- Be open minded.
- Ask anything.
- Enjoy.

Outline

- Definition.
- Examples.

Definition (Quadratics)

A second degree polynomial $P(x)$ is called a *quadratic*. In other words, if there exist a, b, and c real numbers, where $a \neq 0$, such that

$$P(x) = ax^2 + bx + c.$$

Definition (Roots)

A real number r is called a **root** of $P(x)$ if $P(r) = 0$.

Theorem (Existence of roots)

For the quadratic $P(x) = ax^2 + bx + c$, where $a \neq 0$, $\Delta = b^2 - 4ac$ is called the discriminant. $P(x)$ has (i) two distinct roots if $\Delta > 0$; (ii) two same roots if $\Delta = 0$; and (iii) no root if $\Delta < 0$ If $\Delta \geq 0$, the roots of $P(x)$ are $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$.

Theorem (Viete's theorem)

Let x_1, x_2 be the roots of $P(x) = ax^2 + bx + c$, then $x_1 + x_2 = -\frac{b}{a}, x_1 x_2 = \frac{c}{a}$.

Example

For a constant k the polynomial $p(x) = 2x^2 + kx + 117$ is so that $p(1) = p(10)$. Evaluate $p(20)$.

Since $p(x) = 2x^2 + kx + 117$ and $p(1) = p(10)$, so by substitution,
$2 \times (1)^2 + k \times 1 + 117 = 2 \times (10)^2 + k \times 10 + 117 \Rightarrow -9k - 198 = 0 \Rightarrow k = -22.$
$p(20) = 2 \times (20)^2 + (-22) \times 20 + 117 = \boxed{477.}$

Example

Find c, where b and c are constants, $(x + 2)(x + b) = x^2 + cx + 6$, for all x.

The polynomials $x^2 + cx + 6$ and $(x + 2)(x + b) = x^2 + (b + 2)x + 2b$ have the same value for all x, so their coefficients are the same. Thus $c = b + 2,\ 6 = 2b \Rightarrow b = 3,\ c = \boxed{5.}$

Example

Determine the real parameter p so both equations below have real solutions:
$(3 - p)x^2 - 4x + 1 = 0$ and $x^2 + 2x + p = 0$.

Both discriminants should be non-negative, therefore,
$(-4)^2 - 4(3 - p) \geq 0 \Rightarrow 4p + 4 \geq 0 \Rightarrow p \geq -1$; and $2^2 - 4p \geq 0 \Rightarrow p \leq 1$. Thus $\boxed{-1 \leq p \leq 1.}$

Theorem (Roots $0, 1, -1$)

For $P(x) = ax^2 + bx + c$, *where* $a \neq 0$, *then* $P(0) = c,\ P(1) = a + b + c,\ P(-1) = a - b + c$.

- *If* $c = 0$, *then* 0 *is a root of* $P(x)$.
- *If* $a + b + c = 0$, *then* 1 *is a root of* $P(x)$.
- *If* $a - b + c = 0$, *then* -1 *is a root of* $P(x)$.

Example

Prove that if all equations below have a common solution, then it is 1.

$$\begin{cases} ax^2 + bx + c = 0 & (1) \\ bx^2 + cx + a = 0 & (2) \\ cx^2 + ax + b = 0 & (3) \end{cases}$$

Find the other root of (1), (2), and (3).

Let x_0 be the common solution, substitute it into the equations, and add them together, $(a+b+c)x_0^2 + (b+c+a)x_0 + (c+a+b) = 0 \Rightarrow (a+b+c)(x_0^2 + x_0 + 1) = 0$. Since $x_0^2 + x_0 + 1 = (x_0 + \frac{1}{2})^2 + \frac{3}{4} > 0$, so $a + b + c = 0$. By the theorem, $x_0 = \boxed{1}$ is the common root. By Viete theorem, the other roots of (1), (2), and (3) are $\frac{c}{a}$, $\frac{a}{b}$, and $\frac{b}{c}$.

Example

Prove that if all roots of $x^2 + px + q = 0$ are real, then the roots of $x^2 + px + q + (x + a)(2x + p) = 0$ are real for all real number a.

The roots of $g(x) = x^2 + px + q = 0$ are real, so $\Delta_g = p^2 - 4q \geq 0$.
Let $f(x) = x^2 + px + q + (x + a)(2x + p) = 3x^2 + 2(a + p)x + (q + ap)$.
$\Delta_f = 4(a + p)^2 - 12(q + ap) = 4a^2 - 4ap + p^2 + 3p^2 - 12q = (2a - p)^2 + 3(p - 4q) \geq 0$.
Hence, $\boxed{\text{all roots of } f(x) \text{ are real.}}$

Problem

Given a, b are real numbers. Solve the equation system:

$$\begin{cases} x^2 + y^2 = a^2 + b^2 \\ x + y + a + b = 0 \end{cases}$$

Hint: find $x + y$ and xy, then apply Viete theorem. Be careful: the roles of x and y are the same.

- Draw extra (parallel) lines.
- Reflection (over a segment.)
- First synthetic proof - construction by length.
- Second synthetic proof - construction by intersection.
- Trigonometry.

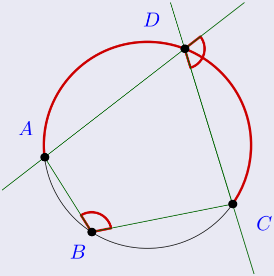
Definition

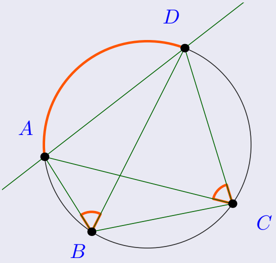
ABCD quadrilateral is cyclic if and only if A, B, C, D are on the perimeter of the same circle.

Theorem

Let ABCD be a convex quadrilateral. The following are equivalent:

- *ABCD is cyclic.*
- $\angle ABC + \angle CDA = 180^\circ$ *(they together subtend the whole circle.)*
- $\angle ABD = \angle ACD$ *(they subtend the same arc AD.)*

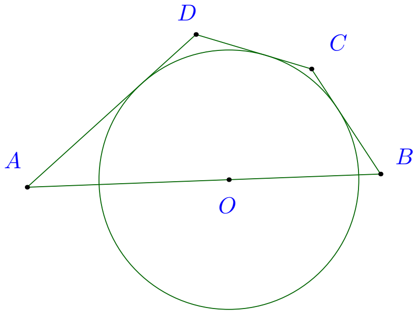




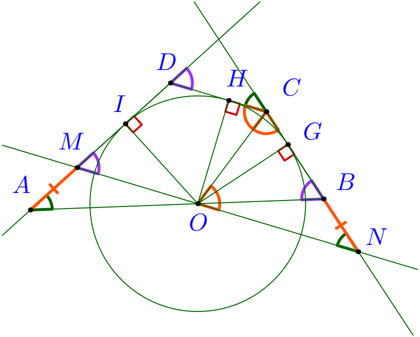
The International Mathematical Olympiad (IMO) is the World Championship Mathematics Competition for High School students and is held annually in a different country. For more information, see https://www.imo-official.org/.

Example

A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

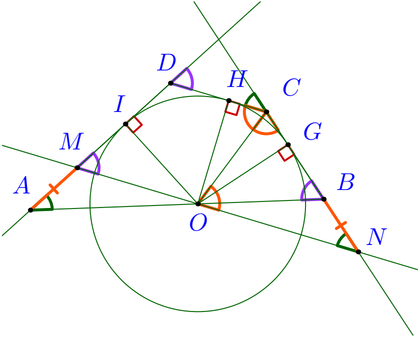


Construct a parallel line to $\overline{CD}$ through O meet AD and BC at M and N, respectively. Then, let G, H and I be the tangent feet of (O). First, we prove that $MN = AB$.



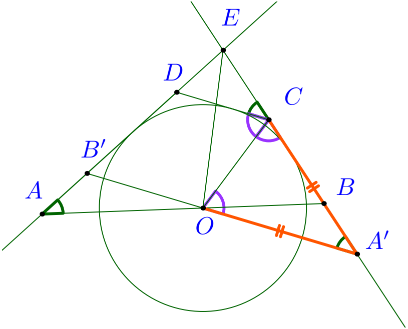
$$\angle ABC = \angle(\overline{AD}, \overline{DC}) \text{ } (ABCD \text{ is yclic}) = \angle OMD \text{ } (CD \parallel MN), \quad \angle BAD = \angle(\overline{BC}, \overline{CD}) = \angle ONB.$$
$$OI = OG, \angle OIM = \angle OGB = 90^\circ, \angle OMD = \angle B \Rightarrow \triangle OIM \cong \triangle OGB \Rightarrow OB = OM.$$
$$OM = OB, \angle A = \angle N, \angle OBN = \angle OAM \Rightarrow \triangle AMO \cong \triangle NBO \Rightarrow AM = BN, OA = ON.$$

Now, we have $AM = BN$, $OA = ON$, $OB = OM$, we will prove that $CN = ON$, $OM = DM$.



$$\angle MDO = \angle IDO = \frac{1}{2}\angle IDH = \angle ODC = \angle MOD, \text{ so } OM = MD. \text{ Similarly } ON = NC.$$
$$AD + BC = (AM + MD) + (CN - BN) = MD + CN = OM + ON = OB + OA = AB.$$
$$\text{Thus } \boxed{AD + BC = AB.}$$

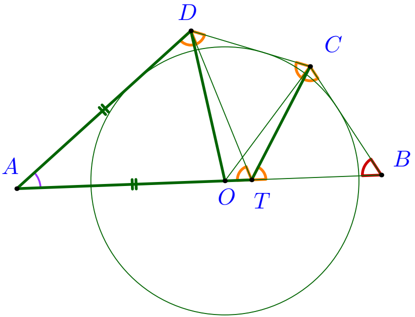
Let E be the intersection of AD and BC. Let A' and B' be the reflections of A and B over $\overline{EO}$



$\angle EA'B' = \angle EAB = \angle ECD$ ($ABCD$ cyclic) $\Rightarrow CD \parallel A'B' \Rightarrow \angle DCO = \angle COA'$.
CO is the angle bisector of $\angle DCB \Rightarrow \angle DCO = \angle OCA' \Rightarrow \angle COA' = \angle OCA' \Rightarrow OA' = CA'$.
Similarly $AD = AO$, since $OA = OA'$ (reflection!), so $AD = AO = OA' = CA'$. Similarly $OB = CB$.

Thus $\boxed{AD + BC = CA' + OB = OA + OB = AB.}$

Synthetic proofs of geometric theorems make use of auxiliary constructs (such as helping lines) and concepts such as equality of sides or angles, similarity and congruence of triangles, equal angles in circles, and cyclic quadrilaterals.

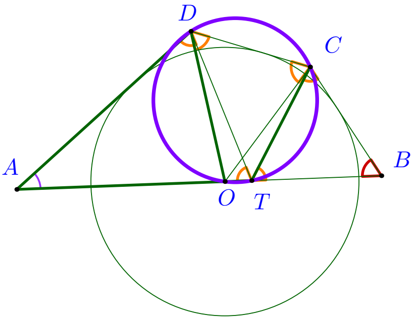


$\triangle DAT$ isosceles, $\angle DTA = 90^\circ - \frac{\angle A}{2} = 90^\circ - \frac{180^\circ - \angle C}{2} = \frac{\angle C}{2} = \angle DCO$. Thus, $DCTO$ is cyclic. $\angle BTC = \angle CDO = \frac{\angle D}{2} = 90^\circ - \frac{\angle B}{2}$, $\angle TCB = 180^\circ - \angle B - (90^\circ - \frac{\angle B}{2}) = 90^\circ - \frac{\angle B}{2} = \angle BTC$. Therefore, $\triangle TBC$ is isosceles at B, $TB = BC$. Hence, $\boxed{\text{AD+BC=AT+BT=AB.}}$

Solution Four

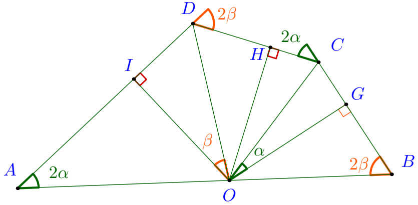
Second synthetic proof - construction by intersection.

Synthetic proofs of geometric theorems make use of auxiliary constructs (such as helping lines) and concepts such as equality of sides or angles, similarity and congruence of triangles, equal angles in circles, and cyclic quadrilaterals.



Let T be the point where (DOC) meets AB again. We prove that $AD = AT$ and $BC = BT$.
$DOTC$ is cyclic, thus $\angle DTO = \angle DCO = \frac{\angle C}{2} = \frac{180^\circ - \angle A}{2} = 90^\circ - \frac{\angle A}{2}$.
So $\angle ADT = 180^\circ - \angle A - (90^\circ - \frac{\angle A}{2}) = 90^\circ - \frac{\angle A}{2} = \angle DTA$. $\triangle DAT$ isosceles, $DA = TA$.
Similarly, $\triangle TBC$ is isosceles, and $BT = BC$. $\Rightarrow$ $\boxed{AD + BC = TA + TB = AB.}$

Let (O) be a unit circle, $OI = OG = OH = 1$. Let G, H, and I be the tangent points to (O) of BC, CD, and AD, respectively. Let $\angle OAD = 2\alpha, \angle OBC = 2\beta$.



It is easy to prove that $\angle COG = \alpha, \angle DOI = \beta$, so

$AD + BC = (AI + ID) + (CG + GB) = (\cot 2\alpha + \tan\beta) + (\cot 2\beta + \tan\alpha)$.

$AD + BC = \left(\frac{\cos 2\alpha}{\sin 2\alpha} + \frac{\sin\alpha}{\cos\alpha}\right) + \left(\frac{\cos 2\beta}{\sin 2\beta} + \frac{\sin\beta}{\cos\beta}\right)$

$= \frac{\cos 2\alpha \cos\alpha + \sin 2\alpha \sin\alpha}{\sin 2\alpha \cos\alpha} + \frac{\cos 2\beta \cos\beta + \sin 2\beta \sin\beta}{\sin 2\beta \cos\beta}$

$= \frac{\cos 2\alpha \cos(-\alpha) - \sin 2\alpha \sin(-\alpha)}{\sin 2\alpha \cos(-\alpha)} + \frac{\cos 2\beta \cos(-\beta) - \sin 2\beta \sin(-\beta)}{\sin 2\beta \cos(-\beta)}$

$= \frac{\cos(2\alpha + (-\alpha))}{\sin 2\alpha \cos\alpha} + \frac{\cos(2\beta + (-\beta))}{\sin 2\beta \cos\beta} = \frac{1}{\sin 2\alpha} + \frac{1}{\sin 2\beta} \quad = AO + OB = AB.$

Theorem (Number of permutations and combinations)

- *We can arrange* n *items in* $n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$ *ways.*
- *We can select* r *items in order from a group of* n *items, when order matters, in* $n \cdot (n-1) \cdot \cdots \cdot (n-r+1) = \frac{n!}{(n-r)!}$ *ways.*
- *The number of ways to choose an* r *person from* n *people, when order does not matter, called* n *choose* r*, is* $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Theorem (Stars and Bars)

The number of ways to distribute n *marbles into* k *piles is* $\binom{n+k-1}{k-1}$.

Definition (Fixed point)

$i \in \{1, 2 \ldots, n\}$ is called fixed point of a permutation f of $\{1, 2 \ldots, n\}$ if $f(i) = i$.

Definition (Derangement)

A permutation of $\{1, 2, \ldots, n\}$ without any fixed point is called derangement.

Example

How many ways are there to sit four boys and three girls in a row if no two boys can sit next to each other? Note that all boys and all girls are considered different.

Since seven persons sit on a row, and no two boys can sit next to each other, so the four boys must occupy positions $1, 3, 5$, and 7 : $B\ G\ B\ G\ B\ G\ B$. There are $4! = 24$ ways to sit the boys, $3! = 6$ ways to sit the girls, in total $\boxed{24 \cdot 6 = 144}$ ways.

Example

How many permutations of $1, 2, \ldots, 9$ are there so that exactly 5 numbers stay in their positions?

First we choose 5 numbers that stay in their original positions, that is $\binom{9}{5}$. We use Principle of Inclusion-Exclusion to count the number of derangements (no number stays in the same position) for 4 remaining numbers by first count all possible permutations, which is $4!$, then

- subtracts permutations where at least one number in its original position, which is $\frac{4!}{1!}$.
- add back permutations where at least two numbers in its original positions, which is $\frac{4!}{2!}$.
- subtracts permutations where at least three numbers in its original positions, which is $\frac{4!}{3!}$.
- add back permutations where all numbers in its original positions, which is $\frac{4!}{4!}$.

The result is $4! - \frac{4!}{1!} + \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!} = 9$. The total number of permutations is $\binom{9}{5} \cdot 9 = 1134$.

Example

Seven people of seven different ages are attending a meeting.

- The seven people leave the meeting one at a time in random order.
- The youngest person leaves the meeting sometime before the oldest person leaves the meeting.

Find the probability that the third, fourth, and fifth people to leave the meeting do so in order of their ages (youngest to oldest).

First, we count the number of orders where the youngest person leaves before the oldest person. By symmetry, the total number is $\frac{7!}{2} = 2520$.
Now, we count the number of orders satisfying both given conditions (lets call them *good* orders).
Let call the youngest person 1 and the oldest 7.
We consider the four following cases:

- 1 and 7 both are not part of the third, fourth, and fifth people to leave;
- 1 is part of but 7 is not;
- 7 is part of but 1 is not;
- both 1 and 7 are part of.

- For the first case, we have $\binom{4}{2} = 6$ ways to chose where 1, 7 are. Then, by symmetry, $\frac{5!}{3!} = \frac{5!}{6}$ of the permutations in this case have the third, fourth, and fifth people leaving in order of ages. This has $\frac{6 \cdot 5!}{6} = 120$ *good* orders.
- For the second case, we have 2 ways to chose where 7 is. 1 is forced to be the third person because no-one else can be younger. By symmetry, $\frac{5!}{2!} = \frac{5!}{2}$ of the permutations have the fourth and fifth people leaving in order of ages. This has $\frac{2 \cdot 5!}{2} = 120$ *good* orders.
- For the third case, we have 2 ways to chose where 1 is. 7 is forced to be the fifth person because no-one else can be older. By symmetry, $\frac{5!}{2!} = \frac{5!}{2}$ of the permutations have the third and fourth people leaving in order of ages. This has $\frac{2 \cdot 5!}{2} = 120$ *good* orders.
- For the fourth case, 1 is forced to be the third and 7 is forced to be the fifth person. Note that among all 5! permutations now it does not matter who is fourth as they have to be younger than 7 and older than 1. So there are 120 good orders.

Our probability is $\boxed{\frac{4 \cdot 120}{2520} = \frac{4}{21}.}$

Problem

Let $p_n(k)$ be the number of permutations of the set $\{1, \dots, n\}$, $n \geq 1$, which have exactly k fixed points (an element i in S is called a fixed point of the permutation f if $f(i) = i$.) Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!.$$

The table below shows an example of all permutations of $\{1, 2, 3\}$ ($n = 3$), fixed points in red.

1	**2**	**3**	1	1	1		3
1	3	2	1	0	0		1
2	1	**3**	0	0	1		1
2	3	1	0	0	0		0
3	1	2	0	0	0		0
3	**2**	1	0	1	0		1
			2	2	2		**6**

There are $n!$ permutations of $1, \dots, n$. Write each in a separate row, then replace each fixed point by 1, and all other numbers by 0. The row sum, the sum of all the number 1s in a row, gives the number of fixed point of the permutation which is represented by the row. The colum sum gives the number of permutations of other elements if the one in that column is a fixed point.

1	**2**	**3**
1	3	2
2	1	**3**
2	3	1
3	1	2
3	**2**	1

1	1	1	3
1	0	0	1
0	0	1	1
0	0	0	0
0	0	0	0
0	1	0	1
2	2	2	**6**

- First, prove that $\sum_{k=0}^{n} k \cdot p_n(k)$ **is the sum of all row sums.**
- Second, prove that **the sum of the column sums is $n(n-1)! = n!$.**

Finally, by the principle of Counting in Two Ways, $\boxed{\sum_{k=0}^{n} k \cdot p_n(k) = n!.}$