School year 2023 - 2024

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Contents

Introduction

Welcome to the second semester of the school year 2023 - 2024!

For the second semester of the 2023-2024 school year, the club plans to organize the program as follows:

- Objective 1: maintain current levels of O, A, I, IC, Pre-algebra.
- Objective 2: strong emphasis on working under pressure.
- Objective 3: solving very challenging problems.

A. Biweekly activity

- On Friday, students will receive 10 problems to solve in that week.
- Detailed solutions must be submitted no later than the following Monday.
- On the following Saturday/Sunday, students receive exam questions already available in the common online folder. The test consists of 10 exercises that are almost identical to the 10 exercises received last week, with some minor changes in the test conditions. Students only need to submit the answer (an integer) for each problem. After 30 minutes, students only need to fill in 10 answers and a file is available in their personal folder.
- If the official solutions of the previous week are available, students must study them and make sure that they master the solutions in order to know how to reuse them, especially when working under duress.
- The submissions, both solutions on Friday and answers on Saturday, are graded, each contributes 50% to the final mark of the problem.

For example:

- *Friday, January 19: Students receive assignments of Session 1*
- *Friday, January 26: Students submit detailed solutions for assignments of Session 1*
- *Saturday, January 27: Students receive the exam of Session 1 (similar problems as received on Friday, January 19), take the exam for 30 minutes, and submit their answers.*
- *Monday, January 29: Students receive the official solutions of Session 1 for both the assignment and the exam Problem. They should study them carefully.*
- *Friday, February 2: Students receive markings of both solutions (submitted on Friday) and answers (submitted on Saturday)*
- *Friday, February 2: Students receive assignments of Session 2*

Note that: students who read and do well on the exercises during the week will have the opportunity to do the exercises correctly during the Purple Comet Competition (with an estimated duration of 3 minutes per exercise).

Below is an example of a problem in a weekly assignment. You will need to submit a solution, similar like the one below.

Example (Problem 1 - Weekly)

(*5 points*) Find the number of integers n for which $\sqrt{\dfrac{(2020-n)^2}{2020-n^2}}$ is a real number.

Solution. The square root is a real number if the expression under the square root sign is non-negative.

$$\frac{(2020-n)^2}{2020-n^2} \geq 0.$$

There are two cases.

Case 1:

$$\frac{(2020-n)^2}{2020-n^2} = 0 \Rightarrow n = 2020.$$

Case 1:

$$\frac{(2020-n)^2}{2020-n^2} > 0 \Rightarrow 2020 - n^2 > 0 \Rightarrow |n| < \sqrt{2020} < 45^2 = 2025 \Rightarrow -44 \leq n \leq 44.$$

There are $1 + 2 \cdot 44 = 89$ such integers between -44 and 44.

Hence, in total there are $1 + 89 = \boxed{90}$ such numbers. □

Then on the exam day, you get a variation of the problem. The correct answer of this problem is $\boxed{92.}$ *Below is the official* **generic solution** *that you don't have to submit.*

Example (Problem 1 - Exam)

(*5 points*) Find the number of integers n for which $\sqrt{\dfrac{(2048-n)^2}{2048-n^2}}$ is a real number.

Solution. Let $m > 1$ is a positive integer.

$$\sqrt{\frac{(m-n)^2}{m-n^2}} \text{ is real number } \Rightarrow \frac{(m-n)^2}{m-n^2} \geq 0.$$

We have two cases:

$$\begin{cases} \dfrac{(m-n)^2}{m-n^2} = 0 \Rightarrow n = m. \\ \dfrac{(m-n)^2}{m-n^2} > 0 \Rightarrow m - n^2 > 0 \Rightarrow |n| < \sqrt{m}. \end{cases} \Rightarrow n \in \{m, -\left\lfloor\sqrt{m}\right\rfloor, \ldots, -1, 0, 1, \ldots, \left\lfloor\sqrt{m}\right\rfloor\}.$$

There are $2 + 2\left\lfloor\sqrt{m}\right\rfloor$ such numbers. For $m = 2048$, there are $2(1 + \left\lfloor\sqrt{2048}\right\rfloor) = \boxed{92.}$ □

If you have successfully solved the problem in time then you will have an advantage at the exam. If you solve the problem in a generic way, the exam will be a breeze.

Solution based on coding can accepted only if

- The code is written in C, C++, Python, MatLab, or Java. No other languages are allowed.
- The code should not use any non-standard libraries.
- The code should compile without extra settings, should result in no compilation errors, or runtime errors.
- Execution time should not exceed 10 minutes.
- Submission must contain both source code, input (if any) and output (printout).

Failure to comply to any of the requirements will lead to rejection of the solution.

B. Semester open competition: Every months for each level there will be 4 problems to solve. Students must submit solutions before given deadlines (will be stated clearly in the text).

C. Purple Comet competition: This year's Purple Comet exam schedule is as follow:

- April 6: middle-school (MS) teams - grade 8 (US, CA), grade 9 (FR, UK, VN) and younger
- April 7: high-school teams - grade 9 (US, CA), grade 10 (FR, UK, VN) and above

Each team has 6 members. Students will be selected based on their weekly activity before April.

- MS team: 60 minutes, 20 problems, middle school math, a team is divided into two groups:
  - M1 - the first ten lessons (1-10): corresponds to level I;
  - M2 - the last ten lessons (11-20): corresponds to level A.
- HS team: 90 minutes, 30 problems, high school math, a team is divided into three groups:
  - H1 - first ten lessons (1-10): corresponds to level I;
  - H2 - the middle ten lessons (11-20): corresponds to level A;
  - H3 - last ten lessons (20-30): corresponds to level O.

Have a great year!

Your teacher,

Nghia Doan

Part I

2024

Chapter 1

Session 1: Jan 19 - Jan 27

1.1 Middle School - Assignment

Middle school students: grade 8 (US, CA), grade 9 (FR, UK, VN) and younger.

- **Submission deadline: Friday, January 26**
- **Test: Saturday, January 27**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 5-14**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 1.1.1 (Problem 1)**.** (*5 points*) Alex launches his boat into a river and heads upstream at a constant speed. At the same time at a point 8 miles upstream from Alex, Alice launches her boat and heads downstream at a constant speed. Both boats move at 6 miles per hour in still water, but the river is flowing downstream at $2\frac{3}{10}$ miles per hour. Alex and Alice will meet at a point that is $\frac{m}{n}$ miles from Alex's starting point, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Relative to the moving water, Alex and Alice are moving at 6 miles per hour, so they are approaching each other at 12 miles per hour. Thus, they meet after $\dfrac{8}{12}=\dfrac{2}{3}$ hours has travel $\dfrac{8}{2}=4$ relative to the moving water. But the water is moving Alex downstream at r miles per hour, so Alex and Alice meet after Alex has traveled:

$$\frac{m}{n}=4-\frac{2}{3}\left(2\frac{3}{10}\right)=\frac{37}{15}.$$

Hence, $m+n=37+15=\boxed{52}$. □

Problem 1.1.2 (Problem 2)**.** (*5 points*) Find a positive integer n such that there is a polygon with n sides where each of its interior angles measures $177°$.

Solution. The sum of the measures of the interior angles in a polygon with n sides is $(n-2)180$ in degree.

$$177n=(n-2)180 \Rightarrow n(180-177)=360 \Rightarrow n=\frac{360}{3}=\boxed{120.}$$

□

Problem 1.1.3 (Problem 3)**.** (*5 points*) Patrick started walking at a constant rate along a straight road from school to the park. One hour after Patrick left, Tanya started running along the same road from school to the park. One hour after Tanya left, Jose started bicycling along the same road from school to the park. Tanya ran at a constant rate of 2 miles per hour faster than Patrick walked, Jose bicycled at a constant rate of 7 miles per hour faster than Tanya ran, and all three arrived at the park at the same time. The distance from the school to the park is $\frac{m}{n}$ miles, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let t be the time it took Patrick to walk from school to the park in hours and let s be the speed that Patrick walked in miles per hour. Then the distance from the school to the park can be measured in different ways:

$$st=(s+2)(t-1)=(s+9)(t-2) \Rightarrow st=st-s+2t-2=st-2s+9t-18$$
$$-s+2t-2=-2s+9t-18=0 \Rightarrow \begin{cases} s=2t-2 \\ 2s=9t-18 \end{cases} \Rightarrow s=\frac{18}{5}, t=\frac{14}{5} \Rightarrow st=\frac{252}{25}$$

Therefore $\frac{m}{n}=\frac{252}{25}$, so $m+n=252+25=\boxed{277.}$ □

Problem 1.1.4 (Problem 4)**.** (*5 points*) Find the number of positive integers less than or equal to 2020 that are relatively prime to 588.

Solution. A number is relatively prime to $588=2^2\cdot 3\cdot 7^2$ if and only if it is not divisible by $2, 3$, or 7. Now, we count the number of positive integers less than or equal to 2020 that are divisible by $2, 3$, or 7, then find the total of those who is not divisible by $2, 3$, or 7 by complementary counting.

There are $\frac{2020}{2}=1010$ positive integers that are divisible by 2.

There are $\left\lfloor\frac{2020}{3}\right\rfloor=673$ positive integers that are divisible by 3.

There are $\left\lfloor \frac{2020}{7} \right\rfloor = 288$ positive integers that are divisible by 7.

There are $\left\lfloor \frac{2020}{2\cdot 3} \right\rfloor = 336$ positive integers that are divisible by $6 = 2 \cdot 3$.

There are $\left\lfloor \frac{2020}{2\cdot 7} \right\rfloor = 144$ positive integers that are divisible by $14 = 2 \cdot 7$.

There are $\left\lfloor \frac{2020}{3\cdot 7} \right\rfloor = 96$ positive integers that are divisible by $21 = 3 \cdot 7$.

There are $\left\lfloor \frac{2020}{2\cdot 3\cdot 7} \right\rfloor = 48$ positive integers that are divisible by $42 = 2 \cdot 3 \cdot 7$.

By the Inclusion-Exclusion principle, there are $(1010 + 673 + 288) - (336 + 144 + 96) + 48 = 1443$ positive integers less than or equal to 2020 that are divisible by 2, 3, or 7. Hence, there are $2020 - 1443 = \boxed{577}$ positive integers less than or equal to 2020 that are relatively prime to 588. ☐

Problem 1.1.5 (Problem 5)**.** (*5 points*) Given that a, b, and c are distinct positive integers such that $a \cdot b \cdot c = 2020$, the minimum possible positive value of $\frac{1}{a} - \frac{1}{b} - \frac{1}{c}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Note that $2020 = 2^2 \cdot 5 \cdot 101$, thus we can assume that c is a multiple of 101, otherwise the expression $\frac{1}{a} - \frac{1}{b} - \frac{1}{c}$ cannot be positive. Furthermore,

$$\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$$

In order for $\frac{1}{a} - \frac{1}{b}$ be as small as possible we make $b - a$ as small as possible and ab as large as possible.

Note that c is a multiple of 101 so $ab \leq 20$ thus its largest value is 20. In addition $1 \leq b - a$, so the smallest possible value for $b - a$ is 1. Both cases happen when $b = 5, a = 4, c = 101$.

Hence

$$\frac{m}{n} = \frac{1}{4} - \frac{1}{5} - \frac{1}{101} = \frac{81}{2020} \Rightarrow m + n = \boxed{2101.}$$

☐

Problem 1.1.6 (Problem 6)**.** (*5 points*) Mary mixes 2 gallons of a solution that is 40 percent alcohol with 3 gallons of a solution that is 60 percent alcohol. Sandra mixes 4 gallons of a solution that is 30 percent alcohol with $\frac{m}{n}$ gallons of a solution that is 80 percent alcohol, where m and n are relatively prime positive integers. Mary and Sandra end up with solutions that are the same percent alcohol. Find $m + n$.

Solution. Let $x = \frac{m}{n}$, then the fraction of each mixture that is alcohol is

$$\frac{0.4(2) + 0.6(3)}{5} = \frac{0.3(4) + 0.8x}{4 + x} \Rightarrow 0.52 = \frac{0.8x + 1.2}{x + 4} \Rightarrow x = \frac{22}{7} \Rightarrow m + n = \boxed{29.}$$

☐

Problem 1.1.7 (Problem 7)**.** (*5 points*) Let a and b be positive integers such that $(a^3 - a^2 + 1)(b^3 - b^2 + 2) = 2020$. Find $10a + b$.

Solution. *Case 1:* $b = 1$, then $a^3 - a^2 + 1 = 1010 \Rightarrow a^2(a - 1) = 1009$, impossible since 1009 is prime.

Case 2: $b = 2$, then $b^3 - b^2 + 2 = 6 \not| 2020$.

Case 3: $b = 3$, then $b^3 - b^2 + 2 = 20 \Rightarrow a^2(a - 1) = 100 \Rightarrow a = 5$.

Case 4: $b = 4$, then $b^3 - b^2 + 2 = 50 \not| 2020$.

Case 5: $b \geq 5$, then $b^3 - b^2 = b^2(b - 1) + 2 \geq 25 \cdot 4 + 2 = 102$.

If $a \geq 4$ then $a^3 - a^2 + 1 = a^2(a - 1) + 1 \geq 16 \cdot 3 + 1 = 49$, thus $1 \leq a \leq 3$. It is easy to test that there is no such positive integer value for a.

Thus $a = 5, b = 3 \Rightarrow 10a + b = \boxed{53.}$ ☐

Problem 1.1.8 (Problem 8)**.** (*5 points*) Find the number of three-digit palindromes that are divisible by 3. Recall that a palindrome is a number that reads the same forward and backward like 727 or 905509.

Solution. Let aba be the three-digit palindromes that are divisible by 3. Since $3 \mid 2a+b$, thus we can choose b based on the remainder of a when divided by 3.

Case 1: $a \in \{3,6,9\}$, then $b = 0,3,6,9$.

Case 2: $a \in \{1,4,7\}$, then $b = 1,4,7$.

Case 3: $a \in \{2,5,8\}$, then $b = 2,5,8$.

Thus the number of three-digit palindromes that are divisible by 3 is $3 \cdot 4 + 3 \cdot 3 + 3 \cdot 3 = \boxed{30.}$ □

Problem 1.1.9 (Problem 9)**.** (*5 points*) Six different small books and three different large books sit on a shelf. Three children may each take either two small books or one large book. Find the number of ways the three children can select their books.

Solution. Consider the number of children who take large books.

Case 1: No child take a large book. Each child can select two small books in $\binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2} = 90$ ways.

Case 2: One child takes one large book. There are 3 ways to select a child who takes a large book, and there are 3 ways for that child to select which large book. Two child can select two small books in $\binom{6}{2} \cdot \binom{4}{2} = 90$ ways. In total there are $3 \cdot 3 \cdot 90 = 810$ ways.

Case 3: Two children takes two large books. There are 3 ways to select a child who takes two small books, and there are $\binom{6}{2} = 15$ ways for that child to select which two small books. There are $\binom{3}{1} \cdot \binom{2}{1} = 6$ ways for the other two children to select two large books. In total there are $3 \cdot 15 \cdot 6 = 270$ ways.

Case 4: There are $\binom{3}{1} \cdot \binom{2}{1} \cdot \binom{1}{1} = 6$ ways for each of the children to select one large book.

Thus the number ways is $90 + 810 + 270 + 6 = \boxed{1176.}$ □

Problem 1.1.10 (Problem 10)**.** (*5 points*) Daniel had a string that formed the perimeter of a square with area 98. Daniel cut the string into two pieces. With one piece he formed the perimeter of a rectangle whose width and length are in the ratio $2:3$. With the other piece he formed the perimeter of a rectangle whose width and length are in the ratio $3:8$. The two rectangles that Daniel formed have the same area, and each of those areas is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let s be the side length of the square then $s^2 = 98$, so $s = 7\sqrt{2}$. So the length of the string is $28\sqrt{2}$. Let the two rectangles made by Daniel be $2x \times 3x$ and $3y \times 8y$, then

$$2x \cdot 3x = 3y \cdot 8y \Rightarrow x^2 = 4y^2 \Rightarrow x = 2y.$$

The total perimeter of the two rectangles is $10x + 22y = 21x = 28\sqrt{2}$, so

$$x = \frac{4\sqrt{2}}{3}, \text{ and the area is } 6\left(\frac{4\sqrt{2}}{3}\right)^2 = \frac{64}{3}.$$

Hence, $m + n = 64 + 3 = \boxed{67.}$ □

Problem 1.1.11 (Problem 11). (*5 points*) Find the number of permutations of the letters ABCDE where the letters A and B are not adjacent and the letters C and D are not adjacent. For example, count the permutations ACBDE and DEBCA but not ABCED or EDCBA.

Solution. There are $5! = 120$ permutations of the letters ABCDE.

The number of permutations where A is adjacent to B can be counted considering the permutations of XCDE, where X represents AB or BA. There are $4! = 24$ permutations of XCDE and 2 possibilities for X. Thus, there are $24 \cdot 2 = 48$ permutations where A is adjacent to B.

Similarly there are 48 permutations where C is adjacent to D.

The number of permutations where A is adjacent to B and C is adjacent to D can be counted considering the permutations of XYE, where X represents AB or BA, Y represents CD or DC. There are $3! = 6$ permutations of XYE, 2 possibilities for X, 2 possibilities for Y. Thus, there are $6 \cdot 2 \cdot 2 = 24$ permutations where A is adjacent to B and C is adjacent to D.

Therefore there are $48 + 48 - 24 = 72$ permutations where A is adjacent to B or C is adjacent to D.

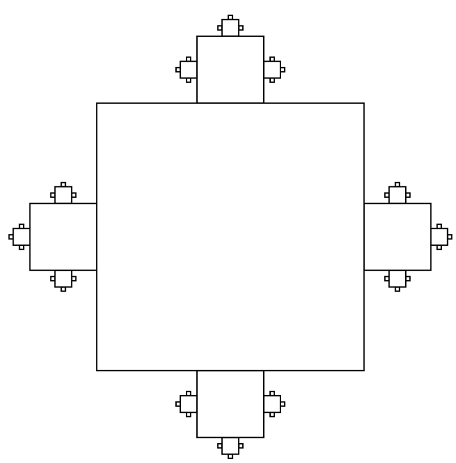
Hence, the number of permutations of the letters ABCDE where the letters A and B are not adjacent and the letters C and D are not adjacent is $120 - 72 = \boxed{48.}$ □

Problem 1.1.12 (Problem 12). (*5 points*) Construct a geometric figure in a sequence of steps. In step 1, begin with a 4×4 square.

In step 2, attach a 1×1 square onto the each side of the original square so that the new squares are on the outside of the original square, have a side along the side of the original square, and the midpoints of the sides of the original square and the attached square coincide.

In step 3, attach a $\frac{1}{4} \times \frac{1}{4}$ square onto the centers of each of the 3 exposed sides of each of the 4 squares attached in step 2. For each positive integer n, in step $n+1$, attach squares whose sides are $\frac{1}{4}$ as long as the sides of the squares attached in step n placing them at the centers of the 3 exposed sides of the squares attached in step n. The diagram shows the figure after step 4.

If this is continued for all positive integers n, the area covered by all the squares attached in all the steps is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. The square in the first step has an area of $4 \cdot 4 = 16$. There are 4 squares added in the second step, each with area $1 \cdot 1 = 1$ for a total of 4. After this, there are 3 times as many squares added in each step, and each square has area $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$ as large as the areas of the squares added in the previous step.

Thus, each step adds $\dfrac{3}{16}$ the area that was added in the previous step. The total area of all the squares added in every step is:

$$16+4+4\cdot\frac{3}{16}+4\cdot\left(\frac{3}{16}\right)^2+\cdots=16+4\left(1+\left(\frac{3}{16}\right)+\left(\frac{3}{16}\right)^2+\cdots\right)=16+4\frac{1}{1-\left(\frac{3}{16}\right)}=16+\frac{64}{13}=\frac{272}{13}.$$

Hence, $m+n=272+13=\boxed{285.}$ □

Problem 1.1.13 (Problem 13)**.** (*5 points*) Wendy randomly chooses a positive integer less than or equal to 2020. The probability that the digits in Wendy's number add up to 10 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. There are 2020 equally likely choices.

Case 1: For the number of choices less than 1000. Selecting a number less than 1000 whose digits add up to 10 is the same as to place 10 stones into three piles. We use the sticks-and-stones method with 10 stones and 2 sticks. The number of ways to place 10 stones into three order piles is given by $\binom{10+2}{2}=66$. This number count the 3 cases when all ten stones end up in the same pile, which means some digit can be equal 10. Therefore the number of possible cases is $66-3=63$.

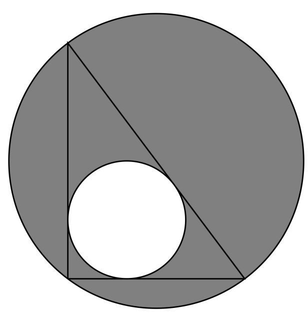
Case 2: For the number of choices at least 1000 and less than 2000. Similarly there are $\binom{9+2}{2}=55$.

Case 3: For the number of choices at least 2000. There are two such numbers: 2008 and 2017.

Therefore there are $63+55+2=120$ such numbers. The required probability is $\frac{120}{2020}=\frac{6}{101}$.

Hence, $m+n=6+101=\boxed{107.}$ □

Problem 1.1.14 (Problem 14)**.** (*5 points*) Right $\triangle ABC$ has side lengths 6, 8, and 10. Find the positive integer n such that the area of the region inside the circumcircle but outside the incircle of $\triangle ABC$ is $n\pi$.



Solution. The hypotenuse of the right $\triangle ABC$ is a diameter of the circumcircle, so the area of the circle is $\pi\left(\frac{10}{2}\right)^2=25\pi$. The area of the right $\triangle ABC$ is $\frac{1}{2}(6\cdot 8)=24$, its semi-perimeter is $\frac{1}{2}(6+8+10)=12$, its inradius is $\dfrac{24}{12}=2$. Thus, the area of the incircle is 4π. The area of the region inside the circumcircle but outside the incircle of $\triangle ABC$ is $25\pi-4\pi=21\pi$. Hence $n=\boxed{21.}$ □

1.2 Middle School - Test

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 5-14**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 1.2.1 (Problem 1)**.** (*5 points*) Alex launches his boat into a river and heads upstream at a constant speed. At the same time at a point 10 miles upstream from Alex, Alice launches her boat and heads downstream at a constant speed. Both boats move at 6 miles per hour in still water, but the river is flowing downstream at 3 miles per hour. Alex and Alice will meet at a point that is $\frac{m}{n}$ miles from Alex's starting point, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Relative to the moving water, Alex and Alice are moving at 6 miles per hour, so they are approaching each other at 12 miles per hour. Thus, they meet after $\dfrac{10}{12}=\dfrac{5}{6}$ hours has travel $\dfrac{10}{2}=5$ relative to the moving water. But the water is moving Alex downstream at r miles per hour, so Alex and Alice meet after Alex has traveled:

$$\frac{m}{n}=5-\frac{5}{6}(3)=\frac{5}{2}.$$

Hence, $m+n=5+2=\boxed{7}$. □

Problem 1.2.2 (Problem 2)**.** (*5 points*) Find a positive integer n such that there is a polygon with n sides where each of its interior angles measures $165°$.

Solution. The sum of the measures of the interior angles in a polygon with n sides is $(n-2)180$ in degree.

$$165n=(n-2)180\Rightarrow n(180-165)=360\Rightarrow n=\frac{360}{15}=\boxed{24.}$$

□

Problem 1.2.3 (Problem 3)**.** (*5 points*) Patrick started walking at a constant rate along a straight road from school to the park. Two hour after Patrick left, Tanya started running along the same road from school to the park. One hour after Tanya left, Jose started bicycling along the same road from school to the park. Tanya ran at a constant rate of 1 miles per hour faster than Patrick walked, Jose bicycled at a constant rate of 6 miles per hour faster than Tanya ran, and all three arrived at the park at the same time. The distance from the school to the park is $\frac{m}{n}$ miles, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let t be the time it took Patrick to walk from school to the park in hours and let s be the speed that Patrick walked in miles per hour. Then the distance from the school to the park can be measured in different ways:

$$st=(s+1)(t-2)=(s+7)(t-3)\Rightarrow st=st-2s+t-2=st-3s+7t-21$$
$$-2s+t-2=-3s+7t-21=0\Rightarrow\begin{cases}t=2s+2\\7t=3s+21\end{cases}\Rightarrow s=\frac{7}{11},t=\frac{36}{11}\Rightarrow st=\frac{252}{121}$$

Therefore $\frac{m}{n}=\frac{252}{121}$, so $m+n=252+121=\boxed{373.}$ □

Problem 1.2.4 (Problem 4)**.** (*5 points*) Find the number of positive **even** integers less than or equal to 2024 that are relatively prime to 15.

Solution. A number is relatively prime to $15=3\cdot5$ if and only if it is not divisible by 3 or 5.

There are $\frac{2024}{2}=1011$ positive integers that are divisible by 2.

Now we find the number of positive **even** integers that are divisible by 3 or 5

There are $\left\lfloor\frac{2024}{2\cdot3}\right\rfloor=337$ positive integers that are divisible by $6=2\cdot3$.

There are $\left\lfloor\frac{2024}{2\cdot5}\right\rfloor=202$ positive integers that are divisible by $10=2\cdot5$.

There are $\left\lfloor\frac{2024}{2\cdot3\cdot5}\right\rfloor=67$ positive integers that are divisible by $30=2\cdot3\cdot5$.

By the Inclusion-Exclusion principle, there are $1012-(337+202)+67=\boxed{540}$ positive even integers less than or equal to 2024 that are relatively prime to 15. □

Problem 1.2.5 (Problem 5)**.** (*5 points*) Given that a, b, and c are distinct positive integers such that $a \cdot b \cdot c = 1236$, the minimum possible positive value of $\frac{1}{a} - \frac{1}{b} - \frac{1}{c}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Note that $1236 = 2^2 \cdot 3 \cdot 103$, thus we can assume that c is a multiple of 103, otherwise the expression $\frac{1}{a} - \frac{1}{b} - \frac{1}{c}$ cannot be positive. Furthermore,

$$\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$$

In order for $\frac{1}{a} - \frac{1}{b}$ be as small as possible we make $b - a$ as small as possible and ab as large as possible.

Note that c is a multiple of 103 so $ab \leq 12$ thus its largest value is 12. In addition $1 \leq b - a$, so the smallest possible value for $b - a$ is 1. Both cases happen when $b = 4, a = 3, c = 101$.

Hence

$$\frac{m}{n} = \frac{1}{3} - \frac{1}{4} - \frac{1}{103} = \frac{91}{1236} \Rightarrow m + n = \boxed{1327.}$$

□

Problem 1.2.6 (Problem 6)**.** (*5 points*) Mary mixes 2 gallons of a solution that is 30 percent alcohol with 3 gallons of a solution that is 70 percent alcohol. Sandra mixes 4 gallons of a solution that is 20 percent alcohol with $\frac{m}{n}$ gallons of a solution that is 90 percent alcohol, where m and n are relatively prime positive integers. Mary and Sandra end up with solutions that are the same percent alcohol. Find $m + n$.

Solution. Let $x = \frac{m}{n}$, then the fraction of each mixture that is alcohol is

$$\frac{0.3(2) + 0.7(3)}{5} = \frac{0.2(4) + 0.9x}{4 + x} \Rightarrow 0.54 = \frac{0.9x + 0.8}{x + 4} \Rightarrow x = \frac{34}{9} \Rightarrow m + n = \boxed{43.}$$

□

Problem 1.2.7 (Problem 7)**.** (*5 points*) Let a and b be positive integers such that $(a^3 - a^2 + 1)(b^3 - b^2 + 2) = 1938$. Find $10a + b$.

Solution. *Case 1:* $b = 1$, then $a^3 - a^2 + 1 = 969 \Rightarrow a^2(a - 1) = 968 = 2^3 \cdot 11^2$. No solution.

Case 2: $b = 2$, then $b^3 - b^2 + 2 = 6 \Rightarrow a^2(a - 1) = 322 = 2 \cdot 6 \cdot 23$. No solution.

Case 3: $b = 3$, then $b^3 - b^2 + 2 = 20 \not| 1938$.

Case 4: $b = 4$, then $b^3 - b^2 + 2 = 50 \not| 1938$.

Case 5: $b = 5$, then $b^3 - b^2 + 2 = 102 \Rightarrow a^2(a - 1) = 18 \Rightarrow a = 3$.

Case 6: $b \geq 6$, then $b^3 - b^2 + 2 = b^2(b - 1) + 3 \geq 36 \cdot 5 + 2 = 182$. Thus $a^3 - a^2 + 1 =\leq \frac{1938}{182} < 11$, thus $a \leq 2$. It is easy to test that there is no such positive integer value for a.

Thus $a = 3, b = 5 \Rightarrow 10a + b = \boxed{35.}$ □

Problem 1.2.8 (Problem 8)**.** (*5 points*) Find the number of *four-digit* palindromes that are divisible by 3. Recall that a palindrome is a number that reads the same forward and backward like 727 or 905509.

Solution. Let $abba$ be the four-digit palindromes that are divisible by 3. Since $3 \mid 2a + 2b$, so $3 \mid a + b$ thus we can choose b based on the remainder of a when divided by 3.

Case 1: $a \in \{3, 6, 9\}$, then $b = 0, 3, 6, 9$.

Case 2: $a \in \{1, 4, 7\}$, then $b = 2, 5, 8$.

Case 3: $a \in \{2, 5, 8\}$, then $b = 1, 4, 7$.

Thus the number of four-digit palindromes that are divisible by 3 is $3 \cdot 4 + 3 \cdot 3 + 3 \cdot 3 = \boxed{30.}$ □

Problem 1.2.9 (Problem 9). (*5 points*) Four different small books and four different large books sit on a shelf. Three children may each take either two small books or one large book. Find the number of ways the three children can select their books.

Solution. Consider the number of children who take large books. Since there are only four different small books, there should be at least one child who takes a large book.

Case 1: One child takes one large book. There are 3 ways to select a child who takes a large book, and there are 4 ways for that child to select which large book. Two child can select two small books in $\binom{4}{2} \cdot \binom{2}{2} = 6$ ways. In total there are $3 \cdot 4 \cdot 6 = 72$ ways.

Case 3: Two children takes two large books. There are 3 ways to select a child who takes two small books, and there are $\binom{4}{2} = 6$ ways for that child to select which two small books. There are $\binom{4}{1} \cdot \binom{3}{1} = 12$ ways for the other two children to select two large books. In total there are $3 \cdot 6 \cdot 12 = 216$ ways.

Case 3: There are $\binom{4}{1} \cdot \binom{3}{1} \cdot \binom{2}{1} = 24$ ways for each of the children to select one large book.

Thus the number ways is $72 + 216 + 24 = \boxed{312.}$ □

Problem 1.2.10 (Problem 10). (*5 points*) Daniel had a string that formed the perimeter of a square. Daniel cut the string into two pieces. With one piece he formed the perimeter of a rectangle whose width and length are in the ratio $2 : 3$. With the other piece he formed the perimeter of a rectangle whose width and length are in the ratio $3 : 8$. The two rectangles that Daniel formed have the same area, and the measure of each of those areas is 2.

The length of the string is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by any perfect square larger than 1. Find $m + n$.

Solution. Let s be the side length of the original square, the length of the string is $4s$.

Let the two rectangles made by Daniel be $2x \times 3x$ and $3y \times 8y$, then

$$2x \cdot 3x = 3y \cdot 8y \Rightarrow x^2 = 4y^2 \Rightarrow x = 2y.$$

The total perimeter of the two rectangles is $10x + 22y = 21x = 4s$. Thus , and the area is $6 \cdot \left(\frac{4s}{21}\right)^2$. Since

$$x = \frac{4s}{21}, \text{ and the area is } 6\left(\frac{4s}{21}\right)^2 = \frac{32s^2}{147} = 2 \Rightarrow s = \frac{7\sqrt{3}}{4} \Rightarrow 4s = 7\sqrt{3}.$$

Hence, $m + n = 7 + 3 = \boxed{10.}$ □

Problem 1.2.11 (Problem 11). (*5 points*) Find the number of permutations of the letters ABCDEF where the letters A and B are not adjacent, the letters C and D are not adjacent, and the letters E and F are not adjacent. For example, count the permutations FACBDE and DEBCAF but not ABCDEF or EDCBAF.

Solution. There are $6! = 720$ permutations of the letters ABCDEF.

The number of permutations where A is adjacent to B can be counted considering the permutations of XCDEF, where X represents AB or BA. There are $5! = 120$ permutations of XCDEF and 2 possibilities for X. Thus, there are $120 \cdot 2 = 240$ permutations where A is adjacent to B.

Similarly there are 240 permutations where C is adjacent to D, 240 permutations where E is adjacent to F.

The number of permutations where A is adjacent to B and C is adjacent to D can be counted considering the permutations of XYEF, where X represents AB or BA, Y represents CD or DC. There are $4! = 24$ permutations of XYEF, 2 possibilities for X, 2 possibilities for Y. Thus, there are $24 \cdot 2 \cdot 2 = 96$ permutations where A is adjacent to B and C is adjacent to D.

Similarly there are 96 permutations where A is adjacent to B and E is adjacent to F. and 240 permutations where C is adjacent to D and E is adjacent to F.

Finally, the number of permutations where A is adjacent to B, C is adjacent to D, and E is adjacent to F can be counted considering the permutations of XYZ, where X represents AB or BA, Y represents CD or DC, Z represents EF. There are $3! = 6$ permutations of XYZ, 2 possibilities for X, 2 possibilities for Y, and 2 possibilities for Z. Thus, there are $6 \cdot 2 \cdot 2 \cdot 2 = 48$ permutations where A is adjacent to B, C is adjacent to D, and E is adjacent to F

Therefore there are $(240 + 240 + 240) - (96 + 96 + 96) + 48 = 480$ permutations where A is adjacent to B, C is adjacent to D, or E is adjacent to F.

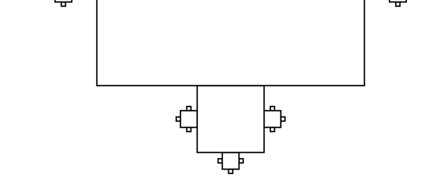
Hence, the number of permutations of the letters ABCDE where the letters A and B are not adjacent, the letters C and D are not adjacent, and the letters E and F are not adjacent is $720 - 480 = \boxed{240.}$ □

Problem 1.2.12 (Problem 12)**.** (*5 points*) Construct a geometric figure in a sequence of steps. In step 1, begin with a 4×4 square.

In step 2, attach a 1×1 square onto the each side of the original square so that the new squares are on the outside of the original square, have a side along the side of the original square, and the midpoints of the sides of the original square and the attached square coincide.

In step 3, attach a $\frac{1}{4} \times \frac{1}{4}$ square onto the centers of each of the 3 exposed sides of each of the 4 squares attached in step 2. For each positive integer n, in step $n + 1$, attach squares whose sides are $\frac{1}{4}$ as long as the sides of the squares attached in step n placing them at the centers of the 3 exposed sides of the squares attached in step n. The diagram shows the figure after step 4.

If this is continued for all positive integers n, let $\frac{m}{n}$, where m and n are relatively prime positive integers, be the area covered by all the squares attached in all the **odd** steps: step 1, step 3, step 5, etc. Find $m + n$.



Solution. The square in the first step has an area of $4 \cdot 4 = 16$. There are 4 squares added in the second step, each with area $1 \cdot 1 = 1$ for a total of 4. After this, there are 3 times as many squares added in each step, and each square has area $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$ as large as the areas of the squares added in the previous step.

Thus, each step adds $\frac{3}{16}$ the area that was added in the previous step. The areas of all the squares added are:

$$16,\ 4,\ 4 \cdot \frac{3}{16},\ 4 \cdot \left(\frac{3}{16}\right)^2,\ \ldots$$

Now, the total area of all the squares added in every **odd** step is:

$$16+4\cdot\frac{3}{16}+4\cdot\left(\frac{3}{16}\right)^3+4\cdot\left(\frac{3}{16}\right)^5+\cdots=16+4\cdot\frac{3}{16}\left(1+\left(\frac{3}{16}\right)^2+\left(\frac{3}{16}\right)^3+\cdots\right)$$
$$=16+4\frac{1}{1-\left(\frac{3}{16}\right)^2}=\frac{4976}{247}$$

Hence, $m+n=4976+247=\boxed{5223.}$ □

Problem 1.2.13 (Problem 13)**.** (*5 points*) Wendy randomly chooses a positive integer less than or equal to 2024. The probability that the digits in Wendy's number add up to 8 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. There are 2024 equally likely choices.

Case 1: For the number of choices less than 1000. Selecting a number less than 1000 whose digits add up to 8 is the same as to place 8 stones into three piles. We use the sticks-and-stones method with 8 stones and 2 sticks. The number of ways to place 8 stones into three order piles is given by $\binom{8+2}{2}=45$.

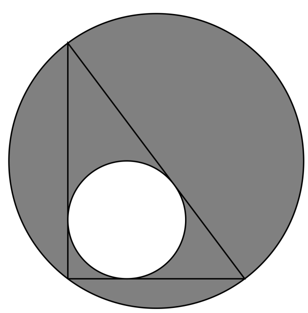
Case 2: For the number of choices at least 1000 and less than 2000. Similarly there are $\binom{7+2}{2}=36$.

Case 3: For the number of choices at least 2000. There are three such numbers: 2006, 2015, 2024.

Therefore there are $45+36+3=84$ such numbers. The required probability is $\frac{84}{2024}=\frac{21}{506}$.

Hence, $m+n=21+506=\boxed{527.}$ □

Problem 1.2.14 (Problem 14)**.** (*5 points*) Right $\triangle ABC$ has integer side lengths a, b, and c, where $\angle BAC=90^\circ$. Let r be the radius of the incircle, R be the radius of the circumcircle, then the ratio $\frac{r+R}{a+b}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution. The hypotenuse of the right $\triangle ABC$ is the diameter of the circle, thus $R=\frac{1}{2}c=\frac{1}{2}\sqrt{a^2+b^2}$.

The area of the right $\triangle ABC$ is $\frac{1}{2}(a\cdot b)$, its semi-perimeter is $\frac{1}{2}(a+b+c)$, therefore its inradius is

$$r=\frac{ab}{a+b+c}=\frac{ab}{a+b+\sqrt{a^2+b^2}}$$
$$r+R=\frac{ab}{a+b+\sqrt{a^2+b^2}}+\frac{1}{2}\sqrt{a^2+b^2}=\frac{2ab+(a+b)\sqrt{a^2+b^2}+a^2+b^2}{2(a+b+\sqrt{a^2+b^2})}$$
$$=\frac{(a+b)^2+(a+b)\sqrt{a^2+b^2}}{2(a+b+\sqrt{a^2+b^2})}=\frac{(a+b)(a+b+\sqrt{a^2+b^2})}{2(a+b+\sqrt{a^2+b^2})}=\frac{a+b}{2}$$
$$\Rightarrow\frac{r+R}{a+b}=\frac{1}{2}\Rightarrow m+n=1+2=\boxed{3.}$$

□

1.3 High School - Assignment

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

- **Submission deadline: Friday, January 26**
- **Test: Saturday, January 27**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 4-13**
- **Olympiad (O) level: Problems 10-19**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 1.3.1 (Problem 1)**.** (*5 points*) There is a complex number K such that the quadratic polynomial $7x^2 + Kx + 12 - 5i$ has exactly one root, where $i = \sqrt{-1}$. Find $|K|^2$.

Note that for a complex number $x = a + bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2 + b^2}$.

Solution. A quadratic polynomial has exactly one root if and only if its determinant is zero, therefore for $7x^2 + Kx + 12 - 5i$, we have:

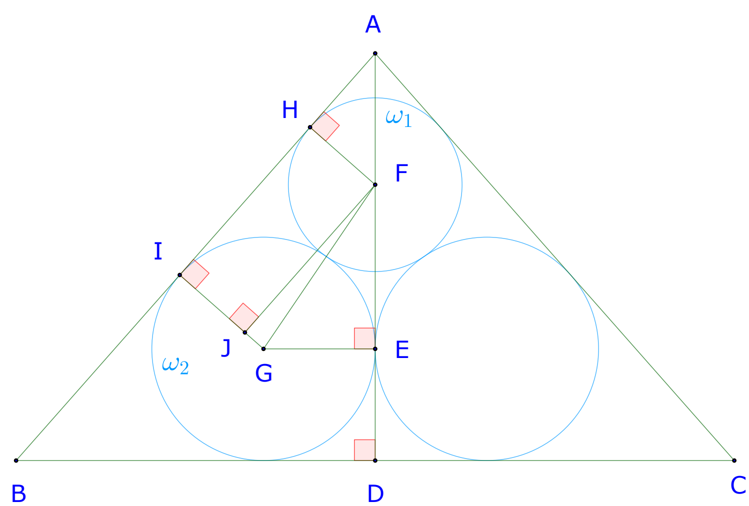
$$K^2 - 4(7)(12 - 5i) = 0 \Rightarrow K^2 = (4 \cdot 7 \cdot 12) - (4 \cdot 7 \cdot 5)i.$$

Note that for a complex number $K = a + bi$, then

$$|K| = \sqrt{a^2 + b^2} \text{ and } K^2 = (a + bi)(a + bi) = a^2 - b^2 + 2abi, \text{ note that}$$
$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2 \Rightarrow |K|^2 = a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + (2ab)^2} = |K^2|$$

Hence, $|K|^2 = |K^2| = \sqrt{(4 \cdot 7 \cdot 12)^2 + (4 \cdot 7 \cdot 5)^2} = 28\sqrt{12^2 + 5^2} = 28 \cdot 13 = \boxed{364.}$. □

Problem 1.3.2 (Problem 2)**.** (*5 points*) Two circles have radius 9, and one circle has radius 7. Each circle is externally tangent to the other two circles, and each circle is internally tangent to two sides of an isosceles triangle, as shown. The sine of the base angle of the triangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. Let ω_1 be the circle with radius 7 with center F and ω_2 be one of the circles with radius 9 with center G. Let label the vertices of the triangles to be A, B, and C as shown in the diagram above. Let AD be the altitude from A to BC, E be the tangent point of the circles with radius 9, and H, I be the feet of the altitudes from F, G to AB, respectively. Let J be the foot of the altitudes from F to GJ.

Then

$$EG = 9, FG = 7 + 9 = 16 \Rightarrow EF = \sqrt{16^2 - 9^2} = \sqrt{175}$$
$$GJ = GI - IJ = GI - FH = 9 - 7 = 2 \Rightarrow FJ = \sqrt{16^2 - 2^2} = \sqrt{252}$$
$$\sin(\angle ABC) = \cos(\angle BAD) = \cos(\angle JFE) = \cos(\angle JFG + \angle GFE)$$
$$= \cos(\angle JFG)\cos(\angle GFE) - \sin(\angle JFG)\sin(\angle GFE)$$
$$= \frac{FJ}{FG}\frac{FE}{FG} - \frac{GJ}{FG}\frac{EG}{FG} = \frac{\sqrt{175} \cdot \sqrt{252} - 2 \cdot 9}{16^2} = \frac{210 - 18}{256} = \frac{3}{4}.$$

Hence, $m + n = \boxed{7.}$ □

Problem 1.3.3 (Problem 3)**.** (*5 points*) There are two distinct pairs of positive integers $a_1 < b_1$ and $a_2 < b_2$ such that both $|(a_1+ib_1)(b_1-ia_1)|$ and $|(a_2+ib_2)(b_2-ia_2)|$ equal 2020, where $i=\sqrt{-1}$. Find $a_1+b_1+a_2+b_2$.

Note that for a complex number $x = a+bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2+b^2}$.

Solution. Note that

$$(a_1+ib_1)(b_1-ia_1) = 2a_1b_1 + i(b_1^2-a_1^2) \Rightarrow |(a_1+ib_1)(b_1-ia_1)| = \sqrt{(2a_1b_1)^2+(b_1^2-a_1^2)^2} = a_1^2+b_1^2$$

Thus $a_1^2+b_1^2 = 2020$. Since a perfect square has a remainder of 0 or 1 when divided by 4, thus both a_1 and b_1 have to be even since $4 \mid 2020$. Therefore $\frac{a_1}{2}, \frac{b_1}{2}$ are positive integers and

$$\left(\frac{a_1}{2}\right)^2 + \left(\frac{b_1}{2}\right)^2 = 505.$$

Since $505 < 23^2 = 529$, by testing we can see that $\left(\frac{a_1}{2}, \frac{b_1}{2}\right) \in \{(12,19),(8,21)\}$.

Similarly for (a_2,b_2). By the given conditions (a_1,b_1) and (a_2,b_2) are distinct pairs, so they must be $(24,38)$ and $(16,42)$.

Hence, $a_1+b_1+a_2+b_2 = 24+38+16+42 = \boxed{120.}$ □

Problem 1.3.4 (Problem 4)**.** (*5 points*) There are relatively prime positive integers s and t such that

$$\sum_{n=2}^{100}\left(\frac{n}{n^2-1} - \frac{1}{n}\right) = \frac{s}{t}.$$

Find $s+t$.

Solution. Note that

$$\frac{n}{n^2-1} - \frac{1}{n} = \frac{n}{(n-1)(n+1)} - \frac{1}{n} = \frac{1}{2}\left(\frac{1}{n-1}+\frac{1}{n+1}\right) - \frac{1}{n} = \frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n}\right) - \frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$\sum_{n=2}^{100}\left(\frac{n}{n^2-1}-\frac{1}{n}\right) = \frac{1}{2}\sum_{n=2}^{100}\left(\frac{1}{n-1}-\frac{1}{n}\right) - \frac{1}{2}\sum_{n=2}^{100}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \frac{1}{2}\left(\frac{1}{1}-\frac{1}{100}\right) - \frac{1}{2}\left(\frac{1}{2}-\frac{1}{101}\right) = \frac{5049}{20200}$$

Hence, $s+t = \boxed{25249.}$ □

Problem 1.3.5 (Problem 5)**.** (*5 points*) Let x be a real number such that $3\sin^4 x - 2\cos^6 x = -\dfrac{17}{25}$. Then $3\cos^4 x - 2\sin^6 x = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $10m+n$.

Solution.

$$-\frac{17}{25} = 3\sin^4 x - 2\cos^6 x = 3\sin^4 x - 2(1-\sin^2 x)^3 = 2\sin^6 x - 3\sin^4 x + 6\sin^2 x - 2$$
$$3\cos^4 x - 2\sin^6 x = 3(1-\sin^2 x)^2 - 2\sin^6 x = 3 - 6\sin^2 x + 3\sin^4 x - 2\sin^6 x$$

Hence, $\dfrac{m}{n} = -\left(-\dfrac{17}{25}+2\right)+3 = \frac{42}{25}$, so $10m+n = 420+25 = \boxed{445.}$ □

Problem 1.3.6 (Problem 6)**.** (*5 points*) Find the sum of all values of x such that the set

$$\{107, 122, 127, 137, 152, x\} \text{ has a mean that is equal to its median.}$$

Solution. First, let's simplify the problem by subtracting 107 from each of the elements, then the statement still stands for the set $\{0, 15, 20, 30, 45, x-107\}$. Then, by dividing each of the elements by 5, it is the same assumption for the set $\{, y = \frac{x-107}{5}\}$. The mean of the set now is $\dfrac{0+3+4+6+9+y}{6} = \dfrac{22+y}{6}$.

Case 1: $y < 3$, then the median of the set is $\dfrac{3+4}{2} = \dfrac{7}{2}$, so $y = -1$.

Case 2: $3 \leq y \leq 6$, then the median of the set is $\dfrac{4+y}{2}$, so $y = 5$.

Case 3: $y < 6$, then the median of the set is $\dfrac{4+6}{2}$, so $y = 8$.

Hence, the sum of all values of x is

$$(-1 \cdot 5 + 107) + (5 \cdot 5 + 107) + (8 \cdot 5 + 107) = 102 + 132 + 147 = \boxed{381.}$$

□

Problem 1.3.7 (Problem 7)**.** (*5 points*) Find the maximum possible value of

$$\left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right)^2$$

where a, b, and c are nonzero real numbers satisfying

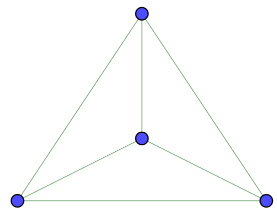
$$a\sqrt[3]{\frac{a}{b}} + b\sqrt[3]{\frac{b}{c}} + c\sqrt[3]{\frac{c}{a}} = 0.$$

Solution.

$$\begin{aligned}
&\text{Let } x = a\sqrt[3]{\frac{a}{b}},\ y = b\sqrt[3]{\frac{b}{c}},\ z = c\sqrt[3]{\frac{c}{a}} \Rightarrow x + y + z = 0\\
&x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx) = 3xyz\\
&x^3 = \left(a\sqrt[3]{\frac{a}{b}}\right)^3 = (a^3)\left(\frac{a}{b}\right) = (abc)\left(\frac{a^3}{b^2c}\right), y^3 = (abc)\left(\frac{b^3}{c^2a}\right), z^3 = (abc)\left(\frac{c^3}{a^2b}\right)\\
&\Rightarrow x^3 + y^3 + z^3 = (abc)\left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right),\ 3xyz = 3a\sqrt[3]{\frac{a}{b}} \cdot b\sqrt[3]{\frac{b}{c}} \cdot c\sqrt[3]{\frac{c}{a}} = 3abc\\
&\Rightarrow \left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right)^2 = \left(\frac{x^3+y^3+z^3}{abc}\right)^2 = \left(\frac{3xyz}{abc}\right)^2 = \left(\frac{3abc}{abc}\right)^2 = \boxed{9.}
\end{aligned}$$

□

Problem 1.3.8 (Problem 8)**.** (*5 points*) The following diagram shows four vertices connected by six edges. Suppose that each of the edges is randomly painted either red, white, or blue. The probability that the three edges adjacent to at least one vertex are colored with all three colors is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution. There are six edges in the diagrams, and there are 3 equally likely ways to paint each edge. Thus, there are 3^6 equally likely ways to paint all the edges.

To count the number of ways to paint the edges so that at least one vertex is adjacent to one edge of each colour, number the vertices 1, 2, 3, and 4 and let A_1, A_2, A_3, and A_4 be the sets of colouring patterns, where each of the vertices 1, 2, 3, and 4, respectively, is adjacent to edges of all three colours. Then, the number of colouring patterns in $A_1 \cup A_2 \cup A_3 \cup A_4$ is the number of ways to colouring the edges such that at least one vertex are colored with all three colors.

There are $3! = 6$ ways to paint the 3 edges adjacent to a particular vertex so that there is one edge of each colour, and there are 3^3 ways to paint the other three edges showing that for each j, the cardinality of A_j is $6 \cdot 3^3$.

A pattern is in $A_j \cap A_k$, for $j \neq k$ if the edges adjacent to vertex j are painted in one of 6 ways, the two edges adjacent to vertex k not adjacent to vertex j are painted in one of 2 ways, and the one edge not adjacent to either vertex j or vertex k is painted in one of 3 ways. This shows that $A_j \cap A_k$ containing $6 \cdot 2 \cdot 3 = 36$ patterns. This is true for all $\binom{4}{2} = 6$ pairs of j and k.

To obtain a pattern in $A_i \cap A_j \cap A_k$ for distinct i, j, and k, the edges adjacent to vertex i can be painted in one of 6 ways, then there is only one way to paint the edge between vertex j and vertex k, and this fixes how all the other edges must be painted, so there are only 6 patterns in $A_i \cap A_j \cap A_k$ for each of the $\binom{4}{3} = 4$ choices of i, j, k.

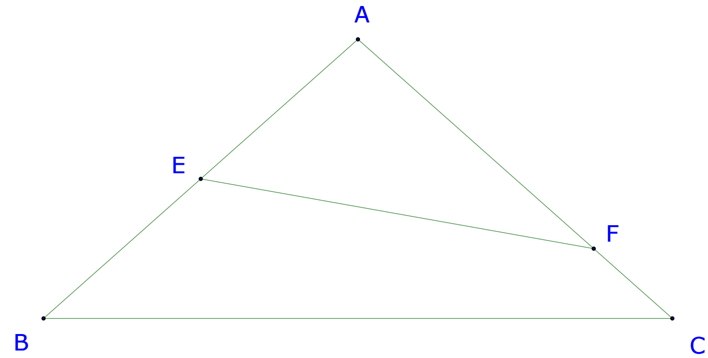
Similarly, there are 6 patterns in $A_1 \cap A_2 \cap A_3 \cap A_4$.

By the Inclusion - Exclusion Principle then the number of colouring patterns in $A_1 \cup A_2 \cup A_3 \cup A_4$ is:

$$\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| \\
&- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\
&+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\
&- |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= 4(6 \cdot 3^3) - 6(6 \cdot 2 \cdot 3) + 4 \cdot 6 - 6 = 6 \cdot 75
\end{aligned}$$

The required probability is $\dfrac{6 \cdot 75}{3^6} = \frac{50}{81}$. Hence, $m + n = 50 + 81 = \boxed{131.}$ $\square$

Problem 1.3.9 (Problem 9)**.** (*5 points*) In isosceles $\triangle ABC, AB = AC, \angle BAC$ is obtuse, and points E and F lie on sides AB and AC, respectively, so that $AE = 10, AF = 15$. The area of $\triangle AEF$ is 60, and the area of quadrilateral $BEFC$ is 102. Find BC.



Solution. Keep in mind that $\angle BAC > 90°$, so $\cos(\angle BAC) < 0$,

$$60 = [AEF] = \frac{1}{2}(AE \cdot AF)\sin(\angle EAF) = \frac{1}{2}(10 \cdot 15)\sin(\angle EAF) \Rightarrow \sin(\angle EAF) = \frac{4}{5}$$

$$[AEF] + [BEFC] = [ABC] = \frac{1}{2}(AB \cdot AC)\sin(\angle BAC) = \frac{1}{2}(AB^2)\frac{4}{5} \Rightarrow AB^2 = \frac{162 \cdot 2 \cdot 5}{4} = 405$$

$$BC^2 = AB^2 + AC^2 - 2(AB \cdot AC)\cos(\angle BAC) = 405(2)(1 - \cos(\angle BAC)) = 405(2)\left(1 + \frac{3}{5}\right) = 1296 = 36^2.$$

Hence, $BC = \boxed{36.}$ □

Problem 1.3.10 (Problem 10)**.** (*5 points*) Find the least prime number greater than 1000 that divides $2^{1010} \cdot 23^{2020} + 1$.

Solution. First, let $n = 2^{1010} \cdot 23^{2020}$, then

$$n = 2^{1010} \cdot 23^{2020} + 1 = (4 \cdot 23^4)^{505} + 1 \Rightarrow 4 \cdot 23^4 + 1 \mid n.$$

By Sophie Germain identity $a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab)$ for $a = 1, b = 23$,

$$4 \cdot 23^4 + 1 = (1 + 2 \cdot 23^2 - 2 \cdot 23)(1 + 2 \cdot 23^2 + 2 \cdot 23) = 1013 \cdot 1105 = 5 \cdot 13 \cdot 17 \cdot 1013.$$

Now, since $7 \mid 1001, 17 \mid 1003, 19 \mid 1007, 3 \mid 1011$, thus the only prime number larger than 1000 and less than 1013 is 1009. We shall prove that it is not a factor of n,

By Fermat's Little Theorem:

$$2^{1010} \equiv 2^2 \pmod{1009},\ 23^{2020} \equiv 23^4 \pmod{1009} \Rightarrow n \equiv 4 \cdot 23^4 + 1 \pmod{1009}.$$

From the factorization of $4 \cdot 23^4 + 1$ shown above, $1009 \nmid n$, hence the required prime is $\boxed{1013.}$ □

Problem 1.3.11 (Problem 11)**.** (*5 points*) Find the maximum possible value of

$$9\sqrt{x} + 8\sqrt{y} + 5\sqrt{z},$$

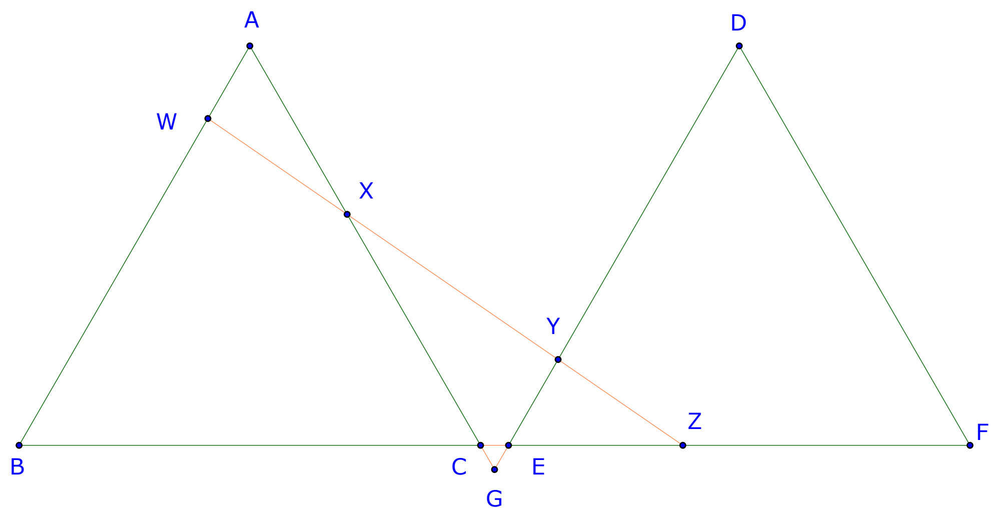
where x, y, and z are positive real numbers satisfying $9x + 4y + z = 128$.

Solution. By the Cauchy-Schwarz inequality,

$$9\sqrt{x} + 8\sqrt{y} + 5\sqrt{z} = (3)(\sqrt{9x}) + (4)(\sqrt{4y}) + (5)(\sqrt{z}) \leq \sqrt{(3^2 + 4^2 + 5^2)(9x + 4y + z)} = \sqrt{50 \cdot 128} = \boxed{80.}$$

□

Problem 1.3.12 (Problem 12)**.** (*5 points*) Two congruent equilateral triangles $\triangle ABC$ and $\triangle DEF$ lie on the same side of line BC so that B, C, E, and F are collinear as shown. A line intersects AB, AC, DE, and EF at W, X, Y, and Z, respectively, such that $\frac{AW}{BW} = \frac{2}{9}$, $\frac{AX}{CX} = \frac{5}{6}$, and $\frac{DY}{EY} = \frac{9}{2}$. The ratio $\frac{EZ}{FZ}$ can then be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. Without loss of generality, let the triangles have side length 11, so

$$AW = 2, BW = 9, AX = 5, CX = 6, DY = 9, EY = 2.$$

Let G be the intersection of lines AC and DE, then $\triangle AWX \sim GYX$, so

$$\frac{GE + EY}{GC + CX} = \frac{GY}{GX} = \frac{AW}{AX} = \frac{2}{5} \Rightarrow \frac{GE + 2}{GC + 6} = \frac{2}{5} \Rightarrow GC = GE = \frac{2}{3}.$$

By Menelaus' Theorem,

$$1 = \frac{AW}{BW} \cdot \frac{BZ}{CZ} \cdot \frac{CX}{AX} = \frac{2}{9} \cdot \frac{11 + CZ}{CZ} \cdot \frac{6}{5} \Rightarrow CZ = 4.$$

$$EZ = 4 - CE = 4 - \frac{2}{3} = \frac{10}{3} \Rightarrow \frac{EZ}{FZ} = \frac{\frac{10}{3}}{11 - \frac{10}{3}} = \frac{10}{23}$$

Thus the desired sum $m + n = 10 + 23 = \boxed{33.}$ □

Problem 1.3.13 (Problem 13)**.** (*5 points*) Find the number of permutations of the letters AAAABBBCC where no letter is next to another letter of the same type. For example, count ABCABCABA and ABABCABCA but not ABCCBABAA.

Solution. Let first consider the letters AAAABBB. There are three cases that could lead to permutations with no repeated letters.

Case 1: there is 1 permutation of As and Bs such that no two like letters are next to each other: $ABABABA$. For these, there are 8 positions around the 7 letters to place the two letters C, thus there are $\binom{8}{2} = 28$ ways to choose two distinct positions for Cs. Thus, in total, there are $1 \cdot 28 = 28$ ways for this case.

Case 2: there is 6 permutations of As and Bs that have one pair of like letters are adjacent. These are obtained by starting with $ABABABA$, removing the beginning or ending A, then placing that A next to one of another A (there are two ways to *insert* right before of or right after it.) Into this permutation, a letter C must be used to insert in between the two neighbouring letters A. For the second letter C there are $9 - 2 = 7$ positions not immediately next to the first letter C. Thus, in total, there are $6 \cdot 7 = 42$ ways for this case.

Case 3: there is 9 permutations of As and Bs such that there are two set of like letters adjacent.

- There are 2 permutations where three letters A appear together: $BAAABAB$ and $BABAAAB$.
- There is 1 permutations where four letters A appear in pairs: $BAABAAB$.
- There are 6 permutations where there is a pair of adjacent letters A and a pair of adjacent letters B:

$$AABBABA,\ AABABBA,\ ABBAABA,\ ABAABBA,\ ABBABAA,\ ABABBAA.$$

Into each of these permutation, two letter C must be used to insert in between the two neighbouring like letters. Thus, in total, there are $(2 + 1 + 6) \cdot 1 = 9$ ways for this case.

Therefore, altogether, there are $28 + 42 + 9 = \boxed{79}$ permutations. □

Problem 1.3.14 (Problem 14)**.** (*5 points*) There is a real number x between 0 and $\dfrac{\pi}{2}$ such that

$$\frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{12}{11}$$

and $\sin x + \cos x = \frac{\sqrt{m}}{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Solution. Let $u = \sin(x), v = \cos(x)$, then $\sin^3 x + \cos^3 x = u^3 + v^3, \sin^5 x + \cos^5 x = u^5 + v^5$, and $u^2 + v^2 = 1$, furthermore

$$\begin{aligned}
&u^3 + v^3 = (u + v)(u^2 - uv + v^2) = (u + v)(1 - uv) \\
&u^5 + v^5 = (u + v)(u^4 - u^3 v + u^2 v^2 + uv^3 + u^4) = (u + v)((u^2 + v^2)^2 - u^2 v^2 - uv(u^2 + v^2)) \\
&\ = (u + v)(1 - uv - u^2 u^2)
\end{aligned}$$

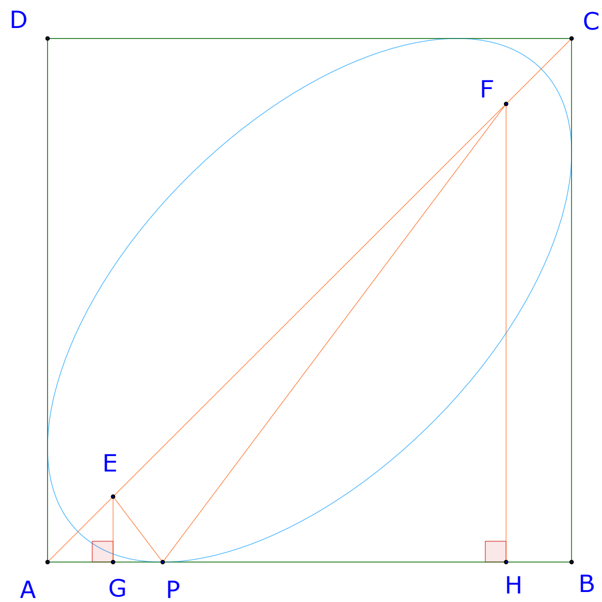
Now, let $s = u + v$, then $uv = \frac{s^2-1}{2}$, therefore

$$\begin{aligned}
&\frac{12}{11} = \frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{s\left(1 - \frac{s^2-1}{2}\right)}{s\left(1 - \frac{s^2-1}{2} - \left(\frac{s^2-1}{2}\right)^2\right)} = \frac{2(3 - s^2)}{5 - s^4} \\
&\Rightarrow 6s^4 - 11s^2 + 3 = 0 \Rightarrow s^2 = \frac{3}{2} \text{ or } \frac{1}{2}.
\end{aligned}$$

Since $0 \le x \le \frac{\pi}{2}$, so $u, v \ge 0$, thus $s^2 = 1 + 2uv \ge 1$, therefore $s^2 = \frac{3}{2} \Rightarrow s = \frac{\sqrt{6}}{2}$. Hence, the desired sum $m + n = 6 + 2 = \boxed{8.}$ □

Problem 1.3.15 (Problem 15). (*5 points*) Points E and F lie on diagonal AC of square $ABCD$ with side length 24, such that $AE = CF = 3\sqrt{2}$. An ellipse with foci at E and F is tangent to the sides of the square. Find the sum of the distances from any point on the ellipse to the two foci.

Hint: Let P be the tangent point on AB. By the properties of the eclipse, a light ray passing from E to P that reflects of the eclipse will pass through point F. Because line AB is tangent to the eclipse, $\angle EPA = \angle FPB$.



Solution. Let P be the tangent point on AB. By the properties of the eclipse, a light ray passing from E to P that reflects of the eclipse will pass through point F. Because line AB is tangent to the eclipse, $\angle EPA = \angle FPB$. Let points G and H be the projections onto AB of points E and F, respectively. Then $\triangle EGP \sim \triangle FHP$,

$$AE = CF = 3\sqrt{2}, \angle CAB = 45^\circ \Rightarrow AG = BH = 3, FH = 24 - 3 = 21, GH = 24 - 2 \cdot 3 = 18$$

$$\frac{PG}{PH} = \frac{EG}{FH} = \frac{3}{21} = \frac{1}{7} \Rightarrow PG = GH \cdot \frac{PG}{PG + PH} = 18 \cdot \frac{PG}{PG + 7PG} = \frac{9}{4}$$

$$EP = \sqrt{EG^2 + PG^2} = \sqrt{3^2 + \left(\frac{9}{4}\right)^2} = \frac{15}{4}$$

$$FP = 7EP \Rightarrow EP + FP = 8EP = \boxed{30.}$$

□

Problem 1.3.16 (Problem 16)**.** (*5 points*) A deck of eight cards has cards numbered $1, 2, 3, 4, 5, 6, 7, 8$, in that order, and a deck of five cards has cards numbered $1, 2, 3, 4, 5$, in that order. The two decks are riffle-shuffled together to form a deck with 13 cards with the cards from each deck in the same order as they were originally. Thus, numbers on the cards might end up in the order 1122334455678 or 1234512345678 but not 1223144553678. Find the number of possible sequences of the 13 numbers.

Solution. There is a one-on-one correspondence between possible orderings and the paths of length 13 in the coordinate plane from $(0, 0)$ to $(8, 5)$ where each step from point (x, y) is one unit to the right to $(x + 1, y)$ or one unit up to $(x, y + 1)$, and there is no point (x, y) on the path where $y > x$.

Indeed, given a possible ordering, construct a path by reading the numbers in order, and each time a number is seen for the first time, have the path take one step right (R), and each time a number is seen for the second time, have the path take one step up (U). This process take every possible number sequence and convert it into a path of the correct type, and the process is reversible showing that this correspondence is one-to-one.

It remains to count the number of paths from $(0, 0)$ to (8.5) that make unit steps to the right or upward that avoid points with $y > x$. Each such path make 8 steps right and 5 steps upward in some order.

Suppose one of these paths crosses a point (x, y) with $y > x$. For example the path $RURRUUURRURRR$ reaches $(3, 4)$ after making the steps $RURRUUU$. If, after this points all U steps are changed to R steps and all R steps are changed to U steps, the path become $RURUUUUURUUU$ which ends up at $(4, 9)$.

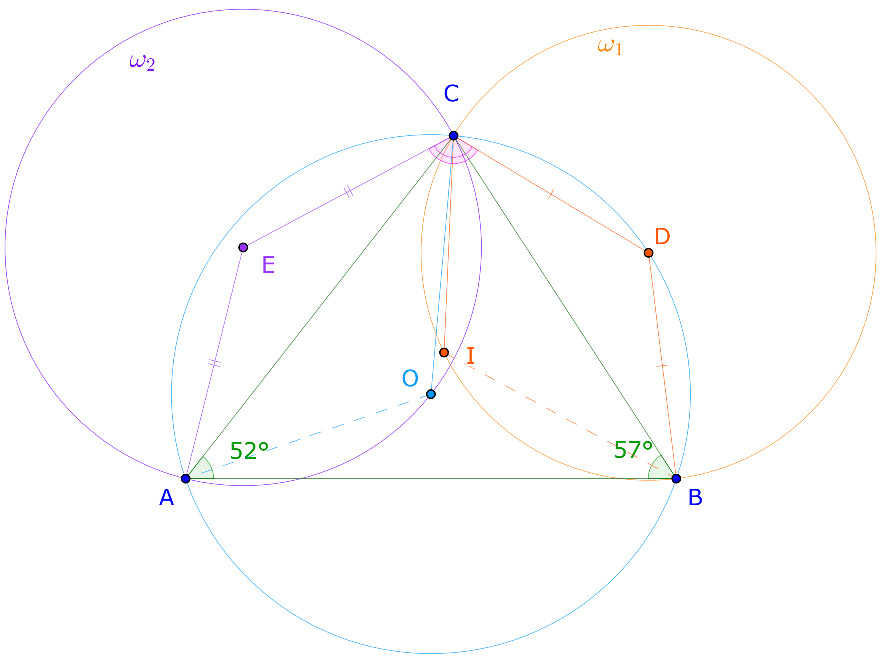
In fact, if any path from $(0, 0)$ to $(8, 5)$ that reaches a point $(x.y)$ where $y > x$ is transformed by interchanging subsequent steps U and R, the path will reach the point $(4, 9)$ instead of point $(8, 5)$. The number of paths from $(0, 0)$ to $(4, 9)$ is $\binom{4+9}{4} = \binom{13}{4}$. This shows that the number of paths to $(8, 5)$ that do not contain point (x, y) with $y > x$ is:

$$\binom{5+8}{5} - \binom{4+9}{4} = \boxed{572.}$$

□

Problem 1.3.17 (Problem 17)**.** (*5 points*) In $\triangle ABC$, $\angle A = 52°$, and $\angle B = 57°$. One circle centred at D passes through the points B, C, and the incenter I of $\triangle ABC$, and a second circle centred at E passes through the points A, C, and the circumcenter O of $\triangle ABC$. Find the degree measure of the acute angle at which the two circles intersect.

Hint: find the measure of the complementary angle of the angle $\angle DCE$.



Solution. First $\angle C = 180° - \angle A - \angle B = 71°$. Furthermore

$$\widehat{BI} = 2\angle BCI = \angle ACB,\ \widehat{CI} = 2\angle CBI = \angle ABC \Rightarrow \angle BDC = \widehat{BC} = \widehat{BI} + \widehat{CI} = \angle ACB + \angle ABC.$$

Since $\triangle BDC$ isosceles, in circle ω_1,

$$\angle BCD = \frac{1}{2}(180° - \angle BDC) = \frac{1}{2}(180° - \angle ACB + \angle ABC) = \frac{1}{2}\angle BAC.$$

On the other hand, in circle $O, \angle AOC = 2\angle AB$, so in circle ω_2,

$$\text{(major) } \widehat{AC} = 2\angle AOC = 4\angle ABC \Rightarrow \angle AEC = 360° - \widehat{AC} = 360° - 4\angle ABC.$$

Since $\triangle AEC$ is isosceles, in circle ω_2,

$$\angle ACE = \frac{1}{2}(180° - \angle AEC) = 2\angle ABC - 90°.$$

The diagram at the right shows both circles ω_1 and ω_2. Because the line tangent to a circle is perpendicular to the radius of the circle that ends at the point of tangency, it follows that the two circles intersect at an angle θ which satisfies $\angle DCE = 180° - \theta$, thus,

$$\theta = 180° - \angle DCE = 180° - (\angle BCD + \angle ACB + \angle ACE) = 180° - \left(\frac{1}{2}\angle BAC + \angle ACB + 2\angle ABC - 90°\right)$$

$$= 270° - (\frac{1}{2}52° + 71° + 2 \cdot 57°) = \boxed{59°.}$$

□

Problem 1.3.18 (Problem 18)**.** (*5 points*) Three doctors, four nurses, and three patients stand in a line in random order. The probability that there is at least one doctor and at least one nurse between each pair of patients is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. The required probability is equal to the probability that if the ten letters $DDDNNNNPPP$ are arranged in random order, then there is at least one D and one N between each pair of Ps. There are

$$\binom{10}{3,4,3} = \frac{10!}{3!4!3!}$$

equally likely ways to arrange the ten letters.

Consider first the possible arrangements of the Ps and Ds that result in at least one D between each pair of Ps. There are four possible arrangements:

$$DPDPDP, PDDPDP, PDPDDP, \text{ and } PDPDPD.$$

Now consider how many ways four Ns can be inserted into one of these lists of six letters.

Without regard to whether there are Ns between each pair of Ps, the number of ways of inserting four Ns into a sequence of six other letters is given by the sticks-and-stones technique as $\binom{4+6}{6} = 210$.

Let X be the set of such arrangements that leave no Ns between the first P and second P, and let Y be the set of such arrangements that leave no Ns between the second P and the third P.

There are now four cases to consider.

Case 1: In the case of inserting four Ns into $DPDPDP$, the sizes of X, Y, and $X \cap Y$ are given by

$$\binom{4+4}{4} = 70, \ \binom{4+4}{4} = 70, \ \binom{4+2}{2} = 15, \text{ respectively.}$$

Thus, by the Inclusion/Exclusion Principle, there are $70+70-15 = 125$ ways to insert four Ns into the sequence and not have at least one N between each pair of Ps. In each of these two cases there are, therefore, $210-125 = 85$ ways of inserting four Ns so that there is at least one N between each pair of Ps.

Case 2: There are also 85 arrangements associated with inserting four Ns into $PDPDPD$.

Case 3: In the case of inserting four Ns into $PDDPDP$, the sizes of X, Y, and $X \cap Y$ are given by

$$\binom{4+3}{3} = 35, \ \binom{4+4}{4} = 70, \ \binom{4+1}{1} = 5, \text{ respectively.}$$

Thus, by the Inclusion/Exclusion Principle, there are $435+70-5 = 100$ ways to insert four Ns into the sequence and not have at least one N between each pair of Ps. In each of these two cases there are, therefore, $210-100 = 110$ ways of inserting four Ns so that there is at least one N between each pair of Ps.

Case 4: There are also 110 arrangements associated with inserting four Ns into $PDPDDP$.

Thus, there are $2(85+110) = 390$ arrangements of the ten letters so that there is at least one D and one N between each pair of Ps. The required probability is

$$\frac{390}{\frac{10!}{3!4!3!}} = \frac{13}{140}.$$

Hence, the desired sum is $13+140 = \boxed{153.}$ □

Problem 1.3.19 (Problem 19)**.** (*5 points*) Let p, q, and r be prime numbers such that $2pqr+p+q+r = 2020$. Find $pq + qr + rp$.

Solution. Note that $p + q + r = 2020 - 2pqr$, thus it is even, so at least one prime is even. WLOG, let $p = 2$.

$$4qr + q + r = 2018 \Rightarrow (4q + 1)(4r + 1) = 4 \cdot 2018 + 1 = 8073 = 3^3 \cdot 13 \cdot 23.$$

Thus, 8073 has 16 divisors. We are looking for those divisors in the form of $4k + 1$ and they are $1, 9, 13, 3 \cdot 23 = 69$. Easy to test and verify that the pair q, r is $17, 29$.

Hence, the desired sum is $pq + qr + rp = 2 \cdot 17 + 17 \cdot 29 + 29 \cdot 2 = \boxed{585.}$ □

1.4 High School - Test

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 4-13**
- **Olympiad (O) level: Problems 10-19**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 1.4.1 (Problem 1)**.** (*5 points*) There is a complex number K such that the quadratic polynomial $5x^2 + Kx + 4 - 3i$ has exactly one root, where $i = \sqrt{-1}$. Find $|K|^2$.

Note that for a complex number $x = a + bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2 + b^2}$.

Solution. A quadratic polynomial has exactly one root if and only if its determinant is zero, therefore for $5x^2 + Kx + 4 - 3i$, we have:

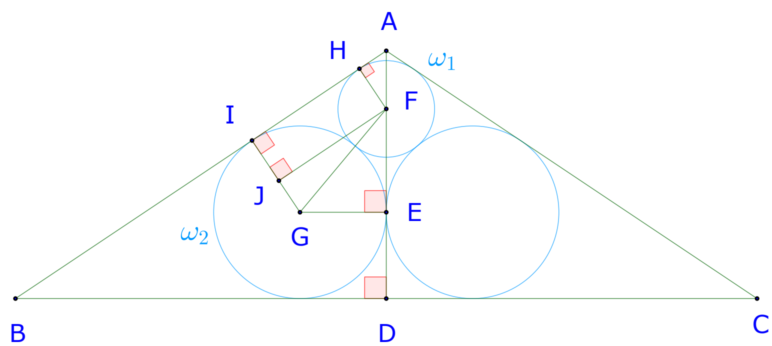
$$K^2 - 4(5)(4 - 3i) = 0 \Rightarrow K^2 = (4 \cdot 5 \cdot 4) - (4 \cdot 5 \cdot 3)i.$$

Note that for a complex number $K = a + bi$, then

$$|K| = \sqrt{a^2 + b^2} \text{ and } K^2 = (a + bi)(a + bi) = a^2 - b^2 + 2abi, \text{ note that}$$
$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2 \Rightarrow |K|^2 = a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + (2ab)^2} = |K^2|$$

Hence, $|K|^2 = |K^2| = \sqrt{(4 \cdot 5 \cdot 4)^2 + (4 \cdot 5 \cdot 3)^2} = 20\sqrt{4^2 + 3^2} = 20 \cdot 5 = \boxed{100.}$. □

Problem 1.4.2 (Problem 2)**.** (*5 points*) Two circles have radius 25, and one circle has radius 14. Each circle is externally tangent to the other two circles, and each circle is internally tangent to two sides of an isosceles triangle, as shown. The sine of the base angle of the triangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. Let ω_1 be the circle with radius 10 with center F and ω_2 be one of the circles with radius 6 with center G. Let label the vertices of the triangles to be A, B, and C as shown in the diagram above. Let AD be the altitude from A to BC, E be the tangent point of the circles with radius 10, and H, I be the feet of the altitudes from F, G to AB, respectively. Let J be the foot of the altitudes from F to GJ.

Then

$$EG = 25, FG = 14 + 25 = 39 \Rightarrow EF = \sqrt{39^2 - 25^2} = \sqrt{896}$$
$$GJ = GI - IJ = GI - FH = 25 - 14 = 11 \Rightarrow FJ = \sqrt{39^2 - 11^2} = \sqrt{1400}$$
$$\sin(\angle ABC) = \cos(\angle BAD) = \cos(\angle JFE) = \cos(\angle JFG + \angle GFE)$$
$$= \cos(\angle JFG)\cos(\angle GFE) - \sin(\angle JFG)\sin(\angle GFE)$$
$$= \frac{FJ}{FG}\frac{FE}{FG} - \frac{GJ}{FG}\frac{EG}{FG} = \frac{\sqrt{1400} \cdot \sqrt{896} - 11 \cdot 25}{39^2} = \frac{5}{9}.$$

Hence, $m + n = 5 + 9 = \boxed{14.}$ □

Problem 1.4.3 (Problem 3). (*5 points*) There are three distinct pairs of positive integers $a_1 < b_1$, $a_2 < b_2$, and $a_3 < b_3$ such that

$$|(a_1 + ib_1)(b_1 - ia_1)| = |(a_2 + ib_2)(b_2 - ia_2)| = |(a_3 + ib_3)(b_3 - ia_3)| = 1025, \text{ where } i = \sqrt{-1}.$$

Find $a_1 + b_1 + a_2 + b_2 + a_3 + b_3$.

Note that for a complex number $x = a + bi$, $|x|$ denotes the absolute value of x and $|x| = \sqrt{a^2 + b^2}$.

Solution. Note that

$$(a_1 + ib_1)(b_1 - ia_1) = 2a_1b_1 + i(b_1^2 - a_1^2) \Rightarrow |(a_1 + ib_1)(b_1 - ia_1)| = \sqrt{(2a_1b_1)^2 + (b_1^2 - a_1^2)^2} = a_1^2 + b_1^2$$

Thus $a_1^2 + b_1^2 = 1025$. Since $1024 = 32^2$, by testing we can see that $(a_1, b_1) \in \{(1, 32), (8, 31), (20, 25)\}$.

Similarly for (a_2, b_2) and (a_3, b_3). By the given conditions they are distinct pairs, so they must be all these three pairs.

Hence, $a_1 + b_1 + a_2 + b_2 + a_3 + b_3 = 1 + 32 + 8 + 31 + 20 + 25 = \boxed{117.}$ □

Problem 1.4.4 (Problem 4). (*5 points*) There are relatively prime positive integers s and t such that

$$\sum_{n=2}^{101} \left(\frac{n}{n^2 - 1} - \frac{1}{n} \right) = \frac{s}{t}.$$

Find $s + t$.

Solution. Note that

$$\frac{n}{n^2 - 1} - \frac{1}{n} = \frac{n}{(n-1)(n+1)} - \frac{1}{n} = \frac{1}{2}\left(\frac{1}{n-1} + \frac{1}{n+1}\right) - \frac{1}{n} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right) - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$\sum_{n=2}^{101} \left(\frac{n}{n^2 - 1} - \frac{1}{n} \right) = \frac{1}{2}\sum_{n=2}^{101}\left(\frac{1}{n-1} - \frac{1}{n}\right) - \frac{1}{2}\sum_{n=2}^{101}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{101}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{102}\right) = \frac{2575}{10302}$$

Hence, $s + t = \boxed{12877.}$ □

Problem 1.4.5 (Problem 5). (*5 points*) Let x be a real number such that $3\sin^4 x - 2\cos^6 x = \dfrac{17}{25}$. Then $3\cos^4 x - 2\sin^6 x = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $10m + n$.

Solution.

$$\frac{17}{25} = 3\sin^4 x - 2\cos^6 x = 3\sin^4 x - 2(1 - \sin^2 x)^3 = 2\sin^6 x - 3\sin^4 x + 6\sin^2 x - 2$$
$$3\cos^4 x - 2\sin^6 x = 3(1 - \sin^2 x)^2 - 2\sin^6 x = 3 - 6\sin^2 x + 3\sin^4 x - 2\sin^6 x$$

Hence, $\dfrac{m}{n} = -\left(\dfrac{17}{25} + 2\right) + 3 = \frac{8}{25}$, so $10m + n = 80 + 25 = \boxed{105.}$ □

Problem 1.4.6 (Problem 6)**.** (*5 points*) Find the sum of all values of x such that the set

$$\{108, 123, 128, 138, 153, x\} \text{ has a mean that is equal to its median.}$$

Solution. First, let's simplify the problem by subtracting 108 from each of the elements, then the statement still stands for the set $\{0, 15, 20, 30, 45, x - 108\}$. Then, by dividing each of the elements by 5, it is the same assumption for the set $\{, y = \frac{x-108}{5}\}$. The mean of the set now is $\dfrac{0+3+4+6+9+y}{6} = \dfrac{22+y}{6}$.

Case 1: $y < 3$, then the median of the set is $\dfrac{3+4}{2} = \dfrac{7}{2}$, so $y = -1$.

Case 2: $3 \le y \le 6$, then the median of the set is $\dfrac{4+y}{2}$, so $y = 5$.

Case 3: $y < 6$, then the median of the set is $\dfrac{4+6}{2}$, so $y = 8$.

Hence, the sum of all values of x is

$$(-1 \cdot 5 + 108) + (5 \cdot 5 + 108) + (8 \cdot 5 + 108) = 103 + 133 + 148 = \boxed{384.}$$

□

Problem 1.4.7 (Problem 7)**.** (*5 points*) Find the **minimum** possible value of

$$\left(\frac{a^2}{b^3c^2} + \frac{b^2}{c^3a^2} + \frac{c^2}{a^3b^2}\right)^2$$

where $a > b > 0 > c$ are real numbers satisfying

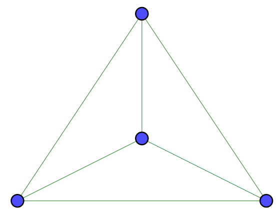
$$a\sqrt[3]{\frac{a}{b}} + b\sqrt[3]{\frac{b}{c}} + c\sqrt[3]{\frac{c}{a}} = 0.$$

Solution.

$$\begin{aligned}
&\text{Let } x = a\sqrt[3]{\frac{a}{b}},\ y = b\sqrt[3]{\frac{b}{c}},\ z = c\sqrt[3]{\frac{c}{a}} \Rightarrow x + y + z = 0\\
&x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx) = 3xyz\\
&x^3 = \left(a\sqrt[3]{\frac{a}{b}}\right)^3 = (a^3)\left(\frac{a}{b}\right) = (abc)\left(\frac{a^3}{b^2c}\right), y^3 = (abc)\left(\frac{b^3}{c^2a}\right), z^3 = (abc)\left(\frac{c^3}{a^2b}\right)\\
&\Rightarrow x^3 + y^3 + z^3 = (abc)\left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right),\ 3xyz = 3a\sqrt[3]{\frac{a}{b}} \cdot b\sqrt[3]{\frac{b}{c}} \cdot c\sqrt[3]{\frac{c}{a}} = 3abc\\
&\Rightarrow \left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right)^2 = \left(\frac{x^3+y^3+z^3}{abc}\right)^2 = \left(\frac{3xyz}{abc}\right)^2 = \left(\frac{3abc}{abc}\right)^2 = \boxed{9.}
\end{aligned}$$

□

Problem 1.4.8 (Problem 8)**.** (*5 points*) The following diagram shows four vertices connected by six edges. Suppose that each of the edges is randomly painted either red, white, or blue. The probability that the three edges adjacent to **at least two** vertices are colored with all three colors is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution. There are six edges in the diagrams, and there are 3 equally likely ways to paint each edge. Thus, there are 3^6 equally likely ways to paint all the edges.

To count the number of ways to paint the edges so that at least one vertex is adjacent to one edge of each colour, number the vertices 1, 2, 3, and 4 and let A_1, A_2, A_3, and A_4 be the sets of colouring patterns, where each of the vertices 1, 2, 3, and 4, respectively, is adjacent to edges of all three colours.

There are $3! = 6$ ways to paint the 3 edges adjacent to a particular vertex so that there is one edge of each colour, and there are 3^3 ways to paint the other three edges showing that for each j, the cardinality of A_j is $6 \cdot 3^3$.

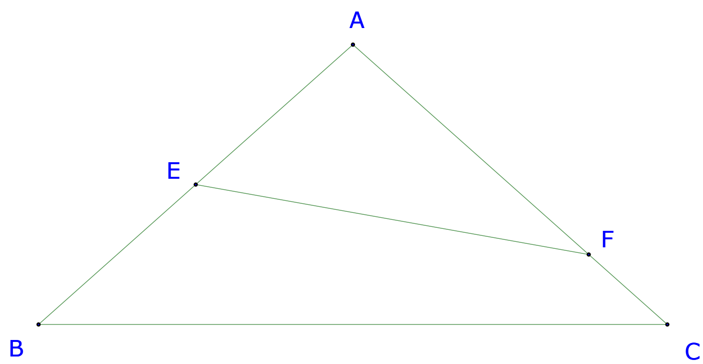
A pattern is in $A_j \cap A_k$, for $j \neq k$ if the edges adjacent to vertex j are painted in one of 6 ways, the two edges adjacent to vertex k not adjacent to vertex j are painted in one of 2 ways, and the one edge not adjacent to either vertex j or vertex k is painted in one of 3 ways. This shows that $A_j \cap A_k$ containing $6 \cdot 2 \cdot 3 = 36$ patterns. This is true for all $\binom{4}{2} = 6$ pairs of j and k.

By the Inclusion - Exclusion Principle, the number of ways to colouring the edges such that *at least two vertices* are colored with all three colors is:

$$\begin{aligned}
&|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4| \\
&- |A_1 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_4| - |A_1 \cap A_3 \cap A_4| - |A_2 \cap A_3 \cap A_4| \\
&+ |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= 6(6 \cdot 2 \cdot 3) - 4 \cdot 6 + 6 = 198
\end{aligned}$$

The required probability is $\dfrac{198}{3^6} = \frac{22}{81}$. Hence, $m + n = 22 + 81 = \boxed{103.}$ □

Problem 1.4.9 (Problem 9)**.** (*5 points*) In isosceles $\triangle ABC, AB = AC, \angle BAC$ is obtuse, and points E and F lie on sides AB and AC, respectively, so that $AE = 20, AF = 30$. The area of $\triangle AEF$ is 240, and the area of quadrilateral $BEFC$ is 408. Find BC.



Solution. Keep in mind that $\angle BAC > 90°$, so $\cos(\angle BAC) < 0$,

$$240 = [AEF] = \frac{1}{2}(AE \cdot AF)\sin(\angle EAF) = \frac{1}{2}(20 \cdot 30)\sin(\angle EAF) \Rightarrow \sin(\angle EAF) = \frac{4}{5}$$
$$[AEF] + [BEFC] = [ABC] = \frac{1}{2}(AB \cdot AC)\sin(\angle BAC) = \frac{1}{2}(AB^2)\frac{4}{5} \Rightarrow AB^2 = \frac{648 \cdot 2 \cdot 5}{4} = 1620$$
$$BC^2 = AB^2 + AC^2 - 2(AB \cdot AC)\cos(\angle BAC) = 1620(2)(1 - \cos(\angle BAC)) = 1620(2)\left(1 + \frac{3}{5}\right) = 5184.$$

Hence, $BC = \sqrt{5184} = \boxed{72.}$ □

Problem 1.4.10 (Problem 10)**.** (*5 points*) Find the least prime number greater than 1000 that divides $2^{1010} \cdot 235^{2020} + 1$.

Solution. First, let $n = 2^{1010} \cdot 235^{2020}$, then

$$n = 2^{1010} \cdot 23^{2020} + 1 = (4 \cdot 235^4)^{505} + 1 \Rightarrow 4 \cdot 235^4 + 1 \mid n.$$

By Sophie Germain identity $a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab)$ for $a = 1, b = 235$,

$$4 \cdot 235^4 + 1 = (1 + 2 \cdot 235^2 - 2 \cdot 235)(1 + 2 \cdot 235^2 + 2 \cdot 235) = (109 \cdot 1009) \cdot (110921).$$

Now, since $7 \mid 1001, 17 \mid 1003, 5 \mid 1005, 19 \mid 1007$, thus $\boxed{1009}$ is the first prime number larger than 1000 that divides n. □

Problem 1.4.11 (Problem 11)**.** (*5 points*) Find the maximum possible value of

$$8\sqrt{x} + 9\sqrt{y} + 12\sqrt{z},$$

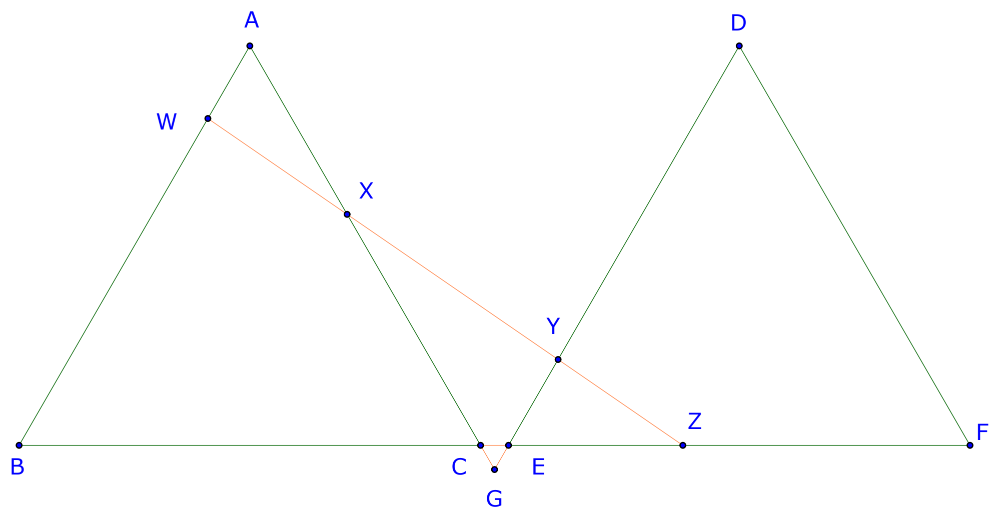
where x, y, and z are positive real numbers satisfying $4x + 9y + z = 256$.

Solution. By the Cauchy-Schwarz inequality,

$$8\sqrt{x}+9\sqrt{y}+12\sqrt{z} = (4)(\sqrt{4x})+(3)(\sqrt{9y})+(12)(\sqrt{z}) \leq \sqrt{(4^2 + 3^2 + 12^2)(4x + 9y + z)} = \sqrt{169 \cdot 256} = \boxed{208.}$$

□

Problem 1.4.12 (Problem 12). (*5 points*) Two congruent equilateral triangles $\triangle ABC$ and $\triangle DEF$ lie on the same side of line BC so that B, C, E, and F are collinear as shown. A line intersects AB, AC, DE, and EF at W, X, Y, and Z, respectively, such that $\dfrac{AW}{BW} = \frac{2}{9}$, $\dfrac{AX}{CX} = \frac{5}{6}$, and $\dfrac{DY}{EY} = \frac{9}{2}$. The ratio $\dfrac{XW}{XZ}$ can then be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution. Without loss of generality, let the triangles have side length 11, so

$$AW = 2, BW = 9, AX = 5, CX = 6, DY = 9, EY = 2.$$

Let G be the intersection of lines AC and DE, then $\triangle AWX \sim GYX$, so

$$\frac{GE + EY}{GC + CX} = \frac{GY}{GX} = \frac{AW}{AX} = \frac{2}{5} \Rightarrow \frac{GE + 2}{GC + 6} = \frac{2}{5} \Rightarrow GC = GE = \frac{2}{3}.$$

By Menelaus' Theorem for $\triangle ABC$ and line WXZ,

$$1 = \frac{AW}{BW} \cdot \frac{BZ}{CZ} \cdot \frac{CX}{AX} = \frac{2}{9} \cdot \frac{11 + CZ}{CZ} \cdot \frac{6}{5} \Rightarrow CZ = 4.$$

Again by Menelaus' Theorem for $\triangle BWZ$ and line AXC,

$$1 = \frac{BA}{AW} \cdot \frac{WX}{XZ} \cdot ZCCB = \frac{11}{2} \cdot \frac{WX}{XZ} \cdot \frac{4}{11} \Rightarrow \frac{WX}{XZ} = \frac{1}{2}.$$

Thus the desired sum $m + n = 1 + 2 = \boxed{3.}$ □

Problem 1.4.13 (Problem 13). (*5 points*) Find the number of permutations of the letters AAABBCC where there are no two like letters adjacent. For example, count ABCABCA and ABABCAC but not ABCCBAA.

Solution. Let first consider the letters $AAABB$. There are three cases that could lead to permutations with no repeated letters.

Case 1: there is 1 permutation of As and Bs such that no two like letters are next to each other: $ABABA$. For these, there are 6 positions around the 5 letters to place the two letters C, thus there are $\binom{6}{2} = 15$ ways to choose two distinct positions for Cs. Thus, in total, there are $1 \cdot 15 = 15$ ways for this case.

Case 2: there is 4 permutations of As and Bs that have one pair of like letters are adjacent. These are obtained by starting with $ABABA$, removing the beginning or ending A, then placing that A next to one of another A (there are two ways to *insert* right before of or right after it.) Into this permutation, a letter C must be used to insert in between the two neighbouring letters A. For the second letter C there are $7 - 2 = 5$ positions not immediately next to the first letter C. Thus, in total, there are $4 \cdot 5 = 20$ ways for this case.

Case 3: there is 9 permutations of As and Bs such that there are two set of like letters adjacent.

- There are 1 permutations where three letters A appear together: $BAAAB$.
- There are 2 permutations where there is a pair of adjacent letters A and a pair of adjacent letters B:

$$AABBA,\ ABBAA.$$

Into each of these permutation, two letter C must be used to insert in between the two neighbouring like letters. Thus, in total, there are $(1 + 2) \cdot 1 = 3$ ways for this case.

Therefore, altogether, there are $15 + 20 + 3 = \boxed{38}$ permutations. □

Problem 1.4.14 (Problem 14). (*5 points*) There is a real number x between 0 and $\dfrac{\pi}{2}$ such that

$$\frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{20}{19}$$

and $\sin x + \cos x = \frac{\sqrt{m}}{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Solution. Let $u = \sin(x), v = \cos(x)$, then $\sin^3 x + \cos^3 x = u^3 + v^3, \sin^5 x + \cos^5 x = u^5 + v^5$, and $u^2 + v^2 = 1$, furthermore

$$\begin{aligned}
&u^3 + v^3 = (u+v)(u^2 - uv + v^2) = (u+v)(1 - uv) \\
&u^5 + v^5 = (u+v)(u^4 - u^3v + u^2v^2 + uv^3 + u^4) = (u+v)((u^2+v^2)^2 - u^2v^2 - uv(u^2+v^2)) \\
&\ = (u+v)(1 - uv - u^2u^2)
\end{aligned}$$

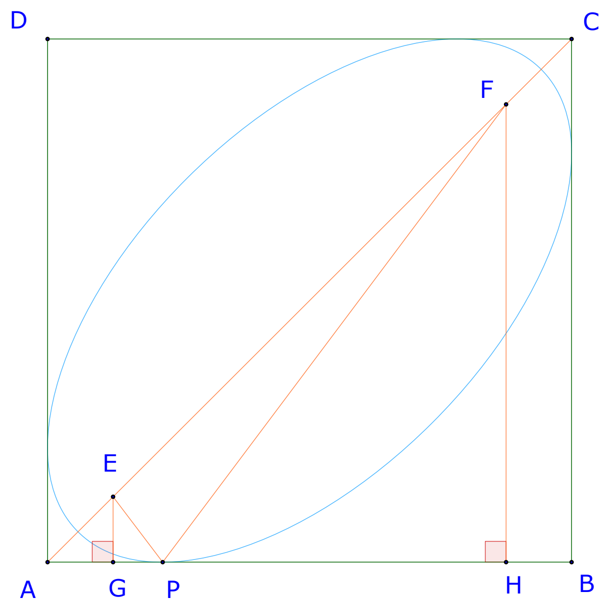
Now, let $s = u + v$, then $uv = \frac{s^2-1}{2}$, therefore

$$\begin{aligned}
&\frac{20}{19} = \frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{s\left(1 - \frac{s^2-1}{2}\right)}{s\left(1 - \frac{s^2-1}{2} - \left(\frac{s^2-1}{2}\right)^2\right)} = \frac{2(3-s^2)}{5-s^4} \\
&\Rightarrow 10s^4 - 19s^2 + 7 = 0 \Rightarrow s^2 = \frac{7}{5} \text{ or } \frac{1}{2}.
\end{aligned}$$

Since $0 \leq x \leq \frac{\pi}{2}$, so $u, v \geq 0$, thus $s^2 = 1 + 2uv \geq 1$, therefore $s^2 = \frac{7}{5} \Rightarrow s = \frac{\sqrt{35}}{5}$. Hence, the desired sum $m + n = 35 + 5 = \boxed{40.}$ □

Problem 1.4.15 (Problem 15)**.** (*5 points*) Points E and F lie on diagonal AC of square $ABCD$ with side length 24, such that $AE = CF = 3\sqrt{2}$. An ellipse with foci at E and F is tangent to the sides of the square. Let P be the tangent point of the eclipse with AB. Let points G and H be the projections onto AB of points E and F, respectively. Let $\frac{m}{n}$ be the value of $\cos EPF$, where m and n are relatively prime positive integers. Find $m + n$.

Hint: By the properties of the eclipse, a light ray passing from E to P that reflects of the eclipse will pass through point F. Because line AB is tangent to the eclipse, $\angle EPA = \angle FPB$.



Solution. Then $\triangle EGP \sim \triangle FHP$,

$$AE = CF = 3\sqrt{2}, \angle CAB = 45° \Rightarrow AG = BH = 3, FH = 24 - 3 = 21, GH = 24 - 2 \cdot 3 = 18$$
$$\frac{PG}{PH} = \frac{EG}{FH} = \frac{3}{21} = \frac{1}{7} \Rightarrow PG = GH \cdot \frac{PG}{PG + PH} = 18 \cdot \frac{PG}{PG + 7PG} = \frac{9}{4}$$
$$EP = \sqrt{EG^2 + PG^2} = \sqrt{3^2 + \left(\frac{9}{4}\right)^2} = \frac{15}{4}, FP = 7EP = 7\left(\frac{15}{4}\right)$$
$$EF^2 = EP^2 + PF^2 - 2(EP \cdot PF)\cos(\angle EPF) \Leftrightarrow 648 = 50\left(\frac{15}{4}\right)^2 - 14\left(\frac{15}{4}\right)^2 \cos(\angle EPF)$$
$$\Leftrightarrow 648 = \left(\frac{15}{4}\right)^2 (50 - 14\cos(\angle EPF)) \Rightarrow \cos(\angle EPF) = \frac{1}{14}\left(50 - \frac{18^2 \cdot 2 \cdot 4^2}{15^2}\right) = \frac{7}{25}.$$

Hence, $m + n = 7 + 25 = \boxed{32.}$ □

Problem 1.4.16 (Problem 16)**.** (*5 points*) A deck of eight cards has cards numbered $1, 2, 3, 4, 5, 6, 7$, in that order, and a deck of five cards has cards numbered $1, 2, 3, 4, 5, 6$ in that order. The two decks are riffle-shuffled together to form a deck with 13 cards with the cards from each deck in the same order as they were originally. Thus, numbers on the cards might end up in the order 1122334455667 or 1234561234567 but not 1223144553667. Find the number of possible sequences of the 13 numbers.

Solution. We show a coding solution. The 13-element *sequence* array is the one that holds a valid sequence. We use 7 loops, each to identify the indexes of the number $1, 2, 3, 4, 5, 6, 7$, when put them into the *sequence*. As you might see, we use the for loops in away that

$$0 \leq i_1 < i_2 < \ldots < i_7 \leq 12.$$

The indexes and the number $1, 2, 3, 4, 5, 6, 7$ are held temporarily in a dictionary d :

```
d = {i0: 1, i1:2, i2: 3, i3:4, i4: 5, i5: 6, i6:7}   # Python dictionary
```

Now, we go through a loop to fill the 13-element *sequence* array with the numbers $1, 2, 3, 4, 5, 6, 7$ by using the dictionary d, the rest of the positions in the array is filled with the numbers $1, 2, 3, 4, 5, 6$ whenever we find an empty slot

```
c = 1
for i in range(0, 13):
    if i in d:  # find an index that is mapped to a number in the dictionary
        sequence[i] = d[i]
    else:   # the dictionary does not has such index, fill it with the current
        number from 1, 2, 3, 4, 5, 6
        sequence[i] = c
        c += 1
```

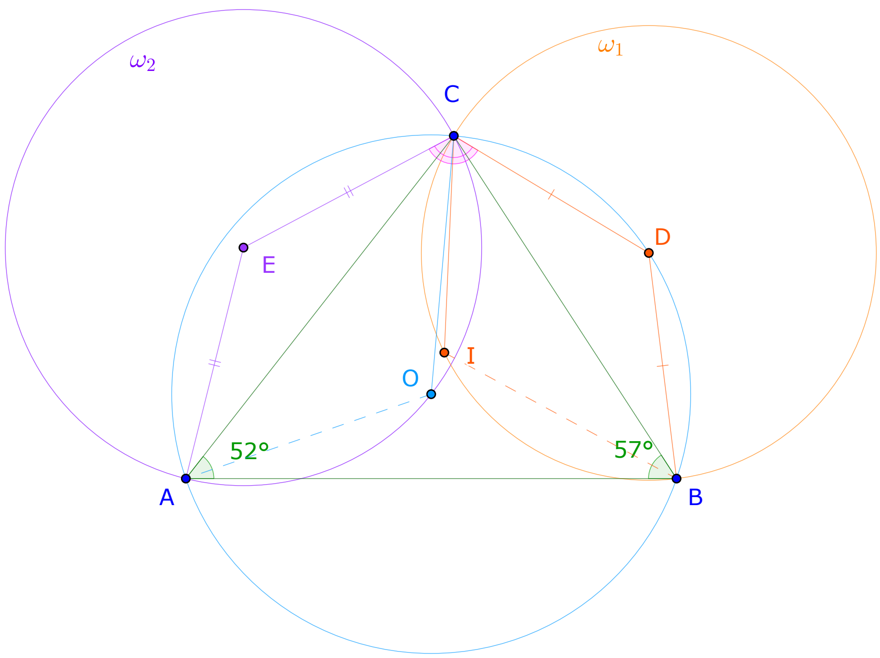
Once the sequence is generated correctly, then we put it into the set *all_sequences* containing all strings representing the sequences. Because it is a set, *all_sequences* does not contain duplicates can will hold correctly the number of found sequences.

```
sequence = [None for i in range(0, 13)]
all_sequences = set()
for i0 in range(0, 6):
    for i1 in range(i0+1, 7):
        for i2 in range(i1+1, 8):
            for i3 in range(i2+1, 9):
                for i4 in range(i3+1, 10):
                    for i5 in range(i4+1, 11):
                        for i6 in range(i5+1, 12):
                            d = {i0: 1, i1:2, i2: 3, i3:4, i4: 5, i5: 6, i6:7}
                            c = 1
                            for i in range(0, 13):
                                if i in d:
                                    sequence[i] = d[i]
                                else:
                                    sequence[i] = c
                                    c += 1
                            all_sequences.add(’’.join(str(e) for e in sequence))
print(len(all_sequences))
```

The answer is $\boxed{297.}$ ∎

Problem 1.4.17 (Problem 17)**.** (*5 points*) In $\triangle ABC$, $\angle A = 50°$, and $\angle B = 59°$. One circle centred at D passes through the points B, C, and the incenter I of $\triangle ABC$, and a second circle centred at E passes through the points A, C, and the circumcenter O of $\triangle ABC$. Find the degree measure of the acute angle at which the two circles intersect.

Hint: find the measure of the complementary angle of the angle $\angle DCE$.



Solution. First $\angle C = 180° - \angle A - \angle B = 71°$. Furthermore

$$\widehat{BI} = 2\angle BCI = \angle ACB,\ \widehat{CI} = 2\angle CBI = \angle ABC \Rightarrow \angle BDC = \widehat{BC} = \widehat{BI} + \widehat{CI} = \angle ACB + \angle ABC.$$

Since $\triangle BDC$ isosceles, in circle ω_1,

$$\angle BCD = \frac{1}{2}(180° - \angle BDC) = \frac{1}{2}(180° - \angle ACB + \angle ABC) = \frac{1}{2}\angle BAC.$$

On the other hand, in circle $O, \angle AOC = 2\angle AB$, so in circle ω_2,

$$\text{(major) } \widehat{AC} = 2\angle AOC = 4\angle ABC \Rightarrow \angle AEC = 360° - \widehat{AC} = 360° - 4\angle ABC.$$

Since $\triangle AEC$ is isosceles, in circle ω_2,

$$\angle ACE = \frac{1}{2}(180° - \angle AEC) = 2\angle ABC - 90°.$$

Therefore

$$\angle ICO = \angle ACB - (\angle BCI + \angle ACO) = 180° - \left(\frac{1}{2}\angle BAC + \angle ACB + 2\angle ABC - 90°\right)$$
$$= 270° - (\frac{1}{2}50° + 71° + 2 \cdot 59°) = \boxed{56°.}$$

□

Problem 1.4.18 (Problem 18)**.** (*5 points*) Three doctors, three nurses, and three patients stand in a line in random order. The probability that there is at least one doctor and at least one nurse between each pair of patients is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. We show a coding solution. The idea is to permute the given string 'DDDNNNPPP', then split it into smaller chunks, using the character P as a separator. The first and last chunks should be ignored. Then we verify if all chunks containing a D and an N letter.

```
from itertools import permutations
sequence = [e for e in 'DDDNNNPPP']
results = set()
for permutation in permutations(sequence):
    found = True
    chunks = (''.join(permutation)).split('P')
    chunks = chunks[1:-1] if len(chunks) == 4 else chunks[:-1] if permutation[0] ==
        'P' else chunks[1:]
    for s in chunks:
        if 'D' not in s or 'N' not in s:
            found = False
            break
    if found:
        results.add(''.join(permutation))
print(len(results))
```

The answer is $\boxed{114.}$ ☐

Problem 1.4.19 (Problem 19)**.** (*5 points*) Let p, q, and r be prime numbers such that $2pqr+p+q+r = 916$. Find $pq + qr + rp$.

Solution. Note that $p + q + r = 916 - 2pqr$, thus it is even, so at least one prime is even. WLOG, let $p = 2$.

$$4qr + q + r = 914 \Rightarrow (4q + 1)(4r + 1) = 4 \cdot 914 + 1 = 3657 = 3 \cdot 23 \cdot 53.$$

We are looking for pair of factors in the form of $4k + 1$ and they are $3 \cdot 23 = 69$. and 53. Easy to test and verify that the pair q, r is $13, 17$.

Hence, the desired sum is $pq + qr + rp = 2 \cdot 13 + 13 \cdot 17 + 17 \cdot 2 = \boxed{281.}$ ☐

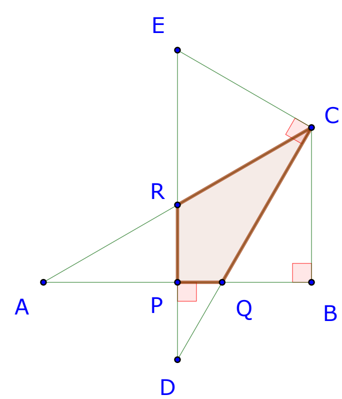
Chapter 2

January Challenges

2.1 I-Level

For both middle school ang high school students. No solution based on coding is allowed.

Problem 2.1.1 (Problem 1)**.** (*10 points*) We divide an equilateral triangle with area 600 into two halves and then arrange them as shown below. Find the area of the $PQCR$.



Solution. First let a be the side length of the equilateral triangle, then $BC = \frac{a}{2}, AB = \frac{a\sqrt{3}}{2}$. Since $\angle EDC = \angle DCB$, so $\triangle QBC \sim \triangle CBA$, thus

$$\frac{[QBC]}{[CBA]} = \left(\frac{CB}{AB}\right)^2 = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} \Rightarrow [QBC] = \frac{[CBA]}{3}.$$

Second, $\angle ECR = \angle PRC - \angle REC = \angle RAP + 90^\circ - 60^\circ = 60^\circ$. Thus $\triangle ECR$ is equilateral, and $RC = EC = CB = \frac{1}{2}AC$. because $RP \parallel CB$, so P is midpoint of AB, thus $[ARP] = \frac{1}{4}[ACB]$.

Therefore

$$[PQCR] = [ACB] - [ARP] - [QBC] = [ACB]\left(1 - \frac{1}{4} - \frac{1}{3}\right) = \frac{5}{12}[ACB] = \frac{5}{12}(300) = \boxed{125.}$$

$\square$

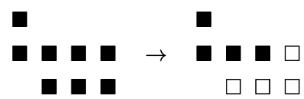
Problem 2.1.2 (Problem 2)**.** (*10 points*) Find all pairs of positive intergers (m, n) such that

$$1 \cdot 2 \cdots (m-1)(m) + 76 = n^2$$

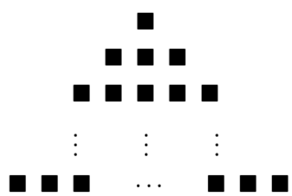
Solution. For $m \geq 7$, we have that $1 \cdot 2 \cdots (m-1)(m) = m! \equiv 0 \pmod 7$, $m! + 76 \equiv 6 \pmod 7$. For any perfect square $n^2 \equiv 0, 1, 2, 4 \pmod 7$, so there is no solution for $m \geq 7$.

For $m \leq 7$, by testing $4! + 76 = 10^2, 5! + 76 = 14^2$, and there is no solution for other value of m. Hence the solutions are $\boxed{(4, 10), (5, 14).}$ ☐

Problem 2.1.3 (Problem 3)**.** (*10 points*) Eight black squares are place on the plane, see the diagram below on the left. Without moving any of them, four of the squares can be coloured white, so that they are divided into *two congruent sets of different coloured squares*: a set of four black squares and a set of four white squares, as show below. *Note that one set is the mirror image of the other.*



Now, 100 black squares are placed on the plane as shown below *(19 black squares in the* 10^{th} *row.)*



In how many ways, without moving any of them, 50 of the squares can be can be coloured white, so that they are divided into two congruent sets of different coloured squares?

Solution. First, the bottom two corners are the only two tiles with one coordinate equal and the other differing by 18. Second, the only pairs in which both coordinates differ by 9 consists of the top corner and one of the bottom corners.

Now, suppose for a contradiction that a partition exists. By the Pigeonhole Principle, one set contains two of the three corners. By the two facts above, the other set cannot contain two tiles equivalently situated. ☐

Problem 2.1.4 (Problem 4)**.** (*10 points*) Let s be the sum of some integers, one of them has at least two digits. Two digits of one of the integers in s is chosen and they change places. The new sum is r. Which prime numbers 2, 3, 5, 7, 9, or 11 are always divisors of the difference $s - r$?

For example

$$s = 1 + 34 + 6 + 752 = 793, \ \underline{7}5\underline{2} \to \underline{2}5\underline{7} \Rightarrow r = 1 + 34 + 6 + 257 = 298 \Rightarrow s - r = 793 - 298 = 495.$$

Solution. First, since all different sums of integers are question, we can try a simple one to narrow down the possibility. Let $s = 21$, then $r = 12$ so $s - r = 9$. Thus the only possible primes are 3 and 9.

Now, for let n be the number in s whose two digits change their places:

$$n = \overline{\dots \underbrace{a \dots b \underbrace{\dots}_{n}}_{m}} \to \overline{\dots \underbrace{b \dots a \underbrace{\dots}_{n}}_{m}} \Rightarrow s - r = \overline{\dots \underbrace{a \dots b \underbrace{\dots}_{n}}_{m}} - \overline{\dots \underbrace{b \dots a \underbrace{\dots}_{n}}_{m}}$$

$$(a10^m + b10^n) - (b10^m + a10^n) = (10^m - 10^n)(a - b) = 10^n(9)(10^{m-n-1} + \cdots + 1).$$

Thus the difference is always divisible by $\boxed{3 \text{ and } 9.}$ ☐

2.2 A-Level

For both middle school ang high school students. No solution based on coding is allowed.

2.3 O-Level

For both middle school ang high school students. No solution based on coding is allowed.

Chapter 3

Entrance Tests

3.1 I-Level Entrance Test

For both middle school ang high school students. You are allowed to use books and calculator. No solution based on coding is allowed. No searching on the Internet for hints or solutions is permitted. **Full and detailed solution is required for every problem.**

Problem 3.1.1 (Problem 1)**.** (*10 points*) You are given a set of 10 positive integers. Summing nine of them in ten possible ways we get only nine different sums

$$86, 87, 88, 89, 90, 91, 93, 94, 95.$$

Find those numbers.

Solution. Let S be the sum of all ten positive integers and suppose x is the repeated sum. Call the elements $a_1, a_2, \ldots, a_10$, Then we have

$$S - a_1 = 86, S - a_2 = 87, \ldots, S - a_9 = 95, S - a_10 = x.$$

Adding these, $9S = 813+x$. The only value of x from $86, 87, 88, \ldots, 95$ which makes $813+x$ divisible by 9 is $x = 87$ and then $S = 100$. It follows that the ten numbers are respectively $\boxed{14, 13, 12, 11, 10, 9, 7, 6, 5, \text{ and } 13.}$ □

Problem 3.1.2 (Problem 2)**.** (*10 points*) Replace the asterisks in the equilateral triangle by the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ so that, starting from the second line, each number is equal to the absolute value of the difference of the nearest two numbers in the line above.

```
*   *   *   *   *   *   *   *   *
  *   *   *   *   *   *   *   *
    *   *   *   *   *   *   *
      *   *   *   *   *   *
        *   *   *   *   *
          *   *   *   *
            *   *   *
              *   *
                *
```

Note that: for this problem, submitting the equilateral triangle with all the asterisks replaced bu the number $1, 2, \ldots, 9$ *is considered as a solution. No additional explanation is required.*

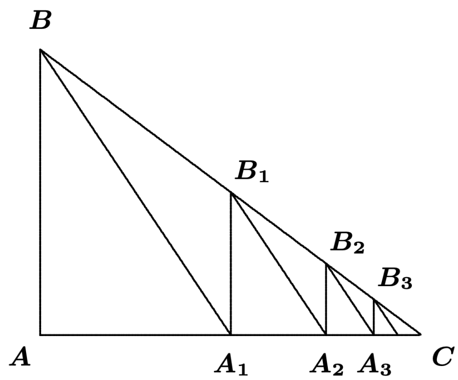
Solution. Below is one of possible solutions.

```
1   7   8   1   9   8   1   7   6
  6   1   7   8   1   7   6   1
    5   6   1   7   6   1   5
      1   5   6   1   5   4
        4   1   5   4   1
          3   4   1   3
            1   3   2
              2   1
                1
```

□

Problem 3.1.3 (Problem 3)**.** (*10 points*) In the diagram below, ABC is a right triangle, where $AB = 4, AC = 4$. Furthermore, each of the line segments $A_1B_1, A_2B_2, \ldots$ is perpendicular to AC, A_1 bisects segment AC, A_2 bisects segment A_1C, A_3 bisects segment A_2C, and so on. Find the sum of the lengths:

$$BA_1 + B_1A_2 + B_2A_3 + \ldots$$



Solution. Since A_1 bisects segment AC, $AA_1 = \frac{1}{2}AC = 2$, and $B_1A_1 = \frac{1}{2}BA = \frac{3}{2}$. By Pythagorean theorem,

$$BA_1 = \sqrt{BA^2 + AA_1^2} = \sqrt{3^2 + 2^2} = \sqrt{13}.$$

Each of the triangles $B_1A_1A_2$, $B_2A_2A_3, \ldots$ is similar to the triangle BAA_1, with the similarity ratio is $\frac{1}{2}$:

$$\frac{1}{2} = \frac{B_1A_1}{BA} = \frac{B_2A_2}{B_1A_1} = \frac{B_3A_3}{B_2A_2} = \cdots \Rightarrow \frac{1}{2} = \frac{B_1A_2}{BA_1} = \frac{B_2A_3}{B_1A_2} = \cdots$$

Therefore

$$BA_1 + B_1A_2 + B_2A_3 + \ldots = BA_1 + \left(\frac{1}{2}\right)BA_1 + \left(\frac{1}{2}\right)^2 BA_1 + \cdots = BA_1\frac{1}{1-\frac{1}{2}} = \boxed{2\sqrt{13}.}$$

□

Problem 3.1.4 (Problem 4)**.** (*10 points*) A four-digit number which is a perfect square is created by writing Lan's age in years followed by Nam's age in years. Similarly, after 31 years, their ages in the same order will again form a four-digit perfect square. Determine the present ages of Lan and Nam.

Note: for example if Lan's is 19 years old and Name is 11 years old then the four-digit number which is a perfect square is created by writing Lan's age in years followed by Nam's age in years is 1911.

Solution. After 31 years, each of these ages is a number with at least two digits, and then their ages in the same order will again form a four-digit perfect square, thus their ages then should be two-digit numbers. Since currently their ages in the same order form a four-digit perfect square, thus their present ages are also two-digit numbers.

Now let $\ell_1\ell_2$ and n_1n_2 be the their present ages, then $\ell_3\ell_4 = \ell_1\ell_2 + 31$, $n_3n_4 = n_1n_2 + 31$ are their ages after 31 years, then

$$\overline{\ell_3\ell_4n_3n_4} - \overline{\ell_1\ell_2n_1n_2} = (\overline{\ell_3\ell_4}\cdot 100 + \overline{n_3n_4}) - (\overline{\ell_1\ell_2}\cdot 100 + \overline{n_1n_2}) = (\overline{\ell_3\ell_4} - \overline{\ell_1\ell_2})\cdot 100 + (\overline{n_3n_4}) - \overline{n_1n_2}) = 31\cdot 100 + 31$$

Now, let the two perfect squares be $x^2 = \overline{\ell_1\ell_2n_1n_2}$, $y^2 = \overline{\ell_3\ell_4n_3n_4}$, then

$$y^2 - x^2 = 3131 = 31 \cdot 101 \Rightarrow \begin{cases} y - x = 31 \\ y + x = 101 \end{cases} \Rightarrow x = 35, y = 66 \Rightarrow x^2 = 1225, y^2 = 4356.$$

Thus, Lan's age now is 12, Nam's age now is 25. □

Chapter 4

Session 2: Feb 02 - Feb 10

4.1 Middle School - Assignment

Middle school students: grade 8 (US, CA), grade 9 (FR, UK, VN) and younger.

- **Submission deadline: Friday, February 9**
- **Test: Saturday, February 10**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 4.1.1 (Problem 1)**.** (*5 points*) Find the number of three-digit positive integers where the digits are three different prime numbers. For example, count 235 but not 553

Solution. There are 4 prime numbers that are digits: 2, 3, 5, and 7. They are all distinct. Thus, there are $4 \cdot 3 \cdot 2 = \boxed{24}$ three-digit positive integers where the digits are three different prime numbers. □

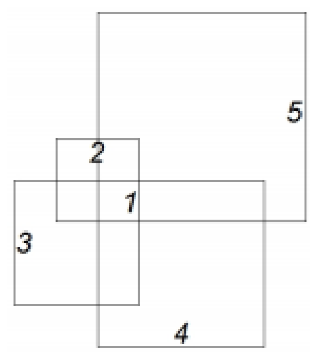
Problem 4.1.2 (Problem 2)**.** (*5 points*) Melanie has $4\frac{2}{5}$ cups of flour. The recipe for one batch of cookies calls for $1\frac{1}{2}$ cups of flour. Melanie plans to make $2\frac{1}{2}$ batches of cookies. When she is done, she will have $\frac{m}{n}$ cups of flour remaining, where m and n are relatively prime positive integers. Find $m+n$.

Solution. The number of cups of flour remaining is:

$$4\frac{2}{5} - 2\frac{1}{2} \cdot 1\frac{1}{2} = \frac{22}{5} - \frac{5}{2} \cdot \frac{3}{2} = \frac{13}{20}.$$

Hence, $m+n = 13+20 = \boxed{33.}$ □

Problem 4.1.3 (Problem 3)**.** (*5 points*) The figure below has a 1×1 square, a 2×2 square, a 3×3 square, a 4×4 square, and a 5×5 square. Each of the larger squares shares a corner with the 1×1 square. Find the area of the region covered by the 1×1, 2×2, 3×3, and 4×4 squares, and the 5×5 square.



Solution. Let $S_1, S_2, \ldots, S_5$ denote the area of the squares 1×1, 2×2, $\ldots$, and 5×5.

It is easy to see that the area cover by the squares 1×1, 2×2 is the square 2×2.

If we add the area cover by the squares 3×3 to it we can see that they overlap a 1×2 area, thus three of them cover $4 + 9 - 2 = 11$.

If we add the area cover by the squares 4×4 to it we can see that they overlap a 1×3 area, thus four of them cover $16 + 11 - 3 = 24$.

If we add the area cover by the squares 5×5 to it we can see that they overlap the squares 1×1, 1×1, and 1×3 thus five of them cover $24 + 25 - (1 + 1 + 3) = \boxed{44.}$ □

Problem 4.1.4 (Problem 4)**.** (*5 points*) Find the value of x such that $2^{x+3} - 2^{x-3} = 2016$.

Solution.

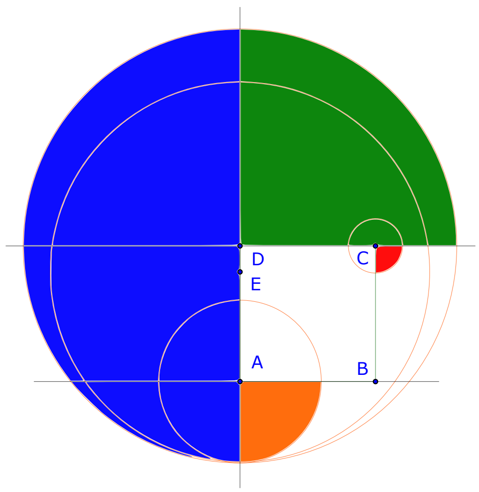
$$2016 = 2^{x+3} - 2^{x-3} = 2^{x-3}(2^6 - 1) \Rightarrow 2^{x-3} = 32 \Rightarrow x = \boxed{8}.$$

□

Problem 4.1.5 (Problem 5)**.** (*5 points*) Mildred the cow is tied with a rope to the side of a square shed with side length 10 meters. The rope is attached to the shed at a point two meters from one corner of the shed. The rope is 14 meters long. The area of grass growing around the shed that Mildred can reach is given by $n\pi$ square meters, where n is a positive integer. Find n.

Solution. The total area is the sum of the areas of:

- half of the blue circle centred at E, where the rope is tied to, with radius 14;
- a quarter of the orange circle centred at A with radius $14 - 8 = 6$;
- a quarter of the green circle centred at D with radius $14 - 2 = 12$;
- a quarter of the red circle centred at C with radius $14 - 2 - 10 = 2$.



Thus, $\left(\frac{14^2}{2} + \frac{6^2}{4} + \frac{12^2}{4} + \frac{2^2}{4}\right)\pi = \boxed{144}\pi$. □

Problem 4.1.6 (Problem 6)**.** (*5 points*) One evening a theater sold 300 tickets for a concert. Each ticket sold for \$40, and all tickets were purchased using \$5, \$10, and \$20 bills. At the end of the evening the theater had received twice as many \$10 bills as \$20 bills, and 20 more \$5 bills than \$10 bills. How many bills did the theater receive altogether?

Solution. Let n be the number of \$20 bills the theater received. Then the number of \$10 bills they received is $2n$, and the number of \$5 bills they received is $2n + 20$, for a total amount collected equalling

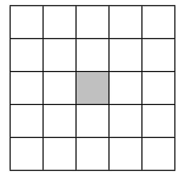
$$5(2n + 20) + 10(2n) + 20n = 300 \cdot 40 \Rightarrow 50n + 100 = 12000.$$

The amount of bills is

$$n + 2n + 2n + 20 = 5n + 20 = \frac{12000}{10} + 10 = \boxed{1210.}$$

□

Problem 4.1.7 (Problem 7)**.** (*5 points*) Find the number of squares such that the sides of the square are segments in the following diagram and where the shaded area is inside the square.



Solution. To count the square let's seletc one of the three vertical line segments to the left of the shaded square and one of the 3 vertical line segments to the right.

Of these $3 \cdot 3 = 9$, ways to select vertical line segments. 1 selection gives segments a distance 1 apart, 2 selections give segments a distance 2 apart, 3 selections give segments a distance 3 apart, 2 selections give segments a distance 4 apart, and 1 selection gives segments a distance 5 apart.

There is the same number of ways of selecting horizontal line segments a distance k apart that make up the side of a square containing the shaded square as there are selecting vertical line segments a distance k apart, so the total number of ways of selecting line segments that make up the sides of a square containing the shaded area is $1^2 + 2^2 + 3^2 + 2^2 + 1^2 = \boxed{19.}$ □

Problem 4.1.8 (Problem 8)**.** (*5 points*) One afternoon Elizabeth noticed that twice as many cars on the expressway carried only a driver as compared to the number of cars that carried a driver and one passenger.

She also noted that twice as many cars carried a driver and one passenger as those that carried a driver and two passengers.

10% of the cars carried a driver and three passengers, and no car carried more than four people.

Any car containing at least three people was allowed to use the fast lane. Elizabeth calculated that $\frac{m}{n}$ of the people in cars on the expressway were allowed to ride in the fast lane, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Let n be the number of the cars that have a driver and two passengers, then

Car type	Number of cars	Number of people
Only a driver	$4n$	$1 \cdot 4n$
A driver and one passenger	$2n$	$2 \cdot 2n$
A driver and two passengers	n	$3 \cdot n$
A driver and three passengers	$\frac{n+2n+4n}{9}$	$4 \cdot \frac{7n}{9}$

The number of people in cars that can use the fast lane is the number of people in cars carrying a driver and two or three passengers, which is $3n + \frac{28n}{9} = \frac{55n}{9}$. The total number of people is $4n + 4n + 3n + \frac{28n}{9} = \frac{127n}{9}$.

The ratio of the people in cars were allowed to ride in the fast lane is $\frac{55n}{9} : \frac{127n}{9} = \frac{55}{127}$.

Hence, $m + n = 55 + 127 = \boxed{182.}$ □

Problem 4.1.9 (Problem 9)**.** (*5 points*) Find the number of positive integers n such that a regular polygon with n sides has internal angles with measures equal to an integer number of degrees.

Solution. The measure of an internal angle, in degrees, of a regular polygon with n sides is:

$$\frac{(n-2)180}{n} = 180 - \frac{360}{n}.$$

This measure is an integer if and only if n is at least 3 and it is a divisor of $360 = 2^3 \cdot 3^2 \cdot 5$.

Since 360 has $(3+1)(2+1)(1+1) = 24$ divisors, and $\boxed{22}$ of them are at least 3. □

Problem 4.1.10 (Problem 10). (*5 points*) The real numbers x, y, and z satisfy the system of equations

$$\begin{cases} x^2 + 27 = -8y + 10z \\ y^2 + 196 = 18z + 13x \\ z^2 + 119 = -3x + 30y. \end{cases}$$

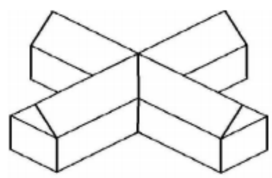
Find $x + 3y + 5z$.

Solution. Adding the equations yields

$$x^2 + y^2 + z^2 + 342 = 10x + 22y + 28z \Rightarrow (x-5)^2 + (y-11)^2 + (z-14)^2 = 0 \Rightarrow x = 5, y = 11, z = 14.$$

Hence, $x + 3y + 5z = 5 + 3 \cdot 11 + 5 \cdot 14 = \boxed{108.}$ □

Problem 4.1.11 (Problem 11). (*5 points*) The figure below shows a barn in the shape of two congruent pentagonal prisms that intersect at right angles and have a common center. The ends of the prisms are made of a 12 foot by 7 foot rectangle surmounted by an isosceles triangle with sides 10 feet, 10 feet, and 12 feet. Each prism is 40 feet long. Find the volume of the barn in cubic feet.



Solution. The volume is the sum of the volume of the 2 pentagonal prisms minus the common center area.

Lets split the pentagonal prism into 2 parts: the top part which is where the isoceles triangle is and the bottom part which is the rectangle underneath the isoceles triangle. The top part is a triangular prism with a triangular base $10, 10, 12$, height 40, so its volume is 1920. The bottom part is a box with side lengths $7, 12, 40$, thus its volume is $7 \cdot 12 \cdot 40 = 3360$. Consequently, the volume of a pentagonal prism is $1920 + 3360 = 5280$.

Now all we need to find the volume of the common center. We can split the common center place into 2 parts as well. The top part which is a pyramid and the bottom which is a box. The top part volume is a pyramid with base side length 12, height 8, so its volume is $\frac{12 \cdot 12 \cdot 8}{3} = 384$. The bottom part is a box which has side lengths $12, 12, 7$ thus its volume is $12 \cdot 12 \cdot 7 = 1008$ Therefore, the total volume of the common center is $1008 + 384 = 1392$.

Thus, the answer is $2 \cdot 5280 - 1392 = \boxed{9168}$. □

Problem 4.1.12 (Problem 12). (*5 points*) Suzie flips a fair coin 6 times. The probability that Suzie flips 3 heads in a row but not 4 heads in a row is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Three consecutive head, but not four:

$$\text{HHHTXX}, \ \text{THHHTX}, \ \text{XTHHHT}, \ \text{XXTHHH}$$

where H denote head, T denote tail and X denote either head or tail.

There are 2 string of the format THHHTX, because X can be fill in 2 ways, 2 string of the format XTHHHT, because X can be fill in 2 ways, $2 \cdot 2$ strings of HHHTXX, and $2 \cdot 2$ strings of XXTHHH. Thus total number of favourable cases are $2 + 2 + 4 + 4 = 12$.

Total possible outcome is $2^6 = 64$. Hence probability is $\frac{12}{64} = \frac{3}{16}$.

Hence $m = 3, n = 16$ and $m + n = \boxed{19}$. □

Problem 4.1.13 (Problem 13)**.** (*5 points*) Find the least positive integer N that is 50 times the number of positive integer divisors that N has.

Solution. Let $N = p_1^a p_2^b \cdots$ for distinct prime numbers $p_1, p_2, \ldots$ and $a, b, \ldots$ positive integers. Then N has $(a+1)(b+1)\cdots$ divisors. Since $50 = 2 \cdot 5^2$ so the *least* N must have

$$p_1^a p_2^b \cdots = 2^1 5^2 (a+1)(b+1)\cdots$$

Thus two of the primes $p_1, p_2, \ldots$ must be 2 and 5.

Case 1: $N = 2^a 5^b$.

$$N = 2^a 5^b = 2^1 5^2 (a+1)(b+1) \Rightarrow b \geq 2.$$

Case 1a: $b = 2$. Then $3 \mid 2^a 5^b$, impossible.

Case 1b: $b = 3$. Then $5^b = 5^3$, thus $5 \mid a+1$, or $a \geq 4$, so $N \geq 2^4 5^3 = 2000$.

Case 1c: $b \geq 4$. It is easy to prove that $5^b \geq 5(b+1)$, thus $a+1 \geq 10$, or $a \geq 9$, so $N \geq 2^9 5^2 > 2000$.

Case 2: $N = 2^a 3^b 5^c$.

$$N = 2^a 3^b 5^c = 2^1 5^2 (a+1)(b+1)(c+1).$$

With the least possible of $c = 2$, then $b = 2, a = 3$, then $N = \boxed{1800}$ is a solution.

For $N = 2^a 5^b p^c$, where $p \geq 7$ leads to $N > 2000$. □

Problem 4.1.14 (Problem 14)**.** (*5 points*) Find the positive integer n such that the least common multiple of n and $n - 30$ is $n + 1320$.

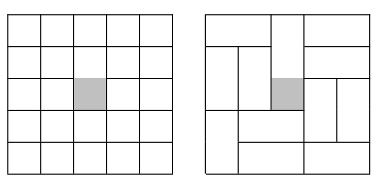
Solution. First, since $n + 1320 = \text{lcm}(n, n-30)$, so $n(n-30) \geq n + 1320$, or $n \geq 55$.

Second, $n \mid n + 1320$, so $n \mid 1320$, the divisors of 1320 that are at least 55 are 55, 60, 66, 88, 110, 120, 132, 165, 220, 264, 330, 440, 660, and 1320.

Third, $n - 30 \mid (n + 1320) - (n - 30) = 1350$. Only $n = 55, 60, 120$, and 165 satisfies $n - 30 \mid 1350$.

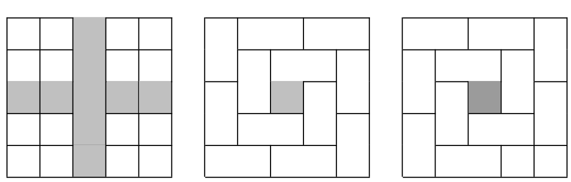
By testing, it is easy to see that only $n = \boxed{165}$ satisfies $n + 1320 = \text{lcm}(n, n-30)$. □

Problem 4.1.15 (Problem 15)**.** (*5 points*) The 24 unshaded squares in the 5×5 grid below can be tiled with twelve 1×2 tiles. One such tiling is shown below.



Find the number of ways the grid can be tiled.

Solution. Call the two-squares pairs that align with the shaded central square *bridges* which are shaded in the left diagram below. An organized way to count the tilings is to consider how many of these four bridges are tiled by a single tile.

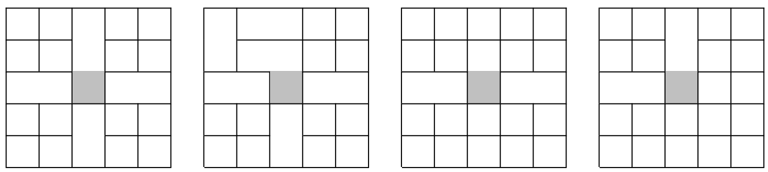


Case 1: First, note that if any one bridge is tiled by two different tiles that do not aligned, then there is only one way to complete the tiling of remaining unshaded squares. Thus, there are only two tilings of this type, both shown above.

Case 2: If all four bridge are each tiled by a single tile as in the first diagram below, it leaves four 2×2 grids which can each be tiled in 2 different ways. Thus, this accounts for $2^4 = 16$ tilings.

Case 3: If exactly three bridges are each tiled by a single tile as shown in the second diagram below, then there are 4 ways to select the one bridge that is not tiled by a single tile. There are two ways to tile the bridge that is tiled by two tiles. This leaves three 2×2 grids that can each be tiled in 2 ways. Thus, this accounts for $4 \cdot 2 \cdot 2^3 = 64$ tilings.

Case 4: If exactly two bridges are each tiled by a single tile, then the two bridges tiled by a single tile can be across the center from each other as shown in the third diagram below, or the two bridges tiled by a single tile can be adjacent bridges as shown in the fourth diagram below.



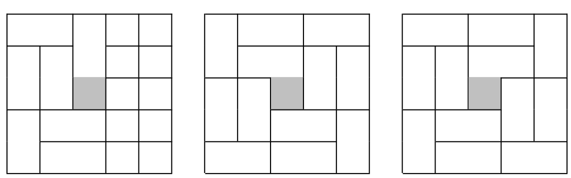
In the first case, there are 2 ways to select the two bridges each tiled by a single tile, and that leaves two 5×2 grids which can each be tiled in one of 4 ways (2 ways to tile the included bridge and 2 ways to tile a 2×2 grid that remains.) This accounts for $2 \cdot 4^2 = 32$ tilings.

In the second case, there are 4 ways to select the two bridges each tile by a single bridge. That leaves 3 ways to tile the remaining two bridges, and each of these 3 ways leaves two 2×2 grids, each to be tiled in one of 2 ways. This accounts for $4 \cdot 3 \cdot 2^2 = 48$. tilings.

Thus, there are $32 + 48 = 80$ tilings where exactly two bridges are each tiled by a single tile.

Case 5: If exactly one bridge is tile by a single tile as shown in the first diagram below, then there are 4 ways to select the one bridge that is tiled by a single tile. There are then two ways to tile the bridge on the opposite side of the center square. This determines how to tile the grid, except for on 5×2 grid that can be tiles in one of 4 ways. This accounts for $4 \cdot 2 \cdot 4 = 32$ tilings.

Case 6: If none of the bridges are tiled by a single tile, there are two ways to tile the grid as shown in the second and third diagrams below accounting for 2 tilings.



Hence, the number of tilings is $2 + 16 + 64 + 80 + 32 + 2 = \boxed{196.}$ □

4.2 Middle School - Test

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 4.2.1 (Problem 1)**.** (*5 points*) Find the number of four-digit positive integers where the digits are four different prime numbers. For example, count 2357 but not 5537

Solution. There are 4 prime numbers that are digits: 2, 3, 5, and 7. They are all distinct. Thus, there are $4 \cdot 3 \cdot 2 \cdot 1 = \boxed{24}$ three-digit positive integers where the digits are three different prime numbers. □

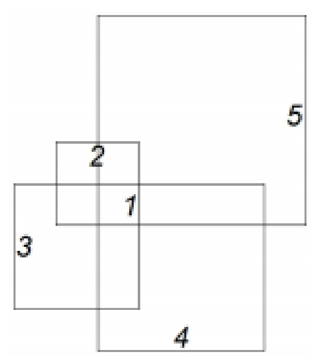
Problem 4.2.2 (Problem 2)**.** (*5 points*) Melanie has $3\frac{4}{5}$ cups of flour. The recipe for one batch of cookies calls for $1\frac{1}{2}$ cups of flour. Melanie plans to make $2\frac{1}{2}$ batches of cookies. When she is done, she will have $\frac{m}{n}$ cups of flour remaining, where m and n are relatively prime positive integers. Find $m + n$.

Solution. The number of cups of flour remaining is:

$$3\frac{4}{5} - 2\frac{1}{2} \cdot 1\frac{1}{2} = \frac{19}{5} - \frac{5}{2} \cdot \frac{3}{2} = \frac{1}{20}.$$

Hence, $m + n = 1 + 20 = \boxed{21.}$ □

Problem 4.2.3 (Problem 3)**.** (*5 points*) The figure below has a 1×1 square, a 2×2 square, a 3×3 square, a 4×4 square, and a 5×5 square. Each of the larger squares shares a corner with the 1×1 square. Find the area of the region covered by the 1×1, 2×2, 3×3, and 4×4 squares, but not covered by the 5×5 square.



Solution. Let $S_1, S_2, \ldots, S_5$ denote the area of the squares 1×1, 2×2, $\ldots$, and 5×5.

It is easy to see that the area cover by the squares 1×1, 2×2 is the square 2×2.

If we add the area cover by the squares 3×3 to it we can see that they overlap a 1×2 area, thus three of them cover $4 + 9 - 2 = 11$.

If we add the area cover by the squares 4×4 to it we can see that they overlap a 1×3 area, thus four of them cover $16 + 11 - 3 = 24$.

Now, we need to remove the area covered by the square 5×5 overlapping with it, and it is a 1×1, a 1×1, and a 1×3 thus $24 - 1 - 1 - 3 = \boxed{19.}$ □

Problem 4.2.4 (Problem 4)**.** (*5 points*) Find the value of x such that $2^{x+5} - 3 \cdot 2^{x-3} = 2024$.

Solution.

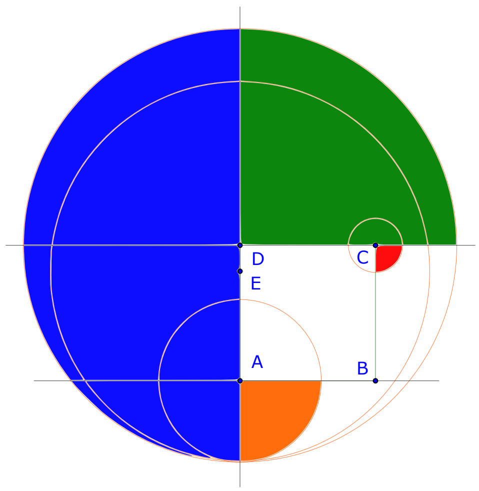
$$2024 = 2^{x+5} - 3 \cdot 2^{x-3} = 2^{x-3}(2^8 - 3) \Rightarrow 2^{x-3} = 8 \Rightarrow x = \boxed{6}.$$

□

Problem 4.2.5 (Problem 5)**.** (*5 points*) Mildred the cow is tied with a rope to the side of a square shed with side length 10 meters. The rope is attached to the shed at a point two meters from one corner of the shed. The rope is 13 meters long. The area of grass growing around the shed that Mildred can reach is given by $\frac{m\pi}{n}$ square meters, where m and n are relatively prime positive integers. Find $m+n$.

Solution. The total area is the sum of the areas of:

- half of the blue circle centred at E, where the rope is tied to, with radius 13;
- a quarter of the orange circle centred at A with radius $13-8=5$;
- a quarter of the green circle centred at D with radius $13-2=11$;
- a quarter of the red circle centred at C with radius $13-2-10=1$.



Thus, $\left(\dfrac{13^2}{2}+\dfrac{5^2}{4}+\dfrac{11^2}{4}+\dfrac{1^2}{4}\right)\pi = \frac{485\pi}{4}$.

Hence $m+n = 485+4 = \boxed{489.}$ □

Problem 4.2.6 (Problem 6)**.** (*5 points*) One evening a theater sold 300 tickets for a concert. Each ticket sold for \$40, and all tickets were purchased using \$5, \$10, and \$20 bills. At the end of the evening the theater had received twice as many \$10 bills as \$20 bills, and 10 more \$5 bills than \$10 bills. How many bills did the theater receive altogether?

Solution. Let n be the number of \$20 bills the theater received. Then the number of \$10 bills they received is $2n$, and the number of \$5 bills they received is $2n+10$, for a total amount collected equalling

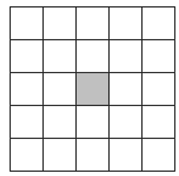
$$5(2n+10)+10(2n)+20n = 300\cdot 40 \Rightarrow 50n+50 = 12000.$$

The amount of bills is

$$n+2n+2n+10 = 5n+10 = \frac{12000}{10}+5 = \boxed{1205.}$$

□

Problem 4.2.7 (Problem 7)**.** (*5 points*) The 5×5 grid below is made up by *unit* squares. The center square is shaded. Find the number of squares with side lengths 1, 3, or 5, such that the sides of the square are segments in the grid, and the shaded area is inside the square.



Solution. To count the square let's select one of the three vertical line segments to the left of the shaded square and one of the 3 vertical line segments to the right.

Of these $3 \cdot 3 = 9$, ways to select vertical line segments. 1 selection gives segments a distance 1 apart, 2 selections give segments a distance 2 apart, 3 selections give segments a distance 3 apart, 2 selections give segments a distance 4 apart, and 1 selection gives segments a distance 5 apart.

There is the same number of ways of selecting horizontal line segments a distance k apart that make up the side of a square containing the shaded square as there are selecting vertical line segments a distance k apart, so the total number of ways of selecting line segments that make up the sides of a square with **odd area** containing the shaded area is $1^2 + 3^2 + 1^2 = \boxed{11.}$ ☐

Problem 4.2.8 (Problem 8)**.** (*5 points*) One afternoon Elizabeth noticed that twice as many cars on the expressway carried only a driver as compared to the number of cars that carried a driver and one passenger.

She also noted that twice as many cars carried a driver and one passenger as those that carried a driver and two passengers.

10% of the cars carried a driver and three passengers, and no car carried more than four people.

Any car containing at least three people was allowed to use the fast lane. Elizabeth calculated that $\frac{m}{n}$ of the people in cars on the expressway were **not** allowed to ride in the fast lane, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Let n be the number of the cars that have a driver and two passengers, then

Car type	Number of cars	Number of people
Only a driver	$4n$	$1 \cdot 4n$
A driver and one passenger	$2n$	$2 \cdot 2n$
A driver and two passengers	n	$3 \cdot n$
A driver and three passengers	$\frac{n+2n+4n}{9}$	$4 \cdot \frac{7n}{9}$

The number of people in cars that can not use the fast lane is the number of people in cars carrying a driver and at most one passenger, which is $4n + 4n = 8n$. The total number of people is $4n + 4n + 3n + \frac{28n}{9} = \frac{127n}{9}$.

The ratio of the people in cars were allowed to ride in the fast lane is $8n : \frac{127n}{9} = \frac{72}{127}$.

Hence, $m + n = 72 + 127 = \boxed{199.}$ ☐

Problem 4.2.9 (Problem 9)**.** (*5 points*) Find the number of **odd** positive integers n such that a regular polygon with n sides has internal angles with measures equal to an integer number of degrees.

Solution. The measure of an internal angle, in degrees, of a regular polygon with n sides is:

$$\frac{(n-2)180}{n} = 180 - \frac{360}{n}.$$

This measure is an integer if and only if n is at least 3 and it is a divisor of $360 = 2^3 \cdot 3^2 \cdot 5$.

Since n must be odd, so n is a divisor of $3^2 \cdot 5$, and there are $(2+1)(1+1) = 6$ such divisors, and $\boxed{5}$ of them are at least 3: $3, 5, 9, 15$, and 45. □

Problem 4.2.10 (Problem 10)**.** (*5 points*) The real numbers x, y, and z satisfy the system of equations

$$\begin{cases} x^2 + 27 = -8y + 10z \\ y^2 + 196 = 18z + 13x \\ z^2 + 119 = -3x + 30y. \end{cases}$$

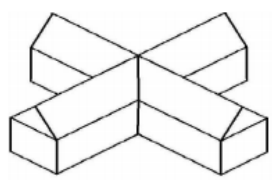
Find $5x + 3y + z$.

Solution. Adding the equations yields

$$x^2 + y^2 + z^2 + 342 = 10x + 22y + 28z \Rightarrow (x-5)^2 + (y-11)^2 + (z-14)^2 = 0 \Rightarrow x = 5, y = 11, z = 14.$$

Hence, $5x + 3y + z = 5 \cdot 5 + 3 \cdot 11 + 14 = \boxed{72.}$ □

Problem 4.2.11 (Problem 11)**.** (*5 points*) The figure below shows a barn in the shape of two congruent pentagonal prisms that intersect at right angles and have a common center. The ends of the prisms are made of a 6 foot by 4 foot rectangle surmounted by an isosceles triangle with sides 5 feet, 5 feet, and 6 feet. Each prism is 40 feet long. Find the volume of the barn in cubic feet.



Solution. The volume is the sum of the volume of the 2 pentagonal prisms minus the common center area.

Lets split the pentagonal prism into 2 parts: the top part which is where the isoceles triangle is and the bottom part which is the rectangle underneath the isoceles triangle. The top part is a triangular prism with a triangular base $5, 5, 6$, height 40, so the area of the triangle is 12, its volume is 480.

The bottom part is a box with side lengths $4, 6, 40$, thus its volume is $4 \cdot 6 \cdot 40 = 960$. Consequently, the volume of a pentagonal prism is $480 + 960 = 1440$.

Now all we need to find the volume of the common center. We can split the common center place into 2 parts as well. The top part which is a pyramid and the bottom which is a box. The top part volume is a pyramid with base side length 6, height 4, so its volume is $\frac{6 \cdot 6 \cdot 4}{3} = 48$. The bottom part is a box which has side lengths $6, 6, 4$ thus its volume is $6 \cdot 6 \cdot 4 = 144$ Therefore, the total volume of the common center is $48 + 144 = 192$.

Thus, the answer is $2 \cdot 1440 - 192 = \boxed{2688}$. □

Problem 4.2.12 (Problem 12)**.** (*5 points*) Suzie flips a fair coin 7 times. The probability that Suzie flips 4 heads in a row but not 5 heads in a row is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Four consecutive heads, but not five:

$$\text{HHHHTXX, THHHHTX, XTHHHHT, XXTHHHH,}$$

where H denote head, T denote tail and X denote either head or tail.

There are 4 string of the format HHHHTXX because XX can be fill in 4 ways, 2 string of the format THHHHTX because X can be fill in 2 ways, 2 strings of XTHHHHT and 4 strings of XXTHHHH. Thus total number of favourable cases are $4 + 2 + 2 + 4 = 12$.

Total possible outcome is $2^7 = 128$. Thus, the probability is $\frac{12}{128} = \frac{3}{32}$.

Hence $m = 3, n = 32$ and $m + n = \boxed{35}$. □

Problem 4.2.13 (Problem 13)**.** (*5 points*) Find the least positive integer N that is 49 times the number of positive integer divisors that N has.

Solution. [Solution 1] Let $N = p_1^a p_2^b \cdots$ for distinct prime numbers $p_1, p_2, \ldots$ and $a, b, \ldots$ positive integers. Then N has $(a+1)(b+1)\cdots$ divisors. Since $50 = 2 \cdot 5^2$ so the *least* N must have

$$p_1^a p_2^b \cdots = 7^2(a+1)(b+1)\cdots$$

Thus one of the primes $p_1, p_2, \ldots$ must be 7. *Case 1*: $N = 2^a 7^b$.

$$N = 2^a 7^b = 7^2(a+1)(b+1).$$

Case 1a: $b = 2$. Then $3 \mid 2^a 7^b$, impossible.

Case 1b: $b = 3$. Then $7^b = 7^3$, thus $7 \mid a+1$, or $a \geq 6$, With $a = 6$ then $N \geq 2^6 7^3 > 2000$.

Case 1c: $b \geq 4$. Then $N > 2000$.

Case 2: $N = 3^a 7^b$.

$$N = 3^a 7^b = 7^2(a+1)(b+1).$$

For $b = 2 : 3^a = (a+1)3$, so $a = 2$ thus $N = \boxed{441}$. □

Solution. [Solution 2] We show a coding solution. The *count_divisor* function count all divisors of n between 2 and $\left\lfloor \frac{n}{2} \right\rfloor$.

```
def count_divisors(n):
    c = 2
    for i in range(2, n+1 // 2):
        if n % i == 0:
            c += 1
    return c

n = 1
while True:
    n_dc = count_divisors(n)
    if n_dc * 49 == n:
        print(n)
        break
    n += 1
```

Thus $N = \boxed{441}$. □

Problem 4.2.14 (Problem 14). (*5 points*) Find the positive integer n such that the least common multiple of n and $n-30$ is $n+4900$.

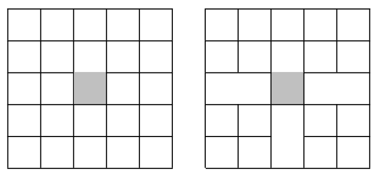
Solution. First, since $n+4900 = \text{lcm}(n, n-30)$, so $n(n-30) \geq n+4900$, or $n \geq 87$.

Second, $n \mid n+4900$, so $n \mid 4900$, the divisors of 4900 that are at least 87 are 98, 100, 140, 175, 196, 245, 350, 490, 700, 980, 1225, 2450, and 4900.

Third, $n-30 \mid (n+4900)-(n-30) = 4930$. The divisors of 4930 that are at least 57 are 58, 85, 145, 170, 290, 493, 986, 3465, and 4930.

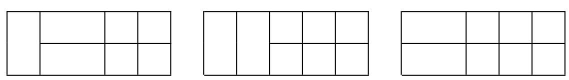
Only $n = \boxed{175}$ satisfies $n-30 \mid 4930$. □

Problem 4.2.15 (Problem 15). (*5 points*) The 24 unshaded squares in the 5×5 grid below (the left diagram) are already tiled by three 1×2 tiles (the right diagram).



Find the number of ways the rest of the grid can be tiled by nine 1×2 tiles.

Solution. For the upper left 2×5 grid as shown below, we can consider three cases, as show below.



The first case has 2 ways to tile the right 2×2 grid. For each of the second and third cases there are 3 ways to tile the right 2×3 grid. Thus, there 8 ways to tile this 2×5 grid. Each of the 2×2 grids on the bottom left and bottom right can each be tiled in 2 ways. Hence, the number of tilings is $8 \times 2 \times 2 = \boxed{32.}$ □

4.3 High School - Assignment

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

- **Submission deadline: Friday, February 9**
- **Test: Saturday, February 10**
- **Official solutions: Monday, January 29**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**
- **Olympiad (O) level: Problems 12-21**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 4.3.1 (Problem 1)**.** (*5 points*) Jeremy wrote all the three-digit integers from 100 to 999 on a blackboard. Then Allison erased each of the 2700 digits Jeremy wrote and replaced each digit with the square of that digit. Thus, Allison replaced every 1 with a 1, every 2 with a 4, every 3 with a 9, every 4 with a 16, and so forth. The proportion of all the digits Allison wrote that were ones is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Jeremy wrote 2700 digits. If he had also written $000, 001, 002, \ldots, 099$, he would have written 3000 digits in all which would have consisted of 300 of each of the 10 digits. In the list of those 100 numbers that he did not write, each nonzero digit appears $\frac{200}{10} = 20$ times, and 0 appear $20 + 100 = 120$ times.

Thus, among the digits Jeremy actually wrote, the digit 0 appears $300 - 120 = 180$ times, and each of the digits each appear $300 - 20 = 280$ times. Allison replaces each $0, 1, 2$, and 3 with one-digit numbers, and each of $4, 5, 6, 7, 8, 9$ with a two-digit number, so Allison wrote a total of

$$180 + 3 \cdot 280 + 2 \cdot 6 \cdot 280 = 180 + 15 \cdot 280 \text{ digits.}$$

Allison wrote one digit 1 each time she squared 1, 4, and 9 accounting for $3 \cdot 280$ digits that were 1. The required proportion is,

$$\frac{3 \cdot 280}{180 + 15 \cdot 280} = \frac{28}{6 + 5 \cdot 28} = \frac{14}{73}.$$

Hence, $m + n = 14 + 73 = \boxed{87.}$ □

Problem 4.3.2 (Problem 2)**.** (*5 points*) Find the number of three-digit positive integers which have three distinct digits where the sum of the digits is an even number such as 925 and 824.

Solution. If the hundred digit is even, then the other two digit must be both odd or even. There are 4 choices for the hundred digit and $5 \cdot 4 + 4 \cdot 3 = 32$ ways to choose the following digits, (5 choices for the odd tens then 4 choices for the odd unit digit or 4 choices for the even tens and 3 choices for the even unit digit), altogether $4 \cdot 32 = 128$ numbers.

If the hundred digit is odd, then it is followed by one even and one odd digit. There are 5 choices for the hundred digit and $5 \cdot 4 + 4 \cdot 5 = 40$ ways to choose the following digits, altogether $5 \cdot 40 = 200$.

Hence, there are $128 + 200 = \boxed{328}$ such numbers. □

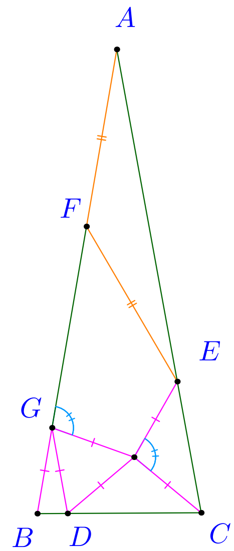
Problem 4.3.3 (Problem 3)**.** (*5 points*) Find the number whose reciprocal is the sum of the reciprocal of $9 + 15i$ and the reciprocal of $9 - 15i$.

Solution.

$$\frac{1}{9+15i} + \frac{1}{9-15i} = \frac{18}{9^2+15^2} = \frac{1}{17}.$$

The number is $\boxed{17.}$ □

Problem 4.3.4 (Problem 4)**.** (*5 points*) In $\triangle ABC$, $AB = AC$, $AF = EF$, $\angle CHE = \angle FGH$, and $EH = CH = DH = GH = DG = BG$. Find $\angle BAC$.



Solution. Let $x = \angle ABC$,

$$\triangle BAC \sim \triangle BGD \Rightarrow \angle BDG = \angle DBG = x \angle BGD + \angle GDH + \angle HDC = 180^\circ \Rightarrow \angle HDC = 120^\circ - x$$
$$\angle HCD = \angle HDC,\ \angle ECH = \angle ACB - \angle HCD \Rightarrow \angle ECH = x - (120^\circ - x) = 2x - 120^\circ$$
$$\angle EHC = 180^\circ - 2(2x - 120^\circ) = 420^\circ - 4x,\ \angle FGH = 180^\circ - \angle HGD - \angle DGB$$
$$\Rightarrow \angle HGF = 180^\circ - 60^\circ - (180^\circ - 2x) = 2x - 60^\circ$$
$$\angle CHE = \angle FGH \Rightarrow 420^\circ - 4x = 2x - 60^\circ \Rightarrow x = 80^\circ$$

Therefore, $\boxed{\angle BAC = 180^\circ - 2 \cdot 80^\circ = 20^\circ.}$ □

Problem 4.3.5 (Problem 5)**.** (*5 points*) Find the greatest possible value of $pq + r$, where p, q, and r are (not necessarily distinct) prime numbers satisfying $pq + qr + rp = 2016$.

Solution. If each of p, q, and r is an odd prime, then $pq + qr + rp$ is odd, thus cannot be equal to 2016.

If $p = 2$, then $qr + 2(q + r) = 2016$, so qr is even, so q or r is even. If $q = 2$ then $4 + 4r = 2016 \Rightarrow r = 503$.

Now, in order to maximize $pq + r$, p or q must be 503, and $pq + r = 2(503 + 1) = \boxed{1008.}$ □

Problem 4.3.6 (Problem 6)**.** (*5 points*) Find the least positive integer of the form $\overline{abaaba}$, where a and b are distinct digits, such that the integer can be written as a product of six distinct primes.

Solution. Let $n = \overline{aba}$, then the given number is $1001n = 7 \cdot 11 \cdot 13 \cdot n$. Thus n must be the least three-digit number that is a product of three distinct primes other than 7, 11, 13.

Now if $a = 1$, then n is odd and $n \geq 3 \cdot 5 \cdot 17 = 255 > \overline{1b1} = n$.

If $a = 2$, then n is even so one of the prime factor of n is 2. 5 cannot be a factor of n since it ends in 2. So $n \geq 2 \cdot 17 \cdot 19 = 646 > \overline{2n2}$. It leaves the only possible case if n has 2 and 3 as prime factors. Now $\frac{300}{6} = 50$, $\frac{200}{6} > 33$, so we can test the primes between 37 and 47. It turns out that $282 = 2 \cdot 3 \cdot 47$ is what we are looking for.

Thus $n = \boxed{282282.}$ □

Problem 4.3.7 (Problem 7)**.** (*5 points*) Henry rolls a fair die. If the die shows the number k, Henry will then roll the die k more times. The probability that Henry will never roll a 3 or a 6 either on his first roll or on one of the k subsequent rolls is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. In order to never roll a 3 or 6, Henry needs to roll $k = 1, 2, 4$, or 5 on his first roll, and then no 3 or 6, on any of the k subsequent rolls.

The probability that Henry does not roll a 3 or 6, on a single throw is $\frac{4}{6} = \frac{2}{3}$. Thus, the probability that Henry will never roll a 3 or a 6 on any of his roll is

$$\frac{1}{6}\left(\frac{2}{3}\right)^1 + \frac{1}{6}\left(\frac{2}{3}\right)^2 + \frac{1}{6}\left(\frac{2}{3}\right)^4 + \frac{1}{6}\left(\frac{2}{3}\right)^5 = \frac{175}{729}.$$

Hence, $m + n = 175 + 729 = \boxed{904.}$ □

Problem 4.3.8 (Problem 8)**.** (*5 points*) The cubic polynomials $p(x)$ and $q(x)$ satisfy

$$\begin{cases} p(1) = q(2) \\ p(3) = q(4) \\ p(5) = q(6) \\ p(7) = q(8) + 13 \end{cases}$$

Find $p(9) - q(10)$.

Solution. Since $p(x)$ and $q(x)$ are cubic polynomials, thus $r(x) = p(x) - q(x+1)$ is also a cubic polynomial. Furthermore $r(1) = r(3) = r(5) = 0$, thus $1, 3$, and 5 are all of its roots, therefore:

$$r(x) = a(x-1)(x-3)(x-5), \text{ where } a \text{ is a real constant.}$$

$r(7) = 13 \Rightarrow a(6)(4)(2) = 13 \Rightarrow a = \frac{13}{48}$.

Hence, $p(9) - q(10) = r(10) = \frac{13}{48}(8)(6)(4) = \boxed{52.}$ □

Problem 4.3.9 (Problem 9)**.** (*5 points*) The Tasty Candy Company always puts the same number of pieces of candy into each one-pound bag of candy they sell. Mike bought 4 one-pound bags and gave each person in his class 15 pieces of candy. Mike had 23 pieces of candy left over. Betsy bought 5 one-pound bags and gave 23 pieces of candy to each teacher in her school. Betsy had 15 pieces of candy left over. Find the least number of pieces of candy the Tasty Candy Company could have placed in each one-pound bag.

Solution. Let n be the number of pieces of candies in each bag. Then Mike's purchase shows that $4n \equiv 23 \pmod{15}$, and Betsy purchase shows that $5n \equiv 15 \pmod{23}$. Multiplying the first congruence by 4 shows that

$$16n = 4 \cdot 4n \equiv 4 \cdot 23 = 92 \pmod{15} \Rightarrow n \equiv 2 \pmod{15}.$$

Dividing the second congruence by 5,

$$n \equiv 3 \pmod{23}.$$

There exists integer k such that

$$n = 3 + 23k \equiv 3 + 8k \pmod{15},\ n \equiv 2 \pmod{15} \Rightarrow 8k + 3 \equiv 2 \pmod{15} \Rightarrow k \equiv 13 \pmod{15}$$

With $k = 13$, then $n = 3 + 23 \cdot 13$ satisfy the required conditions. This solution is unique modulo $15 \cdot 23$ (modulo 345), thus the least number of bad is $3 + 23 \cdot 13 = \boxed{302.}$ □

Problem 4.3.10 (Problem 10)**.** (*5 points*) Jar #1 contains five red marbles, three blue marbles, and one green marble. Jar #2 contains five blue marbles, three green marbles, and one red marble. Jar #3 contains five green marbles, three red marbles, and one blue marble.

You randomly select one marble from each jar.

Given that you select one marble of each color, the probability that the red marble came from jar #1, the blue marble came from jar #2, and the green marble came from jar #3 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let the triple (a,b,c) represent the event that the red marble came from jar #a, the blue marble came from jar #b, and the green marble came from jar #c, then the probability of $(1,2,3)$ is $\left(\frac{5}{9}\right)^3$, the probabilities of $(1,3,2)$, $(2,1,3)$, and $(3,2,1)$ are each $\frac{5}{9}\frac{3}{9}\frac{1}{9}$, the probability of $(3,1,2)$ is $\left(\frac{3}{9}\right)^3$, and the probability of $(2,3,1)$ is $\left(\frac{1}{9}\right)^3$.

Thus, the required probability is

$$\frac{\left(\frac{5}{9}\right)^3}{\left(\frac{5}{9}\right)^3+3\frac{5}{9}\frac{3}{9}\frac{1}{9}+\left(\frac{3}{9}\right)^3+\left(\frac{1}{9}\right)^3}=\frac{125}{198}.$$

Thus $m+n=125+198=\boxed{323.}$ □

Problem 4.3.11 (Problem 11)**.** (*5 points*) Positive integers a,b,c,d, and e satisfy the equations

$$\begin{cases}(a+1)(3bc+1)=d+3e+1\\(b+1)(3ca+1)=3d+e+13\\(c+1)(3ab+1)=4(26-d-e)-1\end{cases}$$

Find d^2+e^2.

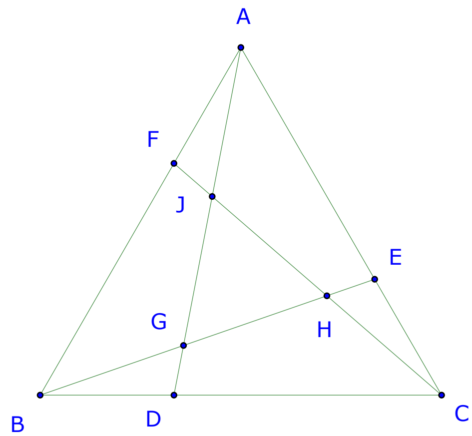
Solution. Adding the three equations together yields

$$9abc+3(ab+bc+ca)+(a+b+c)+3=117\Rightarrow 27abc+9(ab+bc+ca)+3(a+b+c)+1=343$$
$$\Rightarrow(3a+1)(3b+1)(3c+1)=7^3.$$

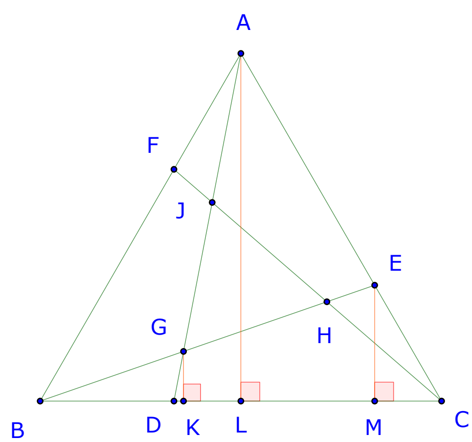
Since a,b,c are positive integers, thus $3a+1,3b+1,3c+1\geq 4$, so the only possible factorization of 7^3 into a product of three factors each at least 4 isosceles $3a+1=3b+1=3c+1=7$, so $a=b=c=4$.

Thus $d+3e+1=38, 3d+e+13=38$, so $d=5, e=11$. Hence $d^2+e^2=25+121=\boxed{146.}$ □

Problem 4.3.12 (Problem 12)**.** (*5 points*) On equilateral $\triangle ABC$ point D lies on BC a distance 1 from B, point E lies on AC a distance 1 from C, and point F lies on AB a distance 1 from A. Segment AD, BE, and CF intersect in pairs at points G, H, and J which are the vertices of another equilateral triangle. The area of $\triangle ABC$ is twice the area of $\triangle GHJ$. The side length of $\triangle ABC$ can be written $\dfrac{r+\sqrt{s}}{t}$, where r,s, and t are relatively prime positive integers. Find $r+s+t$.



Solution. Let d be the side length of $\triangle ABC$. Let point K, L, and M be the projections onto side BC of points G, A, and E, respectively, as shown. Let x and y be the length of BK and GK.



Because $\triangle ECM$ is $30-60-90$ with hypotenuse $CE = 1$, so $EM = \frac{\sqrt{3}}{2}$, and $CM = \frac{1}{2}$.

$$\triangle EMB \sim \triangle GKB \Rightarrow \frac{EM}{MB} = \frac{GK}{KB} \Rightarrow \frac{\frac{\sqrt{3}}{2}}{d-\frac{1}{2}} = \frac{y}{x}$$

$$\triangle ALD \sim \triangle GKD \Rightarrow \frac{AL}{LD} = \frac{GK}{KD} \Rightarrow \frac{\frac{d\sqrt{3}}{2}}{\frac{d}{2}-1} = \frac{y}{x-1}$$

$$\Rightarrow \frac{x\sqrt{3}}{2d-1} = \frac{d\sqrt{3}(x-1)}{d-2} \Rightarrow x = \frac{d(2d-1)}{2(d^2-d+1)},\ y = \frac{d\sqrt{3}}{2(d^2-d+1)}.$$

$$[ABC] = \frac{d^2\sqrt{3}}{4} \Rightarrow [GHJ] = \frac{d^2\sqrt{3}}{8}$$

$$[BCE] = [CAF] = [ABD] = \frac{d\sqrt{3}}{4},\ [BDG] = [CEH] = [AFJ] = \frac{y}{2} = \frac{d\sqrt{3}}{4(d^2-d+1)}$$

$$\Rightarrow \frac{d^2\sqrt{3}}{8} = 3\frac{d\sqrt{3}}{4} - 3\frac{d\sqrt{3}}{4(d^2-d+1)} \Rightarrow d(d^2-7d+7) = 0,\ d > 1 \Rightarrow d = \frac{7+\sqrt{21}}{2}$$

Thus the desired sum $7 + 21 + 2 = \boxed{30.}$ □

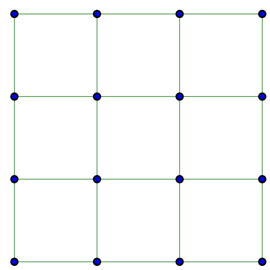
Problem 4.3.13 (Problem 13)**.** (*5 points*) In $\triangle ABC$, $\cos(\angle A) = \frac{2}{3}$, $\cos(\angle B) = \frac{1}{9}$, and $BC = 24$. Find the length AC.

Solution. It is easy to see that $\sin(\angle A) = \sqrt{1-\cos^2(\angle A)} = \frac{\sqrt{5}}{3}$ (note that $\sin(\angle A) > 0$), similarly $\sin(\angle B) = \frac{4\sqrt{5}}{9}$. By the Law of Sines:

$$\frac{AC}{\sin(\angle B)} = \frac{BC}{\sin(\angle A)} \Rightarrow AC = \frac{4\sqrt{5}}{9} \cdot \frac{24}{\frac{\sqrt{5}}{3}} = \boxed{32.}$$

□

Problem 4.3.14 (Problem 14)**.** (*5 points*) Sixteen dots are arranged in a four by four grid as shown. The distance between any two dots in the grid is the minimum number of horizontal and vertical steps along the grid lines it takes to get from one dot to the other. For example, two adjacent dots are a distance 1 apart, and two dots at opposite corners of the grid are a distance 6 apart.



The mean distance between two distinct dots in the grid is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. There are $\binom{16}{2} = 120$ pairs of points in the grid.

For $4 \cdot \binom{4}{2} = 24$ of the pairs of points (the same column), the horizontal distance between the points is 0.

For $3 \cdot 4 \cdot 4 = 48$ of the pairs of points, the horizontal distance between the points is 1.

For $2 \cdot 4 \cdot 4 = 32$ of the pairs of points, the horizontal distance between the points is 2.

For $4 \cdot 4 = 16$ of the pairs of points, the horizontal distance between the points is 3.

Thus, the sum of all the horizontal distances between pairs of points for all the pairs is

$$0 \cdot 24 + 1 \cdot 48 + 2 \cdot 32 + 3 \cdot 16 = 160.$$

This also is the sum of all the vertical distances between pairs of points for all the pairs.

Thus, the mean distances between points is $\dfrac{2 \cdot 160}{120} = \dfrac{8}{3}$. Hence $m + n = 8 + 3 = \boxed{11.}$ □

Problem 4.3.15 (Problem 15)**.** (*5 points*) Find the largest prime p such that p divides $2^{p+1}+3^{p+1}+5^{p+1}+7^{p+1}$.

Solution. Fermat's Little Theorem states that if p is a prime that does not divides a, then $a^{p-1} \equiv 1 \pmod{p}$.

Let's assume that $p > 7$, then

$$2^{p+1} + 3^{p+1} + 5^{p+1} + 7^{p+1} \equiv 2^2 + 3^2 + 5^2 + 7^2 = 87 \pmod{p}.$$

Thus $p \mid 87$. The largest such prime is $\boxed{29}$. □

Problem 4.3.16 (Problem 16)**.** (*5 points*) For n measured in degrees,

$$\text{let } T(n) = \cos^2(30^\circ - n) - \cos(30^\circ - n)\cos(30^\circ + n) + \cos^2(30^\circ + n). \text{ Evaluate } 4\sum_{n=1}^{30} n \cdot T(n).$$

Solution. By the Double Angle formula for cosine and the Prosthaphaeresis formulas, and $\cos(60^\circ) = \frac{1}{2}$, thus

$$\begin{cases} 2\cos^2(30^\circ - n) = 1 + \cos(60^\circ - 2n) = 1 + \cos(60^\circ)\cos(2n) + \sin(60^\circ)\sin(2n) \\ 2\cos(30^\circ - n)\cos(30^\circ + n) = \cos(60^\circ) + \cos(2n) \\ 2\cos^2(30^\circ + n) = 1 + \cos(60^\circ + 2n) = 1 + \cos(60^\circ)\cos(2n) - \sin(60^\circ)\sin(2n) \end{cases}$$

$$\Rightarrow 2T(n) = 2 + 2\cos(60^\circ)\cos(2n) - \cos(60^\circ) - \cos(2n) = \frac{3}{2}$$

Hence

$$4\sum_{n=1}^{30} n \cdot T(n) = 4 \cdot \sum_{n=1}^{30} n \cdot \frac{3}{4} = 3 \cdot \frac{30 \cdot 31}{2} = \boxed{1395.}$$

□

Problem 4.3.17 (Problem 17)**.** (*5 points*) Find the sum of all values of a such that there are positive integers a and b satisfying $(a - b)\sqrt{ab} = 2016$.

Solution. Let $d = \gcd(a, b)$, then $d \mid \sqrt{ab}, d \mid a - b \Rightarrow d^2 \mid 2016 = 2^5 \cdot 3^2 \cdot 7$. Thus d is $1, 2, 3, 4, 6$, or 12.

Furthermore, let $a = dr, b = ds$, then $\gcd(r, s) = 1$. Since $d^2(r - s)\sqrt{rs} = 2016$, thus each of r and s is a perfect square. In other words, there exist positive integers m and n,

$$a = dm^2,\ b = dn^2,\ \gcd(m, n) = 1,\ d^2(m^2 - n^2)mn = 2^5 \cdot 3^2 \cdot 7.$$

Case 1: $d = 1$.

$$(m - n)(m + n)mn = 2^5 \cdot 3^2 \cdot 7.$$

If one of m or n is even, then the other has to be odd and both $m - n, m + n$ are odd, thus 2^5 divides m or n. In that case $mn \geq 2^5 = 32$, $m + n \geq 33$, thus

$$2016 = (m - n)(m + n)mn \geq (m - n)(33)(32) = 1056(m - n) \Rightarrow m - n = 1$$

It is easy to test with m as multiple of 32 that there is no solution.

If both m or n are odd, then one of them is divisible by 7 and the other is divisible by 9. In fact $m = 9, n = 7$ satisfy above equality and $(a, b) = (81, 49)$ is a solution.

Case 2: $d = 2$.

$$(m - n)(m + n)mn = 2^3 \cdot 3^2 \cdot 7.$$

If both m or n are odd, it is easy to test that there is no solution.

If m is divisible by 2^3 then $m + n \geq 9$ and $(m - n)(m + n) \leq 63$, thus $m = 8, n = 1$ and $(a, b) = (128, 2)$ is a solution.

Case 3: $d = 3$.

$$(m - n)(m + n)mn = 2^5 \cdot 7.$$

If both m or n are odd, then $m = 7, n = 1$ leading to no solution.

If m is divisible by 2^5 then $m + n \geq 33$ and $(m - n)(m + n)$ is too large.

Case 4: $d \geq 4$.

$$(m-n)(m+n)mn \leq 2 \cdot 3^2 \cdot 7 = 126.$$

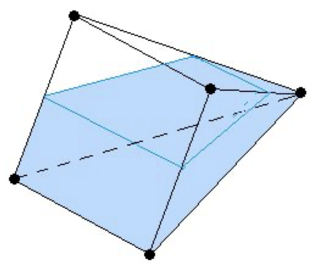
One of $m-n, m+n, m$, or n is divisible by 7.

If $m \geq 7$, then $m+n \geq 8$, $(m-n)n \geq 3$, thus $(m-n)(m+n)mn \geq 8 \cdot 7 \cdot 3 > 126$.

If $m \leq 6$, then $m+n = 7$, $(m-n)mn \leq \frac{126}{7} = 18$. Thus $(m,n) = (4,3)$. It is easy to test that there is no such solution.

Hence, there are two solutions $\{(81,49),(128,2)\}$, and the sum of all values of a is $81 + 128 = \boxed{209.}$ ☐

Problem 4.3.18 (Problem 18)**.** (*5 points*) A container the shape of a pyramid has a 12×12 square base, and the other four edges each have length 11. The container is partially filled with liquid so that when one of its triangular faces is lying on a flat surface, the level of the liquid is half the distance from the surface to the top edge of the container.

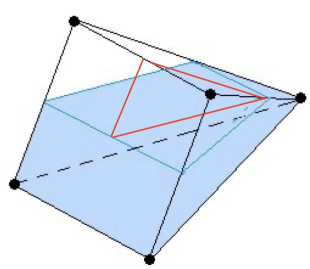


Find the volume of the liquid in the container.

Solution. A square pyramid with base side b and base to the apex height h has a volume $\frac{1}{3}b^2h$. Let s be the length of the other four edges. The height to the apex can be calculated considering the cross section of the pyramid that contains the apex, and two opposite corners of the square base. This cross section is an isosceles triangle with base $b\sqrt{2}$ and two sides equal to s,

$$h^2 = s^2 - \left(\frac{1}{2}b\sqrt{2}\right)^2 = s^2 - \frac{b^2}{2} \Rightarrow h = 7 \Rightarrow V = \frac{1}{3}b^2h = 336.$$

When lying on its side, the portion of the pyramid containing no liquid can be partitioned into two sections, one shaped like the original pyramid by half its size and a skewed triangular prism.



The volume of the pyramid is $\frac{1}{2^3}336 = 42$. The prism has a triangular cross section with base equal to $\frac{b}{2} = 6$ and height equal to $\frac{h}{2} = \frac{7}{2}$ so the area of the triangle is $\frac{21}{2}$. The length of the prism is $\frac{b}{2} = 6$, so the volume of the prism is $6 \cdot \frac{21}{2} = 63$.

This means that the volume of the portion of the container that does not contain liquid is $42 + 63 = 105$. Hence, the volume of the liquid is $336 - 105 = \boxed{231.}$ ☐

Problem 4.3.19 (Problem 19). (*5 points*) Find the sum of all the possible values of the product xy such that x and y are positive integers satisfying

$$(x^2+1)(y^2+1)+2(x-y)(1-xy)=4(1+xy)+140.$$

Solution. Note that

$$\begin{aligned}(x^2+1)(y^2+1) &= (xy)^2+x^2+y^2+1=(xy)^2-2xy+1+x^2-2xy+y^2+4xy\\ &= (1-xy)^2+(x-y)^2+4xy=(1-xy+x-y)^2-2(x-y)(1-xy)+4xy\\ \Rightarrow 144 &= (x^2+1)(y^2+1)+2(x-y)(1-xy)-4xy=(1-xy+x-y)^2 \Rightarrow (1+x)^2(1-y)^2=144.\end{aligned}$$

Since x, y are positive integers, thus

$$(x+1, y-1)\in\{(2,6),(3,4),(4,3),(6,2),(12,1)\} \Rightarrow (x,y)\in\{(1,7),(2,5),(3,4),(5,3),(11,2)\}.$$

Hence, $xy\in\{7,10,12,15,22\}$, and the desired sum is $7+10+12+15+22=\boxed{66.}$ □

Problem 4.3.20 (Problem 20). (*5 points*) Ten square tiles are placed in a row, and each can be painted with one of the four colors red (R), yellow (Y), blue (B), and white (W). Find the number of ways this can be done so that each block of five adjacent tiles contains at least one tile of each color. That is, count the patterns RWBWYRRBWY and WWBYRWYBWR but not RWBYYBWWRY because the five adjacent tiles colored BYYBW does not include the color red.

Solution. Suppose that the colouring $C_1C_2\dots C_9C_{10}$ is an acceptable colouring. Then each sequence of five colours $C_kC_{k+1}C_{k+2}C_{k+3}C_{k+4}$ contains exactly one colour that is repeated.

Call a tile *repeated* if its colour appears among one of the following four tiles immediately to its right. For example, in the pattern $RWBWYRRBWY$, tiles 2 and 6 are repeated tiles:

$$R\underline{W}B\underline{W}YRRBWY,\ RWBWY\underline{RR}BWY.$$

In the pattern $WWBYRWYBWR$, tiles 1, 2, 4, and 6 are repeated tiles:

$$\underline{WW}BYRWYBWR,\ W\underline{W}BYR\underline{W}YBWR, WWB\underline{Y}RW\underline{Y}BWR, WWBYR\underline{W}YB\underline{W}R.$$

Note that each sequence of the four tiles: 1234, 2345, 3456, 4567, 5678, and 6789 must include exactly one repeated tile. It follows that the sequence of repeated tiles is an increasing sequence of k positive integers $a_1<a_2<\cdots<a_k$, where a_1 is one of 1, 2, 3, 4, the difference between two adjacent terms $a_{j+1}-a_j$ cannot exceed 4, the final term a_k is one of 6, 7, 8, 9, and there can be only one term with value greater than 5.

Given any sequence of repeated values that satisfy these conditions, it is easy to construct an acceptable colouring with that sequence of repeated values. For example, the repeated tile sequence 1, 2, 5, 9 comes from the acceptable pattern $\underline{RB}WY\underline{RBR}WW\underline{YY}$.

Also note that if the colour are known for the first tiles, and the repeated sequence is known, then the complete sequence of ten tile colours can be determined. For example, if the colour pattern for the first five tiles is $RBWYR$, and the repeated sequence is 1,2,5,9, then the sixth rile by be a B so that the second tile becomes repeated. Then the seventh tile must be R so the tile 5 becomes repeated. Then the eight tile must be W so the tiles 5678 has a W, the ninth tile must be Y so tiles 6789 has an Y, and finally the tenth tile must be Y so the ninth tile becomes repeated.

Let m_j be the number of ways a repeated sequence containing the number j can be completed by adding terms to the right of j. Then $m_9=m_8=m_7=m_6=1$ because any repeated sequence containing a 6, 7, 8, 9 must end with that term and have no terms to the right of it. For j between 1 and 5, a repeated sequence that includes a term equal to j must be follow by a term with value $j+1, j+2, j+3$, or $j+4$ showing that $m_j = m_{j+1} + m_{j+2} + m_{j+3} + m_{j+4}$. Thus $m_5 = m_6 + m_7 + m_8 + m_9 = 4$, $m_4 = 7$, $m_3 = 13$, $m_2 = 25$, $m_1 = 49$.

Because every repeated sequence must begin with exactly one of 1, 2, 3, or 4, it follows that there are

$$m_1 + m_2 + m_3 + m_4 = 49 + 25 + 13 + 7 = 94 \text{ repeated sequences.}$$

A repeated sequence does not determine a colour pattern. A repeated sequence together with the first five colours of an acceptable colour pattern does determine a full pattern.

If a repeated sequence begins with 4, it means that tile 4 and 5 are coloured the same in the associated colour pattern. There are 4! ways to set a sequence of five tiles using 24 colours when tiles 4 and 5 are colour the same. So there are $4! \cdot a_4 = 24 \cdot 7$ acceptable colour patterns where tiles 4 and 5 are colour the same.

Similarly, a repeated sequence begins with 3, then tile 3 must be coloured the same as either tile 4 or tile 5. Thus, there are $24 \cdot 2 \cdot a_3 = 24 \cdot 26$ acceptable colour patterns where a repeated sequence begins with 3.

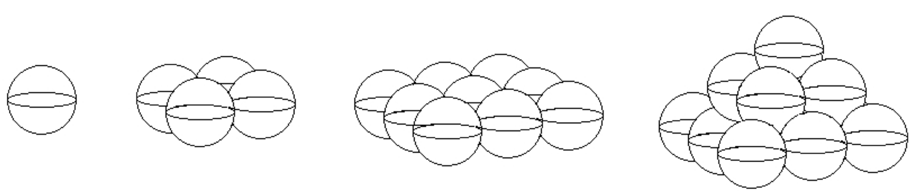
The number of acceptable colour patterns where a repeated sequence begins with 2 is $24 \cdot 3 \cdot a_2 = 24 \cdot 75$.

The number of acceptable colour patterns where a repeated sequence begins with 1 is $24 \cdot 4 \cdot a_1 = 24 \cdot 196$.

Hence, the total number of acceptable colour patterns is $24 \cdot (196 + 75 + 26 + 7) = \boxed{7296.}$ □

Problem 4.3.21 (Problem 21)**.** (*5 points*) Some identically sized spheres are piled in n layers in the shape of a square pyramid with one sphere in the top layer, 4 spheres in the second layer, 9 spheres in the third layer, and so forth so that the bottom layer has a square array of n^2 spheres. In each layer the centers of the spheres form a square grid so that each sphere is tangent to any sphere adjacent to it on the grid. Each sphere in an upper level is tangent to the four spheres directly below it.

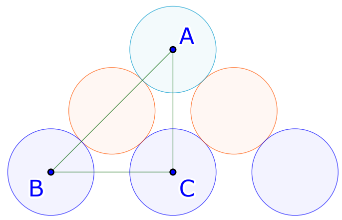
The diagram shows how the first three layers of spheres are stacked.

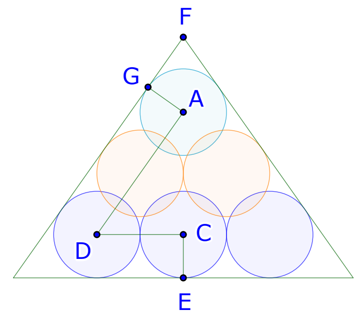


A square pyramid is built around the pile of spheres so that the sides of the pyramid are tangent to the spheres on the outside of the pile. There is a positive integer m such that as n gets large, the ratio of the volume of the pyramid to the total volume inside all of the spheres approaches $\frac{\sqrt{m}}{\pi}$. Find m.

Solution. Let assume that each of the spheres has a radius equal to 1. Then the center of any two adjacent spheres are a distance 2 apart. If four spheres in one layer have their centers at four vertices of a 2×2 square, then the centers of the spheres across the square from each other are a distance of $2\sqrt{2}$ apart.

Consider the diagram below on the left showing a cross section of the pile of spheres passing through the center of the sphere on the top layer labelled A, the center of the sphere in the third layer directly below the top sphere labeled C, and the center of a sphere across a 2×2 square from there labeled B. Because AC is vertical, BC is horizontal, $\triangle ABC$ is right triangle with $AB = 4$, $BC = 2\sqrt{2}$., thus $AC = 2\sqrt{2}$. It follows that the vertical distance between the plane containing the centers of spheres at one level of the pile, and the plane containing the center of spheres at the next lower level of the pile is $\sqrt{2}$.





Now, consider the diagram above on the right showing a cross section of the pile of spheres and the enclosing pyramid that passes through the top vertex of the pyramid, the center of the top sphere labeled A, the center of the sphere directly below A in the third level labeled C, and the center of an adjacent sphere in the third level labeled D. Then ACD is a right triangle with legs $AC = 2\sqrt{2}$, $CD = 2$, thus $AD = 2\sqrt{3}$. Let G be a point where the top sphere is tangent to the side of the pyramid. Since $\triangle AFG \sim \triangle CAD$ and $AG = 1$, thus $AF = \sqrt{3}$.

If the pile of sphere has n layers, then the distance between a horizontal plane passing through point A and a plane passing through the centers of the bottom layers is $(n-1)\sqrt{2}$. Since $AF = \sqrt{3}$, and the centers of the spheres are a distance 1 from the bottom of the pyramid, the height of the pyramid is given by

$$(n-1)\sqrt{2} + 1 + \sqrt{3}.$$

Because the side of the pyramid is parallel to AD, by similarity, so the side of the base square of the pyramid is:

$$2 \cdot \frac{\sqrt{2}}{2}\left((n-1)\sqrt{2} + 1 + \sqrt{3}\right) = 2(n-1) + \sqrt{2} + \sqrt{6}.$$

The volume of the pyramid is equal to $\frac{1}{3}$ times the area of its base, times its height, so the volume is:

$$\frac{1}{3}\left(2(n-1) + \sqrt{2} + \sqrt{6}\right)^2 \cdot \left((n-1)\sqrt{2} + 1 + \sqrt{3}\right).$$

The number of spheres in this pile is given by:

$$1^2 + 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

So the volume of all the spheres is:

$$\frac{4\pi}{3}\frac{n(n+1)(2n+1)}{6}.$$

Now, since the limit of ratio of volumes is:

$$\lim_{n\to+\infty} \frac{\frac{1}{3}\left(2(n-1) + \sqrt{2} + \sqrt{6}\right)^2 \cdot \left((n-1)\sqrt{2} + 1 + \sqrt{3}\right)}{\frac{4\pi}{3}\frac{n(n+1)(2n+1)}{6}} = \frac{\frac{2^2\sqrt{2}}{3}}{\frac{4\pi}{3}\frac{2}{6}} = \frac{\frac{4\sqrt{2}}{3}}{\frac{4\pi}{9}} = \frac{3\sqrt{2}}{\pi} = \frac{\sqrt{18}}{\pi}.$$

Thus $m = \boxed{18.}$ □

4.4 High School - Test

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**
- **Olympiad (O) level: Problems 12-21**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 4.4.1 (Problem 1)**.** (*5 points*) Jeremy wrote all the three-digit integers from 100 to 999 on a blackboard. Then Allison erased each of the 2700 digits Jeremy wrote and replaced each digit with the square of that digit. Thus, Allison replaced every 1 with a 1, every 2 with a 4, every 3 with a 9, every 4 with a 16, and so forth. The proportion of all the digits Allison wrote that were **twos** is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Jeremy wrote 2700 digits. If he had also written $000, 001, 002, \ldots, 099$, he would have written 3000 digits in all which would have consisted of 300 of each of the 10 digits. In the list of those 100 numbers that he did not write, each nonzero digit appears $\frac{200}{10} = 20$ times, and 0 appear $20 + 100 = 120$ times.

Thus, among the digits Jeremy actually wrote, the digit 0 appears $300 - 120 = 180$ times, and each of the digits each appear $300 - 20 = 280$ times. Allison replaces each $0, 1, 2,$ and 3 with one-digit numbers, and each of $4, 5, 6, 7, 8, 9$ with a two-digit number, so Allison wrote a total of

$$180 + 3 \cdot 280 + 2 \cdot 6 \cdot 280 = 180 + 15 \cdot 280 \text{ digits.}$$

Allison wrote one digit 2 each time she squared 5 accounting for $1 \cdot 280$ digits that were 5. The required proportion is,

$$\frac{1 \cdot 280}{180 + 15 \cdot 280} = \frac{14}{219}.$$

Hence, $m + n = 14 + 219 = \boxed{233.}$ □

Problem 4.4.2 (Problem 2)**.** (*5 points*) Find the number of three-digit positive integers which have three distinct digits where the sum of the digits is an odd number such as 935 and 834.

Solution. If the hundred digit is even, then it is followed by one even and one odd digit. There are 4 choices for the hundred digit and $5 \cdot 4 + 4 \cdot 5 = 40$ ways to choose the following digits, (5 choices for the odd tens then 4 choices for the even unit digit or 4 choices for the even tens and 5 choices for the odd unit digit), altogether $4 \cdot 40 = 160$.

If the hundred digit is odd, then the other two digit must be both odd or even. There are 5 choices for the hundred digit and $4 \cdot 3 + 5 \cdot 4 = 32$ ways to choose the following digits, (4 choices for the odd tens then 3 choices for the odd unit digit or 5 choices for the even tens and 4 choices for the even unit digit), altogether $5 \cdot 32 = 160$.

Hence, there are $160 + 160 = \boxed{320}$ such numbers. □

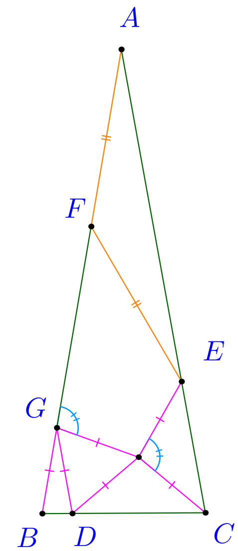
Problem 4.4.3 (Problem 3)**.** (*5 points*) The sum of the reciprocal of $5 + 12i$ and the reciprocal of $5 - 12i$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution.

$$\frac{1}{5 + 12i} + \frac{1}{5 - 12i} = \frac{10}{5^2 + 12^2} = \frac{10}{169}.$$

Hence, $m + n = 10 + 169 = \boxed{179.}$ □

Problem 4.4.4 (Problem 4)**.** (*5 points*) In $\triangle ABC$, $AB = AC$, $AF = EF$, $\angle CHE = \angle FGH$, and $EH = CH = DH = GH = DG = BG$. Find $\angle GFE$.



Solution. Let $x = \angle ABC$,

$$\triangle BAC \sim \triangle BGD \Rightarrow \angle BDG = \angle DBG = x\angle BGD + \angle GDH + \angle HDC = 180^\circ \Rightarrow \angle HDC = 120^\circ - x$$
$$\angle HCD = \angle HDC,\ \angle ECH = \angle ACB - \angle HCD \Rightarrow \angle ECH = x - (120^\circ - x) = 2x - 120^\circ$$
$$\angle EHC = 180^\circ - 2(2x - 120^\circ) = 420^\circ - 4x,\ \angle FGH = 180^\circ - \angle HGD - \angle DGB$$
$$\Rightarrow \angle HGF = 180^\circ - 60^\circ - (180^\circ - 2x) = 2x - 60^\circ$$
$$\angle CHE = \angle FGH \Rightarrow 420^\circ - 4x = 2x - 60^\circ \Rightarrow x = 80^\circ$$

Therefore, $\angle BAC = 180^\circ - 2 \cdot 80^\circ = 20^\circ$, so $\angle GFE = 2\angle FAE = \boxed{40^\circ.}$ □

Problem 4.4.5 (Problem 5)**.** (*5 points*) Find the greatest possible value of $pq + r$, where p, q, and r are (not necessarily distinct) prime numbers satisfying $pq + qr + rp = 2024$.

Solution. If each of p, q, and r is an odd prime, then $pq + qr + rp$ is odd, thus cannot be equal to 2016.

If $p = 2$, then $qr + 2(q + r) = 2024$, so qr is even, so q or r is even. If $q = 2$ then $4 + 4r = 2024 \Rightarrow r = 505$.

But then r is not a prime number. Thus there is $\boxed{\text{no solution.}}$ □

Problem 4.4.6 (Problem 6)**.** (*5 points*) n the least positive integer of the form $\overline{abbabb}$, where a and b are distinct digits, such that the integer can be written as a product of six distinct primes. Find the sum of these primes.

Solution. Let $n = \overline{abb}$, then the given number is $1001n = 7 \cdot 11 \cdot 13 \cdot n$. Thus n must be the least three-digit number that is a product of three distinct primes other than 7, 11, 13.

Now if $a = 1$, then n is odd and $n \geq 3 \cdot 5 \cdot 17 = 255 > \overline{1b1} = n$.

If $a = 2$, and since 3, 5, 17 are the least primes that satisfy $3 \cdot 5 \cdot 17 = 255$, Thus $n = 255255$.

Hence $3 + 5 + 7 + 11 + 13 + 17 = \boxed{56.}$ □

Problem 4.4.7 (Problem 7)**.** (*5 points*) Henry rolls a fair die. If the die shows the number k, Henry will then roll the die k more times. The probability that Henry will never roll a 1, 3 or a 6 either on his first roll or on one of the k subsequent rolls is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. In order to never roll a $1, 3$ or 6, Henry needs to roll $k = 2, 4$, or 5 on his first roll, and then no $1, 3$ or 6, on any of the k subsequent rolls.

The probability that Henry does not roll a $1, 3$ or 6, on a single throw is $\frac{3}{6} = \frac{1}{2}$. Thus, the probability that Henry will never roll a 1, 3 or a 6 on any of his roll is

$$\frac{1}{6}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\left(\frac{1}{2}\right)^4 + \frac{1}{6}\left(\frac{1}{2}\right)^5 = \frac{11}{192}.$$

Hence, $m + n = 11 + 192 = \boxed{203.}$ □

Problem 4.4.8 (Problem 8)**.** (*5 points*) The cubic polynomials $p(x)$ and $q(x)$ satisfy

$$\begin{cases} p(1) = q(2) \\ p(3) = q(4) \\ p(5) = q(6) \\ p(7) = q(8) + 13 \end{cases}$$

Find $p(11) - q(12)$.

Solution. Since $p(x)$ and $q(x)$ are cubic polynomials, thus $r(x) = p(x) - q(x+1)$ is also a cubic polynomial. Furthermore $r(1) = r(3) = r(5) = 0$, thus $1, 3$, and 5 are all of its roots, therefore:

$$r(x) = a(x-1)(x-3)(x-5), \text{ where } a \text{ is a real constant.}$$

$r(7) = 13 \Rightarrow a(6)(4)(2) = 13 \Rightarrow a = \frac{13}{48}$.

Hence, $p(11) - q(12) = r(11) = \frac{13}{48}(10)(8)(6) = \boxed{130.}$ □

Problem 4.4.9 (Problem 9)**.** (*5 points*) The Tasty Candy Company always puts the same number of pieces of candy into each one-pound bag of candy they sell. Mike bought 4 one-pound bags and gave each person in his class 15 pieces of candy. Mike had 19 pieces of candy left over. Betsy bought 5 one-pound bags and gave 19 pieces of candy to each teacher in her school. Betsy had 15 pieces of candy left over. Find the least number of pieces of candy the Tasty Candy Company could have placed in each one-pound bag.

Solution. We show a coding solution. The problem is equivalent to look for the first positive integer n in ascending order such that:

$$\begin{cases} 4n \equiv 19 \pmod{15} \\ 5n \equiv 15 \pmod{19} \end{cases}$$

```
n = 1
while True:
    if (4*n - 19 )% 15 == 0 and (5*n - 15 )% 19 == 0:
        print(n)
        break
    n +=1
```

The answer is $\boxed{136.}$ □

Problem 4.4.10 (Problem 10)**.** (*5 points*) Jar #1 contains five red marbles, three blue marbles, and one green marble. Jar #2 contains five blue marbles, three green marbles, and one red marble. Jar #3 contains five green marbles, three red marbles, and one blue marble.

You randomly select one marble from each jar.

Given that you select one marble of each color, the probability that the red marble came from jar #2, the blue marble came from jar #3, and the green marble came from jar #1 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let the triple (a, b, c) represent the event that the red marble came from jar $\#a$, the blue marble came from jar $\#b$, and the green marble came from jar $\#c$, then the probability of $(1,2,3)$ is $\left(\frac{5}{9}\right)^3$, the probabilities of $(1,3,2)$, $(2,1,3)$, and $(3,2,1)$ are each $\frac{5}{9}\frac{3}{9}\frac{1}{9}$, the probability of $(3,1,2)$ is $\left(\frac{3}{9}\right)^3$, and the probability of $(2,3,1)$ is $\left(\frac{1}{9}\right)^3$.

Thus, the required probability is

$$\frac{\left(\frac{1}{9}\right)^3}{\left(\frac{5}{9}\right)^3 + 3\frac{5}{9}\frac{3}{9}\frac{1}{9} + \left(\frac{3}{9}\right)^3 + \left(\frac{1}{9}\right)^3} = \frac{1}{198}.$$

Thus $m+n = 1+198 = \boxed{199.}$ □

Problem 4.4.11 (Problem 11)**.** (*5 points*) Positive integers a, b, c, d, and e satisfy the equations

$$\begin{cases} (a+1)(3bc+1) = d+3e+1 \\ (b+1)(3ca+1) = 3d+e+13 \\ (c+1)(3ab+1) = 4(26-d-e)-1 \end{cases}$$

Find $a+b+c+d+e$.

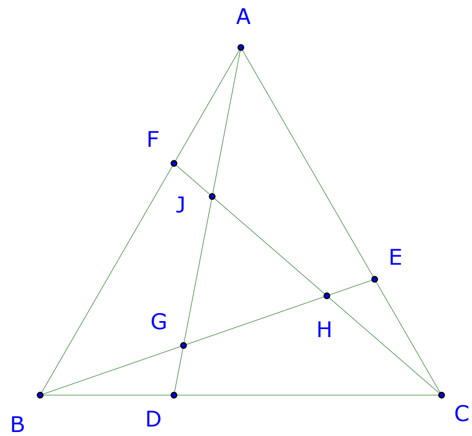
Solution. Adding the three equations together yields

$$9abc + 3(ab+bc+ca) + (a+b+c) + 3 = 117 \Rightarrow 27abc + 9(ab+bc+ca) + 3(a+b+c) + 1 = 343$$
$$\Rightarrow (3a+1)(3b+1)(3c+1) = 7^3.$$

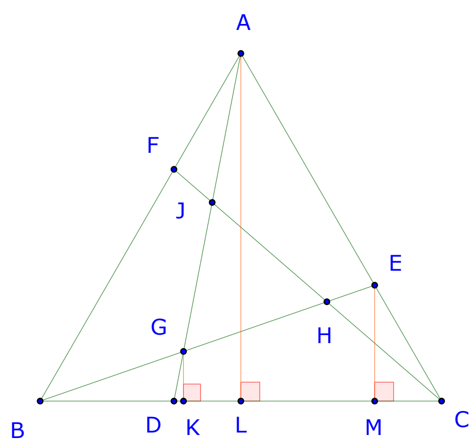
Since a, b, c are positive integers, thus $3a+1, 3b+1, 3c+1 \geq 4$, so the only possible factorization of 7^3 into a product of three factors each at least 4 isosceles $3a+1 = 3b+1 = 3c+1 = 7$, so $a = b = c = 4$.

Thus $d+3e+1 = 38, 3d+e+13 = 38$, so $d = 5, e = 11$. Hence $a+b+c+d+e = 2+2+2+5+11 = \boxed{24.}$ □

Problem 4.4.12 (Problem 12)**.** (*5 points*) On equilateral $\triangle ABC$ point D lies on BC a distance 1 from B, point E lies on AC a distance 1 from C, and point F lies on AB a distance 1 from A. Segment AD, BE, and CF intersect in pairs at points G, H, and J which are the vertices of another equilateral triangle. The area of $\triangle ABC$ is twice the area of $\triangle GHJ$. The side length of $\triangle ABC$ can be written $\frac{r+\sqrt{s}}{t}$, where r, s, and t are relatively prime positive integers. Find the value of the product rst.



Solution. Let d be the side length of $\triangle ABC$. Let point K, L, and M be the projections onto side BC of points G, A, and E, respectively, as shown. Let x and y be the length of BK and GK.



Because $\triangle ECM$ is $30-60-90$ with hypotenuse $CE = 1$, so $EM = \frac{\sqrt{3}}{2}$, and $CM = \frac{1}{2}$.

$$\triangle EMB \sim \triangle GKB \Rightarrow \frac{EM}{MB} = \frac{GK}{KB} \Rightarrow \frac{\frac{\sqrt{3}}{2}}{d-\frac{1}{2}} = \frac{y}{x}$$

$$\triangle ALD \sim \triangle GKD \Rightarrow \frac{AL}{LD} = \frac{GK}{KD} \Rightarrow \frac{\frac{d\sqrt{3}}{2}}{\frac{d}{2}-1} = \frac{y}{x-1}$$

$$\Rightarrow \frac{x\sqrt{3}}{2d-1} = \frac{d\sqrt{3}(x-1)}{d-2} \Rightarrow x = \frac{d(2d-1)}{2(d^2-d+1)},\ y = \frac{d\sqrt{3}}{2(d^2-d+1)}.$$

$$[ABC] = \frac{d^2\sqrt{3}}{4} \Rightarrow [GHJ] = \frac{d^2\sqrt{3}}{8}$$

$$[BCE] = [CAF] = [ABD] = \frac{d\sqrt{3}}{4},\ [BDG] = [CEH] = [AFJ] = \frac{y}{2} = \frac{d\sqrt{3}}{4(d^2-d+1)}$$

$$\Rightarrow \frac{d^2\sqrt{3}}{8} = 3\frac{d\sqrt{3}}{4} - 3\frac{d\sqrt{3}}{4(d^2-d+1)} \Rightarrow d(d^2-7d+7) = 0,\ d > 1 \Rightarrow d = \frac{7+\sqrt{21}}{2}$$

Thus the desired product $7 \cdot 21 \cdot 2 = \boxed{294.}$ □

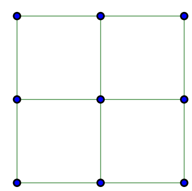
Problem 4.4.13 (Problem 13)**.** (*5 points*) In $\triangle ABC$, $\cos(\angle A) = \frac{\sqrt{5}}{5}$, $\cos(\angle B) = \frac{3}{5}$, $BC = 17$. Find AB.

Solution. It is easy to see that $\sin(\angle A) = \sqrt{1 - \cos^2(\angle A)} = \frac{2\sqrt{5}}{5}$ ($\sin(\angle A) > 0$), similarly $\sin(\angle B) = \frac{4}{5}$.

$$\sin(\angle C) = \sin(\angle A + \angle B) = \sin(\angle A)\cos(\angle B) + \sin(\angle B)\cos(\angle A) = \frac{2\sqrt{5}}{5} \cdot \frac{3}{5} + \frac{4}{5} \cdot \frac{\sqrt{5}}{5} = \frac{2\sqrt{5}}{5}$$

Thus $\angle C = \angle A$, so $AB = BC = \boxed{17.}$ $\square$

Problem 4.4.14 (Problem 14)**.** (*5 points*) Nine dots are arranged in a three by three grid as shown. The distance between any two dots in the grid is the minimum number of horizontal and vertical steps along the grid lines it takes to get from one dot to the other. For example, two adjacent dots are a distance 1 apart, and two dots at opposite corners of the grid are a distance 6 apart.



The mean distance between two distinct dots in the grid is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. There are $\binom{9}{2} = 36$ pairs of points in the grid.

For $3 \cdot \binom{3}{2} = 9$ of the pairs of points (the same column), the horizontal distance between the points is 0.

For $2 \cdot 3 \cdot 3 = 18$ of the pairs of points, the horizontal distance between the points is 1.

For $1 \cdot 3 \cdot 3 = 9$ of the pairs of points, the horizontal distance between the points is 2.

Thus, the sum of all the horizontal distances between pairs of points for all the pairs is

$$0 \cdot 9 + 1 \cdot 18 + 2 \cdot 9 = 36.$$

This also is the sum of all the vertical distances between pairs of points for all the pairs.

Thus, the mean distances between points is $\dfrac{2 \cdot 36}{36} = \dfrac{2}{1}$. Hence $m + n = 2 + 1 = \boxed{3.}$ $\square$

Problem 4.4.15 (Problem 15)**.** (*5 points*) Find the largest prime p such that p divides $2^{p+2}+3^{p+2}+5^{p+2}+7^{p+2}$.

Solution. Fermat's Little Theorem states that if p is a prime that does not divides a, then $a^{p-1} \equiv 1 \pmod{p}$.

Let's assume that $p > 7$, then

$$2^{p+1} + 3^{p+1} + 5^{p+1} + 7^{p+1} \equiv 2^3 + 3^3 + 5^3 + 7^3 = 503 \pmod{p}.$$

Thus $p \mid 503$. The largest such prime is $\boxed{503}$. $\square$

Problem 4.4.16 (Problem 16)**.** (*5 points*) For n measured in degrees,

$$\text{let } T(n) = \cos^2(30^\circ - n) - \cos(30^\circ - n)\cos(30^\circ + n) + \cos^2(30^\circ + n). \text{ Evaluate } 4\sum_{n=1}^{2024} n \cdot T(n).$$

Solution. By the Double Angle formula for cosine and the Prosthaphaeresis formulas, and $\cos(60^\circ) = \frac{1}{2}$, thus

$$\begin{cases} 2\cos^2(30^\circ - n) = 1 + \cos(60^\circ - 2n) = 1 + \cos(60^\circ)\cos(2n) + \sin(60^\circ)\sin(2n) \\ 2\cos(30^\circ - n)\cos(30^\circ + n) = \cos(60^\circ) + \cos(2n) \\ 2\cos^2(30^\circ + n) = 1 + \cos(60^\circ + 2n) = 1 + \cos(60^\circ)\cos(2n) - \sin(60^\circ)\sin(2n) \end{cases}$$

$$\Rightarrow 2T(n) = 2 + 2\cos(60^\circ)\cos(2n) - \cos(60^\circ) - \cos(2n) = \frac{3}{2}$$

$$\Rightarrow 4\sum_{n=1}^{2024} n \cdot T(n) = 4 \cdot \sum_{n=1}^{2024} n \cdot \frac{3}{4} = 3 \cdot \frac{2024 \cdot 2025}{2} = \boxed{6147900.}$$

□

Problem 4.4.17 (Problem 17)**.** (*5 points*) Find the sum of all values of b such that there are positive integers a and b satisfying $(a-b)\sqrt{ab} = 840$.

Solution. Let $d = \gcd(a, b)$, then $d \mid \sqrt{ab}, d \mid a - b \Rightarrow d^2 \mid 840 = 2^3 \cdot 3 \cdot 5 \cdot 7$. This means that d is 1 or 2.

Furthermore, let $a = dr, b = ds$, then $\gcd(r, s) = 1$. Since $d^2(r-s)\sqrt{rs} = 840$, thus each of r and s is a perfect square. In other words, there exist positive integers m and n,

$$a = dm^2,\ b = dn^2,\ \gcd(m, n) = 1,\ d^2(m^2 - n^2)mn = 2^3 \cdot 3 \cdot 5 \cdot 7.$$

Case 1: $d = 1$.

$$(m-n)(m+n)mn = 2^3 \cdot 3 \cdot 5 \cdot 7.$$

If one of m or n is even, then the other has to be odd and both $m-n, m+n$ are odd. Thus one of m or n is divisible by 8.

If n is divisible by 8, then $m - n \geq 1$, $mn \geq 72$, $m + n \geq 15$, which is too large.

Thus m is divisible by 8, then if m is divisible by 3, 5, or 7, then $mn \geq 24$, $m + n \geq 25$ and $(m-n)(m+n)mn \geq 24 \cdot 25(m-n) = 600(m-n)$, so $m - n = 1$, thus

$$(2m-1)m(m-1) = 840 > 2(m-1)^3 \Rightarrow 7 \geq m.$$

Thus m is divisible by 8, and it cannot be divisible by 3, 5, or 7, and so $m = 8$. In that case $n = 7$, and $(a, b) = (64, 49)$ is a solution.

If both m and n are odd, it is easy to see that $n > 1$. So if $m > 7$, then $m \geq 3 \cdot 5 = 15$, thus

$$(m-n)(m+n)mn \geq 2 \cdot (3 + 15) \cdot 15 \cdot 3 = 1620 > 840.$$

Thus $m = 7$, then

$$(7-n)(7+n)7(n) = 840 \Rightarrow n = 5 \text{ or } 3.$$

And $(a, b) = (49, 25)$, or $(a, b) = (49, 9)$.

Case 2: $d = 2$.

$$(m-n)(m+n)mn = 2 \cdot 3 \cdot 5 \cdot 7.$$

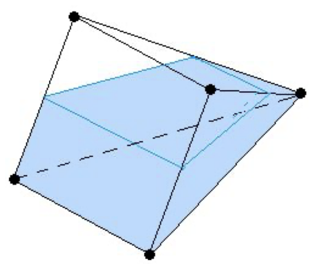
Both m or n cannot be even, and since both cannot be odd, so one is even and the other is odd.

If $n = 1$ then $(m^2 - 1)m = 210 \Rightarrow m = 6$, then $(a, b) = (72, 2)$.

If $n > 1$ then $n, m - n, m, m + n$ are four different pair-wise relatively prime numbers, whose product is $2 \cdot 3 \cdot 5 \cdot 7$. It is easy to see that $m - n$ can only be 3 there for $n = 2$, thus $m = 5$, and $(a, b) = (50, 8)$.

Hence, there are four solutions $\{(64, 49), (49, 25), (49, 9), (72, 2), (50, 8)\}$, and the sum of all values of a is $49 + 25 + 9 + 2 + 8 = \boxed{93.}$ □

Problem 4.4.18 (Problem 18)**.** (*5 points*) A container the shape of a pyramid has a 8×8 square base, and the other four edges each have length 9. The container is partially filled with liquid so that when one of its triangular faces is lying on a flat surface, the level of the liquid is half the distance from the surface to the top edge of the container.

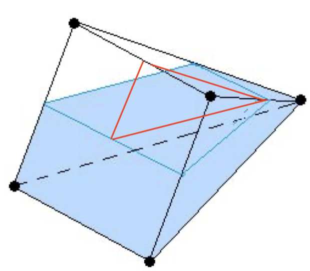


The volume of the liquid in the container is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. A square pyramid with base side b and base to the apex height h has a volume $\frac{1}{3}b^2h$. Let s be the length of the other four edges. The height to the apex can be calculated considering the cross section of the pyramid that contains the apex, and two opposite corners of the square base. This cross section is an isosceles triangle with base $b\sqrt{2}$ and two sides equal to s,

$$h^2 = s^2 - \left(\frac{1}{2}b\sqrt{2}\right)^2 = s^2 - \frac{b^2}{2} = 9^2 - \frac{8^2}{2} \Rightarrow h = 7 \Rightarrow V = \frac{1}{3}b^2h = \frac{448}{3}.$$

When lying on its side, the portion of the pyramid containing no liquid can be partitioned into two sections, one shaped like the original pyramid by half its size and a skewed triangular prism.



The volume of the pyramid is $\frac{1}{2^3}\frac{448}{3} = \frac{56}{3}$. The prism has a triangular cross section with base equal to $\frac{b}{2} = 4$, and height equal to $\frac{h}{2} = \frac{7}{2}$ so the area of the triangle is $\frac{1}{2} \cdot 4 \cdot \frac{7}{2} = 7$. The length of the prism is $\frac{b}{2} = 4$, so the volume of the prism is $4 \cdot 7 = 28$.

This means that the volume of the portion of the container that does not contain liquid is $\frac{56}{3} + 28 = \frac{140}{3}$. Hence, the volume of the liquid is $\frac{448}{3} - \frac{140}{3} = \frac{136}{1}$, and $m + n = 136 + 1 = \boxed{137.}$ □

Problem 4.4.19 (Problem 19)**.** (*5 points*) Find the sum of all the possible values of the product xy such that x and y are positive integers satisfying

$$(x^2+1)(y^2+1)+2(x-y)(1-xy)=4(1+xy)+96.$$

Solution. Note that

$$\begin{aligned}
&(x^2+1)(y^2+1)=(xy)^2+x^2+y^2+1=(xy)^2-2xy+1+x^2-2xy+y^2+4xy\\
&=(1-xy)^2+(x-y)^2+4xy=(1-xy+x-y)^2-2(x-y)(1-xy)+4xy\\
\Rightarrow\ &100=(x^2+1)(y^2+1)+2(x-y)(1-xy)-4xy=(1-xy+x-y)^2\Rightarrow(1+x)^2(1-y)^2=100.
\end{aligned}$$

Since x, y are positive integers, thus

$$(x+1,y-1)\in\{(2,5),(5,2),(10,1)\}\Rightarrow(x,y)\in\{(1,6),(4,3),(9,2)\}\Rightarrow xy\in\{6,12,18\}$$

Hence, the desired sum is $6+12+18=\boxed{36.}$ □

Problem 4.4.20 (Problem 20)**.** (*5 points*) Eight square tiles are placed in a row, and each can be painted with one of the three colors red (R), yellow (Y), and blue (B). Find the number of ways this can be done so that each block of four adjacent tiles contains at least one tile of each color. That is, count the patterns RBYRBYB and BYRYBRY but not RBYYBRYR because the four adjacent tiles colored BYYB does not include the color red.

Solution. We show a coding solution. The problem is equivalent to count all a base-3 number between 0 and 3^8. For the numbers has less then 8 digits, we add as many digits 0 to the left of the numbers as needed to create a 8-digit base-3 numbers Then for each of these base-3 number, we look at every block of 3 consecutive digit to make sure that it contains three different digits.

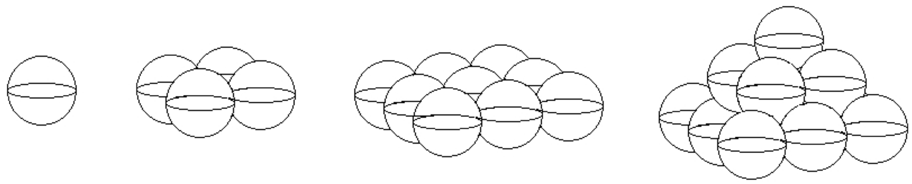
```
def to_base(number, base):
    digits = []
    while number:
        digits.append(number % base)
        number //= base
    return list(reversed(digits))

count = 0
for i in range(0, 3**10):
    s = to_base(i, 3)
    if len(s) < 8:
        s = [0 for j in range(len(s), 8)] + s
    found = True
    for j in range(0, 5):
        if len(set([e for e in s[j: j+4]])) < 3:
            found = False
            break
    if found:
        count +=1
print(count)
```

The answer is $\boxed{444.}$ □

Problem 4.4.21 (Problem 21)**.** (*5 points*) Some **unit** spheres are piled in 10 layers in the shape of a square pyramid with one sphere in the top layer, 4 spheres in the second layer, 9 spheres in the third layer, and so forth so that the bottom layer has a square array of 100 spheres. In each layer the centers of the spheres form a square grid so that each sphere is tangent to any sphere adjacent to it on the grid. Each sphere in an upper level is tangent to the four spheres directly below it.

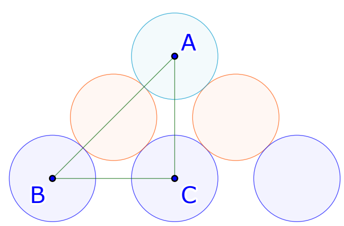
The diagram shows how the first three layers of spheres are stacked.

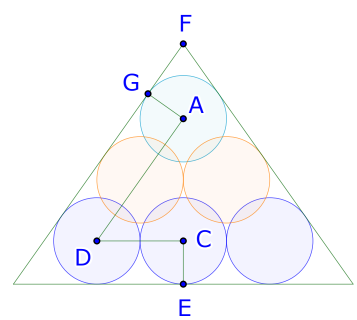


A square pyramid is built around the pile of spheres so that the sides of the pyramid are tangent to the spheres on the outside of the pile. The height of the pyramid is $a + b\sqrt{2} + c\sqrt{3}$, where a, b, and c are positive integers. Find $a + b + c$.

Solution. Let'ss assume that there are n layers. Then the center of any two adjacent spheres are a distance 2 apart. If four spheres in one layer have their centers at four vertices of a 2×2 square, then the centers of the spheres across the square from each other are a distance of $2\sqrt{2}$ apart.

Consider the diagram below on the left showing a cross section of the pile of spheres passing through the center of the sphere on the top layer labelled A, the center of the sphere in the third layer directly below the top sphere labeled C, and the center of a sphere across a 2×2 square from there labeled B. Because AC is vertical, BC is horizontal, $\triangle ABC$ is right triangle with $AB = 4$, $BC = 2\sqrt{2}$., thus $AC = 2\sqrt{2}$. It follows that the vertical distance between the plane containing the centers of spheres at one level of the pile, and the plane containing the center of spheres at the next lower level of the pile is $\sqrt{2}$.





Now, consider the diagram above on the right showing a cross section of the pile of spheres and the enclosing pyramid that passes through the top vertex of the pyramid, the center of the top sphere labeled A, the center of the sphere directly below A in the third level labeled C, and the center of an adjacent sphere in the third level labeled D. Then ACD is a right triangle with legs $AC = 2\sqrt{2}$, $CD = 2$, thus $AD = 2\sqrt{3}$. Let G be a point where the top sphere is tangent to the side of the pyramid. Since $\triangle AFG \sim \triangle CAD$ and $AG = 1$, thus $AF = \sqrt{3}$.

If the pile of sphere has n layers, then the distance between a horizontal plane passing through point A and a plane passing through the centers of the bottom layers is $(n-1)\sqrt{2}$. Since $AF = \sqrt{3}$, and the centers of the spheres are a distance 1 from the bottom of the pyramid, the height of the pyramid is given by

$$(n-1)\sqrt{2} + 1 + \sqrt{3}.$$

Now, since $n = 10$, thus the height is

$$a + b\sqrt{2} + c\sqrt{3} = 1 + 9\sqrt{2} + \sqrt{3} \Rightarrow a = 1, b = 9, c = 1.$$

Thus $a + b + c = \boxed{11.}$ $\square$

Chapter 5

Session 3: Feb 16 - Feb 24

5.1 Middle School - Assignment

Middle school students: grade 8 (US, CA), grade 9 (FR, UK, VN) and younger.

- **Submission deadline: Friday, February 23**
- **Test: Saturday, February 24**
- **Official solutions: Monday, February 26**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 5.1.1 (Problem 1)**.** (*5 points*) A picture with an area of 160 square inches is surrounded by a 2 inch border. The picture with its border is a rectangle twice as long as it is wide. How many inches long is that rectangle?

Solution. Let the width and length of the rectangle be w and $2w$. Then the picture is $w-4$ wide and $2w-4$ long. Its area $(w-4)(2w-4)$ is 160, so $w^2-6w-72=0$, so $w=12$. Thus $2w=\boxed{24.}$ □

Problem 5.1.2 (Problem 2)**.** (*5 points*) Pete's research shows that the number of nuts collected by the squirrels in any park is proportional to the square of the number of squirrels in that park. If Pete notes that four squirrels in a park collect 60 nuts, how many nuts are collected by 20 squirrels in a park?

Solution. The ratio of the number of nuts collected by 20 squirrels to the number collected by four squirrels is $\left(\frac{20}{4}\right)^2=25$. Thus 20 squirrels will collect $60\cdot 25=\boxed{1500.}$ □

Problem 5.1.3 (Problem 3)**.** (*5 points*) How many seven-digit positive integers do not either start or end with 7?

Solution. When selecting the seven-digit positive integer which does not either start or end with 7, there are 8 choices for the first digit (neither 0 or 7), 9 choices for the last digit (not 7), and 10 choices for each of the other five digits. Thus, there are $8\cdot 9\cdot 10^5=\boxed{7\ 200\ 000}$ choices. □

Problem 5.1.4 (Problem 4)**.** (*5 points*) Asheville, Bakersfield, Charter, and Darlington are four small towns along a straight road in that order. The distance from Bakersfield to Charter is one-third the distance from Asheville to Charter and one-quarter the distance from Bakersfield to Darlington. If it is 12 miles from Bakersfield to Charter, how many miles is it from Asheville to Darlington?

Solution. Since the distance from Bakersfield to Charter is one-third the distance from Asheville to Charter, the distance from Asheville to Charter is $3\cdot 12=36$. Similarly, the distance from Bakersfield to Darlington is $4\cdot 12=48$. It follows that the distance from Asheville to Darlington is $36+48-12=\boxed{72.}$ □

Problem 5.1.5 (Problem 5)**.** (*5 points*) Find the sum of all four-digit integers whose digits are a rearrangement of the digits 1, 2, 3, 4, such as 1234, 1432, or 3124.

Solution. There are 4! permutations of the digits 1234. In these permutations each of the four digits appear in each location (thousands, hundreds, tens, unit) exactly 6 times. Thus the sum of all the numbers is $6(1+2+3+4)(1111)=\boxed{66660.}$ □

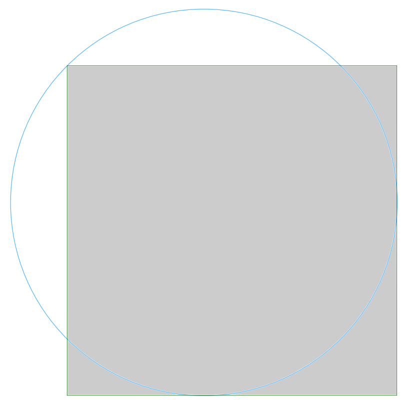
Problem 5.1.6 (Problem 6)**.** (*5 points*) Find the least positive integer k so that the mean of the numbers $k, k+1, k+2, k+3, \ldots, 2k$ exceeds 200.

Solution. The numbers $k, k+1, k+2, k+3, \ldots, 2k$ form an arithmetic sequence, thus the mean is the average of the first and last terms:

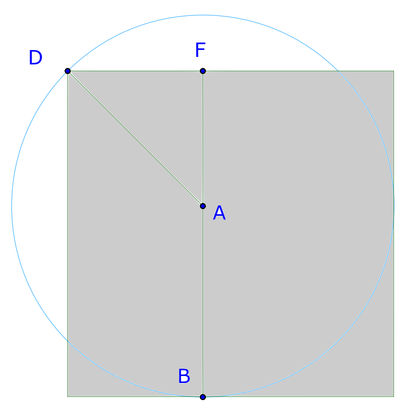
$$\frac{k+2k}{2}=\frac{3k}{2}.$$

If this exceeds 200, then $k>\frac{2\cdot 200}{3}$, or $k\geq\boxed{134.}$ □

Problem 5.1.7 (Problem 7)**.** (*5 points*) In the following diagram two sides of a square are tangent to a circle with diameter 8. One corner of the square lies on the circle. There are positive integers m and n so that the area of the square is $m + \sqrt{n}$. Find $m + n$.



Solution. Let the center of the circle be A, and the corner of the square on the circle be D. Let B and F be points on the sides of the squares so BF perpendicular to the sides of the square so that A lies on BF as shown.



$\triangle ADF$ is isosceles, AD is the radius of length 4, thus $AF = \frac{4}{\sqrt{2}} = 2\sqrt{2}$.

$$BF = AB + AF = 4 + 2\sqrt{2}.$$

So the area of the square is $(4 + 2\sqrt{2})^2 = 24 + 16\sqrt{2} = 24 + \sqrt{512}$. Thus, $m + n = 24 + 512 = \boxed{536.}$ □

Problem 5.1.8 (Problem 8)**.** (*5 points*) Find integer n such that both $n - 86$ and $n + 86$ are perfect squares.

Solution. Let $k^2 = n + 86, \ell^2 = n - 86$, then $k^2 - \ell^2 = 172 = 2^2 \cdot 43$. Thus $(k - \ell)(k + \ell) = 2^2 \cdot 43$. Since both $k - \ell, k + \ell$ have the same parity and their product is even, so both are even: $k - \ell = 2, k + \ell = 86$, so $k = 44, \ell = 42$.

Hence, $n = 42^2 + 86 = \boxed{1850.}$ □

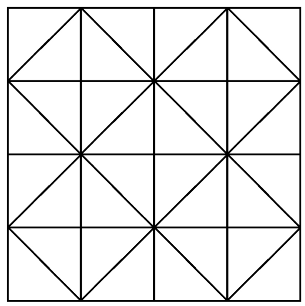
Problem 5.1.9 (Problem 9)**.** (*5 points*) Find n so that $4^{4^{4^2}} = 2^{8^n}$.

Solution.

$$4^{4^{4^2}} = 2^{2\cdot 4^{4^2}},\ 8^n = 2^{3n} \Rightarrow 2\cdot 4^{4^2} = 2^{3n} \Rightarrow 2^{2\cdot 4^2} = 2^{3n-1} \Rightarrow 2\cdot 4^2 = 3n-1 \Rightarrow n = \boxed{11.}$$

□

Problem 5.1.10 (Problem 10)**.** (*5 points*) How many triangles appear in the diagram below?



Solution. First, note that all of the triangles in the diagram are isosceles right triangles whose hypotenuses are either horizontal, vertical, of on an 45° diagonal.

The diagram is made up of 16 small squares which each contain two small triangles. This accounts for 32 small triangles.

Now, count the larger triangles with horizontal hypotenuses, 12 have hypotenuses made up of sides of two of the small squares, and 2 have hypotenuses made up of side of four of small squares for a total of 14 triangles.

Similarly, there are also 14 triangles with vertical hypotenuses.

There are 16 triangles whose hypotenuses are diagonal lines made up of the diagonals from two small squares, and 8 triangles whose hypotenuses are diagonal lines made up of the diagonals from three small squares.

Hence, the number of triangles is $32 + 14 + 14 + 16 + 8 = \boxed{84.}$ □

Problem 5.1.11 (Problem 11)**.** (*5 points*) Let a, b, and c be positive real numbers such that $a^2+b^2+c^2 = 989$ and $(a+b)^2 + (b+c)^2 + (c+a)^2 = 2013$. Find $a+b+c$.

Solution.

$$1024 = 2013-989 = \left[(a+b)^2 + (b+c)^2 + (c+a)^2\right] - \left(a^2+b^2+c^2\right) = a^2+b^2+c^2+2ab+2bc+2ca = (a+b+c)^2$$

Thus, $a+b+c = \boxed{32.}$ □

Problem 5.1.12 (Problem 12)**.** (*5 points*) A quarry wants to sell a large pile of gravel. At full price, the gravel would sell for 3200 dollars. But during the first week the quarry only sells 60% of the gravel at full price. The following week the quarry drops the price by 10%, and, again, it sells 60% of the remaining gravel. Each week, thereafter, the quarry reduces the price by another 10% and sells 60% of the remaining gravel. This continues until there is only a handful of gravel left. How many dollars does the quarry collect for the sale of all its gravel?

Solution. The first week 60% of the gravel is sold for the revenue of 3200(0.6)

The following week the revenue is 3200(0.4)(0.9)(0.6). Each week, 40% of the gravel from the previous week remains, and 60% of it is sold at 90% of the previous price. Thus, during week k, the quarry has a revenue of $3200(0.6)[(0.9)(0.4)]^{k-1}$.

It follows that the total revenue to the quarry is given by the geometric series

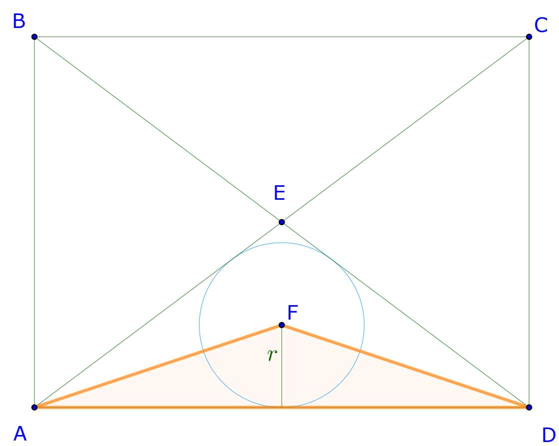
$$\sum_{k=1}^{\infty} 3200(0.6)[(0.9)(0.4)]^{k-1} = 3200(0.6)\sum_{k=0}^{\infty}(0.36)^k = 3200(0.6)\frac{1}{1-0.36} = \frac{3200(0.6)}{0.64} = \boxed{3000.}$$

□

Problem 5.1.13 (Problem 13)**.** (*5 points*) A rectangle has side lengths 6 and 8. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that a point randomly selected from the inside of the rectangle is closer to a side of the rectangle than to either diagonal of the rectangle. Find $m+n$.

Solution. Let the vertices of the rectangle be A, B, C, and D with side $AB = 6$ and $AD = 8$. Let E be the center of the rectangle. Then

$$AE = BE = CE = DE = \frac{1}{2}AC = \frac{1}{2}\sqrt{6^2+8^2} = 5.$$



Let F and r be the incenter and inradius of the $\triangle AED$. Then

$$[AED] = \frac{1}{2}(8\cdot 3) = \frac{8r+5r+5r}{2} \Rightarrow r = \frac{4}{3}.$$

The region with in $\triangle AED$ where the points are closer to side AD to either diagonal AE or DE is the $\triangle AFD$. The area of $\triangle AFD$ is $\frac{1}{2}\cdot 8\cdot\frac{4}{3} = \frac{16}{3}$.

Similarly the area of the region of $\triangle AEB$ of points closer to AB than to either diagonal AE or BE is $\frac{9}{2}$.

The desired probability is

$$\frac{2\frac{16}{3}+2\frac{9}{2}}{6\cdot 8} = \frac{59}{144} \Rightarrow m+n = 59+144 = \boxed{203.}$$

□

Problem 5.1.14 (Problem 14)**.** (*5 points*) Six children stand in a line outside their classroom. When they enter the classroom, they sit in a circle in random order. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that no two children who stood next to each other in the line end up sitting next to each other in the circle. Find $m + n$.

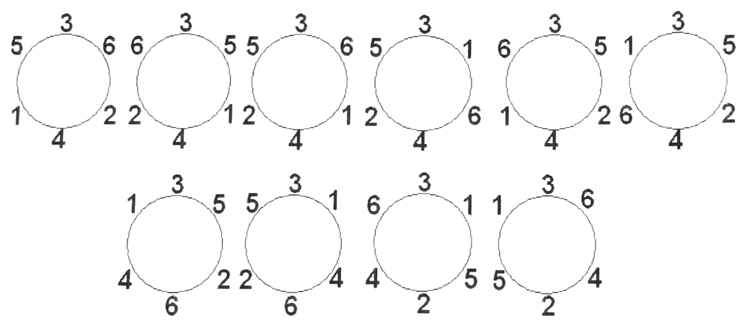
Solution. Let the children be numbered $1, 2, 3, 4, 5,$ and 6 in order that they were standing in the line. There are $5! = 120$ equally likely arrangements of these children when they sit in a circle.

To count the number of arrangements where no child sits next to a child that he/she was next to in the original line, consider where the children number 3 and 4 sit.

If child 3 sits opposite child 4, then children 2 and 5 can either sit opposite each other (2 ways) or next to each other (4 ways).

If child 3 and 4 sit with one child between them, then the child between them must be either 1 (2 ways) or 6 (2 ways).

This gives 10 possible arrangements of the children as shown in the diagram below.



The desired probability is $\frac{10}{120} = \frac{1}{12}$. The sum is $m + n = 1 + 12 = \boxed{13}$. $\square$

Problem 5.1.15 (Problem 15)**.** (*5 points*) For positive integer n let an be the integer consisting of n digits of 9 followed by the digits 488. For example, $a_3 = 999,488$ and $a_7 = 9,999,999,488$. Find the value of n so that a_n is divisible by the highest power of 2.

Solution. Note that

$$a_n = 10^{n+3} - 512 = 2^{n+3}5^{n+3} - 2^9.$$

If $n < 6$, then $a_n = 2^{n+3}(5^{n+3} - 2^{6-n})$, and the highest power of 2 divides a_n is $2^{n+3} < 2^9$.

If $n > 6$, then $a_n = 2^9(2^{n-6}5^{n+3} - 1)$, and the highest power of 2 divides a_n is 2^9.

If $n = 6$, then $a_6 = 2^9(5^9 - 1)$, which is divisible by at least 2^{10}.

Hence, the value of n is $\boxed{6}$. $\square$

5.2 Middle School - Test

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 6-15**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 5.2.1 (Problem 1)**.** (*5 points*) A picture with an area of 180 square inches is surrounded by a 1 inch border. The picture with its border is a rectangle twice as long as it is wide. How many inches long is that rectangle?

Solution. Let the width and length of the rectangle be w and $2w$. Then the picture is $w-2$ wide and $2w-2$ long. Its area $(w-2)(2w-2)$ is 180, so $w^2-3w-88=0$, so $w=11$. Thus $2w=\boxed{22.}$ □

Problem 5.2.2 (Problem 2)**.** (*5 points*) Pete's research shows that the number of nuts collected by the squirrels in any park is proportional to the square of the number of squirrels in that park. If Pete notes that four squirrels in a park collect 15 nuts, how many nuts are collected by 16 squirrels in a park?

Solution. The ratio of the number of nuts collected by 16 squirrels to the number collected by four squirrels is $\left(\frac{16}{4}\right)^2=16$. Thus 16 squirrels will collect $15\cdot 16=\boxed{240.}$ □

Problem 5.2.3 (Problem 3)**.** (*5 points*) How many six-digit positive integers do not either start or end with 7?

Solution. When selecting the six-digit positive integer which does not either start or end with 7, there are 8 choices for the first digit (neither 0 or 7), 9 choices for the last digit (not 7), and 10 choices for each of the other four digits. Thus, there are $8\cdot 9\cdot 10^4=\boxed{720\,000}$ choices. □

Problem 5.2.4 (Problem 4)**.** (*5 points*) Asheville, Bakersfield, Charter, and Darlington are four small towns along a straight road in that order. The distance from Bakersfield to Charter is one-third the distance from Asheville to Charter and one-quarter the distance from Bakersfield to Darlington. If it is 18 miles from Bakersfield to Charter, how many miles is it from Asheville to Darlington?

Solution. Since the distance from Bakersfield to Charter is one-third the distance from Asheville to Charter, the distance from Asheville to Charter is $3\cdot 18=54$. Similarly, the distance from Bakersfield to Darlington is $4\cdot 18=72$. It follows that the distance from Asheville to Darlington is $54+72-18=\boxed{108.}$ □

Problem 5.2.5 (Problem 5)**.** (*5 points*) Find the sum of all three-digit integers whose digits are a rearrangement of the digits 1, 2, 3, 4, such as 123, 142, or 324.

Solution. There are $\binom{4}{3}=4$ ways to choose three digits out of 1,2,3,4, let them be a,b,c.

There are 3! permutations of the three chosen digits a,b,c. In these permutations each of the three digits appear in each location (hundreds, tens, unit) exactly 2 times. Thus the sum of all the numbers is $2(a+b+c)(111)$.

The total is $222((1+2+3)+(1+2+4)+(1+3+4)+(2+3+4))=\boxed{6660.}$ □

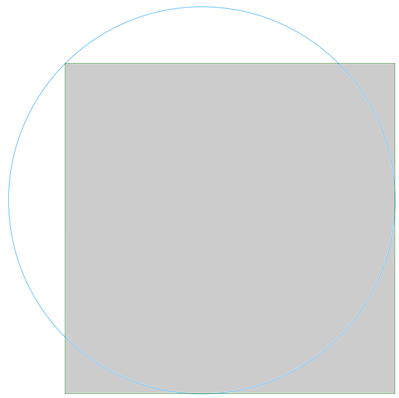
Problem 5.2.6 (Problem 6)**.** (*5 points*) Find the least positive integer k so that the mean of the numbers $k,k+1,k+2,k+3,\ldots,2k$ exceeds 300.

Solution. The numbers $k,k+1,k+2,k+3,\ldots,2k$ form an arithmetic sequence, thus the mean is the average of the first and last terms:

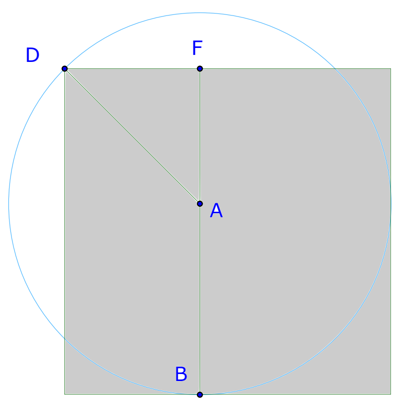
$$\frac{k+2k}{2}=\frac{3k}{2}.$$

If this exceeds 300, then $k>\frac{2\cdot 300}{3}=200$, or $k\geq\boxed{201.}$ □

Problem 5.2.7 (Problem 7)**.** (*5 points*) In the following diagram two sides of a square are tangent to a circle with diameter 8. One corner of the square lies on the circle. There are positive integers m and n so that the area of the square is $m+\sqrt{n}$. Find $n-m$.



Solution. Let the center of the circle be A, and the corner of the square on the circle be D. Let B and F be points on the sides of the squares so BF perpendicular to the sides of the square so that A lies on BF as shown.



$\triangle ADF$ is isosceles, AD is the radius of length 4, thus $AF = \frac{4}{\sqrt{2}} = 2\sqrt{2}$.

$$BF = AB + AF = 4 + 2\sqrt{2}.$$

So the area of the square is $(4+2\sqrt{2})^2 = 24 + 16\sqrt{2} = 24 + \sqrt{512}$. Thus, $n - m = 512 - 24 = \boxed{488.}$ □

Problem 5.2.8 (Problem 8)**.** (*5 points*) Find the sum of all values of n such that both $n-70$ and $n+70$ are perfect squares.

Solution. Let $k^2 = n+70, \ell^2 = n-70$, then $k^2 - \ell^2 = 140 = 2^2 \cdot 5 \cdot 7$. Thus $(k-\ell)(k+\ell) = 2^2 \cdot 5 \cdot 7$. Since both $k-\ell, k+\ell$ have the same parity and their product is even, so both are even:

$$\begin{cases} k-\ell = 2, k+\ell = 70 \Rightarrow k = 36, \ell = 34 \Rightarrow n = 1226 \\ k-\ell = 10, k+\ell = 14 \Rightarrow k = 12, \ell = 2 \Rightarrow n = 74 \end{cases}$$

Hence, the sum of all values of n is $1226 + 74 = \boxed{1300.}$ □

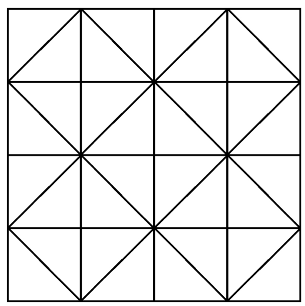
Problem 5.2.9 (Problem 9)**.** (*5 points*) Find n so that $4^{4^{4^2}} = 16^{2^n}$.

Solution.

$$4^{4^{4^2}} = 4^{4^{16}} = 4^{2^{32}} = 4^{2 \cdot 2^{31}} = 16^{2^{31}} \Rightarrow n = \boxed{31.}$$

□

Problem 5.2.10 (Problem 10)**.** (*5 points*) The diagram below is made up of 16 unit squares which each contain two small triangles. How many triangles appear in the diagram below which is **not half of a unit square**?



Solution. First, note that all of the triangles in the diagram are isosceles right triangles whose hypotenuses are either horizontal, vertical, of on an 45° diagonal.

The diagram below is made up of 16 small squares which each contain two small triangles. This accounts for 32 small triangles. These are not the ones we want.

Now, count the larger triangles with horizontal hypotenuses, 12 have hypotenuses made up of sides of two of the small squares, and 2 have hypotenuses made up of side of four of small squares for a total of 14 triangles.

Similarly, there are also 14 triangles with vertical hypotenuses.

There are 16 triangles whose hypotenuses are diagonal lines made up of the diagonals from two small squares, and 8 triangles whose hypotenuses are diagonal lines made up of the diagonals from three small squares.

Hence, the number of triangles is $14 + 14 + 16 + 8 = \boxed{52.}$ □

Problem 5.2.11 (Problem 11)**.** (*5 points*) Let a, b, and c be positive real numbers such that $ab+bc+ca = 432$ and $(a+b)^2 + (b+c)^2 + (c+a)^2 = 2024$. Find $a + b + c$.

Solution.

$$2(a^2 + b^2 + c^2) = \left[(a+b)^2 + (b+c)^2 + (c+a)^2\right] - 2(ab + bc + ca) = 1160$$
$$\Rightarrow (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = \frac{1160}{2} + 2(432) = 1444.$$

Thus, $a + b + c = \boxed{38.}$ □

Problem 5.2.12 (Problem 12)**.** (*5 points*) A quarry wants to sell a large pile of gravel. At full price, the gravel would sell for 1600 dollars. But during the first week the quarry only sells 60% of the gravel at full price. The following week the quarry drops the price by 10%, and, again, it sells 60% of the remaining gravel. Each week, thereafter, the quarry reduces the price by another 10% and sells 60% of the remaining gravel. This continues until there is only a handful of gravel left. How many dollars does the quarry collect for the sale of all its gravel?

Solution. The first week 60% of the gravel is sold for the revenue of $1600(0.6)$

The following week the revenue is $1600(0.4)(0.9)(0.6)$. Each week, 40% of the gravel from the previous week remains, and 60% of it is sold at 90% of the previous price. Thus, during week k, the quarry has a revenue of $1600(0.6)[(0.9)(0.4)]^{k-1}$.

It follows that the total revenue to the quarry is given by the geometric series

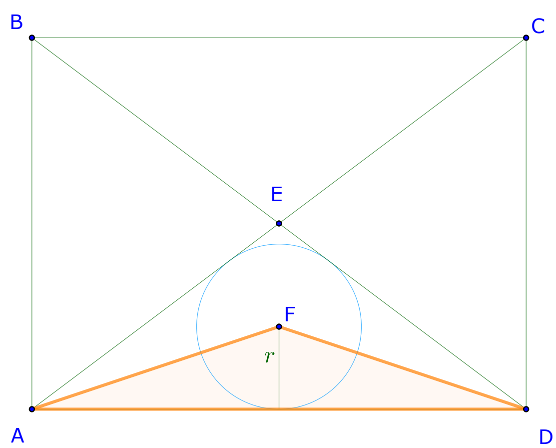
$$\sum_{k=1}^{\infty} 1600(0.6)[(0.9)(0.4)]^{k-1} = 1600(0.6)\sum_{k=0}^{\infty}(0.36)^k = 1600(0.6)\frac{1}{1-0.36} = \frac{1600(0.6)}{0.64} = \boxed{1500.}$$

□

Problem 5.2.13 (Problem 13)**.** (*5 points*) A rectangle has side lengths 6 and 8. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that a point randomly selected from the inside of the rectangle is closer to a side of the rectangle than to either diagonal of the rectangle. Find $n - m$.

Solution. Let the vertices of the rectangle be A, B, C, and D with side $AB = 6$ and $AD = 8$. Let E be the center of the rectangle. Then

$$AE = BE = CE = DE = \frac{1}{2}AC = \frac{1}{2}\sqrt{6^2 + 8^2} = 5.$$



Let F and r be the incenter and inradius of the $\triangle AED$. Then

$$[AED] = \frac{1}{2}(8 \cdot 3) = \frac{8r + 5r + 5r}{2} \Rightarrow r = \frac{4}{3}.$$

The region with in $\triangle AED$ where the points are closer to side AD to either diagonal AE or DE is the $\triangle AFD$. The area of $\triangle AFD$ is $\frac{1}{2} \cdot 8 \cdot \frac{4}{3} = \frac{16}{3}$.

Similarly the area of the region of $\triangle AEB$ of points closer to AB than to either diagonal AE or BE is $\frac{9}{2}$.

The desired probability is

$$\frac{2\frac{16}{3} + 2\frac{9}{2}}{6 \cdot 8} = \frac{59}{144} \Rightarrow n - m = 144 - 59 = \boxed{85.}$$

□

Problem 5.2.14 (Problem 14)**.** (*5 points*) Six children stand in a line outside their classroom. When they enter the classroom, they sit in a circle in random order. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that no two children who stood next to each other in the line end up sitting next to each other in the circle. Find the difference $n - m$.

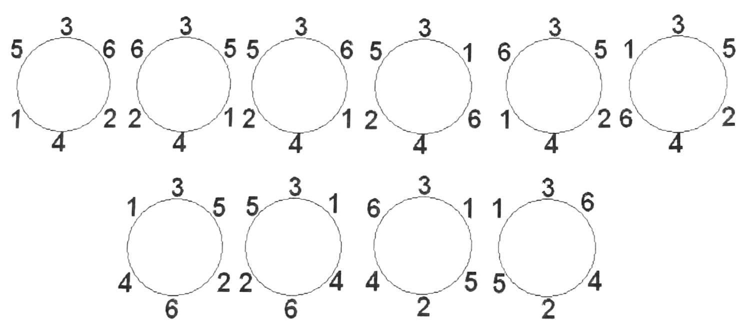
Solution. Let the children be numbered $1, 2, 3, 4, 5$, and 6 in order that they were standing in the line. There are $5! = 120$ equally likely arrangements of these children when they sit in a circle.

To count the number of arrangements where no child sits next to a child that he/she was next to in the original line, consider where the children number 3 and 4 sit.

If child 3 sits opposite child 4, then children 2 and 5 can either sit opposite each other (2 ways) or next to each other (4 ways).

If child 3 and 4 sit with one child between them, then the child between them must be either 1 (2 ways) or 6 (2 ways).

This gives 10 possible arrangements of the children as shown in the diagram below.



The desired probability is $\frac{10}{120} = \frac{1}{12}$. The sum is $n - m = 12 - 1 = \boxed{11}$.

□

Problem 5.2.15 (Problem 15)**.** (*5 points*) For positive integer n let an be the integer consisting of n digits of 9 followed by the digits 1808. For example, $a_2 = 991,808$ and $a_6 = 9,999,991,808$. Find the value of n so that a_n is divisible by the highest power of 2.

Solution. Note that

$$a_n = 10^{n+4} - 8192 = 2^{n+4}5^{n+4} - 2^{13}.$$

If $n < 9$, then $a_n = 2^{n+4}(5^{n+4} - 2^{9-n})$, and the highest power of 2 divides a_n is $2^{n+4} < 2^{13}$.

If $n > 9$, then $a_n = 2^{13}(2^{n-9}5^{n+4} - 1)$, and the highest power of 2 divides a_n is 2^{13}.

If $n = 9$, then $a_9 = 2^{13}(5^{13} - 1)$, which is divisible by at least 2^{14}.

Hence, the value of n is $\boxed{9}$.

□

5.3 High School - Assignment

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

- **Submission deadline: Friday, February 23**
- **Test: Saturday, February 24**
- **Official solutions: Monday, February 26**
- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 8-17**
- **Olympiad (O) level: Problems 16-25**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 5.3.1 (Problem 1)**.** (*5 points*) Find the least positive integer n so that both n and $n+1$ have prime factorizations with exactly four (not necessarily distinct) prime factors.

Solution. One of n and $n+1$ is an odd number. The smallest integer with four prime factors is $3^4 = 81$. But neither 80 or 82 has 4 prime factors. The second smallest is $3^3 \cdot 5 = 135$ and $134 = 2 \cdot 67$, $136 = 2^3 \cdot 17$. The latter one has the desired property. The answer is $\boxed{135.}$ □

Problem 5.3.2 (Problem 2)**.** (*5 points*) Two convex polygons have a total of 33 sides and 243 diagonals. Find the number of diagonals in the polygon with the greater number of sides.

Solution. A convex polygon has n sides then it has $\frac{n(n-1)}{2} - n = \frac{n^2-3n}{2}$ diagonals.

Now

$$243 = \frac{n^2 - 3n}{2} + \frac{(33-n)^2 - 3(33-n)}{2} = n^2 - 33n + 495 \Rightarrow n^2 - 33n + 252 = 0 \Rightarrow n = 12 \text{ or } n = 21$$

The number of diagonals of the polygon with 21 sides is $\frac{21^2 - 3 \cdot 21}{2} = \boxed{189.}$ □

Problem 5.3.3 (Problem 3)**.** (*5 points*) In the tribe of Zimmer, being able to hike long distances and knowing the roads through the forest are both extremely important, so a boy who reaches the age of manhood is not designated as a man by the tribe until he completes an interesting rite of passage. The man must go on a sequence of hikes. The first hike is a 5 kilometer hike down the main road. The second hike is a $5\frac{1}{4}$ kilometer hike down a secondary road. Each hike goes down a different road and is a quarter kilometer longer than the previous hike. The rite of passage is completed at the end of the hike where the cumulative distance walked by the man on all his hikes exceeds 1000 kilometers. So in the tribe of Zimmer, how many roads must a man walk down, before you call him a man?

Solution. The total distance a man walked in the n hikes is

$$5 + 5\frac{1}{4} + 5\frac{1}{2} + \cdots + \frac{19+n}{4} = \sum_{k=1}^{n} \frac{19+k}{4} = \frac{1}{4}\left(19n + \frac{n(n+1)}{2}\right) = \frac{n(n+39)}{8}$$

The smallest n for $n^2 + 39n > 8 \cdot 1000$ is $n = \boxed{73.}$ □

Problem 5.3.4 (Problem 4)**.** (*5 points*) Find the value of x that satisfies $\log_3(\log_9 x) = \log_9(\log_3 x)$.

Solution. Note that

$$\log_9 x = \frac{\log_3 x}{\log_3 9} \Rightarrow \begin{cases} \log_3(\log_9 x) = \log_3\left(\frac{\log_3 x}{\log_3 9}\right) = \log_3\left(\frac{\log_3 x}{2}\right) \\ \log_9(\log_3 x) = \frac{\log_3(\log_3 x)}{\log_3 9} = \frac{\log_3(\log_3 x)}{2} \end{cases}$$

$$\Rightarrow \log_3\left(\frac{\log_3 x}{2}\right) = \frac{\log_3(\log_3 x)}{2} \Rightarrow 2\log_3\left(\frac{\log_3 x}{2}\right) = \log_3(\log_3 x) \Rightarrow \log_3\left(\frac{\log_3 x}{2}\right)^2 = \log_3(\log_3 x)$$

Thus $\left(\frac{\log_3 x}{2}\right)^2 = \log_3 x$. Since $\log_3 x > 0$, $\log_3 x = 4 \Rightarrow x = \boxed{81.}$ □

Problem 5.3.5 (Problem 5)**.** (*5 points*) Consider a sequence of eleven squares that have side lengths $3, 6, 9, 12, \ldots, 33$. Eleven copies of a single square each with area A have the same total area as the total area of the eleven squares of the sequence. Find A.

Solution. It is easy to see that the total area of eleven squares is:

$$3 \cdot 3 + 6 \cdot 6 + \cdots + 33 \cdot 33 = 9(1^2 + 2^2 + \cdots + 11^2) = 9 \cdot \frac{11 \cdot 12 \cdot 23}{6} = 11 \cdot 414 = 11 \cdot A$$

Hence, the area $A = \boxed{414.}$ □

Problem 5.3.6 (Problem 6)**.** (*5 points*) Define $f(x) = 2x + 3$and suppose that $g(x+2) = f(f(x-1) \cdot f(x+1) + f(x))$. Find $g(6)$.

Solution.

$$g(6) = g(4+2) = f(f(4-1) \cdot f(4+1) + f(4)) = f(f(3)f(5) + f(4)) = f(9 \cdot 13 + 11) = f(128) = \boxed{259.}$$

□

Problem 5.3.7 (Problem 7)**.** (*5 points*) Ted flips seven fair coins. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that Ted flips at least two heads given that he flips at least three tails. Find $m + n$.

Solution. The total number of possibilities of flipping k tails with seven flips is $\binom{7}{k}$, thus the probability that Ted flips at least three tails is

$$\frac{\binom{7}{3} + \binom{7}{4} + \binom{7}{5}}{\binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}} = \frac{91}{99}.$$

The answer is $m + n = 91 + 99 = \boxed{190.}$ □

Problem 5.3.8 (Problem 8)**.** (*5 points*) Find the least n for which $n!(n+1)!(2n+1)! - 1$ ends in thirty digits that are all 9's.

Solution. If $n!(n+1)!(2n+1)! - 1$ ends in thirty digits that are all 9's then $n!(n+1)!(2n+1)!$ ends in thirty digits that are all 0's.

Let's try to find if the least $n < 100$. The number of trailing zeros in $n!$ is the number of factors of 5 in $n!$, which is $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor$.

So we are trying to find an n such that the following sum is at least 30:

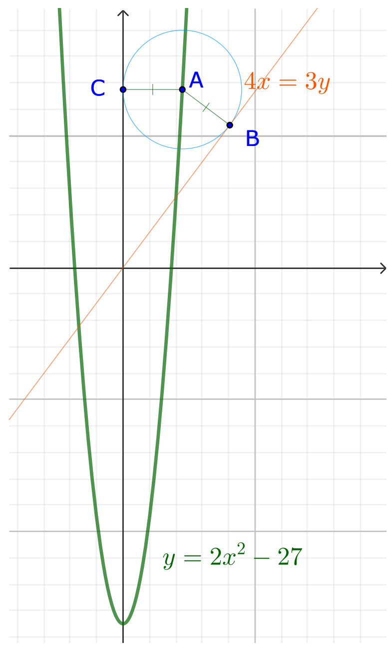
$$S(n) = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n+1}{5} \right\rfloor + \left\lfloor \frac{n+1}{25} \right\rfloor + \left\lfloor \frac{2n+1}{5} \right\rfloor + \left\lfloor \frac{2n+1}{25} \right\rfloor + \left\lfloor \frac{2n+1}{125} \right\rfloor$$

For example $S(25) = 5 + 1 + 5 + 1 + 10 + 2 = 24$, $S(27) = 6 + 6 + 11 + 2 = 25$. By continuing doing so $S(34) = 30$. Hence, the answer is $\boxed{34.}$ □

Problem 5.3.9 (Problem 9)**.** (*5 points*) A circle in the first quadrant with center on the curve $y = 2x^2 - 27$ is tangent to the $y-$axis and the line $4x = 3y$. The radius of the circle is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Solution. The distance from a point $A(x_0, y_0)$ to the line $4x = 3y$ is given by the normal form for the equation of the line $(4)x + (-3)y + 0 = 0$,

$$\frac{|4x_0 + (-3)y_0 + 0|}{\sqrt{(4^2 + (-3)^2)}} = \frac{|4x_0 - 3y_0|}{5}.$$



If point A, in the first quadrant, is on the curve $y = 2x^2 - 27$ at the same distance from the $y-$axis and the line $4x = 3y$, then the distance to the $y-$axis, or its $x-$coordinate (AC), must be equal to the distance to the line $4x = 3y$, thus

$$x_0 = \frac{|4x_0 - 3y_0|}{5} \Rightarrow x_0 = \frac{|4x_0 - 3(x_0^2 - 27)|}{5} = \frac{3(2x_0^2 - 27) - 4x_0}{5} \Rightarrow 6x_0^2 - 9x_0 - 81 = 0 \Rightarrow x_0 = \frac{9}{2}\ (x_0 > 0)$$

Hence, $m + n = 9 + 2 = \boxed{11.}$ □

Problem 5.3.10 (Problem 10)**.** (*5 points*) Let N be a positive integer whose digits add up to 23. What is the greatest possible product the digits of N can have?

Solution. First, digits 0 and 1 would not be among the digits of N.

A digit of 4 can be replaced by two digits of 2 having the same sum and same product.

Any digit k larger then or equal to 5 can be replace by a pair of 2 and $k - 2$, where $2(k - 2) = 2k - 4 > k$.

Therefore N has a digits of 2 and b digits of 3, where a and b are non-negative integers and the product of them is $2^a 3^b$, where $2a + 3b = 23$.

Now, $2 + 2 + 2 = 3 + 3$, and $2 \cdot 2 \cdot 2 < 3 \cdot 3$, thus a can be 0, 1 or 2. Only $a = 1$ gives $b = 7$ integer value.

Thus $N = 2 \cdot 3^7 = \boxed{4374.}$ □

Problem 5.3.11 (Problem 11). (*5 points*) Let a, b, and c be non-zero real numbers such that

$$\frac{ab}{a+b} = 3, \frac{bc}{b+c} = 4, \frac{ca}{c+a} = 5.$$

There are relatively prime positive integers m and n so that

$$\frac{abc}{ab+bc+ca} = \frac{m}{n}.$$

Find $m+n$.

Solution.

$$\frac{ab}{a+b} = 3 \Rightarrow \frac{a+b}{ab} = \frac{1}{3} \Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{1}{3}$$
$$\text{Similarly } \frac{1}{b} + \frac{1}{c} = \frac{1}{4}, \ \frac{1}{c} + \frac{1}{a} = \frac{1}{5}$$
$$\Rightarrow \frac{ab+bc+ca}{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{47}{120}$$

Hence, $m + n = 47 + 120 = \boxed{167.}$ □

Problem 5.3.12 (Problem 12). (*5 points*) How many positive integer solutions are there to $w+x+y+z = 20$ where $w + x \geq 5$ and $y + z \geq 5$?

Solution. Let $k = w + x$, then $y + z = 20 - k$. For a value of $k \geq 2$, there are $k - 1$ ways to assign positive value to w and x so that their sum is k.

Similarly with $20 - k \geq 2$, there are $19 - k$ ways to assign positive value to w and x so that their sum is $20 - k$.

Hence, the number of ways is:

$$\sum_{k=5}^{15}(k-1)(19-k) = \sum_{k=1}^{11}(k+3)(15-k) = \sum_{k=1}^{11}(45+12k-k^2) = 11 \cdot 45 + 12\frac{11 \cdot 12}{2} - \frac{11 \cdot 12 \cdot 23}{6} = \boxed{781.}$$

□

Problem 5.3.13 (Problem 13). (*5 points*) Find the number of three-digit numbers such that its first two digits are each divisible by its third digit.

Solution. It is obvious that the third digit cannot be 0. There are five cases:

3rd digit	1st digit	2nd digit	# of ways
1	1,2,3,4,5,6,7,8,9	0,1,2,3,4,5,6,7,8,9	90
2	2,4,6,8	0,2,4,6,8	20
3	3,6,9	0,3,6,9	12
4	4,8	0,4,8	6
5,6,7,8,9	same as 3rd	0 or same as 3rd	10

Hence, the number of ways is $90 + 20 + 12 + 6 + 10 = \boxed{138.}$ □

Problem 5.3.14 (Problem 14)**.** (*5 points*) Find the remainder when $2^{5^9} + 5^{9^2} + 9^{2^5}$ is divided by 11.

Solution. By Fermat's Little Theorem: $\gcd(n, p) = 1$, then $n^{p-1} \equiv 1 \pmod p$, so:

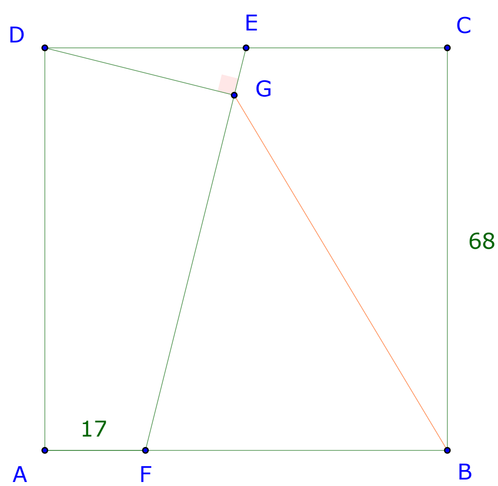
$$2^{10} \equiv 5^{10} \equiv 9^{10} \equiv 1 \pmod{11}.$$

It is easy to see remainders of $5^9, 9^2$, and 2^5 modulo 10 are 5, 1, and 2. Thus:

$$2^{5^9} + 5^{9^2} + 9^{2^5} \equiv 2^5 + 5^1 + 9^2 = 32 + 5 + 81 \equiv \boxed{8} \pmod{11}$$

□

Problem 5.3.15 (Problem 15)**.** (*5 points*) Square $ABCD$ has side length 68. Let E be the midpoint of segment CD, and let F be the point on segment AB a distance 17 from point A. Point G is on segment EF so that EF is perpendicular to segment GD. The length of segment BG can be written as $m\sqrt{n}$, where m and n be are positive integers, and n is not divisible by the square of any prime. Find $m + n$.



Solution. Let vector $\overrightarrow{BC} = u$, and $\overrightarrow{BA} = v$. Then $\overrightarrow{EF} = \frac{1}{4}v - u$. Note that $\overrightarrow{EG}$ is the projection of $\overrightarrow{ED} = \frac{1}{2}v$ onto $\overrightarrow{EF}$. This projection is:

$$\left(\frac{\overrightarrow{ED} \cdot \overrightarrow{EF}}{\overrightarrow{EF} \cdot \overrightarrow{EF}}\right)\overrightarrow{EF} = \frac{\frac{1}{2}v \cdot \left(\frac{1}{4}v - u\right)}{\left(\frac{1}{4}v - u\right) \cdot \left(\frac{1}{4}v - u\right)}\left(\frac{1}{4}v - u\right) = \frac{\frac{1}{8}}{\frac{1}{16} + 1}\left(\frac{1}{4}v - u\right) = \frac{1}{34}v - \frac{2}{17}u$$

Then

$$\overrightarrow{BG} = u + \frac{1}{2}v + \overrightarrow{EG} = \frac{15}{17}u + \frac{9}{17}v$$

The length of this vector is

$$\frac{68}{17}\sqrt{15^2 + 9^2} = 12\sqrt{34} \Rightarrow m + n = 12 + 34 = \boxed{46.}$$

□

Problem 5.3.16 (Problem 16)**.** (*5 points*) Each time you click a toggle switch, the switch either turns from *off* to *on* or from on to *off*. Suppose that you start with three toggle switches with one of them *on* and two of them *off*. On each move you randomly select one of the three switches and click it. Let m and n be relatively prime positive integers so that $\frac{m}{n}$ is the probability that after four such clicks, one switch will be *on* and two of them will be *off*. Find $m+n$.

Solution. Without loss of generality assume that at the beginning the first of the three toggle switches is *on*, and the second and third are *off*.

There are $3^4 = 81$ equally likely ways to select a sequence of four toggle switches to click. After four clicks, there will either be one or three switches in the *on* position.

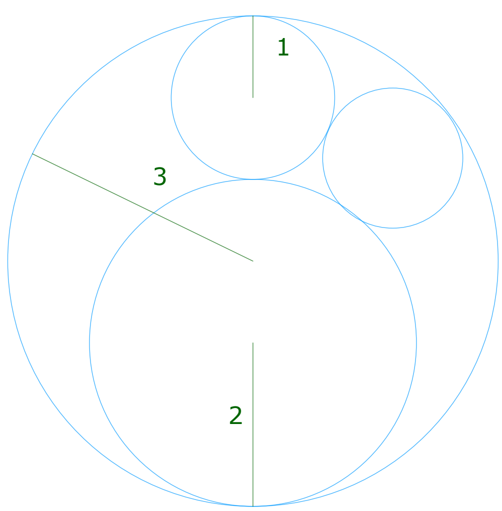
To get three toggle switches in the *on* position, one much click the first switch an even number of times and the second and third switches an odd number of times each. This can be done by clicking the first switch twice and the other switches once each, or by clicking either the second and third switches three times and the other of the two switches one time. The number of ways to do one of these is:

$$\binom{4}{2,1,1} + \binom{4}{0,1,3} + \binom{4}{0,3,1} = \frac{4!}{2!1!1!} + \frac{4!}{0!1!3!} + \frac{4!}{0!3!1!} = 12 + 4 + 4 = 20.$$

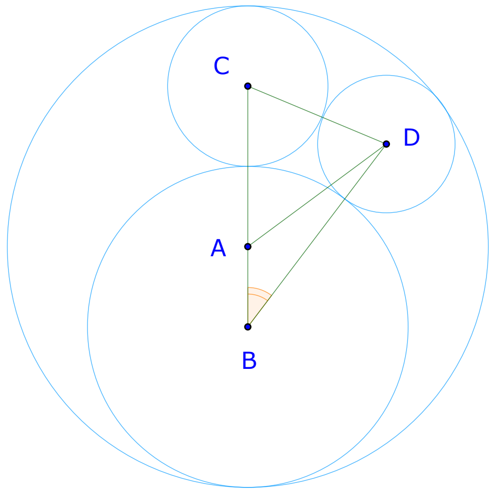
It follows that the desired probability is $\frac{81-20}{81} = \frac{61}{81}$.

Thus the desired sum $61 + 81 = \boxed{142.}$ □

Problem 5.3.17 (Problem 17)**.** (*5 points*) The diagram below shows circles radius 1 and 2 externally tangent to each other and internally tangent to a circle radius 3. There are relatively prime positive integers m and n so that a circle radius $\frac{m}{n}$ is internally tangent to the circle radius 3 and externally tangent to the other two circles as shown. Find $m+n$.



Solution. Let A, B, C, and D be the centres of the circles radius 3, 2, 1 and $r = \frac{m}{n}$.



By the Law of Cosines:

$$\triangle ABD: \ 1^2 + (2+r)^2 - 2 \cdot 1 \cdot (2+r)\cos(\angle ABD) = (3-r)^2$$
$$\triangle CBD: \ 3^2 + (2+r)^2 - 2 \cdot 3 \cdot (2+r)\cos(\angle ABD) = (1+r)^2$$
$$\Rightarrow 3(3-r)^2 - (1+r)^2 = -6 + 2(2+r)^2 \Rightarrow r = \frac{6}{7}.$$

Thus $m + n = 6 + 7 = \boxed{13.}$ □

Problem 5.3.18 (Problem 18)**.** (*5 points*) Find the greatest seven-digit integer divisible by 132 whose digits, in order, are $2, 0, x, y, 1, 2, z$ where x, y, and z are single digits.

Solution. [Solution 1] Let n be the greatest seven-digit integer divisible by 132, whose digits, in order, are $2, 0, x, y, 1, 2, z$ where x, y, and z are single digits.

$$3 \mid 132 \Rightarrow 5 + x + y + z \equiv 0 \pmod 3$$
$$11 \mid 132 \Rightarrow 1 + x - y + z \equiv 0 \pmod{11}$$
$$4 \mid 132 \Rightarrow z \equiv 0 \pmod 4$$

Thus, $z \in \{0, 4, 8\}$. n would be greatest if $x = 9$, in this case $y \equiv 1 - z \pmod 3$, and $y \equiv z - 1 \pmod{11}$.

$z = 0$ means $y \equiv 10 \pmod{11}$, which is impossible.

$z = 8$ means $y \equiv 1 - 8 \equiv 2 \pmod 3$, and $y \equiv 7 \pmod{11}$, which is impossible.

$z = 4$ means $y \equiv 1 - 4 \equiv 0 \pmod 3$, $y \equiv 4 - 1 = 3 \pmod{11}$, thus $y = 3$.

Hence, the number is $\boxed{2093124.}$ □

Solution. [Solution 2] We show a coding solution. The problem is equivalent to find the largest multiple of 132 in the format $\overline{20xy12z}$.

```
for x in range(9, -1, -1):
    for y in range(9, -1, -1):
        for z in range(9, -1, -1):
            i = 2000000 + x * 10000 + y * 1000  + 120 + z
            if i % 132 == 0:
                print(i)
                exit(0)
```

The answer is $\boxed{2093124.}$ □

Problem 5.3.19 (Problem 19)**.** (*5 points*) There are positive integers m and n so that $x = m + \sqrt{n}$ is a solution to the equation

$$x^2 - 10x + 1 = \sqrt{x}(x+1).$$

Find $m + n$.

Solution. It is easy to see that $x = 0$ is not a solution. Divide both side of the equation by x, we have

$$x - 10 + \frac{1}{x} = \sqrt{x} + \frac{1}{\sqrt{x}}.$$

Let $y = \sqrt{x} + \frac{1}{\sqrt{x}} > 0$, then $y^2 = x + \frac{1}{x} + 2$, thus

$$y^2 - 12 = y \Rightarrow (y+3)(y-4) = 0 \Rightarrow y = 4 \Rightarrow \sqrt{x} + \frac{1}{\sqrt{x}} = 4 \Rightarrow x - 4\sqrt{x} + 1 = 0$$
$$\Rightarrow \sqrt{x} = 2 \pm 3 \Rightarrow x = (2+\sqrt{3})^2 = 7 + \sqrt{48}.$$

Thus $m + n = 7 + 48 = \boxed{55.}$ □

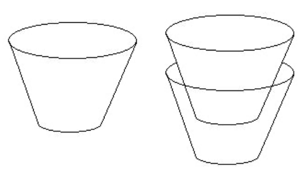
Problem 5.3.20 (Problem 20)**.** (*5 points*) Find the largest prime that divides $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + 44 \cdot 45 \cdot 46$.

Solution. Note that $(k-1)k(k+1) = (k^2-1)k = k^3 - k$, thus the given sum is equivalent to

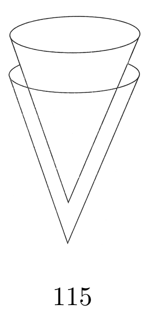
$$\sum_{k=2}^{45} k^3 - k = \sum_{k=1}^{45} k^3 - \sum_{k=1}^{45} k = \left(\frac{45 \cdot 46}{2}^2\right) - \frac{45 \cdot 46}{2} = 45 \cdot 23\,(45 \cdot 23 - 1) = 2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 23 \cdot 47.$$

Thus the largest prime factor is $\boxed{47}$. □

Problem 5.3.21 (Problem 21)**.** (*5 points*) A paper cup has a base that is a circle with radius r, a top that is a circle with radius $2r$, and sides that connect the two circles with straight line segments as shown below. This cup has height h and volume V. A second cup that is exactly the same shape as the first is held upright inside the first cup so that its base is a distance of $\frac{h}{2}$ from the base of the first cup. The volume of liquid that will fit inside the first cup and outside the second cup can be written as $\frac{m}{n} \cdot V$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. The volume between the cups does not change if the sides of each cup are extended so that the cups form a circular cones with radius $2r$ and height $2h$.



The volume of the first cone is:

$$\frac{1}{3}\pi(2r)^2(2h) = \frac{8}{3}\pi r^2 h.$$

The part of the second cone inside the first is a cone that is $\frac{3}{4}$ the height of the first cone, so the volume that it takes up within the first cone is:

$$\left(\frac{3}{4}\right)^3 \frac{8}{3}\pi r^2 h = \frac{9}{8}\pi r^2 h$$

Thus the volume inside the first cup and outside of the second cup is

$$\left(\frac{8}{3} - \frac{9}{8}\right)\pi r^2 h = \frac{37}{24}\pi r^2 h$$

Thus, the volume of the first cup is

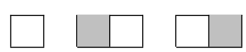
$$\left(1 - \left(\frac{1}{2}\right)^3\right)\frac{4}{3}\pi r^2(2h) = \frac{7}{3}\pi r^2 h$$

The desired fraction is:

$$\frac{\frac{37}{24}}{\frac{7}{3}} = \frac{37}{56} \Rightarrow m + n = 37 + 56 = \boxed{93}.$$

□

Problem 5.3.22 (Problem 22)**.** (*5 points*) You have some white one-by-one tiles and some black and white two-by-one tiles as shown below. There are four different color patterns that can be generated when using these tiles to cover a three-by-one rectangle by laying these tiles side by side (WWW, BWW, WBW, WWB). How many different color patterns can be generated when using these tiles to cover a ten-by-one rectangle?



Solution. Let A_j be the number of colour patterns one can obtain by laying these tiles together to tile a $j \times 1$ rectangle. It is easy to see that $A_1 = 1$, $A_2 = 3$, and $A_3 = 4$.

For $j > 2$, categorize the colouring patterns by the position of the first black square among the j square.

If the colouring patter begins with a black square in its first position, the pattern must have a 2×1 tile followed by one of the A_{j-2} patterns of length $j - 2$.

Similarly if the colouring patter begins with a white square in the first position, and a black square in the second position, the first two squares be followed by any of the A_{j-2} patterns of length $j - 2$.

If the first black square appears in the k^{th} position, where $k > 2$, it can be followed by any of the A_{j-k}, pattern of length $j - k$. Note that if the first black square appears in the last position, it constitutes a pattern. Finally, there is one pattern consisting of all white square. Thus:

$$A_j = A_{j-2} + A_{j-2} + A_{j-3} + A_{j-4} + \ldots + A_1 + 1 + 1.$$

Below is the number of patterns for each of $j \times 1$ rectangles, where $j = 1, 2, \ldots, 10$.

1	2	3	4	5	6	7	8	9	10
1	3	4	9	14	28	47	89	155	286

The answer is $\boxed{286}$.

□

Problem 5.3.23 (Problem 23)**.** (*5 points*) A bag contains 8 green candies and 4 red candies. You randomly select one candy at a time to eat. If you eat five candies, there are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that you do not eat a green candy after you eat a red candy. Find $m+n$.

Solution. The five candies can be eaten in order

$$GRRRR,\ GGRRR,\ GGGRR,\ GGGGR,\ GGGGG.$$

The probability of eating the candies in these orders are, respectively:

$$\begin{cases} GRRRR: \frac{8}{12}\cdot\frac{4}{11}\cdot\frac{3}{10}\cdot\frac{2}{9}\cdot\frac{1}{8} \\ GGRRR: \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{4}{10}\cdot\frac{3}{9}\cdot\frac{2}{8} \\ GGGRR: \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{4}{9}\cdot\frac{3}{8} \\ GGGGR: \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8} \\ GGGGG: \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8} \end{cases}$$

Their sum is

$$\frac{8\cdot4\cdot3\cdot2\cdot1+8\cdot7\cdot4\cdot3\cdot2+8\cdot7\cdot6\cdot4\cdot3+8\cdot7\cdot6\cdot5\cdot4\cdot(2)}{12\cdot11\cdot10\cdot9\cdot8}=\frac{1}{5}.$$

Thus $m+n=1+5=\boxed{6.}$ □

Problem 5.3.24 (Problem 24)**.** (*5 points*) Let $A=1,3,5,7,9$ and $B=2,4,6,8,10$. Let f be a randomly chosen function from the set $A\cup B$ into itself. There are relatively prime positive integers m and n such that $\frac{m}{n}$ is the probability that f is a one-to-one function on $A\cup B$ given that it maps A one-to-one into $A\cup B$ and it maps B one-to-one into $A\cup B$. Find $m+n$.

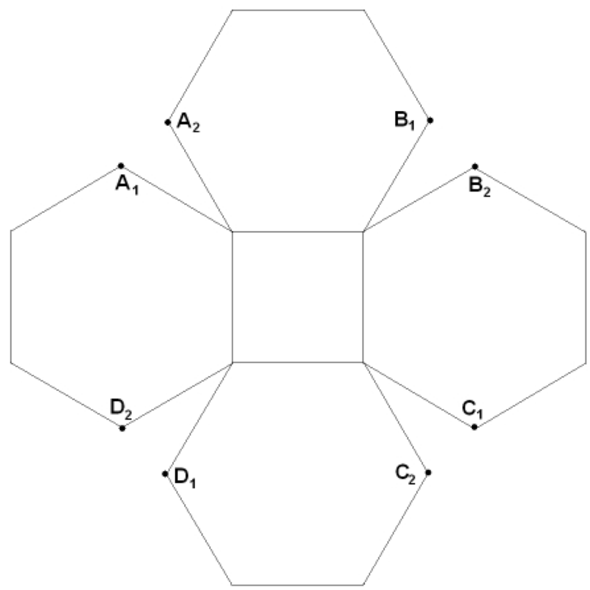
Solution. A one-to-one function f that maps $A\cup B$ into itself is just a permutation of the set $A\cup B$. Thus there are 10! such functions.

The number of one-to-one functions that map A into $A\cup B$ is the number of ways to map each element of A to a distinct element of $A\cup B$, or $10\cdot9\cdot8\cdot7\cdot6$. That is also the number of one-to-one functions that map B into $A\cup B$. Thus the desired probability isosceles

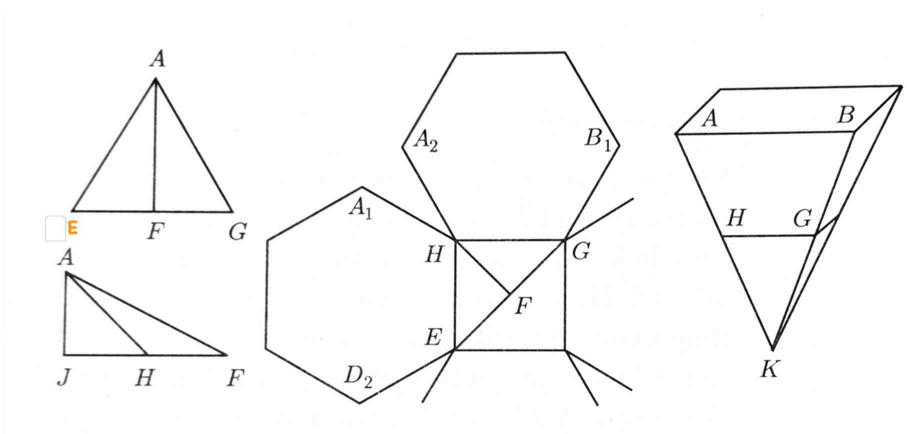
$$\frac{10!}{(10\cdot9\cdot8\cdot7\cdot6)^2}=\frac{5\cdot4\cdot3\cdot2\cdot1}{10\cdot9\cdot8\cdot7\cdot6}=\frac{1}{252}.$$

The answer is $m+n=1+252=\boxed{253.}$ □

Problem 5.3.25 (Problem 25)**.** (*5 points*) The diagram below shows four regular hexagons each with side length 1 meter attached to the sides of a square. This figure is drawn onto a thin sheet of metal and cut out. The hexagons are then bent upward along the sides of the square so that A_1 meets A_2, B_1 meets B_2, C_1 meets C_2, and D_1 meets D_2. The resulting dish is set on a table with the square lying flat on the table. If this dish is filled with water, the water will rise to the height of the corner where the A_1 and A_2 meet. There are relatively prime positive integers m and n so that the number of cubic meters of water the dish will hold is $\sqrt{\frac{m}{n}}$. Find $m+n$.



Solution. Let the point where A_1 and A_2 meet be labeled A, and the point where B_1 and B_2 meet be labeled B. Let the center of the square be labeled F. Let the corner of the square nearest point A be labeled H, the corner of nearest point B be labeled G, and the corner diagonally opposite G be labeled E as shown below.



Let J be the projection of the point A in the plane of the square. Because $\angle AHE = 120°$, the Law of Cosines gives that

$$AE = \sqrt{AH^2 + HE^2 - 2(AH)(HE)\cos(120°)} = \sqrt{1+1+1} = \sqrt{3}.$$

Since $EF = HF = \sqrt{2}2$, the Pythagorean Theorem gives that

$$AF = \sqrt{AE^2 - EF^2} = \sqrt{3 - \frac{1}{2}} = \sqrt{\frac{5}{2}}.$$

The Law of Cosines then gives

$$\cos(\angle AHF) = \frac{AF^2 - AH^2 - HF^2}{-2(AH)(HF)} = \frac{\frac{5}{2} - 1 - \frac{1}{2}}{-\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

Therefore $\angle AHF = 135°$, $\angle AHJ = 45°$. Thus $AJ = \frac{1}{\sqrt{2}}$ is the height of A from the plane of the square. Since the diameter of a regular hexagon is twice the length of one of its sides, it follows that $AB = 2$.

If the vertical edges of the dish AH and BG are extended, they will meet at a point K as shown in the diagram. Note that K is the vertex of a downward pointing square pyramid with base having side length 2 and with height $\sqrt{2}$.

The volume of this pyramid is given by $\frac{1}{3}$ the area of its base times its height $\frac{1}{3} \cdot 4 \cdot \sqrt{2}$. The lower half of the pyramid is another square pyramid with base having side length 1 and with height $\frac{\sqrt{2}}{2}$ so its volume is $\frac{1}{3} \cdot 1 \cdot \sqrt{2}$.

The desired volume is

$$\frac{4}{3} \cdot \sqrt{2} - \frac{1}{3} \cdot \frac{\sqrt{2}}{2} = \frac{7\sqrt{2}}{2} = \sqrt{\frac{49}{18}}.$$

Thus $m + n = 49 + 18 = \boxed{67.}$ $\square$

5.4 High School - Test

High school students: grade 9 (US, CA), grade 10 (FR, UK, VN) and above.

You have **30 minutes** to complete the test. You have to **submit only the answers**. No solution is required.

Note that you have to **follow the instructions by the COs** *for submitting the answers.*

- **Intermediate (I) level: Problems 1-10**
- **Advanced (A) level: Problems 8-17**
- **Olympiad (O) level: Problems 16-25**

If you submit solution based on coding, please make sure that your submission is compliant. Read the introduction chapter for more information.

Problem 5.4.1 (Problem 1)**.** (*5 points*) Find the least positive integer n so that both n and $n+1$ have prime factorizations with exactly three (not necessarily distinct) prime factors.

Solution. One of n and $n+1$ is an odd number. The smallest integer with four prime factors is $3^3 = 27$. And $28 = 2^2 \cdot 7$ has the desired property. The answer is $\boxed{27.}$ □

Problem 5.4.2 (Problem 2)**.** (*5 points*) Two convex polygons have a total of 33 sides and 279 diagonals. Find the number of diagonals in the polygon with the greater number of sides.

Solution. A convex polygon has n sides then it has $\frac{n(n-1)}{2} - n = \frac{n^2-3n}{2}$ diagonals.

Now

$$279 = \frac{n^2-3n}{2} + \frac{(33-n)^2 - 3(33-n)}{2} = n^2 - 33n + 495 \Rightarrow n^2 - 33n + 216 = 0 \Rightarrow n = 9 \text{ or } n = 24$$

The number of diagonals of the polygon with 24 sides is $\frac{24^2-3\cdot 24}{2} = \boxed{252.}$ □

Problem 5.4.3 (Problem 3)**.** (*5 points*) In the tribe of Zimmer, being able to hike long distances and knowing the roads through the forest are both extremely important, so a boy who reaches the age of manhood is not designated as a man by the tribe until he completes an interesting rite of passage. The man must go on a sequence of hikes. The first hike is a 5 kilometer hike down the main road. The second hike is a $5\frac{1}{4}$ kilometer hike down a secondary road. Each hike goes down a different road and is a quarter kilometer longer than the previous hike. The rite of passage is completed at the end of the hike where the cumulative distance walked by the man on all his hikes exceeds **500** kilometers. So in the tribe of Zimmer, how many roads must a man walk down, before you call him a man?

Solution. The total distance a man walked in the n hikes is

$$5 + 5\frac{1}{4} + 5\frac{1}{2} + \cdots + \frac{19+n}{4} = \sum_{k=1}^{n} \frac{19+k}{4} = \frac{1}{4}\left(19n + \frac{n(n+1)}{2}\right) = \frac{n(n+39)}{8}$$

The smallest n for $n^2 + 39n > 8 \cdot 500$ is $n = \boxed{47.}$ □

Problem 5.4.4 (Problem 4)**.** (*5 points*) Find the value of x that satisfies $\log_2(\log_4 x) = \log_4(\log_2 x)$.

Solution. Note that

$$\log_4 x = \frac{\log_2 x}{\log_2 4} \Rightarrow \begin{cases} \log_2(\log_4 x) = \log_2\left(\frac{\log_2 x}{\log_2 4}\right) = \log_2\left(\frac{\log_2 x}{2}\right) \\ \log_4(\log_2 x) = \frac{\log_2(\log_2 x)}{\log_2 4} = \frac{\log_2(\log_2 x)}{2} \end{cases}$$

$$\Rightarrow \log_2\left(\frac{\log_2 x}{2}\right) = \frac{\log_2(\log_2 x)}{2} \Rightarrow 2\log_2\left(\frac{\log_2 x}{2}\right) = \log_2(\log_2 x) \Rightarrow \log_2\left(\frac{\log_2 x}{2}\right)^2 = \log_2(\log_2 x)$$

Thus $\Rightarrow \left(\frac{\log_2 x}{2}\right)^2 = log_2 x$. Since $\log_2 x > 0$, $\log_2 x = 4 \Rightarrow x = \boxed{16.}$ □

Problem 5.4.5 (Problem 5)**.** (*5 points*) Consider a sequence of twelve squares that have side lengths $3, 6, 9, 12, \ldots, 33, 36$. Ten copies of a single square each with area A have the same total area as the total area of the twelve squares of the sequence. Find A.

Solution. It is easy to see that the total area of twelve squares is:

$$3 \cdot 3 + 6 \cdot 6 + \cdots + 36 \cdot 36 = 9(1^2 + 2^2 + \cdots + 12^2) = 9 \cdot \frac{12 \cdot 13 \cdot 25}{6} = 10 \cdot 585 = 10 \cdot A$$

Hence, the area $A = \boxed{585.}$ □

Problem 5.4.6 (Problem 6)**.** (*5 points*) Define $f(x) = 2x + 3$and suppose that $g(x+2) = f(f(x-1) \cdot f(x+1) + f(x))$. Find $g(5)$.

Solution.

$$g(5) = g(3+2) = f(f(3-1) \cdot f(3+1) + f(3)) = f(f(2)f(4) + f(3)) = f(7 \cdot 11 + 9) = f(86) = \boxed{175.}$$

□

Problem 5.4.7 (Problem 7)**.** (*5 points*) Ted flips seven fair coins. There are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that Ted flips at least one head given that he flips at least three tails. Find $m + n$.

Solution. The total number of possibilities of flipping k tails with seven flips is $\binom{7}{k}$, thus the probability that Ted flips at least three tails is

$$\frac{\binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6}}{\binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}} = \frac{98}{99}.$$

The answer is $m + n = 98 + 99 = \boxed{197.}$ □

Problem 5.4.8 (Problem 8)**.** (*5 points*) Find the least n for which $n!(n+1)!(2n+1)! - 1$ ends in thirty-five digits that are all 9's.

Solution. If $n!(n+1)!(2n+1)! - 1$ ends in thirty-five digits that are all 9's then $n!(n+1)!(2n+1)!$ ends in thirty-five digits that are all 0's.

Let's try to find if the least $n < 100$. The number of trailing zeros in $n!$ is the number of factors of 5 in $n!$, which is $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor$.

So we are trying to find an n such that the following sum is at least 35:

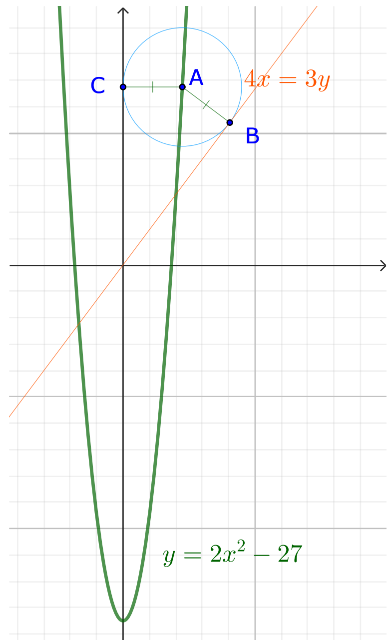
$$S(n) = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n+1}{5} \right\rfloor + \left\lfloor \frac{n+1}{25} \right\rfloor + \left\lfloor \frac{2n+1}{5} \right\rfloor + \left\lfloor \frac{2n+1}{25} \right\rfloor + \left\lfloor \frac{2n+1}{125} \right\rfloor$$

For example $S(25) = 5 + 1 + 5 + 1 + 10 + 2 = 24$, $S(27) = 6 + 6 + 11 + 2 = 25$. By continuing doing so $S(38) = 7 + 1 + 7 + 1 + 15 + 3 = 34$. $S(39) = 7 + 1 + 8 + 1 + 15 + 3 = 35$. Hence, the answer is $\boxed{39.}$ □

Problem 5.4.9 (Problem 9)**.** (*5 points*) A circle in the first quadrant with center on the curve $y = 2x^2 - 27$ is tangent to the $y-$axis and the line $4x = 3y$. The radius of the circle is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m - n$.

Solution. The distance from a point $A(x_0, y_0)$ to the line $4x = 3y$ is given by the normal form for the equation of the line $(4)x + (-3)y + 0 = 0$,

$$\frac{|4x_0 + (-3)y_0 + 0|}{\sqrt{(4^2 + (-3)^2)}} = \frac{|4x_0 - 3y_0|}{5}.$$



If point A, in the first quadrant, is on the curve $y = 2x^2 - 27$ at the same distance from the $y-$axis and the line $4x = 3y$, then the distance to the $y-$axis, or its $x-$coordinate (AC), must be equal to the distance to the line $4x = 3y$, thus

$$x_0 = \frac{|4x_0 - 3y_0|}{5} \Rightarrow x_0 = \frac{|4x_0 - 3(x_0^2 - 27)|}{5} = \frac{3(2x_0^2 - 27) - 4x_0}{5} \Rightarrow 6x_0^2 - 9x_0 - 81 = 0 \Rightarrow x_0 = \frac{9}{2}\ (x_0 > 0)$$

Hence, $m + n = 9 - 2 = \boxed{7.}$ □

Problem 5.4.10 (Problem 10)**.** (*5 points*) Among positive integers whose digits add up to 2024, N is the number whose product of digits is the greatest possible. Find the number of divisors of N.

Solution. First, digits 0 and 1 would not be among the digits of N.

A digit of 4 can be replaced by two digits of 2 having the same sum and same product.

Any digit k larger then or equal to 5 can be replace by a pair of 2 and $k - 2$, where $2(k - 2) = 2k - 4 > k$.

Therefore N has a digits of 2 and b digits of 3, where a and b are non-negative integers and the product of them is $2^a 3^b$, where $2a + 3b = 2024$.

Now, $2 + 2 + 2 = 3 + 3$, and $2 \cdot 2 \cdot 2 < 3 \cdot 3$, thus a can be 0, 1 or 2. Only $a = 1$ gives $b = 674$ integer value.

Thus $N = 2 \cdot 3^{674}$, and the number of divisors of N is $(1 + 1)(674 + 1) = \boxed{1350.}$ □

Problem 5.4.11 (Problem 11)**.** (*5 points*) Let a, b, and c be non-zero real numbers such that

$$\frac{ab}{a+b} = 3, \frac{bc}{b+c} = 4, \frac{ca}{c+a} = 5.$$

There are relatively prime positive integers m and n so that

$$\frac{abc}{ab+bc+ca} = \frac{m}{n}.$$

Find $n - m$.

Solution.

$$\frac{ab}{a+b} = 3 \Rightarrow \frac{a+b}{ab} = \frac{1}{3} \Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{1}{3}$$
$$\text{Similarly } \frac{1}{b} + \frac{1}{c} = \frac{1}{4},\ \frac{1}{c} + \frac{1}{a} = \frac{1}{5}$$
$$\Rightarrow \frac{ab+bc+ca}{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{47}{120}$$

Hence, $n - m = 120 - 47 = \boxed{73.}$ □

Problem 5.4.12 (Problem 12)**.** (*5 points*) How many positive integer solutions are there to $w+x+y+z = 21$ where $w + x \geq 5$ and $y + z \geq 5$?

Solution. Let $k = w + x$, then $y + z = 21 - k$. For a value of $k \geq 2$, there are $k - 1$ ways to assign positive value to w and x so that their sum is k.

Similarly with $21 - k \geq 2$, there are $20 - k$ ways to assign positive value to w and x so that their sum is $21 - k$.

Hence, the number of ways is:

$$\sum_{k=5}^{16}(k-1)(20-k) = \sum_{k=1}^{12}(k+3)(16-k) = \sum_{k=1}^{12}(48+13k-k^2) = 12 \cdot 48 + 13\frac{12 \cdot 13}{2} - \frac{12 \cdot 13 \cdot 25}{6} = \boxed{940.}$$

□

Problem 5.4.13 (Problem 13)**.** (*5 points*) Find the number of three-digit numbers such that the hundreds digit is divisible by the tens digit and the tens digit is divisible by the unit digit.

Solution. It is clear that the unit digit cannot be 0. Similarly the tens digit cannot be 0. We show a coding solution.

```
count = 0
for i in range(100, 1000):
    u = i % 10
    t = (i // 10) % 10
    h = i // 100
    print(h,t,u)
    if u != 0 and t != 0 and h % t == 0 and t % u == 0:
        count +=1
print(count)
```

The answer is $\boxed{44.}$ □

Problem 5.4.14 (Problem 14)**.** (*5 points*) Find the remainder when $2^{5^9} + 5^{9^2} + 9^{2^5}$ is divided by 13.

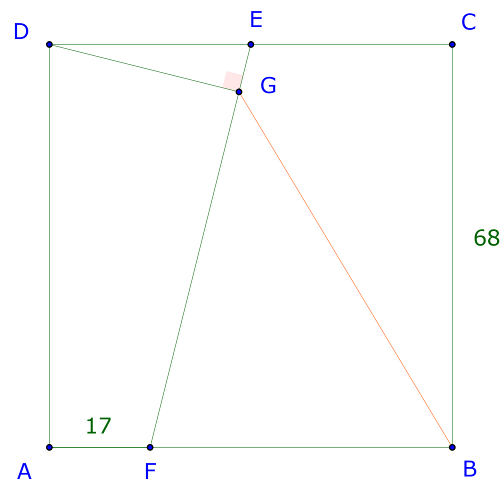
Solution. By Fermat's Little Theorem: $\gcd(n, p) = 1$, then $n^{p-1} \equiv 1 \pmod{p}$, so:

$$2^{12} \equiv 5^{12} \equiv 9^{12} \equiv 1 \pmod{13}.$$

$$5^9 = 5 \cdot (5^2)^4 = 5 \cdot (25)^4 \equiv 5 \cdot 1^4 = 5 \pmod{12} \Rightarrow 2^{5^9} \equiv 2^5 = 32 \equiv 6 \pmod{13}$$
$$9^2 = 81 \equiv 9 \pmod{12} \Rightarrow 5^{9^2} \equiv 5^9 = 5 \cdot (5^2)^4 \equiv 5 \cdot (-1)^4 \equiv 5 \pmod{13}$$
$$2^5 = 32 \equiv 8 \pmod{12} \Rightarrow 9^{2^5} \equiv 9^8 \equiv 4^8 = 4^2 \cdot (4^3)^2 \equiv 3 \pmod{13}$$

Thus $2^{5^9} + 5^{9^2} + 9^{2^5} \equiv 6 + 5 + 3 \equiv \boxed{1} \pmod{13}$. □

Problem 5.4.15 (Problem 15)**.** (*5 points*) Square $ABCD$ has side length 68. Let E be the midpoint of segment CD, and let F be the point on segment AB a distance 17 from point A. Point G is on segment EF so that EF is perpendicular to segment GD. The length of segment BG can be written as $m\sqrt{n}$, where m and n be are positive integers, and n is not divisible by the square of any prime. Find $n-m$.



Solution. Let vector $\overrightarrow{BC}=u$, and $\overrightarrow{BA}=v$. Then $\overrightarrow{EF}=\frac{1}{4}v-u$. Note that $\overrightarrow{EG}$ is the projection of $\overrightarrow{ED}=\frac{1}{2}v$ onto $\overrightarrow{EF}$. This projection is:

$$\left(\frac{\overrightarrow{ED}\cdot\overrightarrow{EF}}{\overrightarrow{EF}\cdot\overrightarrow{EF}}\right)\overrightarrow{EF}=\frac{\frac{1}{2}v\cdot\left(\frac{1}{4}v-u\right)}{\left(\frac{1}{4}v-u\right)\cdot\left(\frac{1}{4}v-u\right)}\left(\frac{1}{4}v-u\right)=\frac{\frac{1}{8}}{\frac{1}{16}+1}\left(\frac{1}{4}v-u\right)=\frac{1}{34}v-\frac{2}{17}u$$

Then

$$\overrightarrow{BG}=u+\frac{1}{2}v+\overrightarrow{EG}=\frac{15}{17}u+\frac{9}{17}v$$

The length of this vector is

$$\frac{68}{17}\sqrt{15^2+9^2}=12\sqrt{34}\Rightarrow n-m=34-12=\boxed{22.}$$

□

Problem 5.4.16 (Problem 16)**.** (*5 points*) Each time you click a toggle switch, the switch either turns from *off* to *on* or from on to *off*. Suppose that you start with three toggle switches with one of them *on* and two of them *off*. On each move you randomly select one of the three switches and click it. Let m and n be relatively prime positive integers so that $\frac{m}{n}$ is the probability that after four such clicks, one switch will be *on* and two of them will be *off*. Find $n-m$.

Solution. Without loss of generality assume that at the beginning the first of the three toggle switches is *on*, and the second and third are *off*.

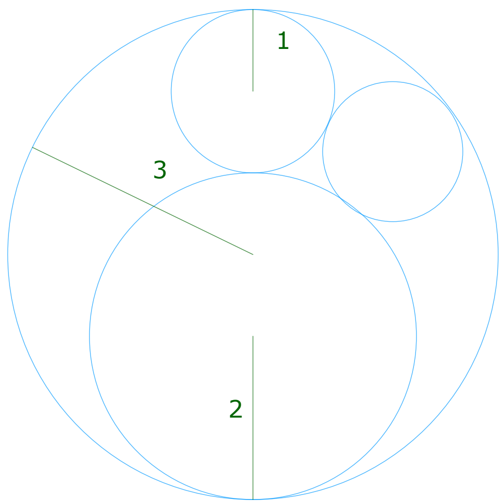
There are $3^4=81$ equally likely ways to select a sequence of four toggle switches to click. After four clicks, there will either be one or three switches in the *on* position.

To get three toggle switches in the *on* position, one much click the first switch an even number of times and the second and third switches an odd number of times each. This can be done by clicking the first switch twice and the other switches once each, or by clicking either the second and third switches three times and the other of the two switches one time. The number of ways to do one of these is:

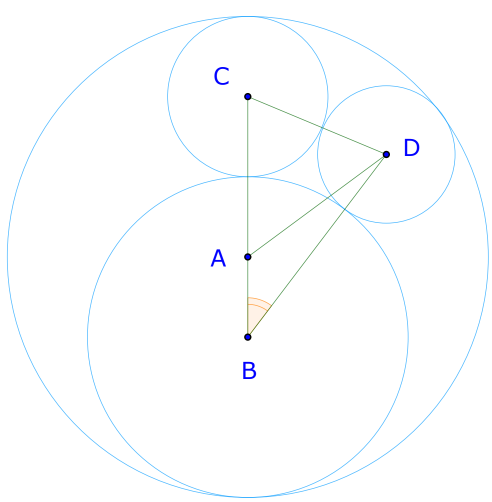
$$\binom{4}{2,1,1} + \binom{4}{0,1,3} + \binom{4}{0,3,1} = \frac{4!}{2!1!1!} + \frac{4!}{0!1!3!} + \frac{4!}{0!3!1!} = 12 + 4 + 4 = 20.$$

It follows that the desired probability is $\frac{81-20}{81} = \frac{61}{81}$. Thus the desired difference is $81 - 61 = \boxed{20.}$ □

Problem 5.4.17 (Problem 17)**.** (*5 points*) The diagram below shows circles radius 1 and 2 externally tangent to each other and internally tangent to a circle radius 3. There are relatively prime positive integers m and n so that a circle radius $\frac{m}{n}$ is internally tangent to the circle radius 3 and externally tangent to the other two circles as shown. Find the product mn.



Solution. Let A, B, C, and D be the centres of the circles radius 3, 2, 1 and $r = \frac{m}{n}$.



By the Law of Cosines:

$$\begin{aligned}
&\triangle ABD: \ 1^2 + (2+r)^2 - 2\cdot 1\cdot (2+r)\cos(\angle ABD) = (3-r)^2 \\
&\triangle CBD: \ 3^2 + (2+r)^2 - 2\cdot 3\cdot (2+r)\cos(\angle ABD) = (1+r)^2 \\
&\Rightarrow 3(3-r)^2 - (1+r)^2 = -6 + 2(2+r)^2 \Rightarrow r = \frac{6}{7}.
\end{aligned}$$

Thus $mn = 6\cdot 7 = \boxed{42.}$ □

Problem 5.4.18 (Problem 18)**.** (*5 points*) Find the greatest seven-digit integer divisible by 132 whose digits, in order, are $x, 0, y, 2, 1, 2, z$ where x, y, and z are single digits.

Solution. We show a coding solution. The problem is equivalent to find the largest multiple of 132 in the format $\overline{x0y212z}$.

```
for x in range(9, -1, -1):
for y in range(9, -1, -1):
    for z in range(9, -1, -1):
        i = x * 1000000 + y * 10000 +  2 * 1000 +  120 + z
        if i % 132 == 0:
            print(i)
            exit(0)
```

The answer is $\boxed{9082128.}$ □

Problem 5.4.19 (Problem 19)**.** (*5 points*) There are positive integers m and n so that $x = m + \sqrt{n}$ is a solution to the equation

$$x^2 - 28x + 1 = \sqrt{x}(x+1).$$

Find $m + n$.

Solution. It is easy to see that $x = 0$ is not a solution. Divide both side of the equation by x, we have

$$x - 28 + \frac{1}{x} = \sqrt{x} + \frac{1}{\sqrt{x}}.$$

Let $y = \sqrt{x} + \frac{1}{\sqrt{x}} > 0$, then $y^2 = x + \frac{1}{x} + 2$, thus

$$\begin{aligned}
&y^2 - 30 = y \Rightarrow (y+5)(y-6) = 0 \Rightarrow y = 6 \Rightarrow \sqrt{x} + \frac{1}{\sqrt{x}} = 6 \Rightarrow x - 6\sqrt{x} + 1 = 0 \\
&\Rightarrow \sqrt{x} = 3 \pm 2\sqrt{2} \Rightarrow x = (3+2\sqrt{2})^2 = 17 + 12\sqrt{2} = 17 + \sqrt{288}
\end{aligned}$$

Thus $m + n = 17 + 288 = \boxed{305.}$ □

Problem 5.4.20 (Problem 20)**.** (*5 points*) Find the largest prime that divides $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + 2023\cdot 2024\cdot 2025$.

Solution. Note that $(k-1)k(k+1) = (k^2-1)k = k^3 - k$, thus the given sum is equivalent to

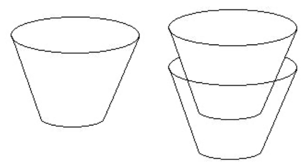
$$\begin{aligned}
&\sum_{k=2}^{n}(k-1)k(k+1) = \sum_{k=2}^{n} k^3 - k = \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} - 1\right) = \frac{(n-1)n(n+1)(n+2)}{4}
\end{aligned}$$

For $n = 2024$,

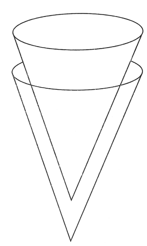
$$\frac{(n-1)n(n+1)(n+2)}{4} = \frac{(2024-1)(2024)(2024+1)(2024+2)}{4} = 2^2 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17^2 \cdot 23 \cdot 1013.$$

Thus the largest prime factor is $\boxed{1013}$. □

Problem 5.4.21 (Problem 21)**.** (*5 points*) A paper cup has a base that is a circle with radius r, a top that is a circle with radius $2r$, and sides that connect the two circles with straight line segments as shown below. This cup has height h and volume V. A second cup that is exactly the same shape as the first is held upright inside the first cup so that its base is a distance of $\frac{h}{2}$ from the base of the first cup. The volume of liquid that will fit inside the first cup and outside the second cup can be written as $\frac{m}{n} \cdot V$, where m and n are relatively prime positive integers. Find $n - m$.



Solution. The volume between the cups does not change if the sides of each cup are extended so that the cups form a circular cones with radius $2r$ and height $2h$.



The volume of the first cone is:

$$\frac{1}{3}\pi(2r)^2(2h) = \frac{8}{3}\pi r^2 h.$$

The part of the second cone inside the first is a cone that is $\frac{3}{4}$ the height of the first cone, so the volume that it takes up within the first cone is:

$$\left(\frac{3}{4}\right)^3 \frac{8}{3}\pi r^2 h = \frac{9}{8}\pi r^2 h$$

Thus the volume inside the first cup and outside of the second cup is

$$\left(\frac{8}{3} - \frac{9}{8}\right)\pi r^2 h = \frac{37}{24}\pi r^2 h$$

Thus, the volume of the first cup is

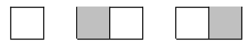
$$\left(1 - \left(\frac{1}{2}\right)^3\right)\frac{4}{3}\pi r^2(2h) = \frac{7}{3}\pi r^2 h$$

The desired fraction is:

$$\frac{\frac{37}{24}}{\frac{7}{3}} = \frac{37}{56} \Rightarrow n - m = 56 - 37 = \boxed{19}.$$

□

Problem 5.4.22 (Problem 22)**.** (*5 points*) You have some white one-by-one tiles and some black and white two-by-one tiles as shown below. There are four different color patterns that can be generated when using these tiles to cover a three-by-one rectangle by laying these tiles side by side (WWW, BWW, WBW, WWB). How many different color patterns can be generated when using these tiles to cover a eight-by-one rectangle?



Solution. Let A_j be the number of colour patterns one can obtain by laying these tiles together to tile a $j \times 1$ rectangle. It is easy to see that $A_1 = 1$, $A_2 = 3$, and $A_3 = 4$.

For $j > 2$, categorize the colouring patterns by the position of the first black square among the j square.

If the colouring patter begins with a black square in its first position, the pattern must have a 2×1 tile followed by one of the A_{j-2} patterns of length $j - 2$.

Similarly if the colouring patter begins with a white square in the first position, and a black square in the second position, the first two squares be followed by any of the A_{j-2} patterns of length $j - 2$.

If the first black square appears in the k^{th} position, where $k > 2$, it can be followed by any of the A_{j-k}, pattern of length $j - k$. Note that if the first black square appears in the last position, it constitutes a pattern. Finally, there is one pattern consisting of all white square. Thus:

$$A_j = A_{j-2} + A_{j-2} + A_{j-3} + A_{j-4} + \ldots + A_1 + 1 + 1.$$

Below is the number of patterns for each of $j \times 1$ rectangles, where $j = 1, 2, \ldots, 10$.

1	2	3	4	5	6	7	8	9	10
1	3	4	9	14	28	47	89	155	286

The answer is $\boxed{89}$. ☐

Problem 5.4.23 (Problem 23)**.** (*5 points*) A bag contains 7 green candies and 4 red candies. You randomly select one candy at a time to eat. If you eat five candies, there are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that you do not eat a green candy after you eat a red candy. Find $m + n$.

Solution. The five candies can be eaten in order

$$GRRRR,\ GGRRR,\ GGGRR,\ GGGGR,\ GGGGG.$$

The probability of eating the candies in these orders are, respectively:

$$\begin{cases} GRRRR: \frac{7}{11} \cdot \frac{4}{10} \cdot \frac{3}{9} \cdot \frac{2}{8} \cdot \frac{1}{7} \\ GGRRR: \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} \\ GGGRR: \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7} \\ GGGGR: \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{4}{7} \\ GGGGG: \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7} \end{cases}$$

Their sum is

$$\frac{7 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 7 \cdot 6 \cdot 4 \cdot 3 \cdot 2 + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 4 + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7} = \frac{19}{110}.$$

Thus $m + n = 19 + 110 = \boxed{129.}$ ☐

Problem 5.4.24 (Problem 24). (*5 points*) Let $A = 1, 3, 5, 7$ and $B = 2, 4, 6, 8$. Let f be a randomly chosen function from the set $A \cup B$ into itself. There are relatively prime positive integers m and n such that $\frac{m}{n}$ is the probability that f is a one-to-one function on $A \cup B$ given that it maps A one-to-one into $A \cup B$ and it maps B one-to-one into $A \cup B$. Find $m + n$.

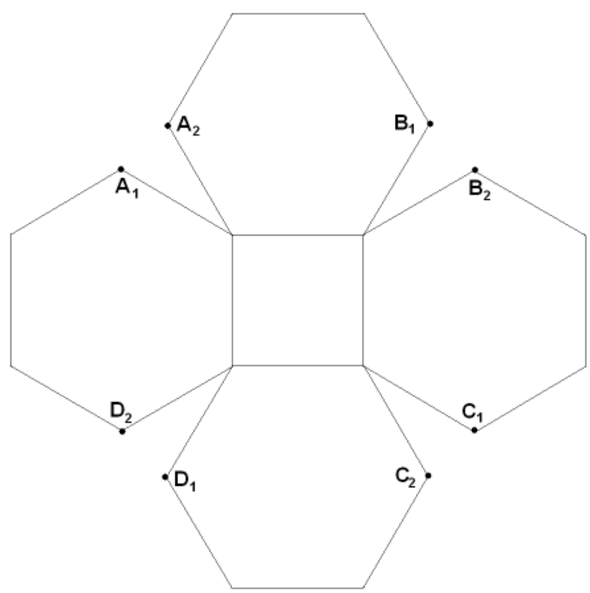
Solution. A one-to-one function f that maps $A \cup B$ into itself is just a permutation of the set $A \cup B$. Thus there are 8! such functions.

The number of one-to-one functions that map A into $A \cup B$ is the number of ways to map each element of A to a distinct element of $A \cup B$, or $8 \cdot 7 \cdot 6 \cdot 5$ That is also the number of one-to-one functions that map B into $A \cup B$. Thus the desired probability isosceles

$$\frac{8!}{(8 \cdot 7 \cdot 6 \cdot 5)^2} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5} = \frac{1}{70}.$$

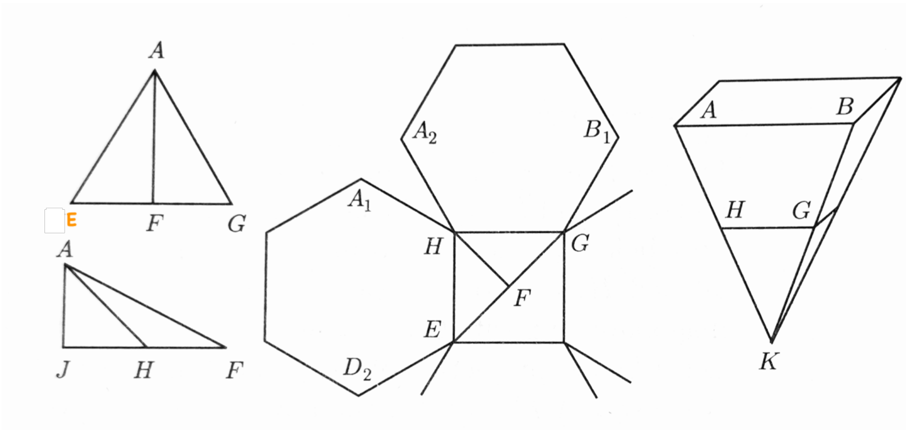
The answer is $m + n = 1 + 70 = \boxed{71.}$ □

Problem 5.4.25 (Problem 25). (*5 points*) The diagram below shows four regular hexagons each with side length 1 meter attached to the sides of a square. This figure is drawn onto a thin sheet of metal and cut out. The hexagons are then bent upward along the sides of the square so that A_1 meets A_2, B_1 meets B_2, C_1 meets C_2, and D_1 meets D_2. The resulting dish is set on a table with the square lying flat on the table. If this dish is filled with water, the water will rise to the height of the corner where the A_1 and A_2 meet.



Let the point where A_1 and A_2 meet be labeled A, the point B_1 and B_2 meet be labeled B, the point C_1 and C_2 meet be labeled C, and the point D_1 and D_2 meet be labeled D. Let the center of the square be labeled F. Let the corner of the square nearest point A be labeled H, the corner of nearest point B be labeled G, and the corner diagonally opposite G be labeled E as shown below.

Let J be the projection of the point A in the plane of the square. If the vertical edges of the dish AH and BG are extended, they will meet at a point K as shown in the diagram. Note that K is the vertex of a downward pointing square pyramid.



Find the measure of $\angle AHF$ in degrees.

Solution. Because $\angle AHE = 120°$, the Law of Cosines gives that

$$AE = \sqrt{AH^2 + HE^2 - 2(AH)(HE)\cos(120°)} = \sqrt{1+1+1} = \sqrt{3}.$$

Since $EF = HF = \frac{1}{\sqrt{2}}$, the Pythagorean Theorem gives that

$$AF = \sqrt{AE^2 - EF^2} = \sqrt{3 - \frac{1}{2}} = \sqrt{\frac{5}{2}}.$$

The Law of Cosines then gives

$$\cos(\angle AHF) = \frac{AF^2 - AH^2 - HF^2}{-2(AH)(HF)} = \frac{\frac{5}{2} - 1 - \frac{1}{2}}{-\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

Therefore $\angle AHF = \boxed{135°.}$ □