-element subsets of S that are missing either a multiple of 2 is $\binom{7}{3}$.

Problem 12.4.7 (23-24-T3-O-P7)**.** (*10 points*) Find $p < q < r < s$ are prime numbers such that

$$pqrs + 1 = 4^{p+q}.$$

Problem 12.4.8 (23-24-T3-O-P8)**.** (*10 points*) A convex polygon is given on the plane. Prove that there exists a triangle formed by three of its vertices such that the sum of squares of its side lengths is at least the sum of squares of the side lengths of the polygon.

Hint if the polygon $A_1A_2 \dots A_n$ has a non-acute interior angle, then by removing its associated vertex, what can you say about the sum of squares of the side lengths of the remaining $(n-1)-gon$.

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Chapter 13

ICLT 2

13.1 IC Level Test Rules

- All test lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *at most one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

13.2 IC Test for Level 1

Problem 13.2.1 (ICLT-2023-SM1-R2-L1-P1)**.** Simplify

$$\sqrt[3]{27p^4} - \frac{2\sqrt[3]{p^7}}{p}.$$

(A) $\sqrt[3]{p^5}$ (B) $25p\sqrt[3]{p}$ (C) $p\sqrt[3]{p}$ (D) $p^{\frac{5}{3}}$ (E) $2p^{\frac{5}{3}}$

Problem 13.2.2 (ICLT-2023-SM1-R2-L1-P2)**.** Find the measure of an angle that is complementary the angle $101°$

(A) $79°$ (B) $11°$ (C) $31°$ (D) $259°$ (E) there is no such angle.

Problem 13.2.3 (ICLT-2023-SM1-R2-L1-P3)**.** How many three-digit numbers are not multiples of 5?

(A) 180 (B) 720 (C) 820 (D) 700 (E) 800

Problem 13.2.4 (ICLT-2023-SM1-R2-L1-P4)**.** Find the smallest composite number that has no prime divisors less than 23.

(A) 24 (B) 46 (C) 361 (D) 529 (E) 841

Problem 13.2.5 (ICLT-2023-SM1-R2-L1-P5)**.** In how many ways can we pick a group of 3 different numbers from the group $1, 2, 3, \ldots, 10$ such that one number is the average of the other two?

(A) 10 (B) 20 (C) 25 (D) 30 (E) 35

Problem 13.2.6 (ICLT-2023-SM1-R2-L1-P6)**.** How many two-digit prime numbers ab are there such that ba is also a two-digit prime numbers?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Problem 13.2.7 (ICLT-2023-SM1-R2-L1-P7)**.** The angles in a triangle are in the ratio $5 : 6 : 7$. What are the measures of the angles?

Problem 13.2.8 (ICLT-2023-SM1-R2-L1-P8)**.** In how many ways can 4 girls and 4 boys be seated in a row of 8 chairs such that all 4 girls sit next to each other?

Problem 13.2.9 (ICLT-2023-SM1-R2-L1-P9)**.** Martin wanted to solve the equation $3x - 5 = 172$, but instead he solved $5x - 3 = 172$. How larger should the answer have been if he would have solved the correct equation?

Problem 13.2.10 (ICLT-2023-SM1-R2-L1-P10)**.** Solve the equation $3t - 7 - 5 + 2t = 4t + 7 - 6t$.

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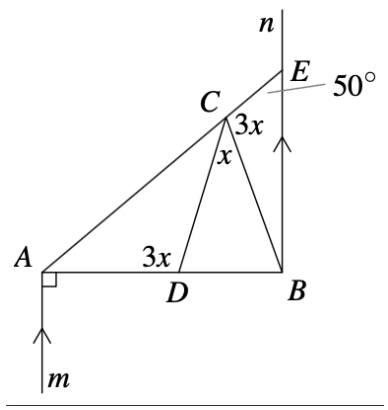
13.3 IC Test for Level 2

Problem 13.3.1 (ICLT-2023-SM1-2-L2-P1)**.** Our math club has 18 members and 3 officers: President, Vice President, and Treasurer. Ali has a huge crush on Brenda, and won't be an officer unless she is one too. Brenda is unaware of Ali's affections, and doesn't care if he is an officer or not; she's perfectly happy to be an officer even if Ali isn't one. In how many ways can the club choose its officers?

(A) 7680 (B) 5922 (C) 5050 (D) 4176 (E) 3212

Problem 13.3.2 (ICLT-2023-SM1-2-L2-P2)**.** In the figure below $m \parallel n$, $AB \perp m$, $\angle ADC = \angle BCE = 3x$, $\angle CEB = 50°$, and $\angle BCD = x$. Find $\angle ACD$.

(A) 35 (B) 30 (C) 25 (D) 20 (E) 15



Problem 13.3.3 (ICLT-2023-SM1-2-L2-P3)**.** How many triples of prime numbers (p, q, r) such that:

$$p \le q \le r,\ p + q + r = 21.$$

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Problem 13.3.4 (ICLT-2023-SM1-2-L2-P4)**.** Evaluate

$$3(2y^2 + 4x - 2) - 6(x^2 - 2y - 7)$$

when $x = -y$.

(A) 36 (B) 24 (C) $2y^2$ (D) 12 (E) $2y + 1$

Problem 13.3.5 (ICLT-2023-SM1-2-L2-P5)**.** My debate club has 10 students: 5 girls and 5 boys. For the upcoming competition I must choose 5 students to form a team. One of the five chosen students is the team captain. Among the remaining four chosen students, two must be girls and two must be boys. How many different team can I choose?

(A) 5400 (B) 1000 (C) 600 (D) 300 (E) 200

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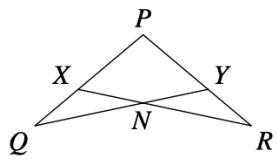
Problem 13.3.6 (ICLT-2023-SM1-2-L2-P6)**.** Two different rectangles have the same width. All four sides of both rectangles have integer lengths. The areas of the rectangles are 1818 and 1746. Find the largest possible value of the common width of the rectangles.

(A) 2 (B) 3 (C) 6 (D) 12 (E) 18

Problem 13.3.7 (ICLT-2023-SM1-2-L2-P7)**.** In the figure below, $PQ = PR$ and $\angle PQY = \angle PRX$.

1. (*5 points*) Prove that $QY = RX$.
2. (*5 points*) Prove that $XN = YN$.



Problem 13.3.8 (ICLT-2023-SM1-2-L2-P8)**.** Tim has four balls of different colours: blue, green, red, and white.

1. (*5 points*) He has two wooden boxes, both look exactly the same. In how many ways can he put the four balls into the two boxes?
2. (*5 points*) If he paints the boxes so that they look completely different, then in how many ways can he put the four balls into these two boxes?

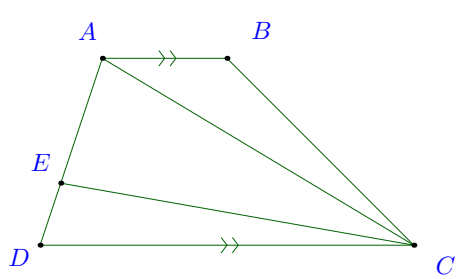
Problem 13.3.9 (ICLT-2023-SM1-2-L2-P9)**.** x and y are positive real numbers such that,

$$\begin{cases} \sqrt{x} + 2\sqrt{y} = \frac{5}{6} \\ 3\sqrt{x} - \sqrt{y} = \frac{1}{6} \end{cases}$$

1. (*5 points*) What is the value of $\sqrt{x}$?
2. (*5 points*) Solve the equation.

Problem 13.3.10 (ICLT-2023-SM1-2-L2-P10)**.** $ABCD$ is a trapezoid such that $AB \parallel CD$, $AB = 2$, $CD = 6$. E is on AD such that $DE = 2EA$.

1. (*4 points*) Prove that $[ADC] = 3[ABC]$.
2. (*5 points*) Prove that $[DEC] = [ABC]$.



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13.4 IC Test for Level 3

Problem 13.4.1 (ICLT-2023-SM1-2-L3-P1)**.** My debate club has 10 students: 5 girls and 5 boys. For the upcoming competition I must choose 5 students to form a team. One of the five chosen students is the team captain. Among the remaining four chosen students, two must be girls and two must be boys. How many different team can I choose?

(A) 200 (B) 300 (C) 600 (D) 1000 (E) 5400

Problem 13.4.2 (ICLT-2023-SM1-2-L3-P2)**.** Two different rectangles have the same width. All four sides of both rectangles have integer lengths. The areas of the rectangles are 1818 and 1746. Find the largest possible value of the common width of the rectangles.

(A) 18 (B) 12 (C) 6 (D) 3 (E) 2

Problem 13.4.3 (ICLT-2023-SM1-2-L3-P3)**.** x and y are positive real numbers such that,

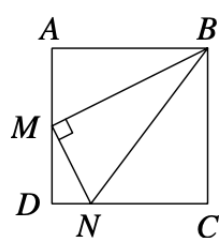
$$\begin{cases} \sqrt{x} + 2\sqrt{y} = \frac{5}{6} \\ 3\sqrt{x} - \sqrt{y} = \frac{1}{6} \end{cases}$$

Find the ratio $\frac{y}{x}$.

(A) 3 (B) 4 (C) 6 (D) 8 (E) 9

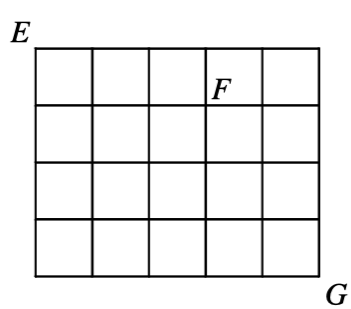
Problem 13.4.4 (ICLT-2023-SM1-2-L3-P4)**.** In the diagram below, $ABCD$ is a square. $AB = 4$. M is the midpoint of AD and $BM \perp MN$. Find BN.

(A) $2 + \sqrt{2}$ (B) 4 (C) $3\sqrt{2}$ (D) 5 (E) $4\sqrt{2}$



Problem 13.4.5 (ICLT-2023-SM1-2-L3-P5)**.** The diagram below shows a grid. How many 9-step paths are there going from G to E through F?

(A) 12 (B) 24 (C) 30 (D) 36 (E) 40



Problem 13.4.6 (ICLT-2023-SM1-2-L3-P6)**.** Let $n = 2^a \cdot 3^b$, where a and b are positive integers. If n has exactly 22 divisors, including 1 and itself, how many divisors does n^2 have?

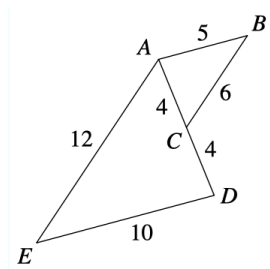
(A) 22 (B) 44 (C) 63 (D) 66 (E) 84

Problem 13.4.7 (ICLT-2023-SM1-2-L3-P7)**.** There are 805 animals in the store. They are dogs, cats, and mice. The ratio of dogs to cats to mice is $2 : 3 : 30$.

1. (*5 points*) How many cats are in the store?
2. (*5 points*) If 23 cats are added to the store, what is then the ratio of cats to dogs?

Problem 13.4.8 (ICLT-2023-SM1-2-L3-P8)**.** Given the side lengths shown in the diagram below.

1. (*5 points*) Prove that $AE \parallel BC$ and $AB \parallel DE$.
2. (*5 points*) What is the ratio of areas $\frac{[\triangle ABC]}{[\triangle ADE]}$?

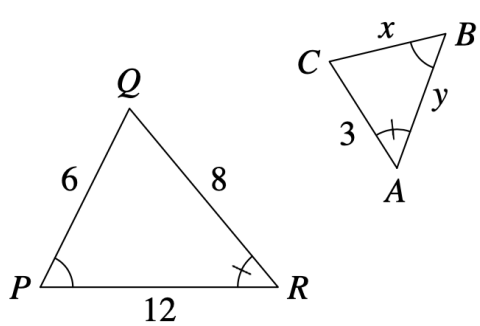


Problem 13.4.9 (ICLT-2023-SM1-2-L3-P9)**.** Let $n = (30)^2 = 900$.

1. (*5 points*) Find the number of distinct positive divisors of n excluding 1 and n.
2. (*5 points*) Find the number of distinct positive divisors of n^2 excluding 1 and n^2.

Problem 13.4.10 (ICLT-2023-SM1-4-L3-P10)**.** In the diagram at below, the angle equalities and side lengths of $\triangle PQR$ and $\triangle ABC$ are given.

1. (*5 points*) What is the value of $y + 2x$?
2. (*5 points*) What is the ratio of the areas $\frac{[PQR]}{[ABC]}$?



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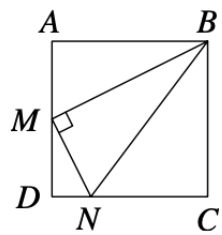
13.5 IC Test for Level 4

Problem 13.5.1 (ICLT-2023-SM1-2-L4-P1). (*6 points*)

In the figure below, $ABCD$ is a square. $AB = 4$. M is the midpoint of AD and $BM \perp MN$. Find BN.

(A) $2+\sqrt{2}$ (B) 4 (C) $3\sqrt{2}$ (D) 5 (E) $4\sqrt{2}$



Problem 13.5.2 (ICLT-2023-SM1-2-L4-P2). Let $n = 2^a \cdot 3^b$, where a and b are positive integers. If n has exactly 22 divisors, including 1 and itself, how many divisors does n^2 have?

(A) 22 (B) 44 (C) 63 (D) 66 (E) 84

Problem 13.5.3 (ICLT-2023-SM1-2-L4-P3). There are 805 animals in the store. They are dogs, cats, and mice. The ratio of dogs to cats to mice is $2:3:30$. How many cats are in the store?

(A) 23 (B) 46 (C) 69 (D) 92 (E) 115

Problem 13.5.4 (ICLT-2023-SM1-2-L4-P4). Suppose 2 distinct integers are chosen from between 0 and 9, inclusive. What is the probability that they are both odd?

(A) $\frac{2}{9}$ (B) $\frac{5}{18}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) $\frac{5}{9}$

Problem 13.5.5 (ICLT-2023-SM1-2-L4-P5). Find the standard form of the equation of the line through $(6,1)$ that is perpendicular to the line $2x = -3y + 7$.

(A) $3x-2y=10$ (B) $3x-2y=20$ (C) $2x-3y=10$ (D) $2x-3y=20$ (E) $3x-4y=20$

Problem 13.5.6 (ICLT-2023-SM1-2-L4-P6). In triangle $\triangle PQR$, $PQ = 3$, $\angle Q = 90^\circ$, and $QR = 4$. Find the inradius of $\triangle PQR$.

(A) 2 (B) $2-\sqrt{2}$ (C) $\frac{3}{2}$ (D) $\frac{1+\sqrt{2}}{2}$ (E) 1

Problem 13.5.7 (ICLT-2023-SM1-2-L4-P7). The Pirates play 4 games with the Mathematicians. The Pirates are 70% likely to win any given game.

1. (*5 points*) What is the probability that The Pirates win the first two games?
2. (*5 points*) What is the probability that the Mathematicians win exactly one game?

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Problem 13.5.8 (ICLT-2023-SM1-2-L4-P8)**.** In $\triangle XYZ$, $XZ = 8$. Point D is on side YZ such that $\angle YXD = \angle ZXD$, $YD = 3$, and $DZ = 4$.

1. (*5 points*) Find XY.
2. (*5 points*) Find the aread of $\triangle XYZ$.

Problem 13.5.9 (ICLT-2023-SM1-2-L4-P9)**.** Find all two-digit number $\overline{ab}$, if

1. (*5 points*) $a^2 + b^2 = 4(a + b) - 8$.
2. (*5 points*) $a^2 + b^2 = 4(a + b) - 3$.

Problem 13.5.10 (ICLT-2023-SM1-2-L4-P10)**.** Three cards are chosen at random from a standard 52-card deck.

1. (*5 points*) What is the probability that they are all the same color?
2. (*5 points*) What is the probability that they are not all the same color?

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Part II

2024

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Chapter 14

Introduction for Semester 2

Welcome to the second semester of the school year 2023 - 2024!

For the second semester of the 2023-2024 school year, the club plans to organize the program as follows:

- Objective 1: maintain current levels of O, A, I, IC, Pre-algebra.
- Objective 2: strong emphasis on working under pressure.
- Objective 3: solving very challenging problems.

A. Biweekly activity

- On Friday, students will receive 10 problems to solve in that week.
- Detailed solutions must be submitted no later than the following Monday.
- On the following Saturday/Sunday, students receive exam questions already available in the common online folder. The test consists of 10 exercises that are almost identical to the 10 exercises received last week, with some minor changes in the test conditions. Students only need to submit the answer (an integer) for each problem. After 30 minutes, students only need to fill in 10 answers and a file is available in their personal folder.
- If the official solutions of the previous week are available, students must study them and make sure that they master the solutions in order to know how to reuse them, especially when working under duress.
- The submissions, both solutions on Friday and answers on Saturday, are graded, each contributes 50% to the final mark of the problem.

For example:

- *Friday, January 19: Students receive assignments of Session 1*
- *Friday, January 26: Students submit detailed solutions for assignments of Session 1*
- *Saturday, January 27: Students receive the exam of Session 1 (similar problems as received on Friday, January 19), take the exam for 30 minutes, and submit their answers.*
- *Monday, January 29: Students receive the official solutions of Session 1 for both the assignment and the exam Problem. They should study them carefully.*
- *Friday, February 2: Students receive markings of both solutions (submitted on Friday) and answers (submitted on Saturday)*
- *Friday, February 2: Students receive assignments of Session 2*

Note that: students who read and do well on the exercises during the week will have the opportunity to do the exercises correctly during the Purple Comet Competition (with an estimated duration of 3 minutes per exercise).

Below is an example of a problem in a weekly assignment. You will need to submit a solution, similar like the one below.

Example (Problem 1 - Weekly)

(*5 points*) Find the number of integers n for which $\sqrt{\dfrac{(2020-n)^2}{2020-n^2}}$ is a real number.

Solution. The square root is a real number if the expression under the square root sign is non-negative.

$$\frac{(2020-n)^2}{2020-n^2} \geq 0.$$

There are two cases.

Case 1:

$$\frac{(2020-n)^2}{2020-n^2} = 0 \Rightarrow n = 2020.$$

Case 1:

$$\frac{(2020-n)^2}{2020-n^2} > 0 \Rightarrow 2020 - n^2 > 0 \Rightarrow |n| < \sqrt{2020} < 45^2 = 2025 \Rightarrow -44 \leq n \leq 44.$$

There are $1 + 2 \cdot 44 = 89$ such integers between -44 and 44.

Hence, in total there are $1 + 89 = \boxed{90}$ such numbers. □

Then on the exam day, you get a variation of the problem. The correct answer of this problem is $\boxed{92.}$ *Below is the official* **generic solution** *that you don't have to submit.*

Example (Problem 1 - Exam)

(*5 points*) Find the number of integers n for which $\sqrt{\dfrac{(2048-n)^2}{2048-n^2}}$ is a real number.

Solution. Let $m > 1$ is a positive integer.

$$\sqrt{\frac{(m-n)^2}{m-n^2}} \text{ is real number } \Rightarrow \frac{(m-n)^2}{m-n^2} \geq 0.$$

We have two cases:

$$\begin{cases} \dfrac{(m-n)^2}{m-n^2} = 0 \Rightarrow n = m. \\ \dfrac{(m-n)^2}{m-n^2} > 0 \Rightarrow m - n^2 > 0 \Rightarrow |n| < \sqrt{m}. \end{cases} \Rightarrow n \in \{m, -\lfloor\sqrt{m}\rfloor, \ldots, -1, 0, 1, \ldots, \lfloor\sqrt{m}\rfloor\}.$$

There are $2 + 2\lfloor\sqrt{m}\rfloor$ such numbers. For $m = 2048$, there are $2(1 + \lfloor\sqrt{2048}\rfloor) = \boxed{92.}$ □

If you have successfully solved the problem in time then you will have an advantage at the exam. If you solve the problem in a generic way, the exam will be a breeze.

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Solution based on coding can accepted only if

- The code is written in C, C++, Python, MatLab, or Java. No other languages are allowed.
- The code should not use any non-standard libraries.
- The code should compile without extra settings, should result in no compilation errors, or runtime errors.
- Execution time should not exceed 10 minutes.
- Submission must contain both source code, input (if any) and output (printout).

Failure to comply to any of the requirements will lead to rejection of the solution.

B. Semester open competition: Every months for each level there will be 4 problems to solve. Students must submit solutions before given deadlines (will be stated clearly in the text).

C. Purple Comet competition: This year's Purple Comet exam schedule is as follow:

- April 6: middle-school (MS) teams - grade 8 (US, CA), grade 9 (FR, UK, VN) and younger
- April 7: high-school teams - grade 9 (US, CA), grade 10 (FR, UK, VN) and above

Each team has 6 members. Students will be selected based on their weekly activity before April.

- MS team: 60 minutes, 20 problems, middle school math, a team is divided into two groups:
  - M1 - the first ten lessons (1-10): corresponds to level I;
  - M2 - the last ten lessons (11-20): corresponds to level A.
- HS team: 90 minutes, 30 problems, high school math, a team is divided into three groups:
  - H1 - first ten lessons (1-10): corresponds to level I;
  - H2 - the middle ten lessons (11-20): corresponds to level A;
  - H3 - last ten lessons (20-30): corresponds to level O.

Have a great year!

Your teacher,

Nghia Doan

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Chapter 15

Session 1

15.1 Middle School - Assignment

Middle school students: grade 8 (US, CA), grade 9 (FR, UK, VN) and younger.

- **Submission deadline: Friday, January 26**
- **Test: Saturday, January 27**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 5-14**

Problem 15.1.1 (Problem 1)**.** (*5 points*) Alex launches his boat into a river and heads upstream at a constant speed. At the same time at a point 8 miles upstream from Alex, Alice launches her boat and heads downstream at a constant speed. Both boats move at 6 miles per hour in still water, but the river is flowing downstream at $2\frac{3}{10}$ miles per hour. Alex and Alice will meet at a point that is $\frac{m}{n}$ miles from Alex's starting point, where m and n are relatively prime positive integers. Find $m+n$.

Problem 15.1.2 (Problem 2)**.** (*5 points*) Find a positive integer n such that there is a polygon with n sides where each of its interior angles measures $177°$.

Problem 15.1.3 (Problem 3)**.** (*5 points*) Patrick started walking at a constant rate along a straight road from school to the park. One hour after Patrick left, Tanya started running along the same road from school to the park. One hour after Tanya left, Jose started bicycling along the same road from school to the park. Tanya ran at a constant rate of 2 miles per hour faster than Patrick walked, Jose bicycled at a constant rate of 7 miles per hour faster than Tanya ran, and all three arrived at the park at the same time. The distance from the school to the park is $\frac{m}{n}$ miles, where m and n are relatively prime positive integers. Find $m+n$.

Problem 15.1.4 (Problem 4)**.** (*5 points*) Find the number of positive integers less than or equal to 2020 that are relatively prime to 588.

Problem 15.1.5 (Problem 5)**.** (*5 points*) Given that a, b, and c are distinct positive integers such that $a \cdot b \cdot c = 2020$, the minimum possible positive value of $\frac{1}{a} - \frac{1}{b} - \frac{1}{c}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 15.1.6 (Problem 6)**.** (*5 points*) Mary mixes 2 gallons of a solution that is 40 percent alcohol with 3 gallons of a solution that is 60 percent alcohol. Sandra mixes 4 gallons of a solution that is 30 percent alcohol with $\frac{m}{n}$ gallons of a solution that is 80 percent alcohol, where m and n are relatively prime positive integers. Mary and Sandra end up with solutions that are the same percent alcohol. Find $m+n$.

Problem 15.1.7 (Problem 7)**.** (*5 points*) Let a and b be positive integers such that $(a^3-a^2+1)(b^3-b^2+2) = 2020$. Find $10a+b$.

Problem 15.1.8 (Problem 8)**.** (*5 points*) Find the number of three-digit palindromes that are divisible by 3. Recall that a palindrome is a number that reads the same forward and backward like 727 or 905509.

Problem 15.1.9 (Problem 9)**.** (*5 points*) Six different small books and three different large books sit on a shelf. Three children may each take either two small books or one large book. Find the number of ways the three children can select their books.

Problem 15.1.10 (Problem 10)**.** (*5 points*) Daniel had a string that formed the perimeter of a square with area 98. Daniel cut the string into two pieces. With one piece he formed the perimeter of a rectangle whose width and length are in the ratio $2:3$. With the other piece he formed the perimeter of a rectangle whose width and length are in the ratio $3:8$. The two rectangles that Daniel formed have the same area, and each of those areas is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 15.1.11 (Problem 11)**.** (*5 points*) Find the number of permutations of the letters ABCDE where the letters A and B are not adjacent and the letters C and D are not adjacent. For example, count the permutations ACBDE and DEBCA but not ABCED or EDCBA.

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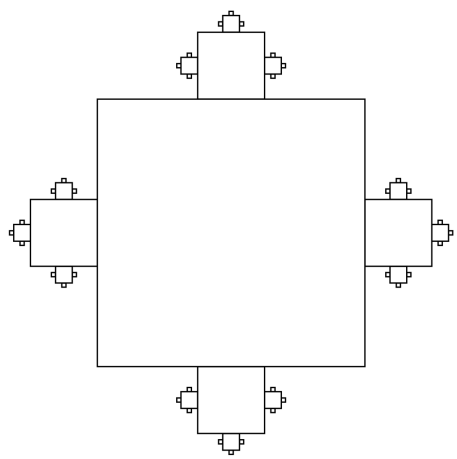
Problem 15.1.12 (Problem 12)**.** (*5 points*) Construct a geometric figure in a sequence of steps. In step 1, begin with a 4×4 square.

In step 2, attach a 1×1 square onto the each side of the original square so that the new squares are on the outside of the original square, have a side along the side of the original square, and the midpoints of the sides of the original square and the attached square coincide.

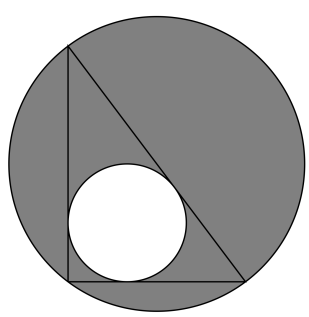
In step 3, attach a $\frac{1}{4} \times \frac{1}{4}$ square onto the centers of each of the 3 exposed sides of each of the 4 squares attached in step 2. For each positive integer n, in step $n+1$, attach squares whose sides are $\frac{1}{4}$ as long as the sides of the squares attached in step n placing them at the centers of the 3 exposed sides of the squares attached in step n. The diagram shows the figure after step 4.

If this is continued for all positive integers n, the area covered by all the squares attached in all the steps is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Problem 15.1.13 (Problem 13)**.** (*5 points*) Wendy randomly chooses a positive integer less than or equal to 2020. The probability that the digits in Wendy's number add up to 10 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 15.1.14 (Problem 14)**.** (*5 points*) Right $\triangle ABC$ has side lengths 6, 8, and 10. Find the positive integer n such that the area of the region inside the circumcircle but outside the incircle of $\triangle ABC$ is $n\pi$.



15.2 Middle School - Test

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 5-14**

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Problem 15.2.1 (Problem 1)**.** (*5 points*) Alex launches his boat into a river and heads upstream at a constant speed. At the same time at a point 10 miles upstream from Alex, Alice launches her boat and heads downstream at a constant speed. Both boats move at 6 miles per hour in still water, but the river is flowing downstream at 3 miles per hour. Alex and Alice will meet at a point that is $\frac{m}{n}$ miles from Alex's starting point, where m and n are relatively prime positive integers. Find $m+n$.

Problem 15.2.2 (Problem 2)**.** (*5 points*) Find a positive integer n such that there is a polygon with n sides where each of its interior angles measures $165°$.

Problem 15.2.3 (Problem 3)**.** (*5 points*) Patrick started walking at a constant rate along a straight road from school to the park. Two hour after Patrick left, Tanya started running along the same road from school to the park. One hour after Tanya left, Jose started bicycling along the same road from school to the park. Tanya ran at a constant rate of 1 miles per hour faster than Patrick walked, Jose bicycled at a constant rate of 6 miles per hour faster than Tanya ran, and all three arrived at the park at the same time. The distance from the school to the park is $\frac{m}{n}$ miles, where m and n are relatively prime positive integers. Find $m+n$.

Problem 15.2.4 (Problem 4)**.** (*5 points*) Find the number of positive **even** integers less than or equal to 2024 that are relatively prime to 15.

Problem 15.2.5 (Problem 5)**.** (*5 points*) Given that a, b, and c are distinct positive integers such that $a \cdot b \cdot c = 1236$, the minimum possible positive value of $\frac{1}{a} - \frac{1}{b} - \frac{1}{c}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 15.2.6 (Problem 6)**.** (*5 points*) Mary mixes 2 gallons of a solution that is 30 percent alcohol with 3 gallons of a solution that is 70 percent alcohol. Sandra mixes 4 gallons of a solution that is 20 percent alcohol with $\frac{m}{n}$ gallons of a solution that is 90 percent alcohol, where m and n are relatively prime positive integers. Mary and Sandra end up with solutions that are the same percent alcohol. Find $m+n$.

Problem 15.2.7 (Problem 7)**.** (*5 points*) Let a and b be positive integers such that $(a^3-a^2+1)(b^3-b^2+2) = 1938$. Find $10a+b$.

Problem 15.2.8 (Problem 8)**.** (*5 points*) Find the number of *four-digit* palindromes that are divisible by 3. Recall that a palindrome is a number that reads the same forward and backward like 727 or 905509.

Problem 15.2.9 (Problem 9)**.** (*5 points*) Four different small books and four different large books sit on a shelf. Three children may each take either two small books or one large book. Find the number of ways the three children can select their books.

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Problem 15.2.10 (Problem 10)**.** (*5 points*) Daniel had a string that formed the perimeter of a square. Daniel cut the string into two pieces. With one piece he formed the perimeter of a rectangle whose width and length are in the ratio $2:3$. With the other piece he formed the perimeter of a rectangle whose width and length are in the ratio $3:8$. The two rectangles that Daniel formed have the same area, and the measure of each of those areas is 2.

The length of the string is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by any perfect square larger than 1. Find $m+n$.

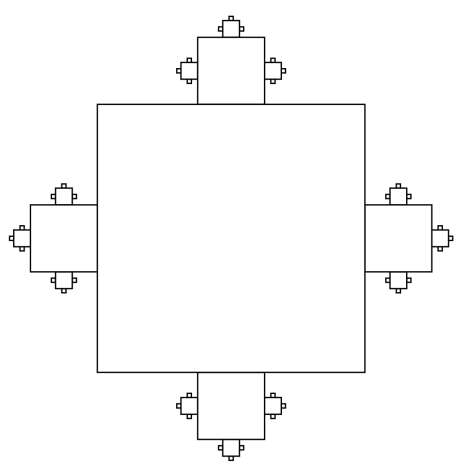
Problem 15.2.11 (Problem 11)**.** (*5 points*) Find the number of permutations of the letters ABCDEF where the letters A and B are not adjacent, the letters C and D are not adjacent, and the letters E and F are not adjacent. For example, count the permutations FACBDE and DEBCAF but not ABCDEF or EDCBAF.

Problem 15.2.12 (Problem 12)**.** (*5 points*) Construct a geometric figure in a sequence of steps. In step 1, begin with a 4×4 square.

In step 2, attach a 1×1 square onto the each side of the original square so that the new squares are on the outside of the original square, have a side along the side of the original square, and the midpoints of the sides of the original square and the attached square coincide.

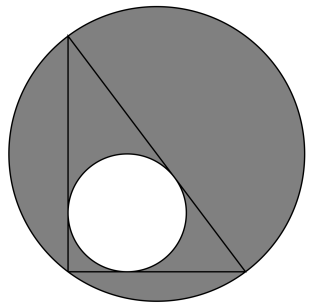
In step 3, attach a $\frac{1}{4} \times \frac{1}{4}$ square onto the centers of each of the 3 exposed sides of each of the 4 squares attached in step 2. For each positive integer n, in step $n+1$, attach squares whose sides are $\frac{1}{4}$ as long as the sides of the squares attached in step n placing them at the centers of the 3 exposed sides of the squares attached in step n. The diagram shows the figure after step 4.

If this is continued for all positive integers n, let $\frac{m}{n}$, where m and n are relatively prime positive integers, be the area covered by all the squares attached in all the **odd** steps: step 1, step 3, step 5, etc. Find $m+n$.



Problem 15.2.13 (Problem 13)**.** (*5 points*) Wendy randomly chooses a positive integer less than or equal to 2024. The probability that the digits in Wendy's number add up to 8 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 15.2.14 (Problem 14)**.** (*5 points*) Right $\triangle ABC$ has integer side lengths a, b, and c, where $\angle BAC = 90°$. Let r be the radius of the incircle, R be the radius of the circumcircle, then the ratio $\frac{r+R}{a+b}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



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15.3 High School - Assignment

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

- **Submission deadline: Friday, January 26**
- **Test: Saturday, January 27**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 4-13**
- **Olympiad (O) level: Problems 10-19**

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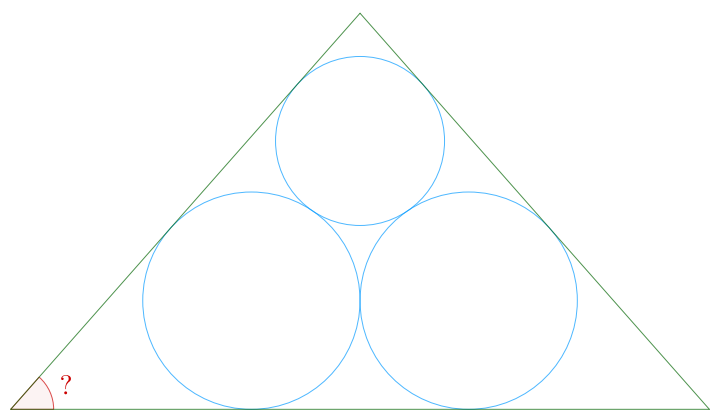
Problem 15.3.1 (Problem 1)**.** (*5 points*) There is a complex number K such that the quadratic polynomial $7x^2 + Kx + 12 - 5i$ has exactly one root, where $i = \sqrt{-1}$. Find $|K|^2$.

Note that for a complex number $x = a + bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2 + b^2}$.

Problem 15.3.2 (Problem 2)**.** (*5 points*) Two circles have radius 9, and one circle has radius 7. Each circle is externally tangent to the other two circles, and each circle is internally tangent to two sides of an isosceles triangle, as shown. The sine of the base angle of the triangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 15.3.3 (Problem 3)**.** (*5 points*) There are two distinct pairs of positive integers $a_1 < b_1$ and $a_2 < b_2$ such that both $|(a_1 + ib_1)(b_1 - ia_1)|$ and $|(a_2 + ib_2)(b_2 - ia_2)|$ equal 2020, where $i = \sqrt{-1}$. Find $a_1 + b_1 + a_2 + b_2$.

Note that for a complex number $x = a + bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2 + b^2}$.

Problem 15.3.4 (Problem 4)**.** (*5 points*) There are relatively prime positive integers s and t such that

$$\sum_{n=2}^{100} \left(\frac{n}{n^2 - 1} - \frac{1}{n} \right) = \frac{s}{t}.$$

Find $s + t$.

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Problem 15.3.5 (Problem 5)**.** (*5 points*) Let x be a real number such that $3\sin^4 x - 2\cos^6 x = -\frac{17}{25}$. Then $3\cos^4 x - 2\sin^6 x = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $10m + n$.

Problem 15.3.6 (Problem 6)**.** (*5 points*) Find the sum of all values of x such that the set

$$\{107, 122, 127, 137, 152, x\} \text{ has a mean that is equal to its median.}$$

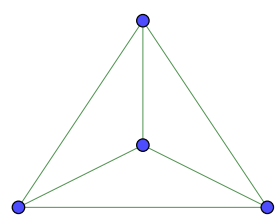
Problem 15.3.7 (Problem 7)**.** (*5 points*) Find the maximum possible value of

$$\left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right)^2$$

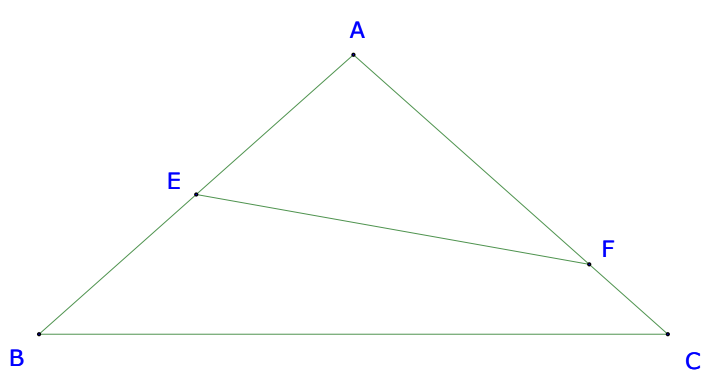
where a, b, and c are nonzero real numbers satisfying

$$a\sqrt[3]{\frac{a}{b}} + b\sqrt[3]{\frac{b}{c}} + c\sqrt[3]{\frac{c}{a}} = 0.$$

Problem 15.3.8 (Problem 8)**.** (*5 points*) The following diagram shows four vertices connected by six edges. Suppose that each of the edges is randomly painted either red, white, or blue. The probability that the three edges adjacent to at least one vertex are colored with all three colors is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 15.3.9 (Problem 9)**.** (*5 points*) In isosceles $\triangle ABC, AB = AC, \angle BAC$ is obtuse, and points E and F lie on sides AB and AC, respectively, so that $AE = 10, AF = 15$. The area of $\triangle AEF$ is 60, and the area of quadrilateral $BEFC$ is 102. Find BC.



Problem 15.3.10 (Problem 10)**.** (*5 points*) Find the least prime number greater than 1000 that divides $2^{1010} \cdot 23^{2020} + 1$.

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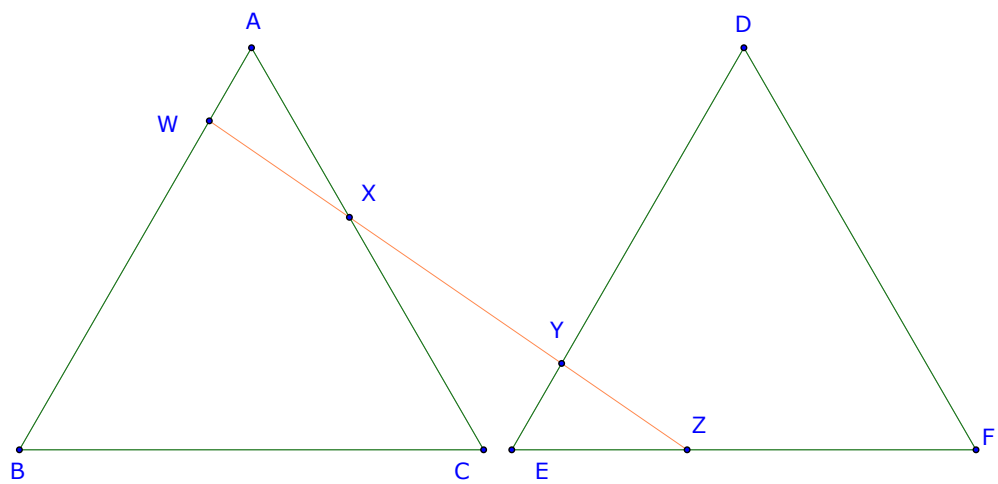
Problem 15.3.11 (Problem 11)**.** (*5 points*) Find the maximum possible value of

$$9\sqrt{x} + 8\sqrt{y} + 5\sqrt{z},$$

where x, y, and z are positive real numbers satisfying $9x + 4y + z = 128$.

Problem 15.3.12 (Problem 12)**.** (*5 points*) Two congruent equilateral triangles $\triangle ABC$ and $\triangle DEF$ lie on the same side of line BC so that B, C, E, and F are collinear as shown. A line intersects AB, AC, DE, and EF at W, X, Y, and Z, respectively, such that $\dfrac{AW}{BW} = \frac{2}{9}$, $\dfrac{AX}{CX} = \frac{5}{6}$, and $\dfrac{DY}{EY} = \frac{9}{2}$. The ratio $\dfrac{EZ}{FZ}$ can then be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 15.3.13 (Problem 13)**.** (*5 points*) Find the number of permutations of the letters AAAABBBCC where no letter is next to another letter of the same type. For example, count ABCABCABA and ABABCABCA but not ABCCBABAA.

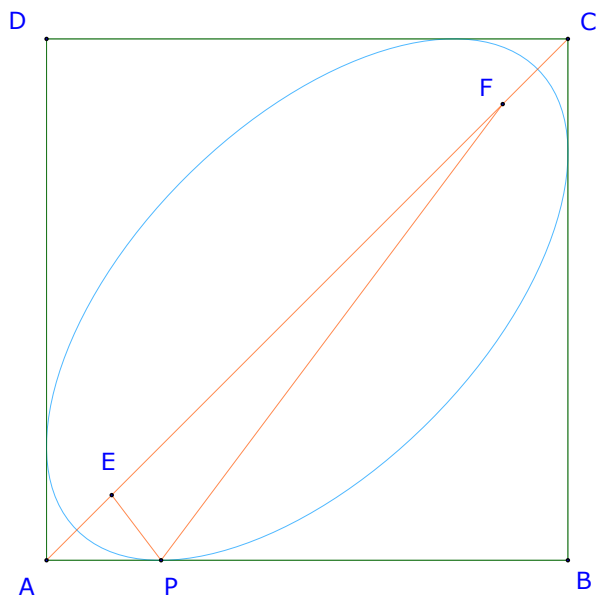
Problem 15.3.14 (Problem 14)**.** (*5 points*) There is a real number x between 0 and $\dfrac{\pi}{2}$ such that

$$\frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{12}{11}$$

and $\sin x + \cos x = \frac{\sqrt{m}}{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Problem 15.3.15 (Problem 15)**.** (*5 points*) Points E and F lie on diagonal AC of square $ABCD$ with side length 24, such that $AE = CF = 3\sqrt{2}$. An ellipse with foci at E and F is tangent to the sides of the square. Find the sum of the distances from any point on the ellipse to the two foci.

Hint: Let P be the tangent point on AB. By the properties of the eclipse, a light ray passing from E to P that reflects of the eclipse will pass through point F. Because line AB is tangent to the eclipse, $\angle EPA = \angle FPB$.



Problem 15.3.16 (Problem 16)**.** (*5 points*) A deck of eight cards has cards numbered $1, 2, 3, 4, 5, 6, 7, 8$, in that order, and a deck of five cards has cards numbered $1, 2, 3, 4, 5$, in that order. The two decks are riffle-shuffled together to form a deck with 13 cards with the cards from each deck in the same order as they were originally. Thus, numbers on the cards might end up in the order 1122334455678 or 1234512345678 but not 1223144553678. Find the number of possible sequences of the 13 numbers.

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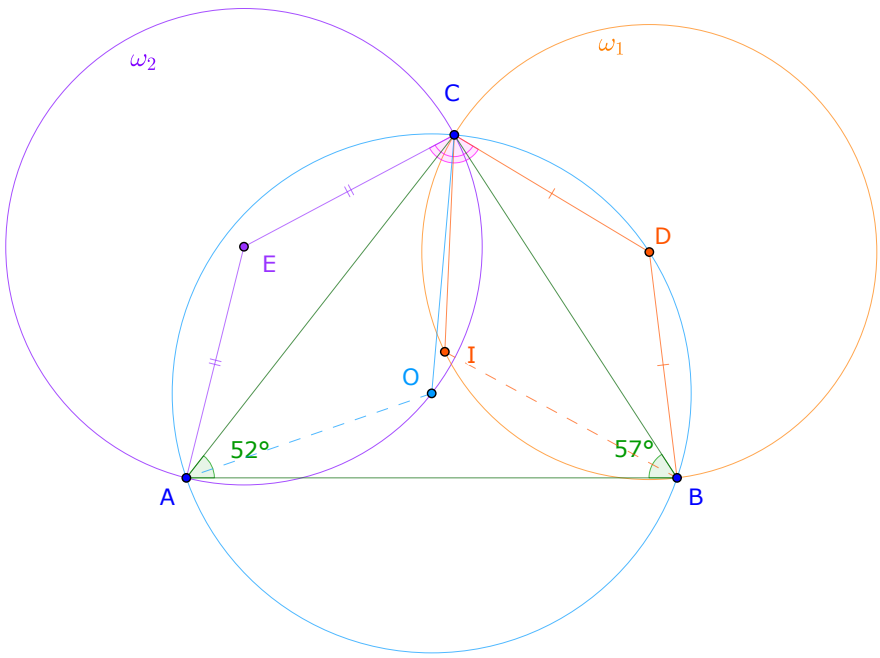
Problem 15.3.17 (Problem 17)**.** (*5 points*) In $\triangle ABC$, $\angle A = 52°$, and $\angle B = 57°$. One circle centred at D passes through the points B, C, and the incenter I of $\triangle ABC$, and a second circle centred at E passes through the points A, C, and the circumcenter O of $\triangle ABC$. Find the degree measure of the acute angle at which the two circles intersect.

Hint: find the measure of the complementary angle of the angle $\angle DCE$.



Problem 15.3.18 (Problem 18)**.** (*5 points*) Three doctors, four nurses, and three patients stand in a line in random order. The probability that there is at least one doctor and at least one nurse between each pair of patients is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 15.3.19 (Problem 19)**.** (*5 points*) Let p, q, and r be prime numbers such that $2pqr+p+q+r = 2020$. Find $pq + qr + rp$.

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15.4 High School - Test

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

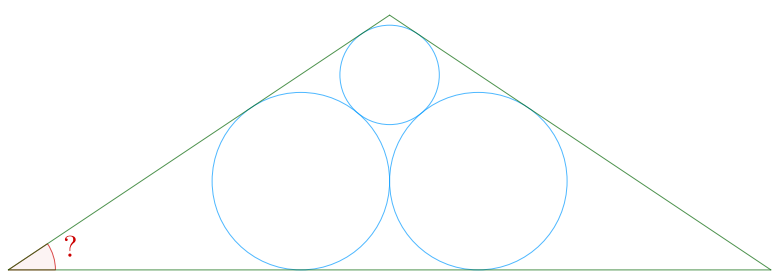
You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 4-13**
- **Olympiad (O) level: Problems 10-19**

Problem 15.4.1 (Problem 1)**.** (*5 points*) There is a complex number K such that the quadratic polynomial $5x^2 + Kx + 4 - 3i$ has exactly one root, where $i = \sqrt{-1}$. Find $|K|^2$.

Note that for a complex number $x = a + bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2 + b^2}$.

Problem 15.4.2 (Problem 2)**.** (*5 points*) Two circles have radius 25, and one circle has radius 14. Each circle is externally tangent to the other two circles, and each circle is internally tangent to two sides of an isosceles triangle, as shown. The sine of the base angle of the triangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 15.4.3 (Problem 3)**.** (*5 points*) There are three distinct pairs of positive integers $a_1 < b_1$, $a_2 < b_2$, and $a_3 < b_3$ such that

$$|(a_1 + ib_1)(b_1 - ia_1)| = |(a_2 + ib_2)(b_2 - ia_2)| = |(a_3 + ib_3)(b_3 - ia_3)| = 1025, \text{ where } i = \sqrt{-1}.$$

Find $a_1 + b_1 + a_2 + b_2 + a_3 + b_3$.

Note that for a complex number $x = a + bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2 + b^2}$.

Problem 15.4.4 (Problem 4)**.** (*5 points*) There are relatively prime positive integers s and t such that

$$\sum_{n=2}^{101} \left(\frac{n}{n^2 - 1} - \frac{1}{n} \right) = \frac{s}{t}.$$

Find $s + t$.

Problem 15.4.5 (Problem 5)**.** (*5 points*) Let x be a real number such that $3\sin^4 x - 2\cos^6 x = \dfrac{17}{25}$. Then $3\cos^4 x - 2\sin^6 x = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $10m + n$.

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Problem 15.4.6 (Problem 6)**.** (*5 points*) Find the sum of all values of x such that the set

$$\{108, 123, 128, 138, 153, x\} \text{ has a mean that is equal to its median.}$$

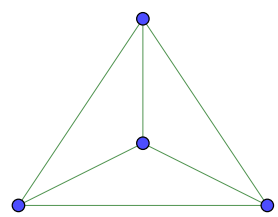
Problem 15.4.7 (Problem 7)**.** (*5 points*) Find the **minimum** possible value of

$$\left(\frac{a^2}{b^3c^2} + \frac{b^2}{c^3a^2} + \frac{c^2}{a^3b^2}\right)^2$$

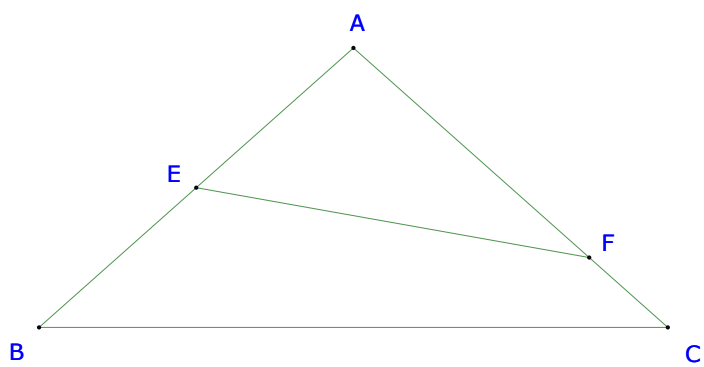
where $a > b > 0 > c$ are real numbers satisfying

$$a\sqrt[3]{\frac{a}{b}} + b\sqrt[3]{\frac{b}{c}} + c\sqrt[3]{\frac{c}{a}} = 0.$$

Problem 15.4.8 (Problem 8)**.** (*5 points*) The following diagram shows four vertices connected by six edges. Suppose that each of the edges is randomly painted either red, white, or blue. The probability that the three edges adjacent to **at least two** vertices are colored with all three colors is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 15.4.9 (Problem 9)**.** (*5 points*) In isosceles $\triangle ABC, AB = AC, \angle BAC$ is obtuse, and points E and F lie on sides AB and AC, respectively, so that $AE = 20, AF = 30$. The area of $\triangle AEF$ is 240, and the area of quadrilateral $BEFC$ is 408. Find BC.



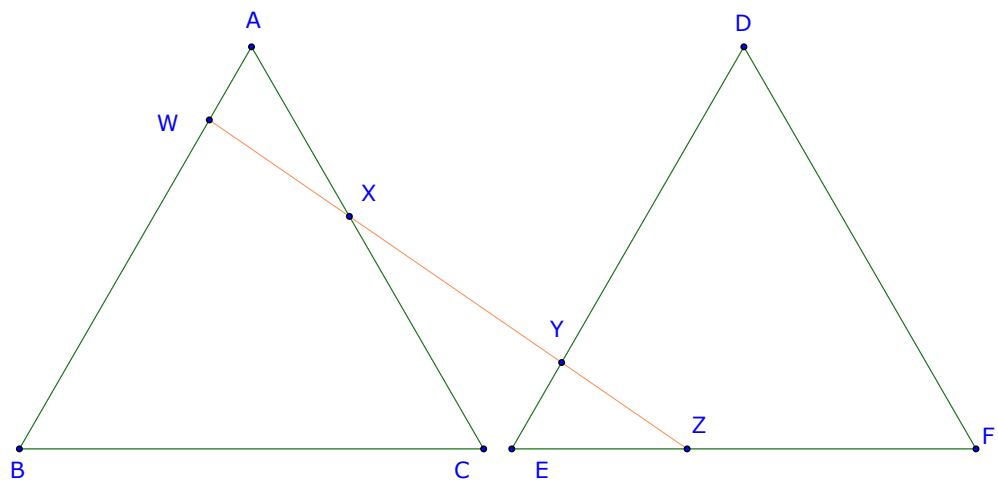
Problem 15.4.10 (Problem 10)**.** (*5 points*) Find the least prime number greater than 1000 that divides $2^{1010} \cdot 235^{2020} + 1$.

Problem 15.4.11 (Problem 11)**.** (*5 points*) Find the maximum possible value of

$$8\sqrt{x} + 9\sqrt{y} + 12\sqrt{z},$$

where x, y, and z are positive real numbers satisfying $4x + 9y + z = 256$.

Problem 15.4.12 (Problem 12)**.** (*5 points*) Two congruent equilateral triangles $\triangle ABC$ and $\triangle DEF$ lie on the same side of line BC so that B, C, E, and F are collinear as shown. A line intersects AB, AC, DE, and EF at W, X, Y, and Z, respectively, such that $\dfrac{AW}{BW} = \frac{2}{9}$, $\dfrac{AX}{CX} = \frac{5}{6}$, and $\dfrac{DY}{EY} = \frac{9}{2}$. The ratio $\dfrac{XW}{XZ}$ can then be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 15.4.13 (Problem 13)**.** (*5 points*) Find the number of permutations of the letters AAABBCC where there are no two like letters adjacent. For example, count ABCABCA and ABABCAC but not ABCCBAA.

Problem 15.4.14 (Problem 14)**.** (*5 points*) There is a real number x between 0 and $\dfrac{\pi}{2}$ such that

$$\frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{20}{19}$$

and $\sin x + \cos x = \frac{\sqrt{m}}{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

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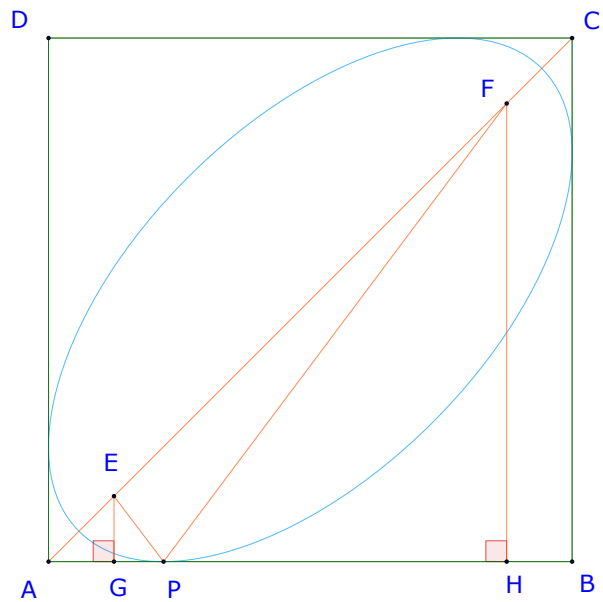
Problem 15.4.15 (Problem 15)**.** (*5 points*) Points E and F lie on diagonal AC of square $ABCD$ with side length 24, such that $AE = CF = 3\sqrt{2}$. An ellipse with foci at E and F is tangent to the sides of the square. Let P be the tangent point of the eclipse with AB. Let points G and H be the projections onto AB of points E and F, respectively. Let $\frac{m}{n}$ be the value of $\cos EPF$, where m and n are relatively prime positive integers. Find $m + n$.

Hint: By the properties of the eclipse, a light ray passing from E to P that reflects of the eclipse will pass through point F. Because line AB is tangent to the eclipse, $\angle EPA = \angle FPB$.



Problem 15.4.16 (Problem 16)**.** (*5 points*) A deck of eight cards has cards numbered $1, 2, 3, 4, 5, 6, 7$, in that order, and a deck of five cards has cards numbered $1, 2, 3, 4, 5, 6$ in that order. The two decks are riffle-shuffled together to form a deck with 13 cards with the cards from each deck in the same order as they were originally. Thus, numbers on the cards might end up in the order 1122334455667 or 1234561234567 but not 1223144553667. Find the number of possible sequences of the 13 numbers.

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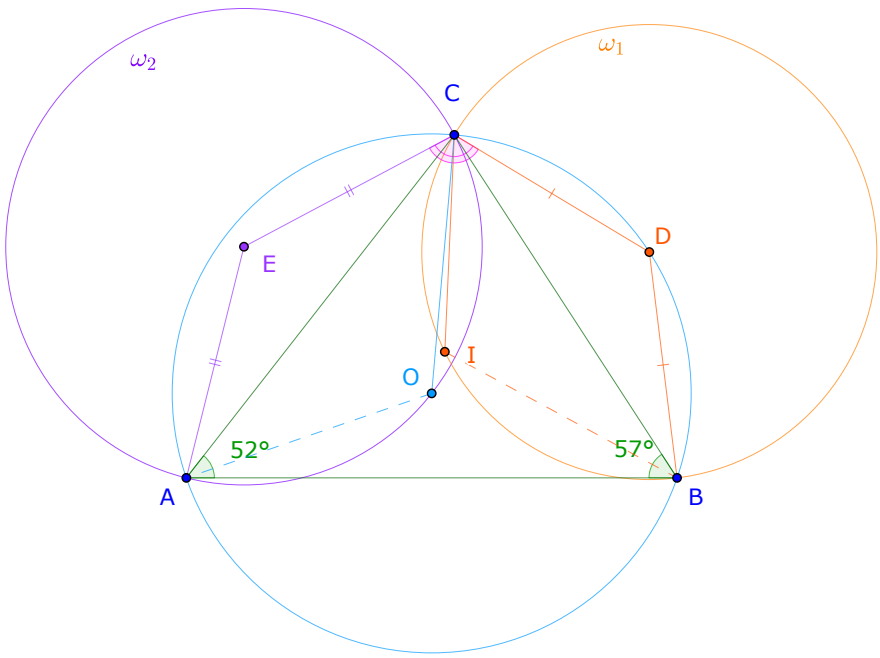
Problem 15.4.17 (Problem 17)**.** (*5 points*) In $\triangle ABC$, $\angle A = 50°$, and $\angle B = 59°$. One circle centred at D passes through the points B, C, and the incenter I of $\triangle ABC$, and a second circle centred at E passes through the points A, C, and the circumcenter O of $\triangle ABC$. Find the degree measure of the acute angle at which the two circles intersect.

Hint: find the measure of the complementary angle of the angle $\angle DCE$.



Problem 15.4.18 (Problem 18)**.** (*5 points*) Three doctors, three nurses, and three patients stand in a line in random order. The probability that there is at least one doctor and at least one nurse between each pair of patients is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 15.4.19 (Problem 19)**.** (*5 points*) Let p, q, and r be prime numbers such that $2pqr+p+q+r = 916$. Find $pq + qr + rp$.

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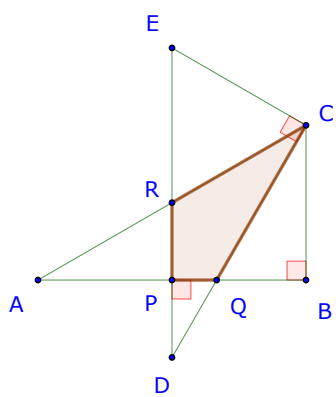
Chapter 16

January Challenges

16.1 I-Level

For both middle school ang high school students. No solution based on coding is allowed.

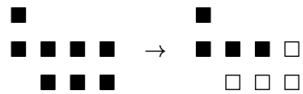
Problem 16.1.1 (Problem 1)**.** (*10 points*) We divide an equilateral triangle with area 600 into two halves and then arrange them as shown below. Find the area of the $PQCR$.



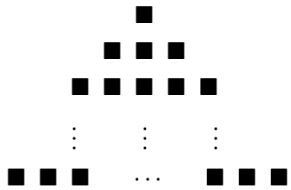
Problem 16.1.2 (Problem 2)**.** (*10 points*) Find all pairs of positive intergers (m, n) such that

$$1 \cdot 2 \cdots (m-1)(m) + 76 = n^2$$

Problem 16.1.3 (Problem 3)**.** (*10 points*) Eight black squares are place on the plane, see the diagram below on the left. Without moving any of them, four of the squares can be coloured white, so that they are divided into *two congruent sets of different coloured squares*: a set of four black squares and a set of four white squares, as show below. *Note that one set is the mirror image of the other.*



Now, 100 black squares are placed on the plane as shown below *(*19 *black squares in the* 10th *row.)*



In how many ways, without moving any of them, 50 of the squares can be can be coloured white, so that they are divided into two congruent sets of different coloured squares?

Problem 16.1.4 (Problem 4)**.** (*10 points*) Let s be the sum of some integers, one of them has at least two digits. Two digits of one of the integers in s is chosen and they change places. The new sum is r. Which prime numbers 2, 3, 5, 7, 9, or 11 are always divisors of the difference $s - r$?

For example

$$s = 1 + 34 + 6 + 752 = 793,\ \underline{7}5\underline{2} \to \underline{2}5\underline{7} \Rightarrow r = 1 + 34 + 6 + 257 = 298 \Rightarrow s - r = 793 - 298 = 495.$$

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Chapter 17

Entrance Tests

17.1 I-Level Entrance Test

For both middle school ang high school students. You are allowed to use books and calculator. No solution based on coding is allowed. No searching on the Internet for hints or solutions is permitted. **Full and detailed solution is required for every problem.**

Problem 17.1.1 (Problem 1)**.** (*10 points*) You are given a set of 10 positive integers. Summing nine of them in ten possible ways we get only nine different sums

$$86, 87, 88, 89, 90, 91, 93, 94, 95.$$

Find those numbers.

Problem 17.1.2 (Problem 2)**.** (*10 points*) Replace the asterisks in the equilateral triangle by the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ so that, starting from the second line, each number is equal to the absolute value of the difference of the nearest two numbers in the line above.

```
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  * * * * * * *
   * * * * * *
    * * * * *
     * * * *
      * * *
       * *
        *
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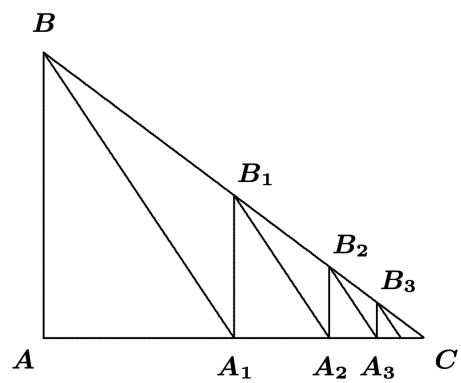
Note that: for this problem, submitting the equilateral triangle with all the asterisks replaced bu the number $1, 2, \ldots, 9$ *is considered as a solution. No additional explanation is required.*

Problem 17.1.3 (Problem 3)**.** (*10 points*) In the diagram below, ABC is a right triangle, where $AB = 4, AC = 4$. Furthermore, each of the line segments $A_1B_1, A_2B_2, \ldots$ is perpendicular to AC, A_1 bisects segment AC, A_2 bisects segment A_1C, A_3 bisects segment A_2C, and so on. Find the sum of the lengths:

$$BA_1 + B_1A_2 + B_2A_3 + \ldots$$



Problem 17.1.4 (Problem 4)**.** (*10 points*) A four-digit number which is a perfect square is created by writing Lan's age in years followed by Nam's age in years. Similarly, after 31 years, their ages in the same order will again form a four-digit perfect square. Determine the present ages of Lan and Nam.

Note: for example if Lan's is 19 years old and Name is 11 years old then the four-digit number which is a perfect square is created by writing Lan's age in years followed by Nam's age in years is 1911.

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Chapter 18

Session 2

18.1 Middle School - Assignment

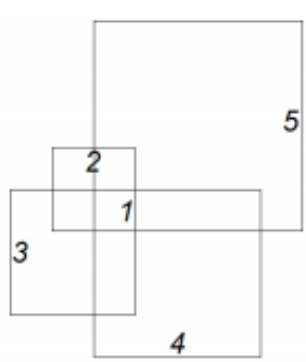
Middle school students: grade 8 (US, CA), grade 9 (FR, UK, VN) and younger.

- **Submission deadline: Friday, February 9**
- **Test: Saturday, February 10**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

Problem 18.1.1 (Problem 1)**.** (*5 points*) Find the number of three-digit positive integers where the digits are three different prime numbers. For example, count 235 but not 553

Problem 18.1.2 (Problem 2)**.** (*5 points*) Melanie has $4\frac{2}{5}$ cups of flour. The recipe for one batch of cookies calls for $1\frac{1}{2}$ cups of flour. Melanie plans to make $2\frac{1}{2}$ batches of cookies. When she is done, she will have $\frac{m}{n}$ cups of flour remaining, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18.1.3 (Problem 3)**.** (*5 points*) The figure below has a 1×1 square, a 2×2 square, a 3×3 square, a 4×4 square, and a 5×5 square. Each of the larger squares shares a corner with the 1×1 square. Find the area of the region covered by the 1×1, 2×2, 3×3, and 4×4 squares, and the 5×5 square.



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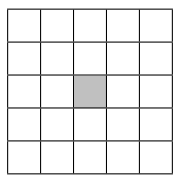
Problem 18.1.4 (Problem 4)**.** (*5 points*) Find the value of x such that $2^{x+3} - 2^{x-3} = 2016$.

Problem 18.1.5 (Problem 5)**.** (*5 points*) Mildred the cow is tied with a rope to the side of a square shed with side length 10 meters. The rope is attached to the shed at a point two meters from one corner of the shed. The rope is 14 meters long. The area of grass growing around the shed that Mildred can reach is given by $n\pi$ square meters, where n is a positive integer. Find n.

Problem 18.1.6 (Problem 6)**.** (*5 points*) One evening a theater sold 300 tickets for a concert. Each ticket sold for \$40, and all tickets were purchased using \$5, \$10, and \$20 bills. At the end of the evening the theater had received twice as many \$10 bills as \$20 bills, and 20 more \$5 bills than \$10 bills. How many bills did the theater receive altogether?

Problem 18.1.7 (Problem 7)**.** (*5 points*) Find the number of squares such that the sides of the square are segments in the following diagram and where the shaded area is inside the square.



Problem 18.1.8 (Problem 8)**.** (*5 points*) One afternoon Elizabeth noticed that twice as many cars on the expressway carried only a driver as compared to the number of cars that carried a driver and one passenger.

She also noted that twice as many cars carried a driver and one passenger as those that carried a driver and two passengers.

10% of the cars carried a driver and three passengers, and no car carried more than four people.

Any car containing at least three people was allowed to use the fast lane. Elizabeth calculated that $\frac{m}{n}$ of the people in cars on the expressway were allowed to ride in the fast lane, where m and n are relatively prime positive integers. Find $m + n$.

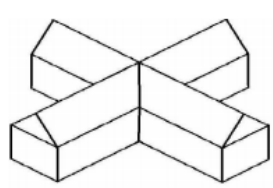
Problem 18.1.9 (Problem 9)**.** (*5 points*) Find the number of positive integers n such that a regular polygon with n sides has internal angles with measures equal to an integer number of degrees.

Problem 18.1.10 (Problem 10)**.** (*5 points*) The real numbers x, y, and z satisfy the system of equations

$$\begin{cases} x^2 + 27 = -8y + 10z \\ y^2 + 196 = 18z + 13x \\ z^2 + 119 = -3x + 30y. \end{cases}$$

Find $x + 3y + 5z$.

Problem 18.1.11 (Problem 11)**.** (*5 points*) The figure below shows a barn in the shape of two congruent pentagonal prisms that intersect at right angles and have a common center. The ends of the prisms are made of a 12 foot by 7 foot rectangle surmounted by an isosceles triangle with sides 10 feet, 10 feet, and 12 feet. Each prism is 40 feet long. Find the volume of the barn in cubic feet.

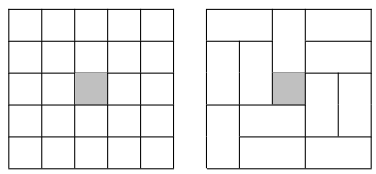


Problem 18.1.12 (Problem 12)**.** (*5 points*) Suzie flips a fair coin 6 times. The probability that Suzie flips 3 heads in a row but not 4 heads in a row is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 18.1.13 (Problem 13)**.** (*5 points*) Find the least positive integer N that is 50 times the number of positive integer divisors that N has.

Problem 18.1.14 (Problem 14)**.** (*5 points*) Find the positive integer n such that the least common multiple of n and $n - 30$ is $n + 1320$.

Problem 18.1.15 (Problem 15)**.** (*5 points*) The 24 unshaded squares in the 5×5 grid below can be tiled with twelve 1×2 tiles. One such tiling is shown below.



Find the number of ways the grid can be tiled.

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18.2 Middle School - Test

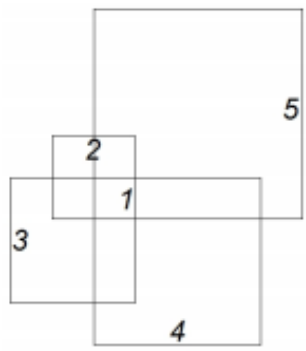
You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

Problem 18.2.1 (Problem 1)**.** (*5 points*) Find the number of four-digit positive integers where the digits are four different prime numbers. For example, count 2357 but not 5537

Problem 18.2.2 (Problem 2)**.** (*5 points*) Melanie has $3\frac{4}{5}$ cups of flour. The recipe for one batch of cookies calls for $1\frac{1}{2}$ cups of flour. Melanie plans to make $2\frac{1}{2}$ batches of cookies. When she is done, she will have $\frac{m}{n}$ cups of flour remaining, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18.2.3 (Problem 3)**.** (*5 points*) The figure below has a 1×1 square, a 2×2 square, a 3×3 square, a 4×4 square, and a 5×5 square. Each of the larger squares shares a corner with the 1×1 square. Find the area of the region covered by the 1×1, 2×2, 3×3, and 4×4 squares, but not covered by the 5×5 square.

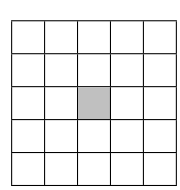


Problem 18.2.4 (Problem 4)**.** (*5 points*) Find the value of x such that $2^{x+5} - 3 \cdot 2^{x-3} = 2024$.

Problem 18.2.5 (Problem 5)**.** (*5 points*) Mildred the cow is tied with a rope to the side of a square shed with side length 10 meters. The rope is attached to the shed at a point two meters from one corner of the shed. The rope is 13 meters long. The area of grass growing around the shed that Mildred can reach is given by $\frac{m\pi}{n}$ square meters, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18.2.6 (Problem 6)**.** (*5 points*) One evening a theater sold 300 tickets for a concert. Each ticket sold for \$40, and all tickets were purchased using \$5, \$10, and \$20 bills. At the end of the evening the theater had received twice as many \$10 bills as \$20 bills, and 10 more \$5 bills than \$10 bills. How many bills did the theater receive altogether?

Problem 18.2.7 (Problem 7)**.** (*5 points*) The 5×5 grid below is made up by *unit* squares. The center square is shaded. Find the number of squares with side lengths 1, 3, or 5, such that the sides of the square are segments in the grid, and the shaded area is inside the square.



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Problem 18.2.8 (Problem 8)**.** (*5 points*) One afternoon Elizabeth noticed that twice as many cars on the expressway carried only a driver as compared to the number of cars that carried a driver and one passenger.

She also noted that twice as many cars carried a driver and one passenger as those that carried a driver and two passengers.

10% of the cars carried a driver and three passengers, and no car carried more than four people.

Any car containing at least three people was allowed to use the fast lane. Elizabeth calculated that $\frac{m}{n}$ of the people in cars on the expressway were **not** allowed to ride in the fast lane, where m and n are relatively prime positive integers. Find $m+n$.

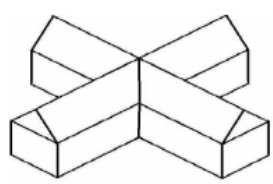
Problem 18.2.9 (Problem 9)**.** (*5 points*) Find the number of **odd** positive integers n such that a regular polygon with n sides has internal angles with measures equal to an integer number of degrees.

Problem 18.2.10 (Problem 10)**.** (*5 points*) The real numbers x, y, and z satisfy the system of equations

$$\begin{cases} x^2 + 27 = -8y + 10z \\ y^2 + 196 = 18z + 13x \\ z^2 + 119 = -3x + 30y. \end{cases}$$

Find $5x + 3y + z$.

Problem 18.2.11 (Problem 11)**.** (*5 points*) The figure below shows a barn in the shape of two congruent pentagonal prisms that intersect at right angles and have a common center. The ends of the prisms are made of a 6 foot by 4 foot rectangle surmounted by an isosceles triangle with sides 5 feet, 5 feet, and 6 feet. Each prism is 40 feet long. Find the volume of the barn in cubic feet.

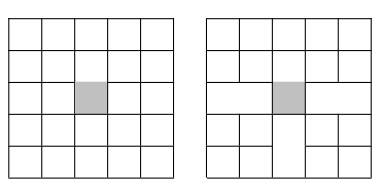


Problem 18.2.12 (Problem 12)**.** (*5 points*) Suzie flips a fair coin 7 times. The probability that Suzie flips 4 heads in a row but not 5 heads in a row is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18.2.13 (Problem 13)**.** (*5 points*) Find the least positive integer N that is 49 times the number of positive integer divisors that N has.

Problem 18.2.14 (Problem 14)**.** (*5 points*) Find the positive integer n such that the least common multiple of n and $n-30$ is $n+4900$.

Problem 18.2.15 (Problem 15)**.** (*5 points*) The 24 unshaded squares in the 5×5 grid below (the left diagram) are already tiled by three 1×2 tiles (the right diagram).



Find the number of ways the rest of the grid can be tiled by nine 1×2 tiles.

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18.3 High School - Assignment

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

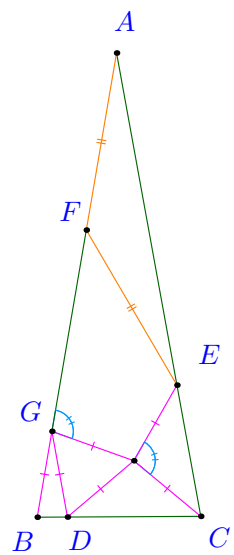
- **Submission deadline: Friday, February 9**
- **Test: Saturday, February 10**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**
- **Olympiad (O) level: Problems 12-21**

Problem 18.3.1 (Problem 1)**.** (*5 points*) Jeremy wrote all the three-digit integers from 100 to 999 on a blackboard. Then Allison erased each of the 2700 digits Jeremy wrote and replaced each digit with the square of that digit. Thus, Allison replaced every 1 with a 1, every 2 with a 4, every 3 with a 9, every 4 with a 16, and so forth. The proportion of all the digits Allison wrote that were ones is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18.3.2 (Problem 2)**.** (*5 points*) Find the number of three-digit positive integers which have three distinct digits where the sum of the digits is an even number such as 925 and 824.

Problem 18.3.3 (Problem 3)**.** (*5 points*) Find the number whose reciprocal is the sum of the reciprocal of $9+15i$ and the reciprocal of $9-15i$.

Problem 18.3.4 (Problem 4)**.** (*5 points*) In $\triangle ABC$, $AB = AC$, $AF = EF$, $\angle CHE = \angle FGH$, and $EH = CH = DH = GH = DG = BG$. Find $\angle BAC$.



Problem 18.3.5 (Problem 5)**.** (*5 points*) Find the greatest possible value of $pq+r$, where p, q, and r are (not necessarily distinct) prime numbers satisfying $pq+qr+rp = 2016$.

Problem 18.3.6 (Problem 6)**.** (*5 points*) Find the least positive integer of the form $\overline{abaaba}$, where a and b are distinct digits, such that the integer can be written as a product of six distinct primes.

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Problem 18.3.7 (Problem 7)**.** (*5 points*) Henry rolls a fair die. If the die shows the number k, Henry will then roll the die k more times. The probability that Henry will never roll a 3 or a 6 either on his first roll or on one of the k subsequent rolls is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18.3.8 (Problem 8)**.** (*5 points*) The cubic polynomials $p(x)$ and $q(x)$ satisfy

$$\begin{cases} p(1) = q(2) \\ p(3) = q(4) \\ p(5) = q(6) \\ p(7) = q(8) + 13 \end{cases}$$

Find $p(9) - q(10)$.

Problem 18.3.9 (Problem 9)**.** (*5 points*) The Tasty Candy Company always puts the same number of pieces of candy into each one-pound bag of candy they sell. Mike bought 4 one-pound bags and gave each person in his class 15 pieces of candy. Mike had 23 pieces of candy left over. Betsy bought 5 one-pound bags and gave 23 pieces of candy to each teacher in her school. Betsy had 15 pieces of candy left over. Find the least number of pieces of candy the Tasty Candy Company could have placed in each one-pound bag.

Problem 18.3.10 (Problem 10)**.** (*5 points*) Jar #1 contains five red marbles, three blue marbles, and one green marble. Jar #2 contains five blue marbles, three green marbles, and one red marble. Jar #3 contains five green marbles, three red marbles, and one blue marble.

You randomly select one marble from each jar.

Given that you select one marble of each color, the probability that the red marble came from jar #1, the blue marble came from jar #2, and the green marble came from jar #3 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18.3.11 (Problem 11)**.** (*5 points*) Positive integers a, b, c, d, and e satisfy the equations

$$\begin{cases} (a+1)(3bc+1) = d + 3e + 1 \\ (b+1)(3ca+1) = 3d + e + 13 \\ (c+1)(3ab+1) = 4(26 - d - e) - 1 \end{cases}$$

Find $d^2 + e^2$.

Problem 18.3.12 (Problem 12)**.** (*5 points*) On equilateral $\triangle ABC$ point D lies on BC a distance 1 from B, point E lies on AC a distance 1 from C, and point F lies on AB a distance 1 from A. Segment AD, BE, and CF intersect in pairs at points G, H, and J which are the vertices of another equilateral triangle. The area of $\triangle ABC$ is twice the area of $\triangle GHJ$. The side length of $\triangle ABC$ can be written $\dfrac{r+\sqrt{s}}{t}$, where r, s, and t are relatively prime positive integers. Find $r+s+t$.

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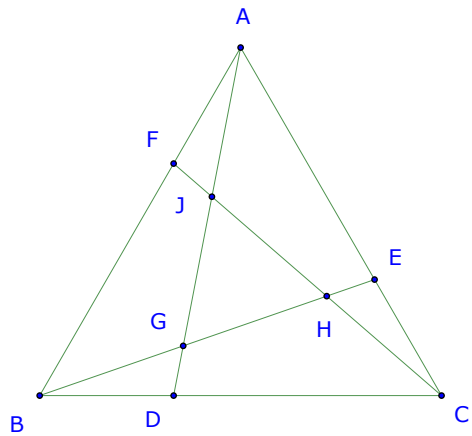
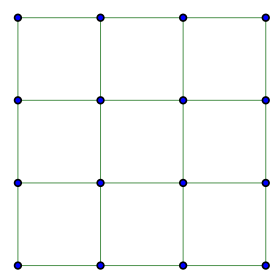


Problem 18.3.13 (Problem 13)**.** (*5 points*) In $\triangle ABC$, $\cos(\angle A) = \frac{2}{3}$, $\cos(\angle B) = \frac{1}{9}$, and $BC = 24$. Find the length AC.

Problem 18.3.14 (Problem 14)**.** (*5 points*) Sixteen dots are arranged in a four by four grid as shown. The distance between any two dots in the grid is the minimum number of horizontal and vertical steps along the grid lines it takes to get from one dot to the other. For example, two adjacent dots are a distance 1 apart, and two dots at opposite corners of the grid are a distance 6 apart.



The mean distance between two distinct dots in the grid is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 18.3.15 (Problem 15)**.** (*5 points*) Find the largest prime p such that p divides $2^{p+1} + 3^{p+1} + 5^{p+1} + 7^{p+1}$.

Problem 18.3.16 (Problem 16)**.** (*5 points*) For n measured in degrees,

$$\text{let } T(n) = \cos^2(30^\circ - n) - \cos(30^\circ - n)\cos(30^\circ + n) + \cos^2(30^\circ + n). \text{ Evaluate } 4\sum_{n=1}^{30} n \cdot T(n).$$

Problem 18.3.17 (Problem 17)**.** (*5 points*) Find the sum of all values of a such that there are positive integers a and b satisfying $(a - b)\sqrt{ab} = 2016$.

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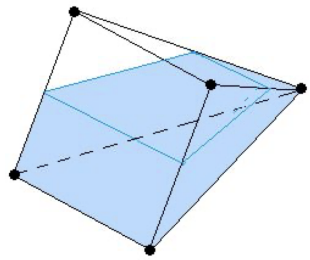
Problem 18.3.18 (Problem 18)**.** (*5 points*) A container the shape of a pyramid has a 12×12 square base, and the other four edges each have length 11. The container is partially filled with liquid so that when one of its triangular faces is lying on a flat surface, the level of the liquid is half the distance from the surface to the top edge of the container.



Find the volume of the liquid in the container.

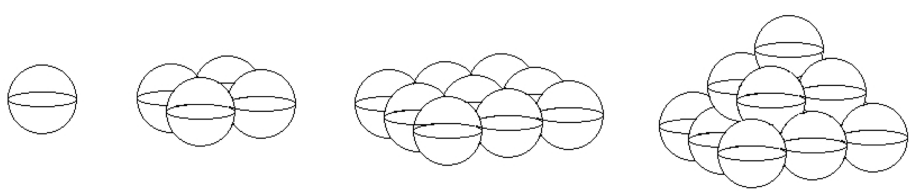
Problem 18.3.19 (Problem 19)**.** (*5 points*) Find the sum of all the possible values of the product xy such that x and y are positive integers satisfying

$$(x^2+1)(y^2+1)+2(x-y)(1-xy)=4(1+xy)+140.$$

Problem 18.3.20 (Problem 20)**.** (*5 points*) Ten square tiles are placed in a row, and each can be painted with one of the four colors red (R), yellow (Y), blue (B), and white (W). Find the number of ways this can be done so that each block of five adjacent tiles contains at least one tile of each color. That is, count the patterns RWBWYRRBWY and WWBYRWYBWR but not RWBYYBWWRY because the five adjacent tiles colored BYYBW does not include the color red.

Problem 18.3.21 (Problem 21)**.** (*5 points*) Some identically sized spheres are piled in n layers in the shape of a square pyramid with one sphere in the top layer, 4 spheres in the second layer, 9 spheres in the third layer, and so forth so that the bottom layer has a square array of n^2 spheres. In each layer the centers of the spheres form a square grid so that each sphere is tangent to any sphere adjacent to it on the grid. Each sphere in an upper level is tangent to the four spheres directly below it.

The diagram shows how the first three layers of spheres are stacked.



A square pyramid is built around the pile of spheres so that the sides of the pyramid are tangent to the spheres on the outside of the pile. There is a positive integer m such that as n gets large, the ratio of the volume of the pyramid to the total volume inside all of the spheres approaches $\frac{\sqrt{m}}{\pi}$. Find m.

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18.4 High School - Test

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

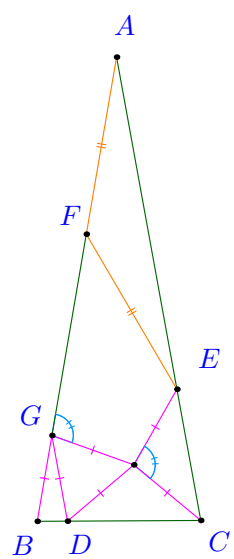
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**
- **Olympiad (O) level: Problems 12-21**

Problem 18.4.1 (Problem 1)**.** (*5 points*) Jeremy wrote all the three-digit integers from 100 to 999 on a blackboard. Then Allison erased each of the 2700 digits Jeremy wrote and replaced each digit with the square of that digit. Thus, Allison replaced every 1 with a 1, every 2 with a 4, every 3 with a 9, every 4 with a 16, and so forth. The proportion of all the digits Allison wrote that were **twos** is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18.4.2 (Problem 2)**.** (*5 points*) Find the number of three-digit positive integers which have three distinct digits where the sum of the digits is an odd number such as 935 and 834.

Problem 18.4.3 (Problem 3)**.** (*5 points*) The sum of the reciprocal of $5+12i$ and the reciprocal of $5-12i$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18.4.4 (Problem 4)**.** (*5 points*) In $\triangle ABC$, $AB = AC$, $AF = EF$, $\angle CHE = \angle FGH$, and $EH = CH = DH = GH = DG = BG$. Find $\angle GFE$.



Problem 18.4.5 (Problem 5)**.** (*5 points*) Find the greatest possible value of $pq+r$, where p, q, and r are (not necessarily distinct) prime numbers satisfying $pq+qr+rp = 2024$.

Problem 18.4.6 (Problem 6)**.** (*5 points*) n the least positive integer of the form $\overline{abbabb}$, where a and b are distinct digits, such that the integer can be written as a product of six distinct primes. Find the sum of these primes.

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Problem 18.4.7 (Problem 7)**.** (*5 points*) Henry rolls a fair die. If the die shows the number k, Henry will then roll the die k more times. The probability that Henry will never roll a 1, 3 or a 6 either on his first roll or on one of the k subsequent rolls is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18.4.8 (Problem 8)**.** (*5 points*) The cubic polynomials $p(x)$ and $q(x)$ satisfy

$$\begin{cases} p(1) = q(2) \\ p(3) = q(4) \\ p(5) = q(6) \\ p(7) = q(8) + 13 \end{cases}$$

Find $p(11) - q(12)$.

Problem 18.4.9 (Problem 9)**.** (*5 points*) The Tasty Candy Company always puts the same number of pieces of candy into each one-pound bag of candy they sell. Mike bought 4 one-pound bags and gave each person in his class 15 pieces of candy. Mike had 19 pieces of candy left over. Betsy bought 5 one-pound bags and gave 19 pieces of candy to each teacher in her school. Betsy had 15 pieces of candy left over. Find the least number of pieces of candy the Tasty Candy Company could have placed in each one-pound bag.

Problem 18.4.10 (Problem 10)**.** (*5 points*) Jar #1 contains five red marbles, three blue marbles, and one green marble. Jar #2 contains five blue marbles, three green marbles, and one red marble. Jar #3 contains five green marbles, three red marbles, and one blue marble.

You randomly select one marble from each jar.

Given that you select one marble of each color, the probability that the red marble came from jar #2, the blue marble came from jar #3, and the green marble came from jar #1 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18.4.11 (Problem 11)**.** (*5 points*) Positive integers a, b, c, d, and e satisfy the equations

$$\begin{cases} (a+1)(3bc+1) = d + 3e + 1 \\ (b+1)(3ca+1) = 3d + e + 13 \\ (c+1)(3ab+1) = 4(26 - d - e) - 1 \end{cases}$$

Find $a+b+c+d+e$.

Problem 18.4.12 (Problem 12)**.** (*5 points*) On equilateral $\triangle ABC$ point D lies on BC a distance 1 from B, point E lies on AC a distance 1 from C, and point F lies on AB a distance 1 from A. Segment AD, BE, and CF intersect in pairs at points G, H, and J which are the vertices of another equilateral triangle. The area of $\triangle ABC$ is twice the area of $\triangle GHJ$. The side length of $\triangle ABC$ can be written $\frac{r+\sqrt{s}}{t}$, where r, s, and t are relatively prime positive integers. Find the value of the product rst.

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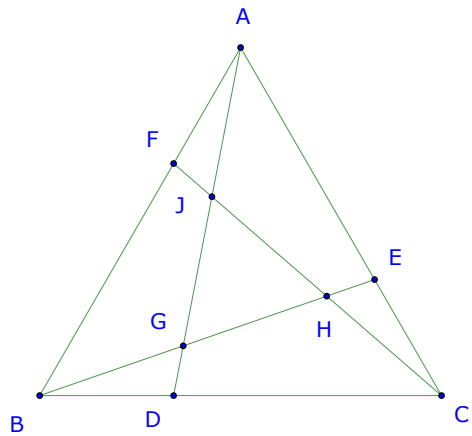
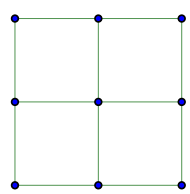


Problem 18.4.13 (Problem 13)**.** (*5 points*) In $\triangle ABC$, $\cos(\angle A) = \frac{\sqrt{5}}{5}$, $\cos(\angle B) = \frac{3}{5}$, $BC = 17$. Find AB.

Problem 18.4.14 (Problem 14)**.** (*5 points*) Nine dots are arranged in a three by three grid as shown. The distance between any two dots in the grid is the minimum number of horizontal and vertical steps along the grid lines it takes to get from one dot to the other. For example, two adjacent dots are a distance 1 apart, and two dots at opposite corners of the grid are a distance 6 apart.



The mean distance between two distinct dots in the grid is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

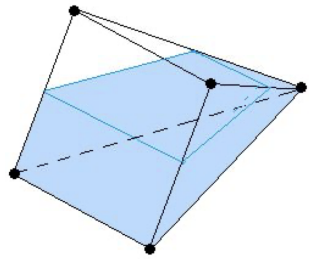
Problem 18.4.15 (Problem 15)**.** (*5 points*) Find the largest prime p such that p divides $2^{p+2} + 3^{p+2} + 5^{p+2} + 7^{p+2}$.

Problem 18.4.16 (Problem 16)**.** (*5 points*) For n measured in degrees,

$$\text{let } T(n) = \cos^2(30^\circ - n) - \cos(30^\circ - n)\cos(30^\circ + n) + \cos^2(30^\circ + n). \text{ Evaluate } 4\sum_{n=1}^{2024} n \cdot T(n).$$

Problem 18.4.17 (Problem 17)**.** (*5 points*) Find the sum of all values of b such that there are positive integers a and b satisfying $(a - b)\sqrt{ab} = 840$.

Problem 18.4.18 (Problem 18)**.** (*5 points*) A container the shape of a pyramid has a 8×8 square base, and the other four edges each have length 9. The container is partially filled with liquid so that when one of its triangular faces is lying on a flat surface, the level of the liquid is half the distance from the surface to the top edge of the container.



The volume of the liquid in the container is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

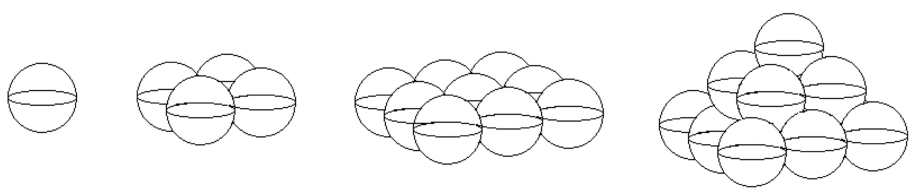
Problem 18.4.19 (Problem 19)**.** (*5 points*) Find the sum of all the possible values of the product xy such that x and y are positive integers satisfying

$$(x^2+1)(y^2+1)+2(x-y)(1-xy)=4(1+xy)+96.$$

Problem 18.4.20 (Problem 20)**.** (*5 points*) Eight square tiles are placed in a row, and each can be painted with one of the three colors red (R), yellow (Y), and blue (B). Find the number of ways this can be done so that each block of four adjacent tiles contains at least one tile of each color. That is, count the patterns RBYRBYB and BYRYBRY but not RBYYBRYR because the four adjacent tiles colored BYYB does not include the color red.

Problem 18.4.21 (Problem 21)**.** (*5 points*) Some **unit** spheres are piled in 10 layers in the shape of a square pyramid with one sphere in the top layer, 4 spheres in the second layer, 9 spheres in the third layer, and so forth so that the bottom layer has a square array of 100 spheres. In each layer the centers of the spheres form a square grid so that each sphere is tangent to any sphere adjacent to it on the grid. Each sphere in an upper level is tangent to the four spheres directly below it.

The diagram shows how the first three layers of spheres are stacked.



A square pyramid is built around the pile of spheres so that the sides of the pyramid are tangent to the spheres on the outside of the pile. The height of the pyramid is $a+b\sqrt{2}+c\sqrt{3}$, where a, b, and c are positive integers. Find $a+b+c$.

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Chapter 19

Session 3

19.1 Middle School - Assignment

Middle school students: grade 8 (US, CA), grade 9 (FR, UK, VN) and younger.

- **Submission deadline: Friday, February 23**
- **Test: Saturday, February 24**
- **Official solutions: Monday, February 26**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

Problem 19.1.1 (Problem 1)**.** (*5 points*) A picture with an area of 160 square inches is surrounded by a 2 inch border. The picture with its border is a rectangle twice as long as it is wide. How many inches long is that rectangle?

Problem 19.1.2 (Problem 2)**.** (*5 points*) Pete's research shows that the number of nuts collected by the squirrels in any park is proportional to the square of the number of squirrels in that park. If Pete notes that four squirrels in a park collect 60 nuts, how many nuts are collected by 20 squirrels in a park?

Problem 19.1.3 (Problem 3)**.** (*5 points*) How many seven-digit positive integers do not either start or end with 7?

Problem 19.1.4 (Problem 4)**.** (*5 points*) Asheville, Bakersfield, Charter, and Darlington are four small towns along a straight road in that order. The distance from Bakersfield to Charter is one-third the distance from Asheville to Charter and one-quarter the distance from Bakersfield to Darlington. If it is 12 miles from Bakersfield to Charter, how many miles is it from Asheville to Darlington?

Problem 19.1.5 (Problem 5)**.** (*5 points*) Find the sum of all four-digit integers whose digits are a rearrangement of the digits 1, 2, 3, 4, such as 1234, 1432, or 3124.

Problem 19.1.6 (Problem 6)**.** (*5 points*) Find the least positive integer k so that the mean of the numbers $k, k+1, k+2, k+3, \ldots, 2k$ exceeds 200.

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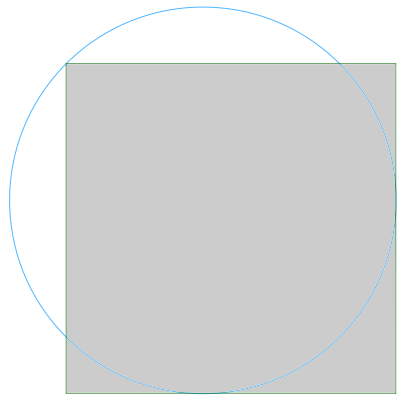
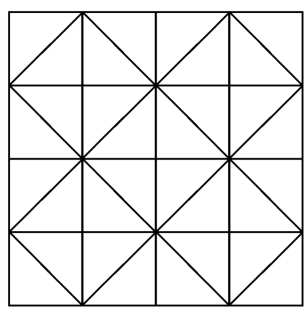
Problem 19.1.7 (Problem 7)**.** (*5 points*) In the following diagram two sides of a square are tangent to a circle with diameter 8. One corner of the square lies on the circle. There are positive integers m and n so that the area of the square is $m + \sqrt{n}$. Find $m + n$.



Problem 19.1.8 (Problem 8)**.** (*5 points*) Find integer n such that both $n - 86$ and $n + 86$ are perfect squares.

Problem 19.1.9 (Problem 9)**.** (*5 points*) Find n so that $4^{4^{4^2}} = 2^{8^n}$.

Problem 19.1.10 (Problem 10)**.** (*5 points*) How many triangles appear in the diagram below?



Problem 19.1.11 (Problem 11)**.** (*5 points*) Let a, b, and c be positive real numbers such that $a^2+b^2+c^2 = 989$ and $(a + b)^2 + (b + c)^2 + (c + a)^2 = 2013$. Find $a + b + c$.

Problem 19.1.12 (Problem 12)**.** (*5 points*) A quarry wants to sell a large pile of gravel. At full price, the gravel would sell for 3200 dollars. But during the first week the quarry only sells 60% of the gravel at full price. The following week the quarry drops the price by 10%, and, again, it sells 60% of the remaining gravel. Each week, thereafter, the quarry reduces the price by another 10% and sells 60% of the remaining gravel. This continues until there is only a handful of gravel left. How many dollars does the quarry collect for the sale of all its gravel?

Problem 19.1.13 (Problem 13)**.** (*5 points*) A rectangle has side lengths 6 and 8. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that a point randomly selected from the inside of the rectangle is closer to a side of the rectangle than to either diagonal of the rectangle. Find $m + n$.

Problem 19.1.14 (Problem 14)**.** (*5 points*) Six children stand in a line outside their classroom. When they enter the classroom, they sit in a circle in random order. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that no two children who stood next to each other in the line end up sitting next to each other in the circle. Find $m + n$.

Problem 19.1.15 (Problem 15)**.** (*5 points*) For positive integer n let an be the integer consisting of n digits of 9 followed by the digits 488. For example, $a_3 = 999,488$ and $a_7 = 9,999,999,488$. Find the value of n so that a_n is divisible by the highest power of 2.

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19.2 Middle School - Test

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

Problem 19.2.1 (Problem 1)**.** (*5 points*) A picture with an area of 180 square inches is surrounded by a 1 inch border. The picture with its border is a rectangle twice as long as it is wide. How many inches long is that rectangle?

Problem 19.2.2 (Problem 2)**.** (*5 points*) Pete's research shows that the number of nuts collected by the squirrels in any park is proportional to the square of the number of squirrels in that park. If Pete notes that four squirrels in a park collect 15 nuts, how many nuts are collected by 16 squirrels in a park?

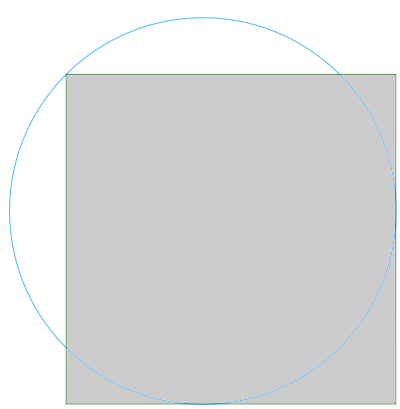
Problem 19.2.3 (Problem 3)**.** (*5 points*) How many six-digit positive integers do not either start or end with 7?

Problem 19.2.4 (Problem 4)**.** (*5 points*) Asheville, Bakersfield, Charter, and Darlington are four small towns along a straight road in that order. The distance from Bakersfield to Charter is one-third the distance from Asheville to Charter and one-quarter the distance from Bakersfield to Darlington. If it is 18 miles from Bakersfield to Charter, how many miles is it from Asheville to Darlington?

Problem 19.2.5 (Problem 5)**.** (*5 points*) Find the sum of all three-digit integers whose digits are a rearrangement of the digits 1, 2, 3, 4, such as 123, 142, or 324.

Problem 19.2.6 (Problem 6)**.** (*5 points*) Find the least positive integer k so that the mean of the numbers $k, k+1, k+2, k+3, \ldots, 2k$ exceeds 300.

Problem 19.2.7 (Problem 7)**.** (*5 points*) In the following diagram two sides of a square are tangent to a circle with diameter 8. One corner of the square lies on the circle. There are positive integers m and n so that the area of the square is $m + \sqrt{n}$. Find $n - m$.



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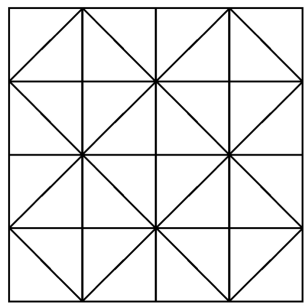
Problem 19.2.8 (Problem 8)**.** (*5 points*) Find the sum of all values of n such that both $n-70$ and $n+70$ are perfect squares.

Problem 19.2.9 (Problem 9)**.** (*5 points*) Find n so that $4^{4^{4^2}} = 16^{2^n}$.

Problem 19.2.10 (Problem 10)**.** (*5 points*) The diagram below is made up of 16 unit squares which each contain two small triangles. How many triangles appear in the diagram below which is **not half of a unit square**?



Problem 19.2.11 (Problem 11)**.** (*5 points*) Let a, b, and c be positive real numbers such that $ab+bc+ca = 432$ and $(a+b)^2 + (b+c)^2 + (c+a)^2 = 2024$. Find $a+b+c$.

Problem 19.2.12 (Problem 12)**.** (*5 points*) A quarry wants to sell a large pile of gravel. At full price, the gravel would sell for 1600 dollars. But during the first week the quarry only sells 60% of the gravel at full price. The following week the quarry drops the price by 10%, and, again, it sells 60% of the remaining gravel. Each week, thereafter, the quarry reduces the price by another 10% and sells 60% of the remaining gravel. This continues until there is only a handful of gravel left. How many dollars does the quarry collect for the sale of all its gravel?

Problem 19.2.13 (Problem 13)**.** (*5 points*) A rectangle has side lengths 6 and 8. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that a point randomly selected from the inside of the rectangle is closer to a side of the rectangle than to either diagonal of the rectangle. Find $n-m$.

Problem 19.2.14 (Problem 14)**.** (*5 points*) Six children stand in a line outside their classroom. When they enter the classroom, they sit in a circle in random order. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that no two children who stood next to each other in the line end up sitting next to each other in the circle. Find the difference $n-m$.

Problem 19.2.15 (Problem 15)**.** (*5 points*) For positive integer n let an be the integer consisting of n digits of 9 followed by the digits 1808. For example, $a_2 = 991,808$ and $a_6 = 9,999,991,808$. Find the value of n so that a_n is divisible by the highest power of 2.

19.3 High School - Assignment

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

- **Submission deadline: Friday, February 23**
- **Test: Saturday, February 24**
- **Official solutions: Monday, February 26**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 8-17**
- **Olympiad (O) level: Problems 16-25**

Problem 19.3.1 (Problem 1)**.** (*5 points*) Find the least positive integer n so that both n and $n+1$ have prime factorizations with exactly four (not necessarily distinct) prime factors.

Problem 19.3.2 (Problem 2)**.** (*5 points*) Two convex polygons have a total of 33 sides and 243 diagonals. Find the number of diagonals in the polygon with the greater number of sides.

Problem 19.3.3 (Problem 3)**.** (*5 points*) In the tribe of Zimmer, being able to hike long distances and knowing the roads through the forest are both extremely important, so a boy who reaches the age of manhood is not designated as a man by the tribe until he completes an interesting rite of passage. The man must go on a sequence of hikes. The first hike is a 5 kilometer hike down the main road. The second hike is a $5\frac{1}{4}$ kilometer hike down a secondary road. Each hike goes down a different road and is a quarter kilometer longer than the previous hike. The rite of passage is completed at the end of the hike where the cumulative distance walked by the man on all his hikes exceeds 1000 kilometers. So in the tribe of Zimmer, how many roads must a man walk down, before you call him a man?

Problem 19.3.4 (Problem 4)**.** (*5 points*) Find the value of x that satisfies $\log_3(\log_9 x) = \log_9(\log_3 x)$.

Problem 19.3.5 (Problem 5)**.** (*5 points*) Consider a sequence of eleven squares that have side lengths $3, 6, 9, 12, \ldots, 33$. Eleven copies of a single square each with area A have the same total area as the total area of the eleven squares of the sequence. Find A.

Problem 19.3.6 (Problem 6)**.** (*5 points*) Define $f(x) = 2x + 3$and suppose that $g(x+2) = f(f(x-1) \cdot f(x+1) + f(x))$. Find $g(6)$.

Problem 19.3.7 (Problem 7)**.** (*5 points*) Ted flips seven fair coins. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that Ted flips at least two heads given that he flips at least three tails. Find $m+n$.

Problem 19.3.8 (Problem 8)**.** (*5 points*) Find the least n for which $n!(n+1)!(2n+1)! - 1$ ends in thirty digits that are all 9's.

Problem 19.3.9 (Problem 9)**.** (*5 points*) A circle in the first quadrant with center on the curve $y = 2x^2 - 27$ is tangent to the $y-$axis and the line $4x = 3y$. The radius of the circle is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

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Problem 19.3.10 (Problem 10)**.** (*5 points*) Let N be a positive integer whose digits add up to 23. What is the greatest possible product the digits of N can have?

Problem 19.3.11 (Problem 11)**.** (*5 points*) Let a, b, and c be non-zero real numbers such that

$$\frac{ab}{a+b} = 3, \frac{bc}{b+c} = 4, \frac{ca}{c+a} = 5.$$

There are relatively prime positive integers m and n so that

$$\frac{abc}{ab+bc+ca} = \frac{m}{n}.$$

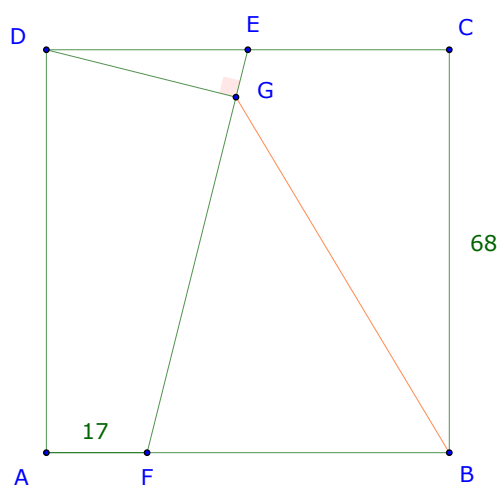
Find $m+n$.

Problem 19.3.12 (Problem 12)**.** (*5 points*) How many positive integer solutions are there to $w+x+y+z = 20$ where $w+x \geq 5$ and $y+z \geq 5$?

Problem 19.3.13 (Problem 13)**.** (*5 points*) Find the number of three-digit numbers such that its first two digits are each divisible by its third digit.

Problem 19.3.14 (Problem 14)**.** (*5 points*) Find the remainder when $2^{5^9} + 5^{9^2} + 9^{2^5}$ is divided by 11.

Problem 19.3.15 (Problem 15)**.** (*5 points*) Square $ABCD$ has side length 68. Let E be the midpoint of segment CD, and let F be the point on segment AB a distance 17 from point A. Point G is on segment EF so that EF is perpendicular to segment GD. The length of segment BG can be written as $m\sqrt{n}$, where m and n be are positive integers, and n is not divisible by the square of any prime. Find $m+n$.



Problem 19.3.16 (Problem 16)**.** (*5 points*) Each time you click a toggle switch, the switch either turns from *off* to *on* or from on to *off*. Suppose that you start with three toggle switches with one of them *on* and two of them *off*. On each move you randomly select one of the three switches and click it. Let m and n be relatively prime positive integers so that $\frac{m}{n}$ is the probability that after four such clicks, one switch will be *on* and two of them will be *off*. Find $m+n$.

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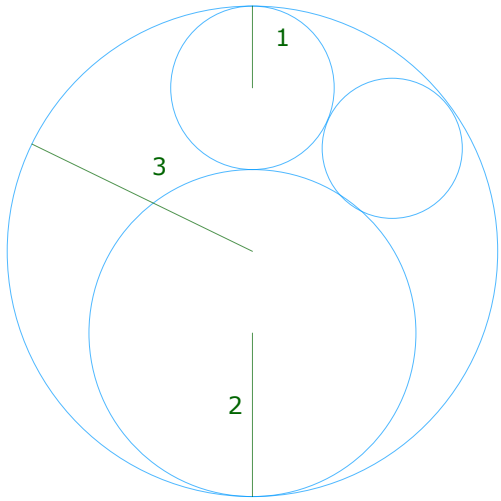
Problem 19.3.17 (Problem 17)**.** (*5 points*) The diagram below shows circles radius 1 and 2 externally tangent to each other and internally tangent to a circle radius 3. There are relatively prime positive integers m and n so that a circle radius $\frac{m}{n}$ is internally tangent to the circle radius 3 and externally tangent to the other two circles as shown. Find $m + n$.



Problem 19.3.18 (Problem 18)**.** (*5 points*) Find the greatest seven-digit integer divisible by 132 whose digits, in order, are $2, 0, x, y, 1, 2, z$ where x, y, and z are single digits.

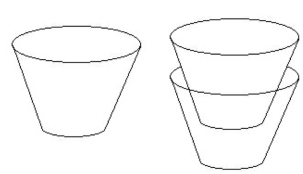
Problem 19.3.19 (Problem 19)**.** (*5 points*) There are positive integers m and n so that $x = m + \sqrt{n}$ is a solution to the equation

$$x^2 - 10x + 1 = \sqrt{x}(x + 1).$$

Find $m + n$.

Problem 19.3.20 (Problem 20)**.** (*5 points*) Find the largest prime that divides $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + 44 \cdot 45 \cdot 46$.

Problem 19.3.21 (Problem 21)**.** (*5 points*) A paper cup has a base that is a circle with radius r, a top that is a circle with radius $2r$, and sides that connect the two circles with straight line segments as shown below. This cup has height h and volume V. A second cup that is exactly the same shape as the first is held upright inside the first cup so that its base is a distance of $\frac{h}{2}$ from the base of the first cup. The volume of liquid that will fit inside the first cup and outside the second cup can be written as $\frac{m}{n} \cdot V$, where m and n are relatively prime positive integers. Find $m + n$.

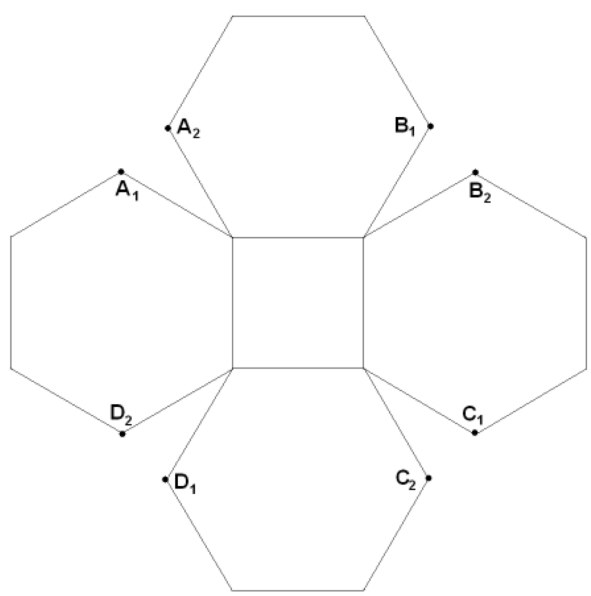


Problem 19.3.22 (Problem 22)**.** (*5 points*) You have some white one-by-one tiles and some black and white two-by-one tiles as shown below. There are four different color patterns that can be generated when using these tiles to cover a three-by-one rectangle by laying these tiles side by side (WWW, BWW, WBW, WWB). How many different color patterns can be generated when using these tiles to cover a ten-by-one rectangle?

Problem 19.3.23 (Problem 23)**.** (*5 points*) A bag contains 8 green candies and 4 red candies. You randomly select one candy at a time to eat. If you eat five candies, there are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that you do not eat a green candy after you eat a red candy. Find $m+n$.

Problem 19.3.24 (Problem 24)**.** (*5 points*) Let $A = 1, 3, 5, 7, 9$ and $B = 2, 4, 6, 8, 10$. Let f be a randomly chosen function from the set $A \cup B$ into itself. There are relatively prime positive integers m and n such that $\frac{m}{n}$ is the probability that f is a one-to-one function on $A \cup B$ given that it maps A one-to-one into $A \cup B$ and it maps B one-to-one into $A \cup B$. Find $m+n$.

Problem 19.3.25 (Problem 25)**.** (*5 points*) The diagram below shows four regular hexagons each with side length 1 meter attached to the sides of a square. This figure is drawn onto a thin sheet of metal and cut out. The hexagons are then bent upward along the sides of the square so that A_1 meets A_2, B_1 meets B_2, C_1 meets C_2, and D_1 meets D_2. The resulting dish is set on a table with the square lying flat on the table. If this dish is filled with water, the water will rise to the height of the corner where the A_1 and A_2 meet. There are relatively prime positive integers m and n so that the number of cubic meters of water the dish will hold is $\sqrt{\frac{m}{n}}$. Find $m+n$.



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19.4 High School - Test

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 8-17**
- **Olympiad (O) level: Problems 16-25**

Problem 19.4.1 (Problem 1)**.** (*5 points*) Find the least positive integer n so that both n and $n+1$ have prime factorizations with exactly three (not necessarily distinct) prime factors.

Problem 19.4.2 (Problem 2)**.** (*5 points*) Two convex polygons have a total of 33 sides and 279 diagonals. Find the number of diagonals in the polygon with the greater number of sides.

Problem 19.4.3 (Problem 3)**.** (*5 points*) In the tribe of Zimmer, being able to hike long distances and knowing the roads through the forest are both extremely important, so a boy who reaches the age of manhood is not designated as a man by the tribe until he completes an interesting rite of passage. The man must go on a sequence of hikes. The first hike is a 5 kilometer hike down the main road. The second hike is a $5\frac{1}{4}$ kilometer hike down a secondary road. Each hike goes down a different road and is a quarter kilometer longer than the previous hike. The rite of passage is completed at the end of the hike where the cumulative distance walked by the man on all his hikes exceeds **500** kilometers. So in the tribe of Zimmer, how many roads must a man walk down, before you call him a man?

Problem 19.4.4 (Problem 4)**.** (*5 points*) Find the value of x that satisfies $\log_2(\log_4 x) = \log_4(\log_2 x)$.

Problem 19.4.5 (Problem 5)**.** (*5 points*) Consider a sequence of twelve squares that have side lengths $3, 6, 9, 12, \ldots, 33, 36$. Ten copies of a single square each with area A have the same total area as the total area of the twelve squares of the sequence. Find A.

Problem 19.4.6 (Problem 6)**.** (*5 points*) Define $f(x) = 2x + 3$and suppose that $g(x+2) = f(f(x-1) \cdot f(x+1) + f(x))$. Find $g(5)$.

Problem 19.4.7 (Problem 7)**.** (*5 points*) Ted flips seven fair coins. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that Ted flips at least one head given that he flips at least three tails. Find $m+n$.

Problem 19.4.8 (Problem 8)**.** (*5 points*) Find the least n for which $n!(n+1)!(2n+1)!-1$ ends in thirty-five digits that are all 9's.

Problem 19.4.9 (Problem 9)**.** (*5 points*) A circle in the first quadrant with center on the curve $y = 2x^2 - 27$ is tangent to the $y-$axis and the line $4x = 3y$. The radius of the circle is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m-n$.

Problem 19.4.10 (Problem 10)**.** (*5 points*) Among positive integers whose digits add up to 2024, N is the number whose product of digits is the greatest possible. Find the number of divisors of N.

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Problem 19.4.11 (Problem 11)**.** (*5 points*) Let a, b, and c be non-zero real numbers such that

$$\frac{ab}{a+b} = 3, \frac{bc}{b+c} = 4, \frac{ca}{c+a} = 5.$$

There are relatively prime positive integers m and n so that

$$\frac{abc}{ab+bc+ca} = \frac{m}{n}.$$

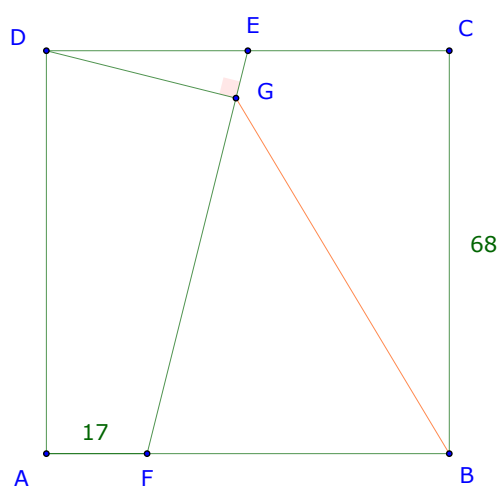
Find $n - m$.

Problem 19.4.12 (Problem 12)**.** (*5 points*) How many positive integer solutions are there to $w+x+y+z = 21$ where $w + x \geq 5$ and $y + z \geq 5$?

Problem 19.4.13 (Problem 13)**.** (*5 points*) Find the number of three-digit numbers such that the hundreds digit is divisible by the tens digit and the tens digit is divisible by the unit digit.

Problem 19.4.14 (Problem 14)**.** (*5 points*) Find the remainder when $2^{5^9} + 5^{9^2} + 9^{2^5}$ is divided by 13.

Problem 19.4.15 (Problem 15)**.** (*5 points*) Square $ABCD$ has side length 68. Let E be the midpoint of segment CD, and let F be the point on segment AB a distance 17 from point A. Point G is on segment EF so that EF is perpendicular to segment GD. The length of segment BG can be written as $m\sqrt{n}$, where m and n be are positive integers, and n is not divisible by the square of any prime. Find $n - m$.



Problem 19.4.16 (Problem 16)**.** (*5 points*) Each time you click a toggle switch, the switch either turns from *off* to *on* or from on to *off*. Suppose that you start with three toggle switches with one of them *on* and two of them *off*. On each move you randomly select one of the three switches and click it. Let m and n be relatively prime positive integers so that $\frac{m}{n}$ is the probability that after four such clicks, one switch will be *on* and two of them will be *off*. Find $n - m$.

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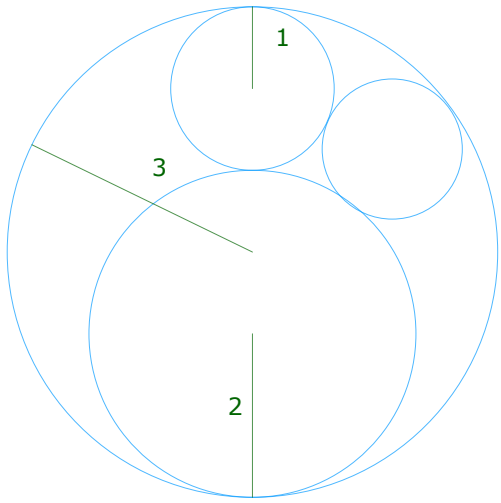
Problem 19.4.17 (Problem 17)**.** (*5 points*) The diagram below shows circles radius 1 and 2 externally tangent to each other and internally tangent to a circle radius 3. There are relatively prime positive integers m and n so that a circle radius $\frac{m}{n}$ is internally tangent to the circle radius 3 and externally tangent to the other two circles as shown. Find the product mn.



Problem 19.4.18 (Problem 18)**.** (*5 points*) Find the greatest seven-digit integer divisible by 132 whose digits, in order, are $x, 0, y, 2, 1, 2, z$ where x, y, and z are single digits.

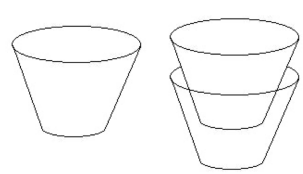
Problem 19.4.19 (Problem 19)**.** (*5 points*) There are positive integers m and n so that $x = m + \sqrt{n}$ is a solution to the equation

$$x^2 - 28x + 1 = \sqrt{x}(x + 1).$$

Find $m + n$.

Problem 19.4.20 (Problem 20)**.** (*5 points*) Find the largest prime that divides $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + 2023 \cdot 2024 \cdot 2025$.

Problem 19.4.21 (Problem 21)**.** (*5 points*) A paper cup has a base that is a circle with radius r, a top that is a circle with radius $2r$, and sides that connect the two circles with straight line segments as shown below. This cup has height h and volume V. A second cup that is exactly the same shape as the first is held upright inside the first cup so that its base is a distance of $\frac{h}{2}$ from the base of the first cup. The volume of liquid that will fit inside the first cup and outside the second cup can be written as $\frac{m}{n} \cdot V$, where m and n are relatively prime positive integers. Find $n - m$.



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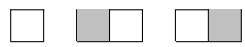
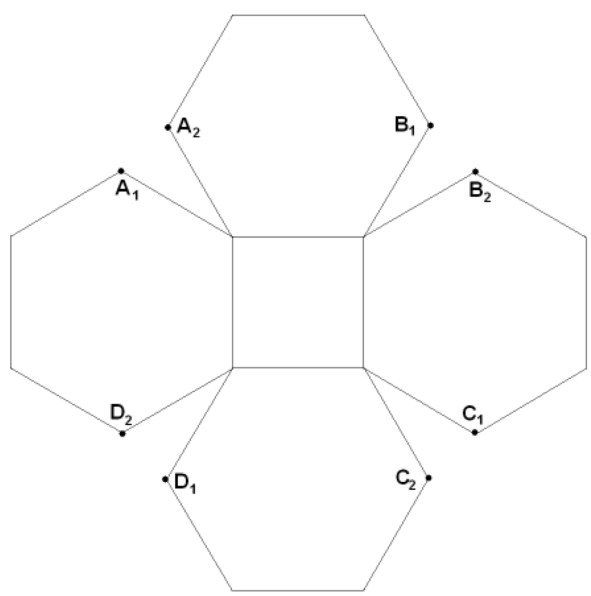
Problem 19.4.22 (Problem 22)**.** (*5 points*) You have some white one-by-one tiles and some black and white two-by-one tiles as shown below. There are four different color patterns that can be generated when using these tiles to cover a three-by-one rectangle by laying these tiles side by side (WWW, BWW, WBW, WWB). How many different color patterns can be generated when using these tiles to cover a eight-by-one rectangle?



Problem 19.4.23 (Problem 23)**.** (*5 points*) A bag contains 7 green candies and 4 red candies. You randomly select one candy at a time to eat. If you eat five candies, there are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that you do not eat a green candy after you eat a red candy. Find $m + n$.

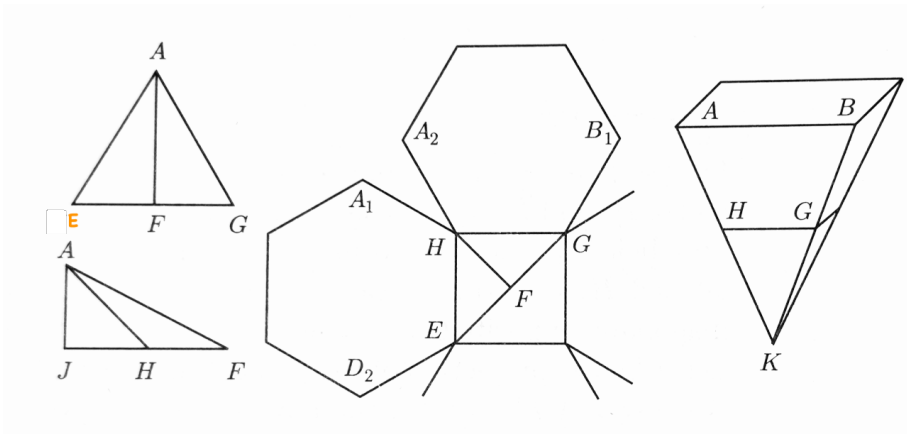
Problem 19.4.24 (Problem 24)**.** (*5 points*) Let $A = 1, 3, 5, 7$ and $B = 2, 4, 6, 8$. Let f be a randomly chosen function from the set $A \cup B$ into itself. There are relatively prime positive integers m and n such that $\frac{m}{n}$ is the probability that f is a one-to-one function on $A \cup B$ given that it maps A one-to-one into $A \cup B$ and it maps B one-to-one into $A \cup B$. Find $m + n$.

Problem 19.4.25 (Problem 25)**.** (*5 points*) The diagram below shows four regular hexagons each with side length 1 meter attached to the sides of a square. This figure is drawn onto a thin sheet of metal and cut out. The hexagons are then bent upward along the sides of the square so that A_1 meets A_2, B_1 meets B_2, C_1 meets C_2, and D_1 meets D_2. The resulting dish is set on a table with the square lying flat on the table. If this dish is filled with water, the water will rise to the height of the corner where the A_1 and A_2 meet.



Let the point where A_1 and A_2 meet be labeled A, the point B_1 and B_2 meet be labeled B, the point C_1 and C_2 meet be labeled C, and the point D_1 and D_2 meet be labeled D. Let the center of the square be labeled F. Let the corner of the square nearest point A be labeled H, the corner of nearest point B be labeled G, and the corner diagonally opposite G be labeled E as shown below.

Let J be the projection of the point A in the plane of the square. If the vertical edges of the dish AH and BG are extended, they will meet at a point K as shown in the diagram. Note that K is the vertex of a downward pointing square pyramid.



Find the measure of $\angle AHF$ in degrees.

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Chapter 20

Session 4

20.1 Middle School - Test 4

You have **60 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 11-20**

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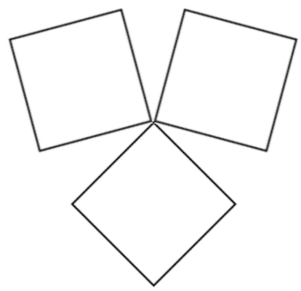
Problem 20.1.1. (*1 point*) There are relatively prime positive integers m and n so that

$$\frac{\frac{1}{2}}{\frac{\frac{1}{3}}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}+\frac{\frac{1}{3}}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}}=\frac{m}{n}.$$

Find $2m+n$.

Problem 20.1.2. (*1 point*) The diagram below shows a 12-sided figure made up of three congruent squares. The figure has total perimeter 60. Find its area.



Problem 20.1.3. (*1 point*) Find the sum of all two-digit integers which are both prime and are 1 more than a multiple of 10.

Problem 20.1.4. (*1 point*) Jerry buys a bottle of 150 pills. Using a standard 12 hour clock, he sees that the clock reads exactly 12 when he takes the first pill. If he takes one pill every five hours, what hour will the clock read when he takes the last pill in the bottle?

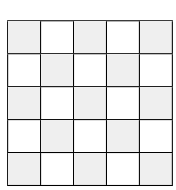
Problem 20.1.5. (*1 point*) Given that $\frac{6}{11}-\frac{10}{19}=\frac{9}{19}-\frac{n}{11}$, find n.

Problem 20.1.6. (*1 point*) The following addition problem is not correct if the numbers are interpreted as base 10 numbers. In what number base is the problem correct?

$$\begin{array}{r} 66 \\ 87 \\ 85 \\ +\,48 \\ \hline 132 \end{array}$$

Problem 20.1.7. (*1 point*) When 12^{18} is divided by 18^{12}, the result is $\left(\frac{m}{n}\right)^3$, where m and n are relatively prime positive integers. Find mn.

Problem 20.1.8. (*1 point*) A square measuring 15 by 15 is partitioned into five rows of five congruent squares as shown below. The small squares are alternately colored black and white as shown. Find the total area of the part colored black.



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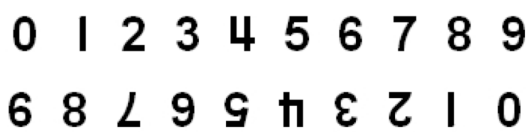
Problem 20.1.9. (*1 point*) A jar contains one quarter red jelly beans and three quarters blue jelly beans. If Chris removes three quarters of the red jelly beans and one quarter of the blue jelly beans, what percent of the jelly beans remaining in the jar will be blue?

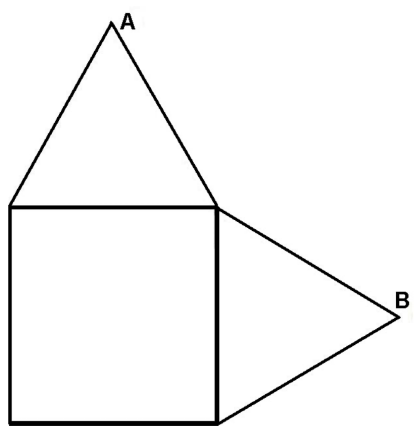
Problem 20.1.10. (*1 point*) Five rays $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD}$, and $\overrightarrow{OE}$ radiate in a clockwise order from O forming four non-overlapping angles such that $\angle EOD = 2\angle COB$, $\angle COB = 2\angle BOA$, while $\angle DOC = 3\angle BOA$. If E, O, A are collinear with O between A and E, what is the degree measure of $\angle DOB$?

Problem 20.1.11. (*1 point*) How many numbers are there that appear both in the arithmetic sequence $10, 16, 22, 28, \ldots 1000$ and the arithmetic sequence $10, 21, 32, 43, \ldots 1000$?

Problem 20.1.12. (*1 point*) When Troy writes his digits, his 0, 1, and 8 look the same right-side-up and upside-down as seen in the figure below. His 6 and 9 look like upside-down images of each other. None of his other digits look like digits when they are inverted. How many different five-digit numbers (which do not begin with the digit zero) can Troy write which read the same right-side-up and upside-down?

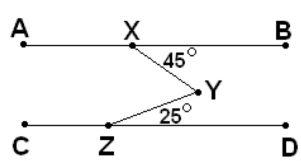


Problem 20.1.13. (*1 point*) The diagram shows two equilateral triangles with side length 4 mounted on two adjacent sides of a square also with side length 4. The distance between the two vertices marked A and B can be written as $\sqrt{m} + \sqrt{n}$ for two positive integers m and n. Find mn.



Problem 20.1.14. (*1 point*) The five-digit number 12110 is divisible by the sum of its digits $1+2+1+1+0 = 5$. Find the greatest five-digit number which is divisible by the sum of its digits.

Problem 20.1.15. (*1 point*) In the diagram below, AB and CD are parallel, $\angle BXY = 45°$, $\angle DZY = 25°$, and $XY = YZ$. What is the degree measure of $\angle YZX$?



Problem 20.1.16. (*1 point*) Let a and b be nonzero real numbers such that

$$\begin{cases} \frac{1}{3a} + \frac{1}{b} = 2011 \\ \frac{1}{a} + \frac{1}{3b} = 1 \end{cases}$$

What is the quotient when $2a + 2b$ is divided by $3ab$?

Problem 20.1.17. (*1 point*) Find the number of ordered quadruples (a, b, c, d) where each of a, b, c, and d are (not necessarily distinct) elements of $\{1, 2, 3, 4, 5, 6, 7\}$ and $3abc + 4abd + 5bcd$ is even. For example, $(2, 2, 5, 1)$ and $(3, 1, 4, 6)$ satisfy the conditions.

Problem 20.1.18. (*1 point*) Find the positive integer n so that n^2 is the least perfect square exceeding $8 + 16 + 24 + \cdots + 8040$.

Problem 20.1.19. (*1 point*) How many ordered pairs of sets (A, B) have the properties:

$$\begin{cases} A \subseteq \{1, 2, 3, 4, 5, 6\} \\ B \subseteq \{2, 3, 4, 5, 6, 7, 8\} \\ |A \cap B| = 3 \text{ (A and B has exactly 3 elements.)} \end{cases}$$

Problem 20.1.20. (*1 point*) Let V be the set of vertices of a regular 25 sided polygon with center at point C. How many triangles have vertices in V and contain the point C in the interior of the triangle?

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20.2 High School - Test 4

You have **90 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 11-20**
- **Olympiad (O) level: Problems 21-30**

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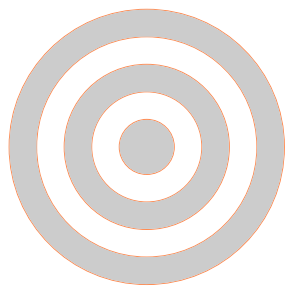
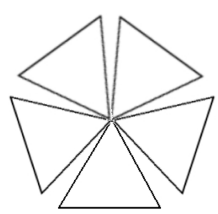
Problem 20.2.1. (*1 point*) The ratio of 3 to the positive number n is the same as the ratio of n to 192. Find n.

Problem 20.2.2. (*1 point*) The target below is made up of concentric circles with diameters 3, 6, 9, 12, and 15. The area of the dark region is $n\pi$. Find n.



Problem 20.2.3. (*1 point*) Shirley went to the store planning to buy 160 balloons for 10 dollars. When she arrived, she was surprised to find that the balloons were on sale for 20 percent less than expected. How many balloons could Shirley buy for her 10 dollars?

Problem 20.2.4. (*1 point*) Five non-overlapping equilateral triangles meet at a common vertex so that the angles between adjacent triangles are all congruent. What is the degree measure of the angle between two adjacent triangles?



Problem 20.2.5. (*1 point*) Let $a_1 = 2$, and for $n \geq 1$, let $a_{n+1} = 2a_n + 1$. Find the smallest value of an $a_n > 100$ that is not a prime number.

Problem 20.2.6. (*1 point*) Working alone, the expert can paint a car in one day, the amateur can paint a car in two days, and the beginner can paint a car in three days. If the three painters work together at these speeds to paint three cars, it will take them $\frac{m}{n}$ days where m and n are relatively prime positive integers. Find mn.

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Problem 20.2.7. (*1 point*) Find the prime number p such that $73p + 1$ is a perfect square.

Problem 20.2.8. (*1 point*) When 126 is added to its reversal, 621, the sum is $126 + 621 = 747$. Find the greatest integer which when added to its reversal yields 1110.

Problem 20.2.9. (*1 point*) There are integers m and n so that $9 + \sqrt{11}$ is the root of the polynomial $x^2 + mx + n$. Find mn.

Problem 20.2.10. (*1 point*) The diagram shows a large circular dart board with four smaller shaded circles each internally tangent to the larger circle. Two of the internal circles have half the radius of the large circle, and are, therefore, tangent to each other. The other two smaller circles are tangent to these circles. If a dart is thrown so that it sticks to a point randomly chosen on the dart board, then the probability that the dart sticks to a point in the shaded area is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find mn.

Problem 20.2.11. (*1 point*) Six distinct positive integers are randomly chosen between 1 and 2023, inclusive. The probability that some pair of the six chosen integers has a difference that is a multiple of 5 is n percent. Find n.

Problem 20.2.12. (*1 point*) Find the area of the region in the coordinate plane satisfying the three conditions

$$\begin{cases} x \leq 2y \\ y \leq 2x \\ x + y \leq 30 \end{cases}$$

Problem 20.2.13. (*1 point*) Let S be the set of all 10-term arithmetic progressions that include the numbers 4 and 10. For example,

$$(-2, 1, 4, 7, 10, 13, 16, 19, 22, 25) \text{ and } (10, 8\frac{1}{2}, 7, 5\frac{1}{2}, 4, 2\frac{1}{2}, 1, -\frac{1}{2}, -2, -3\frac{1}{2})$$

are both members of S. Find the sum of all values of a_{10} for each $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \in S$, that is

$$\sum_{(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10}) \in S} a_{10}.$$

Problem 20.2.14. (*1 point*) The lengths of the three sides of a right triangle form a geometric sequence. The sine of the smallest of the angles in the triangle is $\frac{m+\sqrt{n}}{k}$, where m, n, and k are integers, and k is not divisible by the square of any prime. Find $(m + n)k$.

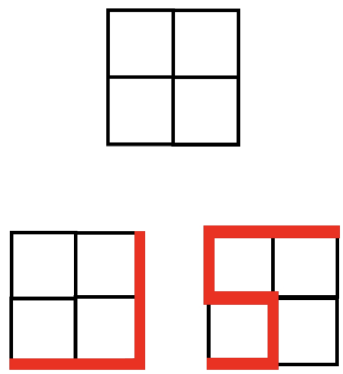
Problem 20.2.15. (*1 point*) A pyramid has a base which is an equilateral triangle with side length 300 centimeters. The vertex of the pyramid is 100 centimeters above the center of the triangular base. A mouse starts at a corner of the base of the pyramid and walks up the edge of the pyramid toward the vertex at the top. When the mouse has walked a distance of 148 centimeters, how many centimeters above the base of the pyramid is the mouse?

Problem 20.2.16. (*1 point*) Evaluate $1^3 - 2^3 + 3^3 - 4^3 + 5^3 - \cdots + 99^3$.

Problem 20.2.17. (*1 point*) In how many distinguishable rearrangements of the letters ABBCCDEEF does the A precede both C's, the F appears between the 2 C's, and the D appears after the F?

Problem 20.2.18. (*1 point*) Let a be a positive real number such that $\dfrac{a^2}{a^4-a^2+1} = \dfrac{4}{37}$. Then $\dfrac{a^2}{a^4-a^2+1} = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find mn.

Problem 20.2.19. (*1 point*) The diagrams below shows a 2 by 2 grid made up of four 1 by 1 squares. Shown are two paths along the grid from the lower left corner to the upper right corner of the grid, one with length 4 and one with length 6. A path may not intersect itself by moving to a point where the path has already been. Find the sum of the lengths of all the paths from the lower left corner to the upper right corner of the grid.



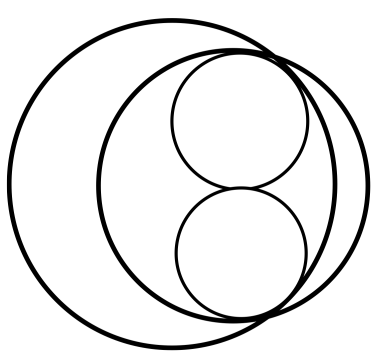
Problem 20.2.20. (*1 point*) Points A and B are the endpoints of a diameter of a circle with center C. Points D and E lie on the same diameter so that C bisects segment DE. Let F be a randomly chosen point within the circle. The probability that $\triangle DEF$ has a perimeter less than the length of the diameter of the circle is $\frac{17}{128}$. There are relatively prime positive integers m and n so that the ratio of DE to AB is $\frac{m}{n}$. Find mn.

Problem 20.2.21. (*1 point*) Let a be the sum of the numbers:

$$99 \times 0.9,\ 999 \times 0.9, 9999 \times 0.9 \ldots, 999 \cdots 9 \times 0.9$$

where the final number in the list is 0.9 times a number written as a string of 100 digits all equal to 9. Find the sum of the digits in the number a.

Problem 20.2.22. (*1 point*) A disk with radius 10 and a disk with radius 8 are drawn so that the distance between their centers is 3. Two congruent small circles lie in the intersection of the two disks so that they are tangent to each other and to each of the larger circles as shown. The radii of the smaller circles are both $\frac{m}{n}$ where m and n are relatively prime positive integers. Find mn.



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Problem 20.2.23. (*1 point*) Let x be a real number in the interval $\left(0, \frac{\pi}{2}\right)$ such that

$$\frac{1}{\sin x \cos x} + 2\cot 2x = \frac{1}{3}.$$

Evaluate

$$\frac{1}{\sin x \cos x} - 2\cot 2x = \frac{1}{2}.$$

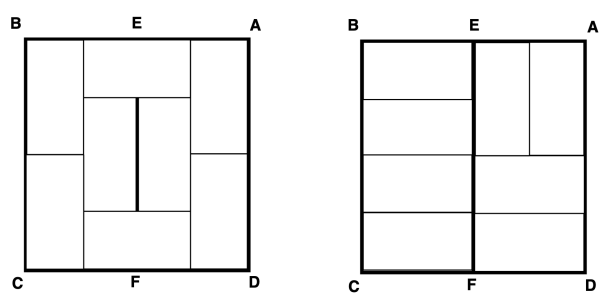
Problem 20.2.24. (*1 point*) The diagram below shows a regular hexagon with an inscribed square where two sides of the square are parallel to two sides of the hexagon. There are positive integers m, n, and p such that the ratio of the area of the hexagon to the area of the square can be written as $\frac{m+\sqrt{n}}{p}$ where m and p are relatively prime. Find $(m+n)p$.

Problem 20.2.25. (*1 point*) Let x_1, x_2, and x_3 be the three roots of the polynomial $x^3 + 3x + 1$. There are relatively prime positive integers m and n so that

$$\frac{m}{n} = \frac{x_1^2}{(5x_2+1)(5x_3+1)} + \frac{x_2^2}{(5x_3+1)(5x_1+1)} + \frac{x_3^2}{(5x_1+1)(5x_2+1)}.$$

Find mn.

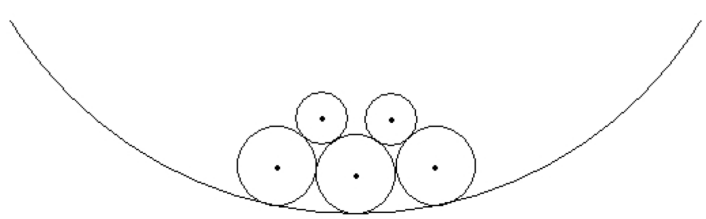
Problem 20.2.26. (*1 point*) Square ABCD has side length 4. Points E and F are the midpoints of sides AB and CD, respectively. Eight 1 by 2 rectangles are placed inside the square so that no two of the eight rectangles overlap (see diagram). If the arrangement of eight rectangles is chosen randomly, then there are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that none of the rectangles crosses the line segment EF (as in the arrangement on the right). Find mn.



Problem 20.2.27. (*1 point*) Find the smallest prime number that does not divide

$$9 + 9^1 + 9^2 + \ldots + 9^{2064}$$

Problem 20.2.28. (*1 point*) Pictured below is part of a large circle with radius 30. There is a chain of three circles with radius 3, each internally tangent to the large circle and each tangent to its neighbors in the chain. There are two circles with radius 2 each tangent to two of the radius 3 circles. The distance between the centers of the two circles with radius 2 can be written as $d\frac{a\sqrt{b}-c}{d}$ where a, b, c, and d are positive integers, c and d are relatively prime, and b is not divisible by the square of any prime. Find $(a+b+c)d$.



Problem 20.2.29. (*1 point*) Let S be a randomly selected four-element subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Let m and n be relatively prime positive integers so that the expected value of the maximum element in S is $\frac{m}{n}$. Find mn.

Problem 20.2.30. (*1 point*) Four congruent spheres are stacked so that each is tangent to the other three. A larger sphere, R, contains the four congruent spheres so that all four are internally tangent to R. A smaller sphere, S, sits in the space between the four congruent spheres so that all four are externally tangent to S. The ratio of the surface area of R to the surface area of S can be written $m + \sqrt{n}$ where m and n are positive integers. Find $m + n$.

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20.3 IC Test for Level 1

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 20.3.1. A bag containing some widgets weighs 77 kg. When 2 of the widgets are removed, the weight of the bag of widgets drops to 69 kg. The bag weighs 1 kg when empty. How many widgets are left in the bag?

Problem 20.3.2. The angle $\angle B$ in $\triangle ABC$ is $60°$. If an exterior angle at A is $170°$, what is $\angle C$?

Problem 20.3.3. Each week, between 60 and 80 students show up for an archery class run by Betty and Wilma. Usually the students break up into groups of equal size for target practice. However, this week, Betty noticed that she could not break the students up into multiple groups of equal size. Wilma noticed that if Betty and Wilma joined the students in practicing, they still could not break the archers up into groups of equal size. How many students showed up to the archery class this week?

Problem 20.3.4. In how many ways can I place 6 different beads on a bracelet if rotating or flipping the bracelet does not change the order of the beads?

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20.4 IC Test for Level 2

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 20.4.1. For what value of the constant c does the system of equations below have no solutions (x, y)?

$$\begin{cases} 3x - 7y = -2.3 \\ 6x = cy + 9.3 \end{cases}$$

Problem 20.4.2. Points P and Q are on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{PQ} \parallel \overline{BC}$. Given $AB = 36$, $PB = 27$, and $AC = 54$, find QA.

Problem 20.4.3. What is the smallest number of marbles that could either be divided up into bags of 12 marbles or into bags of 18 marbles or into bags of 30 marbles with no marbles leftover in each case?

Problem 20.4.4. The student council has 7 boys and 5 girls as class representatives. Two committees, each consisting of 2 boys and 2 girls, are to be created. If no student can serve on both committees, how many different pairs of committees are possible?

20.5 IC Test for Level 3

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 20.5.1. Alice began peeling a pile of 52 potatoes at the rate of 3 potatoes per minute. Four minutes later Bob joined him and peeled at the rate of 5 potatoes per minute. When they finished, how many potatoes had Bob peeled?

Problem 20.5.2. In triangle ABC, let M be the midpoint of $\overline{BC}$. Prove that if $MA = MB = MC$, then $\angle BAC = 90°$.

Problem 20.5.3. How many of the divisors of 10800 are perfect squares?

Problem 20.5.4. Clara is a house painter. She has been given a contract to paint any 3 of the houses that she wants on a street that contains 7 houses. Clara has 4 different colors of paint, in unlimited supply. She can use only one color on any given house, but for each house, she gets to pick which color to use. In how many different ways can Clara fulfill her contract?

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20.6 IC Test for Level 4

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 20.6.1. Find the equation, in standard form, of the line that passes through $(4, 1)$ and has slope $-\frac{2}{3}$. Graph the equation.

Problem 20.6.2. $\triangle XYZ$ is an equilateral triangle with side length 12. M is the midpoint of side $\overline{YZ}$ and N is the midpoint of $\overline{XZ}$. $\overline{YN}$ and $\overline{XM}$ meet at E. Find the inradius of $\triangle XYZ$.

Problem 20.6.3. Find all ordered pairs of positive integers (x, y) where

$$\frac{1}{2x} + \frac{1}{3y} = \frac{1}{12}.$$

Problem 20.6.4. The box A holds 7 red balls and 5 green balls; boxes B and C each hold 2 red balls and 4 green balls. A box is selected at random and then a ball is randomly selected from that box. What is the probability that the ball selected is green?

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20.7 IC Test for Level 5

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 20.7.1. Let $x = 2001^{1002} - 2001^{-1002}$ and $y = 2001^{1002} + 2001^{-1002}$. Find $x^2 - y^2$.

Problem 20.7.2. The bases of a trapezoid are $2x$ and $4x$. The height is $2x$. If the area is 54, what is x?

Problem 20.7.3. Determine which of the following are multiples of 12.

$$717_8,\ 212021_3,\ 14202_5,\ 6234_7,\ C10B_{13}$$

Problem 20.7.4. Given that a and b are real numbers such that $-4 \le a \le -1$ and $-2 \le b \le 4$, and values for a and b are chosen at random, what is the probability that the product ab is positive?

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20.8 IC Test for Level 6

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 20.8.1. Graph the equation $x^2 + 2x + y^2 - 4y - 11 = 0$.

Problem 20.8.2. $A_1A_2A_3A_4A_5$ is a pentagon. Prove that $A_1A_3 > A_1A_2$.

Problem 20.8.3. Find the units digit of $3^{2024} - 2^{2024}$.

Problem 20.8.4. The numbers on a standard six-sided die are arranged such that numbers on opposite faces always add to 7. The die is rolled, and the product of the numbers appearing on the four lateral faces of the die is calculated (ignoring the numbers on the top and bottom). What is the expected value of this product?

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Chapter 21

Session 5

21.1 IC Test for Level 1

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 21.1.1. Stan has an equal positive number of quarters, dimes, and nickels in his bag. The total amount of money Stan has in his bag is a whole number of dollars. What is the smallest amount of money Stan could have?

Problem 21.1.2. Lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are parallel, and $\overleftrightarrow{TV} \perp \overleftrightarrow{PQ}$. If $\overleftrightarrow{TV}$ intersects $\overleftrightarrow{PQ}$ at X and $\overleftrightarrow{RS}$ at Y, find $\angle RYX$.

Problem 21.1.3. Find all two-digit prime numbers whose digits sum is 9.

Problem 21.1.4. In how many ways can 8 people sit around a round table if 3 of the people - Pierre, Rosa, and Thomas - all want to sit together?

21.2 IC Test for Level 2

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 21.2.1. The product of two nonzero numbers is equal to twice the sum. Find the sum of their reciprocals.

Problem 21.2.2. The length of a given rectangle is 2 less than 4 times the width of the rectangle. Its perimeter is 51. What is its area?

Problem 21.2.3. There are 25 primes less than 100. Is their sum even or odd?

Problem 21.2.4. In how many ways can we choose 3 distinct letters of the alphabet, without regard to order, if we must choose 1 vowel (A, E, I, O, or U) and 2 consonants?

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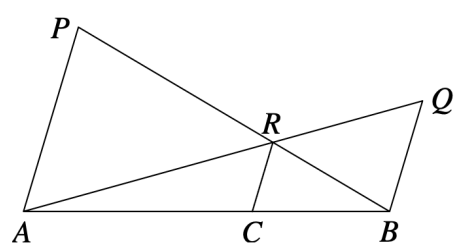
21.3 IC Test for Level 3

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 21.3.1. The tail of a 1-mile long train exits a tunnel exactly 3 minutes after the front of the train entered the tunnel. If the train is moving 60 miles per hour, how long is the tunnel?

Problem 21.3.2. In the diagram below, $\overline{AP} \parallel \overline{BQ} \parallel \overline{CR}$. Prove that

$$\frac{1}{CR} = \frac{1}{AP} + \frac{1}{BQ}.$$



Problem 21.3.3. Find the GCD of 7! and $(5!)^2$.

Problem 21.3.4. How many pairs of diagonals of a regular decagon (10-sided polygon) are parallel?

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21.4 IC Test for Level 4

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 21.4.1. A line passing through the points $(2a+4, 3a^2)$ and $(3a+4, 5a^2)$ has slope $a+3$, where a is nonzero. Find the value of a.

Problem 21.4.2. Points T and U are on a circle centred at K such that K is 2 units from the segment $\overline{TU}$. If $TU = 14$, what is the radius of the circle?

Problem 21.4.3. Let $n = ABC$ be a positive 3-digit integer that is equal to the sum $AB + BA + AC + CA + BC + CB$ of six positive 2-digit integers. Find the largest possible value for n.

Problem 21.4.4. Six green balls and four red balls are in a bag. A ball is taken from the bag, its color recorded. It is NOT placed back in the bag. A second ball is taken and its color recorded. What is the probability the two balls are the same color?

21.5 IC Test for Level 5

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 21.5.1. Alice and Bobby took the same algebra test. The problem asked them to find the solutions to a quadratic equation. Alice misread the constant term. She correctly found that the roots to her incorrect quadratic were -4 and 7. Bobby misread the linear term. He correctly found that the roots to his incorrect quadratic were -4 and 10. What are the roots of the correct quadratic?

Problem 21.5.2. If A, B, C, D are midpoints of the sides of square $EFGH$, what is the ratio of the area of triangle ABC to the area of square $EFGH$?

Problem 21.5.3. What is the smallest integer $b > 3$ for which the base-b number 23_b is a perfect square?

Problem 21.5.4. There are 11 students in Ms. McGinn's Chemistry class, including the Baker triplets: Annika, Billy, and Catherine. The teacher calls all 11 students in random order one at a time to her desk to tell each student his or her final grade.

The first time he calls a student to his desk, Billy is in the bathroom and therefore can't be called up, but he's back in time for the rest of the students to be called. What is the probability that Billy is the first of the triplets to be called to the desk?

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21.6 IC Test for Level 6

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 21.6.1. If $x^2 - 5x + 6 < 0$ and $P = x^2 + 5x + 6$, then what are the possible values of P?

Problem 21.6.2. $A_1A_2A_3A_4A_5$ is a pentagon. Prove that $A_1A_3 > A_1A_2$.

Problem 21.6.3. Find the units digit of each of the following within the indicated number base.

$$314_5 + 2423_5,\ 513_9 \cdot 7121_9,\ (17_{12})^6$$

Problem 21.6.4. Two fair eight-sided dice have their faces numbered from 1 to 8. What is the expected value of the sum of the rolls of both dice?

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21.7 IC Test for Level 7

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 21.7.1. How many square units are in the area of the triangle whose vertices are the x- and y-intercepts of the curve $y = (x-3)^2(x+2)$?

Problem 21.7.2. A and B are two points on a circle with center O, and C lies outside the circle, on ray $\overrightarrow{AB}$. Given that $AB = 24$, $BC = 28$, and $OA = 15$, find OC.

Problem 21.7.3. What is the 4037th digit following the decimal point in the expansion of $\dfrac{1}{17}$?

Problem 21.7.4. Prove that

$$\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \binom{n}{2}\binom{n}{3} + \cdots + \binom{n}{n-1}\binom{n}{n} = \binom{2n}{n-1}.$$

21.8 IC Test for Level 10

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 21.8.1. How many different triples of numbers (a, b, c) satisfy the equation

$$a^2 + bc = b^2 + ac,$$

if a, b, and c are integers from 1 to 5, inclusive? (The numbers a, b, and c are not necessarily different.)

Problem 21.8.2. In $\triangle XYZ$, we have $\angle X = 20^\circ$ and $XY = XZ$. Prove that $XY < 3YZ$.

Problem 21.8.3. How many four-digit integers either leave a remainder of 2 when divided by 7, or a remainder of 4 when divided by 5, but not both?

Problem 21.8.4. In how many ways can we distribute 16 chocolate donuts, 13 glazed donuts, and 15 jelly donuts among 4 people if each person insists on receiving at least 2 of each variety of donut?

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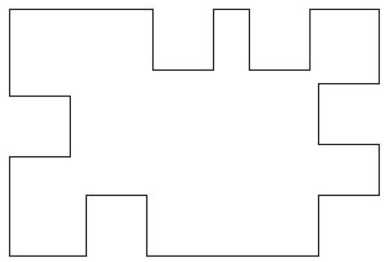
21.9 Middle School - Test 5

You have **60 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 11-20**

Problem 21.9.1. (*1 point*) The diagram shows a polygon made by removing six 2×2 squares from the sides of an 8×12 rectangle. Find the perimeter of this polygon.



Problem 21.9.2. (*1 point*) ohn, Paul, George, and Ringo baked a circular pie. Each cut a piece that was a sector of the circle. John took one-third of the whole pie. Paul took one-fourth of the whole pie. George took one-fifth of the whole pie. Ringo took one-sixth of the whole pie. At the end the pie had one sector remaining. Find the measure in degrees of the angle formed by this remaining sector.

Problem 21.9.3. (*1 point*) There were 891 people voting at precinct 91. There were 20 percent more female voters than male voters. How many female voters were there?

Problem 21.9.4. (*1 point*) Terry drove along a scenic road using 9 gallons of gasoline. Then Terry went onto the freeway and used 17 gallons of gasoline. Assuming that Terry gets 6.5 miles per gallon better gas mileage on the freeway than on the scenic road, and Terry's average gas mileage for the entire trip was 30 miles per gallon, find the number of miles Terry drove.

Problem 21.9.5. (*1 point*) We draw a radius of a circle. We draw a second radius 23 degrees clockwise from the first radius. We draw a third radius 23 degrees clockwise from the second. This continues until we have drawn 40 radii each 23 degrees clockwise from the one before it. What is the measure in degrees of the smallest angle between any two of these 40 radii?

Problem 21.9.6. (*1 point*) We glue together 990 one inch cubes into a 9 by 10 by 11 inch rectangular solid. Then we paint the outside of the solid. How many of the original 990 cubes have just one of their sides painted?

Problem 21.9.7. (*1 point*) Find the sum of all positive integers x such that there is a positive integer y satisfying $9x^2 - 4y^2 = 2021$.

Problem 21.9.8. (*1 point*) The value of

$$\left(1 - \frac{1}{2^2 - 1}\right)\left(1 - \frac{1}{2^3 - 1}\right)\cdots\left(1 - \frac{1}{2^{29} - 1}\right)$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $2m - n$.

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Problem 21.9.9. (*1 point*) Find the positive integer n such that

$$1 + 2 + 3 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + 35).$$

Problem 21.9.10. (*1 point*) For positive integer n, let $s(n)$ be the sum of the digits of n when n is expressed in base ten. For example, $s(2022) = 2 + 0 + 2 + 2 = 6$. Find the sum of the two solutions to the equation:

$$n - 3s(n) = 2022.$$

Problem 21.9.11. (*1 point*) For some integers a and b the function $f(x) = ax + b$ has the properties that $f(f(0)) = 0$ and $f(f(f(4))) = 9$. Find $f(f(f(f(10))))$.

Problem 21.9.12. (*1 point*) A farmer wants to create a rectangular plot along the side of a barn where the barn forms one side of the rectangle and a fence forms the other three sides. The farmer will build the fence by fitting together 75 straight sections of fence which are each 4 feet long. The farmer will build the fence to maximize the area of the rectangular plot. Find the length in feet along the side of the barn of this rectangular plot.

Note that the farmer can only use a whole numbers of sections of fence for any side.

Problem 21.9.13. (*1 point*) Each different letter in the following addition represents a different decimal digit. The sum is a six-digit integer whose digits are all equal.

$$\begin{array}{cccccccc} & & P & U & R & P & L & E \\ + & & & C & O & M & E & T \\ \hline \end{array}$$

Find the greatest possible value that the five-digit number COMET could represent.

Problem 21.9.14. (*1 point*) A square, a regular pentagon, and a regular hexagon are all inscribed in the same circle. The 15 vertices of these polygons divide the circle into at most 15 arcs. Let M be the degree measure of the longest of these arcs. Find the minimum possible value for M.

Problem 21.9.15. (*1 point*) Find the remainder when 7^{7^7} is divided by 1000.

Problem 21.9.16. (*1 point*) On $\triangle ABC$ let D be a point on side AB, F be a point on side AC, and E be a point inside the triangle so that $DE \parallel AC$ and $EF \parallel AB$. Given that $AF = 6, AC = 33, AD = 7, AB = 26$, and the area of quadrilateral $ADEFis14$, find the area of $\triangle ABC$.

Problem 21.9.17. (*1 point*) Find the greatest integer n such that 5^n divides $2019! - 2018! + 2017!$.

Problem 21.9.18. (*1 point*) Three red books, three white books, and three blue books are randomly stacked to form three piles of three books each. The probability that no book is the same color as the book immediately on top of it is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $10m + n$.

Problem 21.9.19. (*1 point*) Given that $a_1, a_2, a_3, \ldots, a_{99}$ is a permutation of $1, 2, 3, \ldots, 99$, find the maximum possible value of

$$|a_1 - 1| + |a_2 - 2| + |a_3 - 3| + \cdots + |a_{99} - 99|.$$

Problem 21.9.20. (*1 point*) Nine light bulbs are equally spaced around a circle. When the power is turned on, each of the nine light bulbs turns blue or red, where the color of each bulb is determined randomly and independently of the colors of the other bulbs. Each time the power is turned on, the probability that the color of each bulb will be the same as at least one of the two adjacent bulbs on the circle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $n - m$.

21.10 High School - Test 5

You have **90 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 11-20**
- **Olympiad (O) level: Problems 21-30**

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Problem 21.10.1. (*1 point*) In eight years Henry will be three times the age that Sally was last year. Twenty five years ago their ages added to 83. How old is Henry now?

Problem 21.10.2. (*1 point*) *The day of the month of Monday, March 11, 2024 is 11. The day of the month of Saturday, March 16, 2024 is 16.*

What is the smallest number that could be the day of month of the first Saturday after the second Monday following the second Thursday of a month?

Problem 21.10.3. (*1 point*) What is the largest integer whose prime factors add to 17?

Problem 21.10.4. (*1 point*) The lengths of the diagonals of a rhombus are, in inches, two consecutive even integers. The area of the rhombus is 24 square inches. Find its perimeter, in inches.

Problem 21.10.5. (*1 point*) Let a, b, and c be nonzero real numbers such that

$$a + \frac{1}{b} = 6,\ b + \frac{1}{c} = 13,\ c + \frac{1}{a} = 15.$$

Find $abc + \frac{1}{abc}$.

Problem 21.10.6. (*1 point*) Evaluate

$$\frac{1}{\log_2\left(\frac{1}{6}\right)} - \frac{1}{\log_3\left(\frac{1}{6}\right)} - \frac{1}{\log_{24}\left(\frac{1}{6}\right)}.$$

Problem 21.10.7. (*1 point*) Find the smallest n such that every subset of $\{1, 2, 3, \ldots, 2024\}$ with n elements contains at least two elements that are relatively prime.

Problem 21.10.8. (*1 point*) Three lines are drawn parallel to each of the three sides of triangle ABC so that the three lines intersect in the interior of ABC. The resulting three smaller triangles have areas 2.4, 5.4, and 15. Find the area of triangle ABC.

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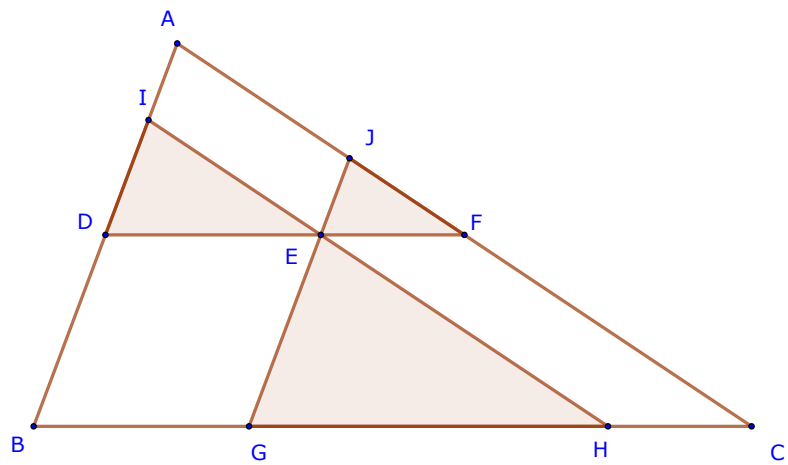


Problem 21.10.9. (*1 point*) Let f be a real-valued function of real and positive argument such that

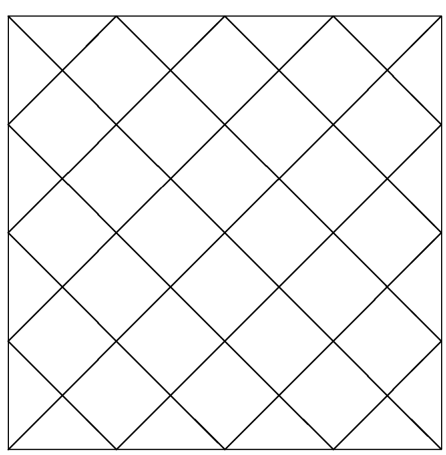
$$f(x) + 3xf\left(\frac{1}{x}\right) = 2(x+1) \text{ for all real numbers } x > 0.$$

Find $2f(2024)$.

Problem 21.10.10. (*1 point*) How far is it from the point $(9, 17)$ to its reflection across the line $3x+4y = 10$?

Problem 21.10.11. (*1 point*) Find the number of nonnegative integers n for which $(n^2 - 3n + 1)^2 + 1$ is a prime number.

Problem 21.10.12. (*1 point*) How many triangles appear in the diagram below?



Problem 21.10.13. (*1 point*) Let $P(x)$ be a polynomial such that, when divided by $x - 2$, the remainder is 3 and, when divided by $x - 3$, the remainder is 2. If, when divided by $(x - 2)(x - 3)$, the remainder is $ax + b$, find $a^3 + b^3$.

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Problem 21.10.14. (*1 point*) Jerry purchased some stock for \$115200 at the same time that Susan purchased a bond for $\$x$. Jerry's investment went up 20 percent the first year, fell 10 percent the second year, and rose another 20 percent the third year. Susan's investment grew at a constant rate of compound interest of 44% for three years. If both investments are worth the same after three years, what is x?

Problem 21.10.15. (*1 point*) We want to paint some identically-sized cubes so that each face of each cube is painted a solid colour and each cube is painted with six different colours. If we have eight different colours to choose from, how many distinguishable cubes can we produce?

Problem 21.10.16. (*1 point*) Five boys and two girls sit at a circular table. How many distinct arrangements are there of these six people if the two girls sit opposite each other?

Problem 21.10.17. (*1 point*) Let $a_k = (k^2+1)k!$ and $b_k = a_1 + a_2 + \cdots + a_k$. Let

$$\frac{a_{49}}{b_{49}} = \frac{m}{n}.$$

where m and n are relatively prime natural numbers. Find $n - m$.

Problem 21.10.18. (*1 point*) Given that $3\sin x + 4\cos x = 5$, where x is in $\left(0, \frac{\pi}{2}\right)$. Find $2\sin x + \cos x + 8\tan x$.

Problem 21.10.19. (*1 point*) In triangle ABC, $\max \angle A, \angle B = \angle C + 30^\circ$, and $\dfrac{R}{r} = 1 + \sqrt{3}$, where R is the radius of the circumcircle and r is the radius of the incircle. Find the smallest angle in $\triangle ABC$.

Problem 21.10.20. (*1 point*) In triangle ABC, $\angle A = 30^\circ$, and $AB = AC = 16$. Let D lie on segment BC such that $DB = 2$. Let E and F be the orthogonal projections of D onto AB and AC, respectively. If $\dfrac{m}{n} = DF - DE$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 21.10.21. (*1 point*) In triangle ABC, $AB = 52$, $BC = 56$, $CA = 60$. Let D be the foot of the altitude from A and E be the intersection of the internal angle bisector of $\angle BAC$ with BC. Find DE.

Problem 21.10.22. (*1 point*) The side lengths of a trapezoid are $\sqrt[4]{3}$, $\sqrt[4]{3}$, $\sqrt[4]{3}$, and $2\sqrt[4]{3}$. Its area is the ratio of two relatively prime positive integers m and n. Find mn.

Problem 21.10.23. (*1 point*) Let a and b be two different real numbers satisfying

$$\begin{cases} a^4 + 8b = 4(a^3 - 1) - 16\sqrt{3} \\ b^4 + 8a = 4(b^3 - 1) + 16\sqrt{3} \end{cases}$$

Find $a^2 + b^2$

Problem 21.10.24. (*1 point*) For each positive integer m and n define function $f(m,n)$ by

$$f(1,1) = 1,\ f(m+1,n) = f(m,n) + m,\ f(m,n+1) = f(m,n) - n.$$

Find the sum of all the values of p such that $f(p,q) = 2024$ for some q.

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Problem 21.10.25. (*1 point*) $P(x)$ is the remainder polynomial when $x^{203} - 1$ is divided by $x^4 - 1$.

Find $P(7)$.

Problem 21.10.26. (*1 point*) Let d_k be the greatest odd divisor of k, for $k = 1, 2, 3, \ldots$ Find

$$d_1 + d_2 + d_3 + \cdots + d_{256}$$

Problem 21.10.27. (*1 point*) Anthony has a fair coin. He writes the number $+1$ on a blackboard.

Then he flips the coin. If it comes up heads (**H**), he writes $+\frac{1}{2}$, and otherwise, if he flips tails (**T**), he writes $-\frac{1}{2}$.

Then he flips the coin again. If it comes up heads, he writes $+\frac{1}{4}$, and otherwise he writes $-\frac{1}{4}$.

Anthony continues to flip the coin, and on the nth flip, if he flips heads, he writes $+\frac{1}{2^n}$, and otherwise he writes $-\frac{1}{2^n}$,.

For example, if Binhao flips **HHTHTHT**, he writes

$$1 + \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128}.$$

The probability that Anthony will generate series whose sum is greater than $\frac{1}{7}$ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find pq.

Note that

$$1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} - \frac{1}{64} + \frac{1}{128} - \frac{1}{256} - \frac{1}{512} + \frac{1}{1024} + \cdots = \frac{1}{7}.$$

Problem 21.10.28. (*1 point*) Rose writes all the 7-digit numbers in which all the digits are different and each digit is greater than the one to its right (so the tens digit is greater than the units, the hundreds greater than the tens, and so on).

Rose circles the middle digit of each number, then adds up all the circled digits. What sum does she get?

Problem 21.10.29. (*1 point*) Two base-120 units digits (not necessarily different) are chosen randomly and their product is written in base 120. $\frac{m}{n}$ is the probability that the units digit of the base-120 product is 1. Find $m + n$.

Problem 21.10.30. (*1 point*) Evaluate

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n},$$

where $\cos\theta = \frac{1}{5}$.

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Chapter 22

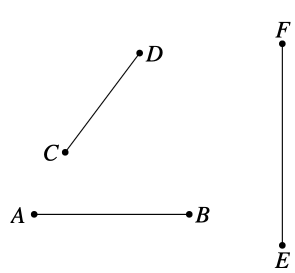
Session 6

22.1 IC Test for Level 1

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 22.1.1. Find all values of x such that $\dfrac{1}{x-1} + \dfrac{2x}{x-1} = 6$.

Problem 22.1.2. Given the segments shown, construct segments with the following length $AB + CD - EF$.



Problem 22.1.3. Lucius, Sarah, Brandon, and Scott take Algebra together. On their first exam, Lucius got a 94, Sarah got a 91, Brandon got a 93, and Scott's score was between 79 and 85 inclusive. If the average of all four of their scores is an integer, what was Scott's score on the first exam?

Problem 22.1.4. Determine the number of arrangements of the letters of the word LONDON.

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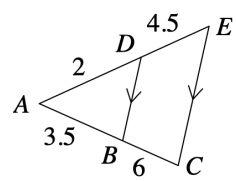
22.2 IC Test for Level 3

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 22.2.1. Suppose a is directly proportional to b, but inversely proportional to c. If $a = 3$ when $b = 7$ and $c = 11$, then what is c when $b = 3$?

Problem 22.2.2. In the diagram below the lengths of the segments AB, AD, and AE are correct. The length of BC is incorrect. Why? What is the correct value of it?



Problem 22.2.3. The largest power of 3 that divides 18 is 3^2. The largest power of 3 that divides 27 is 3^3.

Find the largest power of 3 that divides 81!.

Problem 22.2.4. How many ways are there to put 5 balls in 3 boxes if the balls are not distinguishable but the boxes are distinguishable?

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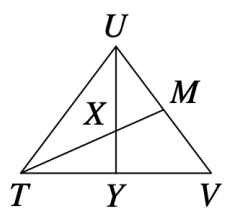
22.3 IC Test for Level 4

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 22.3.1. If $a > b > c > d$, then which is larger, $(a-c)(b-d)$ or $(a-d)(b-c)$

Problem 22.3.2. $\triangle TUV$ is isosceles with $TU = UV = 15$ and $TV = 18$. UY bisects $\angle TUV$ and TM is a median of $\triangle TUV$. UY and TM meet at X. Find XY.



Problem 22.3.3. How many four-digit integers are multiples of 12, 21, and 35?

Problem 22.3.4. Anna has a weighted coin that lands on heads with probability $\frac{1}{3}$, and thus lands on tails with probability $\frac{2}{3}$. If she flips the coin 9 times (where n is a positive integer) then what is the probability that exactly 6 of the flips come up heads.

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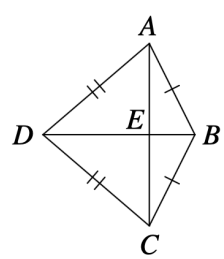
22.4 IC Test for Level 5

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 22.4.1. Simplify

$$(i^{1011} - i^{-1011})^{-1}, \text{ where } i = \sqrt{-1}.$$

Problem 22.4.2. In the convex quadrilateral $ABCD$, $AB = BC = 13$ and $CD = DA$. The diagonals of $ABCD$ meet at E as shown. Find the length AD if $AC = 24, BD = 21$.



Problem 22.4.3. Prove that for positive integer b the number 1331_b (in base b) has at least 4 distinct divisors.

Problem 22.4.4. A real number x is chosen at random such that $0 < x < 100$. What is the probability that

$$x - \lfloor x \rfloor \geq \frac{2}{5}.$$

Note: for any real number y, $\lfloor y \rfloor$ is defined to be the greatest integer less than or equal to y.

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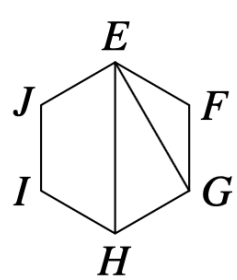
22.5 IC Test for Level 6

- All test lasts for **60 minutes**.
- The test consists of 4 **show-you-work** problems. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{25.}$

Problem 22.5.1. P is a parabola with axis of symmetry parallel to the x-axis passes through the points $(5, 0)$, $(12, -1)$, and $(3, 2)$.

Does P passes through point $(2, 1)$?

Problem 22.5.2. $EFGHIJ$ is a regular hexagon. Find EH/EG.



Problem 22.5.3. Find the possible units digits of a perfect square in base 8.

Problem 22.5.4. Box A has one white ball and four black balls. Box B has three balls labeled "\$2" and one ball labeled "\$14". Box W has five balls labeled "\$4" and one ball labeled "\$460".

A game is played as follows: a ball is randomly selected from box A. If it is black, then a ball is randomly selected from box B; otherwise, if the original ball is white, then a ball is randomly selected from box W.

You win the amount printed on the second ball selected. What is your expected win?

22.6 Middle School - Test 6

You have **60 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 11-20**

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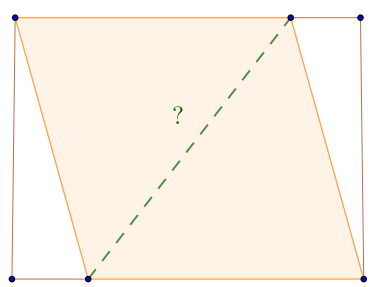
Problem 22.6.1. (*1 point*) Elves and ogres live in the land of Pixie. The average height of the elves is 80 cm, the average height of the ogres is 200 cm, and the average height of the elves and the ogres together is 140 cm. If 36 elves live in Pixie, how many ogres live there?

Problem 22.6.2. (*1 point*) You are given a two-digit positive integer. If you reverse the digits of your number, the result is a number which is 20% larger than your original number. What is your original number?

Problem 22.6.3. (*1 point*) A 9×12 rectangular piece of paper is folded once so that a pair of diagonally opposite corners coincide. The length of the crease is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 22.6.4. (*1 point*) In calm weather, an aircraft can fly from one city to another 200 kilometres north of the first and back in exactly two hours. In a steady north wind, the round trip takes five minutes longer. Find the speed (in kilometres per hour) of the wind.

Problem 22.6.5. (*1 point*) A rectangular floor, 24 feet $\times$ 40 feet, is covered by squares of sides 1 foot. A chalk line is drawn from one corner to the diagonally opposite corner. How many tiles have a chalk line segment on them?

Problem 22.6.6. (*1 point*) How many eight-digit positive integers $abcdefgh$ are there that satisfy the relations

$$a^3 - b^2 = 2,\ c^3 - d^2 = 4,\ 2^e - f^2 = 7, g^3 - h^2 = -1.$$

Problem 22.6.7. (*1 point*) Determine the smallest possible value of $x + y$, if x and y are positive integers with

$$\frac{2022}{2023} < \frac{x}{y} < \frac{2023}{2024}.$$

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Problem 22.6.8. (*1 point*) ABC is an acute triangle, such that $AB = 5, BC = 6, CA = 7$. D, E, and F are the feet of the altitudes from A, B, and C, respectively. Let H be the orthocenter. The sum

$$\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF}$$

is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $4m + 5n$.

Problem 22.6.9. (*1 point*) An autobiographical number is a natural number with ten digits or less in which the first digit of the number (reading from left to right) tells you how many zeros are in the number, the second digit tells you how many 1's, the third digit tells you how many 2's, and so on. For example, 6 210 001 000 is an autobiographical number.

Find the smallest autobiographical number.

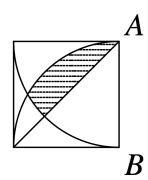
Problem 22.6.10. (*1 point*) What is the sum of all integers less than 100 which have exactly 12 divisors?

Problem 22.6.11. (*1 point*) The probability that a five-card poker hand has exactly two cards of the same value, but no other cards duplicated, is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 22.6.12. (*1 point*) Consider the sets of consecutive integers $S_1 = \{1\}$, $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$, $S_4 = \{7, 8, 9, 10\}$, ..., where each set contains one more element than the preceding one, and where the first element of each set is one more than the last element of the preceding set. Let S_n be the sum of the elements in the nth set. Find S_{21}.

Problem 22.6.13. (*1 point*) In the diagram, the curved paths are arcs of circles centered at vertices A and B of a square of side 6.

The area of the shaded section is $\frac{m}{n}\pi - p\sqrt{q}$, where m and n are relatively prime positive integers, p, q are positive integers and q is not divisible by any perfect square larger than 1. Find $m + n + p + q$.



Problem 22.6.14. (*1 point*) Find $5a + 7b + 9c$ if nonzero a, b, and c satisfy the equations

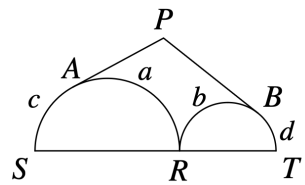
$$\begin{cases} ab &= 2(a+b) \\ bc &= 3(b+c) \\ ca &= 4(c+a) \end{cases}$$

Problem 22.6.15. (*1 point*) Consider the sets of consecutive integers $S_1 = \{1\}$, $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$, $S_4 = \{7, 8, 9, 10\}$, ..., where each set contains one more element than the preceding one, and where the first element of each set is one more than the last element of the preceding set. Let S_n be the sum of the elements in the nth set. Find S_{51}.

Problem 22.6.16. (*1 point*) In the figure PA is tangent to semicircle SAR. PB is tangent to semicircle RBT.SRT is a straight line. The arc measures (not lengths!) are indicated in the figure:

$$a = 122^\circ, b = 131^\circ, c = 58^\circ, d = 49^\circ.$$

Find $\angle APB$ in degrees.



Problem 22.6.17. (*1 point*) The club has 85 students, each of whom plays on at least one of the sports teams: football, baseball, and volleyball. 74 students are on the football; 26 are on the baseball; and 17 are on both the football and volleyball teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and volleyball teams. Compute the number of students playing all three sports, given that twice this number are members of the volleyball team.

Problem 22.6.18. (*1 point*) A ball of radius R is tangent to the floor and one wall of the room. The radius of the largest sphere that can be rolled through the space between the ball, the wall, and the floor, is $R(m - \sqrt{n})$, where m, n are positive integers. Find $m + n$.

Problem 22.6.19. (*1 point*) Find the last three digits of 9^{2024}.

Problem 22.6.20. (*1 point*) A coin is flipped 24 times in a row. Given that exactly 20 heads appeared, the probability that no two consecutive coin flips were both tails is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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22.7 High School - Test 6

You have **90 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 11-20**
- **Olympiad (O) level: Problems 21-30**

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Problem 22.7.1. (*1 point*) Joan has a collection of nickels, dimes, and quarters worth \$2.00. If the nickels were dimes and the dimes were nickels, the value of the coins would be \$1.70. Determine the largest possible number of the dimes.

Problem 22.7.2. (*1 point*) In rectangle $ABCD$, P is on side BC and Q is on side DC so that $BP = 1$, $AP = PQ = 2$, and $\angle APQ = 90°$. The length of QD is $\sqrt{m} - n$, where m and n are positive integers. Find $2m + 3n$.

Problem 22.7.3. (*1 point*) Determine the sum of all real numbers x such that:

$$\sqrt{2 + 4x - 2x^2} + \sqrt{6 + 6x - 3x^2} = x^2 - 2x + 6$$

Problem 22.7.4. (*1 point*) Temperature can be measured in degrees Fahrenheit (F) or in degrees Celsius (C). The two scales are related by the formula $F = 1.8C + 32$. When a two-digit integer degree temperature in Celsius is converted to Fahrenheit and rounded to the nearest integer degree, it turns out the ones and tens digits of the original Celsius temperature sometimes switch places to give the rounded Fahrenheit equivalent. Find the sum of all two-digit integer values of C for which this occurs.

Problem 22.7.5. (*1 point*) The positive odd integers are arranged in a triangular diagram as shown below. Find the sum of the numbers in the seventeenth row.

$$\begin{array}{cccc} & & & 1 \\ & & 3 & 5 \\ & 7 & 9 & 11 \\ 13 & 15 & \dots & \dots \end{array}$$

Problem 22.7.6. (*1 point*) In the cyclic quadrilateral $ABCD$, $AB = 13, BC = 5, CA = 12$, diagonal AC bisects the angle DAB. The side AD is extended beyond D to a point E such that $CE = 12$. The length of AE is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 22.7.7. (*1 point*) Find the sum of all y numbers whereas x, y are positive integers satisfying

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{12}.$$

Problem 22.7.8. (*1 point*) Find the maximum value of $(x^3 + 1)(y^3 + 1)$ if x and y are real numbers such that $x + y = 1$.

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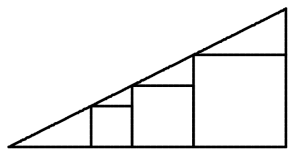
Problem 22.7.9. (*1 point*) Three squares are placed side-by-side inside a right-angled triangle as shown in the diagram. The side length of the smallest of the three squares is 16. The side length of the largest of the three squares is 36. What is the side length of the middle square?



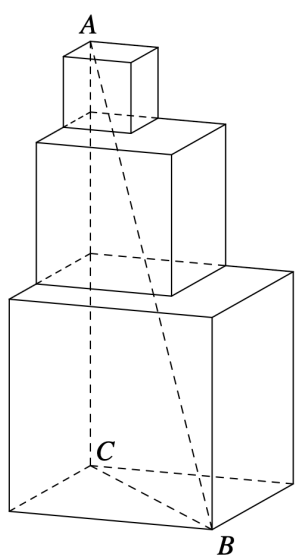
Problem 22.7.10. (*1 point*) Friends Maya and Naya ordered finger food in a restaurant, Maya ordering chicken wings and Naya ordering bite-size ribs. Each wing cost the same amount, and each rib cost the same amount, but one wing was more expensive than one rib. Maya received 20% more pieces than Naya did, and Maya paid 50% more in total than Naya did. The price of one wing was what percentage higher than the price of one rib?

Problem 22.7.11. (*1 point*) The value of the sum

$$\sum_{k=1}^{2024} \sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}}$$

is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 22.7.12. (*1 point*) Three cubes are stacked as shown. The cubes have edge lengths 1, 2, and 3. $m\sqrt{n}$ is the length of the portion of segment AB that is contained in the center cube, where m and n are positive integers and n is not divisible by any perfect square larger than 1. Find $5m + 2n$.



Problem 22.7.13. (*1 point*) E is an ellipse with the lengths of minor and major axis are 21 and 128. ABC is a triangle inscribed in E. The largest possible area of ABC is $m\sqrt{n}$ where m and n are positive integers and n is not divisible by any perfect square larger than 1. Find $m + n$

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Problem 22.7.14. (*1 point*) Consider triangle ABC with medians AD, BE, and CF. If the area of $\triangle ABC$ is 12, what is the area of the triangle $D'E'F'$, where D', E', and F' are the reflections of D, E, and F through A, B, and C respectively?

Problem 22.7.15. (*1 point*) For every n the sum of the first n terms of an arithmetic progression is $2n+3n^2$. What is the 2024th term of the sequence?

Problem 22.7.16. (*1 point*) How many 7-digit numbers, written in decimal notation, have at least one 1, one 2, and one 3 among their digits?

Problem 22.7.17. (*1 point*) The equation $x^4-16x^3+94x^2+px+q=0$ has two double roots. Find $(p+q)^2$.

Problem 22.7.18. (*1 point*) Find the volume of the region in space defined by $|x+y+z|+|x+y-z|\leq 12$, where x, y, $z\geq 0$.

Problem 22.7.19. (*1 point*) Triangle ABC has side lengths $AB=6, AC=5$, and $BC=4$. A point P in the interior of $\triangle ABC$ is a distance x from BC, a distance y from AC, and a distance z from AB.

If $x^2+y^2+z^2=\dfrac{225}{44}$, then $x=\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by any perfect square larger than 1. Find $m+n+p$.

Problem 22.7.20. (*1 point*) In how many ways can a 3×12 rectangle be tiled with tiles of size 1×2?

Problem 22.7.21. (*1 point*) If $f\left(\frac{x}{x-1}\right)=\frac{1}{x}$ for all $x\neq 0,1$ and $0<\theta<\frac{\pi}{2}$, then evaluate

$$\frac{f(\sec^2(\theta))}{\sin^2(\theta)}.$$

Problem 22.7.22. (*1 point*) Find the sum of all pq where p,q are primes such that $p^2+5pq+4q^2$ is a square of an integer.

Problem 22.7.23. (*1 point*) Consider a sequence $\{G_n\}$ with $G_{k+1}=G_k+2G_{k-1}$ for $k\geq 1$, with $G_0=0$ and $G_1=1$. Find $\lfloor \log_{256}(3G_{2024}+1)\rfloor$.

Problem 22.7.24. (*1 point*) Let $ABCD$ be a convex quadrilateral such that $AB+BC=2021$ and $AD=CD$. We are also given that $\angle ABC=\angle CDA=90^o$.

The length of the diagonal BD is $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by any perfect square larger than 1. Find $m+n+p$.

Problem 22.7.25. (*1 point*) a, b, c, d are the solutions of the equation $x^4-38x-3=0$. Let $P(x)$ be the polynomial with leading coefficient 3 whose roots are

$$\frac{a+b+c}{d^2},\ \frac{a+b+d}{c^2},\ \frac{a+c+d}{b^2},\ \text{and}\ \frac{b+c+d}{a^2}.$$

Find $P(13)$

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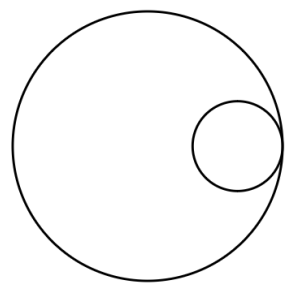
Problem 22.7.26. (*1 point*) Let $A = \{1, 2, 3, \ldots, 16\}$. Find the number of subsets B of A, such that if the sum of two elements of A is a power of 2 then exactly one of them belong to B.

Problem 22.7.27. (*1 point*) A mouse stands at the corner A of a regular ABCD tetrahedron. In a move, the mouse can choose an edge going out from the corner where he stands and move to a different corner. He repeats until he goes back to A. In how many ways can he can end up at A after 8 moves?

Problem 22.7.28. (*1 point*) Two circles are internally tangent. Their radii are 1 and 3. A triangle is formed with one vertex on each circle (not at the tangent point) and the third at the tangent point. The maximum area of such a triangle is $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by any perfect square larger than 1. Find $m + n + p$..



Problem 22.7.29. (*1 point*) In how many ways can we choose 4 vertices of a convex 17-gon to form a convex quadrilateral, such that at least 1 side of the quadrilateral is a side of the 17-gon?

Problem 22.7.30. (*1 point*) A bottle in the shape of a cone lies on its base. Water is poured into the bottle until its level reaches a distance of 8 centimeters from the vertex of the cone (measured vertically). We now turn the bottle upside down without changing the amount of water it contains. This leaves an empty space in the upper part of the cone that is 2 centimeters high.

The height of the bottle is $m + \sqrt{n}$, where m and n are positive integers, and n is not divisible by any perfect square larger than 1. Find $m + n$

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Chapter 23

Club Championships

23.1 Club Championship Individual - Test 1

Rules:

You have **60 minutes** to complete the test. You can submit **the answers, partial-, or full solutions**. Each submission is graded accordingly.

- A correct answer is worth of maximum 1 point.
- A partial solution is worth at most 5 points.
- A full solution is worth 10 points.

Important: if a problem has several questions, students must solve the questions in the order of encounter. Students earn **no points** when solving a later question without an early one.

- **Pre-Algebra (PA) level: Problems 1-4**
- **Introductory Curriculum (IC) level: Problems 5-8**
- **Intermediate (I) level: Problems 9-12**
- **Advanced (A) level: Problems 13-16**
- **Olympiad (O) level: Problems 17-20**

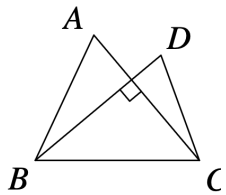
No coding solution is accepted.

Important: Read the rules before proceed!

Problem 23.1.1. (*10 points*) How many odd numbers with middle digit 5 are there between 40000 and 69999 inclusive, with no digits repeated?

Problem 23.1.2. (*10 points*) Adult tickets to a football game were \$3.25 and student tickets were \$1.75. If 1350 fans paid a total of \$2700 to attend the game, how many adults attended?

Problem 23.1.3. (*1 point*) ABC is an isosceles triangle such that $AC = BC$. CBD is an isosceles triangle such that $CB = DB$. BD meets AC at a right angle. If $\angle A = 57°$, what is $\angle D$?



Problem 23.1.4. (*10 points*) On an island, the natives are either knights, who always tell the truth, or liars, who always lie.

Alice and Bob were chatting

Alice: We are both liars, and this is the island of Maya.

Bob: That is true.

Is this the island of Maya? Explain.

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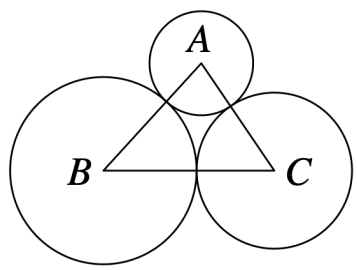
Important: Read the rules before proceed!

Problem 23.1.5. (*10 points*) Alice, Bob, Carol, and Denis were detained for questioning. They were suspected of stealing cookies. Inspector Erica established the following facts:

- (1) If both Alice and Bob are guilty, then Carol was an accomplice
- (2) If Alice is guilty, then at least one of Bob, Carol was an accomplice.
- (3) If Carol is guilty, then Denis was an accomplice
- (4) If Alice is innocent, then Denis is guilty.

Which ones are **definitely** guilty and which ones are doubtful?

Problem 23.1.6. (*10 points*) Circle A, circle B, and circle C are externally tangent. $BC = a, AC = b$, and $AB = c$, Express the radius of circle A in terms of a, b, c.



Problem 23.1.7. (*10 points*) Find the 1275th term of the sequence 1, 3, 5,. . ., 15, 1, 3, 5,. . ., 15, 1, . . .

Problem 23.1.8. (*10 points*) The tens digit of a two-digit number exceeds its units digit by 4. The number exceeds twice the number obtained by reversing the digits of the original number by 10. What is the original number?

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Important: Read the rules before proceed!

Problem 23.1.9. a, b are positive integers. None of $a, b, a-b$ is a multiple of 3.

(*2 points*) Prove that $a+b$ is a multiple of 3.

(*3 points*) Prove that a^3+b^3 is a multiple of 9.

Problem 23.1.10. x, y are real numbers such that $x^2+xy+y^2=2$.

(*5 points*) Find the largest value of x^2-xy+y^2

(*5 points*) Find the smallest value of x^2-xy+y^2

Problem 23.1.11. Diagonal AC and BD of quadrilateral $ABCD$ meet at E. If $AE=2, BE=5, CE=10, DE=4$, and $BC=\frac{15}{2}$.

(*1 point*) Prove that $\triangle AED \sim \triangle BEC$.

(*3 points*) Prove that $ABCD$ is cyclic.

(*6 points*) Find AB.

Problem 23.1.12. (*1 point*) If (x_1, x_2) is a permutation of $(1, 2)$, find the largest value of

$$|x_1-1|+|x_2-2|$$

(*1 point*) If (x_1, x_2, x_3) is a permutation of $(1, 2, 3)$, find the largest value of

$$|x_1-1|+|x_2-2|+|x_3-3|$$

(*8 points*) If $(x_1, x_2, \ldots, x_n)$ is a permutation of $(1, 2, \ldots, n)$, find the largest value of

$$|x_1-1|+|x_2-2|+\cdots+|x_n-n|$$

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Important: Read the rules before proceed!

Problem 23.1.13. Consider the following inequation:

$$\sqrt{x-\frac{1}{x}}-\sqrt{1-\frac{1}{x}}>\frac{x-1}{x}.$$

(*1 point*) Prove that $x=1$ is not a solution.

(*3 points*) Prove that if x is a solution then $x>1$ or $-1\leq x<0$.

(*6 points*) Find all solutions.

Problem 23.1.14. Let $M=\{1,2,\ldots,2005\}$. Let A be a subset of M. It is known that for any $i,j\in A$, $(i\neq j)$, there is exactly one isosceles triangle with side lengths i,j.

(*1 point*) What are the side lengths of a isosceles triangle if two of its side lengths are i,j, if $i<j$?

(*1 point*) If $i,j,k\in A$, $(i<j<k)$ prove that $4i<k$.

(*8 points*) What is the maximum number of elements of A?

Problem 23.1.15. Let ℓ be a straight line. A,B,C, are distinct points on ℓ. P is a point not on ℓ.

Let O_1,O_2, and O_3 be the circumcircles of triangles PAB,PBC, and PCA, respectively.

(*3 points*) Prove that $\angle AO_1P=\angle CO_2P$

(*7 points*) Prove that P,O_1,O_2 and O_3 are concyclic (they are on the same circle).

Problem 23.1.16. a,b,c,d are positive integers such that $ab=cd$.

(*3 points*) Simplify

$$\frac{(a+c)(a+d)}{a+b+c+d}$$

(*7 points*) Prove that $a+b+c+d$ is not a prime.

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Problem 23.1.17. Let $ABCD$ be a convex quadrilateral with an incircle. Let ω_1, ω_2, ω_3, and ω_4 be the excirles of $ABCD$ tangent to segments AB, BC, CD, and DA, respectively.

(*5 points*) Let r_1, r_2, r_3, and r_4 be the radius of ω_1, ω_2, ω_3, and ω_4. Prove that

$$r_1 r_3 = r_2 r_4.$$

(*5 points*) Prove that the lengths of the internal common tangent segments to the pairs of circles (ω_1, ω_3) and (ω_2, ω_4) are equal.

Problem 23.1.18. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

(*2 points*)

$$ab + bc + ca \leq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

(*3 points*)

$$\frac{a}{a+5c} + \frac{b}{b+5a} + \frac{c}{c+5b} \geq \frac{1}{2}.$$

(*5 points*)

$$\frac{a}{b(a+5c)^2} + \frac{b}{c(b+5a)^2} + \frac{c}{a(c+5b)^2} \geq \frac{1}{4(\sqrt{a}+\sqrt{b}+\sqrt{c})}.$$

Problem 23.1.19. Let n be a positive integer. Let a, b be odd positive integers.

(*3 points*) Find all integer triples $(x.y, z)$ such that $x^3 = 2y^3 + 4z^3$.

(*7 points*) Find all integer pairs (m, n) such that

$$\text{each of } 6(m+1)(n-1),\ (m-1)(n+1)+6,\ (m+2)(n-2) \text{ is a perfect cube.}$$

Problem 23.1.20. (*2 points*) Find all pairs of positive integers such that $|n^2 - m^3| = 1$.

(*8 points*) For any two number sets A and B, define their sum and difference as

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\},\ A - B = \{a - b \mid a \in A \text{ and } b \in B\},$$

Let $P = \{2, 3, 5, \ldots\}$ be the set of primes and

$$S = \{0, 1, 8, 16, 27, 64, 81, 125, 216, 256, \ldots\} \text{ be the set of perfect cubes and fourth powers.}$$

Let n be positive integer such that $n \equiv 3 \pmod{7 \cdot 103}$,

Prove that $64n^{12} \notin (P + S) \cup (P - S) \cup (S - P)$.

23.2 Club Championship Team - Test 1

Rules:

You have **60 minutes** to solve **10** problems. Both Middle School and High School Teams solve the same problems. Any team member can choose any problem to work on. Several students can work on the same problem. However for each problem the whole team can submit a single answer. No individual answer is allowed.

The students can use any book, calculator, computer. The students can chat, talk, or use any mean of communications. They can sit next to each other if that is convenient.

No searching on the Internet is allowed. Nobody outside of the team can provide help.

Read carefully what submission (a number, a list of answers, a formula, a diagram, etc) is required by a problem. Each team submit a **single** image, pdf/excel file, or online spreadsheet containing the answers. It is the team responsibility that the answers are readable without any ambiguity. The submission must be done in time (within 60 minutes) and as instructed by the COs.

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Important: Read the rules before proceed!

Problem 23.2.1. (*10 points*) ABC is a right triangle, $\angle ACB = 90°$. D and E are points on AB such that $AD = DE = EB$. Furthermore $CD = 7, CE = 6$. What is the length of AB?

Problem 23.2.2. (*10 points*) In the Forest of Forgetfulness, the Lion lies on Mondays, Tuesdays, and Wednesdays and tells the truth on the other days of the week. On what days of the week is it possible for the Lion to make the following statement: *I lied yesterday and I will lie again tomorrow*?

Problem 23.2.3. (*10 points*) Find all distinct rectangular prisms with integer side lengths having the same volume of 36. For example $(33, 1, 1)$ is such a prism.

Note that $(33, 1, 1)$ *and* $(1, 33, 1)$ *are considered the same.*

Problem 23.2.4. A red bug sits at a vertex of a tetrahedron. At each step, she chooses an *unvisited* edge that leads from where she is to an *unvisited* vertex, then crawls along the edge to that vertex. She *stops* when all vertices are visited. The vertices and edges that she visited in this way form *a path*. It is easy to see that there is a number of different paths that she can crawl in order to visit all the vertices.

A blue bug is doing the same with a cube.

(*3 points*) How many distinct paths the red bug can crawl?

(*7 points*) How many distinct paths the blue bug can crawl?

Problem 23.2.5. (*10 points*) a, b are integers such that $x^2 + x - 1$ is a factor of $x^3 + 4x^2 + ax + b$. What is the value of $a + 10b$.

Problem 23.2.6. (*10 points*) In $\triangle ABC$ $AB = 2AC$. p is the perimeter of $\triangle ABC$. m, n are positive integers such that $m - n = 2$, and

$$\frac{p}{m} < AC < \frac{p}{n}.$$

What is the value of $m + n$?

Problem 23.2.7. (*10 points*) S is a set of positive integers relatively prime to 105. What is the 9999990th smallest element of S?

Problem 23.2.8. (*10 points*) Some blue and green marbles are in a bag. The probability to select any of them is the same for every marble. If two marbles are selected at random without replacement, the probability that both are blue is $\frac{1}{6}$. If three marbles are selected randomly without replacement, the probability that three are blue is $\frac{1}{21}$. What is the fewest number of marbles that must have been in the bag initially?

Problem 23.2.9. (*10 points*) Find all real numbers x, y, z, t such that:

$$\begin{aligned} x^2 + y^2 + z^2 + t^2 &= 50 \\ x^2 - y^2 + z^2 - t^2 &= -24 \\ xz &= yt \\ x - y + z + t &= 0 \end{aligned}$$

Problem 23.2.10. (*10 points*) Let $S = \{1, 2, 3, \ldots, 100\}$, and A be a subset of S. It is known that if $x, y \in A, x < y$, then $y \neq 3x$. Find the maximum number of elements in A.

23.3 Club Championship Individual - Test 2

Rules:

You have **60 minutes** to complete the test. You can submit **the answers, partial-, or full solutions**. Each submission is graded accordingly.

For a problem with multiple questions, each of the questions can be answered independently of others. You don't have to use the result of any preceding question(s) to solve the following ones. However if you use the result of the preceding questions to solve the following ones then you have to provide a correct solution for the preceding questions.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-4**
- **Advanced (A) level: Problems 5-8**
- **Olympiad (O) level: Problems 9-12**

No coding solution is accepted.

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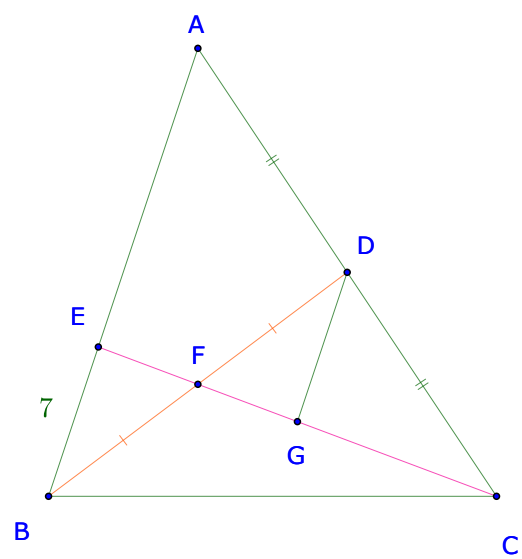
Problem 23.3.1. (*10 points*) Let $a > 1$ be a real number such that

$$22\left(a + \frac{1}{a}\right) = \sqrt{2017}$$

Evaluate $a - \dfrac{1}{a}$.

Problem 23.3.2. (*10 points*) Find the remainder when divided $(257^{33} + 46)^{26}$ by 50.

Problem 23.3.3. (*10 points*) In $\triangle ABC$, BD is the median, E is a point of AB such that CE bisects BD at F. If $BE = 7$, find AB.



Problem 23.3.4. (*10 points*) There are two bags of marbles. Bag A contains four black and three white marbles. Bag B contains three black and six white marbles.

Five marbles were selected randomly from bag A and placed into bag B. One marble is then selected randomly from bag B.

What is the probability that the marble selected from bag B is white?

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Problem 23.3.5. Let x be a root for the equation:

$$2x\lfloor x\rfloor + 2\{x\} = 2017,$$

where $\lfloor x\rfloor$ denotes the largest integer not exceeding x, for example $\lfloor 1.1\rfloor = 1, \lfloor -1.2\rfloor = -2$. Furthermore $\{x\} = x - \lfloor x\rfloor$.

(*2 points*) Prove that if $x \geq 0$, then:

$$2\left(\lfloor x\rfloor\right)^2 \leq 2x\lfloor x\rfloor + 2\{x\} \leq 2x^2 + 2.$$

(*2 points*) Prove that if $x < 0$, then:

$$2x^2 \leq 2x\lfloor x\rfloor + 2\{x\} \leq 2\left(\lfloor x\rfloor\right)^2 + 2.$$

(*6 points*) Find all values for x.

Problem 23.3.6. Let $a < b < c$ be the real roots of the equation

$$x^3 - 3x + 1 = 0.$$

(*2 points*) Let $\alpha = \frac{1}{1+a}$. Prove that α is a root of the equation

$$3x^3 - 3x + 1 = 0.$$

(*3 points*) Find the value of

$$\frac{1-a}{1+a} + \frac{1-b}{1+b} + \frac{1-c}{1+c}$$

(*5 points*) Write an equation with the three roots $a^2 - 2$, $b^2 - 2$, $c^2 - 2$.

Problem 23.3.7. (*2 points*) x, y, z are positive real numbers. Prove that:

$$8xyz \leq (x+y)(y+z)(z+x).$$

(*3 points*) a, b, c are side lengths of a triangle. Prove that:

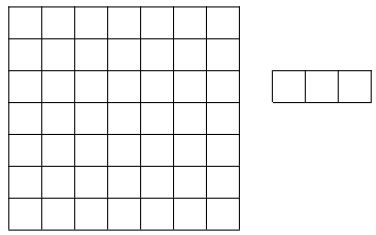
$$(a+b-c)(b+c-a)(c+a-b) \leq abc.$$

(*5 points*) a, b, c are side lengths of a triangle. Prove that:

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc$$

Hint: Use result of the first question for the second and the third.

Problem 23.3.8. You are given a 7×7 board and severals 1×3 trominos as shown below.



(*2 points*) Prove that it is possible to tile the board with the trominos by removing a single square. Show a tiling.

(*8 points*) Which single square can be removed from the board so that the rest can be tiled with trominos?

Hint: To do that, try to fill each of the 1×1 *squares in the board with one of the numbers* $0, 1,$ *and* 2, *then use the sums of the numbers in each of the trominos to arrive at the conclusion which squares can be removed.*

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Problem 23.3.9. n is a positive integer such that

$$\sigma(n) = n + 55, \text{ where } \sigma(n) \text{ is the sum of all divisors of } n, \text{ including 1 and } n.$$

(*1 point*) Prove that the number of distinct prime divisors of n is more than 1.

(*2 points*) Prove that the number of distinct prime divisors of n is less than 4.

(*2 points*) Prove that the number of distinct prime divisors of n cannot be 3.

(*5 points*) Find all n satisfying the given equation.

Problem 23.3.10. $P(x)$ is an odd degree polynomial with real coefficients. The coefficient of x^2 in $P(x)$ is zero. For all real number x:

$$P(P(x)) \leq P(x)^3.$$

(*5 points*) Prove that the degree of $P(x)$ is 3.

(*2 points*) Find the value of the leading coefficient (the coefficient of the highest power of x) in $P(x)$.

(*3 points*) Find all such $P(x)$.

Problem 23.3.11. In $\triangle ABC$, $AB = c, BC = a, CA = b$, m_a, m_b, m_c are medians (connecting A, B, C with midpoints of BC, CA, AB, respectively.)

(*3 point*) Denote $s = \frac{1}{2}(a + b + c)$. Prove that:

$$m_a \geq \sqrt{s(s-a)}$$

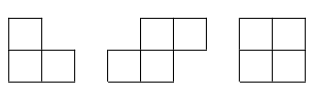
(*3 points*) Prove that:

$$m_a m_b \leq \frac{ab}{4} + \frac{c^2}{2}$$

(*4 points*) Prove that

$$cm_c \leq \frac{am_b + bm_a}{2}.$$

Problem 23.3.12. You are given a $(2n+1) \times (2n+1)$ board and as many as needed pieces of three kinds shown as below:



(*2 point*) Number the columns left-to-right and rows top-to-bottom from 1 to $2n+1$. Colour in black the squares in odd rows and odd columns. Colour in white the rest. What is the number of black squares? Of white squares?

(*3 point*) Let a be the number of pieces of the first kind (from left to right) that cover *one* black square, b be the number of pieces of the first kind that cover *no* black square, c be the number of pieces of the other two kinds. Express the number of white squares in terms of a, b, and c only.

(*5 point*) Prove that if the given board can be tiled with the pieces above, then at least $4n + 3$ pieces of the first kind will be used.

23.4 Club Championship Team - Test 2

Rules:

You have **60 minutes** to solve **10** problems. Both Middle School and High School Teams solve the same problems. Any team member can choose any problem to work on. Several students can work on the same problem. However for each problem the whole team can submit a single answer. No individual answer is allowed.

The students can use any book, calculator, computer. The students can chat, talk, or use any mean of communications. They can sit next to each other if that is convenient.

No searching on the Internet is allowed. Nobody outside of the team can provide help.

Read carefully what submission (a number, a list of answers, a formula, a diagram, etc) is required by a problem. Each team submit a **single** image, pdf/excel file, or online spreadsheet containing the answers. It is the team responsibility that the answers are readable without any ambiguity. The submission must be done in time (within 60 minutes) and as instructed by the COs.

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Important: Read the rules before proceed!

Problem 23.4.1. In the following, different letters represent different digits.

$$\begin{array}{rccccc} & & & O & N & E \\ + & & F & O & U & R \\ \hline & & F & I & V & E \end{array} \qquad \begin{array}{rccccc} & & & O & N & E \\ \times & & & & & 4 \\ \hline & & F & O & U & R \end{array}$$

(*3 points*) What digit the letter O represents?

(*7 points*) What is the number represented by FIVE?

Problem 23.4.2. Each of three coloured cups covers each of three objects.

1. The red cup is somewhere to the left of the white cup.
2. The coin is somewhere left of the bean.
3. The gray cup is somewhere right of the shell.
4. The bean is somewhere right of the gray cup.

(*5 points*) Under which cup is the shell?

(*5 points*) What is the order of the objects from left to right?

Problem 23.4.3. The fraction $\frac{1}{4} = 0.25$ is a decimal written with two decimal places. The fraction $\frac{3}{8} = 0.375$ is a decimal written with three decimal places. Note that the last digit in the decimal must not be a zero.

(*5 points*) Find the number of fractions $\frac{p}{q}$, where p and q are relatively primes, that are decimals written with *two* decimal places, so that

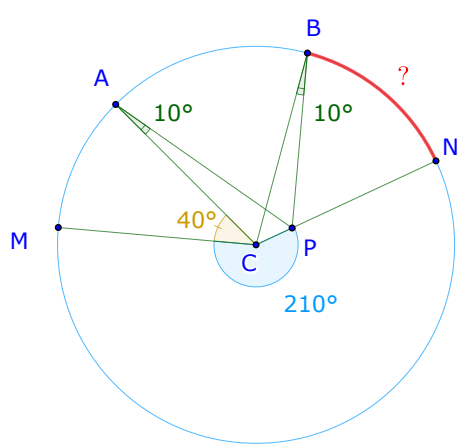
$$\frac{1}{4} \le \frac{p}{q} \le \frac{3}{8}.$$

(*5 points*) Find the number of fractions $\frac{p}{q}$ where p and q are relatively primes, that are decimals written with *three* decimal places, so that

$$\frac{1}{4} \le \frac{p}{q} \le \frac{3}{8}.$$

Problem 23.4.4. M, A, B, N are on the circle centred at C such that $\angle MCN = 210°$, $\angle MCA = 40°$. P is a point on CN such that $\angle CAP = \angle CBP = 10°$.

(*10 points*) What is the measure of the arc $\widehat{BN}$ in degrees?



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Problem 23.4.5. (*10 points*) α is a real number. a and b are the real roots of the equation

$$x^2 + 4x - 480 = \alpha.$$

Find the minimum real root of the equation

$$x^2 - (a+b)x + ab + \alpha = 0.$$

Problem 23.4.6. Two ferries carry passengers over a river. They travel across the river on the same route, back and forth.

At 6AM the red ferry starts from the north bank of the river and repeats the following until 9:30PM: (i) it goes from the north bank to the south bank in 90 minutes; then (ii) it spends 30 minutes on the south bank; then (iii) it takes 90 minutes to go to the north bank; and then (iv) it waits 30 minutes at the north bank.

At 6AM the green ferry starts from the south bank of the river and repeats the following until 9PM: (i) it goes from the south bank to the north bank in 60 minutes; then (ii) it spends 60 minutes on the north bank; then (iii) it takes 60 minutes to go to the south bank; and then (iv) it waits 60 minutes at the south bank.

(*5 points*) How many times the two ferries cross each other, i.e. one goes in a direction while the other in the opposite one?

(*5 points*) In total how many minutes the ferries wait at the opposite banks at the same time, i.e. one waits at the north bank while the other at the south bank?

Problem 23.4.7. Let $S = \{1, 2, 3, \ldots, 2024\}$. M is called a *good* $k-$element $(1 < k < 1012)$ subset of S, if:

1. $|M| = k$, in other words M has k elements;
2. For any $a, b \in M$ then $|a - b| \neq 1$, in other words M does not contain two consecutive integers.

(*2 points*) How many distinct *good* $2-$element subsets are there?

(*8 points*) For $1 < k < 1012$, how many distinct *good* $k-$element subsets are there?

Problem 23.4.8. (*10 points*) a, b, c, d, and e are arbitrary real numbers. Find the maximum value of M such that the following inequality always stands:

$$a^2 + b^2 + c^2 + d^2 + e^2 \geq M(ab + bc + cd + de).$$

Problem 23.4.9. (*10 points*) Let p be an odd prime. Find all integers k such that $\sqrt{k^2 - pk}$ is a positive integer.

Problem 23.4.10. There are m teachers and 2024 students in a school. We know that:

1. Each teacher teaches exactly 100 students;
2. For any two distinct students, there are exactly n teachers teaching them.

(*2 points*) If a teacher t teaches two student $s_1 \neq s_2$ then they form a triples (t, s_1, s_2). How many triples (t, s_1, s_2) are there?

(*8 points*) Find $\frac{m}{n}$.

23.5 Club Championship Individual - Test 3

Rules:

You have **60 minutes** to complete the test. You can submit **the answers, partial-, or full solutions**. Each submission is graded accordingly.

For a problem with multiple questions, each of the questions can be answered independently of others. You don't have to use the result of any preceding question(s) to solve the following ones. However if you use the result of the preceding questions to solve the following ones then you have to provide a correct solutions for the preceding questions.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-4**
- **Advanced (A) level: Problems 5-8**
- **Olympiad (O) level: Problems 9-12**

No coding solution is accepted.

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Problem 23.5.1. (*10 points*) Three aliens and three humans want to get from the moon they are on to another moon close at hand. They must cross between the moons in a space shuttle that will only hold two creatures at a time. Wherever they are, if the aliens outnumber the humans, they will do unspeakable things to them.

How can all three aliens and all three humans cross to the new moon safely?

Problem 23.5.2. a, b, c, d are positive real numbers

(*3 points*) Prove that:

$$\frac{a^2}{b} + \frac{b^2}{a} \geq a + b.$$

(*7 points*) Prove that:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{a} \geq a + b + c + d.$$

Problem 23.5.3. (*2 points*) In an arbitrary $\triangle ABC$, $AB = c, BC = a, CA = b$. Prove that:

$$\frac{a}{b+c} < \frac{2a}{a+b+c}.$$

(*3 points*) In an arbitrary $\triangle ABC$, $AB = c, BC = a, CA = b$. Prove that:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

(*5 points*) Find a $\triangle ABC$ with side lengths $AB = c, BC = a, CA = b$, such that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} > 1.9$$

Problem 23.5.4. The numbers $1, 2, \ldots, 10$ are written on a board. In an operation, any two numbers a and b can be erased and replaced with the number $a - b$.

(*2 points*) Is that possible to have only the numbers 1, 2, 3, 4, 5 remain on the board?

(*8 points*) Is that possible to have only the number 0 remains on the board?

Problem 23.5.5. (*2 points*) In below there are two subtractions, each with two three-digit numbers.

$$\begin{array}{rccc} & 4 & 0 & 0 \\ - & a & b & 4 \\ \hline \end{array} \qquad \begin{array}{rccc} & 4 & a & b \\ - & 4 & 0 & 0 \\ \hline \end{array}$$

Find digits a, b in the following knowing that the results are the same for both subtractions.

(*2 points*) In below there are two subtractions, each with two 2025-digit numbers.

$$\begin{array}{rccccc} & 4 & 0 & \dots & 0 & 0 \\ - & a_{2023} & a_{2022} & \dots & a_0 & 4 \\ \hline \end{array} \qquad \begin{array}{rccccc} & 4 & a_{2023} & \dots & a_1 & a_0 \\ - & 4 & 0 & \dots & & 0 \\ \hline \end{array}$$

Digits $a_0, a_1, a_2, \dots, a_{2023}$ are such that the results are the same for both subtractions.

Find the sum $a_0 + a_1 + \cdots + a_{2023}$.

Problem 23.5.6. (*5 points*) Let $\alpha, \beta, \gamma, \delta \in \left(0, \frac{\pi}{2}\right)$ such that:

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = 1.$$

Prove that:

$$\tan\alpha\tan\beta\tan\gamma\tan\delta \leq \frac{1}{9}$$

(*5 points*) a, b, c, d are positive real numbers such that:

$$\frac{a^2}{1+a^2} + \frac{b^2}{1+b^2} + \frac{c^2}{1+c^2} + \frac{d^2}{1+d^2} = 1$$

Prove that:

$$9abcd \leq 1.$$

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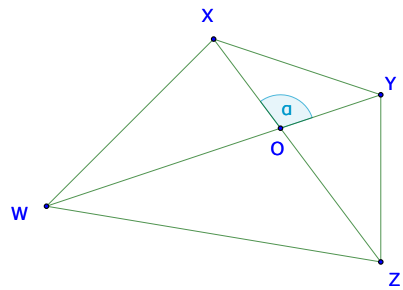
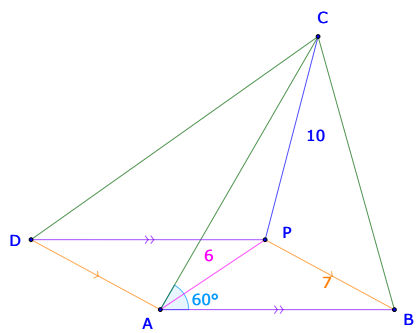
Problem 23.5.7. (*3 points*) $XZYW$ is a convex quadrilateral. Diagonals XZ and YW intersect at O. Let $\angle XOY = \alpha$.



Prove that:

$$-XY^2 + YZ^2 - ZW^2 + WX^2 = 2 \cdot XZ \cdot YW \cdot \cos\alpha.$$

(*3 points*) In $\triangle ABC$, $\angle BAC = 60°$, P is a point inside triangle such that $PA = 6, PB = 7, PC = 10$. D is a point outside of the triangle such that $PD \parallel AB, AD \parallel PB$.



Prove that:

$$CD^2 - 6CD - 183 \leq 0.$$

(*4 points*) Conditions are given in the previous question. Now, prove that

$$[ABC] \leq 36 + 22\sqrt{3}.$$

Problem 23.5.8. If a_1, a_2, a_3, a_4, a_5 is a permutation of $1, 2, 3, 4, 5$ satisfying that for any $i = 1, 2, 3, 4$, each permutation of $a_1, a_2, \ldots, a_i$ is *not* a permutation of $1, 2, \ldots, i$, then we call $(a_1, a_2, a_3, a_4, a_5)$ a *good* permutation.

(*1 points*) Find the number of good permutations for $a_1 = 5$.

(*2 points*) Find the number of good permutations for $a_1 = 4$.

(*7 points*) Find the number of (all) good permutations.

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Problem 23.5.9. a, b, c are non-negative real numbers such that $a + b + c = 1$.

(*2 point*) Let $q = ab + bc + ca$, $r = abc$. Prove that:

$$\begin{cases} a^3 = a^2 - qa + r \\ b^3 = b^2 - qb + r \\ c^3 = c^2 - qc + r \end{cases}$$

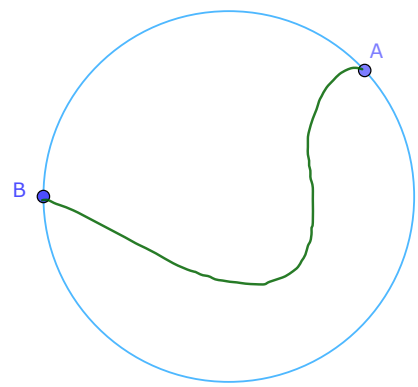
(*4 point*) Prove that:

$$2 \leq (1 - a^2)^2 + (1 - b^2)^2 + (1 - c^2)^2.$$

(*4 point*) Prove that:

$$(1 - a^2)^2 + (1 - b^2)^2 + (1 - c^2)^2 \leq (1 + a)(1 + b)(1 + c).$$

Problem 23.5.10. A, B are two points on the circle. A curve through A, B bisects the area of the circle. Prove that the curve is at least as long as a diameter of the circle.



Problem 23.5.11. In $\triangle ABC$, $AB = c, BC = a, CA = b$. r_a, r_b, r_c are radius of the excircles corresponding to sides BC, CA, AB.

(*2 point*) Prove that if $x = -a + b + c, y = a - b + c, z = a + b - c$, then

$$r_a = \frac{1}{2x}\sqrt{(xyz)(x + y + z)}.$$

(*3 points*) Prove that:

$$\frac{(y + z)^2}{y^2 + z^2} \geq \frac{2xyz(x + y + z)}{x^2y^2 + y^2z^2 + z^2x^2}$$

(*5 points*) Prove that

$$\frac{a^2}{r_b^2 + r_c^2} + \frac{b^2}{r_c^2 + r_a^2} + \frac{c^2}{r_a^2 + r_b^2} \geq 2.$$

Problem 23.5.12. Two players Alice and Bob play a game. Alice goes first then Bob and they alternate.

At the beginning there are 101 numbers 1, 2, . . . , 101 written on a board. At a turn, the player of that turn deletes 9 numbers among the remaining numbers.

After 11 turns, there are two numbers a and b left. The number of points Alice has is the positive difference of a and b.

Prove that there is a way for Alice to reach at least 55 points at the end of the game, regardless what Bob is doing.