Chapter 1

Entrance tests

1.1 Rules

Choice of level:

- There are three test papers, each for one level.
- Each student can chose multiple test levels. If the students choose more than one level, the student must choose multiple different test times.

Registration:

- At registration, student and their parents are responsible for the correct email address and the time, where and when they wish to receive the test problems.
- The club will not verify whether the registered email address is correct.

Entrance test:

- Each level test consisting of 8 problems will be delivered by automatic scheduled email at the registered time. No test paper will be sent twice.
- Students have 270 minutes to solve the problems. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET.
- After the allowed time, the student has 15 minutes to scan the solutions. For example if a student registers for the test at 7:00 AM ET then the student will receive the test at 7:00 AM ET, work on the test until 11:30 AM ET, and has to submit the scanner solutions before 11:45 AM. Any solution submitted later than the allowed time will not be accepted.
- All submissions must be done by email to nghia71@gmail.com.
- Solutions must be written in English. Any solution looks unreadable, or is presented badly will not be considered for grading. Answer without proper solution will not be considered as solution. Answer consists of self-explanatory and easy-to-understand diagrams are acceptable but might not be considered as full solution.
- A problem can contain more than one questions, each with an indication of the maximum number of points can be awarded if a correct solution is provided. Partially correct solution is awarded with a some points depending on its quality. Multiple solutions can be submitted for a single problem. Each different solution, if correct, will be awarded the same number of points. Partially distinct solution also counts.
- Student who performs adequately will be accepted to the registered level. Student who are accepted to different levels can choose any of those levels to attend.

1.2 Problems for Introductory Level

Problem 1.2.1 (23-24-ET-I-P1)**.** (*10 points*) Place algebraic operations $+; -; \times; \div$ between the digits 1 to 9, in that order, so that the total equals 100. You may freely use brackets before or after any of the digits in the expression. No digits may be placed together, such as 123 and 67 are not allowed.

Here is an example for a correct answer: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9 = 100$.

Here is an example for an incorrect answer: $12 + 3 + 4 + 5 - 6 - 7 + 89 = 100$.

Do not submit any of these examples as your solution.

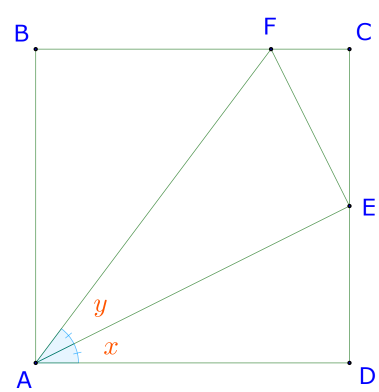
Problem 1.2.2 (23-24-ET-I-P2)**.** (*10 points*) Linh, Minh and Vinh are identical triplet sisters. Linh always tells the truth, Minh always lies and Vinh sometimes lies and sometimes tells the truth.

They are invited to a party. One of them arrives late. The party host asks this sister, who just arrived late, who she is. She answers "I am Vinh". The party host obviously cannot tell the girls apart so she asks the other two sisters the name of the sister who was late. One of them says: "Linh was late", and the other says: "Minh was late".

Which sister was late?

Problem 1.2.3 (23-24-ET-I-P3)**.** (*10 points*) $ABCD$ is a square with side length 4. E is midpoint of CD. $AE \perp EF$. Let $x = \angle EAD, y = \angle FAE$.

Prove that $x = y$.

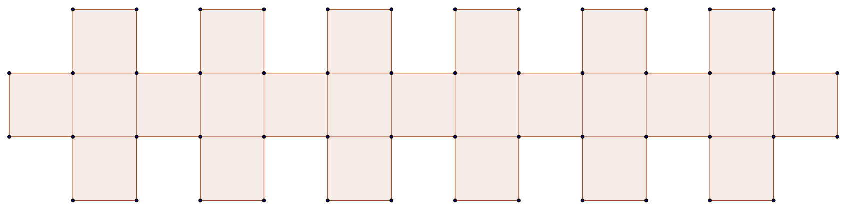


Problem 1.2.4 (23-24-ET-I-P4)**.** (*10 points*) A pair of two-digit numbers has the following properties:

1. The sum of the four digits is 25.
2. The sum of the two numbers is 97.
3. The product of the four digits is 864.
4. The product of the two numbers is 1972.

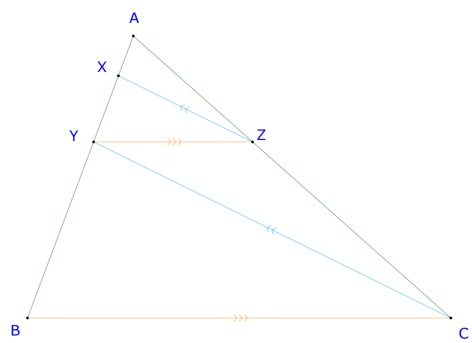
Determine the two numbers.

Problem 1.2.5 (23-24-ET-I-P5)**.** (*10 points*) The shape in the diagram below was created by putting together 25 unit squares. When a similar shape is created with n squares, its perimeter is 100 units. Determine n.



Problem 1.2.6 (23-24-ET-I-P6)**.** (*10 points*) N is the sum of all integers from 1 to n and M is the sum of all integers from 1 to m, where $n > m + 1$. If the difference of the two sum $N - M = 2012$, then what is the value of $n + m$?

Problem 1.2.7 (23-24-ET-I-P7)**.** (*10 points*) The area of $\triangle ABC$ is 1. X, Y are points on AB, Z is a point on AC such that $XY = 2AX$. Furthermore $XZ \parallel YC$, $YZ \parallel BC$. Find the area of $\triangle XYZ$.



Problem 1.2.8 (23-24-ET-I-P8)**.** (*10 points*) Consider a sequence of integers

$$1, 3, 2, -1, \ldots$$

where each term is equal to the term preceding it minus the term before that.

What is the sum of the first 2023 terms?

1.3 Solutions for Introductory Level

Problem 1.3.1 (Mathematics Competitions Vol. 34, #1 2021). (*10 points*) Place algebraic operations $+; -; \times; \div$ between the digits 1 to 9, in that order, so that the total equals 100. You may freely use brackets before or after any of the digits in the expression. No digits may be placed together, such as 123 and 67 are not allowed.

Here is an example for a correct answer: $1+2+3+4+5+6+7+8\times 9 = 100$.

Here is an example for an incorrect answer: $12+3+4+5-6-7+89 = 100$.

Do not submit any of these examples as your solution.

Solution. Following are five solutions, including the correct example:

$$\begin{aligned}
1+2+3+4+5+6+7+8\times 9 &= 100 & \quad 1\times 2\times 3+4+5+6+7+8\times 9 &= 100\\
(1+2+3+4+5)\times 6-7+8+9 &= 100 & \quad 1\times(-2-3-4-5)+6\times 7+8\times 9 &= 100\\
1-2+3\times 4\times 5+6\times 7+8-9 &= 100 & \quad (1-2+3)\times(4+5)\times(-6+7\times 8)\div 9 &= 100\\
1\times(2+3+4-5+6)\times(-7+8+9) &= 100 & \quad 1\times(2+3)\times 4(-5-6+7\times 8)\div 9 &= 100\\
1\times(-2+3+4+5)\times 6\times(7+8)\div 9 &= 100 & \quad (1+2\times 3\times 4)\times 5-6\times 7+8+9 &\\
-1\times 2-3-4-5+6\times 7+8\times 9 &= 100 & \quad 1\times 2\times(3+4)\times 5+6+7+8+9 &= 100
\end{aligned}$$

□

Problem 1.3.2 (Problem Solving for Irish Second level Mathematicians - PRISM 2015/Problem 20). (*10 points*) Linh, Minh and Vinh are identical triplet sisters. Linh always tells the truth, Minh always lies and Vinh sometimes lies and sometimes tells the truth.

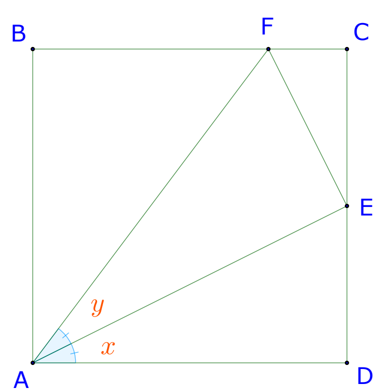
They are invited to a party. One of them arrives late. The party host asks this sister, who just arrived late, who she is. She answers "I am Vinh". The party host obviously cannot tell the girls apart so she asks the other two sisters the name of the sister who was late. One of them says: "Linh was late", and the other says: "Minh was late".

Which sister was late?

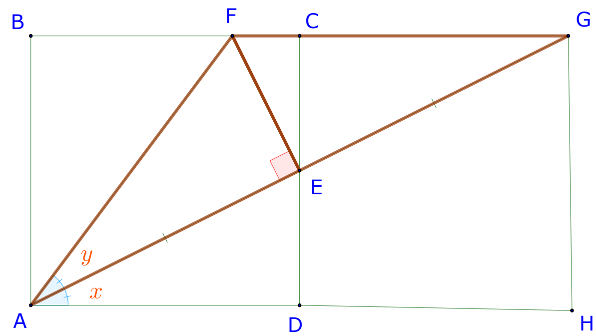
Solution. Obviously the sister in question should not be Linh, since Linh always tells the truth. Thus she could be Minh or Vinh. It means that Linh was one of the other two sisters.

One of these two sisters (both arrived in time) who said that Linh was late could have not been Linh, thus the other one was Linh. Linh told the truth, she said Minh was late, hence $\boxed{\text{Minh was late.}}$ □

Problem 1.3.3 (17th Blundon Mathematics Contest 2000/P5). (*10 points*) $ABCD$ is a square with side length 4. E is midpoint of CD. $AE \perp EF$. Let $x = \angle EAD, y = \angle FAE$. Prove that $x = y$.

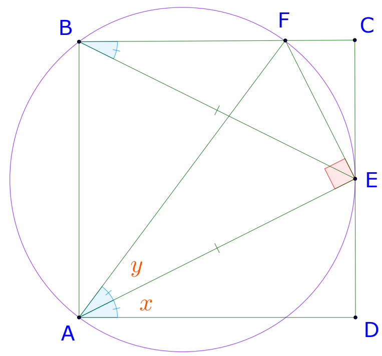


Solution. [Solution 1] Extend AE intersecting BC at G.



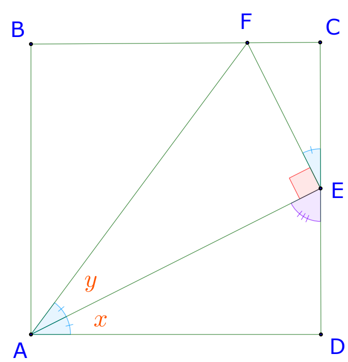
E is midpoint of CD, so $AE = EG$. Thus $\triangle AEF \cong \triangle GEF$. Thus $\angle EAF = \angle EGF = \angle EAD$. □

Solution. [Solution 2] Since both $\triangle AEF$ and ABF are right triangles with the same hypotenuse AF, thus B and E are on the circle diameter AF.



$\angle FAE = \angle EBF(\text{subtend } \widehat{EF}) = \angle EAD$ ($\triangle EAD \cong \triangle EBC$). □

Solution. [Solution 3] Since $\triangle ADE$ is a right triangle at D, so $\angle AED = 90^\circ - \angle EAD$.



Thus $\angle CEF = 90^\circ - \angle AED = \angle EAD$. Therefore $\triangle EAD \sim \triangle FEC$ (angle-angle-angle). $AD = 2EC$, so $AE = 2EF$, or $\frac{AE}{EF} = \frac{AD}{DE} = 2$, $\angle AEF = \angle ADE = 90^\circ \Rightarrow \triangle AEF \sim \triangle ADE \Rightarrow \boxed{\angle FAE = \angle EAD.}$ □

Problem 1.3.4 (Modification to Crux Mathematicorum M266). (*10 points*) A pair of two-digit numbers has the following properties:

1. The sum of the four digits is 25.
2. The sum of the two numbers is 97.
3. The product of the four digits is 864.
4. The product of the two numbers is 1972.

Determine the two numbers.

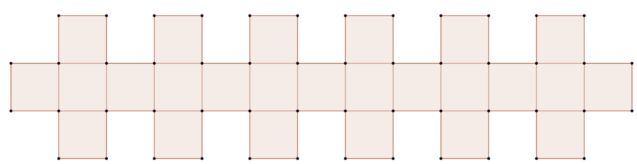
Solution. [Solution 1] Note that $1972 = 2^2 \cdot 17 \cdot 29$, thus it has 5 two-digit divisors $17, 29, 34, 58, 68$:

$$1972 = 17 \cdot 116 = 29 \cdot 68 = 34 \cdot 58.$$

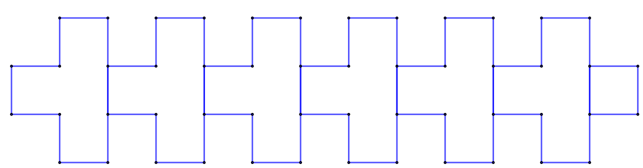
It is easy to verify that only the pair $\boxed{(29, 68)}$ satisfy other conditions. □

Solution. [Solution 2] Let the two-digit numbers be x and y, then $x + y = 97$, $xy = 1972$. Therefore x, y are the roots of $t^2 - 97t + 1972 = 0$. By solving the quadratic equation, we have $(x, y) = \boxed{(29, 68).}$ □

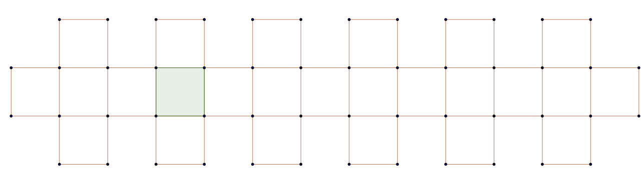
Problem 1.3.5 (2014-2015 Nova Scotia Math League/Game 1/Team Question 2). (*10 points*) The shape in the diagram below was created by putting together 25 unit squares. When a similar shape is created with n squares, its perimeter is 100 units. Determine n.



Solution. The figure shown above consists of k blue block of 4 unit squares and a single unit square at the far right end. Thus altogether $4k + 1$ squares.



The perimeter of the figure is the sum of perimeters of all $4k + 1$ squares, subtracting *twice* the perimeters of all middle squares like the one coloured green.



Thus, the perimeter is:

$$4(4k + 1) - 2(4)(k) = 8k + 4.$$

Now, if $8k + 4 = 100$, then $k = 12$, hence the number of square $n = 4k + 1 = 4 \cdot 12 + 1 = \boxed{49.}$ □

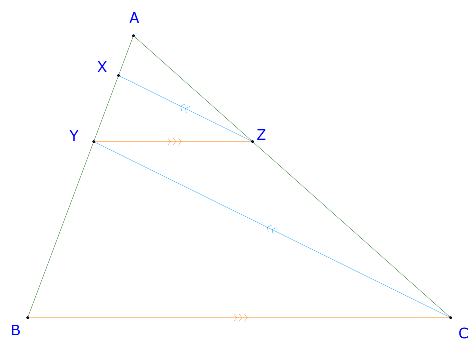
Problem 1.3.6 (2012 BC Secondary School Mathematics Contest, Senior Final, Part A/Problem 5)**.** (*10 points*) N is the sum of all integers from 1 to n and M is the sum of all integers from 1 to m, where $n > m+1$. If the difference of the two sum $N - M = 2012$, then what is the value of $n + m$?

Solution. First, note that

$$N - M = (1 + 2 + \cdots + n) - (1 + 2 + \cdots + m) = \frac{n(n+1)}{2} - \frac{m(m+1)}{2} = \frac{(n-m)(n+m+1)}{2}$$
$$N - M = 2012 \Rightarrow (n-m)(n+m+1) = 4024 \quad (*)$$

Now, $(n - m) + (n + m + 1) = 2n + 1$ is an odd number, so $n - m$ and $n + m + 1$ have different parities (one is odd and the other is even). Because their product, as shown above in (*), is an even number and since $n - m \geq 2$ and $4024 = 2^3 \cdot 503$, therefore the only possibility is $n - m = 8$, $n + m + 1 = 503$, or $n + m = \boxed{502.}$ □

Problem 1.3.7 (2010 Alberta High School Mathematics Competition, Part I/Problem 15)**.** (*10 points*) The area of $\triangle ABC$ is 1. X, Y are points on AB, Z is a point on AC such that $XY = 2AX$. Furthermore $XZ \parallel YC$, $YZ \parallel BC$. Find the area of $\triangle XYZ$.



Solution. Note that $\triangle AYZ \sim \triangle ABC$ and $XZ \parallel YC$, so

$$\frac{AY}{AB} = \frac{AZ}{AC} = \frac{AX}{AY} = \frac{1}{3}$$

Thus $[XYZ] = \frac{2}{3}[AYZ] = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2[ABC] = \boxed{\frac{2}{27}.}$ □

Problem 1.3.8 (2009 Fifth Annual Kansas Collegiate Mathematics Competition/Problem 1)**.** (*10 points*) Consider a sequence of integers

$$1, 3, 2, -1, \ldots$$

where each term is equal to the term preceding it minus the term before that.

What is the sum of the first 2023 terms?

Solution. Lets write down some more terms of the sequence

$1, 3, 2, -1, (-1-2=)-3, (-3-(-1)=)-2, (-2-(-3)) = 1, (1-(-2)=3), (3-1) = 2, (2-3=)-1, \ldots$

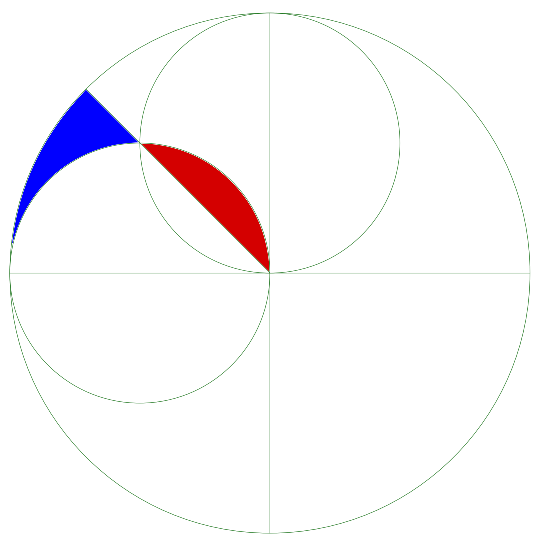
It is easy to see that the sequence consists of repeated blocks of the first six terms. The sum of these six terms is $1 + 3 + 2 + (-1) + (-3) + (-2) = 0$. Since $2023 = 2022 + 1 = 337 * 6 + 1$ thus the sum of the first 2023 terms is the first term of the block, which is $\boxed{1}$. □

1.4 Problems for Advanced Level

Problem 1.4.1 (23-24-ET-A-P1)**.** (*10 points*) Two friends agreed to meet at a club between 3:00 PM and 5:00 PM. What is the probability that they meet at the club within 15 minutes of arrival?

Problem 1.4.2 (23-24-ET-A-P2)**.** (*10 points*) A number n written in base b reads 211, but it becomes 110 when written in base $b+2$. Find n and b in base 10.

Problem 1.4.3 (23-24-ET-A-P3)**.** (*10 points*) In the diagram below two perpendicular diameters divide a circle into four parts. On each of these diameters a circle of half the diameter is drawn, tangent to the original circle and meeting at its centre. A radius to the large circle is drawn through the intersection points of these smaller circles. Show that the red and blue shaded regions are of the same area.



Problem 1.4.4 (23-24-ET-A-P4)**.** (*10 points*) Determine all pairs of integers (not necessarily positive) (x, y) such that

$$(x-8)(x-10) = 2^y.$$

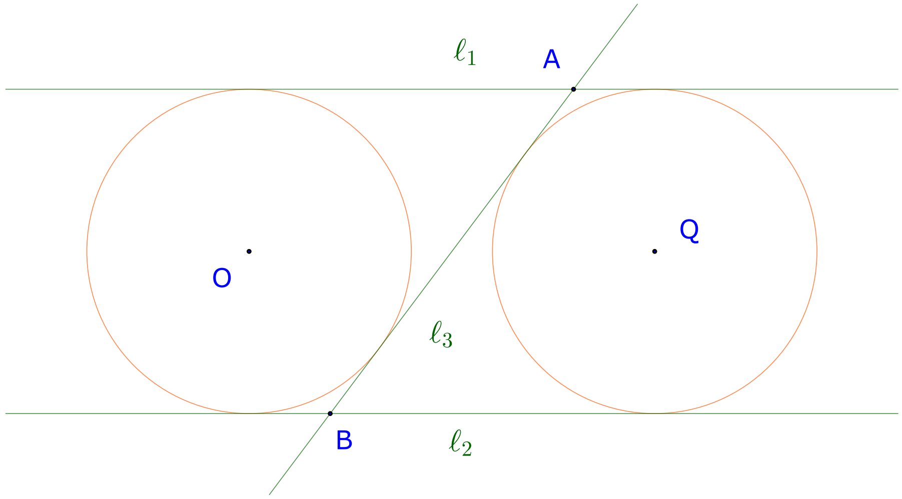
Problem 1.4.5 (23-24-ET-A-P5)**.** (*10 points*) A game is played on a 7×7 board, initially blank. Chi and Mai make alternate moves, with Chi going first. In each of her moves, Chi chooses any four blank squares which form a 2×2 block, and paints these squares brown. In each of her moves, Mai chooses any blank square and paints it green. They take alternate turns until no more moves can be made by Chi. Then Mai paints the remaining blank squares green.

Which player, if either, can guarantee to be able to paint 25 or more squares in her colour, regardless of how her opponent plays?

Problem 1.4.6 (23-24-ET-A-P6)**.** (*10 points*) Determine all sets consisting of an odd number $2m+1$ of consecutive positive integers, for some integer $m \geq 1$ such that the sum of the squares of the smallest $m+1$ integers is equal to the sum of the squares of the largest m integers.

Problem 1.4.7 (23-24-ET-A-P7)**.** (*10 points*) Given two parallel lines ℓ_1 and ℓ_2, the transversal ℓ_3 intersects them at points A and B respectively. Two circles with centres at O and Q lie between the parallel lines on the left and on the right sides of the transversal such that the circles are tangent to all three lines.

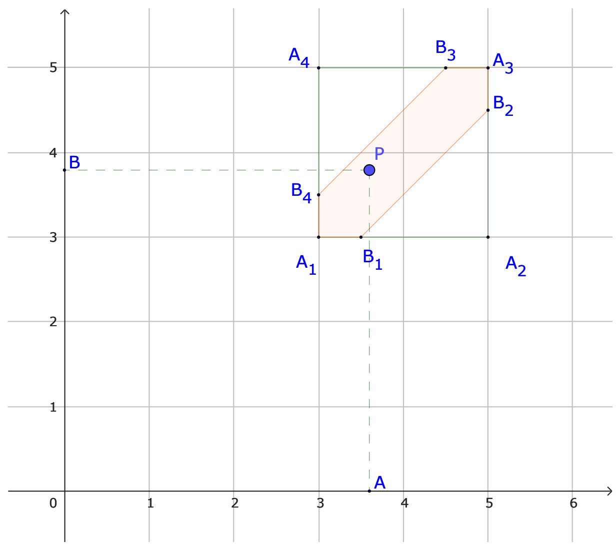
Show that $OQ = AB$.



Problem 1.4.8 (23-24-ET-A-P8)**.** (*10 points*) Find the primes p, q, and r, given that one of the number pqr and $p + q + r$ is 101 times the other.

1.5 Solutions for Advanced Level

Problem 1.5.1 (2016 Kansas MAA Undergraduate Mathematics Competition/Problem 2)**.** (*10 points*) Two friends agreed to meet at a club between 3:00 PM and 5:00 PM. What is the probability that they meet at the club within 15 minutes of arrival?



Solution. Let $3 \leq A \leq 5$ and $3 \leq B \leq 5$ denote the times of arrival of the two friends, shown in the diagram above. Point $P(A, B)$ should be within the square $A_1A_2A_3A_4$ in order to satisfy the given conditions of the problem. Now, if the two friends meet within 15 minutes of arrival, then P point must be within the polygon $A_1B_1B_2A_3B_3B_4$. The probability is the ratio of the two areas, which is

$$\frac{[A_1B_1B_2A_3B_3B_4]}{[A_1A_2A_3A_4]} = 1 - \frac{2[B_1B_2A_2]}{[A_1A_2A_3A_4]} = 1 - \frac{\left(\frac{7}{4}\right)^2}{4} = \boxed{\frac{15}{64}}.$$

□

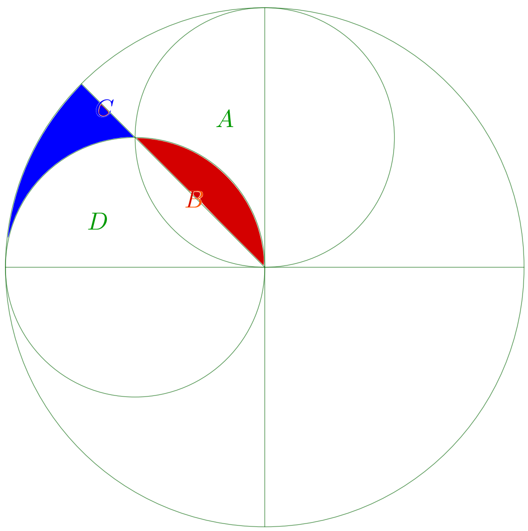
Problem 1.5.2 (Santa Clara University High School Mathematics 2001 Contest/Problem 4)**.** (*10 points*) A number n written in base b reads 211, but it becomes 110 when written in base $b+2$. Find n and b in base 10.

Solution. We have

$$\left.\begin{array}{l} n = 211_b = 2 \cdot b^2 + 1 \cdot b + 1 \\ n = 110_{b+2} = 1 \cdot (b+2)^2 + 1 \cdot (b+2) \end{array}\right\} \Rightarrow 2b^2 + b + 1 = (b+2)^2 + (b+2) \Rightarrow b^2 - 4b - 5 = 0 \Rightarrow b = -1 \text{ or } 5$$

Since $b > 0$, thus $\boxed{b = 5}$. $n = 211_5 = 110_7 = \boxed{56.}$ □

Problem 1.5.3 (2013 Manitoba Mathematical Competition/Problem 6)**.** (*10 points*) In the diagram below two perpendicular diameters divide a circle into four parts. On each of these diameters a circle of half the diameter is drawn, tangent to the original circle and meeting at its centre. A radius to the large circle is drawn through the intersection points of these smaller circles. Show that the red and blue shaded regions are of the same area.



Solution. Let A, B, C, and D denote the regions in the figure above. Let $[X]$ denotes the area of a region X.

$$[A]+[C]=\frac{1}{4}\pi(2r)^2-([B]+[D])=\frac{1}{4}\pi(2r)^2-\frac{1}{2}\pi r^2=\frac{1}{2}\pi r^2=[A]+[B]$$

Thus $[B]=[C]$, or $\boxed{\frac{1}{2}[B]=\frac{1}{2}[C].}$ □

Problem 1.5.4 (1962 examination, MAA Problem Book II (1961-1965)/Problem 36)**.** (*10 points*) Determine all pairs of integers (not necessarily positive) (x, y) such that

$$(x-8)(x-10)=2^y.$$

Solution. First, the right hand side of the equation is an integer, thus $y \geq 0$. Second, since $2^y > 0$, thus both $x-8$ and $x-10$ have the same sign (both are positive or both are negative). Third, their product is a power of two, thus the absolute value of each of them is a power of two.

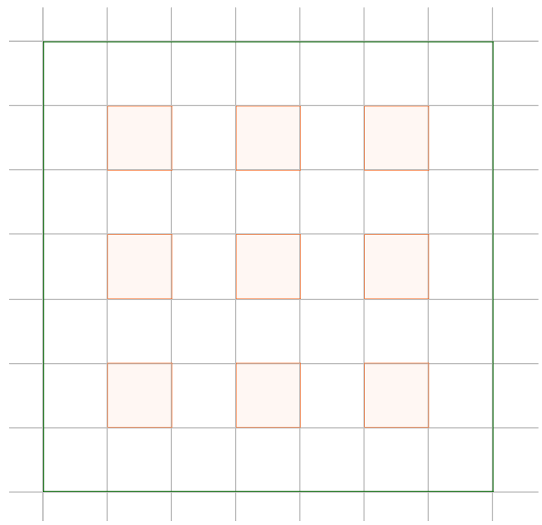
Since $(x-8)-(x-10)=2$, and the only pair of powers of two differ by 2 is 2 and 4, therefore

$$\begin{cases} x-10=2 \Rightarrow x=12, \text{ or} \\ x-10=-4 \Rightarrow x=6 \end{cases} \quad \text{in both cases } y=3$$

Hence, the desired pairs are $(x, y) \in \{(6,3),\ (12,3)\}$. □

Problem 1.5.5 (2009 Alberta High School Mathematics Competi- tion, Part II/Problem 3)**.** (*10 points*) A game is played on a 7×7 board, initially blank. Chi and Mai make alternate moves, with Chi going first. In each of her moves, Chi chooses any four blank squares which form a 2×2 block, and paints these squares brown. In each of her moves, Mai chooses any blank square and paints it green. They take alternate turns until no more moves can be made by Chi. Then Mai paints the remaining blank squares green.

Which player, if either, can guarantee to be able to paint 25 or more squares in her colour, regardless of how her opponent plays?



Solution. There are nine squares at the intersections of even-numbered rows. Any block chosen by Chi will include one of these nine squares. Thus, Mai has to play in those nine squares in her first four moves. This way Chi will have at most five moves and will be able to paint at most twenty squares. Thus, Mai will win. □

Problem 1.5.6 (Crux Mathematicorum/MA 136)**.** (*10 points*) Determine all sets consisting of an odd number $2m+1$ of consecutive positive integers, for some integer $m \geq 1$ such that the sum of the squares of the smallest $m+1$ integers is equal to the sum of the squares of the largest m integers.

Solution. Let $S = \{a, a+1, \dots, a+2m\}$ be the set with integers a, m and $m \geq 1$. Let $s_k = a^2 + (a+1)^2 + \cdots + (a+k)^2$, then

$$\begin{aligned} s_k &= a^2 + (a+1)^2 + \cdots + (a+k)^2 = (k+1)a^2 + 2a(1+2+\cdots+k) + (1^2+2^2+\cdots+k^2) \\ &= (k+1)a^2 + k(k+1)a + \frac{k(k+1)(2k+1)}{6}. \end{aligned}$$

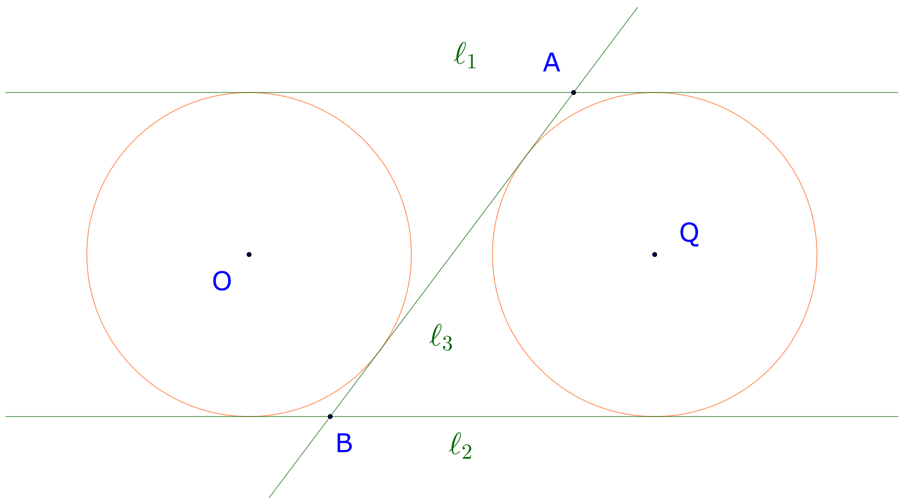
According to the given condition:

$$\begin{aligned} & a^2 + (a+1)^2 + \cdots + (a+m)^2 = (a+m+1)^2 + \cdots + (a+2m)^2 \\ \Rightarrow & 2\left(a^2 + (a+1)^2 + \cdots + (a+m)^2\right) = a^2 + (a+1)^2 + \cdots + (a+2m)^2 \Rightarrow 2s_m = s_{2m} \\ \Rightarrow & 2(m+1)a^2 + 2m(m+1)a + \frac{m(m+1)(2m+1)}{3} = (2m+1)a^2 + (2m)(2m+1)a + \frac{(m)(2m+1)(4m+1)}{3} \\ \Rightarrow & a^2 - 2am^2 - m^2(2m+1) = 0 \Rightarrow (a+m)\left(a - m(2m+1)\right) = 0 \Rightarrow a = m(2m+1) \end{aligned}$$

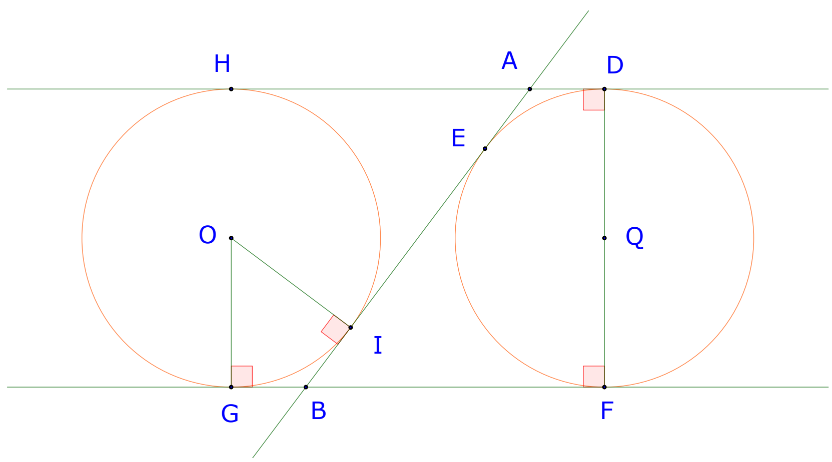
Thus $\boxed{S = \{m(2m+1), m(2m+1)+1, \dots, m(2m+1)+2m\}.}$ □

Problem 1.5.7 (Crux Mathematicorum/MA 190)**.** (*10 points*) Given two parallel lines ℓ_1 and ℓ_2, the transversal ℓ_3 intersects them at points A and B respectively. Two circles with centres at O and Q lie between the parallel lines on the left and on the right sides of the transversal such that the circles are tangent to all three lines.

Show that $OQ = AB$.



Solution. [Solution 1] Let D, E, F, G, H, I be the tangent points as shown in the diagram below.



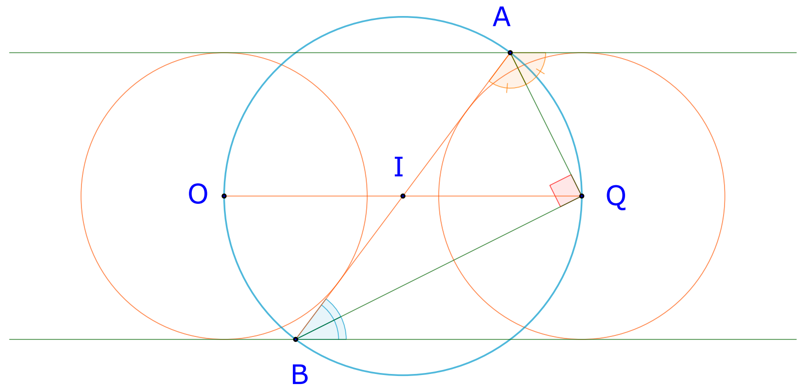
Now,

$$\left.\begin{array}{l} DH = DA + AH = AE + AI = 2AE + EI \\ GF = GB + BF = BI + IE = 2BI + EI \end{array}\right. DH = GF \Rightarrow BI = AE$$

$$\Rightarrow AB = AE + EI + IB = 2AE + EI = DH \Rightarrow \boxed{AB = OQ.}$$

□

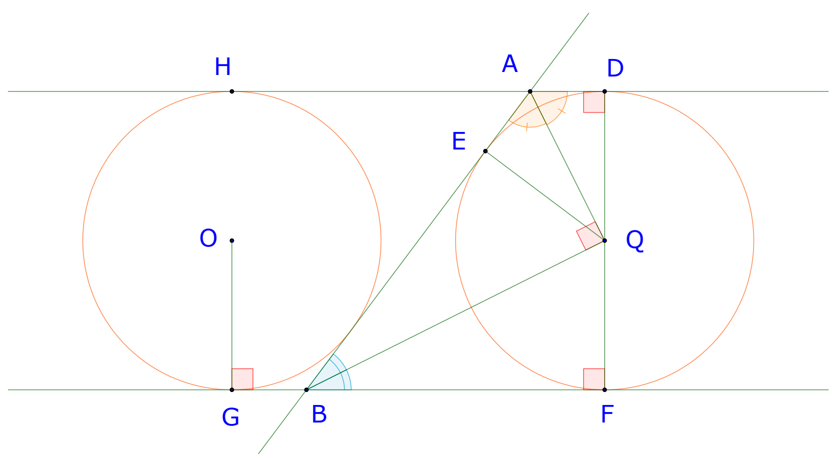
Solution. [Solution 2] Note that A is a point ot two tangents to the circle (Q), so AQ is the angle bisector. Similarly BQ is also an angle bisector as shown below.



Since the two angles are A and B sum up to $180°$, thus half of their sum is $90°$, which means that AQB is $90°$, or Q is on the circle with diameter AB. Similarly O is also on that circle.

Then $IO = IA = IB = IQ$ Note that $\angle IQA = \angle IAQ$, thus IQ parallel to ℓ_1, or I on the segment and is the midpoint of OQ. Therefore $OQ = OI + IQ = AB$. □

Solution. [Solution 3] Let D, E, F, G, H be the tangent points as shown in the diagram above. Let r be the radius of the circles.



Because of the symmetry, $AD = BG$, so $AD + BF = OQ$.

On the other hand, $\angle EAQ = \angle DAQ$, $\angle EBQ = \angle FBQ$, so $\angle EAQ + \angle EBQ = 90°$. Thus $\triangle AQB$ is a right triangle, thus $AB^2 = AQ^2 + BQ^2 = AD^2 + DQ^2 + QF^2 + BF^2$.

Since $\triangle ADQ \sim \triangle QFB$ (angle-angle-angle), so $\frac{AD}{DQ} = \frac{QF}{FB}$, or $AD \cdot BF = DQ \cdot QF = r^2$

Therefore, $AB^2 = AD^1 + BF^2 + 2AD \cdot BF = (AD + BF)^2 = OQ^2$. Hence $\boxed{OQ = AB.}$ □

Problem 1.5.8 (29th Nordic Mathematical Contest 2015/Problem 2)**.** (*10 points*) Find the primes p, q, and r, given that one of the number pqr and $p+q+r$ is 101 times the other.

Solution. [Solution 1] Since the product of two of p, q, r is at least 4, so the product pqr is at least four times the largest of p, q, and r, and therefore larger then $p+q+r$.

Thus,

$$pqr = 101(p+q+r).$$

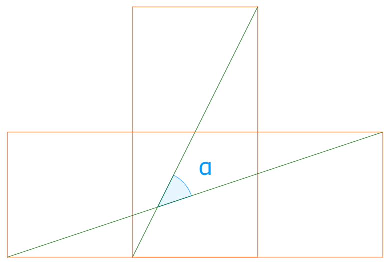
Now, since 101 is a prime, so one of p, q, r must be 101. WLOG, let $p = 101$, then

$$qr = 101 + q + r \Rightarrow q = 1 + \frac{102}{r-1}.$$

Therefore $r-1$ is a divisor of $102 = 2 \cdot 3 \cdot 17$. Since r is prime, so $r = 2, q = 103$ or $r = 103, q = 2$. Thus the three primes p, q, and r are $\boxed{2, 101, \text{ and } 103.}$ $\square$

1.6 Problems for Olympiad Level

Problem 1.6.1 (23-24-ET-O-P1)**.** (*10 points*) The figure below consists of four congruent squares. Find the angle α.



Problem 1.6.2 (23-24-ET-O-P2)**.** Find the largest possible positive integer n such that $n+3$ divides $1^3 + 2^3 + \cdots + n^3$.

Hint 1: Prove that if a prime number p divides $n+3$ and $n+3$ divides $1^3 + 2^3 + \cdots + n^3$, then $p \in \{2, 3\}$.

Hint 2: Analyze the polynomial division $1^3 + 2^3 + \cdots + n^3$ by $n+3$.

Problem 1.6.3 (23-24-ET-O-P3)**.** (*10 points*) Show that if a 5×5 matrix is filled with zeros and ones, there must always be a 2×2 sub-matrix (that is, the intersection of the union of two rows with the union of two columns) consisting entirely of zeros or entirely of ones.

Hint: prove that if we don't have a 2×2 sub-matrix of zeroes, we must have a 2×2 sub-matrix of ones.

Problem 1.6.4 (23-24-ET-O-P4)**.** (*10 points*) The sequence (a_n) is defined by

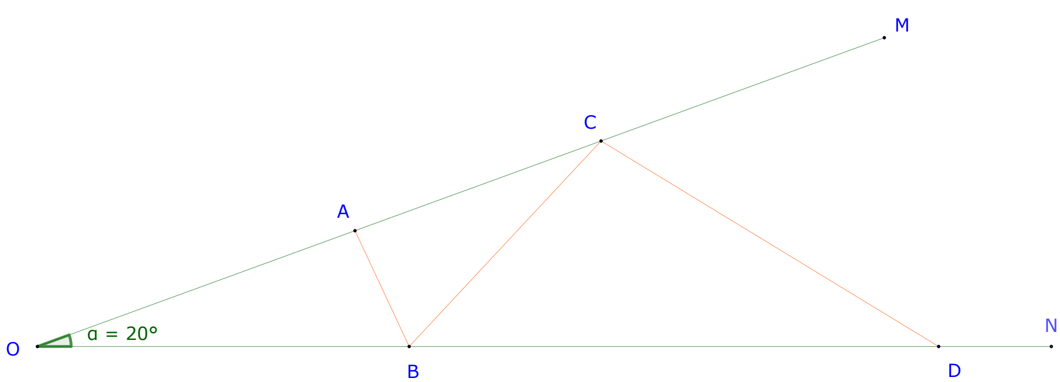
$$a_1 = 1,\ a_n = \sqrt{2a_{n-1} + a_{n-2} + \cdots + a_1},\ \text{if } n > 1.$$

Find a_{2023}. *Hint: compute a_2, a_5.*

Problem 1.6.5 (23-24-ET-O-P5)**.** (*10 points*) Points A, C are on ray OM, and B, D are on ray ON. It is given that $OA = 6, OD = 16, \angle NOM = 20°$.

What is the minimum length of $AB + BC + CD$?

Hint: when is the length of a broken line minimal?



Problem 1.6.6 (23-24-ET-O-P6)**.** (*10 points*) Find all triples of non-negative integers (x, y, z) and $x \leq y$ such that:

$$x^2 + y^2 = 3 \cdot 2016^z + 77.$$

Hint: if $x^2 + y^2 \equiv 0 \pmod 7$, *then what are remainders of* x *and* y *when divided by 7?*

Problem 1.6.7 (23-24-ET-O-P7)**.** (*10 points*) There are 2023 points in the plane such that among any three of them two can be selected so that their distance is less than 1. Prove that there is a circle of radius 1 containing at least 1012 of the given points.

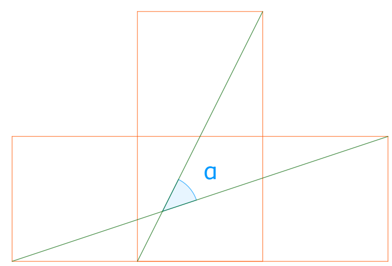
Problem 1.6.8 (23-24-ET-O-P8)**.** (*10 points*) Determine all polynomials $P(x)$ with real coefficients such that

1. $P(2023) = 2022$,
2. $(P(x) + 1)^2 = P(x^2 + 1)$, for all real numbers x.

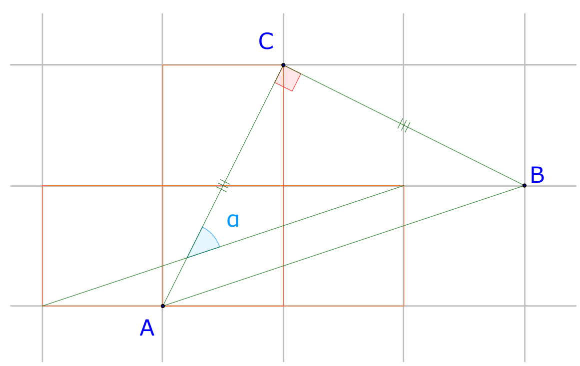
Investigate the sequence (x_n) *where* $x_1 = 2023$, $x_{n+1} = x_n^2 + 1$.

1.7 Solutions for Olympiad Level

Problem 1.7.1 (2016 Kansas MAA Undergraduate Mathematics Competition/Problem 2). (*10 points*) The figure below consists of four congruent squares. Find the angle α.

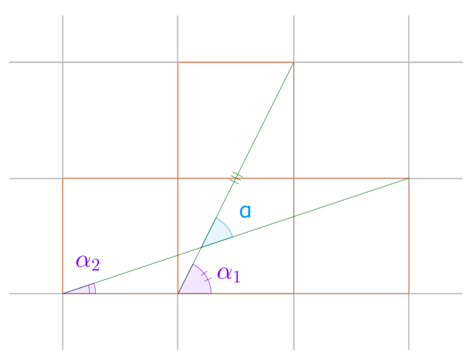


Solution. [Solution 1] It is easy to see that the segments are diagonals of a 1×2 and a 1×3 rectangles. Draw line AB parallel with the longer diagonal as show.



Since $AC^2 + CB^2 = (1^2 + 2^2) + (2^2 + 1^2) = 10 = 1^2 + 3^2 = AB^2$, so $\triangle ABC$ is an isosceles right triangle. Hence $\alpha = \angle CAB = \boxed{45^\circ.}$ $\square$

Solution. [Solution 2] Because the squares are congruent, the steeper line has a slope of 2, which implies that the angle of inclination, α_1, of the steeper line satisfies $\tan \alpha_1 = 2$. On the other hand, the shallower line has a slope of $\frac{1}{3}$, so the angle of inclination, α_2, of the shallower line satisfies $\tan \alpha_2 = \frac{1}{3}$.



Thus, $\alpha = \alpha_1 - \alpha_2 = \tan^{-1}(2) - \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{2-\frac{1}{3}}{1+2\cdot\frac{1}{3}}\right) = \tan^{-1}(1) = \boxed{45^\circ.}$ $\square$

Problem 1.7.2 (Santa Clara University High School Mathematics 2001 Contest/Problem 4)**.** Find the largest possible positive integer n such that $n+3$ divides $1^3+2^3+\cdots+n^3$.

Hint: Prove that if a prime number p divides $n+3$ and $n+3$ divides $1^3+2^3+\cdots+n^3$, then $p \in \{2,3\}$.

Solution. [Solution 1] Note that

$$1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}n^2(n+1)^2.$$

For the first question $n+3 \mid 1^3+2^3+\cdots+n^3 \Rightarrow 4(n+3) \mid n^2(n+1)^2$. Thus if $p \mid n+3$, then $p \mid n$ or $p \mid n+1$.

Case 1: if $p \mid n$ then $p \mid (n+3)-n = 3$, thus $p=3$.

Case 2: if $p \mid n+1$ then $p \mid (n+3)-(n+1) = 2$, thus $p=2$.

Therefore $p \in \{2,3\}$.

Now, for the second question, from the result of the first question, since any prime factor of $n+3$ can only be 2 or 3, thus there exist k, l non-negative integers such that: $n+3 = 2^k 3^\ell$.

$$4(n+3) \mid n^2(n+1)^2 \Rightarrow 2^{k+2}3^\ell \mid (2^k3^\ell-3)^2(2^k3^\ell-2)^2.$$

Let $P = (2^k3^\ell-3)^2(2^k3^\ell-2)^2$.

Case 1: if $k \geq 2$, then $2^k3^\ell - 3 \equiv 1 \pmod 2$, $(2^k3^\ell-2)^2 = 2^2(2^{k-1}3^\ell-1)^2$, and $2^{k-1}3^\ell - 1 \equiv 1 \pmod 2$. This means that the largest power of 2 that divides P is 2^2, but $2^4 \mid 2^{k+2}3^\ell \mid P$, which is a contradiction.

Case 2: if $\ell \geq 3$, then $(2^k3^\ell-3)^2 = 3^2(2^k3^{\ell-1}-1)^2$, where $2^k3^{\ell-1}-1 \equiv 2 \pmod 3$, and $2^k3^\ell-2 \equiv 1 \pmod 3$. Thus the largest power of 3 that divides P is 3^2, but $3^3 \mid 2^{k+2}3^\ell \mid P$, which is a contradiction.

Therefore $k \leq 1$ and $\ell \leq 2$, thus the largest possible value of n is $n = 2^1 3^2 = 18$, or $n = \boxed{15.}$ □

Solution. [Solution 2] Note that

$$1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}n^2(n+1)^2.$$

Thus

$$n+3 \mid \frac{1}{4}n^2(n+1)^2 \exists k \in \mathbb{Z}^+ : \ 4k = \frac{n^2(n+1)^2}{n+3} = n^3 - n^2 + 4n - 12 + \frac{36}{n+3}.$$

Therefore $n+3 \mid 36$, or $n \in \{1,3,6,8,15,33\}$. It is easy to test that only $n \in \{3,6,15\}$ results in positive integer k. Thus the larges n is $\boxed{15.}$ □

Problem 1.7.3 (2014 Science Atlantic Math Contest/Problem 3)**.** (*10 points*) Show that if a 5×5 matrix is filled with zeros and ones, there must always be a 2×2 sub-matrix (that is, the intersection of the union of two rows with the union of two columns) consisting entirely of zeros or entirely of ones.

Solution. [Solution 1] Let call a column 0-type if it has at least 3 zeros, 1-type if it has least 3 ones. Note that if a column is 0-type then it is not 1-type and vice versa. Since there are 5 columns in the given 5×5 matrix, thus at least 3 of them are 0-type or 1-type.

WLOG, lets assume that A, B, and C columns are 0-type. Let call a pair of columns a *perfect-match* if both of them having two same rows where the zeros are, in other words these four zeros form a 2×2 sub-matrix consisting entirely of zeros. Let call a pair of columns *semi-match* if both of them having at most one same row where the zeros are,

Case 1: If any of the three pairs of (A, B), (B, C), (C, A) is a perfect-match, then we have the desired answer.

Case 2: Now, each of the three pairs of (A, B), (B, C), (C, A) is a semi-match. Lets (A, B) share row 1. Let the other two rows containing zeros of A be row 2 and row 3. Then the other two rows containing zeros of B be row 3 and row 4. Two of three rows containing zeros of C must be in one of the two groups of rows $(1, 2, 3)$ or $(1, 3, 4)$. It means that (C, A) or (C, B) is a perfect-match. □

Solution. [Solution 2] Suppose that a 5×5 matrix of zeroes and ones contains no 2×2 sub-matrix of zeroes. We will show that it must contain a sub-matrix of ones.

Let A be the column with the fewest ones, or one of those columns if there are several with the same smallest number of ones, and let n be the number of ones in column A.

Case 1 : $n = 0$. In this case, no other column can contain more than one zero, since otherwise that column and column A would share a 2×2 sub-matrix of zeroes. Then any two columns other than A must each have at least 4 ones, so they share at least three rows of ones. Thus, there are 2×2 sub-matrices of ones using any pair of columns besides A.

Case 2 : $n = 1$. If column A contains only one one, then it has zeroes in 4 rows. In those four rows, no other column will contain a pair of zeroes, so each of the other columns must have at least 3 ones in those four rows. Therefore any two columns (excluding A) will have ones in two of the same rows, so again we have a 2×2 sub-matrix of ones.

Case 3 : $n = 2$. Each of the other columns must contain at least two ones in the rows in which A has zeroes. If any column has three ones in these rows, then it will share a 2×2 sub-matrix of ones with any of the others. If not, each of these four column has two ones among the three rows. But there are only $\binom{3}{2} = 3$ ways to do this, so two columns must have the same arrangement. Then these two contain a 2×2 sub-matrix of ones.

Case 4 : $n \geq 3$. In this case, each column has at least three ones. So, given any two columns, they either share two rows with ones, or between them have at least one one in each row. The first arrangement gives us the 2×2 sub-matrix of ones that we are looking for. In the second arrangement, any third column must share two rows of ones with one of the original two columns, again giving us a 2×2 sub-matrix of ones.

In every case, if we don't have a 2×2 sub-matrix of zeroes, we must have a 2×2 sub-matrix of ones. □

Problem 1.7.4 (2017 Bulgaria Math Olympiad, Grade 10, Second Round/Problem 3)**.** (*10 points*) The sequence (a_n) is defined by

$$a_1 = 1,\ a_2 = 1,\ a_n = \left\lfloor \sqrt{2a_{n-1} + a_{n-2} + \cdots + a_1} \right\rfloor,\ \text{if } n > 1.$$

Find a_{2023}. *Hint: compute a_2, a_5.*

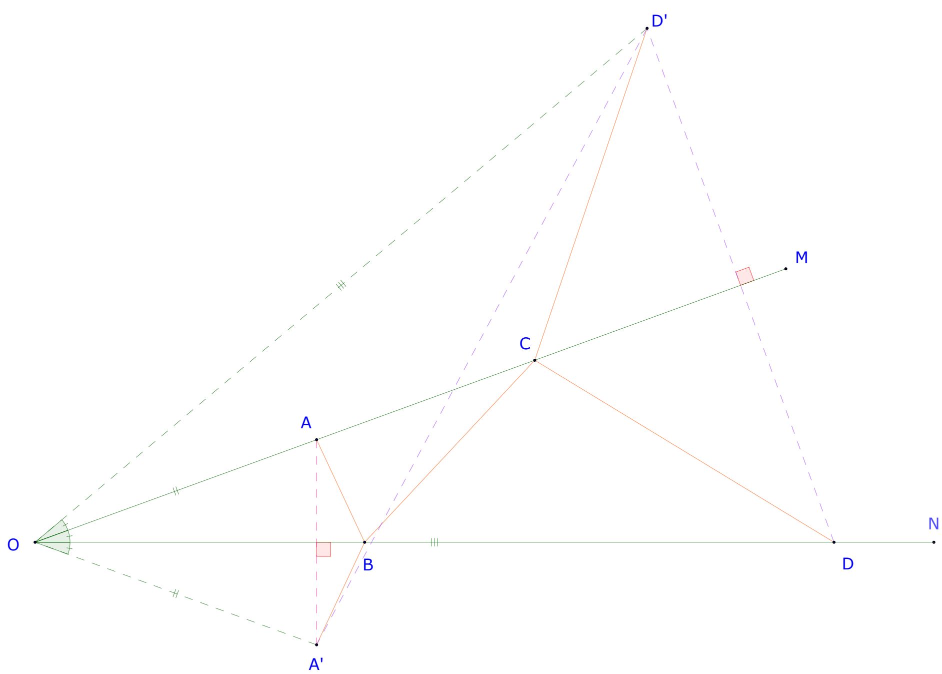
Solution. We prove that $a_{2n+1} = a_{2n} = n,\ \forall n \geq 1$. First, it is easy to verify that $a_2 = \left\lfloor \sqrt{2+1} \right\rfloor = \left\lfloor \sqrt{3} \right\rfloor = 1$. Now, let $a_{2n+1} = a_{2n} = n,\ \forall n \geq 1$.

$$\begin{aligned}
a_{2n+2} &= \left\lfloor \sqrt{2a_{2n+1} + a_{2n} + \cdots + a_1} \right\rfloor = \left\lfloor \sqrt{2n + n + 2(n-1) + 2(n-2) + \cdots + 2\cdot 1 + 1} \right\rfloor \\
&= \left\lfloor \sqrt{3n + (n-1)n + 1} \right\rfloor = \left\lfloor \sqrt{n^2 + 2n + 1} \right\rfloor = n + 1 \\
a_{2n+3} &= \left\lfloor \sqrt{2a_{2n+2} + a_{2n} + 1 + \cdots + a_1} \right\rfloor = \left\lfloor \sqrt{2(n+1) + 2n + 2(n-1) + \cdots + 2\cdot 1 + 1} \right\rfloor \\
&= \left\lfloor \sqrt{2(n+1) + n(n+1) + 1} \right\rfloor = \left\lfloor \sqrt{(n+1)(n+2) + 1} \right\rfloor = n + 1
\end{aligned}$$

Note that $(n+1)^2 = n^2 + 2n + 1 < n^2 + 3n + 3 = (n+1)(n+2) + 1 < n^2 + 4n + 4 = (n+2)^2$. Therefore $a_{2023} = \boxed{1011.}$ $\square$

Problem 1.7.5 (2019 South Africa, Durban, Invitational World Youth Mathematics Intercity Competition, Team Contest/Problem 3)**.** (*10 points*) Points A, C are on ray OM, and B, D are on ray ON. It is given that $OA = 6, OD = 16, \angle NOM = 20°$.

What is the minimum length of $AB + BC + CD$? *Hint: when is the length of a broken line minimal?*



Solution. Let A' be the reflection of A over ON, and D' be the reflection of D over OM. It is easy to see that $AB + BC + CD = A'B + BC + CD' \geq A'D'$. Now $\angle A'OB = \angle BOA = \angle AOD' = 20° \Rightarrow \angle A'OD' = 60°$.

Thus, $(A'D')^2 = A'O^2 + D'O^2 - 2A'O \cdot D'O \cdot \cos 60° = 6^2 + 16^2 - 2 \cdot 6 \cdot 16 \cdot \frac{1}{2} = 196$, hence $A'D' = \boxed{14.}$ $\square$

Problem 1.7.6 (Crux Mathematicorum/MA 136)**.** (*10 points*) Find all triples of non-negative integers (x, y, z) and $x \leq y$ such that:

$$x^2 + y^2 = 3 \cdot 2016^z + 77.$$

Hint: if $x^2 + y^2 \equiv 0 \pmod 7$, *then what are remainders of* x *and* y *when divided by 7?*

Solution. Let consider three cases.

Case 1: $z = 0$. Then $x^2 + y^2 = 80$. Since $x \leq y$, it is easy to verify that $x = 4, y = 8$.

Case 2: $z > 1$. Then $2016^z \equiv 77 \equiv 0 \pmod 7$, and $2016^z \equiv 0 \pmod{7^2}$, but $77 \not\equiv 0 \pmod{7^2}$.

Case 3: $z = 1$. It is easy to see that for any integer x, $x^2 \equiv 0, 1, 2, 4 \pmod 7$, thus if $x^2 + y^2 \equiv 0 \pmod 7$ then the only possibility is $x \equiv y \equiv 0 \pmod 7$. From this claim, let $x = 7x_1, y = 7y_1$, then $x_1^2 + y_1^2 = 125$. Since $x \leq y$, so $x_1 \leq y_1$, and there are two solutions $(x_1, y_1) \in \{(2, 11),\ (5, 10)\}$. Therefore $(x, y) \in \{(14, 77),\ (35, 70)\}$.

The answers are $\boxed{(4, 8, 0),\ (14, 77, 1),\ (35, 70, 1).}$ □

Problem 1.7.7 (2017 Czech-Slovakia Math Olympiad, Category B, Regional Round/Problem 2)**.** (*10 points*) There are 2023 points in the plane such that among any three of them two can be selected so that their distance is less than 1. Prove that there is a circle of radius 1 containing at least 1012 of the given points.

Solution. [Solution 1] Let A be one of the points.

Case 1: if there is no such point B among the rest of the points such that $AB \geq 1$, then all of them are within a circle radius 1 centred at A. Of these, there exist at least 1012 of the given points that are within a circle of radius 1.

Case 2: if there is a point B such that $AB \geq 1$. Then for any point C of the remaining 2021 points: $CA < 1$ or $CB < 1$.

Continue this process, by the Pigeonhole principle, there is at least $\lceil \frac{2021}{2} \rceil = 1011$ points $C_1, C_2, \ldots, C_{1011}$, all of which are within either a circle radius 1 centred at A or a circle radius 1 centred at B. In any case is a circle of radius 1 containing at least 1012 of the given points. □

Solution. [Solution 2] Let A_1 be one of the points. By the Pigeonhole Principle, one of the two situations occur.

Case 1: There exists 1011 point $A_2, A_3, \ldots, A_{1012}$ such that

$$A_1A_2 \leq 1,\ A_1A_3 \leq 1,\ \ldots, A_1A_{1012} \leq 1.$$

Then the circle centred at A_1, radius 1 contain 1012 points $A_1, A_2, \ldots, A_{1012}$.

Case 2: There exists 1012 point $B_1, B_2, \ldots, B_{1012}$ such that

$$A_1B_1 \geq 1,\ A_1B_2 \geq 1,\ \ldots, A_1B_{1012} \geq 1.$$

Now, since among points A_1, B_1, B_2 we can select two so that their distance is less than 1, thus $B_1B_2 < 1$. Similarly $B_1B_3, \ldots, B_1B_{1012} < 1$.

Then the circle centred at B_1, radius 1 contain 1012 points $B_1, B_2, \ldots, B_{1012}$. □

Problem 1.7.8 (29th Nordic Mathematical Contest 2015/Problem 2)**.** (*10 points*) Determine all polynomials $P(x)$ with real coefficients such that

1. $P(2023) = 2022$,
2. $(P(x) + 1)^2 = P(x^2 + 1)$, for all real numbers x.

Investigate the sequence (x_n) *where* $x_1 = 2023$, $x_{n+1} = x_n^2 + 1$.

Solution. We prove the following claim

Claim — For the sequence (x_n) such that $x_1 = 2023$, $x_{n+1} = x_n^2 + 1$, then $P(x_n) = x_n - 1$, for all $n \in \mathbb{Z}^+$.

Proof. By Induction Principle, $x_1 = 2023$, then $P(x_1) = 2022 = x_1 - 1$. Lets assume that the induction hypothesis is correct for n. Then $P(x_{n+1}) = P(x_n^2 + 1) = (P(x_n) + 1)^2 = (x_n)^2 = x_{n+1} - 1$. ■

From the claim it is easy to see that there is an infinite sequence (x_n) where $P(x_n) - x_n + 1 = 0$, or the polynomial $Q(x) = P(x) - x + 1$ has infinite number of distinct roots. Thus $Q(x) \equiv 0$. This means that $P(x) - x + 1 = 0$, hence $\boxed{P(x) = x - 1.}$ □