Review for Mathematical Competitions

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Part I

COMC

Chapter 1

Session 1

1.1 Topics

Algebra

1. Comparing rational and irrational numbers.
2. Identities. Factoring.
3. Equations
4. System of Equations
5. Sums and Products
6. Inequalities

Combinatorics

1. Counting. Correction over Counting.
2. Permutations.
3. Number Partitions
4. Sets
5. Grids. Paths in grids.
6. Combinatorial Geometry
7. Games

Geometry

1. Congruent Triangles, Similar Triangles.
2. Angle Chasing. Angles in Circle. Cyclic Quadrilateral
3. Right triangles. Hypotenuse as Diameter. Midpoint of Hypotenuse to the vertex of the right angle.
4. Simson Line.
5. Incircle/Excircle.

Number Theory

1. Divisibility
2. Modular Arithmetic. Residue Classes (subsets of numbers with same residue modulo a prime). Quadratic Residue (remainders of perfect square modulo a prime)
3. Diophantine Equations

1.2 Problems

Problem 1.2.1 (Moscow MO 2005/A1). (*4 points*) Show at least one integer solution to the equation

$$a^2b^2 + a^2 + b^2 + 1 = 2005$$

How to provide your answer: A *correct answer* with one correct solution earns full marks.

Problem 1.2.2 (Flanders MO 2007/1). (*4 points*) The numbers $1, 2, \ldots$ are placed in a triangle as following:

$$\begin{array}{llll}
1 & & & \\
2 & 3 & & \\
4 & 5 & 6 & \\
7 & 8 & 9 & 10 \\
\ldots & & &
\end{array}$$

What is the sum of the numbers on the n^{th} row?

How to provide your answer: A *correct answer* earns full marks.

Problem 1.2.3 (Korea JMO 2005/2)**.** (*4 points*) In triangle ABC, $AB = 4$, $BC = 5$, and $CA = 6$. P and Q are points (not necessarily inside $\triangle ABC$) such that $\angle BPA + \angle AQC = 90°$. The vertices of the triangle BAP and ACQ both are in clockwise or both in counter-clockwise order. Let the intersection of the circumcircles of the two triangles be N ($A \neq N$, however if A is the only intersection $A = N$), and the midpoint of segment BC be M. What is the length of MN? Does it depend on P and Q?

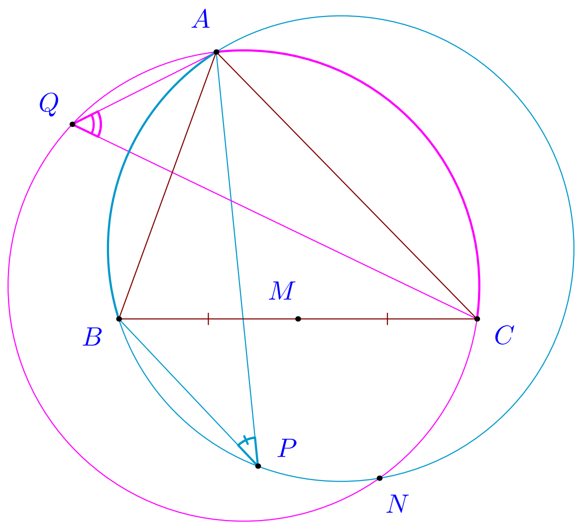


Figure 1.1: Korea JMO 2005/2

How to provide your answer: A *correct answer* earns full marks.

Problem 1.2.4 (Moscow MO 2005/A2). (*4 points*) An 8×8 square of graph paper is folded several times along the grid lines into a small 1×1 square. This square is cut along the line joining the midpoints of the two opposite sides. Into how many pieces can this process cut the initial square?

How to provide your answer: A *correct answer* earns full marks.

Problem 1.2.5 (Belarus MO 2008/1)**.** (*6 points*) x, y, and z are real numbers such that $x^2+y = y^2+z = z^2+x$. Prove that

$$x^3 + y^3 + z^3 = xy^2 + yz^2 + zx^2.$$

How to provide your answer: You must *show all your work*.

Problem 1.2.6 (HC-2021-SM2-HST-P13)**.** (*6 points*) Thanh filled a triangle of squares with the letters of her name, as shown below in the Table 1.1. She counted all the 5−letter paths that form her name T-H-A-N-H, each starts from the T letter in the middle of the bottom row, then goes left, right, or up. An example is shown in the diagram above.

				H				
			H	N	H			
		H	N	A	N	H		
	H	N	A	H	A	N	H	
H	N	A	H	T	H	A	N	H

Table 1.1: HC-2021-SM2-HST-P13

What number did she get?

How to provide your answer: A *correct answer* earns full marks.

Problem 1.2.7 (Korea JMO 2006/1)**.** (*6 points*) $(a_1, a_2, \ldots, a_{2006})$ is a permutation of $(1, 2, \ldots, 2006)$. Prove that the product below is a multiple of 3. (0 is counted as a multiple of 3)

$$\prod_{i=1}^{2006} (a_i^2 - i).$$

How to provide your answer: You must *show all your work*.

Problem 1.2.8 (HC-2021-SM2-HST-P15)**.** (*6 points*) The figure, shown below in Table 1.2, is called a 135−*domino* because it contains 1, 3, and 5 squares on the first, second, and third row, respectively.

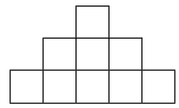


Table 1.2: A 135 domino

Harry tiles $n \times n$ board with 135−*dominoes.* For what values of n can he succeed?

How to provide your answer: A *correct answer* earns full marks.

Problem 1.2.9 (Moscow MO 2002/C2). (*10 points*) a, b, and c are positive real numbers.

1. Prove that $(a+b+c)^2 \geq 3(ab+bc+ca)$.
2. If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \geq a+b+c$, prove that
$$a+b+c \geq 3abc$$

How to provide your answer: You must *show all your work*.

Problem 1.2.10 (Belarus 239 Open MO 2010/G11/2)**.** (*10 points*) The incircle of the triangle ABC touches the sides AC and BC at points K and L, respectively. The B-excircle touches the side AC of this triangle at point P. The segment AL intersects the inscribed circle for the second time at point S. Line KL intersects the circumscribed circle of triangle ASK for the second at point M.

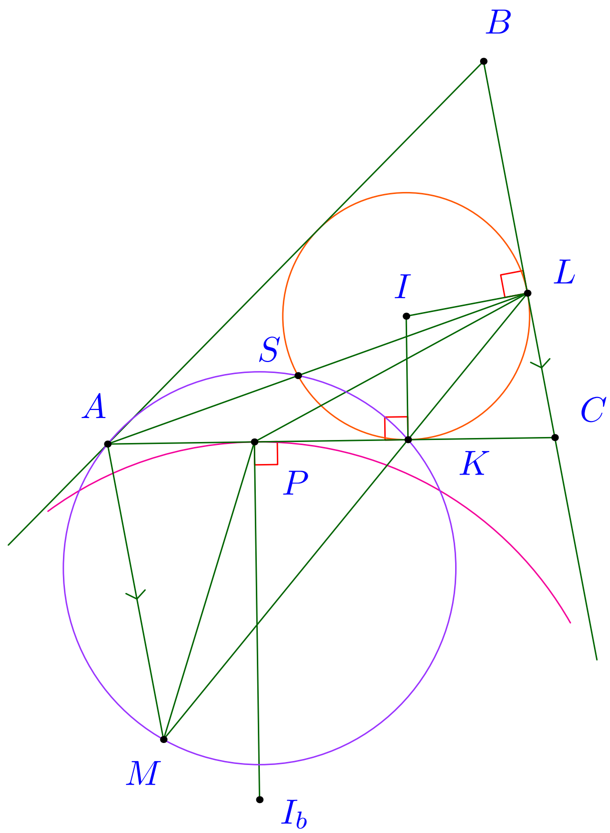


Figure 1.2: Belarus 239 Open MO 2010/G11/2

Prove that,

1. $AM \parallel CL$.
2. $AM = AK = PC$ and $AP = KC = CL$.
3. $PL = PM$.

How to provide your answer: You must *show all your work*.

Problem 1.2.11 (Flanders MO 1994/2)**.** (*10 points*) (a, b, c) is a triple of positive integers, $c \leq 94$, such that,

$$(a + \sqrt{c})^2 + (b + \sqrt{c})^2 = 60 + 20\sqrt{c}$$

1. If c is *not* a perfect square, find all (a, b, c).
2. If c is a perfect square, $c = d^2$, then find all triples of positive integers (a, b, c) for the given equations.

How to provide your answer: You must *show all your work*.

Problem 1.2.12 (HC-2021-SM2-R1-P5)**.** (*10 points*) Berry and Cherry take alternate turns in playing a two-player game removing marbles from a pile of n marbles as follows:

- The player whose turn it is, must remove exactly 2, 4, or 5 marbles from the pile.
- The player who at some point is unable to make a move (cannot remove 2, 4, or 5 marbles from the pile) loses the game.

Note that both players are very smart and they have a ready winning strategy if the situation is suitable.

1. Find S, the set of positive integer values of n, the number of initial marbels in the pile, such that if Cherry starts, she always wins regardless of what Berry does?
2. Does Cherry always lose if $n \notin S$.

1.3 Solutions

Proof. Moscow MO 2005/A1 Note that,

$$a^2b^2 + a^2 + b^2 + 1 = (a^2+1)(b^2+1) = 2005 = 5 \cdot 401$$

Thus, $\boxed{\{(\pm 2, \pm 20), (\pm 20, \pm 2)\}}$ are all solutions. □

First proof. Flanders MO 2007/1 First, it is easy to prove that,

Claim — There are n number on the n^{th} row. The last number of the $(n-1)^{\text{th}}$ row is $\frac{(n-1)(n)}{2}$.

Therefore the numbers placed in the rows from the first to the n^{th} row are

$$\frac{(n-1)(n)}{2} + 1, \frac{(n-1)(n)}{2} + 2, \ldots, \frac{(n-1)(n)}{2} + n$$

Thus the sum of all the numbers from the first to the n^{th} row is

$$n\frac{(n-1)(n)}{2} + \frac{n(n+1)}{2} = \frac{n(n^2+1)}{2}$$

Thus, the desired sum is $\boxed{\frac{1}{2}n(n^2+1).}$ □

Second proof. Flanders MO 2007/1 First, it is easy to prove that,

Claim — There are n number on the n^{th} row. The last number of the n^{th} row is $\frac{n(n+1)}{2}$.

Therefore the numbers placed in the rows from the first to the n^{th} row are

$$1, 2, \ldots, \frac{n(n+1)}{2}$$

Thus the sum of all the numbers from the first to the n^{th} row is

$$\frac{1}{2} \cdot \frac{n(n+1)}{2} \cdot \left(\frac{n(n+1)}{2} + 1\right) = \frac{n(n+1)(n^2+n+2)}{8}$$

Therefore, the sum of all the numbers on the n^{th} row is,

$$\begin{aligned}
&\frac{n(n+1)(n^2+n+2)}{8} - \frac{(n-1)(n)((n-1)^2+(n-1)+2)}{8} \\
=&\frac{n}{8}((n+1)(n^2+n+2) - (n-1)(n^2-n+2)) = \frac{1}{2}n(n^2+1)
\end{aligned}$$

Thus, the desired sum is $\boxed{\frac{1}{2}n(n^2+1).}$ □

Proof. HC-2021-SM2-HST-P13 Consider the *"half"* triangle shown in the Table 1.3 below. It is easy to see that in each path, there is two choices at each steps. One example is shown in the figure, when two As can be chosen after a H. Therefore there are $2^4 = 16$ paths for a half triangle. In total there are $2 \cdot 16 - 1 = 31$,

				H
			H	N
		H	N	A
	H	N	A	H
H	N	A	H	T

Table 1.3: Two choices of A from H

because the vertical path formed by all the squares in the T column is shared by both half triangles.

Thus, the number that Thanh got is $\boxed{31.}$ □

Proof. Korea JMO 2005/2 It is easy to see that both $ABPN$ and $ACNQ$ are cyclic. Thus,

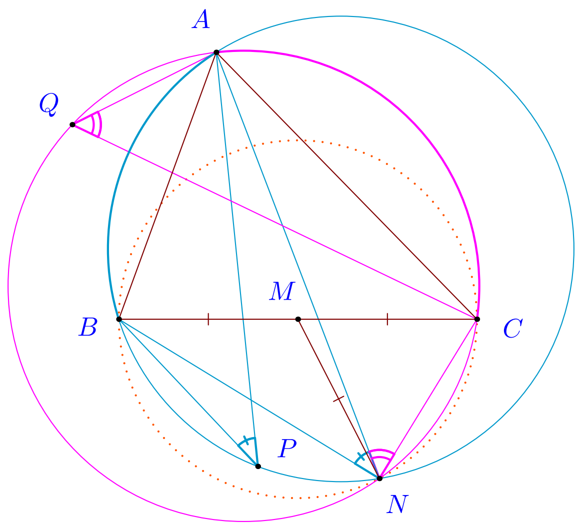


Figure 1.3: $ABPN$ and $ACNQ$ are cyclic

$$90^\circ = \angle BPA + \angle CQA = \angle BNA + \angle CNA = \angle BNC$$

Therefore, $\boxed{MN = \frac{1}{2}BC = 2.5, \text{ and it does not depend on } P \text{ and } Q.}$ □

Proof. Moscow MO 2005/A2 WLOG, suppose that the cut runs vertically. In each column, draw a vertial line joining the midpoints of the top and bottom segments, as shown in the Table 1.4 below. When the paper

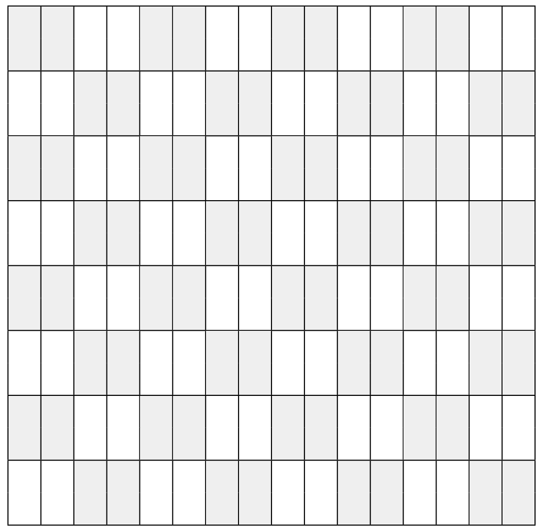


Table 1.4: Moscow MO 2005/A2

is folded along the grid lines, these segments match one another. The square is cut by these lines and only these.

Thus, they divide the square into $\boxed{9}$ pieces. $\square$

Proof. Belarus MO 2008/1 First, if WLOG $x = 0$, then

$$y = y^2 + z = z^2 \Rightarrow y^2 = z^2 - z,\ z^2 = y \Rightarrow y^3 + z^3 = y(z^2 - z) + yz = yz^2.$$

Now, assume that $xyz \neq 0$, since $x^2 + y = y^2 + z$, so $x(x^2 + y) = x(y^2 + z)$, thus

$$x^3 - xy^2 = xz - xy,\ \text{similarly } y^3 - yz^2 = yx - yz,\ z^3 - zx^2 = zy - zx$$

Summing up the equalities obtained, we have the desired equality $\boxed{x^3 + y^3 + z^3 = xy^2 + yz^2 + zx^2.}$ $\square$

Proof. HC-2021-SM2-HST-P15 Assume that a $n \times n$ board can be tiled by a number of $135-$*dominoes*. Since a $135-$*domino* contains 9 squares, so n is divisible by 3. Furthermore, by colouring alternately, as shown in Table 1.5, it is easy to see that the number of black squares and the number of white squares covered by a $135-$*domino* are divisible by 3. Thus, their difference in the $n \times n$ board, tiled by $135-$*dominoes*, is a multiple of 3.

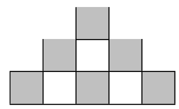
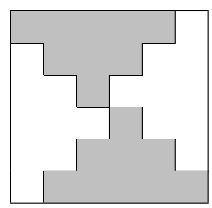


Table 1.5: Alternate colouring

Furthremore, if n is an odd number, the positive difference of black and white coloured squares in a $n \times n$ board is 1. Thus, the tiling can be done if and only if n is divisible by 2. Therefore n is divisible by 6.

Below is a tiling for a 6×6 board with 4 pieces of $135-$*dominoes*. Any $6k \times 6k$ board can be divided into k^2 of 6×6 board.



The values of n between 1 and 20 that Harry can tile $n \times n$ board are $\boxed{6, 12, 18.}$ The answer is $\boxed{C.}$ □

Proof. Korea JMO 2006/1 First, note that a perfect square n^2 has residue 0 or 1 modulo 3. Sinc $2006 = 3 \cdot 668 + 2$, so let $n = 3k + 2$. Among $a_1^2, a_2^2, \ldots, a_{3k+2}^2$, there are k terms with residue 0 and $2k + 2$ term with residue 1. Among $1, 2, \ldots, 3k + 2$, there are k terms with residue 0, $k + 1$ terms with residue 1, and $k + 1$ terms with residue 2 modulo 3.

Therefore in the given product there is a mapping $a_i^2 \to i$ such that both of them has residue 1 modulo 3 and $\boxed{\text{their difference is divisible by 3.}}$ □

Proof. Moscow MO 2002/C2 First, by AM-GM inequality,

$$a^2 + b^2 \geq 2ab,\ b^2 + c^2 \geq 2bc,\ c^2 + a^2 \geq 2ca \Rightarrow (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \geq 3(ab + bc + ca).$$

Now,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a + b + c \Rightarrow ab + bc + ca \geq abc(a + b + c)$$
$$\Rightarrow (a + b + c)^2 \geq 3(ab + bc + ca) \geq 3abc(a + b + c) \Rightarrow a + b + c \geq 3abc$$

Hence, $\boxed{a + b + c \geq 3abc.}$ □

Proof. Belarus 239 Open MO 2010/G11/2 First, since $AMKS$ is cyclic, so $\angle AMK = \angle KSL$. K is the tangent point of (I), so $\angle KSL = \angle LKC$. C is the intersection of two tangent lines, so $\angle LKC = \angle CLK$. Thus $\angle AMK = \angle CLK$. So $AM \parallel LC$.

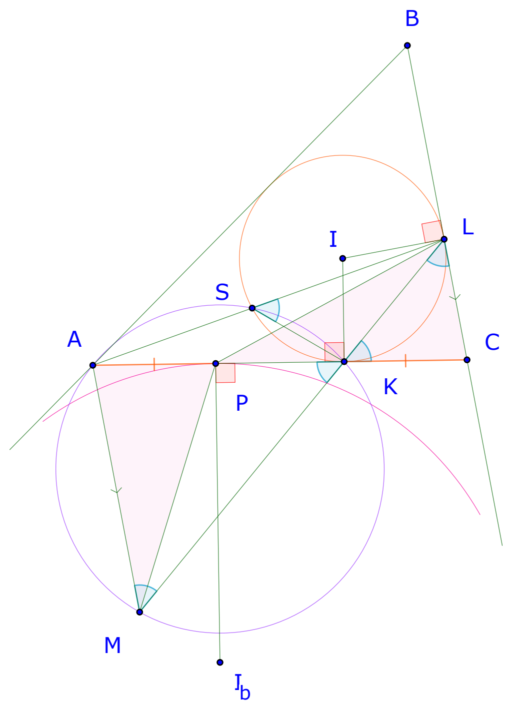


Figure 1.4: $AM \parallel LC$

Now, $\angle AKM = \angle LKC$, so $\angle AMK = \angle AKM$, thus $AM = AK$. By the property of the excircle, $AP = KC$, so $AM = AK = PC$, and $AP = KC = CL$. Furthermore $AM \parallel LC$, so $\angle MAP = \angle PCL$, therefore, $\triangle MAP \cong \triangle PCL$.

Thus, $\boxed{PL = PM}$ as desired. □

Proof. Flanders MO 1994/2 There are two different cases depending on whether c is a perfect square.

Case 1: if c is not a perfect square, then $\sqrt{c}$ is irrational number, thus by expanding both sides,

$$a^2 + c + 2a\sqrt{c} + b^2 + c + 2b\sqrt{c} = 60 + 20\sqrt{c} \Rightarrow \begin{cases} a^2 + b^2 + 2c = 60 \\ a + b = 10 \end{cases} \Rightarrow 10^2 - 2ab + 2c = 60 \Rightarrow ab = c + 20$$

Thus, a and b are positive integer roots of $x^2 - 10x + (c + 20) = 0$,

$$x^2 - 10x + (c + 20) = 0 \Rightarrow (x - 5)^2 = 5 - c \Rightarrow c = 5$$

Therefore, $\boxed{(a, b, c) = (5, 5, 5).}$

Case 2: let $c = d^2$ a perfect square, $c \leq 94$, so $d \in \{1, 2, \ldots, 9\}$, and

$$(a + d)^2 + (b + d)^2 = 60 + 20d$$

Since $4 \mid 60 + 20d$, so both $a + d$ and $b + d$ must be even. Let $a + d = 2k$, $b + d = 2\ell$, then $2k > d,\ 2\ell > d$, and

$$(a + d)^2 + (b + d)^2 = 60 + 20d \Rightarrow k^2 + \ell^2 = 15 + 5d \Rightarrow 20 \leq k^2 + \ell^2 \leq 60$$

A perfect square has remainders 0, 1, or 4 modulo 5. Since $5 \mid k^2 + \ell^2$, so k, ℓ both have remainder 0 or a pair of remainders 0 and 4. We have a few cases,

Case 2a: $5 \mid k,\ 5 \mid \ell$, then $k = \ell = 5$, so $d = 7$, Thus $\boxed{(a, b, c) = (3, 3, 49).}$

Case 2b: $k \equiv 1 \pmod 5),\ \ell \equiv 2 \pmod 5$ *or vice versa,* then $(k, \ell) = (1, 7), (6, 2), (6, 3)$ and vice versa, so $d = 7, 5$, or 6. Each of the cases does not sattify $2k > d,\ 2\ell > d$. There are no solution for these cases.

Case 2c: $k \equiv 3 \pmod 5,\ \ell \equiv 4 \pmod 5$ *or vice versa,* then $(k, \ell) = (2, 4), (3, 4)$ and vice versa, so $d = 1$ or 2. Thus, $\boxed{(a, b, c) \in \{(6, 4, 4), (4, 6, 4), (3, 7, 1), (7, 3, 1).\}}$ □

Proof. HC-2021-SM2-R1-P5 The positive integers from 0 to 21 can be divided into 7 groups of numbers based on their remainders when divided by 7,

$$\begin{aligned} &G_0 = \{0, 7, 14, 21\},\ G_1 = \{1, 8, 15\},\ G_2 = \{2, 9, 16\}, \\ &G_3 = \{3, 10, 17\},\ G_4 = \{4, 11, 18\},\ G_5 = \{5, 12, 19\},\ G_6 = \{6, 13, 20\} \end{aligned}$$

It is easy to verify that,

Claim — If n is a number in G_0 and G_1 then $n - 2$, $n - 4$, and $n - 5$ are in G_2, G_3, G_4, G_5, or G_6.

Now, by the rules of the game, it is obvious that $\{0, 1\}$ are *losing* positions and $\{2, 4, 6\}$ are *winning* positions. Furthermore, because $3 - 2 = 6 - 5 = 1$, so $\{3, 6\}$ are also *winning* positions, too.

Therefore G_0 and G_1 contain all *losing* positions, while G_2, G_3, G_4, G_5, and G_6 containing all *winning* positions. A player, who is in a *winning* position, can always force the game into a *losing* position. Thus, Cherry will win the game if the game starts with a *winning* position.

Now, it is easy to generalize that if the initial marbles $\boxed{n \in \{7k + r \mid k \in {Z_0}^+,\ r \in \{2, 3, 4, 5, 6\}\},}$ then Cherry will win the game and she loses in all other games. □

Chapter 2

Session 2

2.1 Topics

Algebra

1. AM-GM Inequality. Cauchy-Schwarz Inequality.
2. Equations. System of Equations.
3. Quadratic. Polynomials. Factoring of Polynomials with trivial roots $0, \pm 1, \pm 2, \pm 3$.
4. Absolute values.

Combinatorics

1. Counting. Combinations. Complementary Counting.
2. Permutations.
3. Probability. Number of favorable/total outcomes.
4. Combinatorial Geometry.
5. Grids.
6. Counting pairs of elements vs. pairs of sets if no pair of sets share more than a single element.
7. Pigeonhole principle.
8. Constructive solutions

Geometry

1. Angle Chasing. Angles in a circle.
2. Arc. Midpoint of Arc.
3. Similar Triangles, Similarity Ratio.
4. Triangles. Squares. Angle Bisectors.
5. Incircle/Excircle

Number Theory

1. Divisibility, Divisors.
2. Modular Arithmetic, Consecutive Numbers.
3. Number bases.

2.2 Problems

Problem 2.2.1 (CCA Math Bonanza 2018 IR/1). (*4 points*)

Find the remainder, when divided by 100, of the number below

$$(1!)^2 + (2!)^2 + (3!)^2 + \ldots + (2018!)^2$$

How to provide your answer: A *correct answer* earns full marks.

Problem 2.2.2 (CCA Math Bonanza 2018 TR/3). (*4 points*)

In the game of Avalon, there are 10 players, 4 of which are *bad*. A quest is a set of three players who are randomly chosen from the 10 players. A quest fails if *at least* one of its members is a *bad* player.

What is the probability that there is *exactly one bad* player in a failed quest?

How to provide your answer: A *correct answer* earns full marks.

Problem 2.2.3 (CMIMC Geometry 2018/2)**.** (*4 points*)

Let $ABCD$ be a square of side length 1, and let P be a variable point on $\overline{CD}$. Denote by Q the intersection point of the angle bisector of $\angle APB$ with $\overline{AB}$. The set of possible locations for Q as P varies along $\overline{CD}$ is a line segment ℓ.

What is the length of this segment ℓ?

How to provide your answer: A *correct answer* earns full marks.

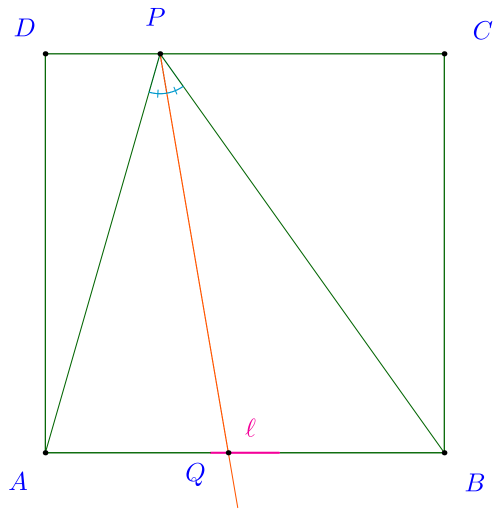


Figure 2.1: CMIMC Geometry 2018/2

Problem 2.2.4 (All Russian MO 2018/G10/1)**.** (*4 points*)

Determine the number of real roots of the equation

$$|x| + |x+1| + \cdots + |x+2018| = x^2 + 2018x - 2019$$

How to provide your answer: A *correct answer* earns full marks.

Problem 2.2.5 (Vietnam MO 1982/4)**.** (*6 points*)

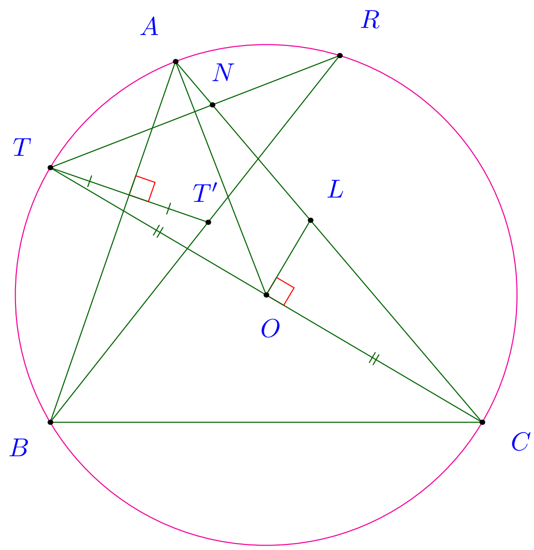
Find all positive integers x, y, z such that $2^x + 2^y + 2^z = 2336$.

How to provide your answer: A *correct answer* earns full marks.

Problem 2.2.6 (Centroamerican and Caribbean MO 2018/2). (*6 points*)

Let ΔABC be a triangle inscribed in the circumference ω of center O. Let T be the symmetric of C respect to O and T' be the reflection of T respect to line AB. Line BT' intersects ω again at R. The perpendicular to CT through O intersects line AC at L. Let N be the intersection of lines TR and AC.

It is known that $AL = 5$ and $TC = 12$.



Find CN.

How to provide your answer: A *correct answer* earns full marks.

Problem 2.2.7 (Germany MOO 2018/1). (*6 points*)

Find all real numbers x, y, z satisfying the following system of equations:

$$\begin{aligned} xy + z &= -30 \\ yz + x &= 30 \\ zx + y &= -18 \end{aligned}$$

How to provide your answer: A *correct answer* earns full marks.

Problem 2.2.8 (Math Prize for Girls Olympiad 2018/3). (*6 points*)

There is a wooden $3 \times 3 \times 3$ cube and 18 rectangular 3×1 paper strips. Each strip has two dotted lines dividing it into three unit squares. The full surface of the cube is covered with the given strips, flat or bent. Each flat strip is on one face of the cube. Each bent strip (bent at one of its dotted lines) is on two adjacent faces of the cube.

What is the greatest possible number of bent strips?

How to provide your answer: A *correct answer* earns full marks.

Problem 2.2.9 (All Russian MO 2018/G11/2)**.** (*10 points*) Prove that,

1. if a and b are positive real numbers, then

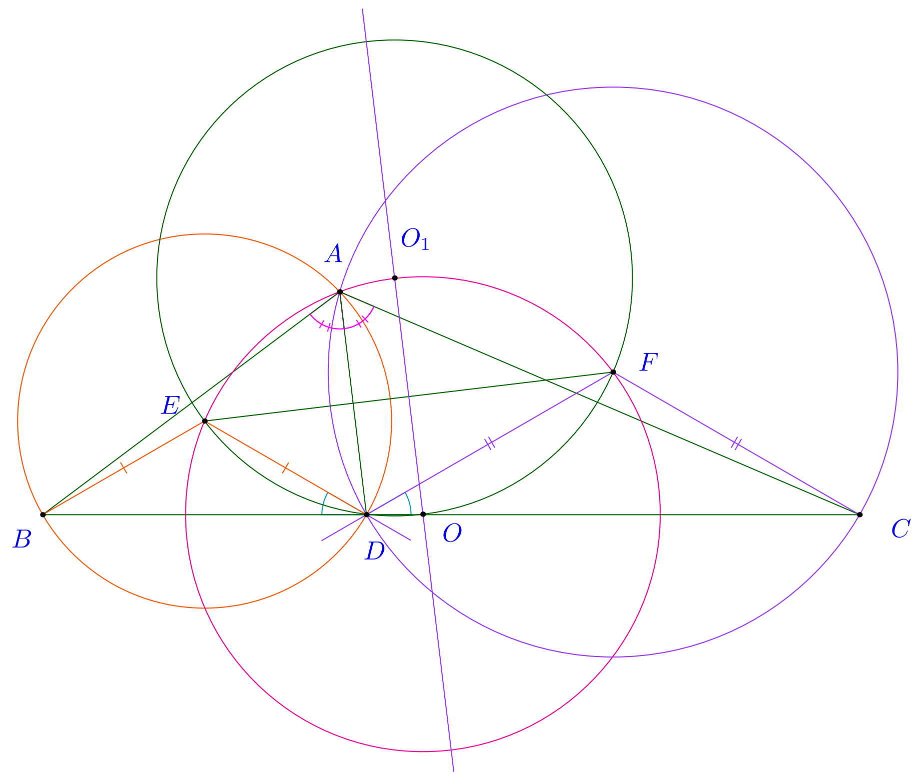
$$1 + ab \leq \sqrt{(1 + a^2)(1 + b^2)}.$$

2. $n \geq 2$, $x_1, x_2, \ldots, x_n$ are positive real numbers, then

$$\frac{1 + x_1^2}{1 + x_1 x_2} + \frac{1 + x_2^2}{1 + x_2 x_3} + \cdots + \frac{1 + x_n^2}{1 + x_n x_1} \geq n.$$

How to provide your answer: You must *show all your work*.

Problem 2.2.10 (Czech and Slovak MO 2018/3). (*10 points*) In triangle ABC let be D an intersection of BC and the A-angle bisector. Denote E, F the circumcenters of (ABD) and (ACD) respectively. The circumcenter O of (AEF) lies on the line BC. Let O_1 be the circumcenter of DEF.



1. Prove that $\angle EBD = \angle FDC$ and $O_1O = O_1E = O_1F$.
2. Prove that $\triangle EOO_1$ and $\triangle FOO_1$ are equilateral.
3. What is the possible size of the angle BAC?

How to provide your answer: You must *show all your work*.

Problem 2.2.11 (Harvard-MIT Math Tournament 2005/A5). (*10 points*) Ten positive integers are arranged around a circle. Each number is one more than the greatest common divisor of its two neighbors.

1. Let n be the largest number. Prove that both its neightbours are $n-1$.
2. For each of these two numbers $n-1$, tind their second neighbours (beside of n).
3. What is the sum of the ten numbers?

How to provide your answer: You must *show all your work*.

Problem 2.2.12 (Taiwan MO 2006/2)**.** (*10 points*)

Ten test papers are to be prepared for the National Olympiad. Each paper has 4 problems, and no two papers have more than 1 problem in common. Let n be the number of problems.

1. Prove that the number of pairs of problems is at least the number of pairs of test papers.
2. Prove that there is no such arrangements for $n = 12$ problems with the given conditions.
3. Give an example when $n = 13$.

How to provide your answer: You must *show all your work*.

2.3 Solutions

Proof. CCA Math Bonanza 2018 IR/1 Note that, for any $n \geq 5$, $(n!)^2$ contains the factor $(2 \cdot 5)^2 = 100$, thus $100 \mid (n!)^2$, and since

$$(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2 = 617 \equiv 17 \pmod{100}$$

Thus, $\boxed{(1!)^2 + (2!)^2 + (3!)^2 + \ldots + (2018!)^2 \equiv 17 \pmod{100}.}$ □

Proof. CCA Math Bonanza 2018 TR/3 The number of quests, containing any three players, is $\binom{10}{3}$. The number of quests, containing only good players, is $\binom{6}{3}$. Therefore, the number of quests, containing at least one bad players is $\binom{10}{3} - \binom{6}{3} = 100$. Furthermore, the number of quests, containing exactly one bad player, is $\binom{6}{2}\binom{4}{1} = 60$.

Thus, the probability that there is exactly one bad player in the failed quest is $\boxed{\frac{60}{100} = \frac{3}{5}.}$ □

Proof. CMIMC Geometry 2018/2 The extreme cases for where Q lies are when P is at C or D. When P is at C, $Q \equiv E$, by the Angle Bisector Theorem,

$$\frac{AC}{AE} = \frac{BC}{BE} \Rightarrow \frac{BC}{BE} = \frac{AC + BC}{AE + BE} = \frac{AC + BC}{AB} = \sqrt{2} + 1 \Rightarrow BE = \frac{1}{\sqrt{2} + 1} = \sqrt{2} - 1$$

Similarly $AF = \sqrt{2} - 1$.

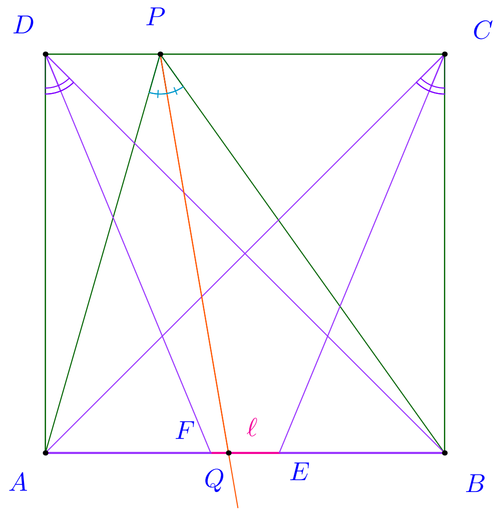


Figure 2.2: CE, DF bisectors

Thus, the answer is $\boxed{EF = 1 - 2(\sqrt{2} - 1) = 3 - 2\sqrt{2}.}$ □

Proof. All Russian MO 2018/G10/1 First,

$$x^2 + 2018x - 2019 = |x| + |x+1| + \cdots + |x+2018| > 0 \Rightarrow (x-1)(x+2019) > 0 \Rightarrow \left[\begin{array}{l} x > 1 \\ x < -2019 \end{array}\right.$$

Case 1: $x > 1$,

$$x + (x+1) + \cdots + (x+2018) = x^2 + 2018x - 2019 \Rightarrow x^2 - x - 1010 \cdot 2019 = 0 \Rightarrow x = \frac{1}{2} + \frac{1}{2}\sqrt{8156761}$$

Case 2: $x < 2019$,

$$-(x+(x+1)+\cdots+(x+2018)) = x^2+2018x-2019 \Rightarrow x^2+4037x-1008\cdot 2019 = 0 \Rightarrow x = -\frac{4037}{2} - \frac{1}{2}\sqrt{24437977}$$

Thus, $\boxed{\text{there are 2 solutions.}}$ □

Proof. Vietnam MO 1982/4 WLOG, $p \geq q \geq r$,

$$2^p + 2^q + 2^r = 2^r(2^{p-r} + 2^{q-r} + 1) = 2^5 \cdot 73$$

Case 1: $q = r$. If $p = r$, then $2^{p-r} + 2^{q-r} + 1 = 3$, but 2236 is not divisible by 3. If $p > r$, then $2^{r+1}(2^{p-r-1} + 1) = 2^5 \cdot 73$, but 72 cannot be a power of 2. So there is no solution for this case.

Case 2: $q > r$. If $p = q$, then $2^{p-r} + 2^{q-r} + 1 = 2 \cdot 2^{p-r} + 1 = 73$, but 36 cannot be a power of 2. If $p > q$, then $2^{p-r} + 2^{q-r} = 72$ is the sum of two powers of 2, that can only be possible with $64 + 8 = 72$.

Thus $\boxed{r = 5, q = 8, p = 11.}$ □

Proof. Germany MOO 2018/1 From the first two equations,

$$0 = xy + z + yz + x = y(x+z) + (x+z) = (y+1)(x+z)$$

Case 1: $y = -1$, then

$$\begin{cases} -x + z = -30 \\ zx - 1 = -18 \end{cases} \Rightarrow \begin{cases} (-x) + z = -30 \\ (-x)z = 17 \end{cases}$$

Therefore, $-x, z$ are roots of $t^2 + 30t + 17 = 0$, $-x, z = 15 \pm 4\sqrt{13}$.

Thus, $\boxed{(x,y,z) = (-15 - 4\sqrt{13}, -1, 15 - 4\sqrt{13}), (-15 + 4\sqrt{13}, -1, 15 + 4\sqrt{13})\}.}$

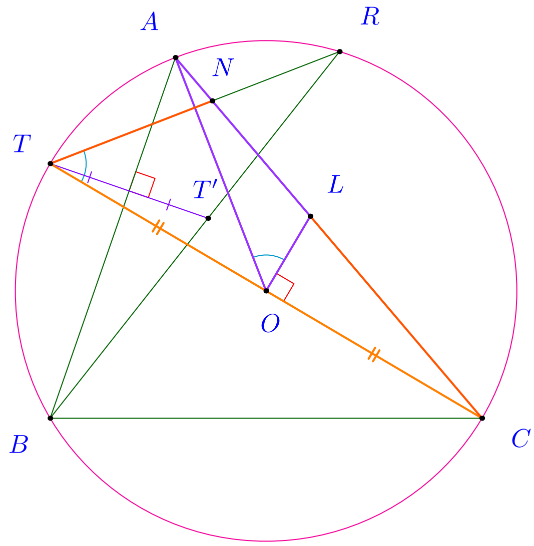
Case 2: $z = -x$, then

$$\begin{cases} xy - x = -30 \\ -x^2 + y = -18 \end{cases} \Rightarrow x(x^2 - 18) - x = -30 \Rightarrow x^3 - 19x + 30 = 0 \Rightarrow (x-3)(x-2)(x+5) = 0$$

Thus, $\boxed{(x,y,z) = (3, -9, -3), (2, -14, -2), (-5, 7, 5).}$ □

Proof. Centroamerican and Caribbean MO 2018/2

$$\angle NTC = \angle RTC = \angle T'BC = \angle B - \angle ABT' = \angle B - \angle ABT = \angle B - \angle ACO = 2\angle B - 90^\circ$$
$$\angle AOC = 2\angle B \Rightarrow 2\angle B = \angle AOL + \angle LOC = \angle AOL + 90^\circ \Rightarrow \angle AOL = 2\angle B - 90^\circ = \angle NTC$$
$$\Rightarrow \angle AOL = \angle NTC,\ \angle NCT = \angle ACO = \angle OAL \Rightarrow \angle NCT = \angle OAL$$
$$\Rightarrow \triangle NCT \sim \triangle AOL \Rightarrow \frac{AO}{TC} = \frac{AL}{NC} \Rightarrow CN = \frac{TC}{AO}AL = 10$$



Thus, $\boxed{CL = 10.}$ □

Proof. Math Prize for Girls Olympiad 2018/3 The answer is $\boxed{14}$ bent strips. To see that at least 14 bent strips is necessary, note that every corner unit cube must have some flat strip touching it; each flat strip touches at most two corner unit cubes, hence there are at least 4 four flat strips.

			3	4	4			
			f	f	e			
			5	5	6			
3	f	5	x	x	x	6	e	4
3	a	a	a	b	7	6	e	d
w	w	w	1	b	7	y	y	y
			1	b	7			
			1	c	8			
			2	c	8			
			2	c	8			
			2	d	d			
			z	z	z			

A construction is shown below, as the net of a cube, with each strip being denoted by a letter or number. The flat strips are colored blue.

To prove the bound, look at all the 8 corners. We have to cover all 3 faces of each corner. However using a bent strip we can only cover only 2 of the faces. This means that the third face must be covered by a flat strip. Hence, since a flat strip has length 3, we will need at least 4 of them to cover all corners.

Finally, six centers can be covered by six bent strips. □

Proof. All Russian MO 2018/G11/2 Since $a,\ b > 0$, by the Cauchy-Schwarz Inequality,

$$1 + ab \leq \sqrt{(1+a^2)(1+b^2)} \Rightarrow \frac{1+a^2}{1 \cdot 1 + a \cdot b} \geq \sqrt{\frac{1+a^2}{1+b^2}}$$

For $n \geq 2$, $x_1, x_2, \ldots, x_n, x_{n+1} = x_1 > 0$,

$$\frac{1+x_i^2}{1+x_ix_{i+1}} \geq \sqrt{\frac{1+x_i^2}{1+x_{i+1}^2}},\ \forall i = 1, \ldots, n \Rightarrow \quad \sum_{i=1}^{n} \frac{1+x_i^2}{1+x_ix_{i+1}} \geq \sum_{i=1}^{n} \sqrt{\frac{1+x_i^2}{1+x_{i+1}^2}}$$

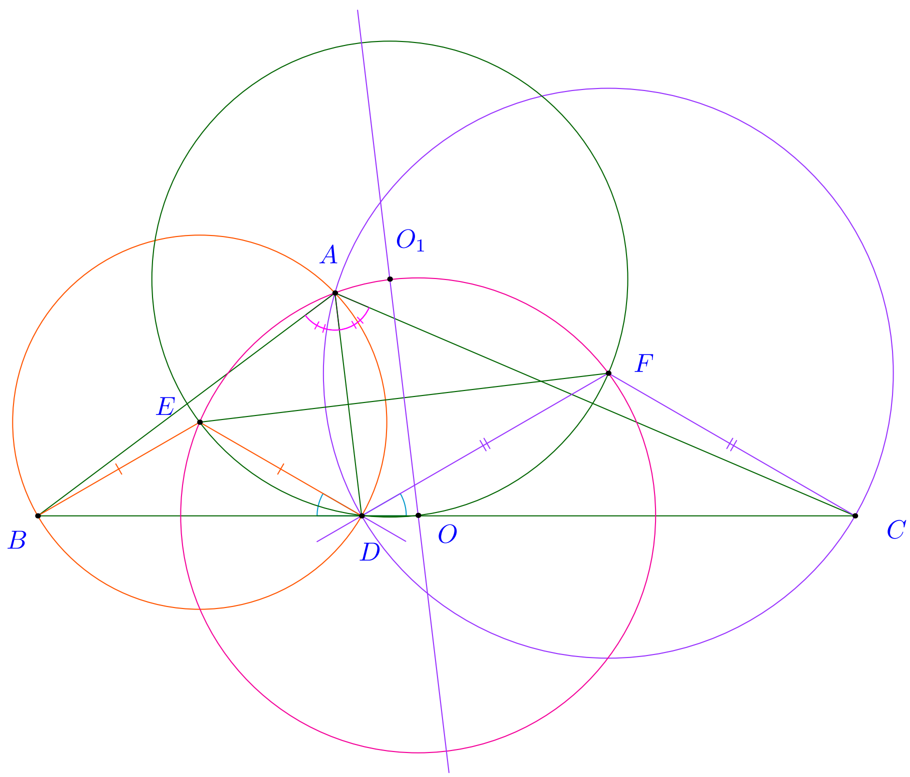
By the AM-GM Inequality,

$$\sum_{i=1}^{n} \sqrt{\frac{1+x_i^2}{1+x_{i+1}^2}} \geq n \sqrt[n]{\prod_{i=1}^{n} \sqrt{\frac{1+x_i^2}{1+x_{i+1}^2}}} = n$$

$\square$

Proof. Czech and Slovak MO 2018/3 Denote by O the circumcenter of AEF and by O_1 the circumcenter of DEF.

$$\angle BAD = \angle DAC = \frac{\angle BAC}{2} < \frac{\pi}{2}$$



Thus, E, F and A on the same side of BC.

$$\angle EDB = \frac{\pi - \angle BED}{2} = \frac{\pi - 2\angle BAD}{2} = \frac{\pi - \angle BAC}{2} = \frac{\pi - 2\angle DAC}{2} = \frac{\pi - \angle DFC}{2} = \angle FDC$$

Therefore BC is the external bisector $\angle EDF$, thus O is on both the external bisector of $\angle EDF$ and the perpendicular bisector of EF, hence O is midpoint of arc EDF, so $O_1O = O_1E = O_1F$.

Furthermore, $\triangle AEF \cong \triangle DEF(AE = DE, AF = DF, EF = EF)$, thus $OE = O_1E = O_1F = OF$, therefore $\triangle EOO_1$ and $\triangle FOO_1$ are equilateral. Thus,

$$\angle EDF = \angle EOF = \angle EOO_1 + \angle O_1OF = \frac{2\pi}{3} \Rightarrow \quad \angle EDB = \frac{\pi - \angle EDF}{2} = \frac{\pi}{6}$$
$$\Rightarrow \angle BAC = \angle BED = \pi - 2\angle BED = \frac{2\pi}{3}$$

$\boxed{\angle BAC = 120^\circ.}$ □

Proof. Harvard-MIT Math Tournament 2005/A5 First note that all the integers must be at least 2 because the greatest common divisor of any two positive integers is at least 1.

Let n be the largest integer in the circle. The greatest common divisor of its two neighbours is $n-1$. Therefore, each of the two neighbours is at least $n-1$ but at most n, so since $n-1$ can't divide n for $n-1 \geq 2$, they must both be equal to $n-1$.

Let m be one of the numbers on the other side of $n-1$ from n. Then $gcd(n,m) = n-2$. Since $n-2 \geq 0$, $n-2 \mid n$ only for $n = 3, 4$. It is easy to see that $m \neq 3$, so $m = 2$, then it is again not hard to find that there is a unique solution up to rotation, namely $4, 3, 2, 2, 3, 4, 3, 2, 2, 3$.

The only possible sum is therefore $\boxed{28.}$ □

Proof. Taiwan MO 2006/2 First, let $P_1, \ldots, P_n$ be the problems. Let $A_1, \ldots, A_{10}$ be the sets of problems in each test, since $|A_i \cap A_j| \leq 1,\ \forall i \neq j$, thus each pair $(P_i,\ P_j)$ can't occur more than one time in these sets.

$$\binom{n}{2} \geq \sum_{i<j} \binom{|A_i|}{2} = 10\binom{4}{2} = 60.$$

Thus, $\boxed{n \geq 12.}$

Since there are 10 test papers, so if there are 12 problem, then one of them, say P_1 must have occured at least $\lceil \frac{40}{12} \rceil = 4$ times. Then the other 3 problems in each of the test papers where P_1 is should be different. Hence, we have at least $3 \times 4 + 1 = 13$ problems. Therefore, $\boxed{n \geq 13.}$

Here's one explicit example when $n = 13$.

1	2	3	4
1	5	6	7
1	8	9	10
1	11	12	13
2	5	8	11
2	6	9	12
3	5	9	11
3	6	8	12
4	5	10	12
4	6	9	11

□

Chapter 3

Session 3

3.1 Topics

Algebra

1. Sums & Products. Third degree identities such as $(x + y + z)^3 - x^3 - y^3 - z^3$.
2. Polynomials with integer coefficients. Polynomial factorization into polynomials with integer coefficients. Degree of divisor polynomial does not exceed degree of division polynomial, except zero polynomial.
3. Even function $P(x) = P(-x)$, $\forall x$ as polynomial has only non-zero coefficient terms with even powersx^{2k}. Odd function $P(x) = -P(-x)$, $\forall x$ as polynomial has only non-zero coefficient terms with odd powersx^{2k-1}.
4. Arithmetic Sequence. Geometric Sequence.
5. Integer Sequence. Periodic Remainders when divided by a prime.
6. Inequality. Rearragement Inequality. Chebysev Inequality.

Combinatorics

1. Combinations. Permutations. Correction over Overcounting.
2. Grids. Paths in grids.
3. Recurrrence Relations.
4. Number partitions
5. Sets.
6. Algorithms.

Geometry

1. Computational Geometry. Line slop, line equation, line intersection. Perpendicular Lines. Equations of line symmetric over the $x-$axis, over the $y-$axis.
2. Paralbola equation. Symmetric axis of parabola.
3. Circle equation. Perpendicular bisector or chords through center of circle.
4. Quadrilateral. Sums of lengths of the two opposite sides of the quadrilateral in tangential quadrilaterals $AB + CD = AC + BD$, and $AB - AC = BD - CD$. (a quadrilateral whose four sides are all tangent to a circle inscribed within it). Circumscribed quadrilateral. Ptolemy theorem.
5. Trigonometry. Law of Sines. Area of triangle. Area of quadrilateral. Law of Cosines. Compute side of triangles. Compute diagonals of quadrilateral in two ways based on two triangles it divides the quadrilateral into.

6. Cyclic quadrilaterals with right opposite angles.

Number Theory

1. Divisibility. Divisors. Sum of digits. Prime Factorization.
2. Prime divisors of a prime. Prime divisor of a power of 2.
3. Modular arithmetic. Residue modulo 3, 5, 6, 13.
4. Remainders of perfect squares when divided by 3.
5. Number bases. Decimal representation. Repeated pattern of digits.
6. Diophantine equations. Integers with known sums and products, e.g. find integers $x \cdot y \cdot z = 24$, $x + y + z = 9$.

3.2 Problems

Problem 3.2.1 (Math Problem Book I/76). (*4 points*)

Perpendiculars from a point P on the circumcircle of $\triangle ABC$ are drawn to lines AB and BC with feet D and E respectively.

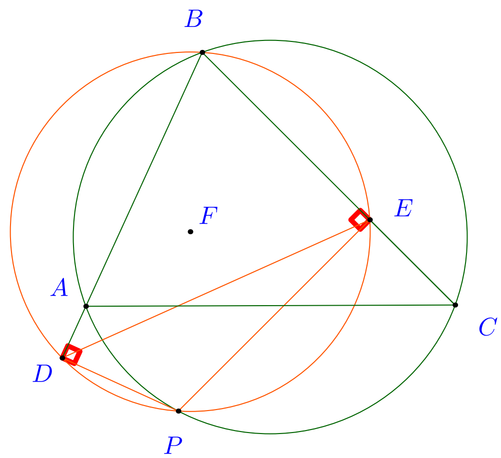


Figure 3.1: Math Problem Book I/76

Find the locust of the circumcenter F of $\triangle PDE$ as P moves around the circle.

How to provide your answer: A *correct answer* earns full marks.

Problem 3.2.2 (RV-COMC-2021-S3-3). (*4 points*)

r and s are the roots of the equation $x^2 + x - 3 = 0$.

Investigate the two sums $S_1 = r^3 - 4s^2 + 19,\ S_2 = s^3 - 4r^2 + 19$.

What is the value of S_1?

How to provide your answer: A *correct answer* earns full marks.

Problem 3.2.3 (China MO 1994). (*4 points*)

In a tennis tournament, each player plays exactly one game against each of the other players. During the tournament, there are 3 players whi have withdrawn from the tournament and each of them participates in exactly 2 matches. The total number of matches is 50.

What is the number of matches the 3 players played among themselves?

How to provide your answer: A *correct answer* earns full marks.

Problem 3.2.4 (RV-COMC-2021-S3-4). (*4 points*)

For n positive integers, find the sum of the digits of the number

$$\sqrt{\frac{1}{27}(\underbrace{33\ldots33}_{2n}-\underbrace{66\ldots66}_{n})}.$$

How to provide your answer: A *correct answer* earns full marks.

Problem 3.2.5 (Math Problem Book I/157)**.** (*6 points*)

Find all **integer** solutions to the system of equations

$$\begin{cases} x + y + z & = 3 \\ x^3 + y^3 + z^3 & = 3 \end{cases}$$

How to provide your answer: A *correct answer* earns full marks.

Problem 3.2.6 (Ireland MO 1997/7). (*6 points*)

Let A be a subset of $\{0, 1, \ldots, 1997\}$ such that,

1. no element of A is a power of 2,
2. no two distinct elements of A, whose sum is a power of 2.

What is the largest possible number of elements of A?

How to provide your answer: A *correct answer* earns full marks.

Problem 3.2.7 (St. Petersburg 1997/12). (*6 points*)

A $n \times n$ square grid is folded several times along grid lines. Two straight cuts are made along grid lines (without refolding the grid).

What is the maximum number of pieces the square can be cut into, if

1. $n = 1$,
2. $n = 2$,
3. $n = 8$.

How to provide your answer: You must *show all your work*.

Problem 3.2.8 (Math Problem Book I/68)**.** (*6 points*)

Let the inscribed circle of $\triangle ABC$ touches side BC at D, side CA at E and side AB at F. Let G be the foot of the perpendicular from D to EF.

Find CE, if $FG = 1$, $GE = 2$, and $BF = 3$.

How to provide your answer: A *correct answer* earns full marks.

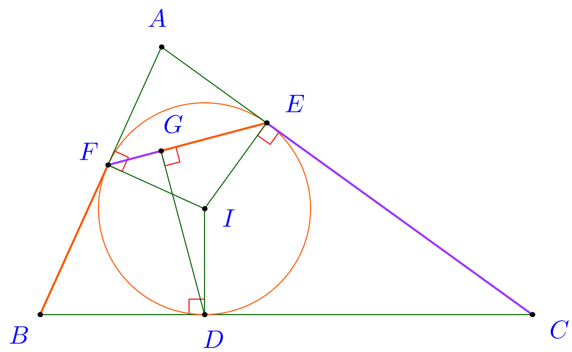


Figure 3.2: Math Problem Book I/68

Problem 3.2.9 (RV-COMC-2021-S3-7). (*10 points*)

There are 1000 points in a square, and among these points and the vertices of the square, no three points are collinear. These points and the vertices of the square are connected by the line segments such that the square is divided completely into several triangles. (The sides of these triangles are all the connecting line segments and the sides of the original square, and any two connecting line segments have no common interior point except the end points).

1. Note that the sum of all interior angles can be counted in two ways, find k, the number of triangles.
2. Note that the number of segments can be counted in two ways, find ℓ, the number of line segments.

How to provide your answer: A *correct answer* earns full marks.

Problem 3.2.10 (Czech and Slovak Match 1995/1). (*10 points*)

Let $a_1 = 2,\ a_2 = 5$ and,

$$a_{n+2} = (2 - n^2)a_{n+1} + (2 + n^2)a_n,\ \forall n \geq 1.$$

1. Prove that $3 \mid a_{n+1} - a_n$, for all $n \geq 1$.
2. Do there exist p, q, r so that $a_p a_q = a_r$?

How to provide your answer: You must *show all your work*.

Problem 3.2.11 (Crux Mathematicorum 7)**.** (*10 points*)

$P(x)$ is a *fifth* degree polynomial, such that $P(x)+1$ is divisible by $(x-1)^3$ and $P(x)-11$ is divisible by $(x+1)^3$.

1. Prove that $(x-1)^3 \mid P(-x)-1$, $(x+1)^3 \mid P(-x)+1$, and $(x-1)^3(x+1)^3 \mid P(x)+P(-x)$.
2. Prove that $P(x)+P(-x) \equiv 0,\ \forall x$.
3. Find all coefficients of $P(x)$?

How to provide your answer: You must *show all your work*.

Problem 3.2.12 (APMO 2000/3)**.** (*10 points*)

Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively. And O the point in which the perpendicular at P to BA meets AN produced.

Let N be the origin, line ANO be the $x-$axis, and the equation of line AB be $y = ax + b$, and line BC be $y = cx$, respectively.

1. Note that the two sides of and angle are symmetric over the angle bisector. Find the coordinates of A, B, P and the equation for PO. Find the equations for AC, and the coordinates of C and M.
2. Find the slop and the equation of the line AM, then the coordinates of Q.
3. Prove that QO is perpendicular to BC.

How to provide your answer: You must *show all your work*.

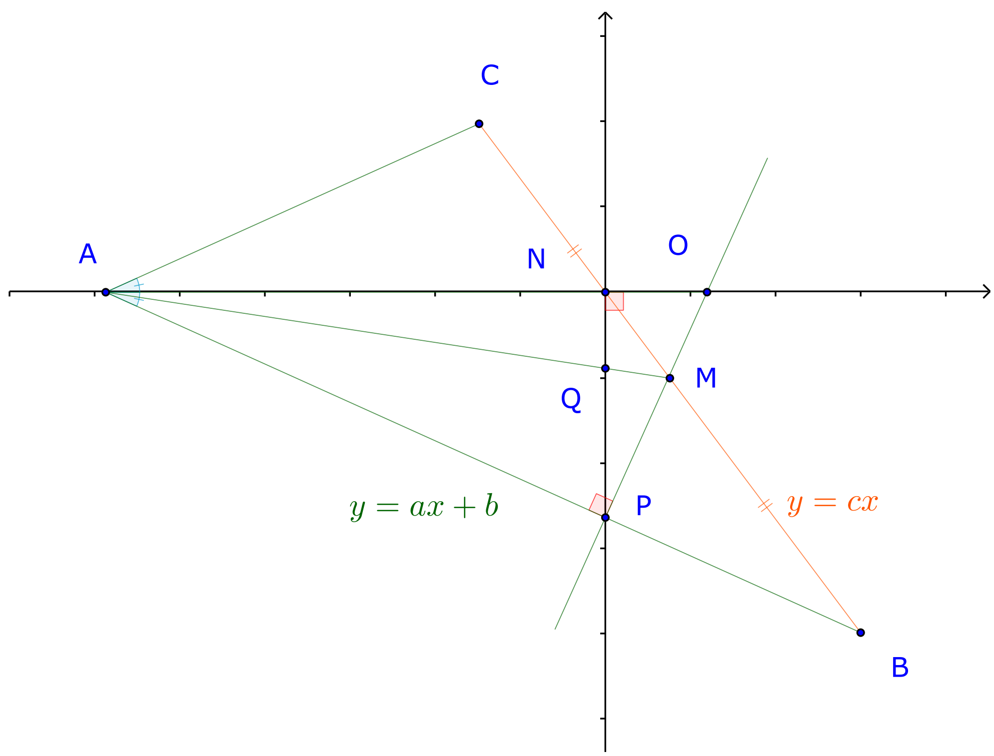


Figure 3.3: APMO 2000/3

3.3 Solutions

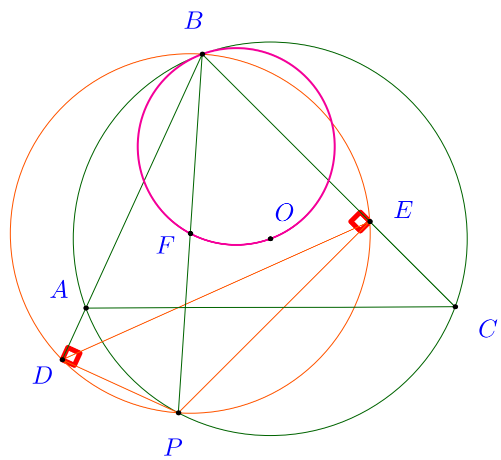


Figure 3.4: Circle diameter OB

Proof. Math Problem Book I/76 It is easy to see that $PDBE$ is cyclic, so F is the midpoint of BP, thus the locus of F is the circle diameter BO, where O is the centre of (ABC). □

Proof. RV-COMC-2021-S3-3 By Vieta's Theorem,

$$\begin{cases} r+s=-1 \\ rs=-3 \end{cases}$$

Now, let $S_1 = r^3 - 4s^2 + 19,\ S_2 = s^3 - 4r^2 + 19,$

$$\begin{cases} S_1 + S_2 = (r^3+s^3) - 4(r^2+s^2) + 38 = (-1))((-1)^2 - 3(-3)) - 4((-1)^2 - 2(-3)) + 38 = 0 \\ S_1 - S_2 = (r^3 - s^3) - 4(r^2 - s^2) = (r-s)((-1)^2 - (-3)) - 4(r-s)(-1) = 0 \end{cases}$$

Thus, $2S_1 = (S_1+S_2) + (S_1 - S_2) = 0$, so $\boxed{S_1 = 0.}$ □

Proof. China MO 1994 The number of games among $n-3$ players is $\binom{n-3}{2}$. Let m be the number of games the three players played among themselves. It is easy to see that $m \in \{0,1,2,3\}$. They played $3 \cdot 2 = 6$ games. In this 6 games, m games counted twice. So the total number of games they contribute to the total number of games is $6-m$, thus

$$50 = \binom{n-3}{2} + (6-m) \Rightarrow 88 + 2m = (n-3)(n-4),\ m \in \{0,1,2,3\}$$

It is easy to test that only $\boxed{m=1}$ is the solution. □

Proof. RV-COMC-2021-S3-4

$$\begin{aligned} \underbrace{33\dots33}_{2n} - \underbrace{66\dots66}_{n} &= \underbrace{33\dots33}_{n}10^n + \underbrace{33\dots33}_{n} - \underbrace{66\dots66}_{n} = \underbrace{33\dots33}_{n}10^n - \underbrace{33\dots33}_{n} \\ &= \underbrace{33\dots33}_{n}(10^n - 1) = \underbrace{33\dots33}_{n}\underbrace{99\dots99}_{n} = 27(\underbrace{11\dots11}_{n})^2 \end{aligned}$$

Thus, $\sqrt{\frac{1}{27}(\underbrace{33\dots33}_{2n} - \underbrace{66\dots66}_{n})}$ is $\underbrace{11\dots11}_{n}$ and its sum of digits is $\boxed{n.}$ □

Proof. Math Problem Book I/157 By the identity $(x+y+z)^3 - (x^3+y^3+z^3) = 3(x+y)(y+z)(z+x)$,

$$\Rightarrow (x+y)(y+z)(z+x) = \frac{3^3-3}{3} = 8 \Rightarrow \begin{cases} (3-x)(3-y)(3-z) & = 8 \\ (3-x)+(3-y)+(3-z) & = 6 \end{cases}$$

Therefore $(3-x, 3-y, 3-z)$ is $(2,2,2)$ or any permutation of $(8,-1,-1)$.

Thus, (x,y,z) are $\{(1,1,1), (-5,4,4), (4,-5,4), (4,4,-5)\}$. ☐

Proof. Ireland MO 1997/7 Suppose that A did not verify the conclusion. Then A would contain at most half of the integers from 51 to 1997, since they can be divided into pairs whose sum is 2048 (with 1024 left over).

Like wise, A contains at most half of the integers from 14 to 50, at most half of the integers from 3 to 13, and possibly 0, for a total of

$$973 + 18 + 5 + 1 = 997$$

integers. ☐

Proof. RV-COMC-2021-S3-7 Let k be the number of triangles and ℓ be the number of lines. Couting by k triangles, the sum of all interior angles is $k \cdot 180°$. Counting per vertices, the sum of these angles is $1000 \cdot 360° + 4 \cdot 90°$. Therefore,

$$k \cdot 180° = 1000 \cdot 360° + 4 \cdot 90°.$$

Hence $k = 2002$. Since each connecting line segment is a common side of two triangles and every side of the square is one side of some triangles, thus,

$$3k = 2l + 4 \Rightarrow \ell = \frac{3k}{2} - 2 = 3001.$$

Therefore, the number of triangles is $\boxed{k = 2001}$ and the number of lines is $\boxed{\ell = 3001.}$ ☐

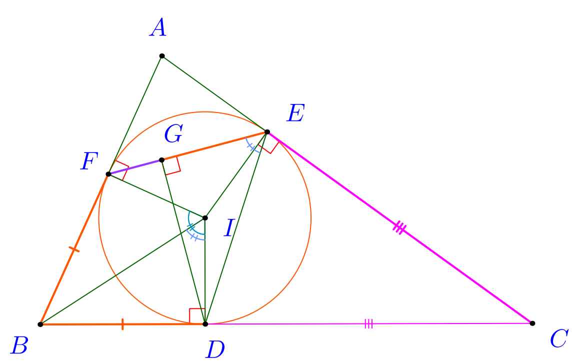


Figure 3.5: $\frac{FG}{EG} = \frac{BF}{CE}$

Proof. Math Problem Book I/68 Let I be the incentre of $\triangle ABC$,

$$\angle DEG = \frac{1}{2}\angle DIF = \angle DIB \Rightarrow \triangle BDI \sim \triangle EGD$$

Similarly $\triangle CDI \sim \triangle FGD$,

$$\frac{FG}{EG} = \frac{FG}{DG} \cdot \frac{DG}{EG} = \frac{ID}{CD} \cdot \frac{BD}{ID} = \frac{BD}{CD} = \frac{BF}{CE}$$

Thus, $\boxed{CE = BF \cdot \frac{EG}{FG} = 6.}$ ☐

Proof. St. Petersburg 1997/12 Note that if we number the $2n \times 2n$ grid lines, then a line can only be folded into a parallel line of the same parity. Thus, a cut break at most n edged in one direction. Having these parallel would create at most $2n+1$ pieces, but making them perpendicular allows at most $(n+1)^2$ pieces.

First we fold the sheet into a $2 \times n$ rectangle, then a 2×2 square, and then cutting along the central grid lines of the square.

Thus, the answer is $(4+1)^2 = 25$.. □

Proof. Czech and Slovak Match 1995/1 By induction, we can prove that

Claim — $3 \mid a_{n+1} - an, \ \forall n \geq 1$.

Proof. $a_2 - a_1 = 5 - 2 = 3$, so the induction hypothesis is true for $n = 1$. Assume that it is true for n. If $3 \not| \, n$, then $n^1 \equiv 1 \pmod 3$, thus $3 \mid n^2 - 1, \ n^2 + 2$, and

$$a_{n+2} - a_{n+1} = (1 - n^2)a_{n+1} + (2 + n^2)a_n \Rightarrow 3 \mid a_{n+2} - a_{n+1}$$

If $3 \mid n$, then $3 \mid a_{n+1} - a_n$, thus

$$a_{n+2} - a_{n+1} = -n^2 a_{n+1} + n^2 a_n + (a_{n+1} - a_n) + 3a_n \Rightarrow 3 \mid a_{n+2} - a_{n+1}$$

■

Therefore $a_n \equiv a_1 \equiv 2 \pmod 3, \ \forall n \geq 1$, therefore there does not exists a triple (p, q, r)

$$a_p a_q = a_r \Leftrightarrow \begin{cases} a_p a_q \equiv 1 \pmod 3 \\ a_r \equiv 2 \pmod 3 \end{cases}, \text{ which is impossible}$$

□

Proof. Crux Mathematicorum 7 First, by substitue $-x$ into x in $(x-1)^3 \mid P(x) + 1$ and $(x+1)^3 \mid P(x) - 1$,

$$(-x-1)^3 \mid P(-x) + 1, \ (-x+1)^3 \mid P(-x) - 1 \Rightarrow (x+1)^3 \mid P(-x) + 1, \ (x-1)^3 \mid P(-x) - 1$$

Therefore

$$(x-1)^3 \mid (P(x)+1) + (P(-x)-1) = P(x) + P(-x), \ (x+1)^3 \mid (P(x)-1) + (P(-x)+1) = P(x) + P(-x)$$

Since $\deg_P = 5$, so $\deg(P(x) + P(-x)) \leq 5$, and $\deg((x-1)^3(x+1)^3) = 6$, so $P(x) + P(-x) \equiv 0$.

Thus, $P(x)$ ia an odd function, $P(x) = -P(-x)$, therefore all of its even power coefficients are zero, thus

$$P(x) + 1 = (x-1)^3(ax^2 + bx + c) \Rightarrow \begin{cases} b - 3a = 0 \\ 3 + 3b - a = 0 \end{cases} \Rightarrow a = \frac{-3}{8}, \ b = -\frac{9}{8}.$$

Therefore $\boxed{P(x) = -\frac{3x^5}{8} + \frac{5x^3}{4} - \frac{15x}{8}.}$ □

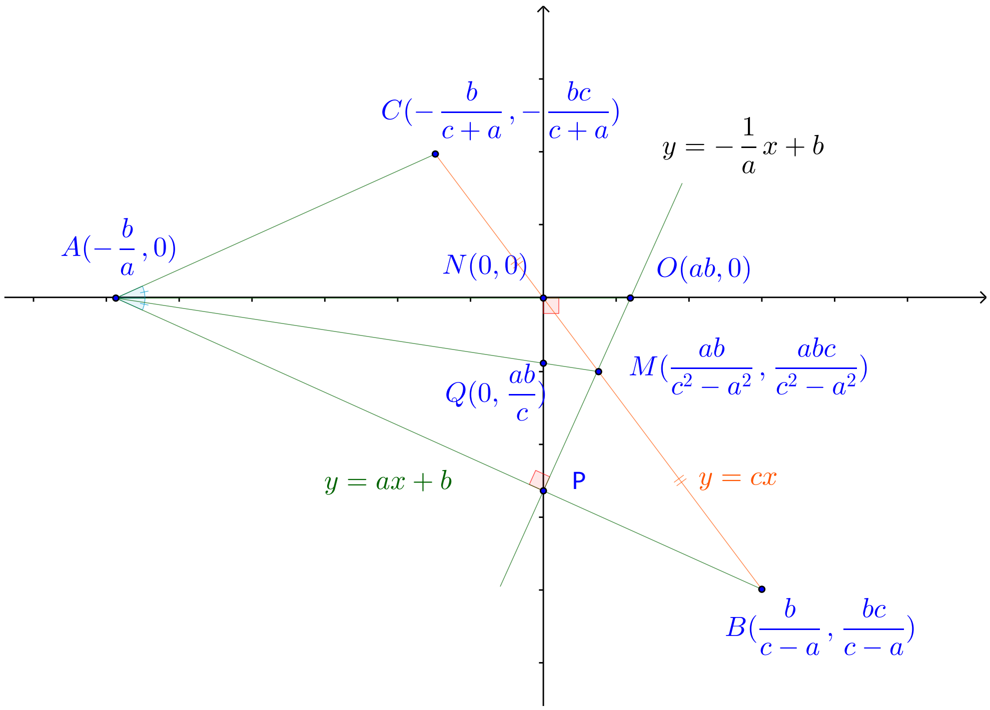


Figure 3.6: APMO 2000/3

Proof. APMO 2000/3 Since the equation of line AB is $y = ax + b$, A is on the $x-$axis, and P is on the $y-$axis, thus $A\left(-\frac{b}{a}\right)$ and $P(0, b)$. Line PO is perpendicular to line AB so its slope is $-\frac{1}{a}$, it is through P, so its equation is $y = -\frac{1}{a}x + b$.

B is the intersection of AB and BC so B coordinates are

$$\left.\begin{aligned} y = ax + b \\ y = cx \end{aligned}\right\} \{ \Rightarrow x = \frac{b}{c-a},\ y = \frac{bc}{c-a} \Rightarrow B\left(\frac{b}{c-a}, \frac{bc}{c-a}\right)$$

Furthermore AC is reflection of AB over the $x-$axis, so the equation of line AC is $y = -ax - b$, thus C coordinates,

$$\left.\begin{aligned} y = -ax - b \\ y = cx \end{aligned}\right\} \Rightarrow x = -\frac{b}{c+\frac{1}{a}},\ y = -\frac{bc}{c-a} \Rightarrow C\left(-\frac{b}{c+a}, -\frac{bc}{c+a}\right).$$

Therefore M coordinates,

$$\left.\begin{aligned} \frac{1}{2}(x_B + x_C) = \frac{1}{2}\left(\frac{b}{c-a} - \frac{b}{c+a}\right) = \frac{ab}{c^2-a^2} \\ \frac{1}{2}(y_B + y_C) = \frac{1}{2}\left(\frac{bc}{c-a} - -\frac{bc}{c+a}\right) = \frac{abc}{c^2-a^2} \end{aligned}\right\} \Rightarrow M\left(\frac{ab}{c^2-a^2}, \frac{abc}{c^2-a^2}\right).$$

O is the intersection of PO with the $x-$axis, so $O(ab, 0)$. The slop of the line through $A\left(-\frac{b}{a}\right)$ and $M\left(\frac{ab}{c^2-a^2}, \frac{abc}{c^2-a^2}\right)$ is

$$\frac{\frac{abc}{c^2-a^2} - 0}{\frac{ab}{c^2-a^2} - \left(-\frac{b}{a}\right)} = \frac{abc}{c^2-a^2}\frac{a(c^2-a^2)}{bc^2} = \frac{a^2}{c}$$

Thus, AM equation is $y - 0 = \frac{a^2}{c}\left(x - \left(-\frac{b}{a}\right)\right)$, or $y = \frac{a^2}{c}x + \frac{ab}{c}$. Therefore $Q\left(0, \frac{ab}{c}\right)$.

Line through QO has the slop $\frac{\frac{ab}{c}-0}{0-ab} = -\frac{1}{c}$. Hence, $QO \perp BC$. $\square$

Chapter 4

Session 4

4.1 Topics

Algebra

1. Radical. Reciprocal. Substitution in expression.
2. Third-degree Identities. Factoring.
3. Minimum and maximum of quadratic expression.
4. Inequality. Rearragement Inequality. Chebysev Inequality.
5. Sums and Products. Sums of pair of numbers in two ways.

Combinatorics

1. Number expressed as different sum of two numbers. Selection of numbers whose difference not matching a constant, for example $1, 2, \ldots, 10$, where $i - j \neq 2$, from pairs $(1, 3), (2, 4), \ldots, (8, 10)$.
2. Counting in two ways.
3. Overcounting, the number of two-digit numbers ab is 10^2 multiplying with $\frac{9}{10}$ since a can only be $1, \ldots, 9$.
4. Counting element in multiple subsets. Estimate number of times each element appear.

Geometry

1. Area of triangles and squares.
2. Paralbola equation. Symmetric axis of parabola.
3. Circle equation. Perpendicular bisector or chords through center of circle.
4. Trigonometry. Law of Sines. Area of triangle. Area of quadrilateral. Law of Cosines. Compute side of triangles. Compute diagonals of quadrilateral in two ways based on two triangles it divides the quadrilateral into.
5. Quadrilaterals with inscribed circle. Sums of lengths of the two opposite sides of the quadrilateral in tangential quadrilaterals $AB + CD = AC + BD$, and $AB - AC = BD - CD$. (a quadrilateral whose four sides are all tangent to a circle inscribed within it).

Number Theory

1. Residue of power modulo prime. Residue of exponent modulo $p - 1$, where p is prime. Little Fermat's Theorem.
2. Number bases. $\overline{abcabc} = 10^3\overline{abc} + abc$.

3. Divisibility. Divisors. Sum of digits. Prime Factorization.
4. Prime divisors of a prime. Prime divisor of a power of 2.
5. Modular arithmetic. Residue modulo 3, 5, 6, 11, 13.
6. Number bases. Decimal representation. Repeated pattern of digits.
7. Diophantine equations. Integers with known sums and products, e.g. find integers $x \cdot y \cdot z = 24$, $x + y + z = 9$.

4.2 Problems

Problem 4.2.1 (RV-COMC-2021-S4-1). (*4 points*)

Let $a = \sqrt[3]{4} + \sqrt[3]{2} + 1$, find $\frac{3}{a} + \frac{3}{a^2} + \frac{1}{a^3}$.

How to provide your answer: A *correct answer* earns full marks.

Problem 4.2.2 (RV-COMC-2021-S4-2). (*4 points*)

Find the remainder of $37^{27^{17}}$ when divided by 11.

How to provide your answer: A *correct answer* earns full marks.

Problem 4.2.3 (Moscow MO 1940/1). (*4 points*)

Factor

$$(b-c)^3 + (c-a)^3 + (a-b)^3$$

How to provide your answer: A *correct answer* earns full marks.

Problem 4.2.4 (RV-COMC-2021-S4-4)**.** (*4 points*)

For $n \geq 2$, let $S = \{1, 2, \ldots, 8n - 1\}$.

What is the maximum number of elements can be chosen from S such that the difference of any two distinct chosen elements is not equal to 4.

How to provide your answer: A *correct answer* earns full marks.

Problem 4.2.5 (RV-COMC-2021-S4-5). (*6 points*)

In triangle ABC, $\angle ACB = 90°$, and $AC = BC = 1$. P is a point on AB. The feet of the perpendiculars from P to AC and BC are Q and R, respectively. Let S be the largest area among $[APQ]$, $[BPR]$, and $[CQPR]$. Note that S varies as P changes its position. Investigate the special cases when $BR = \frac{1}{3}$ and $BR = \frac{2}{3}$.

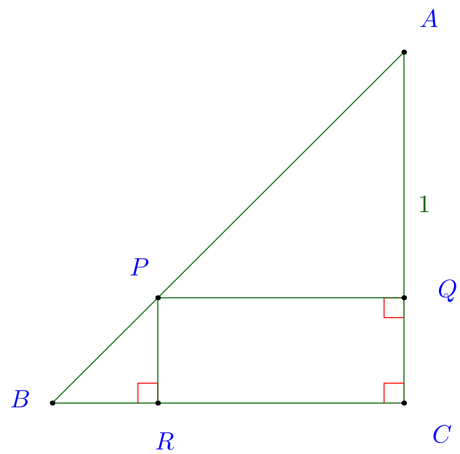


Figure 4.1: RV-COMC-2021-S4-5

What is the minimal value of S?

How to provide your answer: A *correct answer* earns full marks.

Problem 4.2.6 (Chinese Team Training 1994/1). (*6 points*)

For $0 \le a \le b \le c \le d \le e$ and $a + b + c + d + e = 1$, find the maximum of

$$ad + dc + cb + be + ea.$$

Note that

$$ad + dc + cb + be + ea = \frac{1}{2}\left((ad + ae) + (bc + be) + (cb + cd) + (da + dc) + (ea + eb)\right)$$

How to provide your answer: A *correct answer* earns full marks.

Problem 4.2.7 (Russia MO 1998/4). (*6 points*)

A 10−digit number is said to be *interesting* if its digits are all distinct and it is a multiple of 11111.

How many interesting integers are there?

How to provide your answer: A *correct answer* earns full marks.

Problem 4.2.8 (Spain MO 1997/2). (*6 points*)

The parabola $y = x^2 + px + q$ meeting the coordinate axes in three distinct points. A circle ω is draw through these points.

What is the second intersection of ω with the $y-$axis?

How to provide your answer: A *correct answer* earns full marks.

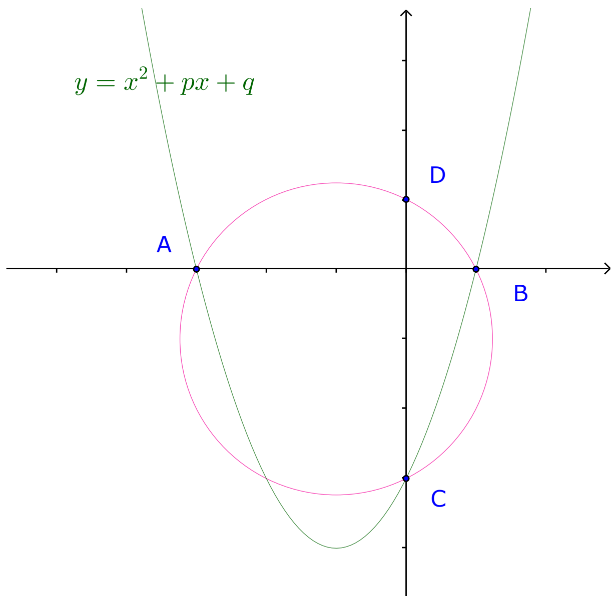


Figure 4.2: Spain MO 1997/2

Problem 4.2.9 (Putnam 1970/B6). (*10 points*)

$ABCD$ is a convex *tangential* quadrilateral (a quadrilateral whose four sides are all tangent to a circle inscribed within it) with $AB = a$, $BC = b$, $CD = c$, $DA = d$, $\angle BAD = \alpha$, and $\angle BCD = \beta$, see Figure 4.6.

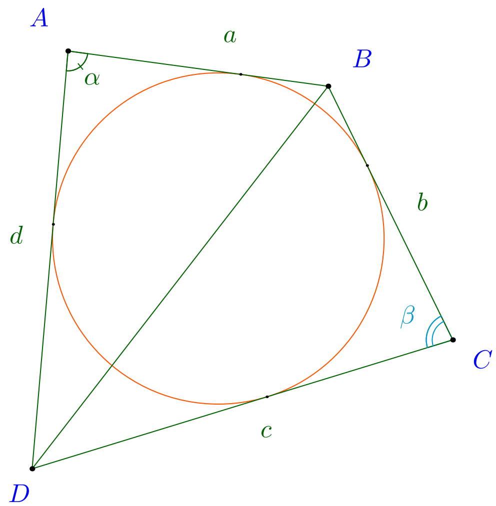


Figure 4.3: Putnam 1970/B6

1. Prove that,

$$ad(1 - \cos\alpha) = bc(1 - \cos\beta)$$

2. $[ABCD]$ denotes the area of the quadrilateral $ABCD$, prove that

$$[ABCD] = \left|\sin\frac{\alpha+\beta}{2}\right| abcd$$

How to provide your answer: You must *show all your work*.

Problem 4.2.10 (Math Problem Book I/167)**.** (*10 points*)

A denotes the set $\{1, 2, \ldots, 2n\}$. Let $B_1, B_2, \ldots, B_m$ be m subsets of A, each has exactly 3 elements and the intersection of any two of them is not a 2−element set,

$$|B_i| = 3,\ |B_i \cap B_j| \neq 2,\ \forall i, j \in \{1, 2, \ldots, m\},\ i \neq j.$$

1. Prove that, for any a of A, the number of subsets B_i containing a cannot exceed $n - 1$.
2. Prove that
$$m \leq \left\lfloor \frac{2n(n-1)}{3} \right\rfloor$$
3. For $n = 4$, show an example when the number of subsets $B_i,\ i = 1, \ldots, m$ is maximal.

How to provide your answer: You must *show all your work*.

Problem 4.2.11 (American Mathematical Monthly E2684). (*10 points*)

Let A_n be the set of positve integers which are less than n and relative prime to n. A_n is also an arithmetic progression.

1. Show that if n is odd, then n is an odd prime.
2. Show that if n is even, n is not divisible by 3, then n is a power of 2.
3. Prove that if $n \geq 3$ is even, $3 \mid n$, the existence of a p prime divisor of n is self-contradictory. Then conclude the solution by determining the set of all n.

How to provide your answer: You must *show all your work*.

Problem 4.2.12 (IMO SL 1990/27). (*10 points*)

For n positive integer, consider the set S_n containing all numbers whose decimal representation has $n-1$ digits 1 and one digit 7,

$$S_n = \{7\underbrace{1\dots1}_{n-1}, 17\underbrace{1\dots1}_{n-2}, \dots \underbrace{1\dots1}_{n-1}7\}.$$

1. For $n = 1$ and $n = 2$, how many prime numbers does S_n contain? For $n = 3$, does S_n contain a prime number?
2. For $n \neq 1$, is $7\underbrace{1\dots1}_{2n}$ a prime number? What can you say S_{2n+1}?
3. For $n \in \{4, 8\}$, does S_n contain a composite numbers? What can you say about S_{12n+4} and S_{12n+8} Finally, for what n, S_n contains only prime numbers?

How to provide your answer: You must *show all your work*.

4.3 Solutions

Proof. RV-COMC-2021-S4-1 Let $b = \sqrt[3]{2}$, then $a = b^2 + b + 1 = \frac{b^3-1}{b-1} = \frac{1}{b-1}$, and $\frac{1}{a} = b - 1$, thus

$$\frac{3}{a} + \frac{3}{a^2} + \frac{1}{a^3} = 3(b-1) + 3(b-1)^2 + (b-1)^3 = (b-1)(3+3b-3+b^2-2b+1) = (b-1)(b^2+b+1) = b^3 - 1 = 1$$

Thus, $\boxed{\frac{3}{a} + \frac{3}{a^2} = 1.}$ □

Proof. RV-COMC-2021-S4-2

$$37^{27^{17}} \equiv 4^{37^{27}} \pmod{11}, \;\; 4^5 = 2^{10} = 1024 \equiv 1 \pmod{11} \Rightarrow 27^{17} \equiv 2^{17} = \left(2^4\right)^4 \cdot 2 \equiv 2 \pmod 5$$

Thus, $4^{37^{27}} \equiv 4^2 \equiv 5 \pmod{11}$, so $\boxed{\text{the remainder is 5.}}$ □

Proof. ??

$$\begin{aligned}(c-a)^3 + (a-b)^3 = & \; ((c-a)+(a-b))\left((c-a)^2 - (c-a)(a-b) + (a-b)^2\right) \\ &= (c-b)(c^2 - 2ca + a^2 - ca + cb + a^2 - ab + a^2 - 2ab + b^2) \\ &= (c-b)(3a^2 + b^2 + c^2 - 3ab + bc - 3ac)\end{aligned}$$

Thus,

$$\begin{aligned}(b-c)^3 + (c-a)^3 + (a-b)^3 &= (b-c)((b-c)^2 - (3a^2 + b^2 + c^2 - 3ab + bc - 3ac)) \\ &= (b-c)(b^2 - 2bc + c^2 - 3a^2 - b^2 - c^2 + 3ab - bc + 3ac) \\ &= (b-c)(3ab + 3ac - 3bc - 3a^2) = 3(b-c)(b(a-c) + a(c-a)) \\ &= 3(a-b)(b-c)(c-a)\end{aligned}$$

Therefore, $\boxed{(b-c)^3 + (c-a)^3 + (a-b)^3 = 3(a-b)(b-c)(c-a).}$ □

Proof. RV-COMC-2021-S4-4 Let $L_i = \{k \in S \mid n \equiv i \pmod 4\}, \;\; i = 0, 1, 2, 3$. Then $|L_0| = 2n - 1, \;\; |L_1| = |L_2| = |L_3| = 2n$, note that the elements in L_1 can be arranged into pairs as,

$$(1, 5), (9, 13), \ldots, (4n-7, 4n-3).$$

By the pigeonhole principle, no more than n elements can be chosen from L_1. It's the same situation with L_2 and L_3. With L_0, there are n odd multiples of 4 that can be chosen

$$4 \cdot 1, 4 \cdot 3, \ldots, 4 \cdot (2n-1) = 8n - 4.$$

Thus, the maximum number of elements that can be chosen is $\boxed{4n.}$ □

Proof. RV-COMC-2021-S4-5 Let $CQ = PR = BR = x$, then $QA = PQ = CR = 1 - x$, we have

$$[PBR] = \frac{x^2}{2}, \;\; [APQ] = \frac{(1-x)^2}{2}, \;\; [QCRP] = x(1-x) = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2$$

Case 1: if $x \geq \frac{2}{3}$, then $S \geq [PBR] \geq \frac{2}{9}$.
Case 2: if $\frac{1}{3} \leq x \leq \frac{2}{3}$, then $1 - x \geq \frac{2}{3}$, so $S \geq [APQ] \geq \frac{2}{9}$.
Case 3: if $x \leq \frac{1}{3}$, then $S \geq [QCRP] \geq \frac{1}{4} - \frac{1}{36} = \frac{2}{9}$.

Thus, the minimal value of S is $\boxed{\frac{2}{9}.}$ □

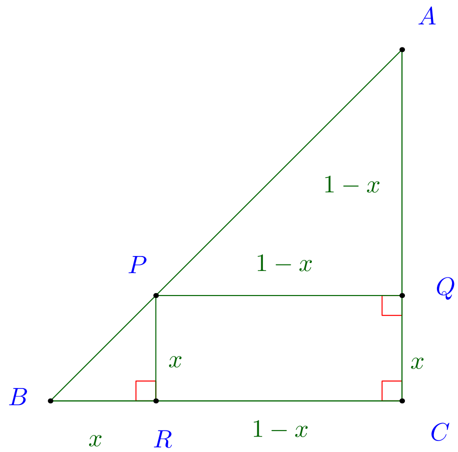


Figure 4.4: $CQ = PR = BR = x$

Proof. Chinese Team Training 1994/1 For $0 \le a \le b \le c \le d \le e$,

$$d + e \ge c + e \ge b + d \ge a + c \ge a + b$$

Thus, by Chebysev Inequality,

$$ad + dc + cb + be + ea = \frac{1}{2}\left((ad + ae) + (bc + be) + (cb + cd) + (da + dc) + (ea + eb)\right)$$
$$= \frac{1}{2}\left(a(d + e) + b(c + e) + c(b + d) + d(a + c) + e(a + b)\right)$$
$$\le \frac{1}{2}\frac{1}{5}(a + b + c + d + e)(d + e + c + e + b + d + a + c + a + b) = \frac{1}{5}$$

Thus, $\boxed{ad + dc + cb + be + ea \le \frac{1}{5}.}$ □

Proof. Russia MO 1998/4 First, note that a 10−digit number with all distinct digits should have the sum of digits divisible by 9. Therefore it is divisible by $9 \cdot 11111 = 99999$. Now, if

$$99999 \mid n = \overline{a_5a_4a_3a_2a_1b_5b_4b_3b_2b_1} \Rightarrow 99999 \mid \overline{a_5a_4a_3a_2a_1} + \overline{b_5b_4b_3b_2b_1}$$
$$0 < \overline{a_5a_4a_3a_2a_1} + \overline{b_5b_4b_3b_2b_1} < 2 \cdot 99999 \Rightarrow \overline{a_5a_4a_3a_2a_1} + \overline{b_5b_4b_3b_2b_1} = 99999$$
$$\Rightarrow b_5 + b_5 = a_4 + b_4 = \ldots = a_1 + b_1 = 9$$

There are 5! ways to distribute the pairs $(0, 9), (1, 8), \ldots, (4, 5)$ among $(a_5, b_5), (a_4, b_4), \ldots, (a_1, b_1)$ and for each pair we can swap the two digits, in 2^5 ways. The exceptions are when $a_5 = 0$, whose cases is one-tenth of all the cases.

Therefore the number of cases is $\boxed{\frac{9}{10} \cdot 2^5 \cdot 5! = 3456.}$ □

Proof. Spain MO 1997/2 Let $(0, q)$, $(r_1, 0)$, and $(r_2, 0)$ $(r_1 > r_2)$ be the three points points, then

$$r_1 + r_2 = -p, \; r_1r_2 = q$$

If $(x - a)^2 + (y - b)^2 = r^2$ is the circle, then the center (a, b) should lie on the symmetric axis of the parabola, thus

$$a = -\frac{p}{2} \Rightarrow r_1 - \frac{p}{2} = \frac{r_1 - r_2}{2}$$

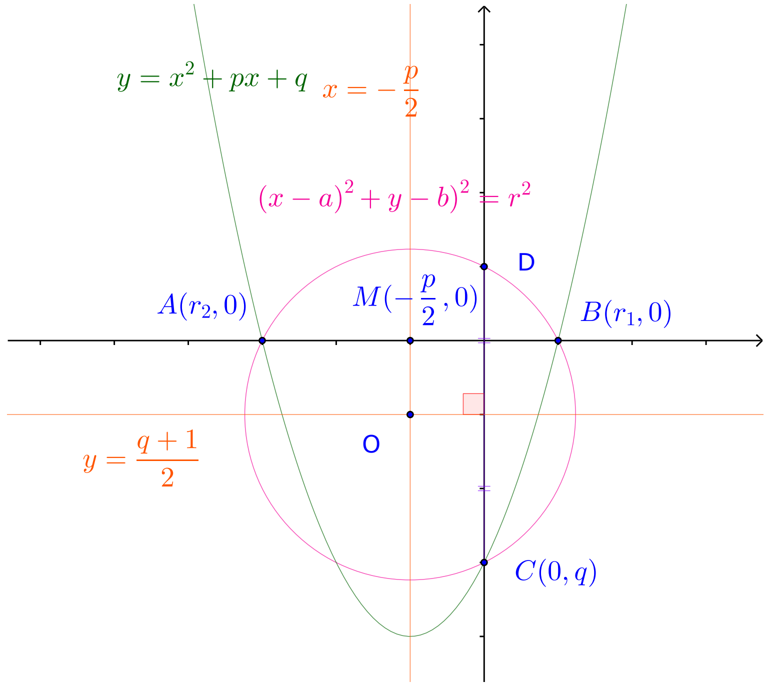


Figure 4.5: $(x-a)^2 + (y-b)^2 = r^2$

Furtheremore, $(0,q)$, $(r_1, 0)$, are on the circle,

$$\frac{p^2}{4} + (q-b)^2 = \left(r_1 - \frac{p}{2}\right)^2 + b^2 = \frac{(r_1 - r_2)^2}{4} + b^2 = \frac{1}{4}\left((r_1+r_2)^2 - 4r_1r_2\right) + b^2$$
$$\Rightarrow \frac{p^2}{4} + q^2 - 2qb + b^2 = \frac{1}{4}(p^2 - 4q) + b^2 \Rightarrow q^2 - 2qb = -q \Rightarrow b = \frac{q+1}{2}$$

Thus, the $y-$ coordinate of second intersection of the circle with the $y-$axis is the reflection of first intersection $0, q$ over lie $y = \frac{q+1}{2}$, which is

$$2 \cdot \frac{q+1}{2} - q = 1$$

Therefore the second intersection is $\boxed{(0,1).}$ □

Proof. Putnam 1970/B6 $ABCD$ is a convex *tangential* quadrilateral, so $a + c = b + d$, thus

$$\left.\begin{aligned} (a-d)^2 = (b-c)^2 \Rightarrow a^2 + d^2 - 2ad = b^2 + c^2 - 2bc \\ BD^2 = a^2 + d^2 - 2ad\cos\alpha = b^2 + c^2 - 2bc\cos\beta \end{aligned}\right\} \Rightarrow ad(1-\cos\alpha) = bc(1-\cos\beta)$$

Since $[ABCD] = [BAD] + [BCD] = \frac{1}{2}(ad\sin\alpha + bc\sin\beta)$, then

$$\begin{aligned} [ABCD]^2 &= \frac{1}{4}\left((ad)^2(1-\cos^2\alpha) + (bc)^2(1-\cos^2\beta) + 2abcd\sin\alpha\sin\beta\right) \\ \Rightarrow \frac{4[ABCD]^2}{abcd} &= \frac{ad(1-\cos\alpha)}{bc}(1+\cos\alpha) + \frac{bc(1-\cos\beta)}{ad}(1+\cos\beta) + 2\sin\alpha\sin\beta \\ &= (1-\cos\beta)(1+\cos\alpha) + (1-\cos\alpha)(1+\cos\beta) + 2\sin\alpha\sin\beta \\ &= 2 - 2(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = 2 - 2\cos(\alpha+\beta) = 4\sin^2\frac{(\alpha+\beta)}{2} \end{aligned}$$

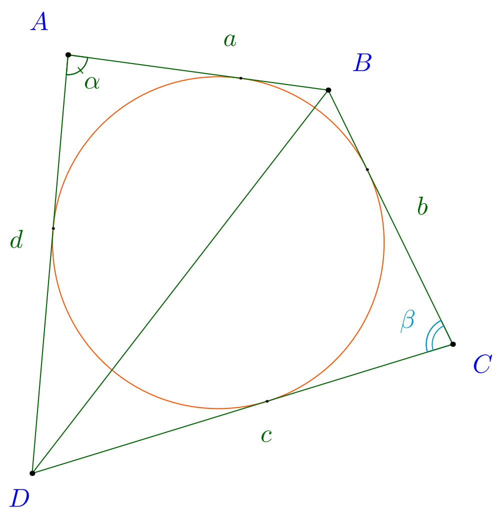


Figure 4.6: Putnam 1970/B6

Thus, $[ABCD] = \left|\sin \frac{\alpha+\beta}{2}\right|$. □

Proof. Math Problem Book I/167 Let $B_1, B_2, \ldots, B_m \subseteq A$, such that $|B_i| = 3, |B_i \cap B_j| \neq 2$, for $i, j \in \{1, 2, \ldots, m\},\ i \neq j$.

If $a \in A$ and $a \in B_1, B_2, \ldots, B_k$, then $|B_i \cap B_j| = 1$, for $i, j \in \{1, 2, \ldots, k\},\ i \neq j$. Since,

$$2n = |A| \geq |B_1 \cup B_2 \cup \ldots \cup B_k| = 1 + 2k \Rightarrow n - 1 \geq k.$$

Therefore each element of A is in at most $n-1$ subset B_i's. Therefore

$$|B_1| + |B_2| + \ldots + |B_m| \leq |A|(n-1) \Rightarrow 3m \leq 2n(n-1) \Rightarrow m \leq \frac{2(n-1)(n)}{3}.$$

Here is one example for $m = 8$ when $2n = 8$,

$$\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{8, 3, 4\}, \{8, 2, 6\}, \{8, 5, 7\}, \{3, 5, 6\}, \{2, 4, 7\}.$$

□

Proof. American Mathematical Monthly E2684

Case 1: if $n \geq 3$ is odd, then $1,\ 2 \in A_n$, thus

$$A_n = \{1, 2, \ldots, n-1\},\ \text{therefore } n \text{ is a prime.}$$

Case 2: if $n \geq 3$ is even, $3 \not| n$, then $1,\ 3 \in A_n$, thus

$$A_n = \{1, 3, 5, \ldots, n-1\},\ \text{therefore } n \text{ is a power of } 2$$

Case 2: if $n \geq 3$ is even, $3 \mid n$, then let p be the smallest prime $p \mid n$, then

Subcase 2a: $p \equiv 1 \pmod 6$, then $A_n = \{1, p, \ldots, n-1\}$,

$$\exists k \in \mathbb{Z}+, n - 1 = 1 + k(p-1) \Rightarrow n \equiv 2 \pmod 6 \text{ impossible}$$

Subcase 2b: $p \equiv 5 \pmod 6$, then $3 \mid 2p-1$, thus $2p-1 \notin A_n$, then

$$A_n = \{1, p\} \Rightarrow n = p+1 \Rightarrow n = 6 \text{ but this is also impossible}$$

Therefore, $\boxed{n \text{ is a prime or a power of } 2}$ □

Proof. IMO SL 1990/27 We go with casework.

Case 1: $n = 1$ and $n = 2$, $S_1 = \{7\}$, $S_2 = \{71, 17\}$. Thus, for $\boxed{n \in \{1,2\},\ S_n \text{ contains only primes.}}$

Case 2: $n = 3$, easy to see that the sum of digits of every element of S_n is $n-1+7 = n+6$, which is a multiple of 3, thus $\boxed{S_{3n} \text{ contains only composite.}}$

Case 3: n is an odd number, so let consider $2n+1$ instead of n,

$$7\underbrace{1\ldots1}_{2n} = 7\cdot 10^{2n} + (10^{2n-1} + \ldots + 10 + 1) = 7\cdot 10^{2n} + \frac{10^{2n}-1}{9} = \frac{1}{9}(64\cdot 10^{2n} - 1) = \frac{1}{9}(8\cdot 10^n - 1)(8\cdot 10^n + 1)$$

It is easy to see that $9 \mid 8\cdot 10^n + 1$, and each of $8\cdot 10^n - 1$, $\frac{1}{9}(8\cdot 10^n + 1)$ are positive integer larger than 1 if $n \geq 1$. Thus it is a composite number. Therefore $\boxed{S_{2n+1} \text{ contains some composite numbers.}}$

Case 4: $n = 4$ and $n = 8$. Note that, all three numbers

$$7111 = 13\cdot 547,\ \underbrace{1\ldots1}_{5}711 = 13\cdot 854747,\ \text{and}\ \underbrace{1\ldots1}_{12} = 3\cdot 7\cdot 11\cdot 13\cdot 37\cdot 101\cdot 9901$$

are multiple of 13. Therefore the numbers

$$\underbrace{1\ldots1}_{12n}7111 \text{ and } \underbrace{1\ldots1}_{12n}\underbrace{1\ldots1}_{5}711$$

are also composite. Thus, $\boxed{S_{12n+4} \text{ and } S_{12n+8} \text{ contain some composite numbers.}}$

For only $\boxed{n \in \{1,2\},}$ S_n contains only prime numbers. □

Chapter 5

Session 5

5.1 Topics

Algebra

1. Quadratic. Discriminant and existence of roots.
2. Factoring. Second-degree Identities.
3. Radical. $\sqrt{a^2 + b - 2a\sqrt{b}} = |a - \sqrt{b}|$.
4. Polynomial long division.

Combinatorics

1. Combinations. Number of ways to choose 2 or 3 items from n.
2. Permutations around table, shifting of positions, reflection of clockwise to anti-clockwise.
3. Casework. Chosing by combinations and permutations of choice.
4. Games. Winning positions. Reduce to games with initial conditions ensuring existence of winning strategy. Change of playing order.

Geometry

1. Right triangles.
2. Orthocentre H reflection over side B.
3. Triangle inequalities.
4. Area. Same base. Square of ratio of similarity.
5. Angle chasing. Perpendicular and Parallel lines. Parallelogram.
6. Radical axis. Power of the Point.
7. Reflection over axis of symmetry.

Number Theory

1. Remainder of perfect squares when divided by 3, 4, or 8.
2. Factorials. Estimation.
3. Number digits.
4. Integer and fractional part $\lfloor a \rfloor + \{b\} = 1.2 \Rightarrow a = 1, b = 0.2$

5.2 Problems

Problem 5.2.1 (RV-COMC-2021-S5-1). (*4 points*)

2021 is the sum of eleven *odd* perfect squares, find them.

How to provide your answer: A *correct answer* earns full marks.

Problem 5.2.2 (AMC 10 2013/11)**.** (*4 points*)

A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly 10 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees.

In how many different ways can a three-person planning committee be selected?

How to provide your answer: A *correct answer* earns full marks.

Problem 5.2.3 (Putnam 1997/1). (*4 points*)

A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ΔABC has H as orthocentre, O as circumcentre, M be the midpoint of BC, F is the feet of altitude from A.

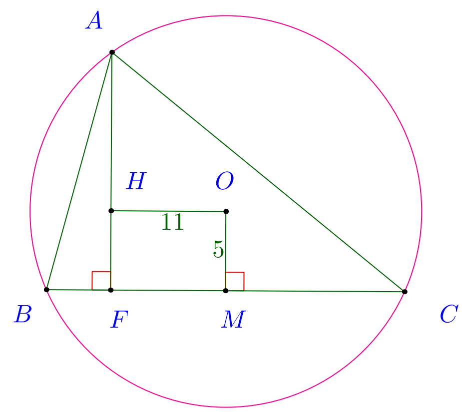


Figure 5.1: Putnam 1997/1

What is the length of BC ?

How to provide your answer: A *correct answer* earns full marks.

Problem 5.2.4 (RV-COMC-2021-S5-4). (*4 points*)

a, b, and c are the lengths of sides of $\triangle ABC$.

Atmost how many real roots can the quadratic $c^2x^2 + (a^2 - b^2 - c^2)x + b^2 = 0$ have?

How to provide your answer: A *correct answer* earns full marks.

Problem 5.2.5 (Moscow MO 1940/2). (*6 points*)

Find all three-digit numbers such that each is equal to the sum of the factorials of its own digits.

How to provide your answer: You must *show all your work*.

Problem 5.2.6 (AMC 12 2014/13). (*6 points*)

A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room.

In how many ways can the innkeeper assign the guests to the rooms?

Note that the rooms and guests are different, so for example if each room has a guest, then there are 5 *guests to choose for the first room,* 4 *for the second, ..., for a total of* $5! = 120$ *assignments.*

How to provide your answer: A *correct answer* earns full marks.

Problem 5.2.7 (RV-COMC-2021-S5-5). (*6 points*)

In $\triangle ABC$, points D and E are on BC and CA, respectively, such that,

$$\frac{BD}{DC} = \frac{3}{2}, \ \frac{AE}{EC} = \frac{3}{4}$$

AD and BE intersect at M. It is known that the area $[ABC]$ is 1.

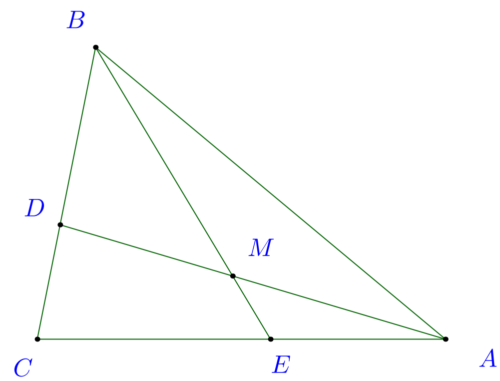


Figure 5.2: RV-COMC-2021-S5-5

Find the area of $\triangle BMD$.

How to provide your answer: A *correct answer* earns full marks.

Problem 5.2.8 (RV-COMC-2021-S5-6). (*6 points*)

$x = \sqrt{19 - 8\sqrt{3}}$, find the value of

$$\frac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15}$$

How to provide your answer: A *correct answer* earns full marks.

Problem 5.2.9 (Lithuania MO 2010/3)**.** (*10 points*)

In an $m \times n$ rectangular chessboard,there is a stone in the lower leftmost square. Two persons A, B move the stone alternately. In each step one can move the stone upward or rightward any number of squares. The one who moves it into the upper rightmost square wins.

1. For $(m, n) = (2, 2)$ and $(m, n) = (3, 3)$, who has a winning strategy?
2. For $(m, n) = (2, 3)$ and $(m, n) = (3, 2)$, who has a winning strategy?
3. Find all (m, n) such that the first person has a winning strategy.

How to provide your answer: You must *show all your work*.

Problem 5.2.10 (RV-COMC-2021-S5-8). (*10 points*)

$\lfloor x \rfloor$ and $\{x\}$ denote the integer and the fractional parts of x. For example, $\lfloor 3 \rfloor = 3$ and $\{3\} = 0$, $\lfloor 2.1 \rfloor = 2$ and $\{2.1\} = 0.1$, and $\lfloor -1.6 \rfloor = -2$ and $\{-1.6\} = 0.4$

Solve the system of equation,

$$\begin{cases} x + \lfloor y \rfloor + \{z\} = 8.3 \\ \lfloor x \rfloor + \{y\} + z = 7.4 \\ \{x\} + y + \lfloor z \rfloor = 9.5 \end{cases}$$

How to provide your answer: You must *show all your work.*

Problem 5.2.11 (China MO 1997/4)**.** (*10 points*)

A convex quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC, BD of $ABCD$ meet at P. Circumcircles of $\triangle ABP$ and $\triangle CDP$, centred at O_1 and O_2, respectively, meet at P and Q (O, P, Q are pairwise distinct).

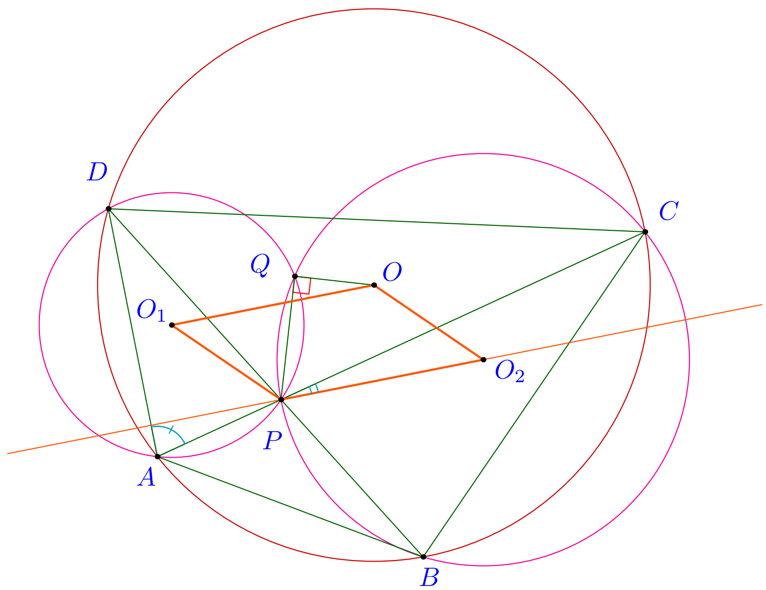


Figure 5.3: China MO 1997/4

1. Prove that $\angle O_2PC + \angle PAD = 90°$.
2. Prove that O_1OO_2P is a parallelogram.
3. Consider P and Q in regard to O_1O_2, prove that $PQO = 90°$.

How to provide your answer: You must *show all your work*.

Problem 5.2.12 (AIME 2012/14)**.** (*10 points*)

Nine people meet in a conference. They are divided into *rings* such that in each *ring*, each member shakes hands with exactly two of the other members. Consider two handshaking arrangements different if and only if at least two people who shake hands under one arrangement do not shake hands under the other arrangement.

1. How many ways are to divide them into 3 *rings* such that each *ring* consists of 3 people?
2. How many ways are to divide them into 2 *rings* such that one *ring* consists of 4 people and the other *ring* of 5 people?
3. Now, let N be the number of ways to divide them into a number of *rings* statifying the given condition. Find the remainder when N is divided by 1000.

How to provide your answer: You must *show all your work*.

5.3 Solutions

Proof. RV-COMC-2021-S5-1 Each odd perfect square has a remainder of 1 when divided by 4, thus the sum of eleven odd perfect squares has remainder 3 when divided by 4. 2021 has remainder 1 when divided by 4, so there are $\boxed{\text{no such eleven odd perfect squares.}}$ □

Proof. AMC 10 2013/11 Since there are $\binom{n}{2} = \frac{n(n-1)}{2} = 10$ ways to choose a two-person committee, so $n = 5$.

Therefore, the number of ways to choose a three-person committee is $\binom{5}{3} = 10$. □

Proof. Putnam 1997/1 Let K be the intersection of AD with (O). It is known that $DK = DH$, thus $DK = OM = 5$. Hence $OK = \sqrt{HK^2 + HO^2} = \sqrt{221}$, thus $OB = OK = \sqrt{221}$, so $MB = \sqrt{221 - 25} = 14$.

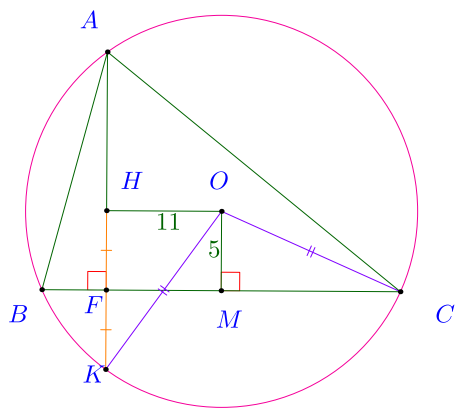


Figure 5.4: $K = AH \cap (O)$

Therefore, $\boxed{BC = 2MB = 28.}$ □

Proof. RV-COMC-2021-S5-4 The discriminant of the quadratic,

$$\Delta = (a^2 - b^2 - c^2)^2 - 4b^2c^2 = (a^2 - b^2 - c^2 - 2bc)(a^2 - b^2 - c^2 + 2bc) = (a^2 - (b+c)^2)(a^2 - (b-c)^2)$$

Since $b + c > a,\ a > b - c$, thus $\Delta < 0$, so the quadratic has $\boxed{\text{real roots.}}$ □

Proof. Moscow MO 1940/2 let $\overline{abc}$ be the number, then

$$\overline{abc} = a! + b! + c! \Rightarrow 100 \le a! + b! + c! \le 999$$

Because $6! = 720$, so atmost one of a, b, or c can be 6. $6! = 720$ implies $a \ge 7$, which is impossible. Thus none of a, b, or c can be 6.

On the other hand $4! = 24$, so one of a, b, or c must be at least 5. Since $3 \cdot 5! = 360$, so $a \le 3$. $a = 3$ implies that $a = b = c = 4$, which is impossible. $a = 2$ implies that $b = c = 5$, which is not possible because $255 \ne 2! + 5! + 5! = 242$. Therefore $a = 1$ and we have two possible cases $\overline{1b5}$ and $\overline{15c}$, where $b, c < 5$. Since,

$$1! + 5! = 121 \Rightarrow \begin{cases} \overline{1b5} = 121 + b! \Rightarrow 10b = 16 + b! \Rightarrow b = 4 \\ \overline{15c} = 121 + c! \Rightarrow 29 + c = c! \Rightarrow \nexists c \end{cases}$$

Thus, $\boxed{\overline{abc} = 145.}$ □

Proof. AMC 12 2014/13 We do casework.

Case 1: Each room houses one guest. In this case, we have 5 guests to choose for the first room, 4 for the second, . . ., for a total of $5! = 120$ assignments.

Case 2: Three rooms house one guest; one houses two. We have $\binom{5}{3}$ ways to choose the three rooms with 1 guest, and $\binom{2}{1}$ to choose the remaining one with 2. There are $5 \cdot 4 \cdot 3$ ways to place guests in the first three rooms, with the last two residing in the two-person room, for a total of $\binom{5}{3}\binom{2}{1} \cdot 5 \cdot 4 \cdot 3 = 1200$ ways.

Case 3: Two rooms house two guests; one houses one. We have $\binom{5}{2}$ to choose the two rooms with two people, and $\binom{3}{1}$ to choose one remaining room for one person. Then there are 5 choices for the lonely person, and $\binom{4}{2}$ for the two in the first two-person room. The last two will stay in the other two-room, so there are $\binom{5}{2}\binom{3}{1} \cdot 5 \cdot \binom{4}{2} = 900$ ways.

In total, there are $\boxed{120 + 1200 + 900 = 2220}$ assignments. ☐

Proof. RV-COMC-2021-S5-5 Let F be the intersection of the line through E and parallel to AD.

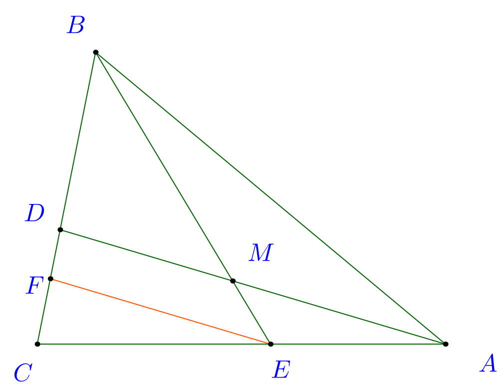


Figure 5.5: $EF \parallel AD$

First, by comparing the area of $\triangle CEF$ with $\triangle CAF$, $\triangle CAF$ with $\triangle CAB$, and $\triangle BFE$ with $\triangle CEF$,

$$\frac{[CEF]}{[CAB]} = \frac{[CEF]}{[CAF]}\frac{[CAF]}{[CAB]} = \frac{CE}{CA}\frac{CF}{CB} \quad \text{and} \quad \frac{[BFE]}{[CEF]} = \frac{BF}{FC}$$

Second, $\triangle BDM \sim \triangle BFE$, thus

$$\frac{[BDM]}{[BFE]} = \left(\frac{BD}{BF}\right)^2$$

Therefore

$$[BDM] = \frac{[BDM]}{[CAB]} = \frac{[BDM]}{[CEF]}\frac{[CEF]}{[CAB]} = \frac{[BDM]}{[BFE]}\frac{[BFE]}{[CEF]}\frac{[CEF]}{[CAB]} = \left(\frac{BD}{BF}\right)^2 \frac{BF}{FC}\frac{CE}{CA}\frac{CF}{CB} = \frac{CE}{CA}\frac{BD}{BF}\frac{BD}{CB}$$

Now,

$$\frac{DF}{DC} = \frac{AE}{AC} = \frac{3}{7} \Rightarrow \frac{DF}{BD} = \frac{DF}{DC}\frac{DC}{DB} = \frac{3}{7}\frac{2}{7} = \frac{2}{7} \Rightarrow \frac{BD}{BF} = 79$$

Therefore, $\boxed{[BDM] = \frac{4}{7}\frac{7}{9}\frac{3}{5} = \frac{4}{15}.}$ ☐

Proof. RV-COMC-2021-S5-6 First, $19 - 8\sqrt{3} = (4^2 + (\sqrt{3})^2 - 2 \cdot 4 \cdot \sqrt{3})^2$, so $x = 4 - \sqrt{3}$, thus

$$x^2 - 8x = 19 - 8\sqrt{3} - 8(4 - \sqrt{3}) = -13 \Rightarrow x^2 - 8x + 13 = 0$$

Therefore, by long division

$$x^4 - 6x^3 - 2x^2 + 18x + 23 = x^2(x^2 - 8x + 13) + 2x(x^2 - 8x + 13) + (x^2 - 8x + 13) + 10 = 10$$

Thus, $\boxed{\dfrac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15} = \frac{10}{2} = 5.}$ □

Proof. Lithuania MO 2010/3 It is easy to see that for $2 \le m, n \le 3$, if $m = n$ then the second player has a winning strategy, and if $m \ne n$, then the first player has the winning strategy.

If $m = n$ then the positions $(k, k),\ 0 \le k \le n$ are the winning positions for the second player. Thus the second player has a winning strategy.

Now WLOG $m > n$ then the first player moves the stone to the position (n, n). Now, the first player becomes the second player and should win the game. □

Proof. RV-COMC-2021-S5-8 By summing up all the equations,

$$\begin{cases} x + \lfloor y \rfloor + \{z\} = 8.3 \\ \lfloor x \rfloor + \{y\} + z = 7.4 \\ \{x\} + y + \lfloor z \rfloor = 9.5 \end{cases} \Rightarrow 2x + 2y + 2z = 25.2 \Rightarrow x + y + z = 12.6$$

Note that, $(x + y + z) - (x + \lfloor y \rfloor + \{z\}) = \{y\} + \lfloor z \rfloor$, thus

$$\begin{cases} \{y\} + \lfloor z \rfloor = 12.6 - 8.3 = 4.3 \\ \{x\} + \lfloor y \rfloor = 12.6 - 7.4 = 5.2 \\ \{z\} + \lfloor x \rfloor = 12.6 - 9.5 = 3.1 \end{cases} \Rightarrow \begin{cases} \lfloor z \rfloor = 4,\ \{y\} = 0.3 \\ \lfloor y \rfloor = 5,\ \{x\} = 0.2 \\ \lfloor x \rfloor = 3,\ \{z\} = 0.1 \end{cases}$$

Thus, $\boxed{(x, y, z) = (3.2, 5.3, 4.1).}$ □

Proof. China MO 1997/4 First,

$$\angle O_2PC + \angle PAD = \frac{1}{2}(180^\circ - \angle PO_2C) + \angle CBD = 90^\circ.$$

Thus, $O_2P \perp DA$, since $OO_1 \perp DA$, then $O_2P \parallel OO_1$. Similarly $O_1P \parallel OO_2$. Therefore O_1OO_2 is a parallelogram.

Q is the reflection of P over O_1O_2 so $PQ \perp OQ$. □

Proof. AIME 2012/14 Given that each person shakes hands with two people, we can view all of these through graph theory as *rings*. This will split it into four cases:

1. three *rings* of three,
2. one *ring* of three and one *ring* of six,
3. one *ring* of four and one *ring* of five,
4. and one *ring* of nine.

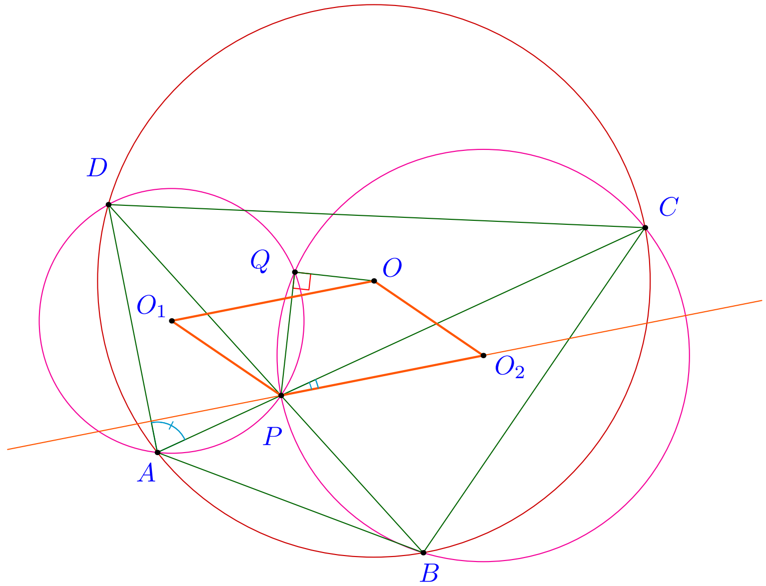


Figure 5.6: $O_2P \perp DA$

All other cases that sum to nine won't work, since they have at least one *ring* of two or fewer points, which doesn't satisfy the handshaking conditions of the problem.

Case 1: To create our groups of three, there are $\dfrac{\binom{9}{3}\binom{6}{3}\binom{3}{3}}{3!}$. In general, the number of ways we can arrange people within the rings to count properly is $\dfrac{(n-1)!}{2}$, since there are $(n-1)!$ ways to arrange items in the circle, and then we don't want to want to consider reflections as separate entities. Thus, each of the three cases has $\dfrac{(3-1)!}{2} = 1$ arrangements. Therefore, for this case, there are $\left(\dfrac{\binom{9}{3}\binom{6}{3}\binom{3}{3}}{3!}\right)(1)^3 = 280$

Case 2: For three and six, there are $\binom{9}{6} = 84$ sets for the rings. For organization within the ring, as before, there is only one way to arrange the ring of three. For six, there is $\dfrac{(6-1)!}{2} = 60$. This means there are $(84)(1)(60) = 5040$ arrangements.

Case 3: For four and five, there are $\binom{9}{5} = 126$ sets for the rings. Within the five, there are $\dfrac{4!}{2} = 12$, and within the four there are $\dfrac{3!}{2} = 3$ arrangements. This means the total is $(126)(12)(3) = 4536$.

Case 4: For the nine case, there is $\binom{9}{9} = 1$ arrangement for the ring. Within it, there are $\dfrac{8!}{2} = 20160$ arrangements.

Summing the cases, we have $280 + 5040 + 4536 + 20160 = 30016 \to \boxed{016}$. $\square$

Chapter 6

Session 6

6.1 Topics

Algebra

1. Equations. Absolute values and casework.
2. Sums and Products. Average. Sums in two ways.
3. Inequalities. Comparison Method.
4. Functional Equations.

Combinatorics

1. Counting.
2. Permutations.
3. Grids. Counting paths on grids.
4. Probability. Number of favorable outcomes. Independent events.
5. Sets. Principle of Inclusion-Exclusion.
6. Recurrent Relations.

Geometry

1. Triangles. Right triangles. $3-4-5$ right triangles.
2. Congruent triangles. Similar triangles.
3. Equilateral triangles.
4. Trapezoids.
5. Area. Areas as sum of sub-areas.
6. Computational Geometry. Line equation. Circle equation. Area bounded by equations. Parabola equations. Graphs of equations with absolute values.

Number Theory

1. Perfect Powers.
2. Divisibility.
3. Modular Arithmetic.

6.2 Problems

Problem 6.2.1 (AMC 8 2017/15). (*4 points*)

In the arrangement of letters and numerals below, by how many different paths can one spell AMC8? Beginning at the A in the middle, a path allows only moves from one letter to an adjacent (above, below, left, or right, but not diagonal) letter. One example of such a path is traced in the picture.

	8	C	8	
8	C	M	C	8
C	M	A	M	C
8	C	M	C	8
	8	C	8	

Figure 6.1: AMC 8 2017/15

How to provide your answer: A *correct answer* earns full marks.

Problem 6.2.2 (AMC 10B 2016/9). (*4 points*)

All three vertices of $\triangle ABC$ are lying on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x-axis. The area of the triangle is 64. What is the length of BC?

How to provide your answer: A *correct answer* earns full marks.

Problem 6.2.3 (AMC 12A 2017/7). (*4 points*)

Define a function on the positive integers recursively by $f(1) = 2$, $f(n) = f(n-1) + 1$ if n is even, and $f(n) = f(n-2) + 2$ if n is odd and greater than 1. What is $f(2017)$?

How to provide your answer: A *correct answer* earns full marks.

Problem 6.2.4 (AMC 10B 2017/13). (*4 points*)

There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

How to provide your answer: A *correct answer* earns full marks.

Problem 6.2.5 (AMC 8 2013/14). (*6 points*)

Abe holds 1 green and 1 red jelly bean in his hand. Bea holds 1 green, 1 yellow, and 2 red jelly beans in her hand. Each randomly picks a jelly bean to show the other. What is the probability that the colors match?

How to provide your answer: A *correct answer* earns full marks.

Problem 6.2.6 (AMC 12B 2017/15). (*6 points*)

Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

How to provide your answer: A *correct answer* earns full marks.

Problem 6.2.7 (AMC 8 2018/25). (*6 points*)

How many perfect cubes lie between $2^8 + 1$ and $2^{18} + 1$, inclusive?

How to provide your answer: A *correct answer* earns full marks.

Problem 6.2.8 (AMC 10A 2019/19)**.** (*6 points*)

What is the least possible value of

$$(x+1)(x+2)(x+3)(x+4)+2019$$

where x is a real number?

How to provide your answer: A *correct answer* earns full marks.

Problem 6.2.9 (AMC 10B 2016/21). (*10 points*)

1. What is the area of the region bounded by the equations $x^2 + y^2 = x + y$, $x = 0$, and $y = 0$?
2. What is the area of the region enclosed by the graph of the equation $x^2 + y^2 = |x| + |y|$?

How to provide your answer: You must *show all your work.*

Problem 6.2.10 (AMC 12B 2021/17)**.** (*10 points*) Let $ABCD$ be an isosceles trapezoid having parallel bases $\overline{AB}$ and $\overline{CD}$ with $AB > CD$. Line segments from a point inside $ABCD$ to the vertices divide the trapezoid into four triangles whose areas are $2, 3, 4$, and 5 starting with the triangle with base $\overline{CD}$ and moving clockwise as shown in the diagram below.

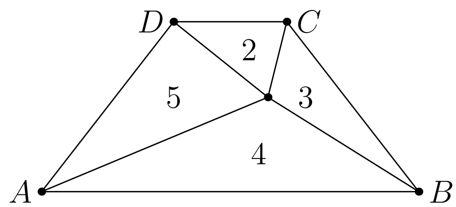


Figure 6.2: AMC 8 2017/15

1. Denote by P the point in the interior of the trapezoid. Let X and Y be the feet of the perpendiculars from P to AB and CD. Prove that $[ABCD] = \frac{1}{2}\left(\frac{8}{AB} + \frac{4}{CD}\right)(AB + CD)$.
2. What is the ratio $\frac{AB}{CD}$?

How to provide your answer: You must *show all your work*.

Problem 6.2.11 (AMC 12B 2017/19)**.** (*10 points*) Let $N = 123456789101112\dots4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other.

1. What is the remainder when N is divided by 9?
2. What is the remainder when N is divided by 45?

How to provide your answer: You must *show all your work*.

Problem 6.2.12 (AMC 12B 2019/23)**.** (*10 points*)

Let $f(n)$ be the number of sequences of length n, each contains 0s and 1s, starts and ends with both 0, and there are no two consecutive 0s and no three consecutive 1s.

1. Find $f(3)$, $f(4)$, and $f(5)$.
2. Prove that $f(n) = f(n-3) + f(n-2)$, with $n \geq 3$.
3. What is $f(19)$.

You can ear all points if you find another way (with a proof) to determine the value for $f(19)$.

How to provide your answer: You must *show all your work*.

6.3 Solutions

Proof. AMC 8 2017/15 From the A, there are four ways to go to an M. From any one of the Ms, then you have three ways to get a C, and from any C, two ways to get an 8. Thus, the answer is $\boxed{4 \cdot 3 \cdot 2 = 4! = 24.}$ □

Proof. AMC 10B 2016/9 Since BC is parallel to the $x-$axis, thus let $B(-b, b^2)$ and $C(b, b^2)$, then

$$[ABC] = \frac{1}{2}b^2 \cdot 2b = b^3 \Rightarrow b^3 = 64 \Rightarrow b = 4.$$

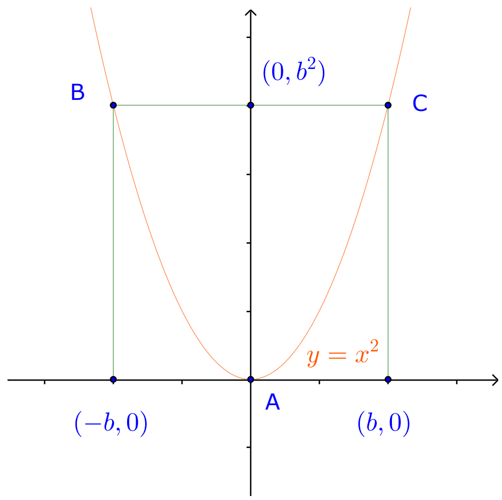


Figure 6.3: AMC 10B 2016/9

Thus, $\boxed{BC = 2b = 8.}$ □

Proof. AMC 12A 2017/7

$$f(2n+1) = f(2n-1) + 2 \cdot 1 = f(2n-3) + 2 \cdot 2 = f(1) + 2n \Rightarrow f(2017) = 2 + 2016 = 2018$$

□

Proof. AMC 10B 2017/13 By PIE (Property of Inclusion/Exclusion), we have

$$|A_1 \cup A_2 \cup A_3| = \sum |A_i| - \sum |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3|.$$

Number of people in at least two sets is

$$\sum |A_i \cap A_j| - 2|A_1 \cap A_2 \cap A_3| = 9.$$

So, $20 = (10 + 13 + 9) - (9 + 2x) + x$, which gives $\boxed{x = 3.}$ □

Proof. AMC 8 2013/14 The probability that both show a green bean is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. The probability that both show a red bean is $\frac{1}{2} \cdot \frac{2}{4} = \frac{1}{4}$. Therefore the probability is $\boxed{\frac{1}{4} + \frac{1}{8} = \frac{3}{8}.}$ □

Proof. AMC 8 2018/25 It is easy to see that $2^8 + 1 = 257 > 216 = 6^3$. Furthermore $2^{18} = 64^3$, thus

$$6^3 < 2^8 + 1 < 7^3 < 8^3 < \ldots < 64^3 < 2^{18} + 1 < 65^3$$

Therefore there are $\boxed{64 - 7 + 1 = 58}$ perfect cubes between $2^8 + 1$ and $2^{18} + 1$, inclusive. □

Proof. AMC 12B 2017/15 First, comparing bases yields that

$$[BA'B'] = 3[AA'B] = 9[ABC] \Rightarrow [AA'B'] = 12$$

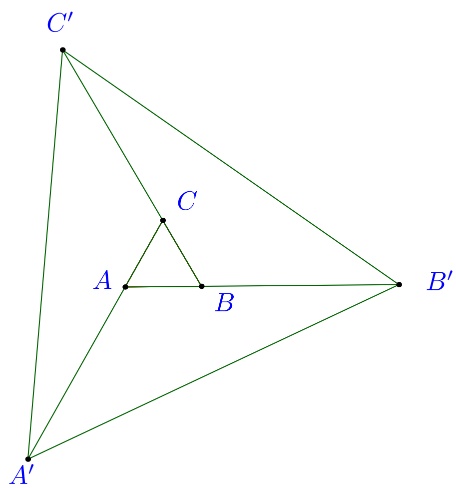


Figure 6.4: AMC 12B 2017/15

By congruent triangles,

$$[AA'B'] = [BB'C'] = [CC'A'] \Rightarrow [A'B'C'] = (12 + 12 + 12 + 1)[ABC],$$

so $\boxed{[A'B'C'] : [ABC] = 37.}$ □

Proof. AMC 10A 2019/19 Note that $(x+1)(x+4) = x^2 + 5x + 4$, $(x+2)(x+3) = x^2 + 5x + 6$, thus let $y = x^2 + 5x + 5$, then

$$(x+1)(x+2)(x+3)(x+4) + 2019 = (y-1)(y+1) + 2019 = y^2 + 2018 \geq 2018$$

So the least value of $(x+1)(x+2)(x+3)(x+4) + 2019$ is $\boxed{2018.}$ □

Proof. AMC 10B 2016/21 For the first question, since

$$x^2 + y^2 = x + y \Rightarrow \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2},$$

is the circle centred at O_1 radius $\frac{1}{\sqrt{2}}$.

Thus, the area bounded by the graphs of the equations $x^2 + y^2 = x + y$, $x = 0$, and $y = 0$ is half the area of the circle (O_1) and the area of $\triangle AOB$, which is

$$\frac{1 \cdot 1}{2} + \frac{\pi \frac{1}{2}}{2} = \frac{2 + \pi}{4}$$

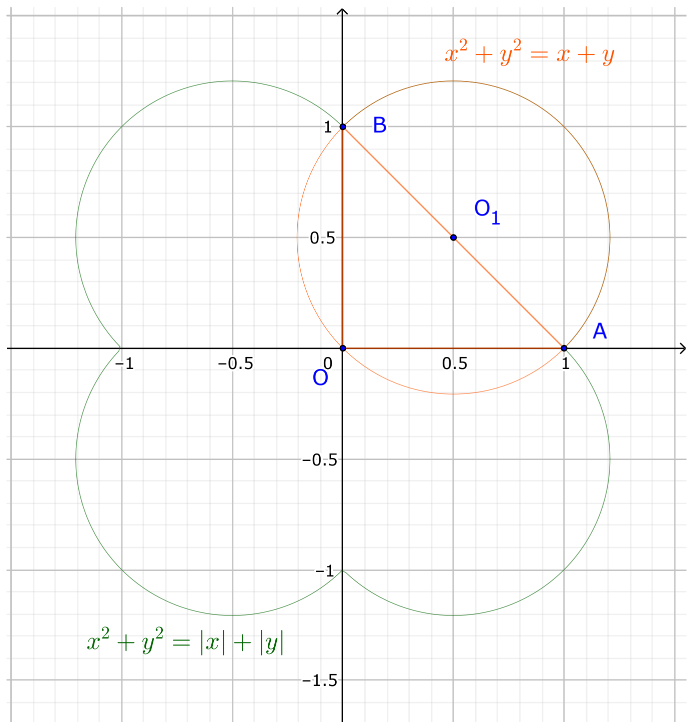


Figure 6.5: **??**

From here the second question, the area enclosed by the graph $x^2 + y^2 = |x| + |y|$ is $\boxed{4 \cdot \frac{2+\pi}{4} = 2 + \pi.}$ □

Proof. AMC 12B 2021/17 Since $PX = \frac{8}{AB}$, $PX = \frac{4}{CD}$, and $[ABCD] = \frac{1}{2}(PX + PY)(AB + CD)$. Thus, $[ABCD] = \frac{1}{2}\left(\frac{8}{AB} + \frac{4}{CD}\right)(AB + CD)$. Now let $r = \frac{AB}{CD} \geq 1$, then

$$14 = 6 + 2r + \frac{4}{r} \Rightarrow r^2 - 4r + 2 = 0 \Rightarrow r = 2 + \sqrt{2}.$$

□

First proof. AMC 12B 2017/19 To calculate the number $\pmod 5$, note that

$$123456 \cdots 4344 \equiv 1+2+3+4+5+6+7+8+9+(1+0)+(1+1)+\cdots+(4+3)+(4+4) \equiv 1+2+\cdots+44 \pmod 9,$$

so it is equivalent to $\boxed{\frac{44 \cdot 45}{2} = 22 \cdot 45 \equiv 0 \pmod 9.}$

Now, we will consider this number $\pmod 5$ and $\pmod 9$. By looking at the last digit, it is obvious that the number is $\equiv 4 \pmod 5$. Let x be the remainder when this number is divided by 45.

$$x \equiv 0 \pmod 9,\ x \equiv 4 \pmod 5, \Rightarrow x \equiv 9 \pmod{45}$$

Thus, the answer is $\boxed{9.}$ □

Second proof. AMC 12B 2017/19 Note that $10^k \equiv 10 \pmod{45}$. Hence

$$N = 44 + 43 \cdot 10^2 + 42 \cdot 10^4 + \cdots + 10^{78} \equiv 44 + 10 \cdot (1 + 2 + 3 + \cdots + 43) \equiv 9 \pmod{45}.$$

□

First proof. AMC 12B 2019/23 It is easy to find $f(3) = 1$ since the only possible valid sequence is 010. $f(4) = 1$ since the only possible valid sequence is 0110. $f(5) = 1$ since the only possible valid sequence is 01010.

We can deduce, from the given restrictions, that any valid sequence of length n will start with a 0 followed by either 10 or 110. Thus, we deduce a recursive function $f(n) = f(n-3) + f(n-2)$, with $n \geq 3$.

It is easy to calculate that $\boxed{f(19) = 65.}$ □

Second proof. AMC 12B 2019/23 After any particular 0, the next 0 in the sequence must appear exactly 2 or 3 positions down the line. In this case, we start at position 1 and end at position 19, i.e. we move a total of 18 positions down the line. Therefore, we must add a series of 2s and 3s to get 18. There are a number of ways to do this:

Case 1: nine 2s - there is only 1 way to arrange them.

Case 2: two 3s and six 2s - there are $\binom{8}{2} = 28$ ways to arrange them.

Case 3: four 3s and three 2s - there are $\binom{7}{4} = 35$ ways to arrange them.

Case 4: six 3s - there is only 1 way to arrange them.

Summing the four cases gives $\boxed{1 + 28 + 35 + 1 = 65.}$ □