MCC Year 2022-2023

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December 10, 2022

Contents

Part I

Fall 2022

Chapter 1

Programs

1.1 Fall 2022 Calendar

Date	Activity	Round
August 27	Entrance Test (ET)	Round 1
August 28	Entrance Test (ET)	Round 2
September 3	No activity	Round 1
September 10	House Championship (HC)	Round 1
September 17	Mathematical Individual Contest (MIC)	Round 1
September 24	Problem Solving Championship (PSC)	Round 1
October 1	Monthly Seminar (MS)	Session 1
October 8	House Championship (HC)	Round 2
October 15	Mathematical Individual Contest (MIC)	Round 2
October 22	Introductory Curriculum Level Test (ICLT)	Round 1
October 29	Problem Solving Championship (PSC)	Round 2
November 5	Monthly Seminar (MS)	Session 2
November 12	House Championship (HC)	Round 3
November 19	Mathematical Individual Contest (MIC)	Round 3
November 26	Introductory Curriculum Level Test (ICLT)	Round 2
December 3	Problem Solving Championship (PSC)	Round 3
December 10	Monthly Seminar (MS)	Session 3

1.2 Entrance Test (ET)

Entrance Test is an activity of Math, Chess, and Coding Club (MCC) at the beginning of the semester. All students must take the test. The problems are designed so that students at various levels - junior (grade under 8), senior (grade 9 and up), or olympiad (prepare for national or international contest) - can participate. The contest problems are selected from algebra, combinatorics, number theory, geometry topics of mathematics, some are puzzles from chess, and some need to be solved by designing a computer program. Some problems do not require specific mathematical knowledge, some require problem solving skills, and some require a more comprehensive understanding of certain mathematical topics.

Each of the students receives a number of points based on their performance, which in turn are used to arrange the student into houses for the club's House Championship.

- Two tests are administered at **10:00 PM Eastern Time (EST) on Saturday 27/8 and Sunday 28/8**. Each test lasts for **90 minutes**. Each student must participate in both tests.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Problem levels: all students must **try to solve all problems**, don't stay within your boundary
  - Junior: Problem 1, 2, and 3 (multiple-choice); Problem 7 (show-your-work).
  - Senior: Problem 4, 5 (multiple-choice); Problem 8, 9 (show-your-work).
  - Olympiad: Problem 6 (multiple-choice); Problem 10 (show-your-work).
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at **an earlier time**. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.

1.3 House Championship (HC)

House Championship is an activity of Math, Chess, and Coding Club (MCC). All students must participate. The students are selected into houses. The championship consists of a number of weekly contests between the teams at various levels. The contest content contains problems and puzzles (including visual, language and chess puzzles) that can be solved by using math, coding, and chess. The houses are ranked after each round. The final standing of the championship is revealed at the end of each semester.

- **Qualification**: *All students* are *qualified* and *required* to participate.
- **Houses**: Initially an even number of $2n$ houses are established depending on the number of students.
  1. Each house consists of **at least 6** students. Initially, house members are selected by the club. Each house will have a captain, a lieutenant, and members. The captain can be elected by the members or appointed by the club.
  2. The members can name their house, design new coats-of-arm, slogan, song, cloths, etc if they wish. Common sense ethics are required.
  3. New members of the club will be assigned to a house at the beginning of each semester. The houses can be reorganized depending on the total number of students.
- **Format**: The championship is contested via $2n-2$ rounds:
  1. In each round all the houses are organized into distinct pairs. The houses in a pair participate in a *round contest*. A round contest contains of a number of problems, grouped by *levels of difficulty* (beginner, intermediate, and advanced). In a few days before the contest, the topics of the contest and the number of difficulty levels.
  2. A house organizes a number of *troops* depending on the number of difficulty levels of the coming round contest. Each troop has an approximately equal number of students. No house member can participate in more than one troop.
  3. A round contest consists of a number of troop games. Each game will be played between *same-level troops*. Each troop receive the same set of problems to be solved.
- **Hit, score, and rank**: In each round:
  1. For each problem in the troop game, each troop will submit a number (i.e 3), a set of numbers (i.e. $\{3,4\}$), or a sequence of letters (i.e *T-Rex dinosaur* or *babyface*), or in a format stated by the problem.
  2. There are two different ways how a troop can score: (i) for some problems if the answer is correct, the troop scores 1 hit; (ii) for some problems, the troop provided the best answer scores 1 hit, the troop that cannot provide the best answer score 0 point. Obviously two equal best answers result in 1 point for both troops.
  3. Each troop can post up to 3 answers until the end of the game. Any answer submitted after the third is not taken into consideration.
  4. For each troop game, if both troops have the same number of hits, the game results in a draw. Otherwise, the troop with more hits is the winning troop while the other becomes the ultimate loser. *For example, House A meets House B and the results of troop games are:* $A1-B1: 2-1, A2-B2: 3-1, A3-B3: 0:3$.
  5. For each game, a house receives 3 points for a game, 1 point for a drawn, 0 point for a loss. The round score of a house is the sum of the points from the three games it plays. *In the example above, House A receives* $3+3=6$ *points because they won two games and lost one game, House B receives* 3 *points because they lost two games and won one game. So A is the round winner and the score is* $A-B: 6-3$.
  6. After each round, all house standings are updated. *Note: It is important to note that since a round consists of three 1-to-1 games, it is wise for a house to organize its troop according to not only the expertises of its members but also the capabilities of the troops of the other team. A relatively weaker, but wiser house can win if it knows itself and it knows its opponent.*

7. The *champion* house of the club is the one who has the most points after all rounds are concluded. In case the two top houses have the same number of points, a final contest will be held to decide the ultimate winner.

- **Problems**: Each troop receives a number of problems for each game. The problems the house receives can be based on any of the following areas:
  1. Logical, math, visual, language, or chess puzzles.
  2. Problems that can be solved by a computer program designed in the contest.
  3. Problems that can be solved with competitive math knowledge and skills.
- **Time**:
  1. All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
  2. The house must announce its troop compositions (what are the troops and who are the members of which troops) to the Contest Organizer. All participants are required to be present 10 minutes before the contest starts.
  3. A house that does not have a troop for a game shall automatically lose, earning 0 points for that game. If the opposing team's troop is present, they shall receive maximum possible points for that game.
- **Contest organization**: The Contest Organizers (CO) are responsible for overseeing the troop games.
  1. The COs allow the troops participating in that game to access the game's problems. The troops can update the answers to the problems according to instructions from the COs. The COs decision on grading is final.
  2. All game scores between the troops are publicly available at any point in the contest, automatically updated when a team has made a hit. The round scores are also automatically updated accordingly.
  3. The contest is carried out via Zoom. All team member must show up on the video. Only the team captains can communicate to the COs: raise your hand when you want to talk; mute your microphone for the rest of the time; use the chat if your audio is not working. Please inform the COs that your mic is not working immediately at the start of the contest.
- **Tools**:
  1. The contest is carried out via Zoom. All team member must show up on the video. Only the team captains can communicate to the COs: raise your hand when you want to talk; mute your microphone for the rest of the time; use the chat if your audio is not working. Please inform the COs that your mic is not working immediately at the start of the contest.
  2. Paper-based books, hand-written notes, calculators are allowed during the contest.
  3. Computers are allowed for programming (coding) only.
  4. Using the Internet to search for similar problems, answers, communicate with outside people, etc, is strictly prohibited.
  5. Help from anyone outside the house is not allowed. Be honest.

  Any violation to the rules, especially using external help (results from the Internet, information from people outside of the team, etc.) can cause the house to lose all of their games in that round.

1.4 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website. The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis. The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	**Introduction to Algebra**	**Introduction to Geometry**	**Introduction to Number Theory**	**Introduction to Counting & Probability**
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

1.5 Mathematical Individual Contest (MIC)

Mathematical Individual Contest is an activity of Math, Chess, and Coding Club (MCC). All students are invited to participate. The problems are designed so that students at various levels - junior (grade under 8), senior (grade 9 and up), or olympiad (prepare for national or international contest) - can participate. The contest problems are selected from algebra, combinatorics, number theory, and geometry topics. Some problems do not require specific mathematical knowledge, some require problem solving skills, and some require a more comprehensive understanding of certain mathematical topics. The students received a number of points after each participation and are ranked after each round, in both overall and grade rankings. The final standing of the contest is concluded at the end of each semester.

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Problem levels: all students must **try to solve all problems**, don't stay within your boundary
  - Junior: Problem 1, 2, and 3 (multiple-choice); Problem 7 (show-your-work).
  - Senior: Problem 4, 5 (multiple-choice); Problem 8, 9 (show-your-work).
  - Olympiad: Problem 6 (multiple-choice); Problem 10 (show-your-work).
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.

1.6 Problem Solving Championship (PSC)

Problem Solving Championship is an activity of Math, Chess, and Coding Club (MCC). All students are invited to participate. The problems are designed so that students at various levels - junior (grade under 8), senior (grade 9 and up), or olympiad (prepare for national or international contest) - can participate. The contest problems are selected from algebra, combinatorics, number theory, geometry topics of mathematics, some are puzzles from chess, and some need to be solved by designing a computer program. Some problems do not require specific mathematical knowledge, some require problem solving skills, and some require a more comprehensive understanding of certain mathematical topics. The students received a number of points after each participation and are ranked after each round, in both overall and grade rankings. The final standing of the contest is concluded at the end of each semester.

- The contest problems become available online at the beginning of the semester or at the end of the previous contest.
- The contest solutions are discussed at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**. All students in the club are invited to the solution discussion.
- All contests start when the contest problems become available. The **solutions must be submitted latest on the last Sunday**, approximately one week **before the solution discussion day on Saturday**.
- There are 4 **show-you-work** problems with multiple steps. For each step of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too. If you solve the problem by designing a computer program, submit that program as the solution to the problem.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading: For each step there are a number of points, highlighted in the problem text, to be awarded. For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Invitation to present own solutions: Students who submitted correct solutions would be invited to present their solutions on the solution day. Students with nearly complete or almost correct solutions would also have a chance for presentation, provided that they studied the comments and suggestions from the COs to modify their solutions.

1.7 Monthly Seminar (MS)

Approximately each month, a seminar will be held to discuss some selected topics in mathematics, chess, and coding. All students are invited to participate. The topics and their problems are selected so that students at various levels - junior (grade under 8), senior (grade 9 and up), or olympiad (prepare for national or international contest) - can participate.

- The topics are discussed at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- Students are encouraged to actively particiate during the session.
- No preparation is required, no homework assigned.

Chapter 2

Entrance Test - Round 1

2.1 Rules

- Two tests are administered at **10:00 PM Eastern Time (EST) on Saturday 27/8 and Sunday 28/8**. Each test lasts for **90 minutes**. Each student must participate in both tests.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Problem levels: all students must **try to solve all problems**, don't stay within your boundary
  - Junior: Problem 1, 2, and 3 (multiple-choice); Problem 7 (show-your-work).
  - Senior: Problem 4, 5 (multiple-choice); Problem 8, 9 (show-your-work).
  - Olympiad: Problem 6 (multiple-choice); Problem 10 (show-your-work).
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at **an earlier time**. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.

2.2 Definitions, Lemmas, and Theorems

Definition (Euler's Totient Function)**.** For $n \in \mathbb{Z}^+$, $\varphi(n)$ is the number of positive integers less than n that are relatively prime to n.

For example, if $n = 9$, *the six numbers* $1, 2, 4, 5, 7$ *and* 8 *that are relatively prime to* 9. *so* $\varphi(9) = 6$.

Theorem (Formula for $\varphi(n)$)

For $n \in \mathbb{Z}^+,\ n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, then

$$\begin{aligned}
&\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right), \text{ or} \\
&\varphi(n) = p_1^{a_1-1} p_2^{a_2-1} \cdots p_k^{a_k-1}(p_1 - 1)(p_2 - 1)\cdots(p_k - 1)
\end{aligned}$$

For example, if $n = 9 = 3^2, \varphi(9) = 3^{2-1} \cdot 3 = 6$. *If* $n = 12 = 2^2 \cdot 3, \varphi(12) = 2^{2-1}3^{1-1} \cdot (2-1)(3-1) = 4$.

Theorem (Euler's Theorem)

$a, n \in \mathbb{Z},\ \gcd(a, n) = 1 \Rightarrow a^{\varphi(n)} \equiv 1 \pmod{n}$, where $\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$.

For example, if $n = 9$ *and* $3 \nmid a$, *then* $a^{\varphi(9)} = a^6 \equiv 1 \pmod{9}$.

2.3 Problems

Problem 2.3.1 (ET-2022-SM2-R1-J1)**.** (Junior, *10 points*)

A sequence of numbers is called **arithmetic** if, except the first and last term, twice of each term is the sum of its two neighbour terms. *For example, the sequence* $1,\ 4,\ 7,\ 10,\ 13$ *is an arithmetic sequence since* $2\times 4 = 1+7,\ 2\times 7 = 4+10,\ 2\times 10 = 7+13.$

A 3×3 square is called an **arithmetic square** if the numbers in **each of its columns and rows form an arithmetic sequence.** *For example, in the diagram below the square on the left is an arithmetic square (see below) and the one on the right is not (look at the numbers coloured red.)*

1	3	5
2	7	12
3	11	19

1	4	5
2	7	12
3	12	19

$$2\times 3 = 1+5,\ 2\times 7 = 2+12,\ 2\times 11 = 3+19,$$
$$2\times 2 = 1+3,\ 2\times 7 = 3+11,\ 2\times 12 = 5+19$$

The 3×3 square below is an arithmetic square. Find the number denoted by the red question mark (?).

9	3	
		4
3	?	

$(A)\quad -3$ $\qquad (B)\quad 5$ $\qquad (C)\quad 6$ $\qquad (D)\quad 7$ $\qquad (E)\quad 11$

Problem 2.3.2 (ET-2022-SM2-R1-J2)**.** (Junior, *10 points*)

Find the value of $\frac{\angle A}{2} + \frac{\angle B}{3} - \frac{\angle C}{4}$.

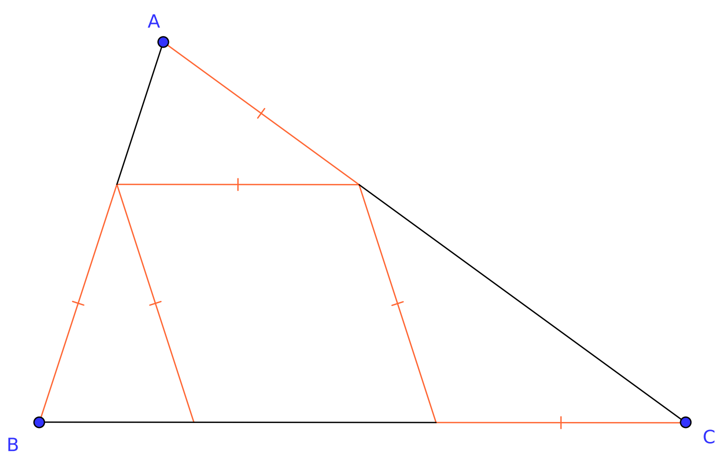


Figure 2.1: ET-2022-SM2-R1-J2

$(A)\quad 35^\circ$ $\qquad (B)\quad 47^\circ$ $\qquad (C)\quad 51^\circ$ $\qquad (D)\quad 63^\circ$ $\qquad (E)\quad 72^\circ$

Problem 2.3.3 (ET-2022-SM2-R1-J3). (Junior, *10 points*)

A word formed by rearranging the letters of the word BACADA is called *A−evenly spaced* if the number of letters (possibly zero) between the first letter A and the second letter A is the same as the number of letters between the second letter A and the third letter A.

For example CABADA is an A−evenly spaced word, but BAACDA is not.

How many different *A−evenly spaced* words are there?

(A) 12 (B) 16 (C) 24 (D) 30 (E) 36

Problem 2.3.4 (ET-2022-SM2-R1-S4). (Senior, *10 points*)

Note that $2n^2 - n - 1 = (n-1)(2n+1)$. Simplify the expression below

$$\frac{5}{9} \times \frac{14}{20} \times \frac{27}{35} \times \cdots \times \frac{2n^2 - n - 1}{2n^2 + n - 1}.$$

(A) $\dfrac{2(2n+1)}{3n(n+1)}$ (B) $\dfrac{2(2n-1)}{n(n+1)}$ (C) $\dfrac{(2n+1)}{3n(n-1)}$ (D) $\dfrac{(2n+3)}{n(n+2)}$ (E) $\dfrac{2(2n-1)}{3n(n-1)}$

Problem 2.3.5 (ET-2022-SM2-R1-S5). (Senior, *10 points*)

Let n be the smallest positive integer such that the remainder of $3n + 45$, when divided by 1060, is 16. Find the remainder of $18n + 17$ upon division by 1920.

(A) 1817 (B) 1043 (C) 921 (D) 1059 (E) 1919

Problem 2.3.6 (ET-2022-SM2-R1-O6). (Olympiad, *10 points*)

Let $x \geq y \geq z$ be positive real numbers such that

$$x^2 + y^2 + z^2 \geq 2xy + 2yz + 2zx.$$

Note that $(x + y - z)^2 = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx + 4xy$.

What is the minimum value of $\dfrac{x}{z}$?

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2 (E) 4

Problem 2.3.7 (ET-2022-SM2-R1-J7)**.** (Junior, *10 points*)

Amy wants to choose 11 positive integers and write them on the perimeter of a circle such that:

- Each number is divisible by 2 or 3 but not divisible by any other prime number.
- The numbers are pair-wise different (any two of them are different).
- Every two neighbouring numbers has a common divisor that is *stricly larger* than 1.

(*5 points*) Can Amy chose such numbers to statisfy the conditions above? Show an example.

(*5 points*) What numbers does Amy choose to make the sum of the numbers smallest?

Problem 2.3.8 (ET-2022-SM2-R1-S8)**.** (Senior, *10 points*)

Let A and B be the two sets of integers. Let C be the set containing all sums $a+b$, where a is an element of A and b is an element of B.

For example if $A = \{1, 2\}$ *and* $B = \{2, 3, 7\}$, *then* $C = \{3, 4, 5, 8, 9\}$.

Now, let $A = \{a_1, a_2, \ldots, a_9\}$, where $a_1 < a_2 < \ldots < a_9$, $B = \{b_1, b_2, \ldots, b_{19}\}$, where $b_1 < b_2 < \ldots < b_{19}$.

(*5 points*) Find 19 elements of C that can be sorted in *strictly* ascending order.

(*5 points*) What is the *smallest* possible number of elements of C? What would be in the A and B sets?

Problem 2.3.9 (ET-2022-SM2-R1-S9)**.** (Senior, *10 points*)

$\triangle ABC$ is a right triangle where $\angle BAC = 90°$ and $\angle ACB = 30°$. M is the midpoint of BC. (Ω) is the circle passing through A and is tangent to BC at M. Let (Γ) be the circumcircle of $\triangle ABC$. (Ω) intersects AC at P and (Γ) at Q.

(*5 points*) Prove that $\triangle AMQ$ is equilateral.

(*5 points*) Prove that PQ is perpendicular to BC.

Problem 2.3.10 (ET-2022-SM2-R1-O10)**.** (Oympiad, *10 points*) *In this problem you first need to prove a lemma, and then solve the problem in two steps.*

Lemma (*2 points*): Prove that if the integer n is not divisible by 3, then the remainder of n^3 when divided by 3 is 1 or 8.

Prove that there do not exist positive integers a and b such that both $a^5b + 3$ and $ab^5 + 3$ are perfect cubes of positive integers.

Step 1 (*3 points*): Prove that if the product ab is divisible by 3, then one of $a^5b + 3$ and $ab^5 + 3$ is not a perfect cube.

Step 2 (*5 points*): Prove that if the product ab is not divisible by 3, apply the lemma to prove that the Euler's Theorem does not stand for the case $(a, n) = (ab, 3)$, then draw the conclusion.

If you are not familiar with the theorem, you can use a short version of the theorem for the $(a, n) = (ab, 3)$ *case as follow*:

$$3 \nmid ab \Rightarrow (ab)^6 \equiv 1 \pmod 9.$$

2.4 Grading

Answers for multiple-choice problems.

Problem 1: D

Problem 2: C

Problem 3: E

Problem 4: A

Problem 5: B

Problem 6: E

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: Separately grading for each part,

(a) 2 points at least half of the neighboring pairs have common divisors larger than 1.

(b) 2 points students recognizes that the numbers are multiple of some powers of 2 and 3.

Problem 8: Separately grading for each part,

(a) 2 points if the student is able to use the given elements of A and B to create some (incomplete) inequality chain.

(b) 2 points students recognizes that the smallest element of C is $a_1 + b_1$ and the largest is $a_9 + b_{19}$.

Problem 9: Separately grading for each part,

(a) 2 points if the student is able to see that M is the centre of the circumcircle Γ of $\triangle ABC$.

(b) 2 points students recognizes that $\angle AMB = \angle AQM$.

Problem 10: Separately grading for each part,

(a) 1 points if the student can consider analyzing numbers in $3k \pm 1$ forms.

(b) 1 point if the student can derive the proof into sub cases of $3 \mid a$ and $3 \mid b$.

(c) 2 points students can combine $a^5b \cdot ab^5 = (ab)^6$ in order to apply the Euler's Theorem.

2.5 Solutions

Solution. ET-2022-SM2-R1-J1

There are several solutions to this problem. In the diagram below we start to find the missing number in the first row, then try to complete the last column, and then the last row.

9	3	
		4
3	?	

9	3	**-3**
		4
3	?	

9	3	-3
		4
3	?	**11**

9	3	-3
		4
3	**7**	11

Each time we use two numbers in that column or row to find the missing number by applying the property of an arithmetic sequence.

$$2 \times 3 = 9 + (-3),\ 2 \times 4 = (-3) + 11,\ 2 \times 7 = 3 + 11.$$

The number is $\boxed{7}$ and answer is $\boxed{D.}$ □

Solution. ET-2022-SM2-R1-J2

First, lets denote the angles $\angle A, \angle B, \angle C.$ as shown in the diagram below.

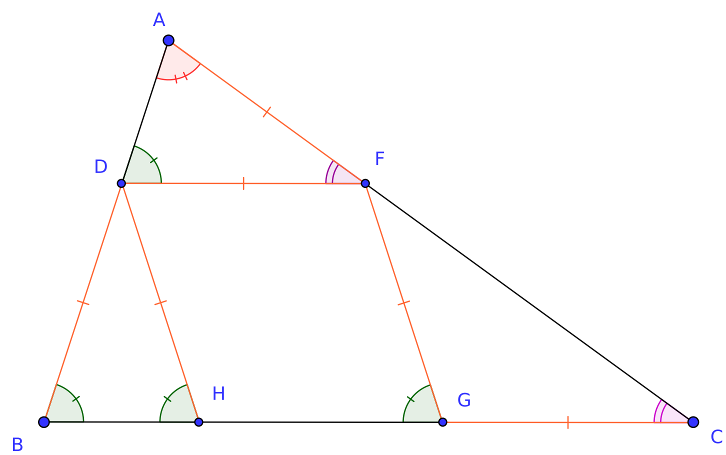


Figure 2.2: ET-2022-SM2-R1-J2

It is easy to see that

$$\angle B = \angle BHD = \angle HGF = 2\angle C$$
$$\angle B = \angle ADF = \angle A,\ \angle C = \angle DFA$$

In $\triangle ADF$,

$$\angle A + \angle ADF + \angle DFA = 180^\circ \Rightarrow 2\angle B + \angle C = 180^\circ \Rightarrow 5\angle C = 180^\circ$$
$$\Rightarrow \angle C = 36^\circ \Rightarrow \angle A = \angle B = 72^\circ$$
$$\Rightarrow \frac{\angle A}{2} + \frac{\angle B}{3} - \frac{\angle C}{4} = \frac{72^\circ}{2} + \frac{72^\circ}{3} - \frac{36^\circ}{4} = \boxed{51^\circ.}$$

The answer is $\boxed{C.}$ □

Solution. ET-2022-SM2-R1-J3

There are six ways of arranging the letters As so that they are evenly spaced:

$$AAA???,\ ?AA??,\ ??AA?,\ ???AAA,\ A?A?A?,\ ?A?A?A.$$

where ? indicate the letter B, C, or D.

Since there are 3! ways to put the letters B, C, and D into the positions of the question marks. Thus, the total number of different words can be created by rearranging the letters of the word BACADA is $6 \cdot 6 = \boxed{36.}$

The answer is $\boxed{E.}$ □

Solution. ET-2022-SM2-R1-S4

Note that $2n^2 - n - 1 = (n-1)(2n+1)$ and $2n^2 + n - 1 = (n+1)(2n-1)$.

$$\frac{5}{9} \times \frac{14}{20} \times \frac{27}{35} \times \cdots \times \frac{2n^2 - n - 1}{2n^2 + n - 1} = \prod_{k=2}^{n} \frac{(k-1)(2k+1)}{(k+1)(2k-1)} = \left(\prod_{k=2}^{n} \frac{k-1}{k+1}\right)\left(\prod_{k=2}^{n} \frac{2k+1}{2k-1}\right) = \frac{1 \cdot 2}{n(n+1)} \cdot \frac{2n+1}{3}$$

The answer is $\boxed{A.}$ □

Solution. ET-2022-SM2-R1-S5

First, we have $3n + 45 \equiv 16 \pmod{1060}$, thus there exist m such that

$$3n + 45 = 16 + 1060m \Rightarrow 3(n+15) = 3(5 + 353m) + m + 1$$

Therefore $3 \mid m + 1$. The smallest such m, which also makes n smallest, is 2. In that case $n = \frac{1}{3}\left((16 + 1060 \cdot 2) - 45\right) = 697$.

Thus, $18n + 17 = 18 \cdot 697 + 17 \equiv \boxed{1043} \pmod{1920}$.

The answer is $\boxed{B.}$ □

Solution. ET-2022-SM2-R1-O6

Note the three-variable identity:

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx.$$

What we have is the expression $x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \geq 0$. So instead of z, let explore the identity with $-z$.

$$(x+y-z)^2 = x^2 + y^2 + (-z)^2 + 2xy + 2y(-z) + 2(-z)x = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx + 4xy$$

Thus the given inequality is equivalent to

$$(x+y-z)^2 - 4xy \geq 0 \Rightarrow x + y - z \geq 2\sqrt{xy} \Rightarrow x + y - 2\sqrt{xy} \geq z \Rightarrow (\sqrt{x} - \sqrt{y})^2 \geq z$$

Therefore $\sqrt{x} - \sqrt{y} \geq \sqrt{z}$, or $\sqrt{x} \geq \sqrt{y} + \sqrt{z}$. This means that

$$\frac{x}{z} \geq \frac{(\sqrt{y} + \sqrt{z})^2}{z} \geq \frac{(2\sqrt{z})^2}{z} = \boxed{4.}$$

The answer is $\boxed{E.}$ □

Solution. ET-2022-SM2-R1-J7

For the first question, see the diagram below. It is easy to verify that every two neighbouring numbers of these 11 positive integers have a common divisor larger than 1.

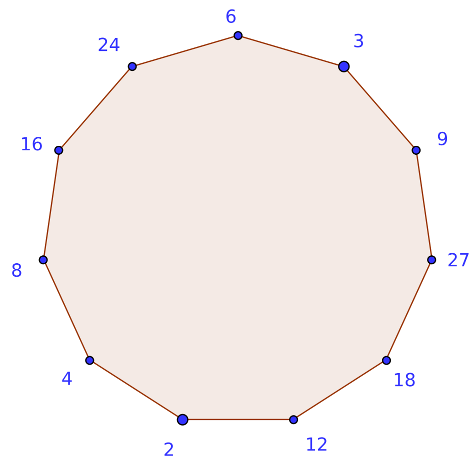


Figure 2.3: ET-2022-SM2-R1-J7

Now, for the second question, since 2 and 3 are the only prime divisors of these numbers so in order to minimize their sum, they should be the first 11 positive integers that are multiple of some powers of 2 and 3,

$$2 = 2^1, 3 = 3^1, 4 = 2^2, 6 = 2 \cdot 3, 8 = 2^3, 9 = 3^2, 12 = 2^2 \cdot 3, 16 = 2^4, 18 = 2 \cdot 3^2, 24 = 2^3 \cdot 3, 27 = 3^3.$$

The diagram above is an example how these 11 numbers can be arranged.

The sum of these number is $2 + 3 + 4 + 6 + 8 + 9 + 12 + 16 + 18 + 24 + 27 = \boxed{129.}$ □

Solution. ET-2022-SM2-R1-S8

The sums below are elements of C and they can be arranged in strictly ascending order,

$$a_1 + b_1 < a_1 + b_2 < \ldots < a_1 + b_{19}.$$

For the second question, note that we can add another 8 numbers into the inequality chain above

$$a_1 + b_1 < a_1 + b_2 < \ldots < a_1 + b_{19} < a_2 + b_{19} < \ldots < a_9 + b_{19}.$$

This chain contains exactly $19 + 8 = 27$ numbers. Thus the number of elements in C must at least be 27. Now, it is easy to see that the smallest element of C is $a_1 + b_1$ and the largest is $a_9 + b_{19}$. thus the number of elements in C cannot exceed $(a_9 + b_{19}) - (a_1 + b_1) + 1$. With $a_9 = a_1 + 8$ and $b_{19} = b_1 + 18$, the number of elements in C shall be $18 + 8 + 1 = \boxed{27.}$

It is easy to show that with $A = \{1, 2, \ldots, 9\}$ and $B = \{1, 2, \ldots, 19\}$, then $C = \{2, 3, \ldots, 28\}$, which has exactly 27 elements. □

Solution. ET-2022-SM2-R1-S9

First, since M is the midpoint of BC and $\triangle ABC$ is a right triangle with $\angle A = 90°$, thus M is the centre of the circumcircle Γ of $\triangle ABC$. Hence, $MB = MA$, so $\triangle ABM$ is an isosceles triangle with $\angle B = 60°$, therefore it is an equilateral triangle, and $\angle BMA = 60°$.

Now, the circle Ω is tangent with BC at M, therefore $\angle AMB = \angle AQM$ (because both subtend the smaller arc $\widehat{AM}$.) This means that $\angle AQM = 60°$. So, the $\triangle AQM$ has an $60°$ angle and two equal sides $AM = MQ$ (since both are radii of the Ω circle), therefore it is an equilateral triangle, hence $\angle AMQ = 60°$.

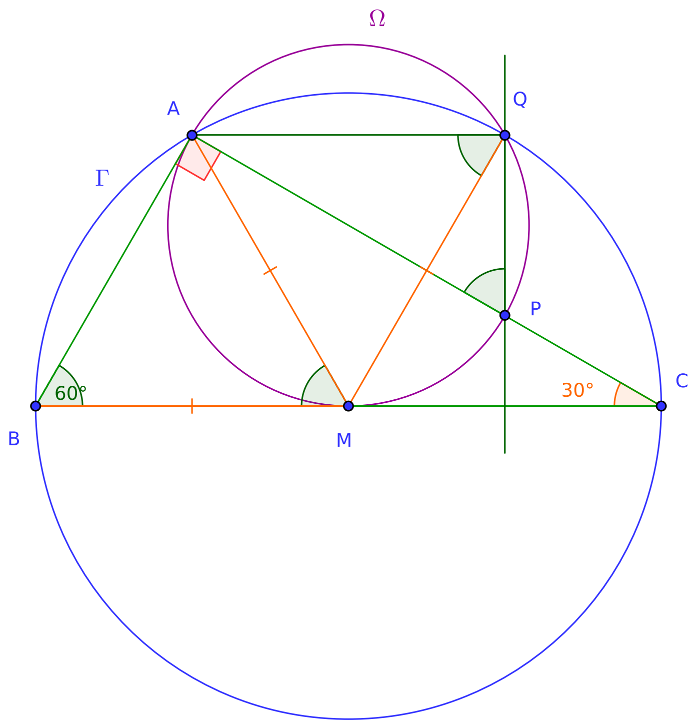


Figure 2.4: ET-2022-SM2-R1-S9

This implies that $\angle APQ = \angle AMQ = 60°$. Angle $PCB = 30°$, so it is trivial to see that $\boxed{QP \perp BC}$. □

Solution. ET-2022-SM2-R1-O10

First, lets prove the lemma,

Claim — a is a positive integer, $3 \nmid a$, then $a^3 \equiv \pm 1 \pmod 9$.

Proof. Let $a = 3k \pm 1$, then $a^3 = (3k)^3 \pm 3(3k)^2 + 3(3k) \pm 1 \equiv \pm 1 \pmod 9$. ■

Now, assume that $3 \mid ab$, then $a^5b + 3$ is divisible by 3, so $3 \mid a$ or $3 \mid b$. WLOG, assum that $3 \mid a$, then $a^5b + 3$ is divisible by 3 but not 27, which contradicts the fact that it is a perfect cube. Similarly if $3 \mid b$, then $ab^5 + 3$ is divisible by 3 but cannot be a perfect cube.

Lets assume that none of $a^5b + 3$ and $ab^5 + 3$ is divisible by 3. By the lemma, their remainders when divided by 3 is 1 or 8. This means that the remainders when divided a^5b and ab^5 can only be 5 or 7, which means that

$$(ab)^6 \equiv 5 \cdot 5 \text{ or } 5 \cdot 7 \text{ or } 7 \cdot 7 \not\equiv 1 \pmod 9.$$

This contradicts the Euler's Theorem, which says

$$3 \nmid ab \Rightarrow (ab)^6 \equiv 1 \pmod 9.$$

Therefore there are no (a, b) positive integers such that both $a^5b + 3$ and $a^5b + 3$ are positive integers. □

Chapter 3

Entrance Test - Round 2

3.1 Rules

- Two tests are administered at **10:00 PM Eastern Time (EST) on Saturday 27/8 and Sunday 28/8**. Each test lasts for **90 minutes**. Each student must participate in both tests.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Problem levels: all students must **try to solve all problems**, don't stay within your boundary
  - Junior: Problem 1, 2, and 3 (multiple-choice); Problem 7 (show-your-work).
  - Senior: Problem 4, 5 (multiple-choice); Problem 8, 9 (show-your-work).
  - Olympiad: Problem 6 (multiple-choice); Problem 10 (show-your-work).
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at **an earlier time**. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.

3.2 Problems

Problem 3.2.1 (ET-2022-SM2-R2-J1)**.** (Junior, *10 points*)

The area of the square $ABCD$ is 100. $AE = BF = CG = HD = 2$.

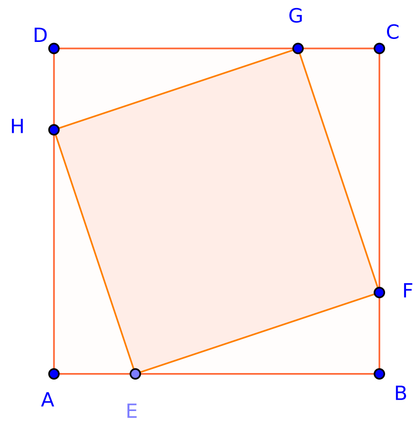


Figure 3.1: ET-2022-SM2-R2-J1

What is the area of $EFGH$?

(A) 52 (B) 56 (C) 64 (D) 68 (E) 76

Problem 3.2.2 (ET-2022-SM2-R2-J2)**.** (Junior, *10 points*)

On a circular running track, Phan stands at point A and Quan is diametrically opposite at point B. Phan starts to run counter-clockwise and Quan runs clockwise. They both run at constant, but different, speeds. After running for a while they notice that when they pass each other it is always at the same three places on the track.

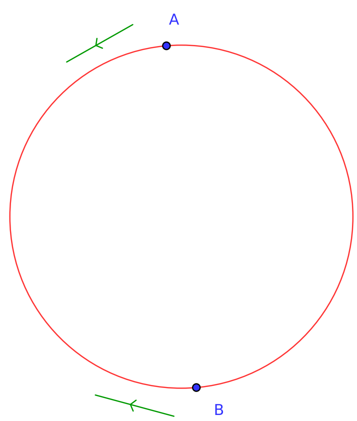


Figure 3.2: ET-2022-SM2-R2-J2

What is the ratio of their speeds?

(A) $3:2$ (B) $3:1$ (C) $2:1$ (D) $4:1$ (E) $5:1$

Problem 3.2.3 (ET-2022-SM2-R2-J3). (Junior, *10 points*)

Six students went to a party. They left their shoes by the doors. It is know that the shoe sizes of all students are different. At night, the students were leaving the party, one by one. Some of them, instead of putting their own shoes on, were putting bigger shoes on.

What was the greatest possible number of students that had to leave barefoot? (A student would leave barefoot if all of the shoes that are left are too small).

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 3.2.4 (ET-2022-SM2-R2-S4). (Senior, *10 points*)

How many 8-digit numbers in base 7 formed of the digits $1, 2, 3$ are divisible by 3?

(A) 128 (B) 256 (C) 243 (D) 729 (E) 2187

Problem 3.2.5 (ET-2022-SM2-R2-S5). (Senior, *10 points*)

Find the sum of all roots of the equation

$$\frac{x^3}{\sqrt{4-x^2}} + x^2 - 4 = 0$$

(A) 2 (B) $\sqrt{2}$ (C) 0 (D) $-\sqrt{2}$ (E) -2

Problem 3.2.6 (ET-2022-SM2-R2-O6). (Olympiad, *10 points*)

Points A, B, C, and D are on a circle of diameter 1, and E is on diameter AD. $BE = CE$ and $3\angle BAC = \angle BEC = 36°$.

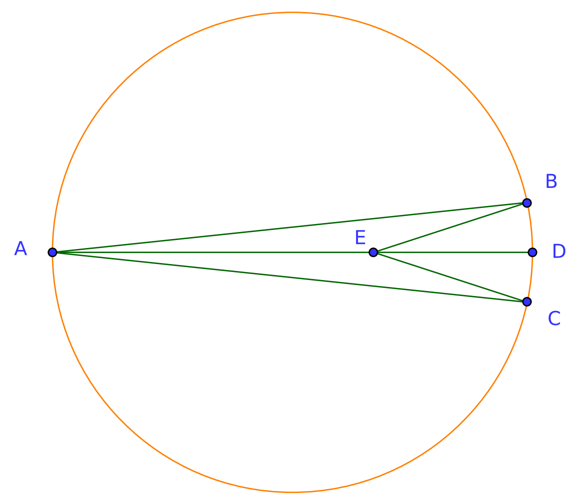


Figure 3.3: ET-2022-SM2-R2-O6

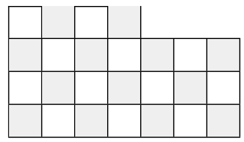
Find AE in terms of trigonometric functions of acute angles.

(A) $\dfrac{\cos 6° \sin 12°}{\sin 18°}$ (B) $\dfrac{\cos 6° \sin 18°}{\sin 12°}$ (C) $\dfrac{\cos 12° \sin 6°}{\sin 18°}$ (D) $\dfrac{\sin 12°}{\cos 6° \sin 18°}$ (E) $\dfrac{\cos 6°}{\sin 12° \sin 18°}$

Problem 3.2.7 (ET-2022-SM2-R2-J7)**.** (Junior, *10 points*)

Minh baked a chocolate-vanilla cake that looks like a 4×7 chessboard. Antoine took a 1×3 piece, so Minh had the cake that is shown below.

After a bit of thinking, Minh managed to cut the cake into three pieces, which she then arranged into a 5×5 square in such a way that the chess pattern was preseved.



How did she do it? Note that oher cuts follow grid lines, the pieces can be rotated but not flipped over.

Problem 3.2.8 (ET-2022-SM2-R2-S8)**.** (Senior, *10 points*)

Let AB and CD be two perpendicular diameters of a circle with centre O. Consider a point M on the diameter AB, different from A and B. The line CM cuts the circle again at N. The tangent at N to the circle and the perpendicular at M to AM intersect at P.

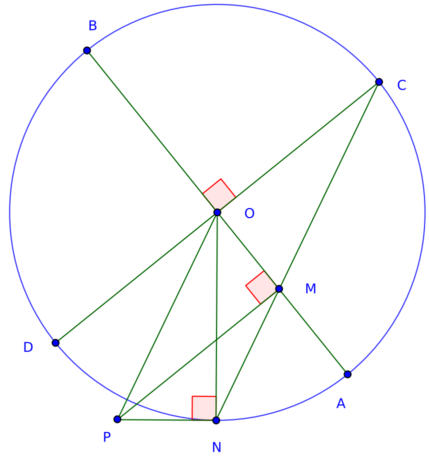


Figure 3.4: ET-2022-SM2-R2-S8

(*5 points*) Show that $OMNP$ is a cyclic quadrilateral.

(*5 points*) Show that $OP = CM$.

Problem 3.2.9 (ET-2022-SM2-R2-S9)**.** (Senior, *10 points*)

Anthony wants to place some rooks on a 8×8 chessboard so that no two of them attack one another.

(*5 points*) In how many ways can he does if he has two rooks?

(*5 points*) In how many ways can he does if he has four rooks?

Problem 3.2.10 (ET-2022-SM2-R2-O10)**.** (Oympiad, *10 points*)

Points A, B, and P are on the perimeter of a circle. Point Q is in the interior of this circle such that $\angle PAQ = 90^\circ$ and $PQ = BQ$.

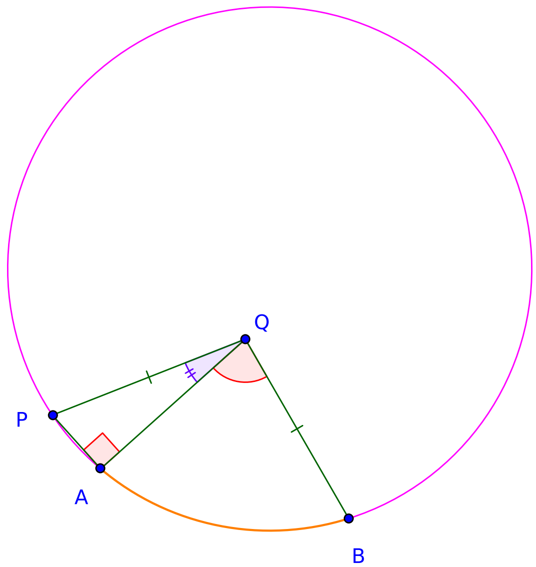


Figure 3.5: ET-2022-SM2-R2-O10

(*5 points*): Prove that $\angle APQ = \angle QPB + \frac{1}{2}\widehat{AB}$.

(*5 points*): Prove that $\angle AQB - \angle PQA = \widehat{AB}$.

3.3 Grading

Answers for multiple-choice problems.

Problem 1: D

Problem 2: C

Problem 3: C

Problem 4: E

Problem 5: B

Problem 6: A

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 point if the student can find a way to combine the pieces with some minor errors.

Problem 8: Separately grading for each part,

(a) 2 points if the student is able to use the right triangle property for M or N.

(b) 2 points students recognizes that $\triangle NOC$ is an isosceles triangle.

Problem 9: Separately grading for each part,

(a) 2 points if the student is able to see that the second rook can freely be chosen from 7×7 grid.

(b) 2 points students recognizes the proof can continue to use the same method from the solution for first question.

Problem 10: Separately grading for each part,

(a) 1 point if the student can see that $\angle BPA = \frac{1}{2}\widehat{AB}$.

(b) 2 points students can recognizes that $\angle AQB = \angle PQB - \angle PQA$.

3.4 Solutions

Solution. ET-2022-SM2-R2-J1

Since the area of $ABCD$ is 100 so $AB = BC = CD = DA = 10$. Thus $EB = FC = GD = HA = 10 - 2 = 8$. Therefore the area of each *corner* right triangle is $\frac{1}{2} \cdot 8 \cdot 2 = 8$. Hence, the area of $EFGH$ is $100 - 4 \cdot 8 = \boxed{68.}$

The answer is $\boxed{D.}$ □

Solution. ET-2022-SM2-R2-J2

Since Phan and Quan always pass each other at the same three places on the track and since they each run at a constant speed, then the three places where they pass must be equally spaced on the track. In other words, the three places divide the track into three equal parts.

We are not told which runner is faster, so we can assume that Quan is the faster runner. Start at one place where Phan and Quan meet. (Now that we know the relative positions of where they meet, we do not actually have to know where they started at the very beginning.) To get to their next meeting place, Quan runs farther than Phan (since she runs faster than he does), so Quan must run $\frac{2}{3}$ of the track while Alphonse runs $\frac{1}{3}$ of the track in the opposite direction, since the meeting placed are spaced equally at $\frac{2}{3}$ intervals of the track. Since Quan runs twice as far in the same length of time, then the ratio of their speeds is $\boxed{2:1}$.

The answer is $\boxed{C.}$ □

Solution. ET-2022-SM2-R2-J3

Let denote the student by the number $1, 2, \ldots, 6$ based on their shoe sizes, from smallest to largest.

Claim — Consider the students $1, 2$, and 3. When one of them was leaving, then at least one of the below statements is true

- Three students already have left the party with their shoes on.
- This student is guaranteed to find a pair of shoes to put on.

Proof. If three students already have left the party with their shoes on, then the first statement is true. If among the students who left, at most two left with their shoes on, then at least four pairs of shoes still remained at the party. In this case a student among $1, 2$, and 3 should be able to use one of those four pairs. ■

Now, it is easy to see that the largest number of students who would leave barefoot is $\boxed{3.}$ This would be done if the first three students who left were $1, 2, 3$, and they took the shoes of $4, 5$, and 6. Then the $4, 5$, and 6 students would leave barefoot.

The answer is $\boxed{C.}$ □

Solution. ET-2022-SM2-R2-S4

Let $n = \overline{a_7 a_6 \ldots a_0}_4$ be an $8-$digit number in base 7, then since $7 \equiv 1 \pmod 3$, so

$$n = a_7 \cdot 7^7 + a_6 \cdot 7^6 + \ldots + a_1 \cdot 7 + a_0 \equiv a_7 + a_6 + \ldots + a_1 + a_0 \pmod 3.$$

Thus, this means that if n is divisible by 3, then the sum of the digits of n is also divisible by 3. Now, if the first 7 digits of n is chosen arbitrary from $1, 2$, and 3, then the remainder of their sum is a number of $0, 1$, and 2, thus there is only one way to choose the 8^{th} digit so that the sum of all 8 numbers is divisible by 3. Therefore the number of ways to choose these digits are $3^7 = \boxed{2187.}$

The answer is $\boxed{E.}$ □

Solution. ET-2022-SM2-R2-S5

First, the existence of $\sqrt{4-x^2}$ and its position as a denominator in the fraction means that $4 > x^2$,

$$\frac{x^3}{\sqrt{4-x^2}} + x^2 - 4 = 0 \Rightarrow x^3 = \left(\sqrt{4-x^2}\right)^3$$

Since x^3 is a cube of a positive number, so $x > 0$. Furthermore,

$$x^3 = \left(\sqrt{4-x^2}\right)^3 \Rightarrow x = \sqrt{4-x^2} \Rightarrow x^2 = 4 - x^2 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

Since $x > 0$, there is a single root for the equation $\boxed{x = \sqrt{2}}$. The answer is $\boxed{B.}$ □

Solution. ET-2022-SM2-R2-O6

By connecting both B and C to the center, O, of the circle we can use SSS congruency to show $\triangle OEB \cong \triangle OEC$, so $\angle BED = \angle DEC$. Using SAS we can then show $\triangle BEA \cong \triangle CEA$ ($\angle BEA = \angle CEA$, $BE = CE$, and $AE = AE$) so that $\angle BAE = \angle CAE = 6°$ (since $\angle BAC = 12°$).

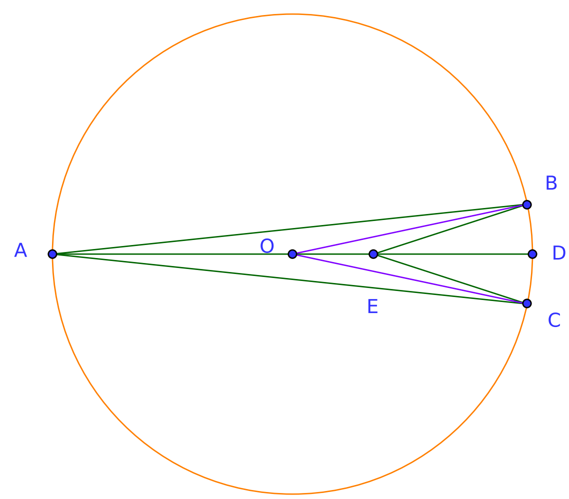


Figure 3.6: ET-2022-SM2-R2-O6

From right triangle ABD, $AB = AD\cos\angle BAD = \cos 6°$. Since $\angle BED = 18°$, $\angle BEA = 162°$ and $\angle ABE = 12°$. Finally, we apply the law of sines to $\triangle ABE$ to find

$$\frac{AE}{\sin 12°} = \frac{AB}{\sin 162°} = \frac{\cos 6°}{\sin 18°} \Rightarrow AE = \boxed{\frac{\cos 6° \sin 12°}{\sin 18°}.}$$

The answer is $\boxed{A.}$ □

Solution. ET-2022-SM2-R2-J7

First, Minh perfomed a cut along the grid line separating the second and the third column from the left. Then, she cut the left piece along the grid line separating the first and the second row from the top. See the Figure 3.7 on the left.

She assembled them by rotated the smallest piece 180° and moved it to the top-left corner of the largest piece, then rotated the middle piece 90° anti-clockwise and put in on the top-right corner of the largest piece. See the Figure 3.8 on the right.

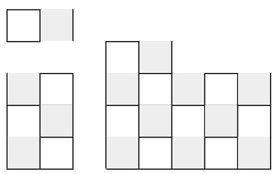


Figure 3.7: Cutting into 3 pieces

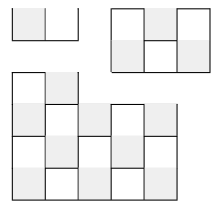


Figure 3.8: Assemblying the pieces

□

Solution. ET-2022-SM2-R2-S8

First, $\angle PMO = \angle PNO = 90°$, so M and N are on the circle diameter OP, thus $\boxed{OMNP \text{ is cyclic.}}$

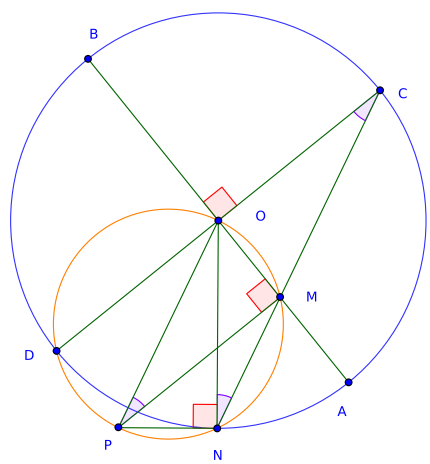


Figure 3.9: $OMNP$ is cyclic

Thus, $\angle OPM = \angle ONM$ (both subtend the same small arc $\overparen{OM}$), $\angle ONM = \angle OCM$ (both side angles in the isosceles $\triangle NOC$). Therefore $OCMP$ is a quadrilateral where $OC \parallel MP$ (both perpendicular to OM), and $\angle OCM = \angle OPM$, so it is a parallelogram. Hence, $\boxed{OP = CM.}$ □

Solution. ET-2022-SM2-R2-S9 For the first question, lets place the first rook to any place we like, we eliminate the other squares in that row and column. The remaining squares form a 7×7 grid, where we can place the second rook anywher we like. Note that once placed on two squares, the rooks can be permuted among those places in 2! ways. Thus, the number of ways we can choose the squares to place the rooks is

$$\frac{8^2 \cdot 7^2}{2!} = \boxed{1568}.$$

For the second question, we continue the argument for the first question. After eliminating the row and column where the second rook stands, we have a 6×6 grid. Similarly, the final rook will be placed on a 5×5 grid.

Note that once placed on four squares, the rooks can be permuted among those places in 4! ways. Thus, the number of ways we can choose the squares to place the rooks is

$$\frac{8^2 \cdot 7^2 \cdot 6^2 \cdot 5^2}{4!} = \boxed{117600}.$$

□

Solution. ET-2022-SM2-R2-O10

First, $\angle QPA = \angle QPB + \angle BPA = \angle QPB + \frac{1}{2}\widehat{AB}$.

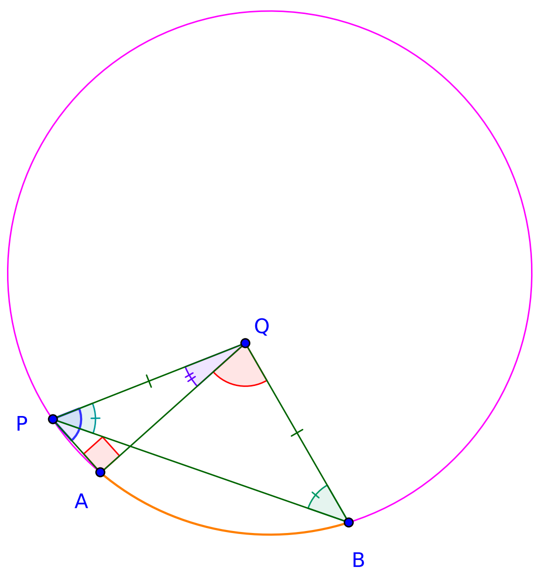


Figure 3.10: ET-2022-SM2-R2-O10

Now, since $\angle AQB = \angle PQB - \angle PQA$, thus

$$\angle AQB - \angle PQA = \angle PQB - 2\angle PQA = (180^\circ - 2\angle QPB) - 2(90^\circ - \angle QPA) = 2(\angle QPA - \angle QPB) = \widehat{AB}.$$

□

Chapter 4

House Championship - Round 1

4.1 Problems

Problem 4.1.1 (HC-2022-SM2-R1-P1)**.** (*Beginner Level*)

A **magic square** is a square array of numbers consisting of the distinct positive integers arranged so that the sum of the numbers in any horizontal, vertical, or main diagonal line is always the same.

For example, the magic square below is constructed by using the numbers $1, 2, \ldots, 9$.

2	7	6
9	5	1
4	3	8

Henry created a magic square by using the numbers $20, 21, \ldots, 28$. He erased the numbers in six cells of the square, as shown below in Figure 4.1. He then challenged Albert to **reconstruct the square.**

23		
		26
	20	

Figure 4.1: HC-2022-SM2-R1-P1

Can you help Albert find the missing number?

How to provide your answer:

- If you think you have found the missing numbers, fill them into the incomplete magic square above, take a picture (or by any other means that can visualize your solution), and submit it.
- If you think that there is no solution, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop does not change regardless of whatever your opponent submitted.

Problem 4.1.2 (HC-2022-SM2-R1-P2). (*Beginner Level*)

Remove **eight matchsticks** from the diagram Figure 4.2 below to obtain exactly **six squares**.

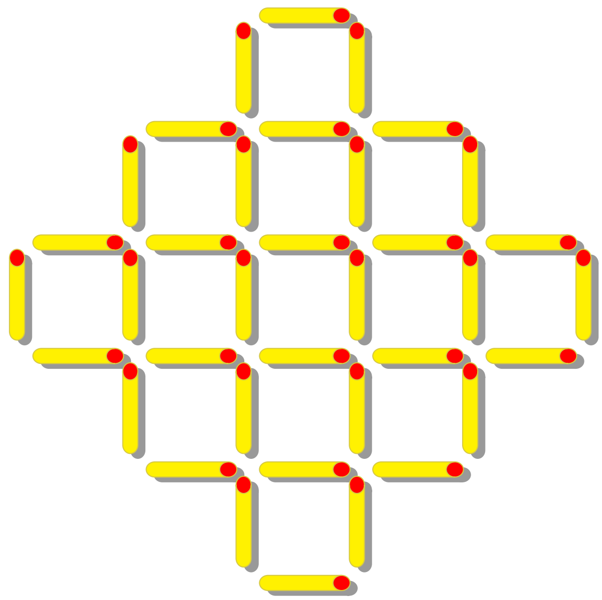


Figure 4.2: HC-2022-SM2-R1-P2

How to provide your answer:

- If you think you have a solution, draw a diagram and take a picture of it (or by any other means that can visualize your solution), and submit it.
- If you think that there is no solution, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop does not change regardless of whatever your opponent submitted.

Problem 4.1.3 (HC-2022-SM2-R1-P3)**.** (*Beginner Level*)

In the Kingdom of Might and Magic, all inhabitants are either paladins or illusionists. For a *paladin*, everything he believes while he is awake is true, and everything he believes while he is asleep is false. An *illusionist* is the opposite: everything he believes while asleep is true, and everything he believes while awake is false.

Minh believes that he and his brother Khoa are both illusionists. At the same time, Khoa believes that both are paladins. As that moment, one of them was awake the other was asleep at the time.

Determine **what each of them was** at that moment.

How to provide your answer:

- If you think that Minh was awake and Khoa was also awake, then submit MW KW, if you think that Minh was asleep and Khoa was also awake, then submit MS KW, and so on.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop does not change regardless of whatever your opponent submitted.

Problem 4.1.4 (HC-2022-SM2-R1-P4)**.** (*Beginner Level*)

After a few moves, the situation on the chessboard looks as follow,

Figure 4.3: HC-2022-SM2-R1-P4

Find the sequence with **least possible number of moves** from both black and white that led to this situation.

How to provide your answer:

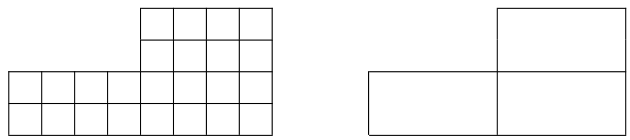
- If you think it is the 4−move sequence 1.e1 e2 2.Nc3 Nf6, then submit 1.e1 e2 2.Nc3 Nf6.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* with a sequence of moves that is *strictly longer* than the one in your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* with a sequence of moves that is *strictly shorter* than the one in your answer.

Problem 4.1.5 (HC-2022-SM2-R1-P5)**.** (*Beginner Level*)

The figure on the left diagram below can be dissected (cut) along the gridlines into three *congruent shapes,* as shown in the diagram on the right.



Note that, congruent shapes are shapes that are exactly the same. The corresponding sides are the same and the corresponding angles are the same. If two shapes are congruent they will fit exactly on top of one another.

Is it possible to dissect the figure on the left into **four congruent** figures with cuts along the gridlines?

How to provide your answer:

- If you think that it is possible, submit a diagram.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop does not change regardless of whatever your opponent submitted.

Problem 4.1.6 (HC-2022-SM2-R1-P6)**.** (*Intermediate Level*)

Four brothers named An, Binh, Chi, and Danh are quadruplets indistinguishable in apperance.

- An is an *acurrate truth-teller.*
- Binh is an *inacurrate truth-teller*, meaning he is totally deluded in all his beliefs but always states honestly what he does believe.
- Chi is an *acurrate lier,* meaning all of his beliefs are correct, but he lies about every one of them.
- Danh is an *inacurrate lier*, he is both deluded and dishonest, meaning he will try to give you false information but is unable to.

An and Binh are both married; the other two brothers are not. An and Chi are both rich; the other two brothers are not.

One day you meet one of the brother. Your task is to find a **three-word question,** for example *Do you lie?,* in order to find out whether he is married or not.

How to provide your answer:

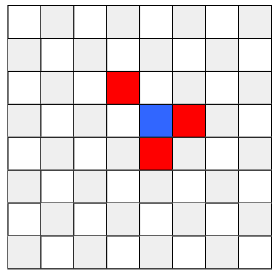
- If you think that the question is *Do you lie?,* submit *Do you lie?*
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop does not change regardless of whatever your opponent submitted.

Problem 4.1.7 (HC-2022-SM2-R1-P7)**.** (*Intermediate Level*)

On a 8×8 standard chessboard we place a new chess piece called **the Tiger**. It can move **horizontally to the right**, **vertically to the bottom**, or **diagonally to the left and up** by exactly **one square.** The figure below shows an example with the blue square indicating where the Tiger stands. The three red squares indicate all the squares the Tiger can legally move into.



Can you choose an initial position (a square) for the Tiger so that it can **visit every square of the chessboard exacly once** and **return to the initial position in its final move**?

How to provide your answer:

- If you think there is a good initial position, then draw a sequence of move depicting the visits of the Tiger from that position, through all the squares, and returning to the initial position. Submit it.
- If you cannot determine it, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop does not change regardless of whatever your opponent submitted.

Problem 4.1.8 (HC-2022-SM2-R1-P8). (*Intermediate Level*)

14 cities are connected by *direct* roads. Each city is connected to either $3, 4$, or 5 other cities. Among them, there are six cities, each of which is connected to exactly 4 other cities. All together, there are 27 direct roads.

How many cities connected to exacly 3 other cities are there? And **how many cities** connected to exacly 5 other cities are there?

How to provide your answer:

- If you think that the answer for the first question is 1 the answer for the second question is 1, submit 1, 1.
- If you cannot determine the answer, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 4.1.9 (HC-2022-SM2-R1-P9)**.** (*Intermediate Level*)

Points $A(-6,3)$, $B(2,-5)$, $C(4,5)$, $D(-4,-3)$, $E(-5,-1)$, and $F(1,-1)$ are shown in the Figure 4.4 below.

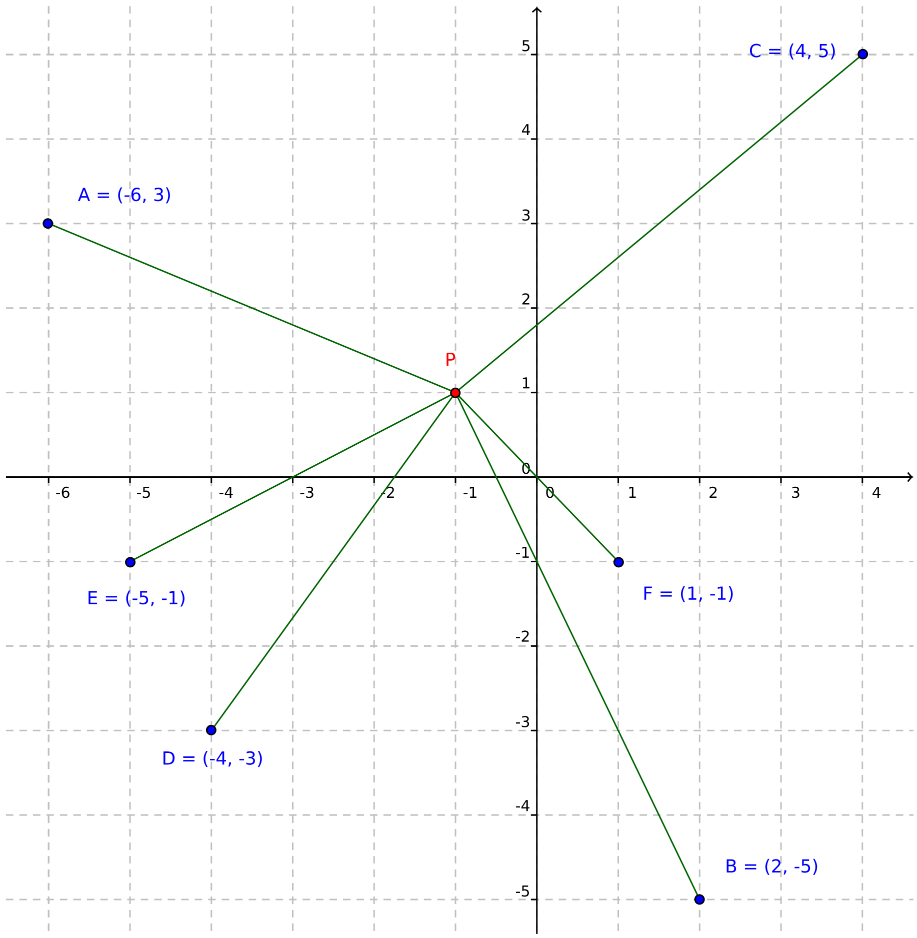


Figure 4.4: HC-2022-SM2-R1-P9

Find point P such that the sum of the distances $PA + PB + PC + PD + PE + PF$ is **minimal**?

How to provide your answer:

- If you think that the point is $P(-1,1)$, and the value of the minimal sum is 1, then submit $P(-1,1),\ 1$.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* whose sum is *strictly larger* than the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* whose sum is *strictly smaller* than your answer.

Problem 4.1.10 (HC-2022-SM2-R1-P10)**.** (*Intermediate Level*)

Four students each own a single distinct ball from a four-piece set of balls (black, green, red, and white). Rare Antique Shop sell these balls individually. Each ball costs a whole number of dollars. In order to complete their sets (by having exactly one ball for every colour), each of the students bought three balls. The costs that they paid are four consecutive numbers, one of which is 46.

Find the cost of **the of most expensive ball.**

How to provide your answer:

- If you think that the cost of the most expensive ball is 1, submit 1.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop does not change regardless whatever submitted by your opponent.

Problem 4.1.11 (HC-2022-SM2-R1-P11)**.** (*Advanced Level*)

In Vietnamese, much like in other languages, mistakes are often made while using similar consonants. For example, *con trâu* is right and *con châu* is wrong. Instead of *trờ đợi*, we must use *chờ đợi*.

How many of such mistakes are there in the text below?

Trong khói mù nhô ra một con heo đầu đàn, cao gần bằng con bò, lông gáy dựng ngược, mũi ngước lên thở phì phì làm tro hai cái nanh dài chỗ khóe mép vươn ra như hai lưỡi dao găm. Rồi vun vút chàn đến một bầy heo rừng, con lớn con bé tranh nhau chạy, sống lưng nhấp nhô chàn tới như một đàn heo mục. Nai co giò phóng bay qua những lùm cây thấp. Hươu, trồn, bông lau, cáo, mèo ... tất cả những con thú bốn trân trong rừng đều nhắm mắt nhắm mũi tranh nhau chạy. Thỉnh thoảng một vài con gì không biết cứ chạy đâm bổ vào người chúng tôi.

Chân tôi đạp lên một khúc lưng con vật gì trơn trơn, lảo đảo chúi tới trước. Một con trăn gió uốn lưng trườn tới, đầu cất cao hơn ngọn sậy, ngoằn ngoèo lướt hút vào bụi cây trầm um tùm. Lâu lâu lại gặp một con rắn to phóng ngược hướng gió, trấn ngang đường chúng tôi. Tía con tôi phải chạy chánh chúng, cũng có lúc cứ mặc kệ, nhảy bừa qua, bất trấp cả những đầu phồng mang dẹt đét đang lắc lư phun nọc phì phì ... Khỉ, vượn, nhọ nồi cuống quít kêu lúc théc trên cây. Một con vượn bạc má bồng con nhảy xuống đất, cố chạy theo vết chúng tôi.

How to provide your answer:

- If you think that there are 2 mistakes, submit 2.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which is *strictly smaller* than the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which is *strictly larger* than your answer.

Problem 4.1.12 (HC-2022-SM2-R1-P12). (*Advanced Level*)

The following diagram shows 6 knights on a standard chessboard such that every knight attacks one and only one other knight.

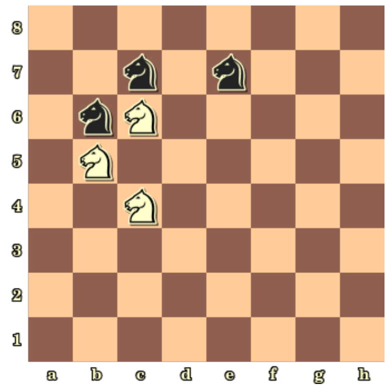


Figure 4.5: HC-2022-SM2-R1-P12

What is the **largest number of knights** that you can put on the standard chessboard such that every knight attacks one and only one other knight?

How to provide your answer:

- If you think the largest number of such knights is 1, then submit 1.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which is *strictly smaller* than the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which is *strictly larger* than your answer.

Problem 4.1.13 (HC-2022-SM2-R1-P13). (*Advanced Level*)

John and Peter are twin brothers and have interesting personalities. John always tells the truth when he is sober, but he lies when he is drunk. Peter, on the other hand, always lies when he is sober, and tells the truth when he is drunk.

One day, one of the twins stayed in the same state the whole day and said that *Tomorrow I will be drunk.* The next day, he was in the same state the whole day and said that *Yesterday I was not truthful.*

Who was he? Was he drunk on the first day? Was he sober on the second day? **How many possible solutions** are there?

How to provide your answer:

- If you think that the man was John, he was drunk on the first day, and he was sober on the second day, submit JDS.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which has *strictly less* number of solutions than the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which has *strictly larger* number of solutions than your answer.

Problem 4.1.14 (HC-2022-SM2-R1-P14). (*Advanced Level*)

In a tennis tournament, there are 7 players. On a certain day during the tournament, Julie noticed that for any 4 players, at least 2 matches have been played amongst them.

At least how many matches have been played?

How to provide your answer:

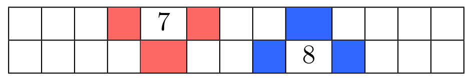
- If you think that the number of matches have been played is at least 1, then submit 1.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* that is *strictly smaller* than your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* that is *strictly larger* than your answer.

Problem 4.1.15 (HC-2022-SM2-R1-P15)**.** (*Advanced Level*)

In a move, a number in a 2×13 board can freely move from the square where it stands to an adjacent square (two squares are adjacent if they share one common border). *Below the number 7 can move to one of the red squares. The number 8 can move to one of the blue squares.*



What is the **minimal number** of moves you need to *reverse* the number $1, 2, \ldots, 13$ in ascending order shown in the Figure 4.6 into the descending order shown in the Figure 4.7?

1	2	3	4	5	6	7	8	9	10	11	12	13

Figure 4.6: Ascending

13	12	11	10	9	8	7	6	5	4	3	2	1

Figure 4.7: Descending

How to provide your answer:

- If you think 1 is the minimal number of move, submit 1. Together with your answer, draw a sequence of move depicting the solution and submit it. Your solution is used to determine whether your answer is correct.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which is *strictly smaller* than the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which is *strictly larger* than your answer.

4.2 Answers

Problem 1: See below

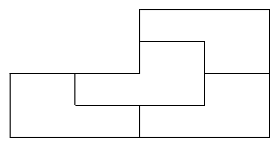
23	28	21
22	24	26
27	20	25

Problem 2: See the solution.

Problem 3: MS, KW.

Problem 4: 1.e3 d6 2.Qh5 Bg4 3.e4 Bd1 4.Q×d1.

Problem 5: See below



Problem 6: *Are you rich?*.

Problem 7: 0

Problem 8: 5, 5

Problem 9: $G(-2,-1),\ 6+16\sqrt{2}$

Problem 10: 17

Problem 11: 8

Problem 12: 32

Problem 13: $JDS,\ JSD$

Problem 14: 12

Problem 15: 108

4.3 Solutions

Solution. HC-2022-SM2-R1-P1 First, note two simple but important facts,

> **Claim —** By subtracting each cell by a same given number, a magic square can remain magical.

Proof. The statement is simple to prove since each sum of a row, a column, or a main diagonal is reduced by three times of that given number, therefore they are still the same. ■

> **Claim —** In a magic square constructed by the numbers $1, 2, \ldots, 9$, the number in the second row and the second column is always 5.

Proof. If a magic square is constructed by the numbers $1, 2, \ldots, 9$, then the sum of the numbers in a row, column, or main diagonal is a third of the sum of all the numbers from $1, 2, \ldots, 9$, which is

$$\frac{1}{3}(1+2+\ldots+9) = \frac{1}{3}\frac{9\cdot 10}{2} = 15.$$

Now, if we add the middle row and column plus the two diagonals, then we have a sum of all the numbers $1, 2, \ldots, 9$ plus three times the middle number (the number in the second row and the second column). Thus the middler number is

$$\frac{1}{3}(4\cdot 15 - (1+2+\ldots+9)) = 5.$$

■

Using the first fact, if we reduce each of the numbers $20, 21, \ldots, 28$ by 19, then we receive a magic square constructed with the numbers $1, 2, \ldots, 9$. Due to the second fact, the middle number shall be 5. With this information and knowing that the sum of a row, a column, or a main diagonal is 15, it is now easy to reconstruct the numbers, as shown in the sequence of diagrams below. Once the magic square with the numbers $1, 2, \ldots, 9$ is reconstructed, we then add back 19 to each number in order to receive a magic square with numbers $20, 21, \ldots, 28$. □

23		
		26
	20	

4		
	5	7
	1	

4	9	
3	5	7
	1	6

4	9	2
3	5	7
8	1	6

23	28	21
22	24	26
27	20	25

Solution. HC-2022-SM2-R1-P2

The desired figured can be obtained by removing 8 matchsticks inside the larger square, leaving four small squares on the outside of a large square, and one small square in it. □

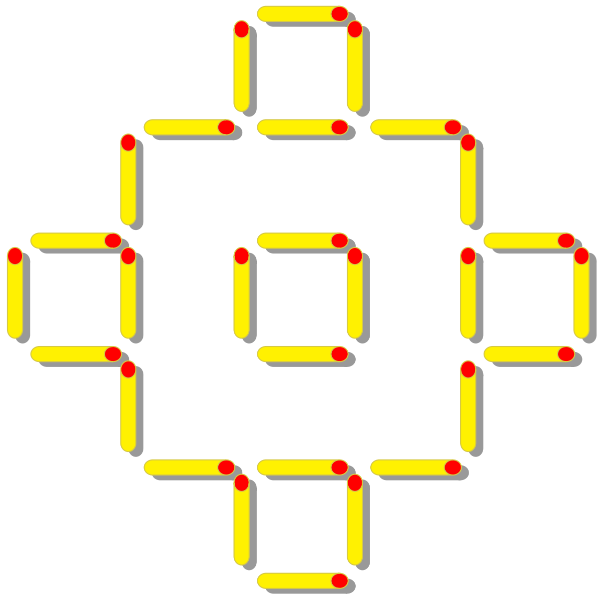


Figure 4.8: 8 matchsticks removed

Solution. HC-2022-SM2-R1-P3 Since one of them was asleep and the other was awake, and since they believed opposite things, **they must be of the same type.** If they were different types, their beliefs would be the opposite when they were both asleep or awake and would coincide when one was asleep and the other was awake.

If both of them were illusionists, then Minh was correct. Thus since he was an illusionist, he must be asleep at the time. Now, if both of them were paladins, then Minh was wrong. Since he was a paladin, he must also be asleep at the time.

Thus, Minh was asleep and Khoa was awake *in every case*. The answer is $\boxed{\text{MS, KW}}$. □

Solution. HC-2022-SM2-R1-P4

1. e3 d6

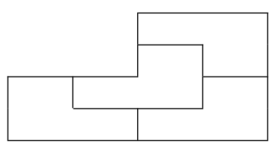
2. Qh5 Bg4

3. e4 Bd1

4. Qxd1

The answer is the $\boxed{7}$–move sequence $\boxed{\text{1.e3 d6 2.Qh5 Bg4 3.e4 Bd1 4.Q}\times\text{d1.}}$ □

Solution. HC-2022-SM2-R1-P5 The cuts can be determined as shown below. □



Solution. HC-2022-SM2-R1-P6 One of the solutions would be $\boxed{\textit{Are you rich?}}$ Note that An and Binh both answer it with *Yes*, while Chi and Danh would answer with *No*. So if the answer is *Yes*, then that brother is married, otherwise he is unmarried. □

Solution. HC-2022-SM2-R1-P7 With each right move the column number where the Tiger stands shall increased by one (assume we enumerate all columns from left to right) and each left-up move decreases the column number by one. The down move does not change the column number. Thus, the total numbers of moves to the right and to the left-up have to be the same.

Similar for the down and left-up numbers. Hence, the total number of all moves have to be divisible by 3. The total number of moves can only be 64, which is not divisible by 3. Thus $\boxed{\text{it is impossible.}}$ □

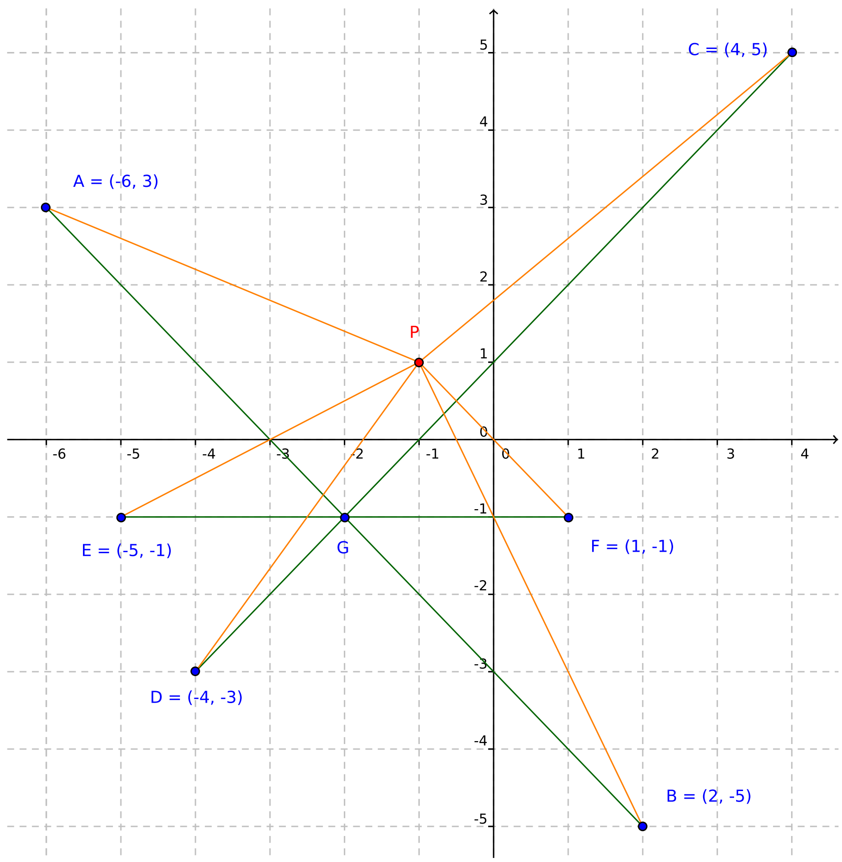
Solution. HC-2022-SM2-R1-P8 Since there are 14 cities with 6 of them connected to 4 other cities, 8 of them are connected to either 3 or 5 other cities. Let x be the number of cities connected to 3 other cities. Then $8 - x$ is the number of cities connected to 5 other cities.

The number of roads is 27, therefore by counting the number of roads going out from a city,

$$6 \cdot 4 + x \cdot 3 + (8 - x) \cdot 5 = 2 \cdot 27 \Rightarrow 64 - 2x = 54 \Rightarrow x = \boxed{5.}$$

□

Solution. HC-2022-SM2-R1-P9 Note that AB, CD, and EF meet at a point $\boxed{G(-2, -1).}$



Thus by triangle inequality,

$$\begin{cases} PA + PB \geq AB = GA + GB \\ PC + PD \geq CD = GC + GD \\ PE + PF \geq EF = GE + GF \end{cases} \Rightarrow PA + PB + PC + PD + PE + PF \geq AB + CD + EF = \boxed{6 + 16\sqrt{2}.}$$

□

Solution. HC-2022-SM2-R1-P10 Let the cost of the balls be $a \le b \le c \le d$. Then, in order to complete the sets, the students have to pay the following sums:

$$a+b+c \le a+b+d \le a+c+d \le b+c+d.$$

As given by the problem, these are four consecutive integers. Therefore the differences between any two consecutive sums is exactly 1. On the other hand, the differences between any two consecutive sums are:

$$\begin{aligned}&d-c,\ c-b,\ b-a \Rightarrow b=a+1,\ c=b+1,\ d=c+1 \Rightarrow b=a+1,\ c=a+2,\ d=a+3\\ &\Rightarrow a+b+c=3a+3,\ a+b+d=3a+4,\ a+c+d=3a+5,\ b+c+d=3a+6.\end{aligned}$$

Now, since one of the sums is 46, which has a remainder of 1 when divided by 3, it has to be $a+b+d=3a+4$, which is the only sum with the same remainder. Hence, $a=14$, and $d=14+3=\boxed{17.}$ □

Solution. HC-2022-SM2-R1-P11

Below is the mistakes are bolded.

This is an excerpt from the chapter Rừng cháy, Đất Rừng Phương Nam, by Đoàn Giỏi.

*Trong khói mù nhô ra một con heo đầu đàn, cao gần bằng con bò, lông gáy dựng ngược, mũi ngước lên thở phì phì làm **tr**o hai cái nanh dài chỗ khóe mép vươn ra như hai lưỡi dao găm. Rồi vun vút **ch**àn đến một bầy heo rừng, con lớn con bé tranh nhau chạy, sống lưng nhấp nhô **ch**àn tới như một đàn heo mục. Nai co giò phóng bay qua những lùm cây thấp. Hươu, **tr**ồn, bông lau, cáo, mèo ... tất cả những con thú bốn **tr**ân trong rừng đều nhắm mắt nhắm mũi tranh nhau chạy. Thỉnh thoảng một vài con gì không biết cứ chạy đâm bổ vào người chúng tôi.*

*Chân tôi đạp lên một khúc lưng con vật gì trơn trơn, lảo đảo chúi tới trước. Một con trăn gió uốn lưng trườn tới, đầu cất cao hơn ngọn sậy, ngoằn ngoèo lướt hút vào bụi cây trầm um tùm. Lâu lâu lại gặp một con rắn to phóng ngược hướng gió, **tr**ắn ngang đường chúng tôi. Tía con tôi phải chạy **ch**ánh chúng, cũng có lúc cứ mặc kệ, nhảy bừa qua, bất **tr**ấp cả những đầu phồng mang dẹt đét đang lắc lư phun nọc phì phì ... Khỉ, vượn, nhọ nồi cuống quít kêu lúc théc trên cây. Một con vượn bạc má bồng con nhảy xuống đất, cố chạy theo vết chúng tôi.*

There are, in total, $\boxed{8}$ mistakes. □

Solution. HC-2022-SM2-R1-P12 First, note that there are 64 squares on the board. If there are more than $\boxed{32}$ knights, then at least one of them will attack more than one other knight.

The diagram below shows a situation with 32 knights obeying the rule. We colour 16 knights black so that it is easy to see that a black knight attacks exactly one white knight. □

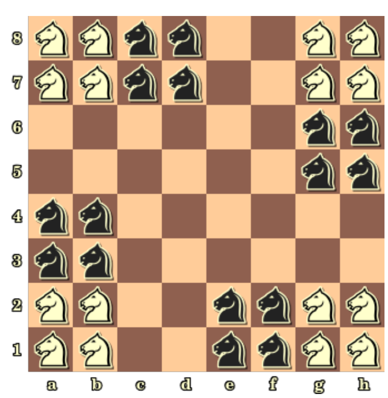


Figure 4.9: 32 knights

Solution. HC-2022-SM2-R1-P13 First, suppose that the man was truthful on the second day. Since he said that he was not truthful on the first day, what he said on the first day must be false. Since on the first day he said he was going to be drunk on the second day and that was false, he must be sober on the second day and telling the truth. So he must be John.

Now, if he lied on the second day, then he was actually truthful on the first day. This means that he was drunk on the second day and since he lied, he must be John.

Thus, there are two possible solutions $\boxed{\{JDS,\ JSD\}.}$ □

Solution. HC-2022-SM2-R1-P14 The total number of matches have been played, S, can be counted by: (i) summing up the number of matches that have been played among all groups of four players; (i) then divide that by the number of times a match can be counted multiple times.

For (i) it is given that the number of matches that have been played among a group of four players is at least 2, and there are $\binom{7}{4}$ ways to chose such a group. For (ii) each match is over counted by $\binom{2}{2} \cdot \binom{7-2}{2}$, the number of ways to choose 2 players who played that match, and another two players from the remaining players. Thus,

$$S \geq \frac{2 \cdot \binom{7}{4}}{\binom{2}{2} \cdot \binom{7-2}{2}} = \boxed{7.}$$

Note that this is a lower boundary, not the minimal. □

Solution. HC-2022-SM2-R1-P15 First, note that all but one of the numbers must move to the top row and then back to the bottom, because if there are two numbers that both remain in the bottom row, there is no way of them reversing their order. Thus, we must have $2 \cdot 12 = 24$ vertical movements.

Then, note that the total number of horizontal movements must be at least

$$12 + 10 + 8 + 6 + 4 + 2 + 0 + 2 + 4 + 6 + 8 + 10 + 12 = 84$$

Thus, we need at least $24 + 84 = \boxed{108}$ movements.

One solution with 108 movements is shown below.

First, lets move the number 13 to the leftmost position on the first row.

13												
1	2	3	4	5	6	7	8	9	10	11	12	

Then in similar way, move $12, 11, \ldots, 2$

13	12	11	10	9	8	7	6	5	4	3	2	
1												

Then move the number 1 to the rightmost position on the second row,

13	12	11	10	9	8	7	6	5	4	3	2	
												1

Now, move all the numbers down to the second row to finish the proof. □

Chapter 5

Math Individual Contest - Round 1

5.1 Rules

Mathematical Individual Contest is an activity of Math, Chess, and Coding Club (MCC). All students are invited to participate. The problems are designed so that students at various levels - junior (grade under 8), senior (grade 9 and up), or olympiad (prepare for national or international contest) - can participate. The contest problems are selected from algebra, combinatorics, number theory, and geometry topics. Some problems do not require specific mathematical knowledge, some require problem solving skills, and some require a more comprehensive understanding of certain mathematical topics. The students received a number of points after each participation and are ranked after each round, in both overall and grade rankings. The final standing of the contest is concluded at the end of each semester.

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Problem levels: all students must **try to solve all problems**, don't stay within your boundary
  - Junior: Problem 1, 2, and 3 (multiple-choice); Problem 7 (show-your-work).
  - Senior: Problem 4, 5 (multiple-choice); Problem 8, 9 (show-your-work).
  - Olympiad: Problem 6 (multiple-choice); Problem 10 (show-your-work).
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.

5.2 Problems

Problem 5.2.1 (MIC-2022-SM2-R1-J1)**.** (Junior, *10 points*)

The population of a town increases 25% during the year 2021. By what percent must it decrease the following year to return to the population it was at the beginning of 2021?

(A) 10% (B) 12.5% (C) 17.5% (D) 20% (E) 22.5%

Problem 5.2.2 (MIC-2022-SM2-R1-J2)**.** (Junior, *10 points*)

In $\triangle ABC$, $\angle A = 60°$. Points K, M, and N lie on AB, BC, and CA, respectively, such that $BK = KM = MN = NC$. Furthermore $AN = 2AK$.

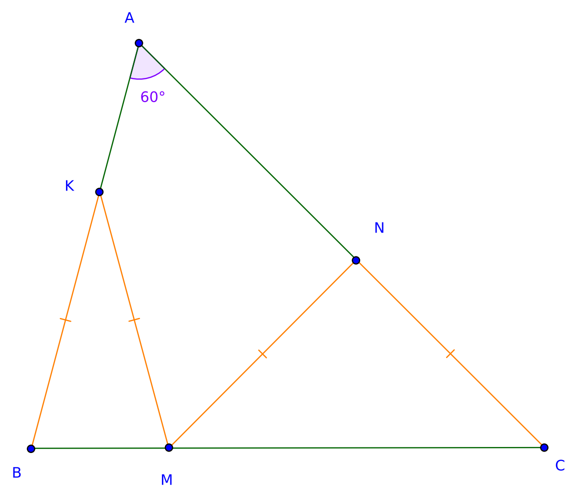


Figure 5.1: HC-2022-SM2-R1-P8

Find the value of $\dfrac{2\angle B - \angle C}{3}$.

(A) 35° (B) 47° (C) 51° (D) 63° (E) 72°

Problem 5.2.3 (MIC-2022-SM2-R1-J3)**.** (Junior, *10 points*)

Let

$$x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}.$$

Compute the value of $(x+1)^{48}$.

(A) $\dfrac{1}{25}$ (B) $\dfrac{1}{5}$ (C) 5 (D) 25 (E) 125

Problem 5.2.4 (MIC-2022-SM2-R1-S4)**.** (Senior, *10 points*)

20 teams played a football tournament. Every team played exactly one match with every other team. A team got 2 points for a win, 1 point for a draw, and 0 for a loss. The tournament ended with a single champion, which is the team that has earned the largest number of points.

What is the largest possible difference between the champion and the runner-up, who has the second most points?

(A) 21 (B) 20 (C) 19 (D) 18 (E) 17

Problem 5.2.5 (MIC-2022-SM2-R1-S5)**.** (Senior, *10 points*)

Steve has a bag with three marbles: one blue, one yellow, and one red. He repeatedly took out randomly one marble one at a time, wrote down its colour, and then put it back into the bag.

The probability that, after 10 drawings, he saw exactly two colours is $\frac{m}{n}$. Find the difference $n - m$.

(A) 1661 (B) 10661 (C) 18661 (D) 20661 (E) 23661

Problem 5.2.6 (MIC-2022-SM2-R1-O6)**.** (Olympiad, *10 points*)

The set A_3, consisting of 3 distinct positive integers is called **good** if the average of the numbers in any subset of A_3 is an integer.

For example, if $A_3 = \{1, 3, 5\}$*, then the average values of the single-element subsets* $\{1\}, \{3\}$*, and* $\{5\}$ *are* $1, 3,$ *and* 5*; the average values of the two-element subsets* $\{1, 3\}, \{3, 5\}$*, and* $\{5, 1\}$ *are* $2, 4,$ *and* 3*; and finally the average of the three-element subsets* $\{1, 3, 5\}$ *is* $\frac{1}{3}(1 + 3 + 5) = 3$*, which also is an integer.*

What is the highest possible value of k such that there is a good set A_k?

(A) 3 (B) 5 (C) 11 (D) 91 (E) there is no upper limit for k

Problem 5.2.7 (MIC-2022-SM2-R1-J7)**.** (Junior, *10 points*)

Note that $a^2 - b^2 = (a - b)(a + b),\ a^2 + 2ab + b^2 = (a + b)^2,\ a^2 - 2ab + b^2 = (a - b)^2.$

(*5 points*) Factor completely the expression $a^3 - ab^2$.

(*5 points*) Factor completely the expression $-a^2b^2 + 2ab^3 - b^4 + a^2c^2 - 2abc^2 + b^2c^2$.

Problem 5.2.8 (MIC-2022-SM2-R1-S8)**.** (Senior, *10 points*)

(*5 points*) Prove that for any $a, b,$ and c real numbers

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

(*5 points*) Now, for an arbitrary positive integer k, let

$$n = \sqrt[3]{k + \sqrt{k^2 - 1}} + \sqrt[3]{k - \sqrt{k^2 - 1}} + 1.$$

Compute $n^3 - 3n^2$ by applying the result above with $a = \sqrt[3]{k + \sqrt{k^2 - 1}}$, $b = \sqrt[3]{k - \sqrt{k^2 - 1}}$, and $c = 1 - n$.

Problem 5.2.9 (MIC-2022-SM2-R1-S9)**.** (Senior, *10 points*)

On a straight line there are 10 distinct points. Anna selects every two points of these 10 points, she then *marks* the midpoint of the segment connected them.

(*5 points*): Let A and B be the two points with the greatest distance between them. Prove that there are 8 *distinct marked* points inside a circle centred at A with radius $\frac{AB}{2}$.

(*5 points*): What is the minimal number of all distinct *marked* points?

In above, only the marked points - midpoints of the segments connected given points - are mattered. Don't consider the 10 *given points.*

Problem 5.2.10 (MIC-2022-SM2-R1-O10)**.** (Oympiad, *10 points*)

Let $a_1, a_2, \ldots, a_{2021}$, and a_{2022} are either -1 or 1 such that

$$a_1 \cdot 2^1 + a_2 \cdot 2^2 + \ldots + a_{2022} \cdot 2^{2022} = 2022.$$

(*5 points*) Prove that

$$(a_1 + 1) \cdot 2^1 + (a_2 + 1) \cdot 2^2 + \ldots (a_{2022} + 1) \cdot 2^{2022} = 2^{2023} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^2.$$

(*5 points*) Find the sum $a_1 + a_2 + \ldots + a_{2022}$.

5.3 Grading

Answers for multiple-choice problems.

Problem 1: D

Problem 2: A

Problem 3: E

Problem 4: B

Problem 5: C

Problem 6: E

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: Separately grading for each part,

(a) 2 points if the student uses $a^2 - b^2 = (a-b)(a+b)$.

(b) 2 points if the student recognizes both $a^2 \pm 2ab + b^2 = (a \pm b)^2$.

Problem 8: Separately grading for each part,

(a) 2 points if the student is able to expand both sides of the equation.

(b) 2 points if the student is able to elimiate the radicals ($\sqrt{}$).

Problem 9: Separately grading for each part,

(a) 2 points if the student is able to see that some distinct marked points are inside the circle centred at A radius $\frac{AB}{2}$.

(b) 2 points if the student is able to see that all distinct marked points are inside two circles radius $\frac{AB}{2}$ centred at A and B.

Problem 10: Separately grading for each part,

(a) 1 points if the student is able to see that $1 + 2 + 2^2 + \ldots 2^{2022} = 2^{2023} - 1$.

(b) 2 points the student recognizes binary (base-2) representation of the number.

5.4 Solutions

Solution. MIC-2022-SM2-R1-J1 One of the easiest way is to assume that there are 100 people in the town. If its population increased by 25% then the population at the end of the year 2021 is $100 + 100 \cdot 0.25 = 125$. Now since its population must be descreased by 25 to be the same at the beginning of 2021, thus it must decreased by $\frac{25}{125} = 0.2$, or $\boxed{20\%.}$

The number answer is $\boxed{D.}$ □

Solution. MIC-2022-SM2-R1-J2 Let P be the midpoint of AN. Since $\angle KAP = 60°$ and $AK = \frac{AN}{2} = AP$, thus $\triangle AKP$ is equilateral. $\triangle KPN$ is isosceles and $\angle KPA = 60°$, so $\angle PNK = 30°$.

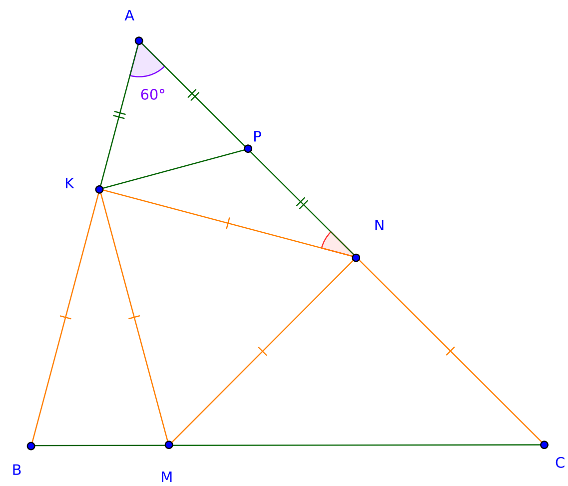


Figure 5.2: $\triangle KMN$ is equilateral

Since $KM = MN$, and in $\triangle KBM$, $\angle KMB = \angle ABC = 180° - (\angle A + \angle C) = 120° - \angle C$, so

$$\angle KMB = 120° - \angle NMC \Rightarrow \angle KMN = 180° - (\angle KMB + \angle NMC) = 60°.$$

Therefore, $\triangle KMN$ is equilateral. Thus $\angle KNM = 60°$, thus $\angle MNC = 90°$. Therefore $\angle C = 45°$ and $\angle B = 180° - 60° - 45° = 75°$. Hence, $\frac{2\angle B - \angle C}{3} = \frac{2(75°) - 45°}{3} = \boxed{35°.}$

The answer is $\boxed{A.}$ □

Solution. Looking at the factors in the denominator, it seems that we can simplify the expression by let $a = \sqrt[16]{5}$, then

$$x = \frac{4}{(a^8 + 1)(a^4 + 1)(a^2 + 1)(a + 1)}.$$

Note that $(a + 1)(a - 1) = a^2 - 1$, so

$$\begin{aligned} x &= \frac{4(a-1)}{(a^8 + 1)(a^4 + 1)(a^2 + 1)(a + 1)(a - 1)} = \frac{4(a-1)}{(a^8 + 1)(a^4 + 1)(a^2 + 1)(a^2 - 1)} \\ &= \frac{4(a-1)}{(a^8 + 1)(a^4 + 1)(a^4 - 1)} = \frac{4(a-1)}{(a^8 + 1)(a^8 - 1)} = \frac{4(a-1)}{(a^{16} - 1)} = \frac{4(a-1)}{4} = a - 1. \end{aligned}$$

Thus, $(x + 1)^{48} = \left(\sqrt[16]{5}\right)^{48} = 5^3 = \boxed{125.}$

The answer is $\boxed{E.}$ □

Solution. MIC-2022-SM2-R1-S4 Let assume that there are n team. The greatest difference in the scores of the champion and the second team(s) may be equal to n. This is achievable when the champion won all the matches, resulted in $2n-2$ points. All other teams drew and each got $n-2$ points. Thus, the difference is $(2n-2)-(n-2)=n$.

Why is it the possible greatest? Note that the maximal value for the champion in any case is $2n-2$. The minimal value for the runner-up is larger than or equal to the number of total collectible points for between all teams except the champion, which is $2\cdot\frac{(n-2)(n-1)}{2}=(n-1)(n-2)$, then divided by the number of all teams except the champion, which is $n-1$, or $\frac{(n-1)(n-2)}{n-1}=n-2$.

Thus, the answer for 20−team tournament is $\boxed{20.}$

The answer is $\boxed{B.}$ □

Solution. MIC-2022-SM2-R1-S5 First, there are $\binom{3}{2}=3$ ways to choose 2 colours from the given three. For any two colours, says A and B, a possible sequence of 10 drawings is

$$\underbrace{XX\ldots X}_{10},\text{ where } X \text{ is } A \text{ or } B \text{ and note that the case } \underbrace{AA\ldots A}_{10} \text{ or } \underbrace{BB\ldots B}_{10} \text{ would not happen.}$$

Thus, the number of favorable cases is $2^{10}-2=1022$. The desired probability is $\frac{3(1022)}{3^{10}}$, thus $n-m=19683-1022=\boxed{18661.}$

The answer is $\boxed{C.}$ □

Solution. MIC-2022-SM2-R1-O6 Consider the following set A_k,

$$\boxed{A_k=\{k!\cdot 1,\ k!\cdot 2,\ \ldots,\ k!\cdot k\}}$$

The average of the elements in an ℓ−element ($\ell\le k$) subset $S=\{k!\cdot i_1,\ k!\cdot i_2,\ \ldots,\ k!\cdot i_\ell\}$ is

$$\frac{k!\cdot i_1+k!\cdot i_2+\ldots+k!\cdot i_\ell}{\ell}=\frac{k!}{\ell}(i_1+i_2+\ldots+i_\ell),\text{ which obviously is an integer.}$$

It is easy to verify that this is a *good* set for $\boxed{\text{any value of } k\ge 5.}$

The answer is $\boxed{E.}$ □

Solution. MIC-2022-SM2-R1-J7

For the first question

$$a^3-ab^2=a(a^2-b^2)=\boxed{a(a-b)(a+b).}$$

For the second question

$$\begin{aligned}&-a^2b^2+2ab^3-b^4+a^2c^2-2abc^2+b^2c^2=-b^2(a^2-2ab+b^2)+c^2(a^2-2ab+b^2)\\&=(a^2-2ab+b^2)(c^2-b^2)=\boxed{(c-b)(c+b)(a-b)^2.}\end{aligned}$$

□

Solution. MIC-2022-SM2-R1-S8 The first question can easily be solved by expanded both side of the given equation.

For the second question,

$$n = \sqrt[3]{k+\sqrt{k^2-1}} + \sqrt[3]{k-\sqrt{k^2-1}} + 1 \Rightarrow \sqrt[3]{k+\sqrt{k^2-1}} + \sqrt[3]{k-\sqrt{k^2-1}} + (1-n) = 0$$

With $a = \sqrt[3]{k+\sqrt{k^2-1}}$, $b = \sqrt[3]{k-\sqrt{k^2-1}}$, and $c = 1-n$, then $a+b+c=0$, and by the given identity $a^3+b^3+c^3 = 3abc$, we have

$$\left(\sqrt[3]{k+\sqrt{k^2-1}}\right)^3 + \left(\sqrt[3]{k-\sqrt{k^2-1}}\right)^3 + (1-n)^3 = 3\left(\sqrt[3]{k+\sqrt{k^2-1}}\right)\left(\sqrt[3]{k-\sqrt{k^2-1}}\right)(1-n)$$

$$\Rightarrow (k+\sqrt{k^2-1}) + (k-\sqrt{k^2-1}) + (1-n)^3 = 3\left(\sqrt[3]{k^2-(\sqrt{k^2-1})^2}\right)(1-n)$$

$$\Rightarrow 2k+(1-n)^3 = 3(1-n) \Rightarrow 2k+1-3n+3n^2-n^3 = 3-3n \Rightarrow n^3-3n^2 = \boxed{2k-2.}$$

□

Solution. MIC-2022-SM2-R1-S9 Let A and B be the points with the greatest distance between them. Connect point A by segments with all other points except B. The midpoints of the obtained $10-2=8$ segments do not coincide, otherwise the second endpoints of the segments would coincide too, and are situated inside a circle centred at A with radius $\frac{AB}{2}$.

The diagram Figure 5.3 below shown an example with the given points colored red and the midpoints slightly moved out of their positions for better visualization.

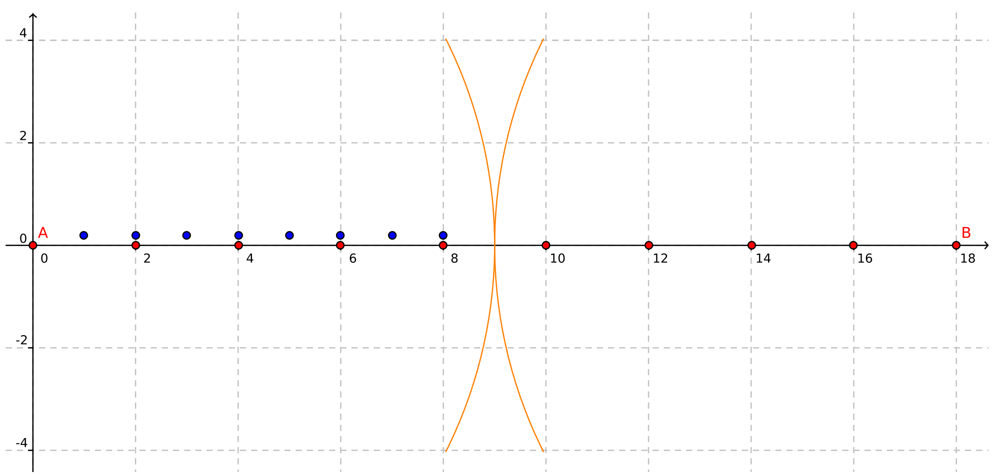


Figure 5.3: $\triangle KMN$ is equilateral

Similarly there are 8 distinct midpoints situated in a circle centred at B of radius with radius $\frac{AB}{2}$. The two circles have exactly one common point, which is the midpoint of the segment AB. Thus, together with the midpoint of AB, we have implicitly constructed $8+8+1 = \boxed{17}$ midpoints of segments.

The example above is to ensure that the minimal number is reached. All given points lie on the same straight line at $0, 2, 4, \ldots, 18$, so with a constant step between them. It is easy to see that there are exactly 17 distinct midpoints, and they are at

$$1, 2, 3, \ldots, 17.$$

□

Solution. MIC-2022-SM2-R1-O10 Since $a_1, a_2, \ldots, a_{2022} = \pm 1$, so $a_1 + 1, a_2 + 1, \ldots, a_{2022} + 1 = 0$ or 2, thus

$$(a_1 + 1) \cdot 2^1 + (a_2 + 1) \cdot 2^2 + \ldots (a_{2022} + 1) \cdot 2^{2022} = 2022 + (2^1 + 2^2 + \ldots + 2^{2022})$$
$$= 2022 + 2^{2023} - 2 = 2^{2023} + 2020 = 2^{2023} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^2$$

Thus, since $a_i + 1 = 0$ or 2, so both the sum on the left and the sum on the right are binary (base-2) representation of the same number. Thus their coefficients must match,

$$a_i + 1 = \begin{cases} 2, \text{ if } i = 1, 4, 6, 7, 8, 9, \text{or } 2022 \\ 0, \text{ otherwise} \end{cases} \Rightarrow a_i = \begin{cases} 1, \ i \in \{1, 4, 6, 7, 8, 9, 2022\} \\ -1, \text{ otherwise} \end{cases}$$

Thus $a_1 + a_2 + \ldots a_{2022} = -2015 + 7 = \boxed{-2008.}$ □

Chapter 6

Problem Solving Championship - Round 1

6.1 Rules

Problem Solving Championship is an activity of Math, Chess, and Coding Club (MCC) open to all students. The problems are designed so that students at various levels - junior (grade under 8), senior (grade 9 and up), or olympiad (prepare for national or international contest) - can participate. The contest problems are selected from algebra, combinatorics, number theory, geometry topics of mathematics, some are puzzles from chess, and some need to be solved by designing a computer program. Some problems do not require specific mathematical knowledge, some require problem solving skills, and some require a more comprehensive understanding of certain mathematical topics. The students received a number of points after each participation and are ranked after each round, in both overall and grade rankings. The final standing of the contest is concluded at the end of each semester.

- The contest problems become available online at the beginning of the semester or at the end of the previous contest. The contest solutions are discussed at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**. All students in the club are invited to the solution discussion. All contests start when the contest problems become available. The **solutions must be submitted latest on the last Sunday**, approximately one week **before the solution discussion day on Saturday**.
- There are 4 **show-you-work** problems with multiple steps. For each step of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too. If you solve the problem by designing a computer program, submit that program as the solution to the problem.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading: For each step there are a number of points, highlighted in the problem text, to be awarded. For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Invitation to present own solutions: Students who submitted correct solutions would be invited to present their solutions on the solution day. Students with nearly complete or almost correct solutions would also have a chance for presentation, provided that they studied the comments and suggestions from the COs to modify their solutions.

6.2 Problems

Problem 6.2.1 (PSC-2022-SM2-R1-P1)**.** (*25 points*)

(*10 points*) There are 15 chameleons in the forest of Hobbiton: 5 of them green, 2 blue, and 8 red. Whenever *two* chameleons of different colors meet, they both change to the third color (i.e., a green and blue would both become red). Is it possible for all chameleons to become one color?

(*15 points*) There are 102 chameleons in the Fangorn Forest: 19 of them green, 25 blue, 28 red, and 30 orange. Whenever *three* chameleons of different colors meet, they both change to the fourth color (i.e., a green, blue, red and would all become orange). Is it possible for all chameleons to become one color?

Problem 6.2.2 (PSC-2022-SM2-R1-P2)**.** (*25 points*)

a and b are positive real numbers such that,

$$\begin{cases} a^2 + b = 1 \\ ab + b^2 = 1. \end{cases}$$

In quadrilateral $ABCD$, $AB = DE = a$, $BD = b$, $AE = AD = BE = 1$.

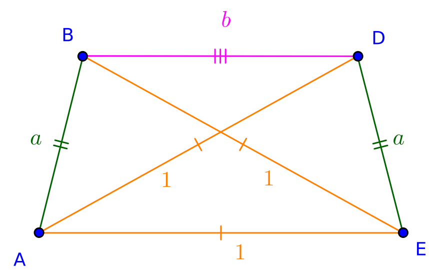


Figure 6.1: PSC-2022-SM2-R1-P2

(*10 points*) Prove that there is a circle Ω pass through four points A, B, D, and E.

(*5 points*) Let F be a point such that $AF = DF = b$, and $BF = 1$. Prove that F is also on the circle Ω.

(*10 points*) Find measures of the angles of the triangle $\triangle DEF$.

Problem 6.2.3 (PSC-2022-SM2-R1-P3)**.** (*25 points*)

Let a and b be positive real numbers such that

$$ab = a + b.$$

(*5 points*) Prove that $ab \geq 4$.

(*10 points*) Prove that $\sqrt{1 + a^2 + b^2 + a^2b^2} \geq 9 - ab$.

(*10 points*) Prove that $\sqrt{1 + a^2} + \sqrt{1 + b^2} \geq \sqrt{20 + (a - b)^2}$.

Problem 6.2.4 (PSC-2022-SM2-R1-P4). (*25 points*)

(p, q, r) is a triple of prime numbers such that

$$p^2 + 2q^2 + r^2 = 3pqr.$$

(*10 points*) Prove that at least one of p or r has to be 3.

(*15 points*) Find all such triples.

6.3 Grading

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 1: Separately grading for each part,

(a) 5 points if the student can guess if it is possible.

(b) 5 points if the student can guess if it is not possible.

Problem 2: Separately grading for each part,

(a) 5 points if the student can use the Ptolemy Theorem or similar triangles.

(b) 2 points if the student can continue with methods similar to the ones in the previous proof.

(c) 5 points if the student can discover the relations between some of the angles.

Problem 3: Separately grading for each part,

(a) 2 points if the student can use any of the well-known identity.

(b) 5 points if the student can apply the $ab = a + b$ relation.

(c) 5 points if the student can reuse the previous inequality.

Problem 4: Separately grading for each part,

(a) 5 points if the student can discover the difference of remainders on both sides of the equation.

(b) 5 points if the student can discover that q has to be even.

6.4 Solutions

Solution. PSC-2022-SM2-R1-P1 For the first question. First, one blue and one red chameleons meet. Both of them become green, the number of green chameleons now is $5+2=7$. Now, 7 green and 7 chameleons meet in pairs and become 14 blue chameleons. Thus, *all become* 15 *blue* chameleons.

For the second question. Note that *the parity of the number of chameleons in each colour is changed* after every time three different coloured chameleons meet. At the beginning there are two odd and two even numbers of chameleons in each colour. Thus, it does not matter how the chameleons meet, there are always two groups of chameleons with odd totals, and two other groups with even totals. Therefore, *it is not possible.* □

Solution. PSC-2022-SM2-R1-P2 For the first question, $ABDE$ is a cyclic quadrilateral, because by the Ptolemy Theorem,

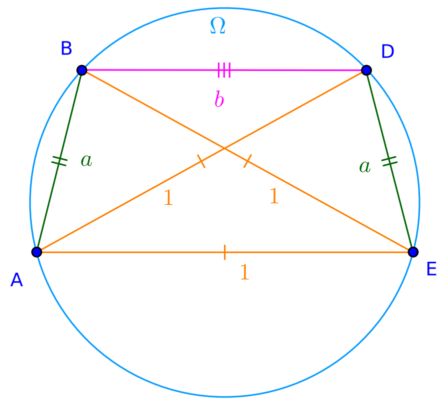


Figure 6.2: $ABDE$ is cyclic

$$AB \cdot DE + BD \cdot AE = a^2 + b = 1 = AD \cdot BE.$$

For the second question, similarly $ABDF$ is a cyclic quadrilateral, because by the Ptolemy Theorem,

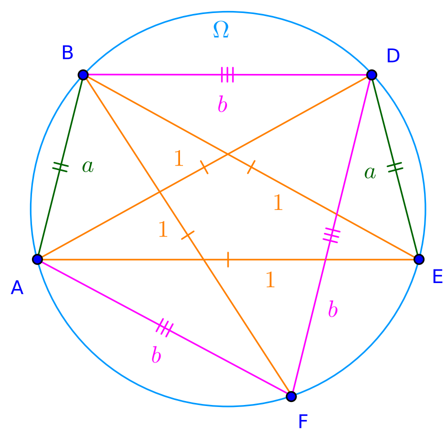


Figure 6.3: $ABDF$ is cyclic

$$AB \cdot DF + BD \cdot AF = ab + b^2 = 1 = AD \cdot BF.$$

This means that F is on the same circle as A, B, D, and E.

Furthermore, since $ABEF$ is cyclic, so according to the Ptolemy Theorem,

$$1 = AE \cdot BF = AF \cdot BE + AB \cdot EF = b + a \cdot EF \Rightarrow EF = \frac{1-b}{a} = a.$$

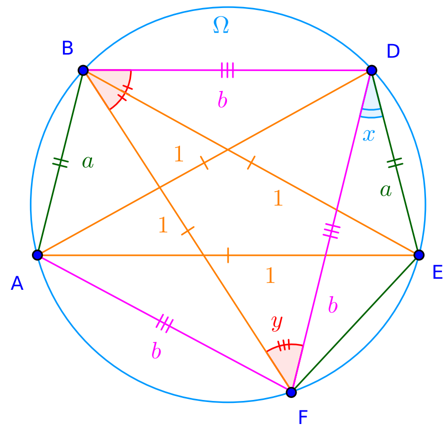


Figure 6.4: $EF = a$

Now, let $\angle EDF = \angle DFE = x$, $\angle DBF = \angle DFB = y$, since $\angle DBF$ subtends the minor arc $\widehat{DF} = \widehat{DE} + \widehat{EF}$, so $y = 2x$.

The angle $\angle DEF$ subtends the major arc $\widehat{DF} = \widehat{DB} + \widehat{BA} + \widehat{AF}$, thus $\angle DEF = y + x + y = 5x$.

Therefore in $\triangle DEF$, $x + x + 5x = 180°$, hence $x = \frac{180°}{7}$.

The angles of the $\triangle DEF$ are $\boxed{\frac{180°}{7}, \frac{180°}{7}, \text{ and } \frac{5 \cdot 180°}{7}.}$ □

Solution. PSC-2022-SM2-R1-P3

For the first question, by the AM-GM inequality,

$$ab = a + b \geq 2\sqrt{ab} \Rightarrow \sqrt{ab} \geq 2 \Rightarrow \boxed{ab \geq 4.}$$

For the second question,

$$(1 + a^2 + b^2 + a^2b^2) - (9 - ab)^2 = a^2 + b^2 + 18ab - 80 = (a + b)^2 + 16ab - 80$$
$$= (ab)^2 + 16ab - 80 = (ab - 4)(ab + 20) \geq 0 \Rightarrow \boxed{\sqrt{1 + a^2 + b^2 + a^2b^2} \geq 9 - ab.}$$

For the third question,

$$\left(\sqrt{1 + a^2} + \sqrt{1 + b^2}\right)^2 = 2 + a^2 + b^2 + 2(\sqrt{1 + a^2})(\sqrt{1 + b^2}) = 2 + a^2 + b^2 + 2\sqrt{1 + a^2 + b^2 + a^2b^2}$$
$$\geq 2 + a^2 + b^2 + 2(9 - ab) = 20 + (a - b)^2 \Rightarrow \boxed{\sqrt{1 + a^2} + \sqrt{1 + b^2} \geq \sqrt{20 + (a - b)^2}.}$$

□

Solution. PSC-2022-SM2-R1-P4

For the first question, note that if neither p nor r is divisible by 3, then their squares have remainder 1 when divided by 3, thus the left side of the equation

$$p^2 + 2q^2 + r^2 = 3pqr$$

has remainder of 2 (if $3 \mid q$), or 1 if $3 \nmid q$, thus it is not divisible by 3, while the right one does. Contradiction.

Thus, since the role of p and r in the equation is the same, so, WLOG, let $\boxed{r = 3.}$ Then

$$p^2 + 2q^2 + 9 = 9pq \Rightarrow p^2 + 2q^2 = 9(pq - 1).$$

If q is an odd prime, then $p^2 + 2q^2$ is odd if p is odd, while $9(pq - 1)$ is even; or $p^2 + 2q^2$ is even if p is even, while $9(pq - 1)$ is odd. Therefore q is even, thus $\boxed{q = 2.}$

Finally $p^2 + 8 = 9(2p - 1) \Rightarrow p^2 - 18p + 17 = 0 \Rightarrow (p - 1)(p - 17) = 0 \Rightarrow \boxed{p = 17.}$

Therefore the desired triples are $\boxed{(17, 2, 3) \text{ and } (3, 2, 17).}$ □

Chapter 7

House Championship - Round 2

7.1 Problems

Problem 7.1.1 (HC-2022-SM2-R3-P1)**.** (*Beginner Level*)

There are four cities in the Kingdom of Four Cities. The cities (green circles) are connected by six rivers (blue arcs), as shown below. There is no bridge over any river. For the people going from one city to another, traveling is simple by taking one or two boat trips. However, for the people living in the country side, visiting their friends and families, also in the country side, is a difficult matter.

Karl the Architect, is promised to get the hand of Hannad the Princess, if he can manage to build **a single road that crosses as many rivers as possible, but each river only once.**

The diagram below shows one of his plan, obviously not the best one.

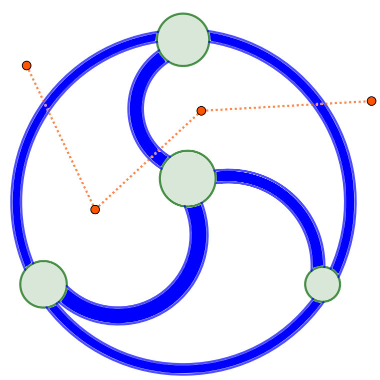


Figure 7.1: HC-2022-SM2-R3-P1

What is **the largest number of river-crossings** a single road can achieve?

How to provide your answer:

- If you think the best road can cross at most 1 river, create a diagram (or by any other means that can visualize your solution), and submit it.
- If you think that there is no solution, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* with *strictly smaller* number of crossings than the one in your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* with *strictly larger* number of crossings than the one in your answer.

Problem 7.1.2 (HC-2022-SM2-R3-P2)**.** (*Beginner Level*)

Can you **cut the shape of the letter** E on the left into at most 5 pieces and assemble them into the square on the right in the figure below?

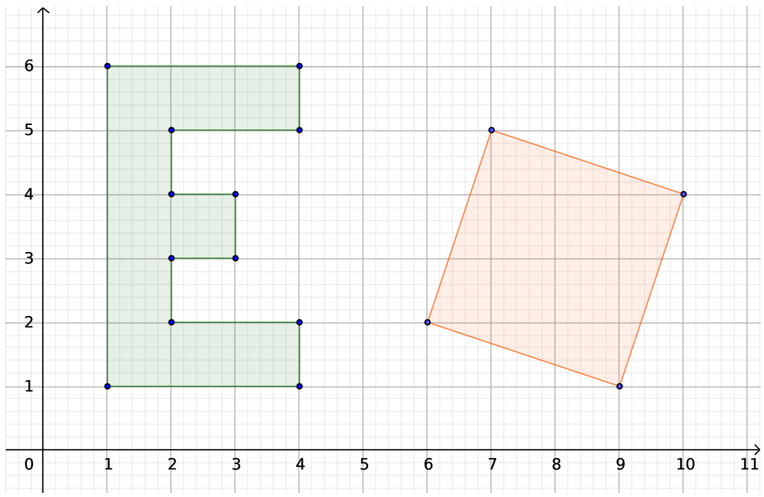


Figure 7.2: HC-2022-SM2-R3-P9

You are free to rotate or turn over any cut pieces.

How to provide your answer:

- If you think that is possible, create a diagram (or by any other means that can visualize your solution), and submit it.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.1.3 (HC-2022-SM2-R3-P3)**.** (*Beginner Level*)

What is **the longest sequence** of *consecutive positive integers* such that *none of them is divisible by any prime number in the form of* $3k+1$?

For example, the terms in the sequence

$$9 = 3^2, 10 = 2 \cdot 5, 11 = 11, 12 = 2^2 \cdot 3$$

are divisible by the primes

$$2 = 3 \cdot 0 + 2,\ 3 = 3 \cdot 1,\ 5 = 3 \cdot 1 + 2,\ 11 = 3 \cdot 3 + 2,$$

and none of them is in the form of $3k+1$.

How to provide your answer:

- If you think the sequence $9, 10, 11, 12$, is one of such sequences, then submit $9, 10, 11, 12$.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* with *strictly shorter* sequence than the one in your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* with *strictly longer* sequence than the one in your answer.

Problem 7.1.4 (HC-2022-SM2-R3-P4). (*Beginner Level*)

Cuc, Dao, Lan, and Mai were hanging around in the kitchen while their mother were making some cookies. When they disappeared, their mother found out that some of cookies did also vanish. In the impromptu court held by their father, the following were discovered:

1. If both Cuc and Dao stole some cookies, then Lan were an accomplice.
2. If Cuc is guilty, then at least one of Dao and Lan was an accomplice.
3. If Lan is a cookie-thief, then Mai was an accomplice.
4. If Cuc is innocent, then Mai is indeed a cookie-stealer.

Which ones of them are definitely guilty and which ones are doubtful (nothing can be said about them)?

How to provide your answer:

- If you think that Cuc and Dao are guilty, nothing can be said about Lan, or Mai, then submit $C1$ $D1$ $L0$ $M0$.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.1.5 (HC-2022-SM2-R3-P5)**.** (*Beginner Level*)

Lena has a $2 \times 2 \times 2$ cube. She divides each face of the cube into 4 identical squares. Thus she receives a total of 24 squares on the faces.

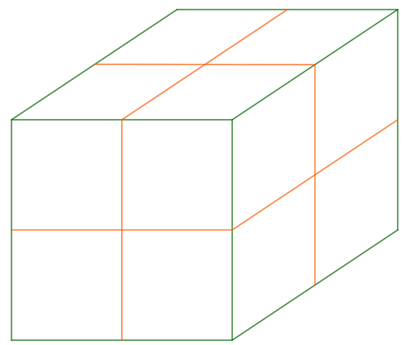


Figure 7.3: HC-2022-SM2-R3-P5

Lena paints the squares, each with one of the three colours, such that *any two squares with a common edge have different colours.* She then counts the number of squares in each of the colours. Let m and n be the maximum and minimum of these numbers.

What is the largest value of $m - n$?

How to provide your answer:

- If you think that 1 is such a number, submit 1.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which has *strictly smaller* than your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which has *strictly larger* than your answer.

Problem 7.1.6 (HC-2022-SM2-R3-P6)**.** (*Intermediate Level*)

How many weeks are there in the years from 2000 to 2399, including both years 2000 and 2399?

How to provide your answer:

- If you think that the sum is 1, submit 1.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which is *strictly farther* to the right answer than your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which has *strictly closer* to the right answer than your answer.

Problem 7.1.7 (HC-2022-SM2-R3-P7)**.** (*Intermediate Level*)

m, n, and p are positive integers. Henry had a $(m+1) \times (n+1) \times (p+1)$ rectangular prism. He painted the entire surface of his prism, then cut it into unit cubes. He found that the number of *unpainted cubes* is exactly $\frac{1}{2}mnp$.

What is the largest value of $m + n + p$?

How to provide your answer:

- If you think there 2 pourings are needed. Submit 2. Submit your solution to make sure you can win.
- If you cannot determine it, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which has *strictly smaller* than your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which has *strictly larger* than your answer.

Problem 7.1.8 (HC-2022-SM2-R3-P8). (*Intermediate Level*)

Samuel met three girls, Emily, Isabella, and Ophelia. The following were his thoughts:

1. I love at least one of the three girls.
2. If I love Emily but not Isabella, then I also love Ophelia.
3. I either love both Ophelia and Isabella or I love neither one.
4. If I love Ophelia, then I also love Emily.

Which of the girls do Samuel love? Which ones you are not sure about?

How to provide your answer:

- If you think Samuel loves Emily but you are not sure if he loves Isabella or Ophelia, submit $E1$ $I0$ $O0$.
- If you cannot determine the answer, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.1.9 (HC-2022-SM2-R3-P9)**.** (*Intermediate Level*)

Points P, Q, and R are chosen on the perimeter of the unit square $ABCD$ such that $\triangle PQR$ **is equilateral.**

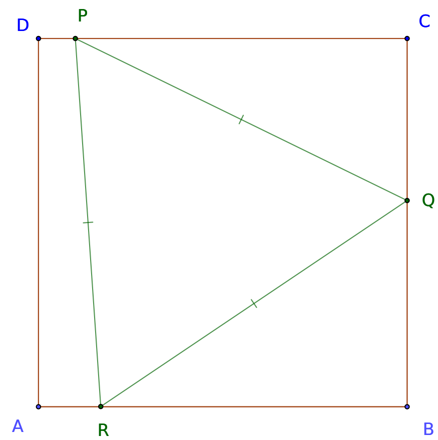


Figure 7.4: HC-2022-SM2-R3-P2

What is **the maximal length** of PQ?

How to provide your answer:

- If you think the length is 1, then submit 1.
- If you think that there is no solution, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* with *strictly smaller* than the one in your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* with *strictly larger* than the one in your answer.

Problem 7.1.10 (HC-2022-SM2-R3-P10). (*Intermediate Level*)

Four (green) cuts can dissect a circle into 11 parts as shown below in the diagram on the left .

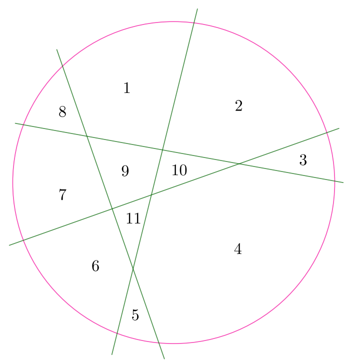


Figure 7.5: 4 cuts, 11 regions

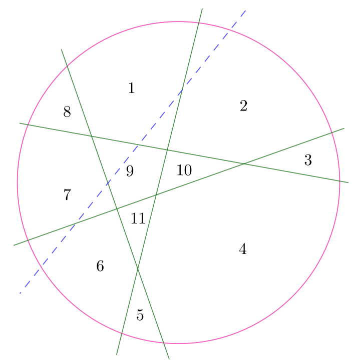


Figure 7.6: 5 cuts, $11 + 5 = 16$ regions

A (blue) cut is *divided* by four existing lines at 4 points into 3 segments and 2 half-lines. These divide 5 existing regions of the circle into 10 regions, thus increasing the number of regions by 5. See the diagram on the right above. The maximal number of regions of a circle that n cuts can produce is $\frac{n(n+1)}{2} + 1$.

What is the maximal number of regions of a crescent moon below that 4 cuts can produce?

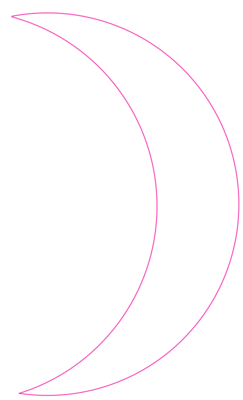


Figure 7.7: HC-2022-SM2-R3-P10

How to provide your answer:

- If you think that the maximum number of regions is 1, submit 1.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which is *strictly smaller* than the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which is *strictly larger* than your answer.

Problem 7.1.11 (HC-2022-SM2-R3-P11)**.** (*Advanced Level*)

There are many ways to partition a unit equilateral $\triangle ABC$ into two distinct sets of points. The figure below shows two different partitions. In each diagram the points of $\triangle ABC$ are divided into two sets: the set G of the green points and the set R containing the rest of the points. Note that each of the points on the perimeter belongs to one of G or R.

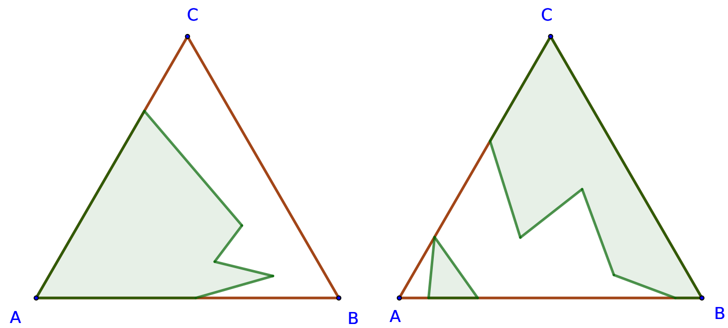


Figure 7.8: Two different partitions

However, it does not matter how the $\triangle ABC$ is partition, because out of the three vertices A, B, and C, two of them should belong to one of the sets, for example, let that be G. This means that in G there exist two points with a distance of 1. It is easy to see that 1 is the maximum value of such distance of any two points in G. This distance is called *the diameter* of G.

So we say, no matter how you *partition the unit equilateral triangle into two distinct sets of points,* **the diameter of one of the sets is at least** 1.

Now, you are given a unit square. **Find the minimal value** m so that no matter how you partition the square into two distinct sets of points, the diameter of one of the sets is at least m.

How to provide your answer:

- If you think that the value is 1, submit 1.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which is *strictly smaller* than the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which is *strictly larger* than your answer.

Problem 7.1.12 (HC-2022-SM2-R3-P12)**.** (*Advanced Level*)

In Transylvania, about half of the inhabitants are humans and another half are vampires. An ongoing infection made some of inhabitants insane. The rest are still sane. All inhabitants look pretty much alike. The only difference is the distinct behavior in belief and truth-telling:

1. sane humans make only true statements,
2. insane humans uncontrollably lie,
3. sane vampires always lie, and
4. insane vampires always tell the truth.

For example, if you ask the inhabitants whether the earth is round, a sane human knows the earth is round and truthfully say so, an insane human believes the earth is not round and says it is not round, a sane vampire knows the earth is round, but will then lie and say it isn't, and an insane vampire believes the earth is not round and then lies and say it is round.

Detective Benny goes to Transylvania. **What question** should he ask a Transylvanian to be answered with a *Yes* regardless of the type the inhabitant is?

How to provide your answer:

- If you think the question is *Do you believe the Earth is round?,* submit it.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.1.13 (HC-2022-SM2-R3-P13)**.** (*Advanced Level*)

The alphabet of the Binary Country has only two letters: 0 and 1. Every word in this langague has 4 letters, where no three consecutive letters can be the same. *For example* 0011 *is a valid word, but not* 0001.

The total 10 distinct words (why?) of this langague can be arranged into a 4×4 board, such that **different word in each of the four rows, four columns, and two diagonals.**

Can you show an example?

Hint: WLOG, you can start with 0 *in the top-left corner. If you reflect about the main diagonal (going from top-left to bottom-right), then the rows and columns swap, the words in those remain. Therefore the word in the other diagonal must be a palindrome. What letter would it start with?*

How to provide your answer:

- If you think that it is possible, submit a diagram.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which has *strictly smaller* number of correct types than the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which has *strictly larger* number of correct types than your answer.

Problem 7.1.14 (HC-2022-SM2-R3-P14)**.** (*Advanced Level*)

Four pairs of brother-sister share 32 cookies. For the sisters, Lan got one, Mai got two, Na got three, and Quynh got four cookies. Danny Tran took as many as his sister, Elvin Nguyen twice as many as his sister, Franklin Pham three times as his sister, and Williams Quach four times as many as his sister.

What are the family names of the girls?

How to provide your answer:

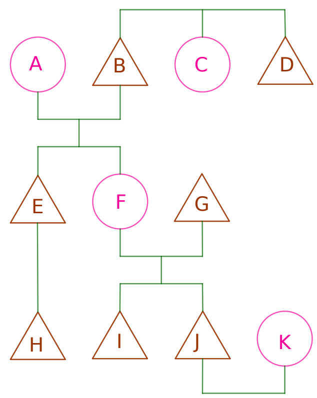
- If you think that Lan Tran, Mai Nguyen, Na Pham, and Quynh Quach are the name of the girls, submit Lan Tran, Mai Nguyen, Na Pham, and Quynh Quach.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 7.1.15 (HC-2022-SM2-R3-P15)**.** (*Advanced Level*)

In below you see an example of *family tree*. The circles denote female members and the triangles males.



A and B are married, as are F and G, and J and K. B, C, and D are siblings, as are E and F. E and F are children of A and B. Similarly, the parents of I and J are F and G. E is the father of H. In addition, A is the grandmother of H, I, and J, F is the aunt of H, and C is the sister-in-law of A.

Inspector Jade asked six children to briefly introduce their brothers, sisters, and first cousins (cousins who share a grandparent.) She had to match the name of the child to each numbered position in the family tree with the responses as below. Note that the relations given are in local languague. *Do not try to guess the genders of the children from the names. It might lead you to the wrong way.*

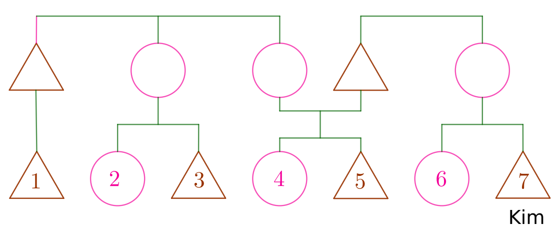


Figure 7.9: HC-2022-SM2-R3-P15

Response from Binh:
- I have three *arawa*: Kim, Minh, Thao
- I have two *surubu*: Oanh and Yen

Response from Kim:
- I have one *arawa*: Dinh
- I have one *surubu*: Binh

Response from Thao:
- I have two *surubu*: Yen and Binh
- I have two *arawa*: Minh and Dinh

Response from Dinh:
- I have two *surubu*: Oanh and Yen
- I have one *ere*: Binh

Response from Minh:
- I have one *ere*: Yen
- I have two *arawa*: Dinh and Thao

How to provide your answer:

- If you think that you have correct mappings for the names of the children and the numbered positions, submit your answer. You can list them for example $1-A, 2-B$, and so on or a detailed diagram if you like. Note that it is vital that *your answer has to be crystal clear.*
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or an which has *strictly smaller* number of correct mappings than your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit an answer which has *strictly larger* number of correct mappings than your answer.

7.2 Answers

Problem 1: 5.

Problem 2: See below for two solutions.

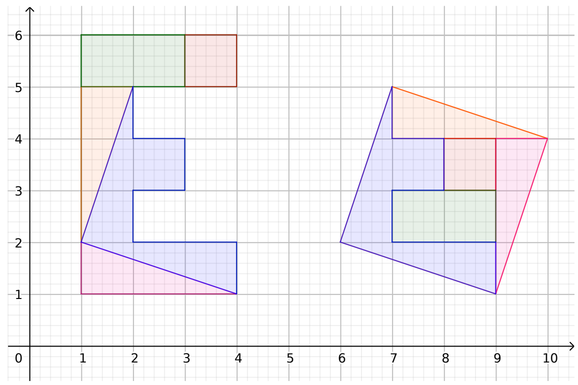


Figure 7.10: 5 pieces

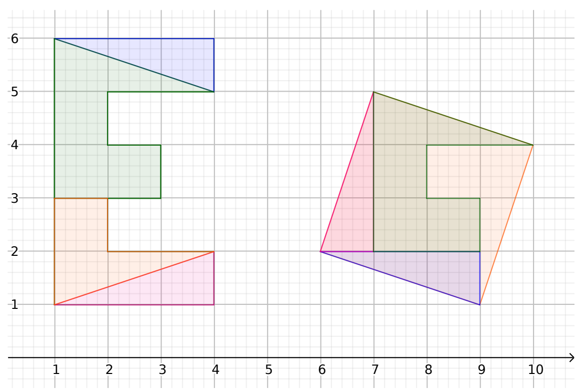


Figure 7.11: 4 pieces

Problem 3: 1, 2, 3, 4, 5, 6.

Problem 4: $C0$ $D0$ $L0$ $M1$.

Problem 5: 0.

Problem 6: 20871.

Problem 7: 24.

Problem 8: $E1$ $I1$ $O1$.

Problem 9: $\sqrt{6} - \sqrt{2}$.

Problem 10: 15.

Problem 11: $\frac{\sqrt{5}}{2}$.

Problem 12: *Do you believe you are a human?* or *Are you truthful?*

Problem 13: Below are all 4 solutions (there are no other solutions). The first two are reflective images of each other over the main diagonal. The last two are from the first two by swapping 0 with 1. *The numbers on the rim the board are decimal values of words as binary numbers. They can be used to verify the existance and arrangement of* 10 *distinct words.*

3	0	0	1	1	
13	1	1	0	1	
2	0	0	1	0	
10	1	0	1	0	
9	5	4	11	12	6

5	0	1	0	1	
4	0	1	0	0	
11	1	0	1	1	
12	1	1	0	0	
9	3	13	2	10	6

1	1	0	0
0	0	1	0
1	1	0	1
0	1	0	1

1	0	1	0
1	0	1	1
0	1	0	0
0	0	1	1

Problem 14: Na Tran, Quynh Nguyen, Lan Pham, and Mai Quach.

Problem 15: 1−Thao, 2−Yen, 3−Minh, 4−Binh, 5−Dinh, 6−Oanh, 7−Kim.

7.3 Solutions

Solution. HC-2022-SM2-R3-P1

The largest number of crossings is $\boxed{5.}$ Here is one solution:

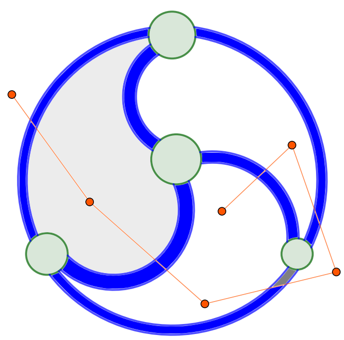


Figure 7.12: 5 crossings

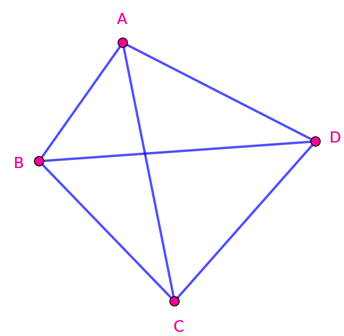


Figure 7.13: Equivalent problem

Note that each region, including the one outer one, has three rivers as borders. Let take one of them, for example the lightly shaded one. If Karl starts from that region he can cross all three rivers and then be outside of that region. However, if he starts outside of that region and crossed into it, then after crossing the other two rivers bordering with the region, he shall be inside the region.

Now, since there are 4 regions, he can start and end at at most two different regions. The river bordered with these two regions cannot be crossed. Therefore he can cross at most 5 rivers.

Another way to prove is to *transform* the graph of the problem. Let

- the regions be vertices, and
- the rivers be edges connecting pairs of vertices.

Then the *equivalent* problem is that given a graph containing 4 vertices, any two vertices are connected by an edge (we call a graph with 4 vertices and any two vertices are connected is *a* $K-4$, *or an complete graph with* 4 *vertices*), is there a *Hamilton path* which is *a way to travel all egdes of the graph, each exactly once.*

It is obvious to see that each vertex has three edges, so you can start and end at most two of them, and by doing so, there is a pair of vertices connected by an edge that is not travelled. Hence, $6-1=5$ is the maximal number of edges can be travelled. □

Solution. HC-2022-SM2-R3-P2

This is a solution cutting the E shape into 5 pieces, rotating some of them without turning over.

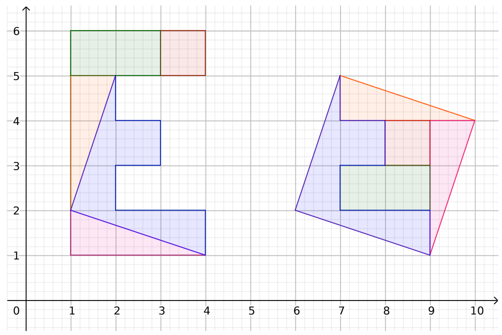


Figure 7.14: 5 pieces

This is another solution cutting the E shape into 4 pieces, rotating and turning some of them over. □

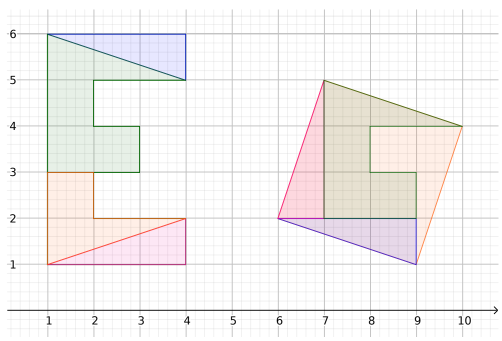


Figure 7.15: 4 pieces

Solution. HC-2022-SM2-R3-P3

First, note that every sequence of 7 consecutive integers is divisible by $7 = 3 \cdot 2 + 1$. Thus, the longest sequence can contain at most $\boxed{6}$ numbers. It is easy to see that the sequence below is one of such sequences

$$1, 2 = 2, 3 = 3, 4 = 2^2, 5 = 5, 6 = 2 \cdot 3$$

The answer is the $\boxed{1,\ 2,\ 3,\ 4,\ 5,\ 6.}$ □

Solution. HC-2022-SM2-R3-P4

First, lets show that if Cuc is guilty, then so is Lan. Suppose that Cuc is guilty, then by (2), either Dao or Lan is guilty. If Dao is innocent, then it must Lan who is guilty. But, suppose Dao is guilty, then both Cuc and Dao are guilty, hence by (1) Lan is guilty too. This proves that if Cuc is guilty, then so is Lan.

Now, by (3) if Lan is guilty, so is Mai. Combining this with the previous argument, if Cuc is guilty, so is Mai. Furtheremore, by (4) if Cuc is innocent, then Mai is guilty. Therefore Mai is guilty regardless of what Cuc is.

Therefore Mai is guilty and the rest are doubtful. The answer is $\boxed{C0\ D0\ L0\ M1.}$ □

Solution. HC-2022-SM2-R3-P5

First note that each of the cube's corner border with 3 squares and each square borders with one corner of the cube. This means that we can divide the squares into 8 groups, each group has exactly three squares sharing the same corner.

Each group requires three different colours for the squares, one for each, since every square shares an common edge with the other two. Therefore the number of squares in each colour is exactly 8.

Thus the difference between the number of squares in one colour to another is $\boxed{0.}$ □

Solution. HC-2022-SM2-R3-P6

First, note that in the Gregorian calendar,

- years that are not divisible by 4 are common years,
- years that are divisible by 4 but not by 100 are leap years,
- years that are divisible by 100 but not by 400 are common years,
- years that are divisible by 400 are leap years,
- a leap year contains 366 days and a common year 365 days.

Now, for any 400 consecutive years, we have 303 common years and 97 leap years. The total number of days is $303 \cdot 365 + 97 \cdot 366 = 146097$ days, which is exactly $\boxed{20871}$ weeks. □

Solution. HC-2022-SM2-R3-P7

First, the number of unpainted unit cubes is $(m-1)(n-1)(p-1)$. WLOG, let $m \le n \le p$,

$$2(m-1)(n-1)(p-1) = mnp \Rightarrow \left(\frac{m-1}{m}\right)\left(\frac{n-1}{n}\right)\left(\frac{p-1}{p}\right) = \frac{1}{2}.$$

Since $m \le n \le p$, so $\frac{m-1}{m} \le \frac{n-1}{n} \le \frac{p-1}{p}$, so

$$\text{ij } m \ge 3 \Rightarrow \left(\frac{m-1}{m}\right)\left(\frac{n-1}{n}\right)\left(\frac{p-1}{p}\right) \ge \left(\frac{3}{4}\right)^3 = \frac{27}{64} > \frac{1}{2},$$
$$\text{if } m = 2 \Rightarrow \frac{m-1}{m} = \frac{1}{2} \Rightarrow \left(\frac{n-1}{n}\right)\left(\frac{p-1}{p}\right) = 1, \text{ impossible.}$$

Thus $m = 3$,

$$\left(\frac{n-1}{n}\right)\left(\frac{p-1}{p}\right) = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}.$$

If $n \ge 8$, then

$$\left(\frac{n-1}{n}\right)\left(\frac{p-1}{p}\right) \ge \left(\frac{7}{8}\right)^2 = \frac{49}{64} > \frac{3}{4}.$$

It is easy to see that $n > 3$, Therefore $n \in \{4, 5, 6, 7\}$, $p \ge n$, and

$$n = 4 \Rightarrow \frac{3}{4} = \left(\frac{3}{4}\right)\left(\frac{p-1}{p}\right) \Rightarrow \text{ no solution.}$$
$$n = 5 \Rightarrow \frac{3}{4} = \left(\frac{4}{5}\right)\left(\frac{p-1}{p}\right) \Rightarrow p = 16$$
$$n = 6 \Rightarrow \frac{3}{4} = \left(\frac{5}{6}\right)\left(\frac{p-1}{p}\right) \Rightarrow \text{ no solution.}$$
$$n = 7 \Rightarrow \frac{3}{4} = \left(\frac{6}{7}\right)\left(\frac{p-1}{p}\right) \Rightarrow p = 8$$

There are two solutions $(3, 5, 16)$ and $(3, 7, 8)$. The solution $(3, 5, 16)$ maximizes the sum $m+n+p = \boxed{24.}$ □

Solution. HC-2022-SM2-R3-P8

Lets examine the given statements.

By (3), Samuel either loves both Ophelia and Isabella or he loves neither. Suppose that he loves neither. Then by (1) he must love Emily. Therefore he loves Emily, he doesn't love Isabella, and he doesn't love Ophelia. This is a contradiction of (2). Hence, Samuel loves both Ophelia and Isabella.

By (4), he loves Ophelia, so he loves Emily. Hence, he loves all the girls.

The answer is $\boxed{E1\ I1\ O1.}$ □

Solution. HC-2022-SM2-R3-P9

Let $DP = a$, $AR = b$ as shown in the diagram below.

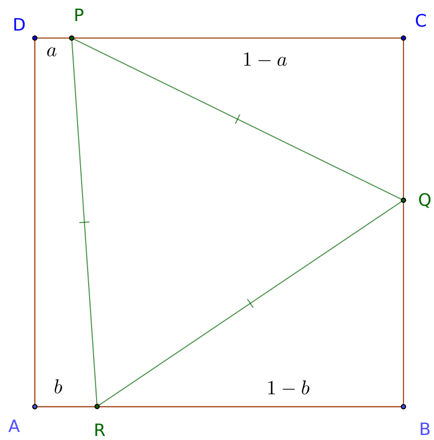


Figure 7.16: $DP = a$, $AR = b$

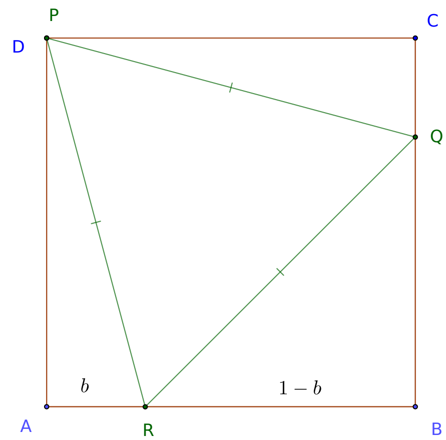


Figure 7.17: $a = 0$, PQ maximal

Assume that $a \le b$, by moving P towards D, while keeping the $\triangle PQR$ being equilateral, the length of PQ is increase and reach maximum value when point P and D are coincident, or $P \equiv D$.

Thus $a = 0$, by the Pythagorean theorem, and note that $0 < b < 1$,

$$PR^2 = AD^2 + AR^2 = 1 + b^2 \Rightarrow PQ^2 = QR^2 = 1 + b^2$$

$$CQ = AR = b,\ BR = BQ = 1 - b \Rightarrow 1 + b^2 = 2(1-b)^2 \Rightarrow b^2 - 4b + 1 = 0 \Rightarrow b = 2 - \sqrt{3}$$

$$PQ = \sqrt{1 + (2-\sqrt{3})^2} = \sqrt{8 - 4\sqrt{3}} = \sqrt{(\sqrt{6})^2 - 2\sqrt{6}\sqrt{2} + (\sqrt{2})^2} = \sqrt{(\sqrt{6}-\sqrt{2})^2} = \boxed{\sqrt{6}-\sqrt{2}.}$$

□

Solution. HC-2022-SM2-R3-P10

Note that the number of regions n cuts will produce for a circle is

$$2+2+3+\ldots+(n-1)+n=\frac{n(n+1)}{2}+1$$

Following the same reasoning, but the n^{th} cut in addition to the number of regions n added to the existing number of regions, it can cut the crescent again, adding one more region. Thus, the number of regions n cuts will produce for a crescent is

$$3+3+\ldots+n+(n+1)=\frac{(n+1)(n+2)}{2}.$$

With $n=4$, we have $\frac{5\cdot 6}{2}=\boxed{15.}$ The diagram below shows an example with alternate colouring for a better visualization.

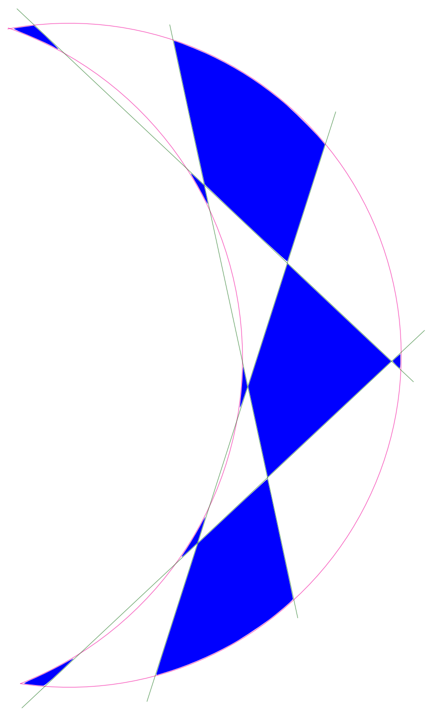


Figure 7.18: 15 regions

□

Solution. HC-2022-SM2-R3-P11

Let $ABCD$ be the unit square, E and F be the midpoints of CD and DA.

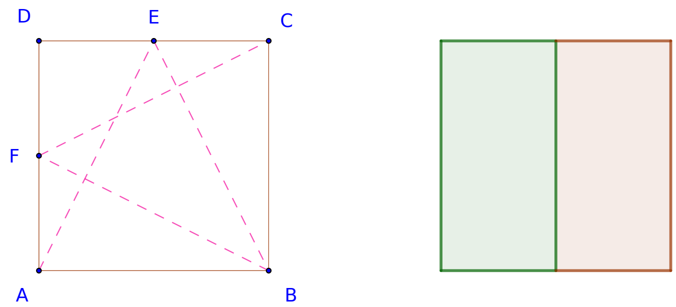


Figure 7.19: $A, B, C \in G$, $E, F \in R$

Let assume that $ABCD$ can be divided into two sets G and R such that *none of the sets has a diameter larger than or equal to* AE. WLOG, let $A \in G$, then $E \notin G$, thus $E \in R$, therefore $B \in G$, so $F \in R$, hence $C \in G$, but then the diameter of G is at least $AC > AE$. Contradiction.

Therefore one of the sets has a diameter larger than or equal to AE.

The diagram on the right of the figure above shows the case when such diameter is equal to $AE = \boxed{\dfrac{\sqrt{5}}{2}.}$ □

Solution. HC-2022-SM2-R3-P12

The question *Do you believe you are a human?* will have *Yes* as an answer.

1. sane humans make only true statements, so the answer must be *Yes.*
2. insane humans uncontrollably lie, he thought that he is not a human, so he lies, thus the answer must be *Yes.*
3. sane vampires always lie, he knew that he is not a human, so he lies, thus the answer must be *Yes.*
4. insane vampires always tell the truth, he thought that he is not a vampire, he tells the truth, hence the answer must be *Yes.*

Another cool solution is to ask *Are you truthful?*. It is interesting, but all of them will answer this question with a *Yes.* □

Solution. HC-2022-SM2-R3-P13

We can start with 0 in the top-left corner. If we reflect about the main diagonal (going from top-left to bottom-right), then the rows and columns swap, the words in those remain. Therefore the word in the other diagonal must be a palindrome and it should start with 0, otherwise we shall have 6 words started with 0 : first and last rows, first and last columns, and both diagonals. This contradicts the fact that out of 10 words, only 5 can start with 0 (the other 5 words start with 1.) So the off-diagonal shall be 1001.

Now, the words in the first row and the first column must be different. Since none of them can contain more than 2 same letters, so each of them should have a distinct permutation of $(0, 1)$. They are shown in the second board from the left. The second column and third row must not repeat the first column and first row, respectively. So they are forced to be as shown in the third board. The rest is simple.

0			1
		0	
	0		
1			

0	0	1	1
1		0	
0	0		
1			

0	0	1	1
1	1	0	
0	0	1	0
1	0		

0	0	1	1
1	1	0	1
0	0	1	0
1	0	1	0

By flipping the board over the main diagonal, or by switching 0 with 1, we receive another three solutions. All together there are 4 solutions as shown below. Note the numbers surrounding the board, they are decimal values of words as binary numbers. We can use them to verify the existance and arrangement of 10 distinct words. $\square$

3	0	0	1	1	
13	1	1	0	1	
2	0	0	1	0	
10	1	0	1	0	
9	5	4	11	12	6

5	0	1	0	1	
4	0	1	0	0	
11	1	0	1	1	
12	1	1	0	0	
9	3	13	2	10	6

1	1	0	0
0	0	1	0
1	1	0	1
0	1	0	1

1	0	1	0
1	0	1	1
0	1	0	0
0	0	1	1

Solution. HC-2022-SM2-R3-P14

Let $x, 2y, 3z$, and $4w$ be the numbers of cookies that Danny Tran, Elvin Nguyen, Franklin Pham, and Williams Quach took. Their sisters took x, y, z, and w cookies respectively. Therefore,

$$x + 2y + 3z + 4w + x + y + z + w = 32 \Rightarrow 2x + 3y + 4z + 5w = 32.$$

Note that x, y, z, w are a permutation of $1, 2, 3, 4$, so $x + y + z + w = 10$. Thus

$$\begin{aligned}&-x + z + 2w = 32 - 3(x + y + z + w) = 2 \Rightarrow x > z, x \text{ and } z \text{ have the same parity}\\&\Rightarrow -x + z = -2 \Rightarrow w = 2 \Rightarrow x = 3, z = 1, y = 4.\end{aligned}$$

Now, Lan got one, Mai got two, Na got three, and Quynh got four cookies.

$x = 3$, so the girl who took three cookies, Na, is the sister of Danny Tran, so she is Na Tran.

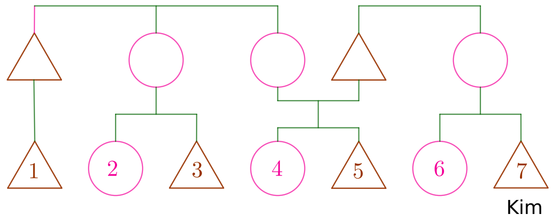
$y = 4$, so the girl who took four cookies, Quynh, is the sister of Elvin Nguyen, so she is Quynh Nguyen.

$z = 1$, so the girl who took one cookie, Lan, is the sister of Franklin Pham, so she is Lan Pham.

$w = 2$, so the girl who took two cookies, Mai, is the sister of Williams Quach, so she is Mai Quach.

Hence, the girls's names are $\boxed{\text{Na Tran, Quynh Nguyen, Lan Pham, and Mai Quach.}}$ $\square$

Solution. HC-2022-SM2-R3-P15



From what Binh said, Binh has the same type of relations to three children. Thus, those cannot be Binh's sisters or brothers, and *arawa* does not mean sister or brother. Only the child number 4 or 5 has three cousins in the same gender.

Look at the cousins of the children 4 or 5, *arawa* or *suburu* rather related to the gender of the cousin than the gender of the cousin's father or the mother. Note that Binh is a *suburu* to Kim and Kim is an *arawa* to Binh. Thus, Binh and Kim are of opposite genders, so Binh is a girl. Hence, $\boxed{\text{Binh is the girl number 4.}}$

Therefore, Kim, Minh, and Thao are the children $1, 3$, and 7, and *arawa* means *male cousin(s).* Furthermore, Binh is a *suburu* to Kim and Thao, in other words, she is a *female cousin* to them.

This means that Binh is not a *suburu* to the boy 5. Obviously, she is not an *arawa* to anyone. Since she is an *ere* to Dinh, Dinh must be her brother. Thus $\boxed{\text{Dinh is the boy number 5.}}$

Now, $\boxed{\text{Thao must be the boy number 1}}$ because he has two female and two male cousins. That leaves $\boxed{\text{Minh must be the boy number 3.}}$

Yen is a *ere* to Minh, so $\boxed{\text{Yen is the girl number 2.}}$ Finally, $\boxed{\text{Oanh is the girl number 6.}}$

The answer is 1−Thao, 2−Yen, 3−Minh, 4−Binh, 5−Dinh, 6−Oanh, 7−Kim. □

Chapter 8

Math Individual Contest - Round 2

8.1 Rules

Mathematical Individual Contest is an activity of Math, Chess, and Coding Club (MCC). All students are invited to participate. The problems are designed so that students at various levels - junior (grade under 8), senior (grade 9 and up), or olympiad (prepare for national or international contest) - can participate. The contest problems are selected from algebra, combinatorics, number theory, and geometry topics. Some problems do not require specific mathematical knowledge, some require problem solving skills, and some require a more comprehensive understanding of certain mathematical topics. The students received a number of points after each participation and are ranked after each round, in both overall and grade rankings. The final standing of the contest is concluded at the end of each semester.

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Problem levels: all students must **try to solve all problems**, don't stay within your boundary
  - Junior: Problem 1, 2, and 3 (multiple-choice); Problem 7 (show-your-work).
  - Senior: Problem 4, 5 (multiple-choice); Problem 8, 9 (show-your-work).
  - Olympiad: Problem 6 (multiple-choice); Problem 10 (show-your-work).
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.

8.2 Problems

Problem 8.2.1 (MIC-2022-SM2-R2-J1)**.** (Junior, *10 points*)

Five identical unit circles (radius 1) and a square are given as shown in the diagram below.

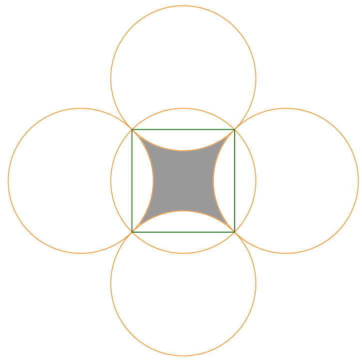


Figure 8.1: MIC-2022-SM2-R2-J2

Find the area of the shaded region.

(A) $\pi - 2$ $\quad$ (B) $4 - \pi$ $\quad$ (C) $2\pi - 4$ $\quad$ (D) $\pi + 2$ $\quad$ (E) $\pi + 4$

Problem 8.2.2 (MIC-2022-SM2-R2-J2)**.** (Junior, *10 points*)

In a convex polygon, any line extending one of its sides does not go through the inside of the polygon. The figure below shows the intersection of a convex pentagon and a convex quadrilateral, which is an octogon (a polygon with 8 sides.)

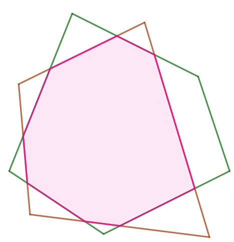


Figure 8.2: MIC-2022-SM2-R2-J3

At most how many sides can the intersection of a convex octogon with a convex decagon have? *A decagon is a polygon with* 10 *sides.*

(A) 14 $\quad$ (B) 15 $\quad$ (C) 16 $\quad$ (D) 17 $\quad$ (E) 18

Problem 8.2.3 (MIC-2022-SM2-R2-J3)**.** (Junior, *10 points*)

There are 12 boxes, each box containing one marble. All the boxes exactly the same. All the marbles look the same. In how many different ways can you colour the marbles of the 12 boxes by black or white?

(A) 0 $\quad$ (B) 11 $\quad$ (C) 12 $\quad$ (D) 13 $\quad$ (E) 14

Problem 8.2.4 (MIC-2022-SM2-R2-S4)**.** (Senior, *10 points*)

All the last four digits of a perfect square are equal to a digit d. How many such d are there?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 8.2.5 (MIC-2022-SM2-R2-S5)**.** (Senior, *10 points*)

Points A, B, and P are on the plane such that $AB = 10$, $AP = 2BP$, and P is not on the segment AB. Point Q is on the line through AB such that B is in between A and Q, and $AB = 3BQ$. Find PQ.

(A) $\frac{17}{3}$ (B) 6 (C) $\frac{20}{3}$ (D) $\frac{13}{2}$ (E) 7

Problem 8.2.6 (MIC-2022-SM2-R2-O6)**.** (Olympiad, *10 points*)

Let S be a set of all positive integers n not exceeding 30 such that

$$\text{if } a \text{ and } b \text{ are divisors of } n \text{ and } \gcd(a, b) = 1 \text{ then } a + b - 1 \text{ is also a divisor of } n.$$

For example, $3 \in S$, *since* $1 \mid 4,\ 3 \mid 3,\ 1 + 3 - 1 = 3 \mid 3$, $6 \notin S$ *because* $2 \mid 6,\ 3 \mid 6$, *but* $2 + 3 - 1 = 4 \nmid 6$.

How many elements are there in S?

(A) 18 (B) 16 (C) 14 (D) 12 (E) 10

Problem 8.2.7 (MIC-2022-SM2-R2-J7)**.** (Junior, *10 points*)

Professor Snape had no money. He went to the bank to cash in a cheque. An absent-minded bank teller switched the dollars and cents when he cashed the cheque and gave the professor twice the money what the cheque is worth for, plus 80 cents.

Note that "switched the dollars and cents" means that instead of $12.34 you have $34.12.

(*5 points*) Let d and c be the number of dollars and cents, resepectively, stated in the cheque. Establish an equation for them.

(*5 points*) Find the value of the cheque (in dollars and cents).

Problem 8.2.8 (MIC-2022-SM2-R2-S8)**.** (Senior, *10 points*)

(*2 points*) Find the complex roots of the polynomial $P(x) = x^2 + 4x + 5$.

(*3 points*) Let x_1, x_2, x_3, x_4 be the roots of the polynomial $P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$.

Is it known that $x_1 + x_2 = 4$. Find the sum $x_1x_2 + x_3x_4$.

(*5 points*) Find the roots x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$.

Problem 8.2.9 (MIC-2022-SM2-R2-S9)**.** (Senior, *10 points*)

At a party no boy danced with all the girls, but each girl dances with at least one boy. Prove that there are two pairs of girl-boy (g_1, b_1) and (g_2, b_2) who danced with each other but g_1 did not dance with b_2 and g_2 did not dance with b_1.

Problem 8.2.10 (MIC-2022-SM2-R2-O10)**.** (Oympiad, *10 points*)

(*5 points*): Let $y_0 = 1, y_1 = 1$, and for $n \geq 2$, $y_n = 4y_{n-1} - 4y_{n-2}$. Find a generic formula for y_n.

(*5 points*): Find the general term x_n of the sequence $x_0 = 3, x_1 = 4$, and for $n \geq 2$,

$$(n+1)(n+2)x_n = 4(n+1)(n+3)x_{n-1} - 4(n+2)(n+3)x_{n-1}.$$

8.3 Grading

Answers for multiple-choice problems.

Problem 1: B

Problem 2: C

Problem 3: D

Problem 4: B

Problem 5: C

Problem 6: B

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: Separately grading for each part,

(a) 2 points if the student can establish $100c + d$ as the mistaken value of the cheque.

(b) 2 points if the student can see that $0 \leq c \leq 99$.

Problem 8: Separately grading for each part,

(a) 1 points if the student can use classical method solving second degree equation for the complex case.

(b) 1 points if the student can use Viete theorem to calculate x_1x_2.

(c) 2 points if the student can find the values for x_1x_2 and x_3x_4.

Problem 9: 2 points if the student can examine a girl g_2, who the b_1 (the boy who danced with the maximum number of girls) did not dance with.

Problem 10: Separately grading for each part,

(a) 2 points if the student can use induction principle.

(b) 2 points if the student can lead the problem to the first question.

Note that,

- *if the student claimed to solve any of the problems by designing some programs, but did not submit the program, then the solution cannot be accepted, thus no point can be given;*
- *if the student submitted a program that produces the same outputs as the answers of the student, then the CO should grade the solution by inspecting the program;*
- *if the student submitted a program that produces some different outputs than the answers of the student, then the CO should ignore the program, and no point can be given.*

8.4 Solutions

Solution. MIC-2022-SM2-R2-J1 Since the radius of a circle is 1, the length of its diameter is 2. The diameter of a circle is the same as the diagonal of the square, thus the side length of the square is $\frac{2}{\sqrt{2}} = \sqrt{2}$. Hence the area of a square is $(\sqrt{2})^2 = 2$.

The sum of the areas of the four circular regions outside of the square and inside of the central circle is the difference of the areas of the circle and the square, or $\pi - 2$.

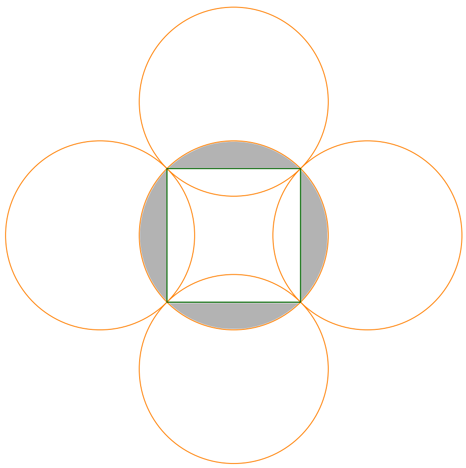


Figure 8.3: difference = cirle − square

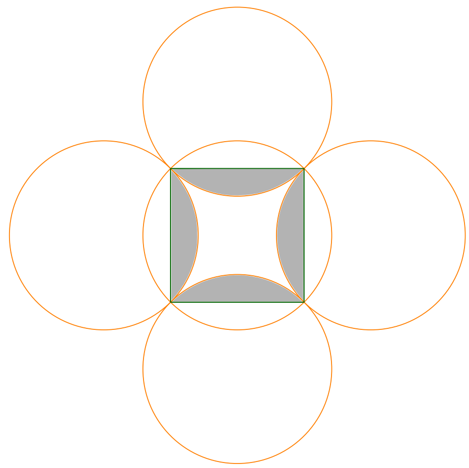


Figure 8.4: desired = square − difference

The four circular regions inside of the square are the same as the ones outside of the square. Thus the area of the shaded region is $2 - (\pi - 2) = \boxed{4 - \pi.}$ The answer is $\boxed{B.}$ □

Solution. MIC-2022-SM2-R2-J2 Consider the 8−sided polygon. Each of its sides can be intersected by at most 2 sides of the 10−sided polygon. It means that the intersection polygon would have at most *the number of sides of the* 8*−sided polygon* plus *the number of sides same as the number of vertices of the* 8*−sided polygon.* See the figure below for an example.

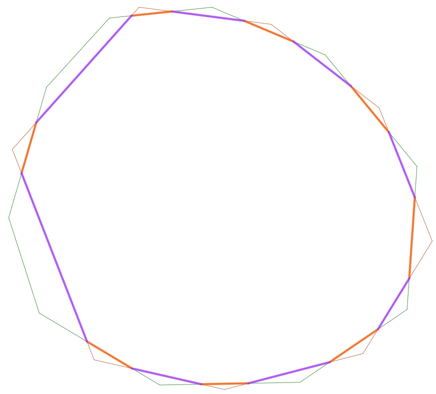


Figure 8.5: MIC-2022-SM2-R2-J3

The number of sides is $\boxed{16.}$. The answer is $\boxed{C.}$ □

Solution. MIC-2022-SM2-R2-J3 Since all the boxes look the same and all the marbles look the same, so the two colourings

$$B\underbrace{W\dots W}_{12} \text{ and } WB\underbrace{W\dots W}_{11}.$$

are the same, thus there is only one distinct colouring if we use a colour a single marble.

Now, it is easy to see that the number of colourings is the number of ways to choose different numbers of marbles from 12 marbles, which is $0, 1, \dots, 12$, or $\boxed{13.}$

The number answer is $\boxed{D.}$ □

Solution. MIC-2022-SM2-R2-S4 We solve the problem by investigating the remainder of n^2 when divided by a number of small integers. First, it is obvious that the last digit of a square can only be $0, 1, 4, 5, 6$, or 9. Let d be the last digit of n^2, then $n^2 \equiv d \cdot 1111 \pmod{10000 = 2^4 \cdot 5^4}$.

It is easy to verify the two claims below,

Claim — The remainder of a perfect square when divided by 8 is $0, 1$, or 4.

Claim — The remainder of a perfect square when divided by 16 is $0, 1, 4$, or 9.

By the first claim, $d \notin \{1, 5, 9\}$ since $1 \cdot 1111 \equiv 7 \pmod 8$, $5 \cdot 1111 \equiv 3 \pmod 8$, and $9 \cdot 1111 \equiv 7 \pmod 8$.

By the second claim $d = 4$ or $d = 6$ cannot be the case since $4 \cdot 1111 \equiv 12 \pmod{16}$, $6 \cdot 1111 \equiv 10 \pmod{16}$.

Therefore, $d = 0$ is the only solution. $100^2 = 10000$ is an example.

The answer is $\boxed{B.}$ □

Solution. [First solution] MIC-2022-SM2-R2-S5 Let $AP = 2b, BP = b, AB = 3a$, then $BQ = a$. Let $\alpha = \angle PBA$. By the Law of Cosines for $\triangle ABP$,

$$AP^2 = AB^2 + PB^2 - 2 \cdot AP \cdot BP \cdot \cos\alpha \Rightarrow 4b^2 = 9a^2 + b^2 - 6ab\cos\alpha \Rightarrow \cos\alpha = \frac{9a^2 - 3b^2}{6ab} = \frac{3a^2 - b^2}{2ab}.$$

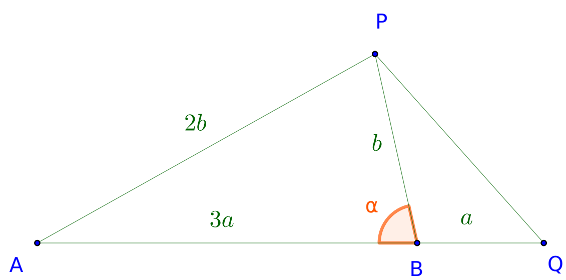


Figure 8.6: MIC-2022-SM2-R2-S5

Since $\angle PBQ = 180^\circ - \angle PBA$, so $\cos\angle PBQ = -\cos\alpha$, thus

$$PQ^2 = QB^2 + PB^2 - 2 \cdot QB \cdot PB \cdot \cos\angle PBQ = a^2 + b^2 + 2ab \cdot \frac{3a^2 - b^2}{2ab} = 4a^2 \Rightarrow PQ = 2a = 2 \cdot \frac{10}{3} = \boxed{\frac{20}{3}.}$$

The answer is $\boxed{C.}$ □

Solution. [Second solution] MIC-2022-SM2-R2-S5 In this solution, we show an application of the Stewart's Theorem,

Theorem (Stewart's Theorem)
Given $\triangle ABC$. D is a point on BC, then

$$AC^2 \cdot BD + AB^2 \cdot CD = BC(AD^2 + BD \cdot CD).$$

By Stewart's Theorem for $\triangle PAQ$ and point B on AQ,

$$PQ^2 \cdot AB + PA^2 \cdot QB = AQ(PB^2 + AB \cdot QB).$$

Let $AP = 2b, BP = b, AB = 3a$, then $BQ = a$, the above equality becomes,

$$PQ^2(3a) + 4b^2 a = 4a(b^2 + 3a^2) \Rightarrow PQ^2 = \frac{12a^2}{3a} = 4a \Rightarrow PQ = 2a = \boxed{\frac{20}{3}}.$$

The answer is $\boxed{C.}$ □

Solution. MIC-2022-SM2-R2-O6 We prove the two claims,

Claim — p is a prime and $k \geq 1$, then $n = p^k \in S$.

Proof. Two divisors of n that are relatively primes can only be 1 and p^ℓ, $(1 \leq \ell \leq k)$, $1 + p^\ell - 1 = p^\ell \mid n$. ■

Claim — If $n \in S$, then there exists p prime and $k \geq 1$, such that $n = p^k$.

Proof. Now, let assume that $n = p^k m$, where p *is the smallest prime dividing* n, where $k \geq 1$, and $p \nmid m$. Then $p + m - 1$ is a divisor of n. Now, *assume that* $q > p$ *is a prime factor of* m, then $m < p + m - 1 < m + q$, so $q \nmid p+m-1$. This means p is the only prime factor of $p+m-1$, there exists $\ell < k$, $p+m-1 = p^\ell \Rightarrow m = p^\ell - p + 1$.

Since $p^\ell \mid n$, so $p^\ell + m - 1 = 2p^\ell - p$ also a divisor of n. Therefore $2p^\ell - p = p(2p^{\ell-1} - 1) \mid p^k m$. Because $\gcd(2p^{\ell-1} - 1, p) = 1$, so $2p^{\ell-1} - 1$ has to be a divisor of m. However

$$p^\ell - p^{\ell-1} - \frac{p-1}{2} < p^\ell - p^{\ell-1} + 1 < p^\ell + p^{\ell-1} - \frac{p+1}{2}$$
$$\frac{p-1}{2}\left(2p^{\ell-1} - 1\right) < p^\ell - p^{\ell-1} + 1 < \frac{p+1}{2}\left(2p^{\ell-1} - 1\right).$$

The last inequality contradicts that m is a multiple of $2p^{\ell-1} - 1$. ■

Therefore only *the powers of primes can be in* S. For $n \leq 20$,

$$S = \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29\} \Rightarrow |S| = \boxed{16.}$$

The answer is $\boxed{B.}$ □

Solution. MIC-2022-SM2-R2-J7 For the first question,

$$100c + d = 2(100d + c) + 80 \Rightarrow \boxed{199d - 98c + 80 = 0.} \quad (*)$$

For the second question, first from (*) it is easy to see that

$$\begin{aligned}
&199d \le 98c - 80 \le 98 \cdot 99 - 80 \Rightarrow d \le 48 \\
&199d - 98c + 80 = 0 \Rightarrow 2 \mid d \\
&7 \mid 196d - 98c + 77 \Rightarrow 7 \mid 3d + 3 \Rightarrow d \equiv 6 \pmod 7
\end{aligned}$$

Thus, d is a positive interger not exceeding 48 and in a format of $14k+6$, Therefore, $d \in \{6, 20, 34\}$, so there are three possible cases:

$$\begin{cases} d = 6 \Rightarrow c = \frac{199d+80}{98} = 13. \\ d = 20 \Rightarrow \text{no solution.} \\ d = 34 \Rightarrow \text{no solution.} \end{cases}$$

Thus, $d = 6, c = 13$, the value of the cheque is $\boxed{\$6.13.}$ □

Solution. MIC-2022-SM2-R2-S8 The first question can easily be solved using the same method for quadratic.

Note that for $i = \sqrt{-1}$, the discriminant for the quadratic $x^2 + 4x + 5$

$$\Delta = 4^2 - 4 \cdot 5 = -4 \Rightarrow \sqrt{\Delta} = \sqrt{-4} = \sqrt{4i^2} = 2i \Rightarrow x_{1,2} = \frac{-4 \pm 2i}{2} = \boxed{-2 \pm i.}$$

For the second question, let x_1, x_2, x_3, x_4 be the roots of the polynomial

$$P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25.$$

Let assume that $x_1 + x_2 = 4$, then by Viete's theorem, the sum of the roots $x_1 + x_2 + x_3 + x_4$ is equal to the coefficient of x^3, thus

$$x_1 + x_2 + x_3 + x_4 = 6 \Rightarrow x_3 + x_4 = 2.$$

Furthermore the sum of the products of every two roots $x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$ is equal to the coefficient of x^2, thus

$$\begin{aligned}
&x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = 18 \Rightarrow x_1x_2 + x_3x_4 + (x_1 + x_2)(x_3 + x_4) = 18 \\
&x_1 + x_2 = 4,\ x_3 + x_4 = 2 \Rightarrow x_1x_2 + x_3x_4 = 18 - 2 \cdot 4 = \boxed{10.}
\end{aligned}$$

Now, let $u = x_1x_2$, $v = x_3x_4$, then from above and by Viete's theorem for the product of the four roots,

$$\begin{cases} u + v = x_1x_2 + x_3x_4 = 10 \\ uv = x_1x_2x_3x_4 = 25. \end{cases}$$

This means that u, v are the two roots of the quadratic

$$y^2 - 10y + 25 = 0 \Rightarrow (y - 5)^2 = 0 \Rightarrow u = v = 5.$$

Thus

$$\begin{aligned}
&x_1x_2 = 5, x_1 \text{ and } x_2 \text{ are roots of } x^2 - 4x + 5 \Rightarrow x_{1,2} = 2 \pm i \\
&x_3x_4 = 5, x_3 \text{ and } x_4 \text{ are roots of } x^2 - 2x + 5 \Rightarrow x_{3,4} = 1 \pm 2i
\end{aligned}$$

Hence, the four roots of the given polynomial are $\boxed{\{2 + i, 2 - i, 1 + 2i, 1 - 2i\}.}$ □

Solution. MIC-2022-SM2-R2-S9 Let b_1 be *the boy who danced with the maximum number of girls.* Then there is a girl g_2 who he did not danced with. For g_2 there is a boy b_2 that (g_2, b_2) danced together. Among the girls who danced with b_1 there is at least one g_1 who did not danced with b_2 (otherwise b_2 danced with g_2 and all the girls that b_1 danced with, meaning b_2 danced with more girls than b_1, contradicting with the choice of b_1.) □

Solution. MIC-2022-SM2-R2-O10 For the first question, note that

$$y_{n+1} = 4y_n - 4y_{n-1} \Rightarrow y_{n+1} - 2y_n = 2(y_n - 2y_{n-1}).$$

Therefore

$$y_{n+1} - 2y_n = 2(y_n - 2y_{n-1}) = \ldots = 2^n y_1 - 2y_0 = -2^n \Rightarrow y_{n+1} = 2y_n - 2^n$$
$$y_{n+1} = 2y_n - 2^n = 2(2y_{n-1} - 2^{n-1}) - 2^n = 2^2 y_{n-1} - 2 \cdot 2^n = 2^3 y_{n-2} - 3 \cdot 2^n = \ldots = 2^{n+1} y_0 - (n+1)2^n.$$

Thus $\boxed{y_n = 2^n - n2^{n-1}, \text{ for all } n \geq 0.}$

Remark. Alternatively we use the induction principle to prove that

Claim — $y_n = 2^n - n2^{n-1}$ for all $n \geq 0$.

Proof. For the base case, it is easy to verify that the hypothesis $y_n = 2^n - n2^{n-1}$ is true for $n = 0$ and $n = 1$. Let assume that $y_k = 2^k - k2^{k-1}$ for all $k \leq n$. We prove that

$$y_{n+1} = 2^{n+1} - (n+1)2^n.$$

By applying the recurrence relation,

$$y_{n+1} = 4y_n - 4y_{n-1} = 4(2^n - n2^{n-1}) - 4(2^{n-1} - (n-1)2^{n-2}) = 4 \cdot 2^{n-1} - 4 \cdot 2^{n-2}(2n - (n-1)) = 2^{n+1} - (n+1)2^n.$$

thus the hypothesis is true for $n + 1$, thus it is true for all n. ■

For the second question, by dividing both sides of the given equation by the product $(n+1)(n+2)(n+3,)$

$$\frac{x_n}{n+3} = 4 \cdot \frac{x_{n-1}}{n+2} - 4 \cdot \frac{x_{n-1}}{n+1}, \text{ for } n \geq 2.$$

By substitue that $y_n = \frac{x_n}{n+3}$, we receive the recurrence relation as above, thus $\boxed{x_n = (n+3)\left(2^n - n2^{n-1}\right).}$ □

Chapter 9

Introductory Curriculum Level Test - Level 1

9.1 Rules

9.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	Introduction to Algebra	Introduction to Geometry	Introduction to Number Theory	Introduction to Counting & Probability
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

9.3 Problems

Problem 9.3.1 (ICLT-2022-SM2-R1-L1-P1)**.** Evaluate

$$(15^{\frac{1}{2}})^{\frac{1}{2}} \times 5^{\frac{1}{3}} \times 3^{\frac{1}{2}}.$$

(A) $(15)^{\frac{3}{4}}$ (B) $3^{\frac{3}{4}} \times 5^{\frac{3}{4}}$ (C) $3^{\frac{3}{4}} \times 5^{\frac{7}{12}}$ (D) $3^{\frac{7}{12}} \times 5^{\frac{7}{12}}$ (E) $15^{\frac{1}{4}}$

Problem 9.3.2 (ICLT-2022-SM2-R1-L1-P2)**.** My team has 14 members. In how many ways can we choose a captain, a lieutenant, a secretary, and a technician, if no member can hold more than one position?

(A) 32760 (B) 24024 (C) 12012 (D) 4004 (E) 1001

Problem 9.3.3 (ICLT-2022-SM2-R1-L1-P3)**.** What is the maximum number of possible points of intersection of a circle and a triangle? (A triangle is formed by connecting three points with line segments.)

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 9.3.4 (ICLT-2022-SM2-R1-L1-P4)**.** Billy puts 24 marbles in b boxes such that each box contains the same number of marbles. If there are at least 2 boxes and each box has more than two marbles, how many possible values of b are there?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 9.3.5 (ICLT-2022-SM2-R1-L1-P5)**.** Factor $6a^4 + 36a^3 - 18a^2$ completely as you can.

(A) $a^2(3a^2 - 18a - 9)$
(B) $2a^2(3a^2 + 18a + 9)$
(C) $6a^2(a^2 + 6a - 3)$
(D) $3a^2(2a^2 - 12a + 6)$
(E) $6a^2(a^2 + 6a - 3)$

Problem 9.3.6 (ICLT-2022-SM2-R1-L1-P6)**.** The class has 7 students: 3 boys and 4 girls. In how many ways can the students seated in a row such that 4 girls seat next to each other?

(A) 144 (B) 288 (C) 384 (D) 576 (E) 720

Problem 9.3.7 (ICLT-2022-SM2-R1-L1-P7)**.** Solve

$$\frac{x}{x-1} + \frac{1}{3} = \frac{3}{x-1}.$$

Problem 9.3.8 (ICLT-2022-SM2-R1-L1-P8)**.** The first angle in a triangle is twice of one of the two other angles. The second angle is twice of of one of the two other angles. What are the possible measures of the smallest angle in the triangle?

Problem 9.3.9 (ICLT-2022-SM2-R1-L1-P9)**.** What is the second largest two-digit prime number whose digits are also each prime?

Problem 9.3.10 (ICLT-2022-SM2-R1-L1-P10)**.** Each of the grades 7 and 8 has 5 clubs. The school decides to hold a competition. A pair of clubs in each grade play each other twice. A pair of clubs in different grades play each other once. What is the total number of matches?

9.4 Grading

Answers for multiple-choice problems.

Problem 1: C

Problem 2: B

Problem 3: E

Problem 4: D

Problem 5: E

Problem 6: D

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can bring it to $\frac{1}{3} = \frac{3-x}{x-1}$.

Problem 8: 2 points if the student can see that there are two different cases of $2x, 2x, x$ and $4x, 2x, x$ angles.

Problem 9: 2 points if the student can find the largest such prime 73.

Problem 10: 2 points if the student can devise a counting method of both within and outside of the grades.

9.5 Solutions

Solution. ICLT-2022-SM2-R1-L1-P1 *Exercise 1.7.5.c, Chapter 1, Introductory to Algebra.*

$$15^{\frac{1}{2}} = (3 \times 5)^{\frac{1}{2}} = 3^{\frac{1}{2}} \times 5^{\frac{1}{2}} \Rightarrow (15^{\frac{1}{2}})^{\frac{1}{2}} = (3^{\frac{1}{2}})^{\frac{1}{2}} \times (5^{\frac{1}{2}})^{\frac{1}{2}} = 3^{\frac{1}{4}} \times 5^{\frac{1}{4}}$$
$$\Rightarrow (15^{\frac{1}{2}})^{\frac{1}{2}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{2}} = 3^{\frac{1}{4}} \times 5^{\frac{1}{4}} \times 5^{\frac{1}{3}} \times 3^{\frac{1}{2}} = 3^{\left(\frac{1}{4}+\frac{1}{2}\right)} \times 5^{\left(\frac{1}{4}+\frac{1}{3}\right)} = \boxed{3^{\frac{3}{4}} \times 5^{\frac{7}{12}}.}$$

The number answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R1-L1-P2 *Exercise 1.4.8, Chapter 1, Introductory to Counting & Probability.*

There are 14 choices for captain, 13 choices for lieutenant, 12 choices for secretary, and 11 choices for treasurer, for a total of $14 \times 13 \times 12 \times 11 = \boxed{24024}$ different choices. The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R1-L1-P3 *Exercise 1.3.3, Chapter 1, Introductory to Geometry.*

A line segment can intersect a circle in at most two points. Since a triangle consists of three line segments, the maximum number of intersections between a triangle and a circle is $3 \times 2 = \boxed{6.}$ The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R1-L1-P4 *Exercise 1.6.1, Chapter 1, Introductory to Number Theory.*

The number of marbles in each box is $\frac{24}{b}$. Thus, b is a divisor of 24. We know that b is at least 2 and $\frac{24}{b}$ is at least 3, thus b can only be $2, 3, 4, 6$, and 8. Hence, there are $\boxed{5}$ possible values for b. The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R1-L1-P5 *Problem 2.10.d, Chapter 2, Introductory to Algebra.*

Note that each terms $6a^4, 36a^3$, and $18a^2$ of the expression contains a factor $6a^2$, thus

$$6a^4 + 36a^3 - 18a^2 = (6a^2)(a^2) + (6a^2)(6a) - (6a^2)(3) = \boxed{6a^2(a^2 + 6a - 3).}$$

The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R1-L1-P6 *Problem 2.12, Chapter 2, Introductory to Counting & Probability.*

We want to deal with the restriction first. Without worrying about which specific boys and girls go in which seats, in how many ways can the girls sit together? There are 4 basic configurations of boys and girls, note that B is a boy and G is a girl,

$$GGGGBBB, BGGGGBB, BBGGGGB, BBBGGGG$$

Then, within each configuration, there are $3! = 6$ ways in which we can assign the 3 boys to seats, and $4! = 24$ ways in which we can assign the 4 girls to seats. Therefore the number of possible seatings is $4 \times 6 \times 24 = \boxed{576}$. The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R1-L1-P7 *Problem 3.18.c, Chapter 3, Introductory to Algebra.*

$$\frac{x}{x-1} + \frac{1}{3} = \frac{3}{x-1} \Rightarrow \frac{1}{3} = \frac{3}{x-1} - \frac{x}{x-1} = \frac{3-x}{x-1} \Rightarrow x - 1 = 3(3-x) = 9 - 3x \Rightarrow 4x = 10 \Rightarrow x = \boxed{\frac{5}{2}.}$$

□

Solution. ICLT-2022-SM2-R1-L1-P8 *Problem 2.18, Chapter 2, Introductory to Geometry.*

There are two possible cases.

Case 1: The first and the second angle are both twice the third angle. Let the measure of the smallest angle, the third angle, be x,

$$2x + 2x + x = 180^\circ \Rightarrow x = \frac{180^\circ}{5}.$$

Case 2: One of the first two angles is twice the other angle of the first two angles, the smaller of of the first two angles is is twice the third angle. Let the measure of the smallest angle, the third angle, be x,

$$4x + 2x + x = 180^\circ \Rightarrow x = \frac{180^\circ}{7}.$$

□

Solution. ICLT-2022-SM2-R1-L1-P9 *Problem 2.11, Chapter 2, Introductory to Number Theory.*

A two-digit number whose digits are prime is made up of the digits $2, 3, 5$, or 7. We check the possible numbers from the largest in this list and work backwards until we find the second largest prime number:

- 77 is divisible by 7.
- 75 is divisible by 3.
- 73 is prime, so it is the largest two-digit prime number whose digits are both prime.
- 57 is divisible by 3.
- 55 is divisible by 5.
- $\boxed{53}$ is prime, so it is the second largest two-digit prime number whose digits are both prime.

□

Solution. [First solution] ICLT-2022-SM2-R1-L1-P10 *Exercise 3.3.3, Chapter 3, Introductory to Counting & Probability.*

We call a match *intra-grade* if it is played between the two clubs from the same grade, and *cross-grade* if it is played between the two clubs from different grades.

It is easy to see that the total number of intra-grade matches is the number of intra-grade matches for grade 7 plus the number of intra-grade matches for grade 8. A pairs of clubs in each grade play each other twice, so the number of intra-grade matches for grade 7 is $2 \cdot \frac{5 \cdot 4}{2} = 20$. Grade 8 has the same number of clubs, so the number of intra-grade matches for grade 8 is also 20. Therefore there are 40 *intra-grade* matches.

For a club in grade 7 there are 5 clubs in grade 8, so the number of *cross-grade* matches (played by all the clubs in grade 7 with all the clubs in grade 8) is $5 \cdot 5 = 25$.

Thus, the total number of matches is $40 + 25 = \boxed{65.}$ □

Solution. [Second solution] ICLT-2022-SM2-R1-L1-P10 The other way to count is to look at each club. Each club plays twice a club in the same grade and once a club in different grade, so each club plays $(5-1) \cdot 2 + 5 = 13$. There are $5 + 5 = 10$ clubs, so there are $13 \cdot 10 = 130$ matches. However each matches counted this way is counted twice (we count both $A - play - B$ and $B - play - A$), hence the total number of matches is $\frac{130}{2} = \boxed{65.}$ □

Chapter 10

Introductory Curriculum Level Test - Level 2

10.1 Rules

10.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	**Introduction to Algebra**	**Introduction to Geometry**	**Introduction to Number Theory**	**Introduction to Counting & Probability**
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

10.3 Problems

Problem 10.3.1 (ICLT-2022-SM2-R1-L2-P1)**.** The first angle in a triangle is twice of one of the two other angles. The second angle is twice of of one of the two other angles. What would be the smallest value of the smallest angle in the triangle?

(A) $15°$ (B) $35°$ (C) $\frac{180°}{5}$ (D) $37.5° \times 5^{\frac{7}{12}}$ (E) $\frac{180°}{7}$

Problem 10.3.2 (ICLT-2022-SM2-R1-L2-P2)**.** Each of the grades 7 and 8 has 5 clubs. The school decides to hold a competition. A pair of clubs in each grade play each other twice. A pair of clubs in different grades play each other once. What is the total number of matches?

(A) 133 (B) 130 (C) 105 (D) 65 (E) 45

Problem 10.3.3 (ICLT-2022-SM2-R1-L2-P3)**.** Simplify as much as possible the following product

$$\frac{2}{3a^2-12b} \cdot \frac{9a^3-36ab}{8a^2}$$

(A) $\frac{3}{5a}$ (B) $\frac{3}{4a}$ (C) $\frac{3a^2-12b}{4a}$ (D) $\frac{2}{3a}$ (E) $\frac{1}{3(3a^2-12b)}$

Problem 10.3.4 (ICLT-2022-SM2-R1-L2-P4)**.** Which pair of triangles must be congruent?

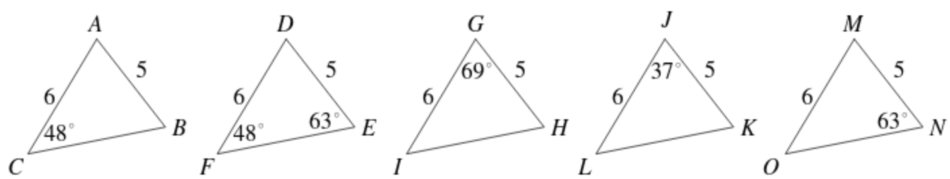


Figure 10.1: ICLT-2022-SM2-R1-L2-P4

(A) $(\triangle ABC, \triangle GHI)$
(B) $(\triangle DEF, \triangle JKL)$
(C) $(\triangle GHI, \triangle DEF)$
(D) $(\triangle MNO, \triangle ABC)$
(E) $(\triangle JKL, \triangle MNO)$

Problem 10.3.5 (ICLT-2022-SM2-R1-L2-P5)**.** Compute the least common multiple of $24, 28$, and 36.

(A) 504 (B) 252 (C) 168 (D) 126 (E) 72

Problem 10.3.6 (ICLT-2022-SM2-R1-L2-P6)**.** Which one is the largest?

(A) $\binom{44}{1}$ (B) $\binom{6}{3}$ (C) $\binom{10}{2}$ (D) $\binom{7}{4}$ (E) $\binom{9}{7}$

Problem 10.3.7 (ICLT-2022-SM2-R1-L2-P7)**.** There are red and blue balls in the Christmas shop. A red ball costs \$5 and a blue ball costs \$3. 8 red balls together weight 1 kg. 2 blue balls together weight 1 kg.

Marianna bought some red and blue balls. Altogether they cost \$111, and weight 10 kg. How many red balls did Marianna buy?

Problem 10.3.8 (ICLT-2022-SM2-R1-L2-P8)**.** AC and BD meet at X as shown. Given $[ABX] = 24$, $[BCX] = 15$, and $[CDX] = 10$, find $[ABCD]$.

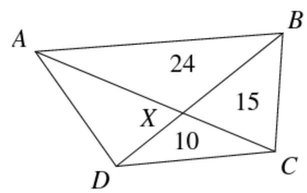


Figure 10.2: ICLT-2022-SM2-R1-L2-P8

Problem 10.3.9 (ICLT-2022-SM2-R1-L2-P9)**.** How many common divisors that 108, 144 and 360 have?

Problem 10.3.10 (ICLT-2022-SM2-R1-L2-P10)**.** The teacher wants to divide 8 students into two groups: one group has 3 students and the other with 5 students, such that two of the 8 students, Lena and Jean, should not be in the same group.

10.4 Grading

Answers for multiple-choice problems.

Problem 1: E

Problem 2: D

Problem 3: B

Problem 4: C

Problem 5: A

Problem 6: C

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can establish the system of equations

$$\begin{cases} 5r + 3b &= 111 \\ \frac{1}{8}r + \frac{1}{2}b &= 10 \end{cases}$$

Problem 8: 2 points if the student can show that $\triangle ABX, \triangle CBX$ or $\triangle ADX, \triangle CDX$ have same height, different bases.

Problem 9: 2 points if the student can see that a common divisor of some numbers is a divisor of the greatest common divisor of them.

Problem 10: 2 points if the student can find the total number of ways to split $\binom{8}{3}$ if use complementary counting (solution 1), or can show casework if use direct counting (solution 2.)

10.5 Solutions

Solution. ICLT-2022-SM2-R1-L2-P1 *Problem 2.18, Chapter 2, Introductory to Geometry.*

There are two possible cases.

Case 1: The first and the second angle are both twice the third angle. Let the measure of the smallest angle, the third angle, be x,

$$2x + 2x + x = 180^\circ \Rightarrow x = \frac{180^\circ}{5}.$$

Case 2: One of the first two angles is twice the other angle of the first two angles, the smaller of of the first two angles is is twice the third angle. Let the measure of the smallest angle, the third angle, be x,

$$4x + 2x + x = 180^\circ \Rightarrow x = \frac{180^\circ}{7}.$$

The number answer is $\boxed{E.}$ □

Solution. [First] ICLT-2022-SM2-R1-L2-P2 *Exercise 3.3.3, Chapter 3, Introductory to Counting & Probability.*

We call a match *intra-grade* if it is played between the two clubs from the same grade, and *cross-grade* if it is played between the two clubs from different grades.

It is easy to see that the total number of intra-grade matches is the number of intra-grade matches for grade 7 plus the number of intra-grade matches for grade 8. A pairs of clubs in each grade play each other twice, so the number of intra-grade matches for grade 7 is $2 \cdot \frac{5 \cdot 4}{2} = 20$. Grade 8 has the same number of clubs, so the number of intra-grade matches for grade 8 is also 20. Therefore there are 40 *intra-grade* matches.

For a club in grade 7 there are 5 clubs in grade 8, so the number of *cross-grade* matches (played by all the clubs in grade 7 with all the clubs in grade 8) is $5 \cdot 5 = 25$.

Thus, the total number of matches is $40 + 25 = \boxed{65.}$ The number answer is $\boxed{D.}$ □

Solution. [Second] ICLT-2022-SM2-R1-L2-P2 The other way to count is to look at each club. Each club plays twice a club in the same grade and once a club in different grade, so each club plays $(5-1) \cdot 2 + 5 = 13$. There are $5 + 5 = 10$ clubs, so there are $13 \cdot 10 = 130$ matches. However each matches counted this way is counted twice (we count both $A - play - B$ and $B - play - A$), hence the total number of matches is $\frac{130}{2} = \boxed{65.}$ The number answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R1-L2-P3 *Exercise 4.3.4, Chapter 4, Introductory to Algebra.*

$$3a^2 - 12b = 3(a^2 - 4b),\ 9a^3 - 36ab = 9a(a^2 - 4b)$$
$$\Rightarrow \frac{2}{3a^2 - 12b} \cdot \frac{9a^3 - 36ab}{10a^2} = \frac{2}{3(a^2 - 4b)} \cdot \frac{9a(a^2 - 4b)}{8a^2} = \boxed{\frac{3}{4a}}$$

The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R1-L2-P4 *Exercise 3.3.1, Chapter 3, Introductory to Geometry.*

In $\triangle DEF$, we have $\angle D = 180° - \angle E - \angle F = 69°$.

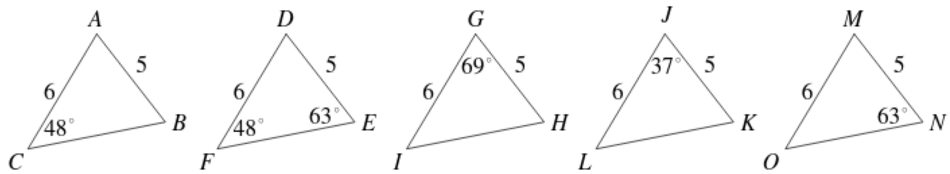


Figure 10.3: ICLT-2022-SM2-R1-L2-P4

Therefore, by SAS we have $\triangle DEF \cong \triangle GHI$. No other two triangles shown need be congruent. The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R1-L2-P5 *Exercise 3.4.1.i, Chapter 3, Introductory to Number Theory.*

Note that

$$24 = 2^3 \cdot 3, 28 = 2^2 \cdot 7, 36 = 2^2 \cdot 3^2 \Rightarrow \text{lcm}(24, 28, 36) = 2^3 \cdot 3^2\ cdot7 = \boxed{504.}$$

The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R1-L2-P6 *Problem 4.7, Chapter 4, Introductory to Counting & Probability.*

$$\binom{44}{1} = 44$$
$$\binom{6}{3} = \frac{6!}{3!3!} = \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} = 20$$
$$\binom{10}{2} = \frac{9 \cdot 10}{1 \cdot 2} = \boxed{45}$$
$$\binom{7}{4} = \frac{7!}{4!3!} = \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3} = 35$$
$$\binom{9}{7} = \frac{8 \cdot 9}{1 \cdot 2} = 36$$

The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R1-L2-P7 *Problem 5.11, Chapter 5, Introductory to Algebra.*

Let r be the number of red balls and b be the number of blue balls that Marianna bought.

$$\begin{cases} 5r + 3b & = 111 \\ \frac{1}{8}r + \frac{1}{2}b & = 10 \end{cases} \Rightarrow \begin{cases} 5r + 3b & = 111 \\ r + 4b & = 80 \end{cases} \Rightarrow 5(r+4b)-(5r+3b) = 160-111 \Rightarrow 17b = 289 \Rightarrow b = 17 \Rightarrow r = \boxed{12.}$$

□

Solution. ICLT-2022-SM2-R1-L2-P8 *Problem 4.12, Chapter 4, Introductory to Geometry.*

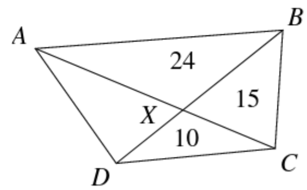


Figure 10.4: ICLT-2022-SM2-R1-L2-P8

Since $\triangle ABX$ and $\triangle CBX$ share an altitude from B, we have

$$\frac{AX}{CX} = \frac{[ABX]}{[CBX]} = \frac{8}{5}.$$

Turning to triangles $\triangle ADX$ and $\triangle CDX$, we have

$$\frac{[ADX]}{[CDX]} = \frac{AX}{CX} = \frac{8}{5}.$$

Therefore, $[ADX] = \frac{8}{5}[CDX] = 16$, thus $[ABCD] = 24 + 15 + 10 + 16 = \boxed{65.}$ ☐

Solution. ICLT-2022-SM2-R1-L2-P9 *Problem 4.4.2, Chapter 4, Introductory to Number Theory.*

A common divisor of 108, 144, and 360 is a divisor of the greatest common divisor of 108, 144, and 360. Thus the number of common divisors of 108, 144, and 360 is the number of divisors of the greatest common divisor of 108, 144, and 360. Now, since

$$108 = 2^2 \cdot 3^3, 144 = 2^4 \cdot 3^2, 360 = 2^3 \cdot 3^2 \cdot 5 \Rightarrow \gcd(96, 108, 360) = 2^2 \cdot 3^2 = 36.$$

It is easy to see that 36 has $\boxed{9}$ distinct divisors. ☐

Solution. [First solution] ICLT-2022-SM2-R1-L2-P10 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

Let's first do the problem by complementary counting. If we have no restrictions on the groups, then we simply need to choose 3 of the 8 students to be in the smaller group, and the rest of the students will make up the larger group. There are $\binom{8}{3}$ ways to do this.

But we can't have Lena and Jean in the same group. So we have to subtract the number of ways that we can form the two groups with Lena and Jean in the same group.

Case 1: Lena and Jean are both in the smaller group. If they are both in the smaller group, then we have to choose 1 more student from the 6 remaining to complete the smaller group, and we can do this in $\binom{6}{1}$ ways.

Case 2: Lena and Jean are both in the larger group. If they are both in the larger group, then we have to choose 3 students from the 6 remaining to compose the larger group, and we can do this in $\binom{6}{3}$ ways.

So to get the number of ways to form groups such that Lena and Jean are both in the same group, we add the counts from our two cases, to get $\binom{6}{1} + \binom{6}{3}$.

But remember that these are the cases that we don't want, so to solve the problem, we subtract this count from the number of ways to form the two groups without restrictions. Thus, our answer is

$$\binom{8}{3} - \left(\binom{6}{1} + \binom{6}{3}\right) = 56 - (6 + 20) = \boxed{30.}$$

☐

Solution. [Second solution] ICLT-2022-SM2-R1-L2-P10 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

We could solve this problem is by direct counting. There are two cases of possible groupings.

Case 1: Lena is in the smaller group, Jean is in the larger group To complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

Case 2: Jean is in the smaller group, Lena is in the larger group Again, to complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

So to count the total number of groupings, we add the counts from our two cases, to get $\binom{6}{2} + \binom{6}{2} = 15 + 15 = \boxed{30}$ as our final answer. $\square$

Chapter 11

Introductory Curriculum Level Test - Level 3

11.1 Rules

11.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	**Introduction to Algebra**	**Introduction to Geometry**	**Introduction to Number Theory**	**Introduction to Counting & Probability**
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

11.3 Problems

Problem 11.3.1 (ICLT-2022-SM2-R1-L3-P1)**.** There are red and blue balls in the Christmas shop. A red ball costs \$5 and a blue ball costs \$3. 8 red balls together weight 1 kg. 2 blue balls together weight 1 kg.

Marianna bought some red and blue balls. Altogether they cost \$111, and weight 10 kg. What is the total number of balls did Marianne buy?

(A) 21 (B) 25 (C) 29 (D) 33 (E) 37

Problem 11.3.2 (ICLT-2022-SM2-R1-L3-P2)**.** AC and BD meet at X as shown. Given $[ABX] = 24$, $[BCX] = 15$, and $[CDX] = 10$, find $[ABCD]$.

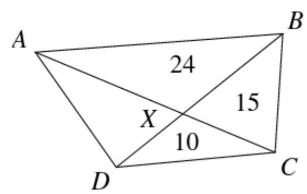


Figure 11.1: ICLT-2022-SM2-R1-L2-P8

(A) 63 (B) 65 (C) 67 (D) 69 (E) 71

Problem 11.3.3 (ICLT-2022-SM2-R1-L3-P3)**.** How many common divisors that 108, 144 and 360 have?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Problem 11.3.4 (ICLT-2022-SM2-R1-L3-P4)**.** The teacher wants to divide 8 students into two groups: one group has 3 students and the other with 5 students, such that two of the 8 students, Lena and Jean, should not be in the same group.

(A) 56 (B) 50 (C) 36 (D) 30 (E) 24

Problem 11.3.5 (ICLT-2022-SM2-R1-L3-P5)**.** What is the ratio of x to y if

$$\frac{10x - 3y}{13x - 2y} = \frac{4}{7}$$

(A) $13 : 18$ (B) $11 : 15$ (C) $9 : 11$ (D) $7 : 10$ (E) $5 : 7$

Problem 11.3.6 (ICLT-2022-SM2-R1-L3-P6). In the diagram below, $DE \parallel BC$, and the segments have the lengths as shown in the diagram.

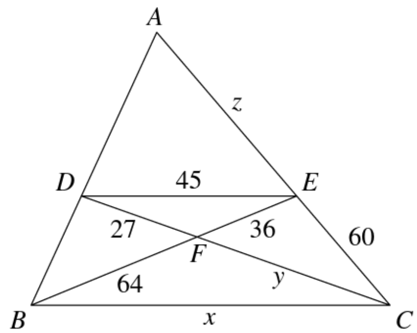


Figure 11.2: ICLT-2022-SM2-R1-L3-P6

Find $x + y + z$.

(A) $\dfrac{1430}{7}$ (B) 188 (C) $\dfrac{1436}{7}$ (D) 205 (E) $\dfrac{1437}{7}$

Problem 11.3.7 (ICLT-2022-SM2-R1-L3-P7). How many of the positive divisors of 540 have 6 positive divisors?

Problem 11.3.8 (ICLT-2022-SM2-R1-L3-P8). Six parallel lines in a plane intersect a set of n parallel lines that go in another direction. The lines form a total of 315 parallelograms, many of which overlap each other. Find n.

Problem 11.3.9 (ICLT-2022-SM2-R1-L3-P9). Six people can mow a lawn in fifteen hours. How many more people are needed to mow the lawn in just three hours, assuming each person mows at the same rate?

Problem 11.3.10 (ICLT-2022-SM2-R1-L3-P10). Find the highest power of 7 that divides $5! + 6! + 7!$.

11.4 Grading

Answers for multiple-choice problems.

Problem 1: C

Problem 2: B

Problem 3: E

Problem 4: D

Problem 5: A

Problem 6: C

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can factor 540 and find how to form a divisor that has 6 factors.

Problem 8: 2 points if the student can see that choosing two pairs of parallel lines will form a parallelogram.

Problem 9: 2 points if the student can find the number of people needed for 3 hours.

Problem 10: 2 points if the student can factor 5! out of the sum.

11.5 Solutions

Solution. ICLT-2022-SM2-R1-L3-P1 *Problem 5.11, Chapter 5, Introductory to Algebra.*

Let r be the number of red balls and b be the number of blue balls that Marianna bought.

$$\begin{cases} 5r+3b & =111 \\ \frac{1}{8}r+\frac{1}{2}b & =10 \end{cases} \Rightarrow \begin{cases} 5r+3b & =111 \\ r+4b & =80 \end{cases} \Rightarrow 5(r+4b)-(5r+3b)=160-111 \Rightarrow 17b=289 \Rightarrow b=17 \Rightarrow r=12.$$

The total number of balls that Marianna bought is $17+12=\boxed{29.}$ The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R1-L3-P2 *Exercise 3.3.3, Chapter 3, Introductory to Counting & Probability.*

Problem 4.12, Chapter 4, Introductory to Geometry.

Since $\triangle ABX$ and $\triangle CBX$ share an altitude from B, we have

$$\frac{AX}{CX}=\frac{[ABX]}{[CBX]}=\frac{8}{5}.$$

Turning to triangles $\triangle ADX$ and $\triangle CDX$, we have

$$\frac{[ADX]}{[CDX]}=\frac{AX}{CX}=\frac{8}{5}.$$

Therefore, $[ADX]=\frac{8}{5}[CDX]=16$, thus $[ABCD]=24+15+10+16=boxed65$. The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R1-L3-P3 *Problem 4.4.2, Chapter 4, Introductory to Number Theory.*

A common divisor of 108, 144, and 360 is a divisor of the greatest common divisor of 108, 144, and 360. Thus the number of common divisors of 108, 144, and 360 is the number of divisors of the greatest common divisor of 108, 144, and 360. Now, since

$$108=2^2\cdot 3^3, 144=2^4\cdot 3^2, 360=2^3\cdot 3^2\cdot 5 \Rightarrow \gcd(96,108,360)=2^2\cdot 3^2=36.$$

It is easy to see that 36 has $\boxed{9}$ distinct divisors. The answer is $\boxed{E.}$ □

Solution. [First solution] ICLT-2022-SM2-R1-L3-P4 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

Let's first do the problem by complementary counting. If we have no restrictions on the groups, then we simply need to choose 3 of the 8 students to be in the smaller group, and the rest of the students will make up the larger group. There are $\binom{8}{3}$ ways to do this.

But we can't have Lena and Jean in the same group. So we have to subtract the number of ways that we can form the two groups with Lena and Jean in the same group.

Case 1: Lena and Jean are both in the smaller group. If they are both in the smaller group, then we have to choose 1 more student from the 6 remaining to complete the smaller group, and we can do this in $\binom{6}{1}$ ways.

Case 2: Lena and Jean are both in the larger group. If they are both in the larger group, then we have to choose 3 students from the 6 remaining to compose the larger group, and we can do this in $\binom{6}{3}$ ways.

So to get the number of ways to form groups such that Lena and Jean are both in the same group, we add the counts from our two cases, to get $\binom{6}{1}+\binom{6}{3}$.

But remember that these are the cases that we don't want, so to solve the problem, we subtract this count from the number of ways to form the two groups without restrictions. Thus, our answer is

$$\binom{8}{3}-\left(\binom{6}{1}+\binom{6}{3}\right)=56-(6+20)=\boxed{30.}$$

The answer is $\boxed{D.}$ □

Solution. [Second solution] ICLT-2022-SM2-R1-L3-P4 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

We could solve this problem is by direct counting. There are two cases of possible groupings.

Case 1: Lena is in the smaller group, Jean is in the larger group To complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

Case 2: Jean is in the smaller group, Lena is in the larger group Again, to complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

So to count the total number of groupings, we add the counts from our two cases, to get $\binom{6}{2} + \binom{6}{2} = 15 + 15 = \boxed{30}$ as our final answer. The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R1-L3-P5 *Problem 6.5, Chapter 6, Introductory to Algebra.*

$$\frac{10x-3y}{13x-2y} = \frac{4}{7} \Rightarrow 7(10x-3y) = 4(13x-2y) \Rightarrow 70x-21y = 52x-8y \Rightarrow 18x = 13y \Rightarrow \frac{x}{y} = \frac{13}{18}$$

Thus, the ratio of x to y is $\boxed{13:18}$. The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R1-L3-P6 *Problem 5.12, Chapter 5, Introductory to Geometry.*

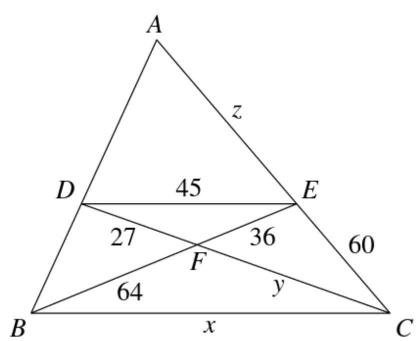


Figure 11.3: ICLT-2022-SM2-R1-L3-P6

Since $DE \parallel BC$, we have $\triangle FBC \sim \triangle FED$ by AA Similarity. Therefore, we have

$$\frac{FC}{FD} = \frac{BC}{DE} = \frac{FB}{FE} = \frac{64}{36} = \frac{16}{9}$$
$$\Rightarrow x = BC = \frac{16}{9}(DE) = 80, \; y = FC = \frac{16}{9}(DF) = 48.$$

Since $\triangle ADE \sim \triangle ABC$ by AA Similarity, we have

$$\frac{AE}{AC} = \frac{DE}{BC} = \frac{45}{80} = \frac{9}{16}, \; AE = z, \; AC = AE + EC = z + 60$$
$$\Rightarrow \frac{z}{z+60} = \frac{9}{16} \Rightarrow 16z = 9z + 540 \Rightarrow z = \frac{540}{7} \Rightarrow x + y + z = 80 + 48 + \frac{540}{7} = \boxed{\frac{1436}{7}.}$$

The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R1-L3-P7 *Problem 5.9, Chapter 5, Introductory to Number Theory.*

The problem requires us to find divisors of 540 with a particular property, so we begin by finding the prime factorization of 540 in order to learn more about its divisors:

$$540 = 2^2 \cdot 3^3 \cdot 5^1.$$

A divisor d of 540 has the form

$$d = 2^a \cdot 3^b \cdot 5^c, \text{ where } a = 0, 1 \text{ or } 2;\ b = 0, 1, 2 \text{ or } 3;\ b = 0, 1;\ c = 0 \text{ or } 1$$

We must now count possible combinations of a, b and c such that d has exactly 6 positive divisors. In other words, we are counting combinations of a, b and c such that

$$(a+1)(b+1)(c+1) = 6.$$

The only ways to get 6 as the product of 3 positive integers are

$$6 = 1 \cdot 1 \cdot 6 = 1 \cdot 2 \cdot 3.$$

These products imply that (a, b, c) must include either two 0's and one 5 or else one 0, one 1, and one 2. Since none of a, b, or c is greater than 3, this leaves us with only four possibilities:

$$(2,1,0), (2,0,1), (1,2,0), (0,2,1) \Rightarrow 2^2 \cdot 3 = 12, 2^2 \cdot 5 = 20, 2 \cdot 3^2 = 18, 3^2 \cdot 5 = 45.$$

It is easy to verify that each of $\boxed{12, 18, 20, 45}$ has exactly six divisors. □

Solution. ICLT-2022-SM2-R1-L3-P8 *Problem 6.4, Chapter 6, Introductory to Counting & Probability.*

The goal here is to find an expression for the number of parallelograms in terms of n. Then we'll set that expression equal to 315, and solve for n.

We start with a constructive approach, wondering how we can create a parallelogram by choosing some of our lines. We need to choose 2 lines from each set - this will give us the 4 sides of a parallelogram, as shown in the figure below.

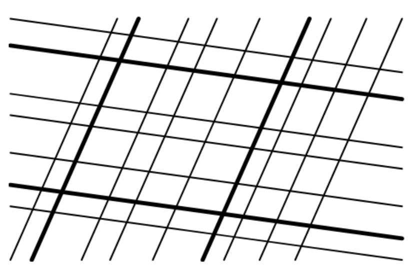


Figure 11.4: ICLT-2022-SM2-R1-L3-P8

There are 6 lines in one set of parallel lines, so the number of ways we can choose 2 lines from that set is $\binom{6}{2} = 15$. There are n lines in the other set of parallel lines, so the number of ways we can choose 2 lines from that set is $\binom{n}{2}$.

Hence there are $15\binom{n}{2}$ parallelograms. But we're told that there are 315 parallelograms, so we have

$$15\binom{n}{2} = 315 \Rightarrow \binom{n}{2} = \frac{315}{15} = 21 \Rightarrow \frac{n(n-1)}{2} = 21 \Rightarrow n = \boxed{7.}$$

□

Solution. ICLT-2022-SM2-R1-L3-P9 *Exercise 7.2.2, Chapter 7, Introductory to Algebra.*

The number of people mowing and the time required to mow are inversely proportional. Letting n be the number of people and t be the amount of time, we have $nt = (6)(15) = 90$ because 6 people can mow a lawn in 15 hours. If m people can mow the lawn in 3 hours, then we must have $m(3) = 90$, so $m = 30$. Therefore, we need to add $30 - 6 = \boxed{24}$ people to the job. □

Solution. ICLT-2022-SM2-R1-L3-P10 *Problem 6.4, Chapter 6, Introductory to Number Theory.*

$5! + 6! + 7! = 5!(1 + 6 + 6 \cdot 7) = 5! \cdot 49 = 5! \cdot 7^2$. The highest power of 7 that divides $5! + 6! + 7!$ is $\boxed{7^2.}$ □

Chapter 12

Introductory Curriculum Level Test - Level 4

12.1 Rules

12.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	Introduction to Algebra	Introduction to Geometry	Introduction to Number Theory	Introduction to Counting & Probability
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

12.3 Problems

Problem 12.3.1 (ICLT-2022-SM2-R1-L4-P1)**.** Six parallel lines in a plane intersect a set of n parallel lines that go in another direction. The lines form a total of 315 parallelograms, many of which overlap each other. Find n.

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 12.3.2 (ICLT-2022-SM2-R1-L4-P2)**.** In the diagram below, $DE \parallel BC$, and the segments have the lengths as shown in the diagram.

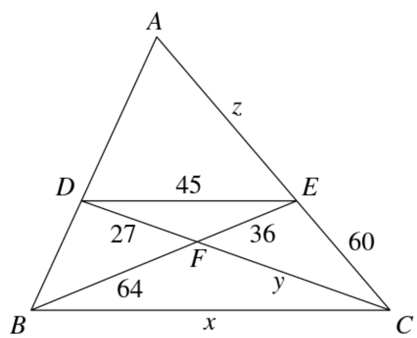


Figure 12.1: ICLT-2022-SM2-R1-L4-P2

Find $x + y + z$.

(A) $\frac{1430}{7}$ (B) 188 (C) $\frac{1436}{7}$ (D) 205 (E) $\frac{1437}{7}$

Problem 12.3.3 (ICLT-2022-SM2-R1-L4-P3)**.** How many of the positive divisors of 540 have 6 positive divisors?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 12.3.4 (ICLT-2022-SM2-R1-L4-P4)**.** A line with slope 4 intersects a line with slope 5 at the point $(10, 20)$. What is the distance between the x-intercepts of these two lines?

(A) 1 (B) 2 (C) $\frac{5}{2}$ (D) 3 (E) $\frac{7}{2}$

Problem 12.3.5 (ICLT-2022-SM2-R1-L4-P5)**.** AD and BC are both perpendicular to AB in the diagram below, and $CD \perp AC$. If $AB = 4$ and $BC = 3$, find AD.

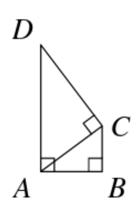


Figure 12.2: ICLT-2022-SM2-R1-L4-P5

(A) $\frac{16}{5}$ (B) $\frac{16}{3}$ (C) 5 (D) $\frac{20}{3}$ (E) $\frac{25}{3}$

Problem 12.3.6 (ICLT-2022-SM2-R1-L4-P6)**.** Let $n = \overline{abc}$ be a positive 3-digit integer that is equal to the sum $\overline{ab} + \overline{ba} + \overline{ac} + \overline{ca} + \overline{bc} + \overline{cb}$ of six positive 2-digit integers.

For the smallest possible n, find the product abc.

(A) 0 $\quad$ (B) 1 $\quad$ (C) 6 $\quad$ (D) 36 $\quad$ (E) 162

Problem 12.3.7 (ICLT-2022-SM2-R1-L4-P7)**.** Two different 2-digit numbers are randomly chosen and multiplied together. What is the probability that the resulting product is not divisible by 53?

Problem 12.3.8 (ICLT-2022-SM2-R1-L4-P8)**.** Sort the following numbers from smallest to greatest

$$2^{600}, 3^{400}, 4^{250}, 5^{200}.$$

Problem 12.3.9 (ICLT-2022-SM2-R1-L4-P9)**.** In $\triangle ABC$, $AB = 10$, $AC = 12$, and $BC = 8$. Point M is on side BC such that $\angle BAM = \angle CAM$. Find BM.

Problem 12.3.10 (ICLT-2022-SM2-R1-L4-P10)**.** Phan and Quan play a number of two-player games. Any game has a winner (no draw). The player who is the first to win 4 games (not necessarily in a row) is the champion. Phan has a $\frac{2}{3}$ probability of winning any individual game. What is the probability that after exactly 7 games Phan is declared champion?

12.4 Grading

Answers for multiple-choice problems.

Problem 1: *D*

Problem 2: *C*

Problem 3: *B*

Problem 4: *A*

Problem 5: *E*

Problem 6: *C*

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can find that the number of ways to choose two numbers so their product is divisible by 53 is 89.

Problem 8: 2 points if the student can simplify the problem by taking the 50^{th} (or similar) roots of the numbers.

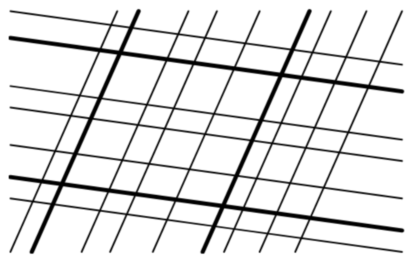
Problem 9: 2 points if the student can use the Angle-Bisector Theorem.

Problem 10: 2 points if the student can see that there is 20 possible sequences of games.

12.5 Solutions

Solution. ICLT-2022-SM2-R1-L4-P1 *Problem 6.4, Chapter 6, Introductory to Counting & Probability.*

The goal here is to find an expression for the number of parallelograms in terms of n. Then we'll set that expression equal to 315, and solve for n. We start with a constructive approach, wondering how we can create a parallelogram by choosing some of our lines. We need to choose 2 lines from each set - this will give us the 4 sides of a parallelogram, as shown in the figure below.

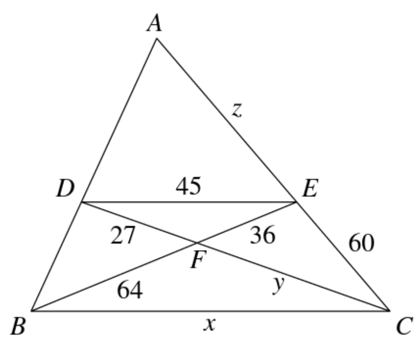


There are 6 lines in one set of parallel lines, so the number of ways we can choose 2 lines from that set is $\binom{6}{2} = 15$. There are n lines in the other set of parallel lines, so the number of ways we can choose 2 lines from that set is $\binom{n}{2}$. Hence there are $15\binom{n}{2}$ parallelograms. But we're told that there are 315 parallelograms,

$$15\binom{n}{2} = 315 \Rightarrow \binom{n}{2} = \frac{315}{15} = 21 \Rightarrow \frac{n(n-1)}{2} = 21 \Rightarrow n = \boxed{7.}$$

The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R1-L4-P2 *Problem 5.12, Chapter 5, Introductory to Geometry.*



Since $DE \parallel BC$, we have $\triangle FBC \sim \triangle FED$ by AA Similarity. Therefore, we have

$$\frac{FC}{FD} = \frac{BC}{DE} = \frac{FB}{FE} = \frac{64}{36} = \frac{16}{9}$$
$$\Rightarrow x = BC = \frac{16}{9}(DE) = 80,\ y = FC = \frac{16}{9}(DF) = 48.$$

Since $\triangle ADE \sim \triangle ABC$ by AA Similarity, we have

$$\frac{AE}{AC} = \frac{DE}{BC} = \frac{45}{80} = \frac{9}{16},\ AE = z,\ AC = AE + EC = z + 60$$
$$\Rightarrow \frac{z}{z+60} = \frac{9}{16} \Rightarrow 16z = 9z + 540 \Rightarrow z = \frac{540}{7} \Rightarrow x + y + z = 80 + 48 + \frac{540}{7} = \boxed{\frac{1436}{7}.}$$

The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R1-L4-P3 *Problem 5.9, Chapter 5, Introductory to Number Theory.*

The problem requires us to find divisors of 540 with a particular property, so we begin by finding the prime factorization of 540 in order to learn more about its divisors:

$$540 = 2^2 \cdot 3^3 \cdot 5^1.$$

A divisor d of 540 has the form

$$d = 2^a \cdot 3^b \cdot 5^c, \text{ where } a = 0, 1 \text{ or } 2;\ b = 0, 1, 2 \text{ or } 3;\ b = 0, 1;\ c = 0 \text{ or } 1$$

We must now count possible combinations of a, b and c such that d has exactly 6 positive divisors. In other words, we are counting combinations of a, b and c such that

$$(a + 1)(b + 1)(c + 1) = 6.$$

The only ways to get 6 as the product of 3 positive integers are

$$6 = 1 \cdot 1 \cdot 6 = 1 \cdot 2 \cdot 3.$$

These products imply that (a, b, c) must include either two 0's and one 5 or else one 0, one 1, and one 2. Since none of a, b, or c is greater than 3, this leaves us with only four possibilities:

$$(2, 1, 0), (2, 0, 1), (1, 2, 0), (0, 2, 1) \Rightarrow 2^2 \cdot 3 = 12, 2^2 \cdot 5 = 20, 2 \cdot 3^2 = 18, 3^2 \cdot 5 = 45.$$

It is easy to verify that each of $\boxed{12, 18, 20, 45}$ has exactly six divisors. The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R1-L4-P4 *Problem 8.21, Chapter 8, Introductory to Algebra.*

The line with slope 3 that passes through $(10, 20)$ is $y - 20 = 4(x - 10)$, or $y = 4x - 20$. The x-intercept of this line is where $y = 0$. Solving $0 = 4x - 20$, we find $x = 5$, so the x-intercept is $(5, 0)$.

Similarly, the line with slope 5 through $(10, 20)$ is $y - 20 = 5(x - 10)$, or $y = 5x - 30$. Setting $y = 0$, we find that the x-intercept of this line is $(6, 0)$. Therefore, the distance between our x-intercepts is $\boxed{1.}$ The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R1-L4-P5 *Exercise 6.4.4, Chapter 6, Introductory to Geometry.*

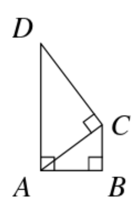


Figure 12.3: ICLT-2022-SM2-R1-L4-P5

We have $\angle ABC = 90°$, so by the Pythagorean Theorem, we have

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

We also have

$$\angle CDA = 90° - \angle DAC = \angle CAB,$$

so triangles ABC and DCA are similar. Therefore,

$$\frac{DA}{AC} = \frac{CA}{BC} \Rightarrow DC = \frac{AC^2}{BC} = \boxed{\frac{25}{3}.}$$

The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R1-L4-P6 *Problem 7.8, Chapter 7, Introductory to Number Theory.*

$$\begin{aligned}
&n = \overline{ab} + \overline{ba} + \overline{ac} + \overline{ca} + \overline{bc} + \overline{cb} \\
\Rightarrow\ &100a + 10b + c = 10a + b + 10b + a + 10a + c + 10c + a + 10b + c + 10c + b \\
\Rightarrow\ &100a + 10b + c = 22(a + b + c) \Rightarrow 78a = 12b + 21c \Rightarrow 26a = 4b + 7c
\end{aligned}$$

Since we want to make n as small as possible, so $a = 1$ would be the smallest value for a. Then $26 = 4b+7c$, so c is even and $c < 4$. c cannot be 0 because $4 \nmid 26$, so $c = 2$, then $b = 3$. Hence $n = \boxed{132.}$ The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R1-L4-P7 *Problem 7.9, Chapter 7, Introductory to Counting & Probability.*

We find the probablity when the product of the two different number is divisible by 53.

There are 90 2-digit numbers, so selecting two different 2-digit numbers, without regard to order, can be done in $\binom{90}{2}$ ways. Two numbers multiply together to give a multiple of 53 number if at least one of the original numbers is a multiple of 53. Between 10 and 99 there is only one such number because 53 is a prime. Thus the number of successful outcomes is the number of ways to choose a number which is not 53, or 89.

Hence, the probability is

$$\frac{89}{\binom{90}{2}} = \frac{89}{\frac{90\cdot 89}{2}} = \frac{1}{45}.$$

Thus, our desired probability is $1 - \frac{1}{45} = \boxed{\frac{44}{45}.}$ □

Solution. ICLT-2022-SM2-R1-L4-P8 *Problem 9.10, Chapter 9, Introductory to Algebra.*

By taking the 50^{th} root of the numbers, we have

$$2^{12} = 4096, 3^8 = 6561, 4^5 = 1024, 5^4 = 625 \Rightarrow 5^4 < 4^5 < 2^{12} < 3^8 \Rightarrow \boxed{5^{200} < 4^{250} < 2^{600} < 3^{400}.}$$

□

Solution. ICLT-2022-SM2-R1-L4-P9 *Exercise 7.3.6, Chapter 7, Introductory to Geometry.*

Let $x = BM$, so $MC = 8 - x$. Since $\angle BAM = \angle CAM$, AM is an angle bisector of $\triangle ABC$. Therefore, by the Angle Bisector Theorem, we have $\frac{BM}{BA} = \frac{CM}{CA.}$ Substitution gives $\frac{x}{10} = \frac{8-x}{12}$, so $x = \boxed{\frac{40}{11}}.$ □

Solution. ICLT-2022-SM2-R1-L4-P10 *Problem 8.11, Chapter 8, Introductory to Counting & Probability.*

It is easy to see that Phan wins 4 games in 7 games, thus Quan must win 3 other games. Let assume that the ways the winners of invidual games are $PPQQPQP$. Then the probability for this sequence is

$$\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} = \left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^3.$$

Thus it does not matter how the games were played. It is easy to see that the last game must be won by Phan. So in the remaining 6 games, there are $\binom{6}{3} = 20$ ways to choose the three games that Phan wins. Thus, the overall probability for Phan to win is

$$20\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^3 = \frac{320}{2187}.$$

□

Chapter 13

Introductory Curriculum Level Test - Level 5

13.1 Rules

13.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	Introduction to Algebra	Introduction to Geometry	Introduction to Number Theory	Introduction to Counting & Probability
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

13.3 Problems

Problem 13.3.1 (ICLT-2022-SM2-R1-L5-P1)**.** AD and BC are both perpendicular to AB in the diagram below, and $CD \perp AC$. If $AB = 4$ and $BC = 3$, find AD.

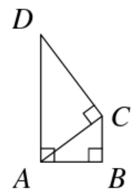


Figure 13.1: ICLT-2022-SM2-R1-L5-P1

(A) $\dfrac{16}{5}$ (B) $\dfrac{16}{3}$ (C) 5 (D) $\dfrac{20}{3}$ (E) $\dfrac{25}{3}$

Problem 13.3.2 (ICLT-2022-SM2-R1-L5-P2)**.** Phan and Quan play a number of two-player games. Any game has a winner (no draw). The player who is the first to win 4 games (not necessarily in a row) is the champion. Phan has a $\frac{2}{3}$ probability of winning any individual game. What is the probability that after exactly 7 games Phan is declared champion?

(A) $\dfrac{40}{729}$ (B) $\dfrac{80}{729}$ (C) $\dfrac{160}{2187}$ (D) $\dfrac{320}{2187}$ (E) $\dfrac{560}{2187}$

Problem 13.3.3 (ICLT-2022-SM2-R1-L5-P3)**.** Let a and b denote the solutions of $18x^2 + 3x - 28 = 0$. Find the value of $(a^2 - 1)(b^2 - 1)$.

(A) $\dfrac{7}{18}$ (B) $\dfrac{13}{324}$ (C) $\dfrac{91}{324}$ (D) $\dfrac{13}{18}$ (E) $\dfrac{2}{3}$

Problem 13.3.4 (ICLT-2022-SM2-R1-L5-P4)**.** The area of trapezoid $ABCD$ is 96. One base is 6 units longer than the other, and the height of the trapezoid is 8. Find the length of the shorter base.

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Problem 13.3.5 (ICLT-2022-SM2-R1-L5-P5)**.** What is the largest base-9 integer that can be written as a three-digit base-4 integer?

(A) 70_9 (B) 63_9 (C) 64_9 (D) 333_9 (E) 1000_9

Problem 13.3.6 (ICLT-2022-SM2-R1-L5-P6). Points A, B, and C are arranged clockwise in that order on a circle, as shown in the figure below. We place a marker on point A. We roll a 6-sided die and move the marker as follows:

- If the die shows a dice-blue-1 or a dice-blue-2, stay put.
- If the die shows a dice-blue-3 or a dice-blue-4, move one step clockwise (for example, from A to B).
- If the die shows a dice-blue-5 or a dice-blue-6, move one step counterclockwise (for example, from A to C).

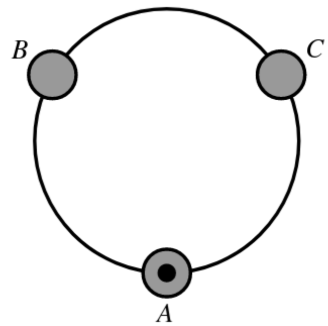


Figure 13.2: ICLT-2022-SM2-R1-L5-P6

What is the probability that after 9 moves the marker is at point A?

$(A)\ \frac{1}{9}$ $\quad (B)\ \frac{2}{9}$ $\quad (C)\ \frac{1}{3}$ $\quad (D)\ \frac{4}{9}$ $\quad (E)\ \frac{2}{3}$

Problem 13.3.7 (ICLT-2022-SM2-R1-L5-P7). Let

$$\begin{cases} x &= z - \sqrt{z^2 - 5} \\ 5y &= z + \sqrt{z^2 - 5} \end{cases}$$

Find x when $y = \frac{3}{4}$.

Problem 13.3.8 (ICLT-2022-SM2-R1-L5-P8). Find the value in base 2 of the sum by performing base-2 addition. *No conversion to any other base is allowed.*

$$101110_2 + 1001_2 + 11011_2?$$

Problem 13.3.9 (ICLT-2022-SM2-R1-L5-P9). Suppose two numbers x and y are each chosen such that $0 < x < 1$ and $0 < y < 1$. What is the probability that $x + y > \frac{1}{2}$?

Problem 13.3.10 (ICLT-2022-SM2-R1-L5-P10). Write $\dfrac{4 - i}{-1 + 5i}$ as a single complex number.

13.4 Grading

Answers for multiple-choice problems.

Problem 1: E

Problem 2: D

Problem 3: B

Problem 4: B

Problem 5: A

Problem 6: C

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can recognize the identity $a^2 - b^2$ for $z \pm \sqrt{z^2 - 5}$.

Problem 8: 2 points if the student can demonstrate how to add in base 2.

Problem 9: 2 points if the student can show the area of desired (x, y).

Problem 10: 2 points if the student can use conjugate of complex number.

13.5 Solutions

Solution. ICLT-2022-SM2-R1-L5-P1 *Exercise 6.4.4, Chapter 6, Introductory to Geometry.*

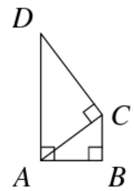


Figure 13.3: ICLT-2022-SM2-R1-L5-P5

We have $\angle ABC = 90°$, so by the Pythagorean Theorem, we have

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

We also have

$$\angle CDA = 90° - \angle DAC = \angle CAB,$$

so triangles ABC and DCA are similar. Therefore,

$$\frac{DA}{AC} = \frac{CA}{BC} \Rightarrow DC = \frac{AC^2}{BC} = \boxed{\frac{25}{3}.}$$

The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R1-L5-P2 *Problem 8.11, Chapter 8, Introductory to Counting & Probability.*

It is easy to see that Phan wins 4 games in 7 games, thus Quan must win 3 other games. Let assume that the ways the winners of invidual games are $PPQQPQP$. Then the probability for this sequence is

$$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^3.$$

Thus it does not matter how the games were played. It is easy to see that the last game must be won by Phan. So in the remaining 6 games, there are $\binom{6}{3} = 20$ ways to choose the three games that Phan wins. Thus, the overall probability for Phan to win is

$$20 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^3 = \frac{320}{2187}.$$

The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R1-L5-P3 *Problem 10.24, Chapter 10, Introductory to Algebra.*

If a and b are the solutions of $18x^2 + 3x - 28 = 0$, then

$$a + b = -\frac{3}{18}, ab = -\frac{28}{18}.$$

Now

$$\begin{aligned}(a^2 - 1)(b^2 - 1) &= [(a-1)(b-1)]\,[(a+1)(b+1)] = (ab - a - b + 1)(ab + a + b + 1) \\ &= \left(-\frac{28}{18} + \frac{3}{18} + 1\right)\left(-\frac{28}{18} - \frac{3}{18} + 1\right) = \boxed{\frac{91}{324}.}\end{aligned}$$

The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R1-L5-P4 *Problem 8.2.3, Chapter 8, Introductory to Geometry.*

Let the length of the shorter base be x, so the longer base is $x+6$. We have $\frac{(x+x+6)(8)}{2} = 96$. Solving, we find that the shorter base has length $x = \boxed{9}$. The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R1-L5-P5 *Problem 8.10, Chapter 8, Introductory to Number Theory.*

The smallest integer that can be written using four digits in base 4 is $1000_4 = 1 \cdot 4^3 = 64$. We count down 1 from 64 to get $333_4 = 63$.

In base-9, $63 = \boxed{70_9}$. The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R1-L5-P6 *Problem 9.3 Chapter 9, Introductory to Counting & Probability.*

We could approach this problem using casework, considering all 6^8 possible sequences of 8 rolls ... On second thought, let's think about it first. On any given turn, there's a $\frac{1}{3}$ chance we stay put, a $\frac{1}{3}$ chance we move clockwise, and a $\frac{1}{3}$ chance we move counterclockwise.

Therefore, after every turn, we're equally likely to end up at each of the 3 points (since the probability is $\frac{1}{3}$ of moving to each of the 3 points). Therefore, it doesn't matter which point we care about, or how many moves we've made - the probability of ending up at any point is always $\boxed{\dfrac{1}{3}.}$

The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R1-L5-P7 *Problem 11.2, Chapter 11, Introductory to Algebra.*

$$\begin{cases} x & = z - \sqrt{z^2 - 5} \\ 5y & = z + \sqrt{z^2 - 5} \end{cases} \Rightarrow (x)(5y) = (z - \sqrt{z^2 - 5})(z + \sqrt{z^2 - 5}) = z^2 - (z^2 - 5) = 5 \Rightarrow xy = 1.$$

When $y = \frac{3}{4}$, then $x = \boxed{\dfrac{4}{3}.}$ □

Solution. ICLT-2022-SM2-R1-L5-P8 *Problem 9.11.c, Chapter 9, Introductory to Number Theory.*

$$\begin{array}{r} 101110_2 \\ 11011_2 \\ + \quad 1001_2 \\ \hline 1010010_2 \end{array}$$

□

Solution. ICLT-2022-SM2-R1-L5-P9 *Problem 10.5, Chapter 10, Introductory to Geometry.*

The "total outcomes" region is the region in the xy-plane with $0 < x < 1$ and $0 < y < 1$. This is the interior of the square with corners $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$. The area of the square is 1.

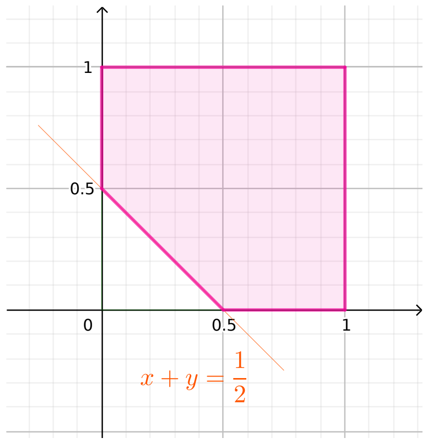


Figure 13.4: ICLT-2022-SM2-R1-L5-P9

Now we need to describe the region corresponding to successful outcomes. We need $x + y > \frac{1}{2}$, so this will be the region inside the square of which is above the line $x + y = \frac{1}{2}$. The area of the region is $1 - \frac{\frac{1}{2}\cdot\frac{1}{2}}{2} = \frac{7}{8}$. Hence, the probability is $\frac{\frac{7}{8}}{1} = \boxed{\frac{7}{8}.}$ □

Solution. ICLT-2022-SM2-R1-L5-P10 *Problem 12.10, Chapter 12, Introductory to Algebra.*

To remove $-1 + 5i$ from the denominator, we multiply it with its conjugate $-1 - 5i$,

$$\frac{4 - i}{-1 + 5i} = \frac{(4 - i)(-1 - 5i)}{(-1)^2 - (5i)^2} = \frac{-4 + i - 20i + 5i^2}{26} = \boxed{-\frac{9}{26} - \frac{19}{26}i.}$$

□

Chapter 14

Introductory Curriculum Level Test - Level 7

14.1 Rules

14.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

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The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	Introduction to Algebra	Introduction to Geometry	Introduction to Number Theory	Introduction to Counting & Probability
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

14.3 Problems

Problem 14.3.1 (ICLT-2022-SM2-R1-L7-P1). Let $f(x) = 3 - 2x$ and $g(f(x)) = 2x^3 - 3x + 5$. Find $g(7)$.

(A) -7 (B) -5 (C) -3 (D) -1 (E) 1

Problem 14.3.2 (ICLT-2022-SM2-R1-L7-P2). The shaded portion of the figure is called a lune. Given that $AB = 1$, $CD = \sqrt{2}$, and that AB and CD are diameters of the respective semicircles shown, find the area of the lune.

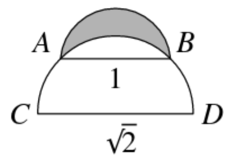


Figure 14.1: ICLT-2022-SM2-R1-L7-P2

(A) $\dfrac{\pi - 1}{8}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{1}{4}$ (D) $\dfrac{3\pi - 4}{16}$ (E) $\dfrac{3}{8}$

Problem 14.3.3 (ICLT-2022-SM2-R1-L7-P3). What is the value of

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r}\binom{n}{0}?$$

(A) $\dbinom{m+n-1}{r-1}$ (B) $\dbinom{m+n}{r-1}$ (C) $\dbinom{m+n-1}{r}$ (D) $\dbinom{m+n}{r}$ (E) $\dbinom{m+n+1}{r+1}$

Problem 14.3.4 (ICLT-2022-SM2-R1-L7-P4). Which statement below is true?

(A) If a graph on the Cartesian plane is the graph of a function, then every vertical line passes through exactly one point.

(B) If f is a function, the graphs of $y = f(x)$ and $y = f(x) + 3$ intersect at more than one point.

(C) The graph of $y = f(3x) + 4$ is the result of stretching the graph of $y = f(x)$ vertically by a factor of 3, then shifting the result 4 units up.

(D) If no horizontal line passes through more than one point on a given graph on the coordinate plane, then the graph is the graph of a function that has an inverse.

(E) If no horizontal line passes through more than one point of the graph of a function, then the function has an inverse.

Problem 14.3.5 (ICLT-2022-SM2-R1-L7-P5). The median AM of $\triangle ABC$ has length 8. Given that $BC = 16$ and $AB = 9$, find the area of $\triangle ABC$.

(A) $\dfrac{45\sqrt{7}}{2}$ (B) 58 (C) $\dfrac{45\sqrt{5}}{2}$ (D) 50 (E) $\dfrac{47\sqrt{5}}{2}$

Problem 14.3.6 (ICLT-2022-SM2-R1-L7-P6)**.** What is the unit digit of n if the units digit of $\binom{2n+2}{2n}$ is 3?

(A) 3 (B) 4 (C) 6 (D) 8 (E) 9

Problem 14.3.7 (ICLT-2022-SM2-R1-L7-P7)**.** Express the repeating decimal $0.\overline{037}$ in reduced form.

Problem 14.3.8 (ICLT-2022-SM2-R1-L7-P8)**.** There are 3 white marbles and k black marbles in a jar. A marble is drawn at random from the jar. If it is white, the player wins \$1, but if a black ball is drawn, the player loses \$1. What is k so that if the expected loss for playing the game is 0?

Problem 14.3.9 (ICLT-2022-SM2-R1-L7-P9)**.** Solve the inequality

$$\frac{3}{x-3} - 4x + 4 > -1.$$

Problem 14.3.10 (ICLT-2022-SM2-R1-L7-P10)**.** A circle is tangent to side BC of equilateral triangle $\triangle ABC$ at point Q as shown below. The circle intersects sides AB and AC in two points each, as shown. Given that $AW = AY$, prove that Q is the midpoint of BC.

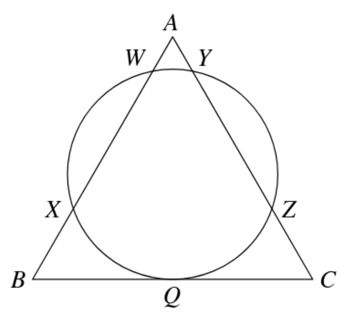


Figure 14.2: ICLT-2022-SM2-R1-L7-P10

14.4 Grading

Answers for multiple-choice problems.

Problem 1: B

Problem 2: C

Problem 3: D

Problem 4: E

Problem 5: A

Problem 6: D

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can eliminate the repeating digits.

Problem 8: 2 points if the student can find the expected value based on k.

Problem 9: 2 points if the student can casework depeding on the signs of $(x+3)$ and $(x+4)$.

Problem 10: 2 points if the student can find that $WX = YZ$.

14.5 Solutions

Solution. ICLT-2022-SM2-R1-L7-P1 *Problem 16.12, Chapter 16, Introductory to Algebra.*

Note that $g(f(x)) = 2x^3 - 3x + 5$, so to find $g(7)$, we need a value of x so that $f(x) = 7$.

$$f(x) = 7 \Rightarrow 3 - 2x = 7 \Rightarrow x = -2 \Rightarrow g(7) = g(f(-2)) = 2(-2)^3 - 3(-2) + 5 = \boxed{-5}.$$

The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R1-L7-P2 *Problem 8.11, Chapter 8, Introductory to Geometry.*

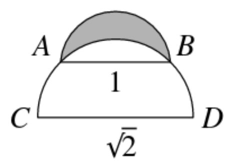


Figure 14.3: ICLT-2022-SM2-R1-L7-P2

$$\begin{aligned}
&\text{Area of lune} = \text{Area of small semicircle} - \text{Area of circular segment } AB \\
&\text{Area of circular segment } AB = \text{Area of sector } AOB - \text{Area of } \triangle AOB \\
\Rightarrow\ &\text{Area of lune} = \text{Area of small semicircle} - \text{Area of sector } AOB + \text{Area of } \triangle AOB
\end{aligned}$$

Now $CD = \sqrt{2}$, so $OC = OA = OB = OD = \frac{\sqrt{2}}{2}$, so $AB^2 = AO^2 + BO^2 = 1$. Thus, The radius of the small semicircle is $\frac{1}{2}$, so its area is $\frac{1}{2} \cdot \pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$. Sector AOB is $\frac{90^\circ}{360^\circ} = \frac{1}{4}$ of a circle with radius $\frac{\sqrt{2}}{2}$, so its area is $\frac{1}{4} \cdot \pi \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{8}$. Finally, $[AOB] = \frac{1}{2} \cdot (AO)(OB) = \frac{1}{4}$, so we have

$$\text{Area of small semicircle} - \text{Area of sector } AOB + \text{Area of } \triangle AOB = \frac{\pi}{8} - \frac{\pi}{8} + \frac{1}{4} = \boxed{\frac{1}{4}.}$$

The answer is $\boxed{C.}$ □

Solution. [First solution] ICLT-2022-SM2-R1-L7-P3 *Problem 12.4.1.b, Chapter 10, Introductory to Counting & Probability.*

We wish to choose an r-person committee from m men and n women. The number of ways to choose an r-person committee from $(m+n)$ people is $\binom{m+n}{r}$.

On the other hand, the committee can have k men and $(r-k)$ women for any k from 0 to r inclusive. There are $\binom{m}{k}$ ways to choose k men (from the m total men) and $\binom{n}{r-k}$ ways to choose $(r-k)$ women (from the n total women). The total number of committees given by this second method of counting is

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r}\binom{n}{0}.$$

The answer is $\boxed{D.}$ □

Solution. [Second solution] ICLT-2022-SM2-R1-L7-P3 *Problem 12.4.1.b, Chapter 10, Introductory to Counting & Probability.*

We are trying to count the number of ways to move r steps to the right out of $m+n$ steps, or $\binom{m+n}{r}$.

After m steps down the Pascals Triangle, we could have taken 0, 1, 2, ..., or r steps to the right. If we take k steps to the right after m steps, we must take $r-k$ steps to the right in the final n steps. There are $\binom{m}{k} \times \binom{n}{r-k}$ ways to do this Summing this over k from 0 to r, we get

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r}\binom{n}{0}.$$

The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R1-L7-P4 *(Parts of) Exercise 17.2.4 & Problem 17.19, Chapter 17, Introductory to Algebra.*

(A) If a graph on the Cartesian plane is the graph of a function, then every vertical line passes through exactly one point. This is $\boxed{\text{False}}$. It is possible that the graph is the graph of a function, but there exist vertical lines that pass through 0 points on the graph.

(B) If f is a function, the graphs of $y = f(x)$ and $y = f(x) + 3$ intersect at more than one point. This is $\boxed{\text{False}}$. The graphs of $y = f(x)$ and $y = f(x) + 3$ cannot intersect. Because the graph of $y = f(x) + 3$ is a 3-unit vertical shift of the graph of $y = f(x)$, any point at which a vertical line hits the graph of $y = f(x) + 3$ must be 3 units above the point where that line hits the graph $y = f(x)$. Therefore, no vertical line meets both graphs at the same point, so there is no point through which both graphs pass.

(C) The graph of $y = f(3x) + 4$ is the result of stretching the graph of $y = f(x)$ vertically by a factor of 3, then shifting the result 4 units up. This is $\boxed{\text{False}}$. The graph of $y = f(3x)$ is the result of scaling the graph horizontally by a factor of $\frac{1}{3}$, not a result of stretching the graph vertically.

(D) If no horizontal line passes through more than one point on a given graph on the coordinate plane, then the graph is the graph of a function that has an inverse. This is $\boxed{\text{False}}$. Just because no horizontal line passes through more than one point on the graph does not mean the graph is the graph of a function.

(E) If no horizontal line passes through more than one point of the graph of a function, then the function has an inverse. This is $\boxed{\text{True}}$. If the graph of a function is such that no horizontal line passes through more than one point on the graph, then the function has an inverse.

The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R1-L7-P5 *Problem 12.23, Chapter 12, Introductory to Geometry.*

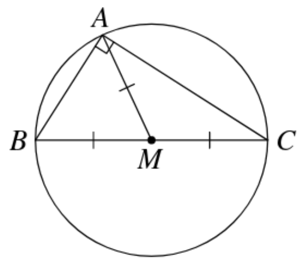


Figure 14.4: ICLT-2022-SM2-R1-L7-P5

When we draw the figure and label all our lengths, we see that $BM = AM = CM = 8$. Therefore, a circle centered at M with radius 8 goes through all three vertices of $\triangle ABC$. Since BC is a diameter of this circle, $\angle BAC$ is inscribed in a semicircle and therefore must be a right angle. So, $AC = \sqrt{BC^2 - AB^2} = 5\sqrt{7}$. $\triangle ABC$ is a right triangle, so its area is half the product of its legs:

$$[ABC] = \frac{(AB)(AC)}{2} = \frac{(9)(5\sqrt{7})}{2} = \boxed{\frac{45\sqrt{7}}{2}.}$$

The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R1-L7-P6 *Problem 10.28, Chapter 10, Introductory to Number Theory*

$$\binom{2n+2}{2n} = \binom{2n+2}{2} = \frac{(2n+2)(2n+1)}{2} = (n+1)(2n+1)$$

1. $n \equiv 0 \pmod{10} \Rightarrow 2n+1 \equiv 1 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 1 \pmod{10}$.
2. $n \equiv 1 \pmod{10} \Rightarrow 2n+1 \equiv 3 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 6 \pmod{10}$.
3. $n \equiv 2 \pmod{10} \Rightarrow 2n+1 \equiv 5 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 5 \pmod{10}$.
4. $n \equiv 3 \pmod{10} \Rightarrow 2n+1 \equiv 7 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 8 \pmod{10}$.
5. $n \equiv 4 \pmod{10} \Rightarrow 2n+1 \equiv 9 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 5 \pmod{10}$.
6. $n \equiv 5 \pmod{10} \Rightarrow 2n+1 \equiv 1 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 6 \pmod{10}$.
7. $n \equiv 6 \pmod{10} \Rightarrow 2n+1 \equiv 3 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 1 \pmod{10}$.
8. $n \equiv 7 \pmod{10} \Rightarrow 2n+1 \equiv 5 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 0 \pmod{10}$.
9. $n \equiv \boxed{8} \pmod{10} \Rightarrow 2n+1 \equiv 7 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 3 \pmod{10}$.
10. $n \equiv 9 \pmod{10} \Rightarrow 2n+1 \equiv 9 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 0 \pmod{10}$.

The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R1-L7-P7 *Problem 11.11, Chapter 11, Introductory to Number Theory.*

$$x = 0.\overline{037} \Rightarrow 1000x = 37.\overline{037} \Rightarrow 999x = 1000x - x = 37 \Rightarrow x = \frac{37}{999} = \boxed{\frac{1}{27}.}$$

□

Solution. ICLT-2022-SM2-R1-L7-P8 *Exercise 11.3.5, Chapter 11, Introductory to Counting & Probability.*

There are 3 white marbles and $3+k$ total marbles, so the probability that a white marble is drawn is $\frac{3}{3+k}$. Similarly, the probability that a black marble is drawn is $\frac{k}{3+k}$. So the expected value of the game is

$$E = \frac{3}{3+k}(1) + \frac{k}{3+k}(-1) = \frac{3-k}{3+k}$$
$$E = 0 \Rightarrow k = \boxed{3.}$$

□

Solution. ICLT-2022-SM2-R1-L7-P9 *Problem 15.7, Chapter 15, Introductory to Algebra.*

$$\frac{3}{x-3} - 4x + 4 > -1 \Rightarrow \frac{3 + (-4x+5)(x-3)}{x-3} > 0 \Rightarrow \frac{4x^2 - 17x + 12}{x-3} < 0$$

Case 1: $x - 3 < 0$, or $x < 3$, then $4x^2 - 17x + 12 > 0$,

$$\Delta = 17^2 - 4 \cdot 4 \cdot 12 = 97 \Rightarrow x < \frac{17-\sqrt{97}}{8} \approx 0.89 \text{ or } x > \frac{17+\sqrt{97}}{8} \approx 3.35 \Rightarrow x < \frac{17-\sqrt{97}}{8}.$$

Case 2: $x - 3 > 0$, or $x > 3$, then $4x^2 - 17x + 12 < 0$, from above,

$$\frac{17-\sqrt{97}}{8} < x < \frac{17+\sqrt{97}}{8} \Rightarrow 3 < x < \frac{17+\sqrt{97}}{8}$$

Hence, $\boxed{x \in (-\infty, \frac{17-\sqrt{97}}{8}) \cup (3, \frac{17+\sqrt{97}}{8}).}$ □

Solution. ICLT-2022-SM2-R1-L7-P10 *Problem 13.2, Chapter 13, Introductory to Geometry.*

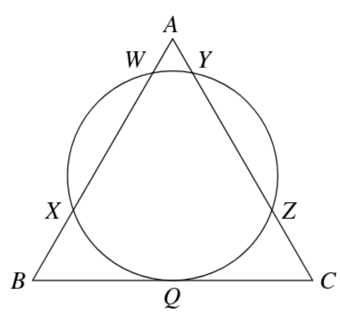


Figure 14.5: ICLT-2022-SM2-R1-L7-P10

The power of point A gives us $(AW)(AX) = (AY)(AZ)$. Since $AW = AY$, we have $AX = AZ$, so

$$WX = AX - AW = AZ - AY = YZ.$$

Since $\triangle ABC$ is equilateral, we have $AB = AC$. Therefore,

$$BX = AB - AX = AC - AZ = CZ \text{ and } BW = BX + XW = CZ + ZY = CY.$$

Finally, we use the powers of points B and C to find:

$$BQ^2 = (BX)(BW) = (CZ)(CY) = CQ^2 \Rightarrow BQ = CQ.$$

$\boxed{\text{Point } Q \text{ is therefore the midpoint of } BC.}$ □

Chapter 15

Introductory Curriculum Level Test - Level 9

15.1 Rules

15.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	**Introduction to Algebra**	**Introduction to Geometry**	**Introduction to Number Theory**	**Introduction to Counting & Probability**
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

15.3 Problems

Problem 15.3.1 (ICLT-2022-SM2-R1-L9-P1)**.** A roll of quarters contains 40 quarters and a roll of dimes contains 50 dimes.

James has a jar that contains 83 quarters and 159 dimes. Lindsay has a jar that contains 129 quarters and 266 dimes. James and Lindsay pool these quarters and dimes and make complete rolls with as many of the coins as possible.

How much are the leftover quarters and dimes worth?

(A) \$3.0 (B) \$2.5 (C) \$3.5 (D) \$5.0 (E) \$5.5

Problem 15.3.2 (ICLT-2022-SM2-R1-L9-P2)**.** A sphere is inscribed in a cylinder, meaning that it is tangent to both bases, and that one great circle of the sphere is along the curved surface of the cylinder.

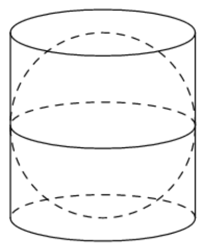


Figure 15.1: ICLT-2022-SM2-R1-L9-P2

Find the ratio of the surface area of the sphere to the lateral surface area of the cylinder.

(A) 1 (B) $\frac{4}{5}$ (C) $\frac{3}{4}$ (D) $\frac{2}{3}$ (E) $\frac{1}{2}$

Problem 15.3.3 (ICLT-2022-SM2-R1-L9-P3)**.** In how many ways can we distribute 13 pieces of candy among Anna, Bele, Cici, and Dodo such that Anna gets at least one candy, Bele gets at least two candy, Cici gets at least three candies, and Dodo gets at least four candies?

(A) 4 (B) 12 (C) 20 (D) 24 (E) 30

Problem 15.3.4 (ICLT-2022-SM2-R1-L9-P4)**.** How many solutions does the equation below has?

$$\sqrt{2x+10}-\sqrt{7-x}=\sqrt{3x-2}-\sqrt{-2x+19}$$

(A) 0 (B) 1 (C) 2 (D) 4 (E) 8

Problem 15.3.5 (ICLT-2022-SM2-R1-L9-P5)**.** How many ordered pairs (A, B) of digits are there such that $25A9B$ is a multiple of 36.

(A) 2 (B) 3 (C) 5 (D) 6 (E) 8

Problem 15.3.6 (ICLT-2022-SM2-R1-L9-P6)**.** What is the coefficient of the term of $(4x + \frac{y^2}{2})^6$ with a y^8 in it?

(A) $\frac{15}{8}$ (B) $\frac{15}{4}$ (C) $\frac{15}{2}$ (D) 15 (E) 240

Problem 15.3.7 (ICLT-2022-SM2-R1-L9-P7)**.** Solve

$$25^{-2} = \frac{(125)^{\frac{8}{x}}}{(25)^{\frac{13}{2x}} \cdot (5)^{\frac{17}{x}}}.$$

Problem 15.3.8 (ICLT-2022-SM2-R1-L9-P8)**.** In the figure, $ABCD$ is a square with side length 1, and M and N are the midpoints of AB and CD, respectively. X and Y are the intersections of DM and BN with AC.

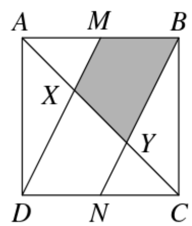


Figure 15.2: ICLT-2022-SM2-R1-L9-P8

Prove that $XYBM$ is the image of $YXDN$ by a rotation about the center of the square.

Problem 15.3.9 (ICLT-2022-SM2-R1-L9-P9)**.** Each term in the sequence

$$a_1 = 1,\ a_2 = 0.2,\ a_3 = 0.04,\ a_4 = 0.008, \ldots$$

is obtained by doubling the previous term and then shifting the decimal point one place to the left. What is the sum of all the terms in the sequence?

Problem 15.3.10 (ICLT-2022-SM2-R1-L9-P10)**.** Prove that any line, that divides a rectangle into two pieces of equal area, should passes through the intersection of the diagonals of the rectangle.

15.4 Grading

Answers for multiple-choice problems.

Problem 1: E

Problem 2: A

Problem 3: C

Problem 4: A

Problem 5: B

Problem 6: D

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can bring all powers to the same base.

Problem 8: 2 points if the student can identify the rotation angle ($180°$) and prove that the intersections of the images of segments are indeed the vertices of the image.

Problem 9: 2 points if the student can find the common ratio $r = \frac{1}{5}$.

Problem 10: 2 points if the student can find establish vertices of the rectangle, the equation of line k, and show some formulas (no necessarily correct) for areas of the pieces divided by the line.

15.5 Solutions

Solution. ICLT-2022-SM2-R1-L9-P1 *Exercise 12.4.5, Chapter 12, Introductory to Number Theory.*

We use residues of the numbers of each type of coin to determine the number of dimes and quarters leftover:

$$\begin{aligned} 83+129 &\equiv 3+9 \equiv 12 \pmod{40} \\ 159+266 &\equiv 9+16 \equiv 25 \pmod{50} \end{aligned}$$

The total value of the leftover quarters and dimes is $12(\$0.25)+25(\$0.10)=\$3.00+\$2.50=\boxed{\$5.50}$. The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R1-L9-P2 *Problem 15.3, Chapter 15, Introductory to Geometry.*

Let r be the radius of the sphere. Then the radius and the height of the cylinder are r and $2r$, respectively. The surface area of sphere is $4\pi r^2$. The lateral surface area of the cylinder is $2\pi r\cdot 2r=4\pi r^2$. Therefore their ratio is $\boxed{1}$.

The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R1-L9-P3 *Problem 12.4.1.b, Chapter 10, Introductory to Counting & Probability.*

Let give Anna one candy, Bele two candy, Cici three candies, and Dodo four candies. Thus we are left with 3 candies and we distribute these for 4 persons

$$\binom{3+4-1}{4-1}=\binom{6}{3}=\boxed{20.}$$

The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R1-L9-P4 *Problem 20.5, Chapter 20, Introductory to Algebra.*

First, in order for the equation stands

$$\sqrt{2x+10}-\sqrt{7-x}=\sqrt{3x-2}-\sqrt{-2x+19}\Rightarrow x\ge -5, x\le 7, x\ge\frac{3}{2}, x\le\frac{19}{2}\Rightarrow\frac{3}{2}\le x\le 7\ (*)$$

Now, by squaring both sides

$$\begin{aligned} &(\sqrt{2x+10}-\sqrt{7-x})^2=(\sqrt{3x-2}-\sqrt{-2x+19})^2 \\ &x+17-2\sqrt{(2x+10)(7-x)}=x+17-2\sqrt{(3x-2)(-2x+19)} \\ \Rightarrow\ &(2x+10)(7-x)=(3x-2)(-2x+19)\Rightarrow 4x^2-57x+108=0\Rightarrow x=\frac{19}{2}, x=12\ (**) \end{aligned}$$

Comparing the possible solutions in (**) with the conditions for existence (*), it implies that there is no solution. The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R1-L9-P5 *Problem 13.9, Chapter 13, Introductory to Number Theory.*

First $25A9B$ is divisible by 4, so $9B$ is divisible by 4, thus B is one of $2,6$. The sum of digits of $25A9B$ is $16+A+B$. This is a multiple of 9 so $7+A+B$ is a multiple of 9. Thus if $B=2$, then A is one of $0,9$. If $B=6$, then $A+4$ is a multiple of 9, so $A=5$. Therefore the desired pairs are $(0,2),(9,2),(5,6)$.

The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R1-L9-P6 *Problem 14.6, Chapter 14, Introductory to Counting & Probability.*

How do we get a term in the expansion of $(4x+\frac{y^2}{2})^8$ with a y^8 in it? The answer is that the term with a y^8 in it is the term that has a $(\frac{y^2}{2})^4$ in it. We know that when expanding $(4x+\frac{y^2}{2})^8$, we have to choose 4 copies of $\frac{y^2}{2}$ from the six $(4x+\frac{y^2}{2})$ terms in order to get a term with y^8 in it. This can be done in $\binom{6}{4}$ ways. We then take an x from each of the remaining two $(4x+\frac{y^2}{2})$ terms that didn't contribute a $\frac{y^2}{2}$. Therefore, the relevant term in the expansion of $(4x+\frac{y^2}{2})^6$ is

$$\binom{6}{4}(4x)^2(\frac{y^2}{2})^4 = 15x^2y^8$$

The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R1-L9-P7 *Problem 19.11, Chapter 19, Introductory to Algebra.*

$$\frac{125^{\frac{8}{x}}}{5^{\frac{13}{x}}\cdot 5^{\frac{17}{x}}} = 5^{\frac{24}{x}-\frac{13}{x}-\frac{17}{x}} = 5^{-\frac{6}{x}}$$
$$25^{-2} = 5^{-\frac{6}{x}} \Rightarrow -4 = -\frac{6}{x} \Rightarrow x = \boxed{\frac{3}{2}.}$$

□

Solution. ICLT-2022-SM2-R1-L9-P8 *Problem 16.16, Chapter 16, Introductory to Geometry.*

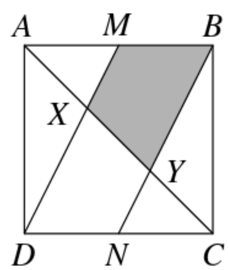


Figure 15.3: ICLT-2022-SM2-R1-L9-P8

Let O be the center of the square. O is the midpoint of diagonal BD, and the midpoint of MN, which connects the midpoints of the opposite sides of the square. Therefore, the image of BN upon 180° rotation about O is DM. Furthermore, the image of AC upon this rotation is CA, so the image of the intersection of BN and AC (point Y) is the intersection of DM and CA (these are the images of BN and AC), which is point X.

Therefore, $DNYX$ is the image of $BMXY$ upon 180° rotation about the center of the square. □

Solution. ICLT-2022-SM2-R1-L9-P9 *Problem 21.4.4, Chapter 21, Introductory to Algebra.*

Every term of the sequence is obtained by doubling the previous term and then shifting the decimal point one place to the left,

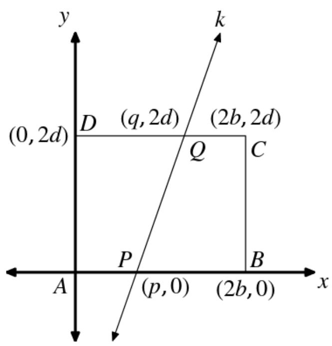
$$a_{n+1} = \frac{2a_n}{10} = \frac{1}{5}a_n \Rightarrow \{a_n\} \text{ is a geometric sequence with } r = \frac{1}{5}.$$

Hence, $\sum_{n=1}^{\infty} a_n = \frac{a_1}{1-r} = \boxed{\frac{5}{4}.}$ □

Solution. [First solution] ICLT-2022-SM2-R1-L9-P10 *Problem 17.17, Chapter 17, Introductory to Geometry.*

First, it is easy to see that if a line divides a rectangle into to pieces of equal areas, then it should intersect two opposite (parallel) sides. Otherwise one of the pieces is less than half of the rectangle.

Let A be the origin, $B = (2b, 0)$, $C = (2b, 2d)$, and $D = (0, 2d)$. Line k intersects AB and CD at $P = (p, 0)$ and $Q = (q, 2d)$ as shown.



If k bisects the area of $ABCD$, then it splits it into two trapezoids with equal area. The bases of trapezoid $APQD$ have lengths p and q, and $APQD$ has height $2d$, so we have

$$[APQD] = (2d)\left(\frac{p+q}{2}\right) = d(p+q).$$

The bases of $PBCQ$ have lengths $2b - p$ and $2b - q$, so we have

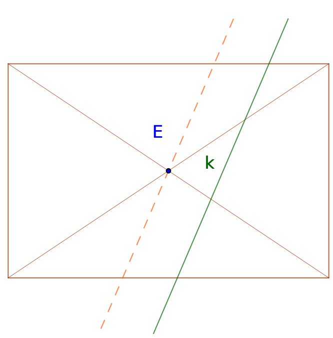
$$[PBCQ] = (2d)\left(\frac{2b-p+2b-q}{2}\right) = d(4b-p-q).$$

Setting these equal gives us $d(p+q) = d(4b-p-q)$. Dividing by d gives us $p + q = 4b - p - q$, so $2(p + q) = 4b$. Finally, we find that $p + q = 2b$. The midpoint of AC is (b, d). The midpoint of $\overline{PQ}$ is

$$\left(\frac{p+q}{2}, \frac{0+2d}{2}\right) = (b, d).$$

Therefore, $\boxed{\text{line } k \text{ passes through the intersection of the diagonals of } ABCD.}$ □

Solution. [Second solution] ICLT-2022-SM2-R1-L9-P10



Let E be the center of the rectangle. If k does not pass through E, then let draw a line through E and parallel to k. It is easy to see that this line divides the rectangle into two equal pieces, thus k can not divide the rectangle into two equal pieces, thus $\boxed{k \text{ must pass through } E.}$ □

Chapter 16

Problem Solving Championship - Round 2

16.1 Rules

Problem Solving Championship is an activity of Math, Chess, and Coding Club (MCC) open to all students. The problems are designed so that students at various levels - junior (grade under 8), senior (grade 9 and up), or olympiad (prepare for national or international contest) - can participate. The contest problems are selected from algebra, combinatorics, number theory, geometry topics of mathematics, some are puzzles from chess, and some need to be solved by designing a computer program. Some problems do not require specific mathematical knowledge, some require problem solving skills, and some require a more comprehensive understanding of certain mathematical topics. The students received a number of points after each participation and are ranked after each round, in both overall and grade rankings. The final standing of the contest is concluded at the end of each semester.

- The contest problems become available online at the beginning of the semester or at the end of the previous contest. The contest solutions are discussed at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**. All students in the club are invited to the solution discussion. All contests start when the contest problems become available. The **solutions must be submitted latest on the last Sunday**, approximately one week **before the solution discussion day on Saturday**.
- There are 4 **show-you-work** problems with multiple steps. For each step of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too. If you solve the problem by designing a computer program, submit that program as the solution to the problem.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading: For each step there are a number of points, highlighted in the problem text, to be awarded. For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Invitation to present own solutions: Students who submitted correct solutions would be invited to present their solutions on the solution day. Students with nearly complete or almost correct solutions would also have a chance for presentation, provided that they studied the comments and suggestions from the COs to modify their solutions.

16.2 Problems

Problem 16.2.1 (PSC-2022-SM2-R2-P1). (*25 points*)

Find the answers for each of the three separate questions. *Note that the result of a solved question, appeared earlier in the order of appearance, can be used in answer to the following one.*

(*5 points*) Given a square $\mathcal{S}$ and a point P outside of the square. Prove, *by contradiction,* that one side of $\mathcal{S}$ can be extended into a line that divides the plane into two half-planes, one contains the square $\mathcal{S}$ and the other contains the point P.

(*5 points*) A square $\mathcal{S}$ is inside a circle $\mathcal{C}$ such that $\mathcal{S}$ does not contains the centre of $\mathcal{C}$. Prove that there exists a diameter of $\mathcal{C}$, parallel to one side of $\mathcal{S}$, divides $\mathcal{C}$ into two half-circles, and one of them contains $\mathcal{S}$.

(*15 points*) A square $\mathcal{S}$ is inside a circle $\mathcal{C}$ radius 1, such that $\mathcal{S}$ does not contains the centre of $\mathcal{C}$. Prove that the side of S cannot be longer than $\sqrt{\frac{4}{5}}$.

Problem 16.2.2 (PSC-2022-SM2-R2-P2)**.** (*25 points*)

Find the answers for each of the four separate questions. *Note that the result of a solved question, appeared earlier in the order of appearance, can be used in answer to the following one.*

Let (a_1, a_2, a_3, a_4) be a permutation of $(1, 2, 3, 4)$, The absolute value $|a_1 - a_2|$ is the positive difference between a_1 and a_2. *For example for* $a_1 = 2, a_2 = 4, a_3 = 3, a_4 = 1$, $|a_1 - a_2| = |2 - 4| = |-2| = 2$.

The tables below list all triples $(a_1, a_2, |a_1 - a_2|)$.

a_1	a_2	$\|a_1 - a_2\|$
1	2	1
1	3	2
1	4	3

a_1	a_2	$\|a_1 - a_2\|$
2	1	1
2	3	1
2	4	2

a_1	a_2	$\|a_1 - a_2\|$
3	1	2
3	2	1
3	4	1

a_1	a_2	$\|a_1 - a_2\|$
4	1	3
4	2	2
4	3	1

(*2 points*) For a pair (a_1, a_2), how many permutations (a_1, a_2, a_3, a_4) of $(1, 2, 3, 4)$ are there?

(*3 points*) For all permutations (a_1, a_2, a_3, a_4) of $(1, 2, 3, 4)$, find the average value of

$$|a_1 - a_2|.$$

(*5 points*) For all permutations (a_1, a_2, a_3, a_4) of $(1, 2, 3, 4)$, find the average value of the sum

$$|a_1 - a_2| + |a_3 - a_4|.$$

(*15 points*) For all permutations $(a_1, a_2, \ldots, a_{2022})$ of $(1, 2, \ldots, 2022)$, find the average value of the sum

$$|a_1 - a_2| + |a_3 - a_4| + \ldots + |a_{2021} - a_{2022}|.$$

Problem 16.2.3 (PSC-2022-SM2-R2-P3)**.** (*25 points*)

Find the answers for each of the three separate questions. *Note that the result of a solved question, appeared earlier in the order of appearance, can be used in answer to the following one.*

(*5 points*) $n+1$ is a composite number, find the greatest common divisor of $n!+1$ and $(n+1)!$.

(*10 points*) $n+1$ is a prime number, find the greatest common divisor of $n!+1$ and $(n+1)!$.

(*10 points*) p is an odd prime number. Note that

$$1 \equiv -(p-1) \pmod{p},\ 3 \equiv -(p-3) \pmod{p},\ \ldots$$

By the Wilson's Theorem,

$$(1 \cdot 3 \cdots (p-2))\,(2 \cdot 4 \cdots (p-1)) \equiv -1 \pmod{p}.$$

Prove that

$$(1 \cdot 3 \cdots (p-2))^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p} \text{ and } (2 \cdot 4 \cdots (p-1))^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}$$

Theorem (Wilson's Theorem)

If integer $p > 1$, then $(p-1)!+1$ is divisible by p if and only if p is prime.

Proof. Wilson's Theorem Let that p be a prime number, we prove that $(p-1)!+1 \equiv 0 \pmod{p}$.

Claim — If $a \in \{1, 2, \ldots p-1\}$ then there exist one and only one $b \in \{1, 2, \ldots p-1\}$ so $ab \equiv 1 \pmod{p}$.

Proof. Assume that there exist $b_1 \neq b_2$ and $b_1, b_2 \in \{1, 2, \ldots, p-1\}$, such that $ab_1 \equiv ab_2 \equiv 1 \pmod{p}$.

First, none of the products $1a, 2a, \ldots, (p-1)a$ should have all residues 0 modulo p, in other words, none of them is divisible by p. Second, for any $b_1 \neq b_2 \in \{1, 2, \ldots, p-1\}$, $ab_1 - ab_2 = a(b_1 - b_2)$, and since since $a \not\equiv 0 \pmod{p}$, $b_1 - b_2 \neq 0$, and $-p < b_1 - b_2 < p$, so $1a, 2a, \ldots, (p-1)a$ should have different residues modulo p, in other words, different remainders when divided by p. ■

Now, since a pair (a, a) has residue 1 modulo p is equivalent to $p \mid a^2 - 1 = (a-1)(a+1)$, or $a = 1$, or $a = p-1$; so the numbers $2, 3, \ldots p-2$ are groupped into $\frac{p-3}{2}$ pairs of distinct numbers such that the product of them has residue 1 modulo p, in other words, has a remainder 1 when divided by p. Therefore

$$(p-1)!+1 \equiv 1(2 \cdot 3 \cdots (p-2))(p-1)+1 \equiv 1(p-1)+1 \equiv 0 \pmod{p}.$$

For the opposite, assume that p is composite, then p has a prime factor q such that $1 < q < p$, thus $q \mid (p-1)!$. Since $q \mid p \mid (p-1)!+1$. Hence, $q \mid ((p-1)!+1) - (p-1)! = 1$. Impossible. Thus, p is a prime. □

Problem 16.2.4 (PSC-2022-SM2-R2-P4). (*25 points*)

(*5 points*) $P(x) = x^2 + bx + 2$ is a second degree polynomial (with the coefficient of x^2 is 1).

The number 1 is a root of $P(x)$. Find the other root of $P(x)$.

(*10 points*) Find a second degree polynomial $P(x)$ such that

$$(x+1)P(x) = (x-2)P(x+1).$$

(*10 points*) For an arbitrary n positive integer, find the polynomial $P(x)$ such that

$$(x+1)P(x) = (x-n)P(x+1).$$

Definition (Polynomial). For n positive integer, $P(x)$ is a $n-$degree polynomial if there exist real number $a_n \neq 0, a_{n-1}, \ldots, a_1, a_0$ such that

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0.$$

The $P(x) = c$, where c is a real number is called zero-degree, or constant polynomial.

Definition (Quadratic Polynomial). A second degree polynomial $P(x)$ is called a *quadratic*. In other words, if there exist $a \neq 0, b$, and c real number such that

$$P(x) = ax^2 + bx + c.$$

Definition (Definition of Root). For n positive integer, $P(x)$ is a $n-$degree polynomial. A real number r is called a **root** of $P(x)$ if $P(r) = 0$.

Fact (Factorization by roots). n is a positive integer, $P(x)$ is a $n-$degree polynomial. If real number $r_1, r_2, \ldots, r_m$ are roots of $P(x)$ then there exist a $(n-m)-$degree polynomial $Q(x)$ such that

$$P(x) = (x-r_1)(x-r_2)\ldots(x-r_m)Q(x).$$

If $n = m$ then $Q(x)$ is a constant polynomial.

Fact (Existence of unique coefficients). $P(x)$ and $Q(x)$ are both $n-$degree polynomials,

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \text{ where } a_n \neq 0.$$
$$Q(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0, \text{ where } n_n \neq 0.$$

If $P(x) = Q(x)$, for all real values of x, then $a_n = b_n,\ a_{n-1} = b_{n-1},\ \ldots,\ a_1 = b_1,\ a_0 = b_0.$

16.3 Grading

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 1: Separately grading for each part,

(a) 2 points if the student can divide the plane into 8 parts.
(b) 2 points if the student can draw a diameter parallel to one of the sides.
(c) 5 points if the student can somehow enlarge the original square in at least one step.

Problem 2: Separately grading for each part,

(a) 1 points if the student can see the number of permutations of (a_3, a_4) is 2!.
(b) 1 points if the student can sum up all coloured values.
(c) 2 points if the student can add up the possible values of $|a_3 - a_4|$, or discover the equality of that with the average value of $|a_1 - a_2|$ with some minor mistakes.
(d) 5 points if the student can discover a way to compute the generic sum based on either assigning $a_1 = k$ for a subcase or try to sum up by correctly generalize all tables from the simple case.

Problem 3: Separately grading for each part,

(a) 2 points if the student can analyze an arbitrary prime $p \leq n$.
(b) 5 points if the student can prove $n + 1$ is a conmmon factor of both numbers.
(c) 5 points if the student can discover roles of products A and B.

Problem 4: Separately grading for each part,

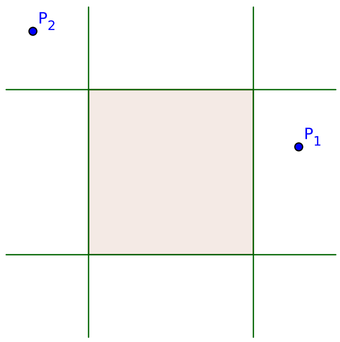
(a) 2 points if the student can compare the coefficients.
(b) 5 points if the student can use substitution to discover either $P(0) = 0$, $P(1) = 0$,
(c) 5 points if the student can discover $P(x) = x(x-1)\dots(x-n)Q(x)$.

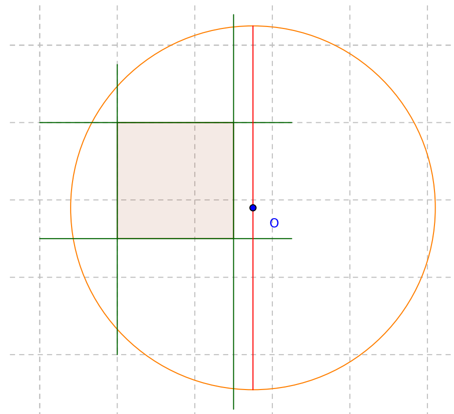
Note that,

- *if the student claimed to solve any of the problems by designing some programs, but did not submit the program, then the solution cannot be accepted, thus no point can be given;*
- *if the student submitted a program that produces the same outputs as the answers of the student, then the CO should grade the solution by inspecting the program;*
- *if the student submitted a program that produces some different outputs than the answers of the student, then the CO should ignore the program, and no point can be given.*

16.4 Solutions

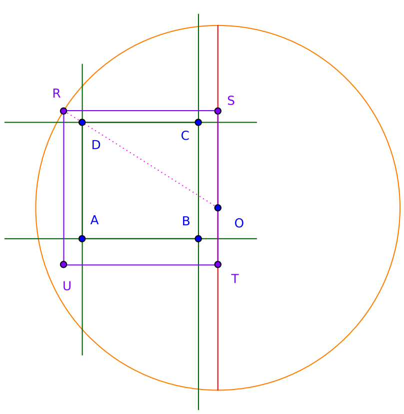
Solution. PSC-2022-SM2-R2-P1 For the first question, lets extend every side of the square $\mathcal{S}$ into a line. It is easy to see that the lines divide the plane into 8 parts, none of them contains $\mathcal{S}$. See below on the left. Assume the opposite, then point P cannot be in any of these parts, because any point in these parts shall be separated from $\mathcal{S}$ by a line extended from a side of $\mathcal{S}$. Thus, P shall be inside of $\mathcal{S}$, which is a contradiction to the given condition.

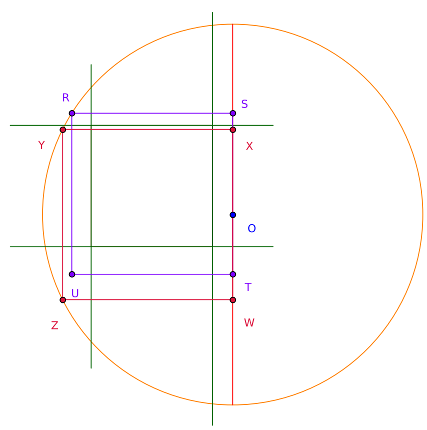




For the second question, by the result of the first, the centre O of $\mathcal{C}$ is outside of $\mathcal{S}$, so there is a line extended from a side of $\mathcal{S}$ and divides $\mathcal{C}$ into two parts, one contain $\mathcal{S}$ and the other contains O. See above on the right. A diameter through O parallel with this line divides $\mathcal{C}$ into two half-circles, one contains $\mathcal{S}$ and the other contains O.

Now, let A, B, C, and D be $\mathcal{S}$'s vertices, such that line through BC iseparates $\mathcal{S}$ from O. Both A and D are inside the circle. R is the intersection of line through OD with the perimeter. See below on the left. *Enlarge* the square $ABCD$ into the square $RSTU$, in other words, construct the square $RSTU$, where vertex S is the foot of the perpendicular line from R to the diameter, note that $BC \leq ST$.





If vertex T is not on the perimeter, the square $RSTU$ can be translated (move) along the diameter and *enlarge* into the square $WXYZ$, such that O becomes the midpoint of ST. Hence $BC \leq ST \leq XW$.

$$1 = OY^2 = XY^2 + XO^2 = 5XO^2 \Rightarrow XO = \frac{1}{\sqrt{5}} \Rightarrow XW = 2XO = \boxed{\sqrt{\frac{4}{5}}}.$$

$\square$

Solution. PSC-2022-SM2-R2-P2 For the first question, note that for each pair (a_1, a_2) listed in the table, there are actually two different permutations with the same pair, for example

$$(1,2,3,4),\ (1,2,4,3),$$

since there are $2! = \boxed{2}$ ways to permute the remaining numbers a_3, a_4.

a_1	a_2	$\lvert a_1 - a_2\rvert$
1	2	1
1	3	2
1	4	3

a_1	a_2	$\lvert a_1 - a_2\rvert$
2	1	1
2	3	1
2	4	2

a_1	a_2	$\lvert a_1 - a_2\rvert$
3	1	2
3	2	1
3	4	1

a_1	a_2	$\lvert a_1 - a_2\rvert$
4	1	3
4	2	2
4	3	1

For the second question, the sum of all the coloured values listed in the tables is

$$\underbrace{1+1+\ldots+1}_{6}+\underbrace{2+\ldots+2}_{4}+\underbrace{3+\ldots+3}_{2} = 1\cdot 6 + 2\cdot 4 + 3\cdot 2 = 20,$$

the sum of all values of $|a_1 - a_2|$ is twice of that, thus it is 40. The number of permutations is 4!, thus the average value of $|a_1 - a_2|$ is $\frac{40}{4!} = \boxed{\dfrac{5}{3}.}$

For the third question, it is easy to see that practically the pair (a_3, a_4) repeats exactly all possible values of (a_1, a_2), thus the average value of $|a_3 - a_4|$ is the same, therefore the average value of the sum $|a_1 - a_2| + |a_3 - a_4|$ is twice the average value of $|a_1 - a_2|$, or $\boxed{\dfrac{10}{3}.}$

The fourth question is just a special case of the generalization, where $n = 1000$. Following the reasoning of the simple case with $n = 2$, we just need to find the average value of $|a_1 - a_2|$, since it is the same as the average value of $|a_3 - a_4|, \ldots, |a_{2n-1} - a_{2n}|$.

Now, consider $a_1 = k$, where $1 \le k \le 2n$. Basically it the same as if we examine the k^{th} table in the simple case, the sum of all possible values of $|a_1 - a_2|$ in this case is,

$$\begin{aligned}
&|k-1| + |k-2| + \ldots + |k-(k-1)| + |k-(k+1)| + \ldots + |k-2n| \\
&= (k-1) + (k-2) + \ldots + 1 + 1 + 2 + \ldots + (2n-k) \\
&= \frac{(k-1)k}{2} + \frac{(2n-k)(2n-k+1)}{2} = k^2 - (2n+1)k + n(2n+1)
\end{aligned}$$

There are $2n-1$ pairs of values (a_1, a_2) ($2n-1$ lines in the k^{th} table), thus the average value of $|a_1 - a_2|$,

$$\frac{k^2 - (2n+1)k + n(2n+1)}{2n-1}$$

For all possible values of a_1 where $a_1 \in \{1, 2, \ldots, 2n\}$, the average value of $|a_1 - a_2|$ is

$$\begin{aligned}
&\frac{1}{2n}\sum_{k=1}^{2n}\frac{k^2 - (2n+1)k + n(2n+1)}{2n-1} = \frac{1}{2n(2n-1)}\left(\sum_{k=1}^{2n}k^2 - (2n+1)\sum_{k=1}^{2n}k + (2n)(n)(2n+1)\right) \\
&= \frac{1}{2n(2n-1)}\left(\frac{2n(2n+1)(4n+1)}{6} - (2n+1)\frac{2n(2n+1)}{2} + (2n)(n)(2n+1)\right) = \frac{2n+1}{3}
\end{aligned}$$

The average of the sum $|a_1 - a_2| + |a_3 - a_4| + \ldots + |a_{2n-1} - a_{2n}|$ is n times the average value of $|a_1 - a_2|$,

$$n \cdot \frac{2n+1}{3},\ n = 1011 \Rightarrow \frac{1011 \cdot 2023}{3} = \boxed{681751.}$$

$\square$

Solution. PSC-2022-SM2-R2-P3 For the first question, if $n+1$ is a composite number, then for any prime number $p < n+1$ or $p \leq n$, so $p \mid n!$, thus $p \nmid n!+1$. However $p \mid (n+1)!$, thus $\gcd(n!+1, (n+1)!) = \boxed{1.}$

For the second question, any prime factor of both numbers $n!+1, (n+1)!$, shall be a prime factor of $(n+1)!$, meaning $n+1$ is a prime number or there exists $q \leq n$ prime number. As in the previous question, for any $q \leq n$ prime number, $q \mid n!$, so $q \nmid n!+1$, so q cannot be a common factor of both numbers $n!+1, (n+1)!$.

Now, $n+1$ is a prime, by Wilson's Theorem $n!+1 \equiv 0 \pmod{n+1}$. It is obvious that $n+1 \mid (n+1)!$, so $n+1$ is a common factor of both $n!+1, (n+1)!$. However $n+1 \nmid n!$, so $(n+1)^2 \nmid (n+1)!$. Thus, the greatest commmon divisor of $n!+1$ and $(n+1)!$ is $\boxed{\text{n+1.}}$

For the third question, let $A = 1 \cdot 3 \cdots (p-2)$, $B = 2 \cdot 4 \cdots (p-1)$, by rearranging the congruence equality in the Wilson's Theorem,

$$AB = (1 \cdot 3 \cdots (p-2))\,(2 \cdot 4 \cdots (p-1)) \equiv -1 \pmod{p}$$

On the other hand,

$$\begin{aligned}
&A = 1 \cdot 3 \cdots (p-2) \equiv (-(p-1))(-(p-3)) \cdots (-(p-(p-2))) = (-1)^{\frac{p-1}{2}}(p-1)(p-3)\cdots 2 \\
&\Rightarrow A \equiv (-1)^{\frac{p-1}{2}} B \pmod{p} \Rightarrow (AB)A \equiv (-1)(-1)^{\frac{p-1}{2}} B \pmod{p}
\end{aligned}$$

Since $p \nmid B$, thus $\boxed{A^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}.}$ Similarly $\boxed{B^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}.}$ $\square$

Solution. PSC-2022-SM2-R2-P4 For the first question, since 1 is a root or $P(x)$, by Factorization by roots, there exists $Q(x) = b_1x + b_0$,

$$x^2 + bx + 2 = (x-1)(b_1x + b_0) \Rightarrow x^2 + bx + 2 = b_1x^2 + (b_0 - b_1)x - b_0.$$

By Existence of unique coefficients,

$$\begin{cases} 1 = b_1 \\ b = b_0 - b_1 \\ 2 = -b_0 \end{cases} \Rightarrow b_0 = -2, b_1 = 1, b = -3 \Rightarrow P(x) = x^2 - 3x + 2 = (x-1)(x-2).$$

Therefore the second root of $P(x)$ is $\boxed{2}$.

For the second question,

$$(x+1)P(x) = (x-2)P(x+1) \Rightarrow \begin{cases} x = -1 & \Rightarrow (0)P(-1) = (-3)P(0) \Rightarrow P(0) = 0 \\ x = 0 & \Rightarrow (+1)P(0) = (-2)P(1) \Rightarrow P(1) = 0 \\ x = +1 & \Rightarrow (+2)P(1) = (-1)P(2) \Rightarrow P(2) = 0 \end{cases}$$

Thus, by Definition of Root $0, 1$ and 2 are roots of $P(x)$. $P(x)$ is a Quadratic Polynomial, there exist a, b, and c real numbers $(a \neq 0)$

$$P(x) = ax^2 + bx + c \quad (*)$$

On the other hand by Factorization by roots, there exist a constant polynomial $Q(x) = d$, where $d \neq 0$ real number, such that

$$P(x) = (x-0)(x-1)(x-2)Q(x) = dx^3 - 3dx^2 + 2dx \quad (**)$$

(*) and (**) imply a contradiction: $P(x)$ cannot be both second- and third-degree polynomial. Hence, $\boxed{\text{there is no such } P(x).}$

For the third question, by substituting $x = -1$, $x = 0, \ldots, x = n-1$, into $(x+1)P(x) = (x-n)P(x+1)$,

$$\begin{aligned} (0)P(-1) = (-1-n)P(0) \Rightarrow &\quad P(0) = 0 \\ (1)P(0) = (-n)P(1) \Rightarrow &\quad P(1) = 0 \\ &\ldots \\ (n)P(n-1) = (-1)P(n) \Rightarrow &\quad P(n) = 0 \end{aligned}$$

So $0, 1, \ldots, n$ are roots of $P(x)$, thus by Factorization by roots, there exist polynomial $Q(x)$, such that

$$P(x) = x(x-1)\ldots(x-n)Q(x).$$

Substituting this into the given equation $(x+1)P(x) = (x-n)P(x+1)$,

$$(x+1)x(x-1)\ldots(x-n)Q(x) = (x-n)(x+1)(x)\ldots(x-n+1)Q(x+1) \Rightarrow Q(x) = Q(x+1).$$

Thus $Q(x) = Q(x+1)$ for all real value of x. This can only be possible if $Q(x)$ is a constant polynomial. This can only be possible if and only if $Q(x) = c$ is a constant polynomial, where c is a real number.

Therefore $\boxed{P(x) = cx(x-1)\ldots(x-n),}$ where c is an arbitrary real number. □

Chapter 17

House Championship - Round 3

17.1 Rules

House Championship is an activity of Math, Chess, and Coding Club (MCC). All students must participate. The students are selected into houses. The championship consists of a number of weekly contests between the teams at various levels. The contest content contains problems and puzzles (including visual, language and chess puzzles) that can be solved by using math, coding, and chess. The houses are ranked after each round. The final standing of the championship is revealed at the end of each semester.

- **Qualification**: *All students* are *qualified* and *required* to participate.
- **Houses**: Initially an even number of $2n$ houses are established depending on the number of students.
  1. Each house consists of **at least 6** students. Initially, house members are selected by the club. Each house will have a captain, a lieutenant, and members. The captain can be elected by the members or appointed by the club.
  2. The members can name their house, design new coats-of-arm, slogan, song, cloths, etc if they wish. Common sense ethics are required.
  3. New members of the club will be assigned to a house at the beginning of each semester. The houses can be reorganized depending on the total number of students.
- **Format**: The championship is contested via $2n - 2$ rounds:
  1. In each round all the houses are organized into distinct pairs. The houses in a pair participate in a *round contest*. A round contest contains of a number of problems, grouped by *levels of difficulty* (beginner, intermediate, and advanced). In a few days before the contest, the topics of the contest and the number of difficulty levels.
  2. A house organizes a number of *troops* depending on the number of difficulty levels of the coming round contest. Each troop has an approximately equal number of students. No house member can participate in more than one troop.
  3. A round contest consists of a number of troop games. Each game will be played between *same-level troops*. Each troop receive the same set of problems to be solved.

- **Hit, score, and rank**: In each round:
  1. For each problem in the troop game, each troop will submit a number (i.e 3), a set of numbers (i.e. $\{3, 4\}$), or a sequence of letters (i.e *T-Rex dinosaur* or *babyface*), or in a format stated by the problem.
  2. There are two different ways how a troop can score: (i) for some problems if the answer is correct, the troop scores 1 hit; (ii) for some problems, the troop provided the best answer scores 1 hit, the troop that cannot provide the best answer score 0 point. Obviously two equal best answers result in 1 point for both troops.
  3. Each troop can submit multiple answers for a problem before the end of the game, but only the last submission is to be graded. Any answer submitted after the end of the game is not taken into consideration. The grading of the submissions is done right after the end of the game. Result is announce shortly after the discussion of the solutions.
  4. For each troop game, if both troops have the same number of hits, the game results in a draw. Otherwise, the troop with more hits is the winning troop while the other becomes the ultimate loser. *For example, House A meets House B and the results of troop games are:* $A1 - B1 : 2 - 1, A2 - B2 : 3 - 1, A3 - B3 : 0 : 3$.
  5. For each game, a house receives 3 points for a game, 1 point for a drawn, 0 point for a loss. The round score of a house is the sum of the points from the three games it plays. *In the example above, House A receives* $3 + 3 = 6$ *points because they won two games and lost one game, House B receives* 3 *points because they lost two games and won one game. So A is the round winner and the score is* $A - B : 6 - 3$.
  6. After each round, all house standings are updated. *Note: It is important to note that since a round consists of three 1-to-1 games, it is wise for a house to organize its troop according to not only the expertises of its members but also the capabilities of the troops of the other team. A relatively weaker, but wiser house can win if it knows itself and it knows its opponent.*
  7. The *champion* house of the club is the one who has the most points after all rounds are concluded. In case the two top houses have the same number of points, a final contest will be held to decide the ultimate winner.
- **Problems**: Each troop receives a number of problems for each game. The problems the house receives can be based on any of the following areas:
  1. Logical, math, visual, language, or chess puzzles.
  2. Problems that can be solved by a computer program designed in the contest.
  3. Problems that can be solved with competitive math knowledge and skills.
- **Time**:
  1. All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
  2. The house must announce its troop compositions (what are the troops and who are the members of which troops) to the Contest Organizer. All participants are required to be present 10 minutes before the contest starts.
  3. A house that does not have troop for a game shall automatically lose, earning 0 points for that game. If the opposing troop is present, they shall receive maximum possible points for that game.
- **Contest organization**: The Contest Organizers (CO) are responsible for overseeing the troop games.
  1. The COs allow the troops participating in that game to access the game's problems. The troops can update the answers to the problems according to instructions from the COs. The COs decision on grading is final.
  2. All game scores between the troops are publicly available at any point in the contest, automatically updated when a team has made a hit. The round scores are also automatically updated accordingly.
  3. The contest is carried out via Zoom. All team member must show up on the video. Only the team captains can communicate to the COs: raise your hand when you want to talk; mute your microphone for the rest of the time; use the chat if your audio is not working. Please inform the COs that your mic is not working immediately at the start of the contest.

- **Tools**:
  1. The contest is carried out via Zoom. All team member must show up on the video. Only the team captains can communicate to the COs: raise your hand when you want to talk; mute your microphone for the rest of the time; use the chat if your audio is not working. Please inform the COs that your mic is not working immediately at the start of the contest.
  2. Paper-based books, hand-written notes, calculators are allowed during the contest.
  3. Computers are allowed for programming (coding) only.
  4. Using the Internet to search for similar problems, answers, communicate with outside people, etc, is strictly prohibited.
  5. Help from anyone outside the house is not allowed. Be honest.

  Any violation to the rules, especially using external help (results from the Internet, information from people outside of the team, etc.) can cause the house to lose all of their games in that round.

17.2 Problems

Problem 17.2.1 (HC-2022-SM2-R2-P1)**.** (*Beginner Level*)

Outside of an arbitrary triangle T, which has a $60°$ angle, three equilateral triangles T_b, T_g, and T_r are constructed on the left of the diagram shown below. Khoa uses three copies of T and one copy of T_g to create a large equilateral triangle, show below on the right of the diagram.

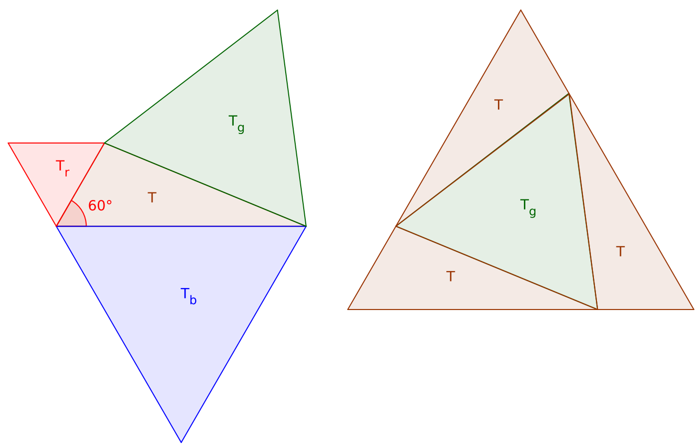


Figure 17.1: HC-2022-SM2-R2-P1

Is there another way for Khoa to build another equilateral triangle, using *a different combination* T, T_b, T_g *and* T_r?

How to provide your answer:

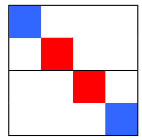
- If you think you have a solution, create a diagram (or by any other means that can visualize your solution), and submit it.
- If you think that there is no solution, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

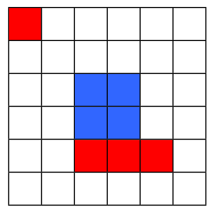
- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop does not change regardless of whatever your opponent submitted.

Problem 17.2.2 (HC-2022-SM2-R2-P2)**.** (*Beginner Level*)

The 4×4 board below is cut along the grid lines into two *congruent shapes* so that each shape contains exactly one red and one blue square.



Cut the board below along the grid lines into four *congruent shapes* so that each shape contains exactly one red and one blue square.



How to provide your answer:

- If you think that the number of matches have been played is at least 1, then submit 1.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 17.2.3 (HC-2022-SM2-R2-P3)**.** (*Beginner Level*)

An, Binh, Chi, and Duy live in the houses A, B, C, and D; and go to the schools $1, 2, 3$, and 4, respectively. (That is, An lives in the house A and goes to the school 1, Binh lives in the house B and goes to the school 2, and so on.) The houses and the schools are inside a village surrounded by four tree lines.

One day in the winter time, the friends wanted to ski to the schools. They started as shown in the diagram below and arrived to their respective schools in a manner such that their tracks did not cross each other. Of course, they cannot ski outside of the boundary of the village.

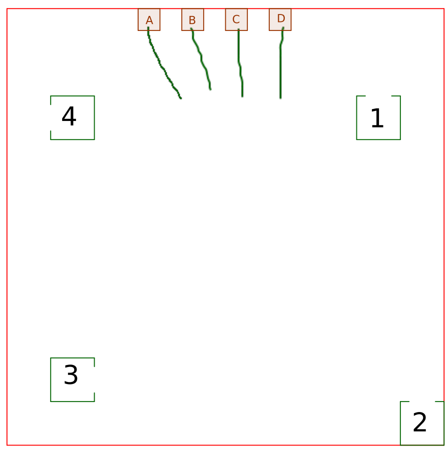


Figure 17.2: HC-2022-SM2-R2-P3

How did they do that?

How to provide your answer:

- If you think that it is possible to do so, submit a diagram.
- If you cannot determine the answer, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

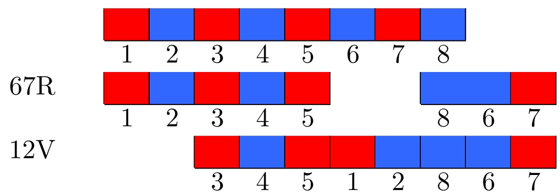
- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 17.2.4 (HC-2022-SM2-R2-P4)**.** (*Beginner Level*)

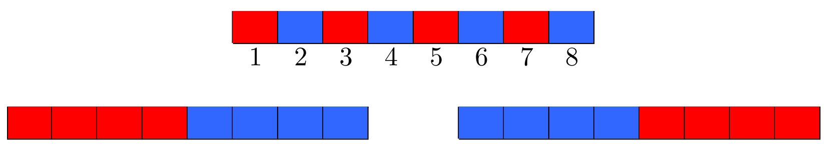
4 red and 4 blue squares form a block as shown below on the first row. In one move, choose freely between

- taking two neighbouring squares and moving them to one end of the block without changing their order, leaving two neighbouring vacant spaces; or
- taking take two neighbouring squares and moving them into two neighbouring vacant spaces.

For example, the diagram below show two moves by first taking the squares 6 *and* 7 *and move them to the right* (67R)*; then taking the squares* 1 *and* 2 *and move them to two neighbouring vacant spaces* (12V).



Find the least number of moves needed to change the block from its initial state on the first row into one of the two end states on the second row in the diagram below?



Note that no vacant spaces are allowed between the coloured squares in any of the end states.

How to provide your answer:

- If you think that 2 moves of 67R 12V is enough, then submit 67R 12V.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* with a sequence of moves that is *at least as long as* the one in your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* with a sequence of moves that is *strictly shorter* than the one in your answer.

Problem 17.2.5 (HC-2022-SM2-R2-P5). (*Beginner Level*)

The pair of numbers (a_1, a_2) are called *one-step apart* if $a_1a_2 + 1$ *is a perfect square.* The pair of numbers (a_1, a_{k+1}) $(k \geq 1)$ is called $k-$*step apart* if there exist $k-1$ numbers $a_2, a_3, \dots, a_k$ so that each of the pairs $(a_1, a_2), (a_2, a_3), \dots, (a_k, a_{k+1})$ $(k \geq 1)$ is one-step apart.

For example:

- $(1, 3)$ *is one-step apart because* $1 \cdot 3 + 1 = 4 = 2^2$;
- $(1, 4)$ *is not one-step apart because* $1 \cdot 4 + 1 = 5$, *which is not a perfect square.*
- $(1, 2)$ *is* $2-$*steps apart, because each of* $(1, 24)$ *and* $(24, 2)$ *is one-step apart,* $1 \cdot 24 + 1 = 25 = 5^2$ *and* $24 \cdot 2 + 1 = 49 = 7^2$.

What is the minimal number of steps needed to reach 9 from 1?

How to provide your answer:

- If you think that 2 steps are needed with the two one-step pairs $(1, 4)$ and $(4, 9)$, submit $1, 4, 9$.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which has *at least as many* number of steps as the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which has *strictly smaller* number of steps than your answer.

Problem 17.2.6 (HC-2022-SM2-R2-P6)**.** (*Intermediate Level*)

$ABCD$ is a parallelogram. E and F are points on AB and BC, respectively. Lines AF, CE, DE, and DF dissect the parallelogram into 8 regions, where some of them have known areas, as shown below in the diagram.

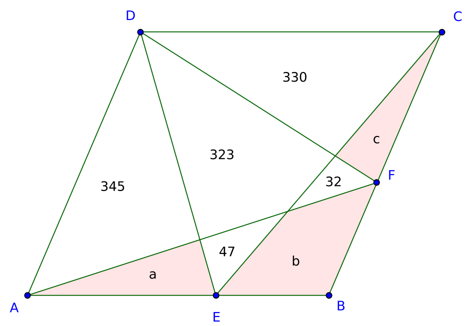


Figure 17.3: HC-2022-SM2-R2-P6

Find the sum of the areas $a + b + c$.

How to provide your answer:

- If you think that the sum is 1, submit 1.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop does not change regardless whatever submitted by your opponent.

Problem 17.2.7 (HC-2022-SM2-R2-P7). (*Intermediate Level*)

The brothers, Billy the bandit and Willy the thug, have stopped a milkman in the middle of the road. The milkman has two $80-$liter cans full of milk. Billy has a $5-$liter jug and Willy has a $4-$liter jug. They demand the milkman to fill each of their jugs with exactly 2 liters of milk, if he values his life.

The problem is that the milkman has no tools to measure how much milk he pours, so he is forced to pour milk between the cans and the jugs and use their volumes as guidance.

What is the minimal number of pourings the milkman has to do in order to save his life from these milk robbers?

How to provide your answer:

- If you think there 2 pourings are needed. Submit 2. Submit your solution to make sure you can win.
- If you cannot determine it, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which has *at least as many* number of pourings as the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which has *strictly smaller* number of pourings than your answer.

Problem 17.2.8 (HC-2022-SM2-R2-P8). (*Intermediate Level*)

Anna, Benny, Chi, Duy, Hannah, Julie, and Khoa played games. Every game had 1 winner and 6 losers.

Whenever a player wins a game, the winner gives each of the losers an amount equal to what that loser has in his pocket, basically doubling the loser's money. They had played 7 games. The winners of these games, in time order, were Anna, Benny, Chi, Duy, Hannah, Julie, and Khoa. When they have finished playing, each of them had exactly $128.

How much had each of them in his pocket before playing?

How to provide your answer:

- If you think that the amounts of money they had before the game were $1, 2, 3, 4, 5, 6, 7$, then submit $1, 2, 3, 4, 5, 6, 7$.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop does not change regardless of whatever your opponent submitted.

Problem 17.2.9 (HC-2022-SM2-R2-P9). (*Intermediate Level*)

In a ceremony of the guild, the master asked the members to form 10 lines, each line consists of 9 persons, so that he can stand in *a place that has the same distance to every row.*

The member were confused because all together, including the masters, **the guild has** 81 **members.** Chi Ton, an invited guest to the ceremony, pointed out how they can do that. Of course, he did neither stand into those lines, nor posed as the master.

What was his solution?

How to provide your answer:

- If you think you have a solution, create a diagram (or by any other means that can visualize your solution), and submit it.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 17.2.10 (HC-2022-SM2-R2-P10). (*Intermediate Level*)

An ordinary die is cubical, with each face showing one of the numbers $1, 2, \ldots, 6$. Each face borders on four other faces; each number is *surrounded* by four other numbers.

A regular dodecahedron is composed of 12 regular pentagonal faces, three meeting at each vertex as seen below.

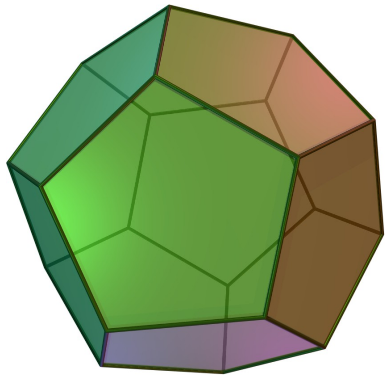


Figure 17.4: HC-2022-SM2-R2-P10

In how many different ways can a regular dodecahedron die be made, where each of the numbers $1, 2, \ldots, 6$ occurs on exactly two different faces and is *surrounded* both times by all of the five other numbers? *Two regular dodecahedron die are the same if one can be rotated to match the other.*

How to provide your answer:

- If you think that the number of ways is 1, submit 1.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 17.2.11 (HC-2022-SM2-R2-P11)**.** (*Advanced Level*)

Triangulating a polygon means partitioning it into non-overlapping triangles with non-intersecting diagonals. The diagram below shows two different ways to triangulate a hexagon.

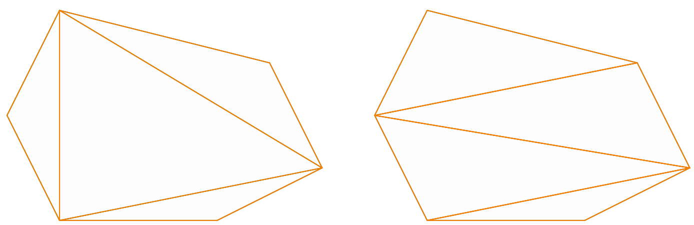


Figure 17.5: Two different triangulations

Two triangulations are called *distinct* if the triangulations can not be identical by any number of *rotations. Note that reflection is NOT allowed when rotating triangulations (don't flip the polygon.)* The diagram below shows two identical triangulations of a *regular* hexagon. By rotating the left triangulation clockwise we can receive the right one.

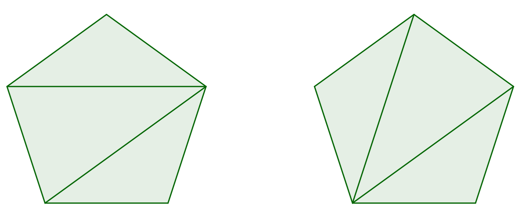


Figure 17.6: Identical triangulations

Find the number of *distinct* triangulations of a regular heptagon (a regular $7-gon$.)

How to provide your answer:

- If you think that the sum is 1, submit 1.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* then your troop scores **1 hit.** In all other cases your troop scores **0 hits**.
- When your opponent submits an answer: The score of your troop is not changed regardless whatever submitted by your opponent.

Problem 17.2.12 (HC-2022-SM2-R2-P12). (*Advanced Level*)

An *arithmetic sequence* is a sequence of numbers where the differences between consecutive terms are constant. *For example* $2, 4, 6, 8$ *and* $7, 5, 3$ *are two arithmetic sequences.*

The arithmetic sequence $5, 16, 27$ is called a $3-$*special arithmetic sequence* since it has 3 terms, the second term is a perfect square, the third term is a perfect cube.

$$5 = 5^1,\ 16 = 4^2,\ 27 = 3^3 \text{ and } 16 = 5 + 9,\ 27 = 16 + 9$$

Find the largest positive integer k less than 17 so that there exists an arithmetic sequence $a_1, a_2, \ldots, a_k$ that is a $k-$*special arithmetic sequence*. In other words, they are positive integers $b_1, b_2, \ldots, b_k$,

$$a_1 = b_1^1,\ a_2 = b_2^2,\ \ldots, a_k = b_k^k.$$

How to provide your answer:

- If you think the largest such number is 1, then submit 1. You have to also *submit a valid sequence with such length.*
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which is *at most as* the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which is *strictly larger* than your answer.

Problem 17.2.13 (HC-2022-SM2-R2-P13)**.** (*Advanced Level*)

In the city of Wisdom, people are very intelligent. They never make statements, they only ask questions.

There are two types of them:

- Yesman, who asks only questions where correct answers can only be yes. *For example a Yesman only asks "Does one plus one equal two?", and not "Does one plus one equal three?"*;
- Nosir, who asks only questions where correct answers can only be no. *For example a Nosir only asks "Does two plus two equal three?", and not "Does two plus two equal four?"*;

There are three students in the local math club: Laetitia, Leah, and Léna. Laetitia asked Leah

"Are you the type who could ask Léna whether she is the type
who could ask you whether you two are of different types?"

Find **the types** of the girls.

How to provide your answer:

- If you think that Laetitia was a Yesman, Leah is a Nosir, and nothing can be said about Léna, submit *Y, N, O;* if you think that Laetitia was a Nosir, nothing can be said about Leah, and Léna is a Yesman, submit *N, O, Y;* and so on.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or a *correct answer* which has *as least as many* number of correct types than the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit a *correct answer* which has *strictly larger* number of correct types than your answer.

Problem 17.2.14 (HC-2022-SM2-R2-P14). (*Advanced Level*)

Match the phrases in Vietnamese on the left of the table below, with their translations into English on the right of the table.

1	băng
2	bó
3	bó hoa
4	cánh hoa
5	đá
6	đá lửa
7	đá phấn
8	đường
9	đường vòng
10	hoa
11	lửa
12	mở
13	mở đường
14	mở mắt
15	núi
16	núi băng
17	núi lửa
18	nước đá
19	nước mắt
20	phấn
21	phấn hoa
22	vòng
23	vòng hoa

A	bouquet (a bunch of flowers)
B	chalk
C	circle
D	cluster
E	detour
F	fire
G	flint (a stone used to make sparks)
H	flower
I	ice
J	iceberg
K	mountain
L	petal
M	pollen
N	powder
O	road
P	rock
Q	tear (as in teardrop)
R	to make aware
S	to open
T	to pave the way
U	volcano
V	wreath

Find **as many matches as possible** and fill the letters of the translations into the table below.

1	2	3	4	5	6	7	8	9	10	11	12

13	14	15	16	17	18	19	20	21	22	23

How to provide your answer:

- If you think you have the matches, take a copy of the answer table and submit it.
- If you think that there is no solution, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or an answer with *at least as many* number of correct matches than the one in your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit an answer with *strictly larger* number of correct matches than the one in your answer.

Problem 17.2.15 (HC-2022-SM2-R2-P15)**.** (*Advanced Level*)

In an annual competition, each of the 10 math clubs sends a 10−member team to compete. In the warm-up game, there is a circle with 100 positions $P_1, P_2, \ldots, P_{100}$ marked (shown as below).

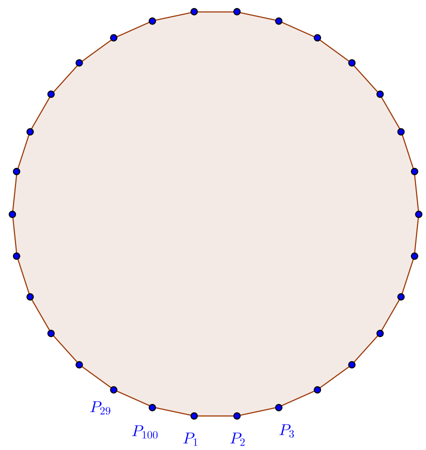


Figure 17.7: HC-2022-SM2-R2-P15

Team MCC can choose 10 places to stand. The other 90 places will then be filled with members from other teams. A referee will go around the circle, anti-clockwise, starting from position P_1, and remove every ninth person. After 90 people are removed, the 10 remaining are declared winners.

Which positions should members of Team MCC choose so that all of them can be winners?

How to provide your answer:

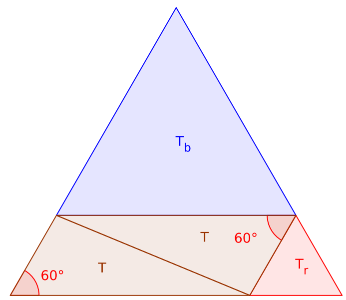
- If you think $1, 2, 3, \ldots, 10$ are the winning positions, submit $1, 2, 3, \ldots, 10$.
- If you cannot determine that, submit 0, and give detailed reasoning for your answer.

How your answer is graded for this problem:

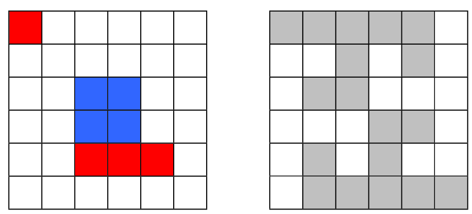
- When your troop submits an answer: If your answer is *correct* and the opponent provides *no answer*, a *wrong answer*, or an answer which has *at least as many* number of correct positions than the one of your answer then your troop scores **1 hit.** In all other cases your troop scores **0 hit**.
- When your opponent submits an answer: The score of your troop can be changed from 1 to 0 if the opponent submit an answer with *strictly larger* number of correct positions than your answer.

17.3 Answers

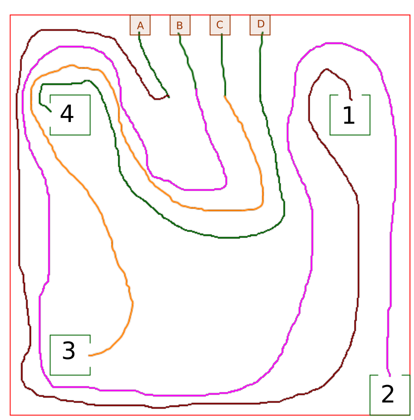
Problem 1: 2 T, 1 T_b, and 1 T_r. See below



Problem 2: See below



Problem 3: See below



Problem 4: 67L 34V 71V 48V. A total of 4 moves.

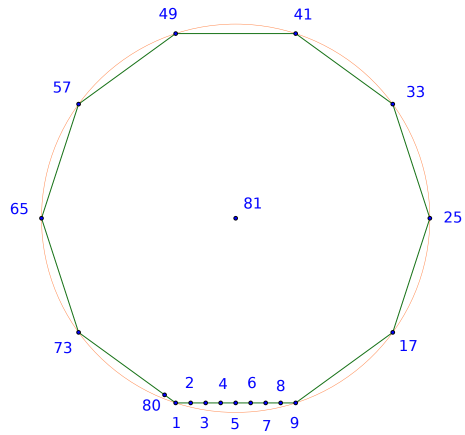
Problem 5: 3−step apart. 1, 8, 136, 9.

Problem 6: 323

Problem 7: 9−pourings.

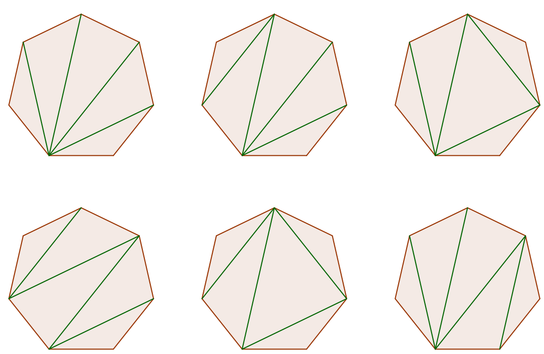
Problem 8: 449, 225, 113, 57, 29, 15, 8.

Problem 9: See below



Problem 10: 12.

Problem 11: 6 ways., see below



Problem 12: 16. Here is an example for any $n \leq 16$,

$$\left(2^{\frac{n!}{1}}\right)^1, \left(2^{\frac{n!}{2}}\right)^2, \ldots, \left(2^{\frac{n!}{n}}\right)^n.$$

Problem 13: Y, O, N.

Problem 14: See below

1	2	3	4	5	6	7	8	9	10	11	12
I	D	A	L	P	G	B	O	E	H	F	S

13	14	15	16	17	18	19	20	21	22	23
T	R	K	J	U	I	Q	N	M	C	V

Problem 15: 7, 15, 16, 46, 49, 55, 71, 73, 87.

17.4 Solutions

Solution. HC-2022-SM2-R2-P1 Yes, he can do it. Here is one solution: □

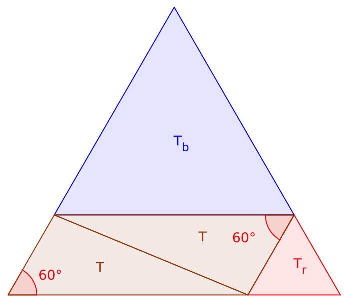
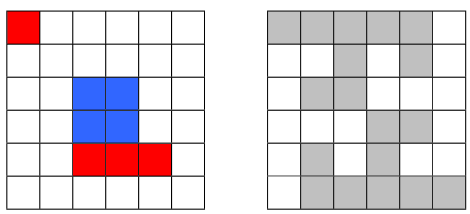


Figure 17.8: 2 T, 1 T_b, and 1 T_r

Solution. HC-2022-SM2-R2-P2 □



Solution. HC-2022-SM2-R2-P3 Below is a solution: □

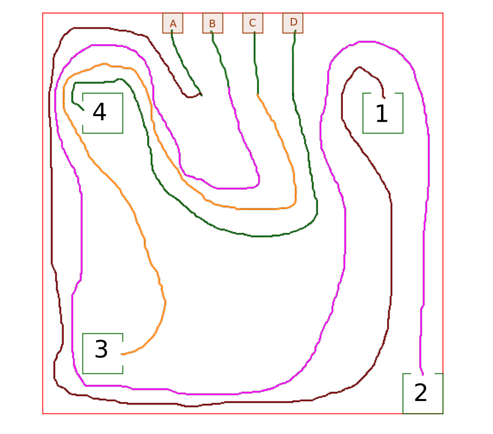
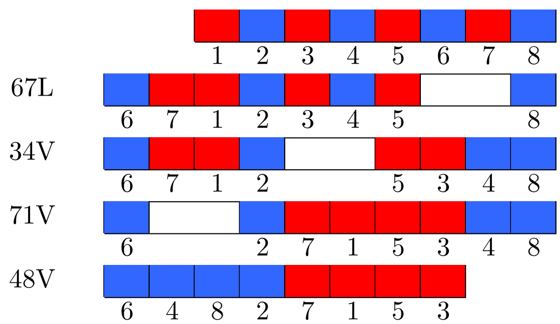


Figure 17.9: HC-2022-SM2-R2-P3

Solution. HC-2022-SM2-R2-P4 Below is a sequence of $\boxed{4}$ moves $\boxed{\text{67L 34V 71V 48V.}}$ □



Solution. HC-2022-SM2-R2-P5

From the example of $(1, 3)$, which is one-step apart, it can be generalized that for any two number with a difference of 2,

$$n(n + 2) + 1 = (n + 1)^2 \Rightarrow (n, n + 2) \text{ is one-step apart.}$$

thus, by following this pattern, $(1, 9)$ *is* $4-$*step apart by the sequence* $(1, 3, 5, 7, 9)$. The sequence $(1, 15, 13, 11, 9)$ is another example to show that $(1, 9)$ is $4-$step apart.

Is it possible to find another sequence to show that $(1, 9)$ is $3-$step apart?

If we look at another example: $(1, 3)$ is one-step apart, or for any n, the pair $(1, n^2 - 1)$,

$$1 \cdot (n^2 - 1) + 1 = n^2 \Rightarrow (n, n + 2) \Rightarrow (1, 8 = 3^2 - 1) \text{ is one-step apart.}$$

Now, how do we reach 9 from 8 in two steps, let take the look at the other example: $(1, 2)$ is $2-$steps apart, because each of $(1, 24)$ and $(24, 2)$ is one-step apart, $1 \cdot 24 + 1 = 25 = 5^2$ and $24 \cdot 2 + 1 = 49 = 7^2$. By the identities, for any n,

$$(4n + 1)^2 = 16n^2 + 8n + 1 = n(16n + 8) + 1 \Rightarrow (8, 16 \cdot 8 + 8) = (8, 136) \text{ is one-step apart}$$
$$(4n + 3)^2 = 16n^2 + 24n + 1 = (n + 1)(16n + 8) + 1 \Rightarrow (8 + 1, 16 \cdot 8 + 8) = (9, 136) \text{ is one-step apart.}$$

Therefore $(1, 9)$ *is* $3-$*step apart by the sequence* $\boxed{1, 8, 136, 9}$.

Is it true that 3 is the minimal number? Let assume the opposite: let a, b, and c positive integers such that

$$\begin{cases} 1 \cdot a + 1 = b^2 \\ a \cdot 9 + 1 = c^2 \end{cases} \Rightarrow 9b^2 - c^2 = 8 \Rightarrow (3b - c)(3b + c) = 4 \cdot 2 \Rightarrow b = c = 1 \Rightarrow a = 0 \text{ this contradicts that } a \geq 1.$$

Thus, the minimal number of steps to reach 9 from 1 is $\boxed{3.}$ □

Solution. HC-2022-SM2-R2-P6 From the properties of the parallelogram $ABCD$, the following triangle pairs, with the same base and height, have the same area

$$\left.\begin{array}{l}[AFB]=[DFB]\\ [CEB]=[DEB]\end{array}\right\} \Rightarrow [AFB]+[CEB]=[DFB]+[DEB]=[DEBF]$$

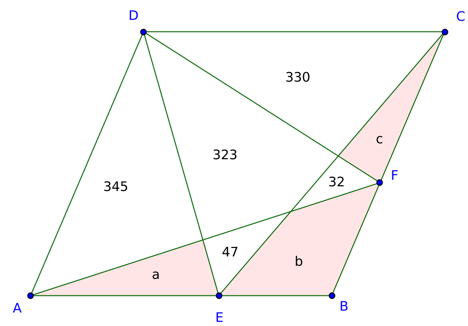


Figure 17.10: $[AFB]=[DFB],[CEB]=[DEB]$

Therefore

$$(a+47+b)+(c+32+b)=(323+47+b+32) \Rightarrow a+b+c=\boxed{323.}$$

□

Solution. HC-2022-SM2-R2-P7 The table below shows a $\boxed{9\text{−pouring}}$ solution. Each row shows the states of the cans and the jugs *at the beginning* of each pouring. The result of the pouring is showed in the following row. The cell coloured in red is the one where the milk is poured from and the cell coloured in blue is the one where the milk is poured into. □

	80-l can	80-l can	5-l jug	4-l jug
1	80	80	0	0
2	75	80	5	0
3	75	80	1	4
4	79	80	1	0
5	79	80	0	1
6	74	80	5	1
7	74	80	2	4
8	78	80	2	0
9	78	76	2	4
	80	76	2	2

Solution. HC-2022-SM2-R2-P8 The winners, in order, were Anna, Benny, Chi, Duy, Hannah, Julie, and Khoa, so we can use backward-counting to solve this problem.

After the last game, Khoa won, so he doubled the money of each player. Thus each of them had exactly \$64 before the 7^{th} game, or after the 6^{th} game, and Khoa had $6 \cdot \$64 + 128 = \$512$.

Thus, Julie won the 6^{th} game. So before that game or after the 5^{th} game, each of Anna, Benny, Chi, Duy, Hannah had \$32, while Khoa had \$256. thus Julie had $5 \cdot \$32 + \$256 + \$64 = \$480$.

Continuing the computation gives us the following table:

After		A	B	C	D	H	J	K		Total
7		128	128	128	128	128	128	128		896
6		64	64	64	64	64	64	512		896
5		32	32	32	32	32	480	256		896
4		16	16	16	16	464	240	128		896
3		8	8	8	456	232	120	64		896
2		4	4	452	228	116	60	32		896
1		2	450	226	114	58	30	16		896
		449	225	113	57	29	15	8		896

The last row indicates what the players had before the play. The last column can serve as a verification sum, the total amount of money that all the players had together, which should be an invariant.

The answer is the $\boxed{\text{449, 225, 113, 57, 29, 15, 8.}}$ □

Solution. HC-2022-SM2-R2-P9

The key here is that *the master is equidistant from each row*, not each member. Thus, there should be 10 rows at the same distant from a point. It reminds us of the circle where the center is at the same distant from 10 chords, if those chords have the same length. To arrange 80 persons into 10 chords, each has 9 persons, we create a regular polygon with 9 persons on each side as show below in the diagram. □

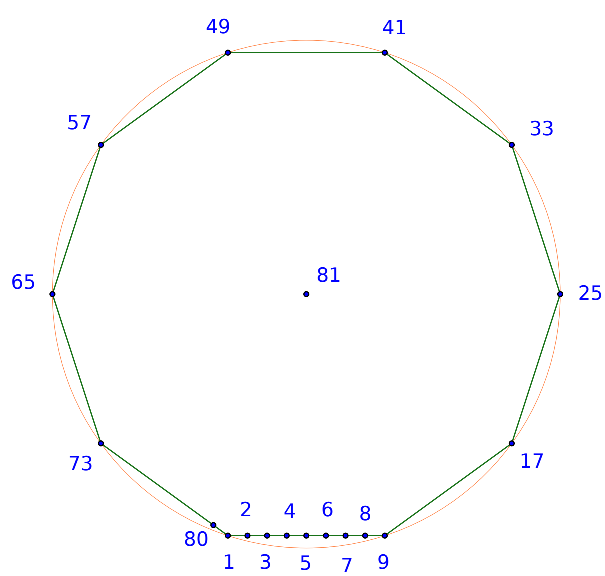
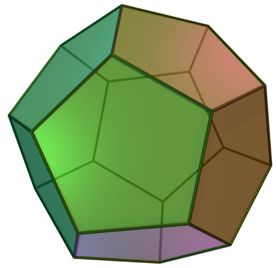


Figure 17.11: 10 chords

Solution. HC-2022-SM2-R2-P10 First, note that the opposite faces of a dodecahedron must have the same numbers, in order to have the same set of surrounded numbers.



Lets fix the front face, its pair the back face and let a number, WLOG, like 1 be on these faces. On one of the five surrounding faces, WLOG let the top-right have the number 2. Then there are $4! = 24$ ways to permute the numbers $3, 4, 5, 6$ on the remaining faces. Since the opposite faces of the faces filled with $2, 3, \ldots, 6$ should have the same numbers, we don't have to choose any other face to fill.

Now, if the permutation of $(2, a, b, c, d)$ (a, b, c, d are a permutation of $3, 4, 5, 6$) surrounds the front face, then the permutation $(2, d, c, b, a)$ surrounds the back face. Thus, the number of ways is $\frac{24}{2} = \boxed{12.}$ □

Solution. HC-2022-SM2-R2-P11 Let T_n be the number of ways to triangulate an $n-$gon. It is easy to see that $T_3 = 1$. It is not difficult to prove the following claim,

Claim — $T_{n+1} = T_n + T_3T_{n-1} + T_4T_{n-2} + \ldots + T_{n-1}T_3 + T_n$, for $n \geq 3$.

Using the above claim, by the Induction Principle, it can be proved that,

Claim — $T_n = \frac{1}{n-1}\binom{2n-4}{n-2}$, for $n \geq 3$.

Note that, T_n is actually known as the Catalan numbers $C_n = T_{n+2}$.

Now, if n is a prime then it is easy to see that only rotations can cause duplicates in triangulations.

$$\frac{1}{n}T_n = \frac{1}{n(n-1)}\binom{2n-4}{n-2}.$$

Thus the number of non-identical triangulations of a $7-$gon is $\frac{1}{7\cdot6}\binom{10}{5} = \boxed{6.}$

Below are six identical triangulations. □

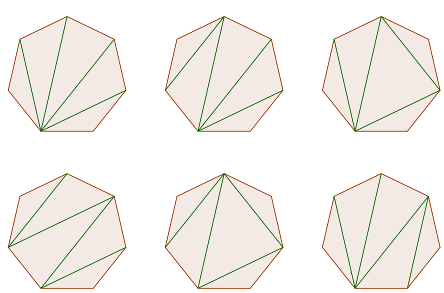


Figure 17.12: 6 identical triangulations

Solution. HC-2022-SM2-R2-P12 Here are the two key ideas for a trivial solutions:

1. a sequence where all terms are equal is an arithmetic sequence where the common difference is 0. This reduces the number of terms to be dealt with from n to 1.
2. the exponent of each perfect power is divisible by $1, 2, \ldots, n$, or if it is divisible by $n!$, then we are done.

In fact, the sequence below satisfies the given requirements.

$$\left(2^{\frac{n!}{1}}\right)^1, \left(2^{\frac{n!}{2}}\right)^2, \ldots, \left(2^{\frac{n!}{n}}\right)^n$$

Hence, the answer is $\boxed{16.}$ □

Solution. HC-2022-SM2-R2-P13 We work backwards on the question.

First we try to find what type is Léna if she can ask a question whether she and someone are of different types. Below are two claims that can easily be verified.

Claim — No girl can ask whether she and a Yesman are of different types.

Proof. If the girl is a Yesman, then answering yes to whether two Yesmen are of different types is a contradiction. If the girl is a Nosir, then answering no to whether a Yesman and a Nosir are of different types is again a contradiction. ■

Claim — Any girl can ask whether she and a Nosir are of different types.

Proof. If the girl is a Yesman, then she could ask whether a Yesman and a Nosir are of different types, and have the answer be yes. If the girl is a Nosir, then she could ask whether two Nosirs are of different types, and have the answer be no. ■

From the claims, if Léna can ask such a question, she must be a Nosir.

Second, we investigate whether Leah can ask a Léna if Léna can ask whether she and Leah are of different types. If Leah is a Yesman, then by the first claim, Léna cannot ask such a question. If Leah is a Nosir, then by the second claim, she *can ask* Léna and receive the answer no. However, the fact that both are Nosirs contradicts the answer no (to the question of whether Leah and Léna are of different types.) Thus, in either cases, Leah cannot ask Léna such a question. In addition, we don't know anything about Leah.

Finally, since Leah cannot ask Léna such a question, the answer that she gives Laetitia is yes. Thus, Laetitia is a Yesman. The answer is $\boxed{\text{Y, O, N.}}$ □

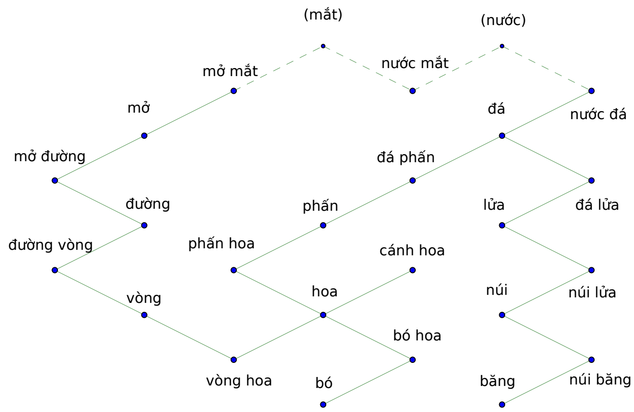
Solution. HC-2022-SM2-R2-P14 Below is the answer table.

The mappings are explained in the table below,

#	Vietnamese	Literal meaning	Answer	English
1	băng	ice	I	ice
2	bó	cluster	D	cluster
3	bó hoa	flower cluster	A	bouquet
4	cánh hoa	flower wing	L	petal
5	đá	rock	P	rock
6	đá lửa	fire rock	G	flint
7	đá phấn	powder rock	B	chalk
8	đường	road	O	road
9	đường vòng	circle road	E	detour
10	hoa	flower	H	flower
11	lửa	fire	F	fire
12	mở	to open	S	to open
13	mở đường	to open a road	T	to pave a way
14	mở mắt	to open eyes	R	to make aware
15	núi	mountain	K	mountain
16	núi băng	ice mountain	J	iceberg
17	núi lửa	fire mountain	U	volcano
18	nước đá	rock water	I	ice
19	nước mắt	eye water	Q	tear
20	phấn	powder	N	powder
21	phấn hoa	flower powder	M	pollen
22	vòng	circle	C	circle
23	vòng hoa	flower circle	V	wreath

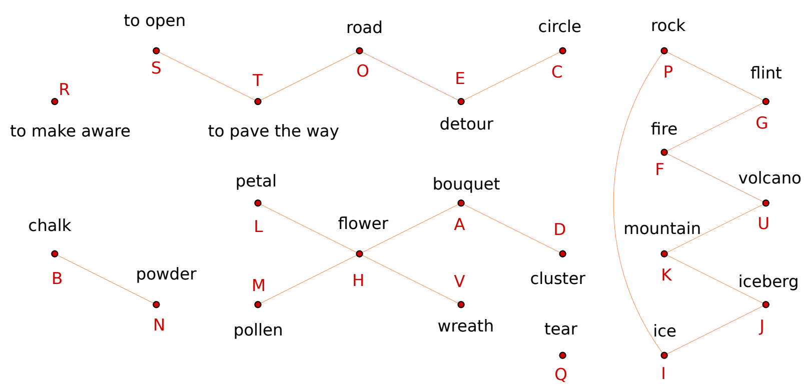
We use a graph theory approach from the point of view of an English speaker to solve the problem.

First, we look at the Vietnamese phrases. They are single- and double-word phrases. We connect the phrases in a graph so that each pair of phrases consists of a single-word phrases and a double-word phrase, the double-word phrase basically contains the single-word phrase. See the diagram below.



Hình 17.13: Graph of Vietnamese phrases

The graph of Vietnamese phrases, we presume, represents connections in *shared meaning* between phrases. Thus, we connect the English phrases in the same way, each phrase with another so that one has a meaning that shall be contained by the meaning of the other. The result is the diagram below.



Hình 17.14: Graph of English phrases

Comparing the graph, only the $5 - vertex$ subgraphs *(phấn hoa, vòng hoa, cánh hoa, bó hoa, hoa)* and *(bouquet, petal, pollen, wreath, flower),* are *topologically* equivalent, thus *hoa=flower*. Note that the relation between *pollen* and *powder*, implies that *powder* is *phấn*. The rest of the vertices then can be paired up *petal - cánh hoa, bouquet - bó hoa, pollen - phấn hoa, wreath - vòng hoa.* Therefor *bó - cluster*, and *chalk - đá phấn.*

Similarly the paths *(băng - núi băng - núi - núi lửa - lửa - đá lửa - đá - nước đá) (ice - iceberg - mountain - volcano - fire - flint - rock)* are very much alike, in addtion, the relation of *đá - đá phấn* is similar to *rock - chalk,* which make both *nước đá* and *băng* to have the meaning of *ice*. Similarly the paths *(vòng - đường vòng - đường - mở đường - mở) (to open - to pave the way - road - detour - circle)* are very much alike.

The rest is simple. □

Solution. HC-2022-SM2-R2-P15 In this solution, we design a program in Python to simulate the game.

```
range = range(0, 99)                # the range of all positions (0, 1, ... 99)
students = [i for i in range]       # each has the initial positions (0, 1, ... 99)
remains = 90                        # at the beginning 90 students to be removed
current = 0                         # the starting position is 0 (indexing in Python)
step = 9                            # every 9th person to be removed

while remains > 0:                                    # loop until 90 removals
    if current + step - 1 < len(students):            # check the next index
        current = current + step - 1                  # if still in range
    else:
        current = current + step - 1 - len(students)  # otherwise reduce

    students.pop(current)                             # remove that student

    if current == len(students):                      # if the current is off
        current = 0                                   # set it be the first

    remains -= 1                                      # one less to be removed

print(' '.join('%s' % (i+1) for i in students))       # print all remaining
```

The remaining positions are $\boxed{7, 15, 16, 46, 49, 55, 71, 73, 87.}$ □

Chapter 18

Math Individual Contest - Round 3

18.1 Rules

Mathematical Individual Contest is an activity of Math, Chess, and Coding Club (MCC). All students are invited to participate. The problems are designed so that students at various levels - junior (grade under 8), senior (grade 9 and up), or olympiad (prepare for national or international contest) - can participate. The contest problems are selected from algebra, combinatorics, number theory, and geometry topics. Some problems do not require specific mathematical knowledge, some require problem solving skills, and some require a more comprehensive understanding of certain mathematical topics. The students received a number of points after each participation and are ranked after each round, in both overall and grade rankings. The final standing of the contest is concluded at the end of each semester.

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Problem levels: all students must **try to solve all problems**, don't stay within your boundary
  - Junior: Problem 1, 2, and 3 (multiple-choice); Problem 7 (show-your-work).
  - Senior: Problem 4, 5 (multiple-choice); Problem 8, 9 (show-your-work).
  - Olympiad: Problem 6 (multiple-choice); Problem 10 (show-your-work).
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.

18.2 Problems

Problem 18.2.1 (MIC-2022-SM2-R3-J1)**.** (Junior, *10 points*)

The discount on a stereo system is \$69, and the rate of discount is 15%. What was the original price of the system?

(A) 10.35 (B) 4.6 (C) 69.15 (D) 70 (E) 460

Problem 18.2.2 (MIC-2022-SM2-R3-J2)**.** (Junior, *10 points*)

Twelve points are equally spaced on the circumference of a circle. How many chords can be drawn that connect pairs of these points and which are longer than the radius of the circle but shorter than its diameter?

(A) 24 (B) 28 (C) 36 (D) 54 (E) 72

Problem 18.2.3 (MIC-2022-SM2-R3-J3)**.** (Junior, *10 points*)

Find the sum of all values of x satisfying

$$\sqrt{x+3}+4=\sqrt{8x+1}.$$

(A) $\frac{316}{49}$ (B) 6 (C) $\frac{272}{49}$ (D) $\frac{22}{49}$ (E) 12

Problem 18.2.4 (MIC-2022-SM2-R3-S4)**.** (Senior, *10 points*)

Find all the sum of possible values of the product ab given that

$$\begin{cases} a+b=2 \\ a^4+b^4=2. \end{cases}$$

(A) 1 (B) 2 (C) 7 (D) 8 (E) 9

Problem 18.2.5 (MIC-2022-SM2-R3-S5)**.** (Senior, *10 points*)

In $\triangle ABC$, $AB=4, BC=8, CA=9$. A line from B intersects AC at D and bisects the arc $\overset{\frown}{AC}$ of the circumcircle (O) at point E. Find DC.

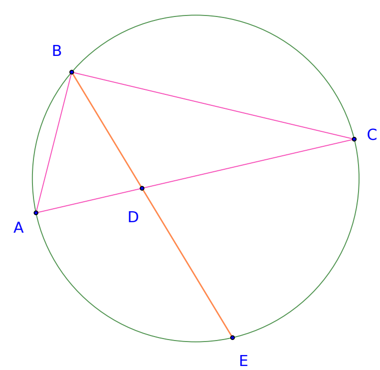


Figure 18.1: MIC-2022-SM2-R3-S5

(A) 6 (B) $6\frac{1}{2}$ (C) $6\frac{3}{4}$ (D) 7 (E) $7\frac{1}{4}$

Problem 18.2.6 (MIC-2022-SM2-R3-O6)**.** (Olympiad, *10 points*)

Find the number of pairs (x, y) of integers (not necessarily positive) such that

$$x^2 + xy + y^2 = 28.$$

(A) 1 $\quad$ (B) 2 $\quad$ (C) 4 $\quad$ (D) 6 $\quad$ (E) 12

Problem 18.2.7 (MIC-2022-SM2-R3-J7)**.** (Junior, *10 points*)

A is a square 9×9 board with the following properties:

1. A is symmetric over the main diagonal (going from top-left to bottom-right), meaning that for any pair (i, j), $i \neq j$, $(1 \leq i, j \leq 9)$ the intersection of the i^{th} row and the j^{th} column has the same value as the intersection of the j^{th} row and the i^{th} column.
2. each row and each column of A consists of a permutation of the integers $1, 2, \ldots, 9$.

Prove that each one of the integers $1, 2, \ldots, 9$, must appear exactly once on the main diagonal of A.

Problem 18.2.8 (MIC-2022-SM2-R3-S8)**.** (Senior, *10 points*)

For $n \geq 0$ non-negative integer, let

$$a_n = (\sqrt{3} + 1)^{2n} + (\sqrt{3} - 1)^{2n}$$

(*2 points*) Compute the value of a_0 and a_1.

(*3 points*) Prove the following recurrence relation

$$a_{n+1} = 8a_n - 4a_{n-1}, \text{ for all } n \geq 1.$$

(*5 points*) Use the Induction Principle, prove that for all $n \geq 1$,

$$\lceil (\sqrt{3} + 1)^{2n} \rceil \text{ is divisible by } 2^{n+1}.$$

Note that $\lceil x \rceil$ *denotes the smallest integer greater than or equal to* x*. For example,* $\lceil 1.1 \rceil = 2$, $\lceil 1 \rceil = 1$.

Problem 18.2.9 (MIC-2022-SM2-R3-S9)**.** (Senior, *10 points*)

In $\triangle ABC$, the angle bisector of the angle A intersects line BC at D and the circumcircle of $\triangle ABC$ at E. The external angle bisector of the angle A intersects line BC at F and the circumcircle of $\triangle ABC$ at G.

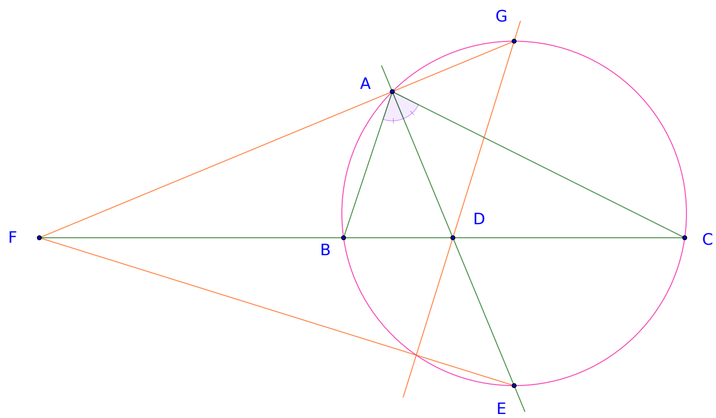


Figure 18.2: MIC-2022-SM2-R3-S9

(*2 points*) Prove that $EA \perp FG$.

(*3 points*) Prove that $GE \perp BC$.

(*5 points*) Prove that $GD \perp EF$.

Problem 18.2.10 (MIC-2022-SM2-R3-O10)**.** (Oympiad, *10 points*)

The following numbers are written on the board

$$2, 3, 6, 7, 10.$$

At each step, Minh picks two numbers, says u and v, where $u \geq v$, and replace them with $u+v$ and $u-v$.

Is there a way to pick the pairs of numbers such that after a finite number of steps, all the numbers of the board return to the initial numbers? Show an example if there is such a way or give a proof if not.

18.3 Grading

Answers for multiple-choice problems.

Problem 1: E

Problem 2: C

Problem 3: B

Problem 4: D

Problem 5: A

Problem 6: E

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the students find that the off-diagonal appearances of any number occur in pairs because of the symmetry.

Problem 8: Separately grading for each part,

(a) 1 point if the student can compute at least one of the numbers.

(b) 1 point if the student can recognize that $(\sqrt{3}+1)(\sqrt{3}-1) = 2$.

(c) 1 point if the student can start and state the induction hypothesis.

Problem 9: Separately grading for each part,

(a) 1 points if the student can recognize the internal and external angle bisectors.

(b) 1 points if the student can see that GE is the diameter, and another 3 points if the can recognize that GE going through the midpoint of arc $\widehat{BC}$.

(c) 2 points if the student see that EA and FD are two altitudes of $\triangle EFG$.

Problem 10: 2 points if the students can see that the sum of squares of the numbers on the table increased.

18.4 Solutions

Solution. MIC-2022-SM2-R3-J1 Let the price be x. Since the discount is \$69 and is 15% of the price, we have

$$0.15x = 69 \Rightarrow x = \frac{69}{0.15} = \boxed{\$460.}$$

The number answer is $\boxed{E.}$ □

Solution. MIC-2022-SM2-R3-J2

We must first determine which diagonals are greater than the radius of the circle and less than the diameter. Diagonals such as AC are equal in length to the radius. This can be seen by noting that $ACEG\cdots$ is a regular hexagon, and the sides of a regular hexagon are equal in length to the radius of the circumscribed circle. Diagonals such as AG are diameters of the the circle.

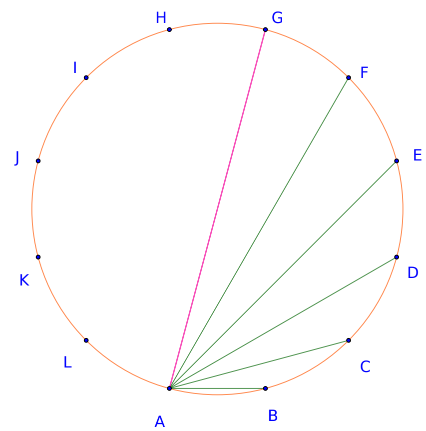


Figure 18.3: MIC-2022-SM2-R3-J2

This leaves us with diagonals like AD, AE, and AF, which are longer than AC and shorter than AG. From each vertex there are 6 such diagonals, for a total of $6(12) = 72$. However, we have counted each diagonal twice, once for each endpoint of the diagonal. Hence, there are actually $\frac{72}{2} = \boxed{36}$ diagonals longer than the radius of the circle and less than the diameter.

The answer is $\boxed{C.}$ □

Solution. MIC-2022-SM2-R3-J3

For the existence of the equation

$$\sqrt{x+3} + 4 = \sqrt{8x+1} \Rightarrow \begin{cases} x+3 \geq 0 \Rightarrow x \geq -3 \\ 8x+1 \geq 0 \Rightarrow 8x \geq -1 \end{cases} \Rightarrow x \geq -\frac{1}{8}.$$

Now

$$\sqrt{x+3} + 4 = \sqrt{8x+1} \Rightarrow (x+3) + 16 + 8\sqrt{x+3} = 8x+1 \Rightarrow 7x - 18 = 8\sqrt{x+3}$$

$$\Rightarrow \begin{cases} 7x - 18 \geq 0 \Rightarrow x \geq \frac{18}{7} > -\frac{1}{8} \ (*) \text{ (the equation still stands)} \\ (7x-18)^2 = 64(x+4) \Rightarrow 49x^2 - 316x + 132 = 0 \Rightarrow x = 6 \text{ or } x = \frac{22}{49} \end{cases}$$

$$\Rightarrow x = 6 \left(\text{by } (*) \ x \geq \frac{18}{7} > \frac{22}{49}\right)$$

The sum of all such values of x is $\boxed{6.}$ The answer is $\boxed{B.}$ □

Solution. MIC-2022-SM2-R3-S4

Since $a^2+b^2=(a+b)^2-2ab=4-2ab$, thus

$$a^4+b^4=(a^2+b^2)^2-2a^2b^2=(4-2ab)^2-2a^2b^2=16-16ab+2a^2b^2$$
$$\Rightarrow 2=16-16ab+2(ab)^2 \Rightarrow (ab)^2-8ab+7=0 \Rightarrow ab=1 \text{ or } ab=7$$

Therefore, the sum of all values of the product ab is $1+7=\boxed{8}$. The answer is $\boxed{D.}$ □

Solution. MIC-2022-SM2-R3-S5

Note that $\widehat{AE}=\widehat{EC}$, thus $\angle ABE=\angle EBC$, hence BE is the angle bisector of $\angle ABC$.

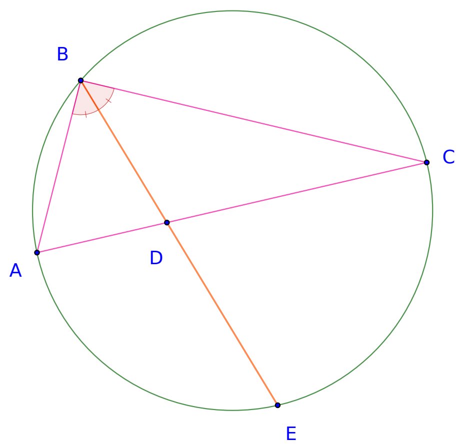


Figure 18.4: BE is angle bisector of $\angle A$

By the Angle Bisector Theorem

$$\frac{AB}{AD}=\frac{CB}{CD}=\frac{AB+BC}{AC}\Rightarrow CD=\frac{8\cdot 9}{4+8}=\boxed{6.}$$

The answer is $\boxed{A.}$ □

Solution. MIC-2022-SM2-R3-O6

By multiplying both sides with 4,

$$4x^2+4xy+4y^2=112\Rightarrow (2x+y)^2+3y^2=112.$$

Three are a few values of y so that $112-3y^2$ is a perfect square

$$\begin{cases} 10^2+3\cdot 2^2=112 \\ 8^2+3\cdot 4^2=112 \\ 2^2+3\cdot 6^2=112 \end{cases}$$

Let take a look at the first equation,

$$2x+y=\pm 10, y=\pm 2\Rightarrow y=\pm 2, 2x=\pm 10\mp 2\Rightarrow (x,y)\in\{(4,2),(-6,2),(6,2),(-4,2)\}.$$

Similarly there are 4 solutions for each of the other two equations. Hence, there are $\boxed{12}$ solutions in total.

The answer is $\boxed{E.}$ □

Solution. MIC-2022-SM2-R3-J7

Each integer of the given set $1, 2, \ldots, 9$ mus appear exactly 9 times in the board A. The off-diagonal appearances occur in pairs because of the symmetry of A.

Since 9 is odd, therefore, each integer must appear at least once on the main diagonal. Hence, it is easy to see that, each integer must appear exactly once on the main diagonal. □

Solution. MIC-2022-SM2-R3-S8

The first question requires simple computation,

$$a_0 = (\sqrt{3}+1)^0 + (\sqrt{3}-1)^0 = \boxed{2}$$
$$a_1 = (\sqrt{3}+1)^2 + (\sqrt{3}-1)^2 = (3+2\sqrt{3}+1) + (3-2\sqrt{3}+1) = \boxed{8.}$$

For the second question, note that $(\sqrt{3}+1)(\sqrt{3}-1) = 2$,

$$\begin{aligned}
&a_{n+1} = (\sqrt{3}+1)^{2(n+1)} + (\sqrt{3}-1)^{2(n+1)} = (\sqrt{3}+1)^{2n+2} + (\sqrt{3}-1)^{2n+2} \\
&= \left[(\sqrt{3}+1)^{2n} + (\sqrt{3}-1)^{2n}\right]\left[(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2\right] - \left[(\sqrt{3}+1)^{2n}(\sqrt{3}-1)^2 + (\sqrt{3}-1)^{2n}(\sqrt{3}+1)^2\right] \\
&= a_n a_1 - \left[4(\sqrt{3}+1)^{2n-2} + 4(\sqrt{3}-1)^{2n-2}\right] \\
&= 8a_n - 4a_{n-1}.
\end{aligned}$$

For the third question, we use the Induction Principle to prove it.

Let the induction hypothesis be that

Claim (Induction hypothesis) —

$$a_n \text{ is divisible by } 2^{n+1}, \text{ for all } n \geq 1 \qquad (*)$$

Proof. Let first prove the base case.

The base case: for $n = 1$, note that $a_1 = 8$ is divisible by $2^{1+1} = 4$. Thus the base case stands.

The inductive step: assume that for some positive integer n,

$$a_k \text{ is divisible by } 2^{k+1}, \text{ for all } 1 \leq k \leq n \qquad (**)$$

(*This is called strong inductive assumption, stating that the assumption is true for all integers from the base case up to n. The usual inductive assumption presumes only for n. Nevertheless, it's the same in our case.*)

Now, with this assumption, we have to prove that

$$a_{n+1} \text{ is divisible by } 2^{n+2}.$$

By the assumption (**), for $k = n$, a_n is divisible by 2^{n+1}, so $8a_n$ is divisible by 2^{n+2}.

Similarly by (**), for $k = n-1$, a_{n-1} is divisible by 2^n, thus $4a_{n-1}$ is divisible by 2^{n+1}, too.

Therefore $a_{n+1} = 8a_n - 4a_{n-1}$ is divisible by 2^{n+2}. With this we complete the proof for (*). ■

Now, it is easy to see that

$$\sqrt{3} - 1 < 1 \Rightarrow (\sqrt{3}-1)^n < 1 \Rightarrow (\sqrt{3}+1)^n + (\sqrt{3}-1)^n = \lceil(\sqrt{3}+1)^{2n}\rceil.$$

Therefore $\lceil(\sqrt{3}+1)^{2n}\rceil$ is divisible by 2^n. □

Solution. MIC-2022-SM2-R3-S9

First, since AE and FG are the internal and external angle bisectors of $\angle A$, so $\boxed{\text{they are perpendicular.}}$

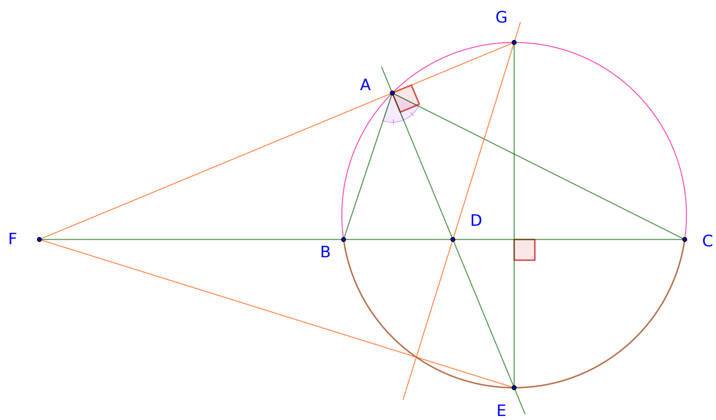


Figure 18.5: $EA \perp DG$, $GE \perp BC$

Second, $\angle EAG = 90°$, thus EG is the diameter of the circumcircle of $\triangle ABC$. Furthermore $\angle BAE = \angle EAC$, so $\widehat{BE} = \widehat{EC}$, thus the diameter GE is through the midpoint of $\widehat{BC}$, therefore it is perpendicular to the chord BC, or $\boxed{EG \perp BC.}$

Finally, if Z is the intersection of EG and BC, then in $\triangle EFG$, EA and FZ are the altitudes to FG and EG, respectively. The intersection of EA and FZ, which is D, should be the orthocenter of $\triangle ABC$. If X is the intersection of GD and EF, then GX is the third altitude in $\triangle ABC$. In other words, $\boxed{DG \perp EF.}$ □

Solution. MIC-2022-SM2-R3-O10

First, note that if Minh picks $u \geq v$, then the sum of squares of the numbers on the board is changed by

$$(u+v)^2 + (u-v)^2 - (u^2+v^2) = \boxed{u^2+v^2.}$$

Now, let S_i be the sum of squares of the numbers on the board after the step i^{th}. Replacing (u, v) with $(u+v, u-v)$ does change the sum by a quantity of u^2+v^2, which is a non-negative number and can be 0 if and only if $u = v = 0$. Therefore, since there is no 0 among the initial numbers, thus

$$S_0 < S_1 \leq S_2 \leq \ldots \leq S_i \leq \ldots$$

Thus, Minh can never obtain the same sum of squares, so $\boxed{\text{the numbers can not return to the initial state.}}$ □

Chapter 19

Problem Solving Championship - Round 3

19.1 Rules

Problem Solving Championship is an activity of Math, Chess, and Coding Club (MCC) open to all students. The problems are designed so that students at various levels - junior (grade under 8), senior (grade 9 and up), or olympiad (prepare for national or international contest) - can participate. The contest problems are selected from algebra, combinatorics, number theory, geometry topics of mathematics, some are puzzles from chess, and some need to be solved by designing a computer program. Some problems do not require specific mathematical knowledge, some require problem solving skills, and some require a more comprehensive understanding of certain mathematical topics. The students received a number of points after each participation and are ranked after each round, in both overall and grade rankings. The final standing of the contest is concluded at the end of each semester.

- The contest problems become available online at the beginning of the semester or at the end of the previous contest. The contest solutions are discussed at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**. All students in the club are invited to the solution discussion. All contests start when the contest problems become available. The **solutions must be submitted latest on the last Sunday**, approximately one week **before the solution discussion day on Saturday**.
- There are 4 **show-you-work** problems with multiple steps. For each step of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too. If you solve the problem by designing a computer program, submit that program as the solution to the problem.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading: For each step there are a number of points, highlighted in the problem text, to be awarded. For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Invitation to present own solutions: Students who submitted correct solutions would be invited to present their solutions on the solution day. Students with nearly complete or almost correct solutions would also have a chance for presentation, provided that they studied the comments and suggestions from the COs to modify their solutions.

19.2 Problems

Problem 19.2.1 (PSC-2022-SM2-R3-P1)**.** (*25 points*)

Find the answers for each of the three separate questions. *Note that the result of a solved question, appeared earlier in the order of appearance, can be used in answer to the following one.*

(*5 points*) Let a, b, c be real numbers such that none of them is equal to 1. Prove that

$$a + b(1 - a) + c(1 - a)(1 - b) = 1 - (1 - a)(1 - b)(1 - c).$$

(*5 points*) Let a, b, c, d be real numbers such that none of them is equal to 1. Prove that

$$a + b(1 - a) + c(1 - a)(1 - b) + d(1 - a)(1 - b)(1 - c) = 1 - (1 - a)(1 - b)(1 - c)(1 - d).$$

(*15 points*) Use the previous results and the Induction Principle below to prove that if $a_1, a_1, \ldots, a_n$ are real numbers such that none of them is equal to 1, then

$$a_1 + a_2(1 - a_1) + \ldots + a_n(1 - a_1)\ldots(1 - a_{n-1}) = 1 - (1 - a_1)(1 - a_2)\ldots(1 - a_n).$$

Definition (Induction Principle)**.** Let a be an integer, and let $P(n)$ be a statement (or proposition) about n for each integer $n \geq a$. The **principle of induction** is a way of proving that $P(n)$ is true for all integers $n \geq a$ in two steps:

1. *Base case*: Prove that $P(a)$ is true.
2. *Inductive step*: Assume that $P(k)$ is true for integer $k \geq a$, and use $P(k)$ to prove that $P(k + 1)$ is true.

Then we may conclude that $P(n)$ is true for all integers $n \geq a$.

Here is an example how to use Induction Principle.

Example (SM-2022-SM1-67-P14)
Show that for all positive integers n, $7 + 6 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \cdots + 6 \cdot 7^n = 7^{n+1}$.

Proof. We use the Induction Principle to prove the equality.

For *the base case* when $n = 1 : 7 + 6 \cdot 7 = 7(1 + 6) = 7^2$, thus the given hypothesis is correct.

Now, for *the inductive step*, lets assume that the hypothesis is correct for $n \geq 1$, or

$$7 + 6 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \cdots + 6 \cdot 7^n = 7^{n+1}. \ (*)$$

We have to prove that it is also correct for $n + 1$, or to prove that

$$7 + 6 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \cdots + 6 \cdot 7^n + 6 \cdot 7^{n+1} = 7^{n+2}.$$

By applying the assumption in the inductive step shown in (*),

$$7 + 6 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \cdots + 6 \cdot 7^n + 6 \cdot 7^{n+1} = 7^{n+1} + 6 \cdot 7^{n+1} = 7^{n+1}(1 + 6) = 7^{n+2}.$$

Thus, the hypothesis is also correct for $n + 1$. Hence, $\boxed{\text{it is correct for all } n \geq 1.}$ □

Problem 19.2.2 (PSC-2022-SM2-R3-P2). (*25 points*)

Diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at point E. Furthermore $\angle BAD = \frac{\pi}{3}$ and $AE = 3CE$.

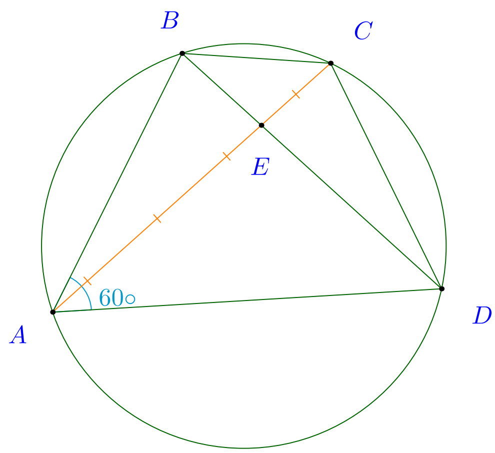


Figure 19.1: PSC-2022-SM2-R3-P2

(*10 points*) Prove that $AB \cdot AD = 3 \cdot CB \cdot CD$.

(*10 points*) Prove that $(CB - CD)^2 = BD^2 - 3 \cdot CB \cdot CD$.

(*5 points*) Prove that the sum of two sides of $ABCD$ is equal to the sum of its other two sides.

Theorem (Law of Cosines)
In $\triangle ABC$, $a^2 = b^2 + c^2 - 2bc \cos A$.

Theorem (Law of Sines)
In $\triangle ABC$, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$.

Problem 19.2.3 (PSC-2022-SM2-R3-P3)**.** (*25 points*)

In a tennis tournament, each of the $2n+1$ players played two games with each of the other players. Each player won exactly half of the games he played (and lost the other half.)

Let $P_1, P_2, \ldots, P_{2n+1}$ vertices represent the players. For any $i \neq j$, $(1 \leq i, j \leq 2n+1)$ players,

1. Assign a $P_i \to P_j$ *directed* edge between the two vertices P_i and P_j Is P_i won both games against P_j.
2. Assign a $P_i - P_j$ *undirected* edge between the two vertices P_i and P_j if each of P_i and P_j won one game against the other.

Now, for every pair of players, there is a single edge connecting them. Below is an example when $n = 2$,

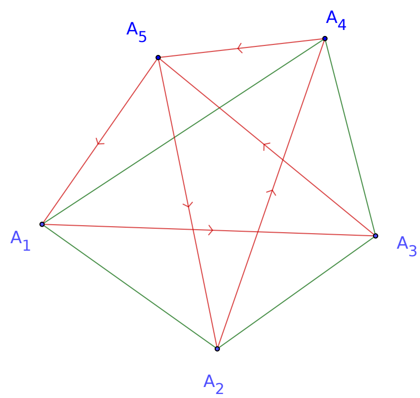


Figure 19.2: PSC-2022-SM2-R3-P3

(*10 points*) Prove, *by contradiction*, that the directed edges form cycles, that is if there is an directed edge $P_{i_1} \to P_{i_2}$, then there exist $i_1, i_2, \ldots, i_k$,

$$P_{i_1} \to P_{i_2} \to \cdots \to P_{i_k} \to P_{i_1}.$$

(*10 points*) Consider the number of directed edges going into or out of a vertex, applying the Euler Cycle, prove that the undirected edges also form cycles, that is if there is an undirected edge $P_{j_1} - P_{j_2}$, then there exist $j_1, j_2, \ldots, j_\ell$,

$$P_{j_1} - P_{j_2} - \cdots - P_{j_\ell} - P_{j_1}.$$

(*5 points*) Prove that it is possible to cancel half of the matches of the tournament such that each player still won exactly half of the games he played (and lost the other half.)

Definition (Paths & Cycles)**.** In a graph,

- path is a sequence of edges, each of which begins at the endpoint of the previous one. In a close path, the start and end points are the same. Simple path is a path that any edge appears only once.
- cycle is a close path that pass a vertex only once.
- an Euler *path* in a graph is a path that includes every edges of the graph.
- an Euler *cycle* in a graph is a cycle that includes every edges of the graph.

Theorem (Euler Cycle)
A connected graph has an Euler *cycle* if and only if all vertices have even number of edges (degree).

Problem 19.2.4 (PSC-2022-SM2-R3-P4). (*25 points*)

Given a circle (O) centred at O with radius 1. Point P_0 is an arbitrary point on the perimeter of the circle. Point P_1 is chosen on the perimeter on the circle in clockwise direction from P_0 such that the arc length

$$\widehat{P_0P_1} = 1.$$

Similarly $P_2, P_3, \ldots$ are chosen in clockwise direction from $P_1, P_2, \ldots$ such that

$$\widehat{P_1P_2} = \widehat{P_2P_3} = \ldots = 1.$$

(*5 points*): Is it possible for a pair of integers m, n such that points P_m and P_n are coincident? (two points are coincident if they are at the same position on the perimeter of the circle.)

(*5 points*): Use the Pigeonhole Principle to prove that there exist two points P_n and P_m, so that the length of $\widehat{P_nP_m}$ is less than $\frac{1}{2022}$.

(*15 points*): Two points A abd B are given on the circle (O) such that $\widehat{AB} = \frac{1}{2022}$. Prove that there an integer n such that point P_n is on the (minor) arc $\widehat{AB}$.

Definition (Pigeonhole Principle). The Pigeonhole Principle (also known as the Dirichlet box principle, Dirichlet principle or box principle) states that if $n+1$ or more holes are placed in n pigeons, then one pigeon must contain two or more holes. Another definition could be phrased as among any n integers, there are two with the same modulo-$n-1$ residue.

The extended version of the Pigeonhole Principle states that if k objects are placed in n boxes then at least one box must hold at least $\left\lceil \frac{k}{n} \right\rceil$ objects. Here $\lceil \cdot \rceil$ denotes the ceiling function.

19.3 Grading

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 1: Separately grading for each part,

(a) 2 points if can expand both sides of the equation and compare them (with some mistakes).
(b) 2 points if can use the previous result (of the first question) even if cannot complete.
(c) 5 points if can follow the induction principle with some issues.

Problem 2: Separately grading for each part,

(a) 2 points if the student use the Law of Sines to find the area of $\triangle ABD$ or $\triangle BCD$.
(b) 2 points if the student use the Law of Cosines for $\triangle ABD$, or $\triangle BCD$.
(c) 2 points if the student can (partially) combining the previous two results.

Problem 3: Separately grading for each part,

(a) 2 points if the student investigate the largest path formed by directed edges, in other words, use the Extremal Principle for the existence of such path.
(b) 2 points if the student sees that the number of directed edges going into and out of a vertex is even.
(c) 2 points if the student tries to find a way to remove half of the edges from any of the cycles.

Problem 4: Separately grading for each part,

(a) 2 point if the student recognize π is irrational.
(b) 2 point if the student can try the Pigeonhole Principle.
(c) 3 points if the students can continue from the previous proof (which might not be correct), to choose the desired point.

19.4 Solutions

Solution. PSC-2022-SM2-R3-P1 For the first question, we expand the expressions on both side of the equation,

$$
\begin{aligned}
&a + b(1-a) + c(1-a)(1-b) \\
&= a + b - ab + c(1 - a - b + ab) \\
&= \boxed{a + b + c - ab - bc - ac + abc} \\
&1 - (1-a)(1-b)(1-c) \\
&= 1 - (1-a)(1 - b - c + bc) \\
&= 1 - (1 - b - c + bc - a + ab + ac - abc) \\
&= \boxed{a + b + c - ab - bc - ac + abc}
\end{aligned}
$$

It is easy to see that both sides (shown in the boxes) are equal.

For the second question, applying the result from the first question,

$$
\begin{aligned}
&a + b(1-a) + c(1-a)(1-b) + d(1-a)(1-b)(1-c) \\
&= [a + b(1-a) + c(1-a)(1-b)] + d(1-a)(1-b)(1-c) \\
&= [1 - (1-a)(1-b)(1-c)] + d(1-a)(1-b)(1-c) \\
&= 1 + (1-a)(1-b)(1-c)(-1+d) \\
&= \boxed{1 - (1-a)(1-b)(1-c)(1-d)}
\end{aligned}
$$

Now let use Induction Principle to prove the generic case. First, for the *base case* $n = 1$ or $n - 2$,

$$
\begin{aligned}
&a_1 = 1 - (1 - a_1). \\
&a_1 + a_2(1 - a_1) = a_1 + a_2 - a_1 a_2 = 1 - (1 - a_1)(1 - a_2).
\end{aligned}
$$

Now, for the *inductive step*, let assume that for some $n \geq 2$, it is true that

$$a_1 + a_2(1 - a_1) + \ldots + a_n(1 - a_1) \ldots (1 - a_{n-1}) = 1 - (1 - a_1)(1 - a_2) \ldots (1 - a_n) \ (*)$$

We shall prove that,

$$a_1 + a_2(1 - a_1) + \ldots + a_{n+1}(1 - a_1) \ldots (1 - a_n) = 1 - (1 - a_1)(1 - a_2) \ldots (1 - a_{n+1}).$$

From the inductive step (*),

$$
\begin{aligned}
&a_1 + a_2(1 - a_1) + \ldots + a_{n+1}(1 - a_1) \ldots (1 - a_n) \\
&= [a_1 + a_2(1 - a_1) + \ldots + a_n(1 - a_1) \ldots (1 - a_{n-1})] + a_{n+1}(1 - a_1) \ldots (1 - a_n) \\
&= 1 - (1 - a_1)(1 - a_2) \ldots (1 - a_n) + a_{n+1}(1 - a_1) \ldots (1 - a_n) \\
&= 1 + (1 - a_1)(1 - a_2) \ldots (1 - a_n)(-1 + a_{n+1}) \\
&= \boxed{1 - (1 - a_1)(1 - a_2) \ldots (1 - a_{n+1})}
\end{aligned}
$$

Thus, it is proven for the case $n + 1$. Hence the proof is complete. $\square$

Solution. PSC-2022-SM2-R3-P2

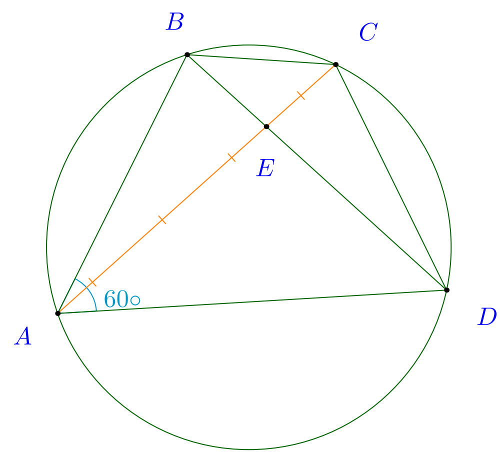


Figure 19.3: PSC-2022-SM2-R3-P2

By the Law of Sines,

$$
\begin{aligned}
&[ABD] = \frac{1}{2}AB \cdot AD \sin BAD = \frac{1}{2}AB \cdot AD \sin 60^\circ,\\
&[CBD] = \frac{1}{2}CB \cdot CD \sin BCD = \frac{1}{2}CB \cdot CD \sin 120^\circ\\
&\Rightarrow \frac{AB \cdot AD}{CB \cdot CD} = \frac{[ABD]}{[CBD]} = \frac{AE}{CE} = 3\\
&\Rightarrow \boxed{AB \cdot AD = 3 \cdot CB \cdot CD} \quad (1)
\end{aligned}
$$

By the Law of Cosines,

$$
\begin{aligned}
BD^2 &= AB^2 + AD^2 - 2 \cdot AB \cdot AD \cos 60^\circ\\
&= AB^2 + AD^2 - AB \cdot AD\\
\Rightarrow (AB - AD)^2 &= BD^2 - AB \cdot AD \quad (2)\\
BD^2 &= CB^2 + CD^2 - 2 \cdot CB \cdot CD \cos 120^\circ\\
&= CB^2 + CD^2 + CB \cdot CD\\
\Rightarrow (CB - CD)^2 &= BD^2 - 3 \cdot CB \cdot CD \quad (3)
\end{aligned}
$$

Thus, from (1), (2), and (3), $(AB - AD)^2 = (CB - CD)^2$, or $\boxed{AB + CD = AD + CB.}$ □

Solution. PSC-2022-SM2-R3-P3 For the first question, let $P_{i_1} \to P_{i_2}$ be a directed edge for which there exist a *longest path*

$$P_{i_1} \to P_{i_2} \to \cdots \to P_{i_k}.$$

Now, if there is no directed edge from P_{i_k}, it means that the number of games that P_{i_k} lost is larger then the number of games he won. This contradicts the given conditions. Thus, there exists P_t, where $P_{i_k} \to P_t$, and P_t is not one of $P_{i_1}, P_{i_2}, \ldots, P_{i_{k-1}}$, otherwise a cycle is formed.

Then

$$P_{i_1} \to P_{i_2} \to \cdots \to P_{i_k} \to P_t.$$

is a longer path, which contradicts the assumption. Hence, $\boxed{\text{directed edges form cycles.}}$

For the second question, note that each vertex has an even number of (directed and undirected) edges, and if the number of directed edges going into and out of a vertex is even. Thus, if we ignore all the directed edges, then each vertex has an even number of undirected edges. By Euler Cycle, in any (sub-)graph where all vertices have even degree. then $\boxed{\text{there exists a cycle,}}$

$$P_{j_1} - P_{j_2} - \cdots - P_{j_\ell} - P_{j_1}.$$

Now, for the last question, on a directed cycle, we remove a match from any directed edge, leaving half of the matches. On an undirected cycle, we also remove half of the matches, keeping each player with a pair of win-lose matches. Thus, this way $\boxed{\text{half of the matches of the tournament can be canceled}}$ such that each player still won exactly half of the games he played (and lost the other half.) □

Solution. PSC-2022-SM2-R3-P4 For the first question, for any two integers $m > n$, the sum of the arcs is

$$\widehat{P_n P_{n+1}} + \ldots + \widehat{P_{m-1} P_m} = m - n.$$

If $P_m \equiv P_n$, then this difference $m - n$, which is an integer, is also a multiple of π, which is $\boxed{\text{impossible}}$ since π is irrational.

For the second question, let choose an integer N so that the arc length

$$\widehat{AB} < \frac{2\pi}{N},$$

Now, lets divide the circle into N equal arcs length $\frac{2\pi}{N}$. Then by the Pigeonhole Principle, at least $\boxed{\text{two of the points } P_1, P_2, \ldots, P_{N+1} \text{ must lie in the same arc length } \frac{2\pi}{N}.}$

For the last question, suppose the points, in the second result, are P_n and P_{n+m}, so that $P_n P_{n+m}$ is an arc of length less than $\frac{2\pi}{N}$. Hence the arcs $P_{n+m}P_{n+2m}, P_{n+2m}P_{n+3m}, \ldots$ ave the same length. So for some integer k, $\boxed{P_{n+km} \text{ will lie in the arc } \widehat{AB}.}$ □

Chapter 20

Introductory Curriculum Level Test 2 - Level 1

20.1 Rules

20.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	**Introduction to Algebra**	**Introduction to Geometry**	**Introduction to Number Theory**	**Introduction to Counting & Probability**
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

20.3 Problems

Problem 20.3.1 (ICLT-2022-SM2-R2-L1-P1)**.** Evaluate

$$(10^{\frac{1}{2}})^{\frac{1}{2}} \times 5^{\frac{1}{3}} \times 2^{\frac{1}{2}}.$$

(A) $(10)^{\frac{3}{4}}$ (B) $10^{\frac{1}{4}}$ (C) $2^{\frac{3}{4}} \times 5^{\frac{3}{4}}$ (D) $2^{\frac{3}{4}} \times 5^{\frac{7}{12}}$ (E) $2^{\frac{7}{12}} \times 5^{\frac{7}{12}}$

Problem 20.3.2 (ICLT-2022-SM2-R2-L1-P2)**.** My team has 13 members. In how many ways can we choose a captain, a lieutenant, a secretary, and a technician, if no member can hold more than one position?

(A) 32760 (B) 24024 (C) 17160 (C) 12012 (D) 4004

Problem 20.3.3 (ICLT-2022-SM2-R2-L1-P3)**.** What is the maximum number of possible points of intersection of a circle and a triangle? (A triangle is formed by connecting three points with line segments.)

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

Problem 20.3.4 (ICLT-2022-SM2-R2-L1-P4)**.** Billy puts 24 marbles in b boxes such that each box contains the same number of marbles. If there are at least 2 boxes and each box has more than two marbles, how many possible values of b are there?

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

Problem 20.3.5 (ICLT-2022-SM2-R2-L1-P5)**.** Factor $6a^4 + 24a^3 - 18a^2$ completely as you can.

(A) $a^2(3a^2 - 12a - 9)$
(B) $2a^2(3a^2 + 12a + 9)$
(C) $6a^2(a^2 + 4a - 3)$
(D) $6a^2(a^2 - 4a - 3)$
(E) $3a^2(2a^2 - 8a + 6)$

Problem 20.3.6 (ICLT-2022-SM2-R2-L1-P6)**.** The class has 7 students: 3 boys and 4 girls. In how many ways can the students seated in a row such that 4 girls seat next to each other?

(A) 720 (B) 576 (C) 384 (D) 288 (E) 144

Problem 20.3.7 (ICLT-2022-SM2-R2-L1-P7)**.** Solve

$$\frac{x}{x-1}+\frac{1}{4}=\frac{4}{x-1}.$$

Problem 20.3.8 (ICLT-2022-SM2-R2-L1-P8)**.** The first angle in a triangle is twice of one of the two other angles. The second angle is twice of of one of the two other angles. What are the possible measures of the smallest angle in the triangle?

Problem 20.3.9 (ICLT-2022-SM2-R2-L1-P9)**.** What is the second largest two-digit prime number whose digits are also each prime?

Problem 20.3.10 (ICLT-2022-SM2-R2-L1-P10)**.** Each of the grades 7 and 8 has 4 clubs. The school decides to hold a competition. A pair of clubs in each grade play each other twice. A pair of clubs in different grades play each other once. What is the total number of matches?

20.4 Grading

Answers for multiple-choice problems.

Problem 1: D

Problem 2: C

Problem 3: A

Problem 4: B

Problem 5: C

Problem 6: B

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can bring it to $\frac{1}{4} = \frac{4-x}{x-1}$.

Problem 8: 2 points if the student can see that there are two different cases of $2x, 2x, x$ and $4x, 2x, x$ angles.

Problem 9: 2 points if the student can find the largest such prime 73.

Problem 10: 2 points if the student can devise a counting method of both within and outside of the grades.

20.5 Solutions

Solution. ICLT-2022-SM2-R2-L1-P1 *Exercise 1.7.5.c, Chapter 1, Introductory to Algebra.*

$$10^{\frac{1}{2}} = (2 \times 5)^{\frac{1}{2}} = 2^{\frac{1}{2}} \times 5^{\frac{1}{2}} \Rightarrow (10^{\frac{1}{2}})^{\frac{1}{2}} = (2^{\frac{1}{2}})^{\frac{1}{2}} \times (5^{\frac{1}{2}})^{\frac{1}{2}} = 2^{\frac{1}{4}} \times 5^{\frac{1}{4}}$$
$$\Rightarrow (10^{\frac{1}{2}})^{\frac{1}{2}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{2}} = 2^{\frac{1}{4}} \times 5^{\frac{1}{4}} \times 5^{\frac{1}{3}} \times 2^{\frac{1}{2}} = 2^{\left(\frac{1}{4}+\frac{1}{2}\right)} \times 5^{\left(\frac{1}{4}+\frac{1}{3}\right)} = \boxed{2^{\frac{3}{4}} \times 5^{\frac{7}{12}}.}$$

The number answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R2-L1-P2 *Exercise 1.4.8, Chapter 1, Introductory to Counting & Probability.*

There are 13 choices for captain, 12 choices for lieutenant, 11 choices for secretary, and 10 choices for treasurer, for a total of $13 \times 12 \times 11 \times 10 = \boxed{17160}$ different choices. The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R2-L1-P3 *Exercise 1.3.3, Chapter 1, Introductory to Geometry.*

A line segment can intersect a circle in at most two points. Since a triangle consists of three line segments, the maximum number of intersections between a triangle and a circle is $3 \times 2 = \boxed{6.}$ The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R2-L1-P4 *Exercise 1.6.1, Chapter 1, Introductory to Number Theory.*

The number of marbles in each box is $\frac{24}{b}$. Thus, b is a divisor of 24. We know that b is at least 2 and $\frac{24}{b}$ is at least 3, thus b can only be $2, 3, 4, 6$, and 8. Hence, there are $\boxed{5}$ possible values for b. The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R2-L1-P5 *Problem 2.10.d, Chapter 2, Introductory to Algebra.*

Note that each terms $6a^4, 24a^3$, and $18a^2$ of the expression contains a factor $6a^2$, thus

$$6a^4 + 24a^3 - 18a^2 = (6a^2)(a^2) + (6a^2)(4a) - (6a^2)(3) = \boxed{6a^2(a^2 + 4a - 3).}$$

The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R2-L1-P6 *Problem 2.12, Chapter 2, Introductory to Counting & Probability.*

We want to deal with the restriction first. Without worrying about which specific boys and girls go in which seats, in how many ways can the girls sit together? There are 4 basic configurations of boys and girls, note that B is a boy and G is a girl,

$$GGGGBBB, BGGGGBB, BBGGGGB, BBBGGGG$$

Then, within each configuration, there are $3! = 6$ ways in which we can assign the 3 boys to seats, and $4! = 24$ ways in which we can assign the 4 girls to seats. Therefore the number of possible seatings is $4 \times 6 \times 24 = \boxed{576}$. The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R2-L1-P7 *Problem 3.18.c, Chapter 3, Introductory to Algebra.*

$$\frac{x}{x-1} + \frac{1}{4} = \frac{4}{x-1} \Rightarrow \frac{1}{4} = \frac{4}{x-1} - \frac{x}{x-1} = \frac{4-x}{x-1} \Rightarrow x - 1 = 4(4-x) = 16 - 4x \Rightarrow 5x = 17 \Rightarrow x = \boxed{\frac{17}{5}.}$$

□

Solution. ICLT-2022-SM2-R2-L1-P8 *Problem 2.18, Chapter 2, Introductory to Geometry.*

There are two possible cases.

Case 1: The first and the second angle are both twice the third angle. Let the measure of the smallest angle, the third angle, be x,

$$2x + 2x + x = 180° \Rightarrow x = \frac{180°}{5}.$$

Case 2: One of the first two angles is twice the other angle of the first two angles, the smaller of of the first two angles is is twice the third angle. Let the measure of the smallest angle, the third angle, be x,

$$4x + 2x + x = 180° \Rightarrow x = \frac{180°}{7}.$$

□

Solution. ICLT-2022-SM2-R2-L1-P9 *Problem 2.11, Chapter 2, Introductory to Number Theory.*

A two-digit number whose digits are prime is made up of the digits $2, 3, 5$, or 7. We check the possible numbers from the largest in this list and work backwards until we find the second largest prime number:

- 77 is divisible by 7.
- 75 is divisible by 3.
- 73 is prime, so it is the largest two-digit prime number whose digits are both prime.
- 57 is divisible by 3.
- 55 is divisible by 5.
- $\boxed{53}$ is prime, so it is the second largest two-digit prime number whose digits are both prime.

□

Solution. [First solution] ICLT-2022-SM2-R2-L1-P10 *Exercise 3.3.3, Chapter 3, Introductory to Counting & Probability.*

We call a match *intra-grade* if it is played between the two clubs from the same grade, and *cross-grade* if it is played between the two clubs from different grades.

It is easy to see that the total number of intra-grade matches is the number of intra-grade matches for grade 7 plus the number of intra-grade matches for grade 8. A pairs of clubs in each grade play each other twice, so the number of intra-grade matches for grade 7 is $2 \cdot \frac{4 \cdot 3}{2} = 12$. Grade 8 has the same number of clubs, so the number of intra-grade matches for grade 8 is also 12. Therefore there are $2 \cdot 12 = 24$ *intra-grade* matches.

For a club in grade 7 there are 4 clubs in grade 8, so the number of *cross-grade* matches (played by all the clubs in grade 7 with all the clubs in grade 8) is $4 \cdot 4 = 16$.

Thus, the total number of matches is $24 + 16 = \boxed{40.}$ □

Solution. [Second solution] ICLT-2022-SM2-R2-L1-P10 The other way to count is to look at each club. Each club plays twice a club in the same grade, so each club plays $(4 - 1) \cdot 2 = 6$ matches with a club in the same grade. Each club plays once a club in different grade, so each club plays $4 \cdot 1 = 4$ matches with a club in the other grade. In total, each club plays $6 + 4 = 10$ matches. There are $4 + 4 = 8$ clubs, so there are $10 \cdot 8 = 80$ matches. However each match counted this way is counted twice (we count both $A - play - B$ and $B - play - A$), hence the total number of matches is $\frac{80}{2} = \boxed{40.}$ □

Chapter 21

Introductory Curriculum Level Test 2 - Level 2

21.1 Rules

21.2 Introductory Curriculum Level Test (ICLT)

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5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
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8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

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- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

21.3 Problems

Problem 21.3.1 (ICLT-2022-SM2-R2-L2-P1). The first angle in a triangle is twice of one of the two other angles. The second angle is twice of of one of the two other angles. What would be the smallest value of the smallest angle in the triangle?

$(A)\ \frac{180^\circ}{7}$ $\quad (B)\ 37.5^\circ$ $\quad (C)\ \frac{180^\circ}{5}$ $\quad (D)\ 35^\circ$ $\quad (E)\ 15^\circ$

Problem 21.3.2 (ICLT-2022-SM2-R2-L2-P2). Each of the grades 7 and 8 has 4 clubs. The school decides to hold a competition. A pair of clubs in each grade play each other twice. A pair of clubs in different grades play each other once. What is the total number of matches?

$(A)\ 76$ $\quad (B)\ 40$ $\quad (C)\ 38$ $\quad (D)\ 36$ $\quad (E)\ 34$

Problem 21.3.3 (ICLT-2022-SM2-R2-L2-P3). Simplify as much as possible the following product

$$\frac{2}{3a^2-12b}\cdot\frac{9a^3-36ab}{8a^2}$$

$(A)\ \frac{3}{5a}$ $\quad (B)\ \frac{2}{3a}$ $\quad (C)\ \frac{3a^2-12b}{4a}$ $\quad (D)\ \frac{3}{4a}$ $\quad (E)\ \frac{1}{3(3a^2-12b)}$

Problem 21.3.4 (ICLT-2022-SM2-R2-L2-P4). Which pair of triangles must be congruent?

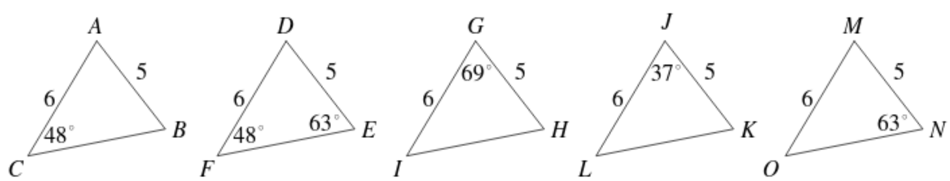


Figure 21.1: ICLT-2022-SM2-R2-L2-P4

(A) $(\triangle ABC, \triangle GHI)$
(B) $(\triangle DEF, \triangle JKL)$
(C) $(\triangle MNO, \triangle ABC)$
(D) $(\triangle JKL, \triangle MNO)$
(E) $(\triangle GHI, \triangle DEF)$

Problem 21.3.5 (ICLT-2022-SM2-R2-L2-P5). Compute the least common multiple of $24, 18$, and 36.

$(A)\ 504$ $\quad (B)\ 252$ $\quad (C)\ 168$ $\quad (D)\ 126$ $\quad (E)\ 72$

Problem 21.3.6 (ICLT-2022-SM2-R2-L2-P6). Which one is the largest?

$(A)\ \binom{44}{1}$ $\quad (B)\ \binom{10}{2}$ $\quad (C)\ \binom{6}{3}$ $\quad (D)\ \binom{7}{4}$ $\quad (E)\ \binom{9}{7}$

Problem 21.3.7 (ICLT-2022-SM2-R2-L2-P7)**.** There are red and blue balls in the Christmas shop. A red ball costs \$5 and a blue ball costs \$3. 8 red balls together weight 1 kg. 2 blue balls together weight 1 kg.

Marianna bought some red and blue balls. Altogether they cost \$111, and weight 10 kg. How many blue balls did Marianna buy?

Problem 21.3.8 (ICLT-2022-SM2-R2-L2-P8)**.** AC and BD meet at X as shown. Given $[ABX] = 24$, $[BCX] = 15$, and $[CDX] = 10$, find $[ADX]$.

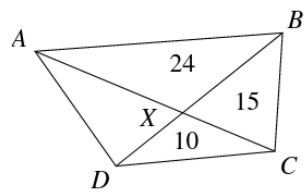


Figure 21.2: ICLT-2022-SM2-R2-L2-P8

Problem 21.3.9 (ICLT-2022-SM2-R2-L2-P9)**.** How many common divisors that 108, 144 and 60 have?

Problem 21.3.10 (ICLT-2022-SM2-R2-L2-P10)**.** The teacher wants to divide 8 students into two groups: one group has 3 students and the other with 5 students, such that two of the 8 students, Lena and Jean, should not be in the same group. In how many ways can the teacher do that?

21.4 Grading

Answers for multiple-choice problems.

Problem 1: A

Problem 2: B

Problem 3: D

Problem 4: E

Problem 5: E

Problem 6: B

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can establish the system of equations

$$\begin{cases} 5r + 3b &= 111 \\ \frac{1}{8}r + \frac{1}{2}b &= 10 \end{cases}$$

Problem 8: 2 points if the student can show that $\triangle ABX, \triangle CBX$ or $\triangle ADX, \triangle CDX$ have same height, different bases.

Problem 9: 2 points if the student can see that a common divisor of some numbers is a divisor of the greatest common divisor of them.

Problem 10: 2 points if the student can find the total number of ways to split $\binom{8}{3}$ if use complementary counting (solution 1), or can show casework if use direct counting (solution 2.)

21.5 Solutions

Solution. ICLT-2022-SM2-R2-L2-P1 *Problem 2.18, Chapter 2, Introductory to Geometry.*

There are two possible cases.

Case 1: The first and the second angle are both twice the third angle. Let the measure of the smallest angle, the third angle, be x,

$$2x + 2x + x = 180^\circ \Rightarrow x = \frac{180^\circ}{5}.$$

Case 2: One of the first two angles is twice the other angle of the first two angles, the smaller of of the first two angles is is twice the third angle. Let the measure of the smallest angle, the third angle, be x,

$$4x + 2x + x = 180^\circ \Rightarrow x = \frac{180^\circ}{7}.$$

The number answer is $\boxed{A.}$ □

Solution. [First] ICLT-2022-SM2-R2-L2-P2 *Exercise 3.3.3, Chapter 3, Introductory to Counting & Probability.*

We call a match *intra-grade* if it is played between the two clubs from the same grade, and *cross-grade* if it is played between the two clubs from different grades.

It is easy to see that the total number of intra-grade matches is the number of intra-grade matches for grade 7 plus the number of intra-grade matches for grade 8. A pairs of clubs in each grade play each other twice, so the number of intra-grade matches for grade 7 is $2 \cdot \frac{4 \cdot 3}{2} = 12$. Grade 8 has the same number of clubs, so the number of intra-grade matches for grade 8 is also 12. Therefore there are $2 \cdot 12 = 24$ *intra-grade* matches.

For a club in grade 7 there are 4 clubs in grade 8, so the number of *cross-grade* matches (played by all the clubs in grade 7 with all the clubs in grade 8) is $4 \cdot 4 = 16$.

Thus, the total number of matches is $24 + 16 = \boxed{40.}$ The number answer is $\boxed{B.}$ □

Solution. [Second] ICLT-2022-SM2-R2-L2-P2 The other way to count is to look at each club. Each club plays twice a club in the same grade, so each club plays $(4-1) \cdot 2 = 6$ matches with a club in the same grade. Each club plays once a club in different grade, so each club plays $4 \cdot 1 = 4$ matches with a club in the other grade. In total, each club plays $6 + 4 = 10$ matches. There are $4 + 4 = 8$ clubs, so there are $10 \cdot 8 = 80$ matches. However each match counted this way is counted twice (we count both $A - play - B$ and $B - play - A$), hence the total number of matches is $\frac{80}{2} = \boxed{40.}$ The number answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R2-L2-P3 *Exercise 4.3.4, Chapter 4, Introductory to Algebra.*

$$3a^2 - 12b = 3(a^2 - 4b),\ 9a^3 - 36ab = 9a(a^2 - 4b)$$
$$\Rightarrow \frac{2}{3a^2 - 12b} \cdot \frac{9a^3 - 36ab}{10a^2} = \frac{2}{3(a^2 - 4b)} \cdot \frac{9a(a^2 - 4b)}{8a^2} = \boxed{\frac{3}{4a}}$$

The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R2-L2-P4 *Exercise 3.3.1, Chapter 3, Introductory to Geometry.*

In $\triangle DEF$, we have $\angle D = 180° - \angle E - \angle F = 69°$.

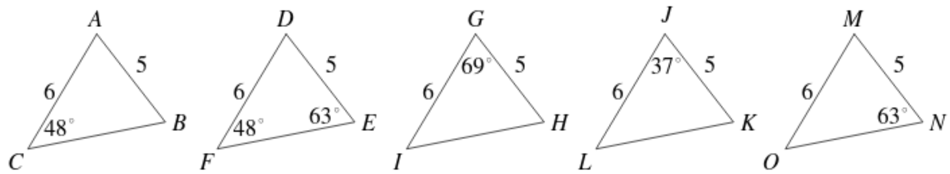


Figure 21.3: ICLT-2022-SM2-R2-L2-P4

Therefore, by SAS we have $\triangle DEF \cong \triangle GHI$. No other two triangles shown need be congruent. The answer is $\boxed{E.}$ $\square$

Solution. ICLT-2022-SM2-R2-L2-P5 *Exercise 3.4.1.i, Chapter 3, Introductory to Number Theory.*

Note that

$$24 = 2^3 \cdot 3, 18 = 2 \cdot 3^2, 36 = 2^2 \cdot 3^2 \Rightarrow \operatorname{lcm}(24, 18, 36) = 2^3 \cdot 3^2 = \boxed{72.}$$

The answer is $\boxed{E.}$ $\square$

Solution. ICLT-2022-SM2-R2-L2-P6 *Problem 4.7, Chapter 4, Introductory to Counting & Probability.*

$$\binom{44}{1} = 44$$
$$\binom{10}{2} = \frac{9 \cdot 10}{1 \cdot 2} = \boxed{45}$$
$$\binom{6}{3} = \frac{6!}{3!3!} = \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} = 20$$
$$\binom{7}{4} = \frac{7!}{4!3!} = \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3} = 35$$
$$\binom{9}{7} = \frac{8 \cdot 9}{1 \cdot 2} = 36$$

The answer is $\boxed{B.}$ $\square$

Solution. ICLT-2022-SM2-R2-L2-P7 *Problem 5.11, Chapter 5, Introductory to Algebra.*

Let r be the number of red balls and b be the number of blue balls that Marianna bought.

$$\begin{cases} 5r + 3b & = 111 \\ \frac{1}{8}r + \frac{1}{2}b & = 10 \end{cases} \Rightarrow \begin{cases} 5r + 3b & = 111 \\ r + 4b & = 80 \end{cases} \Rightarrow 5(r+4b) - (5r+3b) = 160 - 111 \Rightarrow 17b = 289 \Rightarrow b = \boxed{17} \Rightarrow r = 12.$$

$\square$

Solution. ICLT-2022-SM2-R2-L2-P8 *Problem 4.12, Chapter 4, Introductory to Geometry.*

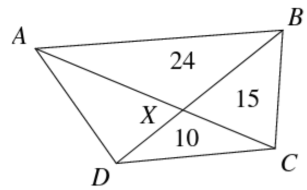


Figure 21.4: ICLT-2022-SM2-R2-L2-P8

Since $\triangle ABX$ and $\triangle CBX$ share an altitude from B, we have

$$\frac{AX}{CX} = \frac{[ABX]}{[CBX]} = \frac{8}{5}.$$

Turning to triangles $\triangle ADX$ and $\triangle CDX$, we have

$$\frac{[ADX]}{[CDX]} = \frac{AX}{CX} = \frac{8}{5}.$$

Therefore, $[ADX] = \frac{8}{5}[CDX] = \boxed{16.}$ □

Solution. ICLT-2022-SM2-R2-L2-P9 *Problem 4.4.2, Chapter 4, Introductory to Number Theory.*

A common divisor of 108, 144, and 60 is a divisor of the greatest common divisor of 108, 144, and 60. Thus the number of common divisors of 108, 144, and 60 is the number of divisors of the greatest common divisor of 108, 144, and 60. Now, since

$$108 = 2^2 \cdot 3^3, 144 = 2^4 \cdot 3^2, 60 = 2^2 \cdot 3 \cdot 5 \Rightarrow \gcd(108, 144, 60) = 2^2 \cdot 3 = 12.$$

It is easy to see that 12 has $\boxed{6}$ distinct divisors. □

Solution. [First solution] ICLT-2022-SM2-R2-L2-P10 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

Let's first do the problem by complementary counting. If we have no restrictions on the groups, then we simply need to choose 3 of the 8 students to be in the smaller group, and the rest of the students will make up the larger group. There are $\binom{8}{3}$ ways to do this.

But we can't have Lena and Jean in the same group. So we have to subtract the number of ways that we can form the two groups with Lena and Jean in the same group.

Case 1: Lena and Jean are both in the smaller group. If they are both in the smaller group, then we have to choose 1 more student from the 6 remaining to complete the smaller group, and we can do this in $\binom{6}{1}$ ways.

Case 2: Lena and Jean are both in the larger group. If they are both in the larger group, then we have to choose 3 students from the 6 remaining to compose the larger group, and we can do this in $\binom{6}{3}$ ways.

So to get the number of ways to form groups such that Lena and Jean are both in the same group, we add the counts from our two cases, to get $\binom{6}{1} + \binom{6}{3}$.

But remember that these are the cases that we don't want, so to solve the problem, we subtract this count from the number of ways to form the two groups without restrictions. Thus, our answer is

$$\binom{8}{3} - \left(\binom{6}{1} + \binom{6}{3}\right) = 56 - (6 + 20) = \boxed{30.}$$

□

Solution. [Second solution] ICLT-2022-SM2-R2-L2-P10 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

We could solve this problem is by direct counting. There are two cases of possible groupings.

Case 1: Lena is in the smaller group, Jean is in the larger group To complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

Case 2: Jean is in the smaller group, Lena is in the larger group Again, to complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

So to count the total number of groupings, we add the counts from our two cases, to get $\binom{6}{2} + \binom{6}{2} = 15 + 15 = \boxed{30}$ as our final answer. $\square$

Chapter 22

Introductory Curriculum Level Test 2 - Level 3

22.1 Rules

22.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	Introduction to Algebra	Introduction to Geometry	Introduction to Number Theory	Introduction to Counting & Probability
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

22.3 Problems

Problem 22.3.1 (ICLT-2022-SM2-R2-L3-P1). There are red and blue balls in the Christmas shop. A red ball costs \$5 and a blue ball costs \$3. 8 red balls together weight 1 kg. 2 blue balls together weight 1 kg.

Marianna bought some red and blue balls. Altogether they cost \$111, and weight 10 kg. How many more blue balls than red balls did Marianna buy?

(A) 5 (B) 7 (C) 9 (D) 11 (E) 13

Problem 22.3.2 (ICLT-2022-SM2-R2-L3-P2). AC and BD meet at X as shown. Given $[ABX] = 24$, $[BCX] = 15$, and $[CDX] = 10$, find $[ABCD]$.

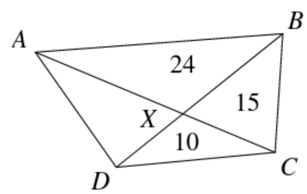


Figure 22.1: ICLT-2022-SM2-R2-L2-P8

(A) 71 (B) 69 (C) 67 (D) 65 (E) 63

Problem 22.3.3 (ICLT-2022-SM2-R2-L3-P3). How many common divisors that 108, 144 and 60 have?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Problem 22.3.4 (ICLT-2022-SM2-R2-L3-P4). The teacher wants to divide 8 students into two groups: one group has 3 students and the other with 5 students, such that two of the 8 students, Lena and Jean, should not be in the same group. In how many ways can the teacher do that?

(A) 24 (B) 30 (C) 36 (D) 40 (E) 44

Problem 22.3.5 (ICLT-2022-SM2-R2-L3-P5). What is the ratio of y to x if

$$\frac{10x - 3y}{11x - 2y} = \frac{4}{7}$$

(A) $13 : 18$ (B) $11 : 15$ (C) $9 : 11$ (D) $7 : 10$ (E) $1 : 2$

Problem 22.3.6 (ICLT-2022-SM2-R2-L3-P6)**.** In the diagram below, $DE \parallel BC$, and the segments have the lengths as shown in the diagram, $x = BC$, $y = CF$, and $z = EA$.

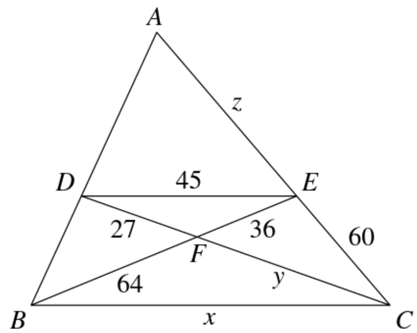


Figure 22.2: ICLT-2022-SM2-R1-L3-P6

Find $x - y + z$.

(A) $\dfrac{1430}{7}$ (B) 188 (C) $\dfrac{1436}{7}$ (D) 109 (E) $\dfrac{764}{7}$

Problem 22.3.7 (ICLT-2022-SM2-R2-L3-P7)**.** How many of the positive divisors of 540 have 6 positive divisors?

Problem 22.3.8 (ICLT-2022-SM2-R2-L3-P8)**.** Six parallel lines in a plane intersect a set of n parallel lines that go in another direction. The lines form a total of 315 parallelograms, many of which overlap each other. Find n.

Problem 22.3.9 (ICLT-2022-SM2-R2-L3-P9)**.** Nine people can mow a lawn in fifteen hours. How many more people are needed to mow the lawn in just three hours, assuming each person mows at the same rate?

Problem 22.3.10 (ICLT-2022-SM2-R2-L3-P10)**.** Find the highest power of 2 that divides $5! + 6! + 7!$.

22.4 Grading

Answers for multiple-choice problems.

Problem 1: *A*

Problem 2: *D*

Problem 3: *B*

Problem 4: *B*

Problem 5: *E*

Problem 6: *E*

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can factor 540 and find how to form a divisor that has 6 factors.

Problem 8: 2 points if the student can see that choosing two pairs of parallel lines will form a parallelogram.

Problem 9: 2 points if the student can find that the number of people mowing and the time required to mow are inversely proportional.

Problem 10: 2 points if the student can factor 5! out of the sum.

22.5 Solutions

Solution. ICLT-2022-SM2-R2-L3-P1 *Problem 5.11, Chapter 5, Introductory to Algebra.*

Let r be the number of red balls and b be the number of blue balls that Marianna bought.

$$\begin{cases} 5r+3b &= 111 \\ \frac{1}{8}r+\frac{1}{2}b &= 10 \end{cases} \Rightarrow \begin{cases} 5r+3b &= 111 \\ r+4b &= 80 \end{cases} \Rightarrow 5(r+4b)-(5r+3b) = 160-111 \Rightarrow 17b = 289 \Rightarrow b = 17 \Rightarrow r = 12.$$

Marianna bought is $17-12=\boxed{5}$ more blue balls than red balls. The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R2-L3-P2 *Exercise 3.3.3, Chapter 3, Introductory to Counting & Probability.*

Problem 4.12, Chapter 4, Introductory to Geometry.

Since $\triangle ABX$ and $\triangle CBX$ share an altitude from B, we have

$$\frac{AX}{CX} = \frac{[ABX]}{[CBX]} = \frac{8}{5}.$$

Turning to triangles $\triangle ADX$ and $\triangle CDX$, we have

$$\frac{[ADX]}{[CDX]} = \frac{AX}{CX} = \frac{8}{5}.$$

Therefore, $[ADX] = \frac{8}{5}[CDX] = 16$, thus $[ABCD] = 24+15+10+16 = boxed65$. The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R2-L3-P3 *Problem 4.4.2, Chapter 4, Introductory to Number Theory.*

A common divisor of 108, 144, and 60 is a divisor of the greatest common divisor of 108, 144, and 60. Thus the number of common divisors of 108, 144, and 60 is the number of divisors of the greatest common divisor of 108, 144, and 60. Now, since

$$108 = 2^2\cdot 3^3, 144 = 2^4\cdot 3^2, 60 = 2^2\cdot 3\cdot 5 \Rightarrow \gcd(108,144,60) = 2^2\cdot 3 = 12.$$

It is easy to see that 12 has $\boxed{6}$ distinct divisors. The answer is $\boxed{B.}$ □

Solution. [First solution] ICLT-2022-SM2-R2-L3-P4 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

Let's first do the problem by complementary counting. If we have no restrictions on the groups, then we simply need to choose 3 of the 8 students to be in the smaller group, and the rest of the students will make up the larger group. There are $\binom{8}{3}$ ways to do this.

But we can't have Lena and Jean in the same group. So we have to subtract the number of ways that we can form the two groups with Lena and Jean in the same group.

Case 1: Lena and Jean are both in the smaller group. If they are both in the smaller group, then we have to choose 1 more student from the 6 remaining to complete the smaller group, and we can do this in $\binom{6}{1}$ ways.

Case 2: Lena and Jean are both in the larger group. If they are both in the larger group, then we have to choose 3 students from the 6 remaining to compose the larger group, and we can do this in $\binom{6}{3}$ ways.

So to get the number of ways to form groups such that Lena and Jean are both in the same group, we add the counts from our two cases, to get $\binom{6}{1}+\binom{6}{3}$.

But remember that these are the cases that we don't want, so to solve the problem, we subtract this count from the number of ways to form the two groups without restrictions. Thus, our answer is

$$\binom{8}{3} - \left(\binom{6}{1}+\binom{6}{3}\right) = 56-(6+20) = \boxed{30.}$$

The answer is $\boxed{B.}$ □

Solution. [Second solution] ICLT-2022-SM2-R2-L3-P4 *Problem 5.5, Chapter 5, Introductory to Counting & Probability.*

We could solve this problem is by direct counting. There are two cases of possible groupings.

Case 1: Lena is in the smaller group, Jean is in the larger group To complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

Case 2: Jean is in the smaller group, Lena is in the larger group Again, to complete the smaller group, we need to choose 2 more students from the 6 remaining. We can do this in $\binom{6}{2}$ ways.

So to count the total number of groupings, we add the counts from our two cases, to get $\binom{6}{2}+\binom{6}{2} = 15+15 = \boxed{30}$ as our final answer. The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R2-L3-P5 *Problem 6.5, Chapter 6, Introductory to Algebra.*

$$\frac{10x-3y}{11x-2y} = \frac{4}{7} \Rightarrow 7(10x-3y) = 4(11x-2y) \Rightarrow 70x-21y = 44x-8y \Rightarrow 26x = 13y \Rightarrow \frac{y}{x} = \frac{1}{2}$$

Thus, the ratio of y to x is $\boxed{1:2}$. The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R2-L3-P6 *Problem 5.12, Chapter 5, Introductory to Geometry.*

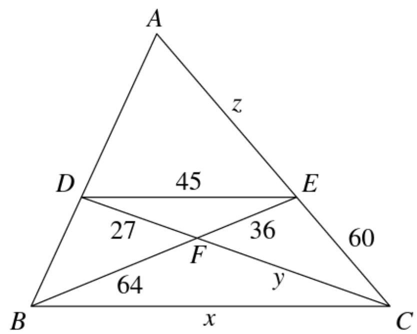


Figure 22.3: ICLT-2022-SM2-R2-L3-P6

Since $DE \parallel BC$, we have $\triangle FBC \sim \triangle FED$ by AA Similarity. Therefore, we have

$$\frac{FC}{FD} = \frac{BC}{DE} = \frac{FB}{FE} = \frac{64}{36} = \frac{16}{9} \Rightarrow x = BC = \frac{16}{9}(DE) = 80,\ \ y = FC = \frac{16}{9}(DF) = 48.$$

Since $\triangle ADE \sim \triangle ABC$ by AA Similarity, we have

$$\frac{AE}{AC} = \frac{DE}{BC} = \frac{45}{80} = \frac{9}{16},\ \ AE = z,\ \ AC = AE + EC = z + 60$$

$$\Rightarrow \frac{z}{z+60} = \frac{9}{16} \Rightarrow 16z = 9z + 540 \Rightarrow z = \frac{540}{7} \Rightarrow x - y + z = 80 - 48 + \frac{540}{7} = \boxed{\frac{764}{7}.}$$

The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R2-L3-P7 *Problem 5.9, Chapter 5, Introductory to Number Theory.*

The problem requires us to find divisors of 540 with a particular property, so we begin by finding the prime factorization of 540 in order to learn more about its divisors:

$$540 = 2^2 \cdot 3^3 \cdot 5^1.$$

A divisor d of 540 has the form

$$d = 2^a \cdot 3^b \cdot 5^c, \text{ where } a = 0, 1 \text{ or } 2;\ b = 0, 1, 2 \text{ or } 3;\ b = 0, 1;\ c = 0 \text{ or } 1$$

We must now count possible combinations of a, b and c such that d has exactly 6 positive divisors. In other words, we are counting combinations of a, b and c such that

$$(a + 1)(b + 1)(c + 1) = 6.$$

The only ways to get 6 as the product of 3 positive integers are

$$6 = 1 \cdot 1 \cdot 6 = 1 \cdot 2 \cdot 3.$$

These products imply that (a, b, c) must include either two 0's and one 5 or else one 0, one 1, and one 2. Since none of $a, b,$ or c is greater than 3, this leaves us with only four possibilities:

$$(2, 1, 0), (2, 0, 1), (1, 2, 0), (0, 2, 1) \Rightarrow 2^2 \cdot 3 = 12, 2^2 \cdot 5 = 20, 2 \cdot 3^2 = 18, 3^2 \cdot 5 = 45.$$

It is easy to verify that each of $\boxed{12, 18, 20, 45}$ has exactly six divisors. □

Solution. ICLT-2022-SM2-R2-L3-P8 *Problem 6.4, Chapter 6, Introductory to Counting & Probability.*

The goal here is to find an expression for the number of parallelograms in terms of n. Then we'll set that expression equal to 315, and solve for n.

We start with a constructive approach, wondering how we can create a parallelogram by choosing some of our lines. We need to choose 2 lines from each set - this will give us the 4 sides of a parallelogram, as shown in the figure below.

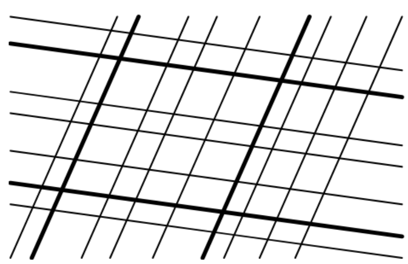


Figure 22.4: ICLT-2022-SM2-R2-L3-P8

There are 6 lines in one set of parallel lines, so the number of ways we can choose 2 lines from that set is $\binom{6}{2} = 15$. There are n lines in the other set of parallel lines, so the number of ways we can choose 2 lines from that set is $\binom{n}{2}$.

Hence there are $15\binom{n}{2}$ parallelograms. But we're told that there are 315 parallelograms, so we have

$$15\binom{n}{2} = 315 \Rightarrow \binom{n}{2} = \frac{315}{15} = 21 \Rightarrow \frac{n(n-1)}{2} = 21 \Rightarrow n = \boxed{7.}$$

□

Solution. ICLT-2022-SM2-R2-L3-P9 *Exercise 7.2.2, Chapter 7, Introductory to Algebra.*

The number of people mowing and the time required to mow are inversely proportional. Letting n be the number of people and t be the amount of time, we have $nt = (9)(15) = 135$ because 9 people can mow a lawn in 15 hours. If m people can mow the lawn in 3 hours, then we must have $m(3) = 135$, so $m = 45$. Therefore, we need to add $45 - 9 = \boxed{36}$ people to the job. □

Solution. ICLT-2022-SM2-R2-L3-P10 *Problem 6.4, Chapter 6, Introductory to Number Theory.*

$5! + 6! + 7! = 5!(1 + 6 + 6 \cdot 7) = 5! \cdot 49 = 5! \cdot 7^2 = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 7^2 = 2^3 \cdot 3 \cdot 5 \cdot 7^2$. The highest power of 2 that divides $5! + 6! + 7!$ is $\boxed{2^3.}$ □

Chapter 23

Introductory Curriculum Level Test 2 - Level 4

23.1 Rules

23.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	**Introduction to Algebra**	**Introduction to Geometry**	**Introduction to Number Theory**	**Introduction to Counting & Probability**
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
    1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
    2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

23.3 Problems

Problem 23.3.1 (ICLT-2022-SM2-R2-L4-P1). Eight parallel lines in a plane intersect a set of n parallel lines that go in another direction. The lines form a total of 280 parallelograms, many of which overlap each other. Find n.

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 23.3.2 (ICLT-2022-SM2-R2-L4-P2). In the diagram below, $DE \parallel BC$, and the segments have the lengths as shown in the diagram, $x = BC$, $y = CF$, and $z = EA$.

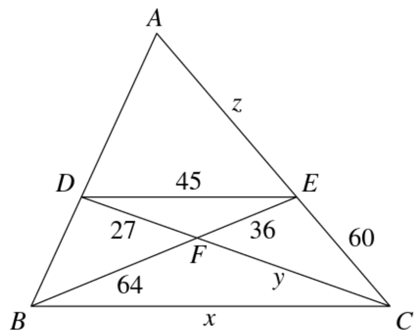


Figure 23.1: ICLT-2022-SM2-R2-L4-P2

Find $x + y - z$.

(A) $\frac{1430}{7}$ (B) 57 (C) $\frac{356}{7}$ (D) 58 (E) $\frac{365}{7}$

Problem 23.3.3 (ICLT-2022-SM2-R2-L4-P3). What is the sum of all positive divisors of 540 whose has 4 positive divisors?

(A) 16 (B) 21 (C) 25 (D) 31 (E) 40

Problem 23.3.4 (ICLT-2022-SM2-R2-L4-P4). A line with slope 4 intersects a line with slope 4 at the point $(10, 20)$. What is the distance between the x-intercepts of these two lines?

(A) $\frac{7}{2}$ (B) 3 (C) $\frac{5}{2}$ (D) 1 (E) 2

Problem 23.3.5 (ICLT-2022-SM2-R2-L4-P5)**.** AD and BC are both perpendicular to AB in the diagram below, and $CD \perp AC$. If $AB = 8$ and $BC = 6$, find AD.

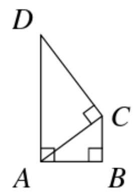


Figure 23.2: ICLT-2022-SM2-R2-L4-P5

$(A)\quad \dfrac{50}{3} \qquad (B)\quad 15 \qquad (C)\quad \dfrac{40}{3} \qquad (D)\quad 12 \qquad (E)\quad \dfrac{32}{3}$

Problem 23.3.6 (ICLT-2022-SM2-R2-L4-P6)**.** Let $n = \overline{abc}$ be a positive 3-digit integer that is equal to the sum $\overline{ab} + \overline{ba} + \overline{ac} + \overline{ca} + \overline{bc} + \overline{cb}$ of six positive 2-digit integers.

For the largest possible n, find the sum $a + b + c$.

$(A)\quad 6 \qquad (B)\quad 8 \qquad (C)\quad 10 \qquad (D)\quad 12 \qquad (E)\quad 18$

Problem 23.3.7 (ICLT-2022-SM2-R2-L4-P7)**.** Two different 2-digit numbers are randomly chosen and multiplied together. What is the probability that the resulting product is not divisible by 53?

Problem 23.3.8 (ICLT-2022-SM2-R2-L4-P8)**.** Sort the following numbers from smallest to greatest

$$2^{600}, 3^{400}, 4^{250}, 5^{200}.$$

Problem 23.3.9 (ICLT-2022-SM2-R2-L4-P9)**.** In $\triangle ABC$, $AB = 10$, $AC = 12$, and $BC = 8$. Point M is on side BC such that $\angle BAM = \angle CAM$. Find BM.

Problem 23.3.10 (ICLT-2022-SM2-R2-L4-P10)**.** Phan and Quan play a number of two-player games. Any game has a winner (no draw). The player who is the first to win 4 games (not necessarily in a row) is the champion. Phan has a $\frac{2}{3}$ probability of winning any individual game. What is the probability that after exactly 7 games Phan is declared champion?

23.4 Grading

Answers for multiple-choice problems.

Problem 1: B

Problem 2: C

Problem 3: E

Problem 4: D

Problem 5: A

Problem 6: E

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can find that the number of ways to choose two numbers so their product is divisible by 53 is 89.

Problem 8: 2 points if the student can simplify the problem by taking the 50^{th} (or similar) roots of the numbers.

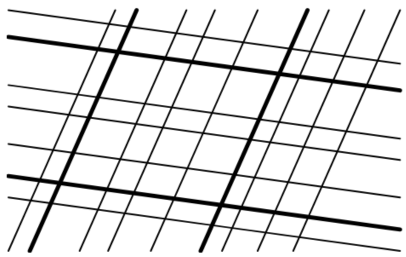
Problem 9: 2 points if the student can use the Angle-Bisector Theorem.

Problem 10: 2 points if the student can see that there is 20 possible sequences of games.

23.5 Solutions

Solution. ICLT-2022-SM2-R2-L4-P1 *Problem 6.4, Chapter 6, Introductory to Counting & Probability.*

The goal here is to find an expression for the number of parallelograms in terms of n. Then we'll set that expression equal to 315, and solve for n. We start with a constructive approach, wondering how we can create a parallelogram by choosing some of our lines. We need to choose 2 lines from each set - this will give us the 4 sides of a parallelogram, as shown in the figure below.

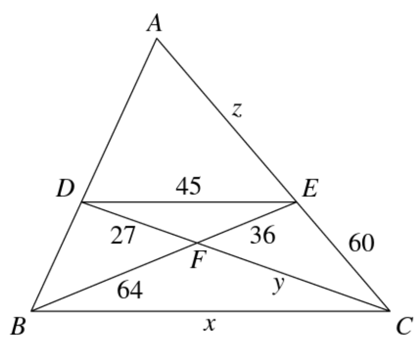


There are 8 lines in one set of parallel lines, so the number of ways we can choose 2 lines from that set is $\binom{8}{2} = 28$. There are n lines in the other set of parallel lines, so the number of ways we can choose 2 lines from that set is $\binom{n}{2}$. Hence there are $28\binom{n}{2}$ parallelograms. But we're told that there are 280 parallelograms,

$$28\binom{n}{2} = 280 \Rightarrow \binom{n}{2} = \frac{280}{28} = 10 \Rightarrow \frac{n(n-1)}{2} = 10 \Rightarrow n = \boxed{5.}$$

The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R2-L4-P2 *Problem 5.12, Chapter 5, Introductory to Geometry.*



Since $DE \parallel BC$, we have $\triangle FBC \sim \triangle FED$ by AA Similarity. Therefore, we have

$$\frac{FC}{FD} = \frac{BC}{DE} = \frac{FB}{FE} = \frac{64}{36} = \frac{16}{9} \Rightarrow x = BC = \frac{16}{9}(DE) = 80,\ y = FC = \frac{16}{9}(DF) = 48.$$

Since $\triangle ADE \sim \triangle ABC$ by AA Similarity, we have

$$\frac{AE}{AC} = \frac{DE}{BC} = \frac{45}{80} = \frac{9}{16},\ AE = z,\ AC = AE + EC = z + 60$$

$$\Rightarrow \frac{z}{z+60} = \frac{9}{16} \Rightarrow 16z = 9z + 540 \Rightarrow z = \frac{540}{7} \Rightarrow x - y - z = 80 + 48 - \frac{540}{7} = \boxed{\frac{356}{7}.}$$

The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R2-L4-P3 *Problem 5.9, Chapter 5, Introductory to Number Theory.*

The problem requires us to find divisors of 540 with a particular property, so we begin by finding the prime factorization of 540 in order to learn more about its divisors:

$$540 = 2^2 \cdot 3^3 \cdot 5^1.$$

A divisor d of 540 has the form

$$d = 2^a \cdot 3^b \cdot 5^c, \text{ where } a = 0, 1 \text{ or } 2;\ b = 0, 1, 2 \text{ or } 3;\ b = 0, 1;\ c = 0 \text{ or } 1$$

We must now count possible combinations of a, b and c such that d has exactly 4 positive divisors. In other words, we are counting combinations of a, b and c such that

$$(a+1)(b+1)(c+1) = 4.$$

The only ways to get 4 as the product of 3 positive integers are

$$4 = 1 \cdot 1 \cdot 4 = 1 \cdot 2 \cdot 2.$$

These products imply that (a, b, c) must include either two 0's and one 3 or else one 0, one 1, and one 1. This leaves us with only five possibilities:

$$(0,3,0), (1,1,0), (1,0,1), (0,1,1) \Rightarrow 3^3 = 9, 2 \cdot 3 = 6, 2 \cdot 5 = 10, 3 \cdot 5 = 15.$$

It is easy to verify that their sum is $9 + 6 + 10 + 15 = \boxed{40}$. The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R2-L4-P4 *Problem 8.21, Chapter 8, Introductory to Algebra.*

The line with slope 4 that passes through $(10, 20)$ is $y - 20 = 4(x - 10)$, or $y = 4x - 20$. The x-intercept of this line is where $y = 0$. Solving $0 = 4x - 20$, we find $x = 5$, so the x-intercept is $(5, 0)$.

Similarly, the line with slope 5 through $(10, 20)$ is $y - 20 = 5(x - 10)$, or $y = 5x - 30$. Setting $y = 0$, we find that the x-intercept of this line is $(6, 0)$. Therefore, the distance between our x-intercepts is $\boxed{1.}$ The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R2-L4-P5 *Exercise 6.4.4, Chapter 6, Introductory to Geometry.*

We have $\angle ABC = 90°$, so by the Pythagorean Theorem, we have

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{8^2 + 6^2} = \sqrt{100} = 10.$$

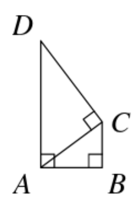


Figure 23.3: ICLT-2022-SM2-R2-L4-P5

We also have

$$\angle CDA = 90° - \angle DAC = \angle CAB,$$

so triangles ABC and DCA are similar. Therefore,

$$\frac{DA}{AC} = \frac{CA}{BC} \Rightarrow DC = \frac{AC^2}{BC} = \boxed{\frac{100}{6} = \frac{50}{3}}.$$

The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R2-L4-P6 *Problem 7.8, Chapter 7, Introductory to Number Theory.*

$$\begin{aligned} & n = \overline{ab} + \overline{ba} + \overline{ac} + \overline{ca} + \overline{bc} + \overline{cb} \\ \Rightarrow\ & 100a + 10b + c = 10a + b + 10b + a + 10a + c + 10c + a + 10b + c + 10c + b \\ \Rightarrow\ & 100a + 10b + c = 22(a + b + c) \Rightarrow 78a = 12b + 21c \Rightarrow 26a = 4b + 7c \end{aligned}$$

Since we want to make n as large as possible, so $a = 3$ since $26a \le 4 \cdot 9 + 7 \cdot 9 = 99$. For $a = 3$, then $4b + 7c = 96$. By direct testing $b = 9, c = 6$. Hence $n = 396, abc = 3 \cdot 9 \cdot 6 = \boxed{18.}$ The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R2-L4-P7 *Problem 7.9, Chapter 7, Introductory to Counting & Probability.*

We find the probablity when the product of the two different number is divisible by 53.

There are 90 2-digit numbers, so selecting two different 2-digit numbers, without regard to order, can be done in $\binom{90}{2}$ ways. Two numbers multiply together to give a multiple of 53 number if at least one of the original numbers is a multiple of 53. Between 10 and 99 there is only one such number because 53 is a prime. Thus the number of successful outcomes is the number of ways to choose a number which is not 53, or 89.

Hence, the probability is

$$\frac{89}{\binom{90}{2}} = \frac{89}{\frac{90 \cdot 89}{2}} = \frac{1}{45}.$$

Thus, our desired probability is $1 - \frac{1}{45} = \boxed{\frac{44}{45}.}$ □

Solution. ICLT-2022-SM2-R2-L4-P8 *Problem 9.10, Chapter 9, Introductory to Algebra.*

By taking the 50^{th} root of the numbers, we have

$$2^{12} = 4096, 3^8 = 6561, 4^5 = 1024, 5^4 = 625 \Rightarrow 5^4 < 4^5 < 2^{12} < 3^8 \Rightarrow \boxed{5^{200} < 4^{250} < 2^{600} < 3^{400}.}$$

□

Solution. ICLT-2022-SM2-R2-L4-P9 *Exercise 7.3.6, Chapter 7, Introductory to Geometry.*

Let $x = BM$, so $MC = 8 - x$. Since $\angle BAM = \angle CAM$, AM is an angle bisector of $\triangle ABC$. Therefore, by the Angle Bisector Theorem, we have $\frac{BM}{BA} = \frac{CM}{CA.}$ Substitution gives $\frac{x}{10} = \frac{8-x}{12}$, so $x = \boxed{\frac{40}{11}}.$ □

Solution. ICLT-2022-SM2-R2-L4-P10 *Problem 8.11, Chapter 8, Introductory to Counting & Probability.*

It is easy to see that Phan wins 4 games in 7 games, thus Quan must win 3 other games. Let assume that the ways the winners of invidual games are $PPQQPQP$. Then the probability for this sequence is

$$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^3.$$

Thus it does not matter how the games were played. It is easy to see that the last game must be won by Phan. So in the remaining 6 games, there are $\binom{6}{3} = 20$ ways to choose the three games that Phan wins. Thus, the overall probability for Phan to win is

$$20 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^3 = \frac{320}{2187}.$$

□

Chapter 24

Introductory Curriculum Level Test 2 - Level 5

24.1 Rules

24.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	Introduction to Algebra	Introduction to Geometry	Introduction to Number Theory	Introduction to Counting & Probability
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
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9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

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- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

24.3 Problems

Problem 24.3.1 (ICLT-2022-SM2-R2-L5-P1). AD and BC are both perpendicular to AB in the diagram below, and $CD \perp AC$. If $AB = 8$ and $BC = 6$, find AD.

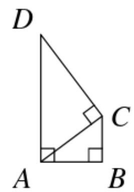

Figure 24.1: ICLT-2022-SM2-R2-L4-P5

$(A)\ \frac{50}{3}$ $(B)\ 15$ $(C)\ \frac{40}{3}$ $(D)\ 12$ $(E)\ \frac{32}{3}$

Problem 24.3.2 (ICLT-2022-SM2-R2-L5-P2). Phan and Quan play a number of two-player games. Any game has a winner (no draw). The player who is the first to win 4 games (not necessarily in a row) is the champion. Phan has a $\frac{2}{3}$ probability of winning any individual game. What is the probability that after exactly 7 games Phan is declared champion?

$(A)\ \frac{560}{2187}$ $(B)\ \frac{320}{2187}$ $(C)\ \frac{160}{2187}$ $(D)\ \frac{80}{729}$ $(E)\ \frac{40}{729}$

Problem 24.3.3 (ICLT-2022-SM2-R2-L5-P3). Let a and b denote the solutions of $6x^2 + x - 2 = 0$. Find the value of $(a^2 - 1)(b^2 - 1)$.

$(A)\ \frac{1}{12}$ $(B)\ \frac{1}{4}$ $(C)\ \frac{5}{12}$ $(D)\ \frac{7}{12}$ $(E)\ \frac{3}{4}$

Problem 24.3.4 (ICLT-2022-SM2-R2-L5-P4). The area of trapezoid $ABCD$ is 96. One base is 6 units longer than the other, and the height of the trapezoid is 8. Find the length of the longer base.

$(A)\ 7$ $(B)\ 8$ $(C)\ 9$ $(D)\ 12$ $(E)\ 15$

Problem 24.3.5 (ICLT-2022-SM2-R2-L5-P5). What is the largest base-9 integer that can be written as a three-digit base-4 integer?

$(A)\ 333_9$ $(B)\ 63_9$ $(C)\ 64_9$ $(D)\ 70_9$ $(E)\ 100_9$

Problem 24.3.6 (ICLT-2022-SM2-R2-L5-P6)**.** Points A, B, and C are arranged clockwise in that order on a circle, as shown in the figure below. We place a marker on point A. We roll a 6-sided die and move the marker as follows:

- If the die shows a dice-blue-1 or a dice-blue-2, stay put.
- If the die shows a dice-blue-3 or a dice-blue-4, move one step clockwise (for example, from A to B).
- If the die shows a dice-blue-5 or a dice-blue-6, move one step counterclockwise (for example, from A to C).

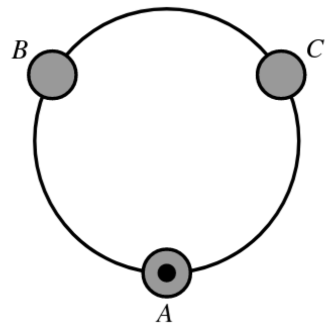

Figure 24.2: ICLT-2022-SM2-R2-L5-P6

What is the probability that after 9 moves the marker is at point A?

$(A)\ \frac{1}{9}$ $(B)\ \frac{1}{3}$ $(C)\ \frac{4}{9}$ $(D)\ \frac{5}{9}$ $(E)\ \frac{7}{9}$

Problem 24.3.7 (ICLT-2022-SM2-R2-L5-P7)**.** Let

$$\begin{cases} x &= z - \sqrt{z^2 - 5} \\ 5y &= z + \sqrt{z^2 - 5} \end{cases}$$

Find x when $y = \frac{3}{4}$.

Problem 24.3.8 (ICLT-2022-SM2-R2-L5-P8)**.** Find the value in base 2 of the sum by performing base-2 addition. *No conversion to any other base is allowed.*

$$101110_2 + 1001_2 + 11011_2?$$

Problem 24.3.9 (ICLT-2022-SM2-R2-L5-P9)**.** Suppose two numbers x and y are each chosen such that $0 < x < 1$ and $0 < y < 1$. What is the probability that $x + y > \frac{1}{2}$?

Problem 24.3.10 (ICLT-2022-SM2-R2-L5-P10)**.** Write $\dfrac{4 - i}{-1 + 5i}$ as a single complex number.

24.4 Grading

Answers for multiple-choice problems.

Problem 1: A

Problem 2: B

Problem 3: C

Problem 4: E

Problem 5: D

Problem 6: B

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can recognize the identity $a^2 - b^2$ for $z \pm \sqrt{z^2 - 5}$.

Problem 8: 2 points if the student can demonstrate how to add in base 2.

Problem 9: 2 points if the student can show the area of desired (x, y).

Problem 10: 2 points if the student can use conjugate of complex number.

24.5 Solutions

Solution. ICLT-2022-SM2-R2-L5-P1 *Exercise 6.4.4, Chapter 6, Introductory to Geometry.*

We have $\angle ABC = 90°$, so by the Pythagorean Theorem, we have

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{8^2 + 6^2} = \sqrt{100} = 10.$$

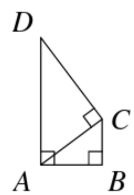

Figure 24.3: ICLT-2022-SM2-R2-L4-P5

We also have

$$\angle CDA = 90° - \angle DAC = \angle CAB,$$

so triangles ABC and DCA are similar. Therefore,

$$\frac{DA}{AC} = \frac{CA}{BC} \Rightarrow DC = \frac{AC^2}{BC} = \boxed{\frac{100}{6} = \frac{50}{3}}.$$

The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R2-L5-P2 *Problem 8.11, Chapter 8, Introductory to Counting & Probability.*

It is easy to see that Phan wins 4 games in 7 games, thus Quan must win 3 other games. Let assume that the ways the winners of invidual games are $PPQQPQP$. Then the probability for this sequence is

$$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^3.$$

Thus it does not matter how the games were played. It is easy to see that the last game must be won by Phan. So in the remaining 6 games, there are $\binom{6}{3} = 20$ ways to choose the three games that Phan wins. Thus, the overall probability for Phan to win is

$$20\left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^3 = \frac{320}{2187}.$$

The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R2-L5-P3 *Problem 10.24, Chapter 10, Introductory to Algebra.*

If a and b are the solutions of $6x^2 + x - 2 = 0$, then

$$a + b = -\frac{1}{6}, ab = -\frac{2}{6} = -\frac{1}{3}.$$

Now

$$\begin{aligned}(a^2-1)(b^2-1) &= [(a-1)(b-1)]\,[(a+1)(b+1)] = (ab-a-b+1)(ab+a+b+1) \\ &= \left(-\frac{1}{3} + \frac{1}{6} + 1\right)\left(-\frac{1}{3} - \frac{1}{6} + 1\right) = \boxed{\frac{5}{12}}.\end{aligned}$$

The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R2-L5-P4 *Problem 8.2.3, Chapter 8, Introductory to Geometry.*

Let the length of the shorter base be x, so the longer base is $x+6$. We have $\frac{(x+x+6)(8)}{2} = 96$. Solving, we find that the longer base has length $x+6 = \boxed{15}$. The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R2-L5-P5 *Problem 8.10, Chapter 8, Introductory to Number Theory.*

The smallest integer that can be written using four digits in base 4 is $1000_4 = 1 \cdot 4^3 = 64$. We count down 1 from 64 to get $333_4 = 63$.

In base-9, $63 = \boxed{70_9}$. The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R2-L5-P6 *Problem 9.3 Chapter 9, Introductory to Counting & Probability.*

We could approach this problem using casework, considering all 6^8 possible sequences of 8 rolls ... On second thought, let's think about it first. On any given turn, there's a $\frac{1}{3}$ chance we stay put, a $\frac{1}{3}$ chance we move clockwise, and a $\frac{1}{3}$ chance we move counterclockwise.

Therefore, after every turn, we're equally likely to end up at each of the 3 points (since the probability is $\frac{1}{3}$ of moving to each of the 3 points). Therefore, it doesn't matter which point we care about, or how many moves we've made - the probability of ending up at any point is always $\boxed{\dfrac{1}{3}.}$

The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R2-L5-P7 *Problem 11.2, Chapter 11, Introductory to Algebra.*

$$\begin{cases} x &= z - \sqrt{z^2-5} \\ 5y &= z + \sqrt{z^2-5} \end{cases} \Rightarrow (x)(5y) = (z - \sqrt{z^2-5})(z + \sqrt{z^2-5}) = z^2 - (z^2-5) = 5 \Rightarrow xy = 1.$$

When $y = \frac{3}{4}$, then $x = \boxed{\dfrac{4}{3}.}$ □

Solution. ICLT-2022-SM2-R2-L5-P8 *Problem 9.11.c, Chapter 9, Introductory to Number Theory.*

$$\begin{array}{r} 101110_2 \\ 11011_2 \\ + \quad 1001_2 \\ \hline 1010010_2 \end{array}$$

□

Solution. ICLT-2022-SM2-R2-L5-P9 *Problem 10.5, Chapter 10, Introductory to Geometry.*

The "total outcomes" region is the region in the xy-plane with $0 < x < 1$ and $0 < y < 1$. This is the interior of the square with corners $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$. The area of the square is 1.

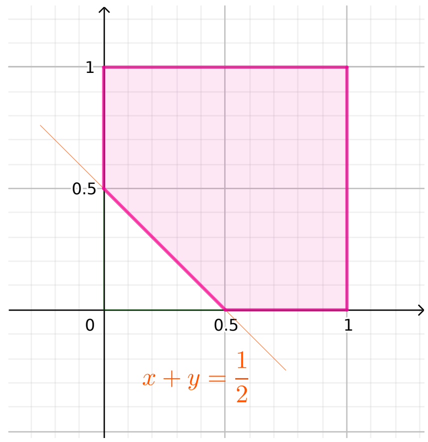

Figure 24.4: ICLT-2022-SM2-R2-L5-P9

Now we need to describe the region corresponding to successful outcomes. We need $x + y > \frac{1}{2}$, so this will be the region inside the square of which is above the line $x + y = \frac{1}{2}$. The area of the region is $1 - \frac{\frac{1}{2}\cdot\frac{1}{2}}{2} = \frac{7}{8}$. Hence, the probability is $\frac{\frac{7}{8}}{1} = \boxed{\frac{7}{8}.}$ □

Solution. ICLT-2022-SM2-R2-L5-P10 *Problem 12.10, Chapter 12, Introductory to Algebra.*

To remove $-1 + 5i$ from the denominator, we multiply it with its conjugate $-1 - 5i$,

$$\frac{4 - i}{-1 + 5i} = \frac{(4 - i)(-1 - 5i)}{(-1)^2 - (5i)^2} = \frac{-4 + i - 20i + 5i^2}{26} = \boxed{-\frac{9}{26} - \frac{19}{26}i.}$$

□

Chapter 25

Introductory Curriculum Level Test 2 - Level 6

25.1 Rules

25.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

The curriculum is intended for students from Grade 5-10 following the Canadian curriculum. Study should be rotated based on the 4 books, each time covering a chapter from each book. A typical chapter will be comprised of theory, exercises, practice problems and challenge problems. Students choose their own learning pace based on the amount of time they can commit to the curriculum on a weekly basis.

The following can serve as a guideline: (i) Basic: 3 hours/week = 30 mins/day x 6 days, (ii) Intermediate: 6 hours/week = 60 mins/day x 6 days, (iii) Advanced: 9 hours/week = 1.5 hour/day x 6 days.

There are several levels for the curriculum, each requires completion of a number of chapters from the text books and successfully pass of completion test. The table below lists the levels and required chapters from each text book.

Level	**Introduction to Algebra**	**Introduction to Geometry**	**Introduction to Number Theory**	**Introduction to Counting & Probability**
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

25.3 Problems

Problem 25.3.1 (ICLT-2022-SM2-R2-L6-P1)**.** Suppose two numbers x and y are each chosen such that $0 < x < 1$ and $0 < y < 1$. What is the probability that $x + y > \frac{1}{2}$?

$(A)\quad \frac{3}{8} \qquad (B)\quad \frac{1}{2} \qquad (C)\quad \frac{5}{8} \qquad (D)\quad \frac{3}{4} \qquad (E)\quad \frac{7}{8}$

Problem 25.3.2 (ICLT-2022-SM2-R2-L6-P2)**.** The shaded portion of the figure is called a lune. Given that $AB = 1$, $CD = \sqrt{2}$, and that AB and CD are diameters of the respective semicircles shown, find the area of the lune.

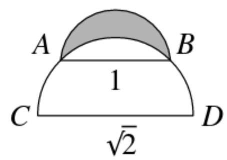

Figure 25.1: ICLT-2022-SM2-R2-L6-P2

$(A)\quad \frac{\pi - 1}{8} \qquad (B)\quad \frac{\pi}{4} \qquad (C)\quad \frac{1}{4} \qquad (D)\quad \frac{3\pi - 4}{16} \qquad (E)\quad \frac{3}{8}$

Problem 25.3.3 (ICLT-2022-SM2-R2-L6-P3)**.** Write $\dfrac{4 - i}{-1 + 5i}$ as a single complex number.

$(A)\quad -9 - 19i \qquad (B)\quad -\frac{9}{26} - \frac{19i}{26} \qquad (C)\quad -\frac{9}{26} + \frac{19i}{26} \qquad (D)\quad \frac{9}{26} - \frac{19i}{26} \qquad (E)\quad \frac{9}{26} + \frac{19i}{26}$

Problem 25.3.4 (ICLT-2022-SM2-R2-L6-P4)**.** What is the unit digit of $3^{2022} - 2^{2022}$.

$(A)\quad 9 \qquad (B)\quad 7 \qquad (C)\quad 5 \qquad (D)\quad 3 \qquad (E)\quad 1$

Problem 25.3.5 (ICLT-2022-SM2-R2-L6-P5)**.** The numbers on a standard six-sided die are arranged such that numbers on opposite faces always add to 7. The die is rolled, and the product of the numbers appearing on the four lateral faces of the die is calculated (ignoring the numbers on the top and bottom). What is the expected value of this product?

$(A)\quad 60 \qquad (B)\quad 72 \qquad (C)\quad 84 \qquad (D)\quad 96 \qquad (E)\quad 120$

Problem 25.3.6 (ICLT-2022-SM2-R2-L6-P6)**.** Let

$$\begin{cases} x &= z - \sqrt{z^2 - 5} \\ 5y &= z + \sqrt{z^2 - 5} \end{cases}$$

Find the product xy.

(A) $\frac{3}{4}$ (B) $\frac{4}{3}$ (C) $\frac{1}{2}$ (D) 1 (E) 5

Problem 25.3.7 (ICLT-2022-SM2-R2-L6-P7)**.** A bin has 6 white balls and k black balls in it, where k is an unknown positive integer. A ball is drawn at random from the bin. If a white ball is drawn, the player wins \$1, but if a black ball is drawn, the player loses \$1. If the expected loss for playing the game is 50 cents, then what is k?

Problem 25.3.8 (ICLT-2022-SM2-R2-L6-P8)**.** What is the unit digit of the sum

$$1! + 2! + 3! + \cdots + 2022!.$$

when written in base-7?

Problem 25.3.9 (ICLT-2022-SM2-R2-L6-P9)**.** Solve the inequality

$$\frac{3}{x-3} - 4x + 4 > -1.$$

Problem 25.3.10 (ICLT-2022-SM2-R2-L6-P10)**.** In isosceles triangle $\triangle ABC$, BC is the base. Given that $BC > AB$, prove that that $\angle BAC > 60^\circ$.

25.4 Grading

Answers for multiple-choice problems.

Problem 1: *E*

Problem 2: *C*

Problem 3: *B*

Problem 4: *A*

Problem 5: *C*

Problem 6: *D*

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can establish the correct formula for the expected value.

Problem 8: 2 points if the student can find that $7! + \ldots + 2022!$ is divisible by 7, thus shall not influence the unit digit of the sum.

Problem 9: 2 points if the student can casework depeding on the signs of $(x - 3)$ and $4x^2 - 17x + 12$.

Problem 10: 2 points if the student can đeduce $\angle A > \angle C$ from $BC > AB$.

25.5 Solutions

Solution. ICLT-2022-SM2-R2-L6-P1 *Problem 10.5, Chapter 10, Introductory to Geometry.*

The "total outcomes" region is the region in the xy-plane with $0 < x < 1$ and $0 < y < 1$. This is the interior of the square with corners $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$. The area of the square is 1.

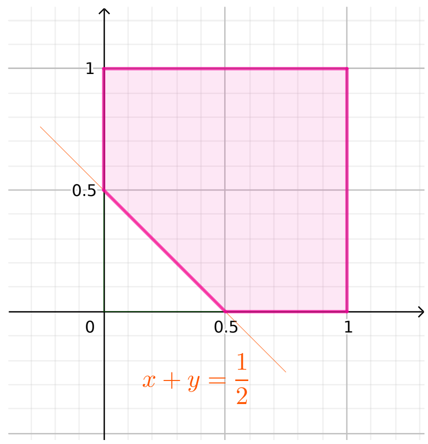

Figure 25.2: ICLT-2022-SM2-R2-L6-P1

Now we need to describe the region corresponding to successful outcomes. We need $x + y > \frac{1}{2}$, so this will be the region inside the square of which is above the line $x + y = \frac{1}{2}$. The area of the region is $1 - \frac{\frac{1}{2}\cdot\frac{1}{2}}{2} = \frac{7}{8}$. Hence, the probability is $\frac{\frac{7}{8}}{1} = \boxed{\frac{7}{8}.}$ The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R2-L6-P2 *Problem 8.11, Chapter 8, Introductory to Geometry.*

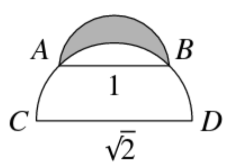

Figure 25.3: ICLT-2022-SM2-R2-L6-P2

$$\begin{aligned}
&\text{Area of lune} = \text{Area of small semicircle} - \text{Area of circular segment } AB\\
&\text{Area of circular segment } AB = \text{Area of sector } AOB - \text{Area of } \triangle AOB\\
\Rightarrow\ &\text{Area of lune} = \text{Area of small semicircle} - \text{Area of sector } AOB + \text{Area of } \triangle AOB
\end{aligned}$$

Now $CD = \sqrt{2}$, so $OC = OA = OB = OD = \frac{\sqrt{2}}{2}$, so $AB^2 = AO^2 + BO^2 = 1$. Thus, The radius of the small semicircle is $\frac{1}{2}$, so its area is $\frac{1}{2} \cdot \pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$. Sector AOB is $\frac{90°}{360°} = \frac{1}{4}$ of a circle with radius $\frac{\sqrt{2}}{2}$, so its area is $\frac{1}{4} \cdot \pi \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{8}$. Finally, $[AOB] = \frac{1}{2} \cdot (AO)(OB) = \frac{1}{4}$, so we have

$$\text{Area of small semicircle} - \text{Area of sector } AOB + \text{Area of } \triangle AOB = \frac{\pi}{8} - \frac{\pi}{8} + \frac{1}{4} = \boxed{\frac{1}{4}.}$$

The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R2-L6-P3 *Problem 12.10, Chapter 12, Introductory to Algebra.*

To remove $-1+5i$ from the denominator, we multiply it with its conjugate $-1-5i$,

$$\frac{4-i}{-1+5i} = \frac{(4-i)(-1-5i)}{(-1)^2-(5i)^2} = \frac{-4+i-20i+5i^2}{26} = \boxed{-\frac{9}{26}-\frac{19}{26}i.}$$

The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R2-L6-P4 *Problem 10.20, Chapter 10, Introductory to Number Theory.*

The units digit of powers of 2 and 3 both repeat in cycles four long, $2,4,8,6$, and $3,9,7,1$. Since $1986 \div 4$ has a remainder of 2, the units digit of $3^{2023}-2^{2023}$ is the same as 3^3-2^3, or $\boxed{9}$. The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R2-L6-P5 *Problem 11.11, Chapter 11, Introductory to Counting & Probability.*

The products of opposite faces are $1\times 6=6$, $2\times 5=10$, and $3\times 4=12$. Therefore, after rolling the die, the outcomes $6\times 10=60$, $6\times 12=72$, and $10\times 12=120$ are equally likely. So the expected value is just the average of these outcomes, namely $\frac{60+72+120}{3}=\boxed{84}$. The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R2-L6-P6 *Problem 11.2, Chapter 11, Introductory to Algebra.*

$$\begin{cases} x &= z-\sqrt{z^2-5} \\ 5y &= z+\sqrt{z^2-5} \end{cases} \Rightarrow (x)(5y) = (z-\sqrt{z^2-5})(z+\sqrt{z^2-5}) = z^2-(z^2-5) = 5 \Rightarrow xy = \boxed{1.}$$

The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R2-L6-P7 *Exercise 11.3.5, Chapter 11, Introductory to Counting & Probability.*

There are 6 white balls and $5+k$ total balls, so the probability that a white ball is drawn is $\dfrac{6}{6+k}$. Similarly, the probability that a black ball is drawn is $\dfrac{k}{5+k}$. So

$$E = \frac{6}{6+k}(1) + \frac{k}{6+k}(-1) = -\frac{1}{2}.$$

Multiply both sides of the equation by $2(6+k)$ to get $12-6k=-6-k$, and we see that $\boxed{k=18}$. □

Solution. ICLT-2022-SM2-R2-L6-P8 *Exercise 10.4.4, Chapter 10, Introductory to Number Theory.*

We can ignore $n!$ when n is 7 or higher, because each is a multiple of 7 and will therefore have a base-7 units digit of 0. There are several ways we can proceed from here. One is to just sum the factorials we want, then find its remainder when divided by 7, which will be the base-7 units digit:

$$1!+2!+3!+4!+5!+6! = 1+2+6+24+120+720 = 873 \Rightarrow 873 \equiv \boxed{5} \pmod{7}.$$

□

Solution. ICLT-2022-SM2-R2-L6-P9 *Problem 15.7, Chapter 15, Introductory to Algebra.*

$$\frac{3}{x-3} - 4x + 4 > -1 \Rightarrow \frac{3 + (-4x+5)(x-3)}{x-3} > 0 \Rightarrow \frac{4x^2 - 17x + 12}{x-3} < 0$$

Case 1: $x - 3 < 0$, or $x < 3$, then $4x^2 - 17x + 12 > 0$,

$$\Delta = 17^2 - 4 \cdot 4 \cdot 12 = 97 \Rightarrow x < \frac{17 - \sqrt{97}}{8} \approx 0.89 \text{ or } x > \frac{17 + \sqrt{97}}{8} \approx 3.35 \Rightarrow x < \frac{17 - \sqrt{97}}{8}.$$

Case 2: $x - 3 > 0$, or $x > 3$, then $4x^2 - 17x + 12 < 0$, from above,

$$\frac{17 - \sqrt{97}}{8} < x < \frac{17 + \sqrt{97}}{8} \Rightarrow 3 < x < \frac{17 + \sqrt{97}}{8}$$

Hence, $\boxed{x \in (-\infty, \frac{17-\sqrt{97}}{8}) \cup (3, \frac{17+\sqrt{97}}{8}).}$ □

Solution. ICLT-2022-SM2-R2-L6-P10 *Problem 10.27, Chapter 10, Introductory to Geometry.*

Since $BC > AB$, we have $\angle BAC = \angle A > \angle ACB = \angle C$. Since $\angle B$ and $\angle C$ are the base angles of isosceles $\triangle ABC$, we have $\angle B = \angle C$. From $\triangle ABC$, we have $\angle A + \angle B + \angle C = 180°$. We also have $\angle A > \angle C$ and $\angle B = \angle C$, so $\angle A + \angle B + \angle C < 3\angle A$. Hence, $180° < 3\angle A$, so $\angle A > 60°$. □

Chapter 26

Introductory Curriculum Level Test 2 - Level 7

26.1 Rules

26.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

Students will follow the Introductory Curriculum of the Art of Problem Solving which is comprised of 71 chapters under 4 subject areas: Algebra, Counting and Probability, Number Theory and Geometry. Students are advised to order these books via Arts of Problem Solving website.

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Level	Introduction to Algebra	Introduction to Geometry	Introduction to Number Theory	Introduction to Counting & Probability
1	1-3	1-2	1-2	1-3
2	4-5	3-4	3-4	4-5
3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
7	16-17	11-13	11	12
8	18-19	14-15	12	13
9	20-21	16-17	13	14
10	22	18-19	14-15	15

Test rules:

- All contests start at **10:00 PM Eastern Time (EST) on a Saturday** and lasts for **90 minutes**.
- The test consists of 6 **multiple-choice** and 4 **show-you-work** problems. To answer each of 6 multiple-choice problems, you can select *atmost one* choice among A, B, C, D, and E. For each of 4 show-you-work problems, you have to provide a complete solution in details. If your solution uses any diagram, please submit them too.
- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
    1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
    2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
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- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

26.3 Problems

Problem 26.3.1 (ICLT-2022-SM2-R2-L7-P1). Let $f(x) = 3 - 2x$ and $g(f(x)) = 2x^3 - 3x + 5$. Find $g(7)$.

$(A)\quad -1 \qquad (B)\quad -2 \qquad (C)\quad -3 \qquad (D)\quad -4 \qquad (E)\quad -5$

Problem 26.3.2 (ICLT-2022-SM2-R2-L7-P2). The shaded portion of the figure is called a lune. Given that $AB = 1$, $CD = \sqrt{2}$, and that AB and CD are diameters of the respective semicircles shown, find the area of the lune.

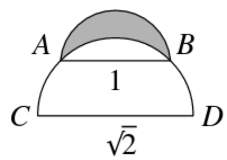

Figure 26.1: ICLT-2022-SM2-R2-L7-P2

$(A)\quad \dfrac{\pi-1}{8} \qquad (B)\quad \dfrac{\pi}{4} \qquad (C)\quad \dfrac{1}{4} \qquad (D)\quad \dfrac{3\pi-4}{16} \qquad (E)\quad \dfrac{3}{8}$

Problem 26.3.3 (ICLT-2022-SM2-R2-L7-P3). What is the value of

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r}\binom{n}{0}?$$

$(A)\quad \dbinom{m+n-1}{r-1} \qquad (B)\quad \dbinom{m+n}{r} \qquad (C)\quad \dbinom{m+n-1}{r} \qquad (D)\quad \dbinom{m+n}{r-1} \qquad (E)\quad \dbinom{m+n+1}{r+1}$

Problem 26.3.4 (ICLT-2022-SM2-R2-L7-P4). Which statement below is true?

(A) If a graph on the Cartesian plane is the graph of a function, then every vertical line passes through exactly one point.

(B) If f is a function, the graphs of $y = f(x)$ and $y = f(x) + 3$ intersect at more than one point.

(C) If no horizontal line passes through more than one point of the graph of a function, then the function has an inverse.

(D) The graph of $y = f(3x) + 4$ is the result of stretching the graph of $y = f(x)$ vertically by a factor of 3, then shifting the result 4 units up.

(E) If no horizontal line passes through more than one point on a given graph on the coordinate plane, then the graph is the graph of a function that has an inverse.

Problem 26.3.5 (ICLT-2022-SM2-R2-L7-P5). The median AM of $\triangle ABC$ has length 8. Given that $BC = 16$ and $AB = 9$, find the area of $\triangle ABC$.

$(A)\quad 58 \qquad (B)\quad \dfrac{45\sqrt{5}}{2} \qquad (C)\quad 50 \qquad (D)\quad \dfrac{45\sqrt{7}}{2} \qquad (E)\quad \dfrac{47\sqrt{5}}{2}$

Problem 26.3.6 (ICLT-2022-SM2-R2-L7-P6)**.** What is the unit digit of n if the units digit of $\binom{2n+2}{2n}$ is 3?

(A) 9 (B) 8 (C) 6 (D) 4 (E) 3

Problem 26.3.7 (ICLT-2022-SM2-R2-L7-P7)**.** Express the repeating decimal $0.\overline{037}$ in reduced form.

Problem 26.3.8 (ICLT-2022-SM2-R2-L7-P8)**.** There are 5 white marbles and k black marbles in a jar. A marble is drawn at random from the jar. If it is white, the player wins \$1, but if a black ball is drawn, the player loses \$1. What is k so that if the expected loss for playing the game is 0 (dollars)?

Problem 26.3.9 (ICLT-2022-SM2-R2-L7-P9)**.** Solve the inequality

$$\frac{3}{x-3} - 4x + 4 > -1.$$

Problem 26.3.10 (ICLT-2022-SM2-R2-L7-P10)**.** A circle is tangent to side BC of equilateral triangle $\triangle ABC$ at point Q as shown below. The circle intersects sides AB and AC in two points each, as shown. Given that $AW = AY$, prove that Q is the midpoint of BC.

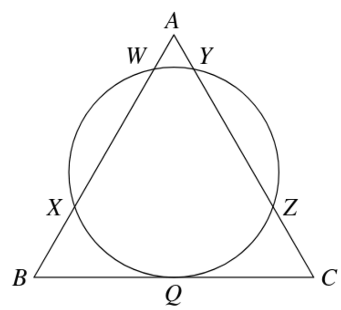

Figure 26.2: ICLT-2022-SM2-R2-L7-P10

26.4 Grading

Answers for multiple-choice problems.

Problem 1: E

Problem 2: C

Problem 3: B

Problem 4: C

Problem 5: D

Problem 6: B

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can eliminate the repeating digits.

Problem 8: 2 points if the student can find the expected value based on k.

Problem 9: 2 points if the student can casework depeding on the signs of $(x-3)$ and $4x^2 - 17x + 12$.

Problem 10: 2 points if the student can find that $WX = YZ$.

26.5 Solutions

Solution. ICLT-2022-SM2-R2-L7-P1 *Problem 16.12, Chapter 16, Introductory to Algebra.*

Note that $g(f(x)) = 2x^3 - 3x + 5$, so to find $g(7)$, we need a value of x so that $f(x) = 7$.

$$f(x) = 7 \Rightarrow 3 - 2x = 7 \Rightarrow x = -2 \Rightarrow g(7) = g(f(-2)) = 2(-2)^3 - 3(-2) + 5 = \boxed{-5}.$$

The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R2-L7-P2 *Problem 8.11, Chapter 8, Introductory to Geometry.*

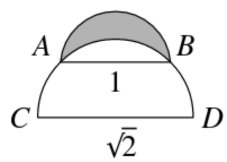

Figure 26.3: ICLT-2022-SM2-R2-L7-P2

$$\begin{aligned} &\text{Area of lune} = \text{Area of small semicircle} - \text{Area of circular segment } AB \\ &\text{Area of circular segment } AB = \text{Area of sector } AOB - \text{Area of } \triangle AOB \\ \Rightarrow &\text{Area of lune} = \text{Area of small semicircle} - \text{Area of sector } AOB + \text{Area of } \triangle AOB \end{aligned}$$

Now $CD = \sqrt{2}$, so $OC = OA = OB = OD = \frac{\sqrt{2}}{2}$, so $AB^2 = AO^2 + BO^2 = 1$. Thus, The radius of the small semicircle is $\frac{1}{2}$, so its area is $\frac{1}{2} \cdot \pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$. Sector AOB is $\frac{90^\circ}{360^\circ} = \frac{1}{4}$ of a circle with radius $\frac{\sqrt{2}}{2}$, so its area is $\frac{1}{4} \cdot \pi \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{8}$. Finally, $[AOB] = \frac{1}{2} \cdot (AO)(OB) = \frac{1}{4}$, so we have

$$\text{Area of small semicircle} - \text{Area of sector } AOB + \text{Area of } \triangle AOB = \frac{\pi}{8} - \frac{\pi}{8} + \frac{1}{4} = \boxed{\frac{1}{4}.}$$

The answer is $\boxed{C.}$ □

Solution. [First solution] ICLT-2022-SM2-R2-L7-P3 *Problem 12.4.1.b, Chapter 10, Introductory to Counting & Probability.*

We wish to choose an r-person committee from m men and n women. The number of ways to choose an r-person committee from $(m+n)$ people is $\binom{m+n}{r}$.

On the other hand, the committee can have k men and $(r-k)$ women for any k from 0 to r inclusive. There are $\binom{m}{k}$ ways to choose k men (from the m total men) and $\binom{n}{r-k}$ ways to choose $(r-k)$ women (from the n total women). The total number of committees given by this second method of counting is

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r}\binom{n}{0}.$$

The answer is $\boxed{B.}$ □

Solution. [Second solution] ICLT-2022-SM2-R2-L7-P3 *Problem 12.4.1.b, Chapter 10, Introductory to Counting & Probability.*

We are trying to count the number of ways to move r steps to the right out of $m+n$ steps, or $\binom{m+n}{r}$.

After m steps down the Pascals Triangle, we could have taken 0, 1, 2, ..., or r steps to the right. If we take k steps to the right after m steps, we must take $r-k$ steps to the right in the final n steps. There are $\binom{m}{k} \times \binom{n}{r-k}$ ways to do this Summing this over k from 0 to r, we get

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r}\binom{n}{0}.$$

The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R2-L7-P4 *(Parts of) Exercise 17.2.4 & Problem 17.19, Chapter 17, Introductory to Algebra.*

(A) If a graph on the Cartesian plane is the graph of a function, then every vertical line passes through exactly one point. This is $\boxed{\text{False}}$. It is possible that the graph is the graph of a function, but there exist vertical lines that pass through 0 points on the graph.

(B) If f is a function, the graphs of $y = f(x)$ and $y = f(x) + 3$ intersect at more than one point. This is $\boxed{\text{False}}$. The graphs of $y = f(x)$ and $y = f(x) + 3$ cannot intersect. Because the graph of $y = f(x) + 3$ is a 3-unit vertical shift of the graph of $y = f(x)$, any point at which a vertical line hits the graph of $y = f(x) + 3$ must be 3 units above the point where that line hits the graph $y = f(x)$. Therefore, no vertical line meets both graphs at the same point, so there is no point through which both graphs pass.

(C) If no horizontal line passes through more than one point of the graph of a function, then the function has an inverse. This is $\boxed{\text{True}}$. If the graph of a function is such that no horizontal line passes through more than one point on the graph, then the function has an inverse.

(D) The graph of $y = f(3x) + 4$ is the result of stretching the graph of $y = f(x)$ vertically by a factor of 3, then shifting the result 4 units up. This is $\boxed{\text{False}}$. The graph of $y = f(3x)$ is the result of scaling the graph horizontally by a factor of $\frac{1}{3}$, not a result of stretching the graph vertically.

(E) If no horizontal line passes through more than one point on a given graph on the coordinate plane, then the graph is the graph of a function that has an inverse. This is $\boxed{\text{False}}$. Just because no horizontal line passes through more than one point on the graph does not mean the graph is the graph of a function.

The answer is $\boxed{C.}$ □

Solution. ICLT-2022-SM2-R2-L7-P5 *Problem 12.23, Chapter 12, Introductory to Geometry.*

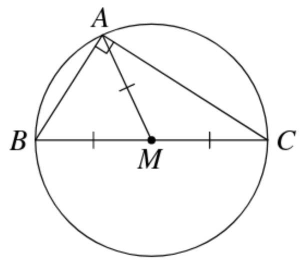

Figure 26.4: ICLT-2022-SM2-R2-L7-P5

When we draw the figure and label all our lengths, we see that $BM = AM = CM = 8$. Therefore, a circle centered at M with radius 8 goes through all three vertices of $\triangle ABC$. Since BC is a diameter of this circle, $\angle BAC$ is inscribed in a semicircle and therefore must be a right angle. So, $AC = \sqrt{BC^2 - AB^2} = 5\sqrt{7}$. $\triangle ABC$ is a right triangle, so its area is half the product of its legs:

$$[ABC] = \frac{(AB)(AC)}{2} = \frac{(9)(5\sqrt{7})}{2} = \boxed{\frac{45\sqrt{7}}{2}.}$$

The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R2-L7-P6 *Problem 10.28, Chapter 10, Introductory to Number Theory*

$$\binom{2n+2}{2n} = \binom{2n+2}{2} = \frac{(2n+2)(2n+1)}{2} = (n+1)(2n+1)$$

1. $n \equiv 0 \pmod{10} \Rightarrow 2n+1 \equiv 1 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 1 \pmod{10}$.
2. $n \equiv 1 \pmod{10} \Rightarrow 2n+1 \equiv 3 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 6 \pmod{10}$.
3. $n \equiv 2 \pmod{10} \Rightarrow 2n+1 \equiv 5 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 5 \pmod{10}$.
4. $n \equiv 3 \pmod{10} \Rightarrow 2n+1 \equiv 7 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 8 \pmod{10}$.
5. $n \equiv 4 \pmod{10} \Rightarrow 2n+1 \equiv 9 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 5 \pmod{10}$.
6. $n \equiv 5 \pmod{10} \Rightarrow 2n+1 \equiv 1 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 6 \pmod{10}$.
7. $n \equiv 6 \pmod{10} \Rightarrow 2n+1 \equiv 3 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 1 \pmod{10}$.
8. $n \equiv 7 \pmod{10} \Rightarrow 2n+1 \equiv 5 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 0 \pmod{10}$.
9. $n \equiv \boxed{8} \pmod{10} \Rightarrow 2n+1 \equiv 7 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 3 \pmod{10}$.
10. $n \equiv 9 \pmod{10} \Rightarrow 2n+1 \equiv 9 \pmod{10} \Rightarrow (n+1)(2n+1) \equiv 0 \pmod{10}$.

The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R2-L7-P7 *Problem 11.11, Chapter 11, Introductory to Number Theory.*

$$x = 0.\overline{037} \Rightarrow 1000x = 37.\overline{037} \Rightarrow 999x = 1000x - x = 37 \Rightarrow x = \frac{37}{999} = \boxed{\frac{1}{27}.}$$

□

Solution. ICLT-2022-SM2-R2-L7-P8 *Exercise 11.3.5, Chapter 11, Introductory to Counting & Probability.*

There are 5 white marbles and $5+k$ total marbles, so the probability that a white marble is drawn is $\dfrac{5}{5+k}$. Similarly, the probability that a black marble is drawn is $\dfrac{k}{5+k}$. So the expected value of the game is

$$E = \frac{5}{5+k}(1) + \frac{k}{5+k}(-1) = \frac{5-k}{5+k}$$
$$E = 0 \Rightarrow k = \boxed{5.}$$

□

Solution. ICLT-2022-SM2-R2-L7-P9 *Problem 15.7, Chapter 15, Introductory to Algebra.*

$$\frac{3}{x-3} - 4x + 4 > -1 \Rightarrow \frac{3 + (-4x+5)(x-3)}{x-3} > 0 \Rightarrow \frac{4x^2 - 17x + 12}{x-3} < 0$$

Case 1: $x - 3 < 0$, or $x < 3$, then $4x^2 - 17x + 12 > 0$,

$$\Delta = 17^2 - 4 \cdot 4 \cdot 12 = 97 \Rightarrow x < \frac{17-\sqrt{97}}{8} \approx 0.89 \text{ or } x > \frac{17+\sqrt{97}}{8} \approx 3.35 \Rightarrow x < \frac{17-\sqrt{97}}{8}.$$

Case 2: $x - 3 > 0$, or $x > 3$, then $4x^2 - 17x + 12 < 0$, from above,

$$\frac{17-\sqrt{97}}{8} < x < \frac{17+\sqrt{97}}{8} \Rightarrow 3 < x < \frac{17+\sqrt{97}}{8}$$

Hence, $\boxed{x \in (-\infty, \frac{17-\sqrt{97}}{8}) \cup (3, \frac{17+\sqrt{97}}{8}).}$ □

Solution. ICLT-2022-SM2-R2-L7-P10 *Problem 13.2, Chapter 13, Introductory to Geometry.*

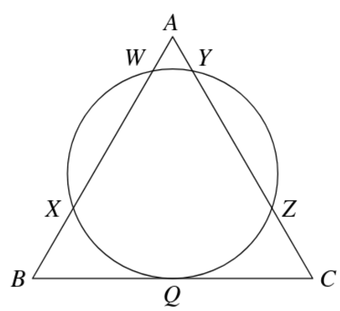

Figure 26.5: ICLT-2022-SM2-R2-L7-P10

The power of point A gives us $(AW)(AX) = (AY)(AZ)$. Since $AW = AY$, we have $AX = AZ$, so

$$WX = AX - AW = AZ - AY = YZ.$$

Since $\triangle ABC$ is equilateral, we have $AB = AC$. Therefore,

$$BX = AB - AX = AC - AZ = CZ \text{ and } BW = BX + XW = CZ + ZY = CY.$$

Finally, we use the powers of points B and C to find:

$$BQ^2 = (BX)(BW) = (CZ)(CY) = CQ^2 \Rightarrow BQ = CQ.$$

$\boxed{\text{Point } Q \text{ is therefore the midpoint of } BC.}$ □

Chapter 27

Introductory Curriculum Level Test 2 - Level 8

27.1 Rules

27.2 Introductory Curriculum Level Test (ICLT)

Introductory Curriculum Level Test is an activity of Math, Chess, and Coding Club (MCC).

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3	6-7	5	5-6	6
4	8-9	6-7	7	7-8
5	10-12	8	8-9	9-10
6	13-15	9-10	10	11
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- Paper-based book, calculators are allowed. Computers are allowed for solving problem with coding. No searching on Internet for ideas, hints, or solutions are allowed. No help from anyone is allowed. Be honest.
- Grading:
  1. For a *multiple-choice problem* if you give a **correct answer**, you get 10 points. For an **unanswered** problem, i.e. if you select no choice, you get 1 point. For a **wrong answer**, i.e. when you select the wrong choice, you get 0 point.
  2. For a *show-you-work problem* the total number of points can be earned is 10**.** For a complete solution for each question of a show-you-work problem, you earn all available points for that question. If your solution for that question is not correct, you may earn some but not all available points for that question. The actual number of points to be awarded is based on the discretion of the grading COs.
- Students, if not able to join the contest at the official designated time, are eligible to write at an earlier time. They must contact the COs in time for arrangement. They must not discuss any of the received contest problems with anyone until the official contest time passed.
- Qualification: The minimum number of points **to pass the level test is** $\boxed{51.}$

27.3 Problems

Problem 27.3.1 (ICLT-2022-SM2-R2-L8-P1)**.** A roll of quarters contains 40 quarters and a roll of dimes contains 50 dimes.

James has a jar that contains 82 quarters and 158 dimes. Lindsay has a jar that contains 128 quarters and 265 dimes. James and Lindsay pool these quarters and dimes and make complete rolls with as many of the coins as possible.

How much are the leftover quarters and dimes worth?

(A) \$3.0 (B) \$2.5 (C) \$3.5 (D) \$5.0 (E) \$5.5

Problem 27.3.2 (ICLT-2022-SM2-R2-L8-P2)**.** A sphere is inscribed in a cylinder, meaning that it is tangent to both bases, and that one great circle of the sphere is along the curved surface of the cylinder.

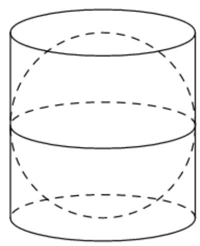

Figure 27.1: **??**

Find the ratio of the surface area of the sphere to the lateral surface area of the cylinder.

(A) 1 (B) $\frac{4}{5}$ (C) $\frac{3}{4}$ (D) $\frac{2}{3}$ (E) $\frac{1}{2}$

Problem 27.3.3 (ICLT-2022-SM2-R2-L8-P3)**.** Find the sum of all solutions to

$$25^{-2} = \frac{(125)^{\frac{8}{x}}}{(25)^{\frac{13}{2x}} \cdot (5)^{\frac{17}{x}}}.$$

(A) $\frac{5}{2}$ (B) 2 (C) $\frac{3}{2}$ (D) 1 (E) $\frac{1}{2}$

Problem 27.3.4 (ICLT-2022-SM2-R2-L8-P4)**.** Find the remainder of $\left(12^{12}\right)^{12}$ when divided by 7.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 27.3.5 (ICLT-2022-SM2-R2-L8-P5)**.** In how many ways can we distribute 5 pieces of candy among 5 kids if some of them have no candies.

(A) 120 (B) 126 (C) 625 (D) 1001 (E) 3125

Problem 27.3.6 (ICLT-2022-SM2-R2-L8-P6)**.** In the circle Ω centred at O, the chords GH and IJ are perpendicular at point M. If $IM = 4$, $MH = 12$, and $GM = 24$, find GJ.

(A) 12 (B) $12\sqrt{10}$ (C) $18\sqrt{10}$ (D) $24\sqrt{10}$ (E) $24\sqrt{5}$

Problem 27.3.7 (ICLT-2022-SM2-R2-L8-P7)**.** Jamie and Billy each think of a polynomial. Each of their polynomials is monic, has degree 3, and has the same positive constant term and the same coefficient of x. The product of their polynomials is

$$x^6 + 2x^5 + 3x^4 + 14x^3 + 17x^2 + 24x + 16.$$

What are the two polynomials?

Problem 27.3.8 (ICLT-2022-SM2-R2-L8-P8)**.** What figure is formed by connecting the centers of the faces of a regulaer octahedron?

Problem 27.3.9 (ICLT-2022-SM2-R2-L8-P9)**.** Find the remainder of $1^3 + 2^3 + \ldots + 2022^3$ when divided by 9.

Problem 27.3.10 (ICLT-2022-SM2-R2-L8-P10)**.** Which one is larger: distribute 15 candies to 5 kids, each have at least two candies, or distribute 5 candies to 5 kids, some can have no candies.

Note that *perform a distribution means that the sum of the candies the kids receive after the distribution is the same as before the distribution.*

27.4 Grading

Answers for multiple-choice problems.

Problem 1: *E*

Problem 2: *A*

Problem 3: *B*

Problem 4: *A*

Problem 5: *D*

Problem 6: *D*

Guideline for show-your-work problems. If the student provides a complete solution for each question in a show-you-work problem, the student earn all available points for that question. If the student does not have a complete solution, follow the guidelines below.

Problem 7: 2 points if the student can find the coefficient of the constant term and the coefficient of x.

Problem 8: 2 points if the student can group the pairs of terms that are divisible by 9.

Problem 9: 2 points if the student can figure out the distance between the centers of the faces are the same.

Problem 10: 2 points if the student can deduce the argument by increase/decrease the number of candies each kid can receive before the distribution.

27.5 Solutions

Solution. ICLT-2022-SM2-R2-L8-P1 *Exercise 12.4.5, Chapter 12, Introductory to Number Theory.*

We use residues of the numbers of each type of coin to determine the number of dimes and quarters leftover:

$$\begin{aligned} 82+128 &\equiv 2+8 \equiv 10 \pmod{40} \\ 158+265 &\equiv 8+15 \equiv 23 \pmod{50} \end{aligned}$$

The total value of the leftover quarters and dimes is $10(\$0.25)+23(\$0.10) = \$2.50+\$2.30 = \boxed{\$4.80}$. The answer is $\boxed{E.}$ □

Solution. ICLT-2022-SM2-R2-L8-P2 *Problem 15.3, Chapter 15, Introductory to Geometry.*

Let r be the radius of the sphere. Then the radius and the height of the cylinder are r and $2r$, respectively. The surface area of sphere is $4\pi r^2$. The lateral surface area of the cylinder is $2\pi r \cdot 2r = 4\pi r^2$. Therefore their ratio is $\boxed{1}$.

The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R2-L8-P3 *Problem 19.11, Chapter 19, Introductory to Algebra.*

$$\frac{125^{\frac{8}{x}}}{5^{\frac{13}{x}} \cdot 5^{\frac{17}{x}}} = 5^{\frac{24}{x}-\frac{13}{x}-\frac{17}{x}} = 5^{-\frac{6}{x}}$$

$$25^{-2} = 5^{-\frac{6}{x}} \Rightarrow -4 = -\frac{6}{x} \Rightarrow x = \boxed{\frac{3}{2}.}$$

The answer is $\boxed{B.}$ □

Solution. ICLT-2022-SM2-R2-L8-P4 *Problem 12.12, Chapter 11, Introductory to Number Theory*

First note that $\left(12^{12}\right)^{12} \equiv \left(5^{12}\right)^{12} \pmod 7$. Now the remainders of the powers of 5, $5^0, 5^1, 5^2, \ldots,$ when divided by 7 form a sequence with a repated pattern of $1,5,4,6,2,3$. Thus $5^{12} \equiv 5^0 \equiv 1 \pmod 7$. Therefore $\left(5^{12}\right)^{12} \equiv \boxed{1} \pmod 7$. The answer is $\boxed{A.}$ □

Solution. ICLT-2022-SM2-R2-L8-P5 *Execise 13.3.1, Chapter 13, Introductory to Counting & Probability.*

If we give each kid one more candy, then each kid has at least one candy. Thus the problem is in how many ways can we distribute 10 candies among 5 kids, which is $\binom{10+5-1}{5-1} = \boxed{1001}$. The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R2-L8-P6 *Problem 13.13, Chapter 13, Introductory to Geometry*

By the power of the point M

$$(GM)(MH) = (IM)(MJ) \Rightarrow (24)(12) = (4)(JM) \Rightarrow JM = 72.$$

Since $GM \perp MJ$, thus GMJ is a right triangle at M, thus

$$GJ = \sqrt{GM^2 + MJ^2} = \sqrt{24^2+72^2} = \boxed{24\sqrt{10}.}$$

The answer is $\boxed{D.}$ □

Solution. ICLT-2022-SM2-R2-L8-P7 *Problem 18.16, Chapter 18, Introductory to Algebra.*

Because the constant terms of both polynomials in the product are positive, are the same, and multiply to 16, they must each equal $\sqrt{16} = 4$.

The coefficients of x in the two polynomials are the same and the polynomials are monic, so the product is of the form

$$x^6 + 2x^5 + 3x^4 + 14x^3 + 17x^2 + 24x + 16 = (x^3 + ax^2 + cx + 4)(x^3 + bx^2 + cx + 4).$$

The x term of the product on the right-hand side is $(cx)(4) + (4)(cx) = 24cx$, thus $c = 3$. Now,

$$\begin{aligned} & x^6 + 2x^5 + 3x^4 + 14x^3 + 17x^2 + 24x + 16 = (x^3 + ax^2 + 3x + 4)(x^3 + bx^2 + 3x + 4) \\ \Rightarrow & \left\{ \begin{array}{l} (x^5):\ 2 = a + b \\ (x^4):\ 3 = 3 + ab + 3 \end{array} \right. \Rightarrow a, b \text{ are roots of } t^2 - 2t - 3 = 0 \Rightarrow (a, b) \in \{(-1, 3), (3, -1)\} \end{aligned}$$

Thus the requested polynomials are $x^3 + 3x^2 + 3x + 2$ and $x^3 - x^2 + 3x + 4$. $\square$

Solution. ICLT-2022-SM2-R2-L8-P8 *Exercise 14.19.b, Chapter 14, Introductory to Geometry.*

There are 8 faces on an octahedron, so our new figure has 8 vertices, since there's one vertex of the new figure on each face of the old figure. By symmetry, we can see that each of the faces of our new figure is a square, and we see that the new figure is a $\boxed{\text{cube}}$. □

Solution. ICLT-2022-SM2-R2-L8-P9 *Exercise 12.6.5, Chapter 12, Introductory to Number Theory.*

Note that for each $0 < r \le 4$, k, ℓ non-negative integers,

$$(9k+r)^3+(9\ell+9-r)^3 = (9(k+\ell)+9)\left[(9k+r)^2-(9k+r)(9\ell+9-r)+(9\ell+9-r)^2\right] \text{ is a multiple of 9.}$$

Thus, $1^3+2024^3 \equiv 2^3+2023^3 = \ldots = 1012^3+1013^3 \equiv 0 \pmod 9$. So $S = 1^3+2^3+\ldots+2022^3 \equiv -2023^3-2024^3 \pmod 9$. But $2023^3 \equiv 7^3 \equiv -2^3 \equiv 1 \pmod 9$, and $2024^3 \equiv 8^3 \equiv -1^3 \equiv 8\ mod 9$.

Thus $S = 1^3+2^3+\ldots+2022^3$ is divisible by 9. □

Solution. ICLT-2022-SM2-R2-L8-P10 *Problem 13.2, Chapter 13, Introductory to Counting & Probability.*

If we distribute 15 candies to 5 kids, each have at least two candies, then it is the same as distribute 10 candies to 5 kids, each have at least 1. Distribute 5 candies to 5 kids, some can have no candies is the same as distribute 10 candies to 5 kids, each have at least 1. So they are the same. □