- Be open minded.
- Ask anything.
- Enjoy.

Outline

- Definition.
- Generic Formula for Terms.
- Characteristic Polynominals.

Example (Factorial)

Find a generic formula for the terms of the sequence,

$$f_1 = 1,\ f_n = nf_{n-1},\ \forall n \geq 2.$$

- $f_n = nf_{n-1}$, is an example of recurrence relation, where the term f_n can be computed based on n and the previous term f_{n-1}.
- We can see that $f_n = nf_{n-1} = n(n-1)f_{n-2} = \ldots = n(n-1)\ldots 1 = n!$.
- It is also easy to prove that $f_n = n!$ based on the Induction Principle.

Here is the formal definition of First-Order Recurrence Relation.

Definition (First-Order Recurrence Relation)

Let $\varphi : \mathbb{N} \times X \to X$ be a function. A **recurrence relation of first order** is defined as

$$u_n = \varphi(n, u_{n-1}) \quad \text{for} \quad n > 0,\ \text{where } u_0 \in X \text{ is the } \textit{initial value}.$$

Note that $f_n = nf_{n-1}$ is a *First-Order Recurrence Relation* with $\varphi(n, f_{n-1}) = nf_{n-1}$.

Example (Fibonacci sequence)

Find a generic formula for the terms of the sequence,

$$f_0 = 0,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2},\ \forall n \geq 2.$$

- $f_n = f_{n-1} + f_{n-2}$, is an example of *second-degree* recurrence relation where the term f_n can be computed based on n and two previous terms f_{n-1} and f_{n-2}.
- It is not difficult to prove the generic formula $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$.
- But it is more difficult to find it. We will show how to do that by using Characteristic Polynomials (or by more advance Generating Functions in later seminar).

Definition (Linear Recurrence Relation)

An *order−d* homogeneous **linear recurrence** with *constant coefficients* is an equation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$, where coefficients c_i $(i = \overline{1, d})$ are *constants*, $c_d \neq 0$.

Note that: $f_n = f_{n-1} + f_{n-2}$ is a *(Linear)) Recurrence Relation* since $f_n = 1 \cdot f_{n-1} + +1 \cdot f_{n-2}$.

Example (The Fibonacci sequence)

$$\text{Prove that } f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \ (*),$$

where $\{f_n\}_{n=0}^{\infty}$ is the Fibonacci sequence: $f_0 = 0,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2},\ \forall n \geq 2.$

Note that $\{f_n\}_{n=0}^{\infty}$ denotes a sequence starts with f_0 and have infinite number of terms.

Proof.

Let $u = \frac{1+\sqrt{5}}{2}$ and $v = \frac{1-\sqrt{5}}{2}$.
First, (*) is true for $n = 0$ and $n = 1$, $f_0 = \frac{1}{\sqrt{5}}(u^0 - v^0) = 0$, $f_1 = \frac{1}{\sqrt{5}}(u^1 - v^1) = \frac{1}{\sqrt{5}} \cdot \sqrt{5} = 1$.
Now, assume (*) is true for all $k < n$, note that $u + v = 1, uv = \frac{1-(\sqrt{5})^2}{4} = -1$, thus

$$f_{n-1} = f_{n-1}(u+v) = \frac{(u^{n-1} + v^{n-1})(u+v)}{\sqrt{5}} = \frac{(u^n - u^{n-2} + v^n - v^{n-2})}{\sqrt{5}} = \frac{u^n + v^n}{\sqrt{5}} - f_{n-2}$$

$$\Rightarrow\ f_n = f_{n-1} + f_{n-2} = \frac{1}{\sqrt{5}}(u^n + v^n). \text{ Thus, } (*) \text{ is true for all } n.$$

□

Problem (LPS V2B/14.5)

Let $a_0 = 1,\ a_n = 3a_{n-1} + 2,\ n = 1, 2, \ldots$. *Prove that* $a_n = 2.3^n - 1,\ \forall n \in \mathbb{N}$.

Note that $\forall n \in \mathbb{N}$ *means for all value of* n *in the set of natural numbers, which is the set of non-negative integers* $0, 1, 2, \ldots$

Problem (IMO 1967/6)

In a sports meeting a total of m *medals were awarded over* n *days. On the first day one medal and* $\frac{1}{7}$ *of the remaining medals were awarded. On the second day two medals and* $\frac{1}{7}$ *of the remaining medals were awarded, and so on. On the last day, the remaining* n *medals were awarded.*

Let m_k *be the number of medals awarded on day* k.

1. *Prove that after day* k, *there were* $6(m_k - k)$ *medals left for day* $k + 1$.
2. *Prove that* $m_{k+1} = \frac{6}{7}m_k + \frac{6}{7}$.
3. *What is the value of* n?

Definition (Characteristic Polynomials)

The characteristic polynomial of a linear recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d},$$

where coefficients c_i ($i \in \{1, 2, \ldots, d\}$) are constants and $c_d \neq 0$, is defined as the polynomial

$$P(x) = x^n - c_1 x^{n-1} - c_2 x^{n-2} - \cdots - c_{d-1} x - c_d.$$

For the recurrence relation $f_n = f_{n-1} + f_{n-2}$, the characteristic polynomial is $P(x) = x^2 - x - 1$.

Theorem (Roots of Characteristic Polynomials)

If all the roots of the polynomial $r_1, r_2, \cdots, r_k$ are distinct, there exist $\alpha_1, \alpha_2, \cdots, \alpha_k$ constant real numbers such that

$$f_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n + \cdots + \alpha_k \cdot r_k^n.$$

If $r_1 = r_2 = \cdots = r_{k_1}$, then

$$f_n = \alpha_1 \cdot r^n + \alpha_2 \cdot r_2^n + \cdots + \alpha_{k_1} \cdot r_{k_1}^n = (\alpha_1 \cdot n^{k_1 - 1} + \alpha_2 \cdot n^{k_1 - 2} + \cdots + \alpha_{k_1}) \cdot r^n.$$

Example (One)

Let $a_0 = 1,\ a_n = 3a_{n-1} + 2,\ n = 1, 2, \dots$. Find a_n.

Solution

Note that $a_n = 3a_{n-1} + 2$ is not a linear recurrence relation. But

$$a_n + 1 = 3(a_{n-1} + 1) \Rightarrow \textit{by letting } b_n = a_n + 1 \Rightarrow b_n = 3b_{n-1} \textit{ is a linear recurrence relation.}$$

The characteristic polynomial is $P(x) = x - 3$, which has a single root $r_1 = 3$. Thus, by the theorem, there exists α_1 constant real numbers such that the generic formula of b_n is

$$b_n = \alpha_1 \cdot 3^n.$$

To obtain the value of α_1, we substitute $n = 0$,

$$b_0 = a_0 + 1 = 2 \Rightarrow 2 = \alpha_1 \cdot 3^0 \Rightarrow \alpha_1 = 2 \Rightarrow b_n = 2 \cdot 3^n \Rightarrow \boxed{a_n = 2 \cdot 3^n - 1.}$$

Example (Two)

Find a generic formula for the terms of the sequence,

$$f_0 = 0,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2},\ \forall n \geq 2.$$

Solution

$f_n = f_{n-1} + f_{n-2}$ is a linear recurrence relation, its characteristic polynomial: $P(x) = x^2 - x - 1$. Solving the quadratics ($\Delta = (-1)^2 - 4(1)(-1) = 5$) to obtain two roots $r_1 = \frac{1+\sqrt{5}}{2},\ r_2 = \frac{1-\sqrt{5}}{2}$. Thus, there exists α_1, α_2 real numbers such that for all n,

$$f_n = \alpha_1 \cdot {r_1}^n + \alpha_2 \cdot {r_2}^n$$

To obtain the value of α_1 and α_2, we substitute $n = 0$ and $n = 1$,

$$\begin{cases} 0 = f_0 = \alpha_1 \cdot {r_1}^0 + \alpha_2 \cdot {r_2}^0 = \alpha_1 + \alpha_2 \Rightarrow \alpha_2 = -\alpha_1, \\ 1 = f_1 = \alpha_1 \cdot {r_1}^1 + \alpha_2 \cdot {r_2}^1 = \frac{1}{2}(\alpha_1 + \alpha_2) + \frac{\sqrt{5}}{2}(\alpha_1 - \alpha_2) = \sqrt{5} \cdot \alpha_1 \end{cases}$$

$$\Rightarrow (\alpha_1, \alpha_2) = \left(\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) \Rightarrow f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

1. You usually start with a recurrence relation such as $f_n = f_{n-1} + f_{n-2}$.
2. (Transform it if necessary and then) Identify the characteristic polynomial, in this case $P(x) = x^2 - x - 1$.
3. Solve the characteristic polynomial to obtain its roots, in this case $r_1 = \frac{1+\sqrt{5}}{2},\ r_2 = \frac{1-\sqrt{5}}{2}$.
4. Establish the generic formula, with unknown co-efficients $\alpha_1, \alpha_2, \ldots$ (same as the number of roots), in this case $f_n = \alpha_1 \cdot {r_1}^n + \alpha_2 \cdot {r_2}^n$.
5. Substitute a few values of n to obtain a system of linear equations, in this case

$$\begin{cases} 0 = f_0 = \alpha_1 \cdot {r_1}^0 + \alpha_2 \cdot {r_2}^0 = \alpha_1 + \alpha_2 \\ 1 = f_1 = \alpha_1 \cdot {r_1}^1 + \alpha_2 \cdot {r_2}^1 = \frac{1}{2}(\alpha_1 + \alpha_2) + \frac{\sqrt{5}}{2}(\alpha_1 - \alpha_2). \end{cases}$$

Finally solve the system of equation to obtain α_1 and α_2, then the formula for f_n.

Problem (LPS V2B/14.9)

Find the generic formula for $\{a_n\}$, *if,*

$$a_0 = 1,\ a_1 = 3,\ a_{n+2} = 3a_{n+1} - 2a_n,\ \forall n \in \mathbb{N}.$$

Problem (IMO SL 1988/1)

An integer sequence is defined by

$$a_n = 2a_{n-1} + a_{n-2},\ n > 1,\ a_0 = 0,\ a_1 = 1.$$

Prove that 2^k *divides* a_n *if and only if* 2^k *divides* n

Hints:

1. *Prove that* $2 \mid a_{2n}$ *and* $a_{2n+1} \equiv 1 \pmod 4$ *for all* n.
2. *Prove that, for an arbitrary* n *and* t, $a_{n+t} = a_{t+1}a_n + a_t a_{n-1}$, *for all* $t \geq 0$.

- Concept & Definition.
- Proof by definition.
- Arithmetic rules.

- We may write an **infinite sequence** as a list of numbers separated by commas, with an ellipsis $(\ldots)$ to indicate that the sequence is infinite, assuming the pattern is clear.
- For example,

$$1, 2, 4, 8, 16, 32, \ldots$$

  is the infinite sequence consisting of all the positive powers of 2.
- We would denote the entire sequence as $\{a_n\}_{n=1}^{\infty}$.
- Sequences need not start at $n = 1$.
- For instance, some sequences might start at $n = 0$ rather than $n = 1$; we would write such as sequence as $\{a_n\}_{n=0}^{\infty}$.

Definition (Convergence of an infinite sequence)

We say that an infinite sequence $\{a_n\}_{n=1}^{\infty}$ converges to L, also written

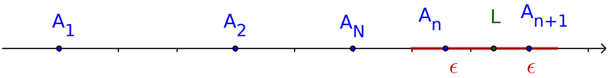
$$\lim_{n\to\infty} a_n = L,$$

if for any $\epsilon > 0$ there exists a positive integer N such that for any $n > N$, we have $|a_n - L| < \epsilon$.

The definition means that we can choose N sufficiently large so that all terms of the sequence after a_N are within the interval $(L - \epsilon, L + \epsilon)$ (in other words, with in a distance ϵ of L.)

$\lim_{n\to\infty} a_n = L$, means that the sequence gets *arbitrarily close* to L as n gets arbitrarily large.

For all $n > N$, $|a_n - L| < \epsilon \Leftrightarrow -\epsilon < a_n - L < \epsilon \Leftrightarrow L - \epsilon < a_n < L + \epsilon \Leftrightarrow a_n \in (L - \epsilon, L + \epsilon)$.



Example

$a_n = 1 - \dfrac{1}{n}$ for all positive integers n; so $a_0 = 0 < a_1 = \frac{1}{2} < a_2 = \frac{2}{3} < a_3 = \frac{4}{5} \ldots$.

Example

$a_n = \left(-\frac{2}{3}\right)^n$, for $n \geq 0$; so $a_0 = 1$, $a_1 = -\frac{2}{3} \approx -0.65$, $a_2 = \left(\frac{2}{3}\right)^2 \approx 0.4$, $a_3 = -\left(\frac{2}{3}\right)^3 \approx -0.3 \ldots$

Example

Let $a_n = 1 - \dfrac{1}{n}$ for all positive integers n. Prove that $\{a_n\}_{n=1}^{\infty}$ converges to 1.

Let get familiar with the way using the definition directly for a proof. First, by observing the terms of the sequence $a_n = 1 - \dfrac{1}{n} = \dfrac{n-1}{n}$,

$$\frac{0}{1} = 0 < \frac{1}{2} < \frac{2}{3} < \ldots < \frac{n-1}{n} < \ldots < 1.$$

$a_n = \dfrac{n-1}{n}$ seems to approach 1. What does that means? For any distance, from a certain term, all the terms of the sequence should be within that $(1-\epsilon, 1)$ (so within $(1-\epsilon, 1+\epsilon)$).

$$\underbrace{\frac{N-1}{N} < \ldots \frac{n-1}{n} < \ldots < 1}_{<\epsilon}.$$

Let, for example, the distance ϵ be $\frac{1}{3}$, we work backward from the condition $|a_n - 1| < \frac{1}{3}$, to find an N (perhaps depended on $\epsilon = \frac{1}{3}$) so that for all $n > N$,

$|a_n - 1| < \dfrac{1}{3} \Rightarrow \dfrac{1}{n} < \dfrac{1}{3} \Rightarrow n > 3$. Thus, by chosing $N = 3,$ then for any $n > N > 3,$ $|a_n - 1| < \epsilon$.

Proof.

Case 1: for any $\epsilon \geq 1$, by choosing $N = 1$, for all $n > N$, $|a_n - 1| = \frac{1}{n} < 1 < \epsilon$.
Case 2: for any ϵ : $1 > \epsilon > 0$, the sequence $\{\frac{1}{n}\}_{n=0}^{\infty}$ decreases to 0, there exists N such that

$$\frac{1}{1} > \ldots > \frac{1}{N-1} > \epsilon > \frac{1}{N} > \ldots > 0 \qquad (*)$$

From both cases, for any $\epsilon > 0$, there exists N, such that for $L = 1$,

$$|a_n - 1| = \frac{1}{n} < \frac{1}{N} < \epsilon.$$

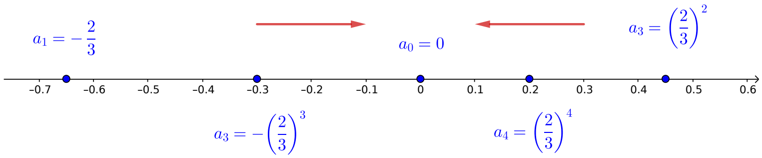
□

- To find (guess) the limit for a sequence, inspect the terms of the sequence, then the positive difference of a term and the (guessed) limit. Make sure you see the positive difference decreases to 0 when n tend to infinity.
- To prove the guess, inspect the positive difference to determine N based on ϵ.
- Note that instead of using a decreasing sequence as in (*) to show the existence of N for an $\epsilon \in (0, 1)$, we can show that any N satisfying $\epsilon > \frac{1}{N}$ or $N > \frac{1}{\epsilon}$ should suffice. However $N > \frac{1}{\epsilon}$ does not provide the insight how to find such N based on an ϵ in general.

Example

Find

$$\lim_{n\to\infty} (-1)^n \cdot \left(\frac{2}{3}\right)^n .$$



Lets examine a first few terms of the sequence,

$$a_0 = 1,\, a_1 = -\frac{2}{3} \approx -0.65,\, a_2 = \left(\frac{2}{3}\right)^2 \approx 0.4,\, a_3 = -\left(\frac{2}{3}\right)^3 \approx -0.3,\, a_4 = \left(\frac{2}{3}\right)^4 \approx 0.2.$$

To prove it, for any $\epsilon > 0$, we look for an N so that, for $n > N$, $|a_n - 0| \leq \epsilon$, or $\left(\frac{2}{3}\right)^n < \left(\frac{2}{3}\right)^N < \epsilon$.

Proof.

Case 1: for any $\epsilon > 0$, if $\epsilon \geq 1$, then for $N = 0$, such that $\left(\frac{2}{3}\right)^N < \epsilon$.
Case 1: for $1 > \epsilon > 0$, the sequence $\{\left(\frac{2}{3}\right)^n\}_{n=0}^{\infty}$ decreases from 1 to zero, there exists N such that,

$$\left(\frac{2}{3}\right)^0 = 1 > \left(\frac{2}{3}\right)^1 = \frac{2}{3} > \ldots > \left(\frac{2}{3}\right)^{N-1} > \epsilon > \left(\frac{2}{3}\right)^N > \ldots > 0.$$

Thus, for all $n > N$, $L = 0$,

$$|a_n - L| = |a_n - 0| = \left(\frac{2}{3}\right)^n < \left(\frac{2}{3}\right)^N < \epsilon.$$

$\square$

Note that: the positive difference, $|a_n - L| = \left(\frac{2}{3}\right)^n$ *ignores the need to deal with terms having different (positive/negative) signs, which helps.*

Problem

Let $a_n = 3.\underbrace{0\ldots0}_{n}1$ *for all integers* $n \geq 0$. *Prove that* $\{a_n\}_{n=1}^{\infty}$ *converges to 3.*

Definition (Sequence Divergence)

If the sequence does not converge, we say it diverges.

Definition (Non-existence of limit)

A sequence does not have a limit if for any number L, there exists an $\epsilon > 0$ such that, for *infinitely many positive integers* n, the distance between a_n and L is at least ϵ.

- If the limit of a sequence exists then for any distance ϵ, there is an index N such that **all terms** a_n after that index ($n > N$) should fall into the interval $(L - \epsilon, L + \epsilon)$.
- The non-existence definition states that no number L can be the limit, because for any L, there always exists a distance ϵ, such that there are **infinitely many terms** are outside of the interval $(L - \epsilon, L + \epsilon)$.
- The $\lim_{n \to \infty} a_n = \pm\infty$ means that the limit does exist.

Example

$$\lim_{n \to \infty} n(-1)^n \text{ does not exist, while } \lim_{n \to +\infty} n = +\infty, \quad \lim_{n \to -\infty} n = -\infty.$$

Theorem (Basic Arithmetic Rules for Sequences)

Assume that c is a constant real number, k is positive integer, and both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

Then the following statements are true,

1. $\lim_{n\to\infty} c = c$
2. $\lim_{n\to\infty}(c \cdot a_n) = c\,(\lim_{n\to\infty} a_n)$
3. $\lim_{n\to\infty}(a_n^k) = (\lim_{n\to\infty} a_n)^k$
4. $\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$
5. $\lim_{n\to\infty}(a_n - b_n) = (\lim_{n\to\infty} a_n) - (\lim_{n\to\infty} b_n)$
6. $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n)\,(\lim_{n\to\infty} b_n)$
7. $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{(\lim_{n\to\infty} a_n)}{(\lim_{n\to\infty} b_n)}$ *if* $\lim_{n\to\infty} b_n \neq 0$.
8. *If* $\lim_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} a_n \neq 0$, *then* $\lim_{n\to\infty}(\frac{a_n}{b_n})$ *does not exists.*
9. *If* $\lim_{n\to\infty} |a_n| = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Example

Find

$$\lim_{n\to\infty} \frac{1-n^3}{1+2n^3}.$$

Proof.

Lets transform the general formula in order to use the basic rules.

$$\frac{1-n^3}{1+2n^3} = \frac{\frac{1}{n^3}-1}{\frac{1}{n^3}+2}.$$

From the example , thus $\lim_{n\to\infty} \frac{1}{n^3} = \left(\lim_{n\to\infty} \frac{1}{n}\right)^3 = 0$, so

$$\lim_{n\to\infty} \frac{1}{n^3} - 1 = -1, \lim_{n\to\infty} \frac{1}{n^3} + 2 = 2 \Rightarrow \lim_{n\to\infty} \frac{1-n^3}{1+2n^3} = \frac{\lim_{n\to\infty} \frac{1}{n^3} - 1}{\lim_{n\to\infty} \frac{1}{n^3} + 2} = \boxed{-\frac{1}{2}.}$$

$\square$

Problem (Five)

Determine the limit of the sequence $a_n = \frac{6n+9}{2n+2}$.

Problem (Six)

Determine the limit of the sequence $a_n = \frac{1+n-2n^2}{1-n+n^2}$.

Problem (Seven)

Determine the limit of the sequence $a_n = \left(\frac{4}{3}\right)^n$.

Problem (Eight)

Determine the limit of the sequence $a_n = 6 + \frac{(-1)^n}{n}$.