Outline

Problem

1. *(10 points) In the forest of Hobbiton*
   - *There are* 15 *chameleons.*
   - 5 *of them green,* 2 *blue, and* 8 *red.*
   - *Whenever two chameleons of different colors meet, they both change to the third color (i.e., a green and blue would both become red).*

   Is it possible for all chameleons to become one color?

2. *(15 points) In the Fangorn Forest:*
   - *There are* 102 *chameleons.*
   - 19 *of them green,* 25 *blue,* 28 *red, and* 30 *orange.*
   - *Whenever three chameleons of different colors meet, they all change to the fourth color (i.e., a green, blue, red and would all become orange).*

   Is it possible for all chameleons to become one color?

Problem

1. *(10 points) In the forest of Hobbiton*
    - *There are 15 chameleons.*
    - *5 of them, 2 blue, and 8 red.*
    - *Whenever two chameleons of different colors meet, they both change to the third color (i.e., a green and blue would both become red).*

Is it possible for all chameleons to become one color?

- Note that we can *reduce both numbers of green* and *red chameleons to* 0 **if they are equal.**
- Each time, *a number of chameleons in a colour is increased by* 2 *and decreased in the other two colours.* In order to to that, we need to increase 5 to 7 and 8 to 7 in order to have **the equal numbers of green and red chameleons.**
- First, one blue and one red chameleons meet. Both of them become green. Thus, the number of green chameleons now is $5 + 2 = 7$, The number of blue and red chameleons are $2 - 1 = 1$ and $8 - 1 = 7$, respectively.
- Now, 7 green and 7 red chameleons meet in pairs and become 14 blue chameleons. All $1 + 14 = 15$ chameleons are now blue.

Thus, it is $\boxed{\text{possible}}$ that all 15 chameleons become blue.

Problem

1. *(10 points) In the forest of Hobbiton*
   - *There are* 15 *chameleons.*
   - 5 *of them,* 2 *blue, and* 8 *red.*
   - *Whenever two chameleons of different colors meet, they both change to the third color (i.e., a green and blue would both become red).*

Is it possible for all chameleons to become red?

- The key here is the remainders of the chameleons in a colour when divided by 3. If 15 of them becomes one colour, then at the beginning the number of chameleons in any colour should be divisible by 3.
- At the start, the remainders of the numbers of the chameleons in a colour are $5 \equiv 2,$, $2 \equiv 2,$ and $8 \equiv 2 \pmod 3.$ This means that by choosing any two colours, the remainders of their numbers are the same. We show that it is possible for all of them to be red.
- First, we want **the numbers of chameleons in green and blue are the same.** Thus, 1 green and 1 red chameleon meet, both become blue. The number of green chameleons now is $5 - 1 = 4,$ The number of blue and red chameleons are $2 + 2 = 4$ and $8 - 1 = 7,$ respectively.
- Finally, 4 green and 4 blue chameleons meet in pairs and become 8 red chameleons. All $7 + 8 = 15$ chameleons are now red.

Thus, it is $\boxed{\text{possible}}$ that all 15 chameleons become red.

Problem

1. *(10 points) In the forest of Hobbiton*
    - *There are* 15 *chameleons.*
    - 5 *of them,* 2 *blue, and* 8 *red.*
    - *Whenever two chameleons of different colors meet, they both change to the third color (i.e., a green and blue would both become red).*

Is it possible for all chameleons to become green?

Problem

2. *(15 points) In the Fangorn Forest:*
   - *There are* 102 *chameleons.*
   - 19 *of them green,* 25 *blue,* 28 *red, and* 30 *orange.*
   - *Whenever three chameleons of different colors meet, they all change to the fourth color (i.e., a green, blue, red and would all become orange).*

   Is it possible for all chameleons to become one color?

- After every time three different coloured chameleons meet, the numbers of chameleons in each colour is reduced by 1, while the number of chameleons in the fourth colour is increased by 3. Therefore **the parity of the number of chameleons in each colour is changed** after every time three different coloured chameleons meet. *This property is called* **the invariant,** *because it does not change after any number of operations.*
- At the beginning there are two odd (19, 25) and two even numbers (28, 30) of chameleons in each colour. Thus, it does not matter how the chameleons meet, there are always *two groups of chameleons with odd totals*, and *the other two groups with even totals.*
- Reaching the end-state, where all 102 chameleons are in one colour, and no chameleons in any of the remaining three colours, means all groups of chameleons haing even totals. This is impossible.

Therefore, $\boxed{\text{it is not possible}}$ for all chameleons to become one color.

The property that *there are always two groups of chameleons with odd totals, and two other groups with even totals* after every step, is called **the invariant.** This is imporant for inspect whether an *end state* (all chameleons in one colour) **is reachable** from a *start state* (19 of them green, 25 blue, 28 red, and 30 orange) which in this case, is not.

Example (MTC 2021/Mar/J3)

At the court of Princess Linh-Chi, dukes, counts, and barons started duelling each other. Anyone who lost a duel cannot participate in another duel. The princess observed that however strangely it was, only dukes can defeat counts, only counts can defeat barons, and only barons can defeat dukes. Furthermore no one won in more than one duel. In the end, only Duke Bill was left alive and victorious.

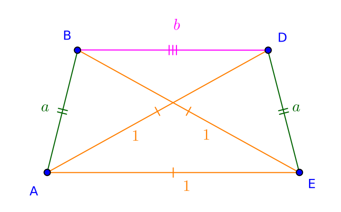
What was the title of the first victim if originally Princess Linh-Chi had 100 courtiers?

Problem

a and b are positive real numbers such that,

$$\begin{cases} a^2 + b = 1 \\ ab + b^2 = 1. \end{cases}$$

In quadrilateral $ABCD$, $AB = DE = a$, $BD = b$, $AE = AD = BE = 1$.

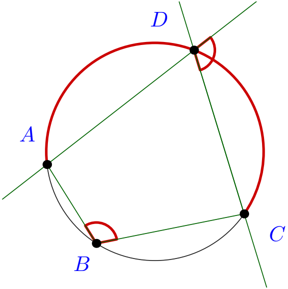


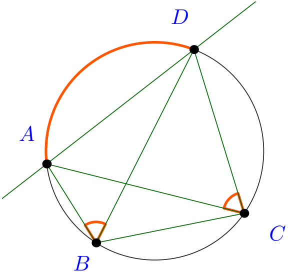
1. *(10 points) Prove that there is a circle Ω pass through four points A, B, D, and E.*
2. *(5 points) Let F be such that $AF = DF = b$, and $BF = 1$. Prove that F is on the circle Ω.*
3. *(10 points) Find measures of the angles of the triangle $\triangle DEF$.*

Theorem

Let $ABCD$ be a convex quadrilateral. Then the following are equivalent:

1. *$ABCD$ is cyclic.*
2. $\angle ABC + \angle CDA = 180^\circ$.
3. $\angle ABD = \angle ACD$.



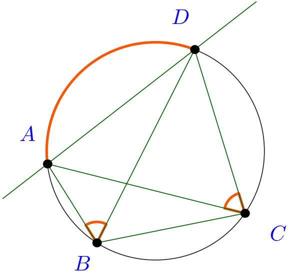


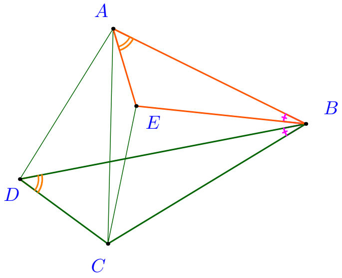
Theorem (Ptolemy Inequality)

The inequality states that in for four points A, B, C, D in the plane,

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD,$$

with equality for any cyclic quadrilateral $ABCD$ with diagonals AC and BD.





Proof.

Let E be the point such that $\angle EAB = \angle CDB$, $\angle EBA = \angle CBD$, therefore

$$\triangle AEB \sim \triangle DCB \Rightarrow \triangle ADB \sim \triangle CEB \Rightarrow \frac{AE}{CD} = \frac{AB}{BD}, \frac{CE}{AD} = \frac{BC}{BD}$$

$$AC \leq AE + CE = \frac{AB \cdot CD + BC \cdot AD}{BD} \Rightarrow AC \cdot BD \leq AB \cdot CD + BC \cdot AD$$

The equality stands if and only if E is on AC, so $\angle CAB = \angle EAB = \angle CDB$, $ABCD$ is cyclic. $\square$

1. (*10 points*) Prove that there is a circle Ω pass through four points A, B, D, and E.

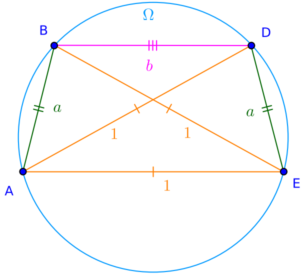


Figure: *ABDE* is cyclic

$$AB \cdot DE + BD \cdot AE = a^2 + b = 1 = AD \cdot BE,$$

By the Ptolemy Theorem, *ABDE* is a cyclic quadrilateral,

2. (*5 points*) Let F be such that $AF = DF = b$, and $BF = 1$. Prove that F is on the circle Ω.

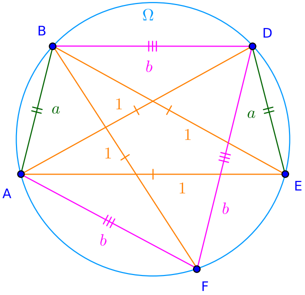
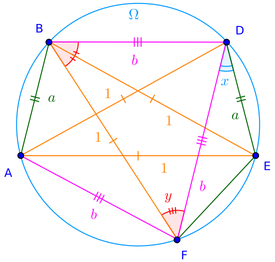


Figure: *ABDF* is cyclic

$$AB \cdot DF + BD \cdot AF = ab + b^2 = 1 = AD \cdot BF,$$

By the Ptolemy Theorem, $ABDF$ is a cyclic quadrilateral,

3. (*10 points*) Find measures of the angles of the triangle $\triangle DEF$.



ABEF is cyclic, so $AE \cdot BF = AF \cdot BE + AB \cdot EF = b + a \cdot EF$, thus $EF = \frac{1-b}{a} = \frac{a^2}{a} = a$, or $\triangle DEF$ is isosceles, so let $x = \angle EDF = \angle DFE$, and let $y = \angle DBF = \angle DFB$.

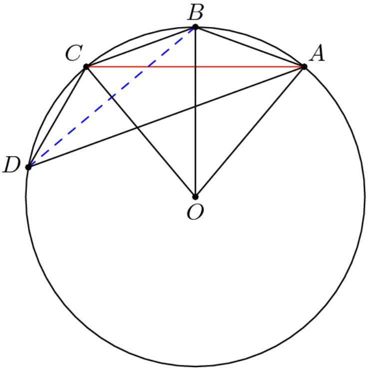
$$\angle DBF = \angle DBE + \angle EBF = \angle DFE + \angle EDF \Rightarrow y = 2x.$$

Since $\angle BED = \angle AEF = \angle DBF = y$, because they *all subend chords with the same length b*. Similarly $\angle BEA = \angle DFE = x$, and $\angle DEF = \angle DEB + \angle BEA + \angle AEF = y + x + y = 5x$.

In the $\triangle DEF$, $x + x + 5x = 180^\circ$, so the angles are $\boxed{\frac{180^\circ}{7}, \frac{180^\circ}{7}, \text{ and } \frac{5 \cdot 180^\circ}{7}.}$

Example (AMC 12 2016/A/21)

A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?



Problem

Let a and b be positive real numbers such that

$$ab = a + b.$$

1. *(5 points) Prove that* $ab \geq 4$.
2. *(10 points) Prove that* $\sqrt{1 + a^2 + b^2 + a^2b^2} \geq 9 - ab$.
3. *(10 points) Prove that* $\sqrt{1 + a^2} + \sqrt{1 + b^2} \geq \sqrt{20 + (a - b)^2}$.

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1. *(5 points) Prove that* $ab \geq 4$.

The AM-GM inequality for a, b positive real numbers

$$\underbrace{\frac{a+b}{2}}_{\text{arithmetic mean of } a,b} \geq \underbrace{\sqrt{ab}}_{\text{geometric mean of } a,b}$$

$$ab = a + b \geq 2\sqrt{ab} \Rightarrow ab \geq 2\sqrt{ab},\ \sqrt{ab} > 0 \Rightarrow \sqrt{ab} \geq 2 \Rightarrow \boxed{ab \geq 4.}$$

Problem

Let a and b be positive real numbers such that

$$ab = a + b.$$

2. *(10 points) Prove that* $\sqrt{1 + a^2 + b^2 + a^2b^2} \geq 9 - ab$.

Note that $9 - ab \geq 0$, by squaring both sides and comparing their difference,

$$\begin{aligned}(1 + a^2 + b^2 + a^2b^2) - (9 - ab)^2 &= 1 + a^2 + b^2 + a^2b^2 - 81 + 18ab - a^2b^2 \\ &= a^2 + b^2 + 18ab - 80\end{aligned}$$

Note that $a^2 + b^2 = (a + b)^2 - 2ab = a^2b^2 - 2ab$, thus

$$a^2 + b^2 + 18ab - 80 = (ab)^2 + 16(ab) - 80 = (ab - 4)(ab + 20) \geq 0.$$

Therefore

$$(1 + a^2 + b^2 + a^2b^2) \geq (9 - ab)^2 \Rightarrow \boxed{\sqrt{1 + a^2 + b^2 + a^2b^2} \geq 9 - ab.}$$

Problem

Let a and b be positive real numbers such that

$$ab = a + b.$$

3. *(10 points) Prove that* $\sqrt{1+a^2} + \sqrt{1+b^2} \geq \sqrt{20+(a-b)^2}$.

Squaring the left side, we receive

$$\left(\sqrt{1+a^2} + \sqrt{1+b^2}\right)^2 = (1+a^2) + (1+b^2) + 2(\sqrt{1+a^2})(\sqrt{1+b^2})$$
$$= 2 + a^2 + b^2 + 2\sqrt{1+a^2+b^2+a^2b^2}$$

From the previous result $\sqrt{1+a^2+b^2+a^2b^2} \geq 9 - ab$, therefore

$$2 + a^2 + b^2 + 2\sqrt{1+a^2+b^2+a^2b^2} \geq 2 + a^2 + b^2 + 2(9-ab) = 20 + (a-b)^2$$
$$\Rightarrow \sqrt{1+a^2} + \sqrt{1+b^2} \geq \sqrt{20+(a-b)^2}.$$

Theorem (AM-GM Inequality)

For any positive real numbers $a_1, \dots, a_n$,

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

Example (MR J560)

Let a, b, c be positive real numbers. Prove that

$$\frac{2}{a^2} + \frac{5}{b^2} + \frac{45}{c^2} \geq \frac{16}{(a+b)^2} + \frac{24}{(b+c)^2} + \frac{48}{(c+a)^2}$$

Problem

(p, q, r) is a triple of prime numbers such that

$$p^2 + 2q^2 + r^2 = 3pqr.$$

1. *(10 points) Prove that at least one of p or r has to be* 3.
2. *(15 points) Find all such triples.*

Problem

(p, q, r) is a triple of prime numbers such that

$$p^2 + 2q^2 + r^2 = 3pqr.$$

1. *(10 points) Prove that at least one of p or r has to be* 3.

- First, note that a perfect square has a remainder 0 or 1 when divided by 3 (we also say has a residue 0 or 1 modulo 3, or congruent to 0 or 1 modulo 3).
- Assume that neither p nor r is 3, then none of them is divisible by 3, thus the sum $p^2 + r^2$ has a remainder 2 when divided by 3.
- Therefore the left-hand side of the equation, $p^2 + 2q^2 + r^2$, when divided by 3 has remainder of 2 if q is divisible by 3, or 1 if q is not divisible by 3.
- The right-hand side of the equation, $3pqr$, is divisible by 3 in any case. It is a contradiction.

Therefore $\boxed{\text{at least one of } p \text{ or } r \text{ has to be 3.}}$

Problem

(p, q, r) is a triple of prime numbers such that

$$p^2 + 2q^2 + r^2 = 3pqr.$$

2. *(15 points) Find all such triples.*

The roles of p and r in the equation are the same, in other words, they are interchangeable. Thus, from the previous result, without loss of generality (WLOG), we can assume that $\boxed{r = 3.}$

$$p^2 + 2q^2 + 9 = 9pq \Rightarrow p^2 + 2q^2 = 9(pq - 1).$$

Now, if q is an odd prime, then

1. $p^2 + 2q^2$ is odd if p is odd, while $9(pq - 1)$ is even; and
2. $p^2 + 2q^2$ is even if p is even, while $9(pq - 1)$ is odd.

Therefore q is even, thus $\boxed{q = 2.}$

$$p^2 + 8 = 9(2p - 1) \Rightarrow p^2 - 18p + 17 = 0 \Rightarrow (p - 1)(p - 17) = 0 \Rightarrow \boxed{p = 17.}$$

Hence, by permuting p and r there are two triples: $\boxed{(17, 2, 3) \text{ and } (3, 2, 17).}$

Theorem

1. *A perfect square has a remainder of* 0 *or* 1 *when divided by* 3.
2. *A perfect square has a remainder of* 0 *or* 1 *when divided by* 4.
3. *A perfect square has a remainder of* $0, 1$ *or* 4 *when divided by* 5.
4. *A perfect square has a remainder of* $0, 1$ *or* 4 *when divided by* 8.
5. *A perfect cube has a remainder of* $0, 1$ *or* 8 *when divided by* 9.

Theorem (Fermat's Little Theorem)

In general, for a positive integer and p prime number, then $a^p \equiv a \pmod{p}$.
In particular, if p is not a divisor of a, then $a^{p-1} \equiv 1 \pmod{p}$.

Example (PCT-2022-SM1-R6-P27)

Suppose $p < q < r < s$ are prime numbers such that

$$pqrs + 1 = 4^{p+q}.$$

Find $r + s$.