Learning Problem Solving

for Middle- and High School Students

Volume 1

Math & Chess Club in Ottawa

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Problems

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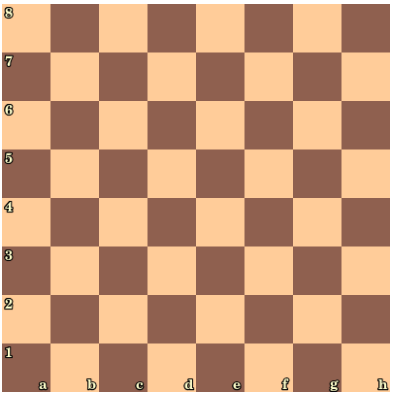
Chapter 1

Insightful Proofs

Examples

Problem 1.1 (A)

A knight stands on square $a1$ of a regular chess table. Is it possible for the knight to go to square $h8$, visiting each remaining square exactly once on the way?



Discussion: At first it seems to be a difficult problem and there is no much else that can be done except trying to move the knight and see if it can reach $h8$ from $a1$. However after a number of trials, you can see that it is not easy at all to try all possibilities. We have to find another way! So it is important in such a case to: *observe what information was given*.

There are three facts we know so far:

(1) A chessboard has 64 squares,

(2) Squares $a1$ and $h8$ are both black squares, and

(3) With every move, a *knight moves from a black to a white square or vice versa*.

The last one is an important condition. It says that the colour of the squares where the knight moves through *alternates between two colours: black and white*. It is the same as if we write the integers 1,2,3,4,5,.... and the number parity changes like this: *odd, even, odd, even, …*

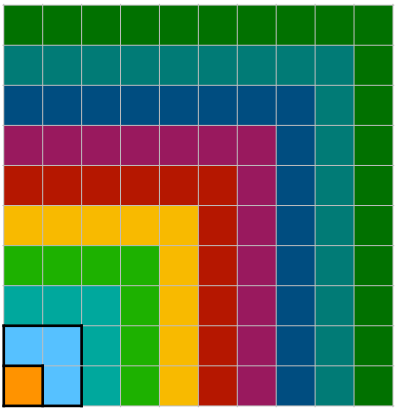
Note: This *two-value alternation is the principle of parity* that we can use to solve a lot of of problems.

Solution: Because there are 64 squares, the knight has to make 63 moves. With each move, the knight goes from a white square to a black square or vice versa. Since $a1$ is a black square, after 63 moves it shall end up on a white square, which cannot be $h8$.

Problem 1.2 (A)

Can you determine the sum of the first 10 odd positive integers without adding them up?

Discussion: Look at the figure below.



This is a 10×10 square.

(1) The square is divided into 'L-shapes', each with different colour

(2) Going from left to right, the numbers of 1×1 squares in each L-shape are:, 2X1-1=1 for the first shape, 2X2-1=3 for the second shape, 2x3-1=5 for the third shape, …, 2X10-1 = 19 for the last shape, respectively. These numbers also happen to be the first 10 odd integers.

Solution 1: It is easy to see that the sum of the first 10 odd integers is the perfect square of 10, which is 100: $1 + 3 + 5 + \ldots + 19 = 10^2 = 100$

We can also generalize this result: <u>the sum of the first n positive odd integers is n^2</u>. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Solution 2: It is also possible to group the numbers in pairs so that each pair has an equal sum:

$$s = 1 + 3 + 5 + \ldots + (2n - 1)$$

$$s = (2n - 1) + (2n - 3) + (2n - 5) + \ldots + 1$$

When adding both equations, side by side:

$$2s = (1 + (2n - 1)) + (3 + (2n - 3)) + (5 + (2n - 5)) + \ldots + ((2n - 1) + 1)$$

$$= 2n \times n = 2n^2 \Rightarrow s = n^2$$

Note: The discussed solution is called *visual proof - proof without words.*

Problem 1.3 (A)

A monk climbed a mountain. He started at 8AM and reached the summit at noon. He spent the night on the summit. The next morning he left at 8AM and went down by the same route that he used the previous day. He reached the bottom at noon. Prove that there was a time between 8AM and noon when the monk was at exactly the same spot of the route on both days.

Discussion: We know nothing about how long is the route, what was the speed of the monk travelling on the route (we don't event know if the monk travelled at a constant speed). Again, *let's observe what was given to us*.

We know two facts:

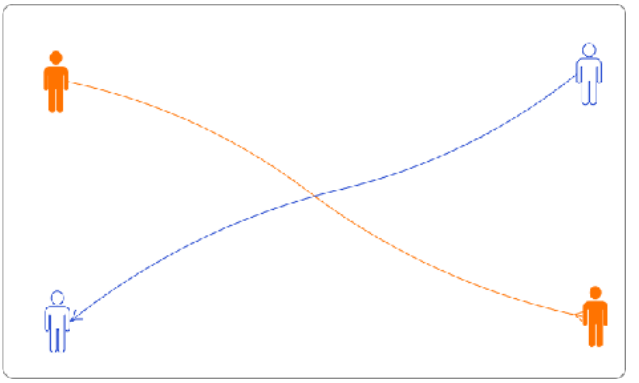
(1) The monk always started at 8AM and finished at noon.

(2) The routes are the same.

The reason we do not know how to compare the routes is because they happened at different time, and naturally we do not compare things outside of the same time frame.

Solution: Assume that on the first day, the *phantom image of the monk would have travelled down the mountain at 8AM in the exact same way as the monk did on the second day*.

When two monks travel on the same route, one from the top and one from the bottom, *they will meet each other at a place on the route, and of course they spend an equal amount of time from the time when they start to the time when they meet*.



In addition, measuring this point from the bottom of the mountain:

(1) Going up: his position's height changes from 0 to the height of the mountain.

(2) Going down: the phantom's position's height changes from the height of the mountain to 0.

At a certain time, both heights have to be equal. That spot when it happened is the spot the monk was at the exact same time on both days.

Note: The principle seems very obvious here for elementary and middle school students, and it is based on the *intermediate value theorem.*

Intermediate value theorem (an oversimplified version)

"When we have two points connected by a continuous curve: one point below the line, the other point above the line, then there will be at least one place where the curve crosses the line!"

Problem 1.4 (B)

Determine the value of $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{99\times 100}$

Discussion: Students, even if armed with calculator, will see that this is hard to hand-calculate by determining decimal value of each fraction: $\frac{1}{1\times 2}=0.5$, $\frac{1}{2\times 3}=0.1666...$

Search for a pattern: If we take a look at each fraction, we can see that the denominator is a product of two *consecutive* integers: (1,2), (2,3), (3,4), ...(99,100). Because each pair consists of two *consecutive* integers, their difference is exactly 1, which is the value of the numerator:

$$\frac{1}{1\times 2}=\frac{2-1}{1\times 2}=\frac{1}{1}-\frac{1}{2}$$
$$\frac{1}{2\times 3}=\frac{3-2}{2\times 3}=\frac{1}{2}-\frac{1}{3}$$
$$\frac{1}{3\times 4}=\frac{4-3}{3\times 4}=\frac{1}{3}-\frac{1}{4}$$
$$\ldots$$
$$\frac{1}{99\times 100}=\frac{100-99}{99\times 100}=\frac{1}{99}-\frac{1}{100}$$

In other words, *each fraction is a difference of reciprocals of consecutive integers.*

Solution: When adding all these fractions together:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{99\times 100}=(\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\ldots+(\frac{1}{99}-\frac{1}{100})$$

Notice that the middle terms are cancelling each other: $-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\ldots-\frac{1}{99}+\frac{1}{99}=0$

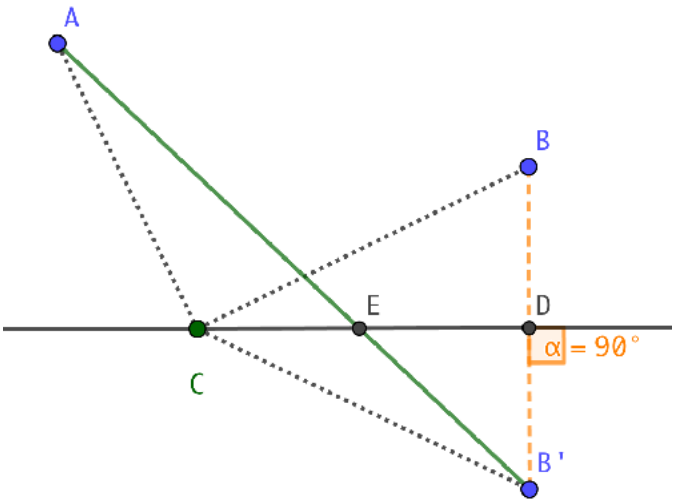
So the value of the sum is $\frac{1}{1}-\frac{1}{100}=\frac{99}{100}$

Note: the *sum of differences principle* is used here to cancel the middle terms.

Problem 1.5 (B)

Elizabeth went on an outdoor trip. To make dinner she needed to gather wood at point A, then she went to the river bank to take some water, and then she returned to the picnic site at point B. What would be the shortest distance that she can travel to complete all her tasks?

Discussion: At first, it seems impossible to determine where on the river bank (point C) should Elizabeth go in order to make $AC+CB$ to be the shortest distance. Let's *observe what information is given*.



The river bank is a *straight line*. Let's *reflect* point B over the river bank to a point B'. Point B' is called the *mirror image* of B and the river bank is called the *line of symmetry*. If we can prove that$CB = CB'$ (which intuitively seems correct because of the refection transformation), then $AC + CB = AC + CB' \geq AB'$.

Why does AB' matter? Because it depends on AB only regardless where point C will be on the line of symmetry.

Solution: Using *reflection over a line* (a type of *geometry transformation*).

It is easy to see that:

$$BD = DB', \angle CDB = \angle CDB' = 90^\circ$$

$$\Rightarrow \text{Triangles CBD and CB'D are congruent. } (\triangle CBD \cong \triangle CB'D)$$

$$\Rightarrow CB = CB'.$$

Therefore:

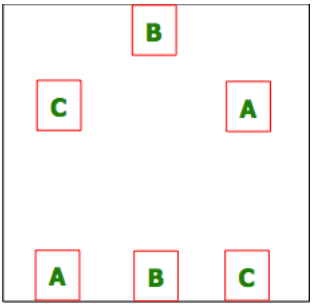
$$AC + CB = AC + CB' \geq AB'.$$

The equality happens when C coincides with E, which is the intersection of AB' with the river bank.

Note: in the example above, the *triangle inequality is used together with the reflection over the line of symmetry* (river bank) in order to find the minimum length of a broken line (ACB).

Problem 1.6 (B)

Can you connect each small box A, B, C on the top with the same-letter box on the bottom with paths that do not cross one another, nor leave the boundaries of the large box?



Discussion: At first glance, this problem seems impossible to solve. It is highly possible that you cannot recall anything similar that could help you to get started with. The troublemaker seems to be the path that connects the two B boxes.

Let's consider the following two ideas:

(1) *The path linked two B boxes does not have to be a straight line.*

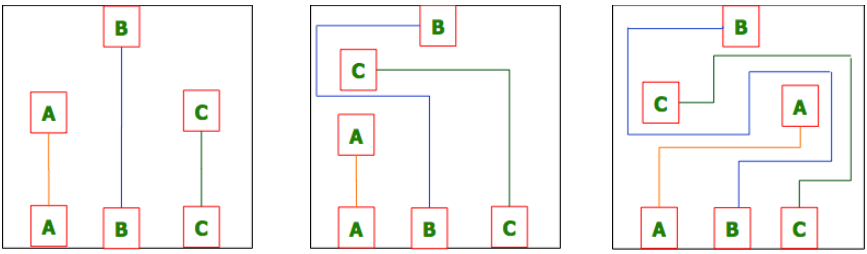
(2) The *working backwards* problem solving strategy can actually help.

Solution: We perform several steps as shown below

(3) Let's redraw a diagram where the top A and C boxes interchange their places. Then, we connect pairwise the two A, two B, and two C boxes with a path, respectively. This represents a working configuration when *all three pairs of boxes are connected*.

(4) Move the top C box into the left column, *bending* the path that connects the two B boxes so that the two paths *do not cross each other*.

(5) Similarly push the top C box into the right column, *bending* the paths connecting the two B boxes and two A boxes, so that *none of the three paths are crossed*.



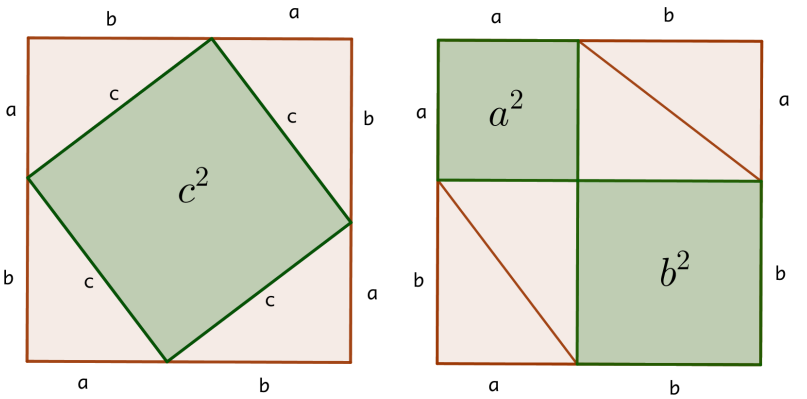
We are done!

Note: Here the paths can be changed - bent, stretched - but not broken. This is call *topological property* or *topological invariant* and it helps to *map the original problem to an equivalent one*.

Problem 1.7 (C)

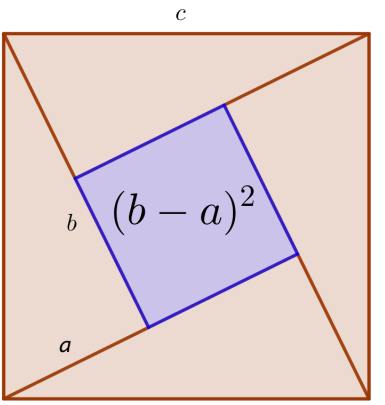
Pythagorean Theorem: in a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides.

Solution 1: On the left, four right triangles with side lengths a, b, c and a square with side length c form the same square as the one on the right side, which is made by the same four right triangles and two squares with a and b side lengths.



Solution 2: In the diagram below, a square with side length c is made from four right triangles with side lengths a, b, c and a a square with $b - a$ side length. In other words:

$$(b-a)^2 + 4 \times \frac{1}{2} a\,b = c^2 \Rightarrow a^2 + b^2 = c^2$$



Note: In both solutions, a *visual proof* is shown to give us an insight. The *divide-and-conquer principle* is used here, which is very popular in determining the area and the perimeter of a geometrical figure.

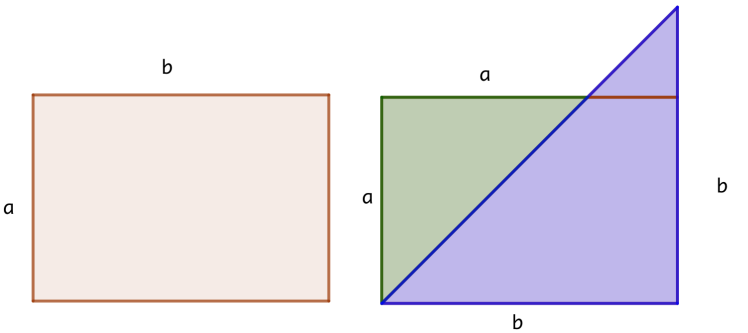
Problem 1.8 (C)

For any a, $b \geq 0$ real numbers: $a^2 + b^2 \geq 2a\,b$

Discussion: This is a trivial inequality that can be derived from the fact that $(a-b)^2 \geq 0$.

Algebra - or algebraic notation, the way we use algebra to describe problems and solutions - is powerful in both, describing the problems, and providing short solutions. However, it does not always help students to have an insight of the problem.

Solution: We can assume $b \geq a$. In the below diagram, the figure on the left is a rectangle with a, b side lengths. The figure on the right is two isosceles right triangles with a, b legs.



It is easy to see that the area of the rectangle is smaller than the total area of the two triangles:

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}.$$

Note: The proof shows a very interesting concept of how to visualize *the relationship of the area of a square to the length of its side*, which can reflect an insight of a well-known inequality.

Problem 1.9 (C)

Prove that the set $\{1, 2, \ldots, 10\}$ has exactly 2^{10} subsets.

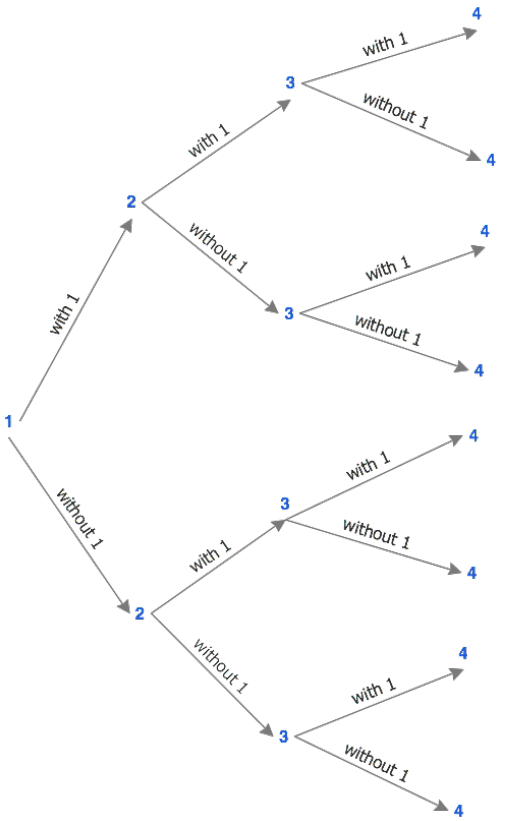
Discussion: What can be the subsets of $\{1, 2, \ldots, 10\}$? Let's list a few of them:

- Subsets that have zero elements: $\{\}$. There is 1 of them.
- Subsets that have 1 element: $\{1\}, \{2\}, \ldots \{10\}$. There are 10 of them.
- Subsets that have 2 elements: $\{1,2\}, \{1,3\}, \ldots \{1,10\}, \{2,3\}, \{2,4\}, \ldots, \{9,10\}$. There are $9 + 8 + \ldots + 1 = 45$ of them.
- Subsets that have 9 elements: there are 10 of them.
- Subsets that have 10 elements: $\{1, 2, \ldots, 10\}$. There is 1 of them.

It is not obvious how we can find the number of subsets that have the same amount of elements. The solutions below show two *different equivalent ways to formulate the problem.*

Solution 1: Let's draw a tree as shown in the diagram on the right.

For each subset composed of the elements 1, 2, … there are two additional branches depending if an element is in the subset or not. In other words, the number of leaves of the tree is the number of possible subsets. It is easy to see that the number of leaves $\underbrace{2 \times 2 \times 2 ... \times 2}_{10} = 2^{10}$.



Solution 2: This solution is more subtle.

For each subset, let's associate a 10-digit number. Each digit at position n ($n = 1, 2, 3, \ldots, 10$) is 1 if the number is in the subset, otherwise it is 0.

For example:

- if the subset is {1,4,6}, then the number is 1001010000

- if the subset is {9,10}, then the number is 0000000011

But these numbers are actually all possible base-2 numbers with maximum 10 digits. They are 0, 1, … $2^{10} - 1$.

The number of 10-digit binary numbers is obviously 2^{10}.

Note:

(1) The solutions are examples how to *formulate a problem to map an equivalent one, perhaps easier to solve or with an already known solution*.

(2) The first solution is based on *properties of a binary tree*. The second solution is based on the *representation of integer numbers in base-2*.

Puzzle 1.10 (Linguistic Challenge, NACLO, Adriana Solovyova): Tajik

Tajik is an Indo-European language related most closely to Farsi (or Persian), the major language of Iran. Most literature in Tajik uses a version of the Cyrillic writing system. This system is used in Russian, Ukrainian, Bulgarian, Eastern Europe and Central Asia. You don't need to know how to read Cyrillic writing to solve this puzzle -- just use your common sense!

Below are three phrases from the Tajik language with their English translations.

дуусти ҳуби ҳамсоай сумо	a good friend of your neighbor
ҳамсоай дуусти ҳуби сумо	a neighbor of your good friend
ҳамсоай ҳуби дуусти сумо	a good neighbor of your friend

Your task is to give the English translations of all four Tajik words. The possibilities are simply "good," "friend," "neighbour," and "your."

дуусти

ҳуби

ҳамсоай

сумо

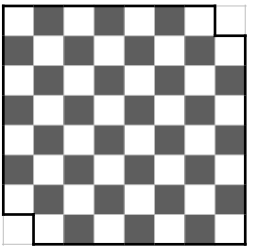
Note: The order of the words is important.

Exercises

Problem 1.11 (A)

Danny the Hooligan cuts the $a1$ and $h8$ squares out chess board. He challenges Kevin the Smart to tile the board with dominos. Can Kevin the Smart do it?

Discussion: Below on the left side, there is the board after Danny cut the two corners. On the right side is a domino.



Note: what is the *parity of the difference of the number of black squares and the number white squares?*

Solution: *Let assume the opposite that the cut board can be tiled by the dominoes.*

Each domino has 1 black and 1 white square. If the board can be tiled by the dominoes then the number of black and white squares of those dominoes must be the same. That is impossible because the cut board has 30 black squares and 32 white squares. This contradicts the assumption (*) above. Hence, the assumption is wrong, or in another word, the board cannot be tiled by the dominoes.

Problem 1.12 (A)

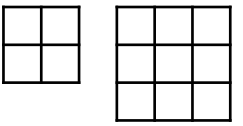
Is it possible to cut a square into:

(a) 6 squares (not necessarily same size)

(b) 7 squares (not necessarily same size)

(c) 8 squares (not necessarily same size)

Discussion: Dividing a square into 4, or 9 squares is simple.



(1) Now since you see how the original square can be divided into 4 or 9 squares, show how you can do that for 6, 7 or 8 squares.

(2) If you have a solution for $n \geq 6$ squares, can you show a solution for $n + 3$ squares?

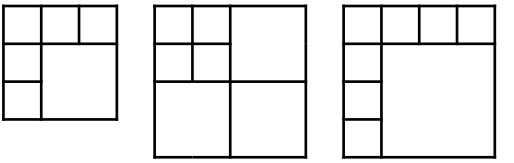
(3) *Generalize*: Is it possible to cut a square into any $n \geq 6$ squares?

Solution: From the ideas in the discussion, the square can be cut into 6, 7, and 8 squares as shown in the box below. Using the solutions for 4, 6, 7, and 8 the following generalization can be obtained.

Generalization:

(i) It is shown as above that a square can be cut into 6, 7, and 8 squares.

(ii) Assume that the square can be cut into n, $n + 1$, $n + 2$ squares. Because a square can be divided into 4 squares, so adding 3 new squares and the number of squares grows to $n + 3$, $n + 4$, $n + 5$ squares.



Therefore by induction principle, ***a square can be cut into any n square, with $n \geq 6$.***

Note: One of the best ways to generalize is to use *induction principle*.

Problem 1.13 (A)

A number has as digits: 2000 ones, 2000 twos, and all other digits are zeros. Can that number be a perfect square?

Discussion*:* Too many digits! Investigate the divisibility of the number by 3. If a perfect square is divisible by 3, should it be divisible by 9? Is the number divisible by 9?

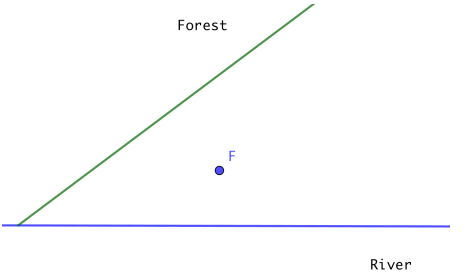
Solution*:* A perfect square is a product of a number with itself $n \times n$. If $n \times n$ is divisible by 3 then n is divisible by 3, thus $n \times n$ is divisible by 9.

The exponent of a prime in the prime factorization of a perfect square must be an even number. For example $54 = 2^1 \times 3^3$ is not, but $36 = 2^2 \times 3^2$ is a perfect number.

The sum of the digits of the given number is $2000 \times 1 + 2000 \times 2 = 6000$ is divisible by 3 but not by 9. Therefore the given number is not a perfect square.

Problem 1.14 (B)

Elizabeth was at point F on a clearcutting patch of an acute angle shape. She had to go to the green tree line to gather wood, then to the blue river bank to fetch some water, and then return to the fireplace.



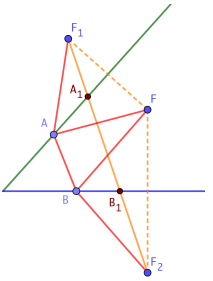
How can she make the shortest trip?

Discussion: Let assume A, B are points on the green, blue lines for wood gathering, water fetching, respectively.

(a) What is the total distance that she had to walk?

(b) Using *line of symmetry*, where would be the mirror images of F? What would be the total distance while using F images?

(c) What would be the minimal? Where would be A and B points?



Solution:

Let F_1, F_2 be reflections of F over the green and blue lines. $AF = AF_1$, $BF = BF_2 \Rightarrow FA + AB + BF = F_1A + AB + BF_2$.

Because $F_1A + AB + BF_2 \geq F_1F_2$, therefore the shortest distance Elizabeth needs to walk is F_1F_2 or $F_1A_1B_1F_2$. It means that she will need to gather wood at A_1 and fetch water at B_1 points.

Note: An exercise of *geometry transformation with refection over a line* to find the shortest distance.

Problem 1.15 (B)

The fast-talking Donkey went up a long spiral staircase. On the first step Donkey found a round pizza. At every step Donkey found half of what he got in the previous step.

Is it possible that at some point the amount of pizza Donkey got is more than 2 pizzas?

Discussion:

(1) All you need is to know is to draw round pizza. Draw what Donkey gets after step 1, 2 and 3.

(2) Can you prove your guess?

Note: It is an example how to obtain *value of an infinite geometric series*.

Solution: Assume that the Donkey was at the nth step, so the numbers of pizza portions he received for each step is $1, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^{n-1}}$. This means that the total amount of pizza he got:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} = \frac{2^{n-1} + 2^{n-2} + \ldots + 2 + 1}{2^{n-1}} = \frac{2^n - 1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$$

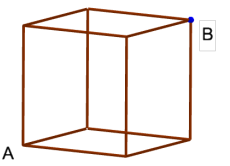
.

Hence, at any step he received an amount smaller to 2 pizza. It can never be 2 pizzas.

Note: It is tempted to conclude that after an *infinity number of steps* the Donkey will get 2 pizzas. That's impossible because there is no stair with infinity steps.

Problem 1.16 (B)

A bug sits on the vertex A of a cube. It crawls to the vertex B. What is the shortest path?

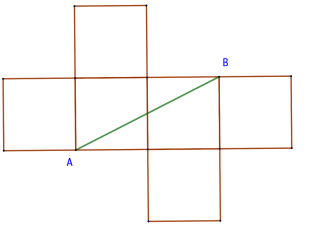


Discussion:

(1) The path consists of a sequence of line segments that are not in the same plane.

(2) Let unfold the cube, such that its faces would be on the same plane.

(3) *U*se the principle of shortest distance.



Note: It is an example how to *reduce the number of dimensions* to a plane geometry problem.

Solution: Let unfold the cube. In another word, we reduce a problem in 3 dimensions in to a better known one in 2 dimensions. Looking at the figure on the right: the shortest distance is the line segment connect A, B.

Problem 1.17 (C)

Determine the value of $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2...}}}}$

Discussion: A real challenge! Square roots in an infinitely repeatedly expression is not a familiar sign for middle school student. Our friend, *observation*, will help.

(1) Let assign x as the value of the expression. $x = \sqrt{2+\sqrt{2+\sqrt{2...}}}$

(2) A closer look at the expression reveal that *the second term of the sum under the square root sign looks like the given expression itself.*

(3) Can you use what is discovered in (2) and establish an equation. Eliminate the square root by squaring both side Now you should have a second-degree equation, try to factor and solve it.

Here's an example:

$x^2 - 5x + 6 = (x^2 - 2x) - (3x - 6) = x(x-2) - 3(x-2) = (x-2)(x-3)$

It means that $x^2 - 5x + 6 = 0$ equation has two solutions $x = 2$, and $x = 3$

Note: It is a typical example of how to *remove infinitely nested expression* to a simpler form.

Solution: $x = \sqrt{2+\sqrt{2+\sqrt{2+\dots}}} \Rightarrow x = \sqrt{2+x} \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x+1)(x-2) = 0 \Rightarrow x = 2 \Rightarrow \sqrt{2+\sqrt{2+\sqrt{2+\dots}}} = 2.$

Problem 1.18 (C)

What is the remainder when $n = \underbrace{20132013...2013}_{2013\ times\ 2013}$ is divided by 333,333?

Discussion: what is the prime factorization of 333,333?

Note: it is *investigation of divisibility by using prime factorization of the divisor.*

Idea 1: From the fact that $1001 = 7 \times 11 \times 13$:

(a) The number $\overline{a_k a_{k-1} \dots a_1 a_0}$ with the digits $a_k, a_{k-1}, \dots, a_1, a_0$ is divisible by 7 if and only if

$$\overline{a_2 a_1 a_0} - \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} - \dots \text{ is divisible by 7}$$

For example 1112444333555444666 is divisible by 7 because:

$$666 - 444 + 555 - 333 + 444 - 112 + 1 = 777 \text{ is divisible by 7}$$

(b) Similarly the number $\overline{a_k a_{k-1} \dots a_1 a_0}$ with the digits $a_k, a_{k-1}, \dots, a_1, a_0$ is divisible by 11 if and only if

$$\overline{a_2 a_1 a_0} - \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} - \dots \text{ is divisible by 11}$$

(c) Similarly the number $\overline{a_k a_{k-1} \dots a_1 a_0}$ with the digits $a_k, a_{k-1}, \dots, a_1, a_0$ is divisible by 13 if and only if

$$\overline{a_2 a_1 a_0} - \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} - \dots \text{ is divisible by 13}$$

Idea 2: For *divisibility by 27 and 37,* from $999 = 27 \times 37$, the number $\overline{a_k a_{k-1} \dots a_1 a_0}$ with the digits $a_k, a_{k-1}, \dots, a_1, a_0$ is divisible by 27 or 37 if and only if the sum: $\overline{a_2 a_1 a_0} + \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} + \dots$ is divisible by 27 or 37.

From Idea 1 & 2: One by one, investigate divisibility of each prime factor of 333,333 by n. If a factor of 333,333 is a power, check if that power is divisible by n.w When all investigations, what can you tell about the remainder of n divided by 333,333?

Solution: Since $333333 = 9 \times 7 \times 11 \times 13 \times 37$

(1) $n = \underbrace{20132013...2013}_{2013\ times\ 2013}$ is divisible by 9 because its sum of digits is $2013(1+2+3)$. This is a number divisible by 9: $2013.6 = 3.11.61.2.3$

(2) n is divisible by 7, 11, 13 because according to Idea 1: $(013 - 132 + 320 - 201) \times 671$ is 0 and therefore divisible by 7, 11, 13.

(3) n is divisible by 37 because according to Idea 2: $(013 + 132 + 320 + 201) \times 671$ is divisible by 37.

From (1), (2), and (3): n is divisible by 333,333.

This means the remainder is 0.

Problem 1.19 (C)

Three man went to a pub to drink beer at the happy hour, which was advertised to cost \$9 per person. After an hour, they each gave the waitress \$10, and asked her to bring back the change. She went to the cashier and learned that the actual charge was only \$25 for any group of three. Feeling lucky, she gave \$25 to the cashier, then she went back to the guests and gave each of them \$1. She decided not to tell them about the actual charge. Thus, the waitress pocketed \$2, each guest spent \$9, for a total of $2 + 3 \times 9 = 29$ dollars. What happened to the other dollar?

Solution: Let's take a look at the money:

(1) There were \$30 given by the guests to the waitress. She gave \$25 to the cashier, pocketed \$2, and returned \$3 to the guests. So nothing was wrong.

(2) The question is a misleading trick. It assumes that what the waitress pocketed plus what the guests paid should add up to the whole payment. That is not correct. What the waitress pocketed plus what the cashier received should be equal to what the guests paid.

Note: It is a typical mistake of *adding unrelated quantities together*.

Puzzle 1.20 (Linguistic Challenge, NACLO, by Raymond Weisling): Learning Indonesian

The major languages spoken in Malaysia, Indonesia, and the small country of Brunei Darussalam, are nearly identical. In English they are known as Indonesian and Malay. These languages are spoken by over 240 million people living among 18,000+ islands that make up these countries.

The only word known is **"satu" (Indonesian) = "one" (English)**.

A. Satu ditambah satu menjadi dua.
B. Dua ditambah dua menjadi empat.
C. Satu ditambah dua menjadi tiga.
D. Dua dikalikan dua menjadi empat.
E. Enam dikurangi tiga menjadi tiga.
F. Sepuluh dikurangi enam menjadi empat.
G. Dua ditambah tiga menjadi lima.
H. Sepuluh dibagi dua menjadi lima.
I. Tiga dikalikan enam menjadi delapan belas.
J. Delapan belas dikurangi satu menjadi tujuh belas.
K. Tiga dikalikan tiga menjadi sembilan.

Can you figure out what are the words for the following numbers in Indonesian:

- number 6 is _____,
- number 9 is _____,
- number 17 is _____

Problems

Problem 1.21 (A)

Clemence said: "The day before yesterday I was 10, but I will turn 13 in the next year." Is this possible?

Discussion: Let put the information that we know into a table of the relevant days, Clemence's age, and the year.

	The day before yesterday	Yesterday	Today	The 12th birthday	The 13th birthday
Clemence's age	10			12	13
The year			This year		Next year

Problem 1.22 (A)

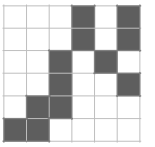
The Island of Knights and Liars has two type of inhabitants: Knights who always tell the truth and Liars who always lie. While visiting the island, Erica meet a group of three islanders: Ariana, Kauri, and Wiremu. Erica asked them the same question: "How many knights are in your group?" Ariana replies "None". Kauri said "One".

Can you find out what Wiremu is? What was his response?

Discussion: Let write down the possible cases in a table. Fill *the answers of the islanders* for each case. Compare with the given conditions.

Ariana	Kauri	Wiremu	Ariana's answer	Kauri's answer	Wiremu's answer
Knight	Knight	Knight			
Knight	Knight	Liar			
Knight	Liar	Knight			
Knight	Liar	Liar			
Liar	Knight	Knight			
Liar	Knight	Liar			
Liar	Liar	Knight			
Liar	Liar	Liar			

Problem 1.23 (A)



Cut the board shown in the picture above into four equal parts so that each one of them contains three shaded squares?

Problem 1.24 (B)

Distribute 127 one-dollar bills into 7 wallets so that any integer amount from 1 through 127 dollars can be paid without opening the wallets.

Discussion:

(1) Let number the wallets with the number from 1 to 7.

(2) For an amount can be paid with a number of wallets, write a *sequence of zeros (0) and ones (1)*: zeros if those wallets are not used and ones if those are used. Below is an example.

W1	W2	W3	W4	W5	W6	W7
0	0	0	1	1	0	1

(3) The sequence can be written now as 0001101. If any amount can be written as a sequence of 0s and 1s, what does that mean?

Problem 1.25 (B)

Find all 3-digit number $\overline{abc}$, with the digits a, b, c: $1 \leq a, b, c \leq 9$ and

$$\overline{ab} + \overline{bc} + \overline{ca} = \overline{abc}$$

Discussion:

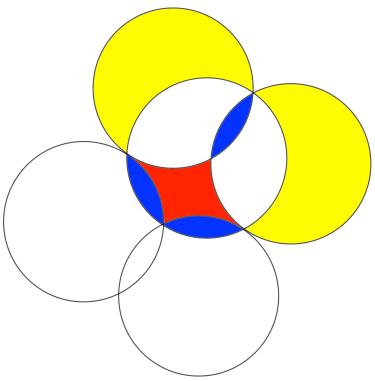
(1) A number such as $\overline{ab}$ can be rewritten as $\overline{ab} = 10a + b$.

(2) A sum like $10c + b$ can always be rewritten as $10c + b = \overline{cb}$

(3) Use estimation to eliminate impossible cases.

Problem 1.26 (B)

Lilian draw five identical circles as below. She coloured some parts of them with blue (**b**), red (**r**), and yellow (**y**). Parts with same colours are identical.



Can you find a formula to calculate a yellow (**y**) part based on blue (**b**), and red (**r**) parts?

Discussion:

(1) Try to find a region that looks same as the yellow one.

(2) Try to fit the blue and red parts into that region.

Problem 1.27 (C)

Determine the value of the expression below:

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3}+\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3}+\frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3}+\ldots\frac{9^{\frac{997}{1000}}}{9^{\frac{997}{1000}}+3}+\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3}+\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}$$

Discussion: Look at the solution 2 of problem 1.2. Try to pair the terms.

(1) Calculate the sums of terms in each pair, for a few pairs.

(2) How many pairs are there?

Problem 1.28 (C)

In the xy-plane, what is the length of the shortest path from $(0, 0)$ to $(12, 16)$ that does not go inside the circle $(x-6)^2+(y-8)^2=25$?

Discussion:

(1) Draw the circle $(x-6)^2+(y-8)^2=25$, mark the two given points $(0, 0)$ to $(12, 16)$

(2) Are the two points are outside of the circle?

(3) What would be shortest path *around* the circle, connecting that two points?

Problem 1.29 (C)

Determine the largest number that is the product of positive integers whose sum is 1976.

Discussion: Let take a look *a simpler problem:* find the largest number that is the product of positive integers, such that:

(1) Their sum is 4. Easy to find that it is $2\times 2=4$

(2) Their sum is 5. Easy to find that it is $3\times 2=6$

(3) Their sum is 6. Easy to find that it is $3\times 3=9$, which is larger than $2\times 2\times 2=8$

Try a few more cases and establish your hypothesis. Then prove it.

Puzzle 1.30 (Linguistic Challenge, NACLO, by N. Bailey): Kurdish

The Kurdish people number at least 25 million. Kurdish is an important member of the Iranian branch of the Indo-European language family. There are several dialects of Kurdish. Kurmanji Kurdish is spoken by about 12-15 million people, living in Turkey, Iran, Iraq and Syria and is represented by three literary standards and several speech varieties. Some speakers also live in the southern parts of the former Soviet Union, especially in Armenia and Georgia. The following puzzle is from the variety of Kurmanji Kurdish spoken in the former Soviet Union. This variety is normally written in Cyrillic script (like Russian), but is presented here in a modified Latin script.

Match the following the Kurdish sentences to their translations:

1. Ez h'irç'ê dibînim
2. Tu dir'evî
3. Tu min dibînî
4. H'irç' dir'eve
5. Ez dir’evim
6. Tu h'irç'ê dibînî

English translations in random order

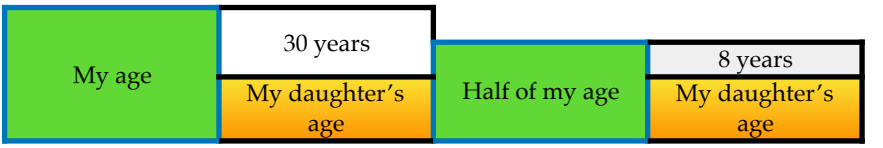
A. You see Bear.
B. You see me.
C. Bear runs.
D. You run.
E. I see Bear.
F. I run.

Challenges

Problem 1.31 (A)

I am 30 years older than my daughter. If I were two times younger than I really am and she was 8 years older than she really is, we would be at the same age. How old are we?

Discussion: *Drawing a picture* is one way to gain insights of a problem.



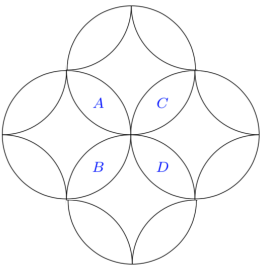
Problem 1.32 (A)

Insert the brackets (,) into the expression below to obtain 1961:

$$4.8 + 5.6.7 + 9 + 2$$

Discussion: *Prime factorization* is a way to write a number as a product of its prime factors. For example $12 = 2^2 \times 3$, $111 = 3 \times 37$. Look at the prime factorization of 1961.

Problem 1.33 (A)



Given four circles, each circle was divided into five parts as in the diagram below. The numbers were entered into each of the hollow parts so that two different parts are filled with two different numbers. Knowing that the sum of the numbers filled in all parts of each circle is 60.

What is $A + B + C + D$?

Discussion: The sum of the numbers filled in all parts of each circle is 60. If we *add all of these sum together*:

(1) How is this sum of sums related to the given numbers 5, 6, 7, ... 19, 20?

(2) How is this sum of sums related to A, B, C, D?

Problem 1.34 (B)

When Han Solo was still relatively unknown, he used to attend problem solving courses. Once he was near the chalkboard, using his super sneaky driver skills he quickly changed the two digits in the multiplication on the board. After he did, the board looked like this:

$$4 \times 5 \times 4 \times 5 \times 4 = 2247$$

Can you reconstruct the original?

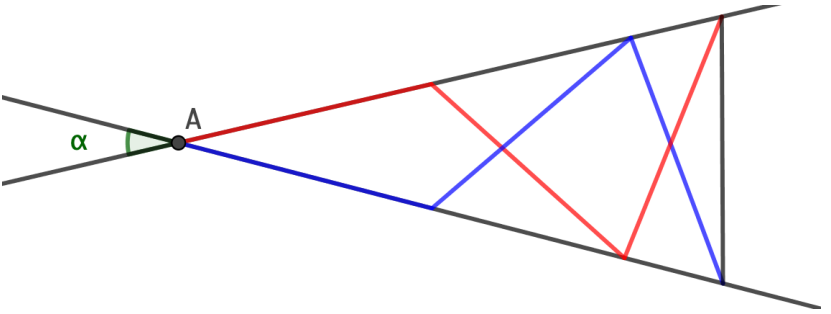
Discussion:

(1) Look at the digit 7. Can you *change two digits on the left side so that the product's unit digit is 7*?

(2) Can both changed digits be on the right side?

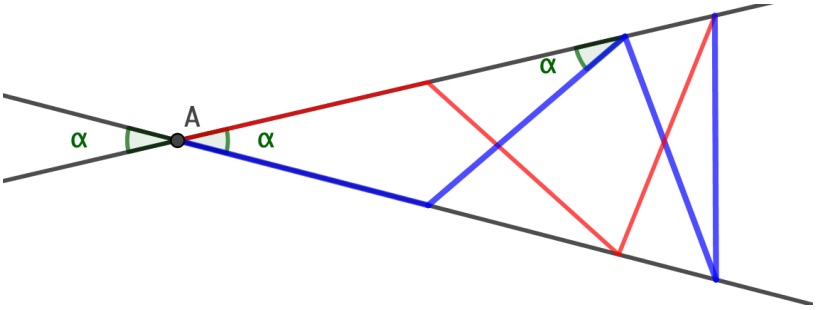
Problem 1.35 (B)

Qi'ra, before becoming the lieutenant in the Crimson Dawn, once was a math competitor. She was required to calculate the value of the angle α in the below figure, that was made from 7 matchsticks with equal length, connected end to end.



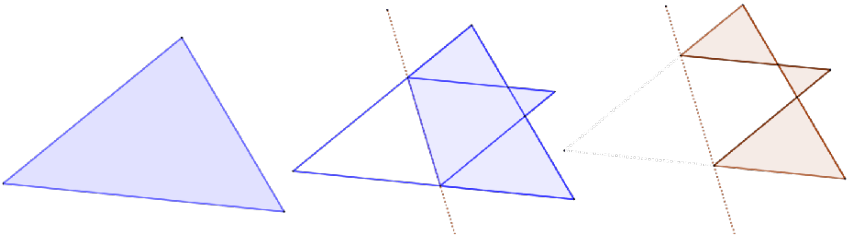
What is the value of α?

Discussion: Change matchsticks to line segment and using the assumption that *all matchsticks have the same length*.



Problem 1.36 (B)

Emma folds the blue triangle on the left below along a dotted line to get a folded figure, which is the blue figure in the middle. Boutheina observes that the triangle area is 1.5 times of the area of the folded figure. Alex says that "it's interesting that the area of the shaded figure formed by three triangles, which is the brown one on the right, is exactly 1 cm^2."



Eui is tasked with calculating "What is the area of the blue triangle?"

Discussion:

(a) What is the difference between the second and third figure

(b) What is the difference between the first and the second figure?

Problem 1.37 (C)

There are 10 bags of coins. One of them contains only counterfeit coins, while other contains only real coins. Each counterfeit coin is 1 gram lighter than a real coin. You have two-pan balance scale with a display that shows the difference between two loads, in grams. Find the counterfeit coin in just one weighing. You are allow to take any number of coins from any bag.

Discussion: Let reduce the problems to:

(1) 2 bags

(2) 3 bags

(3) 4 bags

Problem 1.38 (C)

Prove that the regions formed by 5 circles in a plane can be coloured with black and white such a way that any two regions that share a common arc boundary will be coloured differently.

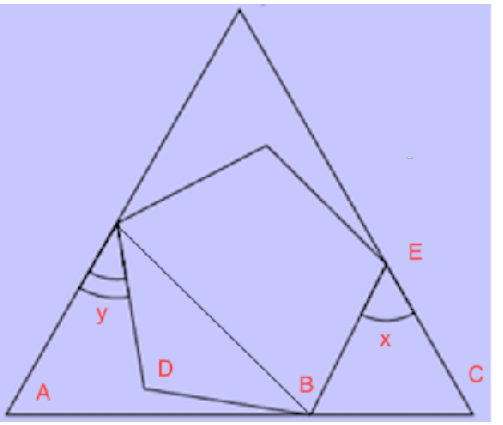
Discussion: Let reduce the problems to 1 circle

(1) Add one more circle, what cases do you have?

(2) Let add another one and analyze the cases. Can you generalize?

Problem 1.39 (C)

The diagram below shows an equilateral triangle and a regular pentagon that touches all three sides of the triangle.



Find $x + y$ in degrees.

Discussion:

(1) What is $\angle EBC + \angle ABD$?

(2) How $\angle EBC$ is related to x?

(3) How is $\angle ABD$ related to y?

Puzzle 1.40 (Linguistic Challenge, NACLO, by Ronnie Sim): Swahili of Eastern Congo

Swahili belongs to a large language family, called Bantu. Bantu languages are spoken by more than 100 million people in Southern and Eastern Africa. Swahili is the mother tongue of about 5 million people, and is the common language of trade along much of the east coast of Africa. However, the variety of Swahili represented in this puzzle is spoken hundreds of kilometres inland, and is quite different from that which is spoken near the coast.

Read the first ten Swahili words and their English translations in the table below.

1. Ninasema.	I speak.'
2. Wunasema.	You speak.'
3. Anasema.	She speaks.'
4. Wanasema.	They speak.'
5. Ninaona.	I see.'
6. Niliona.	I saw.'
7. Ninawaona.	I see them.'
8. Niliwuona.	I saw you.'
9. Ananiona.	She sees me.'
10. Wutakaniona.	You will see me.'

Translate the following three English statements into Swahili.

- She saw them: _____
- I will see you: _____
- She saw me: _____

Chapter 2

Logic

Examples

Problem 2.1 (A)

(a) A big fat Indian and a small thin Indian were sitting outside a teepee, each smoking a pipe. The little Indian was the son of the big Indian, but the big Indian was *not* the little Indian's father. How come?

(b) A visitor points to a portrait on the wall and asks who it is. "Brothers and sisters have I none," says the host, "but that man's father is my father's son." Who is pictured?

(c) How many people is "two pairs of twins twice"?

Solutions*:* The purpose is to show that reasoning cannot be based on *fact-less assumptions*.

(a) The big Indian is the … *mother* of the little Indian. A person who is a *parent* and *smoke* is not necessarily a man. *Hasty conclusion* is what people used to do when they draw a conclusion based on an event with high probability (but not definite). There is a chance, whatever small it is, that the conclusion is not true.

(b) The man is … the host's *son.* "That man's father is my father's son" implies that the man is the grandson of the father of the host. Because the host does not have any brother of sister, so that man has to be his son.

(c) Two (2) pairs (2) of twins (1) twice (2) $= 2 \times 2 \times 2 = 8$ people. *Twin is a person*, not two.

Note: The key is to identify the *right term,* and in some cases the *right quantity*.

Problem 2.2 (A)

While visiting the Knights (who always tell the truth) and Liars (who always lie) Island, Huy passes a beautiful garden where three islanders, Samuel, Tony, and Zecharia are watching the sunset. He asks Samuel, "are you a knight or a liar?". Samuel is shy, Huy cannot hear his quiet answer. So he ask Tony, "what did Samuel say?" Tony answers, "he said that he is a liar." "Don't trust Tony, Tony is a liar!" screams Zecharia.

What do you think, who is what?

Discussion: There is an usual approach this problem, which is based on two principles:

(1) Analyze possible cases and use *process of elimination* to eliminate impossible cases.

(2) *Find an insight or a logic* that leads us to a few number of cases.

Solution: Because Samuel says nothing, so we start with Tony's statement: "*Samuel said that he is a liar*". This simply cannot be true. Why, let use proof by contradiction: assume the opposite and prove that it is wrong. Let assume Tony is right. Samuel cannot be a truth teller because no truth teller states that he is a liar. On the other hand, Samuel cannot be a liar because no liar admits he is a liar. Therefore Tony's statement contradicts itself: it is a lie. Hence Tony is a liar, Zecharia is truth teller because he told Huy that Tony is a liar. What about Samuel? Tony lies about Samuel. So whatever he said about Samuel makes its opposite true. So Samuel is a truth teller.

Note: Identify the *self-contradictory sentence is a great technique*.

Problem 2.3 (A)

A certain small island is inhabited by rabbits of three kinds: white, grey, and spotted.

- Grey mother-rabbits always have two bunnies in their litters: a white and a grey one.
- The same is true for spotted mother-rabbits.
- However, a white rabbit always give birth to three bunnies: a grey one, a white one, and a spotted one.

The first and only rabbit to ever reach the island was a rabbit who is called Great Mother by the rest of the population. Currently, 40 grey, 40 white, and 30 spotted rabbits live on the island.

If no rabbit ever died or left the island, can you tell the colour of the Great Mother?

Note: the problem assumes that there is only one mother-rabbit.

Solution: *Identify the statement that stands out*

(1) If the Great Mother was white, when she arrived to the island, the number of white rabbits is one more the number of grey rabbits.

(2) Similarly, if the Great Mother was grey, the number of grey rabbits is one more the number of white rabbits.

(3) However, the number of grey and white rabbits is the same. It means that the Great Mother rabbit should be a spotted one.

Problem 2.4 (B)

(a) Detective James Smarties walked outside through a pouring rain for half an hour without getting a single hair on his head wet. He did not wear a hat, carry an umbrella, or hold anything over his head. His clothes got really soaked. How could this be possible?

(b) A dog and cat together weigh 27 pounds. If the dog's weight is an odd number, and if he weighs twice as much as she does, how much does each weigh?

(c) When their blind dates arrived to take them to a football game, Connie and Jennie was astonished to see that the two young men looked exactly alike. One of them explained "we are brothers," he said, "we were born on the same day of the same year from the same parents". He added "but we are not twins". How could it be?

Solution*:*

(a) Having no hair getting wet or having *no hair*? It's simple: he is *bald*.

(b) What is he and what is she? If we do not assume that *he* in the second sentence supposes to be the dog, then it's easy. The cat weighs twice the dog, the sum of their weights is 27. So the cat's weight is 18, the dog's 9 (and this is an odd number).

(c) What can be same to twins and *more than* twins: triplets, quadruplets, quintuplets, sextuplets, septuplets, or more! In 2009 the record was set when an octuplets were born in Bellflower, California.

Note: Identify the *logical assumption*, not the writing.

Problem 2.5 (B)

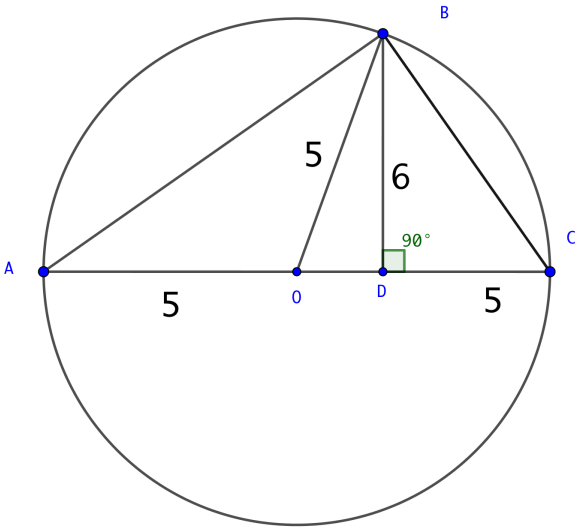
The hypotenuse of a right triangle is 10 inches, and the altitude dropped to it is 6 inches. What is the area of the triangle?

Note: try to draw a right triangle with a 10-inche hypotenuse and a 6-inche altitude from the vertex at the right angel. What do you see?

Solution: Let investigate the original problem by drawing a circumcircle of the right triangle with the length of hypotenuse is 10 inches. What is the role of the hypotenuse in the circumcircle of the right triangle? Show an example where the altitude is 6 inches if the hypotenuse is 10 inches.

Using a formula is simple, you can just calculate the area of the triangle by using the area formula, and obtain 30 square inches as the area of the triangle. However, there is no such a triangle, which makes the whole calculation meaning less.

Note: *Understanding a formula is different. Knowing how and when a formula can be used requires an understanding of the problem nature. For example calculating* $4 \div 2$ *is simple, but* $4 \div x$ *when* $x = 0$ *does not make any sense.*



Problem 2.6 (B)

The inhabitants of Truth Village always tell the truth, while those of the hamlet of Liars, of course, always lie. On the way to visit the Truth Village, you arrive at a T-intersection, where one road leads to the Truth Village, the other one to the hamlet of Liars. There is no sign, except a single person. You have to ask but you do not know whether the man is a citizen of the hamlet of Liars or is from the Truth Village. You are allowed to ask a single question, to learn how to get to the Truth Village. What is your question?

Discussion:

(a) What is the relationship of a person to where that person comes from? What would that person says about where he/she's from? What would then be your question?

(b) Obviously there are multiple possibilities, but not that many. Can you draw a table and show all the cases?

Note: It's a *casework* and listing all the cases carefully does help.

Solution: *Can you point to the village where you come from?* is the question. Both truth tellers and liars will point to the direction of the Truth Village, which is what the you want. Knowing what the person is does not help to solve the problem. Knowing what you want is more important.

Problem 2.7 (C)

Bình was driving through Ottawa when he decided to rest. He stopped at a motel, got a room, and asked the reception girl where he can find a barbershop. "We have two barbers nearby in Ogilvie shopping malls," said the girl. "One of them is at the north entrance and the other close to the south exit."

Bình went to the one next to the north entrance. It looked like as if it had not been cleaned in months. Cut hair was all over the floor. The barber himself would have needed a shave and his haircut looked really terrible.

Bình then walked to the other one. It looked neat and cheerful inside. The floor had been swept. The barber was neatly dressed, freshly shaved, and had a neat haircut.

Why did Bình walk back to the *first* barbershop to get his haircut?

Discussion*:* This problem is actually simple.

(1) Why the first barbershop was not well cleaned, and more importantly the barber had a terrible haircut.

(2) Who would do that to the barber?

(3) Why the second barbershop was so clean?

(4) The barber was so rich to hire somebody to clean the shop?

(5) Why his haircut was so good? Who did that?

Solution:

(a) The key point perhaps is that who did the haircut for the first barber. He definitely could not do that to himself. So that should be the second barber. The haircut was terrible.

(b) And the rest just follows: the person who was busy does not have time to clean up, or the person who has to trash has no need to clean up. In addition he can does himself a favour buy giving himself a shave.

Note: An important conclusion is that thing is not what they look like. Don't fall for some nice looks, great promises, or just kind words. Lots of times, they are just fakes.

Problem 2.8 (C)

The people from the city *No-one-can-be-sure-of-any-thing* are of three different types: those who always tell the truth, those who always lie, and those you *alternate* lie and tell the truth. Quynh Chi meets one of the local people. She is allowed to ask twice so that she can know which group of people that one belongs to.

What are the questions?

Discussion*:* We need to ask a question that the truth tellers and the liars give different answers, and each person gave the same answers twice. The one who alternate gives different answers for the same question!

The *number of outcomes* for two answers determine number of types. Why?

Solution: The point is how to have one question that can help to distinct three different kinds of people: the truth tellers, the liars, and the alternators. Fortunately a single question can do so. Any question whose answer is an universal fact, for example *is one plus one equal two?*

The truth tellers always answer with *True* twice, liars always say *False* twice, alternators answer *True* once and *False* once.

Problem 2.9 (C)

On the island of Knights and Liars, knights and liars live in separate villages. However they often visit their friends in other villages.

This week, the one and only newspaper of the island announced a contest: to come up with a single yes/no question that would allow a tourist to figure out whether he is in a knights' village or a liars' village. The question should be such that it can be addressed to anyone the tourist sees. Of course it is not known in advance that that local person is a knight or a liar, is he from the village or just a visitor. Lilian won the prize.

What was her question?

Discussion*:* Think about a question that you want to know if a person belongs to the place. Draw a table of what the person possibly is and what would be the answer. Based on that you can figure out where that persons's from. *It is important sometime to categorize the information because that would be enough.*

Solution: *Do you live in this village?* is the question that helps. It does not matter who answer the question. *Yes* means this is the knight's village and *No* means this is a liars' village.

Puzzle 2.10: Counting

There is a secret *word* hidden in this matrix of letters. Can you discover it?

R	V	E	O	V	C
S	I	O	V	R	D
V	E	R	C	V	O
R	O	V	E	S	E
E	R	S	C	R	I
C	E	R	E	O	R

Hint:

(a) It is not a usual puzzle, so think out-of-the-box

(b) Look at the title

Exercises

Problem 2.11 (A)

Four students of the Math & Chess Club having a conversation about their favourite food.

- Boy A: Jennie likes fish
- Boy B: I likes cakes, but I am not called Anthony
- Girl C: One of the boys - Anthony and Khang - like chicken
- Girl D: Barbie likes salad, but I don't

Can you figure out who like what?

Discussion: First, identify the names of the girls and the boys.

Solution:

- Since Boy B said he is not Anthony, so he is Khang and Khang likes cakes.
- Boy A should be Anthony and from what girl C said, Anthony likes chicken.
- Jenni should be girl D, and Jennie like fish.
- Barbie is girl C and she likes salad.

Problem 2.12 (A)

Khang plays a game of Ping-Pong in the back yard? He misses the ball. It bounces across the lawn and rolls into a small but deep hole. The hole goes down too far for him to reach the ball with his hand, and the hole bends so much to one side that he cannot get the ball by poking a stick into the hole.

How can Khang get the ball?

Discussion:

(1) If you never hold a Ping-pong ball in your hand, try to get one before trying to solve this.

(2) How can you *put* something into the hole and that can *push* the ball up?

Solution: The ping-pong ball floats over the water, so pour the water into the hole. This is not a strictly math problem. It is a problem to test if you have any relationship with the reality surrounding you. The solution is sometime just from common sense, from physics, or else. It is important to know that solutions can come from very different sources.

Problem 2.13 (A)

Catherine and Clemence got a nice cake from their mother. They were squabbling over who gets to cut the cake. The reason was that each of them thinks the other will cut it in a way to give herself the larger portion. Seeing this, their dad recommended a clever idea and both of them immediately agreed.

What would be the idea?

Discussion*:*

(1) The reason is trust, or lack of it.

(2) Lack of trust if one of them does both: *cuts the cake* <u>*and*</u> *choose her portion*.

(3) Now you know what can be recommended.

Solution: Human tragedy is giving someone to much authority and then regretting it. In this simple case, the solution is also very simple: do not give the girl who cut the case too much authority. She can *cut the cake* or *choose her portion* but not both! Once this is enforced, trust will be reestablished.

So let the senior girl cut the cake and the younger one choose her own portion!

Problem 2.14 (B)

Cindy has two treasure chests, each of them filled either with gold or silver coins. One of them has a label, "I contains gold coins, and the other one contains silver". The other is labelled with "One of us contains gold, while another contains silver". Cindy told Andy that one of the labels is correct and the other is not.

Can Andy figure out which one is correct?

Discussion*:* Draw a table with two columns for two chests, use what Cindy told Andy

(1) Assume that the first label is correct, fill what you think into the first row.

(2) Assume that the second label is correct, fill what you think into the second row.

Solution: The second answer - label - is always true. It's a universal true. The first one is false.

Problem 2.15 (B)

A school teacher in Ottawa had her purse stolen. The suspects were narrowed down to: Loo, Joo, Doo, Too, or Moo. When questioning, each child made three statements:

- Loo: (1) I didn't take the purse (2) I have never in my life stolen anything (3) Too did it
- Joo: (4) I didn't take the purse (5) My daddy is rich enough, and I have a purse of my own (6) Moo knows who did it.
- Doo: (7) I didn't take the purse (8) I didn't know Moo before I enrolled in this school (9) Too did it.
- Too: (10) I am not guilty (11) Moo did it (12) Loo is lying when she says I stole the purse
- Moo: (13) I didn't take the purse (14) Joo is guilty (15) Doo can vouch for me because he knows me since I was born

Later, each child admitted that two of his/her statements were true and one was false.How did Detective James Smarties solve this case?

Discussion*:*

(1) Find the one who has two sentences that both can be true, regardless whatever others said.

(2) Based on this, build a chain of reasoning, for example (1) → (4) → (8) . . . in order to clarify what known and what not.

Solution:

(1) First, it can be easily verified that both (10) and (12) are True, which means (11) is False

(2) (10) is True so (9) is False. (9) is False so (7) and (8) are True.

(3) (8) is True so (15) is False. (15) is False so (13) and (14) are True. So Joo did it (15).

Problem 2.16 (B)

N	E	W		D	O	O	R

Can you rearrange the letters to form one word in the space provided?

Discussion*:* Make sure that you understand what the problem asks.

Solution: The problem asks for forming *one word*, literally.

N	**E**	**W**		**D**	**O**	**O**	**R**

O	**N**	**E**		**W**	**O**	**R**	**D**

Problem 2.17 (C)

Vũ has a dream job as an interplanetary courier at the Orion spaceport. He is responsible for transporting passengers from the ground of the spaceport to the spaceship on a high orbit. His shuttle can carry only two persons: the pilot himself and one more passenger. The spaceship has to wait until all passengers are boarded. Usually it is a simple job. Until one day he has three unusual passengers: a vegetarian Sagittarian, a voracious carnivore Ursan, and a weird-looking quadrupedal creature called a Terrestrial.

(a) The Sagittarians and the Ursans are at war at the moment, so leaving them alone on the ground or in the spaceship can cause an intergalactic incident.

(b) The hapless Terrestrial can be devoured by the Ursan in a second if nobody is around.

Can you help him to get all three passengers into the spaceship without any incident?

Discussion: The key is to *find the pair that can peacefully co-exist*.

(1) Obviously (Sagittarian, Ursan) or (Ursan, Terrestrial) not such a pair.

(2) The Ursan has to be transported such that it is alone on the ground, or in the spaceship.

(3) When Vũ cannot leave two of them somewhere, Vũ can always take one into the shuttle. Vũ can switch two between the ground and the shuttle or the spaceship and the shuttle Can Vũ make it in maximum seven trips?

Solution: It's a *wolf-goat-cabbage puzzle* in a new context. The solution is simple:

(a) Get the Ursan up, come down

(b) Get the Sagittarian up, take the Ursan down

(c) Get the Terrestrial up, come down

(d) Take the Ursan up

Note: this solution might not work nowsaday, since Terrestrial (human) consume so much vegetables so an Sagittarian might be seen as competitor!

Problem 2.18 (C)

All inhabitants of the Knights and Liars Island participated in a survey. In this survey, some of the islanders claimed that the number of knights on the island is even. The rest of the islanders said the number of the liars on the island is odd.

Prove that the total number of knights and liars living on the island is even.

Discussion:

(1) Split all islanders into two groups according to their survey statements.

(2) First, prove that all people belong to the same group should have the same affiliation: either all are knights or all are liars.

(3) Create a table for two groups. Go through all four options of who in each group

Solution:

All the islanders are split up into two groups according to their survey statements. All the people in the same groups with have same affiliation: either they are knights or liars.

	Group 1	Group 2
Option 1	Knights	Knights
Option 2	Knights	Liars
Option 3	Liars	Knights
Option 4	Liars	Liars

Option 1: The number of knights is even, no liars. Nobody in group 2

Option 2: The number of knights is even, even number liars in group 2.

Option 3: The number of knights is odd, so it the number of liars.

Option 4: The number of knights is 0, the number of liars is even.

From the analysis above, the number of knights and liars is even.

Problem 2.19 (C)

Pax has been selected for a game show that offers the chance to win a brand new expensive sport car. The car sat behind one of the three doors, monkeys resided behind the other two. First, Pax has chosen a door, the show host opened one of the remaining two doors, revealing a monkey.

The host then offer Pax a choice:

(a) stick with his original choice, or

(b) switch to the other, still-unopened door.

What did Pax do? Why?

Discussion: The problem might look difficult, however it is not that complicated.

(1) You just need to calculate all the possible cases if Pax chose to stick or switch.

(2) Find out the total number of successful outcomes (when he got the car) and the total number of outcomes for each choice.

(3) Now compare the probabilities of the success in both choices and compare them.

Solution: Pax should switch, the probability is higher. This is the well-known Monty Hall problem.

Puzzle 2.20: Description

Find rule of the sequence and determine the missing term in the box

1, 11, 21, 1211, 111221, $\boxed{}$

Hint:

(a) It is not a usual puzzle, so think out-of-the-box

(b) Look at the title

Problems

Problem 2.21 (A)

Three turtles are crawling along a *straight road* heading in the *same direction*. "Two other turtles are behind me," says the first turtle. "One turtle is behind me and one is ahead," says the second. "Two turtles are ahead of me and one other is behind," says the third turtle.

How can this be possible?

Discussion:

(1) Where is the first turtle?

(2) Where is the second turtle?

(3) Where would be the third turtle? Does what he say make sense?

Problem 2.22 (A)

Three boxes contain some balls. There are two white balls in one box, two black balls in another box, while the third box contains one white and one black ball. The boxes are labeled. On one box the label says WW, on the other it says BB, and the label on the third box is BW. None of the labels describes the contents of the box correctly.

How is it possible to decide which box contains what by *taking exactly one ball out of a box*?

Discussion:

(1) The key is the BW box. What should it contain?

(2) If one ball is taken out of that box, how many possible cases are?

(3) Can you determine the boxes by working out all the possible cases?

Problem 2.23 (A)

The zoos in the Blah Blah Land share some common properties:

(a) If a zoo has both hippos and rhinos, then this zoo has no giraffes

(b) Every zoo has a hippo or a rhino. Some zoos have both.

(c) If a zoo has both a hippo and a giraffe, then this zoo also has a rhino.

The zoo in the capital of Blah Blah Land has a giraffe. Does this zoo has a rhino? How about a hippo?

Discussion:

(1) The second condition says that the zoo in the capital has a hippo or a rhino.

(2) If it has a hippo, then what else should it have according to (c)? If that the case, according to (a) what it should not have?

(3) So beside giraffes, what other animals should that zoo have?

Problem 2.24 (B)

Hobbits, elves and gnomes live in the Kingdom of Godor. The inhabitants are abide by several ancient rules:

(a) If the village is inhabited both by hobbits and elves, the gnomes should not settle in it.

(b) If elves but no gnomes dwell in a village, then hobbits should reside in this village as well.

(c) If both hobbits and gnomes live in a settlement, this settlement should have elf resident as well.

Bilbo the Hobbit would like to establish a "Hobbit Only" retreat near the famous Godor hot springs.

Would he break any ancient rules by establishing this retreat?

Discussion: Assume that Bilbo the Hobbit established such a village. Carefully verify each rules. What do you think?

Problem 2.25 (B)

The son of a professor's father is talking to the father of the professor's son, and the professor does not take part in the conversation.

Is this possible?

Give an example if you think so.

Discussion*:*

(1) Is "son of a professor's father" necessarily the professor?

(2) Is "father of the professor's son" necessarily the professor?

(3) Can you have an example for (1) and another one for (2)

Problem 2.26 (B)

Once upon a time, there was a tribe for which the hippopotamus was sacred. The Chief of the tribe cared for the pet hippo which he fed and coddled incessantly.

Once a year, the Chief loaded his hippo and his Collector of Offerings on a boat and headed up the river to a hut where all of his people must bring their annual offering. The hut contain a huge beam balance. The hippo was placed on one side, and gold ingots offered by the tribesmen were placed on the opposite side until the sides balanced.

One year, the hippo had grown so fat that he broke the beam balance. The Chief grew incensed and demanded that his Collector of Offerings figure out a way to measure the year offering. "If I do not have my offerings by sundown, your head will be mine!" cried the Chief.

The Collector of Offerings thought and thought, and by the end of the day had a solution.

Can you figure out what he thought of?

Discussion:

(a) Perhaps you already know this problem as how to weigh an elephant.

(b) If not, think about the Chief transported the hippo on his boat, the heavier the hippo was, the water line on the outside of the boat's hull got higher.

Problem 2.27 (C)

A box contains marbles of different colours.

How many marbles are there if all of them are red except two, all of them are green except two, all of them are blue except two.

Discussion: Let try a casework with the number of blue marbles: how many possible?

Problem 2.28 (C)

Three tablespoons of milk from a glass of milk are poured into a glass of tea with the same size, and the liquid is throughly mixed. Then three table spoons of this mixture are poured back into the glass of milk.

Which is greater now: the percentage of milk in the tea of the percentage of tea in the milk?

Discussion:

(a) Let assume originally there are each glass of milk and tea contains 10 tablespoons.

(b) You can try to apply algebra or

(c) Think how many tablespoons of milk and tea remain in each glass.

Problem 2.29 (C)

The distance between Ottawa and Kingston is 200 km.

Two cyclists leave from Ottawa and Kingston simultaneously travelling towards one another, one at a speed of 23 km/h, and the other at a speed of 27 km/h. A fly leaves Ottawa together with the first cyclist, and flies towards the second at a speed of 100 km/h. The fly reaches the second cyclist, touches his forehead, the flies back to the first, touches his forehead, return to the second, and so on until the cyclists collide with their foreheads and squash the fly.

How many kilometres has the fly flown altogether?

Discussion:

(1) How the distance between the two cyclists decreased?

(2) How much time is needed so that this distance decrease to zero?

(3) How far the fly *flew in this amount of time*?

Puzzle 2.30: Arriving early

An engineer goes every day by train to the city where he works. At 8:30AM, as soon as he gets off the train, a car picks him up and takes him to the plant. One day the engineer takes a train arriving at 7:00AM, and starts walking toward the plant. On the way, the car picks him up and he arrives at the plant 10 minutes early.

When does he meet the car?

Hint:

(a) Draw a diagram to show how the engineer travels in both cases.

(b) What is that 10 minutes?

Challenges

Problem 2.31 (A)

You are on the Island of Knights and Liars. There was a crowd of people on a plaza, and each person in the crowd declares "You all are liars". How many knights are there?

Discussion:

(1) Would there be no knights?

(2) Would there be at least 2 knight?

Problem 2.32 (A)

In a country there are three towns:

(1) Blah-Blah, where everyone tells the truth.

(2) La-la-la, where everyone always lie.

(3) Nice-nice, where everyone alternate tell the truth and lie.

Once a fire broke out in one of these towns, and the fire department - which is stationed outside of all the towns - is called. Here the conversation:

"There is a fire here"

"Where is it?"

"In Nice-nice."

Where should the firefighter go?

Discussion: Put yourself in the shoe of callers from each city.

Problem 2.33 (A)

A teacher drew several circles on a sheet of paper. Then he asked a student "How many circles are there?" "Seven," was the answer. "Correct! So, how many circles are there?" the teacher asked another student. "Five," answered the student. "Absolutely right!" replied the teacher.

How many circles were really draw on the sheet?

Discussion:

(1) Is it necessary that both students saw the *same page*?

(2) If not, what would be the situation?

Problem 2.34 (B)

How can you build a house that has a windows in all four walls but every window faces south.

Discussion: The actual question would be *where* can you build such a house?

Problem 2.35 (B)

Sasha likes pets.

(a) All but two of her pets are parrots.

(b) All but two are cats.

(c) All but two are dogs.

(d) All the rest are crickets.

How many pet crickets does Sasha have?

Discussion:

(1) Minimum how many pets Sasha has?

(2) Can she has 4 pets, what does what mean?

(3) Could she have no cricket? Could she have only crickets?

Problem 2.36 (B)

A king wants his daughter to marry the smartest of 3 extremely intelligent young princes, and so the king's wise men devised an intelligence test.

The princes are gathered into a room and seated, facing one another, and are shown 2 black hats and 3 white hats. They are blindfolded, and 1 hat is placed on each of their heads, with the remaining hats hidden in a different room.

The king tells them that the first prince to deduce the colour of his hat without removing it or looking at it will marry his daughter. A wrong guess will mean death. The blindfolds are then removed.

You are one of the princes. You see 2 white hats on the other prince's heads. After some time you realize that the other prince's are unable to deduce the colour of their hat, or are unwilling to guess. What colour is your hat?

Discussion:

(1) Make an assumption about your hat.

(2) List the possible cases and reasons why other princes cannot guess.

Problem 2.37 (C)

At a festival on the Island of Knights and Liars, seven local people are sitting around table, and each of them says: "one of my two neighbours is a liar and the other is a knight". Who is sitting at the table?

Discussion:

(1) If there were one knight K1, then next to that knight is a liar L1 and another knight K2. Next to the knight K2 is a liar L2.They will appear in this order: L1 K1 K2 L2. Now think about the two who can be on the left of L1, and on the right of L2. Who will sit in between that two? Is that possible?

(2) If there were no knights, what does that mean?

Problem 2.38 (C)

Todd lives in a magical pond, where all the toads and frogs talk. One of the species always says the truth, while the other one always lie. Figure out the species of each talker.

Todd: My neighbour Tammy is a frog.

Tammy: My neighbour Todd and Ted are both toads.

Ted: I am a frog or a toad.

Terry: Todd is a toad

Discussion:

(1) Assume Todd is a frog. Investigate what would be Tammy.

(2) Would that be contradictory to what Tammy said?

Problem 2.39 (C)

Once upon a time, prince Kalama of Malaka went to ask princess Paqa of Qapa to marry him. The princess listened and decided to put him to a test. She brought him to the throne hall, where there were two doors. A sign on each door said: "Both rooms behind these doors are empty". She informed the prince that each of the rooms is either empty or contains a tiger. In addition, she said "If the first room is empty, then the door sign is true; otherwise, it is false. At the same time, if the second room is empty, then the sign on the door is false; otherwise, it is true."

The princess invited the prince to open the door of his choice. If a hungry tiger jumps out of the door, he will die. However, if he opens the door to the empty room, he will get Paqa's hand, and with her, the whole kingdom of Qapa.

The princess is known as truthful. So what should prince Kalama do?

Discussion:

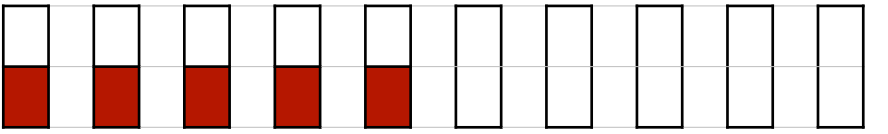
(1) First, let observe that the signs said *both rooms are empty*. Let assume they are true.

(2) But then the princess said *if the second room is empty, then the sign on the door is false; otherwise, it is true*.

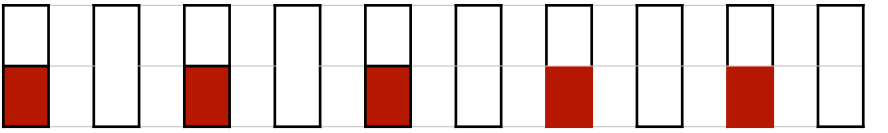
(3) So can you now conclude if the signs are telling the truth and decide which door to open?

Puzzle 2.40: Moving glasses

Ten glasses form a straight line. Five of them on the left are filled with wine, five of them on the right are empty.



What is the minimum number of actions can you do to obtain the below setting:



Hint:

(a) Think out-of-the-box

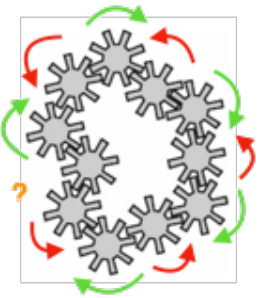
(b) The number of actions can be really small.

Chapter 3

Parity

Examples

Problem 3.1 (A)



Eleven gears are contracted as in the picture below. Can all the gears rotate at the same time?

Discussion: Let assume we number all the gears, and *the first gear rotates clockwise*, then *the second gear must rotate anti-clockwise.*

Solution: Following the discussion: all the odd numbered gears must rotate clockwise and all the even numbered gears rotate anti-clockwise. But then the first and eleven ones must rotate in the same direction, which is impossible.

Problem 3.2 (A)

(a) The Fibonacci sequence begins with 1, 1, 2, 3, 5, and so on. Each term of the sequence after the second term is the sum of two terms before it. How many of the first 50 terms of this sequence are odd numbers?

(b) How can Billy make change of \$20 bill, using nine bills each having a value of \$1, \$5?

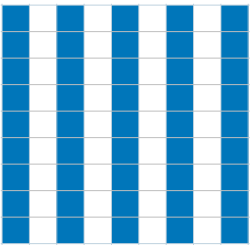
Discussion: If we observe the parity of the terms, easy to see that it is odd, odd, even, odd, even, odd, …

1	1	$1+1=2$	$1+2=3$	$2+3=5$	…
odd	odd	even	odd	even	…

The sum of two odd numbers is an even number, and the sum of an odd and an even number is an odd number.

Solution: From the discussion it shows that except the first two terms, the parity of the terms alternate. Hence the number of odd numbers is $2+\dfrac{50-2}{2}=2+24=26$. The sum of nine odd numbers is an odd number. 20 is an even number. There is no way to make change of \$20 bill.

Problem 3.3 (A)



On each square of a 9×9 board sit a snail. When Hanna plays a sound, each snail moves along the diagonal to a square sharing one corner with the current one. Of course now some squares have more than one snails and some squares have none.

What is the least possible number of empty squares?

Solution:

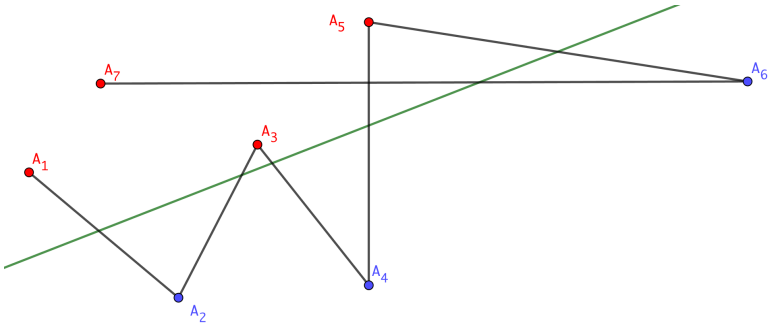
Let colour the board column alternately with two colours black and white as above. Easy to check that the board has 45 black and 36 white coloured squares. Because the snails moves diagonally, so snails in white squares shall be in black squares and vice versa. Since *snails in 45 black squares move to white squares and only snails from 36 white squares move to black square, the number of empty squares shall at least be 9*.

Note: the parity (colours of the squares, where the snails are, change when they move)

Problem 3.4 (B)

A closed path is made up by 11 line segments. Can one line, which does not contain any vertex of the path, intersect all 11 segments?

Discussion: A line divides the plane into two distinct parts. If the line intersects a segment, then two vertices of the segment will be on two different parts of the plane.



Solution: Assume that there is a line intersects all 11 segments. Let number the points along the path as 1, 2, … , 11. Easy to see that the odd numbered points must be on the same part of the plane. See the figure below for an illustrated example when there were 7 points.

This mean *the parity of the numbers on the same part of the plane* (split by the green line) *is the same*, but this is not possible if the line intersect the segment connected 1 with 11.

Problem 3.5 (B)

Anna writes six 0's and five 1's on a paper. She begins to cross pairs of digits: either two 1's, two 0's, or a 1 and a 0. If the digits crossed out are the same, then she writes a new 0, otherwise a 1. She continues to do so until there is only one digit remains.

What is that digit?

Discussion:

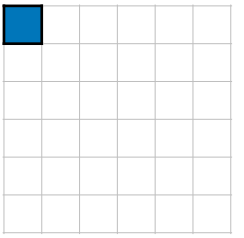
(1) if the pair is (0, 0) or (1, 1) then the new digit is 0

(2) if the pair is (0, 1) (order does not matter here), the new digit is 1.

Solution: After every turn, it is clear that the number of 1's can be reduced by 2 or remains the same. This means the *the parity of the number of 1's remains the same*. At the beginning, the number of 1's is 5, which is an odd number. So this can be only changed to an odd number. At the end, there is only one digit remain, it must be 1.

Problem 3.6 (B)

In a 6×6 chart all but one corner black square are painted white - the diagram below. You are allowed to repaint any column or any row in the chart: i.e., you can select any row or column and change all squares on that row or column to the opposite colour.



In how many operations can it be changed to an entirely white chart?

Discussion: An operation on a row or a column changes all squares to opposite colours. If the number of black squares of that row or column is *odd* or *even* before the operation, then it remains *odd* or *even* after the operation. For example if the first column of the original square is re-painted, then the numbers of black square now is $6 - 1 = 5$

Solution: Based on the observation the number of black squares in the entire chart is always odd, regardless how many operations are done. Therefore it is impossible to change the board into an entirely white coloured chart - which contains 0 - *even* number of black coloured squares.

Problem 3.7 (C)

(1) John and Pete have three pieces of paper. Each of the boys picks one piece, tears it up, and puts the smaller pieces back. John only tears a piece of paper into 3 smaller pieces while Pete only tears a piece of paper into 5 smaller pieces. After a few minutes can there be exactly 100 pieces of paper?

(2) You are cutting a loaf of a bread with a knife. What is the *maximum* number of pieces of bread that you can get if you are allowed to make only three straight cuts?

(1) **Discussion**: Simple to see that the number of pieces are increased by 2 or increased by 4.

Solution: According to the observation, *the parity of the number of pieces* is not changed. So it will remain odd, since the original number is 3. Hence 100 pieces can never be achieved.

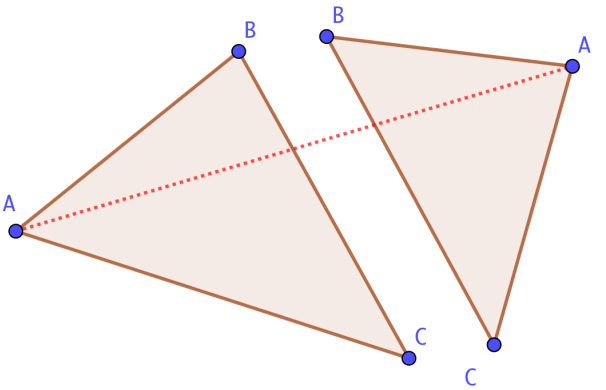
(2) **Discussion**: With each cut, the *maximum* number of pieces can be increased by 2.

Solution: If all the cuts are vertical, then you cannot get more than 7 pieces. Instead, combine a *horizontal* cut with two *vertical* cuts, perpendicular to each other. The result is a 8 pieces.

Problem 3.8 (C)

Three hockey pucks, A, B, and C, lie on a playing field. A hockey player hits one of them in such a way that it passes between the other two. He does this 25 times. Can he return them to the original positions?

Discussion: Assume $\triangle ABC$ is *clockwise* (we can also assume the opposite, of course). It is easy to see that if A puck is hit such that it passes between B and C, then $\triangle ABC$ direction is changed to anti-clockwise.



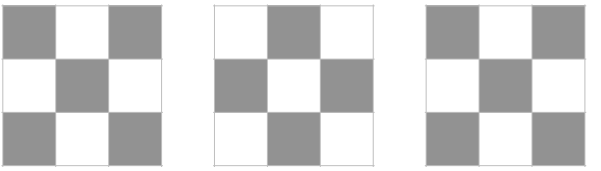
Solution: We can say, *the parity of the number of the moves is reflected by the direction of* $\triangle ABC$ *triangle*. After an *odd* amount of moves, the direction of the triangle is the opposite of the original direction, and therefore cannot be the same.

Problem 3.9 (C)

A cook has a $3 \times 3 \times 3$ cube of cheese containing 27 unit cubes. The central unit cube is poisoned. A very clever mouse is found the cube and start to gnaw. When the mouse is done with a unit cube, it moves to the next unit cube that shares a face with the one it just finished.

Will the mouse finished the entire cube except the central one?

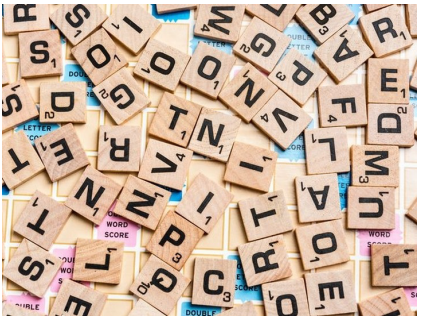
Discussion: Let colour the unit cubes alternately and layer by layer as below.



Solution: First, let count the number of unit cubes by colours. It easy to see that the number of the black ones is 14 and the number of the white ones is 13. Since the mouse wants to avoid the central one, it must eat 14 black and 12 white cubes. However, because *the colours of the unit cubes that the mouse eat changes each step from black to white or from white to black*, so the difference between the number of black and white cubes the mouse can eat is 0 or 1.

This means that the mouse cannot finish the entire cube, except the central one.

Puzzle 3.10: Magic word

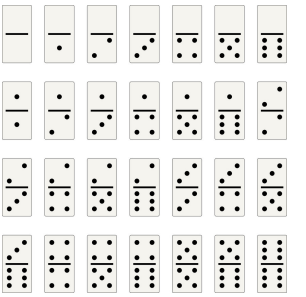


Using Scrabble letters, show on the right side, try to build a word following this instruction "*it is a word that contains itself six times*".

Exercises

Problem 3.11 (A)

Dominoes game: Each domino is a rectangular tile with a line dividing its face into two square ends (the figure on the left, below)



All dominoes in a set are laid out in a chain (the figure in the middle, above) so that the number of spots on the end of adjacent dominoes match.

If one end of the chain is a 5-spot, what is the other end? Why?

Discussion: Think about the number of 5-spots.

Solution: The number of dominoes containing a 5-spots is 8 (the figure on the right, above)., which is an *even* number. According to the rule, the 5-spots, if not at the end of the chain, must be *paired*.

$$5 - \ldots - \underbrace{5-5}_{paired} - \ldots - \underbrace{5-5}_{paired} - \ldots - ?$$

This means that if one-end of the chain is a 5-spot, the the other end of the chain must also contains another 5-spot.

Problem 3.12 (A)

Can a 5×5 square be covered by 1×2 dominoes?

Discussion: Think about the colours of the small squares.

Solution: If a 5×5 square is alternately coloured as above, the number of black squares is 13 and the number of white squares is 12. A domino has one black and one white squares. If a 5×5 square can be covered by 1×2 dominoes then the numbers of white and black squares should be equal. This is impossible.

Problem 3.13 (A)

Aladdin got a book with 96 sheets. The pages are numbered from 1 to 192. Muhammad the friend tore out 25 sheets out of the notebook and added all the number on the tore pages. He got 1900 as the sum. Is that possible? If yes, show an example. If not, explain why not?

Discussion*:* A sheet has two sides - pages. The numbers on the pages are consecutive numbers.

Solution: 25 sheets mean 25 consecutive numbers. Each sheet has an *odd* and an *even* page number. So the sum of the page numbers on each sheet is an *odd* number. *Adding together 25 odd numbers results in an odd number*. 1900 is an even number. It is impossible to get such a result.

Note:

(1) The sum of an odd number of odd integers is always odd.

(2) The sum of an even number of odd integer is an even number.

Problem 3.14 (B)

All natural numbers from 1 to 101 are written in a row. Can the signs "+" and "−" be placed between them so that the value of the resulting expression is 0?

Discussion*:*

(a) What is the value of $1 + 2 + 3 + .. + 100 + 101$?

(b) Does changing + to − necessarily change its parity?

Solution: In the sum $1 + 2 + 3 + .. + 100 + 101$ there are 50 even and 51 odd numbers. The sum of any number of even number is even. The sum of an odd amount of odd numbers is odd. *Changing any operation from + to − does not change the parity of the numbers*.

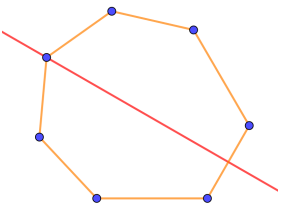
This means that whatever way the + and − signs are used to placed between the number 1, 2, …, 101, the result is still an odd number and that cannot be 0.

Problem 3.15 (B)

Given a convex 101-gon, which has an axis of symmetry, prove that the axis of symmetry passes through one of its vertices.

Discussion*:* An axis of symmetry is a line. It divides the plane into two parts.

Solution: Assume the opposite. We will prove that it contradicts the given conditions.

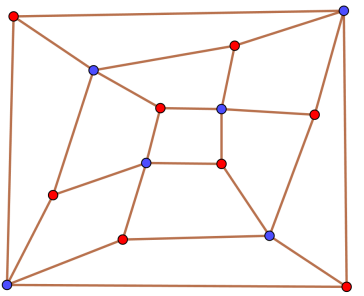


Since an axis of symmetry is a line, it divides the plane into two parts: A and B. Because the 101-gon has an axis of symmetry that does not contains any of its vertex, so each of the vertices can be on A or B.

For each vertex on A, it does have a symmetric image on B, which is also a vertex of the 101-gon and vice versa. So the number of vertices on A is equal with the number of vertices on B, meaning the number of vertices is even. This is impossible for a 101-gon. Therefore the axis of symmetry must pass one of its vertices.

Problem 3.16 (B)

The figure below shows fourteen villages connected by roads.



Is there a path passing through each village once?

Discussion: Let colour the villages by red and blue as above.

Solution: It seems to be possible to *colour the villages by two colours red and blue* such that neighbour villages are coloured differently. Any path through the villages, each once, must alternately visits villages of different colours. It is possible only if the number of blue villages is equal or differs by 1 from the number of red villages.

Since there are six blues and eight red villages, so no such path exists.

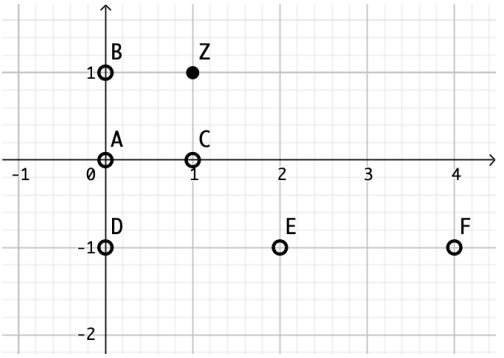
Problem 3.17 (C)

Consider three vertices $A = (0,0)$, $B = (0,1)$, $C = (1,0)$. A refection of B over A is $D = (0, -1)$, of B over C is $E = (2, -1)$, of D over E is $F = (4, -1)$. Let S be a set containing A, B, C and any subsequent reflections via any point in S like for example D, E, F.

Does S contains $Z = (1,1)$?

Discussion: Let colour lattice points (points with both coordinates are integers) by two colours. Observe how the refection can change their colours.

Solution: Let colour all the lattice points, which have at least one even number coordinate, by white. The rest are coloured by black. For example $A = (0,0)$, $B = (0,1)$, $C = (1,0)$ are all white, as well as their reflections and any subsequent reflections D, E, F, … Easy to see that any lattice point with both odd coordinates are black.



So S does not contain Z.

Problem 3.18 (C)

Selena has 101 coins. 50 of them are counterfeit, and they differ from the genuine coins in weight by 1 gram. Selena has a balanced scale that can show the difference in weights on the pans. She chooses one coin, and wants to find out whether it is counterfeit.

Will one weighing be enough?

Discussion:

(1) She can take the remaining 100 coins and split them however she likes into two groups of 50.

(2) If the difference between the weights of two groups is *odd*, then what is the coin? If the difference between the weights of two groups is *even*, then what is the coin?

Solution: Selena choose one coin and puts it aside. She divides 100 coins into two piles and weighs them.

(a) If the coin is genuine, then 50 counterfeit coins are split into two groups of n, m coins. $n + m = 50$. This means that n, m have *same parity*. The difference shows by the scale is the difference between n, m: $|n - m|$ and because of same parity, it must be an *even* number.

(b) If the coin is counterfeit, then 49 counterfeit coins shall be split into two groups of n, m coins. $n + m = 49$. This means that n, m have *different parity*. The difference shows by the scale is the difference between n, m: $|n - m|$ and because of different parity, it must be an *odd* number.

So one weighing is enough to tell what is the coin.

Problem 3.19 (C)

Suppose a, b, and c are integers such that the equation $ax^2 + bx + c = 0$ has a rational solution. Prove that at least one of the integers a, b, and c must be even?

Discussion: Let the rational solution of our equation be $\frac{p}{q}$, where p and q have no factors in common. Assume that all a, b, and c are all odd. Proceed by prove the contradiction.

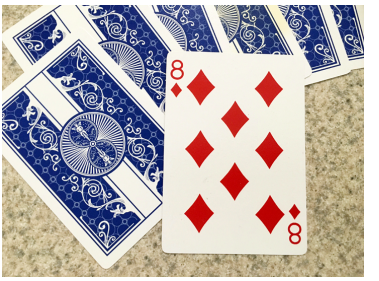
Solution: Assume that $r = \frac{p}{q}$, where p and q are integers with no common integer factor larger than 1. $a(\frac{p}{q})^2 + b\frac{p}{q} + c = 0 \Rightarrow ap^2 + bpq + cq^2 = 0$. Assume that all a, b, and c are all odd. Since $ap^2 + bpq + cq^2$ is 0 and also a sum of three numbers, so:

(a) Each term must be even: it means that p^2, pq, q^2 are even. Therefore p and q has a common integer factor of 2, it is not possible,

(b) One of three terms must be even and the rest is odd. It means one of p^2, pq, q^2 is odd and the other two are even. This is not possible as well.

So at least one of the three co-efficients a, b, and c must be an even number.

Puzzle 3.20: How many 8's



How many figures of 8 on the "8 of diamonds" card?

Problems

Problem 3.21 (A)

The product of 22 integers is equal to 1. Show that their sum cannot be zero.

Discussion:

(1) What are the integers can be a factor of 1?

(2) How can you make a sum *near to zero*?

Problem 3.22 (A)

Can a *magic square* be formed from the first 36 prime numbers? The magic square here means a 6×6 board with a number in each square, such that the sum of the number along any row, column, or diagonal is the same.

Discussion: What is the parity of the first 36 prime numbers?

Problem 3.23 (A)

There are 100 students in grade 5. Every day 3 of them are acting as stewards to help keep the school neat and clean. Can it be arrange so that after a certain period of time each student has shared duty with every other student exactly once?

Discussion: Assume that is possible. Let take a look at student A.

(1) He shares each activity with two other student.

(2) The rest 99 students, each shares with every other student a day of activity exactly once.

Problem 3.24 (B)

Twenty-five checkers are placed on a 25×25 checkerboard in such a way that their positions are symmetric with respect to one of its diagonals. Prove that at least one of the checkers is positioned on that diagonal.

Discussion:

(1) A diagonal divides the boards into two parts, not including its own squares.

(2) If it would not contain through any checkers, how the checkers are placed on the two parts?

Problem 3.25 (B)

The number 1, 2, 3, …, 2017 are written on a blackboard. You can replace any two numbers with their positive difference. This can be done until only one number remains on the board. Can that number be 0?

Discussion*:* The idea is similar to problem 3.5.

Problem 3.26 (B)

The number 1, 2, 3, …, 15 are written into the squares of a 15×15 board. Each square contains a single number. The main diagonal is from top-left to bottom-right.

(a) The squares which are symmetric to the main diagonal contain same numbers.

(b) No row or column contains 2 copies of the same number.

Show that no two of the numbers along the main diagonal are the same.

Discussion*:* Let take a look at the 1's.

(1) How many of them are in a row or a column, in the table

(2) And on the main diagonal?

Problem 3.27 (C)

A grasshopper jumps a long a line. His first jump takes him 1cm, his second takes him 2cm, and so on. Each jump can take him to the right or to the left. Can he return to the starting point after 2017 jumps?

Discussion*:* Take a look at problem 3.14

Problem 3.28 (C)

45 points are chosen on a line AB, all lying outside of segment AB. Prove that the sum of distances from these points to point A is not equal the sum of distances from them to point B.

Discussion*:* Let take a point X.

(1) What is $AX - BX$?

(2) What is the sum of all differences?

Problem 3.29 (C)

Three grasshoppers A, B, and C play leapfrog along a line. At each turn, one grasshopper leaps over another, but not over two others. After 2019 leaps, can they return to the original position?

Discussion*:* Take a look at problem 3.8

Puzzle 3.30: Sorting five bags

Using a beam balance:

Sort 5 bags with different weights, from lightest to heaviest in no more than *seven* weighing!

Challenges

Problem 3.31 (A)

Bethany and her friends stand in a circle. It turns out that both neighbours of each child are of the same gender. If there were five boys in the circle, how many of them are there?

Discussion:

(1) Can any neighbour of a boys be a boy?

(2) Can any neighbour of a girl be a girl?

Problem 3.32 (A)

On a cube, all the vertices and midpoints are marked. All diagonals of faces are drawn. Is it possible to visit all the marked points *once* by walking along the face diagonals?

Discussion: See Problem 3.16

Problem 3.33 (B)

There are seven integer numbers $a_1, a_2, \ldots, a_7$. They are shuffled into $b_1, b_2, \ldots, b_7$.

Can the product $(a_1 - b_1)(a_2 - b_2)\ldots(a_7 - b_7)$ be an odd number?

Discussion: What are the parity of these terms?

Problem 3.34 (B)

Can we draw a close path made up of 9 line segments, each of which intersect exactly one of the other segments?

Discussion:

(1) If such a solution would be possible, then all line segments would be partitioned into pair of intersecting segments.

(2) Then what would be the number of segment and why?

Problem 3.35 (B)

Find m, n integers if $m^2 - 2n^2 = 5$

Discussion:

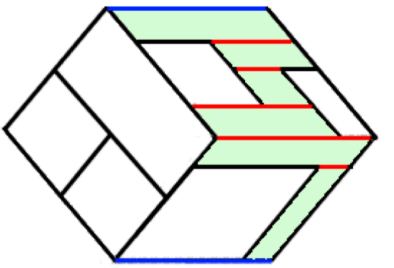
(1) What *parity* does m has? Substitute it with $2m_1$ if it is even, $2m_1 + 1$ if it is odd.

(2) Simplify what you got. Now compare the parities of the two sides.

Problem 3.36 (B)

Can a convex nonagon (a polygon with 9 sides) be cut into parallelograms?

Discussion: Below is an example with the hexagon case to illustrate the ideas.



(1) Start at a side of the nonagon. Since we've split up the nonagon into parallelograms, this side must contain a side of a parallelogram. Jump to the opposite side of this parallelogram.

(2) We're now at a side of a parallelogram. If this is contained in another side of the nonagon, we're done. If not, then it must overlap with a side of another interior parallelogram which is across the line from the one we were at before. Jump to the opposite side of this parallelogram.

(3) This process will create a sequence of parallel sides: we will start at the side of the nonagon, and proceed to jump to sides of interior parallelograms. Since there are finitely many interior parallelograms, and we keep moving the same direction, we will eventually wind up at another side of the nonagon. Pair up this second side with the first one. We have now created a pair of parallel nonagon sides.

Problem 3.37 (C)

A 17-digit number is chosen, and its digits are reversed, forming a new number. These two numbers are added together. Show that their sum contains one even digit.

Discussion:

(1) Assume it is not, look at the ninth digit. How is it formed? Look at the value at 8th column.

(2) Look at the value of the 7th column.

(3) Then look at the smaller numbered odd columns.

Problem 3.38 (C)

Is it possible to place the numbers 1 through 13 on a circle so that every one of them is divisible by the difference between its neighbours?

Discussion: Let take a look at an odd number. If the two neighbours of an odd number have same parity then their difference is even.

(1) How many odd numbers are there?

(2) According to the observation, what would be with their neighbours?

Problem 3.39 (C)

All positive integers are coloured black and white. It is known that:

(1) the sum of two different coloured numbers is black and,

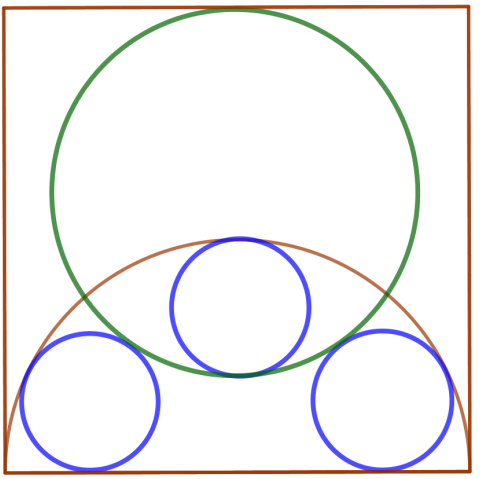
(2) their product is white

What is the colour of the product of two white numbers?

Discussion: Assume that the colour of the product of two white numbers is black. Prove that it leads to contradiction.

Puzzle 3.40: Tangent circles

The four circles and a semi-circle are tangent to each other inside a square. The three small circles have same radius r. The radius of the red circle is R.



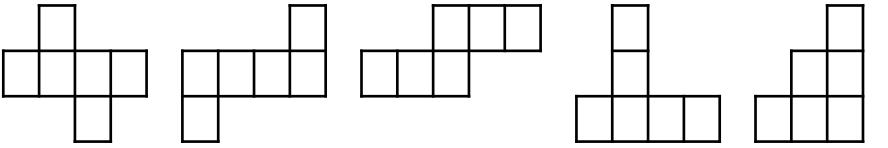
What is the value of the fraction $\frac{r}{R}$?

Chapter 4

Visual patterns

Examples

Problem 4.1 (A)

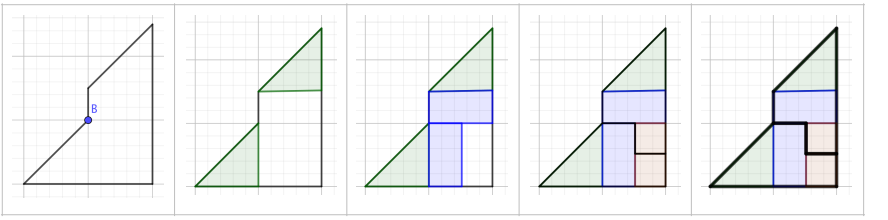


Which of the shapes can be folded into a cube?

Discussion: If you can't imagine at first, use a pencil, a paper sheet and a pair of scissors?

Solution: Easy to see only the first three of them can be folded into a cube.

Problem 4.2 (A)



Cut the figure on the left into two congruent (identical) parts.

Discussion: *Divide the figure into pairs of equal connected parts.*

Solution: From the figure on the left can be cut the two lower-left and top ends, the results are the two equal right green triangle.

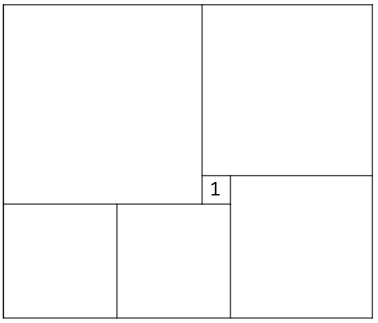
The remaining is a rectangle. This needs to be cut so that we can obtain two equal parts, but those two have to be connected with the two green parts: they are the two blue rectangles.

Finally the small white rectangle needs to be chopped into two parts such that they together with each pair of green triangle and blue rectangle form congruent figures.

The way it is done is shown in the next diagram: two squares. In the most right diagram two final congruent figures obtained by assembling triplets of a green triangle, a blue rectangle, and a brown square.

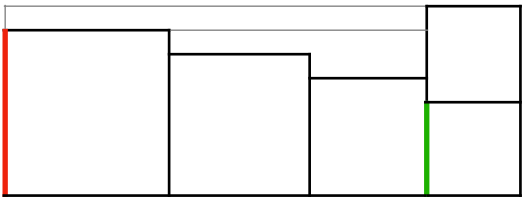
Problem 4.3 (A)

The rectangle contains 6 squares, the smallest one has a side length of 1. Find the side length of the largest square.



Discussion: Compare the two top and bottom lengths made by: the largest square and the second largest one, and the second smallest square and the middle one.

Solution:



Let take the squares out of the rectangles, line them up from largest to smallest (except the unit square) as shown above. It is now easy to see that:

(1) The red side is equal the green side plus three

(2) The red side plus one is twice the green side.

So twice the green side is the green side plus four, hence the green side is 4, and the red side is 7.

Problem 4.4 (B)

Can you solve this crypt-arithmetic? Replace different letters with different digits, empty square can have any digit.

		1	9	9	5
	X		*	*	*
	*	*	*	*	*
*	Y	O	U		
P	L	E	A	S	E

Discussion:

(1) Why the place after U has no square? Try to multiply 1995 by one digit.

(2) How can you get the last two digits differently?

Solution:

(1) It is easy to see that the *middle red star* must be 0 because there is no digit after U.

(2) Since the product of the *left red star* with 1995 is a four-digit number, and the last three digits of this four-digit number are all different, so it should be 3 or 4.

(3) The *right red star* should be at least 6 because the first multiplication has 5 digits. Because the last digit of the result (E) is different from the last digit of the second multiplication (U) so the right red star must be in different parity from the left red star. It cannot be 9 because the last two digits of the result are different (SE)

So we have following possibilities to test: 306, 308, and 407. 306 is the result.

Problem 4.5 (B)

Cut the figure into two congruent parts.

Discussion *Divide the figure into pairs of equal connected parts.*

Solution: The principle here is the same that is used in Problem 4.2.

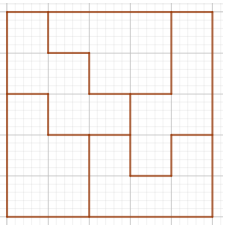
The only different is to chose the starting figures. The two green squares in the middle diagram seem to be best candidates.

From that, if the top left triangle is coloured blue then it is natural to split the middle square into two equal triangles, one of which is also coloured blue. The result is shown in the third diagram from the left.

Finally the remaining two triangles can be coloured brown. Assembling the triplets of (green, blue, brown) together two congruent figures can be shown in the most right diagram.

Problem 4.6 (B)

Colour in the 5×5 board above in different colours, such that every horizontal row, every vertical row, and every region outlined bold in the diagram below have every one of the five colours.



Discussion: Use numbers 1, 2, 3, 4, 5 as if they were colours. Number the squares. The easy way is to number completely each region and move to the next one.

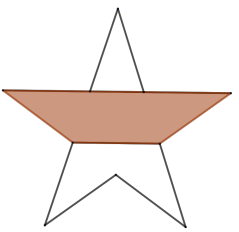
Solution: a, b, c, d, e and 1,2,3,4,5 numbers are used for easy identifying the squares. Let fill the first region with 1, 2, 3, 4, 5 colours. $a3$ must be 3. In the green region $c5$ must be 3 ($d5$ cannot be, otherwise 3 cannot be in the yellow region), hence $d2$ is 3 and $e1$ is 3. Continue by yourself.

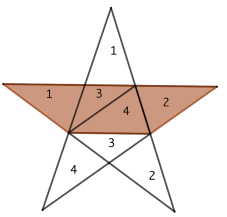
	a	b	c	d	e
5	1		3		
4	2	3			
3	3	4	5		
2				3	
1					3

a	b	c	d	e
1	5^2	3	2^3	4^4
2	3		4^5	
3	4	5		
			3	
			5^1	3

a	b	c	d	e
1	5	3	2	4
2	3	1	4	5
3	4	5	1	2
5	2	4	3	1
4	1	2	5	3

Problem 4.7 (C)





Show that the area of the brown region is exactly half of the star.

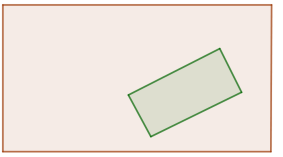
Discussion: Try *divide-and-conquer:* divide both the brown and white areas and compare them.

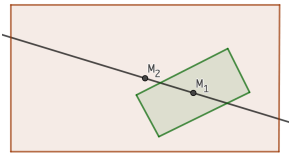
Solution: This *Mexican hat* can easily be divided as shown in the figure on the right. The congruent regions are numbered.

Note: the problem can be rewording as *show that the white area can be cut and reassemble into the brown shape*. If you have the two pieces of the white areas as two separate figures, how many cuts do you need?

Problem 4.8 (C)

A small green rectangle is placed entirely inside a bigger rectangle.





Using only a pencil and a straight edge, draw a line that divides the brown region, outside of the green rectangle, into two parts with equal areas.

Discussion: Think how would you divide a rectangle into two congruent halves. Congruence indicates presence of a symmetric point. What is the symmetric point of a rectangle?

Solution: *A line through the center of a rectangle - its symmetric point - divides it into two congruent parts*. So the line through both centres of the two rectangles will divide both in half, meaning the brown region must be divided into two regions with equal areas.

Problem 4.9 (C)

Let $S = 1 + 11 + 111 + \ldots + \underbrace{1\ldots1}_{111}$. What is the sum of the digits of 9S?

Discussion: What is $9S$? A naive attempt would be to add them up together.

Solution:

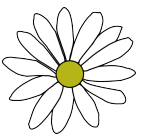
$$S = 1 + 11 + 111 + \ldots + \underbrace{1\ldots1}_{111} \Rightarrow 9S = 9 + 99 + 999 + \ldots + \underbrace{9\ldots9}_{111}$$

Since $9 = 10 - 1$, $99 = 100 - 1, \ldots$

$$9S = (10-1) + (100-1) + (1000-1) + \ldots + (\underbrace{10\ldots0}_{111\ of\ 0's} - 1) = \underbrace{11\ldots10}_{111\ of\ 1's} - 111 = \underbrace{11\ldots1}_{108\ of\ 1's}0999$$

Puzzle 4.10:

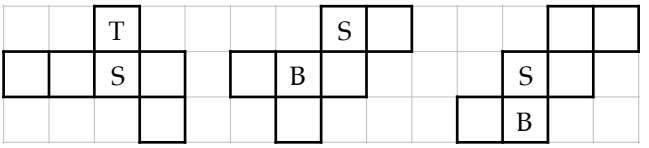
Leila and Quinn play a game with a *thirteen-petal* pink daisy. Each of them take minimum one, maximum two petals off the daisy. Whoever takes the last one win. If Quinn goes first, can Leila win? What is her strategy?



Exercises

Problem 4.11 (A)

Lydia cuts some of the shapes so that she can make paper cubes. To make it easy to assemble, she used pencil to label the faces with letters: T for top, B for bottom, and S for side. Patton, her brother, erase some of the letters. Here are what remain.



Can you help her to restore the letter?

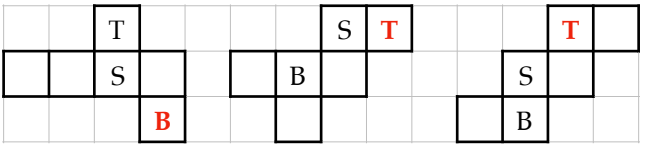
Discussion: If you cannot imagine, use a pencil, a paper sheet and a pair of scissors.

Solution: It is important to note that:

(1) the top is opposite of the bottom, so they do not share any edge

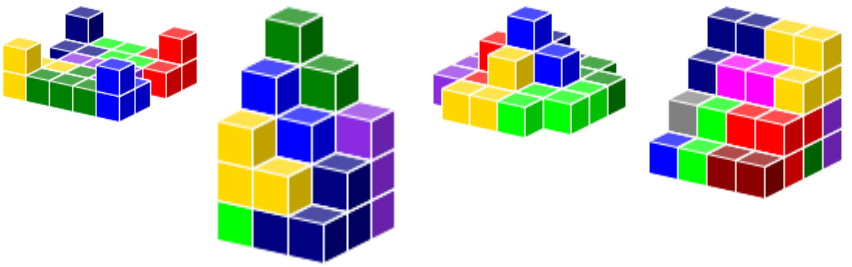
(2) the top and bottom are next to the side, so they each share an edge with the side.

Below are the restored letters:



Problem 4.12 (A)

How many cubes does each figure below contain?



Discussion: Count per each layer from bottom to top and add them together

Solution:

(a) $(5 \times 5 - 2) + 4 = 27$

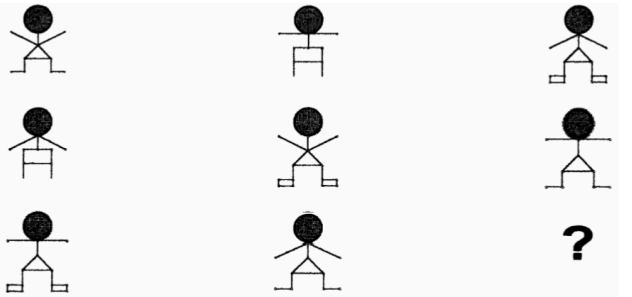
(b) $3 \times 3 + (3 \times 3 - 1) + (3 \times 3 - 3) + 3 + 1 = 27$

(c) $(5 \times 5 - 4) + 5 + 1 = 27$

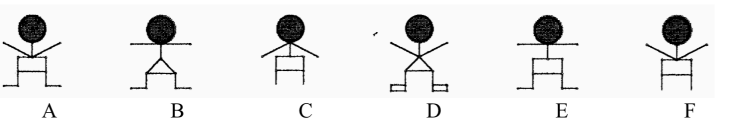
(d) $4 \times 4 + 4 \times 3 + 4 \times 2 + 4 = 16 + 12 + 8 + 4 = 40$

Problem 4.13 (A)

Several dancing figures are shown:



Which one below is the missing figure?



Discussion: Compare the legs then compare the hands.

Solution:

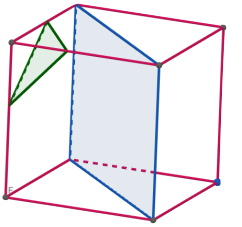
(1) if we first compare the legs, then the first column contains three different types of legs similarly, the second column column contains three different types of legs as well. So the leg of the missing figure in the third column should be the one with a rectangle.

(2) doing comparison of hands in both first two columns, the conclusion is that each column contains three different types of hands. So thew missing figure must contain a pair of hands pointing up.

So the missing figure is F.

Problem 4.14 (B)

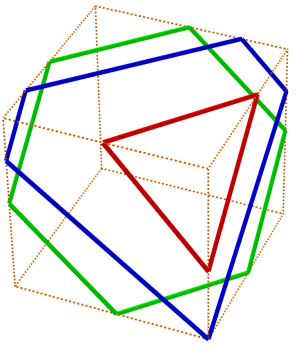
A plane can cut a cube resulting in a cross-section polygon. Below are two examples of a triangle and a square.



What is maximum number of edges a cross-section can have? Draw an example.

Discussion: Move the plane, draw a few examples.

Solution: a cross section is a *convex polygon.* Two examples of a triangle and a quadrilateral are show as in the problem. *Since the cube has at most 6 edges, so a cross section has maximum 6 edges.* The diagram below show another two cases of cross sections with 5, 6 edges.



Problem 4.15 (B)

What is $\dfrac{1.2.3 + 2.4.6 + 4.8.12 + 7.14.21}{1.3.5 + 2.6.10 + 4.12.20 + 7.21.35}$

Discussion:

(1) What is the common factor in the nominator?

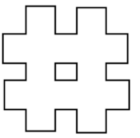
(2) What is the common factor in the denominator?

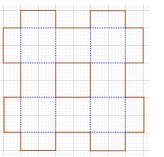
Solution:

$$\frac{1.2.3 + 2.4.6 + 4.8.12 + 7.14.21}{1.3.5 + 2.6.10 + 4.12.20 + 7.21.35} = \frac{2.3.(1 + 2.2.2 + 4.4.4 + 7.7.7)}{3.5.(1 + 2.2.2 + 4.4.4 + 7.7.7)} = \frac{2}{5}$$

Problem 4.16 (B)

Four rectangular strips on the left each measuring 4 by 16 inches are laid out: two vertical strips crossing two horizontal strips forming a single polygon which looks like a tic- tack-toe pattern.





What is the perimeter?

Discussion: What the parts of the perimeters of the four rectangular trips have to be removed?

Solution: The diagram shows that the perimeter of the pattern is exactly four times the perimeter of the rectangle subtract sixteen times its shorter edde lengths. Hence the perimeter is $4 \times 2 \times (16 + 4) - 16 \times 4 = 96$.

Problem 4.17 (C)

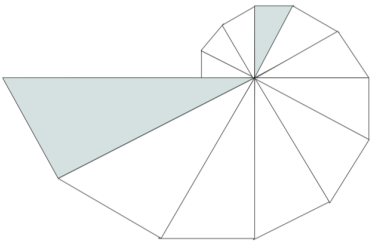
Is $\dfrac{100!}{50!2^{50}}$ an integer?

Discussion: $1 \times 2 = 2$, $2 \times 2 = 4$, $3 \times 2 = 6$, $\ldots$, $50 \times 2 = 100$

Solution: $\dfrac{100!}{50!2^{50}} = \dfrac{1.2.3...99.100}{1.2.3....50\ \underbrace{2.2...2}_{50\ of\ 2's}} = \dfrac{1.2.3...99.100}{4.6.8....100} = 1.3.5...99.$

Problem 4.18 (C)

The diagram shows twelve 30-60-90 triangles placed in a circle so that the hypotenuse of each triangle coincides with the longer leg of the next triangle. The fourth and last triangles are shaded. The ratio of the perimeters of these two triangles can be written as $\dfrac{m}{n}$ where m and n are n relatively prime positive integers. Find $m + n$.



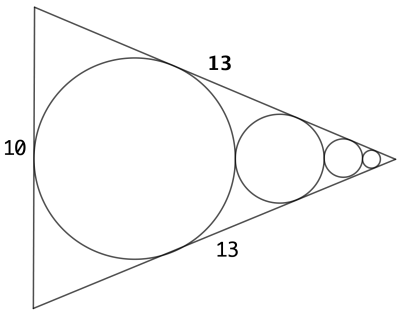
Discussion: What is the ratio of side-length between the first (smallest) and the second triangles? What is the ratio of side-length between the first and the fourth triangles?

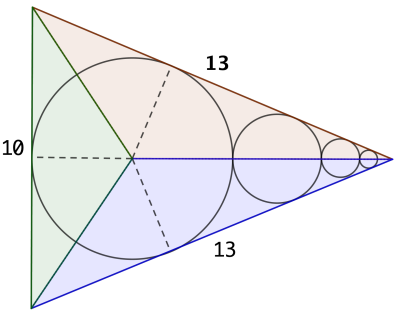
Solution: since all triangles are 30-60-90 right triangles, they are similar. The hypotenuse of the first (smallest) triangle is the long leg of the second triangle. Therefore the edge ratio of the first to the second triangle is $\frac{\sqrt{3}}{2}$. Following this, the ratio between the perimeters of the fourth and the last triangles must be $(\frac{\sqrt{3}}{2})^8 = \frac{3^4}{2^8} = \frac{81}{256} = \frac{m}{n} \Rightarrow m + n = 337$.

Problem 4.19 (C)

What is the sum of the circumferences of the infinite stack of circles inside the triangle below? Note that each circle is tangent to the sides of the triangles and to the circles above or below it.

Discussion:





(1) First approach: find the radius of the first one, and then find a formula for any one. The sum is probably can be calculated based on a generic formula.

(2) Second approach: Think out of the box! What do we need to calculate *actually*?

Solution 1: It's a standard one for anybody with a bit a patience to finish the calculation

(1) Let take a look at the largest circle. It is the incircle of the triangle, so its radius r is $\frac{2S_\triangle}{P_\triangle}$, where $S_\triangle$ is the area and $P_\triangle$ is the perimeter of the triangle. Easy to calculate that the altitude is $\sqrt{13^2 - 5^2} = 12$, so $r = \frac{10.12}{36} = \frac{10}{3}$.

(2) The second circle is the incircle of the second triangle and the ratio of its radius to the first one is is $\dfrac{12 - 2.\frac{10}{3}}{12} = \dfrac{16}{36} = \dfrac{4}{9}$.

Thus, the sum of all radii is:

$$\frac{10}{3}(1 + \frac{4}{9} + \frac{4^2}{9^2} + \ldots) = \frac{10}{3} \cdot \frac{1}{1 - \frac{4}{9}} = \frac{10}{3} \cdot \frac{9}{5} = 6.$$

It means that the sum of all circumferences should be 2π times the sum of radii or 12π.

Solution 2: This solution is based on the following insight *the sum of all diameters of all circles is exactly the length of the altitude.* Hence the sum of circumferences is 12π.

Puzzle 4.20:

In three-dimensional space, a Platonic solid is a regular, convex polyhedron. It is constructed by congruent (identical in shape and size) regular (all angles equal and all sides equal) polygonal faces with the same number of faces meeting at each vertex. There are five of them, four shown as below:

The solid				?	
Name	Tetrahedron	Hexahedron (cube)	Octahedron	Dodecahedron	Icosahedron
Number of faces	Four	Six	Eight	Twelves	Twenty
What is a face	Equilateral triangle	Square	Equilateral triangle	?	Equilateral triangle

You task is to find what regular polygon is a face of the 4th solid.

You are armed with two facts:

(1) It has twelves faces - *twelve a.k.a dodeca in Latin.*

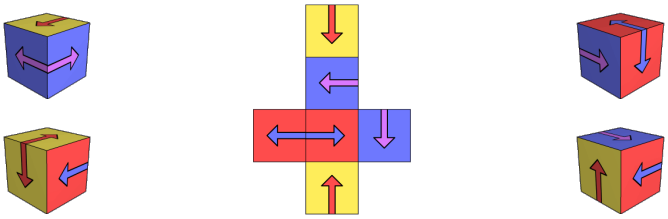
(2) The Euler's formula: If V, E, and F are respectively the numbers of vertices (corners), edges and faces in a polyhedron, then $V - E + F = 2$.

Note: use your imagination and reasoning, don't let Google and Wikipedia work on behalf of you.

Problems

Problem 4.21 (A)

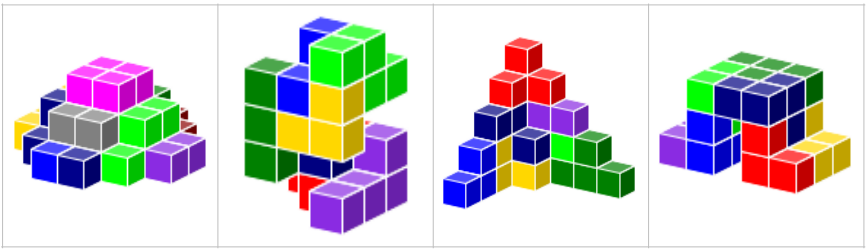
Which of the cubes below could NOT have been folded from the shape in the middle?



Discussion: First identify the double-head arrow. Second look at the remaining arrow direction.

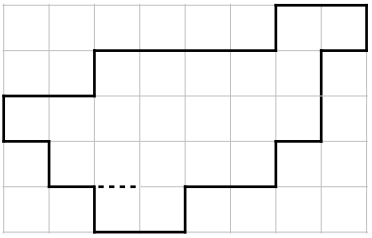
Problem 4.22 (A)

Can you count how many cubes in each of the below figures:



Discussion: Count per each layer, and be careful.

Problem 4.23 (A)



Divide the above figure into *three identical (congruent) parts*

Discussion: The figure give a hint to start.

Problem 4.24 (B)

Find a, b, c, d positive integers such as:

$$ab = c + d$$
$$cd = a + b$$

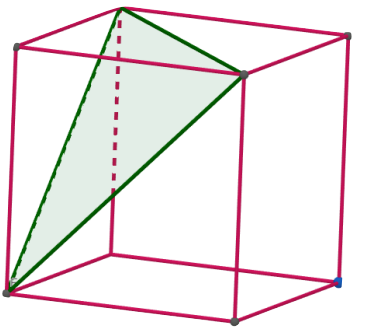
Discussion: It's symmetric!

(1) If you add both sides: $(ab - a - b + 1) + (cd - c - d + 1) = 2$

(2) Think of the values on both sides.

Problem 4.25 (B)

A cube can be cut so that the cross-section is an equilateral triangle.



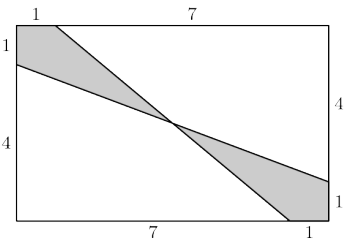
How many cases all the vertices of the triangle are also vertices of the cube?

Discussion:

(1) What are the sides of the triangles in *relation* to the cube faces?

(2) Can you count the number of triple of them *without repeating*?

Problem 4.26 (B)



What is the area of the shaded figure?

Discussion: divide-and-conquer, use the centre of symmetry.

Problem 4.27 (C)

Compute the value of $1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \dots}}}}$

Discussion: Use Problem 1.17

Problem 4.28 (C)

Find a way to pack five unit squares inside a square of side length 3 so that they do not overlap.

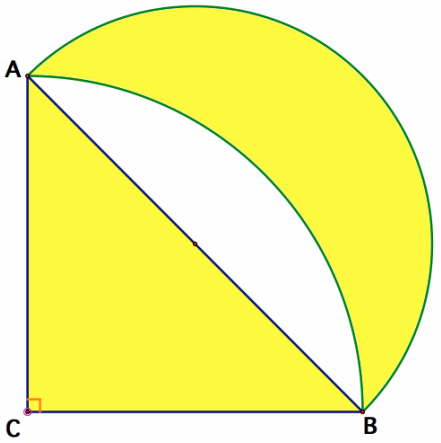
Discussion:

(1) Try to put the four unit squares somewhere so that you can have lots of space for the fifth.

(2) Now try to fit the fifth one without breaking it.

Problem 4.29 (C)

Given an isosceles right triangle ABC. Construct a lune with vertices at A and C with the center of the inner arc being at C and the center of the outer arc being at the midpoint of segment AB.



Prove that the area of the constructed lune is the same as the area of the triangle ABC.

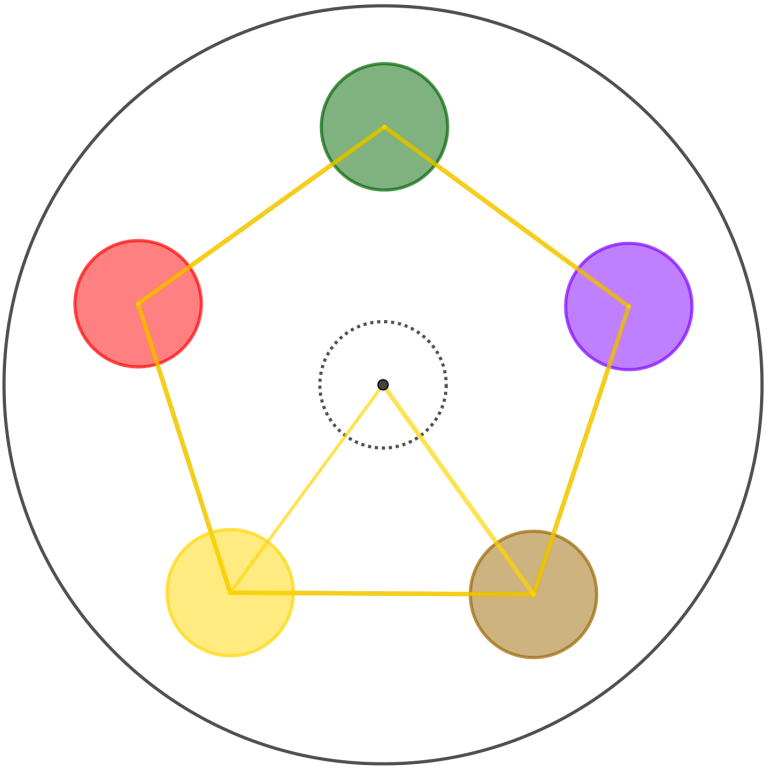
Discussion: Construct the whole square whose quarter is the right triangle.

Puzzle 4.30: Interchange

Five disks with different colours are placed on the five corners of a pentagon. They can be moved to any corners or the center if that place is:

(1) connected to the place where the disk is

(2) is empty (has no disk on it)



Can you exchange the positions of the red and yellow disks? If yes, what is the minimum number of moves you need?

Challenges

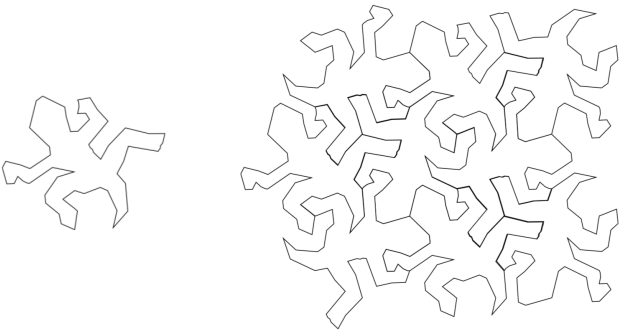
Problem 4.31 (A)

Find the smallest whole number that is not a factor of $1 \times 2 \times 3 \times \ldots \times 22 \times 23$.

Discussion: What would be the next number after 23 which is NOT a factor?

Problem 4.32 (A)

The figure on right below contains a number of chameleons with the pattern on the left.



How many colours needed to colour all of them so that any two chameleons sharing some edges are coloured differently?

Discussion:

(1) Prove that you cannot use only two colours.

(2) Show that how you can use 3 colours.

Problem 4.33 (A)

The numbers are arranged in three rows A, B, and C as following

A:	1	6	7	12	13	18	19	…
B:	2	5	8	11	14	17	20	...
C:	3	4	9	10	15	16	21	...

Which row will contain the number 1000?

Discussion: How the numbers alternate between A, B, and C rows?

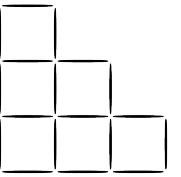
Problem 4.34 (B)

Find the sum of digits of the following number: $234 \times \underbrace{55...5}_{66}$

Discussion: $234 = 26 \times 9$, so $234 \times 55 = 26 \times 5 \times 99$

Problem 4.35 (B)

Sara makes a staircase out of toothpicks as shown

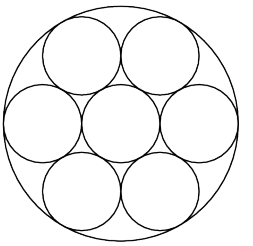


This is a 3-step staircase and uses 18 toothpicks. How many steps would be in a staircase that used 180 toothpicks?

Discussion: count the number of toothpicks to form 1-step staircase, then 2-step staircase, and generalize!

Problem 4.36 (B)

Seven cookies of radius 1 inch are cut from a circle of cookie dough, as shown. Neighbouring cookies are tangent, and all except the center cookie are tangent to the edge of the dough. The leftover scrap is reshaped to form a square cookie of the same thickness.



What is the length in inches of the diagonal of the scrap cookie?

Discussion:

(1) What is the radius of the small cookie?

(2) The difference in areas of the big cookie and the seven small ones is?

Problem 4.37 (C)

For what value of n is

$$S_n = i + 2i^2 + 3i^3 + 4i^4 + \ldots . n\, i^n = 48 + 49i$$

where $i = \sqrt{-1}$

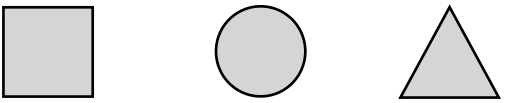
Discussion:

(1) Calculate value of i^{2k}, i^{2k+1}

(2) Generalize value of the sum on the left side

Problem 4.38 (C)

We have three bottles: one has a circular opening, another a square opening, and the third has an opening in a the share of an isosceles triangle. The figure below shows the shape of each opening, seen from above. The side length of the square, the diameter of the circle, the height and base of the triangle are all equal.



(a) Design a bottle stopper that fits all three bottles

(b) Design a bottle stopper that fits all three bottles, and can be pushed inside each of the bottle through the bottle opening.

Discussion:

(1) Think about using a bottle stoppers with *multiple* ends, each can fit a *different* bottle.

(2) Think about a similar one, but now each represents *a cut by a plane*.

Problem 4.39 (C)

S_n is defined as the sum of the digits in the whole number. For example: $S_3 = 3$ and $S_{29} = 2 + 9 = 11$.

Find the value of:

$$S_1 + S_2 + S_3 + \ldots + S_{2010} + S_{2011}$$

Discussion:

(1) Try to calculate $S_1 + S_2 + S_3$

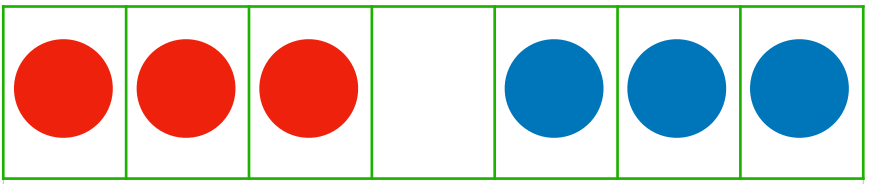
(2) What is $S_1 + S_2 + S_3 + \ldots + S_{10} + S_{11}$

(3) Generalize!

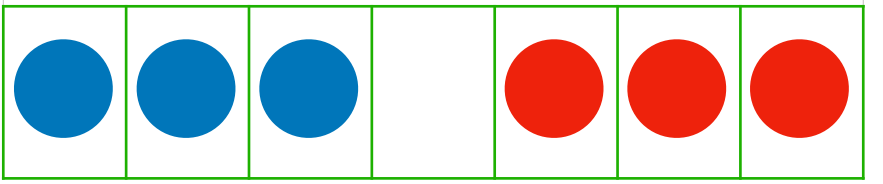
Puzzle 4.40:

There are three red and three blue disks in a line in the *start diagram*. You are requested to move them so that you can reverse the pattern as in the *end diagram*.

Start



End



Five rules must be followed:

(1) Only one disk can be move at a time

(2) A disk may move into an adjacent empty space.

(3) A disk may jump over a disk of the opposite colour into the space immediately beyond it.

(4) A disk may not jump over the disk of the same colour.

(5) No backward moves are permitted.

Note that no jumping off the ends is permitted, and no move might disturb another disk (no pushing).

Can you solve the puzzle in no more than 15 moves?

Chapter 5

Divisibility

Decimal (base-10) representation

A number $\overline{a_k a_{k-1} \ldots a_1 a_0}$ with the digits a_k, a_{k-1}, ..., a_1, a_0 can be written as

$$\overline{a_k a_{k-1} \ldots a_1 a_0} = 10^{k-1} a_k + 10^{k-2} a_{k-1} + \ldots + 10 a_1 + a_0$$

For example: $321 = \overline{321} = 100 \times 3 + 10 \times 2 + 1$

Divisibility rules

(1) Divisible by 2 if the last digit a_0 is an even digit 0, 2, 4, 6, 8

(2) Divisible by 3 if the sum of digits $a_k + a_{k-1} + \ldots + a_1 + a_0$ is divisible by 3

(3) Divisible by 4 if the number formed by the last two digits $\overline{a_1 a_0}$ is divisible by 4

(4) Divisible by 5 if the last digit a_0 is one of 0, 5

(5) Divisible by 6 if it is divisible by both 2 and 3

(6) Divisible by 7 if $\overline{a_2 a_1 a_0} - \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} - \ldots$ is divisible by 7

(7) Divisible by 8 if the number formed by the last three digits $\overline{a_2 a_1 a_0}$ is divisible by 8

(8) Divisible by 9 if the sum of digits $a_k + a_{k-1} + \ldots + a_1 + a_0$ is divisible by 9

(9) Divisible by 10 if the last digit a_0 is 0

(10) Divisible by 11 if $a_0 - a_1 + a_2 - \ldots$ is divisible by 11

(11) Divisible by 11 if $\overline{a_1 a_0} + \overline{a_3 a_2} + \overline{a_5 a_4} + \ldots$ is divisible by 11

(12) Divisible by 11 if $\overline{a_2 a_1 a_0} - \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} - \ldots$ is divisible by 11

(13) Divisible by 12 if it is divisible by both 3 and 4

(14) Divisible by 13 if $\overline{a_2 a_1 a_0} - \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} - \ldots$ is divisible by 13

(15) Divisible by 27 if $\overline{a_2 a_1 a_0} + \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} + \ldots$ is divisible by 27

(16) Divisible by 37 if $\overline{a_2 a_1 a_0} + \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} + \ldots$ is divisible by 37

Factorial

$n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ is called n factorial.

For example: $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

Product of digits

The product of digits of a number $\overline{a_k a_{k-1} \ldots a_1 a_0}$ is $a_k \times a_{k-1} \times \ldots \times a_1 \times a_0$.

Examples

Problem 5.1 (A)

According to the famous astronomer and writer Carl Sagan, *googol* - 10^{100} - is greater than the number of all elementary particles (electrons, protons, neutrons) in our Universe. *googolplex* - the number that the Internet search website Google was named after is $10^{10^{100}}$.

Is *googolplex* divisible by 3?

Discussion: It would take a very very large amount of time to test if the number is divisible by 3 by trying to divide it by 3. Let try the digit sum.

Solution: *googolplex* in decimal representation has the leading 1 and following zero digits. The sum of all these digits is 1, and it can not be divisible by 3.

Problem 5.2 (A)

Find n, if $n! = 40320$

Discussion: The problem ask to find an n where $n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n = 40320$

Solution: The easiest way is keep dividing by 1, 2, 3, … until we got 1.

$40320 \div 1 = 40320$, $40320 \div 2 = 20160$, $20160 \div 3 = 6720$, $6720 \div = 1680$

$1680 \div 5 = 336$, $336 \div 6 = 56$, $56 \div 7 = 8$, $8 \div 8 = 1$. Therefore $n = 8$

Problem 5.3 (A)

Sarah bought 72 pencils. She had the bill, but somehow two digits were worn off both ends.

$_3.5_

Can you help here to find out the price for each pencils?

Discussion: The four-digit number $A35B$ must be divisible by 72,. So it is divisible by 8 and 9.

Solution:

(1) $A35B$ has to be divisible by 4, so the last two digits form a number $5B$ is divisible by 4. It means that $5B$ is 52 or 56.

(2) $A35B$ is divisible by 8 if the number formed by the last three digit $35B$ is divisible by 8. Among 352, 356 only 352 is divisible by 8.

(3) Now $A352$ can be divisible by 9 only if the digit sum $A + 3 + 5 + 2 = A + 10$ is divisible by 9. So A has to be 8.

The total is \$83.72. The price of one pencil is \$1.16.

Problem 5.4 (B)

(a) Find the smallest number, whose product of digits is 216

(b) Find the largest six-digit number, whose product of digits is 216

Discussion: $216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3$

Solution:

(a) The smallest number should have the least number of digits, which is three digits 9, 8, 3. So the smallest number is 389.

(b) The biggest six-digit number when we can create as big leading digits as possible, in this case 9, 8, 3, 1, 1, 1 or 983111.

Problem 5.5 (B)

Find a prime number p such that $p + 2500$ is a perfect square

Discussion: Remember that $2500 = 50^2$, and $a^2 - b^2 = (a - b)(a + b)$

Solution: Let n be a positive integer such that $p + 2500 = n^2$.

$\Rightarrow p = n^2 - 2500 = n^2 - 50^2 = (n - 50)(n + 50)$. On the left hand p is a prime number, therefore $n - 50 = 1$, $n + 50 = p \Rightarrow n = 51$, $p = 101$.

Problem 5.6 (B)

Compute $\dfrac{8! + 6!}{7! + 5!}$

Solution: $8! = 6! \times 7 \times 8 \Rightarrow 8! + 6! = 6! \times 57$

$7! + 5! = 5! \times (6 \times 7 + 1) = 5! \times 43$.

$$\frac{8! + 6!}{7! + 5!} = \frac{6! \times 57}{5! \times 43} = 6 \times \frac{57}{43} = \frac{342}{43}$$

Problem 5.7 (C)

Prove that $145678 + 456781 + 567814 + 678145 + 781456 + 814567$ is product of six different prime numbers.

Discussion: The first term of the above sum can be rewritten as below in using decimal representation:

$145678 = 10^5 \times 1 + 10^4 \times 4 + 10^3 \times 5 + 10^2 \times 6 + 10 \times 7 + 8.$

Solution: By grouping the terms with same exponents $10^5, 10^4, \ldots, 10^1, 1$ we have:

$(1 + 4 + 5 + 6 + 7 + 8)(10^5 + 10^4 + \ldots + 10 + 1) = 31 \times 111111$, and

$31 \times 111111 = 31 \times 111 \times 1001 = 31 \times (3 \times 37) \times (7 \times 11 \times 13)$

$= 3 \times 7 \times 11 \times 13 \times 31$

Problem 5.8 (C)

What is the product of all factors of 400?

Discussion: For each f factor of 400, $\dfrac{400}{f}$ is also a factor of 400, and $f \times \dfrac{400}{f} = 400$.

Solution: The question is to find the value of $P = 1 \times 2 \times 4 \times \ldots \times 100 \times 200 \times 400$.

$P \times P = (1 \times 400) \times (2 \times 200) \times (4 \times 100) \ldots \times (200 \times 2) \times (400 \times 1) = 400^{15} = (20^{15})^2$

Therefore $P = 20^{15}$.

Problem 5.9 (C)

340 tickets are sold for a total \$2019. Among these tickets, some are sold at full price, which is a whole dollar amount, and some are sold at half price.

How much money was made by the full-price tickets?

Discussion: Let p be the full price of the ticket, n be the number of the full-price tickets.

$$pn + \frac{p}{2}(340 - n) = 2019$$

Solution:

$pn + \dfrac{p}{2}(340 - n) = 2019 \Rightarrow p(340 + n) = 4038 \Rightarrow p(340 + n) = 2 \times 3 \times 673$

Because $n \leq 340 \Rightarrow 340 + n \leq 340 \times 2 = 680$.

Therefore $340 + n = 673 \Rightarrow n = 333$, $p = 6$.

The total amount is $6 \times 333 = 1998$.

Puzzle 5.10:

Is it possible for the white knights to exchange their position with the black knights?

If your answer is no, prove it. If it is yes, show how you can do it.

Exercises

Problem 5.11 (A)

For what digit a is the number $\overline{36a958}$ divisible by 18?

Discussion: It is easy to see the number is divisible by 2, regardless what is the value of a. So what we need to do is to investigate what value of a makes the number divisible by 9.

Solution: $3 + 6 + a + 9 + 5 + 8 = 27 + a + 4$ is divisible by 9 if and only if $a = 5$.

Problem 5.12 (A)

Kevin multiplied two two-digit numbers on the blackboard. Then he changed all the digits to letters (different digits were changed to different letters and equal digits to the same letters). He got $AB.CD = EEFF$. Tina told him that he made a mistake somewhere. Why?

Discussion: $EEFF$ is a number that is divisible by 11.

Solution: Because $A \neq B$, $C \neq D$, so AB, CD are not divisible by 11. But $EEFF = E0F \times 11$.

This means that Kevin made a mistake somewhere.

Problem 5.13 (A)

Prove that the product of three consecutive integers is divisible by 6.

Discussion: Why the product of three consecutive integers is divisible by 2? by 3?

Solution: Three consecutive integers have an even number among them, so one of them is even.T hree consecutive integers have remainders of 0, 1, 2 when divided by 3. So one of them is divisible by 3.

From above, the product of three consecutive integers must be divisive by $2 \times 3 = 6$. It is true because 2 and 3 are relative primes.

Problem 5.14 (B)

Find all number in the form $\overline{xyxz}$ that are divisible by 11, 12, and 13.

Discussion: Since 11, 12, and 13 are pair-wise relative primes, so $\overline{xyxz}$ must a multiple of $11 \times 12 \times 13 = 1716$. It is now easy to verify a few of them.

Solution: if $\overline{xyxz}$ is divisible by 11, 12, and 13 then $\overline{xyxz}$ must be divisible by $11 \times 12 \times 13$, because 11, 12, and 13 do not share any common factor. Because $11 \times 12 \times 13 = 1716$, so $\overline{xyxz}$ is a multiple of

1716. Since $1716 \times 6 > 9999$, so values of $\overline{xyxz}$ format are 1716, 3432, 6864, 8580.

Problem 5.15 (B)

Altafini goes to the market to buy apples and oranges. The cost of one apple is \$3 while one orange costs \$5. If Altafini has \$72, he must buy at least one apple and one orange, and he wants to spend all the money. What is the maximum number of apples he can buy?

Discussion: Let a be the number of apples, b be the number of oranges: $3a + 5b = 72$. Because $3a$, 72 are both divisible by 3, so b must be divisible by 3. So $b \geq 3$. Let try $b = 3$.

Solution: Let a be the number of apples, b be the number of oranges: $3a + 5b = 72$. From this equation, the right side is divisible by 3, so the left side is as well. Since the left side is $3a + 5b$, it means that $5b \vdots 3 \Rightarrow b \vdots 3 \Rightarrow b \geq 3$. Because $3a + 5b = 72$, so the smaller b is, the larger a can be. The smallest possible value of b is 3, in that case $a = \dfrac{72 - 15}{3} = \dfrac{57}{3} = 19$. So the maximum amount of apples is 19.

Problem 5.16 (B)

What is the smallest n natural number such that $n!$ has exactly 4 zeros at the end of its decimal representation?

Discussion: How a number end with exactly 1 zero? What should it be divisible by? How a number end with exactly 4 zero? What should it be divisible by?

Solution: If the number has exactly 1 zero, then between 1 and n there must be exactly a single number that is divisible by 5, so n cannot be larger than 9. If the number ends exactly with 4 zeroes, it does not contain more multiples of 5 than 5, 10, 15, 20. The smallest number is 20.

Problem 5.17 (C)

Prove that the number of factors of a perfect square is an odd number, and any other number has an even number of factors.

Discussion: Let n be a positive integer. Let $1 = d_1 < d_2 < \ldots < d_k = n$ be the factors of n. So the two sequence

$$d_1, d_2, \ldots, d_{k-1}, d_k$$
$$d_k, d_{k-1}, \ldots, d_2, d_1$$

can be divided into pairs $(d_1, d_k), (d_2, d_{k-1}), \ldots, (d_{k-1}, d_2), (d_k, d_1)$ such that the product of each pair is n.

Solution: If n is a perfect square, then there is a factor d in the middle of the sequences so $d \times d = n$. It means k is odd. Other number does not have such a factor d in the middle of the factor sequence, it means the number of factors is even.

Problem 5.18 (C)

Find a positive number n such that $n + 72$ and $n + 55$ are perfect squares.

Discussion:

$x^2 = n + 72$, $y^2 = n + 55 \Rightarrow x^2 - y^2 = 17 \Rightarrow ((x - y)(x + y) = 17$.

Solution: continue the discussion, we have

$\Rightarrow x - y = 1$, $x + y = 17$, hence $x = 9$, $y = 8$ or $n = 9$.

Problem 5.19 (C)

Find x integer root of $x^3 + x^2 + x - 3 = 0$

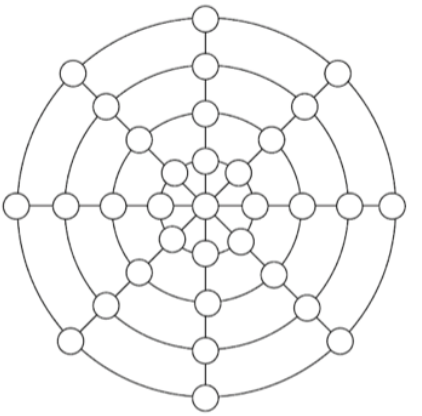
Discussion: Factor the expression on the right-side. It is easier than it looks.

Solution: Easy to see that $1^3 + 1^2 + 1 - 3 = 0$, so the polynomial $x^3 + x^2 + x - 3$ must have $x = 1$ as a root, or in another word it is divisible by $x - 1$, or $x - 1$ is a factor of it.

$$x^3 + x^2 + x - 3 = (x^3 - x^2) + 2(x^2 - x) + 3(x - 1) = (x - 1)(x^2 + 2x + 3) = (x - 1)((x + 1)^2 + 2)$$

Since $(x + 1)^2 + 3 \geq 3 > 0$, so the polynomial has a single root: $x = 1$.

Puzzle 5.20:



Fill the number 1, 2, … , 33 into the small circles such that every circle including its centre and every diameters has the same sum.

Problems

Problem 5.21 (A)

How many possible triplets of different prime numbers such that their sum is 25?

Discussion: Since 25 is odd number, so all three prime numbers must be odd. Let's start with the smallest odd prime number which is 3. The sum of two other different primes and larger than 3 is 22. There are only one possible pair (5, 17). So (3, 5, 17) is one answer.

Problem 5.22 (A)

Find the last digit of $\underbrace{2 \times 2 \times \ldots \times 2}_{50}$

Discussion: Let write down the last digits of a a few similar expressions:

$\underbrace{2}_{1} = 2$, last digit is 2. $\underbrace{2 \times 2}_{2} = 4$, last digit is 4. $\underbrace{2 \times 2 \times 2}_{3} = 8$, last digit is 8, … So if you write only the last digits, you will see 2, 4, 8, 6, 2, 4, 8, 6, … What do you think?

Problem 5.23 (A)

How many positive integers divide the number $1 \times 2 \times 3 \ldots \times 9 \times 10$

Discussion: Let replace the each term of the product of its prime factors. For example $6 = 2 \times 3$, $10 = 2 \times 5$ and so on.

Problem 5.24 (B)

m, n positive integers and $56m = 65n$. Prove that $m + n$ is composite number (not a prime)

Discussion:

$56m = 65n \Rightarrow 56m + 56n = 65n + 56n = 121n \Rightarrow 56(m + n) = 121n$

Assume if $m + n$ is prime number, what should it be? Will there be anything contradictory?

Problem 5.25 (B)

(a) Find all primes numbers can be written as a sum and a difference of two other primes.

(b) If p, q are different prime numbers. How many different divisors p^2q^3 has? How about p^mq^n?

(a) **Discussion**: Assume that there are five primes $p, q \geq r, s \geq t$: $p = q + r = s - t$. Easy to see that p cannot be 2, so it has to be odd, therefore $r = t = 2$, hence $p - 2, p, p + 2$ are all primes. But among three consecutive odd numbers there should be one divisible by 3.

(b) **Discussion**: A divisor of p^2q^3 is a product of an exponent of p and an exponent of q. There are 3 ways to choose exponent of p: $1 = p^0, p^1, p^2$. There are 4 ways to choose exponent of q: $1 = q^0, q^1, q^2, q^3$

Problem 5.26 (B)

Prove that a product of five consecutive integers is divisible by 120

Discussion: $120 = 3 \times 5 \times 8$. Among 5 consecutive numbers which is divisible by 3? By 5? Among 5 consecutive numbers there are at least two even numbers?

Problem 5.27 (C)

Find the last digit of 7^{7^7}

Discussion: Very similar to 5.26

Problem 5.28 (C)

Both p, $8p^2 + 1$ are prime. Find p

Discussion: Evaluate the remainder of p when divided by 3.

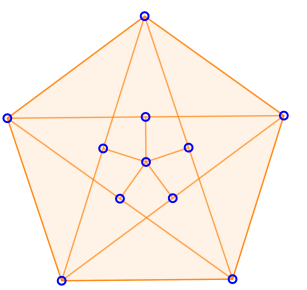
Problem 5.29 (C)

Prove that the sum of the squares of five consecutive integers cannot be a perfect square.

Discussion: Evaluate the remainder of a perfect square divided by 4.

Puzzle 5.30:

Eleven blue villages are connected by the brown road. Can you walk from a village, visit each other village once? You don't have to return to the village where you start.



Challenges

Problem 5.31 (A)

The sum of two prime numbers is 39. What is their product?

Discussion: What would be the largest prime smaller than 39? Find the other, check if it is prime. Work backwards.

Problem 5.32 (A)

a, b, c are prime numbers: $a \times b = 55$, $b \times c = 35$. What is $a \times b \times c$?

Discussion: Factorize 55, 35. Compare.

Problem 5.33 (A)

How many number less than 10,000 are product of four consecutive positive integers.

Discussion: Estimate. What largest number multiplies with itself four times and still smaller than 10,000.

Problem 5.34 (B)

(a) If $a + 1$ is divisible by 3, will $4 + 7a$ be divisible by 3?

(b) If both $2 + a$, $35 - b$ are divisible by 11, will $a + b$ is divisible by 11?

Discussion: $4 + 7a = 4 + 4a + 3a = 4(1 + a) + 3a$

Problem 5.35 (B)

Find the last digit of $1^2 + 2^2 + \ldots + 99^2$

Discussion: Investigate the last digits of the first 10 perfect squares. Establish a pattern.

Problem 5.36 (B)

If p, $2p + 1$, $4p + 1$ are prime numbers. Find p

Discussion: Look at 5.28

Problem 5.37 (C)

Prove that if p, q are both prime numbers, greater than 3, then $p^2 - q^2$ is divisible by 24

Discussion: Very similar to 5.26

Problem 5.38 (C)

Seven natural numbers are given. The sum of any six of them is divisible by 5. Prove that each of these number is divisible by 5.

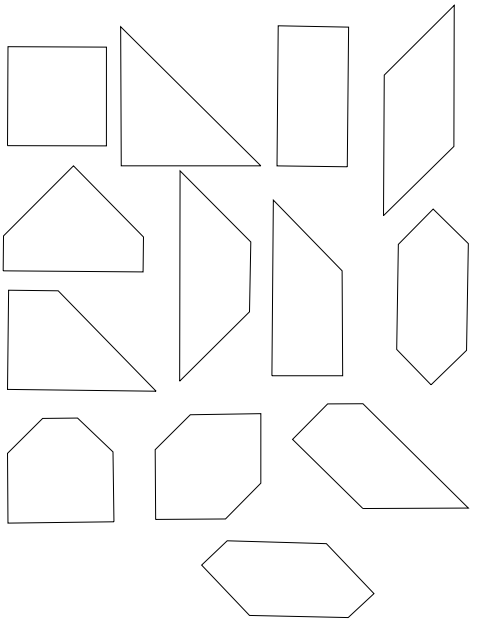
Discussion: Compare two different sums of six numbers. What we know? Compare another two. Try to establish a pattern of remainders.

Problem 5.39 (C)

Prove that $a^3 + b^3 + 4$ is not a perfect cube for any natural number a, b

Discussion: Investigate remainder of $a^3 + b^3 + 4$ when divided by 9.

Puzzle 5.40:



Use the Tangram to construct these figures.

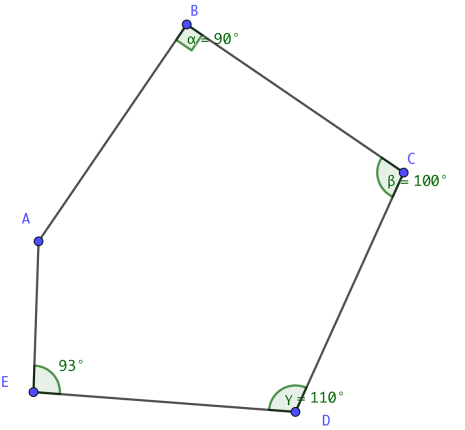
Chapter 6

Basic Geometry

Examples

Problem 6.1 (A)

In the pentagon $ABCDE$ below, four consecutive angles at B, C, D, E are $90°, 100°, 110°, 93°$.



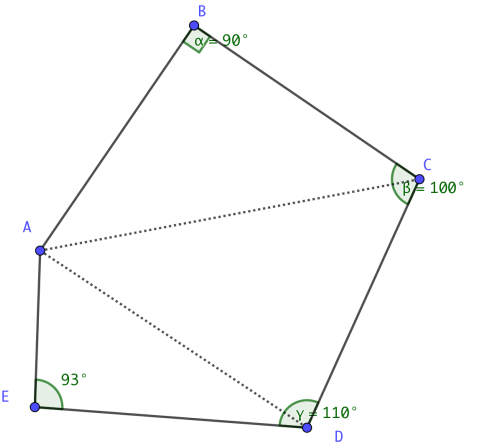
What is the measure of the angle A?

Discussion: Let connect point A with non-neighbours by diagonals. Doing this will *split the pentagon into three triangles*.

Note: the sum of all internal angles in a triangle is $180°$ *(180 degrees).*

Solution:

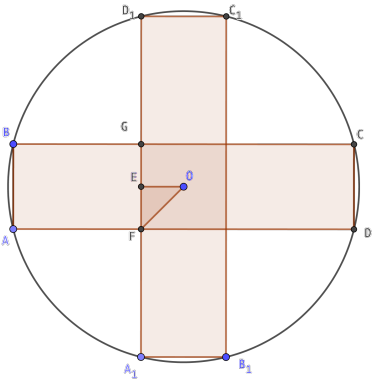
The sum of all internal angles of a pentagon is equal the sum of all angles in the three triangles. It means that $\angle A + 90 + 100 + 110 + 93 =$ $= 3 \times 180 \Rightarrow \angle A = 147°$.



Problem 6.2 (A)

Two congruent rectangle $ABCD$ and $A_1B_1C_1D_1$ form a cross in the diagram.

$BG = 3 \times OE$ and the area of OEF triangle is 1. What is the area of the cross?

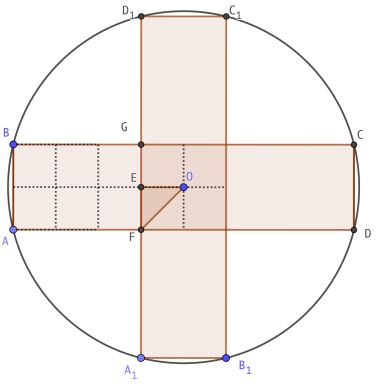


Discussion: Let compare the area of $BGFA$ to OEF by dividing $BGFA$ to smaller pieces.

Solution:

Because $BG = 3 \times OE$, let divide BG into three equal segments. There are two points on BG, let draw the line segments through them and parallel with BA. Similarly because E is midpoint of GF, so GF can is divided into two equal segments, let draw one line segment through E and parallel with BG.

We obtain a mini grid that divides $BGFA$ rectangle into smaller equal squares as shown.



Easy to see that the area of OEF triangle is half of the area of a small square.

(1) the area of $BGFA = 6 \times$ area of a small square $= 12 \times$ area of the OEF triangle $= 12$.

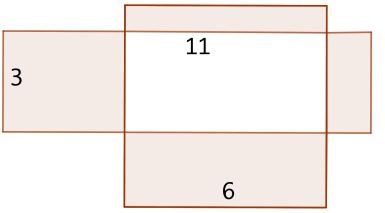
(2) the middle square of the cross is equal to 4 small squares, so its area should be 8.

Hence, the area of the cross is $4 \times 12 + 8 = 56$.

Problem 6.3 (A)

A square side length 6 and a 3×11 rectangle intersect each other as show below.

What is the perimeter of the shaded area?



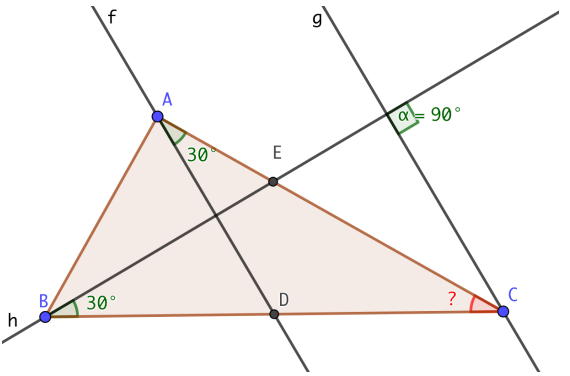
Discussion: At first, you would be tempted to count the length of each segments or the perimeters of the four shaded rectangles. That can help as well. However it would take time and can introduce some mistakes.

Solution: The easiest way is to think about using the perimeters of both square and rectangle and subtracting the overlapping segments. Fortunately, in this case it is easy to see that there is no overlapping. The total perimeter is $2 \times (3 + 11) + 4 \times 6 = 52$.

Note: sometime it is faster to count the total, and removing the repetitions.

Problem 6.4 (B)

ABC is a triangle. D is a point on BC such as $\angle DAC = 30^\circ$. f is a line through AD, g is a line through C and parallel with f. h is a line through B and perpendicular with g and intersects with AC at E. $\angle CBE = 30^\circ$.



What is $\angle BCA$?

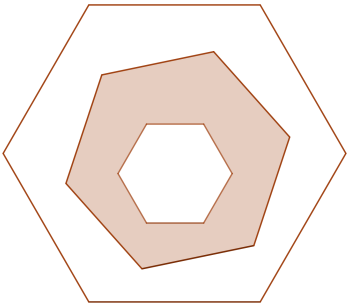
Discussion: $f \parallel g$, $h \perp g \Rightarrow h \perp f$

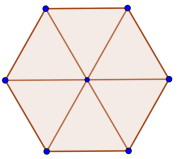
Solution: $BE \perp AD \Rightarrow \angle ADB = 90^\circ - 30^\circ = 60^\circ$

$\angle BDA = \angle DAC + \angle DCA \Rightarrow \angle DCA = 60^\circ - 30^\circ = 30^\circ$.

Problem 6.5 (B)

Three regular hexagons with side lengths 2, 4, 6 are nested inside each other as shown below.





What is the ratio of the area of the shaded figure to the area of the largest hexagon?

Discussion:

A hexagon can be divided into six congruent triangles by connecting its centre with vertices.

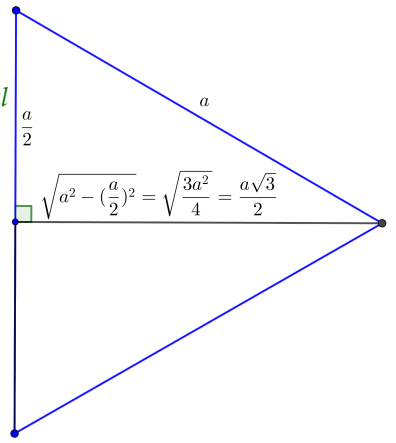
Note: Some properties of equilateral triangle

In an equilateral triangle, side length a:

(1) the height is: $\dfrac{a\sqrt{3}}{2}$

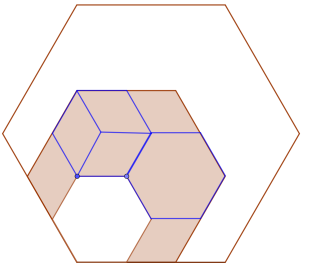
(2) the area is :

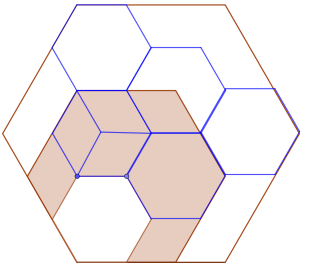
$$\frac{1}{2}\,.\,a\,.\,\frac{a\sqrt{3}}{2} = \frac{a^2\sqrt{3}}{4}$$



Solution 1: The area of the largest hexagon is $6 \times \dfrac{6^2\sqrt{3}}{4}$. The difference between the areas of the medium and the small hexagons: $6 \times \dfrac{4^2\sqrt{3}}{4} - 6 \times \dfrac{2^2\sqrt{3}}{4} = 6 \times \dfrac{12\sqrt{3}}{4}$ is the area of the shaded figure. The answer is $\dfrac{12}{6^2} = \dfrac{1}{3}$.

Solution 2: Let move the middle and small hexagons to one corner of the large hexagon. It is now easy to see that:
(1) the medium hexagon is 4 times the small hexagon.
(2) the large hexagon is 9 times the small hexagon.

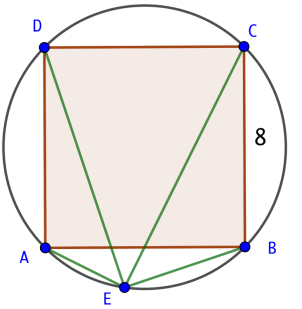




Therefore the ratio of the area of the shaded figure to the area of the largest hexagon is 3 times the small hexagon over 9 times the small hexagon. It is one-third.

Problem 6.6 (B)

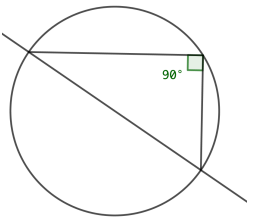
Square ABCD with side length 8 is inscribed in a circle. Point E is on the circle.



Prove that $AE^2 + BE^2 + CE^2 + DE^2$ is a constant and not dependent on the position of E.

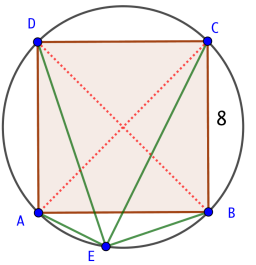
Discussion: $\triangle AEC$, $\triangle ABC$ are right triangles.

Note: If one side of a triangle is the diameter of the circumcircle, then it is a right triangle



Solution:

Because $\triangle AEC$, $\triangle ABC$ are right triangles.



$AE^2 + CE^2 = AC^2 = AB^2 + BC^2 = 2 \times 8^2$

Similarly $DE^2 + BE^2 = 2 \times 8^2$.

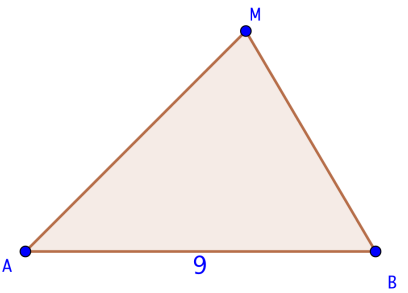
So

$AE^2 + BE^2 + CE^2 + DE^2 = 4 \times 8^2 = 256$

Problem 6.7 (C)

AB is a line segment length 9.

Find all points M such that $AM^2 - BM^2 = 9$.



Discussion: Very tricky problem! The real issue is that what would be the starting point? What given is a line segment only.

However the $AM^2 - BM^2 = 9$ equation remind us that *squares of line segment usually related to Pythagorean Theorem*. That's why it worth to find some right triangles! Perhaps the most logical approach would be to draw a perpendicular line from M to AB and intersects AB at C.

Solution:

For $\triangle ACM$, $\triangle BCM$:

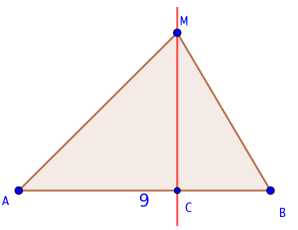
$AM^2 = AC^2 + CM^2$, $BM^2 = BC^2 + CM^2$

$\Rightarrow AM^2 - BM^2 = AC^2 - BC^2$

$\Rightarrow AM^2 - BM^2 = (AC - BC)(AC + BC)$

$\Rightarrow 9 = AM^2 - BM^2 = 9(AC - BC)$

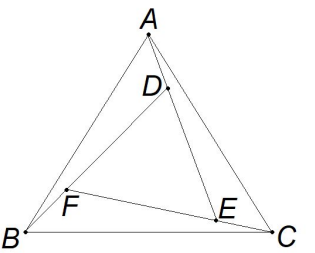
$\Rightarrow AC - BC = 1 \Rightarrow AC = 5,\ BC = 4.$



This means that *the points M satisfy the equation $AM^2 - BM^2 = 9$ are on the perpendicular line of AB, through the C point which is at a distance 5 from A, and a distance 4 from B*. This is the whole **red** line on the diagram on the right.

Problem 6.8 (C)

In the diagram below, points D, E, and F are on the inside of equilateral △ABC such that D is on AE, E is on CF, F is on BD, and the triangles △AEC, △BDA, and △CFB are congruent. Given that AB = 10 and DE = 6, the perimeter of △BDA is $\dfrac{a+b\sqrt{c}}{d}$ where a, b, c, d are positive integers, b, d are relatively prime, and c is not divisible by the square of any prime.



Find $a + b + c + d$.

Discussion:

Let $x = BF = DA$. $\angle BDA = 120°$, $AB = 10$, $AD = x$, $BD = 6 + x$.

Cosine theorem for any XYZ triangle:

$$YZ^2 = XY^2 + XZ^2 - 2.XY.XZ.\cos\angle YXZ.$$

Solution: Applying the Cosine theorem for $\triangle BDA$:

$$10^2 = x^2 + (x+6)^2 - 2x(x+6)(-\frac{1}{2})$$

$$\Rightarrow 3x^2 + 18x - 64 = 0 \Rightarrow x = \frac{-9+\sqrt{273}}{3}$$

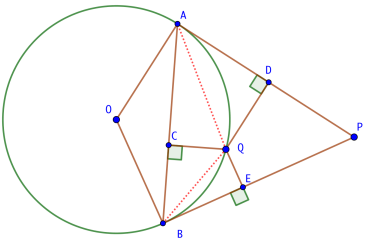
$\triangle BDA$ perimeter is $2x + 10 + 6 = \dfrac{30 + 2\sqrt{273}}{3}$.

This means $a + b + c + d = 308$.

Problem 6.9 (C)

From an external point P, draw two tangent ray PA, PB to a circle centre O. Q is a point on the minor arc AB. C, D, E are the intersections of perpendicular lines from Q to AB, PA, PB.

Prove that $QC^2 = QD.QE$.



Discussion: Here two important properties should be noted:

(1) For $\angle DAQ$, which tangent the circle at A: $\angle DAQ = \frac{1}{2}\widehat{AQ}$

(2) For $\angle ABQ$: $\angle ABQ = \frac{1}{2}\widehat{AQ}$

Solution: $\triangle DAQ \sim \triangle CBQ \Rightarrow \frac{QD}{QC} = \frac{QA}{QB}$, $\triangle EBQ \sim \triangle CAQ$ $\Rightarrow \frac{QE}{QC} = \frac{QB}{QA}$. Therefore $\frac{QD}{QC} \cdot \frac{QE}{QC} = 1 \Rightarrow QC^2 = QD \cdot QE$

Puzzle 6.10:

A telephone call interrupts a man after he has dealt about half of the cards in a bridge game. When he returns to the table, no one can remember where he had deal the last card. Without learning the number of cards in any of the four partly dealt hands, or the number of cards yet to be dealt, how can he continue to deal accurately, everyone getting exactly the same cards he would have had if the deal had not been interrupted?

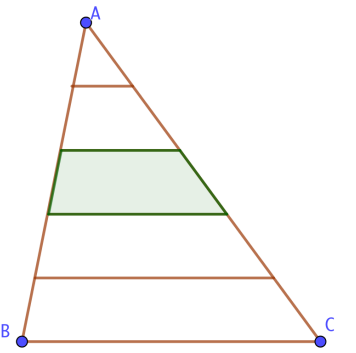
Note: In a bridge game the cards are dealt one at a time, starting with the player to the dealer's left and moving in a clockwise rotation until each player has 13 cards. Wait until the dealer distributes all the cards before you pick up your hand. That's bridge etiquette lesson number one.

Exercises

Problem 6.11 (A)

The triangle ABC in the diagram below has an area of 5. Four lines parallel with BC dividing AB in five equal parts.

What is the area of the **green** shaded figure?



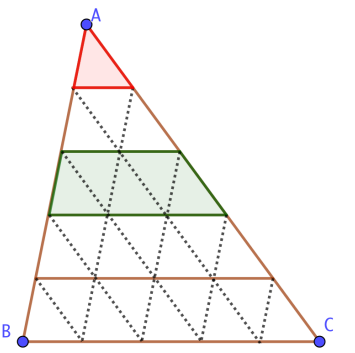
Discussion: When having lots of parallel lines involving in area calculation, a simple way to explore the problem to gain a better understanding is to *draw more parallel lines*! The point is to *divide a certain figure into smaller equal pieces*.

Solution: By drawing parallel lines from the intersection points on AB and AC, the original diagram becomes the figure. Let look at the *layers*, created by the original parallel lines. The number of small triangles with the same area as the **red** one on each layers are: 1, 3, 5, 7, 9.

The shaded green figure is made up by 5 red triangles.

(1) The areas of each layers are consecutive odd times the area of the red triangle: 1, 3, 5, 7, 9.

(2) The area of ABC is the sum of the first consecutive five odd number of the area of red triangle. From chapter 1, we know that the area of ABC is $1 + 3 + 5 + 7 + 9 = 5^2 = 25$ times the area of the red triangle.

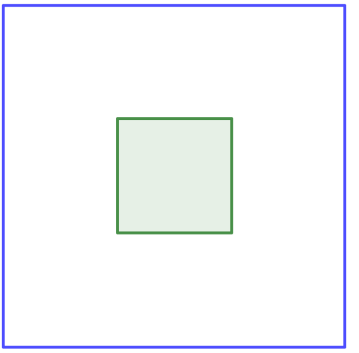


From here it is easy to see that the area of the shaded green figure is one-fifth of the area of the ABC triangle.

Problem 6.12 (A)

Jennie wants to plant flowers in a 3 feet by 3 feet square garden and a stone walkway 3 feet wide around the garden. She wants the outer edges of the walkway to form a square too.

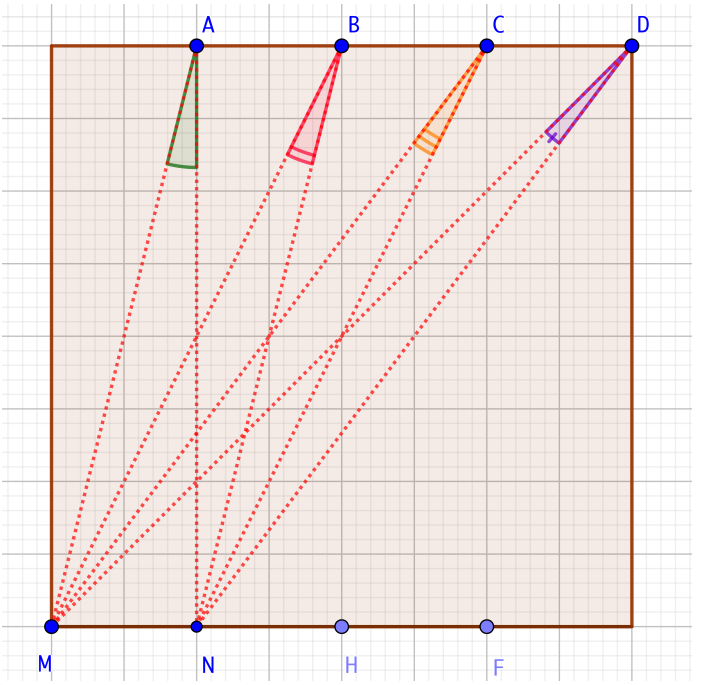
If the flowers cost \$4 per square foot and the walk way cost \$2 per square foot, how much money she has to ask her mom to give her?



Solution: The entire area is a 9×9 square, $3 \times 3 = 9$ square foot of garden, and $81 - 9 = 72$ square foot of walk way. So the cost for the garden is $9 \times 4 + 72 \times 2 = 36 + 144 = 180$.

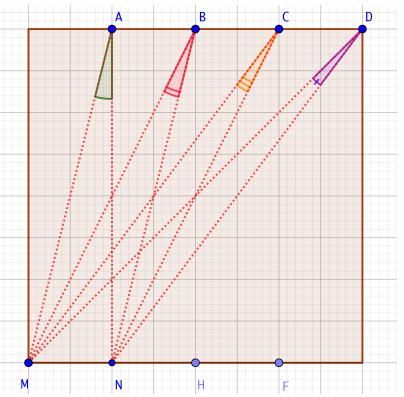
Problem 6.13 (A)

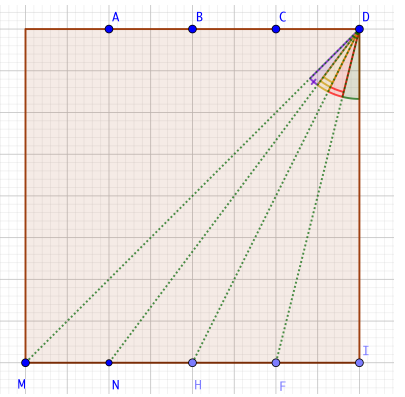
Find the sum of the angles $\angle MAN + \angle MBN + \angle MCN + \angle MDN$ in the right figure below.



Discussion: One of the approaches would to measure each angle. The key to the solution in this problem is to find *similar angles that can complement each other* and their sum can form an easy-to-measure angle.

Solution: If the angles in the figure on the left can be replaced with the angles in the figure on the right as above, then it is very simple to see that the sum of the angles is actually 45°.

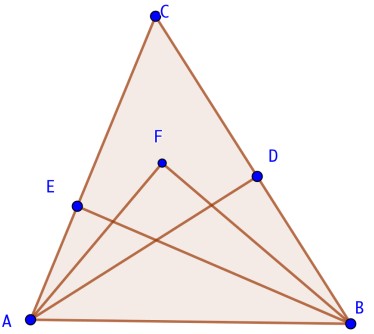




Note: can you generalize this problem?

Problem 6.14 (B)

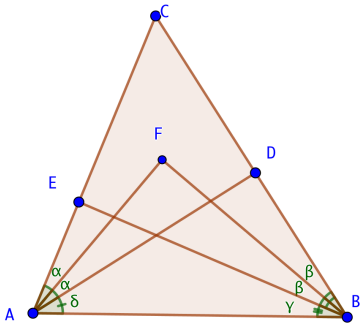
E and D are interior points of the sides AC and BC of the triangle ABC. AF bisects $\angle CAD$, BF bisects $\angle CBE$ as shown in the diagram on the right below.



Prove that: $\angle AEB + \angle ADB = 2\angle AFB$

Discussion: In geometry, *problem solving usually starts with visual recognition of facts*. It is very important that you try to denote (assign symbols to objects) line segments, angles, area, etc by some simple symbols. It helps to realize better the relationships between geometric elements.

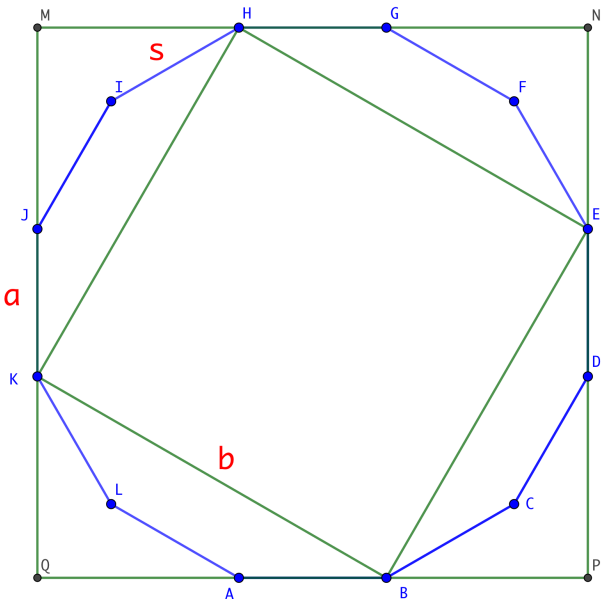
Solution: By denoting the angles as the figure below:



$\angle ADB = 180^\circ - (2\beta + \gamma + \delta)$, $\angle AEB = 180^\circ - (2\alpha + \delta + \gamma)$,
$\angle AFB = 180^\circ - (\alpha + \delta + \beta + \gamma) \Rightarrow \angle AEB + \angle ADB = 2\angle AFB$

Problem 6.15 (B)

A 12-sided polygon with side length s is drawn inside a square with side length a. Four vertices of the polygon are selected and another square with side length b is created inside the polygon.



Prove that $a = b + s$.

Discussion:

The sum of all internal angles of a $n-$sided regular polygon is $\dfrac{(n-2)\times 180^\circ}{n}$.

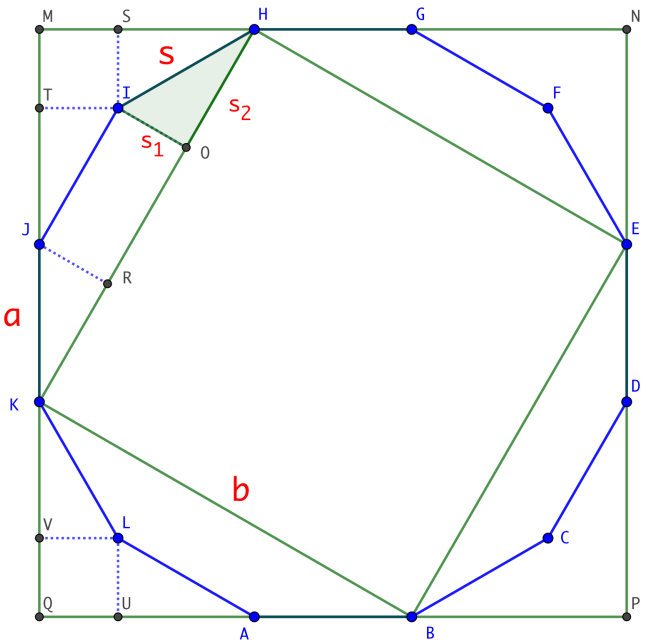
You can prove this by using *triangulation technique (dividing into triangles)* shown in Problem 6.1:

(a) first dividing the $n-$sided regular polygon into $(n-2)$ triangles,

(b) count the sum of all angles in all of those triangles,

(c) then dividing by the number of internal angles of a $n-$sided regular polygon.

Solution:



The key to solve this problem is to compare a, b, s to smaller, common segments:

(1) prove that $\angle GHI = 150^\circ$

(2) prove that $\angle IHK = \angle EHG = 30^\circ$

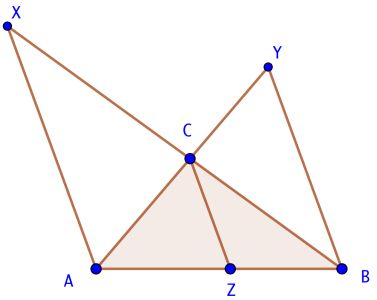
(3) compare $\triangle SHI$, $\triangle TIJ$, $\triangle IHO$

(4) prove $a = s + 2s_1 + 2s_2$, $b = s + 2s_2$.

(5) Use $\triangle IHO$ to prove $s = 2s_1$.

Problem 6.16 (B)

In $\triangle ABC$, Z is a point on AB. Lines through A, B and parallel with CZ intersect line BC and AC at X and Y.



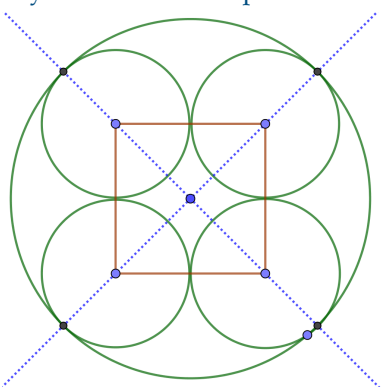
Prove that $\dfrac{1}{AX} + \dfrac{1}{BY} = \dfrac{1}{CZ}$

Discussion: Since $CZ \parallel AX \Rightarrow \triangle BCZ \sim \triangle BXA \Rightarrow \dfrac{BZ}{BA} = \dfrac{CZ}{AX}$.
Similar equality can be obtained by exploring $CZ \parallel BY$.

Solution:
$$\frac{BZ}{BA} = \frac{CZ}{AX}, \frac{AZ}{AB} = \frac{CZ}{BY} \Rightarrow \frac{1}{AX} + \frac{1}{BY} = \frac{1}{CZ}\left(\frac{BZ}{BA} + \frac{AZ}{AB}\right) = \frac{1}{CZ}$$

Problem 6.17 (C)

A carpenter wishes to cut four equal circles from a circular piece of wood whose area is 9π square foot. He wants these circles of wood to be the largest that can possibly be cut from this piece of wood.



Find the measure of the radius of each of the four new circle.

Discussion:

(1) The four equal circles should tangent with the large circle and another two circles

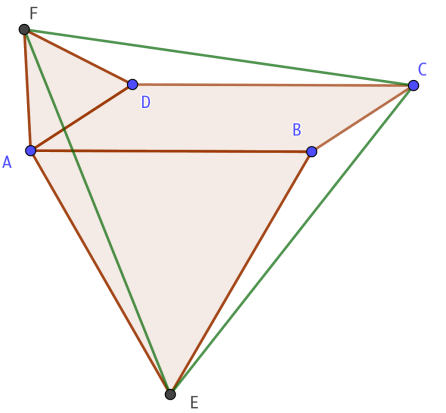
(2) Their centres would form a square

(3) The large circle's centre would be on the line through the centres of two opposite circles.

Solution: The radius of the large circles is $radius = \sqrt{\frac{area}{\pi}} = \sqrt{\frac{9\pi}{\pi}} = 3$. According to the figure, if r is the radius of the small circle, then the side length of the square is $2r$, which makes the diagonal of the square is $2r\sqrt{2}$. Therefore the diameter of the large circle can be calculated in two ways: $2r + 2r\sqrt{2} = 6 \Rightarrow r(1+\sqrt{2}) = 3 \Rightarrow r = \frac{3}{1+\sqrt{2}} = 3(\sqrt{2}-1) = 3\sqrt{2}-3$

Problem 6.18 (C)

ABCD is a parallelogram with equilateral triangles ADF and ABE drawn on the sides AD and AB, respectively.



Prove that $\triangle FCE$ is an equilateral triangle.

Discussion: Compare $\triangle FDC$, $\triangle CBE$, $\triangle FAE$.

Solution: All three triangles $\triangle FDC$, $\triangle CBE$, $\triangle FAE$ are congruent.

For example $\triangle FDC$, $\triangle CBE$ are congruent because

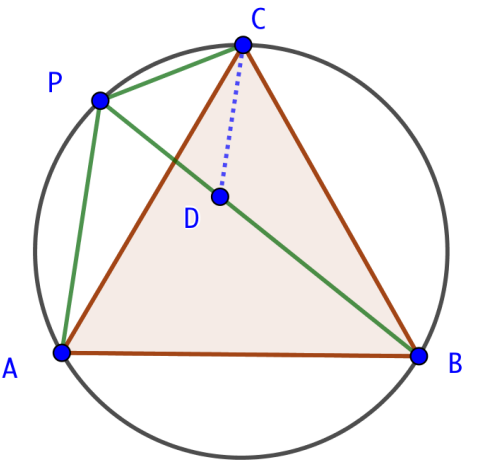
(1) $FD = AD = BC$, $DC = AB = BE$

(2) $\angle FDC = 360^\circ - \angle FDA - \angle ADC = 360^\circ - \angle ABE - \angle ABC = \angle CBE$

Hence $FC = CE$. Similarly $FC = FE$. $\triangle FCE$ is an equilateral triangle.

Problem 6.19 (C)

$\triangle ABC$ is an equilateral triangle and is inscribed in a circle with centre O. P is an point on the minor arc $\widehat{AC}$.



Prove that $PB = PA + PC$.

Discussion: Let D be a point on BP such that $BD = AP$. Compare $\triangle APC$, $\triangle BDC$.

Solution: $\triangle APC$, $\triangle BDC$ are congruent because:

(1) $AP = BD$, $AC = BC$

(2) $\angle PAC = \angle DBC$

Hence $\angle CDB = 120^\circ \Rightarrow \angle PDC = 60^\circ$. Because $\angle DPC = 60^\circ$ too, $\triangle PDC$ is equilateral.

So $PB = PD + DP = AP + PC$.

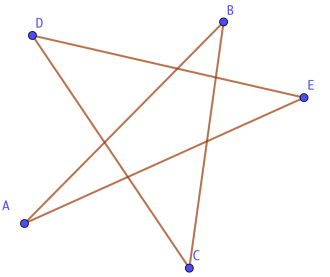
Puzzle 6.20:

The integers 1, 3, 8, 120 form a set with remarkable property: *the product of any two integers is one less than a perfect square*. Find a fifth number that can be added to the set without destroying this property.

Problems

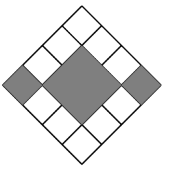
Problem 6.21 (A)

What is the sum of all internal angles of a five pointed stars?



Discussion: Connect the vertices to create an convex pentagon, try complementary counting.

Problem 6.22 (A)

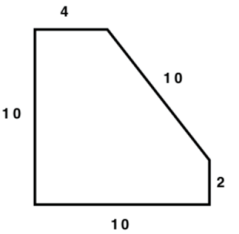


What is the ratio of the grey area to the white area?

Discussion: Count the number of small squares made up the gray area.

Problem 6.23 (A)

The pentagon below has three right angles.

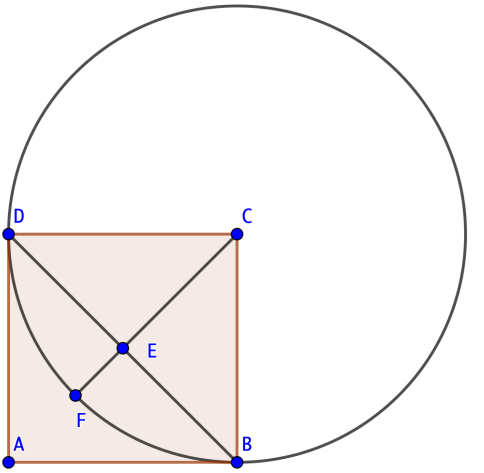


Find its area.

Discussion: The figure looks like a rectangle missing a triangle.

Problem 6.24 (B)

Square ABCD has side length 1. Minor arc $\widehat{BD}$ of a circle with radius 1 centred at C is drawn along with diagonal BD. Let CE be the altitude of $\triangle BCD$ drawn from C, and is extended to intersects BD at F.

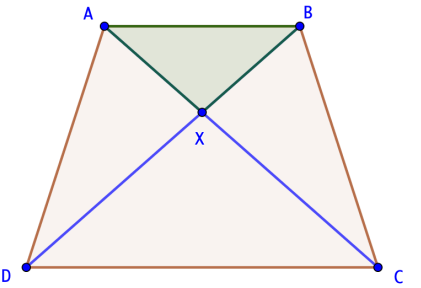


What is the value of 2EF?

Discussion: Since $EF = CF - CE$, let compute CE, CF.

Problem 6.25 (B)

Trapezoid ABCD has base lengths $AB = 30cm$, $CD = 54cm$ and congruent 56cm diagonals. The diagonals intersect each other at X.



What is the area of ABX?

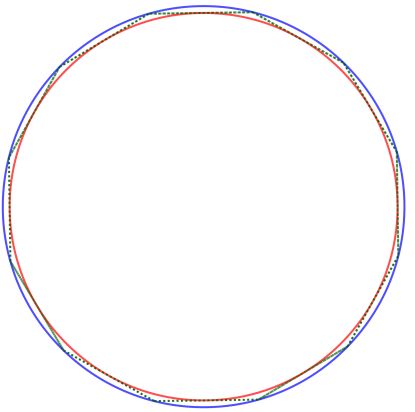
Discussion:

(1) Prove that $\triangle ABX \sim \triangle CDX$. What would be the value of BX?

(2) Knowing BX, try to figure out the height from X of triangle ABX.

Problem 6.26 (B)

Is the difference in area between the inscribed circle and circumscribed circle in a regular polygon of edge length 2cm depending on the number of its vertices? Below is an example of 12-gon.



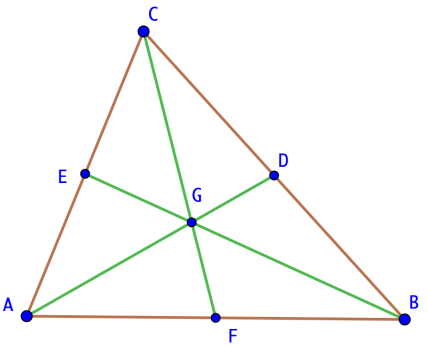
Discussion:

(1) Draw the regular polygon and two circles

(2) Find a right triangle with side lengths equals radiuses of the inscribed circle and circumscribed circle.

Problem 6.27 (C)

For $\triangle ABC$, D, E, F are midpoints of BC, AC, and AB.



Prove that :

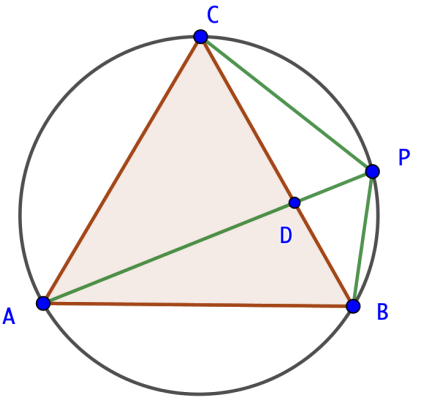
$$\frac{3}{2}(AB + BC + CA) > AD + BE + CF > \frac{3}{4}(AB + BC + CA)$$

Discussion: Use triangle inequality and remember $CF = 3CG$.

Problem 6.28 (C)

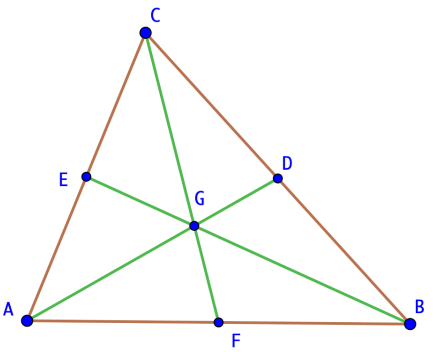
A line is drawn from vertex A of equilateral $\triangle ABC$, meets BC at D and the circumcircle at P.

Prove that $\dfrac{1}{PD} = \dfrac{1}{PB} + \dfrac{1}{PC}$.



Discussion: Compare $\triangle APC$, $\triangle BPD$

Problem 6.29 (C)



In $\triangle ABC$, D is midpoint of BC, E is midpoint of AD, F is midpoint of BE, and G is midpoint of CF.

What is the ratio of the area of $\triangle EFG$ to the area of $\triangle ABC$?

Discussion:

(1) Compare $\triangle EFG$, $\triangle EFC$

(2) Compare $\triangle EFC$, $\triangle AFC$

Puzzle 6.30:

There is a way to place +, - signs between the digits 1 2 3 4 5 6 7 8 9 to obtain 100, for example:

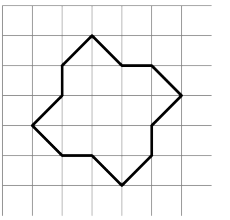
$$1 + 2 + 34 - 5 + 67 - 8 + 9 = 100$$

Find a way to place *a fewest number of +, - signs* between the digits 9 8 7 6 5 4 3 2 1 to obtain 100.

Challenges

Problem 6.31 (A)

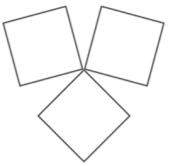
The twelve-sided figure shown has been drawn on $1\ cm \times 1\ cm$ graph paper. What is the area of the figure in cm^2?



Discussion: Count the unit squares, add up the triangles

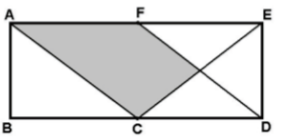
Problem 6.32 (A)

The diagram below shows a 12-sided figure made up of three congruent squares. The figure has total perimeter 60. Find its area.



Discussion: What is the side of each square?

Problem 6.33 (A)



The diagram below shows rectangle ABDE where C is the midpoint of side BD, and F is the midpoint of side AE. If AB=10 and BD=24, find the area of the shaded region.

Discussion: Connect F with C. What can you say about the area of $\triangle AFC$?

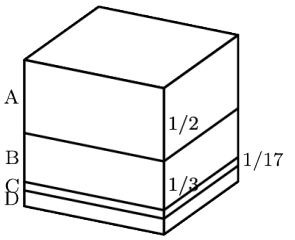
Problem 6.34 (B)

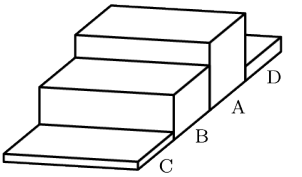
The smallest circle has radius 2 inches, with each successive circle's radius increasing by 2 inches. Approximately what percent of the design is black?

Discussion: Create a table listing the areas of the circles, one by one.

Problem 6.35 (B)

A one-cubic-foot cube is cut into four pieces by three cuts parallel to the top face of the cube. The first cut is $1/2$ foot from the top face. The second cut is $1/3$ foot below the first cut, and the third cut is $1/17$ foot below the second cut. From the top to the bottom the pieces are labeled A, B, C, and D. The pieces are then glued together end to end as shown in the second diagram.





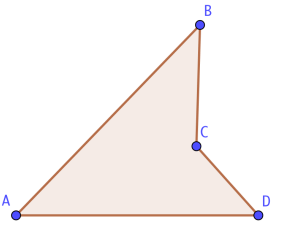
What is the total surface area of this solid in square feet?

Discussion: The top parts and the bottom parts sum to 8. How about the sides ?

Problem 6.36 (B)

A sail, as shown in the diagram, has the form of the quadrilateral ABCD. The angles A, B, D measure 45° each.

If $AC = 4\ m$, find the area of the sail.



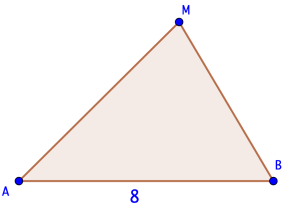
Discussion: Calculate $\angle C$, then create a line through C, perpendicular with AD.

Problem 6.37 (C)

AB is a line segment with length 8.

Find all points M such that:

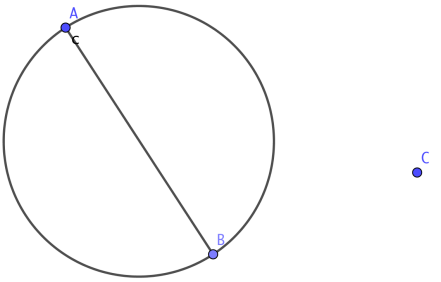
$AM^2 + BM^2 = 50.$



Discussion: Very similar to 6.7

Problem 6.38 (C)

AB is a diameter of a circle. C is a point outside of the circle.

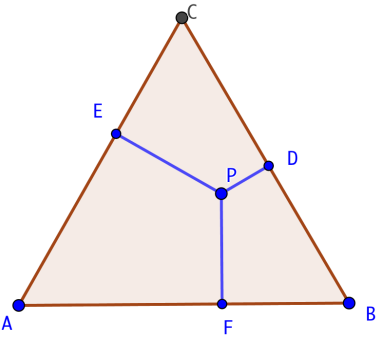


Use only a ruler, draw a line through C perpendicular with the line through AB.

Discussion: In a triangle, three altitude intersect each other at a single point.

Problem 6.39 (C)

$\triangle ABC$ is equilateral. P is a point inside $\triangle ABC$.



Prove that the sum of the distances from P to AB, BC, CA is not dependent on the position of P.

Discussion: The distance from P to side AB can be considered as the height of triangle PAB.

Puzzle 6.40:

Two brothers inherited a herd of sheep. They sold all of them, receiving for each sheep the same number of dollars as there were sheep in the herd.

The money was given to them in $10 bills except for an excess amount, less than $10, that was in silver dollars. They divided the bills between them by placing them on the table and alternately taking a bill until there were none left. "It isn't fair," complaining the younger brother. "You drew first and you also took the last bill, so you got $10 more than I did." To even things up partially the older brother gave the younger on all the silver dollars, but the younger was still not satisfied. "You gave me less than $10," he argued. "You still owe me some money."

"True," said the older brother. "Suppose I write you a cheque that will make the total amounts we each end up with exactly the same." This he did.

What was the value of the cheque?

Chapter 7

Ratio - Average - Percent

Examples

Problem 7.1 (A)

Manuel the oldest, Nicolás, and Olivia the youngest are three siblings in a family. Santiago, their father calculate the age averages of each sibling pair and receives 5, 8, and 11. What is the age average of all of them?

Solution: The average of two numbers is half of their sum. This means that:

$$\text{Manuel's age} + \text{Nicolás' age} = 22$$

$$\text{Manuel's age} + \text{Olivia's age} = 16$$

$$\text{Nicolás' age} + \text{Olivia's age} = 10$$

$$2 \times \text{Manuel's age} + 2 \times \text{Nicolás' age} + 2 \times \text{Olivia's age} = 48$$

The sum $22 + 16 + 10 = 48$ contains twice the ages of each child. The sum of their ages is 24.

$$\text{Manuel's age} + \text{Nicolás' age} + \text{Olivia's age} = 24$$

The average of their ages is one-third of the sum, or 8.

Problem 7.2 (A)

40% of class 5A of No-homework-at-all school are boys. One-third of class 5A girls are members of the school music club. The ratio of boy to girl of class 5A students in the club is $1 : 2$.

If class 5A has 30 students, how many students are music club members?

Solution*:*

The class has 30 students, so a 10% of the class is 3 students, 50% of the class is 15 students. Because 40% of students in class 5A are boys, so $100 - 40 = 60\%$ of them are girls.

It means that 60% of the class are girls and that number (60%=10% +50%) is $3 + 15 = 18$.

One-third of the girls is $18 \div 3 = 6$ girl member of the music club. Because the number of boys in the club is half of the girls - for every 1 boy there are 2 girls - so there are 3 boys.

The total number of class 5A students in the music club is $6 + 3 = 9$.

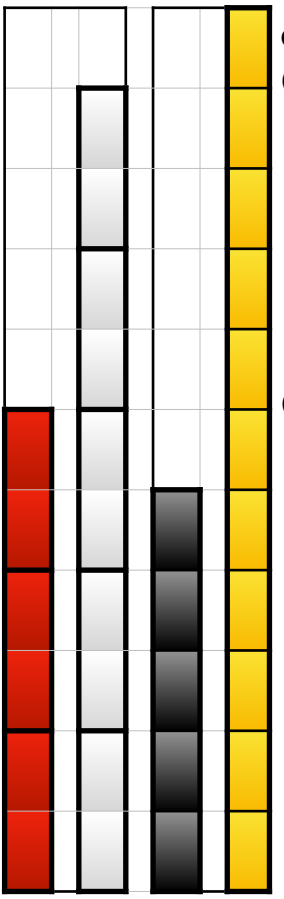
Problem 7.3 (A)

Billy has a pile of some red and white marbles. The ratio of red to white marbles is 3 : 5, in another word for every 3 red marbles there are 5 white marbles. Linda has a pile of some black and yellow marbles. The ratio of black to yellow marbles is 5 : 11, in another word for every 5 black marbles there are 11 yellow marbles.

Tina selects a number of white marbles from Billy's piles such that the remaining white marbles is equal with the number of red marbles. She then selects a number of yellow marbles from Linda piles such that the remaining yellow marbles is equal with the number of black marbles. Now she has a pile of white and yellow marbles.

What is the ratio of white to yellow marbles in Tina's pile if initially the number of marbles in Billy's and Linda's piles is 64?

Solutions*:* Drawing is fun, let do that.



Our drawing on the left shows two piles:

(1) The one on the left contains 3 equal small piles of red marbles, 5 equal small piles of white marbles. This indicates that the ratio of red to white marbles is 3 : 5. There are 8 such small piles. Because there are 64 marbles in total, so a small pile will contain $64 \div 8 = 8$ marbles. Tina likely takes two small piles of white marbles, so the number of white marbles she takes is $8 \times 2 = 16$.

(2) The other on the right consists of 5 equal small piles of black marbles, and 11 equal small piles of yellow marbles. This indicates that the ratio of black to yellow marbles is 5 : 11. There are 16 such small piles. Because there are 64 marbles in total, so a small pile will contain $64 \div 16 = 4$ marbles. Tina likely takes six small piles of yellow marbles, so the number of yellow marbles she takes is $4 \times 6 = 24$.

Now it time to calculate the the ratio of yellow to white marbles in Tina's pile. She has 16 white marbles to 24 yellow marble or for every 2 white marbles there are 3 yellow marbles.

So the ratio is 2 : 3.

Problem 7.4 (B)

The Kartoffel family has different-sized scoops to serve mashed potato for dinner: blobs, globs, and gobs. Three blobs are equal to two glob, and five globs are equal to three gobs of mashed potato. The bowl for dinner holds 12 gobs of mashed potato. A serving contains six blobs.

How many member does the Kartofell family have if:

(1) they comsumes the whole bowl

(2) each of them has exactly one serving?

Note: Kartoffel is potato in German.

Solution:

12 gobs = 4 × (3 gobs) = 4 × (5 globs) = 10 × (2 globs) = 10 × (3 blobs) = 5 × (6 blobs).

So the family has 5 members.

Note: To make more fun, you can invent extra words like clobs, nobs, or anything you like. However, be careful with the proportion (ratio) so that it still does make sense: $\frac{3}{19}$ *knobs sound very unusual.*

Problem 7.5 (B)

Tarantula the spider has to do some exercises in order to get healthy. After a long discussion with all the spider doctors and performing searches on all the *webs* that he knows, he creates a *health building strategy* is as following:

(1) He walks 1 meter on the first day.

(2) Every subsequent day he walks an additional 1 meter comparing to the previous day.

He tells his mother that after a few days the average distance he exercise will be 3 meters per day. However, Tarantula mother insists that Tarantula has to increase the average distance to 5 meters per days.

How many more days Tarantula has to do daily exercise in order to get to that average?

Solution: Let observes a few cases.

The number of days	Total distance (m)	Average (m)
1	1	1
2	$1+2=3$	$\frac{3}{2}$
3	$1+2+3=6$	2
4	$1+2+3+4=10$	$\frac{5}{2}$

It is easy to see that by each day, the average is increased by 0.5m. It means that to achieve $5-3=2$ more meters in average, he has to walk 4 more days. This sounds easy and simple, but can we prove that *the average distance is increased by* $0.5m$ *per day*? Let assume Tarantula walks n days. Easy to see that the total distance he walks is: $1+2+3+\ldots+(n-1)+n=\dfrac{n(n+1)}{2}$

Thus, the average distance after n^{th} days is: $\dfrac{n(n+1)}{2n}=\dfrac{n+1}{2}$

Using this newly found formula, we can calculate that on day $(n+1)^{th}$ the average distance is $\dfrac{(n+1)+1}{2}=\dfrac{n+2}{2}$.

Hence the difference in daily average is $\dfrac{n+2}{2}-\dfrac{n+1}{2}=\dfrac{1}{2}$m.

Problem 7.6 (B)

Victor the local mobster was elected by the gangsters to be a new boss. He wanted to wanted to be rich in a short amount of time but he want *equality* among his gang.

The shops pay him 20% of their monthly income. His top lieutenant takes 20% of what's left.

How many percent of the remaining should he allow the rest of his gang to take so that the total amount of money his top lieutenant and his gang take from the shop is equal to what he takes?

Solution:

Victor takes 20% of the income. His top lieutenant takes 20% of the remaining $100-20=80\,\%$, which is $0.2\times 80\,\%=16\,\%$ The remaining is $80-16=64\,\%$. The gang should take 4% and that is $\dfrac{4}{64}=\dfrac{1}{32}=6.25\,\%$.

Problem 7.7 (C)

How many gallons of a solution which is 15% alcohol do we have to mix with a solution that is 35% alcohol to make 250 gallons of a solution that is 21% alcohol?

Solution: Let x be the number of gallons of 15% alcohol solution that we need to mix. To make 250 gallon we will need $250 - x$ gallons of 35% alcohol solution. This mix contains $0.15x + 0.35(250 - x)$ gallons of alcohol which should be 21% alcohol solution:

$0.15x + 0.35(250 - x) = 0.21(250) \Rightarrow 15x + 35(250) - 35x = 21(250)$

$\Rightarrow 14(250) = 20x \Rightarrow x = 7.25 = 175$ gallons.

Problem 7.8 (C)

Van finished the baseball week with her batting average was 0.36 (36 hits per 100 attempts). After five trainings this week, her average was improved to 0.4. What is the least number of hits Van could have had in last week?

Solution: Suppose that she had been at bat n times during the season. The number of hits she would have made $0.36n$. Suppose that she made m hits during the five trainings: $0.36n + m = 0.4(n + 5) \Rightarrow m = 0.4n + 2$.

Since $n > 0$, so the smallest value of m can be is 3, when $n = 25$. Hence the least number of hits Van could have had in the season is $0.36 \times 25 = 9$.

Problem 7.9 (C)

Given $\frac{a}{b} = \frac{c}{d}$. Prove that: $\frac{ac}{bd} = \frac{a^2 + c^2}{b^2 + d^2} = \frac{(a + c)^2}{(b + d)^2}$

Solution: If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a + b}{b} = \frac{c + d}{d}$, $\frac{a - b}{b} = \frac{c - d}{d}$, and $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{ac}{bd} = \frac{a^2}{b^2} = \frac{c^2}{d^2}$. The equalities can be easily proven.

Puzzle 7.10: Earrings

In a certain African village there live 800 women. Three percent of them are wearing one earring.

Of the other 97 percent, half are wearing two earrings, half are wearing none. How many earrings all together are being worn by the women?

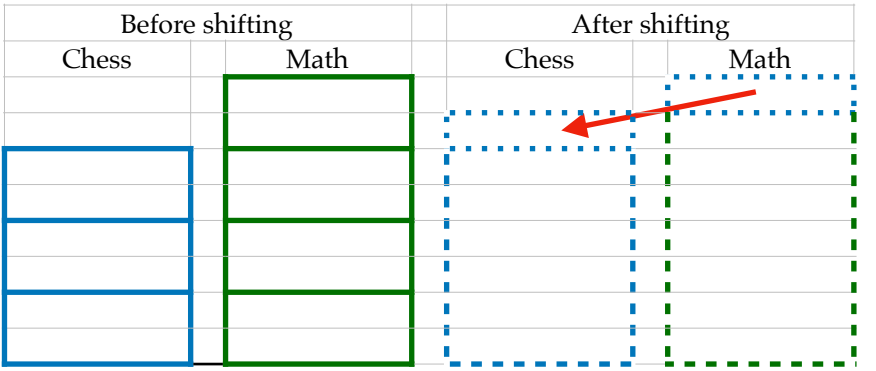
Note: there is a surprisingly simple answer to this puzzle.

Exercises

Problem 7.11 (A)

The ratio of Math and Chess students in a club is $4:3$. If 14 Math students shift to Chess then the ration becomes $1:1$. What is the total number of students in the club?

Discussion: *Let draw a figure.*



It is easy to see that if half of one-quarter of Math students shift to Chess, then the ratio becomes $1:1$, and this is 14 students. Using the diagram above, answer these questions:

(1) What is the number of Math students?

(2) What is the number of Chess students?

Solution: From the above diagram, we see that the students can be divided into 7 boxes, each twice of 14. So the number of students is $7 \times 2 \times 14 = 196$.

Note: drawing a figure is very powerful way to solve problem. In fact, if you are familiar with drawing figure, you will gain insights of the problem and that is the most important thing in problem solving.

Problem 7.12 (A)

What is the average of the first 50 odd integers: $1 + 3 + 5 + \ldots + 99$?

Discussion: We can solve the problems by using what we know from Problem 1.2 (A) how to calculate the sum of the first n odd integers.

Solution: The sum of the first 50 odd integers is $50^2 = 2500$. Their average is $\dfrac{2500}{50} = 50$. Now, can you make a *"formula for calculate the average of the first n odd integers"*?

Problem 7.13 (A)

Business is getting slow in London's Super Shoe Store, running by Clemence. So she decides to have a sale. On Friday, she increases all Thursday's prices by 10%. Over the weekend, she advertises: *"Ten percent off the listed price. Sale starts on Monday."*

How much does a pair of shoes cost on Monday if that costs \$40 on Thursday?

Discussion*:* It's simple, just starts with the Thursday price and follow Clemence's actions.

(1) How much does it cost on Friday after 10% increase in price?

(2) How much does it cost on Monday after reducing 10% off the Friday price?

Solution: On Friday, the price of the shoe is $40 + 4 = 44$. On Monday it is $44 - 4.4 = 39.6$. Now this is fun question *"how much does Clemence lose on that \$40 pair of shoes after this smart campaign?"*.

Problem 7.14 (B)

Minh the skater wants to raise money for a skating events. He purchases a dozen oranges of the same size and a dozen apples of the same size. He can extract 8 cups of apple juice from 3 apples and 8 cups of orange juice from 2 oranges. He makes a juice blend of apple and orange juice from an equal number of apples and oranges.

What percent of the blend is apple juice?

Discussion*:*

(1) From 3 apples he makes 8 cups, so 32 cups of apple juice will be made from 12 apple.

(2) How many cups of orange juice can be made from 12 oranges?

Solution: From 12 apples he makes 32 cups of apples juice. From 12 oranges he makes 48 cups. So the percent of apple juice in the blend is $\frac{32}{32+48} = \frac{32}{80} = \frac{2}{5}$ or 40%.

Problem 7.15 (B)

The average of three different positive integers is 7.

What would be the largest value of the middle number?

Discussion*:*

(1) What is the sum of the three different positive integers?

(2) How to obtain the largest value for the middle number?

Solution:

The sum of the three different positive integers is $3 \times 7 = 21$. Since the middle number can be largest if the difference between it and the largest number is as small as possible and the smallest number of the three number is also as small as possible. Both of these numbers cannot be 1 at the same time, otherwise the middle number is not an integer.

So one of them is 1 and another is 2, in that case the middle number is 9. There are two triplets (1,9,11) and (2,9,10).

Problem 7.16 (B)

Diana orders pizza for her family. Each pizza has 8 slices. Diana knows that:

(a) her brother has twice the slices as she does,

(b) the ratio of the slices that her mother has comparing to her father's is $2 : 3$,

(c) her mother has one slice less than she does,

(d) half a pizza will not be enough for anyone.

What is the minimum number of pizza she needs to order?

Discussion: The key questions are:

(1) Who has the smallest portion?

(2) How many slices does that smallest portion contain?

Solution:

Easy to see Diana's mother has least number of pizza slices. Let it be n be her number of slices. From (d) $n > 4$. Her father has $\frac{3}{2}n$, so n has to be even, so $n = 6$. Diana has $n + 1 = 7$, her brother has $2n + 2 = 14$, her father has 9.

So all together they have $6 + 7 + 14 + 9 = 36$. They must order 5 pizzas.

Problem 7.17 (C)

Jerry purchased some stock for \$14,400 at the same time that Susan purchased a bond for \$6,250. Jerry's investment went up 20% the first year, fell 10% the second year, and rose another 20% the third year. Susan's investment grew at a constant rate of compound interest for three years. If both investment are worth the same after three years, what was the annual percentage increase of Susan's investment?

Discussion:

(1) Jerry's investment value is :

$$14400(1.20)(0.9)(1.20) = 14400.\frac{6}{5}.\frac{9}{10}.\frac{6}{5} = \frac{2^7 3^6}{5}.$$

(2) Susan's investment is worth $6250(1+r)^3 = 2.5^5(1+r)^3$

Solution: $(1+r)^3 = \dfrac{2^6.3^6}{5^6} \Rightarrow 1+r = \dfrac{2^2.3^3}{5^2} = \dfrac{36}{25} \Rightarrow r = \dfrac{11}{25} = 0.44$

(44%)

Problem 7.18 (C)

Cindy's model of the CN Tower is 5 cm taller than Andy's model of the Calgary Tower. The shadow of his model is 3 cm shorter than the shadow of Cindy's model.

If her CN Tower is 120 cm tall, how long is the shadow of his Calgary Tower?

Discussion:

Use the fact that the shadows are proportional in the same way as the heights. Denote x as the length of the Calgary Tower's shadow, establish an equation and solve it.

Solution:

Let x be the length of shadow that the Calgary Tower model cast: $\dfrac{120}{120-5} = \dfrac{x+3}{x}$,

$120x = 115(x+3) \Rightarrow 5x = 3 \times 115 \Rightarrow x = 3 \times 23 = 69.$

Note: Thales (who died around 30 years before the time of Pythagoras and 300 years before Euclid) is often hailed as the first Greek mathematician. One of his famous discovery was when he tried to measure the height of the pyramid.

Problem 7.19 (C)

100 numbers listed as below:

$$99 \times 0.9,\ 999 \times 0.9,\ \ldots,\ \underbrace{99\ldots9}_{101\ 9's} \times 0.9$$

the first one has two 9's and the last one has 101 9's.

What is the average of these numbers?

Discussion: There are several ideas to be considered.

(1) Every number has a factor of 0.9

(2) $99 = 100 - 1 = 10^2 - 1$, $999 = 1000 - 1 = 10^3 - 1$,...

Solution: The sum of all the number is $99 \times 0.9,\ 999 \times 0.9,\ \ldots,\ \underbrace{99...9}_{101\ 9's} \times 0.9$

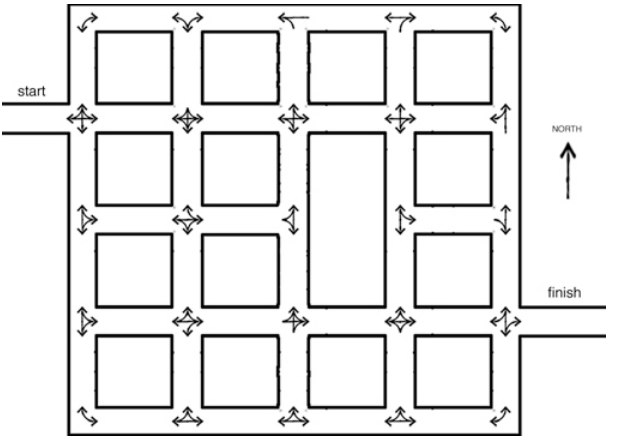
$$= 0.9(99 + 999 + \ldots + \underbrace{99...9}_{101\ 9's}) = 0.9(100 - 1 + 1000 - 1 + \ldots + \underbrace{10...0 - 1}_{101\ 0's})$$

$$= 0.9(\underbrace{11...100}_{100\ 1's} - 100) = \underbrace{9...90}_{100\ 9's} - 90 = \underbrace{9...900}_{99\ 9's}. \text{ So the average is } \underbrace{9...9}_{99\ 9's}.$$

Puzzle 7.20: How to travel

The town had only 37 registered cars. The mayor thought it would be safe to appoint his cousin, who was the town cutup, as its traffic commissioner. But he soon regretted his decision.

When the town awoke one morning, it found that a profusion of signs had been erected establishing numerous one-way streets and confusing restrictions on turns.



The citizens were all for tearing down these signs until the police chief, another cousin of the mayor, made a surprising discovery. Motorists passing through town became so exasperated that sooner or later they made a prohibited turn.

The police chief found that the town was making even more money from these violations than from its speed trap on an outlying country road.

Of course everyone was overjoyed, particularly because the next day was Saturday and Adam, the county's richest farmer, was due to pass through town on his way to the county seat. They expected to extract a large fine from Adam, believing it to be impossible to drive through town without at least one traffic violation.

But Adam had been secretly studying the signs. When Saturday morning came, he astonished the entire town by driving from his farm through town to the county seat without a single violation!

Can you discover the route Adam took?

At each intersection you must follow one of the arrows. That is, you may turn in a given direction only when there is a curved line in that direction, and you may go straight only when there is a straight line to follow. No turns may be made by backing the car around the corner. No U-turns are permitted. You may leave an intersection only at the head of an arrow. For instance, at the first intersection after leaving the farm, you have only two choices: to go north or to go straight. If you go straight, at the next intersection you must either go straight or turn south. True, there is a curved line to the north, but there is no arrow pointing north, so you are forbidden to leave that intersection in a northerly direction.

Problems

Problem 7.21 (A)

The ratio of cherry candies to grape candies in a bag is 3 : 5. After your friend eats half of the grape flavoured candies and only one cherry flavoured candy, the ratio of cherry to grape candies is 7 : 6.

How many candies are left in the bag?

Discussion: Draw a figure, use Problem 7.11 as an example.

(1) What is half of grape flavoured candies? How can you indicate it on the drawing?

(2) How about one cherry flavoured candy? How can you show it on the drawing?

Problem 7.22 (A)

Before a plum is dried to become a prune, it is 92% water. A prune is just 20% water.

If only water is evaporated in the drying process, how many pounds of prunes can be made with 100 pounds of plums?

Discussion:

(1) How much water is in 100 pounds of plum? How much of it *not water*?

(2) How many pounds of prune has that amount of *not water?*

Problem 7.23 (A)

The average age of 10 people in a room is 40 years. There is a teenager who is 13 years old. How much older, on average, are the rest of the people comparing to the teenager?

Discussion:

(1) What is the sum of age of all people?

(2) What is the sum of age of all people except the teenager?

Problem 7.24 (B)

Dennis has three test scores of 84, 90, and 86. There is one more test at the end of the semester. Since Dennis wants to make an A for the grade, he needs to achieve an average to be at least 90.

$$\frac{84 + 90 + 86 + ?}{4} \geq 90$$

What is the least score that he has to make on the fourth test?

Discussion: What is the average of 90 means for the sum of all test scores?

Problem 7.25 (B)

The five interior angles in a pentagon are in the ratio 2 : 3 : 4 : 5 : 6.

What is the measure of the middle angle?

Discussion:

(1) Let draw a diagram to reflect the ratio. Does it so that the smallest angle has 2 "parts".

(2) Think about the sum of all the angles.

Problem 7.26 (B)

Antoinette gets 70% on a 10-problem test. It means that she finds 7 correct solutions and incorrect or no solution for the other 3 problems. She also gets 80% on a 20-problem test and 90% on a 30-problem test. If the three tests are combined into one 60-problem test, and

(1) 5 points given for each correct solution of a problem in the 10-problem test,

(2) 6 points given for each correct solution of a problem in the 20-problem test,

(3) 8 points given for each correct solution of a problem in the 30-problem test.

What is the average in point does she get for each problem?

Discussion: What is the number of points she gets for each test? How many points she gets if all the problems are combined together into one test?

Problem 7.27 (C)

The value of a diamond is proportional to the square of its weight. It is broken into three pieces whose weights are in the ratio 3 : 2 : 5.

If the value of the original diamond was $20000, find the loss incurred due to breaking it up.

Discussion:

(1) Let the weight of the original diamond be w. Find the weights of the smaller diamonds.

(2) Calculate their values.

Problem 7.28 (C)

Ottawa is about 200km from Montreal. Burger the King starts traveling from Ottawa to Montreal and Burger the Queen from Montreal to Ottawa at speeds (not necessarily constant) with a proportion of 3 to 1.

How many miles away from Montreal Burger the Queen will be when she meets Burger King?

Discussion: Think about the amount of time they travel.

Problem 7.29 (C)

In a spaceship arrives to Terra. It's full of bugs. There are beetles (6 legs), spiders (8 legs). There are 162 more legs than bugs and 18 more beetles than spiders.

How many spiders are there?

Discussion: It's a simple two-variable system equation.

Puzzle 7.30: Consecutive letters

What are the four consecutive letters of the alphabet can be arranged to spell a four-letter word?

Bonus: How many words can you make from that four letters

Challenges

Problem 7.31 (A)

The heights of Alex, Brian, Catherine, Debby, and Emily are 147, 149, 150, 151, and 158 cm, respectively. Frank's height is an integer and 3 cm less than the average of five students mentioned before.

What is Frank's height?

Discussion: What is the sum of heights of all five students?

Problem 7.32 (A)

The ratio of the number of dollars Alain has to the number of dollars Benny has is $3:4$. After Alain gives \$40 to Benny, the ratio of the number of dollar Alain has to the number of dollars Benny becomes $1:3$.

How many dollars does Alain have at first?

Discussion: Draw a figure.

(1) Divides what Alain has into *3 parts*, what Benny has into *4 parts*.

(2) Now think *how many parts* you need to move to make what they have to be equal?

Problem 7.33 (A)

Melamine receives 15% commission on every sale she makes.

On the sale of a \$150 calculator (before any discounts), how many more dollars will she receive if her commission is based on the original price of the calculator rather than the price of the calculator after two discounts, each 40%?

Discussion*:* Determine the value of the calculator after two discounts.

Problem 7.34 (B)

Pique paid \$945 to transport his animal by ferry. The costs was \$3, \$2, and \$1 for each cat, dog, and squirrel, respectively. The ratio of cats to dogs was $2:9$, and dogs to squirrel $3:7$.

How many cats were there?

Discussion: Bring the ratios to lowest common denominator, for example $2:3$ and $3:4$ can be rewritten as $8:12$ and $9:12$.

Problem 7.35 (B)

On Friday, a snowboard was discounted 50%. On Monday, that sale price was reduced by 50%. On Tuesday, that sale price was reduced by 20%. On Wednesday, that sale price was reduced by 20%.

How many percent is the price on Wednesday comparing with the price on Thursday of the previous week?

Discussion:

(1) Let x be the price on Thursday. What is the price on Friday?

(2) Continue to do that until Wednesday.

Problem 7.36 (B)

$$\frac{1}{\dfrac{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{24}}{?}}$$

What is the reciprocal of the average of reciprocals of all factors of 24?

Discussion:

(1) What are the factors of 24?

(2) What are the reciprocals of those factors?

(3) What is the average of the sum of the reciprocals?

Problem 7.37 (C)

Alpha runs a publishing company. Zeta wants to publish a book. Alpha offers to pays Zeta \$20000. The cost of publishing a copy of a book is \$15. Alpha also needs to pay the bookstore 30% of the sale price of \$30 each book. He managed to get 10% profit at the end of the year.

At least how many copies of the book did Alpha sell?

Discussion:

Let n be the number of sold copies the books. Establish:

(1) an expression of the total cost, which is a function of n

(2) an expression of the total revenue, which is also a function of n

Problem 7.38 (C)

A cup of water is added to a container of salt water solution. The salt content is 15% alter the addition. The same amount of water is added again to the container. Now the salt content is 12% of the solution in the container.

Find the original salt content of the solution in the container.

Discussion: Let a be the amount of salt originally, m is the amount of solution, b is the amount of water in the container.

(1) What are the values of $\dfrac{a}{m+b}$ and $\dfrac{a}{m+2b}$?

(2) Figure out $\dfrac{a}{m}$ by using the value of two fractions in (1)

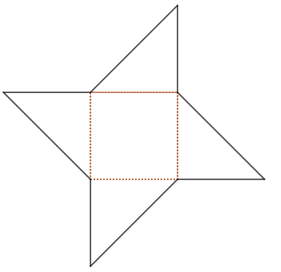
Problem 7.39 (C)

A soccer ball cover is sewn from 32 leather panels: white hexagonal panels (6 corners) and black pentagonal panels (5 corners). Every black panel is surrounded by 5 white panels. Each white panel is surrounded by 3 white and 3 black panels.

How many black panels are there?

Discussion: Count the number of edges that connects black and white panels in two ways.

Puzzle 7.40: How many points



What is the maximum number of points that:

(a) can be placed on the borders or within the figure shown above,

(b) no two points are separated by a distance that is less than the length of the diagonal of the square?

Chapter 8

Counting

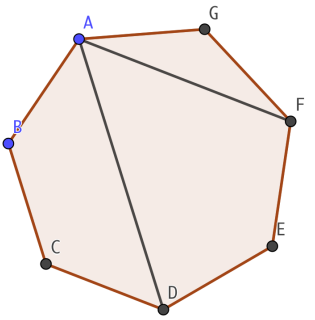
Examples

Problem 8.1 (A)

Viet draw a pentagon. Minh counts the number of diagonals and got 5 of them. Now it's Minh's turn, he draws a decagon - a polygon with 10 vertices.

What is the number of diagonals that Viet counts?

Solution:



If two vertices of a polygon are selected, then the line segment which connects them is an edge or a diagonal of the polygon. On the left is an example of a heptagon with edge AB, BC, … GA and two diagonals AF, AD.

Let count the number of pairs of vertices: there are 10 ways to select the first vertex, 9 ways to select the second vertex. 10×9 would be the total number of them.

However, counting this ways each pair of vertices (A, B) is counted twice: once as AB and once as BA. The total number of them is $\frac{10 \times 9}{2} = 45$. The number of edges of a decagon is 10. So the number of diagonals is $45 - 10 = 35$.

Note: the technique used above is called complementary counting. First count the total number of pieces, then count the number of pieces which are not needed, by subtracting the second from the first the result can be obtained. The solution also uses a sum rule. If a number of pieces are divided into groups then the total number of all pieces is the sum of the counts of pieces in the groups.

Problem 8.2 (A)

A knockout tournament is a type of elimination tournament where at the first round thew players are divided into pairs. The loser of each match is immediately eliminated from the tournament. The winners of the previous round are again divided into pairs and play a match. They continue until the final, whose winner becomes the tournament champion. If the tennis tournament starts with 64 players, how many games are needed to determine the champion?

Solution:

There are two ways to count the number of games:

(1) One game is needed to determine a loser. Since there are 63 losers, there are 63 games.

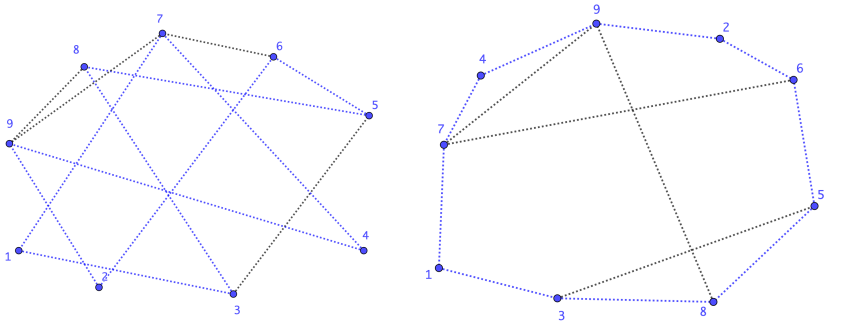
(2) The number of pairs in row 1 is 32, it is also the number of games. There are 16, 8, 4, 2, 1 games in following rounds. So the total number of games is: $32 + 16 + 8 + 4 + 2 + 1 = 64 - 1 = 63$.

Note: above two different ways of counting is shown. It is a very useful technique in later problems.

Problem 8.3 (A)

Can you arrange the number 1, 2, 3, … , 9 along a circle, so that any sum of two neighbours can never be divisible by 3, 5, or 7?

Solutions: In this solution, a different approach is shown by *drawing a graph and finding the solution using the graph*. Let connects any pair of two numbers whose sum is not divisible by 3, 5, or 7. The figure below on the left is the graph. Now rearrange the number so that they form a circle. It can be done by trying to find a sequence of points connected to each other, which is 1, 3, 8, 5, 6, 2, 9, 4, 7. The result is the figure on the right.



Problem 8.4 (B)

A sequence of number has the following properties:

(1) The sum of any 3 consecutive terms is negative (smaller than zero)

(2) The sum of any 4 consecutive terms is positive (larger than zero)

What is the maximum number of terms the sequence can have?

Solution: Let assume that there are at least six terms in the sequence.

(1) Let investigate the *first three sums of four consecutive terms*:

$a_1 + a_2 + a_3 + a_4 > 0$

$a_2 + a_3 + a_4 + a_5 > 0$

$a_3 + a_4 + a_5 + a_6 > 0$

(2) Each column of these sums forms a *sum of three consecutive terms*:

$a_1 + a_2 + a_3 < 0$

$a_2 + a_3 + a_4 < 0$

$a_3 + a_4 + a_5 < 0$

$a_4 + a_5 + a_6 < 0$

(1) and (2) are contradicting each other: a sum of negative numbers is equal a sum of positive numbers. Therefore the maximum number of terms is 5. Here's an example $2, 2, -5, 2, 2$.

Problem 8.5 (B)

Jennie invited Elisa over to an apartment complex of many buildings. She said that she lives in apartment 333 of the tenth building, but forgot to tell which floor. Elisa went to the complex, saw that each building has a total of 9 floors. She knew that the apartments are numbered beginning with 1 on the first floor of the first building, and all apartments in one building are numbered from bottom to top before continuing to the next building. Each floor of each apartment building has the same number of apartments.

Where should Elisa go?

Solution:

If there are 3 or less apartments on each floor of an apartment building, then in 10 apartment buildings there are less than $10.9.3 = 270$ apartment and the number 333 cannot be used.

If there are 5 apartments in each floor, then there will be $9.9.5 = 405$ apartments in the first 9 buildings and the apartment 333 cannot be in the 10th building.

So each floor must have 4 apartment. In the first 9 buildings there are $9.9.4 = 324$ apartment. The apartments in the 10th building start with 325. Since each floor has 4 apartments, so the apartment 333 is the first one on the third floor.

Note: one of the keys in counting is to estimate.

Problem 8.6 (B)

John draws a convex polygon with 100 vertices. He connects all the diagonals of the polygon. Assume that the intersections of all pairs of every two diagonals are different.

How many intersections does John find?

Solution: Let choose any four vertices (A, B, C, D) of the polygon in clockwise order. Easy to see that AC and BD are not edges, so they must be diagonals. Counting the quadruples of vertices this way is the same as counting the number of intersections of every two diagonals.

Since there are $100 \times 99 \times 98 \times 97$ ways to choose A, B, C, D; respectively and each quadruple (A, B, C, D) selected $4 \times 3 \times 2 \times 1$ times, so the number of intersection is $\dfrac{100.99.98.97}{4.3.2.1} = \dfrac{100!}{96! \times 4!}$.

Note: The number of ways 4 objects can be selected from 100 objects is $\binom{100}{4} = \dfrac{100!}{96! \times 4!}$

Problem 8.7 (C)

In how many ways can you take an odd amount of marbles from 100 marbles?

We will prove a generalized version of the problem: *Given any n-element set, the number of its subsets with odd amount of elements is equal the amount of its subsets with even number of elements and it is* 2^{n-1}.

Solution 1: Induction principle based on n, which is the number of elements of the set S.

First, let the set S_1 has 1 element $S_1 = \{a_1\}$. The number of subsets of S_1 that has 0 element is 1: the empty set. The number of subsets of S_1 that has 1 element is 1: the S_1 set itself.

Assume that the statement is true: *Given any* n*-element set* $S_n = \{a_1, a_2, \ldots, a_n\}$, *the number of its subsets with odd amount of elements is equal the amount of its subsets with even number of elements and it is* 2^{n-1}.

We will prove that: *Given any* $(n+1)$ *-element set* $S_{n+1} = \{a_1, a_2, \ldots, a_n, a_{n+1}\}$, *the number of its subsets with odd amount of elements is equal the amount of its subsets with even number of elements and it is* 2^n.

Any subset of S_{n+1} is a subset of S_n if it does not contain a_{n+1}, and any subset of S_{n+1} containing a_{n+1} is a subset of S_{n+1} after removing a_{n+1}. The number of subsets of S_{n+1} having odd number of elements is equal:

(1) the number of subsets of S_n having odd number of elements plus

(2) the number of subsets of S_n having even number of elements (becoming odd number of elements after adding a_{n+1})

So, the number of subsets of S_{n+1} having odd number of elements is $2 \times 2^{n-1} = 2^n$. Similarly the number of subsets of S_{n+1} having even number of elements is also $2 \times 2^{n-1} = 2^n$.

Solution 2: Binomial theorem is a useful tool, shown as below.

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots + \binom{n}{n-2} x^2y^{n-2} + \binom{n}{n-1} + xy^{n-1} + y^n$$

where $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$ is the binomial co-efficient, or the number of ways choose m objects n objects. With $x = 1$, $y = -1$:

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n}, \text{ so}$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$ This means *the total number of ways choosing an odd amount of elements* is same as *the total number of ways choosing an even amount of elements*. In addition the sum of these two amount is 2^n as shown below.

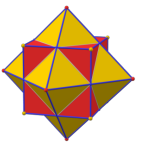
$$2^n = (1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n}$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = \frac{2^n}{2} = 2^{n-1}$$

Solution 3: Let a be an element of S. The subsets of S can grouped into pairs: (a subset X that does not contains a, $X + \{a\}$).

Easy to see in each pair one of the subsets has an odd amount of elements and the other one has an even number of elements. This means the number of subsets that have odd number of elements is equal the number of subsets that have even number of elements. Problem 1.9 shows that the number of subsets of a $n-$element set is 2^n, so half of that is number of subsets that have odd number of elements.

Problem 8.8 (C)



Does a polyhedron (a solid figure with many plane faces) exists with an *odd number of faces*, each face having an *odd number of edges*?

The figure on the left is an example of a polyhedron with triangle faces.

Solution: Let assume that the polyhedron has $(2n+1)$ faces. Let $e_1, e_2, \ldots e_{2n+1}$ be the number of edges of the faces. $e_1 + e_2 + \ldots + e_{2n+1}$ is the number of edges counted twice, meaning that it is an even number. However $e_1 + e_2 + \ldots + e_{2n+1}$ is the sum of odd amount of odd numbers, resulting in an odd number. This is contradictory. So such polyhedron does not exist.

Note: the principle above is two ways of counting resulting in contradiction.

Problem 8.9 (C)

A

B B

C C C

... ...

Z Z Z Z Z

Find the number of ways to form word ABC…Z consisting of 26 letters in the diagram above, staring with the letter A, and always connecting a letter to one of the two below it.

Solution 1: It is the number ways to reach the 26-level in a Pascal Triangle, going from the top:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

For example, for a 5-level triangle, the number of ways is the sum of all number of level 5, which is the binomial sum: $\binom{4}{0} + \binom{4}{1} + \ldots + \binom{4}{4} = 2^{5-1} = 2^4 = 16$. Easy to generalize to reach the conclusion that the number of ways is 2^{25}.

Solution 2: A more insightful approach is to discover that the number of ways going to one level down is always doubled. It means that there is $1 \times \underbrace{2 \times 2 \times \ldots \times 2}_{25} = 2^{25}$ ways.

Puzzle 8.10: Triangles

Show that there are two ways to draw six line segments of equal length to form eight equilateral triangles.

Exercises

Problem 8.11 (A)

How many ways the letters of the word MATH can be rearranged so that A is not next to T?

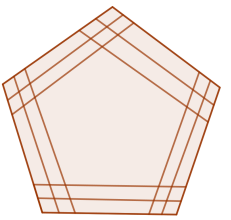
Discussion: Let do *complementary counting*

(1) How many ways are there to arrange all the letters?

(2) Let create a new letter *AT* by putting the two letters together. How many way to create new words with the three letters M, H and *AT*. For example *AT*MH, H*ATM*, and so on.

Solution: Easy to see that there are $4 \times 3 \times 2 \times 1 = 4! = 24$ ways to create words. If we put *AT* together, there are $3 \times 2 \times 1 = 3! = 6$ ways to use M, H, and *AT* to create new words. Similarly 6 other ways with *TA*. This means that the number of ways to arrange the letters so that A is not next to T is $24 - 6 - 6 = 12$.

Problem 8.12 (A)



How many pentagons can be formed by using the lines of the figure on the left?

Discussion: What 5 lines can be used to form a pentagon? Show an example.

Solution: Each pentagon has 5 edges. Each edge can be selected from one of the three parallel lines. This means that for each pentagon, three are $\underbrace{3 \times 3 \times \ldots \times 3}_{5} = 243$ pentagons.

Problem 8.13 (A)

There are six numbers from 1 to 6, each written on one side of a cube. The cube was thrown twice. On the first throw the sum of the numbers on the four side faces was 12, and on the second throw it was 15.

What number is written on the face that is opposite to the face with the number 3?

Solution: The sum of all numbers written on faces of a cube is 21. The first throw showed that there is a pair (top and bottom) whose sum is $21 - 12 = 9$, the second once revealed that there is another pair with sum $21 - 15 = 6$. This also means that the third pair has sum of $21 - 9 - 6 = 6$. There are only two possible pairs summing up to 6, which are (2,4), (1,5). So the remaining pair is (3,6). Hence, 6 is the number written on the face opposite to 3.

Problem 8.14 (B)

There are seven letters in the language of gnomes. A word is a sequence of seven letters, *some pair of which are the same*.

How many words are there in the language?

Discussion*:*

(1) How many sequence of seven letters are there?

(2) How many sequence of seven *different* letters are there?

Solution: Since a word has 7 letters, and there are 7 different letters. So the number of possible 7-letter sequence is 7^7. The number of sequences, each consists of seven *different* letters is $7.6...2.1 = 7!$. So the number of words, having at least two same letters, is $7^7 - 7!$.

Problem 8.15 (B)

Find the number of arrangements of the letters in the word MISSISSIPPI.

Discussion*:*

(1) Find the number of arrangements of the letters in the word ANA?

(2) Find the number of arrangements of the letters in the word ANNA?

Solution: Let take ANA as an example. Because we have two letters A, so let mark them A_1, A_2. It means we will have six arrangements: A_1A_2N, A_2A_1N, A_1NA_2, A_2NA_1, NA_1A_2, NA_2A_1. Now since A_1A_2 and A_2A_1 are practically the same, it means the number of ways is $\frac{3!}{2!} = 3$.

Using the same technique to examine ANNA: A_1A_2 and N_1N_2, the number of ways is $\frac{4!}{2!2!} = 6$. Therefore the number of ways for rearranging MISSISSIPPI is $\frac{11!}{4!4!2!} = 34650$.

Problem 8.16 (B)

A number of students are standing in a circle. The teacher will select one to go to the cafeteria and get cookies for the whole class by the following process *One student is picked and is told to remain standing. The next student, going clockwise, is told to leave the circle and sit down. The next student, is told to remain, and the next one is told to go sit down.*

The teacher continues this way, telling every other students (always advancing clockwise) to go sit down. The circle get smaller and smaller until there is only one student left standing. What place did the last student originally occupy if the number of students is 1000?

Discussion*:*

(1) Let assume that there is 16 students. Who is the remaining one?

(2) Let assume now there is 17 students, what is the situation?

Note: this is a great example of identifying what does not change during the process: an invariant.

Solution:

(1) Let solve the case of 16 students. Let Alice be the first student. At first round, students 2, 4, 6, …, 16 sit down. Alice is the first student again in 8 students. Continually doing this, the number of students is reduced to 4, then 2, and finally 1. Alice is always the first student, regardless of the counting. She is the remaining one.

(2) If there were 17 students, after student number 2 sits down, the number of student is 16, and the counting starts with the 3rd student. According to (1), this is the student who is the remaining one.

Now let solve the case when there were 1000 students. Since the power of 2 closest to 1000 is $2^9 = 512$, let consider when $1000 - 512 = 488$ students leave. Then 512 students remain and the first one on the count is $2 \times 488 + 1 = 977$.

Problem 8.17 (C)

Consider all $2^n - 1$ non-empty subset of $\{1, 2, \ldots, n\}$. For every of such subset, we find the product of the reciprocals of each of its elements.

Find the sum of all these products.

Discussion:

(1) Let take a look at a subset, for example $\{1, 2, 4, 5\}$. The reciprocals of each of its elements are $\frac{1}{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}$ and the product of them is $\frac{1}{1}.\frac{1}{2}.\frac{1}{4}.\frac{1}{5}$.

(2) What we need to look for is the sum of all similar products:

$$\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}+\frac{1}{1}.\frac{1}{2}+\frac{1}{1}.\frac{1}{3}+\ldots+\frac{1}{1}.\frac{1}{n}+\frac{1}{1}.\frac{1}{2}.\frac{1}{3}+\ldots+\frac{1}{1}.\frac{1}{n-1}.\frac{1}{n}+\ldots\frac{1}{1}.\frac{1}{2}\ldots\frac{1}{n}$$

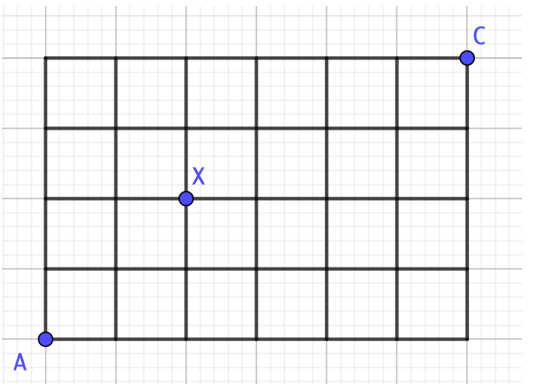
Solution: Let take a look at a smaller case when $n = 3$. Note that:

$$1+\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{1}.\frac{1}{2}+\frac{1}{1}.\frac{1}{3}+\frac{1}{1}.\frac{1}{2}.\frac{1}{3}=(1+\frac{1}{1})(1+\frac{1}{2})\ldots(1+\frac{1}{3})$$

So the given sum is same as :

$$(1+\frac{1}{1})(1+\frac{1}{2})\ldots(1+\frac{1}{n})-1=\frac{2}{1}.\frac{3}{2}\ldots\frac{n}{n-1}\frac{n+1}{n}-1=n+1-1=n$$

Problem 8.18 (C)



How many 10-unit paths from A to C, and not going through X?

Discussion:

(1) How many 10-unit paths from A to C?

(2) How many 4-unit paths from A to X? How many 6-unit paths from X to C?

Solution: Easy to see that going from A to C with 10 steps require 4 steps up and 6 steps right. In another word choosing to take 4 steps up in 10 steps: $\binom{10}{4}=\frac{10.9.8.7}{4.3.2.1}=210$. Since there are $\binom{4}{2}=6$ possible 4-unit paths from A to X and $\binom{6}{2}=15$ possible 6-unit paths from X to C, it means the number of 10-path not through X is: $210-6\times 15=120$.

Problem 8.19 (C)

Elena is on the first floor of a building with 8 floors. On each floor, there are three elevators. One can bring her up one floor, another one takes her two floors up, and the last one can carry her three floors up.

In how many ways Elena can get to the eight floor?

Solution: Let $f(n)$ be the function to count the number of ways Elena can get to the nth floor.

First to get to the nth floor, Elena must take an elevator from either floor $n-1$, $n-2$, or $n-3$. Each of those floors contains exactly one elevator that can take Elena to the nth floor. Therefore:

$$f(n) = f(n-1) + f(n-2) + f(n-3)$$

Note: This is called recursive function and helps to solve a problem by relying on answers from smaller or sub-problems.

Since $f(0) = 0$, $f(1) = 1$, $f(2) = 1$, so:

$f(3) = 1 + 1 + 0 = 2$, $f(4) = 2 + 1 + 1 = 4$, $f(5) = 4 + 2 + 1 = 7$

$f(6) = 7 + 4 + 2 = 13$, $f(7) = 13 + 7 + 4 = 24$, $f(8) = 24 + 13 + 7 = 44$

Puzzle 8.20: Couples

Adam and his wife went to a BBQ party. There were other four married couples. There were a number of handshakes. No one shook hands with himself/herself or with his/her spouse. No one shook hands with the same person more than once.

After all the handshakes were over, Adam asked everyone, including his wife, how many hands he/she had shaken. To his surprise, it turned out that everyone gave a different answer.

How many hand did his wife shake?

Problems

Problem 8.21 (A)

How many three-digit numbers are there such that the middle digit is the average of the other two digits?

Discussion:

(1) How many ways can you choose the left most digit?

(2) If you have chosen the left most digit, how many ways can you choose the right most?

Problem 8.22 (A)

A theatre stage has eight lights and each can be separated turned on or off. In how many way can we illuminate the stage?

Discussion: How many possible states a light can be in?

Problem 8.23 (A)

How many ways to put a king, a queen, two identical bishops, two identical knights, two identical rooks on row?

Discussion:

(1) Let assume that the bishops, the knights, the rooks are different. How many ways to place them in a row?

(2) Since the two bishops are identical, so the number of the ways with one kind of order of the two bishops would be exactly same with the other order to arrange them - in another word they can be switched.

Problem 8.24 (B)

Penelope has bank account with 6-digit PIN as a password. How many different PINs can she create if she uses only two different digits, each appear at least once? For example 333331 and 444555 are valid ones.

Discussion:

(1) How many ways Penelope can choose two different digits?

(2) How can she form a PIN from that two different digits

Problem 8.25 (B)

How many integer triplets (a, b, c) such that $a \geq 5$, $b \geq 6$, $c \geq 1$, and $a + b + c = 17$

Discussion:

(1) Let $x = a - 4 \geq 1$, $y = b - 5 \geq 1$, $z = c \geq 1$ are two non-negative integers.

(2) What is $x + y + z$. How many triplets can we find?

Problem 8.26 (B)

Van Anh splits the set {1, 2, 3, 4, 5, 6, 7} into two disjoint sets such that the sum of digits in each new sets are equal. In how many ways can she do that?

Discussion: What is the sum of all digits? What would be the sum of all digits in one set?

Problem 8.27 (C)

A straight log of length 2m is cut into five pieces with each piece at least 17 cm long. Prove that there are three of these pieces that can be put together to form a triangle.

Discussion:

(1) Let take the two shortest pieces, what would be their minimal length?

(2) What should be the minimal length of the next shortest piece?

Problem 8.28 (C)

How many nine-digit numbers have an even sum of their digits?

Discussion: How can we choose the first 8 digits?

Problem 8.29 (C)

Sets A, B, C, D each have 400 elements. The intersection of any two sets have 115 elements. The intersection of any three of the sets have 53 elements. The intersection of all four sets has 28 elements. How many elements are there is the union of the four sets?

Discussion: Principle of Inclusion - Exclusion.

Puzzle 8.30: Tennis games

Merry beats Anna in a set of tennis, winning six games to Anna' three. Five games were won by the player who did not serve.

Who served first?

Challenges

Problem 8.31 (A)

Last night there was a party and the host's door bell rang 20 times. The first time the door bell rang, only one guest arrived. Each time the door bell rang, two more guests arrived than had arrived on the previous ring.

How many guests were girls if they were a quarter of the total number of guests?

Discussion:

(1) What is the total number of guests arrived at the first ring? At the second? And third?

(2) What is the total number of them?

Problem 8.32 (A)

A total of 6 books, 3 on mathematics and 3 on history, are to be arranged on a bookshelf.

In how many ways can it be done if all the math books are to be placed to the left of all history books?

Discussion:

(1) What would be on the left most position?

(2) What would be on the 4th position from left?

Problem 8.33 (A)

Ten ants march in a line, four of them carrying pieces of red leaves, and the rest carrying pieces of yellow leaves. Assume that there is no difference between pieces of the same colour.

How many ways they can march so that in the first five positions, number of yellow pieces of leaves is at least four?

Discussion:

(1) How many possible values for the number of yellow pieces in the first five positions

(2) Based on that, how many possible cases for the last five?

Problem 8.34 (B)

12 people participated in a chess tournament. Each participant played a game against each other once. 1 point is given for the winner of a game, 0.5 for a tie, and 0 for losing it. At the end of the tournament a number of players, who earned over 70% of the total number of points that can be earned by winning every single game, were awarded with the Master title.

Is that possible if 8 participants gained the Master title?

Discussion: Assume that there were 8 participants, who gained the Master titles.

(1) What is the minimum number of points a player must achieved in order to be aware the Master title?

(2) How many points the Masters can win from the one who had not gained the Master titles?

Problem 8.35 (B)

Mr. and Mrs. Zed want to name their baby Zeta so that its monogram (first, middle, and last initials) will be in alphabetical order with no letter repeated.

How many such monograms are possible?

Discussion: What is a three different letter monogram?

Problem 8.36 (B)

A drawer in a dark room contains 100 red socks, 80 green socks, 60 blue socks, and 40 black socks. A youngster selects socks one at a time from the drawer but is unable to see the colour of the socks drawn.

What is the smallest number of socks that must be selected to guarantee that the selection contains at least 10 pairs?

Note that a pair of socks is two socks of the same colour.

Discussion:

(1) How many socks would be needed: an odd or an even numbers?

(2) Minimum how many socks needed to 10 pairs?

(3) What would be extra?

Problem 8.37 (C)

Nine chairs in a row are to be occupied by six students and Professor Alpha, Beta, and Gamma. These three professors arrive before the six students and decided to choose their chairs so that each professor will be between two students. In how many ways can Professor Alpha, Beta, and Gamma choose their chairs?

Discussion: Who occupy the two end-chairs? Where the three professors can sit?

Problem 8.38 (C)

Call a 7-digit telephone number $d_1d_2d_3 - d_4d_5d_6d_7$ *memorable* if the prefix $d_1d_2d_3$ is exactly same as either $d_4d_5d_6$, or $d_5d_6d_7$ (possibly both). Assuming that each d_i can be one of the ten digits 0, 1, …, 9; find the number of different memorable telephone numbers.

Discussion:

(1) How many ways $d_4d_5d_6d_7$ can be written?

(2) How many ways not all of them are the same?

Problem 8.39 (C)

Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but not divide n?

Discussion: Let $d = 2^k.3^l$. If d is divisor of n^2, less than n but not divide n, what would be k, l?

Puzzle 8.40: Dates

$$\overline{OCT} \times 31 = \overline{DEC} \times 25$$

Find the dates. Each letter is a different digit.

Chapter 9

Distance - Rate - Time

Examples

Problem 9.1 (A)

Elisa rides a bike from home to the nearby hill. She goes up to the top within 30 minutes at 6km per hour. At the top, she turns around immediately and goes home on the same route at a speed of 18km per hour. What is her average speed?

Discussion: The common mistake is to take the average of all the speeds assume that is the average speed. *The average speed is the value of the total distance over the total time*.

Solution 1: She goes to the top of the hill in 30 minutes at 6km per hour, so the distance to the top of the hill is $6 \times \frac{1}{2} = 3$km. The total distance from home to top of the hill back to him is $3 + 3 = 6$km. Since the speed downhill is three times faster, so it takes only $\frac{30}{3} = 10$ minutes. Hence she goes in total 6km in 40 minutes, which is same as 9km in 1 hour. Therefore her average speed should be 9km per hour.

Solution 2: Using a table, the solution is shown in a more organized way

Route	From home to top of the hill	From top of the hill to home	Total
Speed (rate)	6 km/h	18 km/h	**(6)** $\frac{6}{\frac{2}{3}} = 9$ km/h
Time	30 min $= \frac{1}{2}$ hrs	**(4)** $\frac{3}{18} = \frac{1}{6}$ hrs $= 10$ min	**(5)** 40 min $= \frac{2}{3}$ hrs
Distance	**(1)** $6 \times \frac{1}{2} = 3$ km	**(2)** 3km	**(3)** $3 + 3 = 6$ km

Problem 9.2 (A)

7 moles dig 7 holes in 7 minutes. How long will it take 8 moles to dig 8 holes?

Solution: Lot of people would happily give the answer of 8 hours. However that's wrong answer. *If each mole is assigned to a different hole, and assume it works independently.* It takes 7 minutes for each mole to dig one hole. So 8 moles dig 8 holes in 7 minutes.

Note: some important principles:

(1) *The time spent for a number of tasks is equal the time spent for one task times the number of tasks:* if 1 person completes 2 tasks in 6 hours, then 1 person completes 1 task in 3 hours.

(2) *The time required for a number of people to complete a task is equal the time required for one person times the number of people:* if 1 person completes 1 tasks in 3 hours, then 2 person completes 1 task in 1.5 hours

Problem 9.3 (A)

5 hoses fill 10 small tanks in 80 minutes, or 6 large tanks in 90 minutes. What is the least amount of time needed for 10 hoses to fill 8 small tanks and 8 large tanks?

Solutions*:* Let analyze the given situation.

Hose	Small tanks	Minutes		Hose	Large tanks	Minutes
5	**10**	80		5	**6**	90
5	**1**	8		5	**1**	15

5	1	**8**		**5**	1	15
10	1	**4**		**10**	2	15

10	**1**	4		10	**2**	15
10	**8**	32		10	**8**	60

So 10 hoses 8 small and 8 large tank in $32 + 60 = 92$ minutes.

Problem 9.4 (B)

Charles drove to his cottage using the highway with 45 km per hour speed. His speed was 36 km per hour when using a rural route, which was 3 km shorter, but it took him 5 more minutes. How long was the highway route?

Discussion: Charles' speed was 36 km/h on the the rural route. It means in 1 hour he would go 36 km, or it would take 5 minutes for every 3 km.

Solution*:* Charles had to spend 5 more minutes on the rural route means he had to go 3 km farther on this route. If the rural route were $3 + 3 = 6$ km shorter, then Charles would have completed the rural route in the same amount of time he completed the highway. Now, *for every hour* Charles gets $45 - 36 = 9$ *km farther* on the highway comparing with the rural route. To complete a 6 km longer distance, it takes $\frac{6}{9} = \frac{2}{3}$ hours (40 minutes). Hence the distance of the high way is $45 \times \frac{2}{3} = 30$ km.

Problem 9.5 (B)

The full battery in the iPod supposes to last 17 hours without charging if it is used with headphones, and 3 hours if used with speakers.

Ca use headphones and speakers for an equal amount of time, how many minutes will a fully charged iPod last?

Solution:

One hour use of the iPod with headphones will consume $\frac{1}{17}$ of its battery energy. One hour use of the iPod with speaker will consume $\frac{1}{3}$ of its battery energy. Because Ca uses headphones and speakers *for an equal amount of time*, the average consumption is $\frac{1}{2}(\frac{1}{17}+\frac{1}{3})=\frac{10}{51}$ of battery for *an hour*.

A fully charged battery would last $\frac{51}{10}=5.1$ hours, or 5 hours and 6 minutes, or 306 minutes.

Problem 9.6 (B)

Between 7:00 and 8:00, the minute hand and hour hand form a 25 degree angle twice, at m and n minutes after 7:00. What is $m+n$? Express your answer as a common fraction.

Discussion: A circle has 360 degrees.

(1) How many degrees the minute hand moves per minute?

(2) How many degrees the hour hand moves per minute?

(3) What is the value of the angle clockwise from the minute hand to the hour hand at 7:00?

(4) How the angle between the hour hand and the minute hand is changed per minute?

Solution 1:

(1) Since the minute hand moves a full circle in 60 minutes, so it moves 6 degrees per minute.

(2) The hour hand moves one-twelfth of the circle or 30 degrees in 60 minutes, so it moves half (0.5) degree per minute.

At 7:00, the hour hand is $7\times 30=210$ degrees from the minute hand (clockwise from the minute hand to the hour hand). The angle between the hour and minute hands is decreasing by $6-0.5=5.5$ degrees each minute until they meet.

The first time m they form a 25 degree angle is at $\frac{210-25}{5.5} = \frac{185}{5.5} = \frac{37}{1.1} = \frac{370}{11}$ (minutes)

The time when both hand meet angle is zero is $\frac{210-0}{5.5} = \frac{42}{1.1} = \frac{420}{11}$ (minutes).

The second time n when they again form a 25 degree angle is: $\frac{420}{11} + (\frac{420}{11} - \frac{370}{11}) = \frac{470}{11}$

So $m + n = \frac{370+470}{11} = \frac{840}{11}$.

Solution 2: $m + n$ is twice the number of minutes when both hour and minute hand meet. In another word when the angle is zero: $2 \times \frac{210-0}{5.5} = 2.\frac{42}{1.1} = \frac{840}{11}$.

Problem 9.7 (C)

In order to complete a large job, 1000 workers were hired, just enough to complete the job on schedule.

(1) All the workers stayed on the job while the first quarter of the work was done, so the first quarter of the work was completed on schedule.

(2) Then 100 workers were laid off, so the second quarter of the work was completed behind schedule.

(3) Then an additional 100 workers were laid off, so the third quarter of the work was completed still further behind schedule.

Given that all workers work at the same rate, what is the minimum number of additional workers, beyond the 800 workers still on the job at the end of the third quarter, that must be hired after three-quarters of the work has been completed so that the entire project can be completed on schedule or before?

Discussion: one of the factors that might confuse you is the lack of time. Don't be. Let assume that the large job is supposed to be completed with in 4 hours.

Solution:

(1) all workers completed a quarter of the job on schedule, it means that the first quarter of the job was completed in 1 hour.

(2) the second quarter of the work gets completed at 90% of the intended rate because 100 workers (10%) were laid off. It means 90% of the second quarter of the job is done in the second hour. In another

word, the total time required to complete the second quarter of the job is $\frac{10}{9}$ hours.

(3) Because 100 more workers (another 10%) were laid off, 80% of the third quarter of the job was completed in one hour. It means the time requires to complete it was $\frac{10}{8}$ hours. Since the entire project was completed in time, the time left for completing the fourth quarter is: $4 - 1 - \frac{10}{9} - \frac{10}{8} = \frac{23}{36}$. Thus, the work rate for the fourth quarter must be $\frac{36}{23}$ of the usual work rate. So the number of worker is at least $1000 \times \frac{36}{23}$ or 1566.

The additional workers must be hired to complete the fourth quarter of the job in time is 766.

Problem 9.8 (C)

Everyday at 7:00 Dehlila and Samson both go from home to the same workplace. Dehlila takes the train and always arrives at the same time. Samson has one of two choices:

(1) if he takes the red bus at the stop in front of his house, he has to walk to the office after the last stop, and arrive 15 minutes after Dehlila .

(2) if he walks to the blue bus stop, it takes him directly to the office and he arrives at the same time as Dehlila.

In both cases the total distances for Samson in both cases are the same. All the buses travel at an average speed of 42 km/h. Samson walks at 6km/h.

How farther he has to walk if he takes the red bus?

Solution: Let d, d_r, d_b be the total distance, the distance by the red bus, and by the blue bus.

d_r	$d - d_r$	$d - d_b$	d_b

Let s_b, s_w be the speeds of the buses and Samson's. Let t_d be the time Dehlila spent by the train.

$$\frac{d_r}{s_b} + \frac{d - d_r}{s_w} = \frac{1}{4} + t_d, \frac{d - d_b}{s_w} + \frac{d_b}{s_b} = t_d \Rightarrow (d_b - d_r)(\frac{1}{s_w} - \frac{1}{s_b}) = \frac{1}{4} \Rightarrow d_b - d_r = \frac{1}{4}\frac{s_w s_b}{s_b - s_w}$$

$$\Rightarrow d_b - d_r = \frac{1}{4}\frac{42.6}{36} = \frac{7}{4} \Rightarrow (d - d_r) - (d - d_b) = d_b - d_r = 1.75 \text{ km}.$$

This is the extra distance Samson needs to walk he takes the red bus.

Problem 9.9 (C)

How many times in a 12 hours from 00:00 AM to 11:59AM, the hour hand and the minute hand overlap? What are the times?

Solution:

(1) Since the minute hand moves a full circle in 60 minutes, so it moves 6 degrees per minute.

(2) The hour hand moves one-twelfth of the circle or 30 degrees in 60 minutes, so it moves half (0.5) degree per minute.

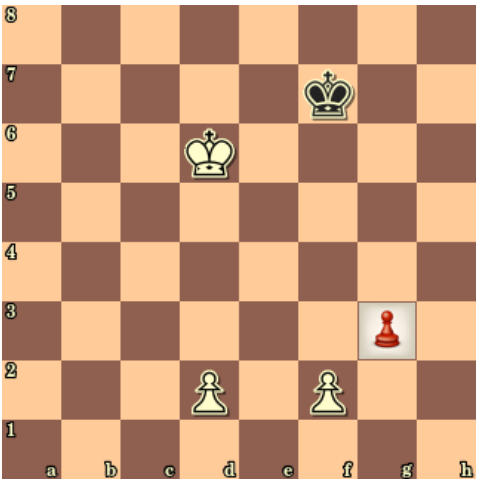
For each hour period, it is easy to see that the two hands overlap only once. From n:00 AM to n:59 AM, for every n between 0 and 11, the hour hand move from n to $n+1$, the minute hand moves from 0 to 59. If m is the minutes they meet:

$$30n + 0.5m = 6m \Rightarrow 30n = 5.5m \Rightarrow m = \frac{60}{11}n$$

So they will meet 11 times: 00:00, 01:06, 02:10, 03:17, 04:22, 05:27, 06:33, 07:38, 08:44, 09:49, 10:55.

Note: the next time after 10:55 they meet at 12:00.

Puzzle 9.10: Which colour is the pawn?



The board shows an end-game with five pieces: two kings, two white pawns, and one pawn of unknown colour, temporarily colour as red. It is known that during the game no piece has moved from a square of one colour to a square of another colour.

What is the actual colour of the red pawn?

Exercises

Problem 9.11 (A)

Erica pumps water into her swimming pool. In the sun, every hour 6 water litres evaporate; in the shade, only 4 litres. In July, the amount of water loss due to evaporation in the sun was equal to the amount lost in the shade. What was the average rate of evaporation from the pool?

Discussion*:* Look at the evaporation of 1 litter of both cases: in the sun and in the shade.

Solution*:*

(1) In the sun 6 litters of water lost every hour. It means 1 litter is lost every 10 minutes.

(2) In the shade 4 litters of water lost every hour. It means 1 litter is lost every 15 minutes.

2 litters lost in 25 min $= \dfrac{5}{12}$ hours. The average rate is $\dfrac{24}{5} = 4.8$ litters.

Problem 9.12 (A)

Huy can solve a Rubik's cube in 4 minutes. Binh can solve 5 cubes in 12 minutes. If they work together to solve 10 cubes, how many minutes they need?

Discussion*:* Find if they each work for a common amount of time, how many does each solve?

Solution: In 12 minutes, Huy solve 3 cubes and Binh solves 5 cubes. Together they can solve 8 cubes in 12 minutes, or 2 cubes in 3 minutes. Hence, they solve $10 = 8 + 2$ cubes in 15 minutes.

Problem 9.13 (A)

Galiena ran three-fourth of the way to the school with the speed of 9 km/h, and she walked at 6km/h for the rest of the way to the school. It took her in total 30 minutes. How far is the school from home?

Solution: The diagram shows:

(1) blue is the time she spent for one-fourth of the way running 9 km/h

(2) green is the time she spent for one-fourth of the way walking 6 km/h

(3) red is the time she spent as if walking 3 km/h

6 blue and 3 green boxes are 30 minutes. So one box is $\frac{0.5}{9}$ hours. So the distance is: 6 blue boxes $\times 9$ km/h $\times \frac{0.5}{9}$ h + 3 green boxes $\times 6$ km/h $\times \frac{0.5}{9}$ h $= 3 + 1 = 4$ km.

Problem 9.14 (B)

Capri can mow a lawn in 50 minutes. If Capri and Basra work together, they can mow the lawn in 30 minutes. How long would it take Basra to mow the lawn alone?

Solution 1: In 10 minutes, Capri mows one-fifth of the lawn. In 30 minutes, he can mows three-fifth of it. So the remaining two-fifth of the lawn is mowed by Basra in 30 minutes. It takes Basra 15 minutes to finished one-fifth of the lawn or 75 minutes for the whole lawn.

Solution 2: Capri can mow the lawn in 50 minutes or $\frac{5}{6}$ hours. So in 1 hour he can mow $\frac{6}{5}$ lawn, or $\frac{3}{5}$ lawn in 30 minutes. From here it similar like the first solution.

Problem 9.15 (B)

Edina arrives at a barber shop at 4:20 PM for an appointment. After less than 1 hour, she leaves the shop. Outside, she notices that the hour and the minute hand point to diametrically opposite point (i.e. their ends are at opposite points on the clock's circle). At least how many minutes does she stays in the shop?

Solution: Since the minute hand moves a full circle in 60 minutes, so it moves 6 degrees per minute. The hour hand moves one-twelfth of the circle or 30 degrees in 60 minutes, so it moves half (0.5) degree per minute. Let m be the minutes where the hands are opposite. The minute hand moves $6m$ degrees with the *12* number. The hour hands moves $0.5m$ degrees, and since the number *4* forms 120 degrees with the *12* number, the hour hand forms $0.5m + 120$ degrees with the *12* number. The two hands can only be opposite if the difference of the degree is 180. Hence the minutes where the hands are opposite is: $6m = 0.5m + 120 \Rightarrow 5.5m = 120 \Rightarrow m = \frac{600}{11}$. So the time Edina stays in the shop is: $\frac{600}{11} - 20 \approx 34$ minutes.

Problem 9.16 (B)

Minh, Anthony, and Viet went from the train station to rock cliff nearby.

(1) Minh walked to the cliff, and climbed to the top.

(2) Anthony ran to the cliff twice as fast as Minh, but climbed only two-third as fast as Minh.

(3) Viet biked to the cliff at speed four times faster than Minh's, and it took two hours longer than Minh to get to the top at the cliff.

All three of them got to the top of the cliff at the same time.

How many minutes did each spend from the train station to the top of the cliff?

Solution 1*:* Let a be the amount of time in minutes Minh went from the train station to the cliff, b is the amount of time he spent climbing to the top of the cliff.

$$a + b = \frac{a}{2} + \frac{b}{\frac{2}{3}} = \frac{a}{4} + (b + 120)$$

From the first and last one: $\frac{3a}{4} = 120 \Rightarrow a = 160$. From the first one and second one: $\frac{a}{2} = \frac{b}{2}$.

So $a = b = 160 \Rightarrow a + b = 320$ minutes.

Solution 2*:*

It is easy to see from the diagram:

(1) comparing the first two line: the amount of blue and red is equal

(2) comparing the first and last line: three-quarter of the first one is 120.

Problem 9.17 (C)

Khang and An bike at equal and constant rates. They also jog equal and constant rates, and they swim at equal and constant rates.

Khang covers 74 km by biking 2 hours, jogging 3 hours, and swimming 4 hours. An covers 91 km by jogging 2 hours, swimming 3 hours, and biking 4 hours.

If all the biking, jogging, and swimming rates are positive integers, what is the sum of biking, jogging, and swimming rates?

Solution: Let b, j, s be the biking, jogging, and swimming rates, respectively.

$2b + 3j + 4s = 74$, $4b + 2j + 3s = 91$. Eliminating b by:

$2(2b + 3j + 4s) - (4b + 2j + 3s) = 2 \times 74 - 91 \Rightarrow 4j + 5s = 57 \Rightarrow 4j = 57 - 5s$

.

So $57 - 5s$ has to be a positive integer and divisible by 4 only if $s = 1, 5, 9 \Rightarrow j = 13, 8, 3$.

For these three possible pairs of (s, j) only $(5, 8)$ resulting in an integer value for $b = 15$.

So $b + j + s = 15 + 8 + 5 = 28$.

Problem 9.18 (C)

Helena and Garibaldi simultaneously jumps from a raft on a river and swim in opposite directions. Garibaldi swims downstream with the current at one rate and Helena swims upstream against the current at possibly a different rate. After 5 minutes, they both turn around and return to the raft with each maintaining their constant rate of swimming throughout.

Who returns first?

Discussion: Does the river effect all objects in it: the raft, Helena and Garibaldi, the same way?

Solution 1: Since the river effects all objects in it the same way, we can ignore the river forces. (Note that it is an important physics observation and if you do not understand it, then check Solution 2). Imagine we a riding a car along the river bank, always as opposite to the raft, and videotaping what happens.

While screening the video, we will see that the raft does not move, two persons jump out of the raft simultaneously. In 5 minutes everyone get to some distance away from the raft and cover those distances again in exactly 5 minutes. So they should arrive back at the same time.

Solution 2: Let r be the speed of the river with respect to the bank, h is Helena's swimming speed. With respect to the bank, after jumping out of the raft, she will swim against the current at speed of $x - v$. After first five minutes, she travels a distance of $5(x - v)$.

The river does downstream a distance of $5v$. Because they move in the opposite way, the distance between the raft and her is $5(x - v) + 5v = 5x$.

When Helena turns and swims downstream towards the raft, her speed is $x + v$. In 5 minutes, she covers a distance of $5(x + v)$. The river still keeps going downstream, and in 5 minutes, it covers a $5v$ distance. They move in the same way, so their distance is $5(x + v) - 5v = 5x$.

For better visualization, the table below depicts the time, positions, and distances between Helena and the raft.

Time	$-10v$	$-5v$	Origin	$5(x - v)$
0 min			← Raft, Helena →	
5 min		← Raft		← Helena
(distance)		$[5(x - v) - 0] + [0 - (-5v)] =$		
10 min	Raft, Helena			
(distance)	$[[5(x - v) - 5(x + v)] - [-5v - 5v] = 0$			

Similarly, Garibaldi would take the same distance both ways and arrive to the raft at the same time as Helena.

Problem 9.19 (C)

Ping and Pong mow a lawn and paint a room.

(1) If Ping does both by herself, it takes 9 hours in total.

(2) If she mows the lawn, and after she finishes, Pong paints the room, it takes 8 hours in total.

(3) If Pong mows the lawn, and after he finishes, Ping paints the room, it takes total 7 hours.

(4) If Pong does both jobs by himself, the total time is 6 hours.

(5) It takes Pong twice the time to paint the room as to mow the lawn

What is the total time needed for both of them to do both jobs?

Discussion: If one person has work rate 1/m and the other has work rate 1/n, in another word the first can complete the job in m hours, the second in n hours, then both of them if work together, can complete the job $\dfrac{1}{\frac{1}{m} + \frac{1}{n}} = \dfrac{mn}{m + n}$ (half of the harmonic mean). The harmonic mean of the two numbers m, n is $\dfrac{2}{\frac{1}{m} + \frac{1}{n}} = \dfrac{2mn}{m + n}$.

Solution: Let r, s be the hours Ping mows the lawn, and paints the room. Similarly u, v be the hours Pong mows the lawn, and paints the room.

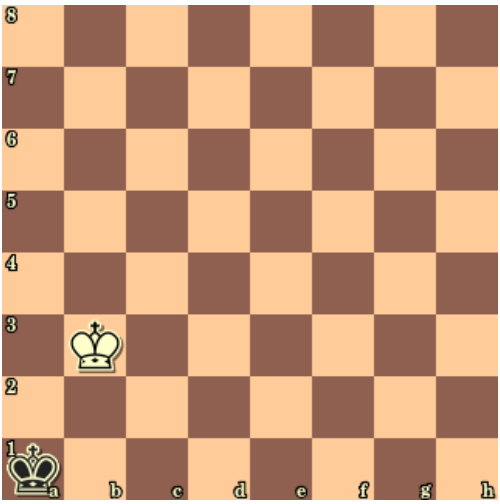
$r + s = 9$, $r + v = 8$, $u + s = 7$, $u + v = 6$, $v = 2u$. Solving $r = 4$, $s = 5$, $u = 2$, $v = 4$.

If both mow the lawn: $\dfrac{1}{\frac{1}{r} + \frac{1}{u}} = \dfrac{ru}{r + u} = \dfrac{4}{3}$ hours, similarly $\dfrac{20}{9}$ hours if both paint the room.

The answer is $\dfrac{4}{3} + \dfrac{20}{9} = \dfrac{32}{9}$ hours.

Puzzle 9.20: Missing piece

Two kings are occupying two positions on the chessboard. Put another chess piece on the board such that:



(1) Neither king is in check.

(2) The position can be reached in a legal game.

(3) It can be proved that neither side has a legal play.

Problems

Problem 9.21 (A)

The Festival of Spring is hosted on the road from the City of Queens to the Town of Kings. Sarah rides bikes from the City and Tyler rides bikes from the Town, and both go to the Festival. They both start at 7:00 AM, and both arrive to the Festival at 10:00 AM. If the road from the City of Queens to the Town of Kings is 105 km long and Tyler rides bike 5 km/h faster than Sarah, what is the average speed of Sarah?

Discussion: Sarah and Tyler ride bikes towards each other.

(1) How far are they from each other at the beginning?

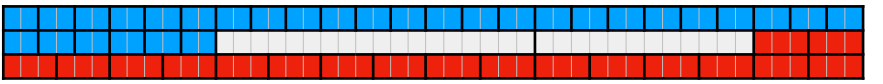
(2) How farther does Tyler ride each hour comparing with Sarah?

(3) What is the distance that does Sarah ride?

Problem 9.22 (A)

Benny the bun and Tongi the tiger play tunnel digging. Benny digs from the left side, and Tongi from the right side of the hill. Working alone Benny can finish the tunnel in 24 minutes, Tongi in 16 minutes. Benny starts 6 minutes before Tongi, then both of them work together, and Tongi continues for 2 more minutes after Benny finishes. How much time passes between the time when Benny starts and Tongi stops?

Discussion: How much can Benny dig alone in 6 minutes? Tongi dig alone in 2 minutes? Take a look at the diagram below:



(a) First row: the tunnel Benny digs alone for 24 minutes

(b) Last row: the tunnel Tongi digs alone for 16 minutes

(c) Middle row: Benny digs from one end for 6 minutes, Tongi from the other end for 2 minutes.

Problem 9.23 (A)

At 1:00 AM the hour and the minute hands of a round clock form an angle of 30 degrees (30°).

From 0:00 AM until 11:59 AM, how many time the hour and the minute hands of a clock form a 30° angle?

Discussion: Estimate

(1) What is the first time the two hands form a 30° angle?

(2) What about the last time?

Problem 9.24 (B)

Two villages on the mountain are connected by a road without any horizontal stretches. A bus travels between the two villages. While going uphill its speed is 15 km/h, and downhill at 30 km/h. What is the distance between the two villages if a round trip takes 4 hours?

Discussion: In a round trip, if the bus goes uphill at a place on one way, it goes downhill at the same place on the way back.

Problem 9.25 (B)

A fuse burns from end to end in an hour, but it does not burn uniformly. How can we measure exactly 45 minutes if we had two such fuses, each of which burn non-uniformly in its own unique ways?

Discussion:

(1) How do you measure 30 minutes?

(2) How do you measure 15 minus if you already measure 30 minutes?

Problem 9.26 (B)

A man was born on May 1, 30 BCE, and died on May 1, 30 CE.

How many years did he live?

Discussion: BCE is Before Common Era. 1 BCE is the year before 1 CE.

Problem 9.27 (C)

Two clocks started to tick at the same time. The first ticks every 2 seconds; the second ticks every 3 seconds. Ticks that occur simultaneously are heard as one tick.

How much time is passed between the first and the 1000th ticks?

Discussion: The ticks are coincide at every 6 seconds.

Problem 9.28 (C)

Last Sunday at noon the date on the calendar was April 15, 2007. What will be the date on the calendar one million minutes after that time?

Discussion: Every hour contains 60 minutes. Every day contains $24 \times 60 = 1440$ minutes.

Problem 9.29 (C)

A factory produces widgets and gadgets. In one hour, 100 workers can produce 300 widgets and 200 gadgets. In two hours, 60 workers can produce 240 widgets and 300 gadgets. In three hours, 50 workers can produce 150 widgets and m gadgets. Find m?

Discussion: Let w be the time needed for one worker produces one widget. Let g the time needed for one worker produces one gadget. Establish an equation system of w, g, solve it and then figure out m. Note that m is a positive integer.

Puzzle 9.30:

A *super queen* in chess is a chess queen that can also move like a knight. Can you put 5 of them on a 5×5 chessboard so that no piece attacks another.

Challenges

Problem 9.31 (A)

Tom and Jerry play a game. They stand back-to-back.

(1) At the first signal each takes 10 steps in opposite directions and stops.

(2) At the second signal both turns around, walk towards each other and meet each other midway. Tom takes six steps.

How large is a step of Tom comparing to a step of Jerry?

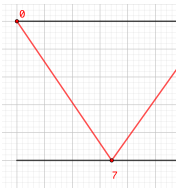
Discussion: Convert the number of Tom's steps to Jerry steps. The figure below shows:

(1) The distances Tom (blue) and Jerry (red) walk after the first signal

(2) The second is the distances Tom and Jerry walk after the second signal.

Problem 9.32 (A)

Two ferries from opposite positions of opposite shores of a lake cross back and forth without stopping. One ferry needs 5 minute to cross the lake, the other 7 minutes. What is the number of time during the 35 minutes that the faster ferry passes the slower one going in the same direction or opposite direction?



Discussion: The figure on the left shows the two shores of the lake. The red line indicate that at the minute 0 the slow ferry leaves a shore. After 7 minutes it reaches the other shore. It it showed as minute 7.

Problem 9.33 (A)

Bao and Dao traveled to a city. Bao agreed to drive two-third of the total distance if Dao agree to drive the remaining distance. After completed his part of driving, Bao slept. When he woke up, Dao still had 20% of her part to drive. What is the fraction that represent the part of the road over the total distance when Bao slept?

Discussion: The blue part is what Bao committed to drive, the red for Dao.

Problem 9.34 (B)

The contest team of MCC has 8 members of the club. While working as a team, they can solve 20 problems in 10 minutes. If MCC needs to send a team to a contest that requires to solve 30 problems in 5 minutes, at least how many club members the team will need?

Discussion: What is the work rate of 2 members?

Problem 9.35 (B)

A clock show 12:00. How long will it be before the hour and minute hand coincide again?

Discussion: Compare the degrees the hour and minute hands move as in previous problems.

Problem 9.36 (B)

Tom can run from his home to Betty's house in 21 minutes. Betty can run from her home to Tom's house in 28 minutes. At noon, Tom starts running from his house toward Betty's house while at the same time Betty starts running to Tom's house. After they meet Betty turns and runs back home. At how many minutes after noon will Betty arrive back home?

Discussion: What is a rate of distance per minute Tom if he run the distance in 21 minutes? What does it means when they meet?

Problem 9.37 (C)

A car's front tires need to be replaced after every 25000 miles, and the rear ones after 15000 miles. When should they be rotated, front to back, so that they can be replace simultaneously? How far will the car travel in total?

Discussion: A distance that divides both 25000 and 15000 is 75000 miles. If we drive this car, we need to change 3 pairs of the front tires and 5 pairs of the rear ones or a total of 8 pairs.

Problem 9.38 (C)

A boat goes with a constant speed downriver from the Elven Town to the Village of Traders in 3 days. It goes upriver from the Village to the Town at a possibly different but constant speed in 5 days. Bilbo Baggins took a raft to float from the Town to the Village.

How long time does that take him?

Discussion: There are two different speeds: the river's and the boat's.

Problem 9.39 (C)

An army battalion forms a row of soldiers marching. It's 20 miles long and it advances 20 miles. During this time, a messenger on a horse travels at constant speed from the rear of the battalion to the front, and immediately turns around and ends up precisely at the rear of the battalion as it completes the 20-mile journey.

How far has the messenger traveled?

Discussion: Let d be the distance travelled by the battalion until the messenger gets to the front of the battalion, and t be the amount of time it takes the messenger to get to the front of the battalion. Let x, y be the rate in mph at which the messenger, and the battalion travelling. So: $t = \frac{d}{y}$, $t = \frac{20+d}{x}$, so $\frac{y}{x} = \frac{d}{d+20}$

Puzzle 9.40:

A *super queen* in chess is a chess queen that can also move like a knight. Can you put 10 of them on a 10×10 chessboard so that no piece attacks another.

Chapter 10

Probability

Examples

Problem 10.1 (A)

What is the probability of:

(1) flipping head with a fair coin

(2) throwing six with a fair dice

(3) select the black marble from a bag containing one black, one white, and one red marble

Discussion: *Probability of an event is the ratio of the number of favourable outcomes to the total number of outcomes.*

Solution:

(1) the number of favourable outcome is 1 - flipping a head. The number of total outcomes is 2: flipping head, flipping tail. The probability is $\frac{1}{2} = 0.5$.

(2) the number of favourable outcome is 1 - throwing six. The number of total outcomes is 6: 1, 2, 3, 4, 5, 6. The probability is $\frac{1}{6}$.

(3) the number of favourable outcome is 1 - the black marble. The number of total outcomes is 3: black, white, or red marble.

Problem 10.2 (A)

(1) What is the probability to roll a six with a pair of fair dices?

(2) There are 3 shirts in the closet: a white, a blue and a red shirt. There are two jeans in the closet: a blue and a grey. What is the probability of selecting a blue shirt and a grey jean?

Discussion*:*

(1) Rolling a six with a pair of fair dices means to roll such that the sum of faces of the dices is 6.

(2) The shirt and the jean can be selected *independently*.

Solution*:*

(1) There are 5 ways to roll a six: (1,5), (2,4), (3,3), (4,2), (5,1). The pairs of numbers are what on the faces of the dices. These are favourable outcomes. Now we count the total number of outcomes: there are 6 ways to roll the first dice and 6 ways to roll the second dice. In total, there are $6 \times 6 = 36$ ways. So the probability is $\frac{5}{36}$.

(2) There are actually two ways to calculate the probability. The first one is easy to see that (blue shirt, grey jean) is the only favourable outcome from 6 ways to combine a shirt with a jean. The second way is to calculate the probability selecting the blue shirt from those three shirts. It's $\frac{1}{3}$. Probability to select the grey jean is $\frac{1}{2}$. And because they are independent, the probability of two independent events can be obtained by multiplying two probabilities.

Problem 10.3 (A)

Three red marbles and four green marbles are placed in a bag. Without looking at them:

(1) first, Ramses pulls one marble, writes down its colour, *puts it back*, takes out one marble, write down its colour.

(2) first, Ramses pulls one marble, writes down its colour, takes out another marble *without putting the first one back*, write down its colour.

What is the probability Ramses has written RG?

Solution*:*

(1) There are 3 ways to select a red marble from 7 marbles. So the probability to write R as the first letter is $\frac{3}{7}$. If the marble is put back, then the probability to write G as the second letter is $\frac{4}{7}$. So the probability to write RG is $\frac{3}{7}\frac{4}{7} = \frac{12}{49}$.

(2) The probability to write R as the first letter is $\frac{3}{7}$. If Ramses does not put back the marble, then the probability to write G as the second letter is $\frac{4}{6} = \frac{2}{3}$. So the probability to write RG is $\frac{3}{7}\frac{2}{3} = \frac{2}{7}$.

Problem 10.4 (B)

Tamara can fire an arrow to hit the bull's eye once every three times. She fires nine arrows, one after another. What is the probability that among those arrows, exactly three of them hit the bull's eye?

Solution*:* Let write the result of the arrows on a row, H means hit, X means miss.

HHH XXX XXX is one of the case when exactly three arrows hit the bull's eye.

The probability to have H is $\frac{1}{3}$, to have X is $\frac{2}{3}$ The probability to get HHH XXX XXX is $\frac{1}{3}\frac{1}{3}\frac{1}{3}.\frac{2}{3}\frac{2}{3}\frac{2}{3}.\frac{2}{3}\frac{2}{3}\frac{2}{3} = \frac{2^6}{3^9}$.

Since HHH XXX XXX and for example XXX HHH XXX are different cases, so the probability we want to calculate is the sum of all the probability of these cases. All probabilities have the same value, we just need to find the number of cases of how do we arrange these HHH: $\frac{9.8.7}{3.2.1}$.

Hence, the probability is $\frac{2^6.9.8.7}{3^9.3.2.1} = \frac{2^8.7}{3^8} = \frac{1792}{6561} \approx 27.3\%$

Problem 10.5 (B)

The digits 1, 2, 3, 4 are randomly arrange to create a four-digit number. What is the probability that the number formed is not divisible by 4?

Solution: There are 4! ways to arrange 4 digits. For a number that is divisible by 4, the last two digits must form a number that is divisible by 4: 12, 24, 32. For each of these, there are two possible combinations with the remaining two digits. So the number of those divisible by 4 is 6.

Hence the probability that the number is not divisible by 4 is $\frac{4!-6}{4!} = \frac{18}{24} = \frac{3}{4}$.

Problem 10.6 (B)

Two circles with radius 4 and 6 share the same centre. A point is selected inside of the large circle. What is the probability that it is outside of the small circle?

Solution:

(1) The area where a point inside the large circle is $\pi \times 6^2 = 36\pi$

(2) The total area where such a point in (1) should be outside of the small circle is $36\pi - 16\pi = 20\pi$.

So the probability is $\frac{20\pi}{36\pi} = \frac{5}{9}$.

Problem 10.7 (C)

Each of the 6 faces of a cube is painted either white or black with equal probability. What is the probability that no two black faces share an edge?

Solution:

There are six faces and 2^6 ways to paint them. Out of these, there are: 1 way to paint all faces white, 6 ways to paint one face black, 3 ways to paint two faces black so that they do not share edge.

So the probability is $\frac{10}{64} = \frac{5}{32}$.

Problem 10.8 (C)

In each of the vertices of a five pointed star, Xerxes placed randomly one of the numbers 1, 2, 3, 4, 5. What is the probability that at least two consecutive integers be connected by a line segment ?

Solution: Let count the complementary cases. Let place 3 somewhere. There are 4 empty places so there are $4! = 24$ possible arrangements. In order for 3 not to connect to 2, or 4, it has to be connected to 1 and 5. Easy to see that there are 2 ways to put 2 and 4 into remaining places. However these are the only two cases. So the probability is $\frac{24-2}{24} = \frac{11}{12}$

Problem 10.9 (C)

X is on the hypotenuse AC of a right triangle ABC such that $AX = 3$, $CX = 2$. Y is chosen inside the triangle ABC. What is the probability that Y is also inside triangle XBC?

Solution: The probability is the ratio of the area of the two triangle. Since both have the same height from B and $CX : AX = 2 : 3$, it means that $\frac{S_{\triangle XBC}}{S_{\triangle ABC}} = \frac{CX}{CA} = \frac{2}{5}$.

Puzzle 10.10: Clever parrot

The pet shop salesman told the buyer that "the parrot will repeat every word it hears." The customer bought it but the parrot would not speak a single word.

Do you think that the salesman lied?

Exercises

Problem 10.11 (A)

What is the probability to draw an ace from the standard deck of cards? What is the probability to draw an even Heart card? How about an odd card?

Solution*:*

(1) There are 4 aces in 52-card deck. So the probability is $\frac{4}{52} = \frac{1}{13}$

(2) There are 13 Heart cards and there are 9 of them with numbers: 2, 3, …, 10. There are 5 even numbers among them. The probability is $\frac{5}{52}$.

(3) There are exactly 4 odd numbers: 3, 5, 7, 9. Each number has 4 cards (heart, diamond, club, spade).

So the total number of them is 16 and the probability is $\frac{16}{52} = \frac{4}{13}$.

Problem 10.12 (A)

Three dices are rolled. What is the probability so that exactly two faces show 1?

Discussion*:* How many cases three faces show 1-1-some thing?

Solution: There are exactly 5 cases of 1-1-something where something is not 1. Similarly for 1-something-1 and something-1-1. So there are 15 favourable cases. The total number of cases is $6 \times 6 \times 6 = 216$.

So the probability is $\frac{15}{216} = \frac{5}{72}$.

Problem 10.13 (A)

Lilian has a piggy bank containing five quarters and five nickels. Her sister shakes it and 2 coins fall out. The sum of them is 30 cents.

What is the probability of such event?

Discussion: What kind of five coins from quarters and nickels can add up to 30 cents.

Solution 1: It is easy to see that the only way to add up 2 coins of quarters or nickels to make 30 cents is $25 + 5 = 30$. So there is two ways: QN or NQ. The probability for QN is $\frac{5}{10}\frac{5}{9}$, for NQ is $\frac{5}{10}\frac{5}{9}$ Because the cases of (1) and (2) independent, so the probability is $2\frac{5}{10}\frac{5}{9} = \frac{5}{9}$.

Solution 2: There are $5 \times 4 = 20$ ways to choose QQ, 20 ways to choose NN, $5 \times 5 = 25$ ways to choose QN, same for NQ. So the probability is $\frac{50}{50+40} = \frac{5}{9}$.

Problem 10.14 (B)

Six students with different heights are chosen from the first row of the classroom. What is the probability five of them standing next to each other from left to right are tallest to shortest?

Solution: Easy to see the number of all cases is $6! = 720$. There are six ways to choose one of the students. One way to arrange the remaining five students by height and then put the chosen student at one of the two ends. So the number of cases is 12. Therefore the probability is $\frac{12}{720} = \frac{1}{60}$.

Problem 10.15 (B)

There are nine marbles in the bag: three reds, three whites, and three blues. Three marbles are selected at random and without replacement (no marble was put back). What is the probability that at least two marbles in different colours?

Solution: Let calculate the case when all of them have the same colour.

The possibility to select three red marbles is: $\frac{3}{9}\frac{2}{8}\frac{1}{7} = \frac{1}{84}$. So to select three marbles of the same colour, the probability is $\frac{3}{84} = \frac{1}{28}$.

So the probability that at least two marbles in different colours is $\frac{27}{28}$.

Problem 10.16 (B)

A point is selected at random on a lie segment AB. What is the probability that twice its distance to A is greater than its distance to B and twice its distance to B is greater than its distance to A

Solution: Let P, Q be the points such that $AP = PQ = QB$.

Any point X such that $2AX > XB$ must be on PB segment. Similarly X must be on AQ. Hence X must be on PQ segment. This means the probability is $\frac{PQ}{AB} = \frac{1}{3}$.

Problem 10.17 (C)

3 digits are selected from. {2, 3, 4, 5, 6, 7, 8} without replacement. What is the probability that the sum of three digits is a prime number?

Solution: The smallest sum is 9 and the largest sum is 21. The prime numbers between them are 11, 13, 17, 19. Easy to find that:

$11 = 2 + 4 + 5 = 2 + 3 + 6,$

$13 = 2 + 5 + 6 = 2 + 4 + 7 = 2 + 3 + 8 = 3 + 4 + 6$

$17 = 2 + 7 + 8 = 3 + 6 + 8 = 4 + 6 + 7 = 4 + 5 + 8,$

$19 = 4 + 7 + 8 = 5 + 6 + 8$

So there are 12 sums out of $\frac{7.6.5}{3.2.1} = 35$ sums. The probability is $\frac{12}{35}$.

Problem 10.18 (C)

What is the probability that the top card in a shuffled deck is an Ace of any colour, and the second card is a spade?

Solution: There are two cases:: the first card is an ace of spades, the second is one of 12 spades: the probability is $\frac{1}{52}\frac{12}{51}$, the first card is an ace from another suite: the probability is $\frac{3}{52}\frac{13}{51}$. So the probability is $\frac{12 + 3.13}{51.52} = \frac{1}{52}$.

Problem 10.19 (C)

Point X is selected at random with in a square ABCD of side length 2. What is the probability that the area of ABXD is greater than 2?

Solution: The area of ABXD is the sum of areas of two triangles ABX and ADX. Each has a base length $AB = AD = 2$, so the sum of heights of X to AB and AD must be greater than 2. Any point on the BD diagonal has the sum of heights to AB and AD equal 2. So X must be inside BCD triangle. The probability is $\frac{1}{2}$.

Puzzle 10.20: Barber again

The barber in London says "I rather cut the hair of two Frenchmen than of one Englishman". Why does he say that?

Problems

Problem 10.21 (A)

A fair coin is flipped three times, and each flip comes up heads. What is the probability that the next flip will also be head?

Discussion*:* What does it mean fair coin? How the fair coin should be have if the previous flip comes up head?

Problem 10.22 (A)

A pair of dices is rolled to see who goes first in a board game. Samuel only needs to beat a 4 to win the chance to go first. What is the probability that he will roll a greater than 4?

Discussion: Remember if a pair of dices are roll, the sum of the faces show is the result of the roll. It means that a 4 is one of the (1,3), (2,2), (3,1) cases.

Problem 10.23 (A)

Three vertices of a regular heptagon (7-side polygon with all equal sides and angles) are connected to form a triangle. What is the probability that the triangle formed is isosceles (having two equal sides)?

Discussion: Select A vertex. Find a pair of vertices that form an isosceles triangle with A.

Problem 10.24 (B)

Ollie rolls a fair dice three times. What is the probability that the third roll is greater than the sum of the first two?

Discussion: Let do casework with the last roll: 6, 5, 4, 3, 2, 1. In each case, how many possible cases for the sum of the first two?

Problem 10.25 (B)

For any date in history, Billy can very quickly tell you what day of the week it was. Emily chooses three dates at random and Billy tells the day of the week for each. What is the probability that two of the days occurred on the same day of the week?

Discussion: Let do complementary counting of the possible cases for different days of the week.

Problem 10.26 (B)

In rectangle ABCD: $AB = 1$, $BC = 2$. X is selected at random inside the rectangle. What is the probability that the area of ABX triangle is more than twice the area of triangle BCX?

Discussion: Draw the rectangle. Connect B with midpoint of CD. Let X be a point on this segment. Investigate the areas of ABX and BCX.

Problem 10.27 (C)

Four vertices of a cube are selected at random. What is the probability that all four selected points will be co-planar (on the same plane)?

Discussion:

(1) Find a few examples where 4 vertices are on the same plane.

(2) How many ways to chose 4 out of all the vertices of a cube?

Problem 10.28 (C)

The weather man has predicted 30% chance of snow for each of the following three days. One nearby listener commented that the chance of snowing is $3 \times 30 = 90\%$, therefore it will likely snow on one of the next three day. What do you think?

Discussion: What is the probability of NOT snowing on all of the next three days?

Problem 10.29 (C)

Andy draws a circle, Benny draw another one completely inside that, Cindy draws a third circle inside the Benny's circle. Andy then colour black the region inside Benny circle but outside Cindy circle.

A point X is selected in Andy's circle at random and the probability of X be in the black region is $\frac{1}{2}$. All circle diameters are integers. What is the smallest possible value of the black region?

Discussion: Let a, b, c be the radii of the circles. What is the relationship between them?

Puzzle 10.30:

Alain said: Freeman owns more than 1000 books.

Betty said: He does not. He owns fewer than that.

Henry said: He surely owns at least one book.

If only one of the statement true, how many books does Freeman own?

Challenges

Problem 10.31 (A)

Four vertices of a regular octagon are selected at random and connected to form a quadrilateral. What is the probability that the quadrilateral NOT a rectangle?

Discussion:

(1) How many ways can you choose 4 vertices out of 8?

(2) What is the condition for 4 vertices so that they form a rectangle?

Problem 10.32 (A)

What is the probability to roll each of the numbers 1 through 6 in any order in six rolls?

Discussion: How many possible rolls are there? How can you have six rolls with each of the numbers 1 through 6 in any order?

Problem 10.33 (A)

The first nine letters of the alphabet are randomly placed into three equal piles, each pile consists of three letters. What is the probability that the letter A and B end up in different piles?

Discussion: How many ways A and B can end up in the same pile?

Problem 10.34 (B)

For three randomly selected two-digit numbers, what is the probability that the sum of unit digits of those number is even?

Discussion: What are the parity cases of the unit digits?

Problem 10.35 (B)

The school cafeteria has chocolate chip cookies and oatmeal cookies for desert. What is the probability that exactly 3 of the next 5 students in line will select chocolate chip cookies, and students prefer chocolate to oatmeal 2 to 1?

Discussion:

(1) What is the probability of a student choosing a chocolate chip cookie if students prefer chocolate to oatmeal 2 to 1?

(2) What is the probability of a student choosing an oatmeal cookie?

Problem 10.36 (B)

The probability that two randomly selected integers from a set, have a product that is odd, is $\frac{1}{5}$.

If the set include 3 odd integers, how many integers are there altogether?

Discussion: What integers are there so their product is odd?

Problem 10.37 (C)

Two distinct pairs of vertices are randomly selected from the vertices of a cube and connected by line segments.

What is the probability that the segments intersect?

Discussion: How many ways to find the first pair? And the second pairs? What kind of segments can intersect?

Problem 10.38 (C)

27 students each flip a fair coin.

What is the probability that there are more heads showing than tails?

Discussion: How many heads would be there?

Problem 10.39 (C)

What is the probability that two real numbers selected at random between 1.0 and 4.0 have sum greater than 4?

Discussion: Let draw a graph of $x + y = 4$

Puzzle 10.40:

Danny and Diana sitting in front of the house.

"I am a boy", said the one with the black hair.

"I am a girl", said the one with the red hair.

If at least one of them is lying, who is which?

Solutions

Chapter 1

Insightful Proofs — Solutions

Problems

Problem 1.21 (A)

Clemence said: "The day before yesterday I was 10, but I will turn 13 in the next year." Is this possible?

Discussion: Let put the information that we know into a table of the relevant days, Clemence's age, and the year.

		(3) The 11th birthday		The 12th birthday	The 13th birthday
	The day before	Yesterday	Today		
Clemence's	10		**(2) 11**	12	13
The year		**(4) Last**	This year	**(1) This year**	Next year

Solution: Let highlight the arguments in the table as below.

- Because the 13th birthday is next year, so the 12th birthday must be this year.
- Today cannot be the 12th birthday, otherwise within 2 days - from the day before yesterday until today Clemence's age changes by 2! So today she is 11.
- The 11th birthday must be yesterday because the day before yesterday she was 10.
- Yesterday was last year.

Hence, Clemence's birthday is December 31 and the situation is real.

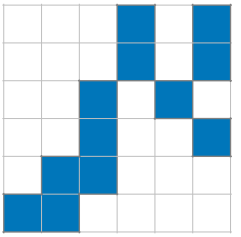
Problem 1.22 (A)

The Island of Knights and Liars has two type of inhabitants: Knights who always tell the truth and Liars who always lie. While visiting the island, Erica meet a group of three islanders: Ariana, Kauri, and Wiremu. Erica asked them the same question: "How many knights are in your group?" Ariana replies "None". Kauri said "One". Can you find out what Wiremu is? What was his response?

Solution: Fill *the answers of the islanders* for each case. Fill the casework table with possible answers. Wiremu will say anything but "one".

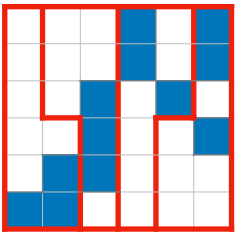
Ariana	**Kauri**	**Wiremu**	**Ariana's answer**	**Kauri's answer**	**Wiremu's answer**
Knight	Knight	Knight	3	3	3
Knight	Knight	Liar	2	2	Not 2
Knight	Liar	Knight	2	Not 2	2
Knight	Liar	Liar	1	Not 1	Not 1
Liar	Knight	Knight	Not 2	2	2
Liar	**Knight**	**Liar**	**Not 1**	**1**	**Not 1**
Liar	Liar	Knight	Not 1	Not 1	1
Liar	Liar	Liar	Not 0	Not 0	Not 0

Problem 1.23 (A)



Cut the board shown in the picture above into four equal parts so that each one of them contains three shaded squares?

Solution: See the figure below.



Problem 1.24 (B)

Distribute 127 one-dollar bills into 7 wallets so that any integer amount from 1 through 127 dollars can be paid without opening the wallets.

Discussion:

(1) Let number the wallets with the number from 1 to 7.

(2) For an amount can be paid with a number of wallets, write a *sequence of zeros (0) and ones (1)*: zeros if those wallets are not used and ones if those are used. Below is an example.

W1	W2	W3	W4	W5	W6	W7
0	0	0	1	1	0	1

(3) The sequence can be written now as 0001101.

(4) If any amount can be written as a sequence of 0s and 1s, what does that mean?

Solution: Any positive integer number between 1 and 127 can be written in binary representation using not more than 7 digits of 0's and 1's. For example:

$0010011 = 2^4 + 2^1 + 2^0 = 16 + 2 + 1 = 19$ Each number is basically a sum of a number of exponent of 2. So we will distribute the one-dollar bills into W1, W2, … W7 such that W1 contains \$1, W2 contains \$2, W3 contains \$4, … W7 contain \$64. Any number between 1 and 127 is equal to a sum of wallets that are chosen based on the binary presentation of the number. For example \$19 is equal to the sum of cash in W1 (1), W2 (2), and W5 (16).

Problem 1.25 (B)

Find all 3-digit number $\overline{abc}$, with the digits a, b, c: $1 \le a, b, c \le 9$ and

$$\overline{ab} + \overline{bc} + \overline{ca} = \overline{abc}$$

Discussion:

(1) A number such as $\overline{ab}$ can be rewritten as $\overline{ab} = 10a + b$.

(2) A sum like $10c + b$ can always be rewritten as $10c + b = \overline{cb}$

(3) Use estimation to eliminate impossible cases.

Solution:

$$\overline{ab} + \overline{bc} + \overline{ca} = \overline{abc}$$

Because $\overline{ab} = 10 \times a + b$

$$\overline{ab} + \overline{bc} + \overline{ca} = \overline{abc} \Rightarrow 10a + b + 10b + c + 10c + a = 100a + 10b + c$$

$$\Rightarrow 11a + 11b + 11c = 100a + 10b + c$$

Think of the equation of both sides as a balanced scale. We need to take off both sides a same number of weights: $11a$, $10b$, c:

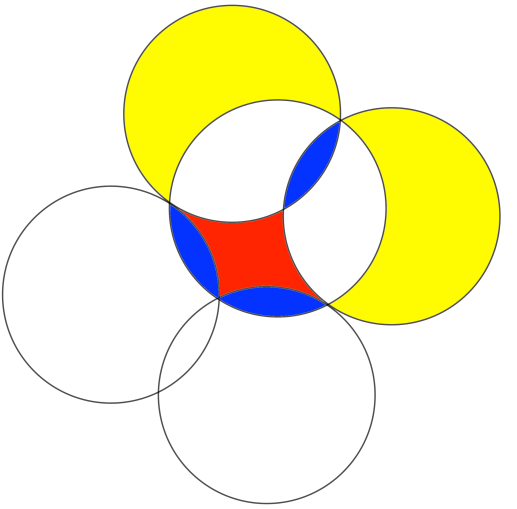
$$\Rightarrow (11a - 11a) + (11b - 10b) + (11c - c) = (100a - 11a) + (10b - 10b) + (c - c)$$

$$\Rightarrow b + 10c = 89a$$

$$\Rightarrow \overline{bc} = 89a \Rightarrow a = 1 \Rightarrow \overline{(}c = 89 \Rightarrow \overline{abc} = 198$$

Problem 1.26 (B)

Lilian draw five identical circles as below. She coloured some parts of them with blue (**b**), red (**r**), and yellow (**y**). Parts with same colours are identical.



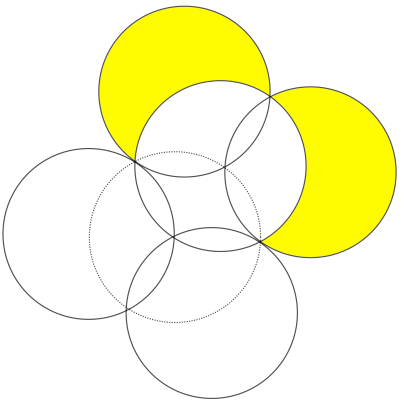
Can you find a formula to calculate a yellow (**y**) part based on blue (**b**), and red (**r**) parts?

Discussion:

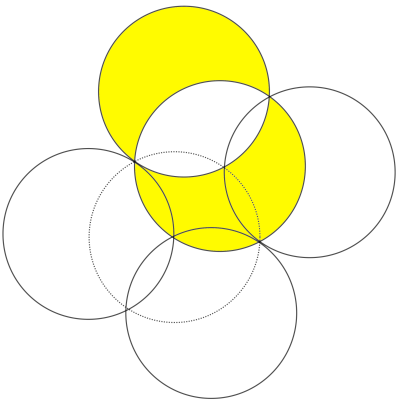
(1) Try to find a region that looks same as the yellow one.

(2) Try to fit the blue and red parts into that region.

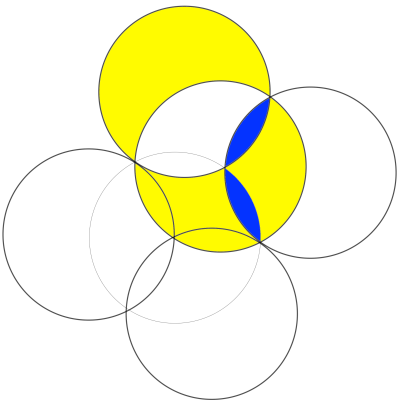
Solution: First let add an extra circle to highlight certain relationships between the regions.



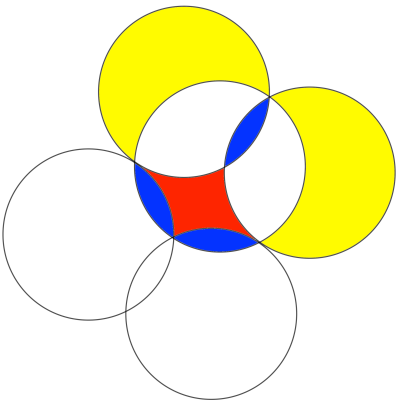
Two original yellow regions



Let change one of the yellow region



The yellow region is split into two equal yellow sub-regions.



Easy now to see that y=**5b+2r**

Problem 1.27 (C)

Determine the value of the expression below:

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3}+\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3}+\frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3}+\ldots\frac{9^{\frac{997}{1000}}}{9^{\frac{997}{1000}}+3}+\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3}+\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}$$

Discussion: Look at the solution 2 of problem 1.2.

(1) Try to pair the terms

(2) Calculate the sums of terms in each pair, for a few pairs.

(3) How many pairs are there?

Solution: For any x real number:

$$\frac{9^x}{9^x+3}+\frac{9^{1-x}}{9^{1-x}+3}=\frac{9^x}{9^x+3}+\frac{9}{9+3\times 9^x}==\frac{9^x}{9^x+3}+\frac{3}{3+9^x}=1\ (*)$$

Therefore let S be the sum.

$$S=\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3}+\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3}+\frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3}+\ldots\frac{9^{\frac{997}{1000}}}{9^{\frac{997}{1000}}+3}+\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3}+\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}$$

$$S=\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}+\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3}+\frac{9^{\frac{997}{1000}}}{9^{\frac{997}{1000}}+3}+\ldots\frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3}+\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3}+\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3}$$

Hence:

$$2S=(\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3}+\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3})+(\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3}+\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3})+(\frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3}+\frac{9^{\frac{997}{1000}}}{9^{\frac{997}{1000}}+3})+\ldots$$

$$+(\frac{9^{\frac{997}{1000}}}{9^{\frac{997}{1000}}+3}+\frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3})+(\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3}+\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3})+(\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}+\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3})$$

Using (*) $2S=999\Rightarrow S=\dfrac{999}{2}$.

Problem 1.28 (C)

In the xy-plane, what is the length of the shortest path from (0, 0) to (12, 16) that does not go inside the circle $(x-6)^2+(y-8)^2=25$?

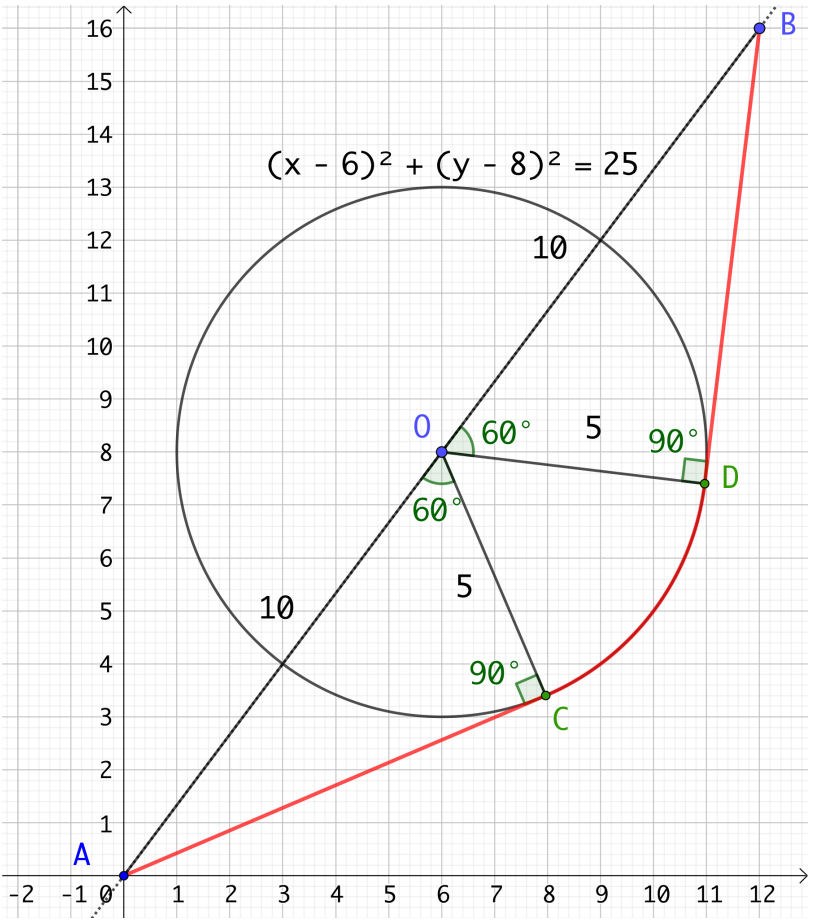
Discussion:

(1) Draw the circle $(x-6)^2+(y-8)^2=25$, mark the two given points (0, 0) to (12, 16)

(2) Are the two points are outside of the circle?

(3) What would be shortest path *around* the circle, connecting that two points?

Solution:



Let A, B, O be (0, 0), (12, 16), and (6, 8). Through A and B draw two lines that are tangent with the circle centre O at C, D, respectively. It is easy to see that the shortest path from A to B and not intersect the circle must be: the line segment AC + the arc CD + the line segment DB.

Because AOC and BOD are both 30-60-90 right triangle, hence:

$$AC = BD = \frac{10\sqrt{3}}{2} = 5\sqrt{3}.$$

$$\angle COD = 60^\circ \Rightarrow \widehat{CD} = \frac{2.\pi.5}{6} = \frac{5\pi}{3} \Rightarrow \overline{A}C + \widehat{CD} + \overline{D}B = 10\sqrt{3} + \frac{5\pi}{3}$$

Problem 1.29 (C)

Determine the largest number that is the product of positive integers whose sum is 1976.

Discussion: Let take a look *a simpler problem:* find the largest number that is the product of positive integers, such that:

(1) Their sum is 4. Easy to find that it is $2 \times 2 = 4$

(2) Their sum is 5. Easy to find that it is $3 \times 2 = 6$

(3) Their sum is 6. Easy to find that it is $3 \times 3 = 9$, which is larger than $2 \times 2 \times 2 = 8$

Try a few more cases and establish your hypothesis. Then prove it.

Solution:

The question is to find a largest product $a_1 . a_2 \ldots . a_n$ where $a_1, a_2, \ldots, a_n$ are positive integers such that $a_1 + a_2 + \ldots + a_n = 1976$.

First, if any a_k number $(1 \leq k \leq n)$ is larger than 4, then we can replace it with two numbers 3, $a_k - 3$. The sum $a_1 + a_2 + \ldots + a_n$ is still the same:

$$a_1 + a_2 + \ldots + a_{k-1} + a_k + a_{k+1} + \ldots + a_n$$
$$= a_1 + a_2 + \ldots + +a_{k-1} + (3) + (a_k - 3) + a_{k+1} + \ldots + a_n$$

But because $a_k > 4 \Rightarrow 3(a_k - 3) > a_k$, so the product of the replaced number is greater:

$$a_1 . a_2 \ldots a_{k-1}.3.(a_k - 3) . a_{k+1} \ldots a_n > a_1 . a_2 \ldots a_{k-1} . a_k . a_{k+1} \ldots a_n$$

Second, if any a_k number $(1 \leq k \leq n)$ is 1, there would be $a_j > 1$ (otherwise the product is 1), then we can replace both with a numbers $a_j + 1$. The sum $a_1 + a_2 + \ldots + a_n$ is still the same

But because $a_j + 1 > 1.a_j$, so the product of the replaced number is greater:

$$a_1 . a_2 \ldots (a_j + 1) \ldots a_n > a_1 . a_2 \ldots 1.a_j \ldots a_n$$

So the $a_1, a_2, \ldots, a_n$ numbers can only be 2, or 3. But $3.3 = 9 > 8 = 2.2.2$. Therefore any triplets of 2 can be replaced with a pair of 3 and the product is increased.

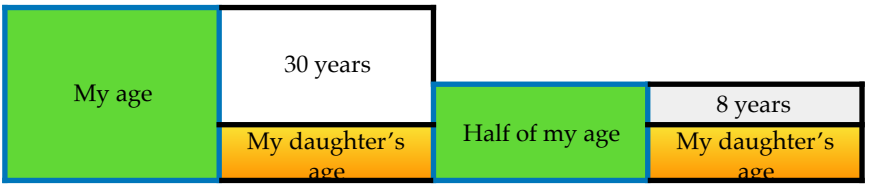
Since $1976 = 3 \times 658 + 2 \Rightarrow 3^{658}.2$ is the largest product.

Challenges

Problem 1.31 (A)

I am 30 years older than my daughter. If I were two times younger than I really am and she was 8 years older than she really is, we would be at the same age. How old are we?

Discussion: *Drawing a picture* is one way to gain insights of a problem.



Solution: From the figure above, it is simple to see that half of my age is 30-8=22. Therefore my age is 44 and my daughter's age is 44-30=14 years.

Problem 1.32 (A)

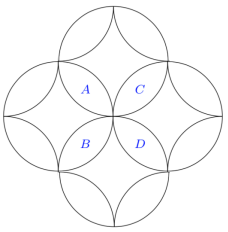
Insert the brackets (,) into the expression to obtain 1961:

$$4.8 + 5.6.7 + 9 + 2$$

Discussion: *Prime factorization* is a way to write a number as a product of its prime factor s. For example $12 = 2^2 \times 3$, $111 = 3 \times 37$. Look at the prime factorization of 1961.

Solution: $(4.8 + 5)\,.\,(6.7 + 9 + 2) = 37.53 = 1961$

Problem 1.33 (A)



Given four circles, each circle was divided into five parts as in the diagram below. The numbers 5, 6, 7, ... 19, 20 were entered into each of the hollow parts so that two different parts are filled with two different numbers. Knowing that the sum of the numbers filled in all parts of each circle is 60. What is $A + B + C + D$?

Discussion: The sum of the numbers filled in all parts of each circle is 60. If we add all of these sum together:

(1) How is this sum of sums related to the given numbers 5, 6, 7, ... 19, 20?

(2) How is this sum of sums related to A, B, C, D?

Solution: Note that A, B, C, D are the parts shared by exactly two circles. Any other part belongs to only one circle. This means that if *the sum of all numbers in all parts of all the circles is the sum of the number* 5, 6, 7, ... 19, 20 *plus the sum of* A, B, C, D. Therefore:

$$4 \times 60 = (5 + 6 + \ldots + 20) + (A + B + C + D)$$
$$\Rightarrow A + B + C + D = 240 - (5 + 6 + \ldots + 20)$$
$$= 240 - ((1 + 2 + .. + 20) - (1 + 2 + 3 + 4))$$
$$= 240 - (\frac{20.21}{2} - 10) = 240 - (210 - 10) = 40$$

Problem 1.34 (B)

When Han Solo was still relatively unknown, he used to attend problem solving courses. Once he was near the chalkboard, using his super sneaky driver skills he quickly changed the two digits in the multiplication on the board. After he did, the board looked like this:

$$4 \times 5 \times 4 \times 5 \times 4 = 2247$$

Can you reconstruct the original?

Discussion:

(1) Look at the digit 7. Can you *change two digits on the left side so that the product's unit digit is 7*?

(2) Can both changed digits be on the right side?

Solution:

(1) On the left side, there are three even numbers. It was impossible to change the digits on the left side such that 7 was the original unit digit of the right side, so it is a changed digit.

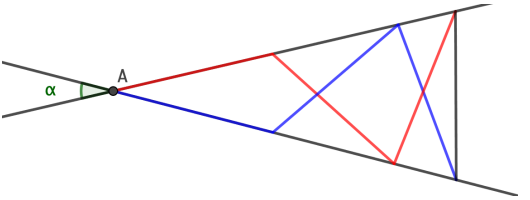
(2) If no digit on the left side was change, its value is 1600 and cannot match the right side. Hence, one digit on the left side was change.

(3) Because there are two 5s on the left side and only one digit was changed, so the unit digit of the product on left side is 0, regardless of which digit got changed. So the unit digit of the right side must be 0.

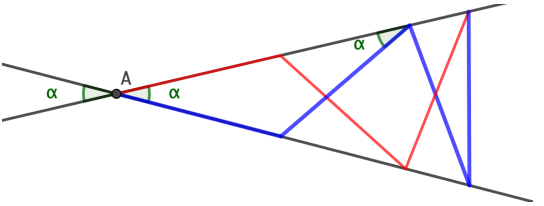
Thus: $4 \times 5 \times 4 \times 7 \times 4 = 2240$

Problem 1.35 (B)

Qi'ra, before becoming the lieutenant in the Crimson Dawn, once was a math competitor. She was required to calculate the value of the angle α in the below figure, that was made from 7 matchsticks with equal length, connected end to end. What is the value of α?



Discussion: Change matchsticks to line segment and using the assumption that *all matchsticks have the same length*.

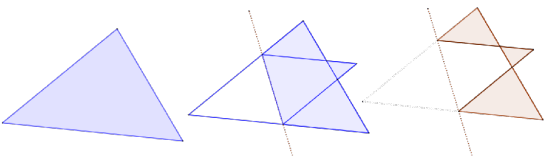


Solution: from the figure above let β be the angle in the most right corner. From the largest triangle: $\alpha + 2\beta = 180^\circ$. The middle triangle is an isosceles triangle with angles 2α, 2α, $180^\circ - 4\alpha$. The triangle on the right is also a isosceles triangle: $\beta, \beta, 180^\circ - 2\beta$: $\alpha + (180^\circ - 4\alpha) + \beta = 180^\circ \Rightarrow \beta = 3\alpha$

$$\Rightarrow \alpha + 6\alpha = 180^\circ \Rightarrow \alpha = \frac{180^\circ}{7}$$

Problem 1.36 (B)

Emma folds the blue triangle on the left below along a dotted line to get a folded figure, which is the blue figure in the middle. Boutheina observes that the triangle area is 1.5 times of the area of the folded figure. Alex says that "it's interesting that the area of the shaded figure formed by three triangles, which is the brown one on the right, is exactly 1 cm^2."



Eui is tasked with calculating "What is the area of the blue triangle?"

Discussion:

(a) What is the difference between the second and third figure

(b) What is the difference between the first and the second figure?

Solution: Let b be the difference of the the blue figure in the middle and the brown one on the right (b is the area of the quadrilateral figure in the middle of the blue figure):

(1) the blue figure = the brown figure plus $b = 1 + b$

(2) the area of the triangle S = the blue figure plus b = the brown figure plus $2b = 1 + 2b$

Because $(1 + 2b) = 1.5(1 + b) \Rightarrow 0.5b = 0.5 \Rightarrow b = 1 \Rightarrow 1 + 2b = 3$.

Problem 1.37 (C)

There are 10 bags of coins. One of them contains only counterfeit coins, while other contains only real coins. Each counterfeit coin is 1 gram lighter than a real coin. You have two-pan balance scale with a display that shows the difference between two loads, in grams. Find the counterfeit coin in just one weighing. You are allow to take any number of coins from any bag.

Discussion: Let reduce the problems to:

(1) 2 bags

(2) 3 bags

(3) 4 bags

Solution:

Let take a simple example with 2 bags. If we take only 1 coin from each bag then it is easy to see which one contains counterfeit coin: the bag that contains the lighter coin. Now let assume there are 3 bags. Let take 1 coin from first bag, 2 coins from the second one and 3 coins from the last one. On the left pan, put the coins from the first two bags and on the other pan the coins from the third bag. If the left is lighter by 1g or 2g, the first bag or the second bag contains the counterfeit coins. If the right side is lighter, then it's the third bag.

We can also repeat the proof with 1, 2, 3, and 6 coins selected from the 1, 2, 3, 4 bags, respectively.

Let solve the general problem: from bag 1, 2, …, $n-1$, n take 1, 2, 3, … $n-1$, $\frac{n(n-1)}{2}$ coins. Put all the coins from the first $n-1$ bags on the left pan, coins from the nth bag onto the right pan. If any of the bag from 1, 2, …, $n-1$contains counterfeit coins, then the left pan will be lighter 1, 2, …, $n-1$ grams, respectively. Otherwise the right pan will be $\frac{n(n-1)}{2}$ grams lighter and the nth bag is the one that contains counterfeit coins.

Problem 1.38 (C)

Prove that the regions formed by 5 circles in a plane can be coloured with black and white such a way that any two regions that share a common arc boundary will be coloured differently.

Discussion: Let reduce the problems to 1 circle

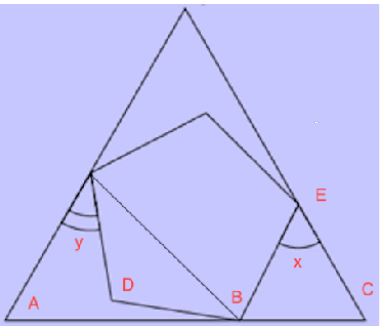
(1) Add one more circle, what cases do you have?

(2) Let add another one and analyze the cases. Can you generalize?

Solution: Let draw the first circle and colour the inner part of the circle black, leaving the outer white. Each time when drawing a new circle, it can cut existing parts into two. Simply change only those parts that are inside of the new circle to the opposite colour. This way the parts outside of the circles remain same, the ones inside the circles have opposite colours and therefore if any part was divided into two new parts then that two parts have different colours.

Problem 1.39 (C)

The diagram below shows an equilateral triangle and a regular pentagon that touches all three sides of the triangle.



Find $x+y$ in degrees.

Discussion:

(1) What is $\angle EBC + \angle ABD$?

(2) How $\angle EBC$ is related to x?

(3) How is $\angle ABD$ related to y?

Solution:

$$y + \angle DFB + \angle ABD + \angle DBF = \angle AFB + \angle ABF = 180 - 60 = 120$$

$$\angle DFB + \angle DBF = 180 - 108 = 72$$

$$y + \angle ABD = 120 - 72 = 48.$$

Since:

$$x + \angle EBC = 180 - 60 = 120$$

$$x + \angle EBC + y + \angle ABD = 120 + 48 = 168$$

Because:

$$\angle EBC + \angle ABD = 180 - \angle DBE = 180 - 108 = 72$$

Hence

$$x + y = 168 - 72 = 96$$

Puzzle 1.10: Tajik

Tajik is an Indo-European language related most closely to Farsi (or Persian), the major language of Iran. Most literature in Tajik uses a version of the Cyrillic writing system. This system is used in Russian, Ukrainian, Bulgarian, Eastern Europe and Central Asia. You don't need to know how to read Cyrillic writing to solve this puzzle -- just use your common sense!

Below are three phrases from the Tajik language with their English translations.

дуусти ҳуби ҳамсоай сумо	a good friend of your neighbor
ҳамсоай дуусти ҳуби сумо	a neighbor of your good friend
ҳамсоай ҳуби дуусти сумо	a good neighbor of your friend

Your task is to give the English translations of all four Tajik words. The possibilities are simply "good," "friend," "neighbour," and "your."

дуусти	ҳуби	ҳамсоай	сумо

Note: The order of the words is important.

Solution:

дуусти	friend
ҳуби	good
ҳамсоай	neighbour
сумо	your

Puzzle 1.20: Learning Indonesian

The major languages spoken in Malaysia, Indonesia, and the small country of Brunei Darussalam, are nearly identical. In English they are known as Indonesian and Malay. These languages are spoken by over 240 million people living among 18,000+ islands that make up these countries.

The only word known is **"satu" (Indonesian) = "one" (English)**.

A. Satu ditambah satu menjadi dua.
B. Dua ditambah dua menjadi empat.
C. Satu ditambah dua menjadi tiga.
D. Dua dikalikan dua menjadi empat.
E. Enam dikurangi tiga menjadi tiga.
F. Sepuluh dikurangi enam menjadi empat.
G. Dua ditambah tiga menjadi lima.
H. Sepuluh dibagi dua menjadi lima.
I. Tiga dikalikan enam menjadi delapan belas.
J. Delapan belas dikurangi satu menjadi tujuh belas.
K. Tiga dikalikan tiga menjadi sembilan.

Can you figure out what are the words for the following numbers in Indonesian:

- number 6 is _____,
- number 9 is _____,
- number 17 is _____

Solution:

- number 6 is **enam**,
- number 9 is **sembilan**,
- number 17 is **tujuh belas**

Puzzle 1.30: Kurdish

The Kurdish people number at least 25 million. Kurdish is an important member of the Iranian branch of the Indo-European language family. There are several dialects of Kurdish. Kurmanji Kurdish is spoken by about 12-15 million people, living in Turkey, Iran, Iraq and Syria and is represented by three literary standards and several speech varieties. Some speakers also live in the southern parts of the former Soviet Union, especially in Armenia and Georgia. The following puzzle is from the variety of Kurmanji Kurdish spoken in the former Soviet Union. This variety is normally written in Cyrillic script (like Russian), but is presented here in a modified Latin script.

Match the following the Kurdish sentences to their translations:

1. Ez h'irç'ê dibînim
2. Tu dir'evî
3. Tu min dibînî
4. H'irç' dir'eve
5. Ez dir’evim
6. Tu h'irç'ê dibînî

English translations in random order

A. You see Bear.
B. You see me.
C. Bear runs.
D. You run.
E. I see Bear.
F. I run.

Solution:

1-E 2-D 3-B 4-C 5-F 6-A

Puzzle 1.40: Swahili of Eastern Congo (Democratic Republic of Congo)

Swahili belongs to a large language family, called Bantu. Bantu languages are spoken by more than 100 million people in Southern and Eastern Africa. Swahili is the mother tongue of about 5 million people, and is the common language of trade along much of the east coast of Africa. However, the variety of Swahili represented in this puzzle is spoken hundreds of kilometres inland, and is quite different from that which is spoken near the coast.

Read the first ten Swahili words and their English translations in the table below.

1. Ninasema.	I speak.'
2. Wunasema.	You speak.'
3. Anasema.	She speaks.'
4. Wanasema.	They speak.'
5. Ninaona.	I see.'
6. Niliona.	I saw.'
7. Ninawaona.	I see them.'
8. Niliwuona.	I saw you.'
9. Ananiona.	She sees me.'
10. Wutakaniona.	You will see me.'

Translate the following three English statements into Swahili.

- She saw them: **Aliwaona**
- I will see you: **Nitakawuona**
- She saw me: **Aliniona**

Chapter 2

Logic — Solutions

Problems

Problem 2.21 (A)

Three turtles are crawling along a *straight road* heading in the *same direction*. "Two other turtles are behind me," says the first turtle. "One turtle is behind me and one is ahead," says the second. "Two turtles are ahead of me and one other is behind," says the third turtle.

How can this be possible?

Discussion:

(1) Where is the first turtle?

(2) Where is the second turtle?

(3) Where would be the third turtle? Does what he say make sense?

Solution: The third turtle is … *lying*.

Note: once every other possibilities are eliminated based on logical reasoning, the only remaining, however improbably it is, is the only possibility.

Problem 2.22 (A)

Three boxes contain some balls. There are two white balls in one box, two black balls in another box, while the third box contains one white and one black ball. The boxes are labeled. On one box the label says WW, on the other it says BB, and the label on the third box is BW. None of the labels describes the contents of the box correctly.

How is it possible to decide which box contains what by *taking exactly one ball out of a box*?

Discussion:

(1) The key is the BW box. What should it contain?

(2) If one ball is taken out of that box, how many possible cases are?

(3) Can you determine the boxes by working out all the possible cases?

Solution:

Take a ball from the BW-labeled box. Since it should not be a BW box:

(1) so if the ball is W then it is the WW box: hence the WW box is the BB and the BB is BW.

(2) otherwise it is the BB box: so the BB box is WW and the WW box is BW.

Problem 2.23 (A)

The zoos in the Blah Blah Land share some common properties:

(a) If a zoo has both hippos and rhinos, then this zoo has no giraffes

(b) Every zoo has a hippo or a rhino. Some zoos have both.

(c) If a zoo has both a hippo and a giraffe, then this zoo also has a rhino.

The zoo in the capital of Blah Blah Land has a giraffe. Does this zoo has a rhino? How about a hippo?

Discussion:

(1) The second condition says that the zoo in the capital has a hippo or a rhino.

(2) If it has a hippo, then what else should it have according to (c)? If that the case, according to (a) what it should not have?

(3) So beside giraffes, what other animals should that zoo have?

Solution:

If the zoo of the capital has a giraffe and a hippo, then according to (c), it should have a rhino. However it is not possible according to (a). So the zoo mush have a giraffe, a rhino, but no hippo.

Problem 2.24 (B)

Hobbits, elves and gnomes live in the Kingdom of Gondor. The inhabitants are abide by several ancient rules:

(a) If the village is inhabited both by hobbits and elves, the gnomes should not settle in it.

(b) If elves but no gnomes dwell in a village, then hobbits should reside in this village as well.

(c) If both hobbits and gnomes live in a settlement, this settlement should have elf resident as well.

Bilbo the Hobbit would like to establish a "Hobbit Only" retreat near the famous Gondor hot springs. Would he break any ancient rules by establishing this retreat?

Discussion: Assume that Bilbo the Hobbit established such a village. Carefully verify each rules. What do you think?

Solution: He would not. No rule is against his establishment.

Problem 2.25 (B)

The son of a professor's father is talking to the father of the professor's son, and the professor does not take part in the conversation.

Is this possible? Give an example if you think so.

Discussion*:*

(1) Is "son of a professor's father" necessarily the professor?

(2) Is "father of the professor's son" necessarily the professor?

(3) Can you have an example for (1) and another one for (2)

Solution: The brother of the professor (son of a professor's father) is talking to the husband of the professor.

Problem 2.26 (B)

Once upon a time, there was a tribe for which the hippopotamus was sacred. The Chief of the tribe cared for the pet hippo which he fed and coddled incessantly.

Once a year, the Chief loaded his hippo and his Collector of Offerings on a boat and headed up the river to a hut where all of his people must bring their annual offering.

The hut contain a huge beam balance. The hippo was placed on one side, and gold ingots offered by the tribesmen were placed on the opposite side until the sides balanced.

One year, the hippo had grown so fat that he broke the beam balance. The Chief grew incensed and demanded that his Collector of Offerings figure out a way to measure the year offering. "If I do not have my offerings by sundown, your head will be mine!" cried the Chief. The Collector of Offerings thought and thought, and by the end of the day had a solution.

Can you figure out what he thought of?

Discussion:

(d) Perhaps you already know this problem as how to weigh an elephant.

(e) If not, think about the Chief transported the hippo on his boat, the heavier the hippo was, the water line on the outside of the boat's hull got higher.

Solution: the Chief transported the hippo on his boat and use the water line on the outside of the boat's hull to mark the hippo's weight. After that he transfers the hippo off and weighs the gold with the mark of the water line. It's the same example as weighing an elephant.

Problem 2.27 (C)

A box contains marbles of different colours. How many marbles are there if all of them are red except two, all of them are green except two, all of them are blue except two.

Discussion: Let try a casework with the number of blue marbles: how many possible?

Solution: Let take a look at the number of blue marbles. For sure it cannot be more than 2, otherwise the statement *all of them are red except two, all of them are green except two* not true.

(1) There are 2 blue marbles. The rest cannot be both red or green. So this is not possible.

(2) There is 1 blue marble. If there is a red marble then the rest should be green. Similar statement is true with a green marble. So there is 1 blue, 1 red, and 1 green marble.

(3) If there is no blue marble: the number of red and green marbles is two. In this case the statement *all of them are red except two, all of them are green except two* will not be true.

The answer is 1 blue, 1 red, and 1 green marble.

Problem 2.28 (C)

Three tablespoons of milk from a glass of milk are poured into a glass of tea with the same size, and the liquid is throughly mixed. Then three table spoons of this mixture are poured back into the glass of milk.

Which is greater now: the percentage of milk in the tea of the percentage of tea in the milk?

Discussion:

(a) Let assume originally there are each glass of milk and tea contains 10 tablespoons.

(b) You can try to apply algebra or

(c) Think how many tablespoons of milk and tea remain in each glass.

Solution: Let say both glasses contain an a amount of tablespoon liquid. At first 3 tablespoons from the milk glass is put into the tea glass. The ratio of milk to tea is $3 : a$. After that 3 tablespoons of the mixed liquid containing $\frac{9}{a+3}$ tablespoon milk and $\frac{3a}{a+3}$ tablespoon tea are poured back into an $a - 3$ tablespoons of milk glass.

In the milk glass now $a - 3 + \frac{9}{a+3} = \frac{a^2}{a+3}$ milk, $\frac{3a}{a+3}$ tea content, and the percentage is $\frac{3}{a}$.

Similarly the tea glass now $3 - \frac{9}{a+3} = \frac{3a}{a+3}$ milk, $a - \frac{3a}{a+3} = \frac{a^2}{a+3}$tea content, and the percentage is also $\frac{3}{a}$. The answer is that the percentages are the same.

Problem 2.29 (C)

The distance between Ottawa and Kingston is 200 km.

Two cyclists leave from Ottawa and Kingston simultaneously travelling towards one another, one at a speed of 23 km/h, and the other at a speed of 27 km/h.

A fly leaves Ottawa together with the first cyclist, and flies towards the second at a speed of 100 km/h. The fly reaches the second cyclist, touches his forehead, the flies back to the first, touches his forehead, return to the second, and so on until the cyclists collide with their foreheads and squash the fly.

How many kilometres has the fly flown altogether?

Discussion:

(4) How the distance between the two cyclists decreased?

(5) How much time is needed so that this distance decrease to zero?

(6) How far the fly *flew in this amount of time*?

Solution: The cyclists travel towards each other and their distance is reduced by 50km every hour. It means that they will meet each other after $\frac{200}{50} = 4$ hours. The fly flies with constant speed of 100km/h for 4 hours or 400km.

Challenges

Problem 2.31 (A)

You are on the Island of Knights and Liars. There was a crowd of people on a plaza, and each person in the crowd declares "You all are liars". How many knights are there?

Discussion:

(1) Would there be no knights?

(2) Would there be at least 2 knight?

Solution: If there is no knights, meaning everyone is liar, then nobody can state that *you are all liars*. So there should be at least one knight. Similarly, if there are at least two knights, then these two can not state that *you are all liars.* So there is only one knight and the statement is true.

Note: it's actually funny if you think that in a crowd there is a single person who is a truth teller and the statement is the same, regardless of whom telling.

Problem 2.32 (A)

In a country there are three towns:

(1) Blah-Blah, where everyone tells the truth.

(2) La-la-la, where everyone always lie.

(3) Nice-nice, where everyone alternate tell the truth and lie.

Once a fire broke out in one of these towns, and the fire department - which is stationed outside of all the towns - is called. Here the conversation:

"There is a fire here"

"Where is it?"

"In Nice-nice."

Where should the firefighter go?

Discussion: Put yourself in the shoe of callers from each city.

Solution: People from Blah-blah do not state that they have fire and then give another town's name. People from Nice-nice can lie about if they have fire but cannot say that it's in their town. So, it should be the people from La-la-la who called. They lie that they have fire, which means that in fact it happens somewhere else. They then again lie about where it would be. Both facts means that the fire should not happen in La-la-la or Nice-nice. Hence, it happens in Blah-blah and is reported by La-la-la people.

Problem 2.33 (A)

A teacher drew several circles on a sheet of paper. Then he asked a student "How many circles are there?" "Seven," was the answer. "Correct! So, how many circles are there?" the teacher asked another student. "Five," answered the student. "Absolutely right!" replied the teacher.

How many circles were really draw on the sheet?

Discussion:

(1) Is it necessary that both students saw the *same page*?

(2) If not, what would be the situation?

Solution: The professor show the students *different sides* of the page. So on one side, it's seven circles, on the other side it's five.

Problem 2.34 (B)

How can you build a house that has a windows in all four walls but every window faces south.

Discussion: The actual question would be *where* can you build such a house?

Solution: At the North Pole.

Problem 2.35 (B)

Sasha likes pets.

(a) All but two of her pets are parrots.

(b) All but two are cats.

(c) All but two are dogs.

(d) All the rest are crickets.

How many pet crickets does Sasha have?

Discussion:

(1) Minimum how many pets Sasha has?

(2) Can she has 4 pets, what does what mean?

(3) Could she have no cricket? Could she have only crickets?

Solution:

The solution of the Problem 2.27 tells us that she have 1 parrot, 1 cat, and 1 dog if only the first three statements are considered. This means the number of her pet crickets is 0.

Problem 2.36 (B)

A king wants his daughter to marry the smartest of 3 extremely intelligent young princes, and so the king's wise men devised an intelligence test.

The princes are gathered into a room and seated, facing one another, and are shown 2 black hats and 3 white hats. They are blindfolded, and 1 hat is placed on each of their heads, with the remaining hats hidden in a different room.

The king tells them that the first prince to deduce the colour of his hat without removing it or looking at it will marry his daughter. A wrong guess will mean death. The blindfolds are then removed.

You are one of the princes. You see 2 white hats on the other prince's heads. After some time you realize that the other prince's are unable to deduce the colour of their hat, or are unwilling to guess. What colour is your hat?

Discussion:

(1) Make an assumption about your hat.

(2) List the possible cases and reasons why other princes cannot guess.

Solution: Let say that there are three princes A, B, C and I am A.

(1) If I see both of other princes wearing black hats. Then I can confidently say that I am wearing the white hat.

(2) If I see one white (B) and one black hat (C). I have to look for. If I see that B does not react immediately, he thinks and then after a while wanted to say, I should say I am wearing the white hat immediately before him.

(3) If I see two white hats:

- I can wait a bit to see if both wanted to take the chance before the other could. I should not risk because it would mean I am wearing black.

- If it is too long time and both then realize something, then it would mean that all three of use wearing white hats. In this case I will try. However it is still a big risk.

Problem 2.37 (C)

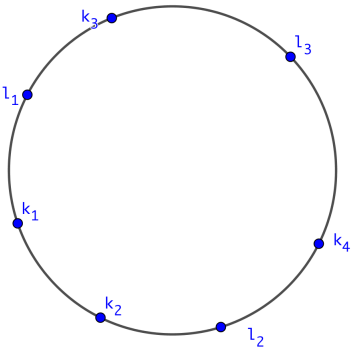
At a festival on the Island of Knights and Liars, seven local people are sitting around table, and each of them says: “one of my two neighbours is a liar and the other is a knight”. Who is sitting at the table?

Discussion:

(1) If there were one knight k_1, then next to that knight is a liar l_1 and another knight K2. Next to the knight K2 is a liar L2.They will appear in this order: L1 K1 K2 L2. Now think about the two who can be on the left of L1, and on the right of L2. Who will sit in between that two? Is that possible?

(2) If there were no knights, what does that mean?

Solution:



(1) If there were one knight k_1, then next to that knight is a liar l_1 and another knight k_2. For example $l_1k_1k_2$. Because l_1 is a liar and one of his neighbour is already a knight (k_1), so the other one has to be a knight as well (k_3). Similarly the other neighbour of l_2 must also be a knight (k_4). Finally the one sits in between k_3 and k_4 cannot be liar (k_4 cannot sit between two liars l_2, l_3).

(2) If there were no knight, then any statement contains two parts: one is false (one of my neighbours is a knight) and the other one is true (one of my neighbours is a liar). Because the statement is using a logical *and* to connect the two part, so it can be true only if both parts are true. This means that any liar is able to tell such a statement, which is a lie.

Problem 2.38 (C)

Todd lives in a magical pond, where all the toads and frogs talk. One of the species always says the truth, while the other one always lie. Figure out the species of each talker.

(a) Todd: My neighbour Tammy is a frog.

(b) Tammy: My neighbour Todd and Ted are both toads.

(c) Ted: I am a frog or a toad.

(d) Terry: Todd is a toad

Discussion:

(1) Assume Todd is a frog. Investigate what would be Tammy.

(2) Would that be contradictory to what Tammy said?

Solution:

(1) Assume that Todd is frog.

   (i) If he is a truth teller, then Tammy should be a frog according to (a), and she is a truth teller as well. However (b) contradicts that Todd is a frog.

   (ii) So Todd is a liar, then Tammy is a toad, and she is a truth teller, but that contradicts (b) saying that Todd is a toad.

(2) Hence Todd is a toad.

   (i) If he is a truth teller, then Tammy should be a frog according to (a), and she is a liar. Since Todd is a toad, Ted must be a frog according to (b), and he is a liar. However this contradicts (c) which is always true.

   (ii) If he is a liar, then Tammy should be a toad according to (a), and she is a liar as well. Since Todd is a toad, Ted must be a frog according to (b). According to (d) Terry is a truth teller and she is a frog.

Hence, Todd and Tammy are both toads and they are liars. Ted and Terry are both frogs and they are truth tellers.

Problem 2.39 (C)

Once upon a time, prince Kalama of Malaka went to ask princess Paqa of Qapa to marry him. The princess listened and decided to put him to a test.

She brought him to the throne hall, where there were two doors. A sign on each door said: "Both rooms behind these doors are empty". She informed the prince that each of the rooms is either empty or contains a tiger. In addition, she said "If the first room is empty, then the door sign is true; otherwise, it is false. At the same time, if the second room is empty, then the sign on the door is false; otherwise, it is true."

The princess invited the prince to open the door of his choice. If a hungry tiger jumps out of the door, he will die. However, if he opens the door to the empty room, he will get Paqa's hand, and with her, the whole kingdom of Qapa.

The princess is known as truthful. So what should prince Kalama do?

Discussion:

(1) First, let observe that the signs said *both rooms are empty*. Let assume they are true.

(2) But then the princess said *if the second room is empty, then the sign on the door is false; otherwise, it is true*.

So can you now conclude if the signs are telling the truth and decide which door to open?

Solution: Let the two statements be

(a) *if the first room is empty, then the door sign is true; otherwise, it is false.*

(b) *if the second room is empty, then the sign on the door is false; otherwise, it is true*.

We consider the following facts:

(i) If Room 1 is empty, then its door sign must be True.

(ii) If Room 1 contains a tiger, then its door sign must be False.

(iii) If Room 2 is empty, then its door sign must be False.

(iv) If Room 2 contains a tiger, then its door sign must be True.

(v) Both signs says *both rooms behind these doors are empty.*

The table below shows what are in the rooms and what the signs would be, using the two

statements (a) and (b) with the fact (i) - (iv):

In Room 1	**In Room 2**	**Sign 1**	**Sign 2**
Empty	Empty	True (i) (v)	False (iii) True (v)
Tiger	Empty	False (ii) (v)	False (iii) (v)
Empty	Tiger	True (i) False (v)	True (iv) False (v)
Tiger	Tiger	False (ii) (v)	True (iv) False (v)

It shows that the only case when the combination of the princess' statements, the signs, and the actual situation are consistent. Hence prince Kalama of Malaka must open the second room in order to get the hand of princess Paqa of Qapa.

Puzzle 2.10: Counting

There is a secret *word* hidden in this matrix of letters. Can you discover it?

R	V	E	O	V	C
S	I	O	V	R	D
V	E	R	C	V	O
R	O	V	E	S	E
E	R	S	C	R	I
C	E	R	E	O	R

Hint:

(a) It is not a usual puzzle, so think out-of-the-box

(b) Look at the title

Solution: Count the number of appearances of each letter and short the letters by the number of appearances. The answer is DISCOVER.

Puzzle 2.20: Description

Find rule of the sequence and determine the missing term in the box

1, 11, 21, 1211, 111221, $\boxed{}$

Hint:

(a) It is not a usual puzzle, so think out-of-the-box

(b) Look at the title

Solution: From the second term, the terms is actually a description of the previous number.

11 - one one-digit, 21 - two one-digits, 1211 - one two-digit one one-digit, …

Puzzle 2.30: Arriving early

An engineer goes every day by train to the city where he works. At 8:30AM, as soon as he gets off the train, a car picks him up and takes him to the plant. One day the engineer takes a train arriving at 7:00AM, and starts walking toward the plant. On the way, the car picks him up and he arrives at the plant 10 minutes early.

When does he meet the car?

Hint:

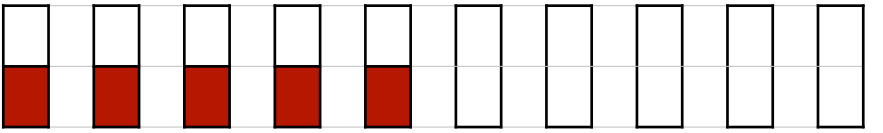
(a) Draw a diagram to show how the engineer travels in both cases.

(b) What is that 10 minutes?

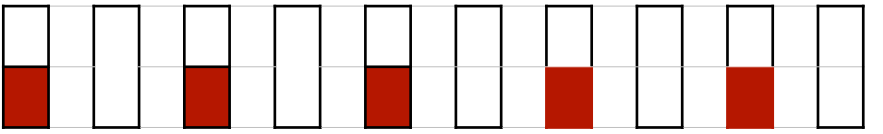
Solution: It was planned such that the car meet him at 8:30AM at the train station. Since it saved 10 minutes on *both directions* to and from the station, so it should meet him 5 minutes earlier than planned. Hence, he meets the car at 8:25AM.

Puzzle 2.40: Moving glasses

Ten glasses form a straight line. Five of them on the left are filled with wine, five of them on the right are empty.



What is the minimum number of actions can you do to obtain the below setting:



Hint:

(a) Think out-of-the-box

(b) The number of actions can be really small.

Solution: Take the second and fourth glasses and pour their wine into the seventh and ninth glasses, respectively.

Chapter 3

Parity — Solutions

Problems

Problem 3.21 (A)

The product of 22 integers is equal to 1. Show that their sum cannot be zero.

Discussion:

(1) What are the integers can be factor of 1?

(2) How can you make a sum *near to zero*?

Solution 1: Because the product of 22 integers is 1, that 22 integers must be factors of 1. The only numbers that are factor of 1 are $1, -1$.

Assume that their sum is zero. *Their sum is zero if and only if the number of* 1*'s is equal to the number of* -1*'s*. So, there are 11 number 1's and 11 number -1's. Their products is -1. It contradicts the given condition that their product is 1. Hence their sum cannot be zero.

Solution 2: The product of 22 integers is 1, meaning that the number of -1 integers must be even, which also has to be half of 22. It is not possible.

Problem 3.22 (A)

Can a *magic square* be formed from the first 36 prime numbers?

The magic square here means a 6×6 board with a number in each square, such that the sum of the number along any row, column, or diagonal is the same.

Discussion: What is the parity of the first 36 prime numbers?

Solution: In the first 36 prime numbers there is one even number (2) and 35 odd numbers. Their sum is an *odd* number (*). If these numbers can form a 6×6 magic square, then their sum must be equal six times the sum of one column. This is an even amount and contradicts (*).

Therefore it is not possible to form an 6×6 magic square with the first 36 prime numbers.

Problem 3.23 (A)

There are 100 students in grade 5. Every day 3 of them are acting as stewards to help keep the school neat and clean.

Can it be arrange so that after a certain period of time each student has shared duty with every other student exactly once?

Discussion: Assume that is possible. Let take a look at student A.

(1) He shares each activity with two other student.

(2) The rest 99 students, each shares with other student a day of activity exactly once.

Solution: Let say student A can act as steward with another two students. The 99 remaining students, each shares stewarding activity with any other student only once, can be split into distinct pairs - a student with whom he/she shares a day of stewarding.

This is a contradiction, since 99 is an odd number and cannot be split into pairs.

Problem 3.24 (B)

Twenty-five checkers are placed on a 25×25 checkerboard in such a way that their positions are symmetric with respect to one of its diagonals. Prove that at least one of the checkers is positioned on that diagonal.

Discussion:

(1) A diagonal divides the boards into two parts, not including its own squares.

(2) If it would not contain through any checkers, how the checkers are placed on the two parts?

Solution:

Suppose that there is no checker positioned on any diagonal of the 25×25 checkerboard. Let connect all pair of checkers that are located symmetrically with respect to one of the diagonals by a *thread*. These threads can connect more than two checkers into a *necklace*. Each necklace contains *two* or *four* checkers, in another word, an *even* number of checkers. It is impossible to divide 25 into groups of even numbers.

So at least one of the checkers must be positioned on one of the diagonals.

Problem 3.25 (B)

The number 1, 2, 3, …, 2017 are written on a blackboard. You can replace any two numbers with their positive difference. This can be done until only one number remains on the board. Can that number be 0?

Discussion: The idea is similar to problem 3.5.

Solution: The sum $1 + 2 + 3 + \ldots + 2017$ is an odd number. When replacing two numbers with their positive difference:

(1) If the two number have same parity, the difference is an even number.

(2) If the two number have different parity, the difference is an odd number.

In both cases, *the parity of the sum of all numbers on the board remains same, or it is odd*.

Therefore it is impossible to obtain the result of 0, which is an even number.

Problem 3.26 (B)

The number 1, 2, 3, …, 15 are written into the squares of a 15×15 board. Each square contains a single number. The main diagonal is from top-left to bottom-right.

(a) The squares which are symmetric to the main diagonal contain same numbers.

(b) No row or column contains 2 copies of the same number.

Show that no two of the numbers along the main diagonal are the same.

Discussion*:* Let take a look at the 1's.

(1) How many of them are in a row or a column, in the table

(2) And on the main diagonal?

Solution: It is simple to see that there are 15 of 1's. Because of (a) and (b) there is a single 1 on the main diagonal, similarly there is a single 2, and so on.

Problem 3.27 (C)

A grasshopper jumps a long a line. His first jump takes him 1cm, his second takes him 2cm, and so on. Each jump can take him to the right or to the left.

Can he return to the starting point after 2017 jumps?

Discussion*:* Take a look at problem 3.14

Solution: Since $1 + 2 + \ldots + 2017$ is an odd number. We know that replacing plus with minus signs - indicating the grasshopper is jumping in the opposite direction - does not change its parity. Because *the distance between the grasshopper's position to the original one is always an odd number*, so it cannot be 0 at any time.

Therefore he cannot return to the starting points.

Problem 3.28 (C)

45 points are chosen on a line AB, all lying outside of segment AB. Prove that the sum of distances from these points to point A is not equal the sum of distances from them to point B.

Discussion: Let take a point X.

(1) What is $AX - BX$?

(2) What is the sum of all differences?

Solution: For any point X on the line but outside of AB: $AX - BX = \pm AB$. The difference of the sum of distances from these points to point A and the sum of distances from them to point B is same as adding an odd amount of $\pm AB$. If this difference is 0 that means the distance AB is 0. This is not possible.

Problem 3.29 (C)

Three grasshoppers A, B, and C play leapfrog along a line. At each turn, one grasshopper leaps over another, but not over two others. After 2019 leaps, can they return to the original position?

Discussion: Take a look at problem 3.8

Solution:

The grasshoppers' positions can be divided into two different groups:

(i) ABC, BCA, and CAB

(ii) ACB, BAC, and CBA

It is important to note that *each leap of any grasshopper will result in a position in the other group.*

Therefore it is not possible after an *odd number* of jumps to return to the same group.

Challenges

Problem 3.31 (A)

Bethany and her girl friends stand in a circle. It turns out that both neighbours of each child have the same gender. If there were five boys in the circle, how many of them are there?

Discussion*:*

(1) Can any neighbour of a boys be a boy?

(2) Can any neighbour of a girl be a girl?

Solution: If Bethany or her girl friends stand next to a girl, that means all children must be girls. However there are five boys, so Bethany and her girl friends have neighbours, who are boys. Similarly all these boys have neighbours who are girls.

From here is it easy to see that there are five boys and five girls.

Problem 3.32 (A)

On a cube, all the vertices and midpoints are marked. All diagonals of faces are drawn. Is it possible to visit all the marked points *once* by walking along the face diagonals?

Discussion*:* See Problem 3.16

Solution: Let colour all the vertices black and all midpoints of faces white. While traveling along face diagonals the vertices and midpoints are visited alternately. However there are 8 vertices and 6 midpoints.

It means that it is not possible to visit all marked points once.

Problem 3.33 (B)

There are seven integer numbers $a_1, a_2, \ldots, a_7$. They are shuffled into $b_1, b_2, \ldots, b_7$. Can the product $(a_1 - b_1)(a_2 - b_2)\ldots(a_7 - b_7)$ be an odd number?

Discussion*:* What are the parity of these terms?

Solution: It is simple to see that among seven integers, there are always *at least four of them with the same parity*. In another word, there are always at least four odd or four even numbers. Let assume that they are a_1, a_2, a_3, a_4. It does not matter how these numbers can be shuffle, there is always *one number among* b_1, b_2, b_3, b_4*, which has the same parity* with a_1, a_2, a_3, a_4. Without loss of generality, assume it is b_1. Hence $a_1 - b_1$is even, meaning the given product is *even*.

Problem 3.34 (B)

Can we draw a close path made up of 9 line segments, each of which intersect exactly one of the other segments?

Discussion:

(1) If such a solution would be possible, then all line segments would be partitioned into pair of intersecting segments.

(2) Then what would be the number of segment and why?

Solution: Assume that there is a way to do so. *The line segments can be divided into distinct pairs of intersecting segments (no pair share any segment).* It is not possible since there are only 9 segments.

Problem 3.35 (B)

Find m, n integers if $m^2 - 2n^2 = 5$

Discussion:

(1) What *parity* does m has? Substitute it with $2m_1$ if it is even, $2m_1 + 1$ if it is odd.

(2) Simplify what you got. Now compare the parities of the two sides.

Solution: Because the left side of $m^2 - 2n^2 = 5$ is odd as the right side. So m must be odd. Let $m = 2m_1 + 1$. Then $4m_1^2 + 4_m 1 + 1 - n^2 = 5 \Rightarrow 4m_1^2 + 4m_1 - 4 = 2n^2 \Rightarrow n = 2n_1$.

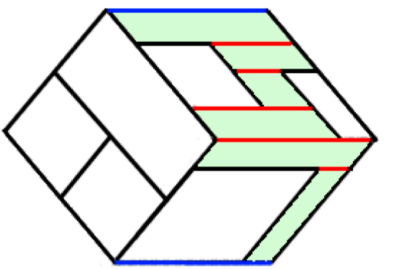
$m_1^2 + m_1 - 1 = 2n_1^2 \Rightarrow m_1(m_1 + 1) = 2n_1^2 + 1$.

It is not possible since the left side is even and the right side is odd.

Problem 3.36 (B)

Can a convex nonagon (a polygon with 9 sides) be cut into parallelograms?

Discussion: Below is an example with the hexagon case to illustrate the ideas.



(1) Start at a side of the nonagon. Since we've split up the nonagon into parallelograms, this side must contain a side of a parallelogram. Jump to the opposite side of this parallelogram.

(2) We're now at a side of a parallelogram. If this is contained in another side of the nonagon, we're done. If not, then it must overlap with a side of another interior parallelogram which is across the line from the one we were at before. Jump to the opposite side of this parallelogram.

(3) This process will create a sequence of parallel sides: we will start at the side of the nonagon, and proceed to jump to sides of interior parallelograms. Since there are finitely many interior parallelograms, and we keep moving the same direction, we will eventually wind up at another side of the nonagon. Pair up this second side with the first one. We have now created a pair of parallel nonagon sides.

Solution: From the discussion, it is not possible since 9 cannot be divided into distinct pairs.

Problem 3.37 (C)

A 17-digit number is chosen, and its digits are reversed, forming a new number. These two numbers are added together.

Show that their sum contains one even digit.

Discussion*:*

(1) Assume it is not, look at the ninth digit. How is it formed? Look at the value at 8th column.

(2) Look at the value of the 7th column.

(3) Then look at the smaller numbered odd columns.

Solution: Let assume the opposite. Let number the columns of digits from right to left.

The ninth digit of the number will be added to itself. This would be an even digit, so there must be something carried from the eighth column. But if there is, then there should be one also carried from column tenth to eleventh (tenth column is identical to eight column except for the order of digits). So the seventh column must have digits of same parity, and requires a carry from the sixth column. Continuing this reasoning, there should be a carry into each *odd* column. However *there cannot be anything carried into the first column*, so this is a contradiction.

This means that their sum contains one even digit.

Problem 3.38 (C)

The number 1, 2, 3, …, 12, 13 are randomly place on a circle by Marvin. John found that every number of these is divisible by the difference between its neighbours. Marvin said it is not possible.

Who is right and why?

Discussion: Let take a look at an odd number. If the two neighbours of an odd number have same parity then their difference is even.

(1) How many odd numbers are there?

(2) According to the observation, what would be with their neighbours?

Solution: An odd number is not divisible by an even number. It means that the neighbours of an odd number cannot have same parity. Assume that John is right, every odd number has to be next to an odd and an even number.

So the odd numbers can be divided into pairs. However that is not possible with 7 odd numbers between 1 and 13.

Problem 3.39 (C)

All positive integers are coloured black and white. It is known that:

(1) the sum of two different coloured numbers is black and,

(2) their product is white

What is the colour of the product of two white numbers?

Discussion: Assume that the colour of the product of two white numbers is black. Prove that it leads to contradiction.

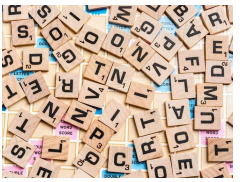
Solution: Let m, n are white number, k is a black number. Assume that mn is black:

(1) m is white, k is a black, so mk is white. mn is black, thus $mk + mn = m(k + n)$ is black.

(2) $n + k$ is black, m is white, so $(n + k)m$ is white.

(1) contradicts (2). Hence the colour of the product of two white numbers is black.

Puzzle 3.10: Magic word



Using Scrabble letters, show on the right side, try to build a word following this instruction "*it is a word that contains itself six times*".

Solution: It's LETTER

Puzzle 3.20: How many 8's

How many figures of 8 on the "8 of diamonds" card?

Solution: The inner part between the diamonds forms another 8. There are 3 of them.

Puzzle 3.30: Sorting five bags

Using a beam balance:

Sort 5 bags with different weights, from lightest to heaviest in no more than *seven* weighing!

Solution: We weigh the first two objects and label the heavier A, the lighter B. We do the same with C and D. So now we have A > B and C > D. 2 weighings are spent.

Let know compare A with C. 1 more weighing is spent.

There are two cases: A > C and C > A.

1/A > C > D:

- Let compare E with C: if E > C then compare E with A: we have E > A > C > D or A > E > C > D; if C > E then compare E with D: we have A > C > E > D or A > C > D > E

=> So after 2 weighings we can sort E into (A, C, D)

- if E > A > C > D: compare B with C; if B > C then we done otherwise compare B with D

- if A > E > C > D: compare B with C; and then B with E or B with D

- if A > C > E > D: compare B with E; and then B with C or B with D

- if A > C > D > E: compare B with D; and then B with C or B with E

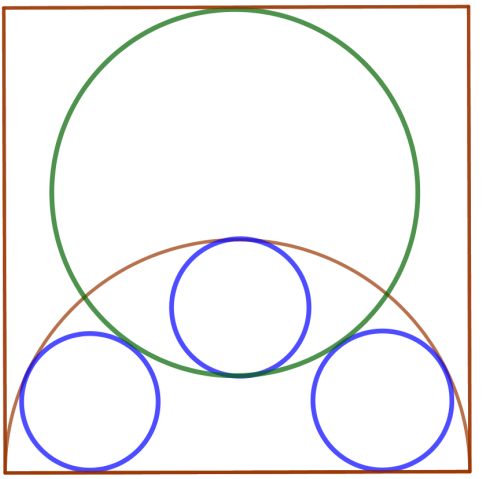
=> So after another 2 weighings we can sort B into (A, C, D, E)

The total number of weighings is 2 + 1 + 2 + 2 = 7.

2/ C > A > B: sort E and then D similar to case 1/

Puzzle 3.40: Tangent circles

The four circles and a semi-circle are tangent to each other inside a square. The three small circles have same radius *r*. The radius of the red circle is *R*.



What is the value of the fraction $\frac{r}{R}$?

Solution: $\frac{3}{8}$. Connect all centres of the circles. Measuring distance between the midpoint of the bottom edge of the squares to the center of the left small square. Center of the largest circle and two others form a right triangles.

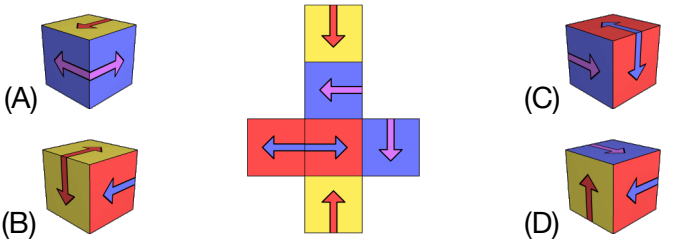
Chapter 4

Visual patterns — Solutions

Problems

Problem 4.21 (A)

Which of the cubes below could NOT have been folded from the shape in the middle?



Discussion: First identify the double-head arrow. Second look at the remaining arrow direction.

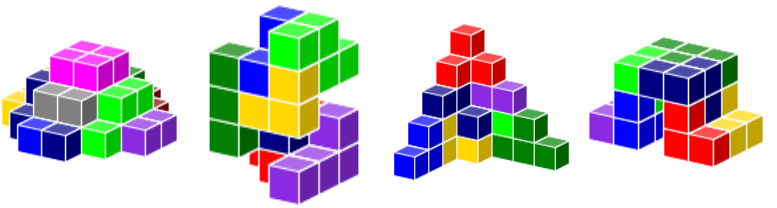
Solution:

(B). Look at the directions of both blue and red arrows.

Note: it would be nice to actually print the sheet, use the middle figure as the cube net, fold it to verify your own theories.

Problem 4.22 (A)

Can you count how many cubes in each of the below figures:



Discussion: Count per each layer, and be careful.

Solution:

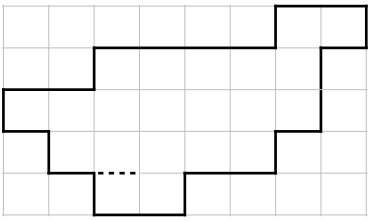
(a) $(6 \times 6 - 4 \times 3) + (4 \times 4 - 4) + 4 = (36 - 12) + (16 - 4) + 4 = 40$

(b) $3 + 6 + 5 + 6 + 3 = 23$

(c) $(2 \times 3 + 4) + (2 \times 2 + 4) + (2 \times 2 + 1) + 3 + 1 = 10 + 8 + 5 + 4 = 27$

(d) $3 \times 9 = 27$

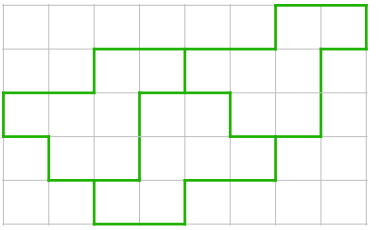
Problem 4.23 (A)



Divide the above figure into *three identical (congruent) parts*

Discussion: The figure give a hint to start.

Solution: Easy to count the number of small square: it's 21. It means each part should have exactly 7 small squares. Uses ideas from previous problem. Below is a solution.



Problem 4.24 (B)

Find a, b, c, d positive integers such as:

$$ab = c + d \qquad cd = a + b$$

Discussion: It's symmetric!

(1) If you add both sides: $(ab - a - b + 1) + (cd - c - d + 1) = 2$

(2) Think of the values on both sides.

Solution: Add both side of the equations together, we have

$(ab - a - b + 1) + (cd - c - d + 1) = 2 \Rightarrow (a-1)(b-1) + (c-1)(d-1) = 2$. Since a, b, c, d positive integers, $(a-1)$, $(b-1)$, $(c-1)$, $(d-1) \geq 0$. Therefore there are three possible cases.

(1) $(a-1)(b-1) = 0$, $(c-1)(d-1) = 2$.
$(a, b, c, d) = (1,5,3,2), (5,1,3,2), (1,5,2,3), (5,1,2,3)$

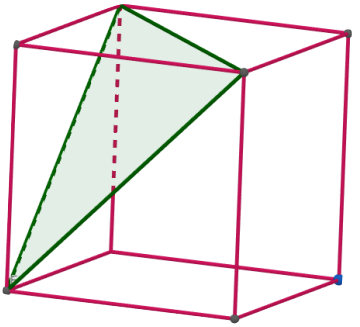
(2) $(a-1)(b-1) = 1$, $(c-1)(d-1) = 1$. $(a, b, c, d) = (2,2,2,2)$

(3) $(a-1)(b-1) = 2$, $(c-1)(d-1) = 0$.
$(a, b, c, d) = (3,2,1,5), (3,2,5,1), (2,3,1,5), (2,3,1,5)$

In general there are two solutions: (2,2,2,2) and (1,5,3,2) where the first two and the last two numbers are interchangeable.

Problem 4.25 (B)

A cube can be cut so that the cross-section is an equilateral triangle.



How many cases all the vertices of the triangle are also vertices of the cube?

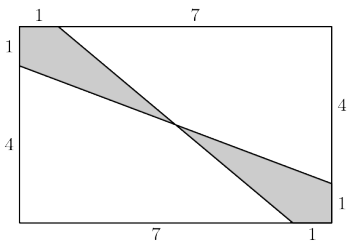
Discussion:

(1) What are the sides of the triangles in *relation* to the cube faces?

(2) Can you count the number of triple of them *without repeating*?

Solution: Each diagonal of a face can be an edge of exactly two different equilateral triangles, whose vertices are also vertices of the cube. So the number of triangles would be 6 (faces) $\times$ 2 (diagonals) $\times$ 2 (triangles). Because each triangle has three edges, so it is counted three times. Therefore the number of triangles is $\dfrac{6 \times 2 \times 2}{3} = 8$.

Problem 4.26 (B)



What is the area of the shaded figure?

Discussion: divide-and-conquer, use the center of symmetry.

Solution: The midpoint is in equidistance of $\dfrac{7+1}{2} = 4$ from both right and left sides, $\dfrac{4+1}{2}$ from top and bottom. The shaded area has two triangles with based 1 and height 4, another two triangles each has a base 1 and height of 2.5. The area of the shaded figure is 4+2.5=6.5

Problem 4.27 (C)

Compute the value of $1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\dots}}}}$

Discussion: Use Problem 1.17

Solution: Let $x=1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\dots}}}} \Rightarrow x=1+\cfrac{1}{2+\frac{1}{x}} \Rightarrow x=1+\cfrac{x}{2x+1}$

$$\Rightarrow x=\frac{3x+1}{2x+1} \Rightarrow 2x^2+x-3x-1=0 \Rightarrow 2x^2-2x-1=0 \Rightarrow x_{1,2}=\frac{2\pm\sqrt{4+8}}{4}=\frac{1\pm\sqrt{3}}{2}$$

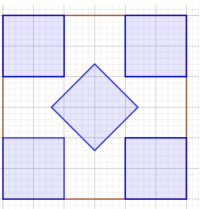
Since $x>0 \Rightarrow x=\dfrac{1+\sqrt{3}}{2}$.

Problem 4.28 (C)

Find a way to pack five unit squares inside a square of side length 3 so that they do not overlap.

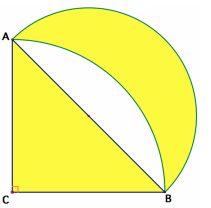
Discussion: Try to put the four unit squares somewhere so that you can have lots of space for the fifth. Now try to fit the fifth one without breaking it.

Solution: Here is a simple solution.



Problem 4.29 (C)

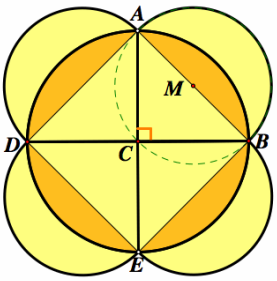
Given an isosceles right triangle ABC. Construct a lune with vertices at A and C with the center of the inner arc being at C and the center of the outer arc being at the midpoint of segment AB.



Prove that the area of the constructed lune is the same as the area of the triangle ABC.

Discussion: Construct the whole square whose quarter is the right triangle.

Solution: Let $a = BC$. From the below diagram, the dark brown area is a quarter of the difference between the circle and the square: $\frac{1}{4}(\pi a^2 - 2a^2) = \frac{(\pi - 2)a^2}{4}$.



The diameter of the small square is $a\sqrt{2}$ so its area is $\pi(\frac{a\sqrt{2}}{2})^2 = \frac{\pi a^2}{2}$. Therefore the lune area is the difference between half of the small circle and the dark brown area:

$$\frac{1}{2}\frac{\pi a^2}{2} - \frac{(\pi - 2)a^2}{4} = \frac{a^2}{2}$$

This is exactly the area of triangle ABC.

Challenges

Problem 4.31 (A)

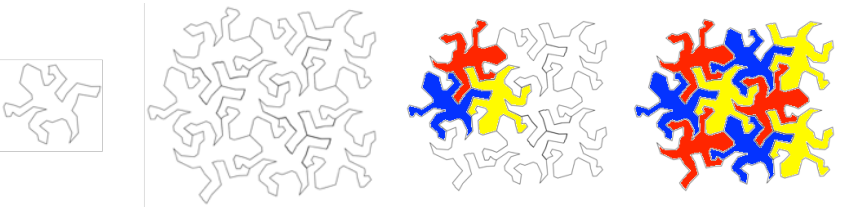
Find the smallest whole number that is not a factor of $1 \times 2 \times 3 \times \ldots \times 22 \times 23$.

Discussion: What would be the next number after 23 which is NOT a factor?

Solution: Obviously it is the next prime number after 23, which is 29. Note that it is very easy to verify that $24 = 3 \times 8$, $25 = \dfrac{5 \times 10}{2}$, $26 = 2 \times 13$, $27 = 3 \times 9$, $28 = 2 \times 14$ are factors.

Problem 4.32 (A)

The figure on right below contains a number of chameleons with the pattern on the left.



How many colours needed to colour all of them so that any two chameleons sharing some edges are coloured differently?

Discussion:

(1) Prove that you cannot use only two colours.

(2) Show that how you can use 3 colours.

Solution:

(1) first, look at the most left chameleon in the pattern. It has two neighbouring chameleons that also share borders. So it means we need at least 3 colours.

(2) Colouring that three chameleons with blue, red and yellow. That's the figure in the middle.

(3) Gradually colour these chameleons, we got the figure on the far right.

Problem 4.33 (A)

The numbers are arranged in three rows A, B, and C as following

A:	1	6	7	12	13	18	19	…
B:	2	5	8	11	14	17	20	...
C:	3	4	9	10	15	16	21	...

Which row will contain the number 1000?

Discussion: How the numbers alternate between A, B, and C rows?

Solution: Note that the numbers are sorted into three rows and 1000 leaves remainder 1 when divided by 3. Let colour the numbers that leaves remainder 1 when divided by 3.

A:	**1**	6	**7**	12	**13**	18	**19**	…
B:	2	5	8	11	14	17	20	...
C:	3	**4**	9	**10**	15	**16**	21	…

It shows that any number:

(1) that leaves remainder 1 when divided by 3 and is an even number is placed on C row.

(2) that leaves remainder 1 when divided by 3 and is an odd number is placed on A row.

From here is it easy to conclude that 1000 is on C row.

Problem 4.34 (B)

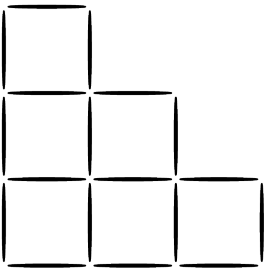
Find the sum of digits of the following number: $234 \times \underbrace{55...5}_{66}$

Discussion: $234 = 26 \times 9$, so $234 \times 55 = 26 \times 5 \times 99$

Solution: $234 \times \underbrace{55...5}_{66} = 26 \times 9 \times 5 \times \underbrace{11...1}_{66} = 26 \times 5 \times \underbrace{99...9}_{66} = 130 \times 1\underbrace{00...0}_{66} - 130$ $= 13\underbrace{000...0}_{67} - 130 = 12\underbrace{9...9}_{64}870$. So the sum of digits is $1 + 2 + 9 \times 64 + 8 + 7 = 594$.

Problem 4.35 (B)

Sara makes a staircase out of toothpicks as shown



This is a 3-step staircase and uses 18 toothpicks. How many steps would be in a staircase that used 180 toothpicks?

Discussion: count the number of toothpicks to form 1-step staircase, then 2-step staircase, and generalize!

Solution:

(1) to built the 4th level, Sara needs 5 verticals and $1 + 4 = 5$ horizontal toothpicks, in total 10.

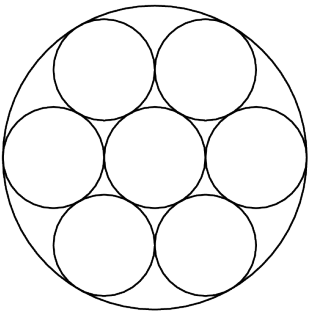
(2) For level n, she would need

$$2(1+1) + 2 \times (2+1) + 2 \times (3+1) + \ldots + 2 \times (n+1) = 2(2 + .. + (n+1)) = n^2 + 3n$$

Since $180 = 12(12 + 3)$ so the staircase contains 12 steps.

Problem 4.36 (B)

Seven cookies of radius 1 inch are cut from a circle of cookie dough, as shown. Neighbouring cookies are tangent, and all except the center cookie are tangent to the edge of the dough. The leftover scrap is reshaped to form a circle cookie of the same thickness.



What is the radius of the scrap cookie?

Discussion:

(1) What is the radius of the small cookie?

(2) The difference in areas of the big cookie and the seven small ones is?

Solution: since 6 times the radius of the small cookie is the diameter of the big circle, so the radius of the big circles is 3. The leftover shall be $\pi.3^2 - 7.\pi.1^2 = 2\pi$. Its radius is $\sqrt{2}$.

Problem 4.37 (C)

For what value of n is:

$$i + 2i^2 + 3i^3 + 4i^4 + \ldots . n\, i^n = 48 + 49i$$

where $i = \sqrt{-1}$

Discussion:

(1) Calculate value of i^{2k}, i^{2k+1}

(2) Generalize value of the sum on the left side

Solution:

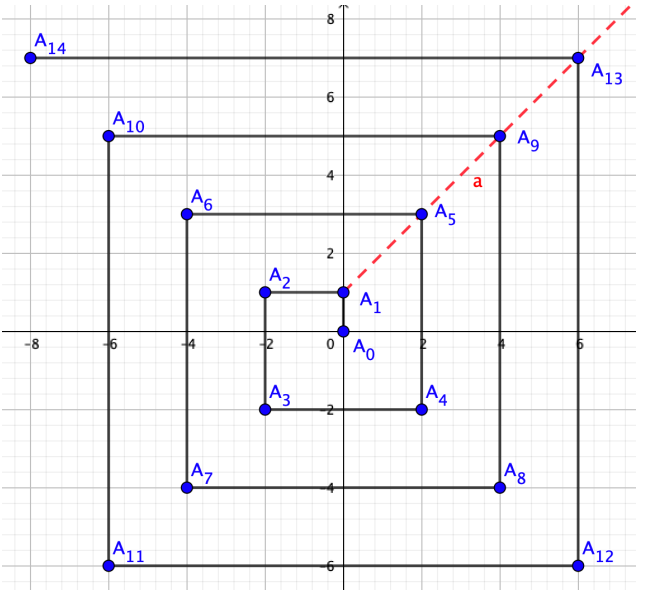
Let $A_0 = 0 + i.0$ or (0,0), the origin on the complex plane, $n \geq 1$:

$$A_n = A_{n-1} + n\,.\,i^n$$

Easy to see that A_n is the sum given by the problem. Let observe few value of A_n. Each A_n is a complex number and therefore can be represented on the complex plane.

Note: The complex plane is a xy-axis with the y-axis is to represent the complex number. For example $2 + 3i$ *is equivalent with point* (2,3) *where the real (x) coordinate 2 represents the real part of the number* $2 + 3i$ *which is 2, and the imaginary (y) coordinate 3 represents the imaginary part of the number* $2 + 3i$ *which is 3.*

- A_1(0,1) is north (increasing on the y-axis) of A_0 by a distance of 1: $A_1 = i$
- $A_2(-2,1)$ is west (decreasing on the x-axis) of A_1 by a distance of 2: $A_2 = -2 + i$
- $A_3(-2, -2i)$ is south (decreasing on the y-axis) of A_2 by a distance of 3: $A_3 = -2 - 2i$
- $A_4(2, -2i)$ is east (decreasing on the y-axis) of A_3 by a distance of 4: $A_4 = 2 - 2i$



Visual Insight: The sum basically is A_n is simply a formula to identify the position of a point after an operation moving north-west-south-east from the origin, each step increased by 1. Knowing this, the number $48 + 49i$ shall be on the red ray from $A_1(0,1)$ as shown. Note that:

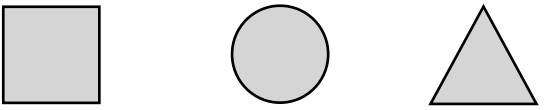
(1) each subsequent point in the sequence of $A_1, A_5, A_9, \ldots$

(2) has both of its coordinates increased by 2 comparing to the previous one.

So $A_{97} = A_{1+24\times4} = (0 + 24 \times 2, 1 + 24 \times 2) = (48,49)$. Hence $n = 97$.

Problem 4.38 (C)

We have three bottles: one has a circular opening, another a square opening, and the third has an opening in a the share of an isosceles triangle. The figure below shows the shape of each opening, seen from above. The side length of the square, the diameter of the circle, the height and base of the triangle are all equal.



(a) Design a bottle stopper that fits all three bottles

(b) Design a bottle stopper that fits all three bottles, and can be pushed inside each of the bottle through the bottle opening.

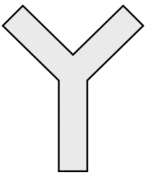
Discussion:

(1) Think about using a bottle stoppers with *multiple* ends, each can fit a *different* bottle.

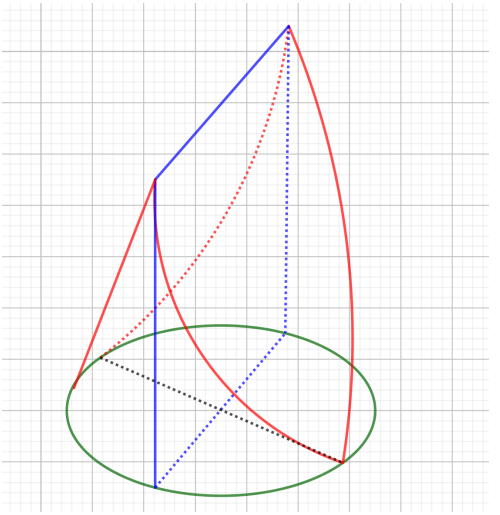
(2) Think about a similar one, but now each represents *a cut by a plane*.

Solution:

(a) It's an Y-share three bottle stopper, each with exactly the opening shape.



(b) It's how they are merged!



Problem 4.39 (C)

S_n is defined as the sum of the digits in the whole number. For example: $S_3 = 3$ and $S_{29} = 2 + 9 = 11$.

Find the value of:

$$S_1 + S_2 + S_3 + \ldots + S_{2010} + S_{2011}$$

Discussion:

(1) Try to calculate $S_1 + S_2 + S_3$

(2) What is $S_1 + S_2 + S_3 + \ldots + S_{10} + S_{11}$

(3) Generalize!

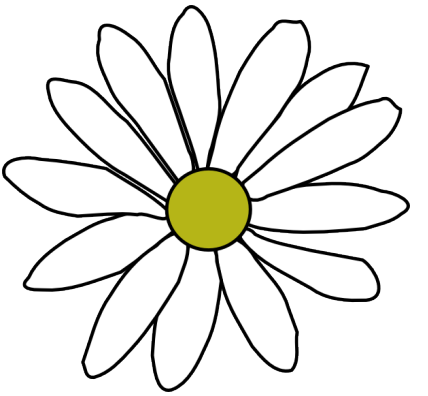
Solution:

From 0 to 9, any digit appear once. So from 0 to 1999 each digit appears $\frac{2000}{10} = 200$ times as units digit, tens or hundreds digits.

Their sum is $(1+2+\ldots+9).(600) = 45.600 = 27000$. In addition from 1000 to 1999, the digit 1 repeats 1000 times. From 2000 to 2011 the sum of sum of digits is $2+3+\ldots+11+3+4 = 72$. So the total is 28072.

Puzzle 4.10:

Leila and Quinn play a game with a *thirteen-petal* pink daisy. Each of them take minimum one, maximum two petals off the daisy. Whoever takes the last one win.



If Quinn goes first, can Leila win? What is her strategy?

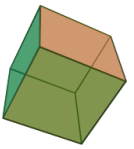
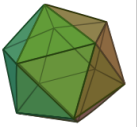
Solution:

1. Do not let the word "her" makes you think that it is Leila.

2. Quinn takes 1 petals. Whatever Leila takes, Quinn always takes an amount of petals equal to 3 subtracts the amount of petals Leila took. Quinn always wins because $13 = 1 + 3 \times 4$.

Puzzle 4.20:

In three-dimensional space, a Platonic solid is a regular, convex polyhedron. It is constructed by congruent (identical in shape and size) regular (all angles equal and all sides equal) polygonal faces with the same number of faces meeting at each vertex. There are five of them, four shown as below:

The solid				**?**	
Name	Tetrahedron	Hexahedron (cube)	Octahedron	Dodecahedron	Icosahedron
Number of faces	Four	Six	Eight	Twelves	Twenty
What is a face	Equilateral triangle	Square	Equilateral triangle	?	Equilateral triangle

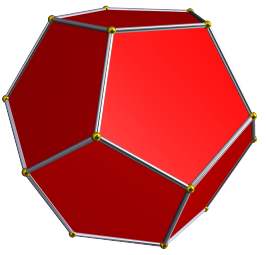
You task is to find what regular polygon is a face of the 4th solid. You are armed with two facts:

(1) It has twelves faces - *twelve a.k.a dodeca in Latin.*

(2) The Euler's formula: If V, E, and F are respectively the numbers of vertices (corners), edges and faces in a polyhedron, then $V - E + F = 2$.

Note: use your imagination and reasoning, don't let Google and Wikipedia work on behalf of you.

Solution: Use the formula: $V - E + F = 2$. The dodecahedron has 12 faces so $F = 12$. $V + 10 = E$. If a face has n edges or in another word n vertices, then $2E = n \times F \Rightarrow E = 6n \Rightarrow V = 6n - 10$. Because V is divisible by n, hence $n = 5$. So A face of a regular dodecahedron is a regular pentagon. It is shown as below.

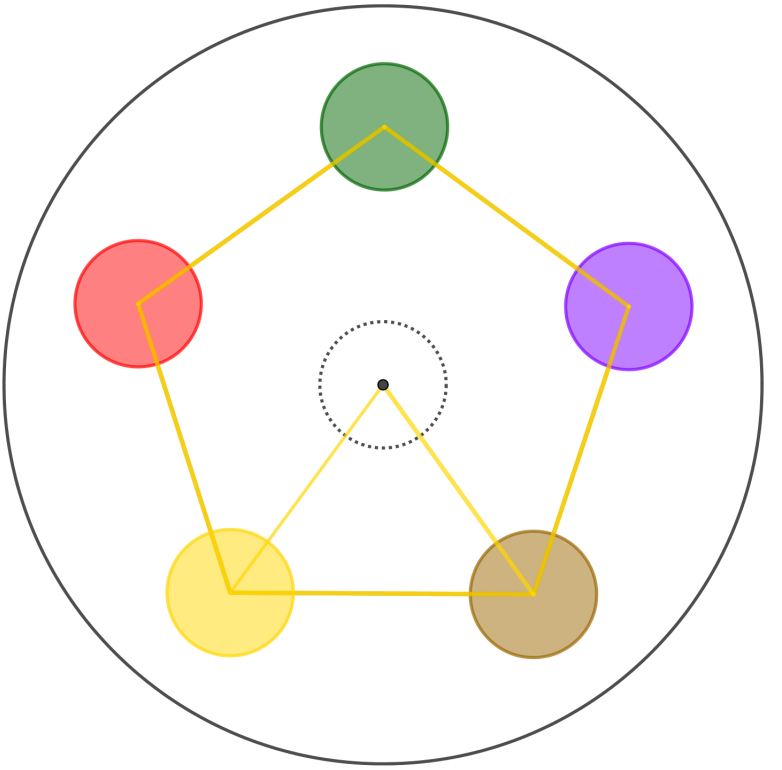


Puzzle 4.30: Interchange

Five disks with different colours are placed on the five corners of a pentagon. They can be moved to any corners or the center if that place is:

(1) connected to the place where the disk is

(2) is empty (has no disk on it)



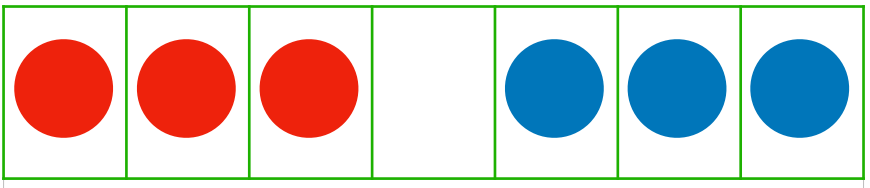
Can you exchange the positions of the red and yellow disks? If yes, what is the minimum number of moves you need?

Solution: Yes, you can. It takes 13 moves. First, move the yellow disk into the center, the red one counter clockwise.

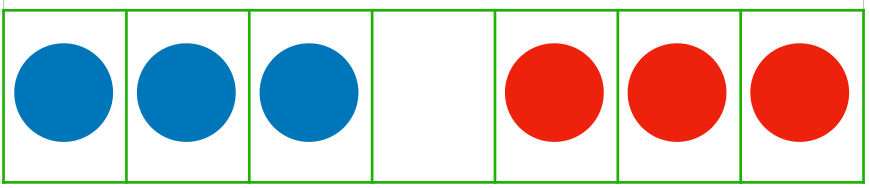
Puzzle 4.40:

There are three red and three blue disks in a line in the *start diagram*. You are requested to move them so that you can reverse the pattern as in the *end diagram*.

Start



End



Five rules must be followed:

(1) Only one disk can be move at a time

(2) A disk may move into an adjacent empty space.

(3) A disk may jump over a disk of the opposite colour into the space immediately beyond it.

(4) A disk may not jump over the disk of the same colour.

(5) No backward moves are permitted.

Note that no jumping off the ends is permitted, and no move might disturb another disk (no pushing).

Can you solve the puzzle in no more than 15 moves?

Solution: it's easy, just move one blue and then one red disk.

Chapter 5

Divisibility — Solutions

Problems

Problem 5.21 (A)

How many possible triplets of different prime numbers such that their sum is 25?

Discussion: Since the sum of three prime numbers is 25 that is odd number, so all three prime numbers must be odd. Hence, the smallest number must be odd.

Solution: Since the smallest number is odd, it should be at least 3. Let estimate how big can it be: $25 \div 3 < 9$, so the smallest number cannot be larger than 3. *It should be* 3, 5, *or 7.*

Now there are three cases:

(1) The smallest odd prime number is 3: The sum of two other different primes and larger than 3 is 22. The middle prime number should be at least 5 and is less than $22 \div 2 = 11$. So it can be 5 or 7. There is only one answer (3, 5, 17) for this case.

(2) The smallest odd prime number is 5: The sum of two different primes larger than 5 is 20. There are only one possible pair (7, 13). So (5, 7, 13) is the second answer.

(3) The smallest odd prime number is 7: The sum of two different primes and larger than 7 is 15. There is no solution for this case.

The answer is two triplets (3, 5, 17) and (5, 7, 13).

Note: this is a typical case when we use estimation to find a range of value and then use casework to find the answers for each case.

Problem 5.22 (A)

Find the last digit of $\underbrace{2 \times 2 \times \ldots \times 2}_{50}$

Discussion: Let write down the last digits of a few similar expressions:

$\underbrace{2}_{1} = 2,\ \underbrace{2 \times 2}_{2} = 4,\ \underbrace{2 \times 2 \times 2}_{3} = 8,\ \underbrace{2 \times \ldots \times 2}_{4} = 16,\ \underbrace{2 \times \ldots \times 2}_{5} = 32,$
$\underbrace{2 \times \ldots \times 2}_{6} = 64, \ldots$

By looking at only the last digits, it is *a periodic sequence with 4-period:* $\underbrace{2, 4, 8, 6}_{4}\ \ 2, 4, 8, 6, \ \ldots$

Solution: a periodic sequence with 4-period has a frequency of 4. The shortest 4-length sequence 2, 4, 8, 6 repeats itself.

$$\underbrace{2 \times 2 \times \ldots \times 2}_{50} = \underbrace{2 \times 2 \times \ldots \times 2}_{4\times 12+2}$$

The last digit of the expression is same as the last digit of 2×2. It is 4.

Problem 5.23 (A)

How many positive integers divide the number $n = 1 \times 2 \times 3 \ldots \times 9 \times 10$

Discussion: Let replace the each term of the by the product of its prime factors. For example $6 = 2 \times 3$, $10 = 2 \times 5$ and so on.

Solution: $1 \times 2 \times 3 \ldots \times 9 \times 10 = 2.3.(2.2).5.(2.3).7.(2.2.2)\,.\,(3.3)\,.\,(2.5)$.

Let rearrange the product terms by grouping same prime factors: $\underbrace{2..2}_{8} \times \underbrace{3..3}_{4}\ \underbrace{5..5}_{2} \times 7$. If an integer is a divisor of n, then it is a product of some 2's, some 3's, some 5's or a 7:

$$4 = 2 \times 2,\ 6 = 2 \times 3,\ 30 = 2 \times 3 \times 5; 28 = 2 \times 2 \times 7.$$

So the number of divisors of n is the way we can create the products of some 2's, some 3's, some 5's or a 7. n contains $\underbrace{2..2}_{8}$ so there are 9 ways to select 2's: none of 2's, one 2, two 2's, …, eight 2's into a divisor. Similarly there are 5 ways to select 3's, 3 ways to select 5's and 2 ways to select 7. So the number of possible ways is $9 \times 5 \times 3 \times 2 = 270$. It is the number of divisors.

Note: if the prime factorization of a number is for example:

$$2^8 \times 3^4 \times 5^2 \times 7^1$$

then the number of divisors is

$$(8+1)(4+1)(2+1)(1+1) = 9 \times 5 \times 3 \times 2 = 270.$$

Problem 5.24 (B)

m, n positive integers and $56m = 65n$. Prove that $m + n$ is composite number (not a prime)

Discussion:

$56m = 65n \Rightarrow 56m + 56n = 65n + 56n = 121n \Rightarrow 56(m+n) = 121n$.

Assume if $m + n$ is prime number, what should it be? Will there be anything contradictory?

Solution: Assume if $m + n$ is not a composite number, so it must be a prime number. Since $56(m+n) = 121n = 11^2 n$, hence $56(m+n)$ is

divisible by 11^2. But 56 and 11 are relative primes, they do not share any common divisor, so $m + n$ should be divisible by 11. Because $m + n$ is a prime number, so it can divides no more than one term of 11's, which makes the equation $56(m + n) = 121n$ to be invalid. Therefore the assumption - that $m + n$ is a prime number - is not valid. Therefore $m + n$ is composite number.

Problem 5.25 (B)

(a) Find all primes numbers can be written as both a sum and a difference of two other primes.

(b) If p, q are different prime numbers. How many different divisors p^2q^3 has? How about p^mq^n?

(a) **Discussion**: Assume that there are five primes $p, q \geq r, s \geq t$: $p = q + r = s - t$.

(1) Easy to see that p cannot be 2, so it has to be odd, therefore $r = t = 2$, hence $p - 2, p, p + 2$ are all primes.

(2) But among three consecutive odd numbers there should be one that is divisible by 3.

(b) **Discussion**: A divisor of p^2q^3 is a product of an exponent of p and an exponent of q

(1) There are 3 ways to choose exponent of p: $1 = p^0, p^1, p^2$

(2) There are 4 ways to choose exponent of q: $1 = q^0, q^1, q^2, q^3$

(a) **Solution:** Assume that there is a prime number p, such that it can be written as both a sum and a difference of two other primes. So there are five primes $p, q > r, s > t$:

$$p = q + r = s - t$$

Since p is odd and $p = q + r$, $q > r$, so one of q, r is even, therefore $r = 2$. Similarly $t = 2$. Now q, p, t are three consecutive odd numbers, it means one of them is divisible by 3 so $q = 3$. The answer is there is one such prime number $5 = 3 + 2 = 7 - 2$.

(b) **Solution:** From the solution of problem 5.23, it is easy to see that a there are 3 ways to choose an exponent of p, 4 ways to choose an exponent of q such that their product is a divisor of p^2q^3. So the number of possible divisors is $3 \times 4 = 12$.

Similarly the number of divisors of p^mq^n is $(m + 1)(n + 1)$

Problem 5.26 (B)

Prove that a product of five consecutive integers is divisible by 120

Discussion: $120 = 3 \times 5 \times 8$

(1) Among 5 consecutive numbers there is one which is divisible by 3? By 5?

(2) Among 5 consecutive numbers there are at least two even numbers?

Solution: Some examples of five consecutive integers

$$1, 2, 3, 4, 5, \text{ or } 8, 9, 10, 11, 12, \text{ or } 500, 501, 502, 503, 504$$

(1) *Among 5 consecutive numbers, there is one number that always divisible by 3*, so the product of them must be divisible by 3.

(2) *Similarly, there is one number that always divisible by 5*, the product of them is divisible by 5.

(3) *Among 5 consecutive numbers there are at least two even numbers, one number is divisible by 2 and the other number is divisible by 4*. So their products is divisible by 8, and the product of five consecutive numbers is divisible by 8.

Because the product of any five consecutive numbers is divisible by 3, 5, and 8; 3, 5, and 8 are pairwise relative prime numbers; so the product of any five consecutive numbers is divisible by $3 \times 5 \times 8 = 120$.

Problem 5.27 (C)

Find the last digit of 7^{7^7}

Discussion: Very similar to 5.22

Solution: First let find a pattern for the last digit of 7^n. To do that we establish a sequence based on the remainder of 7^n while divided by 10: $\underbrace{7, 9, 3, 1}_{length\ 4}, \underbrace{7, 9, 3, 1}_{length\ 4}, \ldots$

This is a *periodic sequence* with a *period* 4. So the last digit depends on where the exponent 7^7 falls in the *period.* In another word, let find the remainder of 7^7 while divided by 4. This is another sequence - similar to the first one - based on the remainder of 7^n while divided by 4 $\underbrace{3, 1}_{length\ 2}, \underbrace{3, 1}_{length\ 2}, \ldots$

So remainder of 7^7 while divided by 4 is 3, hence the last digit of 7^{7^7} is 3.

Problem 5.28 (C)

Both p, $8p^2 + 1$ are prime. Find p

Discussion: Evaluate the remainder of p when divided by 3.

Solution:

(1) if p has remainder 1 while divided by 3, then $8p^2 + 1$ is divisible by 3

(2) if p has remainder 2 while divided by 3, then $8p^2 + 1$ is also divisible by 3

But $8p^2 + 1 > 3$, so p is divisible by 3 or $p = 3$.

Problem 5.29 (C)

Prove that the sum of the squares of five consecutive integers cannot be a perfect square.

Discussion: Evaluate the remainder of a perfect square when divided by 4.

Solution: Let the smallest of five consecutive integers is n.

$$n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 + (n+4)^2 = 5n^2 + 20n + 30 \equiv n^2 + 2 \pmod 4$$

The sum has the same remainder as $n^2 + 2$. Let take a look at the table below:

Remainder of n	0	1	2	3
Remainder of n^2	0	1	0	1
Remainder of $n^2 + 2$	2	3	2	3

Challenges

Problem 5.31 (A)

The sum of two prime numbers is 39. What is their product?

Discussion: What would be the largest prime smaller than 39? Find the other, check if it is prime. Work backwards.

Solution: the sum of two primes number is 39, which is an odd number, so one of the prime numbers must be even, in another word that is 2. The other number is 37. Their product is 74.

Problem 5.32 (A)

a, b, c are prime numbers: $a \times b = 55$, $b \times c = 35$. What is $a \times b \times c$?

Discussion: Factorize 55, 35. Compare.

Solution: $a \times b = 55 \Rightarrow a \times b = 5 \times 11$, $b \times c = 35 \Rightarrow b \times c = 5 \times 7$. Therefore $b = 5$. $a \times b \times c = 11 \times 5 \times 7 = 385$

Problem 5.33 (A)

How many number less than 10,000 are product of four consecutive positive integers.

Discussion: Estimate. What largest number multiplies with itself four times and still smaller than 10,000.

Solution: The smallest of such number obviously is $1 \times 2 \times 3 \times 4 = 24$. The second one is $2 \times 3 \times 4 \times 5 = 120$, and then $3 \times 4 \times 5 \times 6 = 360$, … What would be the largest product of four consecutive numbers but still smaller than 10,000? Here the idea is that *the product of four consecutive numbers* sounds similar to *the product of four same numbers!* Since $10000 = 10 \times 10 \times 10 \times 10$, so the smallest number in the largest product of four consecutive integers is smaller than 10. Easy to test that the largest product of four consecutive integers is $8 \times 9 \times 10 \times 11 = 7920$. Therefore there are 8 such numbers from $1 \times 2 \times 3 \times 4 = 24$, …, $8 \times 9 \times 10 \times 11 = 7920$.

Problem 5.34 (B)

(a) If $a + 1$ is divisible by 3, will $4 + 7a$ be divisible by 3?

(b) If both $2 + a$, $35 - b$ are divisible by 11, will $a + b$ is also divisible by 11?

Solution: The key idea is to use some simple algebra to prove the assumption.

(a) Since we know that $a+1$ is divisible by 3, somehow we need to see how the number $4+7a$ can be formed by using $a+1$? Because $4+7a = 4+4a+3a = 4(1+a)+3a$, and $a+1$ is divisible by 3, $3a$ is also divisible by 3. So $4+7a$ be divisible by 3?

(b) Both $2+a, 35-b$ are divisible by 11, so their difference $(2+a)-(35-b) = a+b-33$ is divisible by 11. 33 is divisible by 11, so $a+b = (a+b-33)+33$ is also divisible by 11.

Note: it is important to note that if any two integers m, n such that m, n are both divisible by 3: both $m+n$, $m-n$ are divisible by 3; $m \times n$ is divisible by 9; the above also true for any p prime (not only 3)

Problem 5.35 (B)

Find the last digit of $1^2+2^2+\ldots+99^2$

Discussion: Investigate the last digits of the first 10 perfect squares. Establish a pattern.

Solution:

Remainder of n	0	1	2	3	4	5	6	7	8	9
Remainder of n^2	0	1	4	9	6	5	6	9	4	1

The remainder of every sum of ten consecutive perfect squares when divided by 10 is 5. Since $1^2+2^2+\ldots+99^2$ is 10 sums of ten consecutive perfect squares, so it must be divisible by 10. So the last digit is 0.

Problem 5.36 (B)

If p, $2p+1$, $4p+1$ are prime numbers. Find p

Solution: Table of remainders when p, $2p+1$, $4p+1$ divided by 3.

Remainder of p	0	1	2
Remainder of $2p+1$	1	0	2
Remainder of $4p+1$	1	2	0

So if p is not divisible by 3 then one of $2p+1$, $4p+1$ is divisible by 3. Hence $p=3$

Problem 5.37 (C)

Prove that if p, q are both prime numbers, greater than 3, then p^2-q^2 is divisible by 24

Discussion: Very similar to 5.26

Solution: if $p, q > 3$ are prime numbers

(1) if p, q has the same remainder (1,1) or (2,2) when divided by 3: $p - q$ is divisible by 3;

(2) if p, q has (1,2) or (2,1) remainders: $p + q$ is divisible by 3.

So in both (1) and (2) cases $p^2 - q^2$ is divisible by 3 (*)

(3) if p, q has the same remainder (1,1) or (3,3) when divided by 4: $p - q$ is divisible by 4, and $p + q$ is divisible by 2;

(4) if p, q has (1,3) or (3,1) as remainders: $p - q$ is divisible by 2, and $p + q$ is divisible by 4.

So in both (3) and (4) cases $p^2 - q^2$ is divisible by 8 (**) From (*) and (**), because 3 and 8 are relative primes, hence $p^2 - q^2$ is divisible by 24.

Problem 5.38 (C)

Seven natural numbers are given. The sum of any six of them is divisible by 5. Prove that each of these number is divisible by 5.

Discussion: Compare two different sums of six numbers. What we know? Compare another two. Try to establish a pattern of remainders.

Solution: $a_1, a_2, \ldots, a_7$ are natural number. $S = a_1 + a_2 + \ldots + a_7$. $S_i = S - a_i$, $i = 1,..,7$ Since $S_1 + S_2 + \ldots + S_7 = 6S$, so S is divisible by 5. For $i = 1,...,7$: $a_i = S - S_i$ is divisible by 5. So each of the given natural numbers is divisible by 5.

Note: 5 can be replaced by any prime number larger or equal 5. Can you prove it?

Problem 5.39 (C)

Prove that $a^3 + b^3 + 4$ is not a perfect cube for any natural number a, b

Discussion: Investigate remainder of $a^3 + b^3 + 4$ when divided by 9.

Solution: The table below show what are the remainders of n^3 while divided by 9.

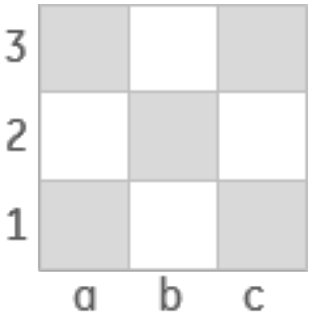
Remainder of n	0	1	2	3	4	5	6	7	8
Remainder of n^3	0	1	8	0	1	8	0	1	8

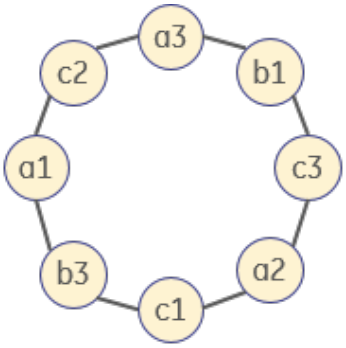
So, for any two integers a, b: $a^3 + b^3 + 4$ can be a perfect cube if it has a remainder of 0, 1, or 8 while divided by 9, or $a^3 + b^3$ has a remainder of 5, 6, or 4 while divided by 9. From the table above, this is impossible.

Puzzle 5.10: Is it possible for the white knights to exchange their position with the black knights?

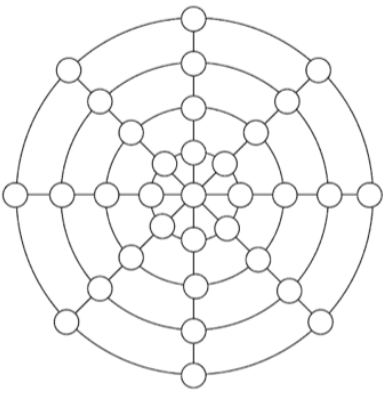
If your answer is no, prove it. If it is yes, show how you can do it.

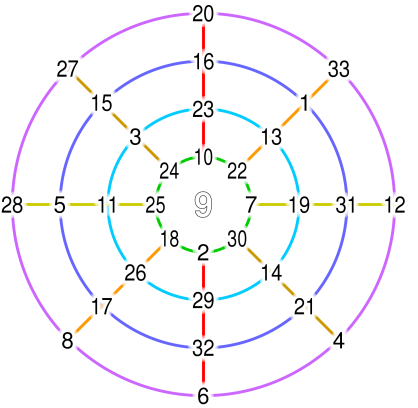
Solution: Let number the board as the left figure below. Easy to see that the knights has to move long the cycle graph on the right.





Puzzle 5.20: Fill the number 1, 2, … , 33 into the small circles such that every circle including its centre and every diameters has the same sum.

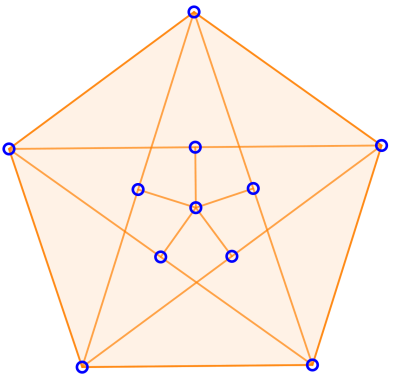


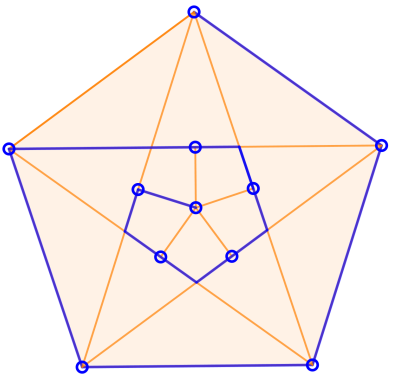


Solution: figure on the right.

Puzzle 5.30:

Eleven blue villages are connected by the brown road. Can you walk from a village, visit each other village once?



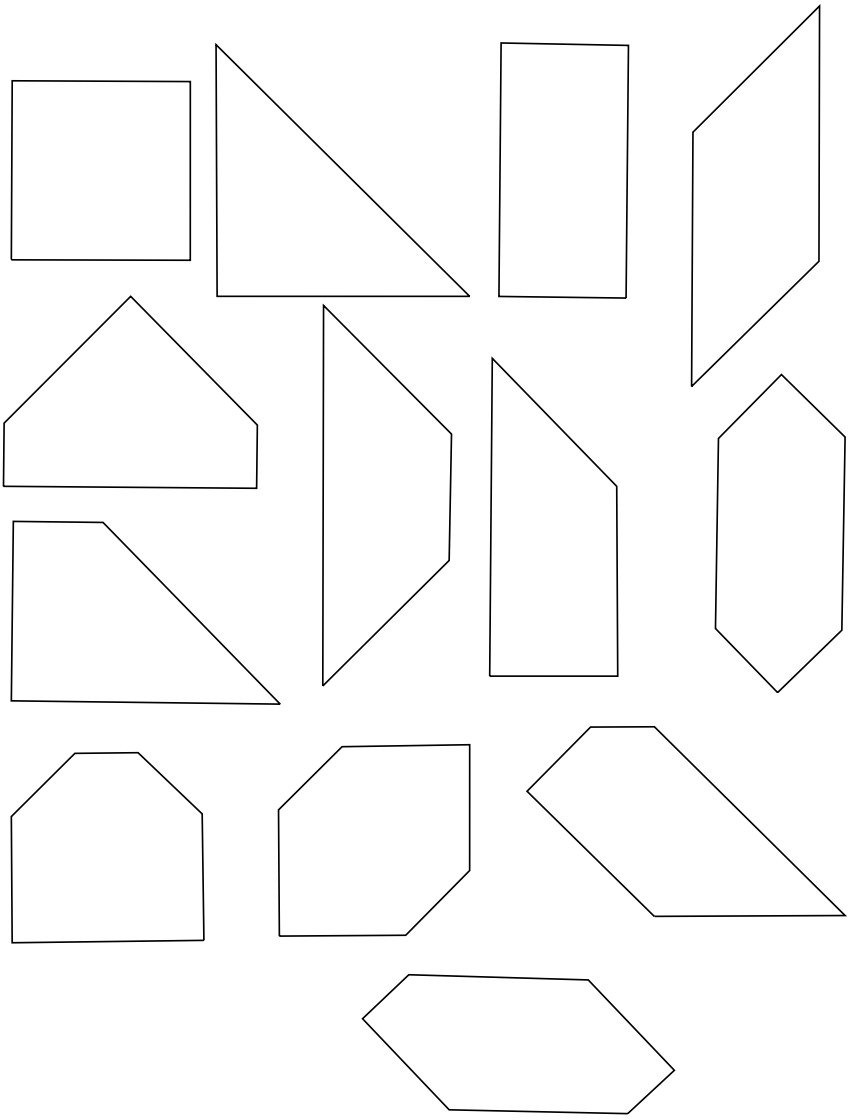


You don't have to return to the village where you start.

Solution: figure on the right.

Puzzle 5.40:

Use the Tangram to construct these figures.



No solution shown for this, it's simple, you just need to practice tangrams. Look for resources on the Internet if you get stuck.

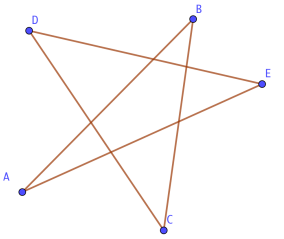
Chapter 6

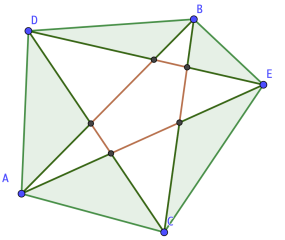
Basic Geometry — Solutions

Problems

Problem 6.21 (A)

What is the sum of all internal angles of a five pointed stars?





Discussion: There are two ways to find the sum:

(1) Connect vertices to create a convex pentagon, complementary counting.

(2) Divide-and-conquer to count the sum of all angles in 5 triangles and then the pentagon.

Solution 1: By creating a convex bound, the sum of all internal angles is

(a) 5 times the full angles 360°: $5 \times 360°$, by the five intersections, plus

(b) the sum of all internal angles of the pentagon bound: $3 \times 360°$, subtracts

(c) the sum of all internal angles of five green-shaded: $5 \times 180°$ triangles

So the answer is $5 \times 360° + 3 \times 360° - 5 \times 180° = 8 \times 180° = 1440°$

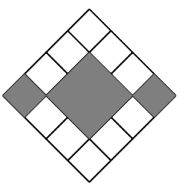
Solution 2: By dividing into triangles and a pentagon, the sum of all internal angles is

(a) the sum of all internal angles of the pentagon bound: $3 \times 360°$, plus

(b) the sum of all internal angles of five triangle: $5 \times 180°$ triangles

So the answer is $3 \times 180° + 5 \times 180° = 8 \times 180° = 1440°$

Problem 6.22 (A)



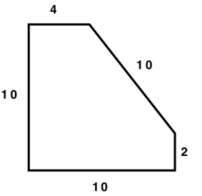
What is the ratio of the grey area to the white area?

Discussion: Count the number of small squares made up the grey area.

Solution: there are 6 grey squares and 10 white ones: $\dfrac{6}{10} = \dfrac{3}{5}$

Problem 6.23 (A)

The pentagon below has three right angles.



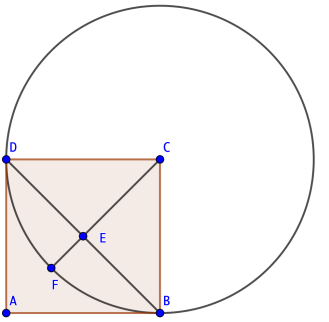
Find its area.

Discussion: The figure looks like a rectangle missing a triangle.

Solution: *Complementary counting helps!* The pentagon is chunked square. The missing part is a right triangle with two legs 6, and 8. So the area of the pentagon is $10 \times 10 - \dfrac{1}{2}.6 \times 8 = 100 - 24 = 76$.

Problem 6.24 (B)

Square ABCD has side length 1. Minor arc $\widehat{BD}$ of a circle with radius 1 centred at C is drawn along with diagonal BD. Let CE be the altitude of $\triangle BCD$ drawn from C, and is extended to intersects BD at F.



What is the value of 2EF?

Discussion: Since $EF = CF - CE$, let compute CE, CF.

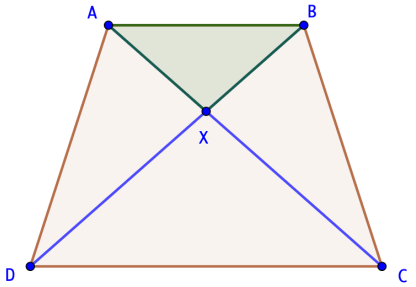
Solution:

$$CB^2 = CE^2 + EB^2 = 2CE^2 \Rightarrow CB = CE\sqrt{2} \Rightarrow CE = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Hence $EF = CF - CE = 1 - \dfrac{\sqrt{2}}{2} = \dfrac{2-\sqrt{2}}{2}$.

Problem 6.25 (B)

Trapezoid ABCD has base lengths $AB = 30cm$, $CD = 54cm$ and congruent 56cm diagonals. The diagonals intersect each other at X.



What is the area of ABX?

Discussion:

(1) Prove that $\triangle ABX \sim \triangle CDX$. What would be the value of BX?

(2) Knowing BX, try to figure out the height from X of triangle ABX.

Solution: Since $\angle XAB = \angle XCD$, $\angle XBA = \angle XDC \Rightarrow \triangle ABX \sim \triangle CDX$.

Since $\frac{AX}{XC} = \frac{AB}{CD} = \frac{30}{54} = \frac{5}{9} \Rightarrow \frac{AX}{5} = \frac{XC}{9} = \frac{AX+XC}{5+9} = \frac{56}{14} = 4 \Rightarrow AX = 20,\ CX = 36$.

Similarly $BX = 20$, $XD = 36$.

So ABX is an isosceles triangle, so the altitude from X to AB meets AB at its midpoint.

Using Pythagorean theorem, the length of the altitude is :

$$\sqrt{AX^2 - (\frac{AB}{2})^2} = \sqrt{20^2 - 15^2} = \sqrt{35.5} = 5\sqrt{7}$$

So the area of triangle ABC is $\frac{AB.5\sqrt{7}}{2} = 15.5\sqrt{7} = 75\sqrt{7}$.

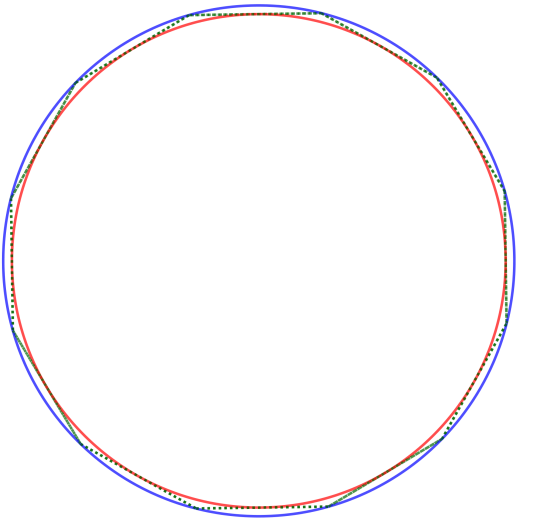
Note: if two triangles T_1, T_2 are similar; b_1, h_1 are base and height of T_1; and b_2, h_2 are base and height of T_2, r is the similarity ratio:

(1) the corresponding elements of both triangle are proportional: $r = \frac{b_1}{b_2} = \frac{h_1}{h_2}$

(2) the ratio of the areas is square of the similarity ratio: $\frac{S_{T_1}}{S_{T_2}} = \frac{\frac{b_1h_1}{2}}{\frac{b_2h_2}{2}} = \frac{b_1}{b_2} \cdot \frac{h_1}{h_2} = r^2$

Problem 6.26 (B)

Is the difference in area between the inscribed circle and circumscribed circle in a regular polygon of edge length 2cm depending on the number of its vertices? Below is an example of 12-gon.

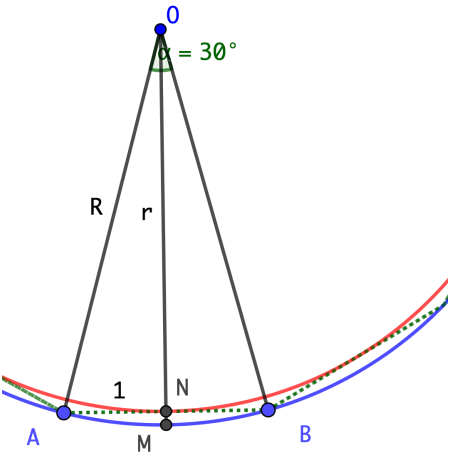


Discussion:

(1) Draw the regular polygon and two circles

(2) Find a right triangle with side lengths equals radiuses of the inscribed circle and circumscribed circle.

Solution:



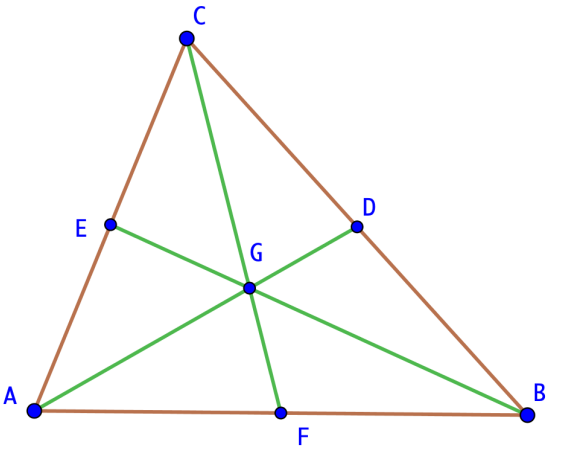
O is the center of both circles. AB is an edge of the n-gon. N is midpoint of AB. R, r are two radii of the circumscribed and inscribed circles.

According to Pythagorean theorem for triangle ANO: $R^2 - r^2 = 1$.

So the difference of the areas of the two circles is $\pi R^2 - \pi r^2 = \pi$. This does not depend on n.

Problem 6.27 (C)

For $\triangle ABC$, D is midpoint of BC, E is midpoint of AC, F is midpoint of AB.



Prove that:

$$\frac{3}{2}(AB + BC + CA) > AD + BE + CF > \frac{3}{4}(AB + BC + CA)$$

Solution:

$AG + CG > AC \Rightarrow \frac{2}{3}(AD + CF) > AC,$

$\frac{2}{3}(AD + BE) > AB, \frac{2}{3}(BE + CF) > BC$

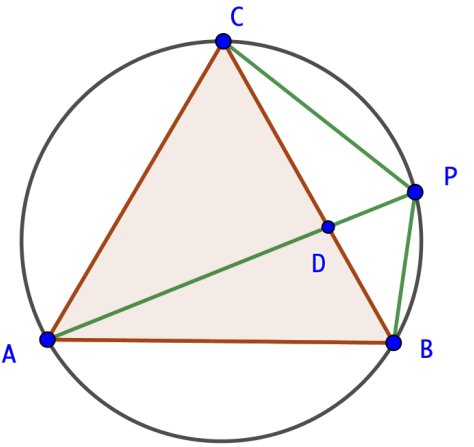
$\Rightarrow \frac{4}{3}(AD + BE + CF) > AB + BC + CA.$

$AG < AF + FG \Rightarrow \frac{2}{3}AD < \frac{1}{2}AB + \frac{1}{3}CF, \ \frac{2}{3}CF < \frac{1}{2}AC + \frac{1}{3}BE$

$\Rightarrow \frac{1}{3}(AD + BE + CF) < \frac{1}{2}(AB + BC + CA)$

Problem 6.28 (C)

A line is drawn from vertex A of equilateral $\triangle ABC$, meets BC at D and the circumcircle at P.



Prove that $\frac{1}{PD} = \frac{1}{PB} + \frac{1}{PC}$.

Discussion: Compare $\triangle ACD$, $\triangle BPD$

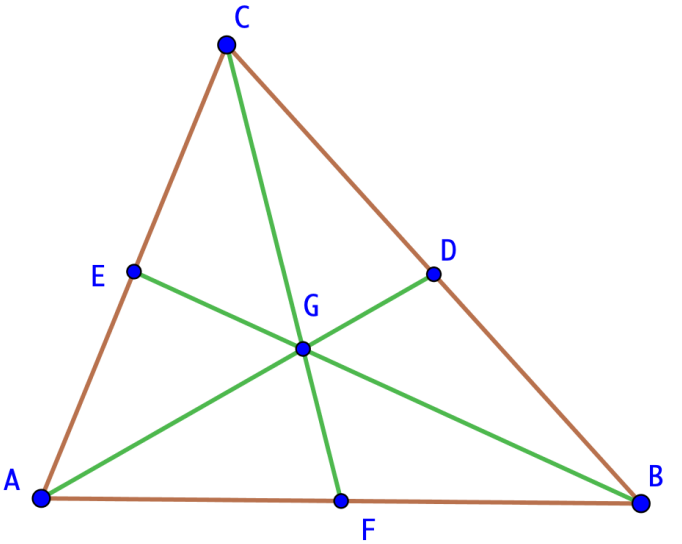
Solution: $\angle ADC = \angle BDP, \angle CAD = \angle PBD$, so $\triangle ACD \sim \triangle BPD$:

$\frac{DP}{BP} = \frac{CD}{AC}$, similarly $\frac{DP}{CP} = \frac{BD}{AB}$.

Therefore: $\frac{1}{PB} + \frac{1}{PC} = \frac{1}{PD}(\frac{CD}{AC} + \frac{BD}{AB}) = \frac{1}{PD}$.

Problem 6.29 (C)

In $\triangle ABC$, D is midpoint of BC, E is midpoint of AC, F is midpoint of AB, and G is intersection of AD, BE, and CF.



What is the ratio of the area of $\triangle EFG$ to the area of $\triangle ABC$?

Discussion:

(1) Compare $\triangle EFG$, $\triangle EFC$

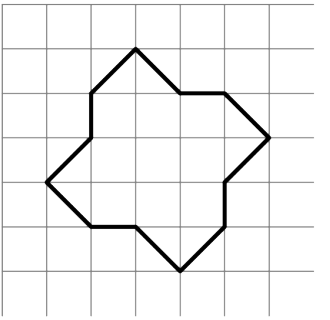
(2) Compare $\triangle EFC$, $\triangle AFC$

Solution: The area of $\triangle EFG$ is one-third of the area of $\triangle EFC$ because GF is one-third of CF. Similarly the area $\triangle EFC$ is half of the area of $\triangle AFC$ because CE is half of AC. Finally the area of $\triangle AFC$ is half of the area of $\triangle ABC$. So the area of $\triangle EFG$ is one-twelfth of the area of $\triangle ABC$.

Challenges

Problem 6.31 (A)

The twelve-sided figure shown has been drawn on $1\ cm \times 1\ cm$ graph paper.



What is the area of the figure in cm^2?

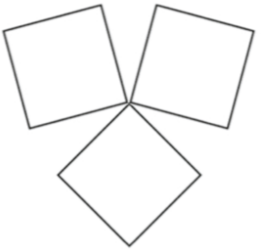
Discussion: Count the unit squares, add up the triangles to make unit squares.

Solution:

The central square has an area of 9 plus four 'bulges', each is an isosceles right triangle with area of 1. So the total area is 13.

Problem 6.32 (A)

The diagram below shows a 12-sided figure made up of three congruent squares. The figure has total perimeter 60.



Find its area.

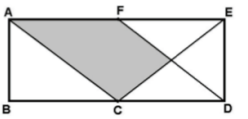
Discussion: What is the side of each square?

Solution: The total perimeter is 12 times the side length of a square.

The area is three times the area of a square. Therefore it is $3 \times 25 = 75$.

Problem 6.33 (A)

The diagram below shows rectangle ABDE where C is the midpoint of side BD, and F is the midpoint of side AE.



If AB=10 and BD=24, find the area of the shaded region.

Discussion: Connect F with C. What can you say about the area of $\triangle AFC$?

Solution: $\triangle AFC$ is half of ABCF, of a quarter of ABDE. The remaining part of the shaded area is a quarter of FEDE or one-eighth of ABDE. So the total area of the shaded figure is three-eight of ABDE: $\frac{3}{8}.10.24 = 90$.

Problem 6.34 (B)

The radius of the smallest circle is 2. From the inner most circle to the outer most, each time the radius of the next circle is larger than the previous one's by 2.

Approximately what percent of the design is black?

Discussion: Create a table listing the areas of the circles, one by one.

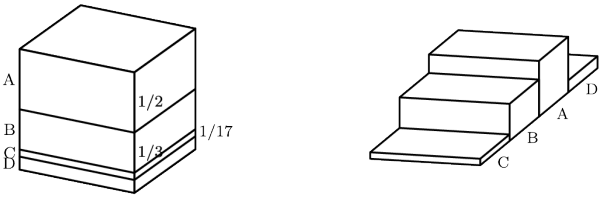
Solution:

The radius of the smallest circle is 2, so the sequence of the radii is 2, 4, 6, 8, 10, 12 . The black area is $(\pi.10^2 - \pi.8^2) + (\pi.6^2 - \pi.4^2) + \pi.2^2 = \pi(100 - 64 + 36 - 16 + 4) = 60\pi$.

So the percent of the design which is black is $\frac{60\pi}{12^2\pi} = \frac{5}{12}$.

Problem 6.35 (B)

A one-cubic-foot cube is cut into four pieces by three cuts parallel to the top face of the cube in the left diagram. The first cut is $\frac{1}{2}$ foot from the top face. The second cut is $\frac{1}{3}$ foot below the first cut, and the third cut is $\frac{1}{17}$ foot below the second cut. From the top to the bottom the pieces are labeled A, B, C, and D. The pieces are then glued together end to end as shown in the right diagram.



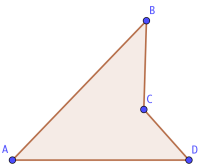
What is the total surface area of this solid in square feet?

Discussion: The top parts and the bottom parts sum to 8.

Solution: The tops of A, B, C, and D in the figure formed has sum 4 as do the bottoms. Thus, the total surface so far is 8. Now, one of the sides has area one, since it combines all of the heights of A, B, C, and D, which is 1. The other side also satisfies this. Thus the total area now is 10. From the front, the surface area is half, because if you looked at it straight from the front it would look exactly like A, with surface area half. From the back it is the same thing. Thus, the total is $10 + \frac{1}{2} + \frac{1}{2} = 11$.

Problem 6.36 (B)

A sail, as shown in the diagram, has the form of the quadrilateral ABCD. The angles A, B, D measure 45° each.



If $AC = 4\ m$, find the area of the sail.

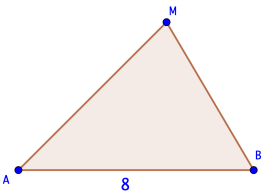
Discussion: Calculate $\angle C$, then create a line through C, perpendicular with AD.

Solution: Extend BC to meet AD at M. Let $x = AM$, $y = CM$. The area of the sail is the area of two triangle AMB and CMD, therefore $\frac{x^2}{2} + \frac{y^2}{2} = \frac{AC^2}{2} = 8$.

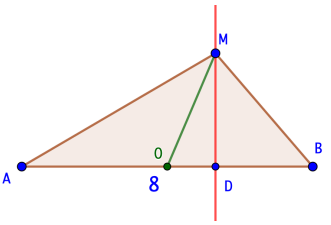
Problem 6.37 (C)

AB is a line segment with length 8.

Find all points M such that $AM^2 + BM^2 = 50$.



Solution: Let D be the intersection of the altitude from M to AB and O be the midpoint of AB.

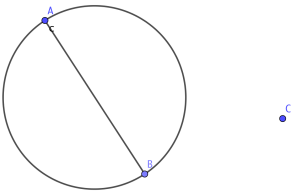


$AM^2 + BM^2 = AD^2 + MD^2 + BD^2 + MD^2 = (AO + OD)^2 + (BO - OD)^2 + 2MD^2$
$= AO^2 + BO^2 + 2OD^2 + 2MD^2 = 2OA^2 + 2OM^2$.
$\Rightarrow OM^2 = \frac{50}{2} - (\frac{8}{2})^2 = 25 - 16 = 9$. So all points of M are on a circle with center at O and radius 3.

Problem 6.38 (C)

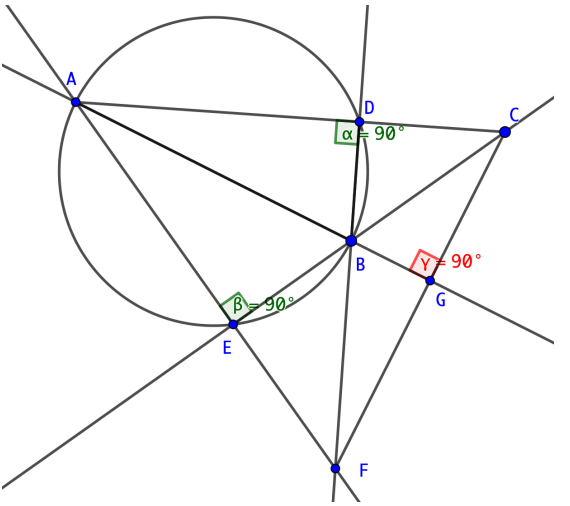
AB is a diameter of a circle. C is a point outside of the circle.

Use only a ruler, draw a line through C perpendicular with the line through AB.



Discussion: In a triangle, three altitude intersect each other at a single point.

Solution:

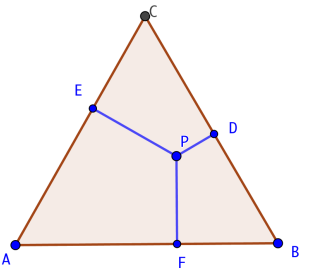


First, line through A, C intersects the circle at D. Since AB is the diameter, ADB is a right angle. Second, line through B, C intersects the circle at E, similarly, AEB is a right angle. Lines through D, B and A, E intersect each other at F. Finally line through A, B intersect CF at G.

Since FD, CE are altitudes of triangle ACF, AG is perpendicular to CF.

Problem 6.39 (C)

$\triangle ABC$ is equilateral. P is a point inside $\triangle ABC$.



Prove that the sum of the distances from P to AB, BC, CA is not dependent on the position of P.

Discussion: The distance from P to side AB can be considered as the height of triangle PAB.

Solution:

$$\frac{1}{2}(PE + PD + PF) \times AB = \frac{1}{2}(PF \,.\, AB) + \frac{1}{2}(PD \,.\, BC) + \frac{1}{2}(PE \,.\, AC) = S_{\triangle ABC}$$

Hence $PE + PD + PC$ is equal to the height of triangle ABC.

Puzzle 6.10:

A telephone call interrupts a man after he has dealt about half of the cards in a bridge game. When he returns to the table, no one can remember where he had deal the last card.

Without learning the number of cards in any of the four partly dealt hands, or the number of cards yet to be dealt, how can he continue to deal accurately, everyone getting exactly the same cards he would have had if the deal had not been interrupted?

Note: In a bridge game the cards are dealt one at a time, starting with the player to the dealer's left and moving in a clockwise rotation until each player has 13 cards. Wait until the dealer distributes all the cards before you pick up your hand. That's bridge etiquette lesson number one.

Solution: he deals from the *bottom* of the deck *backward, himself first.*

Puzzle 6.20:

The integers 1, 3, 8, 120 form a set with remarkable property: *the product of any two integers is one less than a perfect square*. Find a fifth number that can be added to the set without destroying this property.

Solution: It's 0.

Puzzle 6.30:

There is a way to place +, - signs between the digits 1 2 3 4 5 6 7 8 9 to obtain 100, for example:

$$1 + 2 + 34 - 5 + 67 - 8 + 9 = 100$$

Find a way to place *a fewest number of +, - signs* between the digits 9 8 7 6 5 4 3 2 1 to obtain 100.

Solution: $98 - 76 + 54 + 3 + 21 = 100$

Puzzle 6.40:

Two brothers inherited a herd of sheep. They sold all of them, receiving for each sheep the same number of dollars as there were sheep in the herd.

The money was given to them in $10 bills except for an excess amount, less than $10, that was in silver dollars. They divided the bills between them by placing them on the table and alternately taking a bill until there were none left. "It isn't fair," complaining the younger brother. "You drew first and you also took the last bill, so you got $10 more than I did." To even things up partially the older brother gave the younger on all the silver dollars, but the younger was still not satisfied. "You gave me less than $10," he argued. "You still owe me some money."

"True," said the older brother. "Suppose I write you a cheque that will make the total amounts we each end up with exactly the same." This he did.

What was the value of the cheque?

Solution: it's $2.

Note that: the number of dollars received is the square of the number of sheep.

Chapter 7

Ratio - Average - Percent — Solutions

Problems

Problem 7.21 (A)

The ratio of cherry candies to grape candies in a bag is 3 : 5. After your friend eats half of the grape flavoured candies and only one cherry flavoured candy, the ratio of cherry to grape candies is 7 : 6.

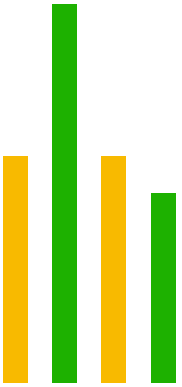
How many candies are left in the bag?

Discussion: Draw a figure, use Problem 7.11 as an example.

(1) What is half of grape flavoured candies? How can you indicate it on the drawing?

(2) How about one cherry flavoured candy? How can you show it on the drawing?

Solution:



After the friend ate half of the grape flavoured candies, the ratio of cherry to grape changed to 6 : 5 as shown in the figure on the left. So for every 6 cherry candies, I have 5 grape candies. In another word, for every 36 cherry candies, I have 30 grape candies.
Because after the friend ate one cherry candies, the ratio changed to 7 : 6, it means that for every 35 cherry candies, I have 30 grape candies.
So it is clear that the remaining candies are 35 cherry candies and 30 grape candies.
All together there are 65 candies.

Problem 7.22 (A)

Before a plum is dried to become a prune, it is 92% water. A prune is just 20% water.

If only water is evaporated in the drying process, how many pounds of prunes can be made with 100 pounds of plums?

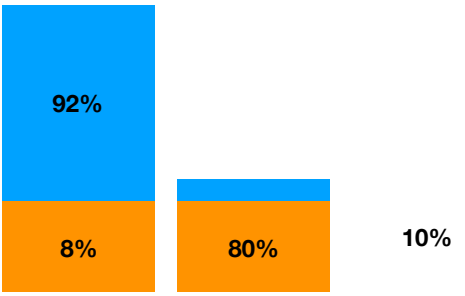
Discussion:

(1) How much water is in 100 pounds of plum? How much of it is *not water*?

(2) How many pounds of prune has that amount of *not water?*

Solution 1: In 100 pounds of plum, there is 92 pounds of water. Or 8 pounds of non-water material. This accountable for 80% of prune. This means that there is 10 pounds of prune.

Solution 2:



The amount non-water material in prune is the same as in plum. The amount of prune is 8% + 2% (20% of 8%) of plum or 10 pounds.

Problem 7.23 (A)

The average age of 10 people in a room is 40 years. There is a teenager who is 13 years old. How much older, on average, are the rest of the people comparing to the teenager?

Discussion:

(3) What is the sum of age of all people?

(4) What is the sum of age of all people except the teenager?

Solution: The sum of ages of all people in the room is is$40 \times 10 = 400$.

After the teenager left, the sum is $400 - 13 = 387$. There are 9 persons left. So their average age is 43 years old. Hence the difference is $43 - 13 = 30$ years.

Problem 7.24 (B)

Dennis has three test scores of 84, 90, and 86. There is one more test at the end of the semester. Since Dennis wants to make an A for the grade, he needs to achieve an average to be at least 90.

$$\frac{84 + 90 + 86 + ?}{4} \geq 90$$

What is the least score that he has to make on the fourth test?

Discussion: What is the average of 90 means for the sum of all test scores?

Solution: The sum of four number must be at least $4 \times 90 = 360$. and the sum of three of them is $84 + 90 + 86 = 260$. It means that the last test must be perfect (100).

Problem 7.25 (B)

The five interior angles in a pentagon are in the ratio $2 : 3 : 4 : 5 : 6$. What is the measure of the middle angle?

Discussion:

(1) Let draw a diagram to reflect the ratio. Does it so that the smallest angle has 2 "parts".

(2) Think about the sum of all the angles.

Solution: The sum of all interior angles of a pentagon is $3 \times 180^\circ = 540^\circ$.

This means that the angles are :

$\frac{2}{20}.540 = 54$, $\frac{3}{20}.540 = 81$, $\frac{4}{20}.540 = 108$, $\frac{5}{20}.540 = 135$, $\frac{6}{20}.540 = 162$

.

So the measure of the middle angle is 108 degrees.

Problem 7.26 (B)

Antoinette gets 70% on a 10-problem test. It means that she finds 7 correct solutions and incorrect or no solution for the other 3 problems. She also gets 80% on a 20-problem test and 90% on a 30-problem test. If the three tests are combined into one 60-problem test, and

(1) 5 points given for each correct solution of a problem in the 10-problem test,

(2) 6 points given for each correct solution of a problem in the 20-problem test,

(3) 8 points given for each correct solution of a problem in the 30-problem test.

What is the average in point does she get for each problem?

Discussion: What is the number of points she gets for each test? How many points she gets if all the problems are combined together into one test?

Solution:

(4) For the 10-problem test, 70% means 7 correct solutions or 7×5 points

(5) For the 20-problem test, 80% means 16 correct solutions or 16×6 points

(6) For the 30-problem test, 90% means 27 correct solutions or 27×8 points

She gets $7 \times 5 + 16 \times 6 + 27 \times 8 = 35 + 96 + 216 = 347$ points for 60 problems. This means that her average is 5.78

Problem 7.27 (C)

The value of a diamond is proportional to the square of its weight. It is broken into three pieces whose weights are in the ratio 3 : 2 : 5.

If the value of the original diamond was \$20000, find the loss incurred due to breaking it up.

Discussion:

(1) Let the weight of the original diamond be w. Find the weights of the smaller diamonds.

(2) Calculate their values.

Solution: Let x be the weight of the original diamond. The weights of the smaller diamonds are $\frac{3}{10}x, \frac{2}{10}x, \frac{5}{10}x$. Let u, v, w be the values of the smaller diamonds.

$$\frac{20000}{x^2} = \frac{u}{(\frac{3}{10}x)^2} = \frac{v}{(\frac{2}{10}x)^2} = \frac{w}{(\frac{5}{10}x)^2} \Rightarrow \frac{20000}{x^2} = \frac{100u}{9x^2} = \frac{100v}{4x^2}$$

$$= \frac{100w}{25x^2} \Rightarrow 200 = \frac{u}{9} = \frac{v}{4} = \frac{w}{25}$$

So $u = 1800$, $v = 800$, $w = 5000 \Rightarrow u + v + w = 7600$.

The lost is 12400.

Problem 7.28 (C)

Ottawa is about 200km from Montreal. Burger the King starts traveling from Ottawa to Montreal and Burger the Queen from Montreal to Ottawa at speeds (not necessarily constant) with a proportion of 3 to 1.

How many km away from Montreal Burger the Queen will be when she meets Burger King?

Discussion: Think about the amount of time they travel.

Solution: For each distance x Burger the Queen travel, Burger the King travels $3x$. When they meet $x + 3x = 200 \Rightarrow x = 50$.

Hence, Burger the Queen will be 50km from Montreal when she meets Burger the King.

Problem 7.29 (C)

In a spaceship arrives to Terra. It's full of bugs. There are beetles (6 legs), spiders (8 legs). There are 162 more legs than bugs and 18 more beetles than spiders. How many spiders are there?

Discussion: It's a simple two-variable system equation.

Solution: Let b be the number of beetles, s be the number of spiders.

$6b + 8s = (b + s) + 162$ and $b - s = 18$.

Substitute b by using the second $b = s + 18$.

$5(s + 18) + 7s = 162 \Rightarrow 12s = 162 - 90 = 72 \Rightarrow s = 6, b = 24$.

So the number of spiders is 6.

Challenges

Problem 7.31 (A)

The heights of Alex, Brian, Catherine, Debby, and Emily are 147, 149, 150, 151, and 158 cm, respectively. Frank's height is an integer and 3 cm less than the average of five students mentioned before. What is Frank's height?

Discussion: What is the sum of heights of all five students?

Solution: The sum of heights of all five students is $\frac{755}{5} = 151$. Frank's height is 148.

Problem 7.32 (A)

The ratio of the number of dollars Alain has to the number of dollars Benny has is $3:4$. After Alain gives \$40 to Benny, the ratio of the number of dollar Alain has to the number of dollars Benny becomes $1:3$.

How many dollars does Alain have at first?

Discussion: Draw a figure.

(1) Divides what Alain has into *3 parts*, what Benny has into *4 parts*.

(2) Now think *how many parts* you need to move to make what they have to be equal?

Solution: Let the number of dollars Benny has is the four boxes on the first line, the second line is Alain's.

Since \$40 should be bigger than one box, otherwise when \$40 is move from the second to the first like it will not make $1:3$. The small brown rectangle indicates that \$40 is more one-third.

So let move \$40 from Alain's row and append it to the end of Benny's row.

Chop the Benny row so that we have two portions look like Alain's row.

Rearrange the brown rectangles, draw a larger border for the portions equal to Alain's.

It can be clearly shown now that 3 brown rectangle equal to the shorter green rectangle, which is made by cut a brown rectangle out of the normal green rectangle. Hence, the area of 4 brown rectangles is equal a green rectangle. It means 5 brown rectangles is \$40. 1 brown rectangle is \$8. 1 normal green rectangle is 32 and the money that Alain has at first is $3 \times 32 = 96$ dollars.

Problem 7.33 (A)

Melamine receives 15% commission on every sale she makes.

On the sale of a \$150 calculator (before any discounts), how many more dollars will she receive if her commission is based on the original price of the calculator rather than the price of the calculator after two discounts, each 40%?

Discussion*:* Determine the value of the calculator after two discounts.

Solution: Because the price is decreased to $150 \times 0.6 \times 0.6 = 54$, so the difference of her commissions is equals the commission of the difference of the prices: $0.15 \times (150 - 54) = 14.4$.

Problem 7.34 (B)

Pique paid \$945 to transport his animal by ferry. The costs was \$3, \$2, and \$1 for each cat, dog, and squirrel, respectively. The ratio of cats to dogs was $2 : 9$, and dogs to squirrel $3 : 7$.

How many cats were there?

Discussion: Bring the ratios to lowest common denominator, for example $2 : 3$ and $3 : 4$ can be rewritten as $8 : 12$ and $9 : 12$.

Solution: Let c, d, s be the number of cats, dogs, and squirrels. Because the ratio of cats to dogs is $2 : 9$, so $\frac{c}{2} = \frac{d}{9}$, similarly $\frac{d}{3} = \frac{s}{7}$. So $\frac{3c}{6} = \frac{2d}{18} = \frac{s}{21} = \frac{3c + 2d + s}{6 + 18 + 21} = \frac{945}{45} = 21$. $c = 2 \times 21 = 41$.

Problem 7.35 (B)

On Friday, a snowboard was discounted 50%. On Monday, that sale price was reduced by 50%. On Tuesday, that sale price was reduced by 20%. On Wednesday, that sale price was reduced by 20%.

How many percent is the price on Wednesday comparing with the price on Thursday of the previous week?

Discussion:

(1) Let x be the price on Thursday. What is the price on Friday?

(2) Continue to do that until Wednesday.

Solution: Let x be the price on Thursday of the previous week.

The price on Wednesday is $x \times 0.5 \times 0.5 \times 0.8 \times 0.8 = 0.16x$ or 16%.

Problem 7.36 (B)

$$\frac{1}{\frac{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{24}}{?}}$$

What is the reciprocal of the average of reciprocals of all factors of 24?

Discussion: What are the factors of 24? What are the reciprocals of those factors? What is the average of the sum of the reciprocals?

Solution:

$$24 \times (\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{24}) = 24+12+8+6+4+3+2+1 = 60$$

$$\Rightarrow \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{24} = \frac{60}{24} = \frac{5}{2} \Rightarrow \frac{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{24}}{8} = \frac{5}{16}.$$

Hence it's reciprocal is $\frac{16}{5}$.

Problem 7.37 (C)

Alpha runs a publishing company. Zeta wants to publish a book. Alpha offers to pays Zeta \$20000. The cost of publishing a copy of a book is \$15. Alpha also needs to pay the bookstore 30% of the sale price of \$30 each book. He managed to get 10% profit at the end of the year.

At least how many copies of the book did Alpha sell?

Discussion:

Let n be the number of sold copies the books. Establish:

(1) an expression of the total cost, which is a function of n

(2) an expression of the total revenue, which is also a function of n

Solution: Let n be the number of sold copies the books.

Total cost $20000 + 15n + 0.3 \times 30n = 20000 + 24n$. Revenue $30n$.

$30n - (20000 + 24n) \geq 0.1 \times 30n \Rightarrow 3n \geq 20000 \Rightarrow n \geq \dfrac{20000}{3}$, so $n = 6667$ is the least.

Note: writing books is more profitable than publishing them?

Problem 7.38 (C)

A cup of water is added to a container of salt water solution. The salt content is 15% alter the addition. The same amount of water is added again to the container. Now the salt content is 12% of the solution in the container.

Find the original salt content of the solution in the container.

Discussion: Let a be the amount of salt originally, m is the amount of solution, b is the amount of water in the container.

(1) What are the values of $\dfrac{a}{m+b}$ and $\dfrac{a}{m+2b}$?

(2) Figure out $\dfrac{a}{m}$ by using the value of two fractions in (1)

Solution: $\dfrac{a}{m+b} = 0.15$, $\dfrac{a}{m+2b} = 0.12$.

$100a = 15(m+b) = 12(m+2b) \Rightarrow 15m - 12m = 24 - 15b \Rightarrow 3m = 9b \Rightarrow m = 3b$

$\Rightarrow 100a = 60b \Rightarrow 5a = 3b \Rightarrow \dfrac{a}{m} = \dfrac{\frac{3}{5}b}{3b} = \dfrac{1}{5}$. It means the original content of salt is 20%.

Problem 7.39 (C)

A soccer ball cover is sewn from 32 leather panels: white hexagonal panels (6 corners) and black pentagonal panels (5 corners). Every black panel is surrounded by 5 white panels. Each white panel is surrounded by 3 white and 3 black panels. How many black panels are there?

Discussion: Count the number of edges that connects black and white panels in two ways.

Solution: Suppose the number of black panel is b and white panel is w. Since every black panel is surrounded by 5 white panels, so the number of black-white edges is $5b$. Every white panel is also touches 3 black panels so the number of black-white edges is $3w$. So $5b = 3w$.

Because $b + w = 32 \Rightarrow b = 12, w = 20$.

Puzzle 7.10: Earrings

In a certain African village there live 800 women. Three percent of them are wearing one earring.

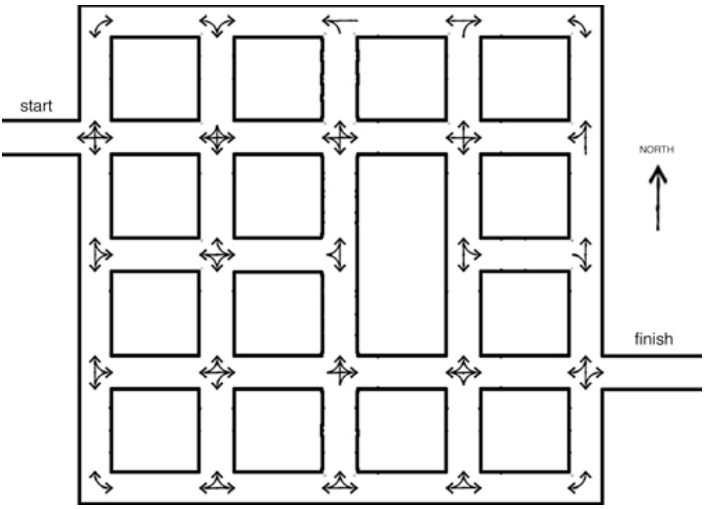
Of the other 97 percent, half are wearing two earrings, half are wearing none. How many earrings all together are being worn by the women?

Note: there is a surprisingly simple answer to this puzzle.

Solution: If the ones who have two earrings give one for the ones that have none, then 97 percent of them has 1 earring, the rest 3% wear one earring. So all of them wear 1 earring. The total is 800.

Puzzle 7.20: How to travel

The town had only 37 registered cars. The mayor thought it would be safe to appoint his cousin, who was the town cutup, as its traffic commissioner. But he soon regretted his decision. When the town awoke one morning, it found that a profusion of signs had been erected establishing numerous one-way streets and confusing restrictions on turns.



The citizens were all for tearing down these signs until the police chief, another cousin of the mayor, made a surprising discovery. Motorists passing through town became so exasperated that sooner or later they made a prohibited turn. The police chief found that the town was making even more money from these violations than from its speed trap on an outlying country road.

Of course everyone was overjoyed, particularly because the next day was Saturday and Adam, the county's richest farmer, was due to pass through town on his way to the county seat. They expected to extract a large fine from Adam, believing it to be impossible to drive through town without at least one traffic violation. But Adam had been secretly studying the signs. When Saturday morning came, he astonished the entire town by driving from his farm through town to the county seat without a single violation!

Can you discover the route Adam took?

At each intersection you must follow one of the arrows. That is, you may turn in a given direction only when there is a curved line in that direction, and you may go straight only when there is a straight line to follow. No turns may be made by backing the car around the corner. No U-turns are permitted. You may leave an intersection only at the head of an arrow. For instance, at the first intersection after leaving the farm, you have only two choices: to go north or to go straight. If you go straight, at the next intersection you must either go straight or turn south. True, there is a curved line to the north, but there is no arrow pointing north, so you are forbidden to leave that intersection in a northerly direction.

Solution: The direction he should take is E-E-S-S-E-N-N-N-N-E-S-W-S-E-S-S-W-W-W-W-N-N-E-S-W-S-E-E-E-E-N-E

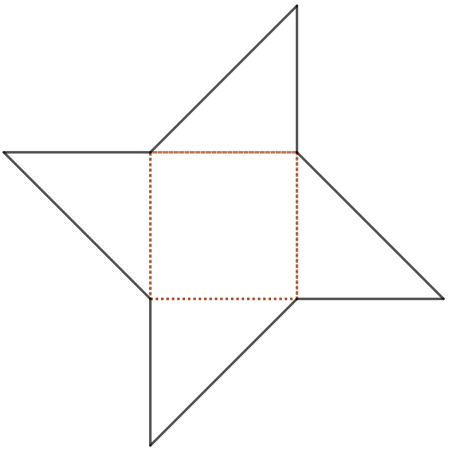
Puzzle 7.30: Consecutive letters

What are the four consecutive letters of the alphabet can be arranged to spell a four-letter word?

Bonus: How many words can you make from that four letters

Solution: R-S-T-U which is RUST. Second word is RUTS.

Puzzle 7.40: How many points



What is the maximum number of points that:

(a) can be placed on the borders or within the figure shown above,

(b) no two points are separated by a distance that is less than the length of the diagonal of the square?

Solution: It's the centre and four points near the tip of each acute angle. Measure the distance between the centre of the square to the tip of an acute angle. Prove that is it larger or equal the length of the diagonal of the square.

Chapter 8

Counting — Solutions

Problems

Problem 8.21 (A)

How many three-digit numbers are there such that the middle digit is the average of the other two digits?

Discussion:

(1) How many ways can you choose the left most digit?

(2) If you have chosen the left most digit, how many ways can you choose the right most?

Solution: There is 9 ways to choose the left most digit (1, 2, .., 9). Depending on the parity of the left most digit, there are always 5 ways to choose the right most one, because it has to have the same parity as the left most. So the number of ways is 45.

Problem 8.22 (A)

A theatre stage has eight lights and each can be separated turned on or off. In how many way can we illuminate the stage?

Discussion: How many possible states a light can be in?

Solution: Each light has two states: ON/OFF. The total number of positions is $\underbrace{2 \times 2 \times \ldots \times 2}_{8}$ or 256. In order to illuminate the stage, the all OFFs position must be discarded. The answer is 255.

Problem 8.23 (A)

How many ways to put a king, a queen, two identical bishops, two identical knights, two identical rooks on row?

Discussion:

(1) Let assume that the bishops, the knights, the rooks are different. How many ways to place them in a row?

(2) Since the two bishops are identical, so the number of the ways with one kind of order of the two bishops would be exactly same with the other order to arrange them - in another word they can be switched.

Solution: Let assume that the bishops, the knights, the rooks are different. There are $8 \times 7 \times 6 ... \times 2 \times 1$ ways to put all eight figures in a row. Since the two bishops can be switched, the number of ways is reduced by half, similarly with the two knights and two rooks. So the total number of ways is $\frac{8.7.6...2.1}{2.2.2} = 7 \times 6 \times \ldots 2 \times 1 = 7!$

Problem 8.24 (B)

Penelope has bank account with 6-digit PIN as a password. How many different PINs can she create if she uses only two different digits, each appear at least once? For example 333331 and 444555 are valid ones.

Discussion:

(1) How many ways Penelope can choose two different digits?

(2) How can she form a PIN from that two different digits

Solution: There are $\binom{10}{2} = 45$ ways she can choose two different digits. For every pair of different digits, she can form 2^6 different sequence of 6-digit PINs. Two sequences containing identical digits must be removed. So the total number of them is $45 \times 2^6 - 10 = 2870$.

Problem 8.25 (B)

How many integer triplets (a, b, c) such that $a \geq 5$, $b \geq 6$, $c \geq 1$, and $a + b + c = 17$

Discussion:

(1) Let $x = a - 4 \geq 1$, $y = b - 5 \geq 1$, $z = c \geq 1$ are two non-negative integers.

(2) What is $x + y + z$. How many triplets can we find?

Solution: $x + y + z = 17 - 9 = 8$. Now think about this as 8 marbles in a row. We have 2 sticks to be put in 7 slots between the 8 marbles so that we can separate them into 3 piles. There are $\binom{7}{2} = 21$ ways.

Problem 8.26 (B)

Van Anh splits the set {1, 2, 3, 4, 5, 6, 7} into two disjoint sets such that the sum of digits in each new sets are equal. In how many ways can she do that?

Discussion: What is the sum of all digits? What would be the sum of all digits in one set?

Solution: The sum of all digits is 28, the sum of digits in one set must be 14.

There are 4 ways to split the set:

{7, 6, 1} and {5, 4, 3, 2}; {7, 5, 2} and {6, 4, 3, 2}; {7, 4, 3} and {6, 5, 2, 1}; {7, 4, 2, 1} and {6, 5, 3}

Problem 8.27 (C)

A straight log of length 2m is cut into five pieces with each piece at least 17 cm long. Prove that there are three of these pieces that can be put together to form a triangle.

Discussion:

(1) Let take the two shortest pieces, what would be their minimal length?

(2) What should be the minimal length of the next shortest piece?

Solution: Assume that it is not possible to find any three of them that form a triangle. Let sort the pieces in length, from shortest to longest. Since they are at least 17 cm long, so the shortest ones are at least that. The next one must be longer than the sum of lengths of the two shortest ones - because they do not form a triangle - so it has to be longer than 34 cm. Similarly the fourth one is longer than $17 + 34 = 51$ cm, the fifth one is longer than $51 + 34 = 85$ cm. So the total length of them is longer than $17 + 17 + 34 + 51 + 85 > 200$ cm. It contradicts the given condition that all of them are cut from a 2m log.

Problem 8.28 (C)

How many nine-digit numbers have an even sum of their digits?

Discussion: How can we choose the first 8 digits?

Solution: The first 8 digits can be chosen arbitrarily. There are 9.10^7 ways to do it. The last one need to be chosen so that the digit sum is even. There is alway 5 ways to choose it. So the answer is 450,000,000.

Problem 8.29 (C)

Sets A, B, C, D each have 400 elements. The intersection of any two sets have 115 elements. The intersection of any three of the sets have 53 elements. The intersection of all four sets has 28 elements. How many elements are there is the union of the four sets?

Discussion: Principle of Inclusion - Exclusion.

Solution: There are four sets, $\binom{4}{2} = 6$ pair of sets, $\binom{4}{3} = 4$ trios of sets. So the number of elements of the union is: $4 \times 400 - 6 \times 115 + 4 \times 53 - 28 = 1094$.

Challenges

Problem 8.31 (A)

Last night there was a party and the host's door bell rang 20 times. The first time the door bell rang, only one guest arrived. Each time the door bell rang, two more guests arrived than had arrived on the previous ring. How many guests were girls if they were a quarter of the total number of guests?

Discussion:

(1) What is the total number of guests arrived at the first ring? At the second? And third?

(2) What is the total number of them?

Solution: The total number of guests is $1 + 3 + 5 + \ldots + 39 = 20^2 = 400$. The girls are 100.

Problem 8.32 (A)

A total of 6 books, 3 on mathematics and 3 on history, are to be arranged on a bookshelf. In how many ways can it be done if all the math books are to be placed to the left of all history books?

Discussion:

(1) What would be on the left most position?

(2) What would be on the 4th position from left?

Solution:

The left most position must have a math book. The second and third from left also have to have a math books. There are $3 \times 2 \times 1 = 6$ ways. From position 4 to 6, there are 6 ways to arrange the remaining 3 history books. So in total, there are 36 ways to arrange all 6 books.

Problem 8.33 (A)

Ten ants march in a line, four of them carrying pieces of red leaves, and the rest carrying pieces of yellow leaves. Assume that there is no difference between pieces of the same colour.

How many ways they can march so that in the first five positions, number of yellow pieces of leaves is at least four?

Discussion:

(1) How many possible values for the number of yellow pieces in the first five positions

(2) Based on that, how many possible cases for the last five?

Solution:

If there were 5 yellow pieces in the first five, then all 1 yellow and 4 red pieces in the last five. There are 5 possible arrangement for that remaining 1 yellow piece.

If there were 4 yellow pieces in the first five, then all 2 yellow and 3 red pieces in the last five. There are 5 possible arrangement for the first yellow, 4 for the second ones, so that 20. The count is duplicated because both yellow pieces are identical, so it is a total of 10.

The answer is $5 + 5 \times 10 = 55$ ways.

Problem 8.34 (B)

12 people participated in a chess tournament. Each participant played a game against each other once. 1 point is given for the winner of a game, 0.5 for a tie, and 0 for losing it. At the end of the tournament a number of players, who earned over 70% of the total number of points that can be earned by winning every single game, were awarded with the Master title.

Is that possible if 8 participants gained the Master title?

Discussion: Assume that there were 8 participants gained the Master titles.

(1) What is the minimum number of points a player must achieved in order to be aware the Master title?

(2) How many points the Masters can win from the one who had not gained the Master titles?

Solution: Let assume there were 8 participants who won the Master titles. Let call them the Masters. In order to gain the title, a Master must win at least $0.7 \times 11 = 7.7$ points. According to how the points were given, a Master must win at least 8 points. So the total number of points they have gained is at least $8 \times 8 = 64$. Now look at the number of points the Master can win against the non-Master ones. Since there are 8 Masters, so the points they won agains non-Master cannot be more than $8 \times 4 = 32$ points. Hence, the number of points they should gain between each other is at least $64 - 32 = 32$ points. However, they played between each other $\frac{8 \times 7}{2} = 28$ matches and no more than 28 points can be won. This is contradiction! So there should not be 8 Masters.

Problem 8.35 (B)

Mr. and Mrs. Zed want to name their baby Zeta so that its monogram (first, middle, and last initials) will be in alphabetical order with no letter repeated. How many such monograms are possible?

Discussion: What is a three different letter monogram?

Solution: Obviously the last letter should be Z. The first two letters can be chosen from 25 letters. So the answer is $\binom{25}{2} = 300$.

Problem 8.36 (B)

A drawer in a dark room contains 100 red socks, 80 green socks, 60 blue socks, and 40 black socks. A youngster selects socks one at a time from the drawer but is unable to see the colour of the socks drawn. What is the smallest number of socks that must be selected to guarantee that the selection contains at least 10 pairs? Note that a pair of socks is two socks of the same colour.

Discussion:

(1) How many socks would be needed: an odd or an even numbers?

(2) Minimum how many socks needed to 10 pairs?

(3) What would be extra?

Solution: At least 20 socks needed for 10 pairs. However there would be an number of unpaired socks, which maximum is 3. Hence 23 socks will be enough (maximum 3 are unpaired). 22 will not be enough (5, 5, 5, 7)

Problem 8.37 (C)

Nine chairs in a row are to be occupied by six students and Professor Alpha, Beta, and Gamma. These three professors arrive before the six students and decided to choose their chairs so that each professor will be between two students. In how many ways can Professor Alpha, Beta, and Gamma choose their chairs?

Discussion:

(1) Who occupy the two end-chairs?

(2) Where the three professors can sit?

Solution: The two end-chairs can only be occupied by the students. If the rest of the chairs are numbered from 2 to 8 then there are 10 possibilities: (2,4,6), (2,4,7), (2,4,8), (2,5,7), (2,5,8), (2,6,8), (3,5,7), (3,5,8), (3,6,8), (4,6,8). Since the professors can switch their chairs in 3! ways. The answer is$10 \times 6 = 60$.

Problem 8.38 (C)

Call a 7-digit telephone number $d_1d_2d_3 - d_4d_5d_6d_7$ *memorable* if the prefix $d_1d_2d_3$ is exactly same as either $d_4d_5d_6$, or $d_5d_6d_7$ (possibly both). Assuming that each d_i can be one of the ten digits 0, 1, …, 9; find the number of different memorable telephone numbers.

Discussion:

(1) How many ways $d_4d_5d_6d_7$ can be written?

(2) How many ways not all of them are the same?

Solution: There are 10000 ways to choose the digits for $d_4d_5d_6d_7$. Out of that 10 ways having all digits same. So $10000 - 10 = 9990$ ways to select $d_4d_5d_6d_7$ with at least one digit different. For each of those there are two ways to form $d_1d_2d_3$ differently. For the 10 ways where all digits are the same, there is one way to form $d_1d_2d_3$. So the number of possible ways is $2 \times 9990 + 10 = 19990$.

Problem 8.39 (C)

Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but not divide n?

Discussion: Let $d = 2^k.3^l$. If d is divisor of n^2, less than n but not divide n, what would be k, l?

Solution: Such d is divisor of n^2, less than n but not divide n can only be:

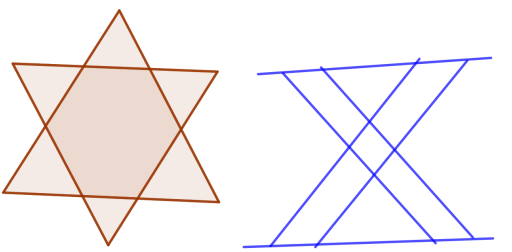
$d = 2^{31+a}.3^{19-b}$, $(2^a < 3^b)$; or $d = 2^{31-a}.3^{19+b}$, $2^a > 3^b$.

and $1 \leq a \leq 31$, $1 \leq b \leq 19$. So there are $19 \times 31 = 589$ such divisors.

Puzzle 8.10: Triangles

Show that there are two ways to draw six line segments of equal length to form eight equilateral triangles.

Solution:



Puzzle 8.20: Couples

Adam and his wife went to a BBQ party. There were other four married couples. There were a number of handshakes. No one shook hands with himself/herself or with his/her spouse. No one shook hands with the same person more than once. After all the handshakes were over, Adam asked everyone, including his wife, how many hands he/she had shaken. To his surprise, it turned out that everyone gave a different answer.

How many hand did his wife shake?

Solution: The persons who shook 8 hands is married to the one who shook 0 hands, similarly with (7, 1), (6, 2), (5, 3) and that's why his wife shook 4 hands.

Puzzle 8.30: Tennis games

Merry beats Anna in a set of tennis, winning six games to Anna' three. Five games were won by the player who did not serve.

Who served first?

Solution: Whoever served first, served five games, and the other player served four. Suppose the first server won x games she served and y games of the other four. The total number of games lost by the player who served them is $5 - x + y$. This is equal 5. So $x = y$ and the first player won $2x$ games. It is Merry because she won an even number of games.

Puzzle 8.40: Dates

$$\overline{OCT} \times 31 = \overline{DEC} \times 25$$

Find the dates. Each letter is a different digit.

Solution: $675 \times 31 = 837 \times 25$

Chapter 9

Distance - Rate -Time — Solutions

Problems

Problem 9.21 (A)

The Festival of Spring is hosted on the road from the City of Queens to the Town of Kings. Sarah rides bikes from the City and Tyler rides bikes from the Town, and both go to the Festival. They both start at 7:00 AM, and both arrive to the Festival at 10:00 AM. If the road from the City of Queens to the Town of Kings is 105 km long and Tyler rides bike 5 km/h faster than Sarah, what is the average speed of Sarah?

Discussion: Sarah and Tyler ride bikes towards each other.

(1) How far are they from each other at the beginning?

(2) How farther does Tyler ride each hour comparing with Sarah?

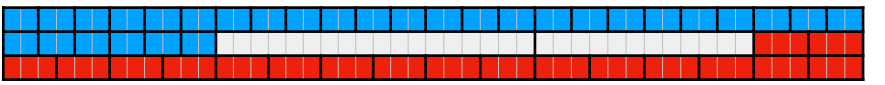
(3) What is the distance that does Sarah ride?

Solution: At 7:00 AM Sarah and Tyler start from their own towns. Sarah is 105 km from Tyler. Their distance is reduce to zero when they meet at the Festival of the Spring. This is done within 3 hours from 7:00 AM to 10:00 AM. Since Tyler rides bike 5 km/h faster than Sarah, so in 3 hours he rides $3 \times 5 = 15$ km more than Sarah. So the distance Sarah rides is $\frac{105 - 15}{2} = 45$. This means her average speed is $\frac{45}{3} = 15$ km/h.

Problem 9.22 (A)

Benny the bun and Tongi the tiger play tunnel digging. Benny digs from the left side, and Tongi from the right side of the hill. Working alone Benny can finish the tunnel in 24 minutes, Tongi in 16 minutes. Benny starts 6 minutes before Tongi, then both of them work together, and Tongi continues for 2 more minutes after Benny finishes and finishes the whole tunnel. How much time passes between the time when Benny starts and Tongi stops?

Discussion: How much can Benny dig alone in 6 minutes? Tongi dig alone in 2 minutes? Take a look at the diagram below:



(a) First row: the tunnel Benny digs alone for 24 minutes

(b) Last row: the tunnel Tongi digs alone for 16 minutes

(c) Middle row: Benny digs from one end for 6 minutes, Tongi from the other end for 2 minutes.

Solution:

Benny works alone for the first 6 minutes. He digs a quarter of the tunnel. Tongi works alone for the last 2 minutes. He digs an eighth of the tunnel. The diagram shows very clearly the empty spaces are the remaining part that both Benny and Tongi dig together.

Since Benny finishes the tunnel for 24 minutes, and Tongi for 16 minutes, it makes more sense to divide the tunnel in a number of tunnel pieces that can be divided both by 24 and 16. Such a number is 48. Let divides the tunnel into 48 pieces as shown in the diagram above.

Using the diagram, it is easy to see that Benny digs 2 tunnel pieces at every minute, Tongi 3 pieces every minutes. In the first 6 minutes, Benny digs 12 pieces; in the last 2 minutes, Tongi digs 6 pieces. If they work together, they can dig $2 + 3 = 5$ pieces per minute. The remaining $48 - 12 - 6 = 30$ pieces are done after 6 more minutes by working together.

So the total amount of time needed is $6 + 6 + 2 = 14$ minutes.

Problem 9.23 (A)

At 1:00 AM the hour and the minute hands of a round clock form an angle of 30 degrees (30°).

From 0:00 AM until 11:59 AM, how many time the hour and the minute hands of a clock form a 30° angle?

Discussion: Estimate

(1) What is the first time the two hands form a 30° angle?

(2) What about the last time?

Solution:

From 0:00-0:59 there are two times the hands form a 30° angle. Similarly in every hour 1, 2, …, 9. From 10:00-10:59 it happens once. From 11:00-11:59 it happens twice So all together 23 times.

Problem 9.24 (B)

Two villages on the mountain are connected by a road without any horizontal stretches. A bus travels between the two villages. While going uphill its speed is 15 km/h, and downhill at 30 km/h. What is the distance between the two villages if a round trip takes 4 hours?

Discussion: In a round trip, if the bus goes uphill at a place on one way, it goes downhill at the same place on the way back.

Solution: Easy to see that the distance the bus goes uphill in one way is the same as the distance the bus goes downhill on the way back. In another word, we can think as if the bus goes all the way uphill from one village to the other and all the way downhill on the wayback.

Let d be the distance between the two villages:

$$\frac{d}{15} + \frac{d}{30} = 4 \Rightarrow \frac{d}{15}(1 + \frac{1}{2}) = 4 \Rightarrow \frac{d}{10} = 4 \Rightarrow d = 40 \text{ km}.$$

Problem 9.25 (B)

A fuse burns from end to end in an hour, but it does not burn uniformly. How can we measure exactly 45 minutes if we had two such fuses, each of which burs non-uniformly in its own unique ways?

Discussion:

(1) How do you measure 30 minutes?

(2) How do you measure 15 minus if you already measure 30 minutes?

Solution:

Let name the fuses as A and B. To measure 30 minutes, we need to jus lit both ends of A. It might not burn uniformly, but once lit from both ends, it burns fully in $\frac{1}{2}$ hour or 30 minutes. We needs to measure the remaining 15 minutes. Note by burning A, we measure 30 minutes, so 15 minutes is half of burning time of something that has 30 minutes remaining to burn.

From this idea:

(1) Period 1: Both ends of A are lit. One end of B is lit. When A is fully burned, it's 30 minutes. Half of burning time for B is spent.

(2) Period 2: The other end of B is lit. The burning time for B is reduced to 30 minutes. B burns from both end. It ends in half of burning time, which is 15 mins.

The total time measure is 30 minutes for period 1 plus 15 minutes for period 2, which is 45 mins.

Problem 9.26 (B)

A man was born on May 1, 30 BCE, and died on May 1, 30 CE.

How many years did he live?

Discussion: BCE is Before Common Era. So 1 BCE is the year before 1 CE.

Solution: The time between May 1, 1 BCE, and died on May 1, 1 CE is exactly 1 year. So the total time between the two dates is $(30-1)+1+(30-1)=59$ years.

Problem 9.27 (C)

Two clocks started to tick at the same time. The first ticks every 2 seconds; the second ticks every 3 seconds. Ticks that occur simultaneously are heard as one tick.

How much time is passed between the first and the 1000th ticks?

Discussion: The ticks are coincide at every 6 seconds.

Solution: If we groups the ticks at every 6 seconds: both clocks ticks once at the same time at second 0, each clock ticks single time at second 2, 3, and 4. So between the 1st and 1000th ticks there will be $\frac{1000}{4}\times 6=1500$ seconds. The time passed between them is 1498 seconds.

Problem 9.28 (C)

Last Sunday at noon the date on the calendar was April 15, 2007.

What will be the date on the calendar one million minutes after that time?

Discussion: Every hour contains 60 minutes. Every day contains $24\times 60=1440$ minutes.

Solution: One million is $\frac{1000000}{1440}=694\frac{4}{9}$. Since $24\times\frac{4}{9}<24\times 12=12$, so the date on the calendar after noon one million minutes is 694 days after April 15. 2008 is a leap year. April 15, 2009 is $366+365=731$ days after April 15, 2007. $731-694=37$ days before April 15 is March 9, 2009.

Problem 9.29 (C)

A factory produces widgets and gadgets. In one hour, 100 workers can produce 300 widgets and 200 gadgets. In two hours, 60 workers can produce 240 widgets and 300 gadgets. In three hours, 50 workers can produce 150 widgets and m gadgets. Find m?

Discussion: Let w be the time needed for one worker produces one widget. Let g the time needed for one worker produces one gadget. Establish an equation system of w, g, solve it and then figure out m. Note that m is a positive integer.

Solution: Since 1 worker can produce 1 widget in w hours, so 1 worker produce 3 widgets in $3w$ hours, or 100 worker produce 300 widgets in $3w$ hours, similarly 100 worker produce 200 gadgets in $2g$. Hence $3w + 2g = 1$. Similarly $4w + 5g = 2$. Therefore $w = \frac{1}{7}$, $g = \frac{2}{7}$.

So for 3 hours, 50 workers produce 150 widgets and m gadgets is equivalent with 1 workers produce 3 widgets and $\frac{m}{50}$ gadgets: $3.\frac{1}{7} + \frac{m}{50}.\frac{2}{7} = 3 \Rightarrow m = 450$.

Challenges

Problem 9.31 (A)

Tom and Jerry play a game. They stand back-to-back.

(1) At the first signal each takes 10 steps in opposite directions and stops.

(2) At the second signal both turns around, walk towards each other and meet each other midway. Tom takes six steps.

How large is a step of Tom comparing to a step of Jerry?

Discussion: Convert the number of Tom's steps to Jerry steps. The figure below shows:

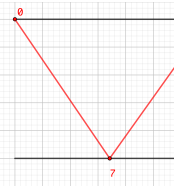
(1) The first is the distances Tom (blue) and Jerry (red) walk after the first signal

(2) The second is the distances Tom and Jerry walk after the second signal.

Solution: the number of steps Jerry takes after the second signal is six. It means that 4 steps of Toms plus 10 steps of Jerry is the same distance as 10 steps of Tom. 2 steps of Tom is same as 10 steps of Jerry. Hence 1 step of Tome is 5 steps of Jerry.

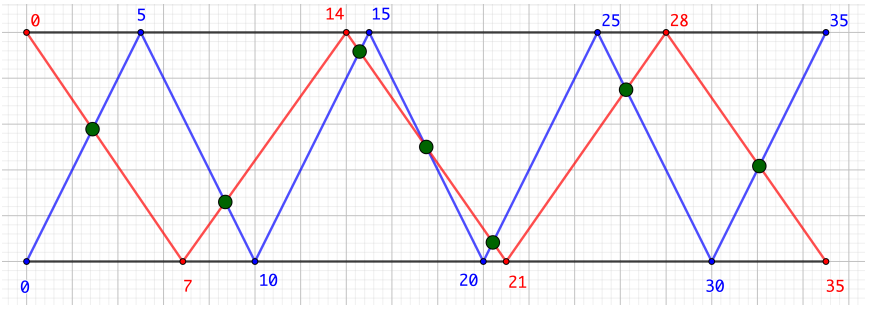
Problem 9.32 (A)

Two ferries from opposite positions of opposite shores of a lake cross back and forth without stopping. One ferry needs 5 minute to cross the lake, the other 7 minutes. What is the number of time during the 35 minutes that the faster ferry passes the slower one going in the same direction or opposite direction?



Discussion: The figure on the left shows the two shores of the lake. The red line indicate that at the minute 0 the slow ferry leaves a shore. After 7 minutes it reaches the other shore. It it showed as minute 7.

Solution: Complete drawing the figure, following the guideline in the discussion.



Now, count the number of green intersection points, they are the times when the faster ferry passes the slower one. The answer is 7.

Problem 9.33 (A)

Bao and Dao traveled to a city. Bao agreed to drive two-third of the total distance if Dao agree to drive the remaining distance. After completed his part of driving, Bao slept. When he woke up, Dao still had 20% of her part to drive.

What is the fraction that represent the part of the road over the total distance when Bao slept?

Discussion: The blue part is what Bao committed to drive, the red for Dao.

Solution: First, if we divide Dao's part into 5 pieces, then because Bao's part is twice Dao's, so it contains 10 pieces. Dao still had 20% of her part to drive, that means she drove all the green distance and had only one red distance left, as shown below.

So the part of distance Bao slept is the green distance over the whole (blue+green+red): $\frac{4}{15}$

Problem 9.34 (B)

The contest team of MCC has 8 members of the club. While working as a team, they can solve 20 problems in 10 minutes. If MCC needs to send a team to a contest that requires to solve 30 problems in 5 minutes, at least how many club members the team will need?

Discussion: What is the work rate of 2 members?

Solution: If we divide 8 members into 4 groups of 2 and also divide 20 problems into 4 groups of 5, then 2 members can solve 5 problems in 10 minutes. By doubling the number of members, the time needed is halved. It means that 4 members can solve 5 problems in 5 minutes. Hence, at least $6 \times 4 = 24$ members will be needed if $6 \times 5 = 30$ problems needed to be solved in 5 minutes.

Problem 9.35 (B)

A clock show 12:00. How long will it be before the hour and minute hand coincide again?

Discussion: Compare the degrees the hour and minute hands move as in previous problems.

Solution: Easy to see that they will meet somewhere after 1:00. Let m be the minutes when they meet $30 + 0.5m = 6m \Rightarrow 30 = 5.5m \Rightarrow m = \frac{60}{11}$. So it's 1 hour and $\frac{60}{11}$ minutes.

Problem 9.36 (B)

Tom can run from his home to Betty's house in 21 minutes. Betty can run from her home to Tom's house in 28 minutes. At noon, Tom starts running from his house toward Betty's house while at the same time Betty starts running to Tom's house. After they meet Betty turns and runs back home.

At how many minutes after noon will Betty arrive back home?

Discussion:

(1) What is a rate of distance per minute Tom if he run the distance in 21 minutes?

(2) What does it means when they meet?

Solution: Tom runs a $\frac{1}{21}$ of the distance between the two houses every minute. Similarly and Betty runs $\frac{1}{28}$ of that distance every minute. Running towards each other: $\frac{1}{21} + \frac{1}{28} = \frac{1}{7}(\frac{1}{3} + \frac{1}{4}) = \frac{1}{12}$ of the distance is decreased for every minute. Easy to see that they need 12 minutes to meet each other. That is also the same amount of time Betty needs to run home. Betty is back at home 24 minutes after noon.

Problem 9.37 (C)

A car's front tires need to be replaced after every 25000 miles, and the rear ones after 15000 miles. When should they be rotated, front to back, so that they can be replace simultaneously? How far will the car travel in total?

Discussion: A distance that divides both 25000 and 15000 is 75000 miles. If we drive this car, we need to change 3 pairs of the front tires and 5 pairs of the rear ones or a total of 8 pairs.

Solution: This means that it is enough rubber on two pairs of tires to cover one-fourth of the 75000 miles distance, which is 18750 miles. So at half of this distance, or 9375 miles, we should switch the tires.

Problem 9.38 (C)

A boat goes with a constant speed downriver from the Elven Town to the Village of Traders in 3 days. It goes upriver from the Village to the Town at a possibly different but constant speed in 5 days. Bilbo Baggins took a raft to float from the Town to the Village. How long time does that take him?

Discussion: There are two different speeds: the river's and the boat's.

Solution: Let b, r be the average speed *per day* of the boat, and of the river, respectively. When the boat goes downstream, the current increases its speed by the speed of the river. The distance going downstream is $3(b+r)$. When going upstream, the current now is against the boat, and decrease the boat's speed by the speed of the river. The distance going upstream is $5(b-r)$. However both distances are the same: $3(b+r) = 5(b-r) \Rightarrow 3b + 3r = 5b - 5r$. Removing $3b$ and adding $5r$ to both side: $8r = 2b \Rightarrow 4r = b$. So the distance is $3(b+r) = 3(4r+r) = 15r$.

The raft, that carries Bilbo Baggins, floats downstream with the speed of the river r and will takes the distance $15r$ in 15 days.

Problem 9.39 (C)

An army battalion forms a row of soldiers marching. It's 20 miles long and it advances 20 miles. During this time, a messenger on a horse travels at constant speed from the rear of the battalion to the front, and immediately turns around and ends up precisely at the rear of the battalion as it completes the 20-mile journey.

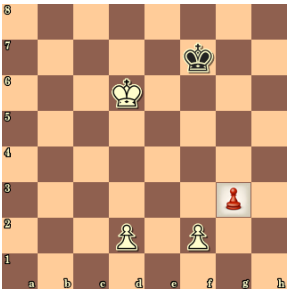
How far has the messenger traveled?

Discussion: Let d be the distance travelled by the battalion until the messenger gets to the front of the battalion, and t be the amount of time it takes the messenger to get to the front of the battalion. Let x, y be the rate in mph at which the messenger, and the battalion travelling. So: $t = \dfrac{d}{y}$, $t = \dfrac{20+d}{x}$, so $\dfrac{y}{x} = \dfrac{d}{d+20}$.

Solution: Now let T be the time it takes the messenger to travel to the back of the battalion. $T = \dfrac{20-d}{y} = \dfrac{d}{x}$. Therefore $\dfrac{20-d}{d} = \dfrac{d}{20+d}$.

$\Rightarrow d^2 = 400 - d^2 \Rightarrow d = 10\sqrt{2}$. So the distance the messenger traveled is $20 + 2d = 20 + 20 \times \sqrt{2}$.

Puzzle 9.10: Which colour is the pawn?

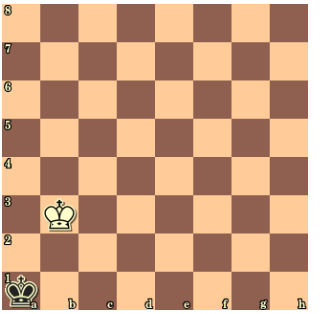


The board shows an end-game with five pieces: two kings, two white pawns, and one pawn of unknown colour, temporarily colour as red. It is known that during the game no piece has moved from a square of one colour to a square of another colour. What is the actual colour of the red pawn?

Solution: The white king can move out of his original position $e1$, but still to a black square, by castling only to $g1$. If the unknown pawn were white, and moved to $g2$ by capturing another piece, then the king can not move to the current position. It means that the unknown pawn must be a black one.

Puzzle 9.20: Missing piece

Two kings are occupying two positions on the chessboard. Put another chess piece on the board such that:



(1) Neither king is in check.

(2) The position can be reached in a legal game.

(3) It can be proved that neither side has a legal play.

Solution: White Bishop on $a2$ and it's Black's move.

Puzzle 9.30:

A *super queen* in chess is a chess queen that can also move like a knight. Can you put 5 of them on a 5×5 chessboard so that no piece attacks another.

Solution: their positions are $a2$, $b5$, $d1$, $e4$.

Puzzle 9.40:

A *super queen* in chess is a chess queen that can also move like a knight. Can you put 10 of them on a 10×10 chessboard so that no piece attacks another.

Solution: their positions are $a7$, $b3$, $c10$, $d6$, $e2$, $f9$, $g5$, $h1$, $i8$, $j4$.

Chapter 10

Probability — Solutions

Problems

Problem 10.21 (A)

A fair coin is flipped three times, and each flip comes up heads. What is the probability that the next flip will also be head?

Discussion: What does it mean fair coin? How the fair coin should be have if the previous flip comes up head?

Solution: It does not matter how any times a fair coin flipped previously. The probability of a flip to be head should always be $\frac{1}{2}$.

Problem 10.22 (A)

A pair of dices is rolled to see who goes first in a board game. Samuel only needs to beat a 4 to win the chance to go first. What is the probability that he will roll a greater than 4?

Solution: It is possible to count the cases when Samuel roll higher than 4. However it is more simpler to count the cases to roll 4 or less: (1,3), (2,2), (3,1), (1,2), (2,1), (1,1). The total number of cases is $6 \times 6 = 36$. So there are $36 - 6 = 30$ ways to roll higher than 4.

Hence, the probability is $\frac{30}{36} = \frac{5}{6}$

Problem 10.23 (A)

Three vertices of a regular heptagon (7-side polygon with all equal sides and angles) are connected to form a triangle. What is the probability that the triangle formed is isosceles (having two equal sides)?

Discussion: Select A vertex. Find a pair of vertices that form an isosceles triangle with A.

Solution: The number of triangles formed by three vertices is the number of ways to choose 3 points out of 7 points: 7 ways to choose the first one, 6 ways the second one, 5 ways the last one. Because there are 6 ways to arrange the three selected points, so the number of ways is $\frac{7.6.5}{6} = 35$. For each vertex, there are 3 ways to choose a pair of vertices to form an isosceles triangle, so in total: $7 \times 3 = 21$. The probability is $\frac{21}{35} = \frac{3}{5}$.

Problem 10.24 (B)

Ollie rolls a fair dice three times. What is the probability that the third roll is greater than the sum of the first two?

Discussion: Let do casework with the last roll: 6, 5, 4, 3, 2, 1. In each case, how many possible cases for the sum of the first two?

Solution: If the last roll is 6, there are 10 possible cases: (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1). Similarly if the last roll is 5, there are 6 possible cases. If the last roll is 4, we have 3 cases, and if it is 3 we have only 1 case.

The probability is $\dfrac{20}{6 \times 6 \times 6} = \dfrac{5}{54}$.

Problem 10.25 (B)

For any date in history, Billy can very quickly tell you what day of the week it was. Emily chooses three dates at random and Billy tells the day of the week for each.

What is the probability that two of the days occurred on the same day of the week?

Discussion: Let do complementary counting of the possible cases for different days of the week.

Solution: The first day can be any day of the week, there are 6 possibilities for the second and 5 for the last: $\dfrac{7.6.5}{7.7.7} = \dfrac{30}{49}$.

Hence, the probability that two of the days occurred on the same day of the week is: $1 - \dfrac{30}{49} = \dfrac{19}{49}$.

Problem 10.26 (B)

In rectangle ABCD: $AB = 1$, $BC = 2$. X is selected at random inside the rectangle. What is the probability that the area of ABX triangle is more than twice the area of triangle BCX?

Discussion: Draw the rectangle. Connect B with midpoint of CD. Let X be a point on this segment. Investigate the areas of ABX and BCX.

Solution: Let E be midpoint of CD. Easy to find that such X on BE: $S_{\triangle ABX} = 2S_{\triangle BCX}$. So X should be inside BCE triangle.

The probability is $\dfrac{1}{4}$.

Problem 10.27 (C)

Four vertices of a cube are selected at random. What is the probability that all four selected points will be co-planar (on the same plane)?

Discussion: Find a few examples where 4 vertices are on the same plane. How many ways to chose 4 out of all the vertices of a cube?

Solution: There are 70 ways to choose 4 objects out of 8. Easy to verify that 4 objects are coplanar if they form a face, or if they form a diagonal plane halving the cube. It's $\frac{6+6}{70} = \frac{6}{35}$.

Problem 10.28 (C)

The weather man has predicted 30% chance of snow for each of the following three days. One nearby listener commented that the chance of snowing is $3 \times 30 = 90\%$, therefore it will likely snow on one of the next three day. What do you think?

Discussion: What is the probability of NOT snowing the next three days?

Solution: Probability of NOT snowing on one of the next three days is 0.70%. So the probability of NOT snowing on all of the next three days is $0.70^3 = 0.343$ or 34.3%. This means that the probability that it will snow on one of the next three days is $0.657 \approx 65.7\%$.

Problem 10.29 (C)

Andy draws a circle, Benny draw another one completely inside that, Cindy draws a third circle inside the Benny's circle. Andy then colour black the region inside Benny circle but outside Cindy circle. A point X is selected in Andy's circle at random and the probability of X be in the black region is $\frac{1}{2}$. All circle diameters are integers. What is the smallest possible value of the black region?

Discussion: Let a, b, c be the radii of the circles. What is the relationship between them?

Solution: Since point selected in a circle at random has $\frac{1}{2}$ probability to be in the black region, so it is half of the c circle: $\pi a^2 - \pi b^2 = \frac{\pi c^2}{2}$.

So $a^2 - b^2 = \frac{c^2}{2}$. Easy to see that the smallest possible c is 2.

So the smallest value of the black region is 2π.

Challenges

Problem 10.31 (A)

Four vertices of a regular octagon are selected at random and connected to form a quadrilateral. What is the probability that the quadrilateral NOT a rectangle?

Discussion:

(1) How many ways can you choose 4 vertices out of 8?

(2) What is the condition for 4 vertices so that they form a rectangle?

Solution: The number of ways to choose 4 vertices out of 8 to form a quadrilateral is $\frac{8.7.6.5}{4.3.2.1} = 70$. Only 6 possible group of 4 vertices can make a rectangle. The answer is $\frac{64}{70} = \frac{32}{35}$.

Problem 10.32 (A)

What is the probability to roll each of the numbers 1 through 6 in any order in six rolls?

Discussion: How many possible rolls are there? How can you have six rolls with each of the numbers 1 through 6 in any order?

Solution: The number of possible rolls is 6^6. The six rolls with each of the numbers 1 through 6 in any order is a way to shuffle these (1, 2, 3, 4, 5, 6) numbers, so the number of rolls is 6!. Therefore the probability is $\frac{6!}{6^6} = \frac{5}{324}$.

Problem 10.33 (A)

The first nine letters of the alphabet are randomly placed into three equal piles, each pile consists of three letters. What is the probability that the letter A and B end up in different piles?

Discussion*:* How many ways A and B can end up in the same pile?

Solution: Let place A in a pile. There are 8 places in A's pile and other piles for B. 2 of those positions are in the same pile as A. the probability for A and B ending up in the same pile is $\frac{2}{8} = \frac{1}{4}$. Therefore the probability that the letter A and B end up in different piles is $1 - \frac{1}{4} = \frac{3}{4}$.

Problem 10.34 (B)

For three randomly selected two-digit numbers, what is the probability that the sum of unit digits of those number is even?

Discussion: What are the parity cases of the unit digits?

Solution: The sum of the unit digits of selected integers can be even if all of them are even of two of them are odd and the third is even. There are 4 such cases out of 8 possible ways. So the probability is $\frac{1}{2}$.

Problem 10.35 (B)

The school cafeteria has chocolate chip cookies and oatmeal cookies for desert. What is the probability exactly 3 of the next 5 students in line will select chocolate chip cookies, students prefer chocolate to oatmeal is 2 to 1?

Discussion: What is the probability of a student choosing a chocolate chip cookie if students prefer chocolate to oatmeal 2 to 1?What is the probability of a student choosing an oatmeal cookie?

Solution: the probability of a student choosing a chocolate chip cookie if students prefer chocolate to oatmeal 2 to 1 is $\frac{2}{3}$, hence the probability of a student choosing an oatmeal cookie is $\frac{1}{3}$. The probability that exactly 3 of the next 5 students in line will select chocolate chip cookies is $(\frac{2}{3})^3(\frac{1}{3})^2 = \frac{8}{243}$. There are 10 ways to arrange these 5 students, so the probability is $\frac{80}{243}$.

Problem 10.36 (B)

The probability that two randomly selected integers from a set, have a product that is odd, is $\frac{1}{5}$. If the set include 3 odd integers, how many integers are there altogether?

Discussion: What integers are there so their product is odd?

Solution: The product of two integers can be odd if both are odd. If there are 3 of them, there are 3 odd products. This is $\frac{1}{5}$ of the total number of possible combinations. So there should be 15 of those and that means the set has 6 integers - 3 odd and 3 even.

Problem 10.37 (C)

Two distinct pairs of vertices are randomly selected from the vertices of a cube and connected by line segments. What is the probability that the segments intersect?

Discussion: How many ways to find the first pair? And the second pairs? What kind of segments can intersect?

Solution: Easy to count that there are 28 ways to choose the first pair, 15 ways to choose the second pairs. So there are $\frac{28.15}{2} = 210$ distance pairs. Since there are only 6 pairs intersect at the faces and 6 combinations of 4 diagonals that can intersect, the probability is $\frac{12}{210} = \frac{2}{35}$.

Problem 10.38 (C)

27 students each flip a fair coin. What is the probability that there are more heads showing than tails?

Discussion: How many heads would be there?

Solution: The probability to have a head and a tail is equal. The probability that there are more heads showing than tails is the same as if there are more tails showing than head. So it is $\frac{1}{2}$.

Problem 10.39 (C)

What is the probability that two real numbers selected at random between 1.0 and 4.0 have sum greater than 4?

Discussion: Let draw a graph of $x + y = 4$

Solution: Easy to see that the $x + y = 4$ will split the (1,1), (4,1), (4,4), (1,4) square into two parts, the region of the square on the top$x + y = 4$ will satisfy the requirement. It's $\frac{7}{9}$.

Puzzle 10.10: Clever parrot

The pet shop salesman told the buyer that "the parrot will repeat every word it hears." The customer bought it but the parrot would not speak a single word.

Do you think that the salesman lied?

Solution: the parrot was deaf.

Puzzle 10.20: Barber again

The barber in London says "I rather cut the hair of two Frenchmen than of one Englishman?"

Solution: Twice the pay is better than one.

Puzzle 10.30:

Alain said: Freeman owns more than 1000 books.

Betty said: He does not. He owns fewer than that.

Henry said: He surely owns at least one book.

If only one of the statement true, how many books does Freeman own?

Solution: Freeman owns 1000 books or no book.

Puzzle 10.40:

Danny and Diana sitting in front of the house.

"I am a boy", said the one with the black hair.

"I am a girl", said the one with the red hair.

If at least one of them is lying, who is which?

Solution: Both of them lying. The boy has red hair, the girl has black hair.