# Learning Problem Solving

for Middle- and High School Students

## Volume 2

Math & Chess Club in Ottawa

Nghia Doan & Catherine Doan

# Club Contest Problems 2020

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# Chapter 1

# Week 22

# Weekly Team Contest

## Problem 1

Clemence counts from 1 to 1000 using the fingers of her left hand as follows. She starts by calling her thumb 1, the first finger 2, middle finger 3, ring finger 4, and little finger 5. Then she reverses direction, calling the ring finger 6, middle finger 7, the first finger 8, and her thumb 9, after which she calls her first finger 10, and so on.

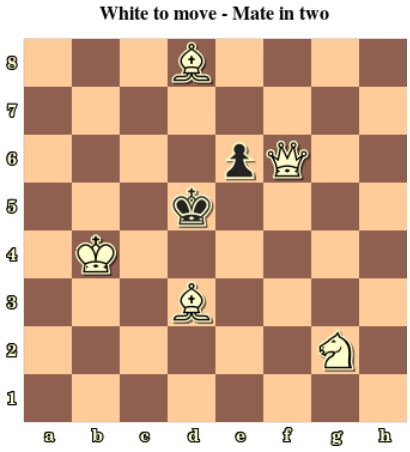
If she continues to count this way, on which finger will she stop?

## Solution

By carefully read the problem text, we can see that she started to count from the thumb to the first finger in step 1 and 2, and then again in step 9 and 10. It means that this is the pattern which repeats every 8 steps, where Clemence counts the first finger. Since 1000 is divisible by 8, so the counting ends when Clement counts the first finger.

## Problem 2

Solve the puzzle below.




## Solution

1 Qe7 Kc6 2 Be4#

if 1... Kd4 2 Qd6#

if 1... Ke6 2 Qc5#

if 1... e5 2 Qd7#

**Problem 3**

Cross out nine letters in the sequence of letters below in such a way that the remaining letter spell a single word.

N A I S N I E N L G E L T E T W E O R R S D

**Solution**

Cross the first and every other letters from the sequence

~~N~~ A ~~I~~ S ~~N~~ I ~~E~~ N ~~L~~ G ~~E~~ L ~~T~~ E ~~T~~ W ~~E~~ O ~~R~~ R ~~S~~ D

The result is as below:

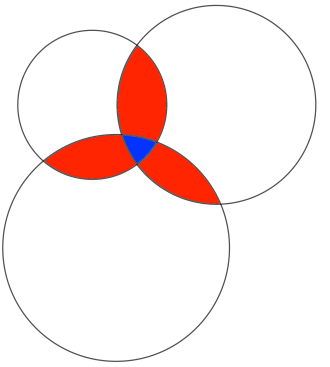
A SINGLE WORD

**Problem 4**

Three rugs have a combined area of 200 m². By overlapping the rugs to cover a floor area of 140 m², the area which is covered by exactly any two rugs is 24 m².

What is the are of floor is covered by all three rugs?

**Solution:**



The total area of the three rugs, not overlapping, is the total area of the three circles, which is 200 m². The area show in the diagram above, when the rugs are overlapping, has a total area of 140 m².

This means that $200 - 140 = 60$ is the area of three red parts (because they overlapped once) plus twice the blue part (because it is overlapped twice). Note that the area of the three red parts is 24 m². So the area of a blue part is $\dfrac{60 - 24}{2} = 18$ m².

**Problem 5**

Let $n$ be a positive integer. A *magic sequence* is created with $n$ as the first term. Then each term is obtained from the previous term as follows: if the previous term is even, the next term is half of the previous term. If the previous term is odd, the next term is 3 times the previous term plus 1. The sequence ends when it reaches 1.

Below are a five sequences when $n = 1, 2, 3, 4, 5$:

| $n$ | **The magic sequence** | **Number of terms** |
|---|---|---|
| **1** | 1 | 1 |
| **2** | 2, 1 | 2 |
| **3** | 3, 10, 5, 16, 8, 4, 2, 1 | 8 |
| **4** | 4, 2, 1 | 3 |
| **5** | 5, 16, 8, 4, 2, 1 | 6 |

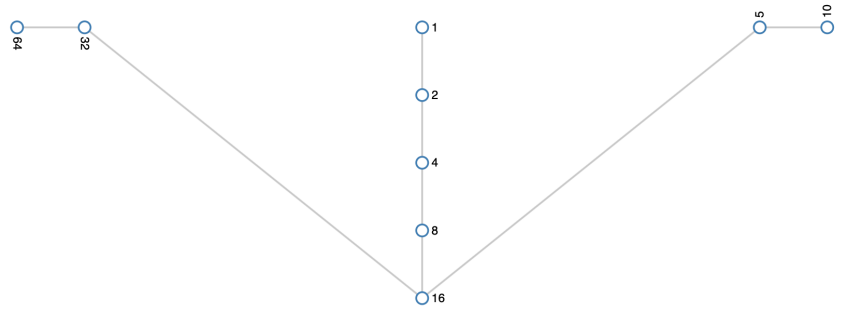
Find all $n \leq 70$ such that its magic sequence has at most 8 terms.

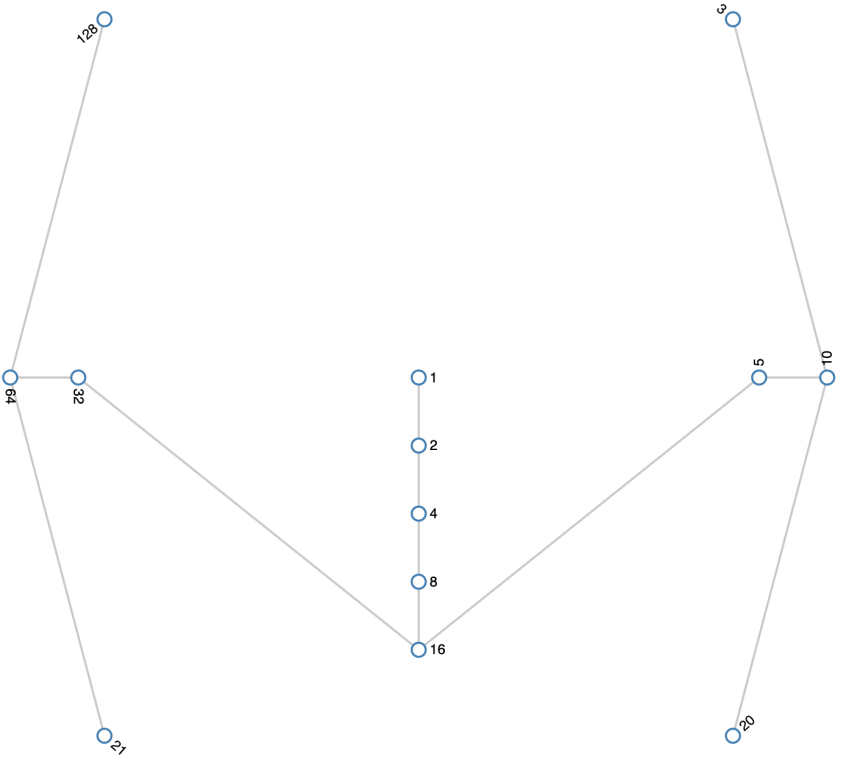
*Extra points for finding a clever algorithm to find those numbers.*

**Solution**

The problem basically can be rephrased to find all the numbers with in 7 *inverse* steps, from the number 1. An inverse step is to multiply the number by 2; or to subtract it by 1 and divide the result by 3 (if the result is still an integer).

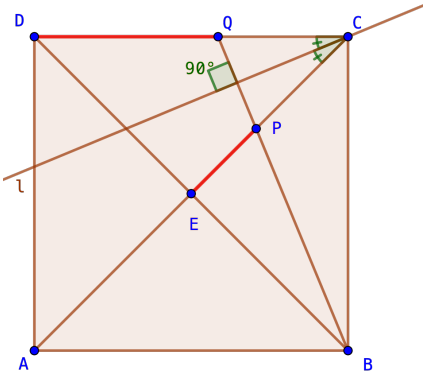
The diagrams show the solution for 7-steps and 8-steps.



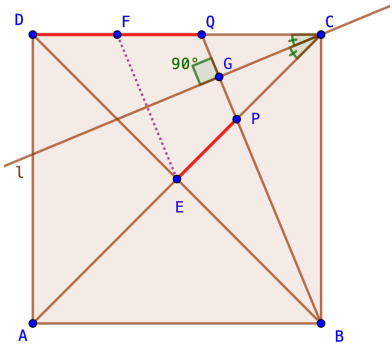
**Problem 6**

$ABCD$ is a square. $E$ is the intersection of the two diagonals $AC$ and $BD$. A line from $B$ perpendicular to the angle bisector $l$ of $\angle ACD$, intersecting $l$ at $P$ and $CD$ at $Q$.




Prove that $DQ = 2 \times EP$.

**Solution:**




$\triangle CGP \cong \triangle CGQ$ by ASA: $\angle PCG = \angle QCG$, $CG$ is shared, $\angle PGC = \angle QGC = 90°$). So $CQ = CP$.

Let $F$ be the midpoint of $DQ$. Since $E$ is midpoint of $DB$, so $EF \parallel BQ$ and $DQ = 2FQ$.

Furthermore $EF \perp CG$. Similarly $\triangle CHE \cong \triangle CHF$ by ASA, so $CF = CE$, hence $EP = FQ$.

Therefore $DQ = 2EP$.

# Problems of the Week

**Level A**

### Problem 1.A

There are 4 cards labeled 1, 1, 4, and 4 respectively. How many different 4-digit numbers can be made using these four cards?

Can you use the "Permutations with Repeated Elements" technique (*Introduction to Counting and Probability, Chapter 3, Section 3.2*) to prove your result?

**Solution**

The number of permutations is $4!$. Since there are two identical 1s and two identical 4s, so we must divide the number of permutations by the numbers of permutations of those identical cards.

Hence, the number of ways is $\dfrac{4!}{2!2!} = 6$.

### Problem 2.A

What is the value of $\dfrac{(2020^2 - 2019) \times 2021}{2020^2 - 2020 \times 2019 + 2019^2}$

Let $x = 2020$. Rewrite the above fraction as a fraction of expressions of $x$. Then use the identity $x^3 + 1 = (x+1)(x^2 - x + 1)$ to simplify it and prove the result obtained in the above question.

**Solution**

Let $x = 2020$.

$(2020^2 - 2019) \times 2021 = (x^2 - (x-1))(x+1) = (x^2 - x + 1)(x+1)$

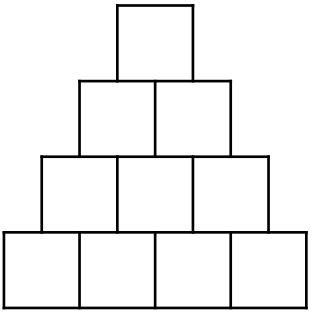
$2020^2 - 2020 \times 2019 + 2019^2 = x^2 - x(x-1) - (x-1)^2$

$= x^2 - (x^2 - x) + (x^2 - 2x + 1) = x^2 - x + 1$

The expression is $\dfrac{(x^2 - x + 1)(x+1)}{x^2 - x + 1} = x + 1 = 2021$

**Problem 3.A**

Squares with sides 1 centimetre long are arranged as below. Each row contains one more square than the row above it. How many centimetres are in the perimeter of the figure that has 10 rows?



How many centimetres are in the perimeter of the figure is contains exactly 210 squares?

**Solution**

The perimeter of a figure consist of:

- the top of 1 cm, and two sides each 1 cm
- the next row having two outside half side parallel with the top, and two sides each 1 cm, and so on
- the bottom having two outside half side parallel with the top, two sides each 1 cm, and as many as the number of squares in the bottom row.

So if we have the bottom having 10 squares, then there are 10 rows and the perimeter is: $3 \times 9 + (3 + 10) = 40$ cm.

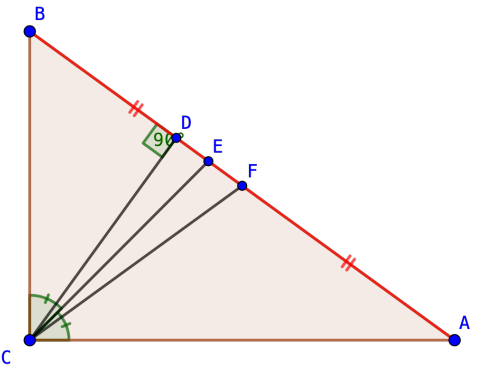
Now let $n$ be the number of rows, note that the number of squares in the figure is $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2} = 210$, then $n = 20$.

By counting similarly as above, the perimeter for a $n-$row figure is $3 \times (n - 1) + (3 + n) = 4n$.

Hence, it is $4 \times 20 = 80$.

**Problem 4.A**

In the right $\triangle ABC$, $CF$ is the median to the hypotenuse $AB$, $CE$ is the angle bisector of $\angle ACB$, and $CD$ is the altitude to $AB$. Prove that $\angle DCE = \angle ECF$.




Does the result still hold for a non-right triangle? Why or why not?

**Solution**

$\angle DCE = \angle BCE - \angle BCD$, $\angle ECF = \angle ACE - \angle ACF$. Because $CE$ is the angle bisector of $\angle ACB$, therefore $\angle BCE = \angle CAE$. So we can prove instead $\angle BCD = \angle ACF$.

$\triangle ACB$ is a right triangle, $CF$ is the median to the hypotenuse $AB$ so $CF = FA = FB$, hence $\triangle CFA$ is isosceles and $\angle ACF = \angle CAF$.

Now it is simple $\angle BCD = 90^\circ - \angle CBD = \angle CAF = \angle ACF$.

If $\triangle ACB$ is a right triangle, then $CF = FA = FB$ is not true any more and the equality will not hold.

## Level B

### Problem 1.B

How many distinguishable arrangements of the letters in OTTAWA have both the vowels last? For example TTOWAA is one such arrangement, but WOTATA is not.

Can you use the "Permutations with Repeated Elements" technique (*Introduction to Counting and Probability, Chapter 3, Section 3.2*) to prove your result?

**Solution**

We can assume all the letters are different, then divide by $2! \cdot 2! = 4$ at the end to get the answer since there are 2 T's and 2 A's.

There are $\binom{3}{2} = 3$choices to pick the last two letters out of the vowels assuming they are all different and there are $4! = 24$ ways to arrange the other 4 letters.

In total, we got $\dfrac{3 \times 24}{4} = 18$ possible permutations.

### Problem 2.B

Calculate $(2^1 + 1)(2^2 + 1) \ldots (2^{32} + 1) + 1$

Let $n = 32$, the number of terms of in the product of the expression above. Use the identity $x^2 - 1 = (x - 1)(x + 1)$ to simplify the expression and then prove the result obtained in the above question.

**Solution**

$x^2 - 1 = (x - 1)(x + 1)$, so $(2^n - 1)(2^n + 1) = 2^{n+1} - 1$, hence:

$$(2^1 - 1)(2^1 + 1) = 2^2 - 1$$
$$(2^2 - 1)(2^2 + 1) = 2^3 - 1$$
$$\ldots$$
$$(2^{32} - 1)(2^{32} + 1) = 2^{64} - 1$$

Therefore $(2^1 + 1)(2^2 + 1) \ldots (2^{32} + 1) + 1 = 2^{64} - 1$.

**Problem 3.B**

How many ways to write the number 3003 as a sum of two or more consecutive positive integers?

Find a formula for the sum of $k$ consecutive integers $n, n+1, \ldots, n+k-1$. Prove that this is a product of two terms that have different parity. Use that to prove the number of solutions in the above question.

**Solution**

The sum of $k$ consecutive integers $n, n+1, \ldots, n+k-1$ is:

$$n + (n+1) + \ldots + (n+k-1) = nk + \frac{(k-1)k}{2} = \frac{(2n+k-1)k}{2}$$

The two numbers $2n+k-1$ and $k$ have different parity (if one is odd then the other is even and vice versa).

So this means that 3003 if can be written as a sum of two or more consecutive integers, then it has to be a product of two odd integers. Because $3003 = 3 \times 5 \times 7 \times 11$, so it has 16 divisors or 16 ways to express it as a product of two odd integers.

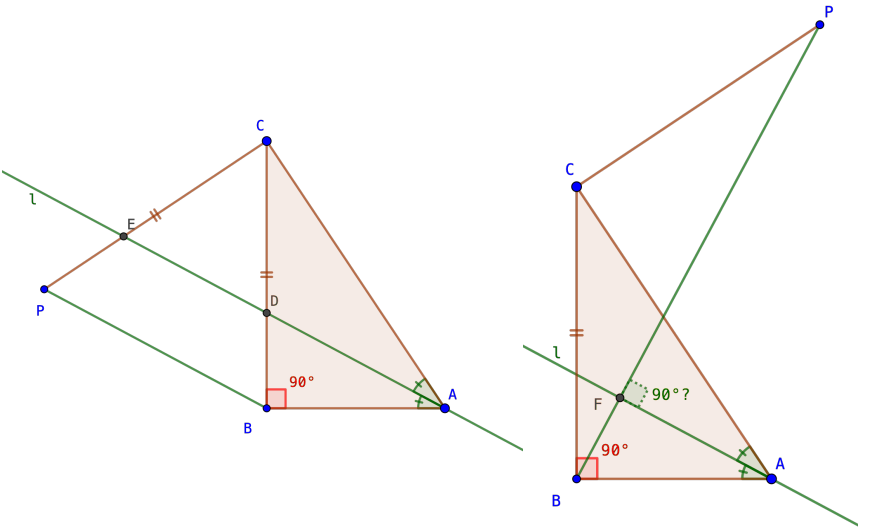
Note that we have to eliminate the trivial case $1 \times 3003$, so the number of solutions can only be 14.

**Problem 4.B**

$AC$ is the hypotenuse of a right $\triangle ABC$. The line through $AC$ divides the plane into two parts. A line segment $PC$ is perpendicular to $AC$ such that $P$ and $B$ are *in the same part of the plane* divided by $AC$. If $PC = BC$, show that $BP$ is parallel to the angle bisector $l$ of $\angle BAC$.

In the case the line segment $PC$ is perpendicular to $AC$ such that $P$ and $B$ are *in the opposite part of the plane* divided by $AC$. If $PC = BC$, show that $BP$ is perpendicular to the angle bisector $l$ of $\angle BAC$.

**Solution**




*First case* *Second case*

For the first case, let $D$, $E$ be the intersections of line $l$ bisecting $\angle A$ with $CB$, $CP$, respectively, so $\angle CAB = \angle CAE$. Since $CP \perp CA$ so $\angle ECA = \angle DBA = 90°$. So $\triangle CEA$ and $\triangle DBA$ are similar. Thus, $\angle EDC = \angle BDA \Rightarrow \angle EDC = \angle DEC$. Therefore $\triangle EDC$ is isosceles. Because $PC = BC$, so triangle CPB is also isosceles. Hence ED is parallel with PB.

For the second case, let E be the intersection of PB and AC. Similarly we can prove that triangle AEB is isosceles: $AE = AB$. Therefore the angle bisector AF is perpendicular with its base, which is also BP.

# Chapter 2

# Week 23

# Weekly Team Contest

## Problem 1

Six girls from MCC sit on a bench from left to right as below.

| | | | | | |
|---|---|---|---|---|---|
| Tiana | Chi | Lily | Bianca | Erica | Paridhi |

Sam can rearrange the girls by selecting any two girls who are not in alphabetical order, and switches them. For example:

| | | | | | |
|---|---|---|---|---|---|
| Tiana | Bianca | Erica | Chi | Lily | Paridhi |

What is **the least number of switches** Sam has to make in order to got all the girls to sit in full alphabetical order shown as below?

| | | | | | |
|---|---|---|---|---|---|
| Bianca | Chi | Erica | Lily | Paridhi | Tiana |

**Solution**

The minimal number of switches is 4. It is demonstrated below:

| | | | | | |
|---|---|---|---|---|---|
| Tiana | Chi | Lily | Bianca | Erica | Paridhi |
| Bianca | Chi | Lily | Tiana | Erica | Paridhi |
| Bianca | Chi | Erica | Tiana | Lily | Paridhi |
| Bianca | Chi | Erica | Paridhi | Lily | Tiana |
| Bianca | Chi | Erica | Lily | Paridhi | Tiana |

## Problem 2

Solve the puzzle at right

**Solution**

1. Re8+ Bxe8 2. Qg8#. If 1… Qxe8 Qxf6#

**Problem 3**

Kevin is invited to play a two-player game: two players take turn breaking a rectangular chocolate bar into pieces. The bar is 6 squares wide by 10 squares long. On his turn, a player snaps an existing piece of chocolate into two smaller pieces by breaking along the line between the squares. The player who cannot take a turn loses the game.

Since Kevin is invited, he can choose to go first or second. Can you suggest a **winning strategy** for Kevin?

**Solution**

Easy to see that a player who can take a turn if there is a piece with more than one square. What the player does it to brake it into two pieces. So the *number of pieces* is increased by one every turn. Since there are $6 \times 10 = 60$ squares, the maximum number of turns is 59.

It means that **the first player wins**.

**Problem 4**

Abma is an Austronesian language spoken in parts of the South Pacific island nation of Vanuatu by about 8000 people. Carefully study the translations below. *Note: there is no separate word for "the" or "he".*

| | |
|---|---|
| Mwami sileng. | He drinks water. |
| Nutsu mwatbo mwamni sileng. | The child keeps drinking water. |
| Nutsu mwegau. | The child grows. |
| Nutsu mwatbo mwegalgal. | The child keeps crawling. |
| Mworob mwabma. | He runs here. |
| Mwerava Mabontare mwisib. | He pulls Mabontare down. |
| Mabontare mwisib. | Mabontare goes down. |
| Mweselkani tela mwesak. | He carries the axe up. |
| Mwelebte sileng mwabma. | He brings water. |
| Mabontare mworob mwesak. | Mabontare runs up. |
| Sileng mworob. | The water runs. |

**Translate** the following into Abma (*"mwegani" means "eat"*):

i. He goes down.

ii. The child keeps eating.

iii. Mabontare keeps growing upwards.

And the following sentences into English:

iv. Sileng mworob mwisib.

v. Mwelebte nutsu mwesak.

**Solution**

| | |
|---|---|
| He goes down. | Mwisib. |
| The child keeps eating. | Nutsu mwatbo mwegani. |
| Mabontare keeps growing upwards. | Mabontare mwatbo mwegau mwesak. |
| Sileng mworob mwisib. | The water runs down. |
| Mwelebte nutsu mwesak. | He brings the child up. |

**Problem 5**

A *magic harshad number* is a positive integer such that:

- All of its digits are different.
- None of its digits is 0.
- The number is divisible by the sum of its digits.

*Note: The word "harshad" comes from the Sanskrit harṣa (joy) + da (give), meaning joy-giver.*

Below are a few magic *harshad* numbers:

| The number of digits | An example of Harshad number |
|---|---|
| **1** | 1 |
| **2** | 24 |
| **3** | 132 |

For each $4 \leq n \leq 9$ find a magic *harshad* $n - digit$ number.

**Solution**

The key idea is to select only those digits whose sum is an integer with easy-to-dealt-with prime divisors as 2, 3, 5.

$1365 \ / \ 15 = 91$

$12345 \ / 15 = 823$

$124675 \ / \ 25 = 4987$

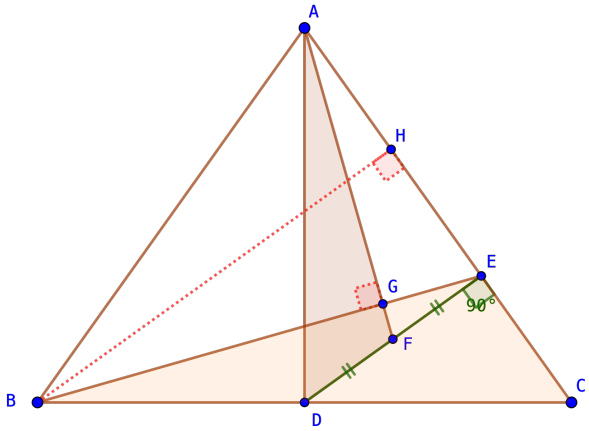
$3579624 \ / \ 36 = 99434$

$12356748 \ / \ 36 = 343243$

$987643215 \ / \ 45 = 21947627$

**Problem 6**

$\triangle ABC$ is isosceles with $AB = AC$. Drop a perpendicular from $A$ to $BC$, meeting $BC$ at $D$. Then drop a perpendicular from $D$ to $AC$, meeting it at $E$. $F$ is midpoint of $ED$.




i. Drop a perpendicular from $B$ to $AC$, meeting $AC$ at $H$. Prove that $\triangle BHC \sim \triangle AED$.

ii. Prove that $\triangle BEC \sim \triangle AFD$.

iii. Prove that $AF \perp BE$.

**Solution**

$\triangle BHC \sim \triangle AED$ by $AAA$ because $\angle BHC = \angle AED = 90°$ and $\angle BCH = \angle ACD = 90° - \angle DAC = \angle ADE$.

$E$ and $F$ are midpoints of $CH$, $DE$ in $\triangle BHC$, $\triangle AED$.

Since $\triangle BHC \sim \triangle AED$ so $\triangle BEC \sim \triangle AFD$.

So $\angle DAF = \angle CBH$ and $AD \perp BC$, hence $AF \perp BE$.

# Problems of the Week

**Level A**

### Problem 1.A

There are five cards labeled 1, 2, 2, 3, and 3, respectively. How many different three-digit numbers can be formed by using any three of five cards?

Now you got an additional card with label 3. Answer the same question.

**Solution**

If all three digits are different. There're $3! = 6$ permutations of the number $123$. If two digits are the same: $221, 223, 331, 332$. The number of permutation for each is $\frac{3!}{2!} = 3$. So there are $4 \times 3 = 12$. All together there are 18 such numbers.

The second question is very much like the first, except that there is a possibility of have a number with all digits are the same. Therefore the number of them is 18.

*Note: The real question here is the different numbers formed by the cards, not the ways the cards can be used to form those numbers.*

### Problem 2.A

What is the value of $\dfrac{20202019^2}{20202018^2 + 20202020^2 - 2}$

Let $n = 20202019$. Rewrite the above fraction as a fraction of expressions of $n$. Simplify it and prove the result obtained in the above question.

**Solution**

Let $n = 20202019$. The expression can be written as:

$$\frac{n^2}{(n-1)^2 + (n+1)^2 - 2} = \frac{n^2}{n^2 - 2n + 1 + n^2 + 2n + 1 - 2}$$

$$= \frac{n^2}{2n^2} = \frac{1}{2}$$

**Problem 3.A**

What is the number that will be listed *beneath* the number 121 when this triangular array is continue in the same manner?

| | | | | |
|---|---|---|---|---|
| | | 1 | | |
| | 2 | 3 | 4 | |
| 5 | 6 | 7 | 8 | 9 |

Where will the number 2020 appear?

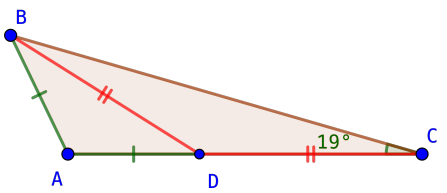
**Solution**

Easy to see that the last number of each row is a perfect square. In fact the last number of the $n^{th}$ row is $n^2$, which has $2n - 1$ numbers. This means that the number beneath 121 is on the $12^{th}$ row, preceding the last number of that row, which is 144. Hence the number is 143.

For the second question: $44^2 = 1936 < 2020 < 2025 = 45^2$. So 2020 is on the $45^{th}$ row. This row starts with $44^2 + 1 = 1937$, so the position of 2020 is $2020 - 1937 + 1 = 84$.

**Problem 4.A**

$AB = AD$, $BD = CD$, $\angle C = 19°$. What is $\angle A$?




Now if $D$ is on the line through $AC$ but not on the same of A as C. Draw the figure and determine what is $\angle BDA$?

**Solution**

If D is between AC: $\angle DBC = \angle DCB = 19°$, so $\angle BDA = 38°$. Hence $\angle BAD = 180° - 2 \cdot 38° = 104°$.

If A is between DC, if we consider $BD = CD$, then A is a point on DC but $AB = AD$, it is impossible because $\angle BDA > 90° > \angle BAD$.

## Level B

### Problem 1.B

Seven MCC students are sitting in a row. Four of them: Anthony, Binh, Chi, and Dan change their position while the other three students remain on the same seats. How many different ways are there to do so?

Can you use the the learned counting techniques to prove your result instead of listing all of possible cases?

**Solution**

4 students have $4!$ ways to permute the ways they sit. That means $4! - 1 = 23$ ways to *change their positions* while other students remain on the same seats.

### Problem 2.B

Calculate $\dfrac{2019^3 - 2018(2019^2 + 2020)}{2018(2017^3 - 2016) - 2017^3}$

Let $x = 2019$, $y = 2017$, simplify the expression and then prove the result obtained in the above question.

**Solution**

$x = 2019$, $y = 2017$. Substitute them into the expression:

$$\frac{2019^3 - 2018(2019^2 + 2020)}{2018(2017^3 - 2016) - 2017^3} = \frac{x^3 - (x-1)(x^2 + x + 1)}{(y+1)(y^3 - (y-1) - y^3}$$

$$= \frac{x^3 - (x^3 - 1)}{y^4 + y^3 - y^2 + y - y + 1 - y^3} = \frac{1}{y^4 - y^2 + 1}$$

$$= \frac{1}{116\ 550\ 971\ 319\ 233}$$

**Problem 3.B**

The sequence shown below was formed by writing the first letter of the alphabet followed by writing the first two letters of the alphabet and continuing the pattern by writing one more letter of the alphabet each time.

$$A, A, B, A, B, C, A, B, C, D, A, B, C, D, E, \ldots$$

Continuing this pattern, what letter is the $280^{th}$ letter in this sequence? What position is the letter Z? (*the alphabet contains 26 letters*)

**Solution**

Note that we have a series of sequence with length 1, 2, 3, …:

$$\underbrace{A}_{1}, \underbrace{A, B}_{2}, \underbrace{A, B, C}_{3}, \underbrace{A, B, C, D}_{4}, \underbrace{A, B, C, D, E}_{5}, \ldots$$

The number of letters between the $23^{rd}$ and $24^{th}$ sequences:

$$1 + 2 + \ldots + 23 = \frac{23 \cdot 24}{2} = 276 < 300 = \frac{24 \cdot 25}{2} = 1 + 2 + \ldots + 24$$

So the $280^{th}$ letter is in the $24^{th}$ sequence, which starts with the $277^{th}$, so the $280^{th}$ is D. The position of letter Z is simple, it is at the end of the $26^{th}$ sequence:

$$1 + 2 + \ldots + 26 = \frac{26 \cdot 27}{2} = 351$$

**Problem 4.B**

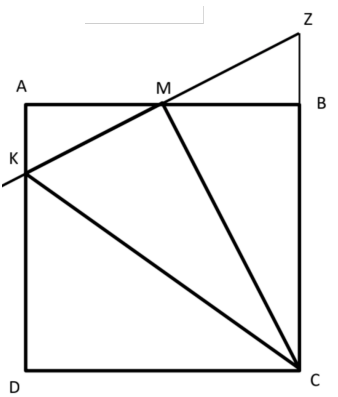
In square $ABCD$, $M$ is midpoint of $AB$. A line perpendicular to $MC$ at $M$ meets $AD$ at $K$. Draw the diagram and prove that $\angle BCM = \angle KCM$.

Prove that the ratios $KD : DC : KC = 3 : 4 : 5$.

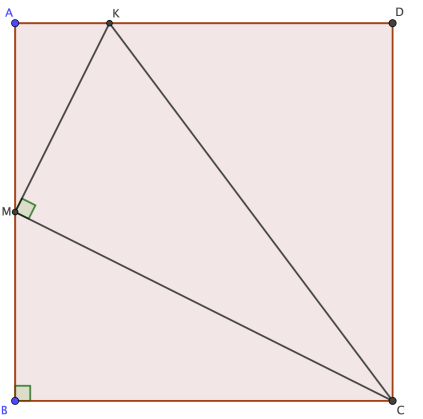
**Solution 1** *(First question)*

Extend $MK$ to meet $CB$ at $Z$. $\angle AMK = \angle ZMB$, because they are both vertical angles. $\angle MBZ = \angle MAK = 90°$. In addition, $AM = MB$ Using ASA we can determine that $\triangle AMK \cong \angle MZB$.

This means $KM = MZ$. Since $\angle CMZ = 90°$, using SAS we can determine that $\triangle KMC \cong \triangle CMZ$. Hence $\angle KCM = \angle MCZ$.




*First question*




*Second question*

**Solution** *(both questions)*

$\angle AMK = \angle MCB$, $\angle KAM = \angle MBC = 90°$

$\Rightarrow \triangle MBC \sim \triangle KAM$. Therefore $\dfrac{KA}{AM} = \dfrac{MB}{BC} = \dfrac{1}{2}$.

So $BC = 2MB = 2MA = 4KA$, $MC^2 = MB^2 + BC^2 = 20AK^2$, $MK^2 = AK^2 + AM^2 = 5AK^2$. $MC = 2MK$, $\angle MBC = \angle KMC = 90°$:

$$\triangle MCK \sim \triangle BCM \Rightarrow \angle MCB = \angle MCK$$

$\triangle KDC$ is a right triangle and the ratio of the two legs:

$$DK : DC = 3 : 4 \Rightarrow DK : DC = 3 : 5$$

# Chapter 3

# Week 24

# Weekly Team Contest

## Problem 1

Two missiles fly in a direct line towards each other, one at 12000 miles per hour and the other at 18000 miles per hour. They start 12583 miles apart.

How far (in miles) they are 3 seconds before they collide?

**Solution**

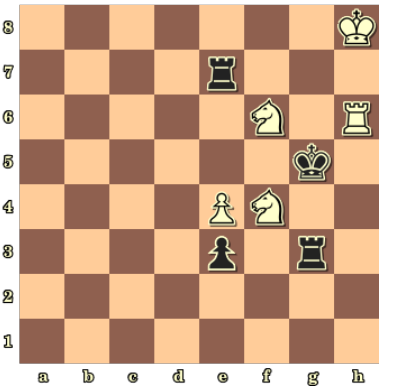
Since they fly directly towards each other, for every hour the distance between them will be reduced 30000 miles. So for every second the distance is decreased by $\frac{30000}{3600} = \frac{300}{36} = \frac{25}{3}$ miles.

In 3 seconds before they collide, they are **25 miles** apart.

## Problem 2

Solve the puzzle below.

**White to move. Mate in two**




**Solution**

1. Rh4+ Kxf4 or Kxf6
2. Rf5#

**Problem 3**

Figure 1 is called a "stack map." The numbers tell how many cubes are stacked in each position. Figure 2 shows these cubes, and Figure 3 shows the view of the stacked cubes as seen from the front.

| | |
|---|---|
| 3 | 4 |
| 2 | 1 |

Figure 1

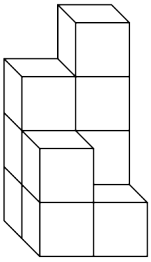


Figure 2

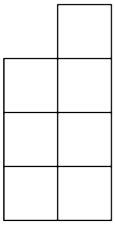
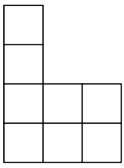


Figure 3

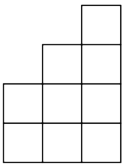
| | | |
|---|---|---|
| 2 | 2 | 4 |
| 1 | 3 | 1 |

Figure 4

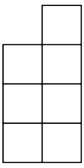
Which one below is the front view for the stack map in Figure 4? Remember, you have to show your proof.



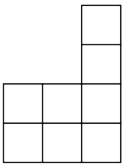
(A)



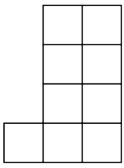
(B)



(C)



(D)



(E)

**Solution 1**

In each 'column' of the map, you take the highest stack and those will be the stacks of cubes we see from the front. The winning combination is 2, 3, then 4, so our front view graph is (B)

**Solution 2**

From the left-most column of figure 4, only B or D are possible. From the second column, only B is possible. We can use the third column to verify. So the answer is B.

**Problem 4**

Professor Merle White of the mathematics department, Professor Leslie Black of philosophy, and Jean Brown, a young stenographer who worked in the university's office of admissions, were lunching together.

'Isn't it remarkable,' observed the lady, 'that our last names are Black, Brown and White and that one of us has black hair, one brown hair and one white.'

'It is indeed,' replied the person with black hair, 'and have you noticed that not one of us has hair that matches his or her name?'

'By golly, you're right!' exclaimed Professor White.

If the lady's hair isn't brown, what is the colour of Professor Black's hair?

**Solution 1**

The colour of Professor Black is white because the last part of the conversation shows that Professor White was talking to a person with black hair, which means she can't have black nor white hair, Professor White has brown hair. Since Professor Black can't have black hair, her hair is white.

**Solution 2**

We've gathered three key pieces of information: *1. No one has the hair colour of their name; 2. Each hair colour only appears once, 3. Professor White doesn't have black hair.*

Professor White doesn't have black nor white hair so she has brown hair. Leslie Black doesn't have black hair, and since Professor White already has brown hair Leslie doesn't have brown either. Hence, she has white hair.

**Solution 3**

The assumption that the "lady" is Jean Brown, the stenographer, quickly leads to a contradiction. Her opening remark brings forth a reply from a person with the black hair, therefore Brown's hair cannot be black. It also cannot be brown, so it must be white. This leaves brown for the colour of Professor Black's hair and black for Professor White. However the statement from the black person cannot prompt an exclamation from White. This is a contradiction.

So Jean Brown is a man. Professor White's hair cannot be white and cannot be black, so it must be brown. If the lady's hair isn't brown, then Professor White is not a lady. Brown is a man, so Professor Black is the lady. Her hair cannot be black, or brown, so **it is white**.

**Problem 5**

Two players play a game with the following 12 numbers:

1, 2, 4, 5, 6, 8, 9, 11, 12, 13, 16, 17

The players take turns to select the numbers from the set. Each turn a player can select only one number. The selected number then is removed from the set. After 12 turns, each player ends up with 6 numbers.

Now each player works separately by **multiplying** their numbers together and then calculate **the sum of the digits** of this product. From this sum, the player calculates again the sum of the digits as well. The player keeps doing it until the result is a single digit.

When done they compare the results. The player with a larger digit wins the game.

Can you suggest a **winning strategy** for Camille?

*Note: A strategy must includes whether she goes first or second and what numbers should she try to select.*

**Solution**

The sum of a number that is divisible by 9 is always divisible by 9. This means that if the number of a player is divisible by 9 then the *invariant* of the operation is *the divisibility by 9 of the number*. This also means that the single digit that the player gets is 9.

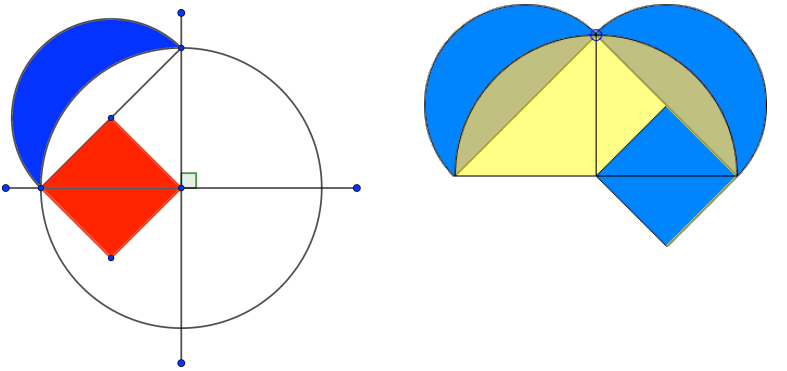
The strategy can be simple: Camille must select the numbers such that the product of those numbers is divisible by 9 and the product of the numbers that her opponent has is not divisible by 9.

Among the 12 given numbers, only 9 is divisible by 9, and the other numbers 6, 12 - is divisible by 3, but not 9.

**Camille needs go first, to select the number 9, and at the second round** one of 6, 12. Whatever else the players choose afterwards, Camille will win.

**Problem 6**

Hippocrates of Chios (c. 470 - c. 410 BCE) has proved that the blue ***lune*** (intersection of a circle and a half-circle) and the red square have the same area in the figure below on the left side.



Can you explain his proof *without showing any calculation* by using the diagram above on the right side?

**Solution**

According to the Pythagorean theorem, the sum of two half circles on the legs of the right triangle is equal to the half yellow circles.

If we *remove the intersection between these half circles*, then the yellow triangle has the same area of the sum of the two blue lunes. Since the blue square is half of the yellow triangle, so it is equal to a lune.

# Problems of the Week

**Level A**

### Problem 1.A

Anthony, Bianca, and Catherine are sitting in a row with the first to third places starting from left to right in that order. How many ways are there to sit them so that Anthony does not sit in the first place, Bianca does not sit in the second place, and Catherine does not sit in the third place?

Now David joins the team and the above problem is extended to four people. Can you use the solution in the previous problem to solve the extended one?

**Solution**

If Anthony does not sit on the first place, then he can sit only on the second or third place. If he sits on the second place, then Catherine can sit only on the first place. If he sits on the third place, then Bianca can sit only on the first place. In each case, there is only one way Bianca and Catherine can sit so that they do not sit on undesired places. Hence, there are two ways to sit.

Again, Anthony can sit on only one of the second, third, or fourth places. If he sits on the second place, in each case when Bianca, Catherine, or David takes the first place, there is only one possible arrangement as below.

| | | | |
|---|---|---|---|
| Bianca | Anthony | David | Catherine |
| Catherine | Anthony | David | Bianca |
| David | Anthony | Bianca | Catherine |

Therefore if Anthony choose a place other than the first place to sit, there are always three possible arrangements for Bianca, Catherine, and David to sit in the other places.

Hence the number of arrangements is $3 \times 3 = 9$.

**Problem 2.A**

Prove that: $(x-1)(x^9+x^8+\ldots+x+1)=x^{10}-1$

What is the value of $x$ if $\dfrac{x^9+x^8+\ldots+x+1}{x^4+x^3+\ldots+x+1}=1025$

*Hint: use the identity* $a^2-1=(a-1)(a+1)$

**Solution**

By expanding the product $(x-1)(x^9+x^8+\ldots+x+1)$

$=x^{10}-x^9+x^9-x^8+\ldots+x-1=x^{10}-1$

$x^9+x^8+\ldots+x+1=\dfrac{x^{10}-1}{x-1}, x^4+x^3+\ldots+x+1=\dfrac{x^5-1}{x-1}$

Hence $\dfrac{x^9+x^8+\ldots+x+1}{x^4+x^3+\ldots+x+1}=\dfrac{x^{10}-1}{x^5-1}=x^5+1\Rightarrow x^5=1024.$

Therefore $x=32$.

**Problem 3.A**

The sum of two prime numbers is 49. What is the sum of the reciprocals of the two prime numbers? *Hint: Use parity.*

$a$, $b$, and $c$ are prime numbers such that $a+b+c=66$ and $ab+bc+ca=1071$. What is $abc$?

**Solution**

The sum of two primes is odd means that one of them is even. This means that one of the primes is 2 and the other is 47. Hence the sum of their reciprocals is $\dfrac{1}{2}+\dfrac{1}{47}=\dfrac{49}{94}$.

$a+b+c=66$ means that one of the primes has to be even Without loss of generality, let assume $a=2$. Substitute this into two given equations:

$$\begin{cases} b+c=64 & (1)\\ 2b+bc+2c=1071 & (2)\end{cases}$$

Again, substitute the sum of $b+c=46$ into (2):

$$2\cdot 64+bc=1071\Rightarrow bc=943=23\cdot 41$$

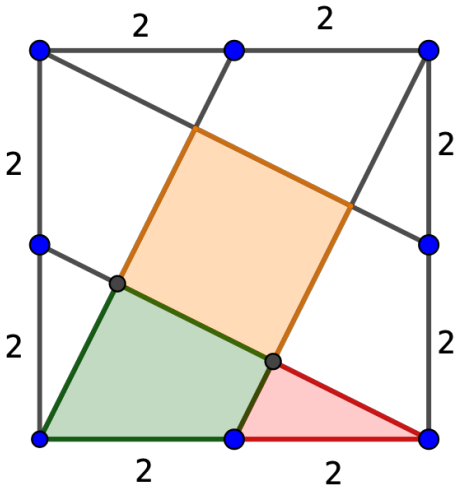
Therefore (2, 23, 41) is the solution: $abc=2\cdot 23\cdot 41=1886$.

**Solution 2** *(for the second question)*

The sum of three primes is even, so let $a=2$, so $b+c=64$ and $ab+bc+ca=1071$. Thus , $bc=943$ and $abc=2\cdot 943=1886$.

**Problem 4.A**

The vertices of a square with side length 4 are connected to the midpoints of the opposite edges as show below.




What is the area of the brown square? Of the red triangle?

**Solution**

Easy to see that the sum of the areas of the green trapezoid and the red triangle is the area of the brown square. Therefore the area of the brown square is $\frac{4 \cdot 4}{5} = \frac{16}{5}$.

The area of the red triangle is a quarter of the area of the triangle made by the green trapezoid and the red triangle or a quarter of the area of the brown square. Hence, the area of the red triangle is $\frac{4}{5}$.

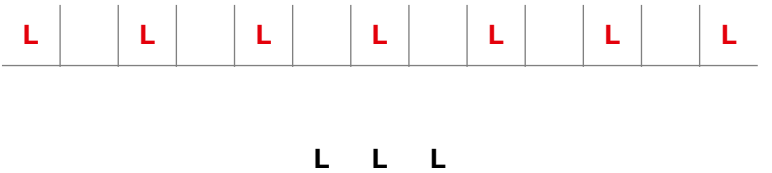
## Level B

### Problem 1.B

There are 10 street lamps labeled 1, 2, 3, …, 10. Each is in front of a house in a row of houses. All of them are turned on. How many ways are there to turn off three lamps such that no two or three adjacent lamps will be turned off and no any lamp in each end will be turned off?

Show a solution that is not based on casework.

**Solution**




In the figure above, the first row consists of seven lamps turned on. A required arrangement lamps is equivalent to a way to *insert* the *three lamps* in the second row into *six empty slots* in the first row. So the number of possible arrangements is $\binom{6}{3} = 20$.

### Problem 2.B

$$A = \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2019}\right)\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2020}\right)$$
$$B = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2019}\right)\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2020}\right)$$

Calculate $A - B$. Can you generalize the result?

**Solution** by **Anthony Ly, Catherine Doan**

Let $C = \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2019}\right)$, then:

$$A = C\left(1 + C + \frac{1}{2020}\right) = C + C^2 + \frac{C}{2020}$$

$$B = (1 + C)\left(C + \frac{1}{2020}\right) = C + \frac{1}{2020} + C^2 + \frac{C}{2020}$$

Therefore $A - B = -\frac{1}{2020}$. The general result is $A - B = -\frac{1}{n}$

**Problem 3.B**

Find all pairs of positive integer $(m, n)$: $m^2 - n^2 = 13$.

Find the value of a positive integer $k$ such that, when 64 is taken away from it, the result is a perfect square; and when 25 is added to it, the result is also a perfect square.

**Solution**

$m^2 - n^2 = 13 \Rightarrow (m+n)(m-n) = 13 = 13 \cdot 1$

$m + n = 13, m - n = 1 \Rightarrow m = 7, n = 6$

$k - 64 = n^2, k + 25 = m^2 \Rightarrow m^2 - n^2 = 64 + 25 = 89 = 89.1$

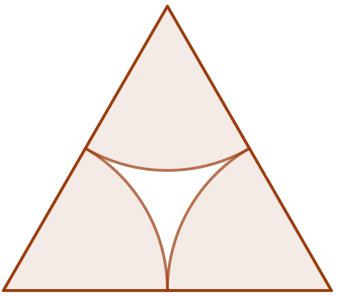
$m + n = 89, m - n = 1 \Rightarrow m = 45, n = 44.$

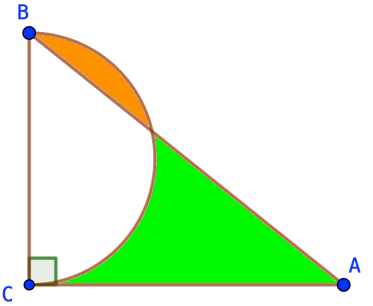
So $k = 44^2 + 64 = 2000$

**Problem 4.B**

Three circles with radius 1 and center at the vertices of an equilateral triangle with side length 2 divide the triangle into 4 regions as shown in the diagram on the left.

What is the area of the unshaded region?






$\triangle ABC$ is a right triangle, $\angle C = 90^\circ$, $BC = 20$. The half-circle with diameter $BC$ split the triangle as shown in the diagram on the right. The green area is larger than the brown area by 20.

Find the length of $AC$.

**Solution**

Easy to see that the unshaded region is the difference between the area of the triangle and three times the area of a sixth of a circle. The area of an equilateral triangle side length $a$ is $\dfrac{a^2\sqrt{3}}{4}$, and the area of a circle length $r$ is $\pi r^2$. Hence the unshaded region has an area of:

$$\frac{2^2\sqrt{3}}{4} - 3 \cdot \frac{1}{6}\pi = \sqrt{3} - \frac{\pi}{2}$$

Let the area of the brown region are is $b$, the area of the green region is $g$, the white area is $w$. We know that:

$$\begin{cases} g + w = \frac{BC\,.\,AC}{2} \\ b + w = \frac{1}{2}\pi\left(\frac{BC}{2}\right)^2 \end{cases}$$

Subtract the second from the first equation, we have:

$$g - b = 5AC - 50\pi \Rightarrow 5AC = 50\pi + 20 \Rightarrow AC = 10\pi + 4$$

# Chapter 4

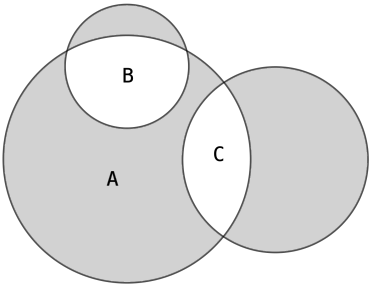
# Week 25

# Weekly Team Contest

## Problem 1

Three circles with radius 2, 3 and 4 are overlapping each other. The largest circle is partitioned into shaded region A and unshaded regions B and C.

If the total area of the shaded region is $17\pi$, what is the area of region A in terms of $\pi$?




## Solution

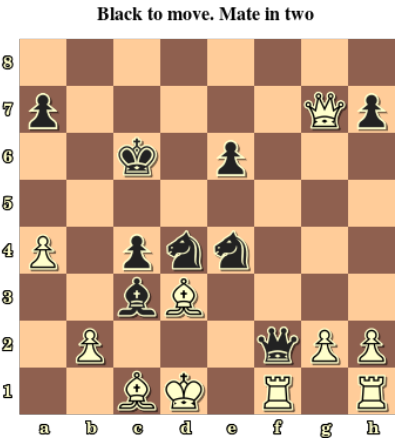
The total area of the circles is the area of the shaded region plus *twice* the area of the unshaded regions:

$$4\pi + 9\pi + 16\pi = 17\pi + 2(B + C) \Rightarrow B + C = 6\pi.$$

Hence $A = 16\pi - (B + C) = 10\pi$.

## Problem 2

Solve the puzzle below.

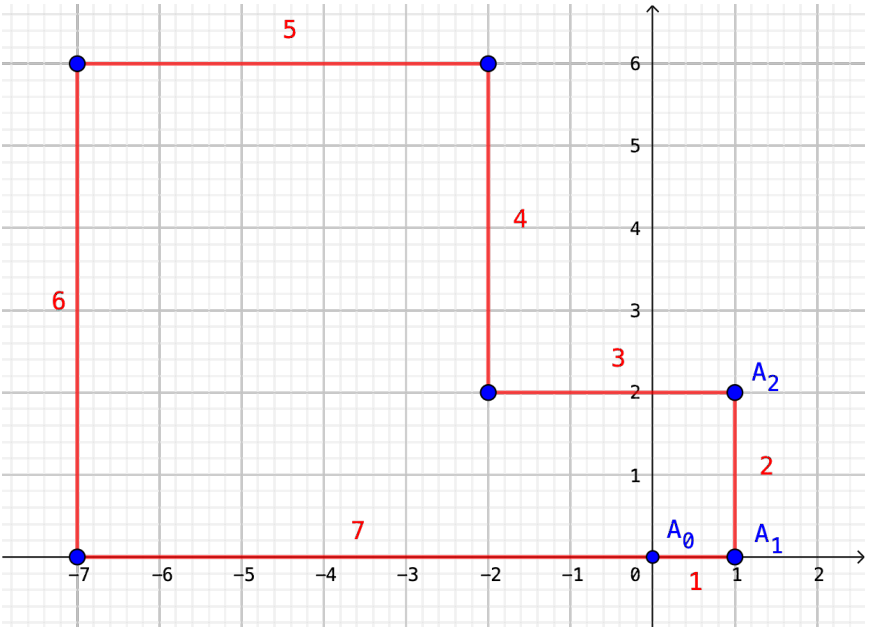



## Solution

1... Qe1+ 2. Rxe1 Nf2#

**Problem 3**

Line segments from a path that starts at $A_0$ (0, 0), is drawn to $A_1$ (1, 2) and then $A_2$ (1, 0). Each new segment form a **right angle** with the previous segment and is **1 unit longer** than that segment. The path ends at $A_0$(0, 0).




How many segments are in the **shortest possible path**?

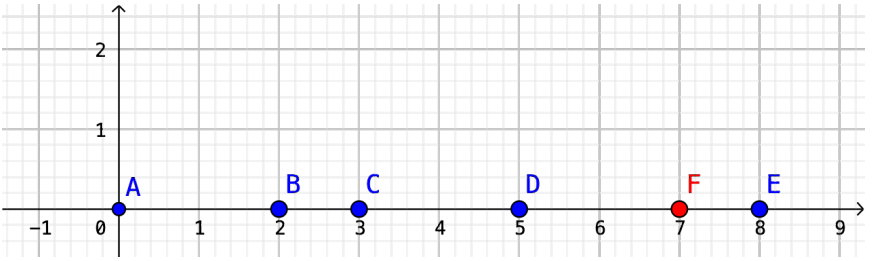
**Solution**

The path can turn either left or right, and each horizontal segment has an odd length while each vertical segment has an even length. So, the shortest possible sequence of horizontal segments leading back to the starting point can be 1, 3, 5, 7 since $1 + 7 = 3 + 5$.

That means the number of segments is 7.

**Problem 4**

Nam plays a game with some friends. Five of his friend stand in a line at points $A$, $B$, $C$, $D$, $E$ as shown below. He has to find the place on the line to stand.




If he stand at F, the total sum of distances from him to all of his friend is $7 + 5 + 4 + 2 + 1 = 19$.

Where should he stand so that **the total sum of distances from him to all of his friend is minimal**?

**Solution**

First, Nam has to stand somewhere between A and E. Standing outside will make the total distances from him to A and E larger. Second, Nam has to stand somewhere between B and D. Standing outside will make the total distances from him to B and D larger. Last, standing at C will make his distance to C minimal.

So he has to stand at C.

**Problem 5**

A stick 100 cm long needs to be cut into 100 shorter sticks, each is 1 cm long.

What is the **minimum number of cuts** required if Dinh Viet are *allowed to cut several stick pieces* at the same time?

**Solution**

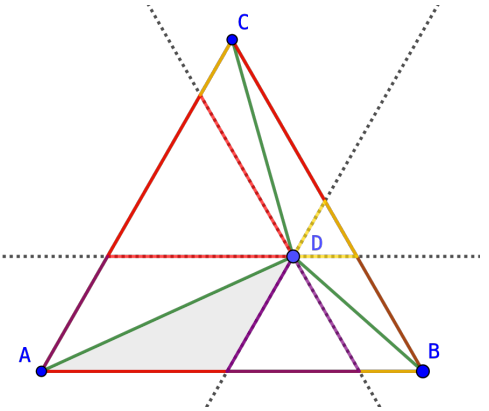
Viet has to cut the stick into 100 pieces. With each cut, he can double the number of pieces. Since:

$$2^6 = 64 < 100 < 128 = 2^7$$

So he has to make at least 7 cuts. It can easily be demonstrated.

**Problem 6**

$\triangle ABC$ is an equilateral triangle. $D$ is a point inside $\triangle ABC$.




i. Prove that $DA < DB + DC$

ii. Prove that $DA + DB + DC < AB + AC$

**Solution**

Colour the segments on the parallel lines through D as above. Since in any triangle the sum of two side lengths is larger the the third.

Therefore for the shaded triangle, *side AD is smaller than the sum of the* ***red*** *and the* ***purple*** *segment:*

$$AD < red + purple$$

Since $BC = red + yellow + purple$:

$$AD < red + purple < red + yellow + purple = BC < DB + DC$$

Similarly:

$$DB < yellow + purple$$

$$DC < red + yellow$$

Hence:

$$DA + DB + DC < 2(red + yellow + purple) = AB + AC$$

# Problems of the Week

**Level A**

**Problem 1.A**

How many triples of positive integers have the product of 24? For example (1, 1, 24) is such a triple.

How many triples of ***distinct*** positive integers have the product of 120? For example (1, 2, 60) is such a triple, but (1, 1, 120) is not.

**Solution**

It's simple case work. We look at the largest number of the triple whose product is 24. This number is a divisor of 24.

- If it is 24: the triple is (1,1,24).
- If it is 12: the triple is (1,2,12).
- If it is 8: the triple is (1,3,8).
- If it is 6: there are two triples (1, 4, 6), and (2, 2, 6).
- If it is 4: the triple is (2, 3, 4).

So the total number of triples is 6.

Now let take a similar approach to the largest number of the triple whose product is 120. Easy to see that it cannot be 120.

- If it is 60, then the triple is (1, 2, 60).
- If it is 40, then the triple is (1, 3, 40).
- If it is 30, then the triples are (1, 4, 30), and (2, 2, 30).
- If it is 24, then the triple is (1, 5, 24).
- If it is 20, then the triple are (1, 6, 20), and (2, 3, 20).
- If it is 15, then the triple are (1, 8, 15), and (2, 4, 15).
- If it is 12, then the triple are (1, 10, 12), and (2, 5, 12).
- If it is 10, then the triple are (2, 6, 10), and (3, 4, 10).
- If it is 8, then the triple are (3, 5, 8)

So the total number of triples in this case is 14.

**Problem 2.A**

$xy + x + y + 1$ can be factored as below:

$$xy + x + y + 1 = (xy + x) + (y + 1) = x(y + 1) + (y + 1) = (x + 1)(y + 1)$$

Factor $xyz + xy + yz + zx + x + y + z + 1$.

Factor $1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xyz}$

**Solution**

$$xyz + xy + yz + zx + x + y + z + 1$$
$$= z(xy + x + y + 1) + (xy + x + y + 1)$$
$$= z(x + 1)(y + 1) + (x + 1)(y + 1) = (x + 1)(y + 1)(z + 1)$$
$$1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xyz}$$
$$= \left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) + \frac{1}{z}\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right)$$
$$= \left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right)\left(1 + \frac{1}{z}\right) = \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{z}\right)$$

**Problem 3.A**

How many positive integers that are divisors of 56 and also multiples of 14? Prove your result.

How many positive integers less than 10000 do not have an odd number of divisors? Prove your result.

**Solution**

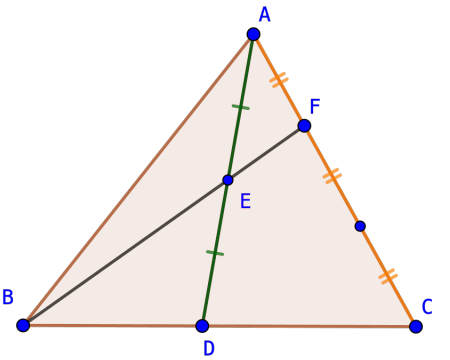
Multiples of 14 are 14, 28, 42, 56, … Only14, 28, and 56 are divisors of 56. So the number of them is 3.

Using complementary counting technique, instead of looking for such numbers, we look for number that have an odd number of divisors. Recall that if $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}$ be the prime factorization of a number $n$, then its number of divisors is $(k_1 + 1)(k_2 + 1)\cdots(k_m + 1)$.

This product can be an odd number if and only if each $k_1, k_2, \cdots, k_m$ is even integer. This means $n$ is a perfect square. Since $10000 = 100^2$, so there are 100 perfect square numbers from 1 to 10000. Hence, the number of positive integers less than 10000 that do not have an odd number of divisors is $10000 - 100 = 9900$.

**Problem 4.A**

$AE = ED, FC = 2AF.$




Determine the following ratios: $\frac{S_{\triangle AEF}}{S_{\triangle CEF}}, \frac{S_{\triangle AEB}}{S_{\triangle BED}}$, and $\frac{S_{\triangle AEB}}{S_{\triangle AEF}}$?

If $S_{\triangle ABC} = 120$, find $S_{\triangle AEF} + S_{\triangle BED}$.

**Solution**

The triangles in these ratios are having the same height, so we only need to compare their base lengths:

$$\frac{S_{\triangle AEF}}{S_{\triangle CEF}} = \frac{AF}{CF} = \frac{1}{2}, \frac{S_{\triangle AEB}}{S_{\triangle BED}} = \frac{AE}{ED} = 1.$$

Now let the mid point of CF be G.

Easy to see $DG = 2EF$, and $BF = 2DG$.

So $BF = 3EF \Rightarrow BE = 3EF, \frac{S_{\triangle AEB}}{S_{\triangle AEF}} = 3.$

$$S_{\triangle BED} + S_{\triangle AEF} = S_{\triangle AEB} + S_{\triangle AEF} = S_{\triangle ABF}$$
$$= \frac{1}{3} S_{\triangle ABC} = 40$$

## Level B

### Problem 1.B

How many ways to sit seven people at a round table?

How many ways to sit seven people at a round table, if Lilian and Viet must not sit in adjacent seats?

**Solution 1**

If all seven people were to sit in a row, we know that the number of permutations is $7 \times 6 ... \times 1 = 7!$. Since they sit in a round table, by rotating clockwise, $ABCDEFG$ is same as $BCDEFGA$. For each sitting, there are 7 such rotations. So the number of ways is $\frac{7!}{7} = 6! = 720$.

Let count the number of ways to sit seven people if Lilian and Viet sit in adjacent seats. Instead of looking at them as separate people, we can represent them as if they were a single person. Similar to above, the number of ways to sit them is $6! = 120$. If we now consider Lilian and Viet as two different persons, there are 2 ways to permute their sittings, meaning the total number of sittings is $2 \times 120 = 240$.

By subtracting this from 720 possible sittings, we obtain the number of ways to sit seven people at a round table, if Lilian and Viet must not sit in adjacent seats. Hence there are $720 - 240 = 480$ ways.

**Solution 2** *(for the second question)*

If two people cannot sit next to each other, then there are 3360 ways to sit 7 people on a round table. There are 7 places for Lilian to sit. Since Viet cannot sit next to her, he only has 4 places to sit. Then, there are 5 places for the next person, 4 for the next, and so on.

So, there are $(7 * 4 * 5 * 4 * 3 * 2 * 1)/7 = 480$ ways for the seven people to sit if Lilian and Viet cannot sit next to each other.

**Problem 2.B**

$xy + x + y + 1$ can be factored as below:

$$xy + x + y + 1 = (xy + x) + (y + 1) = x(y + 1) + (y + 1) = (x + 1)(y + 1)$$

Factor $1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xyz}$

What is the product of positive integers $a$ and $b$, $a > b$:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = 1.$$

**Solution**

$$\begin{aligned}
&1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xyz} \\
&= \left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) + \left(\frac{1}{z} + \frac{1}{xz} + \frac{1}{yz} + \frac{1}{xyz}\right) \\
&= \left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) + \frac{1}{z}\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \\
&= \left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right)\left(1 + \frac{1}{z}\right) \\
&= \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{z}\right)
\end{aligned}$$

First to bring the left side to common denominator and then cross-multiplying them:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = 1 \Rightarrow \frac{b + a + 1}{ab} = 1 \Rightarrow ab - a - b - 1 = 0$$

Using a similar technique to factor the right side:

$$\Rightarrow (ab - a) - (b - 1) - 2 = 0 \Rightarrow (a - 1)(b - 1) = 2$$

Now $a - 1$ and $b - 1$ are two non-negative integers, whose product is 2. There is only a pair of integers (1, 2), which means $(a, b)$ is the pair (2, 3).

Their product is 6.

**Problem 3.B**

How many perfect squares are divisors of 432?

Find the greatest positive integer less than 10000 that has the number of perfect square divisors same as the number of perfect cube divisors. Prove your result.

*Note: A perfect square divisor of a positive integer $n$ is a perfect square that is a divisor of $n$. Similarly a perfect cube divisor of a positive integer $n$ is a perfect cube that is a divisor of $n$.*

**Solution**

$432 = 2^4 \times 3^3$. The only square divisors it can have are the number of ways to choose an even amount of 2's and 3's. There are 3 ways for 2's: 0, 2, 4 and 2 ways for 3's: 0, 2, for a total of $2 \times 3 = 6$.

Now let $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}$ be the prime factorization of a number $n$. We need to find a number where the product of the number of ways to choose even powers and the product of the number of ways to choose powers divisible by three is the same, meaning the numbers of ways to choose each term so that the powers are divisible by twos or threes, a.k.a squares & cubes, are equal:

$$\left(\lfloor\frac{k_1}{2}\rfloor + 1\right)\left(\lfloor\frac{k_2}{2}\rfloor + 1\right)\cdots\left(\lfloor\frac{k_m}{2}\rfloor + 1\right) = \left(\lfloor\frac{k_1}{3}\rfloor + 1\right)\left(\lfloor\frac{k_2}{3}\rfloor + 1\right)\cdots\left(\lfloor\frac{k_m}{3}\rfloor + 1\right)$$

where $\lfloor x \rfloor$ is the integer part of $x$.

If we work at this from 9999 down, we see that the largest number is in fact 9998 as it only has 1 as a cube and square factor.

**Problem 4.B**

$ABCD$ is a rectangle, $AB = 12$, $AE = CF = 6$. The area of the trapezoid $CDEF$ is 120. What is the area of $\triangle BFG$?

Let $AE = CF = x$, $AB = 2x$. What should be the area of the $CDEF$ such that the area of $\triangle BFG$ is $x^2$.

**Solution**

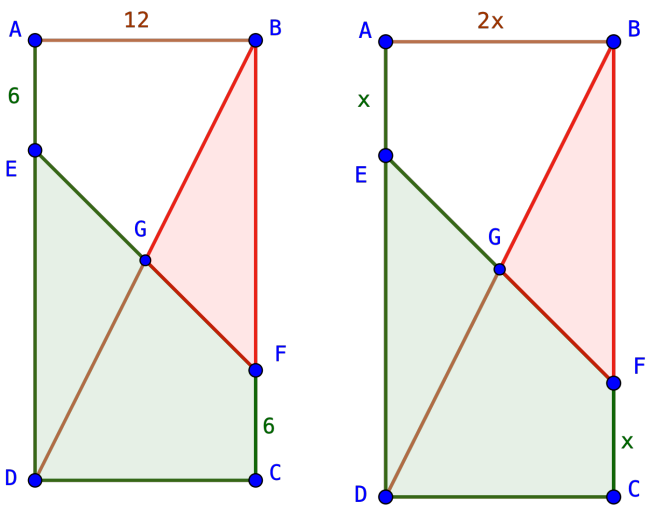
According to the left diagram $\triangle BFG \cong \triangle DEG$ by ASA:

$$\angle BGF = \angle DGE,\ DE = BF,\ \angle EDG = \angle DBG$$

This means that $\triangle BFG$ and $\triangle DEG$ have the same heights because $DE = BF$. Therefore their heights are each half of $AB$, or 6.

$$[CDEF] = 120 \Rightarrow AB \cdot \frac{ED + FC}{2} = 120 \Rightarrow ED + FC = 20.$$

$$BF = ED = 20 - 6 = 14. \text{ So } [BFG] = \frac{6 \times 14}{2} = 42.$$




$\triangle BFG \cong \triangle DEG$, so the height $\triangle BFG$ is $x$.

Since the area of $\triangle BFG$ is $x^2 \Rightarrow BF = \dfrac{2[BFG]}{x} = 2x$.

Therefore: $[CDEF] = AB \cdot \dfrac{ED + FC}{2} = 2x\dfrac{3x}{2} = 3x^2$.

# Chapter 5

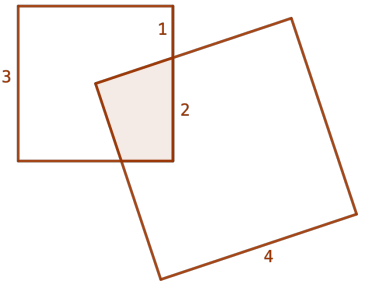
# Week 26

# Weekly Team Contest

## Problem 1

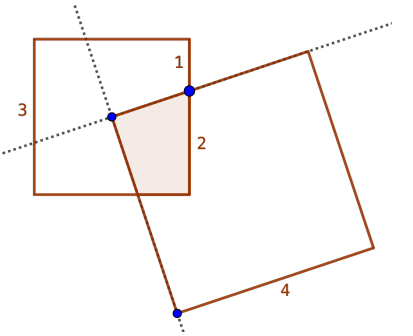
Two squares with side length 3 and 4 are overlapping such that one corner of the large square is at the center of the small square. One side of the small square is *trisected* as shown.

What is the area of the **shaded** region?




## Solution

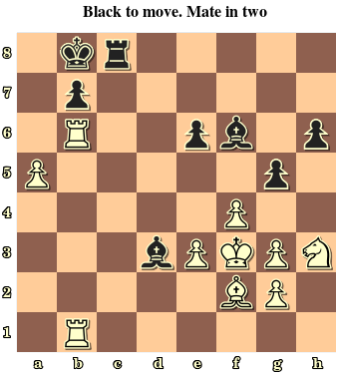
By extending the two sides of the large square that form a corner at the center of the small square, the below figure is obtained. Easy to see that these two lines divide the small square into four identical parts.




Therefore the area of the shaded figure is $\frac{9}{4}$.

## Problem 2

Solve the puzzle on the right.




## Solution

1… g4+ 2. Kxg4 Be2#

**Problem 3**

As a scientist at Innovative Foods Canada, Vu has almost completed the invention of the *strong-like-a-Canadian* egg, i.e. eggs that don't break easily. He has been asked to demonstrate for his boss, Huy, whatever he has got at the moment.

Vu wants to determine **the highest floor** in their 30-floor building from which an egg can fall without breaking. From previous experiments it is known that an egg still breaks:

- if it is dropped from a certain height, or
- if it is dropped the $4^{th}$ times, regardless from which height.

So far, the project has only produced *three eggs*.

What is the highest floor that Vu can **confidently** drop an egg?

**Solution**

Vu can use only *two eggs* to determine the floor and keeps the third for demonstration.

Suppose that he drops the first egg at the $k^{th}$ floor.

If it breaks, then he drops the second egg at the $1^{st}$, then $2^{nd}$, then $3^{rd}$, and so on until $(k-1)^{th}$ to verify which floor from the dropping breaks the egg.

If it does not, then he can drops the first egg at the $\left(2k\right)^{th}$ floor. If it breaks, then he drops the second egg at the $(k+1)^{th}$, then $(k+2)^{th}$, then $(k+3)^{th}$, and so on until $(2k-1)^{th}$ to verify which floor from the dropping breaks the egg.

He can continues to do so one more time for the first egg from the $(3k)^{th}$ floor and use the second egg to test from $(2k+1)^{th}$ to $(3k-1)^{th}$ floor. *Since the eggs always break at the fourth times, so only the third dropping is reliable*. Hence $k-1=3$.

Here's how he tests: He drop the first egg from $4^{th}$ floor, and if that breaks then use the second egg to test which floor from the $1^{st}$, $2^{nd}$, $3^{rd}$. Similarly he continues to test at the $8^{th}$, if that breaks then use the second egg to test which floor from the $5^{th}$, $6^{th}$, $7^{th}$. Last he tests the first egg at the $12^{th}$ floor, if that breaks then use the second egg to test which floor from the $9^{th}$, $10^{th}$, $11^{th}$.

So the highest floor where he can drop the third egg is 12.

**Problem 4**

Apinayé belongs to the Ge language family of Brazil. Currently it is spoken by less than 800 people, and therefore is seriously endangered. The following are some sentences in Apinayé, along with their English translations. You will see some letters here that do not occur in the English or Portuguese writing systems.

| | |
|---|---|
| Kukr˜ɛ kokoi. | The monkey eats. |
| Ape kra. | The child works. |
| Ape kokoi ratš. | The big monkey works. |
| Ape mï mɛtš. | The good man works. |
| Ape mɛtš kra. | The child works well. |
| Ape punui mï piŋetš. | The old man works badly. |

Some Apinayé words have different meanings in different contexts. For example following meanings represented by the *same* word:

"big" and "a lot", "good" and "well", "old" and "long time"

Now, **translate** the following sentences into English:

ii. Ape ratš mï mɛtš.

iii. Kukr˜ɛ ratš kokoi punui.

iv. Ape piŋetš mï.

And the following sentences into Apinayé:

v. The big child works a long time.

vi. The old monkey eats a lot.

**Solution**

| | |
|---|---|
| Ape ratš mï mɛtš. | The good man works a lot. |
| Kukr˜ɛ ratš kokoi punui. | The bad monkey eats a lot. |
| Ape piŋetš mï. | The man works a long time. |
| The big child works a long time. | Ape piŋetš kra ratš. |
| The old monkey eats a lot. | Kukr˜ɛ ratš kokoi piŋetš. |

**Problem 5**

Minh and his friend play a two-player game with a box of matches. Initially the box contains 300 matches. Each turn a player removes *at least one but no more than half of the matches* in the box. The player who cannot move loses.

Should Minh go first or second? How can he win?

**Solution**

First, Minh reduces the number of the matches to $255 = 2^8 - 1$. So there are $2^m - 1$ matches on the table. This is called a **winning position**.

Each time:

- if his friend removes $n$ matches from $2^m - 1$ matches, where $n \leq \lfloor \frac{2^m - 1}{2} \rfloor = 2^{m-1} - 1$
- Minh removes $2^{m-1} - n \geq 1$ matches.

So the total number of matches they remove is $2^{m-1}$, and there is $2^{m-1} - 1$ matches remain, meaning Minh is back to winning position.

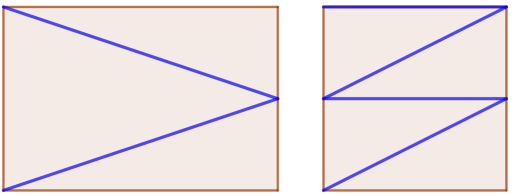
Since soon or later they will run out of matches, and whatever his friend can remove, Minh can still remove some matches.

Hence he will be the last who can move and wins the game.

**Problem 6**

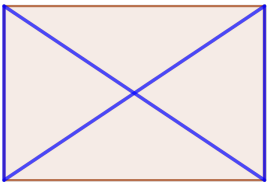
Bill, David, Lily watch a fish swimming in a rectangular fish tank. Bill stands in front of the tank, David stands to the right of it, and Lily is looking from the top of the aquarium.

The path of the fish seen by Bill is shown in the left diagram, and the path seen by David is shown in the right diagram.



Draw the fish swimming's path the way Lily saw it.

**Solution**



We know that the fish swam **along the left-most edg**e because Bill sees it at the top left corner while David sees it moving in a straight line height wise.

Then, we know that the fish is spotted at the bottom right hand corner for Lily, because Bill saw it on the right edge and David saw it on his left edge. We know the fish swam in a straight line from the previous point, from Bill and diagonally down, according to David. This means that to Lily, it's a **straight diagonal from the top left corner to the bottom right corner**.

Then, David sees it moving to his right edge while Bill sees it stay at the same level. To Lily, the fish has moved **along her right edge**.

Then, the fish moves to the left edge again and David and Bill sees it moving downwards. Lily doesn't see the downwards motion, but she sees it moving from her **right hand upper corner to her lower left hand corner**. And thus, the diagram is complete.

# Problems of the Week

## Level A

### Problem 1.A

In a group of 2 cats, 3 dogs, and 10 pigs in how many ways can a committee of 6 animals be chosen such that there must be **exactly** 2 pigs?

In a group of 2 cats, 3 dogs, and 10 pigs in how many ways can a committee of 6 animals be chosen such that there must be **at most** 2 pigs?

*Note: there is no any distinction between animals in each animal type.*

**Solution**

If we want to from a committee of 6 animals with **exactly** 2 pigs, then we need to choose four other animals from 2 cats, and 3 dogs. There are only **two** possibilities: (1 cat, 3 dogs), and (2 cats, 2 dogs).

If we want to from a committee of 6 animals with **at most** 2 pigs, then the number of pigs in the committee can be 0, 1, or 2. Since 0 pig selection is not possible because the number of all other animals is only 5. 1 pig selection means we have to choose 5 other animals. There is only one such choice: (2 cats, 3 dogs). And from the previous question, there is 2 choices for exactly 2 pigs. So there are **three** choices for at most 2 pigs.

### Problem 2.A

If $x + \frac{1}{x} = 1$, what is $x^2 + \frac{1}{x^2}$? And $x^3 + \frac{1}{x^3}$?

**Solution**

Applying $(a+b)^2 = a^2 + 2ab + b^2$ for $\left(x + \frac{1}{x}\right)^2$:

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \cdot x \cdot \frac{1}{x} = x^2 + \frac{1}{x^2} + 2.$$

So if $x + \frac{1}{x} = 1$, then $x^2 + \frac{1}{x^2} = 1 - 2 = -1$.

Applying $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ for $x^3 + \frac{1}{x^3}$:

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 - x \cdot \frac{1}{x} + \frac{1}{x^2}\right) = 1 \cdot (-1 - 1) = -2$$

**Problem 3.A**

If the sum of a square of a prime and an odd positive integer is 125, find the value of the odd positive integer.

$p, q, r$ are prime numbers such that $p+q+r=28$, $pq+qr+rp=185$. Find the greatest among them.

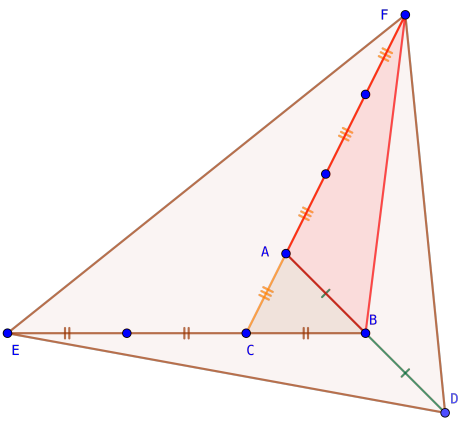
**Solution**

The sum of a square of a prime and an odd positive integer is 125, which is an odd integer. This means that the square of the prime is even, there for the prime is 2, hence the odd number is 121.

$p+q+r=28$ means that one of $p$, $q$, and $r$ is 2, so WLOG let $p=2$, then $q+r=26$. Substitute into $pq+qr+rp=185$, we have $qr+2\cdot 26=185 \Rightarrow qr=133=7.19$. Hence the largest prime is 19.

**Problem 4.A**

$S_{\triangle ABC}=2$. Extend $AB$, $BC$, $CA$ towards $B$, $C$, $A$ such that $BD=BA$, $CE=2CB$, $AF=3AC$. Find $S_{\triangle BAF}$, $S_{\triangle DEF}$.




**Solution**

$AF=3AC \Rightarrow S_{\triangle ABF}=3S_{\triangle ABC}=6. AD=2AB.$

$\Rightarrow S_{\triangle ADF}=2S_{\triangle ABF}=6S_{\triangle ABC}$. So $S_{\triangle BAF}=6$.

$S_{\triangle EDB}=S_{\triangle EAB}=3S_{\triangle ABC}$,

$S_{\triangle ECF}=2S_{\triangle CBF}=8S_{\triangle ABC}$,

$S_{\triangle DEF}=S_{\triangle ADF}+S_{\triangle EBD}+S_{\triangle ECF}+S_{\triangle ABC}$

$=18S_{\triangle ABC}=36$

**Level B**

### Problem 1.B

After collecting for the candy drive, Sally has eight \$1-bills, five \$5-bills, and three \$10-bills. She randomly selected three bills without replacement. What is the total number of ways to select 3 bills, one of each type, from all the bills she has?

What is the probability that she chooses one of each type of bill? Express your answer as a common fraction.

**Solution**

The number of ways to select a \$1 bill from eight \$1-bills is $\binom{8}{1}$, a \$5 bill five \$5-bills is $\binom{5}{1}$, and a \$10 bill from three \$10-bills is $\binom{3}{1}$.

The selections are independent, there are $\binom{8}{1}\binom{5}{1}\binom{3}{1} = 120$ ways to select the three bills.

The total number of ways to choose three any bills from all the bills is $\binom{8+5+3}{3} = 560$.

Hence the probability to choose one from of each type of the bills is $\frac{120}{560} = \frac{3}{14}$.

**Problem 2.B**

Find the value of $\sqrt{15-6\sqrt{6}}+\sqrt{15+6\sqrt{6}}$.

If $m-\dfrac{1}{m}=4$, find the value of $m^4-\dfrac{1}{m^4}$.

**Solution**

Since $15-6\sqrt{6}=9-2\cdot 3\cdot\sqrt{6}+6$

$=3^2-2\cdot 3\cdot\sqrt{6}+(\sqrt{6})^2=(3-\sqrt{6})^2$

Similarly $15+6\sqrt{6}=9+2\cdot 3\cdot\sqrt{6}+6$

$=3^2+2\cdot 3\cdot\sqrt{6}+(\sqrt{6})^2=(3+\sqrt{6})^2$

Therefore $\sqrt{15-6\sqrt{6}}+\sqrt{15+6\sqrt{6}}$

$=3-\sqrt{6}+3+\sqrt{6}=6$

$m-\dfrac{1}{m}=4$, so $\left(m-\dfrac{1}{m}\right)^2=16$

$\Rightarrow m^2-2\cdot m\cdot\dfrac{1}{m}+\dfrac{1}{m^2}=16\Rightarrow m^2+\dfrac{1}{m^2}=18$

Furthermore $\left(m+\dfrac{1}{m}\right)^2=m^2+2\cdot m\cdot\dfrac{1}{m}+\dfrac{1}{m^2}=20$

$\Rightarrow m+\dfrac{1}{m}=\sqrt{20}=2\sqrt{5}$

$m^4-\dfrac{1}{m^4}=\left(m^2-\dfrac{1}{m^2}\right)\left(m^2+\dfrac{1}{m^2}\right)$

$=\left(m-\dfrac{1}{m}\right)\left(m+\dfrac{1}{m}\right)\left(m^2+\dfrac{1}{m^2}\right)$

$=4\cdot 2\sqrt{5}\cdot 18=144\sqrt{5}$

**Problem 3.B**

Two prime numbers $p$, $q$ satisfy the following conditions: $p = m + n$, $q = mn$, where both $m$, $n$ are positive integers.

Find the value of $p^q + q^p$.

For many positive integers $n$ is $n^2 - 4n - 21$ a prime?

**Solution**

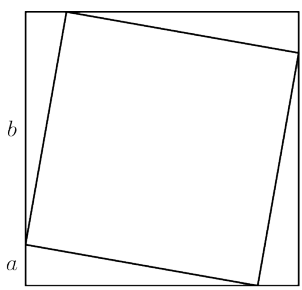
Since $q = mn$, so one of $m$, $n$ must be 1. Hence $q$ and $p = q + 1$ are two consecutive integers that are primes. This is possible if and only if $(p, q) = (3, 2)$. Hence $p^q + q^p = 3^2 + 3^3 = 17$.

The prime $n^2 - 4n - 21 = (n - 7)(n + 3)$ has two factors. Therefore the smaller factor $n - 7$ is 1. Hence $n = 8$. $n^2 - 4n - 21 = 11$.

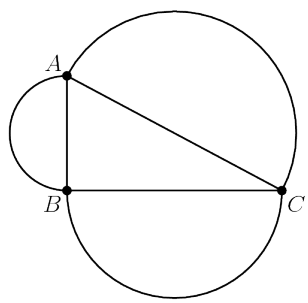
**Problem 4.B**

A square with area $x$ is inscribed in a square with area $y$, with one vertex of the smaller square on each side of the larger square. A vertex of the smaller square divides a side of the larger square into two segments, one of length $a$, and the other of length $b$, as shown in the left diagram below.

Can you express the value of $ab$ using $x$ and $y$?




*First question*




*Second question*

Angle $\angle ABC$ of $\triangle ABC$ is a right angle. The sides of $\triangle ABC$ are the diameters of semicircles as shown in the right diagram above. The area of the semicircle on $AB$ equals $2a^2\pi$, and the arc of the semicircle on $AC$ has length $b\pi$, where $a + b = 20$ and $ab = 80$.

What is the radius of the semicircle on $BC$?

**Solution**

The difference in area of the two square is the area of four right triangles $y - x = 4 \cdot \frac{a \cdot b}{2} = 2ab$. Hence $ab = \frac{y - x}{2}$.

The total perimeter of the circle with diameter $AC$ is $2\pi \cdot \frac{AC}{2}$, which also is $2 \cdot b\pi$. Therefore $AC = 2b$.

The area of the circle with diameter $AB$ is $\pi \cdot \left(\frac{AB}{2}\right)^2$, which also is $2 \cdot 2a^2\pi$. Therefore $\pi \cdot \left(\frac{AB}{4}\right)^2 = 4\pi \cdot a^2 \Rightarrow AB = 4a$.

Because $a + b = 20$, $ab = 80$, so $a$, $b$ are the two roots of

$$x^2 - 20x + 80 = 0$$

The discriminant $\Delta = 20^2 - 4.80 = 80 \Rightarrow \sqrt{\Delta} = 4\sqrt{5}$.

$$x_{1,2} = \frac{20 \pm 4\sqrt{5}}{2} \Rightarrow b = 2(5 + \sqrt{5}),\ a = 2(5 - \sqrt{5})$$

Since $AC^2 = AB^2 + BC^2 \Rightarrow BC^2 = (2b)^2 - (4a)^2 = 4(b^2 - 4a^2)$

$$= 4\left[\left(2(5 + \sqrt{5})\right)^2 - 4\left(2(5 - \sqrt{5})\right)^2\right]$$

$$= 16\left[\left(5 + \sqrt{5}\right)^2 - 4\left(5 - \sqrt{5}\right)^2\right]$$

$$= 16\left[5^2 + 2 \cdot 5 \cdot \sqrt{5} + (\sqrt{5})^2 - 4.5^2 + 8 \cdot 5 \cdot \sqrt{5} - 4(\sqrt{5})^2\right]$$

$$= 16\left(50\sqrt{5} - 90\right) = 4^2\left(50\sqrt{5} - 90\right)$$

Hence the radius of the semicircle on $BC$ is:

$$2\sqrt{50\sqrt{5} - 90}$$

# Chapter 6

# Week 27

# Weekly Team Contest

**Problem Set 1.A**

Coloured $1 \times 1 \times 1$ blocks are to be placed in a $5 \times 5 \times 5$ cardboard box. A **red** block is placed in *one corner* of the box. Next, the least number of **green** blocks are placed to *completely hide* the **red** block from view. Finally the least number of **orange** blocks are placed to completely hide the **green** blocks from view.

At this point, **how many blocks** have been placed in the box?

**Solution**

The **red** block is placed in *one corner* of the box, so 3 of its faces are visible. To hide those 3 faces, 3 **green** blocks are required. One of the **green** blocks is placed on top of the **red** block, and the other two on the same level with the **red** block.

So there are 6 faces of the **green** blocks are visible. 6 **orange** blocks are needed to cover them. One **orange** block is on top of the top **green** block, two on the same level with the top **green** block, and three on the lowest level.

All together there are $1 + 3 + 6 = 10$ blocks.

## Problem Set 1.B

In the 5 × 5 grid shown, each row and each column is to contain the integers 1 through 5 exactly once with one integer per cell. The sum of the two integers in each outlined pair of cells is 5.

| | | | | |
|---|---|---|---|---|
| | | 2 | | |
| 1 | | | | |
| | | | | 1 |
| | | | | |
| | | | | 3 |

What is **the product of the integers** in the four corner cells?

**Solution**

Since the sum of a pair in an outlined rectangle is 5, there are only 2 possibilities: $1 + 4 = 2 + 3 = 5$. Let fill those rectangles step by step (left to right, top to bottom):

| | | | | |
|---|---|---|---|---|
| | | 2 | | |
| 1 | | | | |
| | | | | 1 |
| | | | 3 | |
| 4 | 1 | | 2 | 3 |

| | | | | |
|---|---|---|---|---|
| | | 2 | 1 | |
| 1 | 2 | 3 | 4 | |
| | | | | 1 |
| | | | 3 | |
| 4 | 1 | | 2 | 3 |

| | | | | |
|---|---|---|---|---|
| | | 2 | 1 | |
| 1 | 2 | 3 | 4 | |
| | | 4 | | 1 |
| | | 1 | 3 | |
| 4 | 1 | | 2 | 3 |

| | | | | |
|---|---|---|---|---|
| | | 2 | 1 | |
| 1 | 2 | 3 | 4 | 5 |
| | | 4 | 5 | 1 |
| | | 1 | 3 | |
| 4 | 1 | 5 | 2 | 3 |

| | | | | |
|---|---|---|---|---|
| | | 2 | 1 | 4 |
| 1 | 2 | 3 | 4 | 5 |
| | 3 | 4 | 5 | 1 |
| | | 1 | 3 | |
| 4 | 1 | 5 | 2 | 3 |

| | | | | |
|---|---|---|---|---|
| 3 | 5 | 2 | 1 | 4 |
| 1 | 2 | 3 | 4 | 5 |
| | 3 | 4 | 5 | 1 |
| | | 1 | 3 | |
| 4 | 1 | 5 | 2 | 3 |

Hence the product of the four corners is $3 \times 4 \times 3 \times 4 = 144$.

## Problem Set 1.C

Consider the $4 \times 4$ "multiplication table" below.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 2 | 4 | 6 | 8 |
| 3 | 6 | 9 | 12 |
| 4 | 8 | 12 | 16 |

The numbers in the first column multiplied by the numbers in the first row give the remaining numbers in the table. For example, the 3 in the first column times the 4 in the first row give the 12( $= 3 \cdot 4$) in the cell that is in the 3rd row and 4th column.

We create a path from the upper-left square to the lower-right square by always moving one cell either to the right or down. For example, here is one such possible path, with all the numbers along the path shaded.

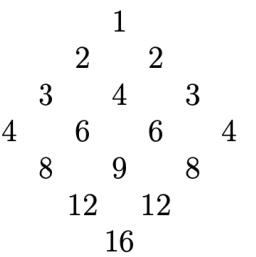
| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 2 | 4 | 6 | 8 |
| 3 | 6 | 9 | 12 |
| 4 | 8 | 12 | 16 |

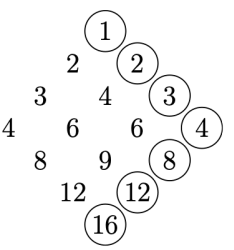
If we add up the circled numbers in the example above (including the start and end squares), we get 48.

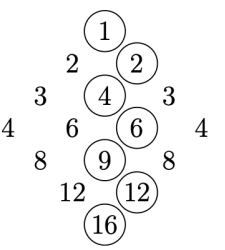
Considering all such possible paths What is the **smallest** sum we can possibly get when we add up the numbers along such a path? What is the **greatest** sum we can possibly get when we add up the numbers along such a path?

**Solution**

To see this more easily, tilt the grid 45 degrees. Now every path must include exactly one number from each row. The smallest and largest numbers in each row are respectively at the edge and in the middle, so the smallest and largest totals are achieved by the paths below.



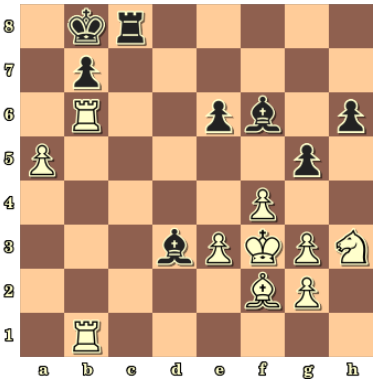




The minimum is 46 and the maximum is 50.

## Problem Set 2.A

Solve the puzzle on the right.

Black to move. Mate in two



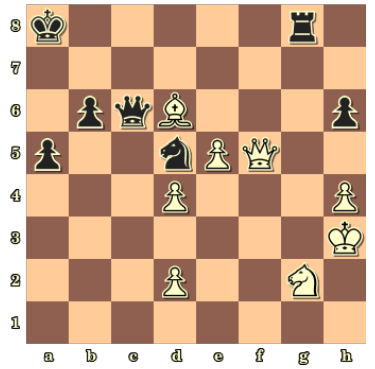
**Solution**

1... g4+ 2. Kxg4 Be2#

## Problem Set 2.B

Solve the puzzle on the right.

Black to move. Mate in two



**Solution**

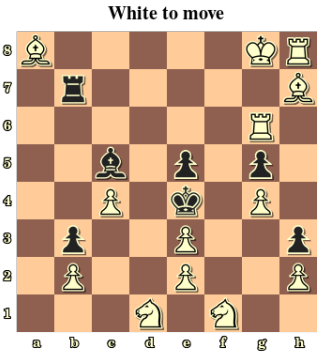
1... Nf4+ 2. Qxf4 Qxg2# or

2. Kh2 Qxg2# or

2. Nxf4 Qh1#

## Problem Set 2.C

Solve the puzzle on the right.



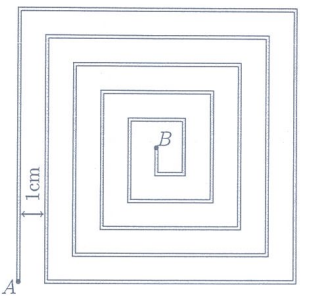

Don't mate the Black in one move

**Solution**

The only white piece that can move **legally** is the rook at g6. Other white pieces cannot move (six pawns, rook at h8, bishop at h7), do not have legal move (king), or mate the Black if move (bot knights, bishop at a8). The only legal move of the rook at g6 that does not result in a mate is **1. Rc6+**, by **1… Rxh7**, Black avoids mate.

## Problem Set 3.A

An ant walks from point A to point B in the below figure.




What is the distance it took?

**Solution**

The path from $B$ to $A$ consists of the segments with lengths:

$$1, 1, 2, 2, \ldots, 9, 9, 10, 10, 10$$

each number from 1 to 9 repeats twice, except 10, which repeats thrice. Therefore the length of the path is:

$$2(1 + 2 + \ldots + 10) + 10 = 2 \cdot \frac{10 \cdot 11}{2} + 10 = 120.$$

**Problem Set 3.B**

Elisa bought two unusual sandglasses. One measures a *nine-minute* interval, and the other measure *thirteen-minute* interval. A certain love potion needs to boil for exactly *thirty minutes*.

Can you help Elisa to measure such a time interval with these two sandglasses?

**Solution**

<table>
<tr><td>13</td><td colspan="3">13</td></tr>
<tr><td></td><td>9</td><td>4</td><td>4</td></tr>
</table>

The key to this solution is to measure 4 minutes.

First, the 13-minutes hourglass is turned.

After 13 mins, both hourglasses are turned.

After 22 mins, the 9-minute hourglass is turned.

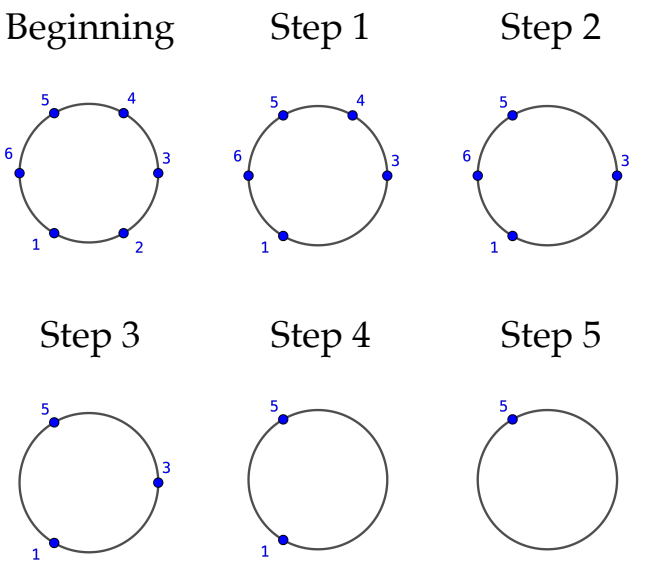
After 22 mins, the 9-minute hourglass is turned **back**.

After 30 mins, the 9-minutes hourglass stops.

*Note: this is the best solution considering you did not start the hourglasses before measurements.*

**Problem Set 3.C**

A group of $n$ persons are standing in a circle, numbered consecutively counter-clockwise from 1 to $n$. Starting with the person number 2, in every step we remove every other person, proceeding counter-clockwise. For example with $n = 6$, the last person who is remaining is the person number 5 and the *last person removed* is the person number 1. See below.




If $n = 67$, who is the *last person removed*?

**Solution**

Easy to see that if $n = 2$, the remaining person is 1. If $n = 4$, then after removing person 2, and 4, the problem is same as $n = 2$. Similarly it can be proved that $n = 2^m$, the remaining person is 1. In every case person $2^m - 1$ who is the *last person removed*.

Now $n = 67$. Let remove three person 2, 4, and 6:

1, **2**, 3, **4**, 5, **6**, 7, 8, 9, …, 66, 67

The number of people is reduced to 64, and person 7 now can be considered as the first one of 64 people:

7, 8, 9, …, 66, 67, 1, 3, 5

Following the above reasoning, the remaining person is person 7, so the last person removed is 5.

**Problem Set 4.A**

There are names of some countries in South America, written in the Georgian language, together with their translations to English:

| | |
|---|---|
| ბრაზილია | Brazil |
| პერუ | Peru |
| ურუგვაი | Uruguay |
| | |
| არგენტინა | |
| კოლუმბია | |

What are the names in English of the two other countries?

**Solution**

In this question, one has only to decipher a different alphabet. For that, one can note that **Peru** and **Uruguay**, in Georgian, have the same amount of characters as their translations; furthermore, the repetition of U in Uruguay assures us that Georgian is written left-to-right. So we can do the relation *one-to-one*.

**Brazil**, nevertheless, has more letters than the version in English, but thanks to the two other names, we already know some letters:

_ R A _ I _ I A

This should probably be *Brasilia* or *Brazilia*. With those letters, we can guess the names of the other two countries:

A R G E _ _ I _ A

_ _ L U _ B I A

which can only be **Argentina** and **Colombia** (Columbia).

**Problem Set 4.B**

In the Iberian Peninsula, there are several languages, apart from Portuguese and Spanish (Castillan), that share the same romanic origin, such as Catalan, Galician, Mirandese, Asturian, Leonese and Aragonese. The list below brings a list of words in Aragonese, their plural forms and their translations to no plural:

| | | |
|---|---|---|
| valley | bal | bals |
| stool | banquet | banquetz |
| hole | clot | clotz |
| stone | cantal | cantals |
| awake | concordau | _____ |
| chocolate | chicolat | _____ |
| union | chunta | _____ |
| unhanded | deixau | deixaus |
| eclipse | eclix | _____ |
| cicada | ferfet | _____ |
| character | personache | personaches |
| fish | peix | peixes |

Write down the **missing** words.

**Solution**

In this question, we have to compare two columns of words: singulars and plurals. With a little examination, one notes that the ending of the plural depends on the ending of the singular. We can thus deduce the following rules:

| if the singular ends in | add for the plural |
|---|---|
| L | -S |
| T | -Z |
| X | -ES |
| vowel | -S |

So the table becomes:

| | | |
|---|---|---|
| valley | bal | bals |
| stool | banquet | banquetz |
| hole | clot | clotz |
| stone | cantal | cantals |
| awake | concordau | concordau-s |
| chocolate | chicolat | chicolat-z |
| union | chunta | chunta-s |
| unhanded | deixau | deixaus |
| eclipse | eclix | eclix-es |
| cicada | ferfet | ferfet-z |
| character | personache | personaches |
| fish | peix | peixes |

**Problem Set 4.C**

Imagine that you have heard these sentences:

Danny is molistic and slatty.

Clara is cluvious and brastic.

Benny and Tony are slatty but danty.

The singer is danty and cloovy.

Anna is blitty but cloovy.

Elvis is not only sloshful but also weasy.

Even though frumsy, James is sloshful.

Strungy and struffy, Donna was a pleasure to watch.

Easy though weasy, Smith is strungy.

Dolly is blitty but struffy.

The managers were cluvious and not slatty.

Which of the following would you be likely to hear? You can select all, some, or none.

a. Van Anh is blitty and brastic.

b. The driver was not only molistic but also cluvious.

c. Tommy found a dog that was danty but sloshful.

What quality or qualities would you be looking for in a person? You can select all, some, or none.

a. blitty

b. weasy

c. sloshful

d. frumsy

**Solution**

None of the adjectives are real English words. There are two classes of adjectives: "bad" and "good". We will refer to this property of adjectives as "polarity".

Each sentence links two or more adjectives as follows: "X and Y" indicates that X and Y have the same polarity. "X but Y" means that they have opposite polarities. Furthermore, "X and not Y" indicates opposite polarities, "even though X, Y" also indicates opposite polarities, while "not only X but also Y" associates adjectives of the same polarity.

The sentence about Donna shows that "strungy" and "struffy" are positive (desirable) quantities. By identifying other occurrences of the same words in other sentences, one can label each adjective as either positive or negative. In the end, one can see that there are seven positive adjectives (strungy, struffy, cloovy, frumsy, danty, cluvious, and brastic) and five negative ones (weasy, blitty, sloshful, slatty, molistic).

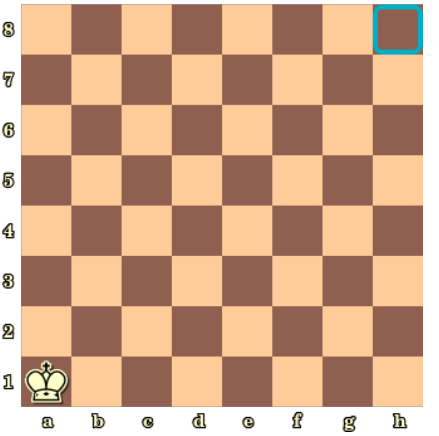
Only **sentence c** includes adjectives of the right polarities, given the structure of the sentence.

Only **answer d** ("frumsy") is on the positive list above.

**Problem Set 5.A**

On the chessboard, a **king** stands on square $a1$. Two players take turns moving the king as many squares as they want, either horizontally to the right, vertically upward, or along the diagonal to the right and upwards. The player who can place the king on square $h8$ wins.

**First player who move the king to h8 wins**




Can you suggest to Catherine a **winning strategy**?

Whether she goes first or second and how she move the king?

**Solution**

Note that if Catherine can keep the King standing on the diagonal from $a1$ to $h8$, she is going to win the game. These positions are called *winning positions*.

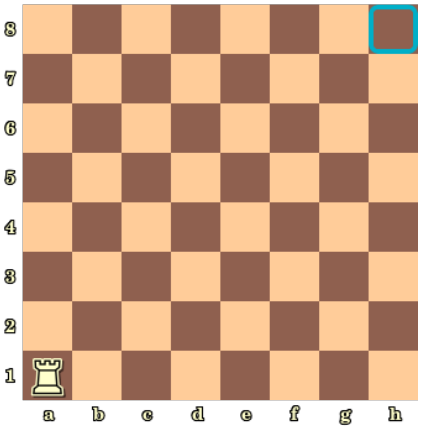
If she starts by moving the King to $b2$, then the opponent always has to move the King *away from the diagonal* and when the next turn, Catherine can move it *back to the diagonal*.

Since the moves are right and upwards, so Catherine will win.

## Problem Set 5.B

On the chessboard, a **rook** stands on square $a1$. Two players take turns moving the rook as many squares as they want, either horizontally to the right or vertically upward. The player who can place the rook on square $h8$ wins.

**First player who moves the rook to h8 wins**




Rook can move to the right or upward only

Can you suggest to Clemence a **winning strategy**?

Whether she goes first or second and how she move the rook?

**Solution**

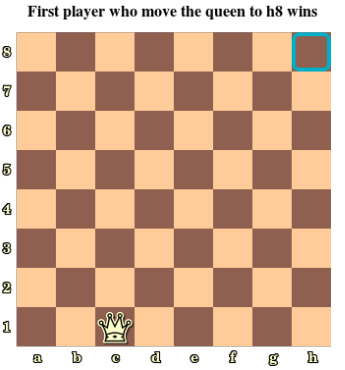
Note that if Clemence can keep the Rook standing on the diagonal from $a1$ to $h8$, she is going to win the game. These positions are called *winning positions*.

If she let the opponent starts first, the Rook is moved *away from the diagonal* to the right or upwards and when the next turn, Clemence can move it *back to the diagonal* by moving upwards or to the right.

Such pair of turns can be repeated, since the moves are always right and upwards, so Clemence will win.

## Problem Set 5.C

On the chessboard, a **queen** stands on square $c1$. Two players take turns moving the queen as many squares as they want, either horizontally to the right, vertically upward, or along the diagonal to the right and upwards. The player who place the queen on square $h8$ wins.




Can you suggest to Camille a **winning strategy**?

Whether she goes first or second and how she move the queen?

**Solution**

Note that in order to move the $h8$, the Queen has to stand on any square but not *none* of the positions that can reach $h8$ in one move.

Let mark $h8$ by a + sign for *winning position* meaning who can get there will win the game and those positions that can reach $h8$ in one move by a − indicated *losing position,* meaning who gets there will lose the game. After that continue for the rest of the board:

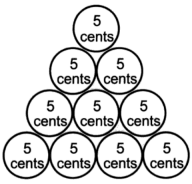
| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| − | − | − | − | − | − | − | + | − | − | − | − | − | − | − | + |
| | | | | | | − | − | − | − | − | − | − | + | − | − |
| | | | | | − | | − | − | − | − | − | − | − | + | − |
| | | | | − | | | − | | | | − | − | − | − | − |
| | | | − | | | | − | | | − | − | − | − | − | − |
| | | − | | | | | − | | − | − | − | | − | − | − |
| | − | | | | | | − | − | − | − | | | − | − | − |
| − | | | | | | | − | − | − | | | | − | − | − |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| − | − | − | − | − | − | − | + |
| − | − | − | − | − | + | − | − |
| − | − | − | − | − | − | + | − |
| − | − | + | − | − | − | − | − |
| + | − | − | − | − | − | − | − |
| − | − | − | − | + | − | − | − |
| − | − | − | − | − | − | − | − |
| − | − | − | + | − | − | − | − |

From here the first player will win by moving the Queen from $c1$ to a *winning position*, for example $c5$, $d1$, or $e3$.

## Problem Set 6.A

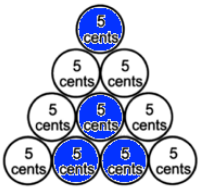
Ten coins are arranged as shown in the figure below.




What is the **minimum** number of coins we must remove so that no three of the remaining coins form an equilateral triangle?

**Solution**

Remove the top coins and three central bottom coins as below:




The number of coins can be removed is 4.

## Problem Set 6.B

Soccer lovers know that a soccer ball is made from pieces of black and white leather. The *black pieces are regular pentagons* and the *white pieces are regular hexagons*. Each pentagon is adjacent to five hexagons and each hexagon is adjacent to three pentagons and three hexagon. The ball has 20 white hexagons.

What is the **number** of black pentagons?

**Solution**

Let call black-white edge is a common edge of two different shapes: a pentagon and a hexagon. Let $b$ be the number of black pentagon and $w$ be the number of white hexagon. $5b$ is the number of black-white edges as edges of black pentagons. Similarly $3w$ the number of black-white edges as edges of white hexagon, so: $5b = 3w$.

$$\text{Since } w = 20 \Rightarrow b = \frac{3 \cdot 20}{5} = 12.$$

**Problem Set 6.C**

Let $\triangle ABC$ be a triangle with $D$ the midpoint of side $AB$, $E$ the midpoint of side $BC$, and $F$ the midpoint of side $AC$.

Let $C_1$ be the circle passing through points $A$, $D$, and $F$. Let $C_2$ be the circle passing through points $B$, $E$, and $D$. The two circles intersect each other at $X$.

Let $C_3$ be the circle passing through $C$, $E$, and $X$.

Prove that $F$ is on $C_3$.

**Solution**

Note that the perpendicular bisector through $D$ of $AB$ and the perpendicular bisector through $F$ of $AC$ shall meet at the centre of the circumference of triangle $\triangle ABC$., which means the centre of the circumference of triangle $\triangle ABC$ is on the circle $C_1$.

Similarly the centre of the circumference of triangle $\triangle ABC$ is on the circle $C_2$, so it is $X$ is the intersection of both circles, which is also where the perpendicular bisectors of a triangle $\triangle ABC$ meets.

So the circle the through $C$, $E$, and $X$ is also through $F$.

# Problems of The Week

**Level A**

### Problem 1.A

How many ways to select in three single draws without replacement two red marbles and one blue marble from a bag containing six red marbles, eight yellow marbles, and seven blue marbles?

What is the *probability* to select in three single draws without replacement two red marbles and one blue marble from a bag containing six red marbles, eight yellow marbles, and seven blue marbles?

**Solution**

There are $\binom{6}{1}$ ways to select in a single draws without replacement *one red marble* from a bag containing six red marbles and other coloured marbles.

So there are $\binom{6}{1}\binom{5}{1}\binom{7}{1}$ ways to select in three single draws without replacement *two red marbles and one blue marble* from a bag containing six red marbles, eight yellow marbles, and seven blue marbles.

The total number of ways to select *three marbles* in three single draws without replacement from a bag containing six red marbles, eight yellow marbles, and seven blue marbles is $\binom{21}{1}\binom{20}{1}\binom{19}{1}$.

Hence the probability is $\dfrac{6 \times 5 \times 7}{21 \times 20 \times 19} = \dfrac{3}{38}$.

**Problem 2.A**

Algebra identities:

$y^2 - x^2 = (y - x)(y + x)$

$y^3 - x^3 = (y - x)(y^2 + yx + x^2)$

Find all positive integer pairs $(x, y)$: $y^2 - x^2 = 2015$.

Find all positive integer pairs $(x, y)$: $y^3 - x^3 = 91$.

**Solution**

$y^2 - x^2 = 2015 \Rightarrow (y - x)(y + x) = 2015$

So $(y - x)(y + x) = 1 \cdot 2015 = 5 \cdot 403 = 13 \cdot 155 = 31 \cdot 65$.

By solving 4 cases, we can obtain four pairs of positive integers:

$(y, x) = (1008, 1007), (204, 199), (84, 71), (48, 17)$.

$y^3 - x^3 = 91 \Rightarrow (y - x)(y^2 + xy + y^2) = 91 = 1 \cdot 91 = 7 \cdot 13$.

If $y - x = 7$, $y^2 + xy + y^2 = 13$.

$y^2 + xy + x^2 = (y^2 - 2xy + x^2) + 3xy = (y - x)^2 + 3xy$

$\Rightarrow 13 = 49 + 3xy$, impossible for positive integer pairs $(x, y)$.

So $y - x = 1$, $y^2 + xy + y^2 = 91$.

$\Rightarrow 91 = 1 + 3xy \Rightarrow xy = 30 \Rightarrow x = 5, y = 6$.

**Problem 3.A**

The least common multiple of 12, 15, 20, and $k$ is 420. What is the least possible value of $k$?

What is the second smallest such value of $k$?

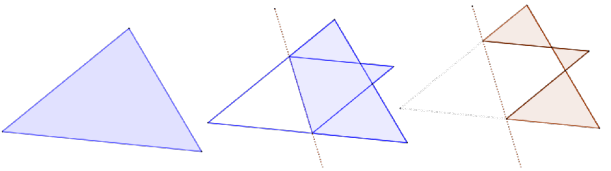
**Solution**

The least common multiple of 12, 15, 20 is 60, so $k = 7$.

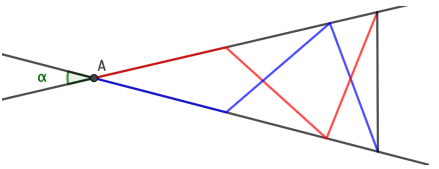
The second smallest such value is $k = 14$.

**Problem 4.A**

Emma folds the blue triangle on the left below along a dotted line to get a folded figure, which is the blue figure in the middle. Boutheina observes that the triangle area is 1.5 times of the area of the folded figure. Alex says that "it's interesting that the area of the shaded figure formed by three triangles, which is the brown one on the right, is exactly 1 $cm^2$." What is the area of the blue triangle?



Qi'ra, before becoming the lieutenant in the Crimson Dawn, once was a math competitor. Once she was required to calculate the value of the angle $\alpha$ in the below figure, that was made from 7 matchsticks with equal length, connected end to end. What is $\alpha$?



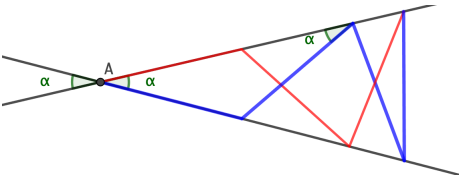

**Solution**

Let $b$ be the difference of the the blue figure in the middle and the brown one on the right:

$$\text{the blue figure} = \text{the brown figure} + b = b + 1$$

$$\text{area of } S = \text{the blue figure} + b = \text{the brow figure} + 2b = 1 + 2b$$

$$(1 + 2b) = 1.5(1 + b) \Rightarrow 0.5b = 0.5 \Rightarrow b = 1 \Rightarrow 1 + 2b = 3$$




From above let $\beta$ be the angle in the most right corner. From the largest triangle: $\alpha + 2\beta = 180°$. The middle triangle is an isosceles triangle with angles $2\alpha$, $2\alpha$, $180° - 4\alpha$ . The triangle on the right is also isosceles:

$$\alpha + (180° - 4\alpha) + \beta = 180° \Rightarrow \beta = 3\alpha \Rightarrow \alpha + 6\alpha = 180° \Rightarrow \alpha = \frac{180°}{7}$$

## Level B

### Problem 1.B

A driver approaches a toll booth and randomly selects two coins from his pocket. If the pocket contain 2 quarters, 2 dimes, and 2 nickels, what is the probability that the value of the two coins he selects will be at least enough to pay the 30-cent toll?

Use *direct* counting to count the number of favourable outcomes.

Use *indirect* counting to count the number of favourable outcome and prove the same result.

**Solution**

The total number of ways to pick two coins from six: $\binom{6}{2} = 15$.

Favourable: 1 quarter, 1 nickel: $\binom{2}{1}\binom{2}{1} = 4$; 1 quarter, 1 dime: $\binom{2}{1}\binom{2}{1} = 4$; 1 quarter, 1 quarter: $\binom{2}{2} = 1$.

The probability is $\frac{9}{15} = \frac{3}{5}$.

If we count the unfavourable outcomes:: 1 nickel, 1 nickel: $\binom{2}{2} = 1$; 1 dime, 1 dime: $\binom{2}{2} = 1$; 1 nickel, 1 dime: $\binom{2}{1}\binom{2}{1} = 4$.

The probability is $1 - \frac{6}{15} = 1 - \frac{2}{5} = \frac{3}{5}$.

### Problem 2.B

Study the factoring below:

$$x^2 + y^2 + 2xy = x^2 + xy + xy + y^2 = x(x+y) + y(x+y) = (x+y)^2$$

Using the same idea, factor $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$.

Factor $4x^2 - 12xy + 9y^2 - 4xz + z^2 + 6yz$.

**Solution**

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x+y)^2 + z^2 + 2z(x+y)$$
$$= (x+y+z)^2$$
$$4x^2 - 12xy + 9y^2 - 4xz + z^2 + 6yz = (2x - 3y - z)^2$$

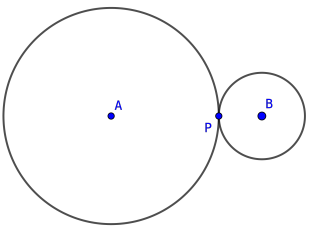
**Problem 3.B**

Two circles, one radius 5 cm, another of radius 2 cm, are tangent at point $P$. Two bugs start crawling at the same time from point $P$, one crawling along the larger circle at $3\pi$ cm per minute, the other crawling along the smaller circle at $\frac{5\pi}{2}$ cm per minute. How many minutes is it before their next meeting at point $P$.

If the first bug finishes the first circle and move over at point $P$ to the smaller circle and continue with the same speed along the circle until it got to the point $P$ and then goes back to the large circle. During the time when the first bug is on the smaller circle, does it meet the second bug at any time?

**Solution**

Let $n$ be the number of minutes they meet again at $P$. The first bug completes a $3\pi n$ distance and the second completes a $\frac{5\pi n}{2}$ distance. The circumference of the first circle is $10\pi$ and the second is $4\pi$.




So $3\pi n$ is a multiple of $10\pi$ and $\frac{5\pi n}{2}$ is a multiple of $4\pi$. Thus, $3n$ is a multiple of 10 and $5n$ is a multiple of 8. Therefore $n$ is a multiple of 10 and 8. The smallest value of such $n$ is 40.

It takes the first bug $\frac{10\pi}{3\pi} = \frac{10}{3}$ minutes to finish its circle. During that time the second bug has traveled $\frac{5\pi}{2} \cdot 103 = \frac{50\pi}{6} = 8\pi + \frac{\pi}{3}$, or two complete circles and a bit more. The remaining distance for the second bug to complete a circle is $4\pi - \frac{\pi}{3} = \frac{23\pi}{6}$.

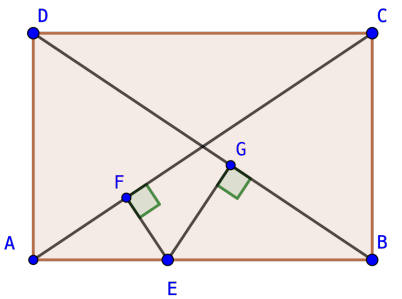
For every minute, the distance between the first bug and the second is reduced by $3\pi - \frac{5\pi}{2} = \frac{\pi}{2} < \frac{23\pi}{6}$, so the first bug will definitely takes over the second one at some point before it returns to $P$.

**Problem 4.B**

$ABCD$ is a rectangle. $E$ is a point on $AB$. Prove that the sum of the altitudes from $E$ to the diagonals $AC$, $BD$ is constant (meaning that it is the same regardless of the position of $E$ on $AB$).

If $E$ is on the extension of $AB$ but not inside $AB$, does the result still hold?

**Solution**




$$\triangle AFE \sim \triangle ABC \Rightarrow \frac{EF}{CB} = \frac{AE}{AC}$$

$$\triangle BGE \sim \triangle BAD \Rightarrow \frac{GE}{AD} = \frac{EB}{BD}$$

Hence $\dfrac{EF}{CB} + \dfrac{GE}{AD} = \dfrac{AE}{AC} + \dfrac{EB}{BD} \Rightarrow \dfrac{EF + GE}{BC} = \dfrac{AE + EB}{BD} = 1$

Therefore $EF + GE = BC$.

Easy to see that when $E$ is on the extension of $AB$ but not inside $AB$, the sum of the distances from $E$ to $AC$ and $E$ to $BD$ can grown to infinity, so the result does not hold.

## Level C

### Problem 1.C

Find all real numbers $x$ that satisfy the equation:

$$\frac{x-2020}{1}+\frac{x-2019}{2}+\ldots+\frac{x-2000}{21}=\frac{x-1}{2020}+\frac{x-2}{2019}+\ldots+\frac{x-21}{2000}$$

Generalize and solve the generic equation.

**Solution**

Consider the generic case, for any $n > 2k$ positive integers:

$$\frac{x-(n-k)}{k}-\frac{x-k}{n-k}=\frac{(n-k)x-(n-k)^2-kx+k^2}{k(n-k)}$$

$$=\frac{(n-2k)x-n(n-2k)}{k(n-k)}=(n-2k)\frac{x-n}{k(n-k)}$$

$$\sum_{k=1}^{m}\frac{x-(n-k)}{k}-\sum_{k=1}^{m}\frac{x-k}{n-k}=\sum_{k=1}^{m}(n-2k)\frac{x-n}{k(n-k)}$$

$$=(n-2k)(x-n)\sum_{k=1}^{m}\frac{1}{k(n-k)}$$

Since $\frac{1}{k(n-k)} > 0$, hence $x = n$ is the only solution.

### Problem 2.C

The integer 202020 is a multiple of 91. Show how an additional 2 may be inserted into the digits of 202020 so that the resulting 7-digit integer is also a multiple of 91. For every positive integer $n \geq 2$, show how $n$ additional 2's may be inserted into the digits of 202020 so that the resulting $(n+6)-$digit integer is also a multiple of 91.

Let $a_n$ be a number with properties described in the previous question. Conjecture a general form for $a_n$ and prove it.

**Solution**

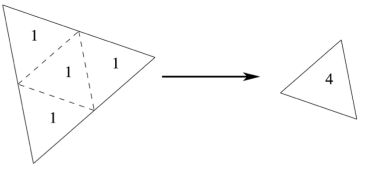
Let $a_0 = 202020$, $a_1 = 2022020 = 10a_0 + 1820$ divisible by 91.

Let prove that all the terms of the sequence $a_0 = 202020$, $a_{n+1} = 10a_n + 1820$ is divisible by 91 by induction principle. The induction hypothesis is true for $n = 0$, $n = 1$. $a_{n+1} = 10a_n + 1820$ has the form of 2022...2020 and divisible by 91.

**Problem 3.C**

Consider a sheet of paper in the shape of an equilateral triangle creased along the dashed lines as in the figure below on the left.

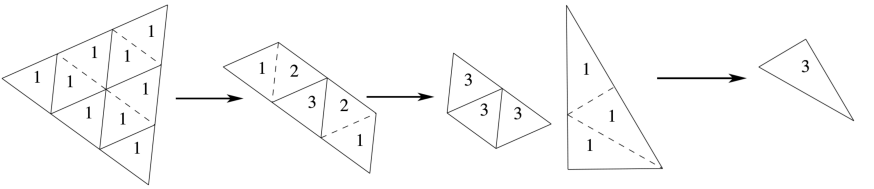
Folding over each of the three corners along the dashed lines creates a new object which is uniformly four layers thick, as in the figure below on the right.

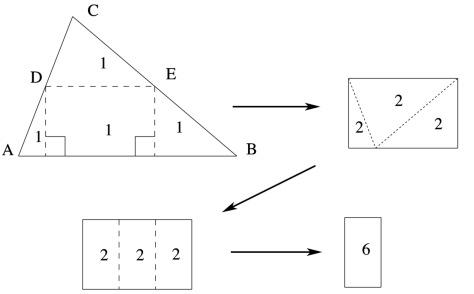



The number in each region indicates that region's thickness (in layers of paper). You can use a *magic ruler* with which you can draw a line through any two given points and you can split any segment into as many equal parts as you wish; and a *right triangle tool* with which you can drop perpendiculars from points to lines and erect perpendiculars to lines from points on them.

Show: 1/ how to fold an equilateral triangle into an object with a uniform thickness of 3 layers; 2/how to fold a $30^\circ - 60^\circ - 90^\circ$ triangle into an object with a uniform thickness of 3 layers; 3/every triangle can be folded into an object with a uniform thickness of 2020 layers.

**Solution by Catherine Doan**




For the last case, first let make rectangle with thickness of 2 layers and then divide the long edge by 3 to get a thickness of 6. If we wish to obtain the 2020 thickness, we need to divide the long edge into 1010 equal segments.

**Problem 4.C**

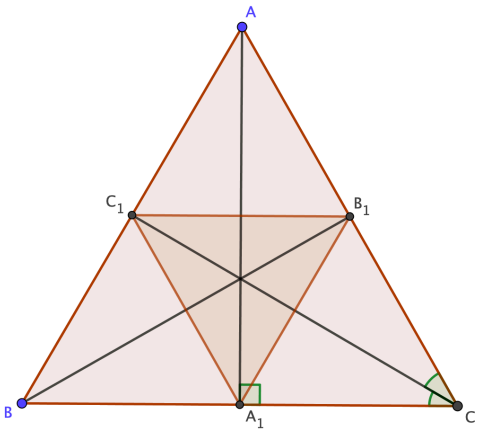
In triangle $\triangle ABC$, points $A_1$ on side $BC$, $B_1$ on side $CA$, and $C_1$ on side $AB$ so that $AA_1$ is an altitude, $BB_1$ is a median, and $CC_1$ is an angle bisector. It is known that $\triangle A_1B_1C_1$ is equilateral.

Prove that $C_1$ is on the circumcircle of $\triangle AA_1C$.

Prove that $C_1$ is midpoint of $AB$.

Prove that $\triangle$ABC is equilateral.

**Solution**




Because $A_1B_1C_1$ is an equilateral triangle $B_1A_1 = B_1C_1$, $B_1A_1$ the radius of the circumcircle of $AA_1C$ as the triangle is a right triangle and $B_1$ the midpoint of its hypotenuse, so $B_1C_1$ is also the radius of this circle. Therefore, point $C_1$ is on the circumcircle of triangle $A_1B_1C_1$.

From the circumcircle of $CC_1A$ where $AC$ is the diameter, we find that $\angle AC_1C = 90^\circ$, so $CC_1$ is both the altitude and the angle bisector of triangle $ABC$. This means that the line is also the median of triangle $ABC$, so $C_1$ is the midpoint of $AB$.

Hence triangle $ABC$ is isosceles at $C$: $BC = AC$. On the other hand $\triangle ABA_1$ is a right triangle with $C_1$ is the midpoint of $AB$, so $AB = 2C_1A_1 = 2C_1B_1 = AC$. Therefore $ABC$ is equilateral.

# Chapter 7

# Week 28

# Weekly Team Contest

## Problem Set 1.A

$AB$, $CD$, $EF$, $GH$, and $JK$ are five 2-digit numbers. *Different letters* represent *different digits*.

Find the greatest possible value of the fraction below:

$$\frac{AB + CD + EF}{GH - JK}$$

**Solution**

The strategy is to *minimize the denominator* and *maximize the numerator*.

The smallest possible denominator is 1, which can be obtained by using $20 - 19$, $30 - 29$, .... In order to save as many large digits as possible for the numerator, use $20 - 19$ for the denominator.

The numerator can be as large as possible if:

- all largest digits are the tens digits (in any order), and
- 3, 4, and 5 are the ones digits (in any order).

Hence, the greatest possible value of the fraction is:

$$\frac{83 + 74 + 65}{20 - 19} = 222$$

**Problem Set 1.B**

In an *up-and-down* counting number, the digits increase to a maximum digit and then decrease. This maximum is not the *first* or the *last* digit. For example 2<u>5</u>68<u>5</u>0 is an *up-and-dow*, but 2<u>55</u>876 is not.

How many different 4-digit *up-and-dow* numbers have the maximum digit of 6?

**Solution**

The digit 6 can be the hundreds digit or the tens digit. It cannot be both. We count the numbers in two cases separately.

<u>Case 1 - 6 is the hundreds digit</u>: The number is in _ 6 _ _ format. There are 5 possible values for the thousands digits: 1, 2, 3, 4, 5.

There are 5 possible values for the tens digits: 5, 4, 3, 2, 1. A tens digit of 4 produces 4 possible ones digits: 3, 2, 1, 0.

Similarly a tens digit of 3, 2, and 1 produces 3, 2, and 1 possible ones digits. Therefore the number of ways for the tens and ones digits is: $5 + 4 + 3 + 2 + 1 = 15$.

Hence the total of all possible numbers in this case is $5 \times 15 = 75$.

<u>Case 2 - 6 is the tens digit</u>: The number is in _ _ 6 _ format.

There are 4 possible values for the thousands digits: 1, 2, 3, 4.

A thousands digit of 1 produces 4 possible hundreds digits: 3, 2, 1, 0. Similarly to case 1, the number of ways to fill the thousands and hundreds digits is $4 + 3 + 2 + 1 = 10$.

There are 6 possible values for the ones digits: 0, 1, 2, 3, 4, 5.

Hence the total of numbers in this case is $10 \times 6 = 60$.

The answer is $75 + 60 = 135$.

**Problem Set 1.C**

Martin house has 7 rooms. Each door of a room opens into another room or is an outside entrance.

Below is the list of the numbers of doors of each room.

- Room #1 has 2 doors.
- Room #2 has 4 doors.
- Room #3 has 6 doors.
- Room #4 has 8 doors.
- Room #5 has 6 doors.
- Room #6 has 4 doors.
- Room #7 has 2 doors.

Martin said that the **number of outside entrance doors is 3**.

Is that possible? If you think it can be, show an example.

If you think it cannot be, prove that.

**Solution**

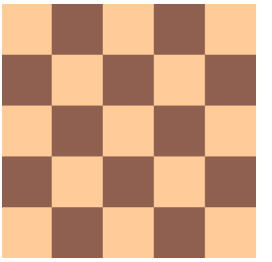
There are 32 doors from the point of view of the rooms.

3 doors connect to the outside, meaning those doors are only ever counted once. The rest of the 29 doors are shared by two rooms, meaning each is counted twice. However, 29 is an odd number and therefore can't be divided by 2.

Therefore, it's impossible to have such a set up.

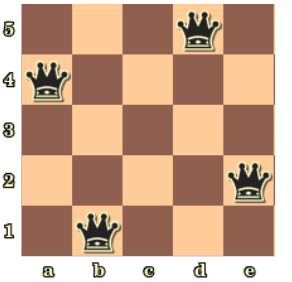
## Problem Set 2.A

A *super-queen* is a chess *queen* that *also moves like a knight*.



Place **four** super-queens on a $5 \times 5$ board so that **no piece attacks another**.

**Solution**

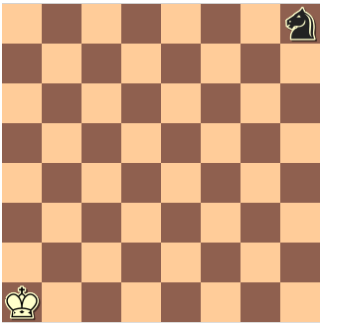



One solution is shown as above.

## Problem Set 2.B

White moves first.

**White to move**



How long the King can avoid the Knight?

In **how many steps** the white King can **avoid being checked** by the black Knight, including the first move?

**Solution**

A knight has to move from one square to a different coloured square i.e. black to white and vice versa.

However, as the king can choose which colour it wants to move to, the strategy is to move to the same colour square the knight is currently standing on. For example, if the knight was on a black square, if the king moved to a black square it can't get checked by the knight.

So there are infinite steps.

## Problem Set 2.C

25 knights on a $5 \times 5$ chessboard *move at the same time*.

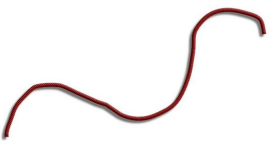
Can each of them land on a **separate square**? In another word, the situations before and after the move is the same: *each square has a single knight standing on it*.

**Solution**

It is impossible. We have 13 knights on the black squares. When they jump, they will switch colour. But we do not have 13 white squares.

**Problem Set 3.A**

Kevin have a string that is two-thirds of a meter long. He needs a shorter string that is exactly half a meter long.



With **only a pair of scissors** and **no any ruler**, can he cut it from the string?

**Solution 1**

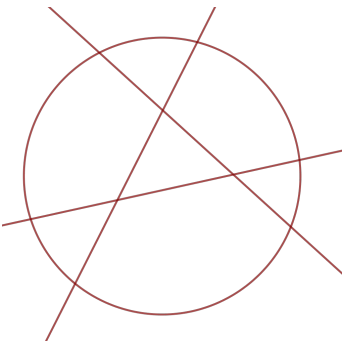
The string is $\frac{2}{3}$ meter, which is $\frac{4}{6}$ meter. In order to get $\frac{1}{2}$ meter, we have to cut $\frac{4}{6} - \frac{1}{2} = \frac{1}{6}$ meter. Since $\frac{1}{4}$ of $\frac{4}{6}$ is $\frac{1}{4} \cdot \frac{4}{6} = \frac{1}{6}$ meter, so we fold it in half twice, give us 4 pieces of $\frac{1}{6}$ meter, remove one of them gives us a $\frac{1}{2}$ meter string.

**Solution 2**

He needs to fold the string into four equal sections, and then cut off a quarter of it. This is because $\frac{2}{3} = \frac{4}{6}$ and $\frac{4}{6} - \frac{1}{6} = \frac{1}{2}$.

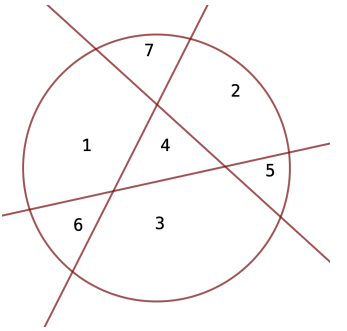
**Problem Set 3.B**

The three lines in the figure divide the circle into seven regions.



Can you help Binh to place the numbers 1 to 7 in these regions so that for each of the lines, the sums of the numbers on either side are all the same?

**Solution**

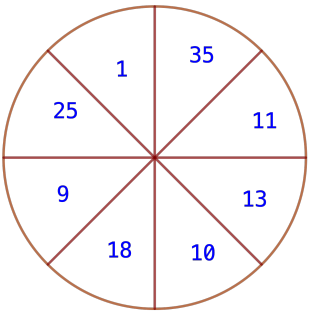



This is one of the four possible solutions of so called the Fano Configurations.

## Problem Set 3

### Level C

9 people throw darts at a target. The number in a sector indicate the points someone can get if that person throws a dart into that sector.




Each person throws 4 darts. No two people have the same set of scores on all of their throws.

Interestingly, each person scores 62 points. How is that **possible**?

**Solution**

Here are the nine possible scores.

(35, 25 , 1, 1) (35, 18, 9, 0) (35, 13, 13, 1)

(35, 9, 9, 9) (25, 25, 11, 1) (25, 18, 18, 1)

(25, 18, 10, 9) (25, 13, 13, 11) (18, 18, 13, 13)

**Problem Set 4.A**

The fraction $\frac{1}{37}$ expended in decimal representation is as follow:

$$\frac{1}{37} = 0.0270270270...$$

In the representation above, after the decimal point "." the first 10 digits are 0270270270. The digit 2 repeats 3 times in these 10 digits.

When the number $\frac{1}{37}$ is expanded into decimal representation, what is the **number of times the digit 7 repeated** in the first 2020 digits?

**Solution**

The digit 7 comes up every 3rd digit. $2019/3 = 673$, so 7 will show up 673 times for the first 2020 digits.

**Problem Set 4.B**

Anthony creates a sequence of fractions as follow:

*If he has fraction* $\frac{m}{n}$, *then the next one is* $\frac{1}{1+\frac{m}{n}}$

For example if he has $\frac{4}{5}$, then the next one is $\frac{1}{1+\frac{4}{5}} = \frac{5}{9}$.

The first fraction of the sequence is $\frac{1}{2}$.

What would be the fraction which has 2584 **as the numerator**?

**Solution**

The fraction $\frac{1}{1+\frac{m}{n}}$ can be simplified into $\frac{n}{m+n}$.

So the numerator is the sum of the previous two numerators.

The sequence begins with $\frac{1}{2}$, which means that the numerators form the Fibonacci sequence $1, 2, 3, 5, \cdots$

Since the term 2584 is followed by 4181 in the Fibonacci sequence so the fraction is $\frac{2584}{4181}$.

**Problem Set 4.C**

There are a 3-tuple of different positive integers (2, 3, 6) such that the sum of their reciprocals is 1:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$$

Find a **5-tuple** of different positive integers $(a, b, c, d, e)$ such that the sum of their reciprocals is also 1:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}=1$$

**Solution**

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{24}=1$$

**Problem Set 5.A**

A group of islands are connected by bridges in such a way that one can walk from any island to any other.

Lily walked around every island, crossing each bridge exactly once. She visited the island of Ott *an odd amount of times*.

How **many bridges are there to Ott**, if Lily neither started, nor ended in Ott?

**Solution**

Lily neither started, nor ended in Ott, and each bridge can be crossed once so each time when she visited the island of Ott and left it, she has crossed two different bridges.

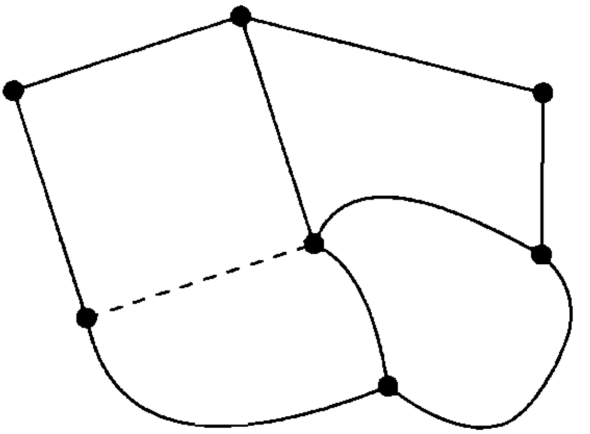
So the number of bridges to Ott is twice the number of times Lily visited Ott.

**Problem Set 5.B**

There are *seven* lakes in Laketown. They are connected by *ten* canals so that anyone can swim from any lake to any other.

How many **islands** are there in Laketown?

**Solution**



To prove the solution correctness, students might have to learn Euler's planar graph formula as below:

*If planar graph is drawn in the plane without any edge intersections, and v is the number of vertices, e is the number of edges and f is the number of faces (regions bounded by edges, including the outer, infinitely large region):*

$$v - e + f = 2$$

If the graph can be drawn as above, it is easy to conclude that there are 4 islands and that would be enough for this problem.

## Problem Set 5.C

Viet was elected to be a mayor of the Bright county, which has 30 towns. The county electricity network supplies the towns. *Every two towns are connected by a single distinct cable.*

In order to save the maintenance bill, Viet decided to cut some cables.

What is the **maximum number of cables** he can cut so that there is still light in the towns?

**Solution**

The total number of cables is $\binom{30}{2} = \frac{30 \cdot 29}{2} = 435.$

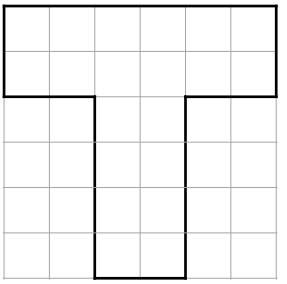
You can start with two towns and see that we need one cable to keep two towns connected. For 3 towns, that is 2, and so on. So the minimum number of cables to keep 30 towns connected is 29.

Therefore he can to cut no more than $435 - 29 = 406$ cables.

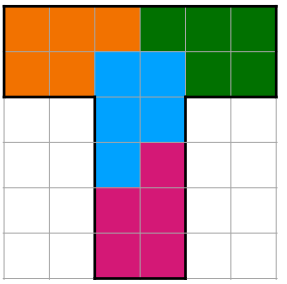
## Problem Set 6

### Level A

Divide the following shape into **4 identical figures**.



**Solution**



## Problem Set 6.B

The clock in Lilian's house has stopped. Lilian wanted to set the correct time, but she was unable to do that, since there is the only one clock in the house. That evening, she went to visit her friend, Bianca, who has an excellent wall clock that always shows correct time. When Lilian came home after the visit, she was able to set the correct time on her clock.

**How did she do it?**

*Note that Lilian's clock remained at Lilian's house, Bianca's clock never left Bianca's house. In addition while Lilian always walks with the same speed and the road is flat, she does not know how long it takes her to get from her house to Bianca's house.*

**Solution**

Lilian checked the time displayed on her clock twice: when she left the house and when she came back. The difference of these two values will be equal to the total time when she was away. Assume that she walked to Bianca house, took a look at the clock and promptly left, the difference above is twice the length of the trip between the two houses. By marking the time shown on Bianca's clock, she should add that to the length of the trip and set the time on her clock.

## Problem Set 6.C

Let $\triangle ABC$ be a triangle with altitude $CD$ and $AE$. Furthermore$BD = 3, DA = 5, BE = 2$.

What is the **area** of $\triangle ABC$?

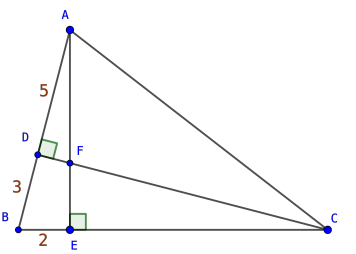
**Solution**

Because $\triangle ABE$ is a right:

$AE = \sqrt{AB^2 - BE^2} = \sqrt{64 - 4} = 2\sqrt{15}$

$\triangle ABE \sim \triangle CBD$ (both right triangle, share the same $\angle B$):

$$\frac{CB}{AB} = \frac{BD}{BE} \Rightarrow CB = \frac{3}{2} \cdot 8 = 12$$




$\triangle CBD$ is a right triangle: $DC = \sqrt{12^2 - 3^2} = 3\sqrt{4^2 - 1} = 3\sqrt{15}$

Therefore $S_{\triangle ABC} = \dfrac{AB \cdot CD}{2} = \dfrac{8 \cdot 3\sqrt{15}}{2} = 12\sqrt{15}$.

# Problems of the Week

**Level A**

### Problem 1.A

There's a box of three hats: one black and two white. Anthony and Bianca (each very smart and very logical) each place a hat on his or her head, while blindfolded. One by one, each student removes his/her blindfold and (without using a mirror) gets one opportunity to guess the colour of the hat on his/her own head. If any of the two guesses correctly, they win. First, Bianca removes her blindfold. She sees the hats that Anthony is wearing, but admits that she is unable to discern her own hat colour. Then Anthony says: "I can answer with my blindfold on! I know what colour hat I am wearing".

What colour is Anthony's hat?

Now the game is played by three students: Anthony, Bianca, and David with a box of five hats: two black and three white. David remove his blindfold first but he doesn't know the colour of his hat. The same happens to Bianca. However, Anthony, with the blindfold still on, can guess it.

Now, what colour is Anthony's hat?

### Solution

Since there is only one black hat. If Anthony wears that, Bianca could know immediately that she has a white hat. She has admitted that she was unable to discern her own hat colour. Hence, Anthony knew that his hat was white.

For the second question, since David doesn't know his hat colour we know that Bianca and Anthony can't both have black hats so he sees one of the following:

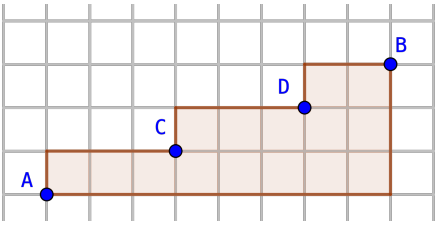
- Bianca and Anthony both have white hats
- Bianca have a black hat and Anthony have white
- Bianca have a white hat and Anthony have black

If Bianca saw that Anthony had a black hat, she would know that she has a white hat but since Bianca doesn't know the colour of her hat we can deduce that Anthony has a white hat.

**Problem 2.A**

A bug moves from A to C, then to D, and finally to B. The bug can only move up or right along the grid lines, on the border or inside the shaded figure.

How many paths are there that the bug can travel?




Find the number of paths from A to B, not necessarily via C or D.

**Solution**

From $A$ to $C$, the bug has to make 4 moves, one of four is up and three to the right. So there are $\binom{4}{1} = 4$ ways to move from $A$ to $C$. Similarly there are $\binom{4}{1} = 4$ ways to move from $C$ to $D$, and $\binom{3}{1} = 3$, from $D$ to $B$ resulted in altogether $4 \times 4 \times 3 = 48$ ways.

The second question can be answer by using the same method: there are three moves upwards in eleven moves, which makes the total number of ways to move from $A$ to $B$ is $\binom{11}{3} = 165$.

**Problem 3.A**

For each pair of real numbers $a \neq b$, define $a \star b = \dfrac{a+b}{a-b}$.

What is the value of $((((1 \star 2) \star 3) \star 4) \star 5)$?

Now define $a@b = ab - b^2$ and $a\#b = a + b - ab^2$.

What is the value of $\dfrac{6@3}{6\#3}$?

**Solution by**

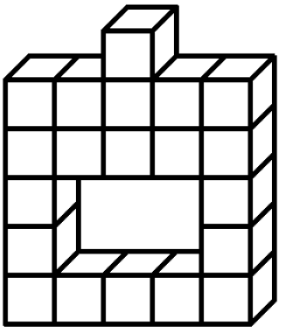
$$1 \star 2 = \frac{1+2}{1-2} = -3, (1 \star 2) \star 3 = (-3) \star 3 = \frac{-3+3}{-3-3} = 0$$

$$(((1 \star 2) \star 3) \star 4) \star 5 = (0 \star 4) \star 5 = \frac{-1+5}{-1-5} = -\frac{4}{6} = -\frac{2}{3}$$

$$\frac{6@3}{6\#3} = \frac{6 \cdot 3 - 3^2}{6 + 3 - 6 \cdot 3^2} = -\frac{9}{-45} = \frac{-1}{5}$$

**Problem 4.A**

In square units, what is the surface area of this figure made from 20 unit cubes?



What is the number of edges of the cubes on the surface?

**Solution**

From the front and the back, there are 20 squares in each view. From the left and the right, there are 6 squares in each view. From the top and bottom, there are 5 squares in each view. Finally the internal hole covered by 8 squares. In total $2 \cdot 20 + 2 \cdot 6 + 2 \cdot 5 + 10 = 72$ squares. So the surface area is 72.

For the second question, similar counting is used. From the front there are 20 squares in view. Each squares has 4 edges. Among these there are 24 edges that are shared by two neighbour squares. Therefore there are $4 \times 20 - 24 = 56$ edges on the front face. Similarly there are 56 edges on the back face.

Note that we already count the edges shared between the front or the back face with the side faces, so on the side faces we need to count only those edges that are not shared by the front or the back face. The number of them is $6 + 6 + 6 + 4 = 22$ on the outside side faces, and $4 + 4 + 1 + 1 = 10$ on the hole faces.

So the total number of visible edges is:

$$2 \times 56 + 22 + 10 = 144$$

## Level B

### Problem 1.B

Kevin tells Linh Chi and Michael that he is thinking of two consecutive integers from 1 to 10. He tells Linh Chi one of the numbers and he tells Michael the other number. Then the following conversation occurs between Linh Chi and Michael.

Linh Chi: I don't know your number.

Michael: I don't know your number either.

Linh Chi: Ah, now I know your number.

Assuming both Linh Chi and Michael used correct logic, what could be one possible pair of number that Linh Chi and Michael have?

What are all possible pairs of number that Linh Chi and Michael could have?

**Solution**

We find all possible pairs of number that Linh Chi and Michael could have.

Because Kevin thought of two consecutive integers, if Linh Chi got 1 or 10, she will know Michael's number. So her number was neither 1 nor 10. Michael knew what Linh Chi's number could not be, but he was unable to tell what it was. This means that his number was neither 2 nor 9, otherwise he knew Linh Chi's number should be 3 or 8.

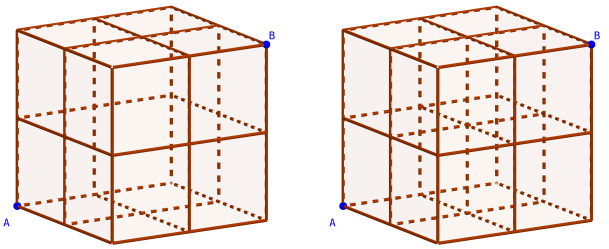
Therefore, if Linh Chi's number was 2 or 9, she knew Michael's number, which was 3 or 8. Similarly if Linh Chi's number was 3 or 8, she knew Michael's number, which was 4 or 7. If Linh Chi's number was 4, 5, 6, or 7, she was unable to tell Michael's number.

Hence the sum of possible pairs is $(2 + 9) + (3 + 8) = 22$.

**Problem 2.B**

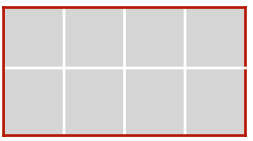
A $2 \times 2 \times 2$ cube is built from 8 unit cubes as in the left diagram.

How many paths of length 6 *on the surface* of the cube are there from $A$ to $B$ along the edges of the component unit cubes?




Now all paths through the inner of the cube are also allowed as in the right diagram. How many paths of length 6 of the cube are there from $A$ to $B$ along the edges of the component unit cubes?

**Solution**



Take two faces, one has A, the other has B as a vertex, share one edge. A *flattened* diagram with A, B at the lower left, upper right corners.

The number of path with length 6 and only two moves upwards from A to B on this is $\binom{6}{2}$. For all six pairs of faces, we have $6 \times \binom{6}{2}$. Any path along the borders as the bold line in above is counted twice by two different pairs of faces, there are $6 \times \binom{4}{2}$ such paths.

Hence, the answer for the first question is:

$$6 \cdot \left( \binom{6}{2} - \binom{4}{2} \right) = 6 \cdot (15 - 6) = 54.$$

The second question requires counting ways length 6, each formed by two moves in a dimension of three dimensions. So we can choose two moves in $z-$dimension from 6 possibilities, but only 2 moves in $y-$dimension from the remaining 4 possibilities, and 2 moves in $x-$dimension from the remaining 2 possibilities. So the total number of ways is: $\binom{6}{2}\binom{4}{2}\binom{2}{2} = \dfrac{6!}{2!2!2!} = 90.$

**Problem 3.B**

For the positive $n$, define $< n >$ as the sum of all positive divisors of $n$, except $n$. For example $< 4 > = 1 + 2 = 3$.

What is $< < < 18 > > >$ ?

Is there a number larger than 18 with the same property?

**Solution**

$$< 18 > = 1 + 2 + 3 + 6 + 9 = 21$$

$$< < 18 > > = < 21 > = 1 + 3 + 7 = 11$$

$$< < < 18 > > > = < 11 > = 1$$

The special property of 18 is that after applied three times the operator $< >$, the result of 1 is obtained.

Since $< 49 > = 1 + 7 = 8$, $< 8 > = 1 + 2 + 4 = 7$, $< 7 > = 1$

So 49 has the same property.

*Note that not all numbers are reducible by the operator* $< >$ *, for example all so called perfect numbers such as* 6, 28, . . . *are* ***not reducible:***

$$< 6 > = 1 + 2 + 3 = 6$$

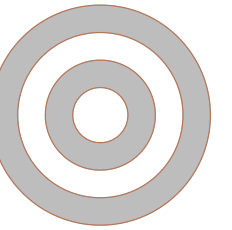
**Problem 4.B**

The dartboard below has a radius of 6 inches. Each of the concentric circles has a radius two inches less than the next larger circle. If nine darts land randomly on the target, how many darts would we expect to land in a non-shaded region? *Think of probability in terms of area.*

Now, let add another circle around the whole figure with radius 2 inches more than the dartboard and let shade it. So now you have two shaded regions. Answer the same question.

**Solution**

The shaded area is the difference of the second largest circles and the smallest circle, which is $4^2\pi - 2^2\pi = 12\pi$.



The total area is the area of the largest circle, which is $6^2\pi = 36\pi$. The probability a dart lands on the shaded area is $\frac{12}{36} = \frac{1}{3}$.

Therefore if 9 darts are throw, $\frac{1}{3} \cdot 9 = 3$ of them will probably landed on the shaded area. Hence 6 of them land on the unshaded area.

The shaded area in the second question is increased by $8^2\pi - 6^2\pi = 28\pi$, so it is $12\pi + 28\pi = 40\pi$. The total area is the area of the largest circle, which is $8^2\pi = 64\pi$. The probability now is $\frac{40}{64} = \frac{5}{8}$.

Therefore if 9 darts are throw, $\frac{5}{8} \cdot 9 = 5.625$ of them will probably landed on the shaded area. Hence not more than 3 of them would land on the unshaded area.

## Level C

### Problem 1.C

You are traveling in a foreign country whose currency consists of five different-looking kinds of coins. You have several of each coin in your pocket. The coins are worth 1, 2, 5, 10, and 20 florins, but you have no idea which coin is which. You find a vending machine where a single candy can be bought for 1 florin: you insert any kind of coin, and receive 1 candy plus any change owed. You can only buy one candy at a time, but you can buy as many as you want, one after the other.

What is the least number of candies that you must buy to ensure that you can determine the values of all the coins?

Give a counter example in that a smaller number of candies is used but cannot help to determine values of the coins.

**Solution**

The answer is four. First we show that three candies are not always enough. If you only buy three candies, then it is possible the three coins you spend will be some combination of 1-, 2-, and 5-florin coins, in which case you definitely won't receive any 10- or 20-florin coins in change. Thus, in this situation, you do not get any information that can be used to distinguish the 10- and 20-florin coins.

Now we show that four candies are enough. In each of these transactions, pay with a different kind of coin. If any of the transactions does not produce change, then you must have paid in that transaction with a 1-florin coin. If all the transactions produce change, then the coin you didn't use as payment is the 1-florin coin. In either case, you can identify the 1-florin coin. Since the change for 2 florins will be 1 florin, you can now tell if you paid for any of the transactions with a 2-florin coin, and thereby identify which of the five coins has a value of 2 florins. Next, we know that the change for 5 florins will be either two 2-florin coins or four 1-florin coins, or one 2-florin and two 1-florin coins; in any event, you can recognize these collections of coins and then deduce which coin is the 5-florin coin (if you see one of these collections, then you know which coin was the 5-florin coin; otherwise, you know it was the coin you didn't use as payment). Likewise, change for 10 florins will be a collection of coins that total 9 florins, and all such collections can now be recognized. Thus we can deduce which coin is the 10-florin coin, and finally we can apply the same procedure to deduce the identity of the 20-florin coin.

**Problem 2.C**

Suppose that 2002 numbers, each equal to $+1$ or $-1$, are written around a circle. For every two adjacent numbers, their product is taken, and the sum of all 2002 such products is negative.

Prove that the number of the $-1$ products is more than 1001.

Prove that the sum of the original numbers has absolute value less than or equal to 1000.

**Solution**

Suppose that $n$ of the original numbers are $+1$, so that the remaining $2002 - n$ are $-1$, and their sum is:

$$1 \cdot n + (-1) \cdot (2002 - n) = 2n - 2002.$$

Also suppose that $m$ of the products equal 1; then the remaining $2002 - m$ equal 1, and the sum of the products is

$$1 \cdot (2002 - m) + (-1) \cdot m = 2002 - 2m$$

Since this is negative, $m > 1001$.

Now, each product of 1 must come from a $+1$ and a 1 among the original numbers, and each 1 among the original numbers can contribute to at most two such products; hence the original numbers include more than $\frac{1001}{2} > 500$ of $-1$'s. Thus,

$$2002 - n \geq 501 \Rightarrow n \leq 1501 \Rightarrow 2n - 2002 \leq 1000$$

Similarly, the original numbers include more than 500 $+1$'s, so

$$n \geq 501 \Rightarrow 2n - 2002 \geq -1000$$

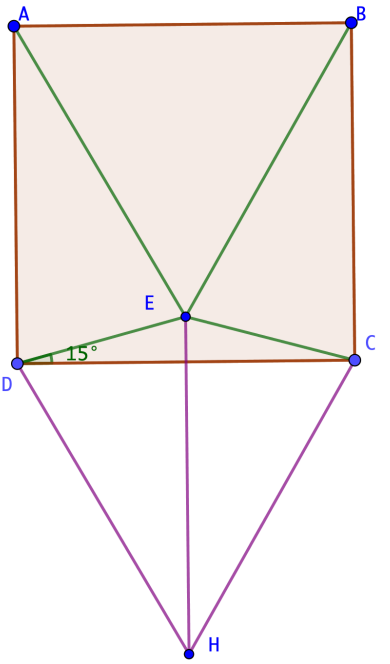
Thus, $-1000 \leq 2n - 2002 \leq 1000$, which is what we need.

**Problem 3.C**

The area of square $ABCD$ is 196. Point $E$ is inside the square, at the same distances from points $D$ and $C$, and such that $\angle DEC = 150°$. What is the perimeter of $\triangle ABE$?

Let $H$ be the foot of the altitude from $E$ to $BC$ in $\triangle EBC$. Use the angles formed by these points. Find the value of $\tan 15°$.

**Solution**




Since $E$ is same distance from $D$ and $C$, so $\triangle DEC$ is isosceles. Because $\angle DEC = 150° \Rightarrow \angle DCE = 15°$.

Draw $DCH$ equilateral triangle outside of $ABCD$. Since $AD = DH$, $\angle ADE = 75° = \angle HDE \Rightarrow \triangle ADE \cong \triangle HDE$. So $\angle DAE = 30°$, or $\angle EAB = 60°$. Similarly $\angle EBA = 60°$. So $\triangle ABE$ is equilateral. Hence its perimeter is $3 \times AB = 3 \times 14 = 42$.

The altitude from $E$ to $DC$ is $14 - \dfrac{14\sqrt{3}}{2} = 7(2 - \sqrt{3})$.

Therefore: $\tan 15° = \dfrac{7(2 - \sqrt{3}}{7} = 2 - \sqrt{3}$

**Problem 4.C**

Consider the $n \times n$ "multiplication table" below. The numbers in the first column multiplied by the numbers in the first row give the remaining numbers in the table.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | ... | $n-1$ | $n$ |
| 2 | 4 | 6 | ... | $2(n-1)$ | $2n$ |
| 3 | 6 | 9 | ... | $3(n-1)$ | $3n$ |
| ... | ... | ... | ... | ... | ... |
| $n-1$ | $2(n-1)$ | $3(n-1)$ | ... | $(n-1)(n-1)$ | $(n-1)n$ |
| $n$ | $2n$ | $3n$ | ... | $(n-1)n$ | $n \cdot n$ |

We create a path from the upper-left square to the lower-right square by always moving one cell either to the right or down. For example, in the case $n = 5$, here is one such possible path, with all the numbers along the path circled:

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 2 | 4 | 6 | 8 | 10 |
| 3 | 6 | 9 | 12 | 15 |
| 4 | 8 | 12 | 16 | 20 |
| 5 | 10 | 15 | 20 | 25 |

If we add up the circled numbers in the example above (including the start and end squares), we get 93. Considering all such possible paths on the $n \times n$ grid:

What is the smallest sum we can possibly get when we add up the numbers along such a path?

What is the largest sum we can possibly get when we add up the numbers along such a path?

**Solution**

The minimum is achieved by staying on the perimeter of the grid, and the maximum by staying as close to the main diagonal as possible. To see why this is true, tilt the grid 45 degrees.

Now every path must include exactly one number from each row. The $n^{th}$ row consists of the number $k(n+1-k)$ for $k = 1, 2, \ldots, n$. As a quadratic function of $k$, this expression is greatest when $k = \dfrac{n+1}{2}$ and gets smaller as $k$ gets farther from in either directions.

Thus the smallest number in row $n$ of the tilted grid is the number on either end of the row, which is $n$, and the largest number in row $n$ is the number closest to the centre, which is $\lfloor \frac{n+1}{2} \rfloor \cdot \lceil \frac{n+1}{2} \rceil$.

By staying on the perimeter of the grid, we may select the smallest entry from every row, and so accumulate the smallest possible total. By staying in the middle, we might select the largest entry from every row and accumulate the largest possible total.

Therefore the minimum total is:

$$(1+2+\ldots+(n-1)) + (n+2n+\ldots+n^2)$$
$$= \frac{n(n-1)}{2} + n\frac{n(n+1)}{2} = \frac{n(n^2+2n-1)}{2}$$

The maximum total is:

$$1^2 + 1\cdot 2 + 2^2 + 2\cdot 3 + \ldots + (n-1)n + n^2$$
$$= (1^2+2^2+\ldots+n^2) + \left(1.2 + 2.3 + \ldots + (n-1)n\right)$$
$$= \frac{n(n+1)(2n+1)}{6} + 2\left(\binom{2}{2} + \binom{3}{2} + \ldots + \binom{n}{2}\right)$$
$$= \frac{n(n+1)(2n+1)}{6} + 2\binom{n+1}{3} = \frac{n(n+1)(4n-1)}{6}$$

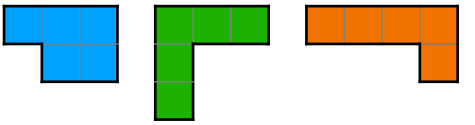
# Chapter 8

# Week 29

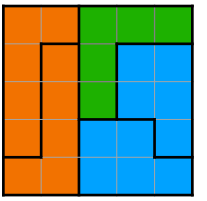
# Weekly Team Contest

**Problem Set 1.A**

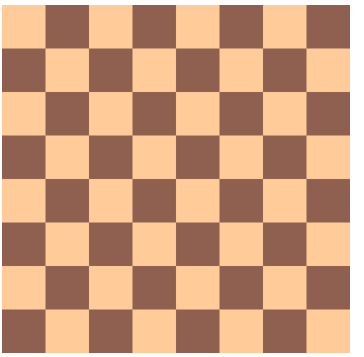
Martin has a lots of these figures.



The maximum number of them that Martin can place on a $5 \times 5$ chessboard is 5. Below is an example how he does it.

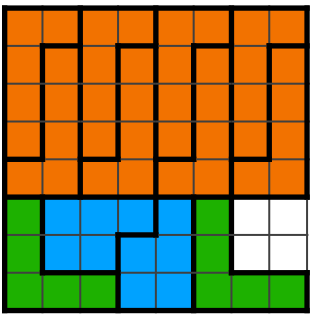


What is the **greatest number** of pieces that he can place on *a standard* $8 \times 8$ *chessboard*?



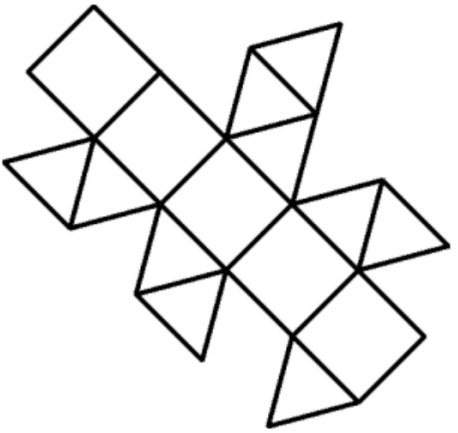
**Solution**

Every figure has 5 squares, and $8 \times 8 = 64$, so the best possible way to cover it is to use 12 figures. One solution is shown as below.



**Problem Set 1.B**

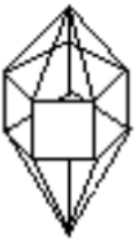
The diagram below shows a net that has *five square* faces and *ten equilateral triangular* faces. This can be folded into a *fifteen*-faced polyhedron.



**How many edges** does the polyhedron have?

**Solution**

The net with 5 squares and 10 equilateral triangles is folded in to a 15-faced polyhedron as shown below.

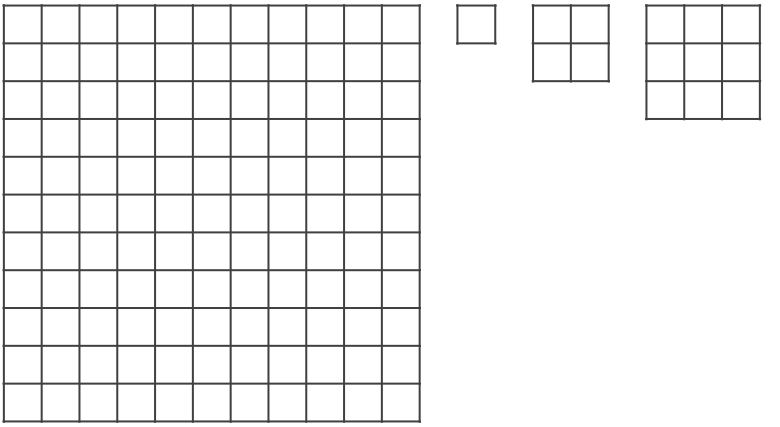


Since there are five squares with 4 sides each, they contributes 20 sides. There are 10 triangles with 3 sides each, which contributes 30 more sides. Notice that every side of a square or triangle hooks up with a side of another square of triangle to form an edge.

Therefore the 50 sides we have will form $\frac{50}{2} = 25$ edges.

**Problem Set 1.C**

Elisa has a $11 \times 11$ board & piles of $1 \times 1$, $2 \times 2$, and $3 \times 3$ tiles.

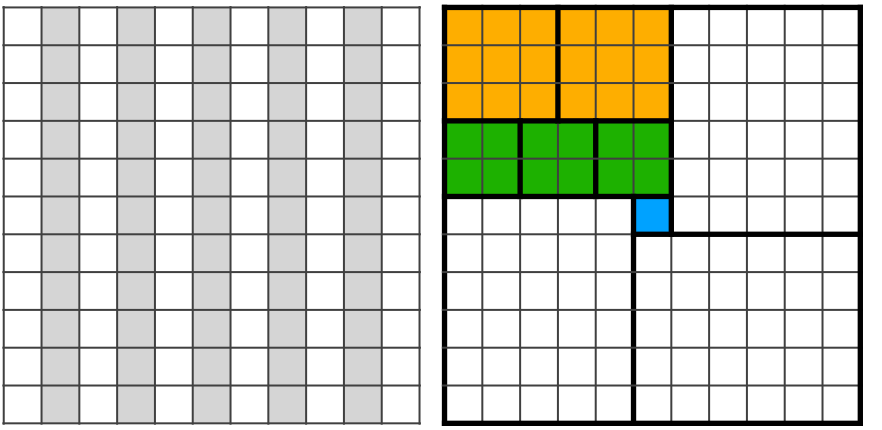


What is the **minimum number of** $1 \times 1$ **tiles** that she must use together with some $2 \times 2$, and $3 \times 3$ tiles to tile the board?

Prove it and show an example of how such tiling can be done.

**Solution**

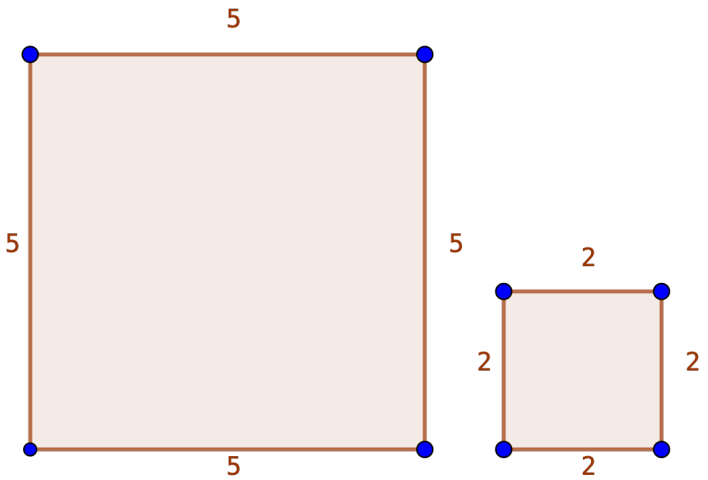
Let colour the columns of the $11 \times 11$ board alternately as left.



Easy to see that the amount of black and white squares in a $2 \times 2$ tiles is the same but differ by 3 in a $3 \times 3$ tiles. So if there is no $1 \times 1$ tile, then the difference of black and white squares must be divisible by 3. Since this difference is 11 (more white squares), so there must be at least one $1 \times 1$ tile. Above on the right is one way to tile it with the $1 \times 1$ in the centre and four $5 \times 6$ rectangles in a nice symmetric pattern.

**Problem Set 2.A**

The diagram below shows two squares with integer side length (the length of a side is an integer number).




The total area of the squares is $25 + 4 = 29$. The difference of their areas is $25 - 4 = 19$.

Find the two squares with integer side lengths such that *the total area of the squares is 325* and **the areas' difference is as small as possible**.

**Solution**

Following pairs of positive integers have the sum of their squares equal to 325: (1, 18), (6, 17), (10, 15).

The least difference of the squares is $15^2 - 10^2 = 125$.

**Problem Set 2.B**

The number 81 has an interesting property: *it has* 2 *digits, and its* $2^{nd}$ *(square) root is equal to the sum of its digit.*

$$\sqrt{81} = 8 + 1$$

Can you find a 3-digit number that has similar property: *the* $3^{rd}$ *(cubic) root of the number is equal to the sum of its digit?*

**Solution**

$$\sqrt[3]{512} = \sqrt[3]{2^9} = \sqrt[3]{8^3} = 2^3 = 8 = 5 + 1 + 2$$

## Problem Set 2.C

Determine the value of the following expression:

$$\left(\sqrt{17+12\sqrt{2}}\right)^3+\left(\sqrt{17-12\sqrt{2}}\right)^3$$

**Solution**

$17+12\sqrt{2}=3^2+2\cdot 3\cdot 2\sqrt{2}+(2\sqrt{2})^2=(3+2\sqrt{2})^2$

Similarly: $17-12\sqrt{2}=(3-2\sqrt{2})^2$. Therefore:

$$\left(\sqrt{17+12\sqrt{2}}\right)^3=(3+2\sqrt{2})^3,\ \left(\sqrt{17-12\sqrt{2}}\right)^3=(3-2\sqrt{2})^3$$

Furthermore:

$(3+2\sqrt{2})^3=3^3+3\cdot 3^2\cdot(2\sqrt{2})+3\cdot 3\cdot(2\sqrt{2})^2+(2\sqrt{2})^3$

$(3-2\sqrt{2})^3=3^3-3\cdot 3^2\cdot(2\sqrt{2})+3\cdot 3\cdot(2\sqrt{2})^2-(2\sqrt{2})^3$

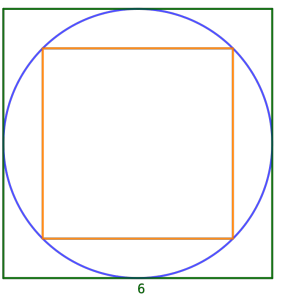
Hence, their sum is:

$2\cdot 3^3+2\cdot 3\cdot 3\cdot(2\sqrt{2})^2=54+18\cdot 8=198$

## Problem Set 3.A

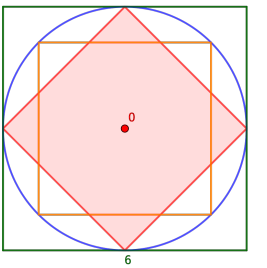
Inside a green square with side length 6, Victor drew a blue circle that is tangent with all sides of the square. He then drew orange square whose sides are parallel with the sides of the green square, and the orange square vertices are on the blue circle.

What is **the area** of the orange square?



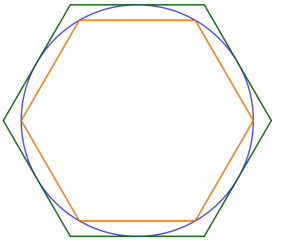

**Solution**

By rotating the orange square 90° around the centre O, we obtain a red square. Easy to see that this red square is exactly half of the green square. So its area is $\dfrac{6\cdot 6}{2}=18$.



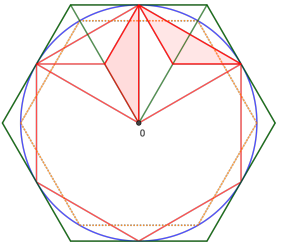
## Problem Set 3.B

Inside a green hexagon with area 24, Victor drew a blue circle that is tangent with all sides of the hexagon, then an orange square whose sides are parallel with the sides of the green hexagon, so the orange hexagon vertices are on the blue circle. What is **the area** of the orange hexagon?



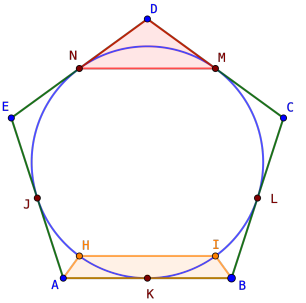
### Solution

When rotating 30° around the centre O, the orange hexagon became a red hexagon. This hexagon is divided into 18 red triangles shown as above, while the green hexagon can be divided into 24 such triangles. Therefore the area of the red hexagon is $\frac{18}{24} \cdot 24 = 18$.



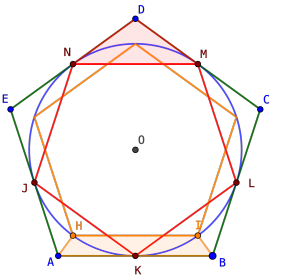

## Problem Set 3.C

Inside a green pentagon $ABCDE$, Victor drew a blue circle that is tangent with all sides of the hexagon at $J$, $K$, $L$, $M$, $N$. $H$ and $I$ are midpoints of the arcs $JK$ and $KL$. The quadrilateral $ABIH$ has an area of 3. What is **the area** of the red triangle $DMN$?




### Solution

By rotating the orange pentagon 36° around the centre O, we obtain a red pentagon. It means that the difference between the green pentagon and the orange, which is 5 times the area of the orange trapezoid, is the same as the difference between the green pentagon and the red pentagon, which is 5 times the area of the red triangle. There for the area of the red triangle is 3.



## Problem Set 4.A

Four frogs sit in a row. Every 5 seconds two neighbouring frogs hop into each other places. Sometime after 80 seconds, but before 100 seconds, the frogs are seen to be in their original order.

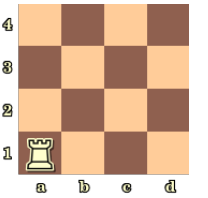
Is it possible to determine on **which jump** that occurred?

**Solution**

Let the positions of the frogs be 0, 1, 2, 3. If a frog changed it position after a jump, then the order of their positions is changed. Therefore if they return to the original position, then the number of jumps must be even. Between 80 and 90 seconds, there are three moments when the number of seconds is divisible by 5: 85, 90, and 95. Only 90 seconds is corresponding to an even number of jumps.

## Problem Set 4.B

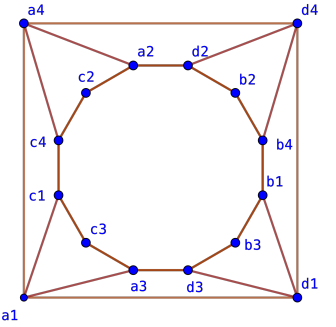
A rook stands on the square $a1$ of a $4 \times 4$ chessboard as on the right. Like the normal rook, this rook can move horizontally and vertically. Unlike the normal rook, this rook cannot move to any adjacent squares, which means it must move at least two squares each time.




Is it possible to move the rook around, **visiting each square exactly once** and return to the original square? If your answer is yes, show an example; if it is a no, prove it.

**Solution**

Let draw a graph by connecting the squares as vertices with of legal moves. From each vertex there is a path leading to any other. This is a connected graph. Furthermore that each vertices has an even amount of edges: 2 or 4. So it is possible to traverse through this graph, visit each vertices exactly once and return to the original vertex.



### Problem Set 4.C

Catherine participated in a game of Snap. Here's how the game is played. The *house* and each of *fifteen players* have shuffled a deck of 17 cards numbered from 1 to 17. After that everyone turned their cards over, one by one. When a player's card matched the house's one, the player shouted "Snap!" and scored a point. At the end, each player counted the number of points and that was the final score of the player.

It turned out that no two players have the same score and no player score 0 or 17 points. Catherine's score was the *second highest*.

What's **her score**?

**Solution**

Note that there is no such score as 16. If a player has 16 cards matched, then the 17th card should also match. Therefore the player's score must be 17, which contradicts the given conditions.

Therefore the only possible scores are 1, 2, …, 15, which means the second highest score is 14.

### Problem Set 5.A

Each day at noon, a steamship leaves UK to Canada while another steamship of the same line leaves Canada to UK. Each ship spends exactly seven 24-hour days at sea, travel along the same route.

**How many** UK-to-Canada ships will a Canada-to-UK meet while underway?

**Solution**

Note that the length of the trip is 7 days. So a ship leaves UK at noon will arrive to Canada also at noon. We can count the first encounter when a ship just departs from Canada meets the one just arrives to Canada. At that moment, there are 6 other ships that has already left UK, not including the one just arrives. During the next 7 days, there are 7 ships leaves UK.

In total the ship from Canada will meet 15 ships if you consider meetings in the ports or 13 if you don't consider meetings in the port.

### Problem Set 5.B

One litre of orange juice is contained in each of the two jugs. Half of the juice from the first jug is transferred to the second jug, then one third of the juice from the second jug is transferred to the first jug, then one fourth of the first is transferred to the second, and so on.

**How much** juice will remain in each jug after 100 transfers?

**Solution**

After some transfers, the amounts of orange juices are as below. After 100 transfers the amount of orange juices in both jugs is 1.

| Transfer | Jug #1 | Jug #2 |
|---|---|---|
| **0** | $1$ | $1$ |
| **1** | $1-\frac{1}{2}=\frac{1}{2}$ | $1+\frac{1}{2}=\frac{3}{2}$ |
| **2** | $\frac{1}{2}+\frac{1}{3}\cdot\frac{3}{2}=1$ | $\frac{3}{2}-\frac{1}{3}\cdot\frac{3}{2}=1$ |
| **...** | ... | ... |
| $2n-1$ | $1-\frac{1}{2n}=\frac{2n-1}{2n}$ | $1+\frac{1}{2n}=\frac{2n+1}{2n}$ |
| $2n$ | $\frac{2n-1}{2n}+\frac{1}{2n+1}\frac{2n+1}{2n}=1$ | $\frac{2n+1}{2n}-\frac{1}{2n+1}\frac{2n+1}{2n}=1$ |

### Problem Set 5.C

In a darkroom there are two tables. The first one is empty and the second one is cover by a layer of nickels (one nickel thick, so no coin is on top of another), in which *31 coins are tails up* and *the rest are heads up*. Minh enters the room with the task of transferring some coins from the second table to the first table (he wears gloves, so he cannot feel the faces of the coins). He *can flip over any number of coins* when transferring them.

Is that possible when he leaves the room **the number of nickels that are tails up is the same on both tables**?

**Solution**

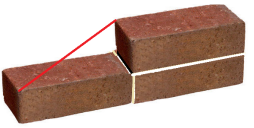
If Minh turns a coin on the second table during transfer to the first one, then the number of coins "tails up" on the second table will be reduce by 1 (if it was a "tails up" coin), or the number of coins "tails up" on the first table is increased by 1 (if it was a "heads up" coin). In any case, the difference between the "tails up" coins on the second and first tables will be reduced by one and after 31 moves, they will be the same.

## Problem Set 6.A

Erica has *three identical rectangular bricks* and one ruler as below.

Can she find **the length of the diagonal** of a brick, i.e. the distance between *two vertices not belonging* to the same face, using *only one measurement* with the ruler?

**Solution**



## Problem Set 6.B

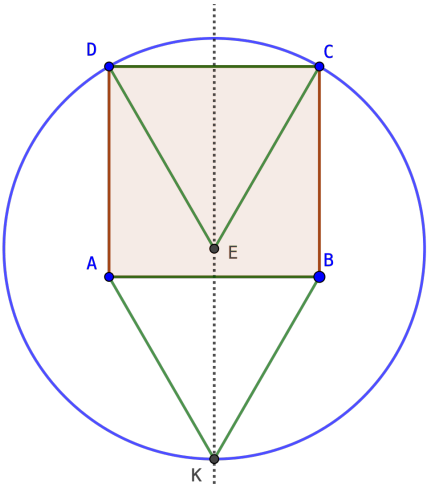
An equilateral triangle $AKB$ has been constructed outside of the *unit* square $ABCD$ on the side $AB$.

Find **the radius** of the circle circumscribed of the triangle $CKD$.

**Solution**

Draw the equilateral triangle $DEC$. $ADEK$ is a parallelogram. Thus, $E$ is the centre of the circle.

Hence the radius is 1.



**Problem Set 6.C**

Let $M$, $N$, $P$, and $Q$ be the midpoints of sides $AB$, $BC$, $CD$, and $DA$ of a convex quadrilateral $ABCD$.

If $MP = NQ$, prove that $AC \perp BD$.

**Solution**

First, the mid-segment in a triangle connecting two sides in a triangle is parallel to the third side and half of its length. In other words, if K and L are the midpoints of sides X Z and Y Z of $\triangle$XYZ, then the mid-segment KL is parallel to side XY and KL is half as long as XY.

As a consequence, the four midpoints M, N, P, and Q in ABCD in our problem form a parallelogram MNPQ. Indeed, since QP is parallel to AC (as a mid-segment in $\triangle$ACD), and MN is parallel to AC (as a mid-segment in $\triangle$ACB), it follows that QP and MN are parallel. They are also half as long as AC and hence equal in length. This means that quadrilateral MNPQ have parallel and equal in length opposite sides, and hence it is a parallelogram.

From our problem we know that the diagonals QN and MP of this parallelogram MNPQ are equal in length. This means that the parallelogram is actually a rectangle (another famous theorem from geometry). So now we know that MNPQ is a rectangle, i.e., PN and PQ are perpendicular.

As mid-segments in $\triangle$ACD and $\triangle$DBC, QP and PN are parallel correspondingly to AC and DB. This implies that AC and BD are perpendicular to each other, completing our proof.

# Problems of the Week

**Level A**

**Problem 1.A**

Joe will randomly select two letters from the word CAMP. What is the probability that he selects M, P from CAMP? *Note: think carefully about "does order matter or not"*

Then, Joe will randomly select two letters from the word CAMP, four letters from the word HERBS, and three letters from the word GLOW.

What is the probability that he will have all the letters to form the word PROBLEM?

**Solution**

The number ways to select M, P from CAMP is 1. The number ways to select two letters from CAMP is $\binom{4}{2} = 6$. The probability is $\frac{1}{6}$.

The number of ways to select E, R, P from HERBS is 1. The number of ways to select E, R, P, and one more letter from HERBS is $\binom{2}{1} = 2$. The total number of ways to select 4 letters from HERBS is $\binom{5}{4} = 5$. So the probability to select E, R, P from HERBS is $\frac{2}{5}$.

Similarly the probability to select 3 letters, two of which are O, L, from GLOW is $\frac{1 \cdot \binom{2}{1}}{\binom{4}{3}} = \frac{2}{4} = \frac{1}{2}$.

Hence the probability to form PROBLEM is $\frac{1}{6} \cdot \frac{2}{5} \cdot \frac{1}{2} = \frac{1}{30}$.

**Problem 2.A**

$x^y = 8192$, where $x$, $y$ are positive integers. How many possible distinct values of $xy$ are there?

$x^y = 4096$, where $x$, $y$ are positive integers. How many possible distinct values of $xy$ are there?

**Solution**

$8192 = 2^{13} = 8192^1$. So there are two different pairs of $(x, y)$: (2, 13), and (8129, 1). The product $xy$ has two possible values: 26, 8192.

$x^y = 4096 = 2^{12} = 4^6 = 8^4 = 16^3 = 64^2 = 4096^1$. So there are six different pairs of $(x, y)$, resulting in 5 different values of $xy$:

(2, 12), (4, 6), (8, 4), (16, 3), (64, 2), (4096, 1)

**Problem 3.A**

$x + y = 5$, $x + z = 8$, and $y + z = 11$.

What is the value of $x + y + z$? What is the value of $xyz$?

**Solution**

Add all equations together: $2(x + y + z) = 5 + 8 + 11 = 24$.

Therefore $x + y + z$ is 12.

Furthermore $x = (x + y + z) - (y + z) = 1$, $y = 4$, $z = 7$.

Hence $xyz$ is $1 \cdot 4 \cdot 7 = 28$.

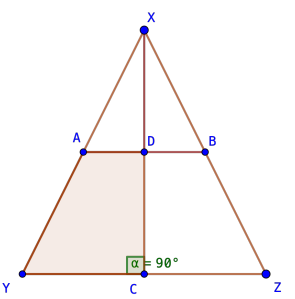
**Problem 4.A**

The area of triangle $XYZ$ is 8. Points $A$ and $B$ are midpoints of congruent segments $XY$ and $XZ$. Altitude $XC$ bisects $YZ$ and intersects $AB$ at $D$. What is the area of $ADCY$? What is the area of $XDB$?

**Solution**

Easy to see that the area of $\triangle XYC$ is is 4. Since $\triangle XAD \sim \triangle XYC$, and $\frac{XA}{XY} = \frac{1}{2}$, the area of $\triangle XAD$ is a quarter of the area of $\triangle XYC$, hence it is 1. Thus, the area of $ADCY$ is $4 - 1 = 3$.

The area of $XDB$ is same as the area of $\triangle XAD$ and is 1.



## Level B

### Problem 1.B

10 distinct points are marked on a circle. What are the maximum number of distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices?

What are the maximum possible number of intersections of diagonals formed by those vertices?

**Solution**

The number of triangles with three points as vertices is $\binom{10}{3}$. Similarly the number of quadrilateral is $\binom{10}{4}$, ..., and the number of decagons is $\binom{10}{10}$. Therefore the total number of such polygons is:

$$\binom{10}{3} + \binom{10}{4} + \ldots + \binom{10}{10} = 968.$$

For the second question, every four points form a distinct pair of diagonals. Thus, the maximum number of intersections is $\binom{10}{4} = 210$.

### Problem 2.B

$p = n^2 - 8n + 15$ is a prime number, where $n$ is a positive integer. What is the next prime number after $p$?

$q = a^3 - 4a + 3$ is a prime number, where $a$ is a positive integer. What is the next prime number after $q$?

**Solution**

$p = n^2 - 8n + 15 = (n-3)(n-5)$.

If this is a prime number, then $n - 15 = 1 \Rightarrow n = 6$.

Therefore $p = 3$. The next prime is 5.

$q = a^3 - 4a + 3 = a^3 - a - 3(a-1) = a(a^2-1) - 3(a-1)$

$= a(a-1)(a+1) - 3(a-1) = (a-1)(a^2+a-3)$.

If this is a prime number, then $a - 1 = 1 \Rightarrow a = 2 \Rightarrow q = 3$.

The next prime is 5.

**Problem 3.B**

$ab = 10$, $bc = 6$, and $ca = 15$.

What is the value of $2abc$? What is the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$?

**Solution**

By multiplying all equations together:

$(ab) \cdot (bc) \cdot (ca) = 10 \cdot 6 \cdot 15 \Rightarrow (abc)^2 = 900 \Rightarrow abc = 30$

Hence $2abc = 60$.

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ca}{abc} = \frac{10 + 6 + 15}{30} = \frac{31}{30}.$$

**Problem 4.B**

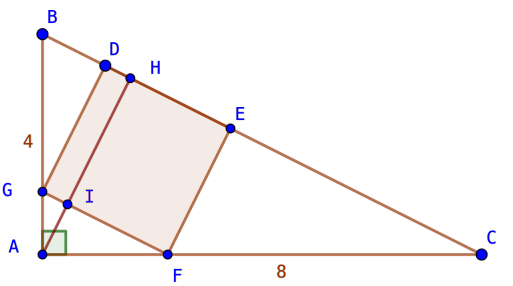
$ABC$ is a right triangle with legs $AB = 4$, $AC = 8$. A square $DEFG$ is drawn inside $ABC$ with $D$, $E$ on $BC$, $F$ on $AC$, and $G$ on $AB$. What is the side length of the square $DEFG$? What is the area of $AFG$?

**Solution**

Let $AH$ be the altitude from $A$, $AH$ intersect $GF$ at $I$.

$$BC = \sqrt{AB^2 + AC^2} = \sqrt{4^2 + 8^2} = 4\sqrt{5}$$

$$AH = \frac{AB \cdot AC}{BC} = \frac{8}{\sqrt{5}}$$




Let $a$ be the side of the square $DEFG$. Since $\triangle GAF \sim \triangle BAC$, so:

$$\frac{AI}{AH} = \frac{GF}{BC} \Rightarrow \frac{AH - a}{AH} = \frac{a}{BC} \Rightarrow 1 = \frac{a}{AH} + \frac{a}{BC} \Rightarrow a = \frac{AH \cdot BC}{AH + BC}$$

$$AH.BC = 4\sqrt{5} \cdot \frac{8}{\sqrt{5}} = 32. AH + BC = \frac{28}{\sqrt{5}} \Rightarrow a = \frac{32}{\frac{28}{\sqrt{5}}} = \frac{8\sqrt{5}}{7}$$

$$\frac{S_{\triangle AFG}}{S_{\triangle ACB}} = \left(\frac{AI}{AH}\right)^2 = \left(\frac{a}{BC}\right)^2 = \frac{4}{49}. \text{ So } S_{\triangle AFG} = \frac{4}{49} . S_{\triangle ABC} = \frac{64}{49}.$$

## Level C

### Problem 1.C

The *distinct prime factors* of an integer are its prime factors listed without repetition. For example, the distinct prime factors of 40 are 2 and 5. Let $A = 2^k - 2$ and $B = 2^k A$, where $k$ is an integer ($k > 1$). Show that $A$ and $B$ have the same set of distinct prime factors. Prove that $A + 1$ and $B + 1$ have the same set of distinct prime factors.

**Solution**

Since $B$ is given as a multiple of $A$, every prime that divides $A$ also divides B. Conversely, suppose $p$ is a prime that divides B. Because $B = 2^k \cdot A$, either $p$ divides $2^k$ or $p$ divides $A$. If p divides $2^k$ , then $p = 2$. But then p divides $A$ anyway, because A = 2(2k−1 − 1). This shows that every prime that divides $B$ also divides $A$. Since every prime that divides $A$ also divides $B$, and vice versa, $A$ and $B$ have the same set of distinct prime factors.

$A + 1 = 2^k - 1$, $B + 1 = 2^k(2^k - 2) + 1$, so $B = 2^{2k} - 2 \cdot 2^k + 1$. $= (2^k - 1)2 = (A + 1)^2$. So every prime that divides $A + 1$ divides $B + 1$ and vice versa. Hence, they have the same set of distinct prime factors.

### Problem 2.C

For $n > 1$, consider an $n \times n$ chessboard and place pieces at the centres of different squares. With $2n$ chess pieces on the board, show that there are 4 pieces among them that form the vertices of a parallelogram. Show that there is a way to place $(2n - 1)$ chess pieces so that no 4 of them form the vertices of a parallelogram.

**Solution**

Since there can be at most $n$ pieces that are left most in their rows (some rows may be empty), there are at least $n$ pieces that are not the leftmost in their row. Record the distances (the number of squares) between the leftmost piece and the other pieces on the same row. There are then at least $n$ distances recorded. The distances range from 1 to $n - 1$, by the Pigeonhole Principle, at least two of these distances are the same. Thus, there are at least two rows each containing two pieces that are the same distance apart. These four pieces yield a parallelogram.

If $(2n - 1)$ pieces are placed, for example, on the squares of the first column and the first row, then there is no parallelogram.

**Problem 3.C**

Find a positive integer $n$ and $a_1, a_2, \ldots, a_n$, where $a_k = 1$or $a_k = -1$ for each $k = 1, 2, \ldots, n$, such that:

$$a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \cdots + a_n \cdot n^3 = 20162016$$

or show that this is impossible.

**Solution**

Note that:

$$4^3 - 3^3 - 2^2 + 1^3 - 0^3 + (-1)^3 + (-2)^3 - (-3)^3 = 48.$$

Therefore it is easy to prove that, for all $u$:

$$-u^3 + (u+1)^3 - (u+2)^3 - (u+3)^3$$
$$+(u+4)^3 - (u+5)^3 - (u+6)^3 + (u+7)^3 = 48$$

Since $20162016 = 420042 \cdot 48$, the number $20162016$ can be expressed as a sum of $420042$ terms, each is a $8-$term sum with the value of 48:

$$20162016 =$$
$$-1^3 + 2^3 + 3^3 - 4^3 + 5^3 - 6^3 - 7^3 + 8^3$$
$$-9^3 + 10^3 + 11^3 - 12^3 + 13^3 - 14^3 - 15^3 + 16^3$$
$$\cdots$$
$$-3360329^3 + 3360330^3 - \cdots - 3360335^3 + 3360336^3$$

Hence, for the $\{a_k\}$ sequence with a repeated pattern of

$$\{-1, +1, +1, -1, +1, -1, -1, +1\}$$
$$a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \cdots + a_n \cdot n^3 = 20162016$$

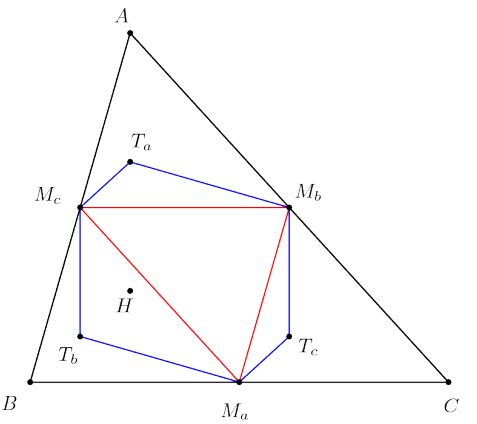
**Problem 4.C**

In an acute triangle ABC let K, L, and M be the midpoints of sides AB, BC, and CA, respectively. From each of K, L, and M drop two perpendiculars to the other two sides of the triangle; e.g., drop perpendiculars from K to sides BC and CA, etc.

The resulting 6 perpendiculars intersect at points Q, S, and T as in the figure to form a hexagon KQLSMT inside triangle ABC.

Prove that the area of the hexagon KQLSMT is half of the area of the original triangle ABC.

**Solution**




Relabel $\triangle M_aM_bM_c$ as the medial triangle and replace $T, Q, S$ with$T_a, T_b, T_c$, respectively. Let $H$ be the orthocenter of $\triangle ABC$.

Note that $\triangle AM_bM_c \sim \triangle ABC$. It is easy to see that $T_a$ is the orthocenter of $\triangle AM_bM_c$ and the role of $T_a$ and $H$ for two similar triangles $\triangle AM_bM_c$ and $\triangle ABC$ are the same.

So $\dfrac{[T_aM_bM_c]}{[AM_bM_c]} = \dfrac{[HBC]}{[ABC]} \Rightarrow [T_aM_bM_c] = \dfrac{1}{4}[HBC]$.

Considering analogous relations:

$$
\begin{aligned}
&[M_cT_aM_bT_cM_aT_b] \\
&= [M_aM_bM_c] + [M_cT_aM_b] + [M_bT_cM_a] + [M_aT_bM_c] \\
&= \frac{[ABC]}{4} + \frac{1}{4}([HBC] + [HAB] + [HCA]) \\
&= \frac{[ABC]}{2}
\end{aligned}
$$

# Chapter 9

# Week 30

# Weekly Team Contest

**Problem Set 1.AB**

Tiana baked some identical pizzas. Each pizza is cut either into 8 small slices or 6 large slices.

When she has done, three friends of her, Catherine, Clemence, and Camille has arrived. They have made three statements and only one of them is true.

- Catherine said *"there are equal numbers of small and large slices, so Tiana has baked less than 7 pizzas"*.
- Clemence said *"there are equal numbers of small and large slices, so Tiana has baked 90 pizzas"*.
- Camille said *"there are 5 small slices for every 3 large slices, so Tiana has baked 90 pizzas"*.

**Which statement** is true**?**

**How many pizzas** are cut into *small slices*?

**Solution**

Let $s$ be the number of pizzas cut into 8 small slices and $l$ be the number of pizzas cut into 6 small slices.

*If there are equal numbers of small and large slices*:

$$8s = 6l \Rightarrow 4s = 3l \Rightarrow 4s + 4l = 7l \Rightarrow 4(s + l) = 7l$$

This means that the number of pizzas is divisible by 7. So the first and the second statements cannot be true.

Hence, Camille's statement is true.

Furthermore since $8s : 6l = 5 : 3$, so $8s \times 3 = 6l \times 5$:

$$\Rightarrow 4s = 5l \Rightarrow 9s = 5(s + l) = 5 \cdot 90 \Rightarrow s = 50$$

**Problem Set 1.C**

Binh, Anthony, and Vu participated in a beauty contest. The contest committee had a number of judges. *Each judge voted for only one* of the three participants.

A spectator called Minh, counted 59 judges, with 15 judges voting for Binh or Anthony, 18 voting for Anthony or Vu, and 20 voting for Vu or Binh.

However Minh was notorious for poor counting, each number he counted in above differs from reality by no more than 13.

**How many judges** voted for Anthony?

**Solution**

Let $a$ be the number judges voted Binh or Anthony, $b$ be the number judges voted for Anthony or Vu , and $c$ be the number judges voted for for Vu or Binh. Let $s$ be the total number of judges.

Easy to see that $a + b + c = 2s$. Because Minh counted poorly, so $a \leq 15 + 13 = 28$, $b \leq 18 + 13 = 31$, and $c \leq 20 + 13 = 33$.

Hence $2s = a + b + c \leq 28 + 31 + 33 \Rightarrow s \leq 46$.

Also because Minh counted poorly, so $s \geq 59 - 13 = 46$.

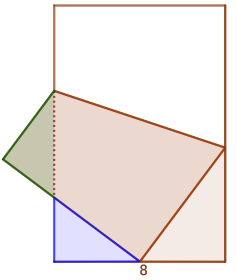
Therefore $s = 46$. In this case $a = 28$, $b = 31$, $c = 33$.

Therefore the number of judges voted for Anthony is:

$$a + b - s = 28 + 31 - 46 = 13$$

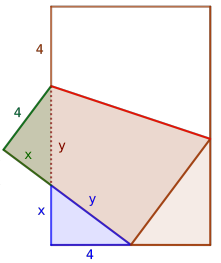
## Problem Set 2.AB

A rectangular sheet of paper is folded so that *one corner bisects the lower side*. Its short side length is 8. The blue and green triangles are congruent. What is **the length of the long side** of the sheet? What is **the length of the hypotenuse of the blue triangle**?




### Solution

Let $x$ be the shorter leg, and $y$ be the hypotenuse of the blue right triangle. Since one corner of the rectangle bisects the lower side, so the other leg of this blue right triangle is 4. Furthermore, the green triangle is congruent to the blue, so we can use the same notation for its sides. So the remaining segment of the right long side of the rectangle has a length of 4.




Now it is easy to see that $x + y = 8$, therefore the length of the long side of the rectangle is 12. Since :

$$4^2 = y^2 - x^2 = (y - x)(y + x) = 8(y - x) \Rightarrow y - x = 2$$

Hence $x = 3$, $y = 5$, so the length of the hypothenuse is 5.

## Problem Set 2

### Level C

Along the path better the houses of Viet and Nam there's a *line of flowers* consisting of 15 tulips and 15 peonies. On the way to visit Nam, Viet took his can and watered all the flowers one by one. After the 10th tulip, the can ran out of water, so 10 flowers got no water. The next morning, coming to see Viet, Nam picked all the flower one by one as he encountered them. After picking the 6th tulip, he decided that they are enough for a bouquet for Viet's mother. **How many flowers were left**?

### Solution

Since there are 10 flowers after the 10th tulip on the way from Viet's house to Nam's house, and the number of tulips is 15, so the 10th tulip is actually the 6th tulip on the way from Nam's house to Viet's house. This means that Nam picked 11 flowers. Hence 19 flowers are left.

**Problem Set 3.AB**

The date of 17/02/2008 has a special property that the sum of the first four digit is equal the sum of the last four digits. Such date is call a *nice date*. **How many other** *nice dates* are there in 2008? **How many years** in the 21st century that have at least one *nice date*? *Note: the 21st century began on January 1, 2001, and will end on December 31, 2100.*

**Solution**

Easy to see that each month has several *nice dates*:

- 09/01, 18/01, 27/01
- 08/02, 17/02, 26/02
- 07/03, 16/03, 25/03
- 06/04, 15/04, 24/04
- 05/05, 14/05, 23/05
- 04/06, 13/06, 22/06
- 03/07, 12/07, 21/07, 30/07
- 02/08, 11/08, 20/08
- 01/09, 10/09
- 09/10, 18/10, 27/10
- 08/11, 17/11, 26/11
- 07/12, 16/12, 25/12

The total of them in 2008 is 36. There are 35 other nice dates.

For generic date $D_1D_2/M_1M_2$ pattern:

$$0+1+0+1=2 \le D_1+D_2+M_1+M_2 \le 2+9+0+9=20$$

So 2 is the smallest sum of digits, and 20 is the highest sum of digits. Since 01/02/2100 is a nice date, and for any year $20ab$:

$$3=2+0+0+1 \le 2+0+a+b \le 2+0+9+9=20$$

So every year in the 21st century has at least one nice date.

The answer is 100.

**Problem Set 3.C**

Six brilliant mathematicians from the MCC club are returning from a party. On the way through the forest they were captured by an enormous ogre.

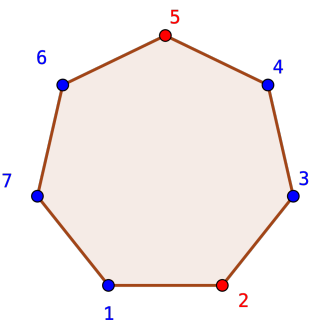
The ogre said "I have *seven caps, each has a different colour*. Before dawn, I will put all of you in a room, blindfold you and put a cap on each of your heads and hide the last cap. Then, I will take the blindfolds off. *You stays where you are*. You will be able to see what colour everyone else is wearing, but not your own. You cannot talk or give any sign to one another. Then, without letting the others hear you, tell me the colour of the hidden cap. If *at least three of you guess correctly*, I will free you all. If not, I will eat you for breakfast."

Did the ogre have a good breakfast or did the six brilliant mathematicians **come up with a winning strategy** to ruin the ogre's breakfast? *Hint: numbering the colours is a simple trick to use the well-ordering principle.*

**Solution**

Since they are brilliant mathematicians, they can number the colours. Let they be 1, 2, …, 7, arranged in a circle with the numbers increased counter-clockwise. Each mathematician knows the two colours that cannot be seen. There are two different counter-clockwise distances from one missing colour to the other. The one with shorter distance is selected as an answer.

Let look at an example: suppose that the cap colour number 5 is hidden. For the mathematician 2 he does not see two colours 2 and 5 and the distance counter-clockwise from 2 to 5 is shorter than the distance from 5 to 2. So he will answer 5.




Similarly the mathematician 3, and 4 will guess correctly. The mathematician 6, 7, and 1 will not guess correctly.

So in every case three mathematicians will have the wrong answers and the other three will answer correctly, so the ogre will be left hungry.

## Problem Set 4.A

The organizers of Math and Chess Club made some bets before the soccer game between Team Girls and Team Boys.

i. Team Boys will score against Team Girls.

ii. Team Boys will win.

iii. Team Boys won't lose.

iv. In total, three goals will be scored.

v. No draw.

It turned out that *exactly three of the bets were true*.

What is **the final score**?

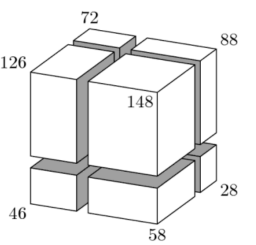
**Solution**

If Team Boys wins then the statements i., ii., iii., and v. are all true. So Team Boys lose. In this case the statements ii., iii. are false so the statements i., iv., and v. are true.

Easy to see that the score is: Team Boys : Team Girls = 1 : 2

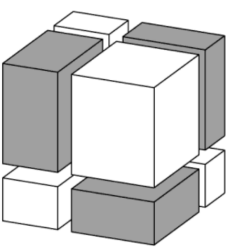
## Problem Set 4.B

Huy cuts a cube of cheese. His three cuts divided the cheese cube into eight smaller blocks. Seven of them are visible with dimensions as shown in the diagram below. The numbers in the diagrams indicate the surface areas of the visible blocks. What is t**he surface area** of the eighth, non-visible block?




**Solution**

Note that for each of the small blocks, the surface area made by the cuts (the shaded faces created by the cuts) is half of the total surface area. If we recolour the blocks as below, the faces, in pairs of congruent black and white rectangles, form 4 white and 4 black blocks. Each group of same coloured blocks has the same surface area.

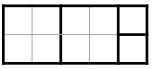


Hence, the surface area of the eight block is:

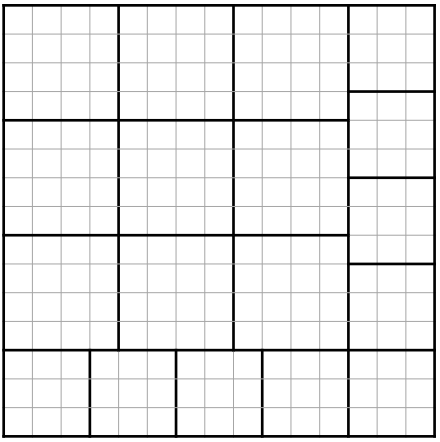
$$(148 + 46 + 72 + 28) - (88 + 126 + 58) = 22$$

**Problem Set 4.C**

The rectangle below can be divided into a group of large squares and another group of small squares. The number of squares in both groups are equal. Can you divide a square into two groups of different sizes as above such that **the number of squares in both groups are equal**?



**Solution**



**Problem Set 5.A**

In a New Year party in Scotland, seven girls and seven boys formed seven girl-boy pairs to dance. Of course, all wore skirts. No skirt has the same length. The boys and girls can pair up in any way they want. However, a pair (Girl-A, Boy-A) can only start dancing if there is at least one other pair (Girl-B, Boy-B) where *Girl-A's skirt is longer* than Girl-B's skirt and *Boy-A's skirt is shorter* than Boy-B's skirt.

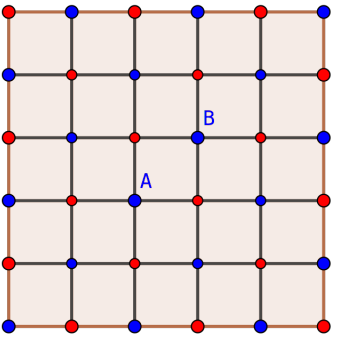
While observing the pairs, Lily saw that no one can dance. After she suggested some rearrangements, six pairs started to dance. What were the **original pairings**? And **what did Lily suggest**?

**Solution**

Let order the girls by the length of their skirts, from longest to shortest: $g_1 > g_2 > \cdots > g_7$, similarly for the boys $b_1 > b_2 > \cdots > b_7$. Originally the situation was: $(g_1, b_1), (g_2, b_2), \cdots, (g_7, b_7)$. This is why nobody can dance. By rearrange them into $(g_1, b_7), (g_2, b_6), \cdots, (g_7, b_1)$, six first pairs now can dance.

**Problem Set 5.BC**

Elisa went to London and went to a famous shopping district. Her friend left her at the fashion shop — point A in the diagram below — and agreed to pick her up at the shoe shop — point B.



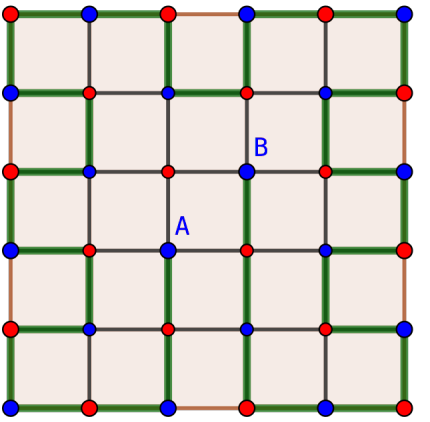

Elisa wanted to visit all of the shops, meaning she wants to walk along all of the streets of the old town show in the diagram.

Of course, she did not want visit the same intersection twice.

Draw the **longest** route. Prove that there is no longer route.

**Solution**

One of the longest route with length 34 block sides, containing 35 intersections is show below.

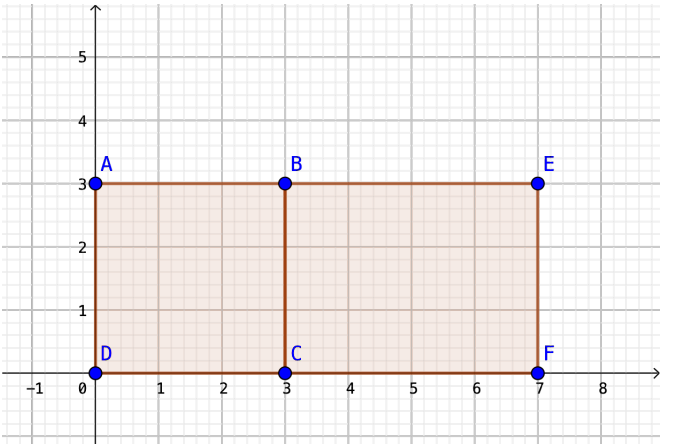



The old town map contains 36 intersections. By colouring the intersections alternately red and blue, A and B has the same blue colour.

A path leads from A to B should contain maximum 35 intersections and therefore 34 block sides.

**Problem Set 6.AB**

In the diagram below, two quadrilaterals (two squares), whose vertices lie on the grid points, are *joined* to form another quadrilateral (a rectangle). Note that joining means the quadrilaterals share the same edges and do not overlap.

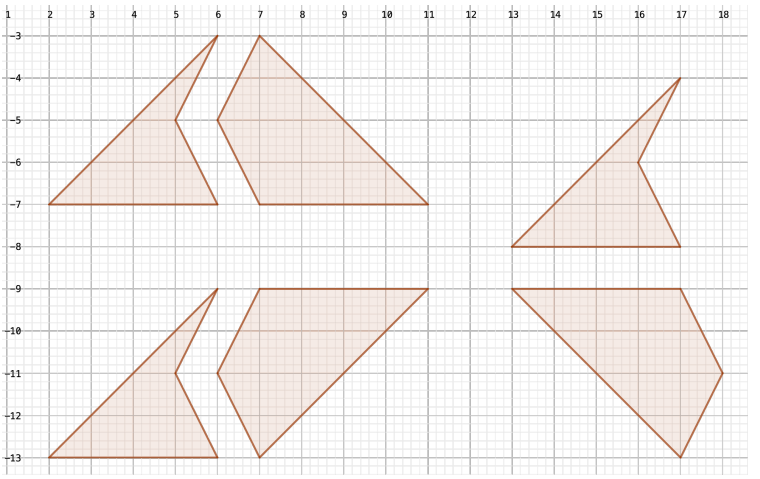



On a sheet of graph paper, draw two *quadrilaterals* whose vertices lie on the grid points, so that they can form **a triangle.**

Use the same two *quadrilaterals* from the solution to Level A question, join them in a different way so that they form **a quadrilateral**.

**Bonus:** Use the same two quadrilaterals from the solution to Level A question, join them in a different way so that they form **a pentagon** (when joined in a different way)

**Solution**



**Problem Set 6.C**

Lilian has a mobile phone. While fully charged, she can use it for 6 hours of talk time or 210 hours of stand-by time. When she got on a train, it was fully charged, but at the moment she arrived at her destination, the battery completely ran out.

**How long did Lilian travel** on the train? It is known that she was talking on the phone for *exactly half of her journey.*

**Solution**

Let the time Lilian has talked on the phone be $t$ hours. The battery capacity that she has used while talking was $\frac{t}{6}$ and the battery capacity that was spent on standby is $\frac{t}{210}$.

Since $\frac{t}{6} + \frac{t}{210} = 1 \Rightarrow \frac{t}{6}\left(1 + \frac{1}{35}\right)$

$= 1 \Rightarrow \frac{t}{6} \cdot \frac{36}{35} = 1 \Rightarrow t = \frac{35}{6}$

So the time she traveled is $\frac{2 \cdot 35}{6} = 11\frac{2}{3}$ or 11 hours 40 minutes.

# Problems of the Week

**Level A**

### Problem 1.A

Strictly speaking, an anagram of a word or phrase is a rearrangement of the letter to form a different word or phrase. In anagrams, spaces are usually ignored. For example one anagram of "anagram" is"nag a ram". In mathematics, and for this problem, we often use "anagram" to mean any permutation of letters in a word, and therefore would consider "aaarngm" an anagram of "anagram".

How many anagram does the word REALSPY have? Can you find one that means herb? One that describes the members of a team? Can you find any other that are English words?

Decipher the following sentence where the correct words are replaced by their anagrams:

VOLES ATTACHELAMMI BERMSLOP AYLID

**Solution**

REALSPY has 7 different letters. To form an anagram, there are 7 possibilities to choose the first letter, 6 possibilities to choose the second letter, …, 1 possibility to choose the last letter. Therefore the number of possible anagrams is $7! = 5040$.

PARSLEY is anagram that is also a herb name.

Furthermore PLAYERS describes the members of a team.

Here are other English words: REPLAYS, SPARELY, PARLEYS.

The sentence was:

SOLVE MATHEMATICAL PROBLEMS DAILY.

### Problem 2.A

Find the sum $6 + 66 + 666 + \cdots + \underbrace{66...6}_{20}$.

*Note that the last term of the sum has 20 digits of 6.*

Find the sum $6 + 66 + 666 + \cdots + \underbrace{66...6}_{2020}$.

*Note that the last term of the sum has 2020 digits of 6.*

**Solution**

$$6+66+666+\cdots+\underbrace{66...6}_{20}=6\left(1+11+111+\cdots+\underbrace{11...1}_{20}\right)$$

$$=\frac{6}{9}\left(9+99+999+\cdots+\underbrace{99...9}_{20}\right)$$

$$=\frac{2}{3}\left(10-1+100-1+\cdots+\underbrace{10...0}_{21}-1\right)=\frac{2}{3}\left(\underbrace{11...10}_{21}-20\right)$$

$$=\frac{2}{3}\left(\underbrace{11...1}_{18}090\right)=2\cdot\underbrace{370\cdots370}_{6}30=\underbrace{740\cdots740}_{6}60$$

$$6+66+666+\cdots+\underbrace{66...6}_{2020}=6\left(1+11+111+\cdots+\underbrace{11...1}_{2020}\right)$$

$$=\frac{6}{9}\left(9+99+999+\cdots+\underbrace{99...9}_{2020}\right)$$

$$=\frac{2}{3}\left(10-1+\cdots+\underbrace{10...0}_{2021}-1\right)=\frac{2}{3}\left(10+\cdots+\underbrace{10...0}_{2021}-2020\right)$$

$$=\frac{2}{3}\left(\underbrace{11...10}_{2021}-2020\right)=\frac{2}{3}\left(\underbrace{11...1}_{2016}09090\right)$$

$$=\frac{2}{3}\left(\underbrace{11...1}_{2016}00000+9090\right)=2\cdot\left(\underbrace{370...3700000}_{2020}+3030\right)$$

$$=\left(\underbrace{740...7400000}_{2020}+6060\right)=\underbrace{740...7406060}_{2020}$$

**Problem 3.A**

Pick a number from 1 to 15 and find all occurrences of it in the table below. Tell the teacher, by row number, which rows your number is in, and the teacher will immediately guess your number without looking at the table.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 7 | 3 | 15 | 13 | 11 | 5 | 9 |
| 2 | 15 | 11 | 6 | 10 | 14 | 7 | 3 |
| 4 | 5 | 15 | 6 | 12 | 7 | 13 | 14 |
| 8 | 12 | 10 | 14 | 9 | 13 | 15 | 11 |

How is that possible? Can you construct the same table for the number 1 to 31?

**Solution**

Note that if we write in the following table in each column: 1 if the number does appear and 0 if the number does not appear.

| Number | Row 4 | Row 3 | Row 2 | Row 1 | Together |
|---|---|---|---|---|---|
| 1 | | | | 1 | 1 |
| 2 | | | 1 | 0 | 10 |
| 3 | | | 1 | 1 | 11 |
| 4 | | 1 | 0 | 0 | 100 |
| | | | | | |
| 14 | 1 | 1 | 1 | 0 | 1110 |
| 15 | 1 | 1 | 1 | 1 | 1111 |

By collating the digits 1 and 0 together, each number corresponds to a string of digit 1s and 0s. They are the base-2 representation of each number:

$$1 = 2^0 = 1 \cdot 2^0 = \overline{1}_2, 2 = 2^1 = 1 \cdot 2^1 + 0 \cdot 2^0 = \overline{10}_2$$

$$3 = 2^1 + 2^0 = 1 \cdot 2^1 + 1 \cdot 2^0 = \overline{11}_2,$$

$$4 = 2^2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 = \overline{100}_2$$

$$\ldots$$

$$14 = 2^3 + 2^2 + 2^1 = 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = \overline{1110}_2$$

$$15 = 2^3 + 2^2 + 2^1 = 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = \overline{1111}_2$$

Hence, it is simple to deduce the number if you know which rows it is in, in another word, if you know the base-2 representation of the number. For example if you know it is in row 4 and row 2, then the number should be: $\overline{1010}_2 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 10$

For the number from 1 to 31, you should be able to construct a similar table following the base-2 presentation as shown above.

**Problem 4.A**

$\triangle ABC$ is a right triangle, $\angle A = 90°$, $\angle B > \angle C$.

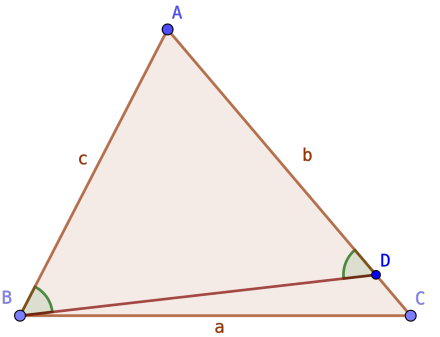
Prove that $AC > AB$.

$\triangle ABC$ is an arbitrary triangle, and $\angle B > \angle C$.

Prove that $AC > AB$?

**Solution**

Let prove: In a triangle $\triangle ABC$, if $AC > AB$, then $\angle B > \angle C$.




$AC > AB$, there exists a point $D$ between $A$, $C$: $AB = AD$:

$$\angle B = \angle BDA = \angle DBC + \angle DCB > \angle DCB = \angle C.$$

Now it is simple to prove the opposite: if $\angle B > \angle C$ then $AC > AB$ by contradiction as follow: suppose that $AC \leq AB$, then using the above proof, $\angle B \leq \angle C$, this contradicts $\angle B > \angle C$.

Hence $AC > AB$.

This solution shows a different approach to solve a problem by proving an equivalent statement, namely: *In any arbitrary triangle* $\triangle ABC$, $AC > AB$ *if and only if* $\angle B > \angle C$.

There are other solutions specifically for the first question ($\angle A = 90°$), however the key approach is similar.

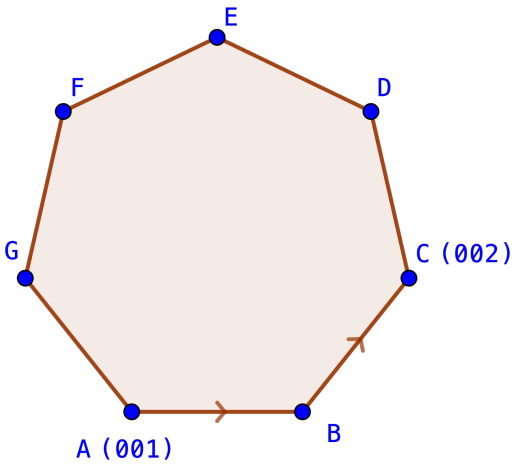
## Problem 1.B

The director of a secret service has put together a list of mutual surveillance assignments for his seven agents, codename 001 through 007. Agent 001 will watch the agent who is watching 002; agent 002 will watch the agent who watches 003; and so on. Agent 007 will watch the agent watching 001. Can you deduce who watches whom?

The service recruits a new agent, and gives the agent codename 008. How could the director change the surveillance assignments to keep it working for all 8 agents?

**Solution**

First, let A, B, C, D, E, F, G be seven points that represent the agents. We will assign the points to agents a long the way as the solution is described. First, let A be agent 001. We knows that A watches B, and B watches agent 002, so let C be agent 002.




By continuing as above the points establish a series of path from 001 to 002, 002 to 003, …, 007 to 001. All these paths are parts of a circle containing all the agents. It means that E is 003, G is 004, B is 005, D is 006, and finally F is 007.

This reasoning requires an *odd* amount of agents. The agent 004 should watch an agent who watches the agent 001, which contradicts the given conditions.

**Problem 2.B**

Cross out one of the factors of the form $n!$ in the product so that the result is a perfect square:

$$(1!)(2!)\cdots(2019!)(2020!)$$

Cross out an *odd amount terms* to obtain a perfect squares.

**Solution**

$(1!)(2!)\cdots(2019!)(2020!)$

$= (2!)(4!)\cdots(2018!)(2020!) \cdot (3!)(5!)\cdots(2017!)(2019!)$

Since $(2!)(4!)\cdots(2018!)(2020!)$

$= (1! \cdot 2)(3! \cdot 4)\cdots(2017! \cdot 2018)(2019! \cdot 2020)$

$= (2 \cdot 4\cdots 2020) \cdot (3!)(5!)\cdots(2017!)(2019!)$

$= \left(2^{1010} \cdot 1 \cdot 2\cdots 1010\right) \cdot (3!)(5!)\cdots(2017!)(2019!)$

$= (1 \cdot 2\cdots 1010) \cdot 2^{1010} \cdot (3!)(5!)\cdots(2017!)(2019!)$

$= 1010! \cdot 2^{1010} \cdot (3!)(5!)\cdots(2017!)(2019!)$

Therefore

$(1!)(2!)\cdots(2019!)(2020!)$

$= 1010! \cdot 2^{1010} \cdot \left((3!)(5!)\cdots(2017!)(2019!)\right)^2$

So if we cross the term $1001!$ then product is a perfect square.

Note that the above solution can be used if the product is ended with a $n!$ factorial, where $n$ is divisible by 4. In the above $n = 2020$.

We can cross any amount equal to a multiple of 4 plus 1 and have a perfect square, for example below is how we can remove 5 terms.

$$(1!)\cdots(1007!)\cancel{(1008!)}(1009!)\cdots(2016!)\cancel{(2017!)(2018!)(2019!)(2020!)}$$

**Problem 3.B**

Factor the following expression into a product of 3 terms:

$$(xy + yz + zx)(x + y + z) - xzy$$

Real numbers $x, y, z \neq 0$ satisfy:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x + y + z}$$

Determine the value of the following expression

$$\frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} - \frac{1}{x^3 + y^3 + z^3}$$

**Solution**

$$\begin{aligned}
&(xy + yz + zx)(x + y + z) - xyz \\
&= x^2y + xy^2 + xyz + yzx + y^2z + yz^2 + zx^2 + zxy + z^2x - xyz \\
&= x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2 + 2xyz \\
&= (x^2y + y^2x) + (y^2z + xyz) + (yz^2 + z^2x) + (zx^2 + xyz) \\
&= xy(x + y) + yz(y + x) + z^2(y + x) + zx(x + y) \\
&= (x + y)(xy + yz + z^2 + zx) \\
&= (x + y)\big(y(x + z) + z(z + x)\big) \\
&= (x + y)(y + z)(z + x)
\end{aligned}$$

Since $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{1}{x + y + z} \Rightarrow \dfrac{xy + yz + zx}{xyz} = \dfrac{1}{x + y + x}$

Hence $(xy + yz + zx)(x + y + z) - xyz = 0$

Therefore $(x + y)(y + z)(z + x) = 0 \Rightarrow x = -y$ or $y = -z$ or $z = -x$.

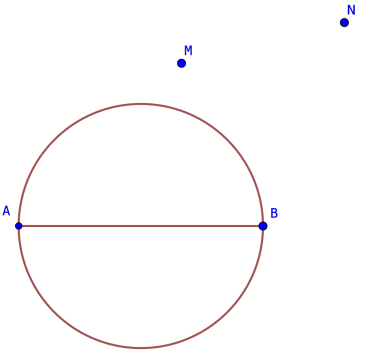
$$\begin{aligned}
S &= \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} - \frac{1}{x^3 + y^3 + z^3} = \frac{x^3y^3 + y^3z^3 + z^3x^3}{x^3y^3z^3} - \frac{1}{x^3 + y^3 + z^3} \\
&= \frac{(x^3 + y^3 + z^3)(x^3y^3 + y^3z^3 + z^3x^3) - x^3y^3z^3}{x^3 + y^3 + z^3} \\
&= \frac{(x^3 + y^3)(y^3 + z^3)(z^3 + x^3)}{x^3 + y^3 + z^3}
\end{aligned}$$

$x = -y$ or $y = -z$ or $z = -x \Rightarrow (x^3 + y^3)(y^3 + z^3)(z^3 + x^3) = 0$.

Hence $S = 0$.

**Problem 4.B**

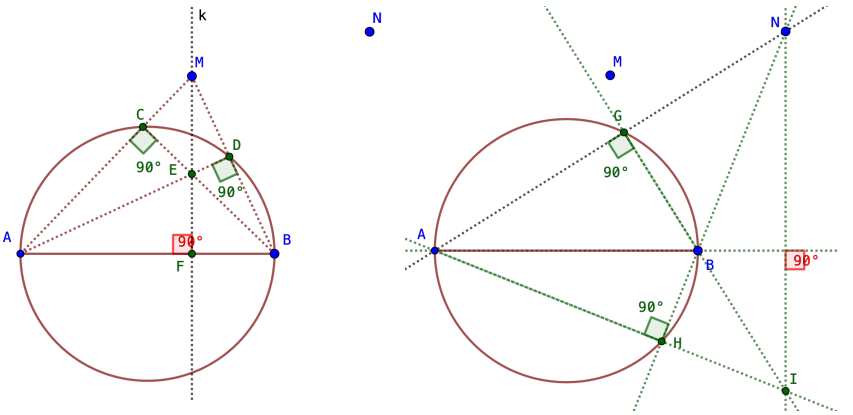
Suppose we have points $M$ and $N$, and a circle with the diameter $AB$ as in the diagram below. Use only a pencil and a straightedge to **drop a perpendicular** from $M$ onto $AB$. Use only a pencil and a straightedge to **drop a perpendicular** from $N$ to the extension of $AB$.




**Solution**

First, use the ruler to draw a line connecting A with M intersects the circle at C. Since $AB$ is the diameter, so $\angle ACB = 90^\circ$. Similarly a line connecting B with M intersects the circle at D and $\angle ADB = 90^\circ$.

This means that $AD$ and $BC$ are the two altitudes of the triangle AMB. Therefore the third altitude MF must goes through point E, which is the intersection of AD and BC.



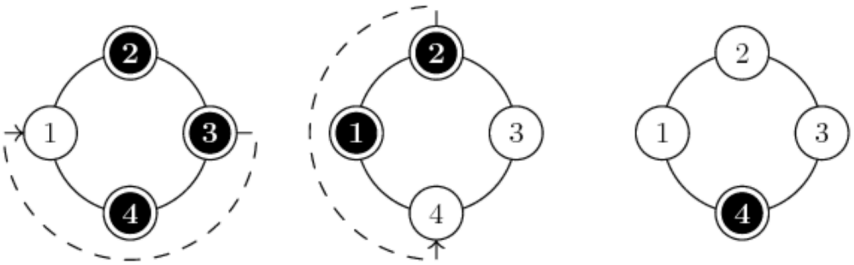

The solution for $N$ is based on the same principle. However it is less obvious. First, use the ruler to draw a line connecting A with N intersects the circle at G. Since $AB$ is the diameter, so $\angle AGB = 90^\circ$. Similarly a line connecting B with N intersects the circle at H and $\angle AHB = 90^\circ$. Extending GB and AH to intersect each other at I.

$IG$ and $NH$ are the two altitudes of the triangle ANI. Therefore the third altitude AB to the side NI must goes through point B, which is the intersection of $NH$ and $IG$.

## Problem 1.C

There are $n$ holes in a circle. The holes are numbered 1, 2, 3 and so on to $n$. In the beginning, there is a peg in every hole except for hole 1. A peg can jump in either direction over one adjacent peg to an empty hole immediately on the other side. After a peg moves, the peg it jumped over is removed. The puzzle will be solved if all pegs disappear except for one.

For example, if $n = 4$ the puzzle can be solved in two jumps: peg 3 jumps peg 4 to hole 1, then peg 2 jumps the peg in 1 to hole 4. (See illustration below, in which black circles indicate pegs and white circles are holes.).




Construct a solution when $n = 2020$.

Amy and Bob play a game. They alternate turns, with Amy going first. At the start of the game, there are 20 cookies on a red plate and 14 on a blue plate. A legal move consists of eating two cookies taken from one plate, or moving one cookie from the red plate to the blue plate (but never from the blue plate to the red plate). The last player to make a legal move wins; in other words, if it is your turn and you cannot make a legal move, you lose, and the other player has won.

Which player can guarantee that they win no matter what strategy their opponent chooses? Prove that your answer is correct.

**Solution**

For even $n$ it is pretty easy to construct a solution: Suppose the holes are numbered 0, 1, 2, ..., $n - 1$ with hole 0 initially empty. Then do the following steps: Jump 2 into 0, 4 into 2, 6 into 4, 8 into 6, ... $n-2$ into $n-4$. Now you're left with a peg in $n-1$ next to 0, and holes 2, 4, 6, ..., $n-4$ are filled, so just jump the peg in $n-1$ over all the even pegs like a successive capture in checkers, and you're done.

Let's write the number of cookies in the red and blue plate, respectively, as an ordered pair $(x,y)$, so that the legal moves are to $(x-2,y)$ or $(x,y-2)$ or $(x-1,y+1)$. Thus the only positions with no legal move are (0, 0) and (0, 1), and since cookies are eaten in pairs, the final position is determined by the original number of cookies.

Starting from (20,14), we know that eventually all the cookies will be eaten, so there are exactly $(20 + 14)/2 = 17$ cookie-eating moves. There may also be some number of moves from the first pile to the second pile, but since an even number of cookies are eaten from each pile, there must be an even number of such moves. Thus, the total number of moves in the game is odd, and the first player gets the last legal move.

For general starting positions, there are a few cases to examine depending on whether the total number of cookies is even and the number of cookies in pile 2 is even, but the logic is similar.

**Problem 2.C**

Let $P(x) = x^2 - 3x - 7$, let $Q(x)$ and $R(x)$ be two quadratic polynomials with the coefficients of $x^2$ equal to 1. Catherine computes each of the three sums $P + Q$, $Q + R$, $R + P$ and surprised to find that each pair of these sum has a common root, and these three common roots are distinct. It is known that $Q(0) = 2$. Find $R(0)$.

**Solution**

Let $Q(x) = x^2 + ax + 2$ and $R(x) = x^2 + bx + c$.

$P(x) + Q(x) = 2x^2 + (a - 3)x - 5$

$P(x) + R(x) = 2x^2 + (b - 3)x + (c - 7)$

$Q(x) + R(x) = 2x^2 + (a + b)x + (c + 2)$

Let the common root of $P(x) + Q(x)$, $P(x) + R(x)$ be $r$; $P(x) + R(x)$, $Q(x) + R(x)$ be $s$; and $P(x) + Q(x)$, $Q(x) + R(x)$ be $t$. We then have that the roots of $P(x) + Q(x)$ are $r$, $t$, the roots of $P(x) + R(x)$ are $r$, $s$, and the roots of $Q(x) + R(x)$ are $s$, $t$. By Vieta's formula:

$$r + t = \frac{3 - a}{2} \qquad (1)$$

$$r + s = \frac{3 - b}{2} \qquad (2)$$

$$s + t = \frac{-a - b}{2} \qquad (3)$$

$$rt = \frac{-5}{2} \qquad (4)$$

$$rs = \frac{c - 7}{2} \qquad (5)$$

$$st = \frac{c + 2}{2} \qquad (6)$$

From (1), (3): $r - s = \dfrac{3 + b}{2}$. Since (2): $2r = 3 \Rightarrow r = \dfrac{3}{2}$.

From (4): $t = \dfrac{-5}{3}$. From (5), (6): $s = \dfrac{c - 7}{3} = \dfrac{-3c - 6}{10}$.

$\Rightarrow c = \dfrac{52}{19}$.

So $R(0) = c = \dfrac{52}{19}$.

**Problem 3.C**

For a $p$ prime number and $n$ positive integer, let $v_p(n)$ be the highest power of $p$ that divides $n$. For example: $v_3(63) = 2$; and for $m$ positive integer, $q \neq p$ is prime, $v_p(p^m) = m$, $v_p(p^m q^n) = m$.

Prove that $v_p(a^n - b^n) = v_p(a - b)$, whereas $a, b \in \mathbb{Z}$, $p \in \mathbb{P}$, $n \in \mathbb{Z}^+$, $(n, p) = 1$, $p \mid (a - b)$, $p \nmid (a, b)$.

Let $n$ be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$.Find the number of positive divisors of $n$.

**Solution**

Because $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$

We have to prove that $p \nmid a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}$.

Because $a \equiv b \pmod{p}$:

$a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1} \equiv na^{n-1} \pmod{p}$

Since $\gcd(n, p) = 1$, $p \nmid a$, therefore $p \nmid na^{n-1}$.

Note that $v_3(149^3 - 2^3) = 2$, similarly $v_3(149^9 - 2^9) = 3$.

If $n = a \cdot 3^b$, $3 \nmid a$, $b < 2$, then:

$v_3(149^n - 2^n) = v_3(149^{3^b} - 2^{3^b}) = b + 1 < 3$

So $3^2$, a factor of $n$ would be the least possible choice such that:

$$v_3(149^n - 2^n) \geq 3$$

Similarly $7^5$, a factor of $n$ would be the least possible choice so:

$$v_7(149^n - 2^n) \geq 5$$

Similarly $4 \cdot 5^4$, a factor of $n$ would be the least possible choice:

$$v_5(149^n - 2^n) \geq 4$$

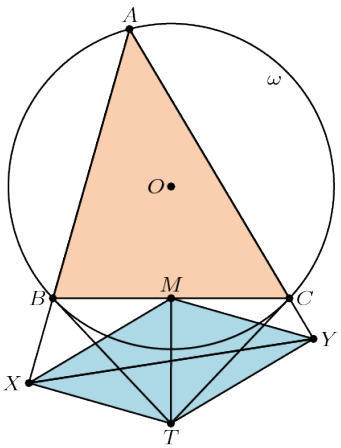
The least possible $n$ is: $2^2 \cdot 3^2 \cdot 5^4 \cdot 7^5$, which has 270 factors.

**Problem 4.C**

Let $\triangle ABC$ be an acute scalene triangle (*a triangle in which all three sides have different lengths*) with circumcircle $\omega$. The tangents to $\omega$ at $B$ and $C$ intersects at $T$. $X$ and $Y$ are the projections of $T$ onto lines $AB$ and $AC$, respectively. Let $M$ be the midpoint of $BC$. $BT = CT = 16$, $BC = 22$, and $TX^2 + TY^2 + XY^2 = 1143$.

- Prove that the quadrilaterals $BMTX$ and $CMTY$ are both cyclic.
- Prove that $MYTX$ is a parallelogram.
- Prove that $2(TX^2 + TY^2) = XY^2 + MT^2$.
- Now find the value of $XY^2$.

**Solution**




$\angle BXT = \angle BMT = 90^\circ$, so $MBXT$ is cyclic.

So: $\angle MXA = \angle MXB = \angle MTB = 90^\circ - \angle TBM = 90^\circ - \angle A$.

So $MX \perp AC$. Similarly $MY \perp AB$.

So $MYTX$ is a parallelogram.

Easy to prove the Parallelogram Law by the law of cosines.

So: $2(TX^2 + TY^2) = XY^2 + MT^2$.

Since $TM^2 = TB^2 - BM^2 = 16^2 - 11^2 = 135$

Hence $2(1143 - XY^2) = XY^2 + 135 \Rightarrow XY^2 = 717$

# Chapter 10

# Week 31

# Weekly Team Contest

**Problem Set 1 Level A**

Evaluate the following expression

$$\frac{(6!+5!)(5!+4!)(4!+3!)(3!+2!)(2!+1!)}{(6!-5!)(5!-4!)(4!-3!)(3!-2!)(2!-1!)}$$

**Problem Set 1 Level B**

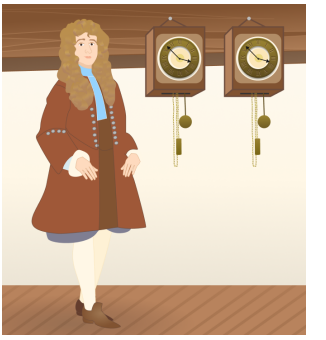
Two palm trees grow on opposite sides of a river. One is 10 m tall and the other is 15 m. The distance between the bases of the trees is 25 m. On the crown of each tree sits a bird. A fish surfaces in the river between the trees, and the two birds dive towards it simultaneously.

If the birds fly along straight lines at equal rates and reach the fish at the same time, **how far from the base of the right palm tree** did the fish surface?

*Note: the artist drew the trees bent, but for the purposes of the problem, you should assume that they are vertical.*

**Problem Set 1 Level C**

Tiana has two clocks. Both starts to tick at the same time. The first clock ticks every 2 seconds, while the second ticks every 3 seconds. Ticks that occur simultaneously are heard as one tick. After the first tick **how much time will pass** before Tiana hears the $2020^{th}$ tick?



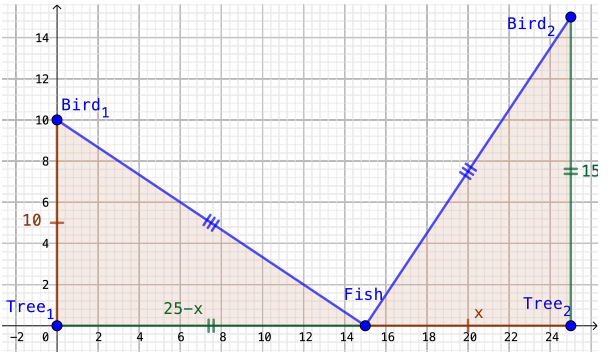
**Problem Set 1 Level A Solution**

$$\text{Since } \frac{6!+5!}{6!-5!} = \frac{(6+1)5!}{(6-1)5!} = \frac{7}{5}, \frac{5!+4!}{5!-4!} = \frac{(5+1)4!}{(5-1)4!} = \frac{6}{4}, \ldots$$

$$\frac{(6!+5!)(5!+4!)(4!+3!)(3!+2!)(2!+1!)}{(6!-5!)(5!-4!)(4!-3!)(3!-2!)(2!-1!)} = \frac{7\cdot 6\cdots 3}{5\cdot 4\cdots 1} = 21$$

**Problem Set 1 Level B Solution**

The situation is reconstructed in a diagram as below.




Let $x$ be the distance between the fish and the right tree. If the distance the birds fly to the fish is the same, then according to the Pythagorean theorem:

$$\sqrt{10^2+(25-x)^2} = \sqrt{x^2+15^2} \Rightarrow 10^2+(25-x)^2 = x^2+15^2$$

$$(25-x)^2 - x^2 = 15^2 - 10^2 \Rightarrow 25^2 - 50x = 125 \Rightarrow x = 10$$

**Problem Set 1 Level C Solution by Team Baby Sharks**

Let write down a first several seconds to see how the clocks tick.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Clock 1** | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 |
| **Clock 2** | | | 1 | | | 1 | | | 1 | | | 1 |
| **# of ticks** | | 1 | 1 | 1 | | 1 | | 1 | 1 | 1 | | 1 |

From the table above, the ticks heard can be divided into block of four ticks that happen within 6 seconds. Because $2020 \div 4 = 505$, so the total length of 505 blocks is 3030. The first tick happens at the $2^{nd}$ second and the last one at $3030^{th}$ second.

Therefore the time passed is $3030 - 2 = 3028$ seconds.

**Problem Set 2 Level A**

Sam, a clever boy, is counting envelopes for his dad. Every package contains one hundred envelopes. Sam can count 10 envelopes in 10 seconds. He counted and gave his dad 60 envelopes. **How many seconds** did he take? His dad then asked for another 240 envelopes. **How much time** did he need to count all those?

**Problem Set 2 Level B**

Ha-Anh Le has two cups. There are 150 millilitres of milk in the first cup and 150 millilitres of coffee in the second cup. She wants to do the following experiment:

- First she takes *15 millilitres of milk from the first cup* and then pours into the second cup.
- The liquid is then mixed and *15 millilitres of the mixture is taken from the second cup* and is poured into the first cup.

Is there **more coffee in the** $1^{st}$ **cup** or **more milk in the** $2^{nd}$ **cup**?

***Bonus:*** *Instead of 15 millilitres, what amount of liquid to be transferred between the cups as above so that at the end the amount of milk and the amount of coffee in the first cup are the same?*

***Bonus****: Let $y$, instead of 150 millilitres, be the amount of milk at the beginning in each cup, and $x$, instead of 15 millilitres, be the amount of liquid transfer between the cup. Can you prove the above answer in this generic case?*

**Problem Set 2 Level C**

Evaluate the following expression

$$\frac{\binom{20}{0}}{1} + \frac{\binom{20}{1}}{2} + \cdots + \frac{\binom{20}{20}}{21}$$

*Note that:*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \textit{ for example } \binom{5}{3} = \frac{5!}{3! \cdot 2!}$$

**Problem Set 2 Level A Solution**

Sam took a package of 100 envelopes and counted 40 envelopes. He gave the rest of the envelopes to his dad. It took 40 seconds. For 240 envelopes, he gave his dad 2 packages, each contains 100 envelopes, and the 40 envelopes he has counted earlier. It took almost no time.

**Problem Set 2 Level B Solution**

Let $y$ be the amount of milk at the beginning in both cup

| $1^{st}$ cup | | | $2^{nd}$ cup | | |
|---|---|---|---|---|---|
| **Milk** | **Coffee** | **Total** | **Milk** | **Coffee** | **Total** |
| $y$ | 0 | $y$ | 0 | $y$ | $y$ |
| $y-x$ | 0 | $y-x$ | $x$ | $y$ | $y+x$ |

The liquid in $2^{nd}$ cup has the volume $y+x$, so in $x$ amount went to the $1^{st}$ cup, $\frac{x}{y+x}\cdot x=\frac{x^2}{x+y}$ is milk, $\frac{x}{y+x}\cdot y=\frac{xy}{x+y}$ is coffee. The coffee volume in the $1^{st}$ and $2^{nd}$ cup are $\frac{xy}{y+x}$, $x-\frac{x^2}{x+y}=\frac{xy}{x+y}$.

They are the same.

For the bonus question, Ha-Anh Le just needs to empty the $1^{st}$ cup into the $2^{nd}$, then pour half of the $2^{nd}$ cup into the $1^{st}$ cup.

**Problem Set 2 Level C Solution**

$$\binom{20}{3}=\frac{20!}{3!\cdot 17!}\Rightarrow\frac{\binom{20}{3}}{4}=\frac{20!}{3!\cdot 17!\cdot 4}=\frac{20!}{4!\cdot 17!}=\frac{1}{21}\binom{21}{4}$$

$$\frac{\binom{20}{0}}{1}+\frac{\binom{20}{1}}{2}+\cdots+\frac{\binom{20}{20}}{21}=\frac{1}{21}\left(\binom{21}{1}+\binom{21}{2}+\ldots+\binom{21}{21}\right)$$

Since $\binom{21}{0}+\binom{21}{1}+\binom{21}{2}+\ldots+\binom{21}{21}=2^{21}$ (both is the number of ways to choose some objects from 21 objects:

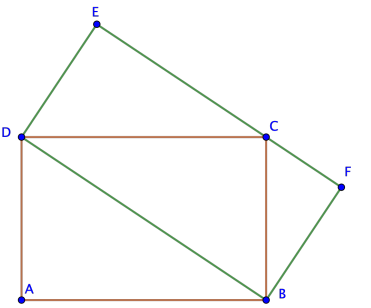
$$\binom{21}{1}+\binom{21}{2}+\ldots+\binom{21}{21}=2^{21}-\binom{21}{0}=2^{21}-1.$$

$$\frac{\binom{20}{0}}{1}+\frac{\binom{20}{1}}{2}+\cdots+\frac{\binom{20}{20}}{21}=\frac{2^{21}-\binom{21}{0}}{21}=\frac{2^{21}-1}{21}$$

### Problem Set 3 Level A

Vu drew the rectangle $ABCD$, $AD = 4$, $AB = 6$. Huy drew another rectangle $BDEF$, where $EF$ contains point $C$ as shown below.

**What is the area** of $BDEF$? Prove your answer.

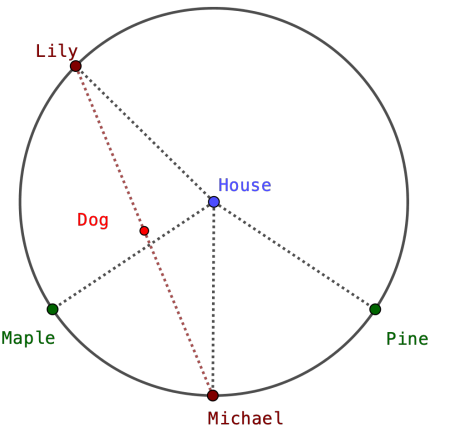



### Problem Set 3 Level B

$ABC$ is a triangle with $\angle B = 90°$, $AB = BC = 1$. Point $M$ is randomly chosen on $AC$. It is possible to tell **the sum of the distances** from $M$ to $AB$ and $M$ to $BC$?

### Problem Set 3 Level C

A maple and a pine tree are at the same distance from Lily's house, so they are on a circle centred at the house. In the winter time, she and her younger brother Michael play a game with their dog. Michael stands at the middle of the arc formed by the two trees. The dog stays on the straight line from Michael to Lily, the same distance from both of them.



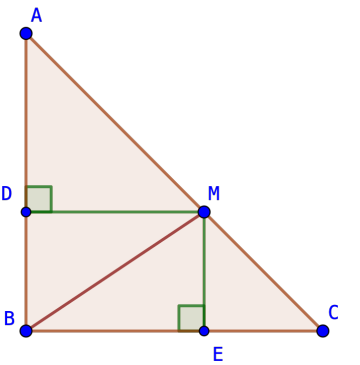

If Lily stomps her foot clockwise around the circle, **what kind of shape** is the dog's paw prints on the snow?

**Problem Set 2 Level A Solution**

Easy to see that the triangle $DCB$ has an area that is equal to half of the area of the rectangle $BDEF$ and it is also half of the area of the rectangle $ABCD$.

Hence the area of $BDEF$ is 24.

**Problem Set 2 Level B Solution**

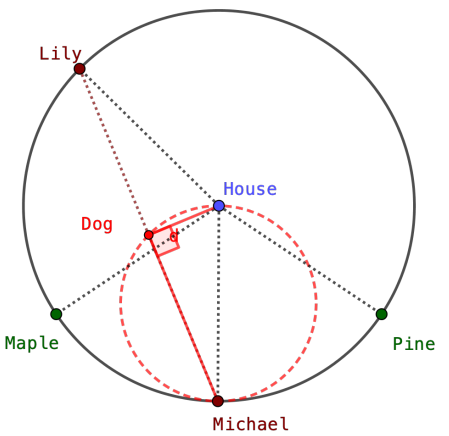



Note that the sum of the area of $\triangle AMB$ and $\triangle CMB$ is the area of $\triangle ABC$, which is $\frac{1}{2}$ and not dependent on where $M$ is. Furthermore, the sum of the area of $\triangle AMB$ and $\triangle CMB$ is:

$$\frac{AB \cdot MD}{2} + \frac{BC \cdot ME}{2} = \frac{MD + ME}{2}$$

Hence the sum of the distances from $M$ to $AB$ and $M$ to $BC$ is 1 and it is not dependent on where $M$ is.

**Problem Set 2 Level C Solution**




Note that the Dog is at the midpoint of the segment connect Lily and Michael, so the segment connect the Dog and the House is perpendicular to it and also bisects it.

Hence the angle form by the House, the Dog, and Michael is always a right angle. That means the Dog moves on the circle with diameter formed by the House and Michael as shown above.

**Problem Set 4 Level A**

The following problem is based on a language that uses algebra to express words, phrases, and sentences. There are some symbols to indicate basic concepts such as; mouth, letter, hammer, pencil, heart, etc. You can see there are familiar mathematical operators such as $+, -, \times, \div, =$, and some other such as exponents and square roots ($\sqrt{\ }$). They help connect the symbols and create meanings.

Several sentences are translated in to English as below.

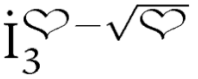
1. $(\frac{\dot{\Lambda}\dot{\Delta}\dot{\imath}\dot{\Delta}}{\dot{\Delta}\dot{\imath}\dot{\Delta}} + \frac{\dot{\imath}\dot{\Delta}}{\dot{\Delta}})^{\leqslant}$ — The father and the brother are
2. $n(> \dot{I})^{[\!]\!-\!-t}$ — The giants are working
3. $(\frac{\dot{\imath}\dot{\Delta}(-\dot{\Lambda}\dot{\Delta})}{(-\dot{\Lambda}\dot{\Delta})})^{\diagup} = \boxtimes$ — The orphans are writing a
4. $(-n\dot{I}_1)^{\diagup} - t = \dot{I}_2$ — It wasn't us who wrote about
5. $\boxtimes^{\sqrt{\diagup}} - t = -\dot{\Delta}_3$ — It was not by her that the
6. $(\frac{\dot{\Lambda}\dot{\Delta}\dot{\imath}\dot{\Delta}}{\dot{\Delta}\dot{\imath}\dot{\Delta}})^{-\heartsuit} = [\!]\!-$ — The father doesn't like the
7. $((> \dot{I}) - \heartsuit)^{\cap} - t = \frac{\dot{\Lambda}\dot{\Delta}\dot{\imath}\dot{\Delta}}{\dot{\imath}\dot{\Delta}}$ — The wicked giant ate the
8. $\dot{\Delta}_3^{-t}$ — She is not in a hurry.

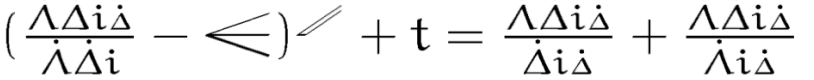
Determine **what are the symbols** for the following words or phrases. *Each correct answer will earn 2 points. No solution is required.*

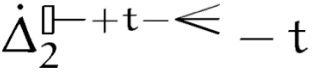
1. Man
2. Woman
3. Boy
4. Girl
5. Husband + Wife
6. Family

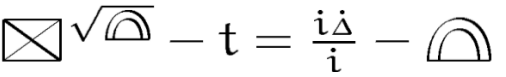
**Problem Set 4 Level B**

**Translate into English** the following four sentences. *Each correct sentence will earn 3 points. No solution is required.*

7 

8 

9 

10 

**Problem Set 4 Level C**

**Translate** the following sentences **into the algebraic language.** *Each correct sentence will earn 5 points. No solution is required.*

11. It wasn't about then that I and my husband talked.
12. The people are working reluctantly.
13. The good widow loves the unemployed dwarf.
14. You will be talked about.

## Problem Set 4 Level A Solution

### Analysis

Nouns:

- $\dot{\Lambda}$ 'man', $\dot{\Delta}$ 'woman', $\mathfrak{i}$ 'boy', $\dot{\mathfrak{i}}$ 'girl', $\boxtimes$ 'letter', $\vdash$ 'work'.
  - Combinations: $\dot{\Lambda}\dot{\Delta}$ 'man + woman = husband + wife', $\mathfrak{i}\dot{\mathfrak{i}}$ 'boy + girl = brother + sister', $\dot{\Lambda}\dot{\Delta}\mathfrak{i}\dot{\mathfrak{i}}$ 'man + woman + boy + girl = family'.
  - Family members are singled out by division and cancellation: $\frac{\dot{\Lambda}\dot{\Delta}\mathfrak{i}\dot{\mathfrak{i}}}{\dot{\Delta}\mathfrak{i}\dot{\mathfrak{i}}}$ 'family/(woman + kids) = father', $\frac{\mathfrak{i}\dot{\mathfrak{i}}}{\dot{\mathfrak{i}}}$ 'kids/girl = brother', $\frac{\dot{\Lambda}\dot{\Delta}\mathfrak{i}\dot{\mathfrak{i}}}{\mathfrak{i}\dot{\mathfrak{i}}}$ 'family/kids = parents'.
  - Missing (deceased) family members are preceded by a minus sign: $\frac{\mathfrak{i}\dot{\mathfrak{i}}(-\dot{\Lambda}\dot{\Delta})}{(-\dot{\Lambda}\dot{\Delta})}$ 'kids (−parents)/(−parents) = orphans' (apparently orphaned children of one and the same family).
- $\dot{\mathrm{I}}$ 'person', $(>\dot{\mathrm{I}})$ 'giant'.

Pronouns are composed of the character $\dot{\mathrm{I}}$ or $\dot{\Delta}$ (for feminine gender) and the subscripts 1 to 3, which indicate the person.

The plural of nouns and pronouns is expressed by the coefficient $\mathfrak{n}$. The plus sign plays the part of the conjunction 'and'.

Verbs: $\lessgtr$ 'talk', $\vdash$ 'work', $\mathfrak{t}$ 'hurry', $\diagup$ 'write', $\heartsuit$ 'like, love', $\cap$ 'eat'. If what the verb denotes is absent or uncharacteristic, a minus sign expresses that: $-\heartsuit$ 'not inclined to affection = wicked'. (We can assume that a characteristic property is expressed by a plus sign, hence $+\heartsuit$ 'good', a concept we need.)

Sentence structure:

- the subject is the base of the power;
- the predicate is the exponent, whereby negation is expressed by a minus sign ($-\heartsuit$ 'not like') and passive voice by a radical sign ($\sqrt{\diagup}$ 'be written'); additional activities can be added or subtracted ($\dot{\mathrm{I}}_3^{\vdash-\mathfrak{t}}$ 'he is working and doesn't hurry = he is working without haste');
- past tense is marked by $-\mathfrak{t}$ ($\dot{\mathrm{I}}_3^{\vdash} - \mathfrak{t}$ 'he worked'), future tense by $+\mathfrak{t}$;
- the direct object, if there is one, follows an equals sign.

## Problem Set 4 Level B Solution

7 He loves with an requited love (i.e. loves without being loved)
8 The taciturn daughter will write about the father and the mother
9 You (sg. fem.) worked quickly (or hastily) and silently
10 The letter was eaten by the hungry sister.

## Problem Set 4 Level C Solution

| | |
|---|---|
| It wasn't about then that I and my husband talked. | $(\dot{\Delta}_1 + \frac{\dot{\Lambda}\dot{\Delta}}{\dot{\Delta}})^{\lessgtr} - \mathfrak{t} = -\mathfrak{n}\dot{\mathrm{I}}_3$ |
| The people are working reluctantly. | $(\mathfrak{n}\dot{\mathrm{I}})^{\vdash-\heartsuit}$ |
| The good widow loves the unemployed dwarf. | $(\frac{\dot{\Delta}(-\dot{\Lambda})}{(-\dot{\Lambda})} + \heartsuit)^{\heartsuit} = (<\dot{\mathrm{I}}) - \vdash$ $(\mathfrak{n}\dot{\mathrm{I}}_2)^{\sqrt{\lessgtr}} + \mathfrak{t}$ |
| You will be talked about. | |

**Problem Set 5 Level A**

David went to the bazaar in Cairo, Egypt. He wanted to buy some nice earrings for his mother but was challenged by the local sellers.

There are eight identical-looking coins on the table. One of them is a counterfeit (a fake coin) and lighter than the others.

What is the **minimum number of weighings** needed to identify the fake coin with a two-pan balance scale and without weights?

**Problem Set 5 Level B**

Bianca is practicing tiling with dominoes and chessboards so that she can redecorate the house for Christmas.

**Can you help her** to to tile an $8 \times 8$ standard chessboard by $1 \times 2$ dominoes so that no two dominoes form a $2 \times 2$ square?

**Problem Set 5 Level C**

**Evaluate** the following expression

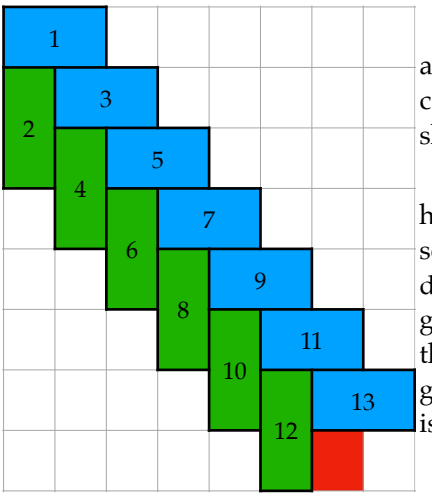
$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{2019^2}+\frac{1}{2020^2}}$$

**Problem Set 5 Level A Solution**

Two weighings will be needed: first let split the coins in two groups of three and one group of two. The first weighing is used to compare the first two groups:

- if they are genuine coins, then the second weighing is used to find the fake coin in the third group.
- if one group is lighter, then the second weighing is used to compare any two coins in that group. The fake coin is shown by comparison is it is among the two coins, or it is the remaining one.

**Problem Set 5 Level B Solution**




If such tiling exists, we can assume that the upper left corner can be till with a horizontal $1 \times 2$ as shown. Let this be the domino 1.

The untiled upper left corner has to be tiled by a vertical domino, so let this be the domino 2. Continue doing so, we have a set of blue and green dominos as shown. However the red square cannot be tiled by any green domino. Hence, no such tiling is possible.

**Problem Set 5 Level C Solution**

Since $\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} = \sqrt{\frac{k^2(k+1)^2+(k+1)^2+k^2}{k^2(k+1)^2}}$, and

$$k^2(k+1)^2+(k+1)^2+k^2 = k^4+2k^3+k^2+k^2+2k+1+k^2$$
$$= k^4+2k^3+3k^2+2k+1 = (k^4+k^3+k^2)+(k^3+k^2+k)+(k^2+k+1)$$
$$= k^2(k^2+k+1)+k(k^2+k+1)+(k^2+k+1) = (k^2+k+1)^2$$

$$\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} = \frac{k^2+k+1}{k(k+1)} = 1+\frac{1}{k(k+1)} = 1+\frac{1}{k}-\frac{1}{k+1}$$

Hence the sum is equal to: $2019+\frac{1}{1}-\frac{1}{2020} = 2019\frac{2019}{2020}$

**Problem Set 6 Level A**

In a cross-country skiing competition, Lilian placed exactly in the middle among all participants. Martin placed lower, in tenth place. Nam place sixteenth.

Is it possible to figure out **how many skiers** took part in the competition?

**Problem Set 6 Level B**

People in went to vote in an election. The proportion of male voters to female voters was 17 : 15. Had 80 fewer woman and 90 fewer men taken part, the proportion would have been 8 : 7.

Bill is asked by the election organizers to determine **how many people** went to vote. What was his conclusion?

**Problem Set 5 Level C**

Kevin has developed a taste for cookie. He has been collecting cookies to celebrate the coming Christmas. He has 15 cookie jars containing 1, 2, 3, …, 15 cookies, respectively.

On a cold dark night in December, the cookie monster came into the house. He decided to choose a subset of jars (some number of jars) and take the same number of cookies from each jar and then left the house. This hideous act was repeated for another three nights before Christmas.

**Is it possible** that the heartless monster ate all the cookies that Kevin has spent great effort to collect leaving him devastated on the Christmas Eve?

**Problem Set 6 Level A Solution**

Since Lilian finished exactly in the middle, so there is an odd amount of competitors. Because Nam finished at $16^{th}$, so there must have at least 17 skiers, which means Lilian's position is at least $9^{th}$.

One the other hand the position of Martin is $10^{th}$ and it is lower position than Lilian's, so Lilian's position cannot be more than $9^{th}$.

That means Lilian's position is $9^{th}$ and there are 17 skiers.

**Problem Set 6 Level B Solution**

Let there be $17n$ men and $15n$ women participating in the election. We have:

$$\frac{17n-90}{15n-80}=\frac{8}{7}\Rightarrow 119n-630=120n-640\Rightarrow n=10.$$

So the number of participants is $170+150=320$.

**Problem Set 6 Level C Solution**

The key idea is to realize that jars with the same number of cookies can be treated as if they were just one jar: for if you've got two (or more) jars with $k$ cookies, the cookie monster might as well do the same thing with them in each subsequent step - those jars are depleted at the same rate.

With this in mind, we just need to focus on the *different numbers of cookies* in the jars at each night. Below is a solution:

$$\{15\ 14,\cdots,\ 1\}\xrightarrow{\text{remove 8 cookies}}\{7\ 6,\cdots,\ 1\}$$

$$\{7,\ 6,\cdots,\ 1\}\xrightarrow{\text{remove 4 cookies}}\{3,\ 2,\ 1\}$$

$$\{3,\ 2,\ 1\}\xrightarrow{\text{remove 2 cookies}}\{1\}$$

$$\{1\}\xrightarrow{\text{remove 1 cookie}}\varnothing$$

*Note: $\varnothing$ means the empty set.*

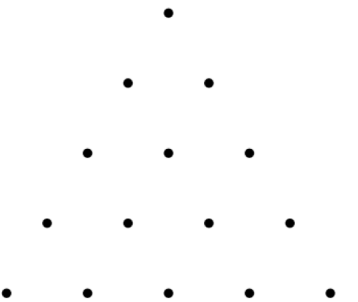
# Problems of the Week

**Level A**

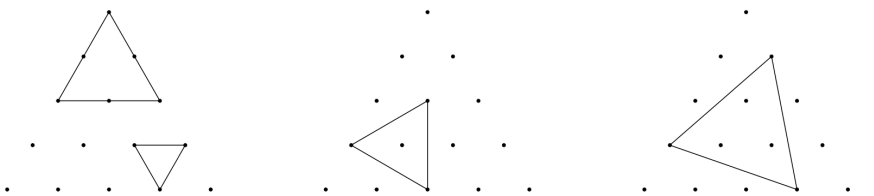
### Problem 1.A

How many different sets of three points in this equilateral-triangular grid are the vertices of an equilateral triangle?

How many different sets of four points in the above equilateral-triangular grid are the vertices of a parallelogram with non-zero area? Prove your result.



**Solution**



Call the distance between two adjacent dots on the same row 1 unit. Call an equilateral triangle size $x$ if each of its sides are $x$ units long. So the smallest triangle that can be drawn on the grid is size 1, and the largest triangle has size 4.

It is easy to see that in this grid there are only a few possible distances between pairs of points: 1, 2, 3, 4, $\sqrt{3}$, $\sqrt{7}$, $2\sqrt{3}$, and $\sqrt{13}$, so every equilateral triangle that fits in the grid must have its sides equal to one of these lengths.

First, consider triangles with one side horizontal. They can point up or down, and the figure on the left above illustrates a size 1 triangle pointing down and a size 2 triangle pointing up. Each such triangle is completely determined by its uppermost or lowermost point. For upward-pointing triangles there are $4 + 3 + 2 + 1 = 10$ size 1 triangles, $3+2+1 = 6$ size 2 triangles, $2+1 = 3$ size 3 triangles, and 1 size 4 triangle, for a total of 20 upward- pointing triangles. Next, consider triangles with one side horizontal pointing down. There are $3 + 2 + 1 = 6$ size 1 triangles and 1 size 2 triangle of this type, for a total of 7 such triangles.

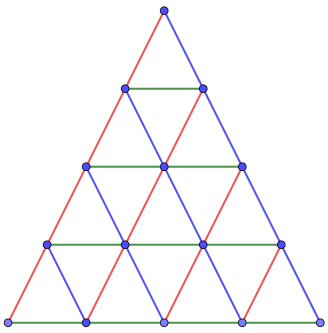
Next, consider tilted triangles that do not have any horizontal sides. There are two possible sizes illustrated by the middle and right-most figures above. The smaller type with side $\sqrt{3}$ have a vertical edge and point either right or left. There are 3 right-pointing triangles a three left-pointing triangles for a total of 6. There are only two of the larger triangles with no horizontal line having side $\sqrt{7}$.

The only other distances between pairs of points in the grid are $2\sqrt{3}$ and $\sqrt{13}$ with these side lengths will not fit.

Thus there are 20+7+6+2 = 35 total triangles.

For the second question, we use a different counting approach.

First, easy to see that the the triangular grid can be divided into groups of parallel lines: red, green, blue as shown right.



Case 1: If a side of the parallelogram is parallel with a side of the border of the triangular grid, then its vertices are on the same pair of lines.

Let count the parallelograms with sides on the blue lines.

If the left side is on the most left blue line, there is only one choice, if the right side is on the second most left blue line, there are two choices, all together $1 \times 2$ ways. Similarly if the right side on subsequent lines: $1 \times 3$, $1 \times 4$. The total number of them is :

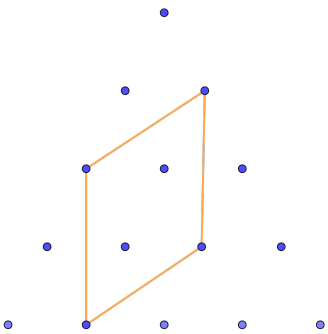
$$1 \times 2 + 1 \times 3 + 1 \times 4 + 2 \times 3 + 2 \times 4 + 3 \times 4 = 35$$

By counting for all three colours, there are 105 such parallelograms.

Case 2: If a side of the parallelogram is not parallel with a side of the border of the triangular grid.

There are only three such parallelograms, one is shown as right.

Hence all together there are 108 parallelograms.



**Problem 2.A**

In the Rhind mathematical papyrus from ancient Egypt, among other information you will find the expansion of a fraction into a sum of fractions with the numerator 1, for example:

$$\frac{2}{73} = \frac{1}{60} + \frac{1}{219} + \frac{1}{292} + \frac{1}{x}$$

What is the value of $x$?

Find $m > n$ positive integers such that:

$$\frac{13}{20} = \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{48n} + \frac{1}{48m}$$

**Solution**

By completing simple operations with fractions:

$$\frac{1}{x} = \frac{2}{73} - \left(\frac{1}{60} + \frac{1}{219} + \frac{1}{292}\right) = \frac{1}{365}. \text{ So } x = 365.$$

Similarly:

$$\frac{13}{20} - \left(\frac{1}{2} + \frac{1}{8} + \frac{1}{48}\right) = \frac{1}{240}$$

We have to find $m$, $n$ positive integers such that $m > n$ and:

$$\frac{1}{48n} + \frac{1}{48m} = \frac{1}{240}$$

The above is equivalent with:

$$\frac{1}{48}\left(\frac{1}{n} + \frac{1}{m}\right) = \frac{1}{48} \cdot \frac{1}{5}. \text{ Hence } \frac{1}{n} + \frac{1}{m} = \frac{1}{5}$$

There are several ways to prove that $n = 6$, $m = 30$. Here is an important identity that can be used later in several cases.

For every positive integer $k$:

$$\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}$$

It can easily be proved by bringing the right side to common denominator. For a few values of $k$:

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{2 \cdot 3}, \frac{1}{3} = \frac{1}{4} + \frac{1}{3 \cdot 4}, \frac{1}{5} = \frac{1}{6} + \frac{1}{5 \cdot 6}$$

Therefore $n = 6$, $m = 30$ is the solution.

**Problem 3.A**

Find all solutions of the following cryptogram, where each distinct letter encodes a distinct digit:

$$I + HE + HE + HE + HE + HE + HE + HE + HE = US$$

The book Magic for Dummies contains the following passages:

*"If you replace each of the distinct letters in* GLOBALHELLFRY *with distinct digits, and you get a prime number, then the world will suffer a terrible heat wave."*

Are you holding the key to create this terrible heat wave?

**Solution**

$$I + 8 \cdot HE = US.$$

Since $US < 100 \Rightarrow 8 \cdot HE < 100 \Rightarrow HE \leq 12$. Since $I$ and $S$ must represent different digits, so $E$ cannot be 0.

Therefore $H = 1$, $E = 2$.

$8 \cdot HE = 96$, so $US$ can be $96, 97$ or $98$ (99 is not possible because $U \neq S$). The least two cases are not possible because then $I$ is equal to 1 or 2. So $US = 96$, $I = 0$.

The original equation is:

$$0 + 12 + 12 + 12 + 12 + 12 + 12 + 12 + 12 = 96$$

For the second question, the word GLOBALHELLFRY has 10 different letters. Each letter appears once except the letter L appears 4 times. The sum of all the letters is the sum of all the digits from 0 to 9 plus three times the digit that the letter L represents:

$$0 + 1 + 2 + \ldots + 9 + 3L = 45 + 3L$$

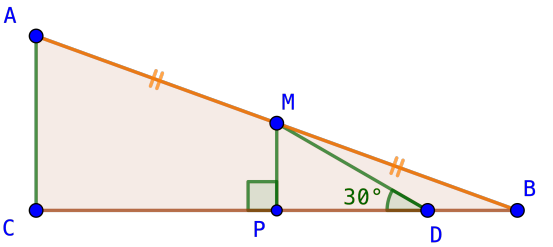
This sum is always divisible by 3, so unfortunately you do not hold the key to create a terrible heat wave.

**Problem 4.A**

Let triangle $ABC$ have a right angle at $C$, and let $M$ be the midpoint of the hypotenuse $AB$. Choose a point $D$ on line $BC$ so that angle $CDM$ measures 30 degrees. Prove that the segments $AC$ and $MD$ have equal lengths.

Let $P$ be the foot of the altitude from $M$ to $BC$. Furthermore $MP = 4$, $BC = 15$. Find the value of $MC - MD$.

**Solution**




Let $P$ be the foot of the altitude from $M$ to $BC$. $MP \parallel AC$ so $P$ is also midpoint of $BC$ and $AC = 2 \cdot MP$.

$\triangle MDP$ is a $30 - 60 - 90$ triangle, therefore $MD = 2 \cdot MP$.

Hence $AC = MD$.

Since $M$ is the midpoint of the hypotenuse $AB$:

$MA = MB = MC$. $BC = \sqrt{15^2 + 8^2} = 17$, so $MC = \dfrac{17}{2}$.

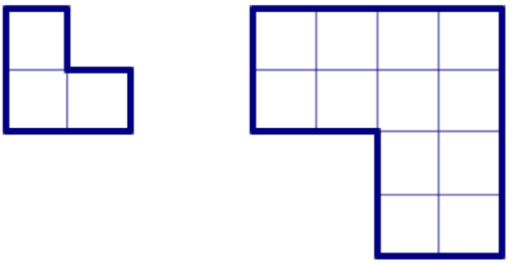
$MD = AC = 2 \cdot MP = 8$.

Hence $MC - MD = \dfrac{17}{2} - 8 = \dfrac{1}{2}$.

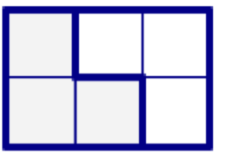
## Level B

### Problem 1.B

Define a $size - n$ *tromino* to be the shape you get when you remove one quadrant from a $2n \times 2n$ square. In the figure below, a $size - 1$ tromino is on the left and a $size - 2$ tromino is on the right.



We say that a shape can be tiled with $size - 1$ trominos if we can cover the entire area of the shape—and no excess area—with non-overlapping $size - 1$ trominos. For example, a $2 \times 3$ rectangle can be tiled with $size - 1$ trominos as shown below, but a $3 \times 3$ square cannot be tiled with $size - 1$ trominos.



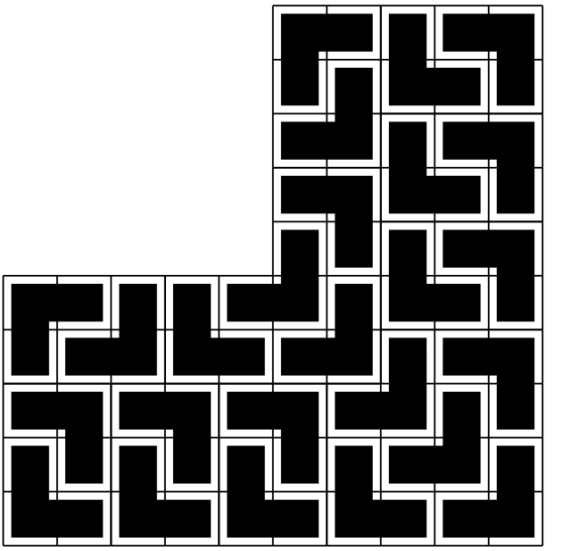
Can a $size - 5$ tromino be tiled by $size - 1$ trominos?

Can a $3 \times k$ rectangle be tiled by $size - 1$ trominos?

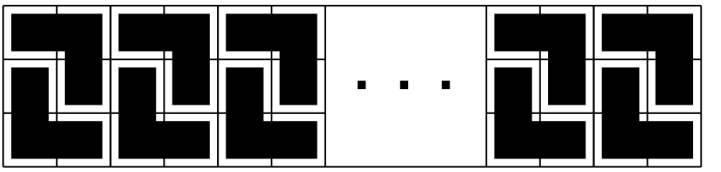
Can a $size - 2020$ tromino be tiled by $size - 1$ trominos?

**Solution**

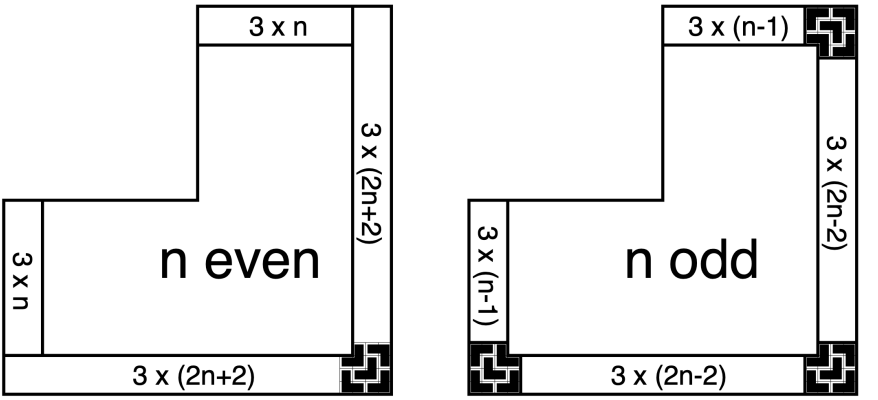
A $size - 5$ tromino can be tiled by $size - 1$ trominos as below.



A $3 \times k$ rectangle with $k$ even can be tiled by $size - 1$ trominos:



We can constructively tile a $size - n$ tromino be tiled by $size - 1$ trominos as below.




size n+3 from size n

**Problem 2.B**

$n$ is a positive integer and $p = 2^n - 1$ is a prime number. Prove that $\dfrac{p(p+1)}{2}$ is a perfect number.

$n$ is a positive integer such that $p = 3 \cdot 2^{n-1} - 1$, $q = 3 \cdot 2^n - 1$, and $r = 9 \cdot 2^{2n-1} - 1$ are prime numbers. Prove that the sum of all proper divisors of $2^n pq$ is $2^n r$ and the sum of all proper divisors of $2^n r$ is $2^n pq$

*Note: A perfect number is an integer that is equal to the sum of its proper divisors. For example* $6 = 1 + 2 + 3$ *is a perfect number.*

**Solution**

Let $m = \dfrac{p(p+1)}{2} = 2^{n-1}p$

This number has following proper divisors:

$$1,\ 2,\ \cdots,\ 2^{n-1},\ p,\ 2p,\ \cdots,\ 2^{n-2}p$$

The sum of them is:

$t(m) = 1 + 2 + \cdots + 2^{n-1} + p + 2p + \cdots + 2^{n-2}p$

$= (1 + 2 + \cdots + 2^{n-1}) + p(1 + 2 + \cdots + 2^{n-2})$

$= 2^n - 1 + p(2^{n-1} - 1)$

$= p + p(2^{n-1} - 1)$

$= 2^{n-1}p$

$= m$

Hence it is a perfect number.

For the second question $2^n pq$ has the following proper divisors:

$1,\ 2,\ \cdots,\ 2^n,\ p,\ 2p,\ \cdots,\ 2^n p,\ q,\ 2q,\ \cdots,\ 2^n q,\ pq,\ 2pq,\ \cdots,\ 2^{n-1}pq$

Similar as above their sum is:

$t(2^n pq) = 2^{n+1} - 1 + p(2^{n+1} - 1) + q(2^{n+1} - 1) + pq(2^n - 1)$

$p(2^{n+1} - 1) = (3 \cdot 2^{n-1} - 1)(2^{n+1} - 1) = 3 \cdot 2^{2n} - 7 \cdot 2^{n-1} + 1$

$q(2^{n+1} - 1) = (3 \cdot 2^n - 1)(2^{n+1} - 1) = 6 \cdot 2^{2n} - 10 \cdot 2^{n-1} + 1$

$pq(2^n - 1) = (3 \cdot 2^{n-1} - 1)(3 \cdot 2^n - 1)(2^n - 1)$

$= 9 \cdot 2^{3n-1} - 9 \cdot 2^{2n} + 11 \cdot 2^{n-1} - 1$

$t(2^n pq) = 9 \cdot 2^{3n-1} - 2^n = 2^n(9 \cdot 2^{2n-1} - 1) = 2^n r.$

Continuing in a similar way, we can prove that: $t(2^n r) = 2^n pq$

**Problem 3.B**

Below is a cryptarithm:

$$250 \cdot \text{EIK} + \text{FOX} = 2005 \cdot \text{OWL}$$

Note that different letters stand for different digits, and the same letter stands for the same digit. Some letter might stand for the digits that already present: 0, 2, or 5.

Find one solution for the cryptarithm.

Find all other solutions of the cryptarithm or prove there is only one solution.

**Solution**

If $E \leq 7$, the left side is less than $125 \cdot 800 + 1000 = 201000$, but the right side is at least $2005 \cdot 102 = 204510$.

So $E$ is 8 or 9.

If $OWL \geq 124$, the right side is at least $2005 \cdot 124 = 248620$, but the left side is smaller than $250 \cdot 987 + 1000 = 247750$. So $OWL \leq 123$.

So $O = 1$ and $W$ is 0 or 2.

Furthermore, note that $250 \cdot ELK$ and 2005 are divisible by 5 so $FOX$ is also divisible by 5, therefore $X$ is 0 or 5. If $X$ is 0, then $L$ should be even and $W$ ( $\neq X$) should not be 0. So $OWL \geq 124$. This is not possible.

So $X$ is 5, and of course $L$ is odd: 3, 7, or 9.

Since $O = 1$, $X = 5$, if we divide both side by 50 then the remainder of the left size is 15, hence the right size must also has remainder 15 when divided by 50.

This can only possible if $L = 3$.

Since $W$ is 0 or 2, let test W with 0. $2005 \cdot 103 = 206515$, which means $K$ is even (otherwise the tens digit will not be a 1 on the left side). Hence $EIK \geq 824$, and $F \geq 6$ (the rest of the digits are taken), and the right side is smaller than the left side.

It means that $W = 2$.

So $OWL = 123$. By testing: $E = 9$, $I = 8$, $K = 4$, $F = 6$.

$$250 \cdot 984 + 615 = 2005 \cdot 123$$

**Problem 4.B**

Elisa has a special ruler. She can draw as usual lines, also measure segments, and duplicate them. She can also divide a segment into as many equal parts as she wants.

$ABCD$ is drew as parallelogram. The midpoint $M$ of side $CD$ is determined by the ruler, and side $CB$ is extended to $N$ so that segments $CB$ and $BN$ were equal in length. She happily went for a snack.

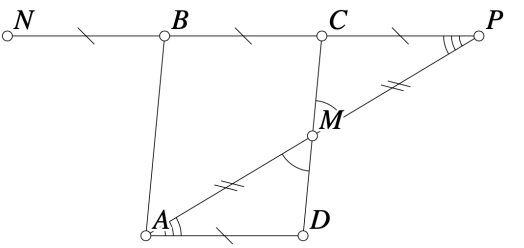
When she was away, somebody came, saw the diagram and erased everything on the diagram except for points $A$, $M$ and $N$.

Can the original parallelogram $ABCD$ be reconstructed the special ruler only? If yes, write down the steps and prove that it is possible. If not, show a proof.

Can you show another way to reconstruct it, of course with the special ruler only.

**Solution**

Elisa should extend the line AM beyond M. Measure AM and find the point P on the extension of AM beyond M such that AM = MP. Vertical angles ∠CMP = ∠DMA, CM = MD and AM = MP so △PMC is congruent to △AMD by SAS.




Because of the triangle congruence, ∠CPM = ∠DAM. This means that the transversal AP makes equal angles with PC and AD so PC will be parallel to AD. The line BC is another line through C that is parallel to AD so it is the same as line PC, so P lies on the line containing B, C, and N.

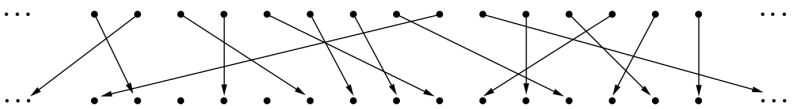
Again, by the congruence of the triangles, CP = AD and AD = BC = BN, so if we use the magic ruler to divide PN into three equal parts, the division points must correspond to the missing points B and C. By extending CM and measuring off an additional length of CM on the other side of M, Elisa can construct the final missing point D. The other possible construction is to construct P, then connect N with M, find the midpoint of MN, draw a line from A via the midpoint of MN to intersect NP at B. The rest is similar.

## Level C

### Problem 1.C

Two infinite rows of evenly-spaced dots are aligned as in the figure below. Arrows point from every dot in the top row to some dot in the lower row in such a way that:

- No two arrows point at the same dot; and
- No arrow can extend right or left by more than 1010 positions.



Show that at most 2020 dots in the lower row could have no arrow pointing to them.

**Solution**

Call dots in the lower line that lie at the endpoint of arrows "target dots" and those that are not, "missed dots". If an arrangement has 2021 or more missed dots, pick a contiguous set S of dots in the lower line that includes exactly 2021 missed dots and t target dots.

Consider the set of $t + 2020$ dots directly above the dots in S from which $t + 2020$ arrows must initiate. At most $t$ of them can terminate in S, so at least 2020 of them terminate outside S. But since arrows can only extend to dots 1010 outside of S on either side, there are only $1010 + 1010 = 2020$ possible targets for those 2021 or more arrows, which is impossible.

Therefore it is impossible to have 2021 or more missed dots in a valid configuration.

**Problem 2.C**

$\{F_n\}$ is the Fibonacci sequence:

$F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$

Prove that there is a term that is divisible by 2020.

How many such terms are there?

**Solution**

Let take a look at the sequence of remainders when divided the Fibonacci sequence's terms by 2020.

There are on 2020 possible values of remainders: 0, 1, …, 2019. Therefore only $2020^2$ possible pairs of these values. By the pigeonhole principle, within $2020^2 + 1$ pairs, a pair should be repeated.

This would also mean that the rest of the sequence are repeating blocks, for ever, ad infinitum.

So let the beginning of the block $F_m = a$, $Fm + 1 = b$, and the block length is $l$. In another word $F_{m+l} = a$, $F_{m+l+1} = b$ .By going backward $F_{m+l-1} = F_{m-1} = b - a$, $F_{m+l-2} = F_{m-2}$, and so on until $F_l = F_0$. This mean that the periodicity starts at $F_0$, repeats every $l$ steps.

**Problem 3.C**

Suppose that for $x, y \in \mathbb{R}$:

$$\left(x + \sqrt{1 + x^2}\right)\left(y + \sqrt{1 + y^2}\right) = 1$$

Determine the value of $x + y$.

**Solution**

Multiplying both sides by $-y + \sqrt{1 + y^2}$:

$$\left(x + \sqrt{1 + x^2}\right)\left(-y^2 + (1 + y^2)\right) = -y + \sqrt{1 + y^2}$$

$$x + y = \sqrt{1 + y^2} - \sqrt{1 + x^2} \qquad (1)$$

Multiplying both sides by $-x + \sqrt{1 + x^2}$:

$$\left(-x^2 + \sqrt{1 + x^2}\right)\left(y + \sqrt{1 + y^2}\right) = -x + \sqrt{1 + x^2}$$

$$x + y = \sqrt{1 + x^2} - \sqrt{1 + y^2} \qquad (2)$$

From (1) and (2): $x + y = 0$

**Problem 4.C**

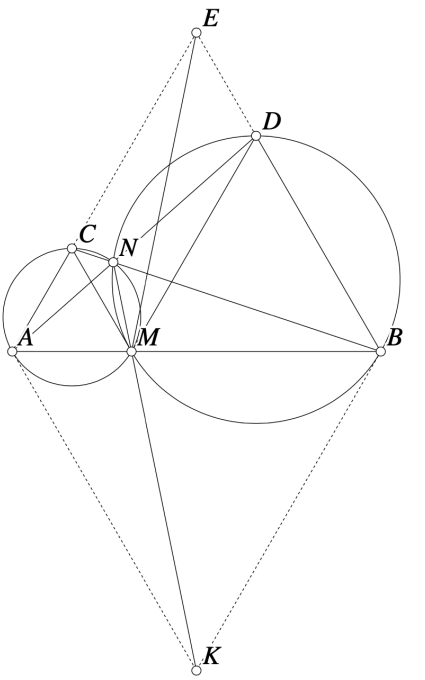
Given a segment $AB$ in the plane, a point $M$ on $AB$ and different from $A$ and $B$. Two equilateral triangles $\triangle AMC$ and $\triangle BMD$ in the plane are constructed on the same side of segment $AB$. The circumcircles of the two triangles intersect in point $M$ and another point $N$.

Prove that lines $AD$ and $BC$ pass through point $N$.

By choosing several locations $M$ (also special and edge cases), observe how $N$ change locations, and how the line $MN$ is changed. Prove that no matter where one chooses the point $M$ along segment $AB$, all lines $MN$ will pass through some fixed point $K$ in the plane.

**Solution**

It is not hard to show that point $N$ is on the same side of segment $AB$ as the two triangles, and that $N$ is inside $\angle CMD$ so that $\{A, M, N, C\}$, and $\{B, M, N, D\}$, are arranged in these orders correspondingly on the circumcircles, as shown at right. (*The reason is essentially that side $MC$ is tangent to the circumcircle of $\triangle BMD$, because of the angle it makes with $AB$*).




Since $A$, $M$, $N$, and $C$ are concyclic, and $C$ and $N$ are on the same side of line $AB$:

$$\angle ANM = \angle ACM = 60^\circ.$$

Since $B$, $M$, $N$ and $D$ are con-cyclic, and $B$ and $N$ are on opposite sides of chord $MD$, $\angle MND = 180^\circ - \angle MBD = 180^\circ - 60^\circ = 120^\circ$.

Thus, the sum $\angle ANM + \angle MND = 60^\circ + 120^\circ = 180^\circ$, which proves that $A$, $N$, and $D$ lie on a line. One can prove analogously that $B$, $N$, and $C$ also lie on a line.

# Chapter 11

# Week 32

# Weekly Team Contest

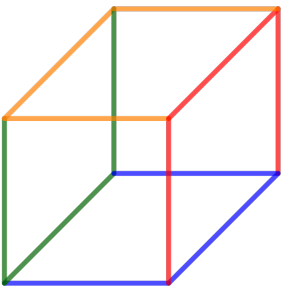
**Problem Set 1 Level A**

Each member of MCC club either swims or runs. One-seventh of the swimmers also run, and one-ninth of the runners also swim.

What is **the proportion** of the members who swim only to the members who run only?

**Problem Set 1 Level B**

A string length 12 is cut into a number of parts. The parts are used to construct a frame of a cube side length 1. Below is an example how the cube is constructed when the string is cut to 4 equal parts (*coloured differently for better visualization*), each has a length of 3.



For each of the 14 ways to cut the string as below, show how you construct the frame. 6321 means that 4 parts have length 6, 3, 2, and 1.

| | | | | |
|---|---|---|---|---|
| 9111 | 8211 | 7311 | 7221 | 6411 |
| 6321 | 6222 | 5511 | 5421 | 5331 |
| 5322 | 4431 | 4422 | 4332 | |

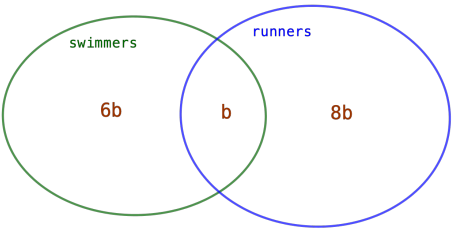
**Problem Set 1 Level C**

Tina brought a parrot. On the first day it said "O" and on the second day "OK". The third day the parrot said "OKKO", the fourth day "OKKOKOOK" and so on.

What is **the** $2020^{th}$ **letter** on the sixteenth day?

**Problem Set 1 Level A Solution**

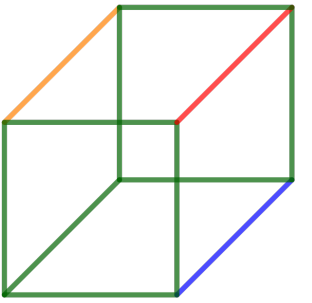
Let $b$ be the number of members who both swim and run. Then seven times as many, $7b$, swim, and nine times as many, $9b$, run. From the diagram, $6b$ swim only and $8b$ run only.




Therefore the proportion of of the members who swim only to the members who run only is $6b : 8b = 3 : 4$.

**Problem Set 1 Level B Solution**

It is possible to use any cut combination for construction. Below is an example for the first one.



Students are encouraged to find all the remaining ones.

**Problem Set 1 Level C Solution**

Replace the "O" and "K" by 0 and 1, let $d(k)$ be the $k^{th}$ digit. We have $d(1) = 0$, $d(2) = 1$, $d(3) = 1$, $d(4) = 0$, $d(8) = 1$.

Furthermore for $n$ not power of 2, $2^t < b < 2^{t+1}$:

$$d(n) = 1 - d(n - 2^t)$$

So $d(2020) = 1 - d(2020 - 1024) = 1 - d(996) = d(484)$

$= 1 - d(228) = d(100) = 1 - d(36) = d(4) = 0$

**Problem Set 2 Level A**

Binh plays a two-player game with one of his friend. They take turns to place rooks on a standard chessboard. Each turn a player place a rook so that none of the existing rooks can capture it. The player who cannot place a rook is the loser. Can you suggest a **winning strategy** for Binh?

**Problem Set 2 Level B**

Decrypt the following **cryptarithm.** Same letter always stands for the same digit, different letters stands for different digits.

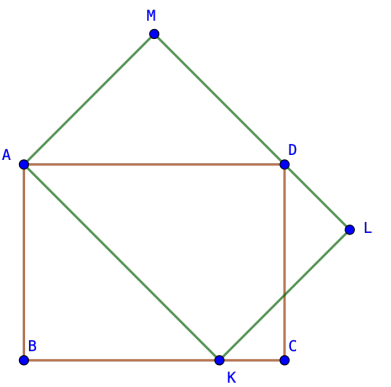
$$\mathrm{BAO} \times \mathrm{BA} \times \mathrm{B} = 2002$$

**Bonus**: Solve the following: $\mathrm{BAO} \times \mathrm{BC} \times \mathrm{B} = 2002$

**Problem Set 2 Level C**

In the rectangle $ABCD$, $AB = 3$, $BC = 4$. Another rectangle $AMLK$ is drawn such that $ML$ contains point $D$ and point $K$ is on $BC$ as shown below.

What is **the area** of $AMLK$?




**Problem Set 2 Level A Solution**

Each turn, the number of unoccupied rows (as well as the number of unoccupied columns) is decreased by 1. Since there are 8 rows, the second player will win.

**Problem Set 2 Level B Solution**

$$2002 = 2 \times 7 \times 11 \times 13 = 143 \times 14 \times 1 = 154 \times 13 \times 1.$$

**Problem Set 2 Level C Solution**

Connect $D$ with $K$. The area of the triangle $ADK$ is half of $AMLK$, and is also half of $ABCD$. So the area of $AMLK$ is 12.

### Problem Set 3 Level A

Solve the following **cryptarithm**. Same letter always stands for the same digit, different letters stands for different digits.

$$\begin{array}{cccc} & & M & E \\ + & & & M \\ \hline A & S & & A \end{array}$$

### Problem Set 3 Level B

Minh and Penn take turns placing bishops (each turn only one) on the squares of a standard chessboard so that they cannot capture each other. The player who cannot move loses.

What would be the **winning strategy** for Minh?

### Problem Set 3 Level C

Which number is larger: $\sqrt[19]{19!}$ or $\sqrt[20]{20!}$

### Problem Set 3 Level A Solution

$$\begin{array}{cccc} & & 9 & 2 \\ + & & & 9 \\ \hline 1 & 0 & & 1 \end{array}$$

### Problem Set 3 Level B Solution

Use the line $L$ separating the fourth and fifth column as a line of symmetry. Anytime a player places a bishop on a square $S$, the other players can place the other bishop on the square symmetric to $S$ in regard to $L$. Note that two squares have different colours! So the second player will win.

### Problem Set 2 Level C Solution

$(\sqrt[19]{19!})^{19\times 20} = (19!)^{20} = (19!)^{19} \cdot (19!)$

$(\sqrt[20]{20!})^{19\times 20} = (20!)^{19} = (19! \cdot 20)^{19} = (19!)^{19} \cdot 20^{19}$

Obviously $20^{19} > 19!$ so the second number is larger.

**Problem Set 4 Level A**

Victoria found two treasure chests in a river of Ottawa. The first chest has a label that reads: "*One of us contains gold, while another contains silver*". The label on the second chess reads: "*I contain gold and the other one contains silver*." Unable to decide, she asks her friend Lilian, who quickly obtain some hints from an urban legend: "*each of the chests is filled either with gold coins or with silver coins. One of their label is correct and the other one not. If one chest is open, the person who opens it can keep all the coins, but the content of the second chess will vanish*." Obviously gold coins are more worthy than the silver ones. How does Victoria **decide**?

**Problem Set 4 Level B**

Catherine counts her money as follow: first she counts how many bills she has regardless of denomination, then she adds the number of bills with denomination greater than \$1, then the number of bills with denomination greater than \$2, then the number of bills with denomination greater than \$3, and so on. **W**ill she have the correct total?

**Problem Set 4 Level C**

Elisa and Betsy play a two-player game. Twenty points are placed around a circle. The girls take turns joining two of the points with a line segment which does not cross a segment already drawn in. The player who cannot make a move loses. What is **winning strategy**?

**Problem Set 4 Level A Solution**

The second label cannot be correct, because if it is, then the first label is also correct. Hence the second label is incorrect and the first one is correct. So she has to open the first chest, which contains gold coins.

**Problem Set 4 Level B Solution**

It's simple. Each \$$n$ bill is counted (1 is added to the sum) $n-$ times: first time with all the bills, second times if $n > 1$, third time if $n > 2$, … $n-$time if $n > (n-1)$.

**Problem Set 4 Level C Solution**

Elisa should go first. She must draw a chord which separates the points into two groups of 9. She then replies symmetrically to each move of Betsy.

### Problem Set 5 Level A

Linh Chi has a grasshopper. Each second it can hop 80 cm forward or 50 cm backward along a straight line. Is it possible that the grasshopper **stops at a distance exactly 170 cm** from the starting point in a fewer than 7 seconds?

### Problem Set 5 Level B

Erica has visited the Island of Knights and Liars. She was asked to conduct a survey.

All people on the Island of Knights and Liars have participated in the survey. Some of the islanders claimed that the number of the Knights on the island was even. The rest of the islanders said that the number of the Liars on the island was odd.

What can Elisa say about **the parity of total population** - the total number of Knights and Liars on the island?

### Problem Set 5 Level C

Consider the expression:

$$S = 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + 2020 \cdot 2020!$$

What would be the value of $S$?

### Problem Set 5 Level A Solution

There is only one solution for 6 seconds:

$$170 = 50 \cdot 5 - 80 \cdot 1$$

*Note: There is one solution for 7 seconds:* $170 = 80 \cdot 4 - 50 \cdot 3$, *however the problem asks for a solution with* ***fewer than 7 seconds****, so this is not acceptable.*

**Problem Set 5 Level B Solution**

Let Group 1 be the people who said the number of Knights is even and Group 2 be the people who said the number of Liars is odd. Note that each group can have only one type of people: all Knights or all Liars.

Case 1: Both groups have only Knights. This means Group 1's statement is correct, therefore the number of Knights is even. Group 2 is empty and there are no Liars. So the total population is an even number.

Case 2: Group 1 has Knights, Group 2 has Liars. This means Group 1's statement is correct, therefore the number of Knights is even. Group 2's statement is incorrect, therefore the number of Liars is also even. So the total population is an even number.

Case 3: Group 1 has Liars, Group 2 has Knights. This means Group 1's statement is incorrect, therefore the number of Knights is odd. Group 2's statement is correct, therefore the number of Liars is odd. So the total population is an even number.

Case 4: Group 1 has Liars, Group 2 has Liars. This means Group 1's statement is incorrect, therefore the number of Knights is odd. Since there is no Knights, so Group 1 has to be empty. Group 2's statement is incorrect, therefore the number of Liars is even. So the total population is the number of Liars and that is an even number

**Problem Set 5 Level C Solution**

You can prove by using the induction principle:

$$S_n = 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1 \quad (*)$$

Since $S_1 = 1! = 2! - 1$, the initial hypothesis (*) is correct.

Let assume the hypothesis (*) is correct for $n$. Since:

$$S_{n+1} = S_n + (n+1) \cdot (n+1)! = (n+1)! - 1 + (n+1) \cdot (n+1)!$$
$$= (n+1)! \cdot (1 + n + 1) - 1 = (n+2)! - 1$$

Hence the hypothesis is correct for $n + 1$. So it is correct for all $n$.

**Problem Set 6 Level A**

Nam has two jugs. One is a 5-litter jug, but for the other he doesn't remember if it 4- or 3-litter jug. It is impossible to determine by just looking into the jug. Of course it is possible to pour the water from one jug into the other and see if the water overflows.

Help Nam to **determine the capacity** of his second jug!

**Problem Set 6 Level B**

Clemence wrote six digits on a blackboard. Is it possible to **arrange them into a single row** such that the difference between the sum of the first three and the sum of the last three is less than 10?

**Problem Set 5 Level C**

If $x > 0$, $y > 0$ and $x = 3 + \dfrac{1}{3 + \frac{1}{x}}$ and $y = 3 + \dfrac{1}{3 + \frac{1}{3 + \frac{1}{y}}}$.

What is the **value** of $|x - y|$?

**Problem Set 6 Level A**

Let first fill the second jug, pour it into the first jug, then fill the second jug again and pour it into the first jug until it is full. Then empty the first jug, and pour in what's left from the second jug. We have two different cases of how much water is in the $1^{st}$ jug $3 + 3 - 5 = 1$, $4 + 4 - 5 = 3$. Now fill the second jug and start to pour into the first one. If it is a 4-litter jug, it will cause the water to overflow, otherwise not.

**Problem Set 6 Level B Solution**

Let the six number be in this order $a \geq b \geq c \geq d \geq e \geq f$.

Arrange them as $a, c, e, b, d, f$. The difference of the sums is:

$$D = (a + c + e) - (b + d + f) = (a - f) + (c - b) + (e - d)$$

Since $a - f \leq 9 - 0$, $c - b \leq 0$, $e - d \leq 0$ so $D \leq 9$.

**Problem Set 6 Level C Solution**

By simplifying the given equation, both $x, y$ are roots of the quadratic $f(t) = t^2 - 3t - 1$. This equation has two different roots $\dfrac{3 \pm \sqrt{13}}{2}$, only one is positive. Hence $x = y = \dfrac{3 + \sqrt{13}}{2} \Rightarrow |x - y| = 0$.

# Problems of the Week

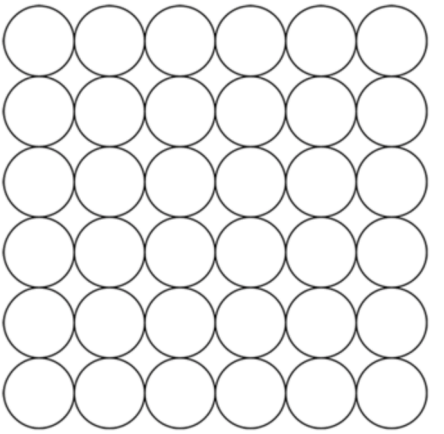
**Level A**

### Problem 1.A

Thirty-six canon balls are placed on a flat surface in the shape of a square to form the base on a display beside the canon. How many additional canon ball are needed to form a *pyramid* with the given base?

If you have a $n \times n$ square formed by $n^2$ cannon ball, can you generalize the above result?

**Solution**



The 36 cannon balls form a square as above. Easy to see that we can put another layer on top of them and that form by $5 \times 5 = 25$ cannon balls. Continuing this, the amount of additional balls needed is:

$$5^2 + 4^2 + 3^2 + 2^2 + 1^2 = \frac{5 \cdot (5+1) \cdot (2 \cdot 5 + 1)}{6} = 55$$

The above result can be generalized as below:

$$1^2 + 2^2 + \cdots + (n-1)^2 = \frac{(n-1)(n)(2n-1)}{6}$$

For $n = 6$, $\dfrac{(n-1)(n)(2n-1)}{6} = 55.$

**Problem 2.A**

If all alligators are ferocious creatures and some creepy crawlers are alligators, which statements below must be true?

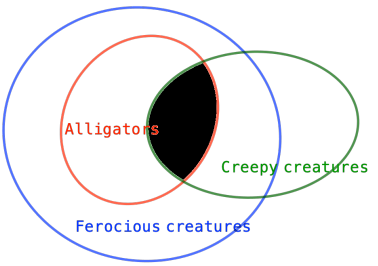
- All alligators are creepy crawlers
- Some ferocious creatures are creepy crawlers.
- Some alligators are not creepy creatures.

Which statements are true, which are false, which are undecidable (you can't say it is true or false)?

- Some of creepy creatures are ferocious.
- Some ferocious creatures are creepy crawlers.
- Some alligators are not creepy creatures.
- Some ferocious crawlers are not alligators.
- Alligators are ferocious and creepy crawlers.

**Solution**

Looking at the Venn diagram above, it is obvious that neither the statement i., nor statement iii. is always correct. On the other hand statement ii. is *always correct*. So the ii. statement must be true.




For the second question:

*Some of creepy creatures are ferocious*: since some creepy crawlers are alligators and because all alligators are ferocious creatures so those creepy crawlers are ferocious creatures. This statement is *true*.

*Some ferocious creatures are creepy crawlers*: because all alligators are ferocious creatures and some of them are creepy crawlers, therefore this is also *true*.

*Some alligators are not creepy creatures*: this is not necessarily true if all alligators are creepy. This is *undecidable*.

*Some ferocious crawlers are not alligators*: this is *undecidable*. We don't know if there are other ferocious creatures beside the alligators. *Note that it is important that we are reasoning with what we know.*

*Alligators are ferocious and creepy crawlers*: this is *undecidable*, only some of them are creepy.

**Problem 3.A**

Five lamps $A$, $B$, $C$, $D$, and $E$ are arranged in a row from left to right. Each lamp has its own switch. All five lamps are off. Erica starts to turn each switch from $A$ to $E$ and repeats the pattern (always from $A$ to $E$ and then back to $A$) until she turns the switches 126 times. Which lamps are on at the end?

If five lamps $A$, $B$, $C$, $D$, and $E$ are in the first row from left to right, and five lamps $I$, $J$, $K$, $L$, and $M$ are in the second row from left to right. Each lamp has its own switch. All ten lamps are off. Now Erica starts to turn each switch from $A$ to $E$, then $M$ to $I$ and repeats the pattern (always from $A$ to $E$, then $M$ to $I$ and then back to $A$) until she turns the switches 333 times. Which lamps are on at the end?

**Solution**

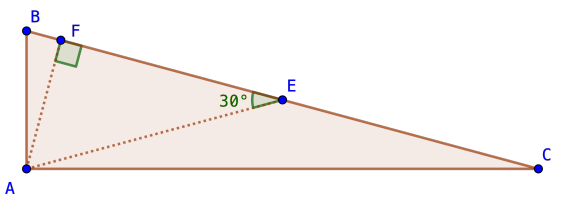
If a lamp is switched an even number of times, it does not change its state. Since $126 = 5 \times 25 + 1$, so all the lamps should be on, except the first one, which is $A$.

The second question is similar to the first one: $333 = 33 \times 10 + 3$, so all lamps are on except $A$, $B$, $C$.

**Problem 4.A**

In triangle $ABC$, $\angle A = 90°$, $\angle B = 75°$, and $BC = \sqrt{3}$. The area of triangle $ABC$ is $\frac{m}{n}$, where $m$, $n$ are prime relative positive integers. Find $m + n$.

**Solution**




Let $E$ be midpoint of $BC$. Because $\angle A = 90°$: $AE = \frac{BC}{2} = \frac{\sqrt{3}}{2}$.

$\triangle EAF$ is a $30 - 60 - 90$ triangle, so $AF = \frac{AE}{2} = \frac{\sqrt{3}}{4}$.

Hence: $[ABC] = \frac{AF \cdot BC}{2} = \frac{3}{8}$.

## Level B

### Problem 1.B

In right triangle $\triangle A_1B_1C_1$, $\angle B_1A_1C_1 = 90°$, $A_1B_1 = 3 \times 2^{50}$, and $A_1C_1 = 2^{52}$. $A_2$, $B_2$, $C_2$ are the midpoints of $B_1C_1$, $C_1A_1$, and $A_1B_1$ respectively. Similarly $A_3$, $B_3$, $C_3$ are the midpoints of $B_2C_2$, $C_2A_2$, and $A_2B_2$ respectively. This process goes for 100 times, so $A_{101}$, $B_{101}$, $C_{101}$ are the midpoints of $B_{100}C_{100}$, $C_{100}A_{100}$, and $A_{100}B_{100}$ respectively.

Now all triangles $B_2C_2A_1$, $B_3C_3A_2$, …, and $B_{101}C_{101}A_{100}$ are shaded. What is the total area of the shaded triangle $B_2C_2A_1$? What is the total area of all shaded triangles?

**Solution**

Easy to see: $[B_2C_2A_1] = \frac{1}{4}[A_1B_1C_1]$, so:

$$[B_2C_2A_1] = \frac{1}{4}[A_1B_1C_1] = \frac{1}{4}\frac{3 \cdot 2^{50} \cdot 2^{52}}{2} = 3 \cdot 2^{99}$$

Similarly: $[B_3C_3A_2] = \frac{1}{4}[A_2B_2C_2] = \frac{1}{4}3 \cdot 2^{99} = 3 \cdot 2^{97}$

So: $[B_{n+1}C_{n+1}A_n] = \left(\frac{1}{4}\right)^{n-1}[B_2C_2A_1] = 3 \cdot 2^{99-2(n-1)}$

$= 3 \cdot 2^{101-2n}$

Therefore the total area of all shaded triangles is:

$3 \cdot 2^{99} + 3 \cdot 2^{97} + \cdots + 3 \cdot 2^{101-2\cdot 100}$

$= 3 \cdot 2^{99} + 3 \cdot 2^{97} + \cdots + 3 \cdot 2^{-99}$

$= \frac{3}{2^{99}}\left(2^{198} + 2^{196} + \cdots + 1\right) = \frac{3}{2^{99}} \cdot (4^{99} + 4^{98} + \cdots + 1)$

$= \frac{3}{2^{99}}\frac{4^{100} - 1}{4 - 1} = \frac{4^{100} - 1}{2^{99}} = \frac{2^{200} - 1}{2^{99}} = 2^{101} - \frac{1}{2^{99}}$

**Problem 2.B**

It is known that "*all boys in the math club are good at math.*" Which of the following statements must be true?

*i.* *No boy whose math is not good is a member of the math club*

*ii.* *All boys whose math is good are members of the math club*

*iii.* *All boys who are not members of the math club are not good at math.*

*iv.* *Every member of the math club whose math is good is a boy.*

*v.* *There is one boy in the math club whose math is not good.*

Now decide that the following is true or false.

*"A girl is good at math as a boy who is not member of the math club is not good as math as a boy who is member of the math club"*

**Solution**

We know that *all boys in the math club are good at math*. We see that it says in the 2nd *all boys that are good at math are in the math club*. That obviously isn't true because not all boys that are good at math in the world would be in the club.

The 3rd remark says that *all boys who are not members of the math club are not good at math.* That's not true because we know that not all boys that are good at math are in the club which means there must be many boys out there who are good at math but not in the club. The 4th remark says that *every member of the math club whose math is good is a boy.* That's not true because there can also be girls in the club who are good at math. Finally for the 5th remark it says that *there is one boy in the math club whose math is not good*. This cannot be true because it is said that all boys in the club are good at math.

This leaves us with the 1st remark which says that *no boy whose math is not good is a member of the math club*. This is true because we know there is no boy in the club who is not good at math. Therefore the 1st remark is the *true* one.

For the second question, let split the statement into 2 parts, one containing the girl and the other containing the 2 boys. We see that it says *a boy who is not member of the math club is not good as math as a boy who is member of the math club.* That is obviously false because we know there are boys who are good at math outside the club, that boy could be one of the ones who are good at math. This means that the girl could also be good at math. Therefore that statement is *false*.

**Problem 3.B**

All positive integers greater than 1 are arranged in five columns $A$, $B$, $C$, $D$, and $E$ from left to right as shown below.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | 2 | 3 | 4 | 5 |
| 2 | 9 | 8 | 7 | 6 | |
| 3 | | 10 | 11 | 12 | 13 |
| 4 | 17 | 16 | 15 | 14 | |
| ... | ... | | | | |

What row and what column contains the number 800?

All positive integers from 1 are arranged in five columns $A$, $B$, $C$, $D$, and $E$ from left to right as shown below.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | 1 | 2 | | 3 | |
| 2 | 7 | 6 | | 5 | 4 |
| 3 | | 8 | 9 | 10 | 11 |
| 4 | 14 | | 13 | 12 | |
| 5 | 15 | 16 | 17 | | 18 |
| 6 | 22 | 21 | 20 | | 19 |
| 7 | 23 | | 24 | 25 | 26 |
| ... | ... | | | | |

What row and what column contains the number 2020?

**Solution**

Easy to see that for every 4 numbers from 2, there is an empty cell. So for $800 \div 4 = 200$ quadruples, counting from 2, all together they make 200 rows. The last row should contain 798, 799, 800, and 801. Note that every odd rows have the numbers increased from the left and every even rows have the numbers increased from the right. So the number 800 is on the row 200 and second column from the left.

The second question is trickier, after every prime number there is an empty cell. Since there are 306 prime number less than 2020, there are in total $2020 + 306 = 2326$ cells containing the numbers from 1 to 2020. The number 2020 is in the last cell $2326 = 465 \times 5 + 1$ which is on the row 466 and column 1.

**Problem 4.B**

In triangle $ABC$, $AB = 12$, $AC = 9$.

Point $D$ is on segment $BC$, so $BD : DC = 2 : 1$, and $AD = 6$.

Find the length of $BC$.

**Solution**

Let $CD = x$, $DB = 2x$.

Let calculate the area of triangle $ACD$. Its semi-perimeter is:

$$\frac{1}{2}(9 + 6 + x) = \frac{1}{2}(15 + x)$$

So it's area is:

$$\sqrt{\frac{15+x}{2}\frac{x+3}{2}\frac{x-3}{2}\frac{15-x}{2}}$$

Nou let calculate the area of triangle $ABD$. Its semi-perimeter is:

$\frac{1}{2}(6 + 12 + 2x) = 9 + x$.

So it's area is:

$$\sqrt{(9+x)(x-3)(x+3)(9-x)}$$

Since $\frac{[ABD]}{[ACD]} = \left(\frac{BD}{CD}\right) = 2$

$$2 = \frac{\sqrt{(9+x)(x-3)(x+3)(9-x)}}{\sqrt{\frac{15+x}{2}\frac{x+3}{2}\frac{x-3}{2}\frac{15-x}{2}}} = 4\frac{\sqrt{(9+x)(9-x)}}{\sqrt{(15+x)(15-x)}}$$

Therefore $15^2 - x^2 = 4(9^2 - x^2) \Rightarrow x^2 = 33 \Rightarrow x = \sqrt{33}$.

Hence, $BC = 3\sqrt{33}$.

## Level C

### Problem 1.C

There are 997 distinct points on the plane. Prove that there are at least 1991 pairs of points so that their midpoints are all distinct. *Hint: consider the largest distance pair $A$, $B$.*

Show that there is a way to construct 997 distinct points such that there are exactly 1991 distinct midpoints from all possible pairs of points.

**Solution**

Consider two points $A, B$ such that their distance is largest among all distances between pairs of given points. Let $M$ be midpoint of $A, B$. Consider $X, Y$, two distinct points from 997 given points, and both different from $A, B$.

The midpoints of $AX, AY$ are different. The midpoints of $BX, BY$ are different. The midpoints of $AX, AY, BX, BY$ cannot be $M$. Furthermore the midpoints of $AX, BY$ are different, because otherwise $ABXY$ is a parallelogram and $\max\{AX, BY\} > AB$ which contradicts the selection of $AB$.

Now consider $A, B$ and all $X_1, X_2, \cdots X_{995}$ points. All the midpoints of these $AX_i, BX_i, i = \overline{1, 997}$ and $M$ are different, so the number of distinct midpoints is at least $2 \times 995 + 1 = 1991$.

For the second question, let the points be (0,1), (0,3), $\cdots$, (0,1993). They line on the $x-$axis and have exactly 1991 different midpoints at (0,2), (0,3), $\cdots$, (0,1992).

**Problem 2.C**

Is it possible to tile a $10 \times 10$ board by $1 \times 4$ figures?

The standard $8 \times 8$ chessboard can be recoloured by selecting a row or a column, then change all squares in that row or column to the opposite colour. Is it possible to change the board to have a single black square?

**Solution**

Colour the board by four colours as below

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 |
| 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 |
| 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 |
| 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 |
| 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 |
| 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 |
| 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 |
| 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 |

If a $10 \times 10$ board can be tiled $1 \times 4$ figures, then the number of squares by each colours should be divisible by 4, but that is not the case with the the colour 2, which have 25 squares.

For the second question, if a row (or a column) has $k$ black squares and $8 - k$ white squares, then after the recolouring, it has $k$ white squares and $8 - k$ black squares. So the number of black squares changed is $(8 - k) - k = 8 - 2k$, which is an even number.

Since the number of black squares at the beginning is 32, it is not possible to have only 1 black square at the end.

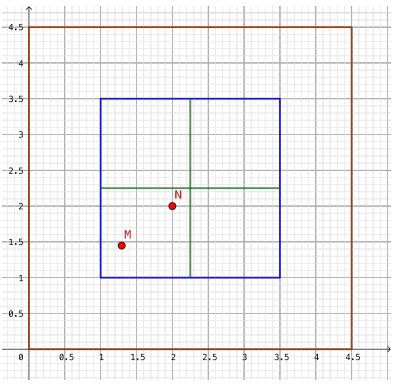
**Problem 3.C**

It is possible to cut five unit circle figures from a square with side length 4.5?

It is an unproven fact that $12^{th}$ planet exists and has the shape of a perfect sphere. It is covered by continents and oceans, and the total area of all oceans is more than the total area of all continents. Prove that there are two points on the oceans that are symmetric to each other with regard to the centre of the planet.

**Solution**

Assume that it is possible. Since the five circles can be cut from the brown square, their centres must be inside the blue square which sides are 1cm from the perimeter of the brown square shown at right.




The blue square is divided into four smaller squares with side length half of the original square. Since there are 5 centres, according to the pigeonhole principle, there are two of them in the same square.

WLOG let they represented by the red point $M$, $N$ and their distance cannot exceed the diagonal of the small square:

$$\frac{(4.5-2)}{2}\sqrt{2} = 1.25\sqrt{2} < 2$$

But the distance between two non-overlapping centres of circles with 1cm radius should not be less than 2. It is a contradiction.

For the second question, let $A$ be the set of all points on land and $B$ be the set of all points on water. Since $A \cap B$ is empty set and:

$$|A \cup B| = |A| + |B| - |A \cap B| = |A| + |B|$$

Let $C$ be the set of all points symmetric a point $B$ with regard to the centre of the planet, if $C \cap B$ is empty, then:

$$|C \cup B| = |C| + |B| - |C \cap B| = |C| + |B| > |A| + |B| = |A \cup B|$$

This is impossible since $C \cup B$ is still a subset of $A \cup B$.

So $C \cap B$ is not empty. This is what needs to be proven.

**Problem 4.C**

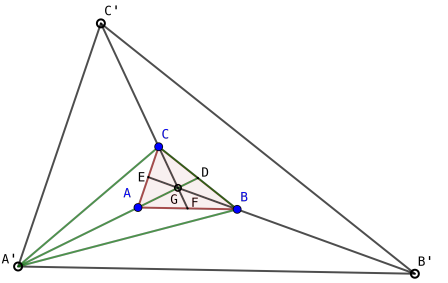
Every point on the plane is coloured by one of the two colours. Prove that there exists a non-trivial triangle such that all of its vertices have the same colour. Prove that there exists a non-trivial triangle such that all of its vertices and the centroid have the same colour.

**Solution 1**

First, selecting 5 points on the plane so no any three point are collinear, there are three points that are coloured in the same way. They are vertices of a non-trivial triangle.

For the second question let choose three points $A$, $B$, $C$ are blue-coloured. Let $G$ be the centroid of $\triangle ABC$ with $D$, $E$, $F$ are midpoints of $BC$, $CA$, $AB$. Let extend $DG$, $EG$, $FG$:

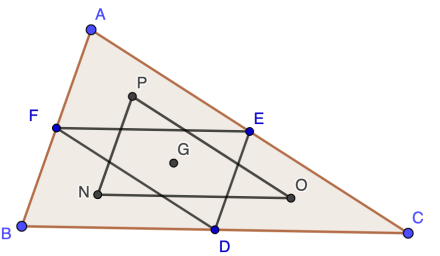
$AA' = 2AD$, $BB' = 2BE$, $CC' = 2CF$




Now $A$, $B$, $C$ are centroids of $\triangle A'BC$, $\triangle AB'C$, $\triangle ABC'$ (*). Now if any point $A'$, $B'$, $C'$, or $G$ is blue, then the corresponding triangle in (*) has all vertices and the centroid coloured by blue. Otherwise all vertices of the triangle $\triangle A'B'C'$ and its centroid $G$ are coloured by red.

**Solution 2**

Assume there does not exist such a triangle. Let's take a random triangle with all three of its vertices as the same colour. Since the median triangle and the outer triangle have the same centroid, the midpoints of the triangle must have the same colour as the vertices.




The other three triangles (shown below) also have centroids, and since the centroids can't be the same colour as the vertices, they must have the same colour as the big triangle's centroid. The centroid of these centroids though, is the same as the big triangle's centroid, for each of its vertices lie on a median of the bigger triangle, thus making its medians the same as the bigger triangle. This is a contradiction.

# Chapter 12

# Week 33

# Weekly Team Contest

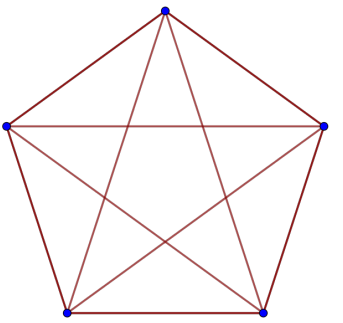
**Problem Set 1 Level A**

October 25 was Thursday. **What day of the week** was August 1?

**Bonus**: If February 5 of the leap year $n$ was a Sunday, **what day of the week** was February 5 of the year $n + 6$?

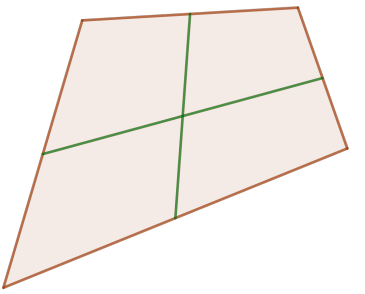
**Problem Set 1 Level B**

**How many triangles** are there in the below diagram?



**Problem Set 1 Level C**

Erica cuts a convex quadrilateral into 4 parts along the two green medians (connecting midpoints of opposite sides) of the quadrilateral as show below.



Is it possible to reassemble the parts into a parallelogram?

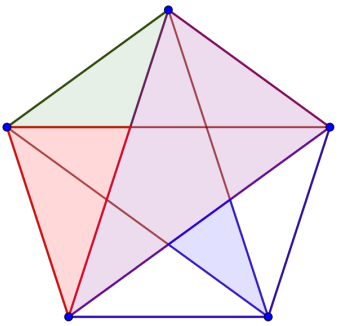
**Problem Set 1 Level A Solution**

After August 1, there are 30 more days in August, 30 additional days in Septembers, and 25 more days to October 5. It's a total of 85 days. Since $85 = 7 \cdot 12 + 1$, so if October 5 was a Thursday, then August 1 is a Wednesday. For the bonus question: the number of days between them is: $6 \cdot 365 + 2 = 2192 = 313 \cdot 7 + 1$.

Therefore February 5, the year $n + 6$ is a Monday.

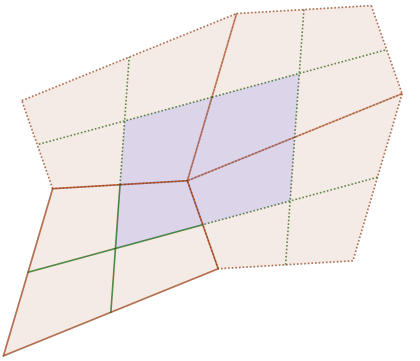
**Problem Set 1 Level B Solution**

Let count the different size triangles highlighted as below.



There are 5 green triangles, 5 blue triangles, 10 red triangles , 10 blue+ red triangles, 5 purple triangles. So in total 35 triangles.

**Problem Set 1 Level C Solution**



By reflecting the quadrilateral a few times, the above diagram shows that a parallelogram can be formed as shown above. It is possible because the sum of all internal angles is a full angle.

**Problem Set 2 Level A**

Minh drives a car that has a faulty odometer. It registers 4.6 kilometres for every 5 kilometres actually driven. When traveling to visit his friend, the faulty odometer registers 92 kilometres. **How far** did Minh drive?

**Problem Set 2 Level B**

A train is 250 meter long and travels at constant speed of 8 kilometres per hour. David on a bicycle, traveling at constant speed of 11 kilometres per hour, rides on a road next to the rail track. From the moment he passes the rear of the train, it takes him $n$ minutes to reach the front. **Find** $n$.

**Bonus**: If David on a bicycle, traveling at constant speed of 12 kilometres per hour, rides on a road next to the rail track, but in the ***opposite*** direction, then **how much time it takes him** from the moment he passes the front to the moment he read the rear of the train?

**Problem Set 2 Level C**

Clemence bought a toy for her sister. It is placed inside a box with dimension $1 \times 1 \times 1$. She has a $3 \times 3$ wrapping paper.

Can she **use the paper to wrap** the toy box?

**Problem Set 2 Level A Solution**

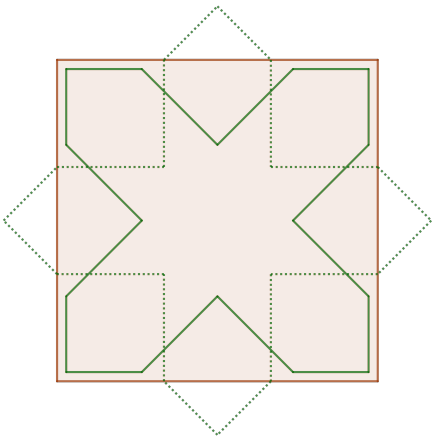
It's simple $\frac{92}{4.6} = 20$. In fact Minh has driven $20 \cdot 5 = 100$ km.

**Problem Set 2 Level B Solution**

Let assume that the train does not move. It means that David travels at the speed of $11 - 8 = 3$ kilometres per hours or 1 kilometre in 20 minutes or 250 meter in 5 minutes.

For the bonus question, the same technique can be applied. If the train does not move then David travels at the speed of $8 + 12 = 20$ kilometres per hours or 1 kilometre in 3 minutes or 250 meter in 45 seconds.

**Problem Set 2 Level C Solution**



It is possible. One example is a shown as above.

**Problem Set 3 Level A**

The average of five positive integers is 30.

What is the **greatest possible value** of their median?

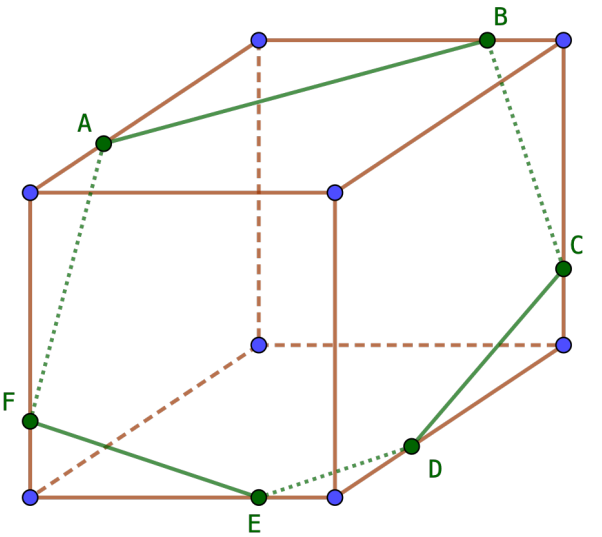
**Problem Set 3 Level B**

A $10 \times 10 \times 10$ cube is painted red on all faces and then cut into ten $10 \times 10 \times 1$ slices. Each slice is then cut into twenty-five $2 \times 2 \times 1$ blocks. **How many** of the 250 blocks have **exactly one face painted** red?

**Bonus**: **How many** of the 250 blocks have either no faces painted or 3 faces painted?

**Problem Set 3 Level C**

Lilian cuts a *unit* cube. She got a with a hexagon cross-section as shown below.




What is the **minimal value** of the hexagon **perimeter**? In another word, what is the minimal value of the sum:

$$AB + BC + CD + DE + EF + FA?$$

**Problem Set 3 Level A Solution**

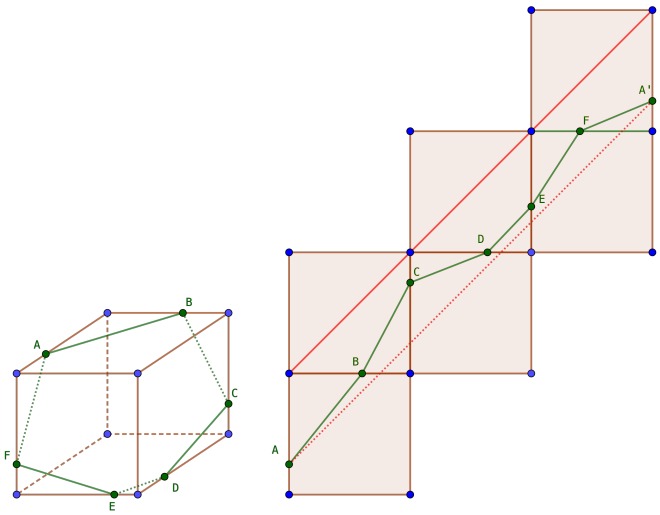
The minimal values for the third two number is 1, 1. So in order to maximize the median, the remaining three numbers have to be as nearly as equal as possible. Since $\dfrac{5 \cdot 30 - 2}{3} = 49.\overline{3}$, so they are 49, 49, 50, and the median is 49.

**Problem Set 3 Level B Solution**

Suppose the slices are made horizontally. The top and bottom $10 \times 10 \times 1$ slices each have 9 middle blocks with size $2 \times 2 \times 1$ that have only one face painted red. So there are $2 \times 9$ blocks. On each of the eight remaining slices on each side there are three $2 \times 2 \times 1$ blocks with only one face painted red. So there are $8 \times 3 \times 4 = 96$ such blocks. Hence the total number of such blocks is $18 + 96 = 114$.

For the bonus question: we just need to count the 9 middle blocks with size $2 \times 2 \times 1$ of the eight layers which is $9 \times 8 = 72$. There are 8 corner blocks that have three faces painted. Hence the number of them is 80.

**Problem Set 3 Level C Solution**




The right diagram is obtained by unfolding the cube into a net. The hexagon perimeter forms a broken line $ABCDEFA'$. This is always larger or equal the distance $AA'$, which is same as three times the diagonal of the unit square. Hence the perimeter is always at least $3\sqrt{2}$.

**Problem Set 4 Level A**

Mickey Mouse, Minnie Mouse, Goofy, and Pluto are sitting on a bench. If Goofy, who is sitting in the rightmost spot, gets up and then sits down *somewhere between* Mickey Mouse and Minnie Mouse , then Minnie Mouse will be the leftmost one.

In **which order** are they sitting?

**Bonus**: Is there any **other possible order**?

**Problem Set 4 Level B**

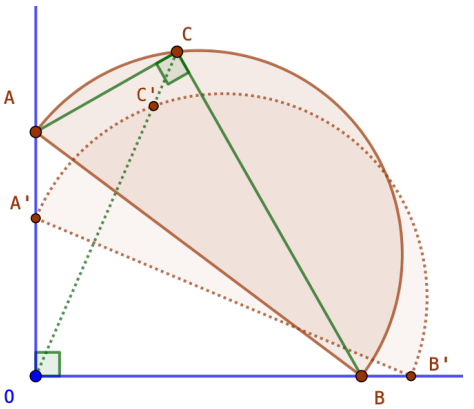
Sharks, Demons, Rhinos, and Unicorns are contesting in MCC soccer championship. Each team played against every other team once, with 3 points given for a victory, 1 point for a draw, and 0 for a loss. Sharks got the most points and Unicorns got the least.

Is it possible that Sharks got two more points than Unicorns?

**Bonus**: Suppose that Sharks still got the most points and Unicorns got the least. **What is the possible largest number of points** that Sharks can have than Unicorns?

**Problem Set 4 Level C**

A brown semi-circle with diameter $AB$ glides along the two legs of a blue right angle $O$. C is a fix point on the perimeter of the semi-circle. $A'B'C'$ shows a situation when the diameter of the semi-circle glides from $AB$ to $A'B'$ as shown below.




What is the **path** that the point C **draws** when the brown semi-circle glides?

**Problem Set 4 Level A Solution**

Easy to see that originally the order is:

Minnie Mouse, Mickey Mouse, Pluto, Goofy

And it could be changed to:

Minnie Mouse, Goofy, Mickey Mouse, Pluto

For the bonus question, the following order can be considered:

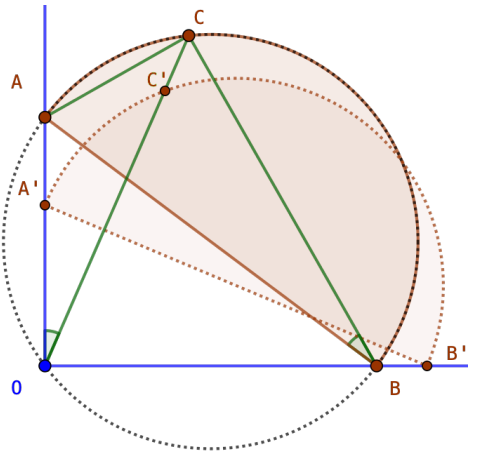
Minnie Mouse, Pluto, Mickey Mouse, Goofy

**Problem Set 4 Level B Solution**

| Team | Sharks | Demons | Rhinos | Unicorns | Total |
|---|---|---|---|---|---|
| **Sharks** | - | 1 | 1 | 3 | 5 |
| **Demons** | 1 | - | 3 | 0 | 4 |
| **Rhinos** | 1 | 0 | - | 3 | 4 |
| **Unicorns** | 0 | 3 | 0 | - | 3 |

For the bonus questions: the largest difference is 9, when Sharks won all three games, Unicorns lost all three.

**Problem Set 4 Level C Solution**




Since $\angle ACB = \angle AOB = 90^\circ$ so $O$ is on the circle diameter $AB$. Hence $\angle ABC = \angle AOC$.

This means that $C$ moves on the ray from $O$, which forms an angle with $OA$ equal to $\angle ABC$.

**Problem Set 5 Level A**

A notepad cost a whole number of cents. Nine notepads cost less than \$10, and ten notepads cost more than \$11.

How much does **one notepad cost**?

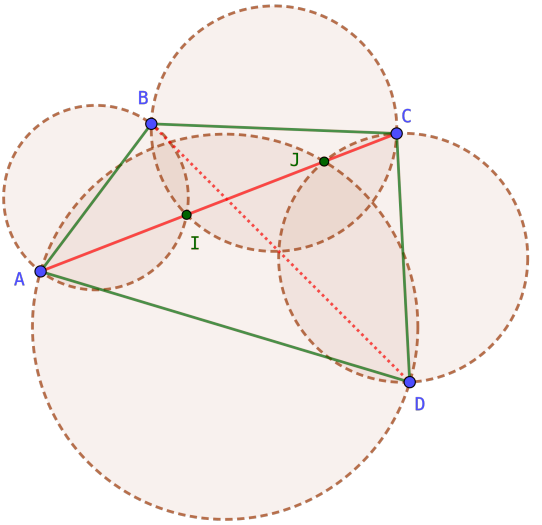
**Problem Set 5 Level B**

A marble has been placed in the square of the far right end of a $1 \times 20$ grid. Bianca and her friend take turns moving the marble right or left a number of squares as long as that number has not been used in any prior turn. The winner is the last to make a move.

What is the **winning strategy** for Bianca?

**Problem Set 5 Level C**

$ABCD$ is a convex quadrilateral. Catherine draws draw four circles with $AB$, $BC$, $CD$, $DA$ as diameters.




It seems that the four circles cover the whole quadrilateral.

Is that **possible**?

**Problem Set 5 Level A Solution**

Let $n$ be the number of cents that a notepad costs. We have:

$$9n < 1000 \Rightarrow n < \frac{1000}{9} \Rightarrow n \leq 111$$

$$10n > 1100 \Rightarrow n > 110 \Rightarrow n \geq 111$$

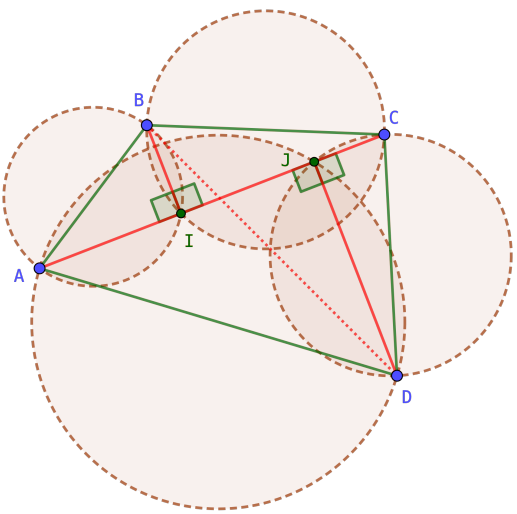
Hence $n = 111$.

**Problem Set 5 Level B Solution**

Bianca goes first and move the marble 19 squares to the left, to the opposite end of the grid. Afterwards, she moves the marble to the end of the board in the direction where her friend just move it. Let prove that she will always be able to move.

Suppose that her friend moved the marble through $x$ squares and the move of this length has not been done prior to that. Then Bianca would need to move the marble through $19 - x$. Since $x$ and $19 - x$ have different parity so they are different. Furthermore $x \neq 19$ and no move with such length has happened before, otherwise a move with $19 - (19 - x) = x$ should have happened before.

**Problem Set 5 Level C Solution**




$BI$, $DJ$ are perpendicular from $B$, $D$ to $AC$. Since $AB$, $BC$, $CD$ and $DA$ are diameters, so $\triangle ABI$, $\triangle BCI$, $\triangle ADJ$, $\triangle CDJ$ are right triangles and covered by their corresponding circles.

**Problem Set 6 Level A**

The number of gift cards that fifteen MCC students individually collected is 100. Is it true that there are **two students who have the same number** of gift cards?

**Bonus**: Is it true if fourteen MCC students have 100 gift cards?

**Problem Set 6 Level B**

On a stormy night, ten MCC students came to a party. They left their shoes in the foyer in order to keep the carpet clean. After the dinner, there was a power outage. So the students, leaving one by one, put on, at random, any pair of shoes big enough for their feet (each pair of shoes stay together). Any student who couldn't find a pair of shoes big enough spent the night at the place of the party.

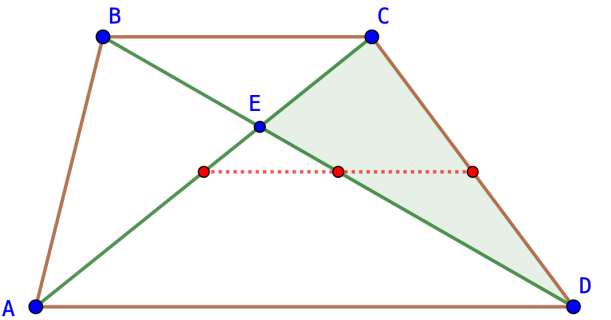
Can you guess **the largest number of students** who might have had to spend the night? Show an example for this number.

*Note: no two students wear the same size shoes.*

**Bonus**: Prove your guess.

**Problem Set 6 Level C**

$ABCD$ is a trapezoid with area 12. $BC \parallel AD$.




Prove that the area of $ECD$ is not more than 3.

**Problem Set 6 Level A Solution**

If 15 students have different number of gift cards, then the number of gift cards is at least $0 + 1 + 2 + \cdots + 14 = 105 > 100$. So at least two of them must have the same number of gift cards.

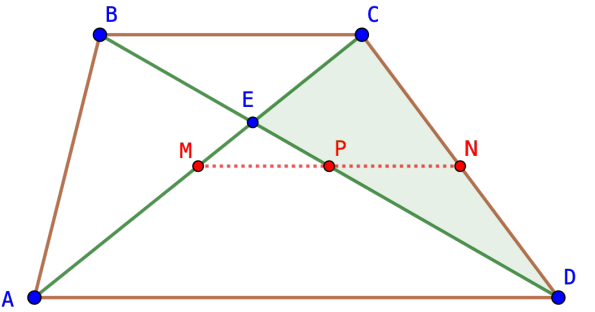
For the bonus question: (0, 1, 2, 3, 4, 6, 8, 9, 10, 11, 12, 13, 14) is a possible case.

**Problem Set 6 Level B Solution**

Easy to see that if the five students with smallest feet left first wearing all five largest pairs of shoes then none of the remaining student can find a pair of shoes to leave.

If there were 6 students spent the night and there were 6 pairs of shoes left, then because $6 + 6 > 10$, so there is a student with his pair of shoes is still at the place of the party. This is not possible!

**Problem Set 6 Level C Solution**




Let $M$, $N$, $P$ be the midpoint of $AC$, $CD$, $BD$.

$$S_{ECD} \leq S_{MCDP} = S_{MCN} + S_{PND} = \frac{S_{ACD}}{4} + \frac{S_{BCD}}{4}$$

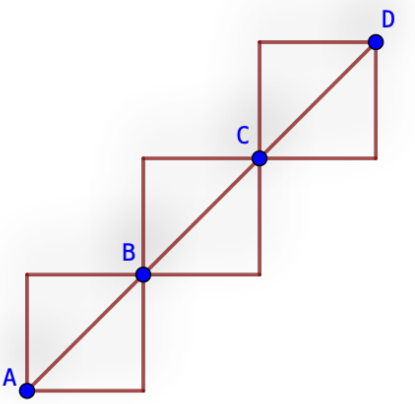
$$= \frac{S_{ACD}}{4} + \frac{S_{ABC}}{4} = \frac{S_{ABCD}}{4} = 3$$

# Problems of the Week

**Level A**

**Problem 1.A**

Anthony is walking from $A$ to $D$ in the figure below.




If any point of intersection and any line cannot be walked twice in one trip. How many ways can he complete the trip?

If he can pass any point of intersection as many times as he want, but he cannot walk any line twice, then how many ways can he can?

**Solution**

If Anthony cannot walked twice on any intersection, then he has to travel from A to B to C and to D, in this order. In another word, his path is ABCD. There are three ways to walk from A to B, B to C, or C to D. So the number of different ways is $3^3 = 27$.

For the second question, let do some casework by the path:

<u>Case 1 - Straight from A to D or ABCD</u>: from the first question, there are 27 ways.

<u>Case 2 - ABABCD (similarly for ABCBCD or ABCDCD):</u> Anthony travels from A to B, then goes back to A, and then travels again to B: he can has $\binom{3}{1}$ ways to choose the segment from A to B, then $\binom{2}{1}$ ways to choose a segment to go back to A and use the remaining one in order to reach B again. So all together he has $3 \cdot 2 = 6$ ways to reach B at the second time. Now he can go from B to C to D in $3^2 = 9$ ways. So the total number of ways in this case is $6 \cdot 9 = 54$ ways.

<u>Case 3 - ABABCBCD (similarly for ABCBCDCD or ABABCDCD):</u> Anthony travels from A to B, then goes back to A, and then travels again to B: 6 ways. Then he travels from B to C, then goes back to B, and then travels again to C: 6 way. Now he can go from C to D in 3 ways. The total number of ways in this case is $6 \cdot 6 \cdot 3 = 108$ ways.

<u>Case 4 - ABABCBCDCD:</u> there are $6^3 = 216$ ways.

<u>Case 5 - ABCBABCD (similarly for ABCDCBCD):</u> Anthony travels from A to B to C then goes back to B to A, and then travels again to B to C: he can has 3 ways to choose the segment from A to B, then 3 to C, 2 ways to B, 2 to A, and then remaining one to reach B and then C again. So all together he has $3 \cdot 3 \cdot 2 \cdot 2 = 36$ ways to reach B at the second time. Now he can go from C to D in 3 ways. So the total number of ways in this case is $36 \cdot 3 = 108$ ways.

<u>Case 6 - ABCDCBABCD:</u> Anthony travels from A to B to C to D then goes back to C to B to A, and then travels again to B to C to D: he can has 3 ways to choose the segment from A to B, 3 to C, 3 to D, then 2 ways to C, 2 ways to B, 2 ways to A, and then remaining one to reach B, C, then D again. So the total number of ways in this case is $3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 216$ ways.

The total number of ways is :

$$27 + 3 \cdot 54 + 3 \cdot 108 + 216 + 2 \cdot 108 + 216 = 1161$$

**Problem 2.A**

Find the remainder of $300^6 - 1$ when divided by 7.

Find the remainder of $300^{3000} - 1$ when divided by 1001.

*Hint:* $1001 = 7 \times 11 \times 13$.

**Solution**

$300 \equiv 6 \pmod 7$, $300^2 \equiv 6^2 \equiv 1 \pmod 7$, $300^6 \equiv 1 \pmod 7$.

Sso $300^6 - 1$ leaves remainder 0 when divided by 7.

For the second question $300^{3000} = (300^6)^{500} \equiv 1 \pmod 7$.

So $300^{3000} - 1$ is divisible 7.

$300 \equiv 3 \pmod{11}$, $300^5 \equiv 3^5 = 243 \equiv 1 \pmod{11}$.

So $300^5 - 1$ is divisible by 11 and so $300^{3000} - 1$ is divisible 11.

$300 \equiv 1 \pmod{13}$, so so $300^5 - 1$ is divisible by 13,

$300^{3000} - 1$ is divisible 13.

Hence $300^{3000} - 1$ is divisible by $1001 = 7 \times 11 \times 13$.

**Problem 3.A**

Find $x^2 + y^2 + z^2$ if $xy = 2$, $yz = 3$, and $zx = 4$?

What is the value of $x^2 + y^2 + z^2$ if $xy = 2$, $yz = 3$, $zx = -4$?

**Solution**

$xy \cdot yz \cdot zx = (xyz)^2 \Rightarrow (xyz)^2 = 2 \cdot 3 \cdot 4 = 24$

Because $x^2 = \dfrac{(xyz)^2}{(yz)^2}$, $y^2 = \dfrac{(xyz)^2}{(zx)^2}$, $z^2 = \dfrac{(xyz)^2}{(xy)^2}$, so

$$x^2 + y^2 + z^2 = \frac{24}{4} + \frac{24}{9} + \frac{24}{16} = 24\left(\frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right) = \frac{61}{6}$$

For the second question

$xy \cdot yz \cdot zx = (xyz)^2 \Rightarrow (xyz)^2 = 2 \cdot 3 \cdot (-4) = -24$

Hence $x^2 + y^2 + z^2 = \dfrac{-61}{6}$.

The result is exactly the same by using the above solution.

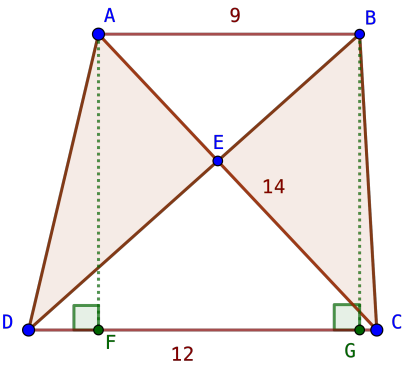
**Problem 4.A**

The convex quadrilateral $ABCD$ has $AB = 9$, $CD = 12$. The diagonals $AC$, $BD$ intersect at $E$. Furthermore $AC = 14$ and the two triangles $AED$ and $BEC$ have the same area.

Prove that $AB \parallel CD$.

What is the length of $AE$?

**Solution**




Let $AF$, $BG$ be the altitudes from $A$, $B$ to $DC$, respectively. The area of $\triangle AED$ and $\triangle BEC$ are the same, so the area of $\triangle ACD$ and $\triangle BDC$ are also the same, hence $AF = BG$. This mean $A$ and $B$ are at the same distance from line through $DC$, or $AB \parallel CD$.

Now $AB \parallel CD$, $\triangle AEB \sim \triangle CED$, hence $\dfrac{AE}{AB} = \dfrac{CE}{DC}$, so:

$$\frac{AE}{9} = \frac{CE}{12} = \frac{AE + EC}{9 + 12} = \frac{14}{21} = \frac{2}{3} \Rightarrow AE = 6$$

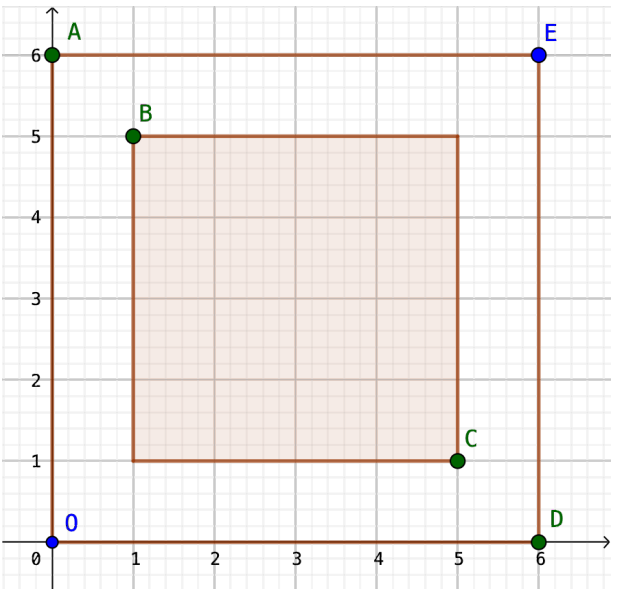
## Problem 1.B

A 12-step path is to go on the grid from (0,0) to (6,6) with each step increasing either the $x-$coordinate or $y-$coordinate by 1.

How many such paths stay outside or on the boundary of the square containing $(x, y)$ grid points such that $1 \leq x \leq 5$, $1 \leq y \leq 5$, at each step?

**Solution**

Let A, B, C, D, E, and O be the grid points as shown below.




Note that a 12-length path must go from O to E through exactly one of the points A, B, C, D.

The number of O-A-E paths: 1

The number of O-B-E paths: $\binom{6}{1}$ ways from O to B, and B to E.

The number of O-C-E paths: $\binom{6}{1}$ ways from O to B, and B to E.

The number of O-A-E paths: 1

So the total number of paths is: $1 + 6 \cdot 6 + 6 \cdot 6 + +1 = 74$.

**Problem 2.B**

What day of the week will be in $9^{2000}$ days from Sunday?

Find the remainder of $7^{120}$ when divided by 143.

**Solution**

A week is 7 days. $9 \equiv 2 \pmod 7$, $9^3 \equiv 2^3 = 8 \equiv 1 \pmod 7$

So $9^{1998} = (9^3)^{666} \equiv 1 \pmod 7$

Therefore $9^{2000} \equiv 9^2 \cdot 9^{1998} \equiv 4 \pmod 7$.

Hence 4 days from Sunday, the day of the week is Thursday.

For the second question since $143 = 11 \cdot 13$, let find the remainder of $7^{120}$ when divided by 143.

$7^3 \equiv 49 \cdot 7 \equiv 5 \cdot 7 = 35 \equiv 2 \pmod{11}$

$7^{10} = 7 \cdot 7^9 = 7 \cdot (7^3)^3 \equiv 7 \cdot (2)^3 = 7 \cdot 8 = 56 \equiv 1 \pmod{11}$

So $7^{120} = (7^{10})^{12} \equiv 1 \pmod{11}$

$7^{12} = (7^2)^7 \equiv 10^6 \cdot 1000^2 \equiv 12^2 = 144 \equiv 1 \pmod{13}$

So $7^{120} = (7^{12})^{10} \equiv 1 \pmod{13}$

From here $7^{120}$ has remainder 1 when divided by $143 = 11 \cdot 13$.

**Problem 3.B**

For $a$, $b$ real positive numbers: $a^2 + b^2 = 20$, $a^4 + b^4 = 328$.

What is the value of $ab$?

What is the value of $a^3 - b^3$?

**Solution**

$a^2 + b^2 = 20$, $a^4 + b^4 = 328 \Rightarrow 2a^2b^2 = (a^2 + b^2)^2 - (a^4 + b^4)$

$\Rightarrow a^2b^2 = 36$.

Because $a, b > 0$, so $ab > 0$, therefore $ab = \sqrt{36} = 6$.

$a^3 - b^3 = (a - b)(a^2 + ab + b^2) = 26(a - b)$

$(a - b)^2 = a^2 + b^2 - 2ab = 20 - 12 = 8 \Rightarrow a - b = \pm 2\sqrt{2}$

Hence $a^3 - b^3 = \pm 52\sqrt{2}$.

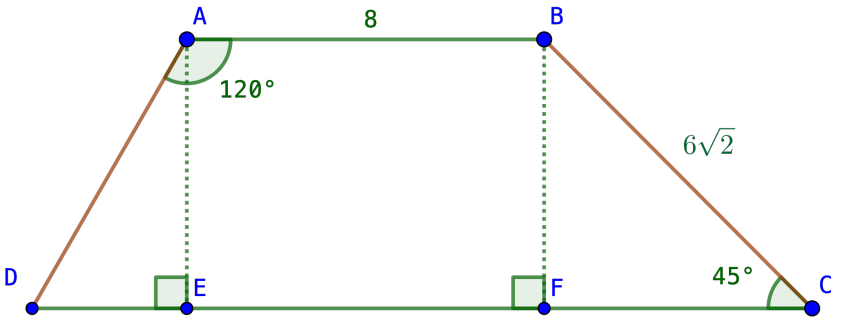
**Problem 4.B**

$ABCD$ is a convex quadrilateral. $AB \parallel CD$. $AB = 8$, $BC = 6\sqrt{2}$, $\angle BCD = 45^\circ$, and $\angle BAD = 120^\circ$.

What is the area of $ABCD$?

What is the value of $AC - BD$?

**Solution**




$E$, $F$ are the foots of the attitudes from $A$, $B$ to $CD$ respectively.

$$\angle BCF = 45^\circ \Rightarrow BF = \frac{BC}{\sqrt{2}} = 6$$

$$\angle DAE = 30^\circ \Rightarrow DE = \frac{AE}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

Therefore $DC = 2\sqrt{3} + 8 + 6 = 14 + 2\sqrt{3}$

$$[ABCD] = \frac{AB + DC}{2} \cdot BF = (11 + \sqrt{3})6 = 66 + 6\sqrt{3}$$

$$AC = \sqrt{AE^2 + EC^2} = \sqrt{6^2 + 14^2} = 2\sqrt{58}$$

$$BD = \sqrt{BE^2 + DE^2} = \sqrt{6^2 + (8 + 2\sqrt{3})^2} = 4\sqrt{7 + 2\sqrt{3}}$$

So $AC - BD = 2\sqrt{58} - 4\sqrt{7 + 2\sqrt{3}}$

## Level C

### Problem 1.C

Prove without algebra that:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

For $m \leq n$ positive integers, simplify:

$$\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k}$$

**Solution**

Imagine that you have $2n$ distinct pieces of candy, and you wish to pick $n$ of them. The easiest way to count the number of ways you could do this is $\binom{2n}{n}$. Alternatively, you could split the $2n$ pieces of candy into two groups of $n$ candy each. If you picked a $k$ amount of candy from one pile, you'll have to pick $n-k$ pieces of candy from the second pile. Combined together, this means there are $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$ ways to do this. Repeating this with all possible amount of candy you can pick from the first pile (0 to $n$) means that there is $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$ ways to pick n pieces of candy from a pile of $2n$ pieces of candy.

Therefore, $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

For the second question imagine you have a large pile of $n$ candy. You want to take a certain amount of candy from this pile (say, all of the chocolate ones), and then choose $m$ pieces of candy from the second pile (for example, the chocolate ones in the shape of a bunny). However, you know for sure that there are m chocolate bunnies, but you have no idea how many chocolate candies there are. The goal is to find the number of possible ways to choose all of the chocolate candies and the chocolate bunnies.

Because chocolate bunnies are definitely made of chocolate and that there can't be more than $n$ pieces of chocolate because we only have n pieces of candy, we conclude that the pile of chocolates has $k$ pieces of

candy such that $m \leq k \leq n$. Therefore, there are $\binom{n}{k}$ ways to choose the chocolates, and $\binom{k}{m}$ ways to choose the chocolate bunnies. As a result, there are $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k}$ ways to choose the chocolate and the chocolate bunnies.

As neat as this method is, there is a different way to solve this problem. First, we determine from the get-go that there are $\binom{n}{m}$ ways to choose all of the chocolate bunnies. From here, we simply have to count the number of possible amounts of non-bunny chocolates. They can number from 0 (where all of the chocolate are bunnie) to $n - m$ (where all of the candy is chocolate), making the total number of ways to choose non-bunny chocolates $\sum_{k=0}^{n-m} \binom{n}{k} = 2^{n-m}$.

Therefore, $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k} = 2^{n-m}\binom{n}{m}$.

*Note on algebraic manipulation (for the second question)*

For the second question, note that $(k - m) + (n - k) = n - m$, so: $\binom{k}{m}\binom{n}{k} = \frac{k!}{m!(k-m)!} \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{m!(k-m)!(n-k)!}$

$$= \frac{n!}{m!(n-m)!}\frac{(n-m)!}{(k-m)!(n-k)!} = \binom{n}{m}\binom{n-m}{k-m}$$

Therefore $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k} = \binom{n}{m} \cdot \sum_{k=m}^{n} \binom{n-m}{k-m}$, since:

$$\sum_{k=m}^{n} \binom{n-m}{k-m} = \binom{n-m}{0} + \binom{n-m}{1} + \ldots \binom{n-m}{n-m} = 2^{n-m}$$

Hence $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k} = 2^{n-m}\binom{n}{m}$

**Problem 2.C**

Prove that for any $x, y, z \geq 0$: $(x+y)(y+z)(z+x) \geq 8xyz$

$a, b, c$ are sides of a triangle. Prove that:

$abc \geq (a+b-c)(b+c-a)(c+a-b)$, and

$(a+b)(b+c)(c+a) \geq 8(a+b-c)(b+c-a)(c+a-b)$

*Hint: there are inequalities related to the triangle sides.*

**Solution**

Note that for $x, y, \geq 0$, by the AM-GM inequality:

$$\frac{x+y}{2} \geq \sqrt{xy}$$

so the first inequality can simply be obtained by the application of three similar AM-GM inequalities.

Let $x = \dfrac{a+b-c}{2}$, $y = \dfrac{b+c-a}{2}$, $z = \dfrac{c+a-b}{2}$.

Since $a, b, c$ are sides of a triangle so $x, y, z \geq 0$.

Because $(x+y)(y+z)(z+x) = abc$, so by applying the inequality proven in the previous question:

$$abc \geq (a+b-c)(b+c-a)(c+a-b)$$

The last equation is simple, since $a, b, c$ are sides of a triangle, so $a, b, c > 0$ and by the first and second proven inequalities:

$$(a+b)(b+c)(c+a) \geq 8abc$$

$$abc \geq (a+b-c)(b+c-a)(c+a-b)$$

Hence:

$$(a+b)(b+c)(c+a) \geq 8(a+b-c)(b+c-a)(c+a-b)$$

**Problem 3.C**

$a \le b$ are positive integers such that:

$$\frac{a^2+b}{b^2-a} \text{ and } \frac{b^2+a}{a^2-b}$$

are both integers.

Prove that $(a+b)(a-b+1) \ge 0$.

Find all such positive integers $a$, $b$.

**Solution**

Since $a \le b$ are positive integers so $b^2 \ge b \ge a$.

Therefore $\dfrac{a^2+b}{b^2-a}$ is a positive integer so $\dfrac{a^2+b}{b^2-a} \ge 1$, and:

$$a^2+b \ge b^2-a \Rightarrow a^2+a-b^2+b \ge 0$$

$$(a-b)(a+b)+(a+b) \ge 0 \Rightarrow (a+b)(a-b+1) \ge 0$$

From this inequality $a \ge b-1$, thus $b \ge a \ge b-1$.

<u>Case $a = b$</u>:

$$\frac{a^2+b}{b^2-a} = \frac{b^2+a}{a^2-b} = \frac{a+1}{a-1}.$$

Because $\dfrac{a+1}{a-1} \le 3 \Leftrightarrow a \ge 2$ so $a = 2$. In this case $a = b = 2$.

<u>Case $a = b-1$</u>:

$$\frac{a^2+b}{b^2-a} = \frac{a^2+a+1}{a^2+a+1} = 1,$$

$$\frac{b^2+a}{a^2-b} = \frac{a^2+3a+1}{a^2-a-1} = 1 + \frac{4a+2}{a^2-a-1}$$

1/ If $a^2-a-1 < 0 \Rightarrow a = 1, b = 2$

2/ If $a^2-a-1 \ge 1 \Rightarrow a \ge 2$.

If $a = 2, b = 3$.

If $a \ge 3$. Because $\dfrac{4a+2}{a^2-a-1} < 4 \Leftrightarrow 4a^2-4a-4 > 4a+2$

$\Leftrightarrow 4a^2-8a+4 > 10 \Leftrightarrow 4(a-1)^2 > 6$.

This always true if $a \ge 3$.

Hence there are three solutions (1, 2), (2, 2), and (2, 3)

**Problem 4.C**

Let $ABC$ be a triangle and $D$ be a point on side $BC$. A line through $D$ intersect $AB$ at $X$ and ray $AC$ at $Y$. The circumcircle of triangle $BXD$ intersects the circumcircle $\omega$ of triangle $ABC$ again at a point $Z \neq B$. The line $ZD$ and $ZY$ intersect $\omega$ again at $V$ and $W$ respectively. $XY$ intersects $\omega$ at points $P$ and $Q$, where $Q$ lines between $X$ and $Y$.

Prove that $V$ and $W$ are the reflections of $A$ and $B$ with respect to the perpendicular bisector of $PQ$.

Prove that $AVWB$ is an isosceles trapezoid and then$AB = VW$.

*Hint: you can also prove that $BXDZ$, $ABZV$, and $ZDCY$ are cyclic.*

**Solution**

Using the cyclic nature of circumcircles $BZXD$ & $BCAZ$:

$\angle ZDY = \angle ZBX = \angle ZBA = \angle ZCA = \angle ZCY$

Therefore $CDZY$ is cyclic. From this, we can find that :

$\angle AXB = 180^\circ - \angle ACB = 180^\circ - \angle YCD$,

$\angle WZV = 180^\circ - \angle VZY = 180^\circ - \angle DZY = \angle YCD = \angle ACB$ .

Therefore, $\angle WZV = 180^\circ - \angle AZB$.

The arcs these two circles represent is minor arcs AB & WV

Hence $AB = WV$, so trapezoid $ABWV$ is isosceles.

# Chapter 13

# Week 34

# Weekly Team Contest

**Problem Set 1 Level A**

Three coloured cups each cover one of three objects.

i. The **red** cup is somewhere to the left of the **orange** cup.

ii. The coin is somewhere to the left of the bean.

iii. The **blue** cup is somewhere to the right of the shell.

iv. The bean is somewhere to the right of the **blue** cup.

Which cup is the shell under?

**Problem Set 1 Level B**

Lily has 1995 unit cubes of granite rock and a wooden plank with total area 958. She wants to build a closed (6-face) rectangular box that can hold all of these cubes.

**Can** she do it?

**Bonus**: She found an additional 25 unit cubes of rocks. Can she reuse all the material from the existing wooden rectangular box and an additional piece of rectangle wooden plank with area 4 to **build another** closed (6-face) rectangular box that can hold these additional cubes?

**Problem Set 1 Level C**

What is the **value** of:

$$\frac{(20+14\sqrt{2})^{\frac{1}{3}}+(20-14\sqrt{2})^{\frac{1}{3}}}{(3+2\sqrt{2})^{\frac{1}{2}}-(3-2\sqrt{2})^{\frac{1}{2}}}$$

**Problem Set 1 Level A Solution**

Let determine the order of the cups. From i. the order could be ROB, RBO, or BRO. From iii. B cannot be the leftmost, and from iii. B cannot be the leftmost. Therefore the order is RBO.

According to iii. the shell is on the left of the blue cup, so it is under the red cup.

**Problem Set 1 Level B Solution**

The $11 \times 13 \times 14$ box has volume $11 \cdot 13 \cdot 14 = 2002$ and a surface area of $2(11 \cdot 13 + 11 \cdot 14 + 13 \cdot 14) = 958$ can host all 1995 cubes.

Furthermore a $12 \times 13 \times 13$ box with volume$12 \cdot 13 \cdot 13 = 2028$ and a surface area of $2(12 \cdot 13 + 12 \cdot 13 + 13 \cdot 13) = 962$ can easily host all 2020 cubes.

**Problem Set 1 Level C Solution**

Since $20 + 14\sqrt{2} = 8 + 12\sqrt{2} + 12 + 2\sqrt{2}$

$= 8 + 3 \cdot 4 \cdot \sqrt{2} + 3 \cdot 2 \cdot (\sqrt{2})^2 + (\sqrt{2})^3 = (2 + \sqrt{2})^3$

Therefore the numerator is $(2 + \sqrt{2}) + (2 - \sqrt{2}) = 4$.

Furthermore $3 + 2\sqrt{2} = (\sqrt{2})^2 + 2 \cdot \sqrt{2} \cdot 1 + 1 = (\sqrt{2} + 1)^2$

Therefore the denominator is $(\sqrt{2} + 1) - (\sqrt{2} - 1) = 2$.

The value of the given fraction is 2.

**Problem Set 2 Level A**

Vu travels from home to school at 60 kilometres per hour (kph). He returns along the same route. His average speed for the round trip is 72 kph.

**What is his speed** from school to home?

**Problem Set 2 Level B**

David gets apple and grape juice in identical jugs and makes a mixed drink, which it then packages in same-sized bottles. One jug of apple juice is enough for exactly 6 bottles of the mix, and one jug of grape juice is enough for exactly 10 bottles of the mix. Then the recipe was changed, and one jug of apple juice is now sufficient for exactly 5 bottles of the mix.

How **many bottles of mix** is one jug of the grape juice good for now?

**Bonus**: If the recipe is changed again such that one jug of apple juice is now sufficient for exactly 4 bottles of the mix, then how **many bottles of mix** is one jug of the grape juice enough for now?

**Problem Set 2 Level C**

**Prove** that

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cdots\cfrac{1}{2019+\cfrac{1}{2020}}}}}+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4+\cdots\cfrac{1}{2019+\cfrac{1}{2020}}}}}}=1$$

**Problem Set 2 Level A Solution**

Let pick a distance easy to deal with. Since lcm(60,72) = 360, let assume the distance for the whole trip is 360 km, hence the distance one-way from home to school is 180 km.

The round trip requires $\frac{360}{72} = 5$ hours. Since travelling to school requires $\frac{180}{60} = 3$ hours, so it takes him 2 hrs on the way back. This means that his average speed on the return trip is $\frac{180}{2} = 90$ kph.

**Problem Set 2 Level B Solution**

To make 30 bottles of the drink according to the old recipe, one needed 5 jugs of apple juice and 3 of grape juice: a total of 8 jugs. After the change, 30 bottles required 6 jugs of apple juice, and hence $8 - 6 = 2$ jugs of grape juice.

Hence now one jug of grape juice serve $\frac{30}{2} = 15$ bottles.

For the bonus question, 30 bottles required $\frac{30}{4} = 7.5$ jugs of apple juice, and hence $8 - 7.5 = 0.5$ jugs of grape juice.

Hence now one jug of grape juice serve $\frac{30}{0.5} = 60$ bottles.

**Problem Set 2 Level C Solution**

$$\text{Let } x = \cfrac{1}{3 + \cfrac{1}{4 + \cdots \cfrac{1}{2019 + \cfrac{1}{2020}}}}$$

The left hand side of the expression is:

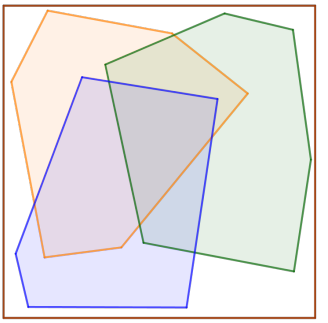
$$\frac{1}{2+x} + \frac{1}{1 + \frac{1}{1+x}} = \frac{1}{2+x} + \frac{1}{\frac{2+x}{1+x}} = \frac{1}{2+x} + \frac{1+x}{2+x} = 1$$

### Problem Set 3 Level A

In a square area 6, Lilian draws 3 polygons, each has an area 3.

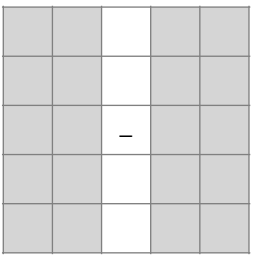
Prove that Lilian can always find **two polygons** whose **intersection's area** is at least 1.

*Hint: Use the diagram as a Venn diagram.*



### Problem Set 3 Level B

Huy fills the cells of a $5 \times 5$ board with one minus "$-$" and 24 plus "$+$" signs. In a move, he can choose a square on the board, consisting of more than one cell, and simultaneously change all the signs in this square to the opposite sign.




Prove that if he starts with the minus "$-$" sign at the **centre of the board**, then he can make **all pluses on the board** after several moves.

**Bonus**: Find **all other positions** of the minus such that it is possible for Huy to obtain all pluses on the board after several moves.

### Problem Set 3 Level C

A subset of the set $\{1, 2, 3, \cdots, n\}$ is called a *sparse* subset if it does not contain any two consecutive integers. For example $\{1, 3, 5\}$ is a *sparse* subset of $\{1, 2, 3, \cdots, 7\}$, but $\{1, 2, 4, 5, 7\}$ is not.

Let $a_n$ be the number of *sparse* subsets of $\{1, 2, 3, \cdots, n\}$. What is the value of $a_{10}$?

*Hint: Find a relation between consecutive terms $a_{n-1}$, $a_n$, $a_{n+1}$.*

**Problem Set 3 Level A Solution**

Look at the diagram as a Venn diagram, let $A, B, C$ be the polygons. The union of the areas of the three polygon is at most the area of the square, which is 6: $|A \cup B \cup C| \leq 6$

*The union of the three polygons* is equal to *the area of the three polygons* subtracts *the sum of the areas of the intersection between any two polygons* plus *the area of the intersection of all three polygons:*

$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$-\big(|A \cap B| + |B \cap C| + |C \cap A|\big) + |A \cap B \cap C|$$
$$|A \cap B \cap C| \geq 0\text{:}\ 6 \geq 9 - \big(|A \cap B| + |B \cap C| + |C \cap A|\big)$$

Hence $|A \cap B| + |B \cap C| + |C \cap A| \geq 3$

This means that one of them is at least 1.

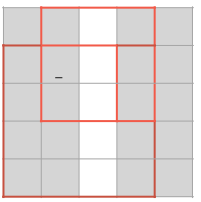
**Problem Set 3 Level B Solution**

The solution is shown as below.

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| + | + | + | + | + | | + | + | + | − | − | | + | + | + | − | − |
| + | + | + | + | + | | + | + | + | − | − | | + | + | + | − | − |
| + | + | − | + | + | | + | + | − | + | + | | + | + | − | + | + |
| + | + | + | + | + | | + | + | + | + | + | | − | − | + | + | + |
| + | + | + | + | + | | + | + | + | + | + | | − | − | + | + | + |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| − | − | − | − | − | | − | − | − | − | − |
| − | − | − | − | − | | − | − | − | − | − |
| − | − | + | + | + | | − | − | − | − | − |
| − | − | + | + | + | | − | − | − | − | − |
| − | − | + | + | + | | − | − | − | − | − |

For the bonus question: note that if the minus sign is in any of shaded squares, then any square consisting more than one cell, shall contain an even number of shaded squares. Three square examples with red border are shown above. Therefore the initial minus should not be in those cells.




We can also mark all cells in the two upper and lower rows in the similar way and prove that the initial minus cannot be in those rows. Hence the only possible position is the centre square.

**Problem Set 3 Level C Solution**

Let $S$ be a sparse subset of $\{1, 2, 3, \cdots, n+1\}$.

If $n+1$ is in $S$, then $n$ cannot be, so the set $S$ with $n+1$ removed is a sparse subset of $\{1, 2, 3, \cdots, n-1\}$. On the other hand adding $n+1$ to a sparse subset of $\{1, 2, 3, \cdots, n-1\}$ results in a sparse subset of $\{1, 2, 3, \cdots, n+1\}$.

If $n+1$ is not in $S$, then it is a sparse subset of $\{1, 2, 3, \cdots, n\}$. On the other hand a sparse subset of $\{1, 2, 3, \cdots, n\}$ is a sparse subset of $\{1, 2, 3, \cdots, n+1\}$ without $n+1$.

By the reasons above, there is a bijective mapping (bijection): for every sparse subset of $\{1, 2, 3, \cdots, n+1\}$ there exists one and only one either sparse subset of $\{1, 2, 3, \cdots, n-1\}$ or a sparse subset of $\{1, 2, 3, \cdots, n\}$.

Therefore $a_{n+1} = a_n + a_{n-1}$.

Since $a_1 = 1, a_2 = 2$. So $\{a_n\}$ is the Fibonacci sequence.

Hence $a_{10} = F_{10} = 55$.

**Problem Set 4 Level A**

It is possible to arrange 4 same-size balls in the space so that **every ball touches the other three**?

**Bonus**: It is possible to arrange 7 same-size circles on the plane so that every circle touches at least three others?

**Problem Set 4 Level B**

$m > n, a > b$ are four positive integers such that

$$mn + m = (n+a)^2$$
$$mn + n = (n+b)^2$$

**Prove** that $m > 3n$.

**Bonus**: Can you find an **example** of $m, n, a, b$ that satisfies the given conditions?

**Problem Set 4 Level C**

There are 5 points in the plane where no three are collinear. Prove that you can pick **4 points** that are the vertices of a **convex quadrilateral**.

**Problem Set 4 Level A Solution**

Put three balls on the table, let each touch another two. Put the four ball on top of these three.

For the bonus question: six unit circles are centred at the vertices of a hexagon side length 2 and the seven unit circle is centred at the centre of the hexagon.

**Problem Set 4 Level B Solution**

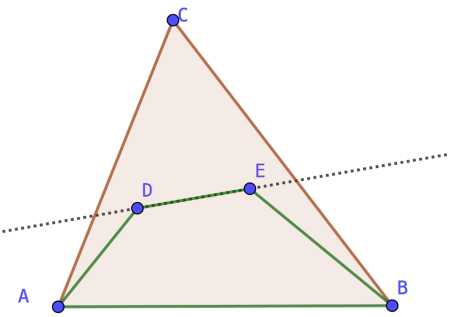
$$m - n = (mn + m) - (mn + n) = (n + a)^2 - (n + b)^2$$
$$= 2n(a - b) + a^2 - b^2 > 2n \cdot 1 = 2n$$

Hence $m > 3n$.

Example: $m = 8$, $n = 1$, $a = 3$, $b = 2$.

**Problem Set 4 Level C Solution**

Let consider the convex hull of the 5 points. There are two cases. If the convex hull is a pentagon or a convex quadrilateral, then we are done. If it is a triangle, let it be $ABC$ triangle as shown below with two remaining points $D$ and $E$ are inside the triangle.




From the given conditions the line through $DE$ cannot pass through any vertex of the triangle, so there is an side of the triangle that the line does not cross. Let it be $AB$. Since both $D$ and $E$ are on the same side of $AB$ and the line through $DE$ does not intersect $AB$ so $ABED$ is a convex quadrilateral.

**Problem Set 5 Level A**

Kevin draws four straight lines on the plane such that every two of them intersect, but no three of them have a point in common. Each line is divided into two rays and two segments by three intersection points. The total number of segments is eight. Is it possible that the lengths of these segments are equal to 1, 2, 3, 4, 5, 6, 7, and 8?

**Bonus**: Can the lengths be **distinct positive integers**?

**Problem Set 5 Level B**

Elisa and Erica went to visit each other. They left at the same time and each walked at a steady pace, but, because they had been preoccupied with counting squirrels that were nearby, they did not notice each other when they met, and they ended up at each other's houses. Erica got to Elisa's home 40 minutes after they passed each other, and Elisa got to Erica's house 10 minute after they passed each other.

**How long did each take** to get to the other's house?

**Problem Set 5 Level C**

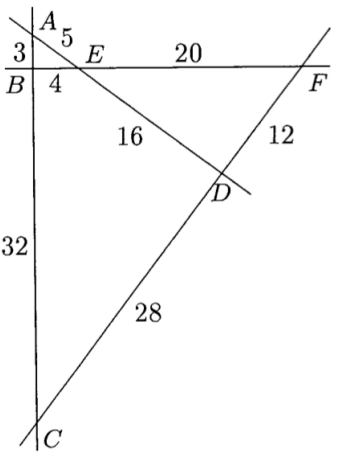
Catherine builds a palace consists of $2019 \times 2019$ rooms with doors connecting pairs of adjacent rooms (the rooms are $1 \times 1$ squares and two rooms are adjacent if they had a common edge).

Is it possible for **every room** to have **exactly 2 doors**?

**Problem Set 5 Level A Solution**

It is not possible because according to the triangle inequality the side length 1 should be larger than the difference of two other side lengths. However this is not possible with different side lengths.

For the second question here is an example.



**Problem Set 5 Level B Solution**

Easy to see that Elisa walked faster than Erica. Let say Elisa walked $x$ times as fast as Erica.

Before the meeting she had covered $x$ times that Erica did.

After the meeting Erica took a distance $x$ times longer than Elisa had, at a speed $\frac{1}{x}$ times slower. So the time Erica spent is $\frac{x}{\frac{1}{x}} = x^2$ times greater than the time Elisa spent walking after the meeting.

Therefore $x^2 = \frac{40}{10} = 4 \Rightarrow x = 2$.

Erica took 40 minutes to go from the meeting to Elisa's house, so it took Elisa 20 minutes to cover that same distance. Hence Elisa spent $20 + 10 = 30$ minutes, while Erica spent 60 minutes in total.

**Problem Set 5 Level C Solution**

Let suppose that it is possible. Let each room be a vertex of a graph. Two rooms share the same door are connected by an edge. Easy to see that the graph is a set of disjoint cycles.

Furthermore, each cycle has even length because of the number of vertical steps is the same in both directions and the same holds for horizontal steps. Therefore the number of vertices of the graph, which is the sum of the lengths of these cycles is even.

This contradicts the given plan.

**Problem Set 6 Level A**

Albert cuts a $4 \times 4 \times 4$ cube of cheese into 64 unit cheese cubes.

**What is the least number of cuts** he must do so if he can rearrange pieces after each cut? Show that how the cuts are done.

*Note that every cut must be parallel to a face of the big cube.*

**Problem Set 6 Level B**

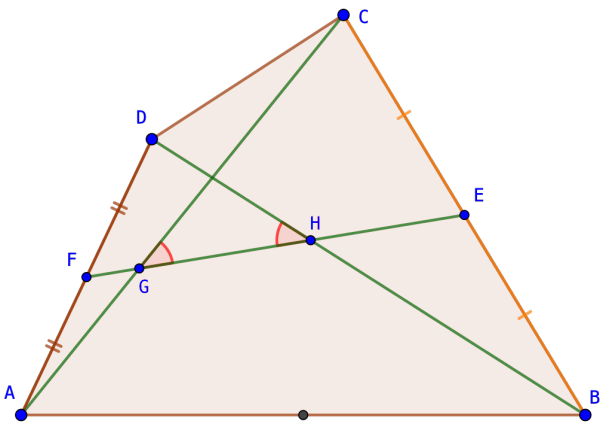
2020 numbers are written on a board. Tina can erase any two numbers $a$ and $b$ and writes $\dfrac{a+b}{4}$ instead. After 2019 moves there is only one number remaining on the board.

Now, at the beginning, all of the numbers are 1. Is it possible that **the sum of reciprocals of all the numbers** on the board **decreases** over time?

**Bonus**: Can the last number less than $\dfrac{1}{2020}$?

**Problem Set 6 Level C**

$ABCD$ is a convex quadrilateral. Segment $EF$ connecting midpoints of the two opposite sides $AD$ and $BC$ forms the same angles with the diagonals $AC$ and $BD$ as shown below.




Prove that $AC = BD$.

**Problem Set 6 Level A Solution**

Note that each cut can double the number of cut pieces. Since $2^6 = 64$, he has to perform at least that number of cut.

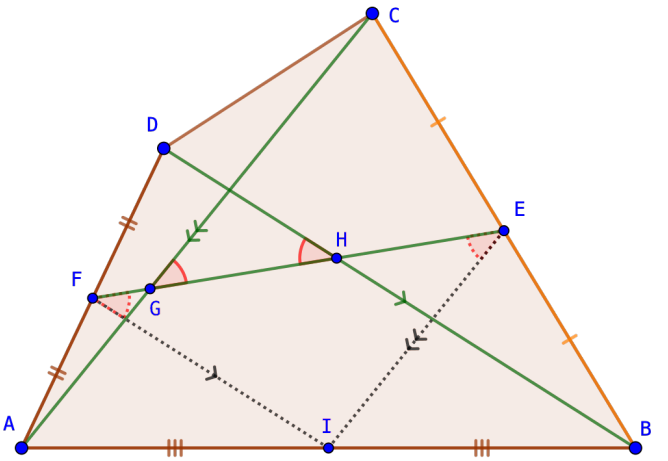
It is easy to demonstrate how to perform the six cuts.

**Problem Set 6 Level B Solution**

Since $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b} \Leftrightarrow (a+b)^2 \geq 4ab$ which is true, so the sum of the reciprocals of the numbers does not decrease.

At the beginning, it was $n$, so the last number at least $\frac{1}{n}$.

**Problem Set 6 Level C Solution**




Let $I$ be the midpoint of $AB$. Since $EI \parallel AC$, $FI \parallel DB$ so:

$$\angle FEI = \angle FGC = \angle DHG = \angle IFE$$

So $\triangle EFI$ is isosceles at $I$: $FI = EI \Rightarrow DB = CA$

# Problems of the Week

**Level A**

### Problem 1.A

The numbers 1, 2, 3, …, 2003 are written on a board. Erica erases some numbers and writes the remainder of the sum of those numbers when divided by 25 on the board. At some points, she has two numbers on the board. One of them is 1964, what is the other number?

**Solution**

The sum of all the numbers on the board is $\frac{2003 \cdot 2004}{2}$, which has remainder 6 when divided by 25. Since 1964 has remainder 14 when divided by 25, so the other number is the remainder of the sum of all numbers, except 1964, when divided by 25. That sum has the remainder 17 when divided by 25 because: $17 + 14 = 31 \equiv 6 \pmod{25}$

So 17 is the value of the second number.

### Problem 2.A

$10! = 36288\underbrace{00}_{2}$ has two trailing zeros.

Find the least $n$ such that $n!$ has at least 123 trailing zeros.

Find the largest $n$ such that $n!$ has at least 123 trailing zeros.

**Solution**

The trailing zeros are produced by the product of prime factor 2s and 5s in the product: $n! = 1 \cdot 2 \cdot 3 \cdots n$. Since the number of factors of 2 are much more than 5's, so we only need to figure out the number of factors of 5. Now let take a look at $500!$.

The number of factors of 5 in $500!$ is:

$$\lfloor \frac{500}{5} \rfloor + \lfloor \frac{500}{25} \rfloor + \lfloor \frac{500}{125} \rfloor = 100 + 20 + 4 = 124.$$

Since $\lfloor \frac{499}{5} \rfloor + \lfloor \frac{499}{25} \rfloor + \lfloor \frac{499}{125} \rfloor = 99 + 19 + 3 = 121$, so 500 is the smallest such integer.

Obviously it is also the largest $n$ such that $n!$ has at most 123 trailing zeros is 499.

**Problem 3.A**

$p$, $q$ are prime numbers such that $p > q > 3$ and $p - q = 2$.

Prove that $p + q$ is divisible by 3.

Prove that $p + q$ is divisible by 12.

**Solution**

Let take a look at the remainder of $p$ when divided by 3. It cannot be 0 because $p > q > 3$. If it is 1, because $p - q = 2$, so the remainder of $q$ when divided by 3 is 2. In that case the remainder of $p + q$ when divided by 3 is 0. If it is 2, because $p - q = 2$, so the remainder of $q$ when divided by 3 is 1. In this case the remainder of $p + q$ when divided by 3 is also 0. In both case $p + q$ is divisible by 3.

For the second question, easy to see that both $p$ and $q$ are consecutive odd numbers. So if one of them has remainder of 1 when divided by 4, then the other has remainder of 3 when divided by 4. Thus, their sum has remainder of 0 when divided by 4. Hence $p + q$ is divisible by both 3 and 4, and because 3 and 4 are relatively prime numbers, so $p + q$ is divisible by $3 \cdot 4 = 12$.

**Problem 4.A**

$ABC$ is an arbitrary triangle. Use only a straightedge and a compass construct a triangle $MNP$ such that:

$$S_{\triangle MNP} = \frac{3}{4} S_{\triangle ABC}$$

**Solution**

First construct the perpendicular bisector of $BC$ as follow: construct a circle through $C$ centred at $B$ and a circle through $B$ centred at $C$ intersect each other at $X$ and $Y$.

The line through $XY$ bisects $BC$ at $D$, which is the midpoint of $BC$. Similarly constructing $E$, $F$ as midpoint of $AC$, $AB$, respectively.

The segment connecting $EF$ intersect the segment connecting $AD$ at $H$. Because $\triangle AEF \sim \triangle ACB$, so $H$ is midpoint of $AD$. Similarly construct $I$ as midpoint of $AH$.

Because $\dfrac{ID}{AD} = \dfrac{3}{4}$, so $S_{IBC} = \dfrac{3}{4} S_{ABC}$.

## Level B

### Problem 1.B

$A$ and $B$ are two distinct $7-$digit numbers. The digits of each of them are $1, 2, \cdots, 7$, each digit appears once. Is it possible that $A$ is a divisor $B$ or $B$ is a divisor of $A$?

$M$ and $N$ are two distinct $8-$digit numbers. All the digits of $M$ are distinct. All the digits of $N$ are also distinct. Is it possible that $M$ is a divisor $N$ or $N$ is a divisor of $M$?

**Solution**

Without loss of generality, assume that $A = nB$, where $n$ is a positive integer and $n > 1$ (*)

Since $n \leq \dfrac{7654321}{1234567} \Rightarrow n \leq 6 \Rightarrow 2 \leq n \leq 6$.

Because the sums of digits of $A$ and $B$ are the same and equal to:

$$1 + 2 + \cdots + 7 = \frac{7 \cdot 8}{2} = 28$$

which has remainder of 1 when divided by 9. So $A = nB$ has the remainder $n$ when divided by 9, which cannot be 1. This contradicts the assumption (*). Hence it is impossible that $A$ is a divisor $B$ or $B$ is a divisor of $A$.

For the second question, there are several answer, here is one:

$$12345678 \cdot 5 = 61728390$$

**Problem 2.B**

$a_1, a_2, a_3, a_4, a_5$ are five distinct integers. Let $P$ be the product of the following differences: $(a_1 - a_2)$ , $(a_1 - a_3)$ , $(a_1 - a_4)$ , $(a_1 - a_5)$ , $(a_2 - a_3)$, $(a_2 - a_4)$, $(a_2 - a_5)$, $(a_3 - a_4)$, $(a_3 - a_5)$, $(a_4 - a_5)$

Prove that $P$ is divisible by 9.

Prove that $P$ is divisible by 288.

**Solution**

Among $a_1, a_2, a_3, a_4, a_5$, according to the pigeonhole principle there are at least two numbers that have the same remainder when divided by 3. Without loss of generality, let those be $a_1$ and $a_2$. Their difference $a_1 - a_2$ is divisible by 3.

Among the remaining three numbers $a_3, a_4, a_5$, again according to the pigeonhole principle there are at least two numbers that have the same remainder when divided by 3. Without loss of generality, let those be $a_3$ and $a_4$. Their difference $a_3 - a_4$ is divisible by 3.

Hence the product $P$ is divisible by 9.

For the second question, let investigate the parities of those five given numbers.

If there are four of them with same parity, then $\binom{4}{2} = 6$ differences of them are even and therefore divisible by $2^6 = 64$.

If there are exactly three of them with the same parity, then without loss of generality, let those be $a_1$, $a_2$, and $a_3$. Then the following differences are even: $(a_1 - a_2)$, $(a_1 - a_3)$, $(a_2 - a_3)$, $(a_4 - a_5)$, so they are divisible by $2^4 = 16$.

In addition since there are given 5 numbers, so according to the pigeonhole principle, there are at least two of them have the same remainder when divided by 4. This is one of the four even differences : $(a_1 - a_2)$, $(a_1 - a_3)$, $(a_2 - a_3)$, $(a_4 - a_5)$. Thus, their product is divisible by $2^5 = 32$.

Because 9 and 32 are relatively prime numbers, and $P$ is divisible by 9 both 9 and 32, so $P$ is divisible by $9 \cdot 32 = 288$.

**Problem 3.B**

How many triples of three consecutive prime numbers $(a, b, c)$ are there such that $a^2 + b^2 + c^2$ is also a prime number?

How many triples of three consecutive prime numbers $(a, b, c)$ are there such that $a^2 + 2b^2 + 3c^2$ is also a prime number?

**Solution**

Let assume that $c > b > a$. If $a > 3$, note that the square of any integer has remainder 1 when divided by 3. So $a^2 + b^2 + c^2$ has remainder 0 when divided by 3, therefore cannot be a prime number.

Since $a$ cannot be 2, otherwise $a^2 + b^2 + c^2$ is even number, so $a = 3$ and $a^2 + b^2 + c^2 = 3^2 + 5^2 + 7^2 = 83$ is prime number.

So the answer is (3,5,7).

For the second question if the smallest prime if larger than 3 then $a^2 + 2b^2 + 3c^2$ has a remainder same as $1 + 2 \cdot 1 + 3 \cdot 1$ or 0 when divided by 3. While $a = 2$: $a^2 + 2b^2 + 3c^2 = 2^2 + 2 \cdot 3^2 + 3 \cdot 5^2 = 97$ is a prime number.

Note that it is also simple to see that $a = 2$ is the only possible case to guarantee that $a^2 + 2b^2 + 3c^2$ is an odd number.

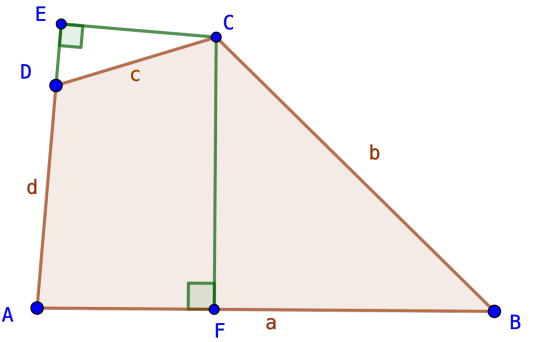
So the answer is (2,3,5).

**Problem 4.B**

What is the maximum area a quadrilateral side lengths 1, 4, 7, 8.

Let $a, b, c, d$ be the consecutive sides, clockwise of a quadrilateral area $S$. Prove that: $S \leq \dfrac{ab+cd}{2}$.

**Solution**




Let prove the second question first. In triangle $ABC$ the altitude $CF < CB = b$ and in triangle $ACD$ the altitude $CE < CD = c$.

Therefore the quadrilateral area:

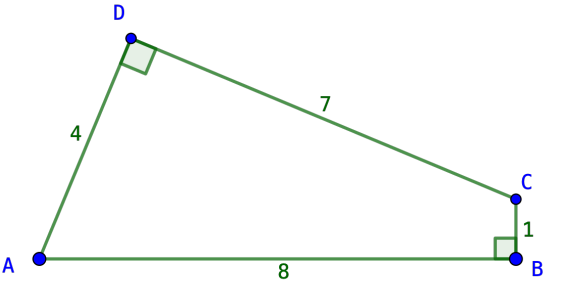
$$S = S_{ABCD} = S_{ABC} + S_{ACD} = \frac{CF \cdot a}{2} + \frac{CE \cdot d}{2} \leq \frac{ab+cd}{2}$$

Note that the equality does happen when $CB \perp CA$, $CD \perp AD$.

For the first question, use the above results:

$$S \leq \left(\frac{ab+cd}{2} + \frac{ad+bc}{2}\right) = \frac{(a+c)(b+d)}{4}$$

If $a, b, c, d$ are 1, 4, 7, 8, then $S$ is maximal if side lengths 7 and 8 are opposite: $S \leq \dfrac{(1+4)(7+8)}{4} = \dfrac{75}{4}$, $CB \perp CA$, $CD \perp AD$ as below.



## Level C

### Problem 1.C

$n$ children sit in a row of $n$ seats. How many ways are there to rearrange them such that each child may move by at most one seat?

A binary 10− word has 10 letters, each letter is 0 or 1. For example 1001110101 is a binary 10− word. Furthermore this binary 10−word has exactly three 01-blocks: $10\underbrace{01}_{1^{st}}11\underbrace{01}_{2^{nd}}\underbrace{01}_{3^{rd}}$.

Prove that the number of binary $n-$words with exactly $m$ 01−blocks is $\binom{n+1}{2m+1}$

*Note: the first question is not related to the second one.*

**Solution**

Let $a_n$ be the number of ways to move $n$ children such that each child may move by at most one seat. Consider $n+1$ children. If the first one stays in his/her place, then there are $a_n$ ways to move the rest. If the first move to the second's place, then the second must move to the first's place, so there are $a_{n-1}$ ways to move the rest. Hence:

$$a_{n+1} = a_n + a_{n-1}$$

Since $a_1 = 1, a_2 = 2$, so $\{a_n\} = \{F_n\}$ is the Fibonacci sequence.

For the second question consider $0-1$ transitions, there are exactly $m$ of these. We don't know if the number of $1-0$ transitions is $m-1$, $m$or $m+1$. It would be nice to have exactly $m+1$ transitions from 1 to 0. However we can always extend the word by a 1 at the beginning and a 0 at the end. This way, we always have exactly $m+1$ from 1 to 0. All together we have $n+2$-word with $n+1$ slots between 0 and 1 letters. We have to choose $2m+1$ places for a switch from 0 to 1 or from 1 to 0.

Hence, there are $\binom{n+1}{2m+1}$ ways or that is the number of binary $n-$words with exactly $m$ 01−blocks.

**Problem 2.C**

Let $\sigma(n)$ be the sum of positive divisors of $n$ (including 1 and $n$) . Prove that $n$ is a composite number if and only if:

$$\sigma(n) > n + \sqrt{n}$$

Let $\varphi(n)$ be the number of positive integers that are relatively prime to and not exceeding $n$. Prove that:

$$\sigma(n) + \varphi(n) \geq 2n, \forall n \in \mathbb{N}$$

**Solution**

First let assume $n$ is a composite number. There exist $a, b$ positive integers, such that $a > 1$, $b > 1$ and $n = a \cdot b$.

If $a \neq b$: since $a + b > 2\sqrt{ab}$ so:

$\sigma(n) = \sigma(ab) \geq 1 + a + b + ab > ab + \sqrt{ab} = n + \sqrt{n}$

If $a = b, n = a^2$:

$\sigma(n) = \sigma(ab) \geq 1 + a + a^2 > a + a^2 = n + \sqrt{n}$

The opposite is quite simple, if $\sigma(n) > n + \sqrt{n}$ then $n$ cannot be a prime number because $\sigma(n) = 1 + n < n + \sqrt{n}$, so $n$ is a composite number.

For the second question, let $1 = d_1 < d_2 < \cdots < d_k = n$ be the positive divisors of $n$. Since $\sigma(n)$ is the sum of positive divisors of $n$ (including 1 and $n$), so

$\sigma(n) = d_1 + d_2 + \cdots + d_k = d_1 + d_2 + \cdots + d_{k-1} + n.$

Furthermore $d_1 = \dfrac{n}{d_k}$, $d_2 = \dfrac{n}{d_{k-1}}$, $\cdots$, $d_k = \dfrac{n}{d_1}$ so:

$$\frac{n}{d_2} + \cdots + \frac{n}{d_k} = d_{k-1} + \ldots + d_1$$

The left hand side of the equations contains the sum of the numbers of multiples of $d_2, \ldots, d_k$ not exceeding $n$. Obviously this is at least the number of integers not exceeding $n$ and not relatively prime with $n$, which is $n - \varphi(n)$. Thus:

$$\sigma(n) - n = d_1 + d_2 + \cdots + d_{k-1} = \frac{n}{d_2} + \cdots + \frac{n}{d_k} \geq n - \varphi(n)$$

$$\sigma(n) + \varphi(n) \geq 2n$$

**Problem 3.C**

Let $\{a_i\}$, $\{b_i\}$be real numbers. Prove the inequality:

$$\left(a_1^2+a_2^2+\cdots+a_n^2\right)\left(b_1^2+b_2^2+\cdots+b_n^2\right)\geq\left(a_1b_1+a_2b_2+\cdots+a_nb_n\right)^2$$

Prove that the equality occurs if and only if the two sequences $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are proportional: $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}$

Applied the above, prove that if $a_1, a_2, \cdots, a_n$ are real numbers and $b_1, b_2, \cdots, b_n$ are positive real numbers, then:

$$\left(\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n}\right)\geq\frac{(a_1+a_2+\cdots+a_n)^2}{(b_1+b_2+\cdots+b_n)}$$

Applying the Cauchy-Schwarz inequality to prove that:

$$\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}\geq\frac{3(a+b+c)}{3+a+b+c}$$

**Solution**

The first question can easily be solved by expanding and rearranging the terms into the sum of squares as below, which is non-negative: $\sum_{1\leq i<j\leq n}(a_ib_j-a_jb_i)^2$ (*). The other solution is to use induction principle and note that:

$$\left(a_1^2+a_2^2+\cdots+a_n^2+a_{n+1}^2\right)\left(b_1^2+b_2^2+\cdots+b_n^2+b_{n+1}^2\right)$$
$$\geq\left(\sqrt{a_1^2+a_2^2+\cdots+a_n^2}\sqrt{b_1^2+b_2^2+\cdots+b_n^2}+a_{n+1}b_{n+1}\right)^2$$
$$\geq\left(a_1b_1+a_2b_2+\cdots+a_nb_n+a_{n+1}b_{n+1}\right)^2$$

The equality, according to (*), can happen if and only if:

$$\forall\ 1\leq i<j\leq n:\ a_ib_j-a_jb_i=0\Leftrightarrow\frac{a_i}{a_j}=\frac{b_i}{b_j}$$

iii. is just a simple application of the Cauchy-Schwarz inequality.

For the second question:

$$\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}=\frac{a^2}{a(b+1)}+\frac{b^2}{b(c+1)}+\frac{c^2}{c(a+1)}$$
$$\geq\frac{(a+b+c)^2}{a(b+1)+b(c+1)+c(a+1)}=\frac{(a+b+c)^2}{(ab+bc+ca)+(a+b+c)}$$
$$\geq\frac{(a+b+c)^2}{\frac{(a+b+c)^2}{3}+(a+b+c)}=\frac{3(a+b+c)}{a+b+c+3}$$

**Problem 4.C**

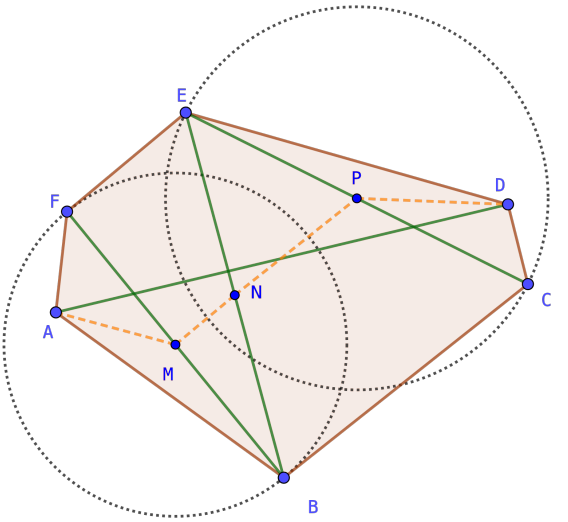
$ABCDEF$ is a convex hexagon, $\angle A \geq 90^\circ$, $\angle D \geq 90^\circ$.

Prove that: $BC + CE + EF + FB \geq 2AD$.

Prove that in any polygon there are two sides $a$ and $b$:

$$1 \leq \frac{b}{a} < 2.$$

**Solution**




Let $M, N, P$ be the midpoints of $BF, BE, CE$, respectively. Since any broken line is longer or equal the distance between two endpoints:

$$AD \leq AM + MN + NP + PD$$

Because $\angle A \geq 90^\circ$, $\angle D \geq 90^\circ$, so $AM \leq \dfrac{BF}{2}$, $DP \leq \dfrac{CE}{2}$.

$MN = \dfrac{EF}{2}$ and $NP = \dfrac{BC}{2}$, so $BC + CE + EF + FB \geq 2AD$.

For the second question, let: $a_1 \leq a_2 \leq \cdots \leq a_n$ be the side lengths of the polygon. Assume the opposite, we have: $a_2 \geq 2a_1$, $a_3 \geq 2a_2$, $\cdots$, $a_n \geq 2a_{n-1}$, so:

$a_n \geq 2a_{n-1} \geq 2^2 a_{n-2} \geq \cdots \geq 2^{n-1} a_1$

$a_1 + a_2 + \cdots + a_{n-1} \leq a_1 + 2a_1 + \cdots + 2^{n-2} a_1 = (2^{n-1} - 1)a_1 < 2^{n-1} a_1 \leq a_n$

This contradicts $a_1 + a_2 + \cdots + a_{n-1} > a_n$.

# Chapter 14

# Week 35

# Weekly Team Contest

**Problem Set 1 Level A**

Once upon a time, prince Kelvin decided to seek the hand of the beautiful princess Tina from the kingdom of Smarties-pants. The king of Smarties-pants decided to put the prince's smarts to the test. He brought him into a hall with two doors. A sign on each door said: "*Both rooms behind these doors are empty*." The king informed the prince that each of the rooms is either empty or contain a tiger. Moreover, if the first room is empty, then the door sign is true; otherwise it is false. At the same time, if the second room is empty, then the sign on the door is false; otherwise it is true. The king invited Kelvin to open the door of his choice. If a hungry tiger jumps out of the door, Kelvin will lose. However, if Kelvin opens the door to en empty room, he will get Tina's hand in marriage. Assuming the king is truthful, can you help Kelvin pass this challenge?

**Problem Set 1 Level B**

Vu and Anthony takes turns placing $1 \times 2$ dominoes on a $10 \times 10$ chessboard. The player who cannot place a domino loses. Vu goes first. Who has **a winning strategy** and what is it?

**Problem Set 1 Level C**

There are ten death-water wells in the far, far away Dark Forest. The wells are marked by signs with number from 1 to 10. The water from these wells look and taste like normal water; however, even a single gulp of it is lethal - unless one has an antidote. The only known antidote is water from a higher-numbered well. (For example, if you had a sip of water from the well number 6, you could save yourself by drinking from the well 7, 8, 9, or 10.) Unfortunately, there is no antidote for the water from the tenth well. While the first nine wells are easily accessible, the tenth one is located in the castle of King Haggard, the ruler of the Dark Forest. Prince Martin challenged the King to the death-water duel. By the rules of the duel, each participant should drink a cup of water offered by the opponent. King Haggard eagerly agreed. His plan was to use his tenth death well; he reasoned that water from this well will neutralize any drink offer by Martin, and will definitely kill the prince. The duel took place as planned. Each participant drank the liquid offered by the opponent. To everybody surprise and delight. Prince Martin lived and King Haggard died. **How did this happen**? Explain.

**Problem Set 1 Level A Solution**

Notice that the two signs assert the same thing: that both rooms are empty. Can these signs be true? Assuming that they are.

Then, since the king said "if the second room is empty, then the sign on the door is false; otherwise it is true", the sign is false. By assuming that the signs are true, we ran into contradiction, so the signs cannot be true. They are false.

Now recall the rest of what the king said about the second room, "otherwise" [meaning, if the room is not empty], it [the door sign] is true". Since we know the door sign is false, so the second room must be empty.

Therefore the prince can safely choose the second room.

**Problem Set 1 Level B Solution**

The second player wins by mirroring the moves of the first player with respect to the central point of the board. With such a strategy, the second player will always be restoring central symmetry. Therefore wherever the first player is able to find a position to place his tile, the second player will be able to follow up by selecting a symmetric position for his tile.

Therefore the second player will make the last move.

**Problem Set 1 Level C Solution**

Prince Martin gave him a drink of clear water. Thus instead of curing him, the water from the tenth well poisons him.

Prince Martin, before the duel, drank some water from a death well with a number smaller than 10. Therefore Haggard's drink worked as antidote for him.

**Problem Set 2 Level A**

There are three types of coins in the country of Sugarland: sugriks, tugriks, and shmollars. Sugricks are the lowest denomination coins; both tugriks and shmollars are worth an integer number of sugriks. The price of a magic lollipop at Sugarland's Sunday Market is 11 sugriks. Lily has 4 sugriks, 1 tugrik, and 9 shmollars in her pocket. She uses *all of this money* to pay for several magic lollipops. Linh Chi has 15 sugriks, 12 tugrik, and 9 shmollars with her.Prove that she can also use *all these coins* to pay for several magic lollipops as well.

**Problem Set 2 Level B**

A vicious giant captured Sam, Elisa, and Catherine. He intends to eat them, but he decides to play a game with them first. The giant opens a box containing 2 red hats and three black hats. Then he gives each of them a piece of paper, turns off the lights, and puts 3 of the 5 hats on their heads. After the giant turns the light on, each person is able to see the colour of all the others' hats, but not the colour of his/her hat. They cannot communicate with each other.

The giant explains that each of them will be given a chance to guess the colour of his/her hat. If a person makes the right guess, he/she is free to go (you don't want to think about the result of a wrong guess). The guessing works as follow: the giant will wave his hand three times. At each wave, every person will have a choice: either write the colour of his/her own hat on the piece of paper and hand it immediately to the giant, or do nothing. After each wave, the giant will read aloud the notes he had received. After three waves, the giant will release those who guessed the colours correctly.

What shall Sam, Elisa, and Catherine do to avoid being served as sushi, steak, or stews?

**Problem Set 2 Level C**

Kevin is feasting upon a cheese cub that is $3 \times 3 \times 3$ cubic inches. This cube consists of 27 small cubes of side length 1 inch. As soon as he finishes one small cube, he starts working on another cube that *has a common face* with the one that was just eaten. Kevin knows that the middle one contain a nice present that he just doesn't want to open yet. Is there away for him to **eat all the cubes except for the one in the middle**?

**Problem Set 2 Level A Solution**

The difference between their money is 11 sugriks and 11 tugriks, so surely it is possible to purchase exactly an integer amount of lollipops since their price is a multiple of 11.

**Problem Set 2 Level B Solution**

Let consider three scenarios of the hats: RRB, RBB, and BBB.

RRB: the person in the B hat immediately hands in his guess as B. The other two persons will give their guesses as R at the $2^{nd}$ round.

RBB: The one has R hat is not sure if he has a R or a B hat. Since both of his friend are not able to move , so none of them can move in the first round. Since the ones wearing B knows that his/her hat is B and therefore could be able to announce that in the second round. The last person now know that he wears a R one.

BBB: Similarly no one can do anything in the first or second round. In the third round they have realized the situation and would guess at the same time.

**Problem Set 2 Level C Solution**

Let colour all the cubes alternately red and blue like a chess board. Let assume that all cube at the corners are red. Then Kevin has 14 cubes as red and 13 as blue. Note that the central cube is blue.

Since the colour is changed as he eat, if the central is the last he eats, then the number of eaten red cubes cannot exceed the number of eaten blue ones.

**Problem Set 3 Level A**

A pirate ship's flag has a rectangular shape and made of alternating vertical black and white stripes of fabric of the same width. The total number of stripes on the flag is equal to the number of captives currently held on the ship. Last week, there were 12 captives, and the pirates displayed a flag with 12 stripes. However, two captives escaped during the weekend.

Nam, the new ship's cabin boy was given the task of modifying the flag. The new flag should be *rectangular* with the *same area*, and 10 alternating black and white vertical *stripes of the original width*.

Can you help Nam to cut the flag into *two parts* that can be sewn together to form the new flag?

**Problem Set 3 Level B**

Drago the Dragon has 15 sane and 20 insane heads. Prince Bill's magic sword can cut either one or two of the Dragon's heads in a single strike. New heads grow immediately in place of the removed ones:

- If a pair of identical heads is cut off, a new insane head grows back.
- If a pair of different heads is cut off, a new sane head grows back.
- If a single head is cut off, it is replaced with a new head of the same kind.

If Drago is left with one head, he will lose his courage and give up the fight. **Show the sequence of a strikes** that will leave the dragon with just one head. **Bonus**: Will this last head be sane or insane?

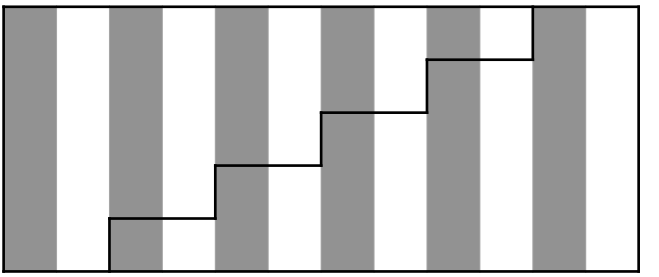
**Problem Set 3 Level C**

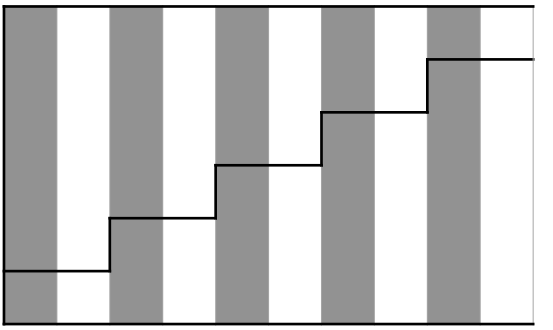
Lilian caught this encrypted message. Help her to decode:

$$\text{COW} + \text{COW} + \cdots + \text{COW} = \text{HERD}$$

She has to look for the **maximum number** of "cows" in the "herd". *Note: different letters stand for different digit.*

**Problem Set 3 Level A Solution**





**Problem Set 3 Level B Solution**

It is not difficult to come up with the sequence.

For the second question, note that each strike will preserves or reduce the number of sane heads by 2. So its parity is not change. Since at the beginning it was 15, so the last head is the sane one.

**Problem Set 3 Level C Solution**

Note that HERD $\leq 9876$, and COW $\geq 102$. So the maximum number of "cows" at most 96. Easy to see that 96 is not possible, because in that case COW = 102, 103, 104 $(105 \cdot 96 > 9999)$, and by testing each case, none is possible.

Similarly testing with 95 and COW = 102, 103, 104, 105:

$103 \cdot 95 = 9785$

**Problem Set 4 Level A**

Clemence's father wrote an equation on the board and left the house. Her younger sister came and erased all the signs, except for the equality one. Thus, the equation on the board was as below:

$$8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 = 3$$

Desperate, Clemence called her father. He told her that all the signs which were deleted were minus "−" signs.

Can you help her to **reconstruct** the equation?

**Problem Set 4 Level B**

The temple of the god Pi stood in the ancient Egyptian city of Alexandria. A circle of 13 magical oil lamps surrounded the central altar in this temple. Whenever a priest walked inside the circle of lamps, the two lamps on his sides would magically change their state. If a lamp was lit, it would go off. If a lamp was off, it would light up. Moreover, walking between lamps was the only way to light them up and to put them off.

On the morning of the Annual MCC Team Contest, two of the magical lamps located next to each other were lighted, and the rest of the lamps were not.

A young priest wanted to light all the lamps, but he does not know whether it is possible; if yes, how to do it; if not, why not. He asked Erica, who's a member of a contest team, for the solution.

**What did she told him?**

**Problem Set 4 Level C**

Ha-Anh Le visits the Grand Bazaar of Istanbul. She wanted to find some beautiful coins for her friends. The seller showed her a pair of gold coins, a pair of silver coins, and a pair of bronze coins. He told her that in each pair, one coin is counterfeit - it is lighter than the real coin. All real coins have the same weight, and the counterfeit coins are all of the same weight as well.

If she can use a balance scale to *find out all three counterfeit coins* in just **two weighings**, she can have all of them for free.

Can she do it? If yes, show a way. If not, explain why.

**Problem Set 4 Level A Solution**

$$87 - 6 - 54 - 3 - 21 = 3.$$

**Problem Set 4 Level B Solution**

When she walked between two lit lamps, they go off. So the number of "on" lamp goes down by 2. When she walks between two lamps that are not lit, the number of "on" lamps goes up by 2. When she walks between an "on" and an "off" lamp, the number of "on" lamps is not changed.

Hence from the original 2 lit lamps, she would not be able to lit all the lamps.

**Problem Set 4 Level C Solution**

Let the pair of gold coins, the pair of silver coins, and the pair of bronze coins be $G_1$, $G_2$, $S_1$, $S_2$, $B_1$, $B_2$.

On the first weighing, compare $(S_1, B_1)$ with $(S_2, G_2)$.

Case $(S_1, B_1) < (S_2, G_2)$: $B_1$ is light and $G_2$ is heavy.

Obviously $S_1$ and $G_1$ are also light.

Case $(S_1, B_1) > (S_2, G_2)$: $B_1$ is heavy and $G_2$ is light.

Obviously $S_2$ and $B_2$ are also light.

Case $(S_1, B_1) = (S_2, G_2)$: let now compare $B_1$ and $G_2$. From that we know the counterfeit between them. So we know which is the counterfeit between $S_1$ and $S_2$; as well $B_2$ and $G_1$.

**Problem Set 5 Level A**

The MCC soccer team is participating in a soccer tournament. According to the rules of this tournament, every victory brings the team one point, every loss reduces the score by one point, and a draw does not affect the score. After the first 20 games, the MCC team has 10 points. After 25 more games, the score is 20 points.

Huy, the MCC newspaper's journalist, was away for the last 25 games. He looked at the score and said that at least one game out of the last 25 resulted in a draw.

**How did he know**?

**Problem Set 5 Level B**

A long time ago on the Island of Knights and Liars, there lives three friends, all mighty warriors. Their names were Albert, David, and Victor. Two of them were Liars, and one was Knight. The friends keep their affiliations secret (nobody know who's what).

One of their great deeds was a battle with a terrible dragon that was terrorizing the MCC club. Not much is known about this battle except that the dragon was slayed by a Knight.

In a recently discovered old letter, Albert states that Victor had slayed the dragon and David were merely watching.

W**ho actually killed** the dragon?

**Problem Set 5 Level C**

A robot drew a painting consisting of a $4 \times 4$ white square with the top-left corner cell painted black.

Victoria commands the robot to select a row or a column of her choice and repaints every single square in this row or column into the opposite colour.

Can she **create a sequence of commands** that turns the entire $4 \times 4$ square black?

**Problem Set 5 Level A Solution**

If there were no draw in the last 25 games, then each of them is a loss or win. Since the numbers of losses and wins add up to 25, so one of them is odd and another is even. Moreover, the difference of the number of wins and loss is equal to the number of point achieved, which is obviously an odd number. However they have gained $20 - 10 = 10$ more points, so that contradicts the assumption.

**Problem Set 5 Level B Solution**

If we assume that Albert was a Knight, then Victor slayed the dragon. Because the dragon slayer was a Knight, hence Victor was a Knight, that's impossible because only one of them is a Knight. Therefore Albert is a Liar, which means that Victor did not kill the dragon and David was the slayer.

**Problem Set 5 Level C Solution**

Whenever a row or a column is repainted, the parity of the number of black cells does not change. It is easy by examine the cases or by assume that there are $k$ black unit squares and the number of changed unit squares is $(4 - k) - k = 4 - 2k$, which is an even number.

Since the number of black square at the beginning is 1. It is not possible to paint all of them to be black.

**Problem Set 6 Level A**

An airport has fifteen gates and various moving walkways, each of which connects exactly two gates in both directions. From each gate Bianca reach at least seven others via a single walkway.

Is it possible for her to go from any gate to any other via either one or two walkways?

**Problem Set 6 Level B**

There are 9 towns in a small country, and the distances between any two towns are all different. A person starts in each town and walks towards the closest town.

Prove that t**here are two towns** $A$ **and** $B$ that a person walks from A to B and a person from B walks to A.

**Bonus**: Prove that there exists a town that nobody walks to.

**Problem Set 6 Level C**

A certain country has several airfields. The distances between all of them are different. An airplane takes off from each of the airfields and flies to the closest one.

Prove that at most 5 airplanes will land at each fields.

**Problem Set 6 Level A Solution**

Take any two A and B gates, if they are connected by a walkway, then it is done. Otherwise at least 7 walkways serve A, and 7 walkways serve B. Since there are only 15 gates, so two of them should lead to a same gate and that is the one she needs to find

**Problem Set 6 Level B Solution**

A and B are the two towns with the smallest distance.

For the second question, nobody else visit A or B. Remove them and consider two towns C and D with smallest distance. There are two distinct persons, one walks from C to D and the other from D to C. Continues this until we have only one town left.

This is the one nobody visits.

**Problem Set 6 Level C Solution**

If A is the airfield receiving 6 airplanes, then there exists two airfields that form an at most 60 degrees angle with A. Let them be B and C. In triangle ABC, the sum of two angles B and C are at least 120 degree, and because none of the distances are the same, so one of the angles is larger than 60 degrees, let say that is is the angle C. In that case AB is larger than BC, which is impossible for the plane at B landed at A instead of C.

# Problems of the Week

**Level A**

### Problem 1.A

There are 12 distinct points on circle. How many distinct convex polygons can be constructed using these points as vertices?

**Solution**

By selecting any set of at least three vertices, it is possible to construct a distinct polygon. Therefore the number of possible polygons:

$$S = \binom{12}{3} + \binom{12}{4} + \cdots + \binom{12}{12}$$

According to the Binomial Theorem:

$$\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \cdots + \binom{12}{12} = 2^{12}$$

Hence $S = 2^{12} - \left(\binom{12}{0} + \binom{12}{1} + \binom{12}{2}\right) = 4017$

**Problem 2.A**

Note that the expression $x^6 - 1$ can be factored in two different ways as $(x^2 - 1)(x^4 + x^2 + 1)$ and $(x^3 - 1)(x^3 + 1)$.

Factor the expression $x^{15} - 1$ in two different ways.

Factor the expression $x^{10} + x^5 + 1$.

**Solution**

$$x^{15} - 1 = (x^3)^5 - 1 = (x^3 - 1)\left((x^3)^4 + (x^3)^3 + (x^3)^2 + (x^3)^1 + 1\right)$$
$$= (x^3 - 1)(x^{12} + x^9 + x^6 + x^3 + 1)$$
$$= (x - 1)(x^2 + x + 1)(x^{12} + x^9 + x^6 + x^3 + 1)$$
$$x^{15} - 1 = (x^5)^3 - 1 = (x^5 - 1)\left((x^5)^2 + (x^5)^1 + 1\right)$$
$$= (x^5 - 1)(x^{10} + x^5 + 1)$$
$$= (x - 1)(x^4 + x^3 + x^2 + 1)(x^{10} + x^5 + 1)$$

For the second question, from the first question we have:

$$(x^2 + x + 1)(x^{12} + x^9 + x^6 + x^3 + 1)$$
$$= (x^4 + x^3 + x^2 + 1)(x^{10} + x^5 + 1)$$

So the task here is to factor $x^{12} + x^9 + x^6 + x^3 + 1$ into a product which has $x^4 + x^3 + x^2 + 1$ as a factor.

$$\begin{aligned}
x^{12} + x^{11} + x^{10} + x^9 + x^8 &= +x^8(x^4 + x^3 + x^2 + x + 1)\\
-x^{11} - x^{10} - x^9 - x^8 - x^7 &= -x^7(x^4 + x^3 + x^2 + x + 1)\\
x^9 + x^8 + x^7 + x^6 + x^5 &= +x^5(x^4 + x^3 + x^2 + x + 1)\\
-x^8 - x^7 - x^6 - x^5 - x^4 &= -x^4(x^4 + x^3 + x^2 + x + 1)\\
x^7 + x^6 + x^5 + x^4 + x^3 &= +x^3(x^4 + x^3 + x^2 + x + 1)\\
-x^5 - x^4 - x^3 - x^2 - x^1 &= -x^1(x^4 + x^3 + x^2 + x + 1)\\
x^4 + x^3 + x^2 + x^1 + 1 &= +x^4 + x^3 + x^2 + x + 1
\end{aligned}$$

By summing up both side:

$$x^{12} + x^9 + x^6 + x^3 + 1$$
$$= \left(x^4 + x^3 + x^2 + x + 1\right)\left(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1\right)$$

Hence $x^{10} + x^5 + 1$

$$= \left(x^2 + x + 1\right)\left(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1\right)$$

**Problem 3.A**

Find all 3−digit number $\overline{abc}$ such that: $\overline{abc} = a! + b! + c!$

**Solution**

Since $7! = 5040 > 999$, so any of $a, b, c$ should not be larger than 6. If one of them is 6, then $a$ is at least 7 because $6! = 720$. So none of them is more than 5.

In addition since $4! + 4! + 4! = 72 < 100$ so at least one is 5.

Because $5! + 5! + 5! = 360 < 500$, so $a$ cannot be more than 3.

Since $3! + 5! + 5! = 246$ so $a$ cannot be more than 2.

Case $a = 1$:

If $b = 5$, $\overline{15c} = 121 + c! \Rightarrow 29 = c! - c$, no such $0 \leq c \leq 5$.

If $c = 5$, $\overline{1b5} = 121 + b! \Rightarrow 10b = 16 + b! \Rightarrow b = 4$.

Case $a = 2$:

If $b = 5$, $\overline{25c} = 122 + c! \Rightarrow 128 = c! - c$, impossible.

If $c = 5$, $\overline{2b5} = 122 + b! \Rightarrow 83 = b! - 10b$, this is impossible.

The answer is 145.

**Problem 4.A**

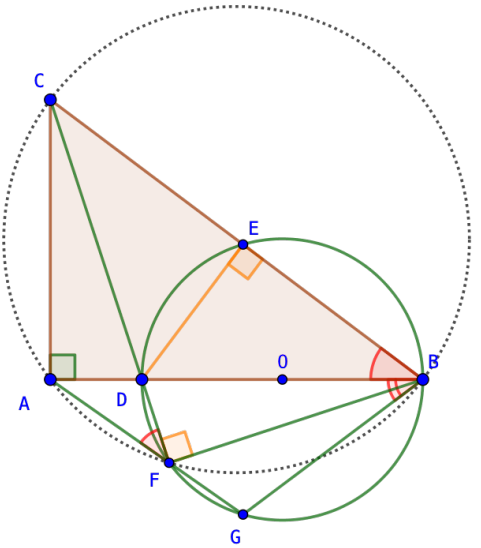
$ABC$ is a right triangle at $A$ and $AB = 4$, $AC = 3$. Point $D$ is on $AB$ such that $AD < DB$. The circle $\omega$ with diameter $BD$ intersects $CB$ at $E$. The line through $CD$ intersects the circle $\omega$ at $F$. $BF = 3$.

Prove that there is a circle through $A$, $C$, $E$, $D$.

Find $BC$ and the area of $BFC$.

The line through $AF$ intersects the circle $\omega$ at $G$. Prove that $BA$ bisects the angle $\angle CBG$.

**Solution**




Since both angle $\angle DAC = \angle DEC = 90^\circ$, so $A$, $C$, $E$, $D$ are on the circle with diameter $CD$.

$BC = \sqrt{AB^2 + AC^2} = 5$.

Since $BF = CA$, so $\triangle BFC \cong \triangle CAB$, so $S_{BFC} = 6$.

Because $A$, $C$, $E$, $D$ are on the circle with diameter $CD$, so both angles $\angle ABC$, $\angle AFC$ subtends arc $AC$, thus $\angle CBA = \angle AFC$.

Furthermore $B$, $G$, $F$, $D$ are on the circle with diameter $BD$, so $\angle DBG = 180^\circ - \angle DFG = \angle AFD$, $\angle CBA = \angle DBG = \angle ABG$, so $BA$ bisects the angle $\angle CBG$.

## Level B

### Problem 1.B

There are 15 empty boxes. At each step, some boxes are chosen and some marbles are put into them such that the number of marbles put into a box is a power of 2 and no any two boxes receive the same number of marble (within that step).

Find the smallest $k$ such that after $k$, all boxes have the same number of marbles.

**Solution**

Assume that after $k$ steps, each box has $n$ marbles. The total number of marbles is $15n$. Let $2^{m_i}$ be the maximum number of marbles put into some boxes at step $i$, $0 \le i \le k$. Let $m$ be the maximum of $\{m_i\}$. So at each step the number of marbles put into all of the boxes is not more than $2^m + 2^{m-1} + \cdots + 2^1 + 2^0 = 2^{m+1} - 1$. After $k$ steps:

$$15n \le k(2^{m+1} - 1) < k \cdot 2^{m+1}$$

Since $n \ge 2^m$:

$$15 \cdot 2^m \le 15n < k \cdot 2^{m+1} \Rightarrow k > 7.5 \Rightarrow k \ge 8$$

We show now that it can be done in 8 steps, below is the sequence of the numbers marbles put into each box at each step

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | $2^2$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ | $2^7$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 2 | $2^2$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ | 0 | $2^7$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | $2^2$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ | 0 | 0 | 0 | $2^7$ | 0 | 0 | 0 | 0 | 0 |
| $2^2$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ | 0 | 0 | 0 | 0 | 0 | $2^7$ | 0 | 0 | 0 | 0 |
| $2^3$ | $2^4$ | $2^5$ | $2^6$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $2^7$ | 0 | 0 | 0 |
| $2^4$ | $2^5$ | $2^6$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $2^7$ | 0 | 0 |
| $2^5$ | $2^6$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $2^7$ | 0 |
| $2^6$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $2^7$ |
| $2^7$ | $2^7$ | $2^7$ | $2^7$ | $2^7$ | $2^7$ | $2^7$ | $2^7$ | $2^7$ | $2^7$ | $2^7$ | $2^7$ | $2^7$ | $2^7$ | $2^7$ |

Note that the sum of all marbles in each box now is $2^7$.

**Problem 2.B**

$x$, $y$ are positive real numbers and $5(x-y)^2 \le x^2+y^2$.

Prove that $\frac{1}{2} \le \frac{x}{y} \le 2$.

**Solution**

$5(x-y)^2 \le x^2+y^2 \Leftrightarrow 4x^2-10xy+4y^2 \le 0 \Leftrightarrow 2x^2-5xy+2y^2 \le 0$

$2x^2-5xy+2y^2 = 2x^2-xy-4xy+2y^2 = x(2x-y)-2y(2x-y)$

$= (x-2y)(2x-y)$

Hence $(x-2y)(2x-y) \le 0$.

We have two cases:

Case $x-2y \ge 0$, $2x-y \le 0$: $2y \le x \le \frac{y}{2}$, this is impossible.

Case $x-2y \le 0$, $2x-y \ge 0$: then $\frac{1}{2} \le \frac{x}{y} \le 2$.

**Problem 3.B**

Let $n$ be a positive integer.

Prove that $n^2+3n+11$ is not divisible by 49.

Find all of positive integers $(a, b)$ such that: $a^2+9b^2 = 6(a+6)$

**Solution**

Assume that there is an $n$ positive integer such that $n^2+3n+11$ is divisible by 49.

Therefore $4(n^2+3n+11) = (2n+3)^2+35$ is divisible by 49. Since 35 is divisible by 7, so $2n+3$ is also divisible by 7. Thus $(2n+3)^2$ is divisible by 49.

$4(n^2+3n+11) = (2n+3)^2+35$ is divisible by 49 and $(2n+3)^2$ is divisible by 49, so 35 is divisible by 49, which is impossible. Hence, there is no $n$ positive integer such that $n^2+3n+11$ is divisible by 49.

For the second question:

$a^2+9b^2 = 6(a+6) \Leftrightarrow a^2-6a+9 = 45-9b^2$

$\Leftrightarrow (a-3)^2 = 9(5-b^2) \ge 0$. So $b = 1$ or $b = 2$.

There are two answers (9, 1), (6, 2).

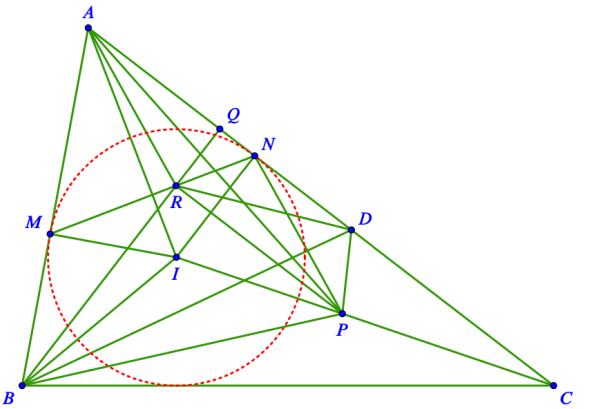
**Problem 4.B**

$ABC$ is an acute triangle, i.e. all of its angles are acute. $AB < BC$. The incircle centred at $I$ tangent with $AB$, $AC$ at $M$, $N$. $BQ$ is the altitude from $B$ to $AC$ and intersects $AC$ at $Q$. $D$ is point on $QC$ such that $AQ = QD$. $MN$ intersects $BQ$ at $R$. $P$ is the centre of the incircle of the triangle $BCD$. Prove that:

$$\angle BMR = 90^\circ + \frac{\angle BAC}{2} = \angle BIC,$$

$$\angle BRM = \frac{1}{2}\angle BAC \text{ and } \triangle BMR \sim \triangle BPI.$$

**Solution**




Since $AM$, $AN$ tangent with the circle centred at $I$: $AM = AN$, therefore $\triangle AMN$ is isosceles at $A$.

$$\angle BMR = 180^\circ - \angle AMN = 180^\circ - \frac{(180^\circ - \angle BAC)}{2} = 90^\circ + \frac{\angle BAC}{2}$$

$$\angle BIC = 180^\circ - \angle IBC - \angle ICB = 180^\circ - \frac{(\angle ABC + \angle BCA)}{2}$$

$= 90^\circ + \dfrac{\angle BAC}{2}$. Hence: $\angle BMR = \angle BIC$.

Because $QA = QD$ and $BQ \perp AD$ so $\triangle ABD$ is isosceles at $B$:
$\angle ABR = \angle DBR = 90^\circ - \angle BAC$, $\angle BRM = 180^\circ - \angle BMR - \angle MBR$

$$= 180^\circ - (90^\circ + \frac{\angle BAC}{2}) - (90^\circ - \angle BAC) = \frac{\angle BAC}{2}$$

Because $C$, $P$, $I$ are collinear:

$$\angle BPI = \angle PBC + \angle PCB = \frac{\angle DBC}{2} + \frac{\angle DCB}{2} = \frac{\angle ADB}{2} = \frac{\angle BAC}{2}$$

Hence $\triangle BMR \sim \triangle BPI$.

## Problem 1.C

A group of $n$ chickens are having a tournament where every chicken plays one game against every other competing chicken (life as a chicken gets boring). Every match ends in a win or loss for the players (i.e. there are no ties), the winner being the chicken who pecks the other first. We say a chicken $u$ virtually pecks another chicken $v$ if $u$ pecks some other chicken $w$ who pecked $v$. A chicken is called King if, when the tournament ends, that chicken either pecked or virtually pecked every other chicken in the tournament.

Prove that every tournament must have at least one King.

A group of $n$ people play a round-robin tournament. Each game ends in either a win or loss (there is no ties). Show that it is possible to label the player $P_1, P_2, \cdots, P_n$ in such a way that $P_1$ defeated $P_2$, $P_2$ defeated $P_3$, …, $P_{n-1}$ defeated $P_n$.

**Solution**

Base case: For $n = 2$, the smallest reasonable tournament is one with 2 chickens. One must pec the other, this is the King chicken. Thus, any tournament with two chickens has a King.

Induction Hypothesis: Suppose that for $n = k$, any tournament of $k$ chickens has a King.

Induction Step: Consider a tournament with $k + 1$ chickens. The tournament with the first $k$ chickens should have a King. If the $k + 1$ chicken is pecked by the King, then that King is the King for the whole tournament. If the $k + 1$ chicken is pecked by some chicken that the King of the sub-tournament pecked, then it is virtually pecked by the King and hence the King is the King for the whole tournament. The remaining case is where the $k + 1$ chicken pecked the King of the sub-tournament by is not packed any other chicken from the sub-tournament. Then obviously the $k + 1$ chicken is the King for the whole tournament. By the principle of mathematical induction, the assumption is correct for all value of $n \geq 2$.

For the second question, a similar induction can be done by establishing an order for the first $n$ player of a sub-tournament $P_1, P_2, \cdots, P_n$ and insert the $P_{n+1}$ into the position on the left of $P_1$ if he defeats all the players $P_1, P_2, \cdots, P_n$, or right to the player $P_j$ whose has the largest index $j$ among the players who defeat $P_{n+1}$.

**Problem 2.C**

Show that for any 3 integer points chosen on the line, some pair of them average to an integer. Show that for any 5 points in the two-dimensional space $\mathbb{R}^2$with integer coordinates, there exists some pair of them such that the midpoint of the line joining the two points also has integer coordinates. How many points with integer coordinates do you need for the corresponding result in $\mathbb{R}^n$?

Suppose $A$ is a collection of subsets of the set $\{1, 2, \cdots, n\}$ with the property that any two sets in $A$ have a non-empty intersection. Show that $A$ has at most $2^{n-1}$ elements.

**Solution**

If we have 3 points, by the pigeonhole principle, at least two numbers must have the same parity, the average of these two is another integers.

By the pigeonhole principle, there are $2 \cdot 2 = 4$ possible parity combinations of (even, even), (even, odd), (odd, even), (odd, odd). By having 5 points, two of them must be in the same category, hence their midpoint must have all integer coordinates.

Similarly it can be proved for $2^n + 1$ points in $\mathbb{R}^n$.

For the second question: there are $2^n$ subset of $\{1, 2, \cdots, n\}$. Let create $2^{n-1}$ pairs of subsets by select one subset of $\{1, 2, \cdots, n\}$ with its complement. If $A$ has more than $2^{n-1}$ elements, then two elements of $A$ must form one of those $2^{n-1}$ pairs. However that is not possible since every two elements of $A$ has a non-empty intersection.

**Problem 3.C**

$x, y, z \in \mathbb{R}$ such that $x + y + z = 0$, $x^2 + y^2 + z^2 = 6$.

What is the maximal value of $|(x - y)(y - z)(z - x)|$?

*Hint:* $\big((x - y)(y - z)(z - x)\big)^2 = 27(1 - xy)(1 - yz)(1 - zx)$

Solve the system of equation:

$$\begin{cases} x^3 + y^3 = 4y^2 - 5y + 3x + 4 & (1) \\ 2y^3 + z^3 = 4z^2 - 5z + 6y + 6 & (2) \\ 3z^3 + x^3 = 4x^2 - 5x + 9z + 8 & (3) \end{cases}$$

*Hint:* (1) $\Leftrightarrow x^3 - 3x - 2 = -y^3 + 4y^2 - 5y + 2$. *Factor both sides.*

Let $\left(\sqrt{2017}\cdot x-\sqrt{2027}\right)^{2017}$

$=a_{2017}x^{2017}+a_{2016}x^{2016}+\cdots+a_1x+a_0$

Find the value of :

$(a_1+a_3+\cdots+a_{2017})^2-(a_0+a_2+\cdots+a_{2016})^2$

*Hint: investigate the given polynomial values at special values of x.*

**Solution**

$z=-(x+y)\Rightarrow x^2+y^2+z^2=x^2+2xy+y^2+x^2+y^2$

$\Rightarrow x^2+xy+y^2=3$

$(x-y)^2=x^2-2xy+y^2=3-xy-2xy=3-3xy$

Hence $\left((x-y)(y-z)(z-x)\right)^2=27(1-xy)(1-yz)(1-zx)$

Furthermore:

$$xy+yz+zx=\frac{(x+y+z)^2-(x^2+y^2+z^2)}{2}=-3$$

According to AM-GM inequality:

$$\sqrt[3]{(1-xy)(1-yz)(1-zx)}\leq\frac{(1-xy)+(1-yz)+(1-zx)}{3}$$

$$=\frac{3-(xy+yz+zx)}{3}=2$$

Therefore $\left((x-y)(y-z)(z-x)\right)^2\leq 27\cdot 8$

$\Rightarrow |(x-y)(y-z)(z-x)|\leq 6\sqrt{6}$

For the second question:

$$\begin{cases} x^3+y^3=4y^2-5y+3x+4 & (1)\\ 2y^3+z^3=4z^2-5z+6y+6 & (2)\\ 3z^3+x^3=4x^2-5x+9z+8 & (3)\end{cases}$$

can be written as:

$$\begin{cases} x^3-3x-2=-y^3+4y^2-5y+2\\ 2y^3-6y-4=-z^3+4z^2-5z+2\\ 3z^3-9z-6=-x^3+4x^2-5x+2\end{cases}$$

By factoring both sides:

$$\begin{cases} (x-2)(x+1)^2 = -(y-2)(y-1)^2 & (4) \\ 2(y-2)(y+1)^2 = -(z-2)(z-1)^2 & (5) \\ 3(z-2)(z+1)^2 = -(x-2)(x-1)^2 & (6) \end{cases}$$

If $x \neq 2$, $y \neq 2$, $z \neq 2$, by multiplying them together and canceling non-zero terms:

$$6\left((x+1)(y+1)(z+1)\right)^2 = -\left((x-1)(y-1)(z-1)\right)^2$$

Thus both sides are zero.

WLOG, let assume $x = 1$. (6) $\Rightarrow z = -1$.

From (4) & (5): $2 = (y-2)(y-1)^2$ and $(y-2)(y-1)^2 = 6$.

This is impossible.

If $x = 2$: if $y = 2$ then from (5) & (6): $z = 2$.

If $y \neq 2$, from (4) $y = 1$, from (6) $z = 2$ or $z = -1$.

In any case (5) cannot stand.

Hence there is a single solutions (2, 2, 2).

Let $P(x) = \left(\sqrt{2017} \cdot x - \sqrt{2027}\right)^{2017}$

$$P(x) = a_{2017}x^{2017} + \cdots + a_1 x + a_0$$

$$(a_1 + a_3 + \cdots + a_{2017})^2 - (a_0 + a_2 + \cdots + a_{2016})^2$$

$$= (a_1 + a_3 + \cdots + a_{2017} + a_0 + a_2 + \cdots + a_{2016})$$

$$\times (a_1 + a_3 + \cdots + a_{2017} - a_0 - a_2 - \cdots - a_{2016})$$

$$= P(1)\left[-P(-1)\right]$$

$$= -\left(\sqrt{2017} - \sqrt{2027}\right)^{2017}\left(-\sqrt{2017} - \sqrt{2027}\right)^{2017}$$

$$= -10^{2017}$$

**Problem 4.C**

Prove that:

$$16\sin^4\frac{\pi}{18}+8\sin^3\frac{\pi}{18}-12\sin^2\frac{\pi}{8}-4\sin\frac{\pi}{8}+1=0$$

*Hint: prove the identity* $\sin 3x = 3\sin x - 4\sin^3 x$.

$\alpha, \beta$ are two acute angles: $\alpha, \beta \in \left[0, \frac{\pi}{2}\right)$, such that:

$\sin(\alpha+\beta)-1=\sin^2\alpha-\cos^2\beta$

Prove that: $\alpha+\beta=\frac{\pi}{2}$.

*Hint: prove that* $\sin\alpha=\cos\beta$.

**Solution**

Since $\sin 3x = 3\sin x - 4\sin^3 x$, therefore:

$$3\sin\frac{\pi}{18}-4\sin^3\frac{\pi}{18}=\sin\frac{3\pi}{18}=\sin 30^\circ=\frac{1}{2}$$

Therefore $16\sin^4\frac{\pi}{18}+8\sin^3\frac{\pi}{18}-12\sin^2\frac{\pi}{8}-4\sin\frac{\pi}{8}$

$$=4\sin\frac{\pi}{8}\left(4\sin^3\frac{\pi}{18}-3\sin\frac{\pi}{18}\right)+2\left(4\sin^3\frac{\pi}{18}-3\sin\frac{\pi}{18}\right)+2\sin\frac{\pi}{18}$$

$$=-2\sin\frac{\pi}{8}-1+2\sin\frac{\pi}{18}=-1$$

So: $16\sin^4\frac{\pi}{18}+8\sin^3\frac{\pi}{18}-12\sin^2\frac{\pi}{8}-4\sin\frac{\pi}{8}+1=0$

For the second question: $\sin(\alpha+\beta)-1=\sin^2\alpha-\cos^2\beta$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$, $1-\cos^2\beta=\sin^2\beta$:

$\sin\alpha\cos\beta+\cos\alpha\sin\beta=\sin^2\alpha+\sin^2\beta$

$\sin\beta(\cos\alpha-\sin\beta)=\sin\alpha(\sin\alpha-\cos\beta)$

If $\sin\alpha<\cos\beta$, the right hand side is negative: $\cos\alpha<\sin\beta$.

But $\sin^2\alpha+\cos^2\alpha=\sin^2\beta+\cos^2\beta=1$, so it is not possible.

Similarly $\sin\alpha>\cos\beta$ also leads to contradiction.

Hence $\sin\alpha=\cos\beta$.

Easy to see that $\alpha+\beta=\frac{\pi}{2}$.

# Chapter 15

# September

# Monthly Team Contest

**Problem Set 1 Level A**

A number is called *magical* if it can be written as *the sum of two* ***odd*** *composite integers.* For example 39 is not a *magical* number but $18 = 9 + 9$ is. The first forty positive integers 1, 2, 3, $\cdots$, 40 are written on a board. Catherine and her friend play a two-player game. They took turns to remove *a single magical number* from the board. Who can not move loses the game. Can you suggest a **winning strategy** for Catherine?

*Note: A composite number is a positive integer that can be formed by multiplying two smaller positive integers (that are larger than 1). Equivalently, it is a positive integer that has at least one divisor other than 1 and itself.*

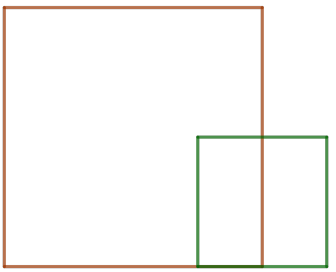
**Problem Set 1 Level B**

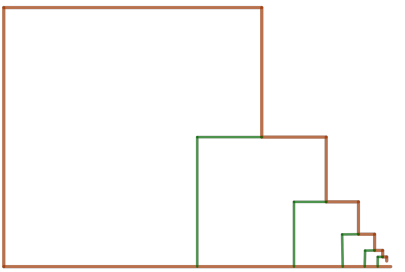
Vu-Lam is one of the volunteers to help out at a school event attending by students grade 5, 6, 7, and 8. He has to sit 11 students at a round table so that among *any five successive students* there are *representatives of all four grades.* Can he do that? If yes, show an example, if not, prove your reason.

**Bonus**: Can he do that if he has to sit 12 students, 3 students from each grade? If yes, show an example, if not, prove your reason.

**Problem Set 1 Level C**

Lilian starts with a square side length 1. She draws a square with side length 1/2 so that it bisected by the side of the original square, as shown below in the figure on the left.





This process is repeated with a square of side length 1/4 is bisected by the square of side length 1/4, and so on without end as shown on the right. What would be the area of the figure on the right?

**Problem Set 1 Level A Solution**

A magical number is even since it is the sum of two odd numbers. Furthermore a composite odd number is an odd number that is neither 1 nor prime. Easy to see that the first one is 9 and the one that is less than $40-9=31$ are: $9, 15, 21, 25, 27$. Direct checking $18=9+9$, $24=9+15$, $30=9+21$, $34=9+25$, $36=9+27$, $40=15+25$ show there are only 6 magical numbers. Hence, Catherine should go second.

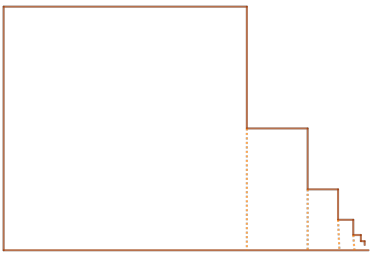
**Problem Set 1 Level B Solution**

If each grade can has at least three students then there are at least $3\cdot 4=12>11$ students. So there is a grade with not more than 2 students. It is not possible to sit them such that among any five successive students there is at least one representative of this grade. It is easy to sit 12 students, 3 from each grades such that among any five successive students there are representatives of all grades.

$$5, 6, 7, 8, 5, 6, 7, 8, 5, 6, 7, 8$$

**Problem Set 1 Level C Solution**

As seen from the diagram, half the area of the second square is added to the entire figure, as is half the area of the third square, etc.



Half of the area of the second square is $\frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{2}\right)=\frac{1}{2^3}$, half of the area of the third square is $\frac{1}{2}\left(\frac{1}{4}\cdot\frac{1}{4}\right)=\frac{1}{2^5}$, half of the area of the third square is $\frac{1}{2}\left(\frac{1}{8}\cdot\frac{1}{8}\right)=\frac{1}{2^7}$, and so on. So the area of the figure is

$$S=1+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^7}+\cdots$$

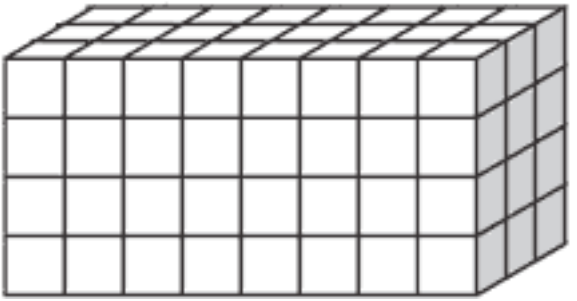
$$S-\frac{S}{2^2}=1+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^7}+\cdots-\frac{1}{2^2}-\frac{1}{2^5}-\frac{1}{2^7}-\cdots\Rightarrow S=\frac{7}{6}.$$

**Problem Set 2 Level A**

Linh Chi wants to place 4 pawns, 4 knights, 4 bishops, and 4 rooks on a $4 \times 4$ board such that *no row, no column, and none of the two main diagonals* would contain *two same chess pieces*. Can she do that? If yes, show an example, if not, prove your reason.

**Problem Set 2 Level B**

A rectangular prism is a 3-dimensional object which has six faces that are rectangles. Below is one example.



Victoria has a rectangular prism which is composed of unit cubes. She paints the prism such that the outside faces of the prism are painted blue and *the seven unit cubes in the interior* remain entirely unpainted. **How many unit cubes** have exactly one painted face?

**Problem Set 2 Level C**

Albert works at the observatory to watch the Edziza volcano. If he travels to the top of the volcano he has to take a road and then a passway, each takes 4 hours.

On the top there are two craters. The big crater keeps repeating an eruption pattern: it erupts for 1 hour and then it is silent for 17 hours; and then the pattern repeats.

The small crater's cycle is different: it erupts for 1 hour and then it is silent for 9 hours, then it erupts for 1 hour, then is silent for 17 hours.; and then the pattern repeats. The eruption of the big crater is so dangerous that it is not advised to take any of the passway or the road. However the eruption of the small crater is only dangerous while taking the passway. At noon Albert saw that both the craters *simultaneously started to erupt*. Will it be ever possible for him to **reach the top of Edziza** without risking his life?

**Problem Set 2 Level A Solution**

Below is one example.

**Problem Set 2 Level B Solution**

Since the interior is 7 unit cubes, so the prism is $9 \times 3 \times 3$. Note that all cubes at the corners or along the edges have at least two faces painted blue. Only the one on the face but not at any edge has one faces painted blue. So there are $4 \times 7 + 2 = 30$ cubes:

- 7 on each face, along the 7 unpainted interior cubes
- 2 at the two ends of the 7 unpainted interior cubes

**Problem Set 2 Level C Solution**

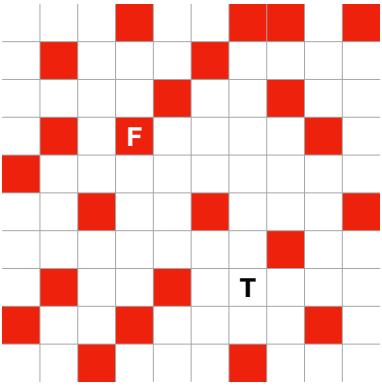
The trip along the road and the passway (tour and retour) takes 16 hours. Hence, if one departs immediately after the first crater finished eruption, it will not be dangerous. The trip along the passway (there and back) takes 8 hours. Hence, if one departs immediately after the second crater finished eruption, it will not be dangerous.

To ensure safety, it suffices to Albert to find a moment of time when by the time he takes the road the first crater stopped to erupt, and 4 hours later, by the time he takes the passway, the second crater stopped to erupt. Let us find such a moment.

The first crater starts to erupt at the beginning of the 1-st, 19-th, 37-th, etc., hours. The second crater starts to erupt at the beginning of the 1-st, 11-th, 21-th, 31-th, 41-th, etc., hours. Hence, Albert can depart at the beginning of the 38-th hour.

**Problem Set 3 Level A**

On an infinite grid there are some squares coloured red and some white. A frog jumps between red squares and a toad on white ones. Each jump can be made over any distance, vertically or horizontally. Below is one example where the frog (F) and the toad (T) sit.




Can Clemence, well-versed in amphibian tongues, help the frog and the toad so that after *at most three jumps* they would **be side-by-side**?

**Problem Set 3 Level B**

Lily places the numbers 1 and $-1$ at the vertices of a cube, one number per vertex. So there are 8 numbers at the vertices. For each face of the cube she multiplies all four numbers at four vertices of the face and write the result in the centre of the face. So now there are 6 numbers at the centres of the faces.

Her brother Michael look at the figure and said that the sum of all 14 numbers is 7. **Is that possible**? If yes, show an example. If not, prove your answer. **Bonus**: Is that possible that that the sum of all 14 numbers is 0. If yes, show an example. If not, prove your answer. *Hint: Assume that at all vertices of the cube, Lily wrote only numbers 1. How the sum changes if she changes one number at a single vertex from* 1 *to* $-1$.

**Problem Set 3 Level C**

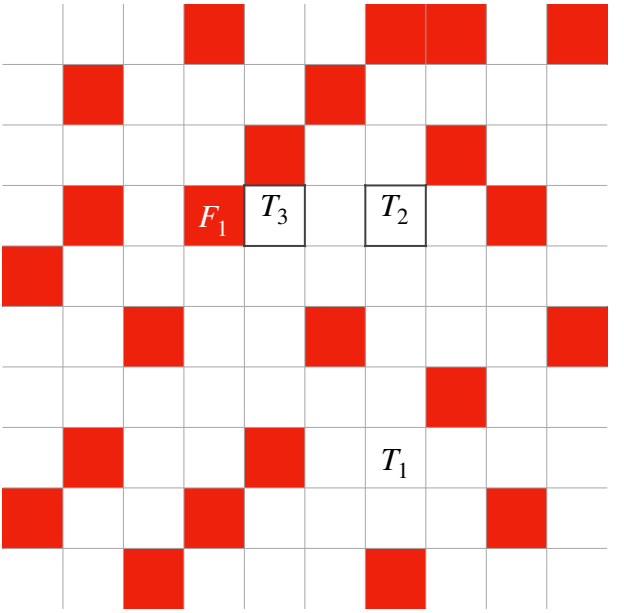
Erica draws and cuts 4 identical right triangles from a paper sheet. In one move she can cut one of the triangles into two right triangles by cutting *along the height from the right angle to the hypotenuse.* This way she gets one more right triangles.

Is that true that at all time she has **two identical triangles**?

**Problem Set 3 Level A Solution**

The square at the intersection point of the horizontal, where the frog stands, and the vertical, where the toad rests, is either red or white.

If it is red, the grasshopper should jump on it and if the square is white the toad jumps there. So the two amphibians find themselves either on the same horizontal or on the same vertical. Any of them must have two adjacent squares of different colours, the red square being for the frog and the white for the flea. Thus, they will be side by side after 3 jumps. In the example below, the toad jumps twice $T_1 \to T_2 \to T_3$.




**Problem Set 3 Level B Solution**

Following the hint, if Lily wrote only numbers 1, then the sum is 14. If she changes only one number at a single vertex from 1 to $-1$, then the sum is changed by 8, meaning that it is 6 now.

By observing this pattern, each time if Lily changed only one number then the sum changes by a multiple of 4. If she completes the changes so that she can have a desired number of $-1$ numbers at the vertices that she wants, the sum is still changed by a multiple of 4.

At the beginning, it was 14.

It doesn't matter where the numbers written, it cannot be 0 or 7.

**Problem Set 3 Level C Solution**

Assume the contrary and consider the least number $n$ such that having made $n$ cuts we can we can obtain pairwise non-equal triangles from a foursome of identical right triangles.

Observe for the future that the order in which we cut the triangles is irrelevant (in the sense that the final result does not depend on the order of cutting).

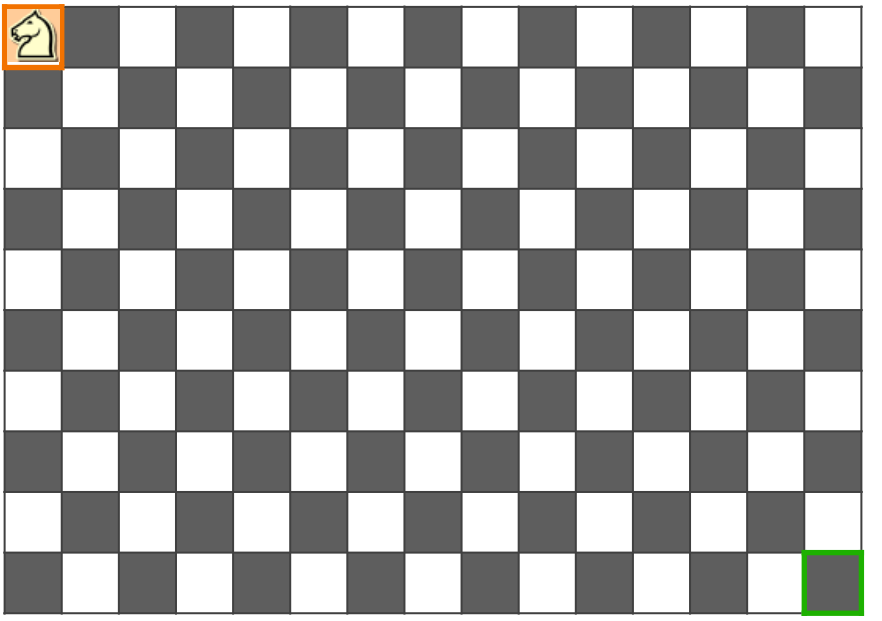
Since initially we had four equal triangles, we have to cut three of them. Assume that the first three cuts go across these three triangles. As a result we get two triples of identical triangles. In each triple we have to cut two of the triangles. Having done the cuts we again get on our hands a foursome of equal triangles, as is easy to see.

We have $n-7$ cuts at our disposal to cut the triangles into pairwise non-equal but this contradicts to the minimality of $n$.

The contradiction implies that at every move there are two equal triangles.

**Problem Set 4 Level A**

Bill has to move the knight from the top left corner of the $15 \times 10$ board to the lower right corner as show in the figure below.



What is the **minimum number** of moves should he use? Can you show a sequence of move with that number of moves?

**Problem Set 4 Level B**

Ha-Anh Le wrote down a row of six 3−digit numbers on a board. These six numbers form an *arithmetic sequence*. While she was absent for a few minutes, her playful younger sister came and erased some digits, leaving the following on the board:

$$\underbrace{1\,_\,_}\,,\ \underbrace{_\,_\,9}\,,\ \underbrace{2\,_\,2}\,,\ \underbrace{_\,6\,_}\,,\ \underbrace{2\,_\,_}\,,\ \underbrace{_\,3\,_}$$

Can you help her to figure out the **next number in the sequence**? **Bonus**: can you restore the whole sequence?

*Note: An arithmetic sequence is a sequence of numbers such that the difference of any two successive members of the sequence is a constant, which is also called as a common difference. For example* $2, 4, 6, 8, 10, \cdots$ *is an arithmetic sequence with the common difference 2.*

**Problem Set 4 Level C**

Bianca investigates *good triple* $(n, x, y)$ of positive integers:

$$2^n = 7x^2 + y^2$$

She knows that by using the identities:

$$7\left(\frac{x+y}{2}\right)^2 + \left(\frac{|7x-y|}{2}\right)^2 = 14x^2 + 2y^2 \qquad (*)$$

$$7\left(\frac{|x-y|}{2}\right)^2 + \left(\frac{7x+y}{2}\right)^2 = 14x^2 + 2y^2 \qquad (*)$$

By applying the identity (*) for the *good triple* (3, 1, 1) she obtains another *good triple* (4, 1, 3)*:*

$$2^4 = 7 \cdot 1^2 + 3^2 = 7\left(\frac{1+1}{2}\right)^2 + \left(\frac{|7 \cdot 1 - 1|}{2}\right)^2$$

By applying the identity (*) or (**) for the *good triple* (4, 1, 3) she obtains another *good triple* (5, 1, 5)*:*

$$2^5 = 7 \cdot 1^2 + 5^2 = 7\left(\frac{|1-3|}{2}\right)^2 + \left(\frac{7 \cdot 1 + 1}{2}\right)^2$$

$$2^5 = 7 \cdot 2^2 + 2^2 = 7\left(\frac{1+3}{2}\right)^2 + \left(\frac{|7 \cdot 1 - 3|}{2}\right)^2$$

Can you help Bianca to **find** $x, y$ **are positive integers for a *good triple*** $(10, x, y)$, such that:

$$2^{10} = 7x^2 + y^2$$

**Problem Set 4 Level A Solution**

The minimum number of moves is 9: 9 down and 14 to the right.

**Problem Set 4 Level B Solution**

$$\underbrace{1__}_{1^{st}},\ \underbrace{__9}_{2^{nd}},\ \underbrace{2_2}_{3^{rd}},\ \underbrace{_6_}_{4^{th}},\ \underbrace{2__}_{5^{th}},\ \underbrace{_3_}_{6^{th}}$$

By looking at the unit digit of the $2^{nd}$ and $3^{rd}$ term, we see that the unit digit of the common difference is 3. By looking at the hundreds digit of the $3^{rd}$ and $5^{th}$ term, we see that the hundreds digit of the $4^{th}$ is 2. From these, the $4^{th}$ term is 265, the $5^{th}$ is 2_8.

The $6^{th}$ is 331 otherwise the common difference is too large. Hence the common difference half of the difference between the $6^{th}$ and $4^{th}$, which is $\dfrac{331-265}{2} = 33$.

Hence the next number in the sequence is 364.

The whole sequence is 166, 199, 232, 265, 298, 331, 364.

**Problem Set 4 Level C Solution**

Following the identities that:

$$7\left(\frac{|1-5|}{2}\right)^2 + \left(\frac{7\cdot 1+5}{2}\right)^2 = 7\cdot 2^2 + 6^2 = 2^6$$

$$7\left(\frac{2+2}{2}\right)^2 + \left(\frac{|7\cdot 2-2|}{2}\right)^2 = 7\cdot 2^2 + 6^2 = 2^6$$

So similarly:

$$7\left(\frac{|2-6|}{2}\right)^2 + \left(\frac{7\cdot 2+6}{2}\right)^2 = 7\cdot 2^2 + 10^2 = 2^7$$

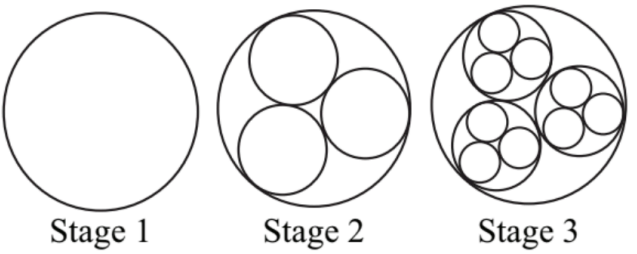
$$7\left(\frac{|2-10|}{2}\right)^2 + \left(\frac{7\cdot 2+10}{2}\right)^2 = 7\cdot 4^2 + 12^2 = 2^8$$

$$7\left(\frac{|4-12|}{2}\right)^2 + \left(\frac{7\cdot 4+12}{2}\right)^2 = 7\cdot 4^2 + 20^2 = 2^9$$

$$7\left(\frac{|4-20|}{2}\right)^2 + \left(\frac{7\cdot 4+20}{2}\right)^2 = 7\cdot 8^2 + 24^2 = 2^{10}$$

**Problem Set 5 Level A**

Tiana draws a fractal. The diagram below shows three stages of her drawing.




What is **number of circles** are there in the tenth stage?

**Problem Set 5 Level B**

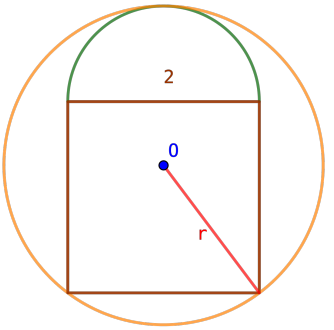
Huy is used to spend one dollar a day for a cookie and a candy at the school cafeteria. At the new school year, all prices go up by $\%20$, but he can still buy half of a cookie and a candy for the same dollar.

Will a dollar still be enough for him to buy **at least a candy** if the school cafeteria decides to increase all prices by $\%20$?

**Bonus**: if the school cafeteria decides to increase all prices by $\%20$, at least how many whole cookies can he buy?

**Problem Set 5 Level C**

Victor places a semi-circle above a square to form a silo-shaped figure. The diameter of the semicircle is 2 and coincides with the top of the square. He draws the circle contains the silo-shaped figure below.




What is the **radius** $r$ of the circle? **Bonus**: if the diameter of the semicircle is given as $a$, what is $r$?

**Problem Set 5 Level A Solution**

In stage 1: only 1, stage 2: $1 + 3 = 4$, stage 3: $1 + 3 + 9 = 13$.

In stage 4: each of the smaller circles would have 3 circles inscribed in each of the 9 medium sized circles of Stage 3. That's a total of 27 more circles, so there are $1 + 3 + 9 + 27 = 40$. Easy to see that the number of circles in each stage has a pattern: $1 + 3^1 + 3^2 + \cdots + 3^{n-1}$

So the number of circles at stage 10 is:

$$1 + 3^1 + 3^2 + \cdots + 3^9 = \frac{3^{10} - 1}{3 - 1} = \frac{3^{10} - 1}{2} = 29524.$$

**Problem Set 5 Level B Solution**

Let $a$, $b$ be the prices of a cookie and a candy. He used to spend for a cookie and a candy: $a + b = 1 \Rightarrow a = 1 - b$

When the price went up, then:

$$1 = \left(\frac{a}{2} + b\right) \times 1.2 \Rightarrow \frac{1}{1.2} = \left(\frac{1-b}{2} + b\right) \Rightarrow \frac{1}{1.2} = \frac{1+b}{2}$$

Thus, $b = \dfrac{2}{1.2} - 1 = \dfrac{0.8}{1.2}$. $a = 1 - b = \dfrac{0.4}{1.2}$.

After the next inflation, a candy is $1.2^2 \cdot \dfrac{0.8}{1.2} = 0.96$, and a cookie is $1.2^2 \cdot \dfrac{0.4}{1.2} = 0.48$. He can get a candy or get two cookies.

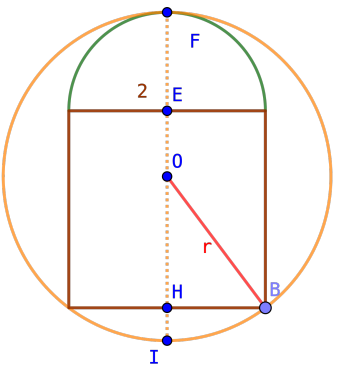
**Problem Set 5 Level C Solution**

$HI = 2r - EH - EF = 2r - 3$

$OH = r - HI = r - (2r - 3) = 3 - r$

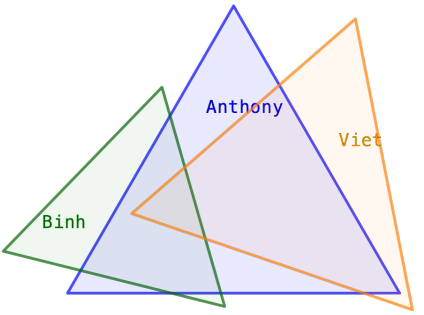
$OB^2 = OH^2 + HB^2 \Rightarrow r^2 = (3 - r)^2 + 1$

$\Rightarrow 10 - 6r = 0 \Rightarrow r = \dfrac{5}{3}$



**Problem Set 6 Level A**

Anthony draws an equilateral triangle. He challenges Binh and Viet, each to draw one equilateral triangle with smaller side length such that the two equilateral triangles can cover his triangle. Below shows one unsuccessful attempt. Who will win? Why?




**Problem Set 6 Level B**

2020 speakers from various nations came to the Festival of Languages. Every speaker could speak several languages and it was known that any three speakers could have a common conversation without assistance. (A speaker could serve as a translator for a pair of his colleagues.) Prove that it was possible to distribute all speakers in 1010 rooms, so that in every room there were 2 speakers and they can understand each other.

**Problem Set 6 Level C**

Cindy had a colony of 1000 bacteria lived in a beaker the laboratory. She left the lab at 4:00 PM on Monday.

Immediately, a virus got into the beaker. In the first minute the virus destroyed one bacterium and immediately after that *both the virus and the remaining bacteria split in halves*. In the second minute the two viruses destroyed two bacteria, and then the viruses and the remaining bacteria again split in halves. In the third minute the four viruses destroyed four bacteria, and then the viruses and the remaining bacteria again split in halves, and so on. When Cindy returned to the lab to continue at 9:00 AM on Tuesday, she saw the problem and went to save the remaining bacteria.

**How many bacteria were there** at that time? **Bonus**: How many bacteria were destroyed by the virus in total?

**Problem Set 6 Level A Solution**

If it were possible, then two of three vertices of Anthony's equilateral triangles must be inside one of the equilateral triangles, which is impossible.

**Problem Set 6 Level B Solution**

Of any three speakers, there are two who can always speak with one another. Dismiss them and go ahead with the procedure for the remaining speakers. After 1008 such operations there are only 4 speakers left, call them A, B, C, D. If any two of them can understand with each other, let distribute them in pairs at random. If there exist two speakers A and B, who can not understand each other, this means that C can speak with both A and B: otherwise the triple A, B, C could not communicate; similarly, D can speak with A and B. So, let A share room with C and let B share room with D.

**Problem Set 6 Level C Solution**

Let $n$ be the number of bacteria at 4:00 PM, Monday (Time Zero). The number of bacteria $k$ minutes after Time Zero will be $2^k(n-k)$.

| Time | Bacteria | Virus |
|---|---|---|
| 0 | $n$ | 1 |
| 1 | $2(n-1)$ | 2 |
| 2 | $2(2n-2-2) = 4(n-2)$ | 4 |
| 3 | $2(4n-8-4) = 8(n-3)$ | 8 |
| … | | |
| $k$ | $2(2^{k-1}(n-(k-1)) - 2^{k-1}) = 2^k(n-k)$ | $2^k$ |

The last bacteria would be destroyed 1000 minutes after 4:00 PM, Monday. Since From 4:00 PM Monday to 9:00 AM Tuesday there were 16 hours or 960 minutes.

The number of bacteria destroyed is equal to the sum of the virus at each minutes, up to the last minute, so it is:

$$1 + 2 + 4 + \cdots + 2^{959} = 2^{960} - 1.$$

# Problems of the Month

**Level A**

**Problem 1.A**

You are given a convex quadrilateral. Can you tile a plane, without gaps or overlapping with the tiles same as the quadrilateral?

**Problem 2.A**

All integers are written in a row: $1234567891911121314\cdots$

Find the digit $500000^{th}$ from the left.

**Problem 3.A**

Find all the positive integer ordered pairs $(x, y)$ such that:

$0 < x, y \leq 1000$ and $x^2 + y^2 \vdots 49$.

**Problem 4.A**

Given a square of side 1, find the set of points such that for any point the sum of distances from the point to the sides of this square (or their extensions) equals 4.

**Problem 5.A**

Which is larger $A = \dfrac{1}{6}$ or $B = \dfrac{1}{3033} + \dfrac{1}{3036} + \cdots + \dfrac{1}{6060}$?

**Problem 6.A**

Construct a triangle $ABC$ given the length of the altitude from $A$ is 4, the length of the median from $A$ to the midpoint of $BC$ 5 , and the radius of the circumscribed circle of the triangle $ABC$ is 6.

**Problem 7.A**

In every square of a $8 \times 10$ board a number 1 or $-1$ is written. The number of 1's is at least two, the number of $-1$'s is also at least two. Is it possible to choose two rows and two columns such that the sum of the four numbers in the squares at their intersections is equal to 0.

**Problem 8.A**

In city Rush-People, the bus routes are organized to satisfy three conditions: a bus-stop can be reached from any other bus-stop without changing buses; each pair of routes has a single bus-stop at which travellers can change buses; each bus route has exactly three stops.

How many bus routes are there in total? Show an example.

## Level B

### Problem 1.B

There are 12 seats on a circle. 4 students dressed in distinct colours: one red, one yellow, one green and one blue. They are sitting at neighbouring seats.

In one move a student can move four seats to the left or right, onto the fifth seat, if it is empty. If after several moves the students appear again at the four original seats, how that might happen?

### Problem 2.B

$ABCD$ is a convex quadrilateral. $M, N$ are the midpoints of $AB$, $CD$ respectively. It is known that: $MN = \frac{1}{2}(AD + BC)$.

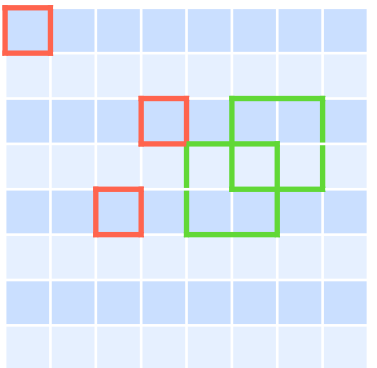
Prove that the $AD \parallel BC$.

### Problem 3.B

Prove that if for some $n$ positive integer, the number $\frac{2^n - 2}{n}$ is an integer, then $\frac{2^{2^n-1} - 2}{2^n - 1}$ is also an integer.

### Problem 4.B

On a standard $8 \times 8$ chessboard how many squares different in size or location are there? For example there are 5 different squares in the below diagram.



**Problem 5.B**

Solve the equation system for the real numbers $x$, $y$:

$$\begin{cases} 27(x^3+y^3)(x^2+y^2) = 5\cdot 2020^5 & (1) \\ x+y = 2020 & (2) \end{cases}$$

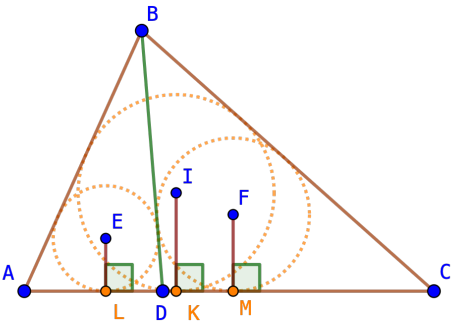
**Problem 6.B**

Prove that:

$$\frac{2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{2020\ \sqrt{}}}{2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{2019\ \sqrt{}}} > \frac{1}{4}$$

**Problem 7.B**

In triangle $ABC$, let $D$ be a point on $AC$. $E$, $F$ are the centres of the incircles of triangles $ABD$, $DBC$. $I$ is the centre of the incircle of triangle $ABC$. $K$, $L$, $M$ are the tangent points of the circles centred at $I$, $E$, $F$ with $AC$. Prove that $EL + FM > IK$.




**Problem 8.B**

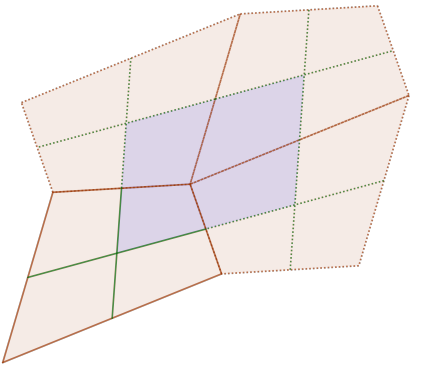
Minh wrote wrote 2025 digits on a circle. From a certain point, Minh read the digits clockwise and write them down. He obtained a 2025-digit number, which is divisible by 27.

Is it possible that if Nam begins reading the digits at any other place, also clockwise then the 2025-digit number which Name obtains also divisible by 27?

## Level A

### Solution 1.A

By reflecting the quadrilateral a few times, the below diagram shows how to tile the plane. Note the lines connecting opposite midpoints form parallelograms.



### Solution 2.A

Consider the following:

$$\underbrace{12...9}_{1\times 9 \text{ digits}} \quad \underbrace{1011...99}_{2\times 90 \text{ digits}} \quad \underbrace{100...999}_{3\times 900 \text{ digits}} \quad \underbrace{1000...999}_{4\times 9000 \text{ digits}} \quad \underbrace{10000...9999}_{5\times 90000 \text{ digits}} \quad \underbrace{100000...99999}_{6\times 900000 \text{ digits}}$$

Hence the number of digits from 1 to 99999 is:

$$1 \cdot 9 + 2 \cdot 90 + 3 \cdot 900 + 4 \cdot 9000 + 5 \cdot 90000 + 6 \cdot 900000 = 488889$$

The remaining 11111 digits are in the following block:

$\underbrace{1000000...1001586}_{7\times 1587=11109 \text{ digits}}\ 1\underline{0}01587$. So the $500000^{th}$ digit is 0.

### Solution 3.A

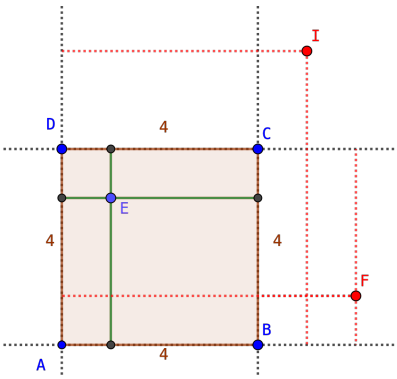
If a positive integer has remainder 0, 1, 2, 3, 4, 5, 6 when divided by 7, then its perfect square has the remainder 0, 1, 4, 2, 2, 4, 1 when divided by 7.

Therefore if $x^2 + y^2$ is divisible by 49, both of them has to be divisible by 7. On the other hand if both of them are divisible by 7, then $x^2 + y^2$ is divisible by 49.

Hence $x, y$ are from $7 \cdot 1 = 7, 7 \cdot 2 = 14, \ldots, 7 \cdot 142 = 994$. Because each pair of (7, 7), (14, 14), $\cdots$, (994, 994) should be counted one, so there are $142^2 - 142 = 20022$ such pairs.

**Solution 4.A**

It is easy to prove that the points should be inside or on the borders of the given square. $E$ is one such arbitrary point




**Solution 5.A**

Let $n = 1010$. Note that:

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{2n} > \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots \frac{1}{2n}}_{n \text{ terms}} = \frac{n}{2n} = \frac{1}{2}$$

$$\text{So } B = \frac{1}{3 \cdot 1011)} + \frac{1}{3 \cdot 1012} + \cdots + \frac{1}{3 \cdot 2020}$$

$$= \frac{1}{3}\left(\frac{1}{1010+1} + \frac{1}{1010+2} + \cdots \frac{1}{2 \cdot 1010}\right) > \frac{1}{3} \cdot \frac{1}{2} = A$$

**Solution 6.A**

Following are the construction steps:

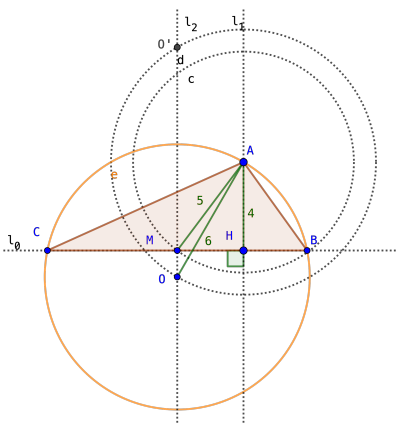
First let $H$, the foot of the altitude from $A$ to $BC$ be a point on a line $l_0$.

The line $l_1$ is constructed to be perpendicular to $l_0$ at $H$. $A$ is selected on $l_1$ such that $AH = 4$.

Circle centred at $A$ with radius 5 intersects line $l_0$ at $M$.

Line $l_1$ through $M$ and perpendicular to $l_0$.

Circle centred at $A$ with radius 6 intersects line $l_1$ at $O$ (and $O'$).




Circle centred at $O$ with radius 6 intersects line $l_0$ at $B$ and $C$. $ABC$ is the desired triangle.

**Solution 7.A**

Let us consider a table with the sum of numbers at vertices of any rectangle (with sides directed along the paper's sides) is not equal to each other. Let us prove that such a table cannot have two 1′ or two $-1$'s.

Let us consider 4 small squares — the vertices of a rectangle — with not all numbers equal to each other. Let them, for example be:

| | |
|---|---|
| 1 | $-1$ |
| 1 | 1 |

Let us consider two more numbers (squares) placed as follows:

| | | |
|---|---|---|
| $x$ | 1 | $-1$ |
| $y$ | 1 | 1 |

Clearly, the only possibility is $x = y = 1$ since otherwise this is a rectangle with a zero sum. Therefore, both rows and both of the initially considered columns contain only 1's (except one $-1$). Taking instead of the second row any other one, we again obtain :

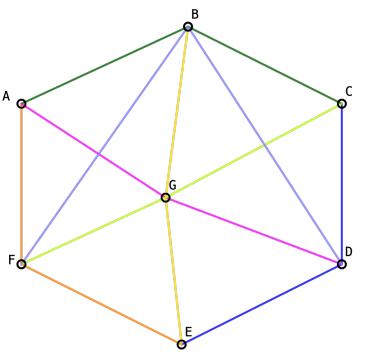
| | |
|---|---|
| 1 | $-1$ |
| 1 | 1 |

and, therefore, this other row also consists of 1's only.

Therefore, there is only one $-1$ in the whole table.

**Solution 8.A**

Let $r$ be the total number routes. Since each routes has $n$ stop and each stop has exactly $n$ routes go through it. For a route, the number of stops on it is $n$ and there are $n-1$ routes go through each stops. Therefore $r = n(n-1)+1 = 3 \cdot 2 + 1 = 7$.




Below is one example how a bus network with 7 routes, each with a different colour, satisfying the given requirements.

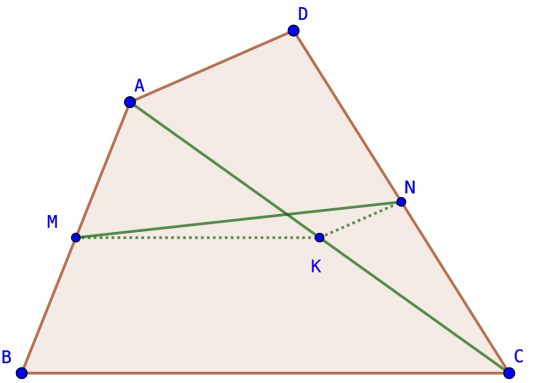
## Level B

### Solution 1.B

Change the order of points on the circle so that a student can get from one seat to an adjacent one in one move. That is, place 6 after 1, then place 11, 4, 9, 2, 7, 12, 5, 10, 3, 8 and again 1. Now, it is clear that the only way to rearrange students is to move them along the circle in one direction or the other.

### Solution 2.B

Let $K$ be the midpoint of $AC$. In triangle $MNK$: $MN \leq KN + NM$. In triangle $ABC$: $K$, $M$ are midpoints of $AC$, $BC$ so: $KM = \dfrac{BC}{2}$, similarly $KN = \dfrac{AD}{2}$. So $MN \leq \dfrac{AD + BC}{2}$.




The equality happens if and only if $K$, $M$, $N$ are collinear, which means $AD \parallel BC$.

### Solution 3.B

Let $m = \dfrac{2^n - 2}{n} \Rightarrow 2^n - 2 = mn$. Then according to the identity:

$a^m - 1 = (a-1)(a^{m-1} + a^{m-2} + \ldots + a + 1)$ with $a = 2^n$:

$$\frac{2^{2^n-1} - 2}{2^n - 1} = \frac{2 \cdot (2^{2^n-2} - 1)}{2^n - 1} = 2 \cdot \frac{2^{mn} - 2}{2^n - 1}$$
$$= 2\left((2^n)^{m-1} + (2^n)^{m-2} + \cdots + 2^n + 1\right)$$

This, obviously, is an integer.

**Problem 4.B**

For every $1 \le n \le 8$ by placing a $n \times n$ square in order from top to bottom down, from left top right, it is easy to prove that the number of $n \times n$ square is $(9-n)^2$. The number of them is $1^2 + 2^2 + \cdots + 8^2 = 204$.

**Solution 5.B**

Let $z = xy$. Instead of 2020, we use generic $a = 2020$.

From (2): $x + y = a \Rightarrow x^2 + y^2 = a^2 - 2z$, furthermore:

$x^3 + y^3 = (x+y)(x^2 - xy + y^2) = a(a^2 - 3z)$.

$27a(a^2 - 3z)(a^2 - 2z) = 5a^5 \Rightarrow 162z^2 - 135za^2 + 22a^4 = 0$

$\Rightarrow (9z - 2a^2)(18z - 11a^2) = 0$. Hence $z = \dfrac{2a^2}{9}$ or $z = \dfrac{11a^2}{18}$.

Case 1: $x + y = a$, $xy = \dfrac{2a^2}{9}$. Substitute $y = a - x$:

$x(a-x) = \dfrac{2a^2}{9} \Rightarrow 9x^2 - 9ax + 2a^2 = 0$

$\Rightarrow (3x - 2a)(3x - a) = 0 \Rightarrow (x, y) = (\dfrac{2a}{3}, \dfrac{a}{3})$

Case 2: $x + y = a$, $xy = \dfrac{11a^2}{18}$. Substitute $y = a - x$:

$x(a-x) = \dfrac{11a^2}{18} \Rightarrow 18x^2 - 18ax + 11a^2 = 0$

But $18x^2 - 18ax + 11a^2 = 18x^2 - 18x + \dfrac{9a^2}{2} + \dfrac{13a^2}{2}$

$= 18(x^2 - 2 \cdot x \cdot \dfrac{a}{2} + \dfrac{a^2}{4}) + \dfrac{13a^2}{2} = 18(x - \dfrac{a}{2})^2 + \dfrac{13a^2}{2} > 0$

Hence the only solution is $(\dfrac{2a}{3}, \dfrac{a}{3})$.

**Problem 6.B**

Let $x = \underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}_{2020\ \sqrt{\ }}$, then $x^2 - 2 = \underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}_{2019\ \sqrt{\ }}$

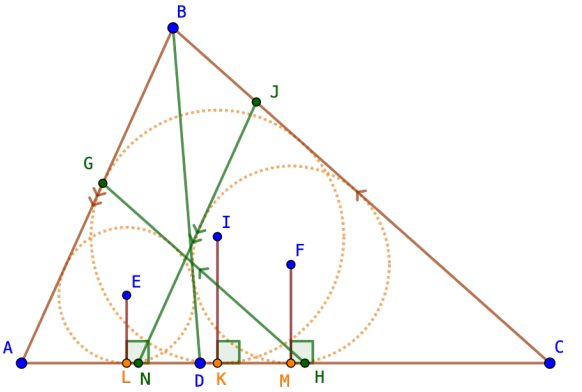
Easy to see that $x < 2$. So:

$$\frac{2-x}{2-(x^2-2)} = \frac{2-x}{4-x^2} = \frac{2-x}{(2-x)(2+x)} = \frac{1}{2+x} > \frac{1}{2+2} = \frac{1}{4}.$$

**Problem 7.B**

Let $G$, $H$ be points in $AB$, $AC$ such that $GH \parallel BC$ and $GH$ is tangent to the circle centred at $E$. Because the two circles at $E$ and $F$ are tangent, the circle centred at $E$ is completely inside the triangle $AGH$, therefore since the line $BD$ tangent with the circle centred at $E$, it must intersect $AC$ at point $D$ inside $AH$.

Hence $\dfrac{EL}{IK} = \dfrac{AH}{AC} > \dfrac{AD}{AC}$. Similarly $\dfrac{FM}{IK} = \dfrac{CN}{CA} > \dfrac{CD}{CA}$.




Therefore $\dfrac{EL}{IK} + \dfrac{FM}{IK} > \dfrac{AD}{AC} + \dfrac{CD}{CA} = 1 \Rightarrow EL + FM > IK$.

**Problem 8.B**

Let the original number $A = \overline{a_1a_2\cdots a_{2025}} \vdots 27$. Since:

$10 \cdot \overline{a_{2025}a_1a_2\cdots a_{2024}} = a_{2025} \cdot 10^{2025} + 10 \cdot \overline{a_1a_2\cdots a_{2024}}$

$= a_{2025} \cdot \underbrace{99\cdots9}_{2025} + \overline{a_1a_2\cdots a_{2025}}$

Since $999 = 27 \cdot 37 \vdots 27$, $2025 = 3 \cdot 675$, $\Rightarrow \underbrace{99\cdots9}_{2025} \vdots 27$,

$A = \overline{a_1a_2\cdots a_{2025}} \vdots 27$, thus $10 \cdot \overline{a_{2025}a_1a_2\cdots a_{2024}} \vdots 27$.

Therefore, $\overline{a_{2025}a_1a_2\cdots a_{2024}} \vdots 27$.

Similarly $\overline{a_{2024}a_{2025}a_1a_2\cdots a_{2023}} \vdots 27$, and so on.

This means that Nam's number is also divisible by 27.

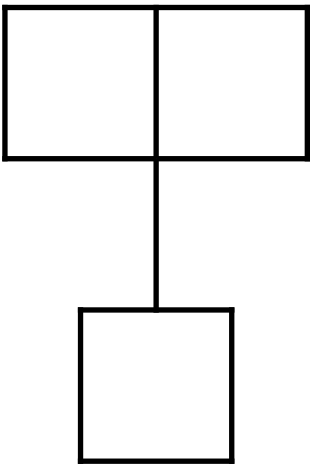
# Chapter 16

# October

# Monthly Team Contest

**Problem Set 1 Level A**

Twelves matches are used to form three squares as show below.



Can Binh **move six matches** to form *exactly 5 squares*?

**Problem Set 1 Level B**

Big Ben, a famous 12-hour clock in London, England, sounds four notes on the first quarter of the hour, eight notes on half-an-hour, twelve notes on the third quarter, and sixteen notes on the hour. Furthermore, it strikes a number of notes equal to the hour every hour; for example, it strikes five additional notes at 5 o'clock.

Lilian spends a full day in London near the Big Ben. **How many notes** does she hear in this twenty-four hour period?

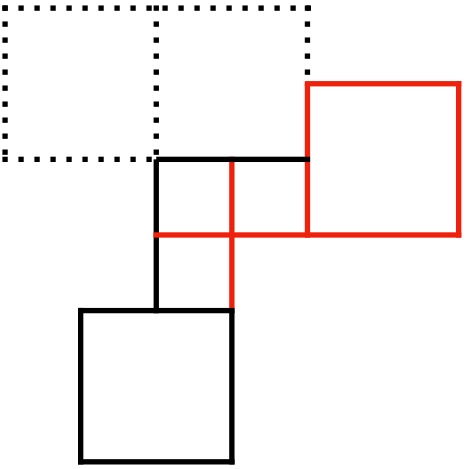
**Bonus**: how many times does Big Ben strike on the full-hour marks in 24 hours?

**Problem Set 1 Level C**

A snail crawls over a table at a constant speed. Every 15 minutes it turns 90◦ and in between these turns it crawls along a straight line.

Can it return to its starting point after 7h30 minutes of crawling?

**Problem Set 1 Level A Solution**



The red colour matches are the ones that have been moved.

**Problem Set 1 Level B Solution**

Since Big Ben is a 12-hour clock, let's first deal with 12 hours. Then we can multiply by 2. In one hour, not counting the number of notes it strikes for the hour, Big Ben will strike $4 + 8 + 12 + 16 = 40$ notes.

Therefore, in a 12-hour period, Big Ben will strike:

$(40 \times 12) + (1 + 2 + \ldots + 11 + 12)$

$= 480 + (13 \times 6) = 480 + 78 = 558$

For 24 hours worth: $558 \times 2 = 1116$

The bonus question: $2(1 + 2 + \ldots + 11 + 12) = 156$.

**Problem Set 1 Level C Solution**

Let a link be a segment that the snail covers in-between turns. The number of horizontal links along which the snail moves away from the origin is equal to the number of horizontal links along which it moves towards the origin.

This means that the number of all horizontal links is even. The number of all vertical links is even by the same argument. But the numbers of vertical and horizontal links are equal. This means that the number of all links is divisible by 4.

**Problem Set 2 Level A**

While wandering in the Black Forest, Prince Sam found the legendary House of the Wicked Witch. Inside of the house are three treasure chests. *Each chest has a label and two sacks*. The labels of the first chest said "*a sack of diamonds and a sack of silver coins*"; the second chest has a label of "*diamonds and some of useless toys*"; and the third chest with the label "*silver coins and useless toys*".

According to legend, all the labels correctly described contents in the chests, but all the chests are **wrongly labeled**. (i.e. the chest labeled 'diamonds' will not have diamonds in it, but there are diamonds in another chest.)

Knowing the Witch will be back soon, Prince Sam only has time to open *one* chest. *Which chest* should he choose?

**Problem Set 2 Level B**

In a game of chess, is it possible for each of the 30 diagonals of the chess-board to have an odd number of pieces on it? (The number of pieces on each diagonal could be equal, but not necessarily.)

**Problem Set 2 Level C**

The distance between the two cities is 999 km. At each kilometre of the road connecting the cities is a sign showing the distances to both cities as follow:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 999 | 1 | 998 | 2 | 997 | … | 998 | 1 | 999 | 0 |

**How many signs** are there where both distances are written using only *two distinct digits*?

**Bonus:** How many signs are there where both distances are written using only *two distinct digits*?

**Problem Set 2 Level A Solution**

He should chose the chest that has a label of *"a sack of silver coins and useless toys"*. In that case he shall get the diamonds.

**Problem Set 2 Level B Solution**

If the answer would have been yes, then on the black cells that constitute 8 diagonals in one direction as well as 7 diagonals in the other direction, there would have stood simultaneously an even and an odd number of pieces — the sum of an even and simultaneously of an odd number of odd numbers.

**Problem Set 2 Level C Solution**

Since the last digit has to be 9, there are only 9 pairs of digits:

$$(0, 9), (1, 8), (2, 7), (3, 6), (4, 5)$$

Case $(0, 9)$: There are 8 cases: $(0, 999)$, $(9, 990)$, $(90, 909)$, $(99, 900)$, $(900, 99)$, $(909, 90)$, $(990, 9)$, $(999, 0)$

Case $(1, 8)$: There are 8 cases: $(111, 888)$, $(118, 881)$, $(181, 818)$, $(188, 811)$, $(811, 188)$, $(818, 181)$, $(881, 118)$, $(888, 111)$

Similarly for the rest of the cases. So there are 40 signs.

For the bonus question only the digits $d$. Since $\overline{dd} + \overline{dd}$ is too small, so: $d + \overline{ddd} = 999 \Rightarrow d \cdot 112 = 999$, which is impossible.

**Problem Set 3 Level A**

Lily and her friend play a two-player game. They begin with 59 sticks and, on her turn, each player removes at least one, but no more than five sticks. The loser is the one forced to take the last stick.

What is the **winning** strategy for Lily?

**Bonus**: if there are 60 sticks, should Lily change her strategy?

**Problem Set 3 Level B**

$x$, $y$ are two positive real number such that:

$$\sqrt{x} - \sqrt{y} = 10$$

Prove that $x - 2y \leq 200$.

**Problem Set 3 Level C**

To guard a military installation around the clock, a day shift and a night shift are required. A sentry guard may take a day or a night shift or work around the clock. Depending on their choice, the guard is given a leave of absence of no less than 1, 1.5 or 2.5 full days, respectively.

**What is the least number** of guards necessary to ensure the security of this installation?

**Problem Set 3 Level A Solution**

The player who chooses from among 7 sticks will always lose. If he chooses  sticks, the other player chooses  sticks, and the first player is left with only 1 stick. Similarly, the player choosing from among 13 sticks will always lose, for if he chooses  sticks, the other player chooses , leaving our first player with 7.

From above, this player loses since he is choosing from among 7 sticks. Continuing in this manner, we find that the player choosing from 19, from 25, from 31, from 37, from 43, from 49, and from 55 is the loser.

Thus if Lily goes first, takes 4 sticks, leaving her friend with 55, she will win.

For the bonus question, she should go second and takes an amount of sticks so the sum of that and what he friend picks earlier is 6.

**Problem Set 3 Level B Solution**

$\sqrt{x} = \sqrt{y} + 10$. Substitute this in to $x - 2y$:

$$x - 2y = \left(\sqrt{y} + 10\right)^2 - 2y = -y + 2\sqrt{10}y + 100$$

$$= -\left(y - 2\sqrt{10}\sqrt{y} + 100\right) + 200 = 200 - (\sqrt{y} - 10)^2 \leq 200$$

**Problem Set 3 Level C Solution**

Whatever the schedule, the sentries earn 2.5 full days off for 1 full working day, therefore, the mean number of sentries having their days off should be $\geq 2.5$.

Hence, the total number of sentries should be $\geq 3.5$.

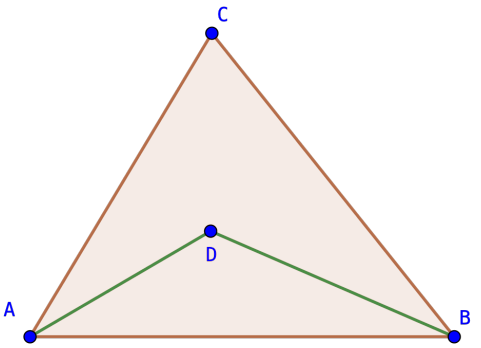
On the other hand, it is clear that 4 sentries suffice: let each work 24 hours around the clock and take three full days off.

**Problem Set 4 Level A**

How many 3-digit numbers start or end with an even digit?

**Problem Set 4 Level B**

$D$ is a point inside a triangle $ABC$.




Prove that: $AD + DB < AC + CB$

**Bonus**: prove that $DA + DB + DC < AB + BC + CA$.

**Problem Set 4 Level C**

Lily is in the capital city of the country of Train-and-Plane. In this country, any two cities are connected either by a direct railroad or a direct air route.

Is it possible for her to *choose a type of transportation* so that she can **reach any other city from the capital** with *no more than one transfer* (changing from one route to another at a third city), using only her chosen transportation type?

**Problem Set 4 Level A Solution**

3-digit numbers that start with an even digit: There are 4 choices for the first digit (2, 4, 6, or 8), and 10 choices for each remaining digit, so there are $4 \times 10 \times 10 = 400$ of these numbers. 3-digit numbers that end with an even digit: There are 9 choices for the first digit (since it can't be 0), 10 choices for each of the second digit, and 5 choices for the last digit (2, 4, 6, 8, or 0), so there are $9 \times 10 \times 5 = 450$ of these numbers. 3-digit numbers that start and end with an even digit: There are 4 choices for the first digit, 10 choices for each of the four middle digits, and 5 choices for the last digit, so there are $4 \times 10 \times 5 = 200$ of these numbers.

Therefore, the number of 6-digit numbers that start or end with an even digit is: $400 + 450 - 200 = 650$.

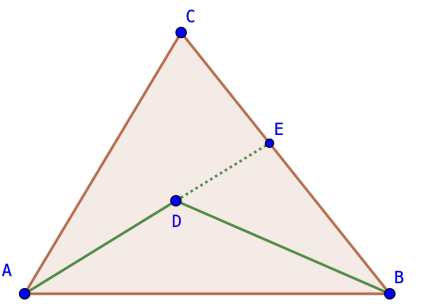
**Problem Set 4 Level B Solution**

Let $E$ be the intersection point of $AD$ and $BC$.

$DB < DE + EB, AE < AC + CE$:

$AD + DB < AD + (DE + EB)$

$= AE + EB < (AC + EC) + EB$
$= AC + CB$




The bonus question is simple by applying the proven equations with different vertices of triangle ABC.

**Problem Set 4 Level C Solution**

Let assume the opposite. Let $C$ be the capital city. Let $A$ be the city that cannot be reached from $C$ by airplane with no more than one transfer, and $B$ be the city that cannot be reached from $C$ by train with no more than one transfer. Since $A$ and $B$ are connected by some kind of transportation. Without loss of generality, we can assume that $A$ and $B$ are connected by train. Since $A$ and $C$ are connected by train, so $B$ can be reached from $C$ via $A$ by train.

**Problem Set 5 Level A**

Which number is greater?

$$(1.01)^{1000} \text{ and } 1000$$

**Problem Set 5 Level B**

Each square of the $3 \times 3$ board is filled with one of the numbers $-1, 0, +1$. For example:

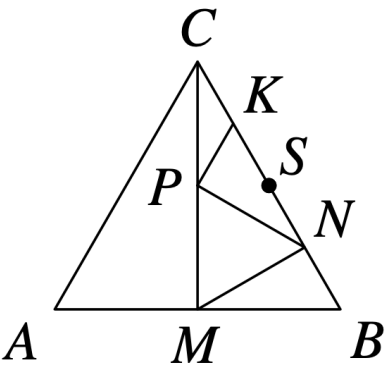
| | | |
|---|---|---|
| $-1$ | $+1$ | $+1$ |
| $0$ | $-1$ | $+1$ |
| $-1$ | $+1$ | $0$ |

Viet calculates the sums of the rows, columns, and two main diagonals. He found that there are at least two equal sums. For example, in the example above, both the sums of the second and third rows are 0.

**Why?** *Note: Solve the generic case, not the example above.*

**Problem Set 5 Level C**

Let $M$ be the midpoint of side $AB$ of equilateral triangle $ABC$, and let points $N$, $S$ and $K$ divide side $BC$ into *four equal segments*. Given that $P$ is the midpoint of $CM$, prove that $\angle KPN = 90°$?




**Bonus:** Prove that $\angle MNB = 90°$

**Problem Set 5 Level A Solution**

Note that the exponent $1000 = 10 \cdot 10 \cdot 10$, so:

$(1.01)^{10} = 1.104... > 1.1$

$(1.1)^{10} = 2.593... > 2$

$2^{10} = 1024 > 1000$.

**Problem Set 5 Level B Solution**

The largest possible sum is 3 and the smallest is -3. So there are 7 possible values for 8 sums. Obviously two of them should be equal.

**Problem Set 5 Level C Solution**

Connect $SP$. $SP = \dfrac{MB}{2} = \dfrac{BC}{4} = SK = SN$ so $\angle NPK = 90°$.

In $\triangle MNB : MB = 2NB$, $\angle MBN = 60°$, so it is a 30-60-90.

Therefore $\angle MNB = 90°$

**Problem Set 6 Level A**

In Live-in-Cloudland, the only way to get to somewhere is to hop on a magic carpet and go. Each carpet line connects two cities. Twenty one carpet lines serve the capital city. Only one line flies to Far-and-Insignificant-Ville, and every other city is served by exactly 20 carpet lines.

Is it always possible to travel from the capital to Far-and-Insignificant-Ville by magic carpet (directly or by transferring from one carpet line to another)?

**Problem Set 6 Level B**

Find *two four-digit numbers* whose product is :

$$4^8 + 6^8 + 9^8.$$

*Hint:* $4^8 = (2^8)^2$, $9^8 = (3^8)^2$

**Bonus**: the larger four-digit number has three prime factors. What is the sum of them?

**Problem Set 6 Level C**

Prove that among any 27 different positive integers less than 100 each there are two integers that are not relatively primes.

*Hint: investigate their prime factors.*

**Problem Set 6 Level A Solution**

Let assume that it is not possible. Then there are two disjoint carpet networks, one containing the far city and the other contains the city. The number of lines in the second network is 21 plus a multiple of 20, which is an odd number. However these lines are counted twice because each connect two cities, meaning the sum of them is even. This is contradictory.

**Problem Set 6 Level B Solution**

$$4^8 + 6^8 + 9^8 = (2^8)^2 + (2^8)(3^8) + (3^8)^2$$
$$= (2^8)^2 + 2(2^8)(3^8) + (3^8)^2 - (2^8)(3^8) = (2^8 + 3^8)^2 - (6^8)$$
$$= (2^8 + 3^8)^2 - (6^4)^2 = (2^8 + 3^8 - 6^4)(2^8 + 3^8 + 6^4)$$
$$= 8113 \cdot 5521$$

$8113 = 7 \times 19 \times 61$, so $7 + 19 + 61 = 87$.

**Problem Set 6 Level C Solution**

There are exactly 25 primes between 1 and 100. Adding 1 to this set we get 26 relatively prime numbers. By hypothesis, 27 numbers, each less than 100, are selected and the total number of different prime divisors of these numbers does not exceed 25. If there is no number 1 among them, then at least 2 of these 26 numbers are not relatively prime; if there is a 1 among them, it is possible to select two of the 27 numbers with the same prime divisor and then these two numbers are not relatively prime.

# Problems of the Month

**Level A**

**Problem 1.A**

2020 "+" signs and 2021 "−" signs are written on a board. At each step two signs are erased, a "+" is written instead if two erased signs are the same and a "−" is written instead if erased signs are the different. After 4040 steps, what sign remains on the board?

**Problem 2.A**

Factor the following expression for all $a$, $b$ real numbers:

$$a(a+b)(a^2+b^2) - \frac{a^4-b^4}{2}$$

Use the result to prove that for all $a$, $b$, $c$ real numbers:

$$a(a+b)(a^2+b^2) + b(b+c)(b^2+c^2) + c(c+a)(c^2+a^2) \geq 0$$

**Problem 3.A**

$a-2$, $a-1$, $a$, $a+1$, $a+2$ are five consecutive integers.

Prove that: $(a-2)^5 + (a-1)^5 + a^5 + (a+1)^5 + (a+2)^5 \vdots 25$.

Prove that the sum of the $5^{th}$ powers of the first 2020 positive integers is divisible by 25:

$$1^5 + 2^5 + 3^5 + \cdots + 2020^5 \vdots 25$$

**Problem 4.A**

Points $M$, $N$ are on sides $AC$, $AB$ of $\triangle ABC$. $BM$ and $CN$. Intersects are $P$. The area of $\triangle BPN$, $\triangle BPC$, $\triangle CPM$ are 3, 4, and 5, respectively. What is the area of $\triangle ABC$?

**Problem 5.A**

On each vertex of a regular hexagon is a butterfly. At one moment all the butterflies fly off, and one moment later, each of them lands at a vertex of the hexagon, not necessarily the same one where they flew off.

Prove that there are three butterflies where the triangles of vertices where they flew off is congruent to the triangle of vertices where they land on.

**Problem 6.A**

Find all pairs of positive integers $(a, b)$ such that among the statements below exactly three of them are true.

i. $a = 5b + 9$

ii. $a + 6$ is divisible by $b$

iii. $a + 2017b$ is divisible by 5

iv. $a + 7b$ is a prime number

**Problem 7.A**

What is the largest amount of integers can be chosen from the first 2020 positive integers 1, 2, 3, …, 2020 such that the sum of the cubes of any two integers is not divisible by 4?

**Problem 8.A**

$ABC$ is a right triangle triangle at $A$ and $AB = AC$. Let $D$ be a point on $AB$. $E$ be a point on $AC$ such that $AE = BD$. Line through $A$, perpendicular to $DE$, intersects $BC$ at $K$. Prove that $DE = AK$.

## Level B

### Problem 1.B

One hundred bears went to collect nuts in the forest. The $1^{st}$ bear got 1 nut, the $2^{nd}$ one got 2 nuts, ..., the $100^{th}$ bear got 100 nuts. On the way home they met an old fox. He offered to help the bear to "equally" dividing the nuts among the bears as follow: the fox choose any two bears, add the nuts together, divide the nuts equally between the two bears, and if there is a leftover (in case the total number of nuts is odd), then he eats that nut. The process is continued until each of the bear has the same amount of nuts.

What is the maximum amount of nuts the fox can eat?

### Problem 2.B

Find all real number $x$, such that:

$$25^{-2} = \frac{5^{\frac{48}{x}}}{5^{\frac{26}{x}}.25^{\frac{17}{x}}}$$

### Problem 3.B

What is the sum of digit of $P$ if $P = (\underbrace{3...3}_{333})^2$?

### Problem 4.B

$\triangle ABC$ is a right triangle at $A$ and $AB = AC$. Points $D, E$ are on $BC$ such that $\angle DAE = 45^\circ$ ($D$ is between $B$ and $E$).

Prove that $BD^2 + CE^2 = DE^2$.

**Problem 5.B**

Each square of the $4 \times 4$ board contains a "+" or "−" signs. At each step, you can change all the signs in a row, a column, or a diagonal to their opposite ones, i.e. "+" to "−", and "−" to "+". An example of how a column is changed shown as below:

| | | | |
|---|---|---|---|
| + | − | + | + |
| + | + | + | + |
| + | + | + | + |
| + | + | + | + |

| | | | |
|---|---|---|---|
| + | + | + | + |
| + | − | + | + |
| + | − | + | + |
| + | − | + | + |

Is it possible to obtain a board with the "+" signs only?

**Problem 6.B**

$x$, $y$, $z$ are positive real numbers: $xy + yz + zx = 1$.

Factor $1 + x^2$ as a product of two terms.

Compute the following expression:

$$x\sqrt{\frac{(1+y^2)(1+z^2)}{1+x^2}} + y\sqrt{\frac{(1+z^2)(1+x^2)}{1+y^2}} + z\sqrt{\frac{(1+x^2)(1+y^2)}{1+z^2}}$$

**Problem 7.B**

Let $ABC$ be a triangle and let $P$, $Q$, $R$ be any points on the sides $BC$, $CA$, $AB$, respectively. The circumcircles of the triangles $ARQ$ and $BPR$ intersect each other at point $D$. Prove that the circumcircle of the $\triangle CQP$ also passes through $D$.

*Hint: a $XYZW$ quadrilateral is circumscribed by a circle $K$ - all vertices $A$, $B$, $C$, $D$ are on the circle - if $\angle ABC + \angle ADC = 180°$.*

**Problem 8.B**

$a$, $b$ are positive integers such that $a\sqrt{3} > b\sqrt{7}$.

Prove that: $a\sqrt{3} - b\sqrt{7} > \dfrac{1}{a+b}$.

## Level A

### Solution 1.A

If we replace the $+$ sign by $+1$ and the $-$ sign by $-1$, then at the beginning the product of all numbers is $-1$. Each operation does not change the sign of the product. Hence the remaining number shall be a $-1$.

### Solution 2.A

$$a(a+b)(a^2+b^2)-\frac{a^4-b^4}{2}=a(a+b)(a^2+b^2)-\frac{(a^2+b^2)(a^2-b^2)}{2}$$
$$=(a^2+b^2)\frac{2a^2+2ab-a^2+b^2}{2}=(a^2+b^2)\frac{a^2+2ab+b^2}{2}$$
$$=\frac{(a^2+b^2)(a+b)^2}{2}$$

Therefore for all $a$, $b$:

$$a(a+b)(a^2+b^2)-\frac{a^4-b^4}{2}\geq 0\Rightarrow a(a+b)(a^2+b^2)\geq\frac{a^4-b^4}{2}\ (*)$$

Apply (*) for $b$, $c$ and $c$, $a$:

$$b(b+c)(b^2+c^2)\geq\frac{b^4-c^4}{2},\ c(c+a)(c^2+a^2)\geq\frac{c^4-a^4}{2}$$

Hence:

$$a(a+b)(a^2+b^2)+b(b+c)(b^2+c^2)+c(c+a)(c^2+a^2)$$
$$\geq\frac{a^4-b^4}{2}+\frac{b^4-c^4}{2}+\frac{c^4-a^4}{2}=0.$$

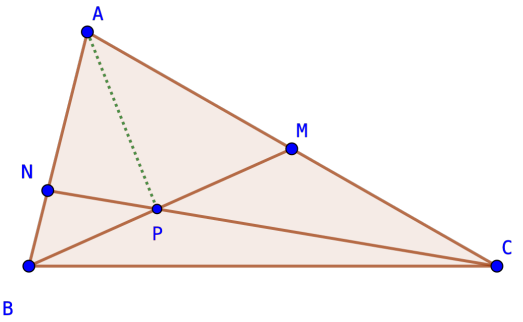
### Solution 3.A

$$(a-2)^5+(a-1)^5+a^5+(a+1)^5+(a+2)^5=5a(a^4+20a^2+34)$$
$$=5(a^5-a)+25a(4a^2+7)=5a(a-1)(a+1)(a^2+1)+25a(4a^2+7)$$

Since $a(a-1)(a+1)(a^2+1)$ is always divisible by 5 regardless of the remainder when divided $a$ by 5, hence the given expression.

Since $2020=5\times 404$, the second question is trivial application of the solution for the first one.

**Solution 4.A**




Let $S_{APN} = s$, since $\dfrac{S_{APN}}{S_{APC}} = \dfrac{PN}{PC} = \dfrac{S_{BPN}}{S_{BPC}} = \dfrac{3}{4}$, $\Rightarrow S_{APC} = \dfrac{4s}{3}$.

Thus, $S_{APB} = S_{APN} + S_{BPN} = s + 3$,

$S_{APM} = S_{APC} - S_{CPM} = \dfrac{4s}{3} - 5$

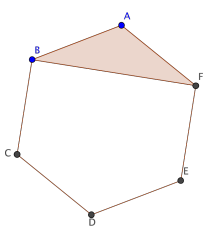
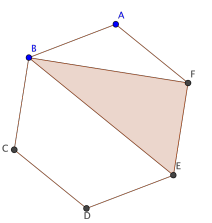
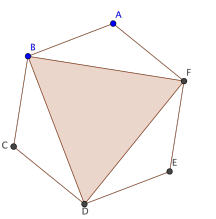
Furthermore: $\dfrac{S_{APB}}{S_{APM}} = \dfrac{PB}{PC} = \dfrac{S_{BPC}}{S_{CPM}} = \dfrac{4}{5}$, therefore:

$5(s+3) = 4(\dfrac{4s}{3} - 5) \Rightarrow 5s + 15 = \dfrac{16s}{3} - 20 \Rightarrow s = 105$

$S_{ABC} = S_{APN} + S_{APM} + 3 + 4 + 5 = 105 + 135 + 12 = 252.$

**Solution 5.A**

First, we find out how many types of triangles there are. See the diagrams below. For Type 1, there are 6 triangles. Type 2, 12 triangles. Type 3, 2 triangles. Each triangle represents a combination of butterflies. There's 20 in total. There's 12 for the second type, which means after max $6 + 2 = 8$ combinations can change to other types, there are always some that return to the same combination type.

| | | |
|---|---|---|
|  |  |  |
| Type 1 | Type 2 | Type 3 |

**Solution 6.A**

Assume i and iv are true: $a + 7b = 12b + 9$ cannot be a prime that is divisible by 3 and larger than 3. Contradiction.

Assume iii and iv both true: $a + 2107b = (a + 7b) + 2010\text{b}$ both divisible by 5, hence $a + 7\,\text{b}$ is divisible by 5 and larger than 3. Contradiction. So iv cannot be true. Otherwise only two statements ii and iv can be true.

By i and ii: $5b + 15$ is divisible by $b$, so 15 is divisible by 3, so $b$ can only be 1, 3, 5, 15. (*) By i and iii: $2022b + 9$ is divisible by 5, so $2b + 4$ is divisible by 5. So $b$ has a remainder 3 when divided by 5. (**)

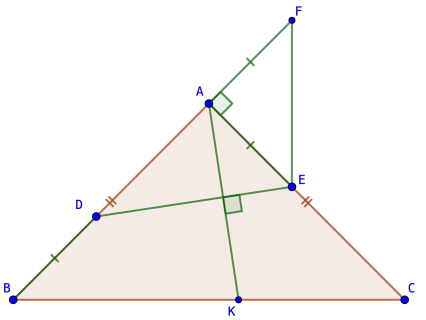
From (*) and (**): $b = 3$, $a = 5b + 9 = 24$. Easy to check that all i-iii statements are true and iv is not true.

**Solution 7.A**

For all positive integers $a$: if $a$ is even then $a^3$ is divisible by 4; if $a$ is odd then $a^3$ has a remainder 1 (if $a$ has remainder 1) or 3 (if $a$ has remainder 3) when divided by 4. If we choose $n$ integers from the first 2020 positive integers, we cannot chose more than one even integer.

Furthermore there are 505 integers (1, 5, … 2017) that have remainder 1 when divided by 4, 505 integers (3, 7, … 2019) that have remainders 3 when divided by 4. We should not choose a pair that have different remainders when divided by 4. Hence the maximum number of choosable integers is $1 + 505 = 506$.

**Solution 8.A**




Let $F$ be on the extension of $BA$ towards $A$ and $AF = AE$.

Since $AC = DF$, $\angle DEF = \angle ACK$, $\angle EDF = \angle KAC$:

$$\triangle DEF \cong \triangle AKC \Rightarrow DE = AK$$

## Level B

### Solution 1.B

The total number of nuts is $1 + 2 + \cdots + 100 = 5050$.

The process stops when each of the bear has at least one nut, so the fox cannot eat more than $5050 - 100 = 4950$ nuts. We prove that the fox can reach that maximum.

Let number the bears as $1, 2, , \cdots, 100$.

The fox proceeds through 100 stages. At each stage $k$, $1 \le k \le 100$, with the bears numbered $1, 2, , \cdots, k$ have $1, 2, , \cdots, k$ nuts, the fox takes $k - 1$ steps to deal with them.

In each step $t$, $1 \le t \le k - 1$, the fox takes the nuts from the $k - t + 1$ and $k - t$ bears, add them together, divide equally between the two bears, and eat one leftover nut.

By completing this, the nuts of the $1, 2, , \cdots, k$ numbered bears now are $1, 1, 2, , \cdots, k - 1$.

Easy to see that each bear ends up with one nut and the fox eats the rest.

### Solution 2.B

$$5^{-4} = 25^{-2} = \frac{5^{\frac{48}{x}}}{5^{\frac{26}{x}}.25^{\frac{17}{x}}} = 5^{\frac{48}{x}-(\frac{26}{x}+2.\frac{17}{x})}$$

$$= 5^{\frac{48-(26+2.17)}{x}} = 5^{\frac{-12}{x}} \Rightarrow -\frac{12}{x} = -4 \Rightarrow x = 3$$

### Solution 3.B

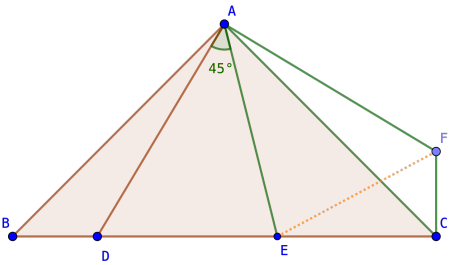
$$P = (\underbrace{3...3}_{333})^2 = \underbrace{3...3}_{333} . \underbrace{3...3}_{333} = \frac{1}{9} \underbrace{9...9}_{333} . \underbrace{9...9}_{333}$$

$$= \frac{1}{9}(10^{333} - 1)(10^{333} - 1) = \frac{1}{9}(10^{666} - 2.10^{333} + 1)$$

$$= \frac{1}{9}(\underbrace{9...9}_{332} 8 \underbrace{0...0}_{332} 1) = \underbrace{1...1}_{332} 0 \underbrace{8...8}_{332} 9.$$

Hence the sum of digits of $P$ is $332 + 8.332 + 9 = 2997$.

**Problem 4.B**




Let $CF \perp CE$ and $CF = BD$.

$AB = AC$, $\angle ABD = \angle ACF$, $BD = CF$,

so by $SAS$: $\triangle ABD \cong \triangle ACF$.

Therefore $AD = AF$.

Furthermore $AD = AF$, $\angle DAE = \angle EAF$:

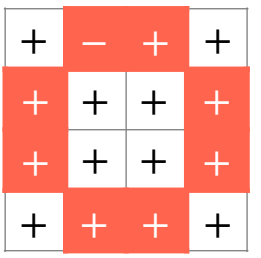
$\triangle DAE \cong \triangle FAE$.

Thus, $DE = EF$.

Because $EF^2 = EC^2 + CF^2 : BD^2 + CE^2 = DE^2$

**Solution 5.B**

Take a look at the following red squares.




If we replace the $+$ sign by $+1$ and the $-$ sign by $-1$, then at the beginning the product of all numbers in these red square is $-1$.

Easy to see that each operation to change all the signs in a row, a column or a diagonal shall change exactly two red squares. Whatever the numbers in the two red squares, the product of their negates remain same. Since the product of the red squares at the beginning is $-1$, so it cannot be changed at all.

**Problem 6.B**

Since $xy + yz + zx = 1$: $1 + x^2 = xy + yz + zx + x^2$

$= y(x+z) + x(z+x) = (x+y)(x+z)$

Therefore: $x\sqrt{\dfrac{(1+y^2)(1+z^2)}{1+x^2}}$

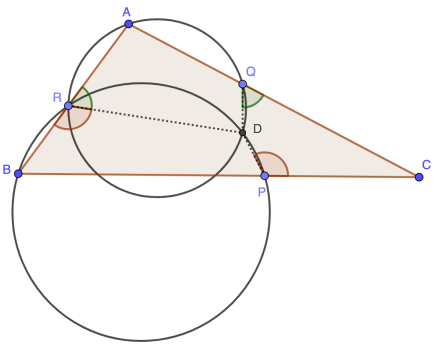
$= x\sqrt{\dfrac{(y+x)(y+z)(z+x)(z+y)}{(x+y)(x+z)}} = x(y+z)$

Similarly with the remaining terms of the expression:

$x(y+z) + y(z+x) + z(x+y) = 2(xy + yz + zx) = 2.$

**Problem 7.B**

Let two of the circles intersects at D. $RAQD$ is cyclic, so $\angle DQC = 180 - \angle DQA = \angle ARD$. Similarly, $\angle DPC = \angle BRD$. Hence $\angle ARD + \angle BRD = 180$, or $PDQC$ cyclic.




Therefore, the circumcircle of $\triangle PQC$ also goes through point D, meaning all three circles meet at one point.

**Problem 8.B**

$a\sqrt{3} > b\sqrt{7} \Rightarrow 3a^2 > 7b^2 \Rightarrow 3a^2 - 7b^2 \geq 1$

$3a^2 - 7b^2$ has the same remainder when divided by 7 as the same remainder then divided $3a^2$ by 7, or 0, 3, 5, 6.

Hence $3a^2 - 7b^2 \geq 3$.

$$a\sqrt{3} - b\sqrt{7} = \frac{3a^2 - 7b^2}{a\sqrt{3} + b\sqrt{7}} \geq \frac{3}{a\sqrt{3} + b\sqrt{7}} > \frac{1}{a\frac{\sqrt{3}}{3} + b\frac{\sqrt{7}}{3}} > \frac{1}{a+b}$$

# Chapter 17

# November

# Monthly Team Contest

**Problem Set 1 Level A**

Nam has a balance scale and a weight sets of 1, 2, 3, …, 10 kg. His task is to choose three weights from the weight set so that he can measure 13 different bags of 1, 2, …, 13 kg. *What are the weights* that Nam chooses? Show how can he weigh each bag.

**Problem Set 1 Level B**

Prove: $3^{29} < 2^{47}$. *Hint:* $3^3 = 27 < 2^5 = 32$, $3^5 = 243 < 2^8 = 256$.

**Problem Set 1 Level C**

Clemence swims 8 kilometres downstream in 2 hours. She returns upstream in 16 hours. How fast does she swim in still water? **Bonus**: how fast is the current?

**Problem Set 1 Level A Solution**

W1, W3, and W9 are the three weights of 1kg, 3kg, and 9kg are the ones he should choose. To measure 1kg: W1, 2kg: W3-W1, 3kg: W3, 4kg: W3+W1, 5kg: W9-W5-W1, 6kg:W9-W3, 7kg:W9+W1-W3, 8kg: W9-W1, 9kg: W9, 10kg: W9+W1, 11kg: W9+W3-W1, 12kg:W9+W3, 13kg: W9+W3+W1.

**Problem Set 1 Level B Solution**

$3^5 = 243 < 256 = 2^8 \Rightarrow 3^{20} < 2^{32}$ and $3^3 < 2^5 \Rightarrow 3^9 < 2^{15}$.

Hene $3^{29} = 3^{20} \cdot 3^9 < 2^{32} \cdot 2^{15} = 2^{47}$

**Problem Set 1 Level C Solution**

The current helps Clemence swim downstream and hinders her swimming upstream. Let Clemence's rate in still water be $x$ and the current be $y$. Thus, swimming downstream Clemence's rate is $x + y$, while swimming upstream it is $x - y$. Now apply rate times time equals distance: downstream, we have: $(x + y)2 = 8 \Rightarrow x + y = 4$, and upstream gives: $(x - y)16 = 8 \Rightarrow x - y = 2$. Solving these equations, we find that $(x, y) = (3, 1)$. Thus, Clemence's rate in still water is 3 kilometres per hour. The rate of the current is 1.

**Problem Set 2 Level A**

Kelvin goes shopping. Four pounds of onions costs the same as 2 pounds of string beans. At the same time, 1 pound of string beans costs 3 times as much as a pound of potatoes, while 1 pound of onions costs 4 cents less than 2 pounds of potatoes. Kelvin buys 1 pound of each of the vegetables.

**How much** does he have to pay?

**Bonus**: what is the ratio of the price per pound of :

potato : onion : spring bean

*Note: Express the ratio in simplest form. For example* $12 : 3$ *should be expressed as* $4 : 1$.

**Problem Set 2 Level B**

Which number is greater? $2^{100} + 3^{100}$ or $4^{100}$

**Problem Set 2 Level C**

How many 6-digit binary numbers (numbers with 0's and 1's as digits) have a string of three consecutive 1's appearing in them?

*Note: For example, 101110 and 111100 both have a string of three consecutive 1's, but 100110 doesn't.*

**Problem Set 2 Level A Solution**

Let $x$ be the cost of a pound of potatoes. From the question, a pound of string beans costs $3x$. Since we can get twice as many pounds of onions as pounds of string beans for the same price, the price of a pound of onions is half that of the string beans, or $\frac{3x}{2}$. From the information on onions and potatoes we have $3x/2 = 2x - 4$, so $x/2 = 4$ and $x = 8$. Thus potatoes are 8 cents a pound, string beans 24 cents a pound, and onions 12 cents a pound. Thus a pound of each together costs 44 cents. The ratio if $8 : 12 : 24$ or $2 : 3 : 6$.

**Problem Set 2 Level B Solution**

$4^3 > 2 \cdot 3^3 \Rightarrow 4^{100} > 2.3^{100} > 2^{100} + 3^{100}$.

**Problem Set 2 Level C Solution**

There are 4 possible position arrangement if three 1s appearing together: 111???, ?111??, ??111?, ???111

Note that ?111?? is the same as 1111??, so every number in ?111?? is also a number in 111??? format.

By similar reasons, we have only three types of numbers:

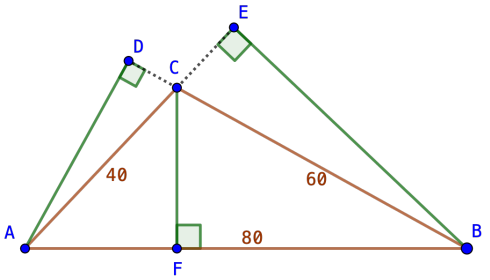
111???, 1?111??, 1??111

- The numbers in one of the three formats:
  - There are 8 numbers with 111??? format (1).
  - There are 4 numbers with 1?111? format (2).
  - There are 4 numbers with 1??111 format (3).
- The numbers in two of the three formats:
  - There are 2 numbers with 11111? format (1 & 2).
  - There are 2 numbers with 1?1111 format (2 & 3).
  - There are 1 numbers with 111111 format (1 & 3).
- The numbers in all three formats:
  - There are 1 numbers with 111111 format (1, 2, & 3)

The total number of them is $8 + 4 + 4 - (2 + 2 + 1) + 1 = 12$.

**Problem Set 3 Level A**

If triangle $PQR$ has sides 40, 60, and 80, then the shortest altitude is $K$ times the longest altitude.




Find the value of $K$.

**Problem Set 3 Level B**

Find the unit digit of $3^{2021} - 2^{2021}$

**Bonus**: what is the unit digit of $3^{2020} - 2^{2020}$?

**Problem Set 3 Level C**

Minh goes to a soccer match. He wants to bring *two socks of the same size and colour*.

There are 4 pairs of socks of two different sizes and of two colours lying pell-mell on a shelf in his closet. It's dark in the morning so Minh has to choose a number of socks randomly.

**What is the minimal number of socks** from the shelf that he should grab?

**Problem Set 3 Level A Solution**

Since the area of the triangle is half the product of an altitude and the side to which it is drawn, the product of the altitude and the side to which it is drawn is the same for all three sides. Thus the shortest altitude is drawn to the longest side and the longest altitude to the shortest side. Letting $x$ be the longest and $y$ the shortest altitude, we have $40x = 80y$, so $y = \frac{1}{2}x$ and the shortest altitude is half the longest.

Thus $K = \frac{1}{2}$

**Problem Set 3 Level B Solution**

The units digit of the difference $3^{2021} - 2^{2021}$ will be the same as the units digit of the difference of the units digits of $3^{2021}$ and $2^{2021}$. We thus compute the two units digits separately. For powers of 3, the units digits go 3, 9, 7, 1, 3, …, repeating in cycles of 4. Since the sequence repeats in cycles of 4, the $2021^{st}$ term will be the same as the $1^{st}$ term (dividing by 4 and taking the remainder, or 1), which is 3. For the powers of 2, the units digit goes 2, 4, 8, 6, 2, 4, , again repeating in fours. Modding out by 4, the $2021^{st}$ digit is the same as the $1^{st}$, which is 2.

The difference is $3 - 2 = 1$.

Then the bonus question is simple, it's 5 because of the difference between 1 and 6.

**Problem Set 3 Level C Solution**

If the sizes are $A$ and $B$, and the colours are $a$ and $b$, then all 8 socks are as follows:

$$(A, a),\ (A, a),\ (B, a),\ (B, a),\ (A, b),\ (A, b),\ (B, b),\ (B, b)$$

Socks can be divided into four pairs:

$$(A, a) - (B, b),\ (A, a) - (B, b),\ (A, b) - (B, a),\ (A, b) - (B, a)$$

If we choose 5 socks than there will necessarily be one of these pairs among them.If you choose only 4 socks, they may be $(A, a)$, $(A, a)$ and $(B, a)$, $(B, a)$. There will be no two socks of different sizes and colours among them.

So the minimum number is 5.

**Problem Set 4 Level A**

$p$ is the product of the first 25 prime numbers:

$$p = 2 \cdot 3 \cdot 5 \cdots 97$$

Can $p + 1$ be a perfect square? Why or why not?

**Bonus**: give two examples when $p$ is a product of two consecutive odd prime numbers, and $p + 1$ be a perfect square?

**Problem Set 4 Level B**

Let $a = 201920202021$. Compute:

$$\left(\frac{a^3 - 1}{a^2 - 1}\right)\left(\frac{a^2 + 2a + 1}{a^3 + 1}\right)\left(\frac{a^2 - a + 1}{a + 1}\right) - \frac{a^2}{a + 1}$$

**Bonus**: Let $a$ now be 20192020202120222, what is the value of the expression?

**Problem Set 4 Level C**

Dogs in the GoodDog obedience school win a *blue ribbon* for learning how to sit, a *green ribbon* for learning how to roll over, and a *white ribbon* for learning how to stay. A new class of 180 dogs has enrolled in the GoodDog school.

We have the following facts:

- An equal number of dogs have each of the ribbons.
- An equal number of dogs have each pair of ribbons.
- 15 dogs have all three ribbons.
- All dogs have at least one ribbon.
- The number of dogs with exactly one ribbon equals twice the number of dogs with more than one ribbon.

**How many dogs** have blue ribbons?

**Problem Set 4 Level A Solution**

If there is such perfect square $n^2 = p + 1 = 2 \cdot 3 \cdot 4 \cdots 97 + 1$:

$$n^2 - 1 = (n-1)(n+1) = 2 \cdot 3 \cdot 4 \cdots 97$$

Since $n^2$ is odd, so both $n-1$, $n+1$ are even. The left side is divisible by 4, the right side is not.

**Problem Set 4 Level B Solution**

Since: $a^3 - 1 = (a-1)(a^2+a+1)$ , $a^2 - 1 = (a-1)(a+1)$ , and $a^2 + 2a + 1 = (a+1)^2$, $a^3 + 1 = (a+1)(a^2 - a + 1)$. Therefore:

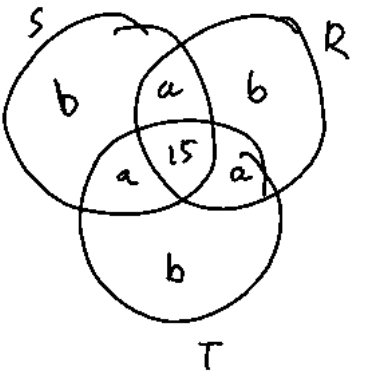
$$\left(\frac{(a-1)(a^2+a+1)}{(a-1)(a+1)}\right)\left(\frac{(a+1)^2}{(a+1)(a^2-a+1)}\right)\left(\frac{a^2-a+1}{a+1}\right)$$

$= \dfrac{a^2+a+1}{a+1}$. So the value of the given expression is:

$$\frac{a^2+a+1}{a+1} - \frac{a^2}{a+1} = 1$$

**Problem Set 4 Level C Solution**

Let the number of dogs who have each of the three ribbons be $S, R, T$ for each category. Let the number of dogs who *have only one ribbons* be $b$ and the number of dogs with *exactly two ribbons* be $a$.




The number of dogs that have exactly two ribbons is $3a$. The number of dogs with only one ribbon is $3b$.

Since the total number of dogs is 180 and any dog has at least one ribbon, so: $3a + 3b + 15 = 180 \Rightarrow a + b = 55$.

From the last fact given:

$3b = 2(3a + 15) \Rightarrow b = 2a + 10 \Rightarrow 3a + 10 = 55$

$\Rightarrow a = 15, b = 40$

So the number of dogs have blue ribbon is:

$b + 2a + 15 = 40 + 30 + 15 = 85$

**Problem Set 5 Level A**

Anthony forgot his first three exam scores. He did remember that his first score was 3 less than the second, and that the third grade was 11 more than the second. His teacher told him that he needed a grade of 100 to raise his average to 81. What was his third exam grade?

**Problem Set 5 Level B**

Chairs are equally spaced around a table and numbered from 1 to $2n$. Every odd chair is directly opposite another odd numbered chair. Prove that the number of chairs $2n$ is a multiple of 4.

**Problem Set 5 Level C**

The number 2010 and 2020 are written on a black board. Lilian and her friend takes turn to write a positive difference between two numbers already on the board, if this number does not already appear on the board. The loser is the one who cannot write a number. Can you suggest a **winning strategy** for Lilian? **Bonus**: how many turns the game will last if Lilian does right?

**Problem Set 5 Level A Solution**

Let his second score be $x$. The first score was $x - 3$, the third was $x + 11$. If he scores 100 on the fourth test, the sum of his four scores will be $4 \cdot 81 = 324$. Thus $x + x - 3 + x + 11 + 100 = 324$ and $x = 72$. Thus his third exam grade was $x + 11 = 83$.

**Problem Set 5 Level B Solution**

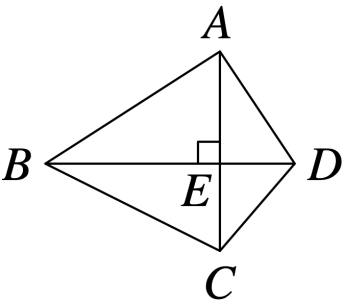
The difference between the number on a chair and the number on the chair across from it is $2n/2 = n$. If every odd numbered chair is directly across from another odd numbered chair, the difference between them, $n$, is even. Since $n$ is even, $2n$, is divisible by 4.

**Problem Set 5 Level C Solution**

In this game the greatest common divisor of the two initial numbers must eventually be written down. Therefore, every multiple of the greatest common divisor, and not greater than the original numbers, will also appear. Since the greatest common divisor of the original numbers in this game is 10, so every number from 10 to 2020 will appear. There are 200 turns and the second player will win.

**Problem Set 6 Level A**

In quadrilateral $ABCD$ is $AC \perp BD$ as shown below.




Prove that: $AB^2 + CD^2 = BC^2 + DA^2$.

**Problem Set 6 Level B**

In one Berwyn high school, there are 75 students. If 30 students are studying Czech, 20 are studying Polish, 15 are studying German, 11 are studying Czech and Polish, 9 are studying Czech and German, 34 are not studying any language, and 4 are studying all three languages, then how many students are studying Polish and German?

**Bonus**: how many students learn a single language only?

**Problem Set 6 Level C**

A set of points is chosen on the circumference of a circle so that *the number of different triangles* with all three vertices among the points is equal to *the number of pentagons* with all five vertices in the set.

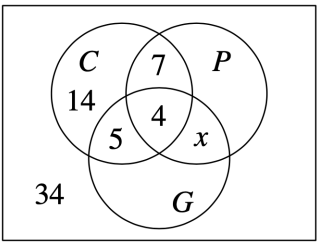
**How many points** are there?

**Problem Set 6 Level A Solution**

Using the Pythagorean Theorem to express the squares of the sides of  in terms of the distances from $E$ to the vertices:

$$AB^2 + CD^2 = (AE^2 + EB^2) + (EC^2 + ED^2)$$
$$= (AE^2 + ED^2) + (BE^2 + EC^2) = AD^2 + BC^2$$

**Problem Set 6 Level B Solution**




Draw a Venn diagram and denote $x$ as above. Note that:

- Polish only: $20 - 7 - 4 - x = 9 - x$, and
- German only: $15 - 4 - 5 - x = 6 - x$.

Therefore $34 + (9 - x) + 7 + 4 + x + 14 + 5 + (6 - x) = 79 - x$.

Hence $x = 4$. So 8 students study both Polish and German is 8.

For the bonus question, it's :

$$14 + (9 - 4) + (6 - 4) = 14 + 5 + 2 = 21$$

**Problem Set 6 Level C Solution**

We form triangles by just choosing three vertices from among the points, so the number of triangles that can be drawn from $n$ points is just $\binom{n}{3}$. Similarly the number of pentagons is $\binom{n}{5}$. Thus we have:

$$\frac{n!}{5!\,(n-5)!} = \frac{n!}{3!\,(n-3)!} \Rightarrow \frac{(n-3)!}{(n-5)!} = \frac{5!}{3!} = 20$$

Hence $(n - 3)(n - 4) = 20 \Rightarrow n = 8$.

# Problems of the Month

**Level A**

**Problem 1.A**

What is the minimum number of weights which enable us to weigh any integer number of grams of gold from 1 to 100 on a standard balance with two pans?

*Note: Weights may be placed on the either pan of the balance.*

**Problem 2.A**

For $k$, $l$, $m$ positive integers, prove that:

$$2^{k+l+m} > 2^k + 2^l + 2^m$$

**Problem 3.A**

There are eight white pawns in the first row of a chessboard, and eight black pawns in the eighth row. Each of two players, in turn, can move one of his or her pawns towards the other end of the board in a vertical direction for any number of boxes. It is not allowed to jump over a pawn of the opposite colour. The player who cannot make a move loses.

What is the winning strategy?

**Problem 4.A**

In $\triangle ABC$, $\angle A = 90^\circ$. Median $AM$, angle bisector $AK$, and altitude $AH$ are drawn.

Prove that $\angle MAK = \angle KAH$.

**Problem 5.A**

Construct the triangle $ABC$ if you know the midpoints $D$, $E$, $F$ of $BC$, $CA$, $AB$, respectively.

**Problem 6.A**

Which number is greater?

$$\frac{1\underbrace{0\cdots0}_{2019}1}{1\underbrace{0\cdots0}_{2020}1} \text{ and } \frac{1\underbrace{0\cdots0}_{2020}1}{1\underbrace{0\cdots0}_{2021}1}$$

**Problem 7.A**

In a certain country each town is connected with every other by a one-way road. Prove that there is a town from which you can drive to any other.

**Problem 8.A**

Dr. Gizmo has invented a coin changing machine which can be used in any country in the world. No matter what the system of coinage, the machine takes any coin, and, if possible, returns exactly five others with the same total value.

Prove that no matter how the coinage system works in a given country, you can never start with a single coin and end up with 26 coins.

## Level B

### Problem 1.B

An evil king wrote three secret two-digit numbers $a$, $b$, and $c$. A handsome prince must name three numbers $X$, $Y$, and $Z$, after which the king will tell him the sum $aX + bY + cZ$.

The prince must then name all three of the King's numbers. Otherwise he will be executed. Help him out of this dangerous situation

### Problem 2.B

For $x, y > 0$ real numbers, prove that:

$$\frac{x^6}{y^2} + \frac{y^6}{x^2} \geq x^4 + y^4$$

### Problem 3.B

There are eight white pawns in the first row of a chessboard, and eight black pawns in the eighth row. Each of two players, in turn, can move one of his or her pawns towards or backwards to one end of the board in a vertical direction for any number of boxes. It is not allowed to jump over a pawn of the opposite colour. The player who cannot make a move loses.

What is the winning strategy?

### Problem 4.B

Square $ABCD$ is given. A circle with radius $AB$, centred at $A$ is drawn. This circle intersects the perpendicular bisector of $BC$ at two points, of which $O$ is the closest to $C$.

Find the value of $\angle AOC$.

### Problem 5.B

Twenty teams played a volleyball tournament in which each team played every other team exactly once.

Prove that the teams can be numbered 1 through 20 in such a way that team 1 defeated team 2, team 2 defeated team 3, team 19 defeated team 20.

**Problem 6.B**

Which number is greater?

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots-\frac{1}{99}+\frac{1}{100} \text{ or } \frac{1}{5}$$

**Problem 7.B**

Segment $AB$ is given in the plane. An arbitrary point $M$ is chosen on the segment, and isosceles right triangles $AMC$ and $BMD$ are constructed on segments $AM$ and $MB$ (as their hypotenuses) so that points $C$ and $D$ are on the same side of $AB$.

Find the set of midpoints of all such segments $CD$.

**Problem 8.B**

There are three printing machines. The first accepts a card with any two numbers $a$ and $b$ on it and returns a card with the numbers $a+1$ and $b+1$. The second accepts only cards with two even numbers $a$ and $b$ and returns a card with the numbers $a/2$ and $b/2$ on it. The third accepts two cards with the numbers $a$, $b$ and $b$, $c$respectively, and returns a card with the numbers $a$, $c$. All these machines also return the cards given to them.

Is it possible to obtain a card with numbers 1, 1988, if we originally have only a card with the numbers 5, 19?

## Level A

### Solution 1.A

To explain the solution to this problem we need the following interesting property of the base 3 system. Every natural number can be represented as the difference of two numbers whose base 3 representations contain only 0's and 1's. We can prove this property by writing the original number in the base 3 system and constructing the required numbers digit by digit from right to left.

Now it is clear that it suffices to have only 5 weights that weigh 1, 3, 9, 27, 81. Four weights are insufficient since we cannot weigh more than $3^4 - 1 = 80$ different weights on them (each weight is either placed on the left pan, or on the right, or not placed on the balance at all).

### Solution 2.A

WLOG assume $k \geq l \geq m$. Hence:

$$2^{k+l+m} > 2^{k+2} = 4 \cdot 2^k > 2^k + 2^l + 2^m$$

### Solution 3.A

The second player maintains the line symmetry of the board with respect to the line separating the fourth and fifth column. He just need to duplicate whatever the first player does.

### Solution 4.A

$M$ is the circumcenter: $MA = MC, \angle MAC = \angle MCA$.

### Solution 5.A

Consider $D$. Draw line $l$ through $D$, parallel with $EF$. Draw circle $\omega$ centred at $A$, radius $EF$. $l$ and $\omega$ intersect at $B$, $C$.

### Solution 6.A

$$\text{Let } x = 1\underbrace{0\cdots0}_{2020}. \quad \frac{1\underbrace{0\cdots0}_{2019}1}{1\underbrace{0\cdots0}_{2020}1} = \frac{x+1}{10x+1}, \quad \frac{1\underbrace{0\cdots0}_{2020}1}{1\underbrace{0\cdots0}_{2021}1} = \frac{10x+1}{100x+1}$$

$$(x+1)(100x+1) = 100x^2 + 101x + 1 > 100x^2 + 20x + 1 = (10x+1)^2$$

$$\text{So } \frac{1\underbrace{0\cdots0}_{2019}1}{1\underbrace{0\cdots0}_{2020}1} > \frac{1\underbrace{0\cdots0}_{2020}1}{1\underbrace{0\cdots0}_{2021}1}$$

**Solution 7.A**

We proceed by induction on the number of towns. The base of the induction is obvious. To prove the inductive step we remove one of the towns. For the rest of them, using the inductive step, we can find a town A possessing the required property. Now we replace the removed town B. If there is at least one road going to B, then A is the required town for the original problem. If all the roads lead from B, then B is the town we need.

**Solution 8.A**

Since after every operation of Dr. Gizmo's machine the number of coins increases by 4, then the remainder of the number of coins when divided by 4 is invariant. But 26 and 1 have different remainders when divided by 4. Therefore we cannot end up with 26 coins.

## Level B

**Solution 1.B**

The prince can name the numbers 1, 100, and$100^2 = 10000$. The numbers $a$, $b$, and $c$ will then just be the digits of the sum $aX + bY + cZ$ in the base 100 system.

**Solution 2.B**

$$\frac{x^6}{y^2} + \frac{y^6}{x^2} - (x^4 + y^4) = \frac{x^8 + y^8 - x^2y^2(x^4 + y^4)}{x^2y^2}$$

$$= \frac{x^6(x^2 - y^2) - y^6(x^2 - y^2)}{x^2y^2} = \frac{(x^2 - y^2)(x^6 - y^6)}{x^2y^2} \text{ (*)}$$

Since $x^6 - y^6 = (x^2 - y^2)(x^4 + x^2y^2 + y^4)$, so:

$(x^2 - y^2)(x^6 - y^6) = (x^2 - y^2)^2(x^4 + x^2y^2 + y^4)$

Because $x^4 + x^2y^2 + y^4 = (x^2 + \frac{y^2}{2})^2 + \frac{3y^4}{4} > 0$, $(x^2 - y^2)^2 \geq 0$

So the fraction (*) is non-negative.

**Solution 3.B**

The second player maintains the line symmetry of the board with respect to the line separating the fourth and fifth column. He just need to duplicate whatever the first player does. Note that the game can last forever.

**Solution 4.B**

$\triangle AOD$ is an equilateral. The rest is simple.

**Solution 5.B**

We prove this by induction on the number of teams $n$. The base case $n = 2$ is easy. To prove the inductive step let us temporarily remove one of the teams $X$ and number the other $n - 1$ teams as required. If $X$ has defeated team 1 or has lost to team $n - 1$, then we can easily add $X$ to the chain. Assuming the opposite we have that $X$ lost to team 1 and defeated team $n - 1$. Therefore there must exist such an integer k that $X$ lost to team $k$ and defeated team $k + 1$ - otherwise $X$ must have lost to team 2, therefore to team 3, therefore to team 4, et cetera. Having found such a $k$ we can insert $X$ into the existing chain by "cutting in" between teams $k$ and $k + 1$ and create the chain of teams we need.

**Solution 6.B**

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots-\frac{1}{99}+\frac{1}{100}$$
$$=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{98}-\frac{1}{99}\right)+\frac{1}{100}$$
$$=\frac{1}{6}+\frac{1}{20}+\cdots+\frac{1}{100}>\frac{1}{6}+\frac{1}{20}>\frac{1}{5}$$

**Solution 7.B**

Draw perpendicular line from the midpoint to $AB$. Observe the distance of the point to $AB$.

**Solution 8.B**

The first operation maps $(a,b)$ to $(a+1,b+1)$. What is invariant under this operation? Certainly, the difference between the numbers on the cards, since !~+l)-(b+l) = a-$(a+1)-(b+1)=a-b$b. But the second operation changes the difference $a/2-b/2=(a-b)/2$ the difference is divided by two. The third operation adds the differences: $a-c=(a-b)+(b-c)$..

These observations make us think that it is not the difference between the numbers on the card which is invariant. However, it is very likely that the difference has something to do with this (so far unknown) invariant. So what can it be? Let 115 look more closely and try to obtain some cards from the given one:

(5,19) → (6,20), (6,20) → (3,10),(3,10) → (20,27), (20,27) → (6,27)

We have the following cards:(5,19), (6,20), (3,10), (20,27), (6,27)

The differences for the pairs of number on the cards are: 14, 14, 7, 7, 21. Finally, we know what we must prove! The most plausible conjecture is that our difference$a-b$ is always divisible by 7. This fact can be proved quite easily.

The difference of (1, 1988) is not divisible by 7.