Olympiad Team 2022-2023

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December 10, 2022

Contents

The Olympiad Team Program

Part I

Fall 2022

Chapter 1

Divisibility

1.1 Examples

Example 1.1.1 (OT-22-23-S1-E1)

$n > 1$ is a positive integer. Express

1. 2^n as a sum of two odd consecutive integers.
2. 3^n as a sum of three consecutive integers.

Remark. The simple approach for the first question is to ask how *to denote two odd consecutive integers* and that is simple $2k-1$ and $2k+1$. It is basically to solve the equation for k $2^n = (2k-1)+(2k+1)$.

Solution. For the first question, let k be an integers such that:

$$2^n = (2k-1)+(2k+1) \Rightarrow 2^n = 4k \Rightarrow k = 2^{n-2} \Rightarrow \boxed{2^n = (2^{n-1}-1)+(2^{n-1}+1).}$$

For the second question, let k be an integers such that:

$$3^n = (k-1)+k+(k+1) = 3k \Rightarrow k = 3^{n-1} \Rightarrow \boxed{3^n = (3^{n-1}-1)+3^{n-1}+(3^{n-1}+1).}$$

□

Theorem (Properties of Divisibility)

a, b, and c are integers, $a \neq 0$, then

- $a \mid a$ (reflexivity);
- $a \mid b$, $b \mid c$, then $a \mid c$ (transitivity);
- if $a \mid b$, $b \neq 0$, then $|a| \leq |b|$;
- if $a \mid b$, $b \mid c$, then $a \mid \alpha b + \beta c$, for any integers α and $beta$;
- if $a \mid b$, $a \mid b \pm c$, then $a \mid c$;
- $a \mid b$, $b \mid a$, then $|a| = |b|$.

Example 1.1.2 (OT-22-23-S1-E2)

Find all integers n such that $3n-4,\ 4n-5,\ 5n-3$ are all prime numbers.

Remark. If a prime number is divisible by a prime, then that number is equal to the prime divisor. How about investigate divisibility by 2, namely parity? Note that if n is even, then $3n-4$ is even. If n is odd then $5n-3$ is odd. So is there any relationship between these numbers? Or is there any *invariant* about them? Their sum?

$$(3n-4)+(4n-5)+(5n-3)=12(n-1).$$

Thus, all three of primes are even or only one of them is even. And if a prime is even, then it is 2.

Solution. We consider two cases based on the parity of n.

$2 \mid n$: $3n-4$ is even. The only even prime is 2, so $3n-4=2$, $n=2$, $4n-5=3$, $5n-3=7$ are primes.

$2 \nmid n$: $5n-3$ is even. $5n-3=2$, so $n=1$, but $4n-5=-1<0$, so there is no solution for this case.

Thus $\boxed{n=2}$ is the solution. □

Example 1.1.3 (OT-22-23-S1-E3)

The number 1 can be expressed as a sum of the reciprocals of 9 odd integers, as show below,

$$1=\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{35}+\frac{1}{45}+\frac{1}{231}.$$

Is it possible to write 1 as a sum of the reciprocals of 8 odd integers?

Remark. What if we assume that 1 can be expressed as a sum of reciprocals of 8 odd integers?

$$1=\frac{1}{n_1}+\frac{1}{n_2}+\ldots+\frac{1}{n_8}.$$

Then by bringing the fractions on the right side to the denominator, and then cross-multiplying the equation, and *compare the parities of both sides,* what do you get?

Solution. Let assume that

$$\begin{aligned}1&=\frac{1}{n_1}+\frac{1}{n_2}+\ldots+\frac{1}{n_8}=\frac{n_2n_3\ldots n_8+n_1n_2n_4\ldots n_8+\ldots+n_1n_2\ldots n_7}{n_1n_2\ldots n_8},\\ \Rightarrow n_1n_2\ldots n_8&=\underbrace{n_2n_3\ldots n_8}_{odd}+\underbrace{n_1n_2n_4\ldots n_8}_{odd}+\ldots+\underbrace{n_1n_2\ldots n_7}_{odd}.\end{aligned}$$

The left-hand side of the last equation is the product of odd numbers, so it is odd. The right-hand side is the sum of even number of odd numbers, so it is even. Thus, the parities of both sides do not match.

Hence, $\boxed{\text{there are no such odd integers.}}$ □

Example 1.1.4 (OT-22-23-S1-E4)

Prove that, for all n positive integer, $6 \mid n^3 - n$.

> **Remark.** By factoring $n^3 - n = n(n^2 - 1) = (n-1)n(n+1)$. Note that $n-1, n$, and $n+1$ are consecutive integers. Investigate the parity (divisibility by 2) and divisibility by 3

Solution. Since $n^3 - n = n(n^2 - 1) = (n-1)n(n+1)$ is the product of three consecutive integers, so one of them is even and one of them is divisible by 3, thus the product is divisible by $\boxed{6.}$ $\square$

Theorem (Division Theorem)

For any integer a and positive integer b, there exists exactly one pair of integers q and r such that

$$a = bq + r, \text{ where } 0 \leq r < b$$

We call a the dividend, b the divisor, q the quotient, and r the remainder.

Theorem (Euclidean Algorithm)

For two natural $a, b, a > b$, to find $\gcd(a, b)$ we use the division algorithm repeatedly:

$$\begin{aligned} a &= bq_1 + r_1 \\ b &= r_1 q_2 + r_2 \\ r_1 &= r_2 q_3 + r_3 \\ &\cdots \\ r_{n-2} &= r_{n-1} q_n + r_n \\ r_{n-1} &= r_n q_{n+1} \end{aligned}$$

Then we have $\gcd(a, b) = \gcd(b, r_1) = \gcd(r_1, r_2) = \cdots = \gcd(r_{n-1}, r_n) = r_n$.

Theorem (Extended Euclidean Algorithm)

Let m and n be integers such that $m = qn + r$, where $0 \leq r < n$. Then

$$\gcd(m, n) = \gcd(r, n)$$

The algorithm applies this simplification repeatedly to determine the GCD of two integers.

Example 1.1.5 (OT-22-23-S1-E5)

Let x and y be integers. Prove that $17 \mid 2x + 3y$ if and only if $17 \mid 9x + 5y$.

> **Remark.** $13 \cdot 2 - 1 \cdot 9 = 17 \Rightarrow 13(2x + 3y) = 26x + 39y = (17x + 34y) + (9x + 5y)$.

Solution. Since $13(2x + 3y) = 26x + 39y = (17x + 34y) + (9x + 5y)$, so the statement is true. $\square$

Example 1.1.6 (OT-22-23-S1-E6)

Let $p > 2$ be an odd number and n be a positive integer. Prove that

$$p \mid 1^{p^n} + 2^{p^n} + \ldots + (p-1)^{p^n}.$$

Remark. There are two important facts based on the parity of p.

First, p is odd, so the number of terms in $1^{p^n} + 2^{p^n} + \ldots + (p-1)^{p^n}$ is even. This means that we can pair up the terms into a number of sums $d^{p^n} + (p-d)^{p^n}$, where $1 \le d \le \frac{p-1}{2}$. Note that $d + (p-d) = d$.

Furthermore p^n is odd, so $d^{p^n} + (p-d)^{p^n}$ is factorablable by using the identity

$$a^{2n+1} + b^{2n+1} = (a+b)(a^{2n} + a^{2n-1}b + \ldots + b^{2n}).$$

Solution. The number of terms of $1^{p^n} + 2^{p^n} + \ldots + (p-1)^{p^n}$ is $p-1$. Since p is odd, so $p-1$ is even, thus we can pair up the terms of the sum as follow

$$\begin{aligned} 1^{p^n} + 2^{p^n} + \ldots + (p-1)^{p^n} &= \left(1^{p^n} + (p-1)^{p^n}\right) + \left(2^{p^n} + (p-2)^{p^n}\right) + \ldots + \left(\left(\frac{p-1}{2}\right)^{p^n} + \left(\frac{p+1}{2}\right)^{p^n}\right) \\ &= \sum_{d=1}^{\frac{p-1}{2}} \left(d^{p^n} + (p-d)^{p^n}\right) \quad (1) \end{aligned}$$

It is well-known that for any a, b real numbers and k non-negative integer,

$$a^{2k+1} + b^{2k+1} = (a+b)(a^{2k} + a^{2k-1}b + \ldots + b^{2k}) \quad (*)$$

Since p^n is also odd, by (*), for $d \in \{1, 2, \ldots, \frac{p-1}{2}\}$,

$$d^{p^n} + (p-d)^{p^n} = \left(\underbrace{(d) + (p-d)}_{p}\right)(d^{p^n-1} + d^{p^n-2}(p-d) + \ldots + (p-d)^{p^n-1}) \Rightarrow p \mid d^{p^n} + (p-d)^{p^n} \quad (2)$$

From (1) and (2) $p \mid \sum_{d=1}^{\frac{p-1}{2}} \left(d^{p^n} + (p-d)^{p^n}\right)$. Hence, $\boxed{p \mid 1^{p^n} + 2^{p^n} + \ldots + (p-1)^{p^n}.}$ ☐

Theorem (Euclidean Algorithm for Polynomials)

If $a(x) = b(x)q(x) + r(x)$, where the degree of $r(x)$ is less than the degree of $b(x)$, then

$$\gcd(a(x), b(x)) = \gcd(b(x), r(x))$$

Find $q(x)$ and $r(x)$ in the divison $a(x) = b(x)q(x) + r(x)$, where $a(x) = x^4 + 3x^3 + 10,\ b(x) = x^2 - x$.
$a(x) = x^4 + 3x^3 + 10,\ b(x) = x^2 - x$, so

$$\begin{aligned} &x^4 + 3x^3 + 10 = x^2(x^2 - x) + 4x^3 + 10 = x^2(x^2-x) + 4x(x^2-x) + 4x^2 + 10 \\ &= (x^2-x)(x^2+4x) + 4(x^2-x) + 4x + 10 = (x^2-x)(x^2+4x+4) + (4x+10) \\ &\Rightarrow q(x) = x^2 + 4x + 4, r(x) = 4x + 10 \end{aligned}$$

Example 1.1.7 (OT-22-23-S1-E7)
What is the sum of all integers n such that $n^2 + 2n + 2$ divides $n^3 + 4n^2 + 4n - 14$?

Remark. By polynomial division

$$n^3 + 4n^2 + 4n - 14 = (n^2 + 2n + 2)(n + 2) + (-2n - 18).$$

Solution. By polynomial division,

$$n^3 + 4n^2 + 4n - 14 = (n^2 + 2n + 2)(n + 2) + (-2n - 18) \quad (*)$$

Since $n^2 + 2n + 2 \mid n^3 + 4n^2 + 4n - 14$, then by (*), $n^2 + 2n + 2 \mid -2n - 18$.

By Properties of Divisibility there are two possible cases.

Case 1: $-2n - 18 = 0$, then $n = -9$.

Case 2: $|-2n - 18| \geq n^2 + 2n + 2$,

$$|-2n - 18| \geq n^2 + 2n + 2 \Rightarrow \begin{cases} -2n - 18 \geq n^2 + 2n + 2 \Rightarrow (n+2)^2 + 16 \leq 0 & (*) \\ +2n + 18 \geq n^2 + 2n + 2 \Rightarrow (n-4)(n+4) \leq 0 & (**) \end{cases}$$

It is easy to see that (*) is impossible and (**) implies that $-4 \leq n \leq 4$,

By direct testing for $n \in \{-4, -3, \ldots, 4\}$ so that $n^2 + 2n + 2 \mid -2n - 18$, we have $n \in \{-4, -2, -1, 0, 1, 4\}$.

Combine the results of both cases, $n \in \{-9, -4, -2, -1, 0, 1, 4\}$.

Hence, the sum of all values of n is $(-9) + (-4) + (-2) + (-1) + 0 + 1 + 4 = \boxed{-11.}$ □

Lemma (Polynomial Division of $a^m + b^m$ by $a^n + b^n$ $(m > n)$)

$$a^m + b^m = (a^n + b^n)(a^{m-n} + b^{m-n}) - a^n b^{m-n} - a^{m-n} b^n.$$

Example 1.1.8 (OT-22-23-S1-E8)
Find all positive integers n such that $3^{n-1} + 5^{n-1}$ divides $3^n + 5^n$.

Remark. Let $s_n = 3^n + 5^n$, then we perform a so-called polynomial division

$$s_n = (3 + 5)s_{n-1} - 3 \cdot 5 \cdot s_{n-2},$$

What is the relation between s_{n-1} and s_{n-2}?

Solution. Let $s_n = 3^n + 5^n$. Applying the lemma for dividing $s_n = 3^n + 5^n$ by $3^1 + 5^1$, we have

$$s_n = 3^n + 5^n = (3^1 + 5^1)(3^{n-1} + 5^{n-1}) - (3)5^{n-1} - 3^{n-1}(5) = (3 + 5)s_{n-1} - (3)(5)s_{n-2}.$$

Since $s_{n-1} \mid s_n$, so $s_{n-1} \mid (3)(5)s_{n-2}$ (*)

It is easy to see that, for all $n : \gcd(s_n, 3) = \gcd(s_n, 5) = 1$. Thus (*) implies that $s_{n-1} \mid s_{n-2}$ (**)

However if $n > 1$, then $s_{n-1} > s_{n-2}$. thus (**) is impossible.

Hence, the only possible value for n is $\boxed{1.}$ This can easily be verified. □

Lemma (Bézout's Lemma)

If $a, b \in \mathbb{Z},\ a, b \neq 0$ and $d = \gcd(a, b)$, then $\exists m, n \in \mathbb{Z}$ such that:

$$am + bn = d \quad (\textbf{Bézout's Identity})$$

In other words, *there exists a linear combination* of a and b equal to d. m and n are **Bézout coefficients**.

Proof. Bézout's Lemma Run the Euclidean Algorithm backward:

$$\gcd(a, b) = r_{n-2} - r_{n-1}q_n = r_{n-2} - (r_{n-3} - r_{n-2}q_{n-1})q_n = \ldots = am + bn$$

where m and n are some combination of the quotients. The two variables run through at every step in the equation are: $(r_{n-2}, r_{n-1}) \to (rn-2, r_{n-3}) \to \cdots \to (b, r_1) \to (a, b)$. $\square$

Example 1.1.9 (OT-22-23-S1-E9)

Prove that the expression

$$\frac{\gcd(m, n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

Solution. By Bézout's Lemma, if $d = \gcd(m, n)$, then $\exists a, b \in \mathbb{Z}$, such that $am + bn = d$, therefore,

$$\frac{d}{n}\binom{n}{m} = \frac{am + bn}{n}\binom{n}{m} = \frac{am}{n}\frac{n!}{m!(n-m)!} + b\binom{n}{m} = a\frac{(n-1)!}{(m-1)!(n-m)!} + b\binom{n}{m} = a\binom{n-1}{m-1} + b\binom{n}{m}.$$

Hence, $\boxed{\frac{\gcd(m, n)}{n}\binom{n}{m} \text{ is an integer for all } n \geq m \geq 1.}$ $\square$

Example 1.1.10 (OT-22-23-S1-E10)
Find all positive integers n such that for all odd integers a, where $a^2 < n$, then $a \mid n$.

Solution. Assume that n is such an integer. Let a be the largest odd integer such that $a^2 < n$, then

$$a^2 < n \leq (a+2)^2 \quad (*)$$

Case 1: If $a \geq 7$, then since each of $a-4, a-2, a$ are odd integers larger than 1, and each of the squares of them is less than n, so each of them divides n. It is easy to see that they are *three consecutive odd numbers, so pair-wise relatively primes.* Thus their product divides n, or

$$a(a-2)(a-4) \leq n \Rightarrow a(a-2)(a-4) \leq (a+2)^2 \Rightarrow a^3 - 7a^2 + 4a - 4 \leq 0 \Rightarrow a^2(a-7) + 4(a-1) \leq 0.$$

The last inequality is false because $a \geq 7$. Thus there is no solution for this case.

Case 2: Now, for $a \in \{1, 3, 5\}$, let test n where $a^2 < n \leq (a+2)^2$ and $a(a-2)(a-4) \leq n$,

$$a = 1 \Rightarrow 1^2 < n \leq 3^2, (1) \mid n \Rightarrow n \in \{2, 3 \ldots, 9\}$$
$$a = 3 \Rightarrow 3^2 < n \leq 5^2, (1)(3) \mid n \Rightarrow n \in \{12, 15, 18, 21, 24\}$$
$$a = 5 \Rightarrow 5^2 < n \leq 7^2, (1)(3)(5) \mid n \Rightarrow n \in \{30, 45\}$$

Hence, $\boxed{n \in \{2, 3, \ldots, 8, 9, 12, 15, 18, 21, 24, 30, 45\}.}$ □

Remark. In the this example, we perform a so-call establishment of upper and lower bounds for n with the largest possible a as a parameter. By doing so we arrive at an inequality for a.

For the upper bound, we assume that a is the largest odd integer such that $a^2 < n$, so implicitly it means that

$$a^2 < n \leq (a+2)^2, \text{ and } (a+2)^2 \text{ now is an upper bound of } n.$$

For the lower bound, we look at values similar to a and collect a bunch of them. Note that the upper bound is $(a+2)^2$ which is a second-degree polynomial of a, so if we manage to collect more than 2 $a-$like values, then we can establish a nice lower bound *parameterized by* a. Luckily, by the given condition of the problem, $a-2$ and $a-4$ are also odd, thus

$$a - 2 \mid n \text{ and } a - 4 \mid n.$$

The extra bonus came in when we recognize that $a, a-2$, and $a-4$ are *three consecutive odd numbers, so pair-wise relatively primes.* This means that *if each of them divides n then their product divides n.* Thus, by the Properties of Divisibility, the lower bound can be established,

$$a(a-2)(a-4) \mid n \Rightarrow a(a-2)(a-4) \leq n.$$

We don't know why $a \geq 7$ is a right choice for case work. This was actually derived by comparing the bounds

$$a(a-2)(a-4) - (a+2)^2 = a^3 - 7a^3 + 4a - 4 = a^2(a-7) + 4(a-1).$$

It is obvious that $a - 1 \geq 0$, thus if $a \geq 7$, then the inequality

$$\text{lower bound } = a(a-2)(a-4) < (a+2)^2 = \text{ upper boud}$$

would not stand, meaning there is no such solution for a, thus no solution for n. Therefore we don't have to deal with any $a \geq 7$, which leaves us with a few cases when a is odd and $a \leq 7$, namely $a \in \{1, 3, 5\}$.

Example 1.1.11 (OT-22-23-S1-E11)

The positive integers $a_1, a_2, \ldots, a_n$ are such that each less than 1000, and their least common multiple $\operatorname{lcm}(a_i, a_j) > 1000$, for all i, j such that $i \neq j$. Show that

$$\sum_{i=1}^{n} \frac{1}{a_i} < 2.$$

Solution. First, it is obvious to see that for any positive integers m and a,

Claim — If $\frac{1000}{m+1} < a \leq \frac{1000}{m}$, then m multiples of a, namely: $a, 2a, \ldots, ma$ do not exceed 1000.

Let consider *the distribution of* $\{a_i\}_{i=1}^n$ within the interval $[1, 1000]$ by dividing this interval into

$$\left(\frac{1000}{2}, 1000\right] \cup \left(\frac{1000}{2}, \frac{1000}{3}\right] \cup \left(\frac{1000}{3}, \frac{1000}{4}\right] \cup \ldots,$$

and invesigating the amount of $\{a_i\}_{i=1}^n$ falling within each of $\left(\frac{1000}{2}, 1000\right]$, $\left(\frac{1000}{2}, \frac{1000}{3}\right]$, and so on.

Let k_1 be the number of $\{a_i\}_{i=1}^n$ within $\left(\frac{1000}{2}, 1000\right]$, k_2 be the number of $\{a_i\}_{i=1}^n$ within $\left(\frac{1000}{3}, \frac{1000}{2}\right]$, and so on. This means that for an i, such that $1 \leq i \leq n$, if a_i is within $\left(\frac{1000}{m+1}, \frac{1000}{m}\right]$, for some integer m,

$$a_i > \frac{1000}{m+1} \Rightarrow \frac{1}{a_i} < \frac{m+1}{1000} \quad (1)$$

For an m positive integer, since k_m is the number of a_i within $\left(\frac{1000}{m+1}, \frac{1000}{m}\right]$, by (1) we have,

$$\sum_{a_i \in \left(\frac{1000}{m+1}, \frac{1000}{m}\right]} \frac{1}{a_i} < k_m \frac{m+1}{1000} \quad (*)$$

Furthermore, for an m positive integer, by the claim above, for an i, such that $1 \leq i \leq n$, if a_i within $\left(\frac{1000}{m+1}, \frac{1000}{m}\right]$, then there are m multiples of a_i within $\left(\frac{1000}{m+1}, \frac{1000}{m}\right]$. Thus,

$$\text{the number of all multiples of all } \{a_i\}_{i=1}^n \text{ within } \left(\frac{1000}{m+1}, \frac{1000}{m}\right] \text{ is } mk_m \quad (2)$$

Now, by considering $m = 1, 2, \ldots$, by (2), there are $k_1 + 2k_2 + 3k_3 + \ldots$ integers within

$$\left(\frac{1000}{2}, 1000\right] \cup \left(\frac{1000}{3}, \frac{1000}{3}\right] \cup \ldots = [1, 1000],$$

(meaning each of the integers does not exceed 1000), that are multiples of at least one a_i.

The multiples are all distinct, so the number of them is less than the number of integers within $[1, 1000]$,

$$k_1 + 2k_2 + 3k_3 + \ldots < 1000 \Rightarrow 2k_1 + 3k_2 + 4k_3 + \ldots < 1000 + (k_1 + k_2 + k_3 + \ldots) = 1000 + n < 2000 \quad (**)$$

Therefore from (*) and (**),

$$\sum_{i=1}^{n} \frac{1}{a_i} \leq k_1 \frac{2}{1000} + k_2 \frac{3}{1000} + k_3 \frac{4}{1000} + \ldots = \frac{2k_1 + 3k_2 + 4k_3 + \ldots}{1000} < 2.$$

□

Example 1.1.12 (OT-22-23-S1-E12)
Determine all pairs (a, b) of positive integers: $ab^2 + b + 7 \mid a^2b + a + b$.

Proof. First, we use a linear combination to simplify $ab^2 + b + 7 \mid a^2b + a + b$,

$$(a^2b + a + b) - a(ab^2 + b + 7) = b^2 - 7a \Rightarrow ab^2 + b + 7 \mid b^2 - 7a.$$

By the Properties of Divisibility

$$ab^2 + b + 7 \leq |b^2 - 7a|.$$

Case 1: $b^2 - 7a \geq 0$,

$$ab^2 + b + 7 > b^2 > b^2 - 7a \Rightarrow b^2 - 7a = 0 \Rightarrow \exists c \in \mathbb{Z}^+ : \ a = 7c^2, b = 7c \quad (*)$$

Let verify the result (*) by substituting into the original divisibility equation,

$$a^2b + a + b = 7^3c^5 + 7c^2 + 7c = c(7^3c^4 + 7c + 7) = c(ab^2 + b + 7) \Rightarrow ab^2 + b + 7 \mid a^2b + a + b.$$

Case 2: $b^2 - 7a < 0$, so $7a - b^2 > 0, ab^2 + b + 7 \mid 7a - b^2$.

$$ab^2 + b + 7 < 7a - b^2 \Rightarrow 7a > 7a - b^2 > ab^2 + b + 7 > ab^2 \Rightarrow 7 > b^2 \Rightarrow b \in \{1, 2\}.$$

Sub-case 2a: $b = 2$, then $ab^2 + b + 7 = 4a + 2 + 7 \mid 7a - b^2 = 7a - 4 \Rightarrow 4a + 9 \mid 79$. There is no solution.
Sub-case 2b: $b = 1$, then $ab^2 + b + 7 = a + 1 + 7 \mid 7a - b^2 = 7a - 1, \Rightarrow a + 8 \mid 57$. Thus, $a \in \{11, 49\}$ (**)
Let verify the result (**) by substituting into the original divisibility equation,

$$\begin{aligned} a = 11 :\ & ab^2 + b + 7 = (11)1 + 1 + 7 = 19 \mid a^2b + a + b = (11^2)1 + 11 + 1 = 133 \\ a - 49 :\ & ab^2 + b + 7 = (49)1 + 1 + 7 = 57 \mid a^2b + a + b = (49^2)1 + 11 + 1 = 2451. \end{aligned}$$

Hence, the solutions for (a, b) are $\boxed{(11,\ 1), (49,\ 1), \text{ and } \{(7c^2, 7c) \mid c \in \mathbb{Z}^+\}.}$ ∎

1.2 Problems

Problem 1.2.1 (OT-22-23-S1-P1)**.** Five numbers are chosen from the set $\{1, 2, 3, 4, 5, 6, 7\}$. If we know what the product of the chosen numbers was, that would not be enough to figure out whether the sum of the chosen numbers was even or odd. What is the product of the chosen numbers?

> **Remark.** *Providing the product of the chosen numbers is equivalent to telling the product of the two unchosen numbers.*

Problem 1.2.2 (OT-22-23-S1-P2)**.** Prove that for all integers n,

1. $120 \mid n^5 - 5n^3 + 4n$.
2. $121 \nmid n^2 + 3n + 5$.

> **Remark.** Note that n is integer and not positive integer. So your solution must covers cases when n is negative.
> 1. For $n \geq 3$, compare $n^5 - 5n^3 + 4n$ with $\binom{n+2}{5}$.
> 2. Compare $n^2 + 3n + 5$ with $(n + 7)(n - 4) + 33$.

Problem 1.2.3 (OT-22-23-S1-P3)**.** Find all positive integers d such that d divides both n^2+1 and $(n+1)^2+1$ for some integer n.

> **Remark.** The problem does not requires d to be divisor of both $n^2 + 1$ and $(n + 1)^2 + 1$ for *all* n. For example d should be a divisor of $\left((n + 1)^2 + 1\right) - (n^2 + 1) = 2n + 1$, and of $4\left((n + 1)^2 + 1\right) - (2 + 1)^2 = 4n + 7$.

Problem 1.2.4 (OT-22-23-S1-P4)**.** Let n be a positive integer larger than 2. Prove that among the fractions

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$$

an even number of them are irreducible.

> **Remark.** The fraction $\frac{k}{n}$ is irreducible if and only if $\gcd(k, n) = 1$.

Problem 1.2.5 (OT-22-23-S1-P5)**.** Let r be the remainder when $1059, 1417$ and 2312 are divided by $d > 1$. Find the value of $d - r$.

> **Remark.** By Division Algorithm
> $$\begin{cases} 1059 = q_1 d + r, \\ 1417 = q_2 d + r, \\ 2312 = q_3 d + r \end{cases}$$
> What numbers can be multiples of d?

Problem 1.2.6 (OT-22-23-S1-P6)**.** Prove that, if a and b integers such that $3 \mid a^2 + b^2$, then $3 \mid a$ and $3 \mid b$.

> **Remark.** Proof by contradiction?

Problem 1.2.7 (OT-22-23-S1-P7)**.** Prove that no term of the sequence $11, 111, 1111, \ldots$ is a perfect square.

> **Remark.** Study the previous problem.

Problem 1.2.8 (OT-22-23-S1-P8)**.** What is the largest integer n such that $n + 10 \mid n^3 + 100$

> **Remark.** If $n + 10 \mid n^3 + 100$ then there exist a, b, c real constants such that the polynomial division of $n^3 + 100$ by $n + 10$ is $n^3 + 100 = (n + 10)(n^2 + an + b) + c$, for all n.

Problem 1.2.9 (OT-22-23-S1-P9)**.** Suppose that $a_1, a_2, \ldots, a_{2n}$ are distinct integers such that the equation

$$(x - a_1)(x - a_2) \ldots (x - a_{2n}) - (-1)^n (n!)^2 = 0$$

has an integer solution r. Prove that

$$r = \frac{a_1 + a_2 + \ldots + a_{2n}}{2n}$$

> **Remark.** Assume that r is the integer solution of the given equation. It is obvious that $r \neq a_i$, for $i = 1, 2, \ldots, 2n$, so the numbers $r - a_1, r - a_2, \ldots, r - a_{2n}$ are all distinct. Then what would be the minimal value of
>
> $$(r - a_1)(r - a_2) \ldots (r - a_{2n})?$$

Problem 1.2.10 (OT-22-23-S1-P10)**.** Let m and n be relatively prime postive integers $\gcd(m, n) = 1$. Find all possible values of $\gcd(5^m + 7^m, 5^n + 7^n)$.

> **Remark.** Let assume that $m > n$. Perform a division
>
> $$5^m + 7^m = (5^n + 7^n)(5^{m-n} + 7^{m-n}) - 5^n 7^{m-n} - 5^{m-n} 7^n.$$
>
> Consider the case when $m < 2n$, then compare
>
> $$\gcd(5^m + 7^m, 5^n + 7^n) \text{ and } \gcd(5^n + 7^n, 5^{2n-m} + 7^{2n-m})$$
>
> Then investigate the case when $m > 2n$.

1.3 Solution

Example 1.3.1 (OT-22-23-S1-P1)

Five numbers are chosen from the set $\{1, 2, 3, 4, 5, 6, 7\}$. If we know what the product of the chosen numbers was, that would not be enough to figure out whether the sum of the chosen numbers was even or odd. What is the product of the chosen numbers?

Solution. First, it is important to note that we choose 5 numbers, thus 2 numbers are unchosen.

Since it is not possible **with certainty** to tell the sum of the chosen numbers was even or odd, therefore

1. the product of the two unchosen numbers is **the same** as the product of *some* chosen numbers.
2. the sum of the two unchosen number has **different parity** as the sum of *those* chosen numbers.

By direct testing, there are only two possible products that can be factored in different ways:

$$12 = 2 \cdot 6 = 3 \cdot 4 \quad \text{and} \quad 6 = 1 \cdot 6 = 2 \cdot 4$$

Since $8 = 2 + 6$ and $7 = 3 + 4$ have different parities, $7 = 1 + 6$ and $5 = 2 + 3$ have the same parity, thus one of the pairs $(2, 6)$ and$(3, 4)$ was not chosen. *Regardless of which pair was not chosen, their product is* 12, hence the prodct of the chosen numbers is:

$$\frac{1 \cdot 2 \cdots 7}{12} = \boxed{420.}$$

□

Example 1.3.2 (OT-22-23-S1-P2)

Prove that for all integers n,

1. $120 \mid n^5 - 5n^3 + 4n$.
2. $121 \nmid n^2 + 3n + 5$.

Remark (Quadratic factorization by finding roots)**.** It is easy to see that

$$n^5 - 5n^3 + 4n = n(n^4 - 5n^2 + 4).$$

But, how do we deal with $n^4 - 5n^2 + 4$?

Note that *all exponents of* n *powers are even*, so by letting $x = n^2$, we have a *quadratic* $x^2 - 5x + 4$. In order to analyze if it is divisible by a number, in this case 120, we want to factor the quadratic, in orther words to find a and b real numbers such that,

$$\boxed{x^2 - 5x + 4 = (x-a)(x-b).}$$

We prove the following claim.

Claim — a and b are the root of $x^2 - 5x + 4$.

Proof. Since $x^2 - 5x + 4 = (x-a)(x-b)$, so for $x = a$, or $x = b$, then $x^2 - 5x + 4 = 0$, in other words, a are b are a root of the quadratic $x^2 - 5x + 4 = 0$. thus ■

This is why we use the discriminant to find the factorization of a quadratic.

$$\Delta = (-5)^2 - 4 \cdot 1 \cdot 4 = 9 \Rightarrow \sqrt{\Delta} = 3 \Rightarrow x_{1,2} = \frac{-(-5) \pm \sqrt{\Delta}}{2} = \frac{-(-5) \pm 3}{2} \Rightarrow x_1 = 1,\ x_2 = 4.$$

Thus $x^2 - 5x + 4 = (x-1)(x-4)$, or

$$n^5 - 5n^3 + 4n = n(n^4 - 5n^2 + 4) = n(n^2 - 1)(n^2 - 4)$$

Remark (Quadratic factorization by Viete's theorem)**.** By comparing the two polynomials

$$x^2 - 5x + 4 = (x-a)(x-b) \Rightarrow x^2 - 5x + 4 = x^2 - (a+b)x + ab.(*)$$

Since the factorization is $x^2 - 5x + 4 = (x-a)(x-b)$, then *(*) is true for all* x. This means that the quadratic on the left-hand side and the quadratic on the right-hand side are identical (they are the same for all x!). Thus,

$$\begin{cases} a + b = 5 \\ ab = 4 \end{cases}$$

This system of equations, by Viete's theorem, implies that a and b are roots of the quadratic $x^2 - 5x + 4$.

Remark (Quadratic factorization is unique)**.** Now, the assumption is that somehow, with the keen eyes or with practic, we know that $x^2 - 5x + 4 = (x-1)(x-4)$, thus

$$(x-1)(x-4) = (x-a)(x-b).$$

the two factorizations containing the same number of linear terms $x-1$, $x-4$ or $x-a$, $x-b$, and since *every polynomial has a unique factorization* (like the integer has a unique prime-factorization.) thus the two sets of roots are the same $\{a, b\} = \{1, 4\}$.

Remark (The Example LPS V2B/4.7.2 (LPS Vol 3, Chapter Divisibility))**.** The example in the book shows a technique using consecutive integers to prove that $3 \mid f(n), 5 \mid f(n)$, and $8 \mid f(n)$, thus $120 = 3 \cdot 5 \cdot 8 \mid f(n)$. In this solution we explore the relationship between $(n-2)(n-1)n(n+1)(n+2)$ and $\binom{n+2}{5}$.

Solution. [For the first question] Since we want to prove that $120 \mid n^5 - 5n^3 + 4n$, lets factor the expression.

Now, it is easy to recall that $a^2 - b^2 = (a-b)(a+b)$, and since $n^2 - 1 = n^2 - 1^2, n^2 - 4 = n^2 - 2^2$,

$$n^5 - 5n^3 + 4n = n(n^4 - 5n^2 + 4) = n(n^2-1)(n^2-4) = n(n-1)(n+1)(n-2)(n+2)$$

Rearrange the terms into a product of five consecutive integers, and let denote it by $f(n)$:

$$f(n) = (n-2)(n-1)n(n+1)(n+2).$$

We have to prove that $120 \mid f(n)$.

Case 1: the *trivial case* $f(n) = 0$, then

$$n \in \{-2, -1, 0, 1, 2\} \Rightarrow 120 \mid f(n) \qquad (*)$$

Case 2: $f(n) \neq 0$, we have now two choices $n \geq 3$ or $n \leq -3$.

Case 2a: $n \geq 3$, then

$$f(n) = (n-2)(n-1)n(n+1)(n+2) = \frac{(n+2)!}{(n-3)!} = 5!\frac{(n+2)!}{5!(n-3)!} = 5!\binom{n+2}{5}$$

$$\binom{n+2}{5} \text{ is an integer } \Rightarrow 5! = 120 \mid f(n) \quad (**)$$

Case 2b: $n \leq -3$, then let $m = -n$,

$$f(n) = f(-m) = -(m+2)(m+1)m(m-1)(m-2) = -5!\binom{m+2}{5} \Rightarrow 120 \mid f(n) \quad (***)$$

(*), (**), and (***) imply that $\boxed{120 \mid f(n), \text{ for all integer } n.}$

Remark (Nice result!)**.**

$$(n-2)(n-1)n(n+1)(n+2) = 5!\binom{n+2}{5}$$

□

Remark (Reducing to remainders)**.** The second question seems to be difficult. The quadratic n^2+3n+5 cannot be factored without complex numbers.

$$n^2+3n+5=\left(n-\frac{-3-i\sqrt{11}}{2}\right)\left(n-\frac{-3+i\sqrt{11}}{2}\right), \text{ where } i=\sqrt{-1}.$$

Lets take another approach. To prove that $121 \nmid n^2+3n+5$, by divison algorithm $n=121k+r$, where $0\le r\le 120$. Thus n^2+3n+5 has the same remainder when divided by 121 as r^2+3r+5. So we can go test if $121 \mid r^2+3r+5$ for all cases of $0\le r<121$.

This technique is an important one. Instead of dealing with an infinite amount of all possible integers that n can be, as usually given in the problem context, we have to deal with only a finite number of remainders.

It is still a huge amount of work. So instead of dealing with remainders of 121, we shall *look into the remainders of its prime divisors,* in this case we are luckly because $121=11^2$. So lets assume that $n=11k+r, 0\le r<11$. We are looking now for the values of the remainder r such that r^2+3r+5 is divisible by 11.

By direct testing $r^2+3r+5\equiv 0 \pmod{11}$ results in $r=4$. Hence, $n=11k+4$. By substitution,

$$n^2+3n+5=(11k+4)^2+3(11k+4)+5=\boxed{121k^2+121k+33.}$$

On a side note you can use this piece of Python code to verify the remainder of r^2+3r+5 when divided by 11. Read the description of the *divmod* function to understand its return values.

```
for r in range(0, 11):
    print(divmod(r*r + 3*r + 5, 11))
```

Now, it is clear that $121k^2+121k+33$ is divisible by 11 but not 121. *The hard work is paid off.* Now we can concentrating on writing elegant solution.

Solution. [Solution One for the second question]

Let k and r be the quotient and the remainder of the division of n by 11,

$$n=11k+r,\ 0\le r<11 \Rightarrow n^2+3n+5\equiv r^2+3r+5 \pmod{11}.$$

Assume that there exists n such that $121 \mid n^2+3n+5$, then

$$11 \mid n^2+3n+5 \Rightarrow 11 \mid r^2+3r+5.$$

By direct testing, the only value of r such that $r^2+3r+5\equiv 0 \pmod{11}$ is $r=4$.

Thus $n=11k+4$, therefore

$$n^2+3n+5=(11k+4)^2+3(11k+4)+5=121k^2+121k+33 \Rightarrow 121 \nmid 121k^2+121k+33.$$

Hence, n^2+3n+5 is not divisible by 121 (even when it is divisible by 11.)

Remark. Lots of work, and the solution is still not that clean, because *it assumes that the jury will perform the lengthy calculation as we did,* which unfortunately is not always welcome.

□

Remark. Consider two the claims of the previous analysis,

Claim — If n^2+3n+5 is divisible by 121 then it is divisible by 11.

This is trivial.

Claim — If n^2+3n+5 is divisible by 11 then it has a remainder of 33 when divided by 121.

This we will use factorization to prove divisibility of $(n^2+3n+5)-33 = n^2+3n-28$ by 121, instead of direct testing for all remainders of n modulo 11. Thus, we shall need to factor $n^2+3n-28$, which we already learned how to do

$$\Delta = 3^2 - 4(1)(-28) = 81 = 9^2 \Rightarrow n_{1,2} = \frac{-3\pm 9}{2} \Rightarrow n_1 = -7, n_2 = 4 \Rightarrow n^2+3n-28 = (n+7)(n-4).$$

Solution. [Solution Two for the second question]

Lets assume that $121 \mid n^2+3n+5$, then of course $11 \mid n^2+3n+5$.

$$n^2+3n+5 = (n+7)(n-4)+33 \Rightarrow 11 \mid (n+7)(n-4) \Rightarrow 11 \mid n+7 \text{ or } 11 \mid n-4.$$

If $11 \mid n+7$, then $11 \mid (n+7)-11 = n-4$, and if $11 \mid n-4$, then $11 \mid (n-4)+11 = n+7$. Thus, both cases lead to

$$11 \mid n+7 \text{ and } 11 \mid n-4, \text{ therefore } 121 \mid (n+7)(n-4).$$

Now,

$$\begin{gathered}121 \mid n^2+3n+5 \text{ and } 121 \mid (n+7)(n-4) = (n^2+3n+5)-33 \\ \Rightarrow 121 \mid (n^2+3n+5) - \big((n^2+3n+5)-33\big) = 33, \text{ which is impossible.}\end{gathered}$$

Thus the assumption is incorrect, so $\boxed{\text{there does not exists } n \text{ such that } 121 \mid n^2+3n+5.}$ □

Example 1.3.3 (OT-22-23-S1-P3)

Find all positive integers d such that d divides both n^2+1 and $(n+1)^2+1$ for some integer n.

Remark. The task basically is to find all possible values of $\gcd(n^2+1,(n+1)^2+1)$.

For example, if we need to find $d \mid \gcd(21n+4, 14n+2)$, then

$$\begin{aligned} d \mid 21n+4,\ d \mid 14n+2 &\Rightarrow d \mid (21n+4)-(14d+2) = 7d+2 \\ d \mid 7d+2 \Rightarrow d \mid 14d+4 &\Rightarrow d \mid (14n+4)-(14d+2) = 2 \end{aligned}$$

Thus d is a divisor of 2, $d=1$ or $d=2$. It is easy to see that if n is even, then $d=2$, if n is odd, then $d=1$.

It is easy to see that $d \mid \gcd(n^2+1,(n+1)^2+1)$, then $d \mid ((n+1)^2+1)-(n^2+1) = 2n+1$. So basically we need to play around with these expressions and find a way *to reduce the right-hand side of the expression* $d \mid (\ldots\text{something}\ldots)$ to be some constant such as $d \mid 4$, or $d \mid 9$.

Theorem (Properties of Divisibility)

a, b, and c are integers, $a \neq 0$, then

- $a \mid a$ (reflexivity);
- $a \mid b$, $b \mid c$, then $a \mid c$ (transitivity);
- if $a \mid b$, $b \neq 0$, then $|a| \leq |b|$;
- if $a \mid b$, $b \mid c$, then $a \mid \alpha b + \beta c$, for any integers α and *beta*;
- if $a \mid b$, $a \mid b \pm c$, then $a \mid c$;
- $a \mid b$, $b \mid a$, then $|a| = |b|$.

Remark. By Properties of Divisibility,

$$\begin{aligned} &d \mid (n^2+2n+2)-(n^2+1) = 2n+1 \\ &d \mid (2n+1)(2n+1) = 4n^2+4n+1 \\ &d \mid 4(n^2+2n+2)-(4n^2+4n+1) = 4n+7 \\ &d \mid (4n+7)-2(2n+1) = \boxed{5} \end{aligned}$$

Solution.

$$\begin{aligned} d \mid n^2+1,\ d \mid (n+1)^2+1 = n^2+2n+2 &\Rightarrow d \mid (n^2+2n+2)-(n^2+1) = 2n+1 \\ d \mid 2n+1,\ d \mid n^2+2n+2 &\Rightarrow d \mid 4(n^2+2n+2)-(2n+1)^2 = 4n+7 \\ d \mid 2n+1,\ d \mid 4n+7 &\Rightarrow d \mid (4n+7)-2(2n+1) = 5 \end{aligned}$$

Thus $d=1$ or $d=5$. For $n=2$, $n^2+1=5$, $(n+1)^2+1=10$, so both $\boxed{1}$ and $\boxed{5}$ are desired values. □

Example 1.3.4 (OT-22-23-S1-P4)
Let n be a positive integer larger than 2. Prove that among the fractions

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n},$$

an even number of them are irreducible.

Remark (Get your hand dirty)**.** Let take a look at some sample values of n,

$$n = 3 \Rightarrow \underbrace{\frac{1}{3}}, \underbrace{\frac{2}{3}} \Rightarrow 2 \text{ irreducibles.}$$

$$n = 4 \Rightarrow \underbrace{\frac{1}{4}}, \frac{2}{4}, \underbrace{\frac{3}{4}} \Rightarrow 2 \text{ irreducibles.}$$

$$n = 5 \Rightarrow \underbrace{\frac{1}{5}}, \underbrace{\frac{2}{5}}, \underbrace{\frac{3}{5}}, \underbrace{\frac{4}{5}} \Rightarrow 4 \text{ irreducibles.}$$

$$n = 6 \Rightarrow \underbrace{\frac{1}{6}}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \underbrace{\frac{5}{6}} \Rightarrow 2 \text{ irreducibles.}$$

$\ldots$

For $n = 3$, each in the pair $(\frac{1}{3}, \frac{2}{3})$ is irreducible. For $n = 4$, each in the pairs $(\frac{1}{4}, \frac{3}{4})$ is irreducible. For $n = 5$, each in the pairs $(\frac{1}{5}, \frac{4}{5})$, and $(\frac{2}{5}, \frac{3}{5})$ is irreducible. For $n = 6$, each in the pairs $(\frac{1}{6}, \frac{5}{6})$ is irreducible.

In general $\boxed{(\frac{k}{n}, \frac{n-k}{n})}$?

Solution. Note that

$$\gcd(k, n) = gcd(n-k, n) \Rightarrow \text{ if } \frac{k}{n} \text{ is irreducible then } \frac{n-k}{n} \text{ is irreducible, too.}$$

Now, if $\frac{k}{n} = \frac{n-k}{n}$, then $n = 2k$ so $\frac{k}{n}$ is reducible. Thus, the irreducible fractions can be paired up. Hence, their number is $\boxed{\text{even.}}$ □

Example 1.3.5 (OT-22-23-S1-P5)

Let r be the remainder when $1059, 1417$ and 2312 are divided by $d > 1$. Find the value of $d - r$.

Remark (Division Algorithm)**.** Lets perform the division algorithm by d for all three numbers $1059, 1417$ and 2312. We know that each time we receive the same remainder r, so

$$\begin{cases} 1059 = q_1 d + r, \\ 1417 = q_2 d + r, \\ 2312 = q_3 d + r \end{cases}$$

Then lets explore by using by the Properties of Divisibility,

$$\begin{cases} 2312 - 1059 = (q_3 - q1)d, \\ 2312 - 1417 = (q_3 - q2)d, \\ 1417 - 1059 = (q_2 - q_1)d \end{cases}$$

Here the equations show what are the multiple of d.

Solution. By Division Algorithm, let q_1, q_2, and q_3 be the quotients, and r be the same remainder,

$$1059 = q_1 d + r, 1417 = q_2 d + r, 2312 = q_3 d + r \Rightarrow \begin{cases} 358 & = 1417 - 1059 = d(q_2 - q_1), \\ 1253 & = 2312 - 1059 = d(q_3 - q_1), \\ 895 & = 2312 - 1417 = d(q_3 - q2) \end{cases}$$

$$\Rightarrow d \mid \gcd(358 = 2 \cdot 179, 1253 = 7 \cdot 179, 895 = 5 \cdot 179) = 179, \text{ since } d > 1 \text{ so } d = 179.$$

$$\text{Because } 1059 = 5 \cdot 179 + 164 \Rightarrow r = 164 \Rightarrow d - r = \boxed{15.}$$

□

Example 1.3.6 (OT-22-23-S1-P6)

Prove that, if a and b integers such that $3 \mid a^2 + b^2$, then $3 \mid a$ and $3 \mid b$.

Remark (Remainder of perfect square when divided by a prime)**.** This is a simple problem to remind the residue classes (remainders) of perfect squares when divided by a prime.

Solution. It is known that a perfect square can only be in $3k$ or $3k + 1$ form, if $3 \mid a^2 + b^2$ then both a and b have to be multiples of 3. $\square$

Example 1.3.7 (OT-22-23-S1-P7)

Prove that no term of the sequence $11, 111, 1111, \ldots$ is a perfect square.

Remark (Remainder of perfect square when divided by 4)**.** In order to investigate if some number can be a perfect square, we might just investigate their remainder when divided by some primes. Note that $\overline{1\ldots11} = \overline{1\ldots12} - 1 = 4k + 3$.

Solution. Each number in the sequence has the format of $\overline{1\ldots11}$, since $\overline{1\ldots12}$ is divisible by 4, so $\overline{1\ldots12} - 1 = 4k + 3$. It is known that an odd perfect square can only be in $4k + 1$ form (or having a remainder 1 when divided by 4,) so no a_n can be a perfect square. $\square$

Example 1.3.8 (OT-22-23-S1-P8)

What is the largest integer n such that $n + 10 \mid n^3 + 100$

Remark (Using identity $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$)**.** Note that $n^3 + 1000 = n^3 + 10^3$, which is divisible by $n + 3$ by the identity

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2), \text{ with } a = n, \; b = 10.$$

Thus $n^3 + 100 = k(n+10) - 900$, hence $n + 10 \mid 900$.

Solution. [The short one] Note that

$$n + 10 \mid n^3 + 100 = (n^3 + 10^3) - 900 \Rightarrow n + 10 \mid 900.$$

By the Properties of Divisibility, $n + 10 \leq 900$. Thus the maximum value for n is $\boxed{890.}$ □

Remark (Polynomial Division)**.** In a more generic case, if $n + 10 \mid n^3 + 100$, then there exist a, b, c real constants such that

$$n^3 + 100 = (n+10)(n^2 + an + b) + c, \text{ for all } n.$$

So now, what we need to do is to find the values for a, b, and c and attempt to find the values for n.

Solution. [The long one] By polynomial division, there exist a, b, c real constants such that,

$$n^3 + 100 = (n+10)(n^2 + an + b) + c \Rightarrow n^2(10 + a) + n(b + 10a) + (10b + c - 100) = 0$$

Since the polynomial $n^2(10 + a) + n(b + 10a) + (10b + c - 100)$ is identical to 0 for all n, thus all of its coefficients shall be 0, or

$$\begin{cases} 10 + a = 0 \\ b + 10a = 0 \\ 10b + c - 100 = 0 \end{cases} \Rightarrow a = -10, b = 100, c = -900$$

Therefore $n + 10 = \gcd(n^3 + 100, n + 10) = \gcd(c, n + 10) = \gcd(-900, n + 10) = \gcd(900, n + 10)$.

Hence, the maximum value for n is $900 - 10 = \boxed{890}$. □

Remark. The solution based on polynomial division seems to be a waste of time, while in fact it is a better approach because it investigate the divisibility of the polynomials in context. Most of the times, it is hard to recognize or find some algebra tricks as in the first solution.

Example 1.3.9 (OT-22-23-S1-P9)

Suppose that $a_1, a_2, \ldots, a_{2n}$ are distinct integers such that the equation

$$(x - a_1)(x - a_2) \ldots (x - a_{2n}) - (-1)^n (n!)^2 = 0$$

has an integer solution r. Prove that

$$r = \frac{a_1 + a_2 + \ldots + a_{2n}}{2n}$$

Definition (Zeros)**.** A **zero**, also sometimes called a **root** of a function f:

- is a member x of the domain of f such that $f(x)$ *vanishes* at x;
- or a solution to the equation $f(x) = 0$.

Remark (Properties of Root)**.** If r is the root of the equation, then

$$(r - a_1)(r - a_2) \ldots (r - a_{2n}) - (-1)^n (n!)^2 = 0.$$

It is easy to see that r cannot be equal to any a_i, $i = 1, 2, \ldots, 2n$. This is interesting, because it makes $2n$ numbers $r - a_1,\ r - a_2,\ \ldots,\ r - a_{2n}$ to be all distinct.

In the same equation, their product is equal to $(-1)^n (n!)^2$, which is the product of another $2n$ distinct number $(-n) \ldots (-1)(1) \ldots (n)$.

Solution. Let r be the integer solution of the given equation.

$$(r - a_1)(r - a_2) \ldots (r - a_{2n}) - (-1)^n (n!)^2 = 0. \quad (1)$$

From (1), it is easy to see that r is not equal to any of the $\{a_i\}_{i=1}^{2n}$ numbers,

$$r \neq a_i,\ \forall i = 1, 2, \ldots, 2n \quad (*)$$

Also from (1), the product of all $\prod_{i=1}^{2n}(r - a_i)$ has the same absolute value as $(n!)^2$,

$$|(r - a_1)(r - a_2) \ldots (r - a_{2n})| = (n!)^2 \quad (**)$$

Now by (*), the numbers $r - a_1, r - a_2, \ldots, r - a_{2n}$ are all distinct, thus the absolute value of their products

$$|(r - a_1)(r - a_2) \ldots (r - a_{2n})| \geq |(1)(2) \ldots (n)(-1)(-2) \ldots (-n)| = (n!)^2 \quad (***)$$

Because of (**), the equality in (***) stands, so the two sets below are the same,

$$\{r - a_1, r - a_2, \ldots, r - a_{2n}\} = \{-n, -(n-1), \ldots, -1, 1, 2, \ldots, n\}$$

Therefore

$$\frac{(r - a_1) + (r - a_2) + (r - a_{2n})}{2n} = 0 \Rightarrow 2nr - \sum_{k=1}^{2n} a_k = 0 \Rightarrow r = \boxed{\frac{\sum_{k=1}^{2n} a_k}{2n}}$$

$\square$

Example 1.3.10 (OT-22-23-S1-P10)
Let m and n be relatively prime postive integers $\gcd(m,n)=1$.

Find all possible values of $\gcd(5^m+7^m, 5^n+7^n)$.

Remark (Polynomial Division of a^m+b^m by a^n+b^n $(m>n)$). Often needed,

$$a^m+b^m=(a^n+b^n)(a^{m-n}+b^{m-n})-a^nb^{m-n}-a^{m-n}b^n.$$
$$\text{e.g. } a^4+b^4=(a^3+b^3)(a+b)-a^3b-ab^3 \quad a^5+b^5=(a+b)(a^4+b^4)-a^4b-ab^4$$

Solution. Let assume that $m>n$. Perform a division of 5^m+7^m by 5^n+7^n,

$$5^m+7^m=(5^n+7^n)(5^{m-n}+7^{m-n})-5^n7^{m-n}-5^{m-n}7^n \quad (*)$$

Case 1: $m<2n$, then by factoring out $5^{m-n}7^{m-n}$ from the last two terms of (*),

$$\begin{aligned}5^m+7^m&=(5^n+7^n)(5^{m-n}+7^{m-n})-5^{m-n}7^{m-n}(5^{2n-m}+7^{2n-m})\\ \Rightarrow \gcd(5^m+7^m,5^n+7^n)&=\gcd(5^n+7^n,5^{2n-m}+7^{2n-m})\end{aligned}$$

Case 2: $m>2n$, then by factoring out 5^n7^n from the last two terms of (*),

$$\begin{aligned}5^m+7^m&=(5^n+7^n)(5^{m-n}+7^{m-n})-5^n7^n(5^{m-2n}+7^{m-2n})\\ \Rightarrow \gcd(5^m+7^m,5^n+7^n)&=\gcd(5^n+7^n,5^{m-2n}+7^{m-2n})\end{aligned}$$

Note that $m\neq 2n$ because $\gcd(m,n)=1$. Now, let $a_{m,n}=\gcd(5^m+7^m,5^n+7^n)$, then if $m>n$,

$$a_{m,n}=\begin{cases}a_{n,2n-m} & \text{if } m<2n,\\ a_{n,m-2n} & \text{if } m>2n,\end{cases}$$

Hence, by starting with a pair (m,n) and descend to $(n,2n-m)$ if $m<2n$ or to $(n,m-2n)$ if $m>2n$, the sum of the exponents $m+n$ is decreasing $m+n>n+(2n-m)$, and $m+n>n+(m-2n)$. Since m and n are relatively prime and the process is invariant modulo 2, So

Case 1: $m+n$ is odd, then $a_{m,n}=a_{0,1}=\gcd(5^0+7^0,5^1+7^1)=\boxed{2.}$

Case 2: $m+n$ is even, then $a_{m,n}=a_{1,1}=\gcd(5^1+7^1,5^1+7^1)=\boxed{12.}$

Thus

$$\gcd(5^m+7^m,5^n+7^n)=\begin{cases}2 \text{ if } 2\nmid m+n,\\ 12 \text{ if } 2\mid m+n\end{cases}$$

□

Remark. Such method like this

$$a_{m,n}=\begin{cases}a_{n,2n-m} & \text{if } m<2n,\\ a_{n,m-2n} & \text{if } m>2n,\end{cases}$$

and similar methods based on reduction of positive integers until reach irreducible states (in this case $(0,1)$ or $(1,1)$) is an effective one to find an **invariant**, in this case the common divisors.

Note that, with some assumption if the descent can be continued forever, then it leads to a contradiction to the assumption. This famous method, called *infinite descent*, first used by Euclid, later Fermat while proving *Fermat's theorem on sums of two squares.*

Chapter 2

The Principle of Inclusion-Exclusion (PIE)

2.1 Sets - Concepts and Definitions

Roughly speaking, a **set** is a collection of objects. The objects can be anything: numbers, functions, other sets, any combination of these, or nothing at all. *The order of the objects in the set is unimportant.* All that matters is what objects are in the set.

There might only be a finite number of objects in the set in which case the set is called a **finite set.** Otherwise we call it an **infinite set.** The objects in the set are called the **elements** or **members** of the set.

$\mathbb{R}$ denotes the set of all real numbers, $\mathbb{Z}$ the set of all integers and $\mathbb{Z}^+$ the set of all positive integers.

Definition (Universal quantifier)**.** The universal quantifier is the equivalent of the words **for all** or **every.** It is sometimes denoted by $\forall$.

For example, we can write the (false) statement "All integers are divisible by 2" as

$$\forall x \in \mathbb{Z},\ 2 \mid x.$$

In English, we would read this as "For all elements x of the set of integers, 2 divides x." This means, naturally, that 2 divides every element of $\mathbb{Z}$, the integers. Of course, this is a false statement, but it is a legal statement nonetheless.

Definition (Existential quantifier)**.** The existential quantifier is the equivalent of the words **there exists.** It is sometimes denoted by $\exists$.

For example, we can write the statement "There exists an integer x such that $x^2 - 5x + 6 = 0$" as

$$\exists x \in \mathbb{Z},\ x^2 - 5x + 6 = 0.$$

In English, this reads "There exists an element x in the set of integers such that $x^2 - 5x + 6 = 0$." This means that there is some element of $\mathbb{Z}$ that makes the statement true.

Definition (Element of a set)**.** If x is an element of A, we write $x \in A$, otherwise $x \notin A$.

For example,

- $A = \{2, 3, 9\}$ *is a set of three elements* $2, 3,$ *and* 9.
- $B = \{x \mid x \textit{ is integer } \}$ *is a set of all integers,*
- $C = \{x \mid x \in \mathbb{R},\ 2 < x < 3\}$ *is a set of all real numbers between* 2 *and* 3*, excluding* 2 *and* 3.

Definition (Cardinality)**.** For a set S, $|S|$ denotes the **cardinality** of S, in other words the number of elements in S, *for example* $|\{1, 3, 5, 7\}| = 4$. The **empty** set is denoted by $\emptyset$.

Definition (Subsets)**.** A set A is called **subset** of a set B, if every element of A is also an element of B. The notation is $A \subseteq B$ (in this case A is a subset or might be the same as B) Formally,

$$A \subseteq B \Leftrightarrow \forall x \in A \Rightarrow x \in B.$$

For example

$$\{1, 2\} \subseteq \{1, 2\}$$
$$\{1, 2, 3\} \subset \{-1, 1, 2, 3, 4\}$$
$$\{x \in \mathbb{Z}^+ \mid n \textit{ is multiple of 6 } \} \subset \{x \in \mathbb{Z}^+ \mid n \textit{ is multiple of 2 } \}$$

Definition (Proper Subsets)**.** $A \subset B$ is A is a **proper subset** of B, meaning A cannot be equal to B. If A is not a subset of B, we write $A \not\subset B$.

Definition (Supersets)**.** If $A \subseteq B$, then B is a **superset** of A.

Definition (Equal sets are the same set)**.** If A and B are the same set, meaning if they are both emptysets or if they contain the same elements, then $A = B$.

Theorem (Properties of Subsets)

and C are sets,

- $A \subseteq A$ for any set A;
- $\emptyset \subseteq A$ for any set A;
- For sets A and B, if $A \subseteq B$, and $B \subseteq A$, then $A = B$.
- For sets A and B, both $A \subset B$, and $B \subset A$, cannot happen.
- For sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Definition (Power Set)**.** For every set S, $\mathcal{P}(S)$ is called the **power set** of S, where

$$\mathcal{P}(S) = \{T \mid T \subseteq S\}.$$

In other words, $\mathcal{P}(S)$ contains all subsets of S (including S itself.)

For example $\mathcal{P}(\emptyset) = \{\emptyset\}$, $\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$, *and* $\mathcal{P}(\{a, B\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Theorem (Cantor's theorem)

For any set A, $|A| < |\mathcal{P}(A)|$.

The real surprise here is that in case $|A|$ is infinite, for example $A = \mathbb{Z}$, then the cardinalty of $\mathcal{P}(A)$ is *strictly larger* then the cardinality of A. How can an infinity be strictly larger than the other infinity? If you finds it is interesting, check this out `https://en.wikipedia.org/wiki/Cantor%27s_theorem`.

2.2 Sets - Operations

Definition (Union of sets)**.** The union $A \cup B$ of two sets A and B is the set of all objects that are elements of at least one of A and B. Formally

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Definition (Intersection of sets)**.** The intersection $A \cap B$ of two sets A and B is the set of all objects that are elements of both of A and B. Formally

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Definition (Disjoint sets)**.** If $A \cap B = \emptyset$, then we say that A and B are disjoint.

Example 2.2.1 (OT-22-23-S2-E1)

Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Solution. We prove the two sets are the same by using the definition

$$A = B \text{ is equivalent to } \forall x \in A \Rightarrow x \in B \text{ and } \forall x \in B \Rightarrow x \in A.$$

By equivalent statements,

$$\begin{aligned}
\forall x \in A \cap (B \cup C) &\Leftrightarrow \begin{cases} x \in A \\ x \in (B \cup C) \end{cases} \\
&\Leftrightarrow \begin{cases} x \in A \\ x \in B \text{ or } x \in C \end{cases} \\
&\Leftrightarrow \text{ or } \left[\begin{array}{l} x \in A \text{ and } x \in B \\ x \in A \text{ and } x \in C \end{array}\right. \\
&\Leftrightarrow \text{ or } \left[\begin{array}{l} x \in A \cap B \\ x \in A \cap C \end{array}\right. \\
&\Leftrightarrow x \in (A \cap B) \cup (A \cap C)
\end{aligned}$$

Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

Definition (Complement sets)**.** The complement of a set A with respect to a universal set U,

$$\overline{A} = U \setminus A = \{x \in U \mid x \notin A\}.$$

Definition (Set-theoretic Difference)**.** The set-theoretic difference of $A \setminus B$

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

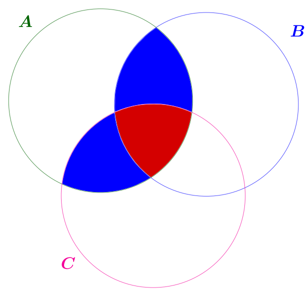
Lemma ($|A \cap (B \cup C)|$)

If A, B, and C are three sets, then

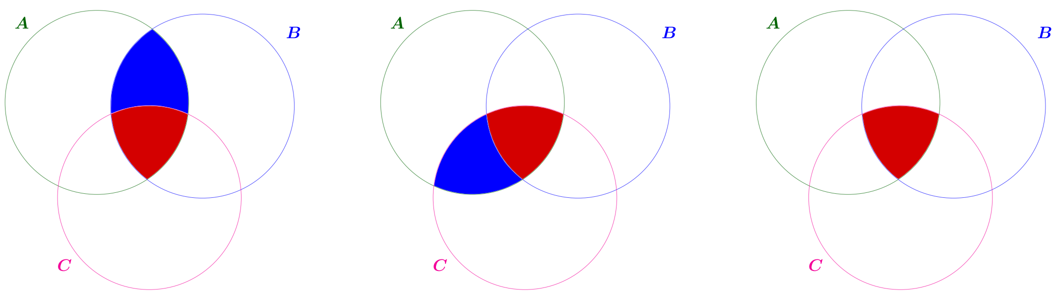
$$|A \cap (B \cup C)| = |A \cap B| + |A \cap C| - |A \cap B \cap C|.$$

Proof. We give a proof based on Venn-diagram and counting principles.

Consider three sets A, B, C, the cardinal of $|A \cap (B \cup C)|$, as shown in the diagram below, contains two blue regions and a red region.



This cardinality can be computed base on the carinalities of $|A \cap B|$, $|A \cap C|$, and $A \cap B \cap C|$ as below.



$$|A \cap (B \cup C)| = |A \cap B| + |A \cap C| - |A \cap B \cap C|$$

□

The need to find the cardinality of an union: let take an example.

Theorem (The number of primes between 1 and n)

Let $p_1 = 2 < p_2 = 3 < \ldots < p_k \leq \lfloor \sqrt{n} \rfloor$ be prime numbers not exceeding $\lfloor \sqrt{n} \rfloor$. Let A_i, $(i = 1, 2, \ldots, k)$ be the set of multiples of p_i not exceeding n, then the number of prime numbers between 1 and n is

$$n + k - 1 - \left| \bigcup_{i=1}^{k} A_i \right|.$$

So, how to we compute $\left| \bigcup_{i=1}^{k} A_i \right|$?

2.3 The Principle of Inclusion-Exclusion

Theorem (The Principle Inclusion-Exclusion for two/three sets)
If A and B are two sets, then

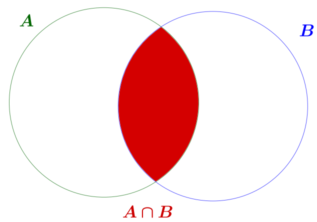
$$|A \cup B| = |A| + |B| - |A \cap B|.$$

If A, B, and C are three sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Proof. We first show the proof for two sets A and B, and then for three sets A, B and C.

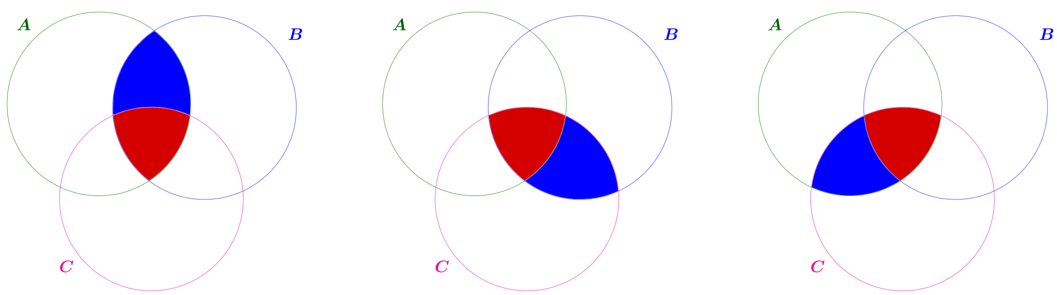
The Venn-diagram below describes sets A, B, and their intersection $A \cap B$, coloured red.



It is easy to see that when *counting* the elements of set A ($|A|$) and then of set B ($|B|$) we *overcount* each of the element of the intersection $A \cap B$ ($|A \cap B|$) *twice.* Thus, the number of elements of the union $A \cup B$

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Now, if we consider the three sets A, B, and C. Following the same reasoning, if we add up the elements in A, B, and C, then each element in the intersections $A \cap B$, $B \cap C$, and $C \cap A$ is counted *twice.*



To be precise, each of the blue regions is counted *twice,* and the red region is counted *thrice.* Thus, if we remove the numbers of the elements of the intersections $A \cap B$, $B \cap C$, and $C \cap A$, from the sum of $|A| + |B| + |C|$, then in the number

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \quad (*)$$

each of elements in the blue regions is counted exactly once (which is good), but none of the elements in the red region is not counted. Therefore by adding to (*) the number of the elements in the red region $A \cap B \cap C$, each of the elements in A, B, and C now counted exactly once, thus

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

□

Definition (Induction Principle)**.** Let a be an integer, and let $P(n)$ be a proposition (or a statement)) about n for each integer $n \geq a$. The **principle of induction** is a way of proving that $P(n)$ is true for all integers $n \geq a$ in two steps:

1. *The base case:* we have to explicitly prove that $P(a)$ is true.
2. *The inductive step:* assume that $P(k)$ is true for some integer $k \geq a$, and using that assumption to prove that $P(k+1)$ is true.

Then we may conclude that $P(n)$ is true for all integers $n \geq a$.

Theorem (Principle of Inclusion-Exclusion)

If A_1, A_2, ..., A_n are sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = (|A_1| + \ldots + |A_n|) - (|A_1 \cap A_2| + \ldots + |A_{n-1} \cap A_n|) + (-1)^{n+1}|A_1 \cap \cdots \cap A_n| \quad (*)$$

Proof. We use the Principle of Inclusion-Exclusion to prove (*). The base cases for $n = 2$ and $n = 3$ have been established previously. For the inductive step, let assume that for any n sets A_1, A_2, ..., A_n, we have

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = (|A_1| + \ldots + |A_n|) - (|A_1 \cap A_2| + \ldots + |A_{n-1} \cap A_n|) + (-1)^{n+1}|A_1 \cap \cdots \cap A_n| \quad (*)$$

Then, we have to prove that for any $n+1$ sets $A_1, A_2, \ldots, A_{n+1}$,

$$|A_1 \cup A_2 \cdots A_{n+1}| = (|A_1| + \ldots + |A_{n+1}|) - (|A_1 \cap A_2| + \ldots + |A_n \cap A_{n+1}|) + (-1)^{n+1}|A_1 \cap \cdots \cap A_{n+1}| \quad (**)$$

Remark. Although the proof for this inductive step is not complicated as you might find, the way the indexes of the sets are dealt with is not exactly simple. That's why we show here the inductive step going the $n = 3$ case to the $n = 4$ case, so that it would be easier for you to understand how it works from n to $n+1$.

For any 3 sets A_1, A_2, A_3, we have

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3| \quad (1)$$

Then, we have to prove that for any 4 sets A_1, A_2, A_3, A_4,

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\ & - |A_1 \cap A_2 \cap A_3 \cap A_4| \quad (2) \end{aligned}$$

Now, it is easy to see that $A_1 \cup A_2 \cup A_3 \cup A_4 = A_1 \cup A_2 \cup (A_3 \cup A_4)$, and the right-hand side is actually an union of 3 sets A_1, A_2, and $(A_3 \cup A_4)$. By applying (1),

$$(3) \quad \begin{cases} |A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1 \cup A_2 \cup (A_3 \cup A_4)| = & \\ + |A_1| + |A_2| + |(A_3 \cup A_4)| & (\to 1^{\text{st}} \text{ sum}) \\ - (|A_1 \cap A_2| + |A_2 \cap (A_3 \cup A_4)| + |(A_3 \cup A_4) \cap A_1|) & (\to 2^{\text{nd}} \text{ sum}) \\ + |A_1 \cap A_2 \cap (A_3 \cup A_4)| \quad (3) & (\to 3^{\text{rd}} \text{ sum}) \end{cases}$$

Lets take a look at the 1^{st}, 2^{nd}, and 3^{rd} sums. First, by the Principle of Inclusion-Exclusion for $n = 2$,

$$\begin{aligned} |A_3 \cup A_4| &= |A_3| + |A_4| - |A_3 \cap A_4| \\ (1^{\text{st}} \text{ sum}) : |A_1| + |A_2| + |(A_3 \cup A_4)| &= |A_1| + |A_2| + |A_3| + |A_4| - |A_3 \cap A_4| \end{aligned}$$

Second, by the lemma $|A \cap (B \cup C)|$,

$$\begin{aligned}
&|A_2 \cap (A_3 \cup A_4)| = |A_2 \cap B_3| + |A_2 \cap A_4| - |A_2 \cap A_3 \cap A_3| \\
&|(A_3 \cup A_4) \cap A_1| = |A_1 \cap (A_3 \cup A_4)| = |A_1 \cap B_3| + |A_1 \cap A_4| - |A_1 \cap A_3 \cap A_4| \\
(2^{\text{nd}}\ \text{sum}) :\ &|A_1 \cap A_2| + |A_2 \cap (A_3 \cup A_4)| + |(A_3 \cup A_4) \cap A_1| \\
&= |A_1 \cap A_2| + |A_2 \cap B_3| + |A_2 \cap A_4| + |A_1 \cap B_3| + |A_1 \cap A_4| \\
&- |A_2 \cap A_3 \cap A_3| - |A_1 \cap A_3 \cap A_4|
\end{aligned}$$

Third, by the same lemma $|A \cap (B \cup C)|$,

$$(3^{\text{rd}}\ \text{sum}) :\ |A_1 \cap A_2 \cap (A_3 \cup A_4)| = |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

Finally, by substituting the three sums in new forms into (3), we receive (2), which is what we want to prove. Now let continue with the generic inductive step.

Note that $A_1 \cup A_2 \cup \cdots \cup A_{n+1}| = |A_1 \cup A_2 \cdots A_{n-1} \cup (|A_n \cup A_{n+1})|$. By applying (*) for n sets $A_1, A_2, \ldots, A_{n-1}, (A_n \cup A_{n+1})$. (the n^{th} set is $A_n \cup A_{n+1}$,) the cardinality $|A_1 \cup A_2 \cup \cdots \cup A_{n+1}|$ is the sum

$$(1) \quad \begin{cases} + \sum_{i=1}^{n-1} |A_i| + |A_n \cup A_{n+1}| & (\to 1^{\text{st}}\ \text{sum}) \\ - \sum_{1 \le i \le j \le n-1} |A_i \cap A_j| + \sum_{i=1}^{n-1} |A_i \cap (A_n \cup A_{n+1})| & (\to 2^{\text{nd}}\ \text{sum}) \\ \ldots & \\ + (-1)^{n+2} \left| \left(\bigcap_{i=1}^{n-1} A_i \right) \cap (A_n \cup A_{n+1}) \right| & (\to (n+1)^{\text{th}}\ \text{sum}) \end{cases}$$

The 1^{st} sum of (1),

$$\sum_{i=1}^{n-1} |A_i| + |A_n \cup A_{n+1}| = \sum_{i=1}^{n-1} |A_i| + |A_n| + |A_{n+1}| - |A_n \cap A_{n+1}| = \underbrace{\sum_{i=1}^{n+1} |A_i|}_{\text{the } 1^{\text{st}} \text{ first term of (**)}} \quad \underbrace{-|A_n \cap A_{n+1}|}_{\text{extra in } 1^{\text{st}}}$$

Note that in the 2^{nd} sum of (1), $\forall i :\ |A_i \cap (A_n \cup C)| = |A_i \cap A_n| + |A_i \cap A_{n+1}| - |A_i \cap A_n \cap A_{n+1}|$.

Thus 2^{nd} sum of (1), after added the extra sum $-|A_n \cap A_{n+1}|$ from 1^{st}, becomes

$$- \sum_{1 \le i \le j \le n-1} |A_i \cap A_j| - \sum_{i=1}^{n-1} |A_i \cap (A_n \cup C)| - |A_n \cap A_{n+1}| = \underbrace{- \sum_{1 \le i \le j \le n+1} |A_i \cap A_j|}_{\text{the } 2^{\text{nd}} \text{ term of (**)}} + \underbrace{\sum_{i=1}^{n-1} |A_i \cap A_n \cap A_{n+1}|}_{\text{extra in } 2^{\text{nd}}}$$

By continuing doing so the k^{th} sum of (1) $(k = 1, 2, \ldots, n+1)$,

$$\underbrace{(-1)^{k+1} \sum_{1 \le i_1 \ldots \le i_k \le n+1} |\bigcap_{\ell=1}^{k} A_{i_\ell}|}_{\text{the } k^{\text{th}} \text{ term of (**)}} + \underbrace{\sum_{1 \le j_1 \ldots \le j_{k-1} \le n-1} \left| \left(\bigcap_{\ell=1}^{k} A_{j_\ell} \right) \cap A_n \cap A_{n+1} \right|}_{\text{extra in } k^{\text{th}}}$$

The last extra sum, when $k = n+1$, is actually the last ($(n+1)^{\text{th}}$) term of (**).

Thus, the right-hand side of (1) is a sum of the same terms of the right-hand sides of (**). Hence (**) stands. The proof is complete. □

Example 2.3.1 (OT-22-23-S2-E2)

How many positive integers less than 180 are relatively prime to 180?

Remark. This is the application of the Principle of Inclusion-Exclusion out of the box. Since $180 = 2^2 \cdot 3^2 \cdot 5$, thus if a number is relatively prime to 180, then it is *not* divisible by 180.

We can consider two approaches: (i) *direct counting:* find all the sets, each contains integers that are relatively primes to a divisor of 180, then use Principle of Inclusion-Exclusion to count the sum of their union. or (i) *complementary counting:* count the integers that are not relatively primes to 180, in other words, not divisible by $2, 3$, or 5, and then use Principle of Inclusion-Exclusion to count the sum of their unions, then subtract that total from 180.

The first approach requires requires to deal with a number of sets equivalent to the number of divisors of 180, which is much larger then the second approach, which deals with only three prime divisorr of 180.

Solution. Let $S_i = \{n \in \mathbb{Z}^+ \mid n \text{ is divisible by i}\}$, where $i = 2, 3, 5$. then we need to find is $|S_2 \cup S_3 \cup S_5|$. By Principle of Inclusion-Exclusion,

$$\begin{aligned}
&|S_2 \cup S_3 \cup S_5| = |S_2| + |S_3| + |S_5| - |S_2 \cap S_3| - |S_2 \cap S - 5| - |S_3 \cap S - 5| + |S_2 \cap S_3 \cap S_5| \\
&= \frac{180}{2} + \frac{180}{3} + \frac{180}{5} - \frac{180}{6} - \frac{180}{10} - \frac{180}{15} + \frac{180}{30} = 90 + 60 + 36 - 30 - 18 - 12 + 6 = 132
\end{aligned}$$

Thus, the number of positive integers less than 180 are relatively prime to 180 is $180 - 132 = \boxed{48.}$ □

By following the same method, we can prove the formula for Euler's Totient Function $\varphi(n)$.

Theorem (Formula for $\varphi(n)$)

For $n \in \mathbb{Z}^+$, let $\varphi(n)$ be the number of positive integers less than n that are coprime to n.

Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of n. Prove that,

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right) = p_1^{a_1-1} p_2^{a_2-1} \cdots p_k^{a_k-1}(p_1 - 1)(p_2 - 1)\cdots(p_k - 1).$$

Proof. Let $A = \{1, 2, \ldots, n\}$. Let A_i $(i = 1, 2, \ldots, k)$ be the subset of A containing all multiples of p_i. Then, by the Principle of Inclusion-Exclusion

$$\varphi(n) = n - |\bigcup_{i=1}^{k} A_i| = n + \sum_{I \subseteq A, I \neq \emptyset} (-1)^{|I|+1} \left|\bigcap_{j \in I} A_j\right|.$$

Since $\bigcap_{j \in I} A_j$ contaning all numbers in A that are divisible by $\prod_{j \in I} A_j$, hence $\left|\bigcap_{j \in I} A_j\right| = n \prod_{j \in I} \frac{1}{p_j}$. Therefore, in the same process as in the solution of OT-22-23-S2-E2,

$$\varphi(n) = n\left[1 + \sum_{I \subseteq A, I \neq \emptyset} (-1)^{|I|+1} \prod_{j \in I} \frac{1}{p_j}\right] = n \prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right).$$

□

Here is another example of straight application of the Principle of Inclusion-Exclusion, abeit with a little of attention to details.

Example 2.3.2 (OT-22-23-S2-E3)

How many 6-digit binary numbers (numbers with 0's and 1's as digits) have a string of three consecutive 1's appearing in them? (For example, 101110 and 111100 both have a string of three consecutive 1's, but 100110 doesn't).

Solution. There are four possible positions for a run of three ones: 111???, ?111??, ??111?, and ???111. Let

$$A_1 = \{111???\},\ A_2 = \{?111??\},\ A_3 = \{??111?\},\ A_4 = \{???111\}.$$

Note that every 6-digit number begins with a 1, so actually

$$A_1 = \{111???\},\ A_2 = \{1111??\},\ A_3 = \{1?111?\},\ A_4 = \{1??111\}.$$

Now

$$\begin{aligned}
&A_1 \cap A_2 = \{1111??\},\ A_1 \cap A_3 = \{11111?\},\ A_1 \cap A_4 = \{111111\} \\
&A_2 \cap A_3 = \{11111?\},\ A_2 \cap A_4 = \{111111\}, A_3 \cap A_4 = \{1?1111\}, \\
&A_1 \cap A_2 \cap A_3 = \{11111?\},\ A_1 \cap A_2 \cap A_4 = \{111111\}, \\
&A_1 \cap A_3 \cap A_4 = \{111111\},\ A_2 \cap A_3 \cap A_4 = \{111111\}, \\
&A_1 \cap \cap A_2 \cap A_3 \cap A_4 = \{111111\}
\end{aligned}$$

By Principle of Inclusion-Exclusion,

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = (8 + 4 + 4 + 4) - (4 + 2 + 1 + 2 + 1 + 2) + (2 + 1 + 1 + 1) - 1 = \boxed{12}.$$

It is also pretty easy to list them: $100111, 101110, 101111, 110111$, and all 8 numbers of the form 111???. $\square$

Theorem (Intersection of Complement Sets)

If A_1, A_2, …, A_n are subsets of S, then the intersection of their complements,

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |S| - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

Proof. Note that $\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} = \overline{A_1 \cup A_2 \cup \cdots \cup A_n}$. For any $A \subseteq S$, $|A| + |\overline{A}| = |S|$. The rest follows. $\square$

Example 2.3.3 (OT-22-23-S2-E4)

Compute the number of positive integers less than 1000 and divisible by neither 5 nor 7.

Remark. Find the formula for the number of positive integers less than 1000 and divisible by a prime p is simple. It is $\left\lfloor \frac{1000}{p} \right\rfloor$. Thus the formula for the number of positive integers less than 1000 and *not* divisible by a prime p is $1000 - \left\lfloor \frac{1000}{p} \right\rfloor$. However, when the number of dividends is increased, for example the formula for the number of positive integers less than 1000 and divisible by two prime p and q is still $\left\lfloor \frac{1000}{pq} \right\rfloor$, but the the formula for the number of positive integers less than 1000 and *neither* divisible by p *nor* by q is not that simple anymore.

To find the number of positive integers less than 1000 and *neither* divisible by p *nor* by q, we need to consider the application of Intersection of Complement Sets and find the number of positive integers less than 1000 and divisible by p *or* by q by the Principle of Inclusion-Exclusion.

Hence, when you see such a statement like *find the number of elements that neither X nor Y* then think about Intersection of Complement Sets.

Solution. Let $S = \{1, 2, \ldots, 999\}$, $A_5 = \{k \in S \mid k \text{ is divisible by } 5\}$, similar we define A_7, then

$$|\overline{A_5} \cap \overline{A_7}| = |S| - |A_5 \cup A_7| = |S| - |A_5| - |A_7| + |A_5 \cap A_7|.$$

The rest is simple. $\square$

Example 2.3.4 (OT-22-23-S2-E5)

How many 6-digit numbers, written in decimal notation, have *at least* one 1, one 2, and one 3 among their digits?

Solution. Let S be the set of all 6-digit numbers.

Let A_1, A_2, and A_3 be subsets S, whose elements *do not contain* the digits 1, 2, and 3, respectively.

Then we need to find the value $s = |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$. By the Intersection of Complement Sets theorem,

$$s = |S| - |A_1 \cup A_2 \cup A_3| = |S| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

First we count $|A_1|$, the number of 6-digit numbers that do not contain the digit 1. The first digit has 8 possible values (2 through 9), and each of the remaining 5 digits has 9 possible values, so $|A_1| = 8 \cdot 9^5$. Similarly, $|A_2| = |A_3| = 8 \cdot 9^5$.

Next we count $|A_1 \cap A_2|$, the number of 6-digit numbers that do not contain the digit 1 or 2. The first digit has 7 possible values (3 through 9), and each of the remaining 5 digits has 8 possible values, so $|A_1 \cap A_2| = 7 \cdot 8^5$. Similarly, $|A_1 \cap A_3| = |A_2 \cap A_3| = 7 \cdot 8^5$.

Finally, we count $|A_1 \cap A_2 \cap A_3|$, the number of 6-digit numbers that do not contain the digits 1, 2, or 3. The first digit has 6 possible values (4 through 9), and each of the remaining 5 digits has 7 possible values, so $|A_1 \cap A_2 \cap A_3| = 6 \cdot 7^5$.

Thus, $s = 900,000 - 3 \cdot 8 \cdot 9^5 + 3 \cdot 7 \cdot 8^5 - 6 \cdot 7^5 = \boxed{70,110}$. $\square$

Remark. When you have to deal with counting with some conditions containing *at least* or *at most*, it is very probably that Principle of Inclusion-Exclusion is the right tool. How so? Let consider the example below.

Example 2.3.5 (OT-22-23-S2-E6)

In how many ways can we give five candies to three kids so each kid must have *at least* one candy?

Solution. Let S be the set of all cases.

$\boxed{S_i \text{ be the set of the cases when the kid } i\ (i = A, B, C) \text{ have } \textit{no candies.}}$

Then what we look for is the cardinality of $\overline{S_A} \cap \overline{S_B} \cap \overline{S_C}$.

Now, each candy can go to any of the three kids, so the total possible cases is $|S| = 3^5 = 243$.

If the kid, say A, receives no candies, then each candy can go to any of the two remaining kids, B and C, or $|S_A| = 2^5 = 32$. Similarly $|S_B| = |S_C| = 32$.

But if any of the two kids A and B receive no candies, then kid C receives all candies, and there only one way to do it. It means that $|S_A \cap S_B| = 1$. Similarly $|S_B \cap S_C| = |S_C \cap S_A| = 1$.

It is simple to verify that $|S_A \cap S_B \cap S_C| = 0$.

$$\begin{aligned}
|\overline{S_A} \cap \overline{S_B} \cap \overline{S_C}| &= |S| - |S_A \cup S_B \cup S_C| \\
&= |S| - |S_A| - |S_B| - |S_C| + |S_A \cap S_B| + |S_B \cap S_C| + |S_C \cap S_A| - |S_A \cap S_B \cap S_C| \\
&= 243 - 3 \cdot 32 + 3 \cdot 1 - 0 = \boxed{150}.
\end{aligned}$$

□

Remark. Here we can try to do casework (i) either two kids each have two candies and the other has one, or (ii) two kids each have one candies, and the other has three. However, each problem has a different type of casework. The PIE approach seems straight and the result can logically be verified.

Example 2.3.6 (OT-22-23-S2-E7)

In how many ways can we arrange the letters in the word OTTAWA if *no two of the same letter* may appear consecutively in an arrangement?

Solution. Let S be the set of all cases. Since we have two same letters T and two same letters A, $|S| = \frac{6!}{2!2!}$.

Now let $\boxed{S_T \text{ be the set of cases when } \textit{two letter letters } T \textit{ can appear consecutively.}}$

Since we can make a *block* of the two letters, thus there are 5 letters remain with two same letter A, and $|S_T| = \frac{5!}{2!}$. Similarly $|S_A| = \frac{5!}{2!}$.

Next, the set $S_T \cap S_A$ has $4!$ cases since we make two *block* letters from two T and two A letters.

Hence the desired number of cases

$$|\overline{S_T} \cap \overline{S_A}| = |S| - |S_A \cup S_T| = |S| - |S_A| - |S_T| + |S_A \cap S_T| = \frac{6!}{2!2!} - (2)\frac{5!}{2!} + 4! = \boxed{84}.$$

□

Example 2.3.7 (OT-22-23-S2-E8)

Each unit square of a 3-by-3 unit-square grid is to be colored either blue or red. For each square, either color is equally likely to be used. Find the probability of obtaining a grid that does not have a 2-by-2 red square.

Remark. All counting problems are ... counting problems. The counting principles applied all the same. The question is how to recognize which counting principle can you use. Casework is probably the most *tempting* one since it gets you started in notime. However if you recognize that casework division is not that clear (we want independent and distinct cases.) Then you can stop a bit and think what else can you do.

In this problem, the question is *the union of the sets where the gird contains a* $2-by-2$ *red square?* How can we define each of the sets? How do they intersect? If we can answer these questions, then Principle of Inclusion-Exclusion is the right counting method.

Solution. In below, $p(E)$ denotes the probability that event E occur. Let number the squares as show below.

1	2	3
4	5	6
7	8	9

For $i = 1, 2, 4,$ and 5, let $\boxed{Q_i \text{ be the event that } i \text{ is } \textit{the upper left corner 2-by-2 red square,}}$

Now

$$p(Q_1) = p(Q_2) = p(Q_4) = p(Q_5) = \left(\frac{1}{2}\right)^4 \quad \text{(4 squares can be coloured red.)}$$

$$p(Q_1 \cap Q_2) = p(Q_1 \cap Q_4) = p(Q_2 \cap Q_5) = p(Q_4 \cap Q_5) = \left(\frac{1}{2}\right)^6$$

$$p(Q_1 \cap Q_5) = p(Q_2 \cap Q_4) = \left(\frac{1}{2}\right)^7$$

$$p(Q_1 \cap Q_2 \cap Q_4) = \ldots = p(Q_2 \cap Q_4 \cap Q_5) = \left(\frac{1}{2}\right)^8$$

$$p(Q_1 \cap Q_2 \cap Q_4 \cap Q_5) = \left(\frac{1}{2}\right)^9$$

The probability that the grid has at least one 2-by-2 red square is equal to $p(Q_1 \cup Q_2 \cup Q_4 \cup Q_5)$.

By Principle of Inclusion-Exclusion,

$$\begin{aligned} p(Q_1 \cup Q_2 \cup Q_4 \cup Q_5) &= \sum p(Q_i) - \sum p(Q_i \cap Q_j) + \sum p(Q_i \cap Q_j \cap Q_k) - p(Q_1 \cap Q_2 \cap Q_4 \cap Q_5) \\ &= (4)\left(\frac{1}{2}\right)^4 - \left[(4)\left(\frac{1}{2}\right)^6 + (2)\left(\frac{1}{2}\right)^7\right] + (4)\left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^9 = \frac{95}{512}. \end{aligned}$$

Thus, the probability of obtaining a grid that does not have a 2-by-2 red square is $1 - \frac{95}{512} = \boxed{\frac{417}{512}}$. □

Example 2.3.8 (OT-22-23-S2-E9)

How many ways are there to seat 5 pairs of twins in a row of 10 chairs, so *no pair of twins sit together*?

Solution. $\boxed{\text{Let } A_i \text{ be the set arrangements such that the pair of twins } i\ (i = 1, 2, \ldots, 5) \text{ sit together.}}$

Then there are 9! ways to seat the twins (thought of as a block) and the other 8 people, then 2 ways to seat the twins within their block. Thus, $|A_i| = 2 \cdot 9!$, for $i = 1, 2, \ldots, 5$.

Now, any two pairs of twins, there are 8! ways to seat the two pairs and the other 6 people, and 2 ways to seat each of the two twins within each pair, so $|A_i \cap A_j| = 2^2 \cdot 8!$, for $1 \leq i < j \leq 5$.

Similarly, $|A_i \cap A_j \cap A_k| = 2^3 \cdot 7!$, for $1 \leq i < j < k \leq 5$; $|A_i \cap A_j \cap A_k \cap A_k| = 2^4 \cdot 6!$, for $1 \leq i < j < k < l \leq 5$; and $A_1 \cap A_2 \cap \ldots \cap A_5| = 2^5 \cdot 5!$. Thus

$$|A_1 \cup A_2 \cup \ldots \cup A_5| = \binom{5}{1} 2 \cdot 9! - \binom{5}{2} 2^2 \cdot 8! + \binom{5}{3} 2^3 \cdot 7! - \binom{5}{4} 2^4 \cdot 6! + \binom{5}{5} 2^5 \cdot 5! = 2,365,440.$$

The number of ways to seat them so that no pair of twins is together is $10! - 2,365,440 = \boxed{1,263,360.}$ □

Example 2.3.9 (OT-22-23-S2-E10)

An alphabet consists of the letters $a_1, a_2, \ldots, a_n$. Prove that the number of all words that contain each of these letters twice, but with *no consecutive identical letters,* is equal to

$$\frac{1}{2^n}\left[(2n)! - \binom{n}{1} 2(2n-1)! + \binom{n}{2} 2^(2n-2)! - \ldots + (-1)^n 2^n n!\right].$$

Solution. The number of such words without imposing the restriction about consecutive letters is

$$\frac{(2n)!}{(2!)^n} = \frac{(2n)!}{2^n}. \text{ (This is so because the identical letters can be permuted.)}$$

(*OT-22-23-S2-E10 seems to be a general case of OT-22-23-S2-E9, while in fact there is a suble difference: two twins can permute, but two identical letters not.*)

Now $\boxed{\text{let } A_i \text{ the number of words formed with the } n \text{ letters, each occurring twice,}}$ for which the two letters a_i appear next to each other. The answer to the problem is then

$$\frac{(2n)!}{2^n} - |A_1 \cup A_2 \cup \ldots \cup A_n|.$$

We evaluate $|A_1 \cup A_2 \cup \ldots \cup A_n|$ using the Principle of Inclusion-Exclusion. To this end, let us compute $|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$, for some indices $i_1, i_2, \ldots, i_k$, $k \leq n$. Collapse the consecutive letters a_{i_j}, $j = 1, 2, \ldots, k$. As such, we are, in fact, computing the number of words made of the letters $a_1, a_2, \ldots, a_n$ in which $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ appear once and all other letters appear twice. This number is clearly equal to

$$\frac{(2n-k)!}{2^{n-k}},$$

since such a word has $2n - k$ letters, and identical letters can be permuted. There are $\binom{n}{k}$ $k-$tuples $(i_1, i_2, \ldots, i_k)$. Thus,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_k \sum_{i_1, i_2, \ldots, i_k} (-1)^k |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| = \sum_k (-1)^{k-1} \binom{n}{k} \frac{(2n-k)!}{2^{n-k}}.$$

□

2.4 Problems

Remark. To prove two sets A and B are equivalent (or the same) $A = B$, then prove that

$$(\Rightarrow) \quad \forall x \in A \Rightarrow x \in B$$
$$(\Leftarrow) \quad \forall x \in B \Rightarrow x \in A$$

There is no secret sauce when proving two sets are equivalent. The only trickly issue might be if you can recognize the relations of an element to different sets, their intersections, or their unions.

Problem 2.4.1 (OT-22-23-S2-P1)**.** Suppose that B is a set such that $B \subseteq \emptyset$. What would be B?

Problem 2.4.2 (OT-22-23-S2-P2)**.** Prove that, if A, B, and C are three sets, then

$$|(A \cap B) \setminus C| + |(B \cap C) \setminus A| + |(C \cap A) \setminus B| = |A \cap B| + |B \cap C| + |C \cap A| - 3|A \cap B \cap C|$$

Problem 2.4.3 (OT-22-23-S2-P3)**.** Show that for any sets A, B, and C,

$$C \subseteq A \text{ if and only if } (A \cap B) \cup C = A \cap (B \cup C)$$

Problem 2.4.4 (OT-22-23-S2-P4)**.** $\mathcal{P}(S)$ is the power set of S.

- What is $\mathcal{P}(\mathcal{P}(\emptyset))$?
- Prove that if $|S| = n$, then $|\mathcal{P}(S)| = 2^n$.

Problem 2.4.5 (OT-22-23-S2-P5)**.** For sets S and T, prove that

- $\mathcal{P}(S) \cup \mathcal{P}(T) \subseteq \mathcal{P}(S \cup T)$.
- $\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T)$.

Problem 2.4.6 (OT-22-23-S2-P6)**.** Dogs in the GoodDog obedience school win a blue ribbon for learning how to sit, a green ribbon for learning how to roll over, and a white ribbon for learning how to stay. A new class of 180 dogs has enrolled in the GoodDog school. We have the following facts:

- An equal number of dogs have each of the ribbons.
- An equal number of dogs have each pair of ribbons.
- 15 dogs have all three ribbons.
- All dogs have at least one ribbon.
- The number of dogs with exactly one ribbon equals twice the number of dogs with more than one ribbon.

How many dogs have blue ribbons?

Remark. Venn-diagram for OT-22-23-S2-P6 can do wonder. However practicing intersections and unions can really help you to get better understanding or be better with tools.

Problem 2.4.7 (OT-22-23-S2-P7)**.** If A, B, and C are sets for which

$$|\mathcal{P}(A)| + |\mathcal{P}(B)| + |\mathcal{P}(C)| = |\mathcal{P}(A \cup B \cup C)| \text{ and } |A| = |B| = 100,$$

Then what is the maximal possible value of $|A \cap B \cap C|$?

Problem 2.4.8 (OT-22-23-S2-P8)**.** 15 students are each going to enroll in exactly one of economics, psychology, or sociology. In how many ways can they enroll, provided that no class is left empty?

> **Remark.** Application of Intersection of Complement Sets for OT-22-23-S2-P7? Let E, P, and S be the sets of enrollments where the economics class, psychology class, and sociology class is empty, respectively.

Problem 2.4.9 (OT-22-23-S2-P9)**.** Lan has a deck of cards consisting of the 2 through 5 of hearts and the 2 through 5 of spades. She deals two cards (at random) to each of four players. What is the probability that no player receives a pair?

> **Remark.** Straight application of Principle of Inclusion-Exclusion for OT-22-23-S2-P8? Let A_1, A_2, A_3, and A_4 be the sets of deals where the first, second, third, and fourth player receives a pair, respectively.

Problem 2.4.10 (OT-22-23-S2-P10)**.** *We say a graph contains a triangle if there exist three vertices A, B, and C, such that all pairs (A, B), (B, C), and (C, A) are connected by some edges.*

Given a graph with n vertices, prove that either it contains a triangle, or there exists a vertex that is the endpoint of at most $\left\lfloor \frac{n}{2} \right\rfloor$ edges.

> **Remark.** For an x vertices, let A_x be the set of vertices that are connected to x by some edges. Then what does the second condition say?
>
> Now assume that there is no such x so that the second condition satisfy, then the fact that there is a triangle in the graph meaning that there are two vertices x and y, such that the sum of the numbers of vertices that each x and y connected to must somehow *overlap*, resulting in a common vertex?

2.5 Solution

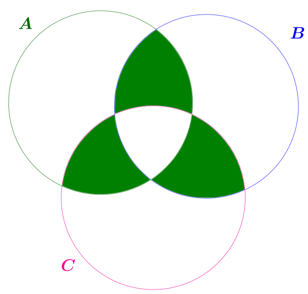
Example 2.5.1 (OT-22-23-S2-P1)
Suppose that B is a set such that $B \subseteq \emptyset$. What would be B?

Solution. We prove that $B = \emptyset$. Assume the opposite: $\exists x$, such that $x \in B$. Since $B \subseteq \emptyset$, $x \in B$, thus $x \in \emptyset$, which is impossible. Thus $B = \emptyset$. ☐

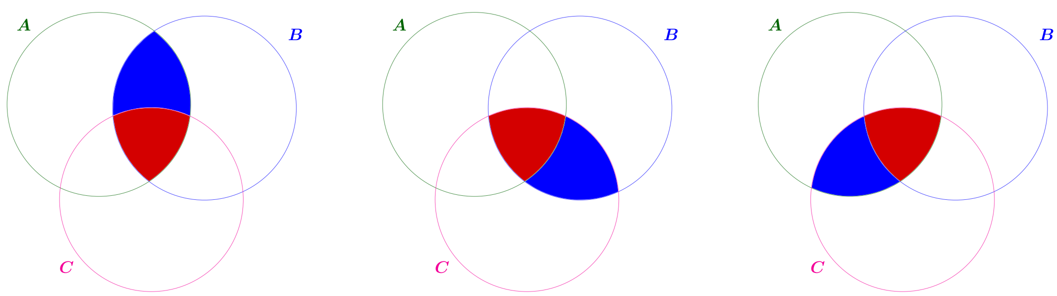
Example 2.5.2 (OT-22-23-S2-P2)
Prove that, if A, B, and C are three sets, then

$$|(A \cap B) \setminus C| + |(B \cap C) \setminus A| + |(C \cap A) \setminus B| = |A \cap B| + |B \cap C| + |C \cap A| - 3|A \cap B \cap C|$$

Solution. $|(A \cap B) \setminus C| + |(B \cap C) \setminus A| + |(C \cap A) \setminus B|$ is the cardinality of the three green regions.



which can easily be computed as $|A \cap B| + |B \cap C| + |C \cap A|$ removing *thrice* the red region $|A \cap B \cap C|$.



Thus,

$$|(A \cap B) \setminus C| + |(B \cap C) \setminus A| + |(C \cap A) \setminus B| = |A \cap B| + |B \cap C| + |C \cap A| - 3|A \cap B \cap C| \quad (*)$$

☐

Example 2.5.3 (OT-22-23-S2-P3)

Show that for any sets A, B, and C,

$$C \subseteq A \text{ if and only if } (A \cap B) \cup C = A \cap (B \cup C)$$

Solution. We prove both directions of the statement.

Claim — ($\Rightarrow$) First, we assume that $C \subseteq A$. We prove that $(A \cap B) \cup C = A \cap (B \cup C)$.

Proof. If $x \in (A \cap B) \cup C$, then $x \in A \cap B$ or $x \in C$.

If $x \in A \cap B$, then $x \in A$ and $x \in B$, so $x \in B \cup C$, thus $x \in A \cap (B \cup C)$. If $x \in C$, then $x \in B \cup C$, and $x \in A$ (since $C \subseteq A$) thus $x \in A \cap (B \cup C)$. In both cases $x \in A \cap (B \cup C)$.

Now, let $x \in A \cap (B \cup C)$, so $x \in A$ and $x \in B \cup C$, which implies that $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$, so $x \in (A \cap B) \cup C$. If $x \in C$, then $x \in (A \cap B) \cup C$. Either way, $x \in (A \cap B) \cup C$.

We conclude that if $C \subseteq A$, then $(A \cap B) \cup C = A \cap (B \cup C)$. ■

Claim — ($\Leftarrow$) Now, assume that $(A \cap B) \cup C = A \cap (B \cup C)$. We want to show that $C \subseteq A$.

Proof. Let $x \in C$. Then $x \in (A \cap B) \cup C$, so $x \in A \cap (B \cup C)$. It follows that $x \in A$. In other words, every element in C is also in A, so $C \subseteq A$. ■

□

Example 2.5.4 (OT-22-23-S2-P4)

$\mathcal{P}(S)$ is the power set of S.

- What is $\mathcal{P}(\mathcal{P}(\emptyset))$?
- Prove that if $|S| = n$, then $|\mathcal{P}(S)| = 2^n$.

Solution. For the first question, by definition, $\mathcal{P}(X)$ is the set of all subsets of X. The empty set has only one subset, namely itself, so $\mathcal{P}(\emptyset) = \{\emptyset\}$. Then the subsets of $\{\emptyset\}$ are $\emptyset$ and $\{\emptyset\}$, so $\mathcal{P}(\mathcal{P}(\emptyset)) = \boxed{\{\emptyset, \{\emptyset\}\}}$.

For the second question, if S is the empty set, then $|S| = |\emptyset| = 0$, and $\mathcal{P}(S) = \mathcal{P}(\emptyset) = \{\emptyset\}$, so $|\mathcal{P}(S)| = |\{\emptyset\}| = 1 = 2^0$. Hence, the result is true when S is the empty set. Otherwise, S contains at least one element.

Let x be an element of S. Then for any subset of S, either x is in the subset, or x is not in the subset. Thus, we can construct a subset of S as follows: For each element of S, we choose whether to include the element in the subset or not. So for each element, there are two possible choices, and there are n elements. Furthermore, each combination of choices leads to a different subset, so the number of subsets of S is $|\mathcal{P}(S)| = 2^n$. □

Example 2.5.5 (OT-22-23-S2-P5)

For sets S and T, prove that

- $\mathcal{P}(S) \cup \mathcal{P}(T) \subseteq \mathcal{P}(S \cup T)$.
- $\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T)$.

Solution. For the first question, suppose that $A \in \mathcal{P}(S) \cup \mathcal{P}(T)$. Then $A \in \mathcal{P}(S)$ or $A \in \mathcal{P}(T)$, which means that $A \subseteq S$ or $A \subseteq T$. Thus every element of A is an element of S or every element of A is an element of T, which means that every element of A is an element of $S \cup T$. Therefore, $A \subseteq (S \cup T)$, which means that $A \in \mathcal{P}(S \cup T)$.

Remark. The solution above might lead you to conjecture that $\mathcal{P}(S \cup T) = \mathcal{P}(S) \cup \mathcal{P}(T)$. However, this is not true; for example, if $S = \{1\}$ and $T = \{2\}$, then $\mathcal{P}(S \cup T) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, whereas $\mathcal{P}(S) \cup \mathcal{P}(T) = \{\emptyset, \{1\}, \{2\}\}$.

For the second question, we prove it by proving two claims.

Claim — Suppose that $A \in \mathcal{P}(S \cap T)$. We prove that $A \in (\mathcal{P}(S) \cap \mathcal{P}(T))$.

Proof. $A \in \mathcal{P}(S \cap T)$ means that $A \subseteq (S \cap T)$. Then every element of A is an element of $S \cap T$, which means that every element of A is an element of S and an element of T. This means that $A \subseteq S$ and $A \subseteq T$, which means that $A \in \mathcal{P}(S)$ and $A \in \mathcal{P}(T)$. Therefore, $A \in (\mathcal{P}(S) \cap \mathcal{P}(T))$. ■

The same argument runs in reverse.

Claim — Suppose that $A \in (\mathcal{P}(S) \cap \mathcal{P}(T))$. We prove that $A \in \mathcal{P}(S \cap T)$.

Proof. $A \in (\mathcal{P}(S) \cap \mathcal{P}(T))$ means that $A \in \mathcal{P}(S)$ and $A \in \mathcal{P}(T)$, which means that $A \subseteq S$ and $A \subseteq T$. But then every element of A is an element of S and an element of T, which means that every element of A is an element of $S \cap T$. Therefore, $A \subseteq (S \cap T)$, which means that $A \in \mathcal{P}(S \cap T)$. ■

This shows that

$$\boxed{\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T)}.$$

□

Example 2.5.6 (OT-22-23-S2-P6)

Dogs in the GoodDog obedience school win a blue ribbon for learning how to sit, a green ribbon for learning how to roll over, and a white ribbon for learning how to stay. A new class of 180 dogs has enrolled in the GoodDog school. We have the following facts:

1. An equal number of dogs have each of the ribbons.
2. An equal number of dogs have each pair of ribbons.
3. 15 dogs have all three ribbons.
4. All dogs have at least one ribbon.
5. The number of dogs with exactly one ribbon equals twice the number of dogs with more than one ribbon.

How many dogs have blue ribbons?

Solution. [Solution with PIE] Let B, G, and W be the sets of dogs that have won a blue ribbon, green ribbon, and white ribbon, respectively. By conditions (1) and (2),

$$x = |B| = |G| = |W| \text{ and } y = |B \cap G| = |B \cap W| = |G \cap W|.$$

Now, we use Principle of Inclusion-Exclusion to count *the number of dogs with at least one ribbon,* which we know to be 180 (since every dog in the school has at least one ribbon). By conditions (3) and (4)

$$180 = (x + x + x) - (y + y + y) + 15 = 3x - 3y + 15 \Rightarrow x - y = 55.$$

On the other hand, we can also count the number of dogs that have more than one ribbon. Each such dog has two or three ribbons. The sum $|B \cap G| + |B \cap W| + |G \cap W|$ counts each dog with two ribbons exactly once, but counts dogs with three ribbons three times, so the number of dogs with more than one ribbon is

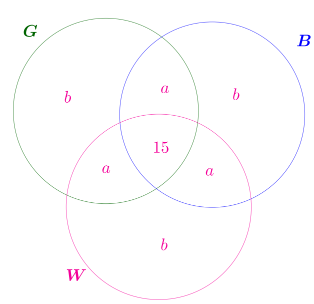
$$|B \cap G| + |B \cap W| + |G \cap W| - 2|B \cap G \cap W| = y + y + y - 2 \cdot 15 = 3y - 30.$$

Since each dog has at least one ribbon, the number of dogs with exactly one ribbon is

$$180 - (3y - 30) = 210 - 3y.$$

Thus, by condition (5), $210 - 3y = 2(3y - 30)$, which means that $y = 30$. Combining this with $x - y = 55$ gives us $x = \boxed{85}$ dogs with any one ribbon, in particular a blue ribbon. □

Solution. [By Venn-diagram]



We can read off of the diagram that the total number of dogs is $3a + 3b + 15 = 180$, and that the relationship between dogs with 1 ribbon and dogs with 2 or more ribbons is $3b = 2(3a + 15)$. This solves to give $a = 15$ and $b = 40$, and the answer is then $b + 2a + 15 = \boxed{85.}$ □

Example 2.5.7 (OT-22-23-S2-P7)
If A, B, and C are sets for which

$$|\mathcal{P}(A)| + |\mathcal{P}(B)| + |\mathcal{P}(C)| = |\mathcal{P}(A \cup B \cup C)| \text{ and } |A| = |B| = 100,$$

Then what is the maximal possible value of $|A \cap B \cap C|$?

Solution. First $\mathcal{P}(A) = \mathcal{P}(B) = 2^{100}$. Since $\mathcal{P}(C)$ and $\mathcal{P}(A \cup B \cup C)$ are integral powers of 2, let $2^m = |\mathcal{P}(C)|, 2^n = |\mathcal{P}(A \cup B \cup C)|$.

$$2^n - 2^m = 2 \cdot 2^{100} \Rightarrow 2^m(2^{n-m} - 1) = 2^{101} \Rightarrow m = 101, n = 102.$$

Now

$$\begin{aligned} &|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| + |C \cap A| - |A \cap B \cap C| \\ &\Rightarrow |A \cap B \cap C| = (|A \cap B| + |B \cap C| + |C \cap A|) - 199. \end{aligned}$$

Since

$$\begin{aligned} &|A \cap B| = |A| + |B| - |A \cup B| = 200 - |A \cup B|, \text{ and similarly for pairs } (B, C),\ (C, A) \\ &A \cup B, B \cup C, C \cup A \subseteq A \cup B \cup C \Rightarrow \max\{|A \cup B|, |B \cup C|, |C \cup A|\} \leq 102. \end{aligned}$$

Thus

$$|A \cap B \cap C| = 200 + 201 + 201 - 199 - (|A \cup B| + |B \cup C| + |C \cup A|) \leq 403 - 3 \cdot 102 = \boxed{97.}$$

$A = \{1, 2, \ldots, 100\}$, $B = \{3, 4, \ldots, 102\}$, and $C = \{1, 2, 4, 5, \ldots, 102\}$, satisy the given condition and achieve the maximal value for $|A \cap B \cap C|$. □

Example 2.5.8 (OT-22-23-S2-P8)
15 students are each going to enroll in exactly one of economics, psychology, or sociology. In how many ways can they enroll, provided that no class is left empty?

Solution. Let E, P, and S be the sets of enrollments, where he economics class, psychology class, and sociology class $\boxed{\textit{is empty,}}$ respectively.

Then by Principle of Inclusion-Exclusion, the number of possible enrollments where at least one of the classes is empty is

$$|E \cup P \cup S| = |E| + |P| + |S| - |E \cap P| - |E \cap S| - |P \cap S| + |E \cap P \cap S|$$

First we count $|E|$, the number of possible enrollments where the economics class is empty. This means each of the 15 students is in psychology or sociology. Hence, $|E| = 2^{15}$. Similarly, $|P| = |S| = 2^{15}$.

Next we count $|E \cap P|$, the number of possible enrollments where both the economics class and psychology class is empty. This means all 15 students are in the sociology class. Hence, $|E \cap P| = 1$. Similarly, $|E \cap S| = |P \cap S| = 1$.

Finally, the set $|E \cap P \cap S|$ is the set of enrollments where all three classes are empty. But every student is in some class, so $|E \cap P \cap S| = 0$.

Therefore, $|E \cup P \cup S| = 3 \cdot 2^{15} - 3 + 0$. The total number of possible enrollments is 3^{15}, so there are $3^{15} - 3 \cdot 2^{15} + 3 = \boxed{14,250,606}$ possible enrollments where no class is empty. □

Example 2.5.9 (OT-22-23-S2-P9)

Lan has a deck of cards consisting of the 2 through 5 of hearts and the 2 through 5 of spades. She deals two cards (at random) to each of four players. What is the probability that no player receives a pair?

Solution. Let A_1, A_2, A_3, and A_4 be the sets of deals where the first, second, third, and fourth player receives a pair, respectively. Then by Principle of Inclusion-Exclusion, the number of hands where at least one player receives a pair is

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\ & - |A_1 \cap A_2 \cap A_3 \cap A_4|\end{aligned}$$

First lets count $|A_1|$, the number of ways the first player can receive a pair. There are four possible pairs (one for each denomination from 2 through 5), and the number of ways the remaining six cards can be dealt to the remaining three players is $\binom{6}{2}\binom{4}{2} = 90$, because we have $\binom{6}{2}$ choices for cards for the first remaining player, then $\binom{4}{2}$ choices for cards for the next remaining player. Therefore, $|A_1| = 4 \cdot 90 = 360$. Similarly,

$$(|A_1| =)|A_2| = |A_3| = |A_4| = 360.$$

Next we count $|A_1 \cap A_2|$, the number of ways both the first player and second player can receive a pair. There are four possible pairs for the first player, then three possible pairs for the second player. Then the number of ways the remaining four cards can be dealt to the remaining two players is $\binom{4}{2} = 6$. Therefore, $|A_1 \cap A_2| = 4 \cdot 3 \cdot 6 = 72$. Similarly,

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4| = 72.$$

Finally, if three players receive a pair, then the fourth remaining player must also receive a pair. This can occur in $4! = 24$ ways. Therefore,

$$|A_1 \cap A_2 \cap A_3| = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = |A_1 \cap A_2 \cap A_3 \cap A_4| = 24.$$

So the number of hands where at least one player receives a pair is

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 4 \cdot 360 - 6 \cdot 72 + 4 \cdot 24 - 24 = 1080.$$

The total number of possible hands is

$$\frac{8!}{2!2!2!2!} = 2520.$$

Hence, the probability that no player receives a pair is

$$1 - \frac{1080}{2520} = 1 - \frac{3}{7} = \boxed{\frac{4}{7}}.$$

□

Example 2.5.10 (OT-22-23-S2-P10)

In a graph, a triangle consist of three vertices A, B, and C, such that all pairs (A, B), (B, C), and (C, A) are connected by some edges.

Given a graph with n vertices, prove that either it contains a triangle, or there exists a vertex that is the endpoint of at most $\left\lfloor \frac{n}{2} \right\rfloor$ edges.

Solution. For an x vertices, $\boxed{\text{let } A_x \text{ be the set of vertices that are connected to } x \text{ by some edges.}}$ First, if there exists a vertex x such that

$$|A_x| \leq \left\lfloor \frac{n}{2} \right\rfloor$$

then this is a desired vertex. Thus, for all vertex x,

$$|A_x| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Now, let consider a pair of x and y vertices, since $|A_x \cup A_y| \leq n$,

$$|A_x \cap A_y| = |A_x| + |A_y| - |A_x \cup A_y| \geq 2 \left\lfloor \frac{n}{2} \right\rfloor + 2 - n \geq 1.$$

This implies that there is a triangle with x, y, and $z \in A_x \cap A_y$. $\square$

Chapter 3

Inequalities I

3.1 Elementary Inequalities

Fact (Trivial facts)**.** There are many trivial facts that form the basis for inequalities. Here are a few of them.

1. $x \geq y$ and $y \geq z$, then $x \geq z$, $\forall x, y, z \in \mathbb{R}$.
2. $x \geq y$ and $a \geq b$, then $x + a \geq y + b$, $\forall x, y, a, b \in \mathbb{R}$.
3. $x \geq y \geq 0$ and $c > 0$, then $xc \geq yc$, $\forall x, y, c \in \mathbb{R}$.
4. $x \geq y \geq 0$ and $c < 0$, then $xc \leq yc$, $\forall x, y, c \in \mathbb{R}$.
5. $x \geq y \geq 0$ and $a \geq b \geq 0$, then $xa \geq yb$, $\forall x, y, a, b \in \mathbb{R}$.
6. $x^2 \geq 0$, $\forall x \in \mathbb{R}$, with equality if and only if $x = 0$.
7. $x_1, x_2, \ldots, x_n \in \mathbb{R}$, and $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$, then $a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 \geq 0$.
8. $x < 0$, and $y < 0$, then $xy > 0$, $\forall x, y \in \mathbb{R}$.
9. $x < 0$, and $y < 0$, then $xy > 0$, $\forall x, y \in \mathbb{R}$.
10. $x < y$, then $\frac{1}{x} > \frac{1}{y}$, if x and y have the same sign, otherwise $\frac{1}{x} < \frac{1}{y}$.

Note that $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+, \mathbb{R}^-$, and $\mathbb{R}_0^+$ are the sets of positive, negative, non-negative real numbers, respectively.

Definition (Absolute value)**.** $|x|$ denotes the absolute value of x :

$$|x| = \begin{cases} +x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Fact (Absolute values)**.** $|x|$ denotes the absolute value of x.

1. $|x| \geq 0$, $\forall x \in \mathbb{R}$.
2. $|x| = |-x|$ $\forall x \in \mathbb{R}$.
3. $|x|^2 = x^2$ $\forall x \in \mathbb{R}$.
4. $|ab| = |a||b|$ $\forall a, b \in \mathbb{R}$.
5. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $\forall a, b \in \mathbb{R}, b \neq 0$.
6. $|a + b| \leq |a| + |b|$, $\forall a, b \in \mathbb{R}$.
7. $|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$, $\forall a_1, \ldots, a_n \in \mathbb{R}$.

Example 3.1.1 (OT-22-23-S3-E1)

Show that if $0 \le a \le b \le 1$, then

1. $0 \le \frac{b-a}{1-ab} \le 1$,
2. $0 \le \frac{a}{1+b} + \frac{b}{1+a} \le 1$
3. $0 \le ab^2 - ba^2 \le \frac{1}{4}$.

Remark. Note that we use the *equivalent* statements. $A \Leftrightarrow B$ means that A is true if and only if B is true. This means that instead of proving A, we have to prove B. If B is true, then A becomes true. If B is false, then A cannot be true.

Solution. For the first question, $0 \le a \le b \le 1 \Rightarrow b - a \ge 0, 1 - ab \ge 0 \Rightarrow 0 \le \frac{b-a}{1-ab}$, which is true.

$\frac{b-a}{1-ab} \le 1 \Leftrightarrow b - a \le 1 - ab \Leftrightarrow 0 \le 1 - b + a - ab \Leftrightarrow (1-b)(1+a)$. The last inequality is always true.

For the second question, $0 \le a \le b \le 1 \Rightarrow 0 \le \frac{a}{1+b}, 0 \le \frac{b}{1+a} \ge 0$, thus $0 \le \frac{a}{1+b} + \frac{b}{1+a}$.

$\frac{a}{1+b} + \frac{b}{1+a} \le 1 \Leftrightarrow a + a^2 + b + b^2 \le 1 + a + b + ab \Leftrightarrow 0 \le 1 + ab - a^2 - b^2 \Leftrightarrow 0 \le (1-b^2) + a(b-a)$.

For the third question, $0 \le a \le b \le 1 \Rightarrow 0 \le ab^2 - ba^2 = ab(b-a)$. The last inequality is always true.

$ab^2 - ba^2 \le ab^2 - b^2a^2 = b^2(a - a^2) \le a - a^2 = \frac{1}{4} - \left(\frac{1}{2} - a\right)^2 \le \frac{1}{4}$. □

Example 3.1.2 (OT-22-23-S3-E2)

Show that if a, b, c are real numbers, then

1. $||a| - |b|| \le |a - b|$,
2. $|a| + |b| + |c| - |a+b| - |b+c| - |c+a| + |a+b+c| \ge 0$.

Solution. It is easy to prove the claim below.

Claim — $x \le |y|, -x \le |y|$, then $|x| \le |y|$.

For the first question, by the claim. $|b| + |a-b| \le |a|$, so $|a| - |b| \le |a-b|$, similarly $|b| - |a| \le |a-b|$, thus $||a| - |b|| \le |a-b|$.

For the second question, if any of a, b, or c is 0, then we have the equality. Now, WLOG lets assume that $|a| \ge |b| \ge |c|$, and dividing both side by $|a|$, then

$$1 + \left|\frac{b}{a}\right| + \left|\frac{c}{a}\right| - \left|1 + \frac{b}{a}\right| - \left|\frac{b}{a} + \frac{c}{a}\right| - \left|1 + \frac{c}{a}\right| + \left|1 + \frac{b}{a} + \frac{c}{a}\right| \ge 0 \quad (*)$$

Now $\left|\frac{b}{a}\right| \le 1$, so $\left|1 + \frac{b}{a}\right| = 1 + \frac{b}{a}$. Similarly $\left|\frac{c}{a}\right| \le 1$, $\left|1 + \frac{c}{a}\right| = 1 + \frac{c}{a}$.

Thus (*) becomes

$$\left|\frac{b}{a}\right| + \left|\frac{c}{a}\right| - \left|\frac{b}{a} + \frac{c}{a}\right| - \left(1 + \frac{b}{a} + \frac{c}{a}\right) + \left|1 + \frac{b}{a} + \frac{c}{a}\right| \ge 0.$$

This is true because

$$\left|\frac{b}{a} + \frac{c}{a}\right| \le \left|\frac{b}{a}\right| + \left|\frac{c}{a}\right| \text{ and } 1 + \frac{b}{a} + \frac{c}{a} \le \left|1 + \frac{b}{a} + \frac{c}{a}\right|.$$

□

Example 3.1.3 (OT-22-23-S3-E3)

Prove the following statements,

1. If $a \geq b \geq c \geq d$, then $ac + bd \geq ad + bc$.
2. If $a \geq b > 0$, and $c \geq d > 0$, then $ac \geq bd$.
3. If $a \geq b \geq c \geq d$, and $a + d = b + c$, then $bc \geq ad$.

Remark. This example shows how to use some identities for four numbers a, b, c, and d to compare the pair-wise products.

Solution.

$$\begin{aligned}
&(1) \quad \underbrace{(ac+bd)-(ad+bc) = ac-ad+bd-bc}_{\text{different groupping of terms}} = a(c-d)+b(d-c) = (a-b)(c-d) \geq 0,\\
&(2) \quad \underbrace{ac-bd = ac-bc+bc-bd}_{\text{add and remove the same term}} = (a-b)c+b(c-d) \geq 0,\\
&(3) \quad \underbrace{bc-ad = bc-bd+bd-ad}_{\text{add and remove the same term}} = \underbrace{b(c-d)+d(b-a) = (b-d)(c-d)}_{a+d=b+c \Rightarrow a-b=c-d} \geq 0.
\end{aligned}$$

□

Example 3.1.4 (OT-22-23-S3-E4)

Prove the following statements,

1. If $x \geq y > 0$, then $\frac{x}{y} > 1$.
2. If $x, y \in \mathbb{R}_0^+$, then $x \geq y$ if and only if $x^2 \geq y^2$.
3. If $x, y \in \mathbb{R}_0^+$, then $\frac{x+y}{2} \geq \sqrt{xy}$.

Solution.

$$\begin{aligned}
&(1) \quad x-y \geq 0, y > 0 \Rightarrow \frac{x}{y} - 1 = \frac{x-y}{y} \geq 0,\\
&(2) \quad x, y \in \mathbb{R}_0^+ \Rightarrow x+y > 0 \Rightarrow (x^2-y^2) = (x-y)(x+y) \geq 0 \Leftrightarrow x-y \geq 0,\\
&(3) \quad \frac{x+y}{2} \geq \sqrt{xy} = \frac{1}{2}\left((\sqrt{x})^2 + (\sqrt{y})^2 - 2\sqrt{xy}\right) = \frac{1}{2}(\sqrt{x}-\sqrt{y})^2 \geq 0.
\end{aligned}$$

□

Example 3.1.5 (OT-22-23-S3-E5)

Prove that, if $\frac{x_1}{y_1} \geq \frac{x_2}{y_2} \geq \ldots \geq \frac{x_n}{y_n}$, and $y_1, y_2, \ldots, y_n \in \mathbb{R}^+$, then

$$\frac{x_1}{y_1} \geq \frac{x_1+x_2+\ldots+x_n}{y_1+y_2+\ldots+y_n} \geq \frac{x_n}{y_n}.$$

Solution.

$$\begin{aligned}
&(1) \quad \frac{x_1}{y_1}(y_1+y_2+\ldots+y_n) = \frac{x_1}{y_1}y_1 + \frac{x_1}{y_1}y_2 + \ldots + \frac{x_1}{y_1}y_n \geq \frac{x_1}{y_1}y_1 + \frac{x_2}{y_2}y_2 + \ldots \frac{x_n}{y_n}y_n = x_1+x_2+\ldots+x_n\\
&(2) \quad \frac{x_n}{y_n}(y_1+y_2+\ldots+y_n) = \frac{x_n}{y_n}y_1 + \frac{x_n}{y_n}y_2 + \ldots + \frac{x_n}{y_n}y_n \leq \frac{x_1}{y_1}y_1 + \frac{x_2}{y_2}y_2 + \ldots \frac{x_n}{y_n}y_n = x_1+x_2+\ldots+x_n
\end{aligned}$$

□

Example 3.1.6 (OT-22-23-S3-E6)

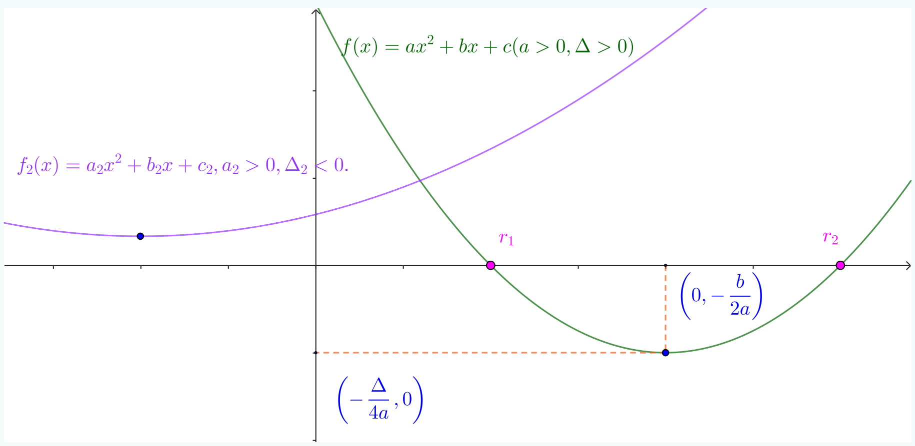
Show that
$$a^2 + b^2 + c^2 - ab - bc - ca \geq \frac{3}{4}(a-b)^2.$$

Remark. $a^2 + b^2 - ab - \frac{3}{4}(a-b)^2 = \frac{1}{4}(4a^2 + 4b^2 - 4ba - 3a^2 - 3b^2 + 6ab) = \frac{1}{4}(a^2 + b^2 + 2ab) = \left(\frac{a+b}{2}\right)^2.$

Solution. $a^2 + b^2 + c^2 - ab - bc - ca - \frac{3}{4}(a-b)^2 = c^2 - c(a+b) + \left(\frac{a+b}{2}\right)^2 = \left(c + \frac{a+b}{2}\right)^2 \geq 0.$ □

Theorem (Quadratic Method)

Let $f(x) = ax^2 + bx + c$, where a, b, and c are constants. Let $\Delta = b^2 - 4ac$.



Case $a > 0$, $f_{min} = \min_{x \in \mathbb{R}} f(x) = f\left(-\frac{b}{2a}\right) = \frac{-\Delta}{4a}$. If $\Delta < 0 \Rightarrow f(x) \geq f_{min} > 0$.

If $\Delta \geq 0 \Rightarrow f$ has $r_1 \leq r_2$ roots, $f(x) > 0$, if $x < r_1$ or $x > r_2$; and $0 \geq f(x) \geq f_{min}$, if $r_1 < x < r_2$.

f decreases on $\left(-\infty, -\frac{b}{2a}\right)$ and increases on $\left(-\frac{b}{2a}, +\infty\right)$.

Case $a < 0$, $f_{max} = \max_{x \in \mathbb{R}} f(x) = f\left(-\frac{b}{2a}\right) = \frac{-\Delta}{4a}$. If $\Delta < 0 \Rightarrow f(x) \geq f_{min} > 0$.

If $\Delta \geq 0 \Rightarrow f$ has $r_1 \leq r_2$ roots, $f(x) < 0$, if $x < r_1$ or $x > r_2$; and $0 \leq f(x) \leq f_{max}$, if $r_1 < x < r_2$.

f increases on $\left(-\infty, -\frac{b}{2a}\right)$ and decreases on $\left(-\frac{b}{2a}, +\infty\right)$.

Example 3.1.7 (OT-22-23-S3-E7)

Prove that if $0 \leq a \leq b \leq 1$, then $0 \leq ab^2 - ba^2 \leq \frac{1}{4}$.

Solution. Let $f(x) = xb^2 - bx^2 = (-b)x^2 + xb^2$, then the highest coefficient of x in f is $-b < 0$, $\Delta = b^4 > 0$, thus f will attain the maximal value at $-\frac{b^2}{2(-b)} = \frac{b}{2}$, which is $f(\frac{b}{2}) = (-b)\frac{b^2}{4} + \frac{b}{2}b^2 = \frac{b^3}{4}$.

Hence, $f(x) \leq f_{max} = \frac{b^3}{4} \leq \frac{1}{4}$, so $f(a) = ab^2 - ba^2 \leq \frac{1}{4}$. □

Example 3.1.8 (OT-22-23-S3-E8)

Prove that $3(x+y+1)^2+1 \geq 3xy$.

Solution. Let $f(x) = 3(x+y+1)^2+1-3xy$,

$$\begin{aligned} f(x) &= 3x^2+3y^2+3+6xy+6x+6y+1-3xy \\ &= 3x^2+3x(y+2)+(3y^2+6y+4)) \\ \Delta &= 9(y+2)^2-12(3y^2+6y+4) = -27y^2-36y-12 = -3(3y+2)^2 \leq 0 \end{aligned}$$

Thus, the second-degree coefficient of f is $3>0$, its discriminant $\Delta \leq 0$, therefore by Quadratic Method $f(x) \geq 0$, for all x. The equality stands if and only if $f(x)=0$, or $\Delta = 0$, thus $x=y=-\frac{2}{3}$. ☐

Example 3.1.9 (OT-22-23-S3-E9)

If $a<b<c<d$, prove that $(a+b+c+d)^2 > 8(ac+bd)$.

Remark. Note that if we consider a as the variable, then we can rewrite the inequality as

$$x<b<c<d, f(x) = (x+b+c+d)^2-8(xc+bd) \geq 0.$$

We need to investigate the second-degree coefficient of f, its discriminant, and its roots.

Solution. [First solution] Let $f(x) = x^2+x(2(b+c+d)-8c)$,

$$\begin{aligned} f(x) &= (x+b+c+d)^2-8(xc+bd) = x^2+x(2(b+c+d)-8c)+(b+c+d)^2-8bd \\ &= x^2+2x(b+d-3c)+(b^2+c^2+d^2+2bc+2cd-6bd) \\ \Delta &= 4\left[(b+d-3c)^2-(b^2+c^2+d^2+2bc+2cd-6bd)\right] \\ &= 4(b^2+d^2+9c^2+2bd-6bc-6cd-b^2-c^2-d^2-2bc-2cd+6bd) \\ &= 4(8c^2-8cb-8cd+8bd) = 32(c-b)(c-d) < 0. \end{aligned}$$

Thus, the second-degree coefficient of f is $1>0$, its discriminant $\Delta < 0$, therefore by Quadratic Method $f(x) > 0$, for all x, including the case $x<b<c<d$. ☐

Theorem (Intermediate Zero Theorem)

Let $f(x)$ be a function which is continuous on the closed interval $[a,b]$ and suppose that $f(a)$ and $f(b)$ have opposite signs or one of them is zero.

$$f(a)f(b) \leq 0$$

Then there is at least one c with $a \leq c \leq b$ such that $f(c)=0$.

Solution. [Second solution] Now let $A=2, B=(a+b+c+d), C=ac+bd$. The exrpession $(a+b+c+d)^2-8(ac+bd)$ is the same as B^2-4AC, thus it is a discriminant of the polynomial

$$g(x) = 2x^2-(a+b+c+d)x+(ac+bd) = (x-a)(x-c)+(x-b)(x-d).$$

Note that $g(a)=(a-b)(a-d)>0$, $g(b)=(b-a)(b-c)<0$, thus according to the Intermediate Zero Theorem, $g(x)$ has a root between a and b, thus its discriminant $\Delta = B^2-4AC \geq 0$. ☐

3.2 AM-GM Inequality

Theorem 3.2.1 (AM-GM Inequality)
For any positive real numbers $a_1, \ldots, a_n$,

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

Example 3.2.2 (OT-22-23-S3-E10)
For $a_1, a_2, \ldots, a_n$ positive real numbers. Prove that

$$(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \geq n^2.$$

Solution. First, by AM-GM Inequality for n numbers $a_1, a_2, \ldots, a_n$, $a_1 + a_2 + \ldots + a_n \geq n\sqrt[n]{a_1 \cdot a_2 \cdots a_n}$.

Similarly, by AM-GM Inequality for n numbers $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$, $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \geq n\sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdots \frac{1}{a_n}}$.

Thus $(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \geq n\sqrt[n]{a_1 \cdot a_2 \cdots a_n} \cdot n\sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdots \frac{1}{a_n}} = n^2$. □

Corollary (Arithmetic Mean - Harmonic Mean Inequality)
As a consequence of AM-GM Inequality: if $x_1, x_2, \cdots, x_n$ are positive reals, then

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots \frac{1}{x_n}\right) \geq n^2$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$;

Example 3.2.3 (OT-22-23-S3-E11)
Prove the following statements,

1. If $x, y, z > 0$, then $xy + yz + zx \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$.
2. If $\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq x + y + z$.
3. If $x, y \in \mathbb{R}_0^+$, then $x^4 + y^4 + 8 \geq 8xy$.

Remark. The way the expression $x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$ is written called *cyclic*, since we can change $x \to y \to z$ and the expression remains same.

In this example, we show some *groupping techniques*, that can be very useful for AM-GM Inequality.

Solution. By AM-GM Inequality, the sum of the first two terms $xy + yz \geq 2\sqrt{y^2xz} = 2y\sqrt{xz}$. Thus by following the groupping the second and the third, then the third and the first and sum all up, we shall receive the desired inequality.

By AM-GM Inequality, the sum $\frac{xy}{z} + \frac{yz}{x} \geq 2\sqrt{\frac{xy}{z} \cdot \frac{yz}{x}} = y$. Following the same process as in the solution for the first question.

By AM-GM Inequality, $x^4 + y^4 + 4 + 4 \geq 4\sqrt[4]{x^4 \cdot y^4 \cdot 4 \cdot 4} = 8xy$. □

Example 3.2.4 (OT-22-23-S3-E12)

For a, b, c positive real numbers. Prove that

$$\frac{a}{\sqrt{(a+b)(a+c)}} + \frac{b}{\sqrt{(b+c)(b+a)}} + \frac{c}{\sqrt{(c+a)(c+b)}} \leq \frac{3}{2}.$$

Solution. By AM-GM Inequality for $\frac{a}{a+b}$ and $\frac{a}{c+a}$: $\frac{a}{a+b} + \frac{a}{a+c} \geq 2\sqrt{\frac{a^2}{(a+b)(a+c)}} = \frac{2a}{\sqrt{(a+b)(a+c)}}$.

Thus the left-hand side of the given expression

$$\begin{aligned}&\frac{a}{\sqrt{(a+b)(a+c)}} + \frac{b}{\sqrt{(b+c)(b+a)}} + \frac{c}{\sqrt{(c+a)(c+b)}}\\ \leq\ &\frac{1}{2}\left(\frac{a}{a+b} + \frac{a}{a+c} + \frac{b}{b+a} + \frac{b}{b+c} + \frac{c}{c+a} + \frac{c}{c+b}\right) = \frac{3}{2}.\end{aligned}$$

□

Example 3.2.5 (OT-22-23-S3-E13)

Show that

$$n\left(1 - \frac{1}{\sqrt[n]{n}}\right) + 1 > 1 + \frac{1}{2} + \ldots + \frac{1}{n} > n(\sqrt[n]{n+1} - 1).$$

Solution. By AM-GM Inequality for n numbers $1, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-1}{n}$, we have

$$\begin{aligned}&1 + \frac{1}{2} + \frac{2}{3} + \ldots + \frac{n-1}{n} \geq n\sqrt[n]{1 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n-1}{n}} = n\sqrt[n]{\frac{1}{n}}\\ \Rightarrow\ &n - \left(\frac{1}{2} + \ldots + \frac{1}{n}\right) \geq n\sqrt[n]{\frac{1}{n}} \Rightarrow n\left(1 - \frac{1}{\sqrt[n]{n}}\right) + 1 > 1 + \frac{1}{2} + \ldots + \frac{1}{n}.\end{aligned}$$

For the right-hand side inequality, note that $1 + \frac{1}{2} + \ldots + \frac{1}{n} + n = 2 + \frac{3}{2} + \ldots + \frac{n+1}{n}$, thus by AM-GM Inequality for n numbers $2, \frac{3}{2}, \ldots, \frac{n+1}{n}$,

$$2 + \frac{3}{2} + \ldots + \frac{n+1}{n} \geq n\sqrt[n]{2 \cdot \frac{3}{2} \cdots \frac{n+1}{n}} = n\sqrt[n]{n+1}.$$

□

Example 3.2.6 (OT-22-23-S3-E14)

Show that for a, b, c positive real numbers such that $a^2 + b^2 + c^2 = 1$,

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{3}{2}.$$

Solution. By result of OT-22-23-S3-E6 for three number $1 + ab, 1 + ac, 1 + ca$,

$$\begin{aligned}&\left(\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca}\right)\left(\frac{1}{\frac{1}{1+ab}} + \frac{1}{\frac{1}{1+bc}} + \frac{1}{\frac{1}{1+ca}}\right) \geq 9\\ \Rightarrow\ &\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{9}{3+ab+bc+ca}.\end{aligned}$$

Since $a^2 + b^2 + c^2 \geq ab + bc + ca$, thus $\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{9}{3+a^2+b^2+c^2} = \frac{3}{2}$. □

3.3 Cauchy-Schwarz Inequality

Theorem (Cauchy-Schwarz Inequality)
For any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

with equality when there exists a nonzero constant μ such that for all $1 \leq i \leq n$, $\mu a_i = b_i$.

Example 3.3.1 (OT-22-23-S3-E15)
x, y, z are positive real numbers such that $x + y + z = 1$, prove that

$$\sqrt{xy+z} + \sqrt{yz+x} + \sqrt{zx+y} \geq \sqrt{xy} + \sqrt{yz} + \sqrt{zx} + x + y + z$$

Solution. By the Cauchy-Schwarz Inequality for $a_1 = \sqrt{x}$, $a_2 = \sqrt{z}$, and $b_1 = \sqrt{y}$, $b_2 = \sqrt{z}$ we have

$$\begin{aligned}
&(\sqrt{x}\sqrt{y} + \sqrt{z}\sqrt{z})^2 \leq (x+z)(y+z) \\
&\Rightarrow \sqrt{xy} + z \leq \sqrt{(x+z)(y+z)} = \sqrt{xy + yz + xz + z^2} = \sqrt{xy + z(x+y+z)} = \sqrt{xy+z} \\
&\Rightarrow \sqrt{xy} + z \leq \sqrt{xy+z}.
\end{aligned}$$

Similarly $\sqrt{yz} + x \leq \sqrt{yz+x}$, $\sqrt{zx} + y \leq \sqrt{xz+y}$.
By summing up the results, we shall receive the desired inequality. □

Example 3.3.2 (OT-22-23-S3-E16)
a, b, c are positive real numbers such that $a + b + c = 1$, prove that

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} < 5$$

Solution. By the Cauchy-Schwarz Inequality for $a_1 = \sqrt{4a+1}$, $a_2 = \sqrt{4b+1}$, $a_3 = \sqrt{4c+1}$, and $b_1 = b_2 = b_3 = 1$ we have

$$(\sqrt{4a+1} \cdot 1 + \sqrt{4b+1} \cdot 1 + \sqrt{4c+1} \cdot 1)^2 \leq ((4a+1) + (4b+1) + (4c+1))(1+1+1) = 21.$$

Thus $\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} \leq \sqrt{21} < 5$. □

Example 3.3.3 (OT-22-23-S3-E17)
Show that for a, b, x, y positive real numbers,

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y.}$$

Solution. By the Cauchy-Schwarz Inequality for $a_1 = \frac{a}{\sqrt{x}}$, $a_2 = \frac{b}{\sqrt{y}}$, and $b_1 = \sqrt{x}$, $b_2 = \sqrt{y}$, we have

$$\left[\left(\frac{a}{\sqrt{x}}\right)^2 + \left(\frac{b}{\sqrt{y}}\right)^2\right]((\sqrt{x})^2 + (\sqrt{y})^2) \geq \left(\frac{a}{\sqrt{x}} \cdot \sqrt{x} + \frac{b}{\sqrt{y}} \cdot \sqrt{y}\right)^2 \Leftrightarrow \left(\frac{a^2}{x} + \frac{b^2}{y}\right)(x+y) \geq (a+b)^2.$$

□

Corollary (T2 Lemma)

For any positive real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \ldots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \ldots + a_n)^2}{b_1 + b_2 + \ldots + b_n}$$

with equality when there exists a nonzero constant μ such that for all $1 \leq i \leq n$, $\mu a_i = b_i$.

Example 3.3.4 (OT-22-23-S3-E18)

Show that for a, b real numbers, $a^4 + b^4 \geq \frac{1}{8}(a+b)^4$.

Solution. By the T2 Lemma with $a_1 = a^2, a_2 = b^2$ and $b_1 = b_2 = 1$, $a^4 + b^4 = \frac{a^4}{1} + \frac{b^4}{1} \geq \frac{(a^2+b^2)^2}{2}$.

Similarly $a^2 + b^2 \geq \frac{(a+b)^2}{2}$, thus $a^4 + b^4 \geq \frac{1}{8}(a+b)^4$. □

Example 3.3.5 (OT-22-23-S3-E19)

Show that for $a_1, \ldots, a_n$ real numbers,

$$n(a_1^2 + a_2^2 + \ldots + a_n^2) \geq (a_1 + a_2 + \ldots + a_n)^2.$$

Solution. By the Cauchy-Schwarz Inequality with $a_1, \ldots, a_n$ and $b_1 = b_2 = \ldots = b_n = 1$,

$$(a_1^2+a_2^2+\ldots+a_n^2)(1^2+1^2+\ldots+1^2) \geq (a_1 \cdot 1+a_2 \cdot 1+\ldots+a_n \cdot 1)^2 \Rightarrow n(a_1^2+a_2^2+\ldots+a_n^2) \geq (a_1+a_2+\ldots+a_n)^2.$$

□

Example 3.3.6 (OT-22-23-S3-E20)

Show that for a, b, c positive real numbers, then

$$\frac{a}{a+2b} + \frac{b}{b+2c} + \frac{c}{c+2a} \geq 1.$$

Solution. Note that by multiplying both nominators and denominators of the fractions ith a, b, and c, respectively.

$$\frac{a}{a+2b} = \frac{a^2}{a^2+2ab}, \quad \frac{b}{b+2c} = \frac{b^2}{b^2+2bc}, \quad \frac{c}{c+2a} = \frac{c^2}{c^2+2ca}$$

Then by the T2 Lemma,

$$\frac{a^2}{a^2+2ab} + \frac{b^2}{b^2+2bc} + \frac{c^2}{c^2+2ca} \geq \frac{(a+b+c)^2}{a^2+b^2+c^2+2ab+2bc+2ca} = 1.$$

□

Example 3.3.7 (OT-22-23-S3-E21)
Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Prove that,

$$\frac{a_1}{\sqrt{1-a_1}} + \frac{a_2}{\sqrt{1-a_2}} + \frac{a_n}{\sqrt{1-a_n}} \geq \frac{1}{\sqrt{n-1}}(\sqrt{a_1} + \sqrt{a_2} + \ldots + \sqrt{a_n}).$$

Solution. First, note that for any $0 < a < 1$ real number:

$$\frac{a}{\sqrt{1-a}} = \frac{1}{\sqrt{1-a}} - \sqrt{1-a}.$$

Let $S_1 = \frac{1}{\sqrt{1-a_1}} + \ldots + \frac{1}{\sqrt{1-a_n}}$ and $S_2 = \sqrt{1-a_1} + \ldots + \sqrt{1-a_n}$, then the left-hand side of the inequality:

$$\frac{a_1}{\sqrt{1-a_1}} + \ldots + \frac{a_n}{\sqrt{1-a_n}} = \left(\frac{1}{\sqrt{1-a_1}} + \frac{1}{\sqrt{1-a_n}}\right) - \left(\sqrt{1-a_1} + \ldots + \sqrt{1-a_n}\right) = S_1 + S_2.$$

By AM-GM Inequality,

$$S_1 = \frac{1}{\sqrt{1-a_1}} + \ldots + \frac{1}{\sqrt{1-a_n}} \geq n\sqrt[n]{\frac{1}{\sqrt{(1-a_1)\cdots(1-a_n)}}} = \frac{n}{\sqrt{\sqrt[n]{((1-a_1)\cdots(1-a_n))}}}.$$

By AM-GM Inequality,

$$\sqrt[n]{(1-a_1)\cdots(1-a_n)} \leq \frac{1}{n}\left[(1-a_1) + \ldots + (1-a_n)\right] = \frac{n-1}{n} \Rightarrow S_1 \geq \frac{n}{\sqrt{\frac{n-1}{n}}} = \frac{n\sqrt{n}}{\sqrt{n-1}}.$$

Now, by Cauchy-Schwarz Inequality,

$$S_2^2 = \left(\sqrt{1-a_1} + \ldots + \sqrt{1-a_n}\right)^2 \leq \left((1-a_1) + \ldots + (1-a_n)\right)(1 + \ldots + 1) = (n-1)n$$
$$\Rightarrow S_2 \leq \sqrt{n(n-1)} \Rightarrow S_1 - S_2 \geq \frac{n\sqrt{n}}{\sqrt{n-1}} - \sqrt{n(n-1)} = \frac{\sqrt{n}}{\sqrt{n-1}} \quad (*)$$

And finally, by Cauchy-Schwarz Inequality,

$$(\sqrt{a_1} + \ldots + \sqrt{a_n})^2 \leq (a_1 + \ldots + a_n)(1 + \ldots + 1) = (1)(n) \Rightarrow \sqrt{a_1} + \ldots + \sqrt{a_n} \leq \sqrt{n} \quad (**)$$

From(*) and (**),

$$S_1 - S_2 \geq \frac{1}{\sqrt{n-1}}(\sqrt{a_1} + \sqrt{a_2} + \ldots + \sqrt{a_n}).$$

□

3.4 The Rearrangement and Chebyshev Inequality

Theorem (Rearrangement Inequality)

For real numbers $x_1 \leq \cdots \leq x_n$, and $y_1 \leq \cdots \leq y_n$, let $z_1, \ldots, z_n$ be a permutation of $x_1, \ldots, x_n$, then

$$x_n y_1 + \ldots + x_1 y_n \leq z_1 y_1 + \ldots + z_n y_n \leq x_1 y_1 + \ldots + x_n y_n$$

$(z_1, \ldots, z_n) \equiv (x_n, \ldots, x_1)$ and $(z_1, \ldots, z_n) \equiv (x_1, \ldots, x_n)$ for the left and right equalities.

Corollary (Chebyshev Inequality)

For real numbers $x_1 \leq \cdots \leq x_n$, and $y_1 \leq \cdots \leq y_n$,

$$\frac{x_1 y_1 + \ldots + x_n y_n}{n} \geq \frac{x_1 + \ldots + x_n}{n} \cdot \frac{y_1 + \ldots + y_n}{n}$$

Example 3.4.1 (OT-22-23-S3-E22)

Show that for a, b, c positive real numbers, then

$$\frac{b^2 - a^2}{c + a} + \frac{c^2 - b^2}{a + b} + \frac{a^2 - c^2}{b + c} \geq 0.$$

Solution. WLOG $a \geq b \geq c$, then $a^2 \geq b^2 \geq c^2$, and $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$, thus by the Rearrangement Inequality for these sequences,

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a^2}{c+a} + \frac{b^2}{a+b} + \frac{c^2}{b+c}.$$

By subtracting the right-hand side from the left one, we receive the desired inequality. □

Example 3.4.2 (OT-22-23-S3-E23)

Show that for a, b, c positive real numbers, then

$$3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right)\left(3 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Solution. WLOG $a \geq b \geq c$, then $\frac{1}{a} \leq \frac{1}{b}$, so $\frac{1}{a+1} \leq \frac{1}{b+1}$, and $\frac{a+1}{a} \leq \frac{b+1}{b}$. thus by the Chebyshev Inequality for the two sequences $\frac{1}{a+1} \leq \frac{1}{b+1} \leq \frac{1}{c+1}$ and $\frac{a+1}{a} \leq \frac{b+1}{b} \leq \frac{c+1}{c}$.

$$\begin{aligned}\left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right)\left(\frac{a+1}{a} + \frac{b+1}{b} + \frac{c+1}{c}\right) &\leq 3\left(\frac{1}{a+1}\frac{a+1}{a} + \frac{1}{b+1}\frac{b+1}{b} + \frac{1}{c+1}\frac{c+1}{c}\right) \\ &= 3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).\end{aligned}$$

□

3.5 Problems

Problem 3.5.1 (OT-22-23-S3-P1)**.** x, y, and z are real numbers. Prove that

$$x^4 + y^4 + z^2 + 1 \geq 2x(xy^2 - x + z + 1).$$

> **Remark.** The left-hand side is a sum of squares $(x^2)^2, (y^2)^2, z^1, 1^2$. Are the terms on the right-hand side be related to those? For example $2x^2y^2 = 2 \cdot x^2 \cdot y^2$.

Problem 3.5.2 (OT-22-23-S3-P2)**.** a, b, c are positive real numbers such that $a > 1, b > 1, c > 1$. Prove that

$$abc + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > a + b + c + \frac{1}{abc}.$$

> **Remark.** It might help if you first consider the inequality with 2 variables (a and b). Note that $ab - a - b + 1 = (a-1)(b-1)$.

Problem 3.5.3 (OT-22-23-S3-P3)**.** a, b, c are positive real numbers. Prove that at least one of the following inequalities is true.

$$\begin{cases} \frac{3}{a} \geq 2 - b \\ \frac{2}{b} \geq 4 - c \\ \frac{6}{c} \geq 6 - a. \end{cases}$$

> **Remark.**
>
> $$\frac{3}{a} \geq 2 - b \Leftrightarrow 3 \geq (2-b)a$$
>
> The right-hand side is not always true, thus it might not be easy to casework to figure out which is true. However, why don't we assume the opposite? Then none of them is true, but then these three can be combined and investigateed.
>
> What would be the maximum value of $a(6-a)$?

Problem 3.5.4 (OT-22-23-S3-P4)**.** x, y, z are positive real numbers. Prove that

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq x + y + z$$

> **Remark.** How about this *simplified* inequality?
>
> $$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x + y + z.$$
>
> Note that $\sqrt{\frac{x^2}{y} \cdot y} = x$.

Problem 3.5.5 (OT-22-23-S3-P5)**.** a, b, c, d are positive real numbers. Prove that

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \geq 0.$$

> **Remark.** By adding 1 to each fraction on the left, we can bring their numerators to be positive. Then investigate the numerators and denominators. Use the result of Arithmetic Mean - Harmonic Mean Inequality when needed.

Problem 3.5.6 (OT-22-23-S3-P6)**.** a, b, c are positive real numbers less than 1. Prove that

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

> **Remark.** Note that for all $0 < x < 1 : \sqrt{x} < \sqrt[3]{x}$.

Problem 3.5.7 (OT-22-23-S3-P7)**.** a, b, c, d are positive real numbers. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

> **Remark.** Note that the denominator of the fraction on the right is the sum of the denominators of the fractions on the left.

Problem 3.5.8 (OT-22-23-S3-P8)**.** a, b, c are positive real numbers. Prove that

$$\frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1} \geq \frac{3(a+b+c)}{3+a+b+c}.$$

> **Remark.** First, prove that
>
> $$\frac{a^2}{a(b+1)} + \frac{b^2}{b(c+1)} + \frac{c^2}{c(a+1)} \geq \frac{(a+b+c)^2}{ab+bc+ca+a+b+c}.$$

Problem 3.5.9 (OT-22-23-S3-P9)**.** a, b, c are positive real numbers. Prove that

$$a^5 + b^5 + c^5 \geq a^4 b + b^4 c + c^4 a.$$

> **Remark.** Consider two cases $a \geq b \geq c$, and $a \geq c \geq b$.

Problem 3.5.10 (OT-22-23-S3-P10)**.** a, b, c are positive real numbers. Prove that

$$\frac{a^2+c^2}{b} + \frac{b^2+a^2}{c} + \frac{c^2+b^2}{a} \geq 2(a+b+c)$$

> **Remark.** WLOG, consider $a \geq b \geq c$.

3.6 Solution

Example 3.6.1 (OT-22-23-S3-P1)

x, y, and z are real numbers. Prove that

$$x^4 + y^4 + z^2 + 1 \geq 2x(xy^2 - x + z + 1).$$

Solution.

$$\begin{aligned} x^4 + y^4 + z^2 + 1 - 2x(xy^2 - x + z + 1) &= (x^4 + y^4 - 2x^2y^2) + (x^2 + z^2 - 2xz) + (x^2 + 1^2 - 2x) \\ &= (x^2 - y^2)^2 + (x - z)^2 + (x - 1)^2 \geq 0 \end{aligned}$$

□

Example 3.6.2 (OT-22-23-S3-P2)

a, b, c are positive real numbers such that $a > 1, b > 1, c > 1$. Prove that

$$abc + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > a + b + c + \frac{1}{abc}.$$

Remark. Consider the inequality with two variables

$$ab + \frac{1}{a} + \frac{1}{b} > a + b + \frac{1}{ab}.$$

Bringing both sides to the same denominator and subtract the right-hand side from the left one,

$$(ab)^2 + b + a - (a + b)ab - 1 = ab(ab - a - b + 1) - (ab - a - b + 1) = (ab - 1)(ab - a - b + 1) = (ab - 1)(a - 1)(b - 1).$$

The last inequality is true because $a > 1, b > 1$.

Solution.

$$abc + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > a + b + c + \frac{1}{abc} \Leftrightarrow \frac{1}{abc}\big((abc)^2 + ab + bc + ca - abc(a + b + c) - 1\big) > 0$$

$$\Leftrightarrow (ab)(bc)(ca) - [(ab)(bc) + (bc)(ca) + (ca)(ab)] + (ab + bc + ca) - 1 > 0 \Leftrightarrow (ab - 1)(bc - 1)(ca - 1) > 0$$

The last inequality is true because $a > 1, b > 1$, and $c > 1$. □

Example 3.6.3 (OT-22-23-S3-P3)

a, b, c are positive real numbers. Prove that at least one of the following inequalities is true.

$$\begin{cases} \frac{3}{a} \geq 2 - b \\ \frac{2}{b} \geq 4 - c \\ \frac{6}{c} \geq 6 - a. \end{cases}$$

Solution. Lets assume that none of them is true. Then

$$\begin{cases} \frac{3}{a} < 2 - b \Rightarrow 3 < (2-b)a \\ \frac{2}{b} < 4 - c \Rightarrow 2 < (4-c)b \\ \frac{6}{c} < 6 - a \Rightarrow 6 < (6-c)c \end{cases} \Rightarrow 36 < a(6-a)b(2-b)c(4-c) \quad (*)$$

Now, by AM-GM Inequality for two numbers,

$$\begin{cases} a(6-a) \leq \left(\frac{a+(6-a)}{2}\right)^2 = 3^2 \\ b(2-a) \leq \left(\frac{b+(2-b)}{2}\right)^2 = 1^2 \\ c(4-c) \leq \left(\frac{c+(4-c)}{2}\right)^2 = 2^2 \end{cases} \Rightarrow a(6-a)b(2-b)c(4-c) \leq 6^2 = 36 \quad (**)$$

Since (**) contradicts (*), thus the assumption is false. Hence, one of the inequality must be true. $\square$

Example 3.6.4 (OT-22-23-S3-P4)

x, y, z are positive real numbers. Prove that

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq x + y + z$$

Remark. For the *simplified* inequality, we can pair up $\frac{x^2}{y}, y$ so by AM-GM Inequality,

$$\frac{x^2}{y} + y \geq 2\sqrt{\frac{x^2}{y}y} = 2x.$$

Thus, by cycling through x, y, z and summing up the result inequalities,

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x + y + z.$$

So, we can follow the same technique in the more coomplicated problem

Solution. By AM-GM Inequality, $\frac{x^3}{yz} + y + z \geq 3\sqrt[3]{\frac{x^3}{yz}yz} = 3x$. Thus $\frac{y^3}{zx} + z + x \geq 3y$, $\frac{z^3}{xy} + x + y \geq 3z$. Hence, $\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq x + y + z$. $\square$

Example 3.6.5 (OT-22-23-S3-P5)

a, b, c, d are positive real numbers. Prove that

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b} \geq 0.$$

Solution.

$$\begin{aligned}4+\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b} &= \frac{a+c}{b+c}+\frac{b+d}{c+d}+\frac{c+a}{d+a}+\frac{d+b}{a+b} \\ &= (a+c)\left(\frac{1}{b+c}+\frac{1}{d+a}\right)+(d+b)\left(\frac{1}{c+d}+\frac{1}{a+b}\right)\end{aligned}$$

Note that, from the result of Arithmetic Mean - Harmonic Mean Inequality, for $x, y > 0$,

$$\frac{1}{x}+\frac{1}{y} \geq \frac{4}{x+y}.$$

Thus

$$\begin{aligned}&(a+c)\left(\frac{1}{b+c}+\frac{1}{d+a}\right) \geq \frac{4(a+c)}{a+b+c+d},\ (d+b)\left(\frac{1}{c+d}+\frac{1}{a+b}\right) \geq \frac{4(b+d)}{a+b+c+d} \\ \Rightarrow\ & 4+\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b} \geq 4 \Rightarrow \frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b} \geq 0.\end{aligned}$$

□

Example 3.6.6 (OT-22-23-S3-P6)

a, b, c are positive real numbers less than 1. Prove that

$$\sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)} < 1.$$

Solution. Note that for all $0 < x < 1 : \sqrt{x} < \sqrt[3]{x}$, thus

$$\begin{aligned}&\sqrt{abc} \leq \sqrt[3]{abc} < \frac{a+b+c}{3}, \\ &\sqrt{(1-a)(1-b)(1-c)} \leq \sqrt[3]{(1-a)(1-b)(1-c)} < \frac{(1-a)+(1-b)+(1-c)}{3} \\ \Rightarrow\ & \sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)} < \frac{a+b+c}{3}+\frac{(1-a)+(1-b)+(1-c)}{3} = 1.\end{aligned}$$

□

Example 3.6.7 (OT-22-23-S3-P7)

a, b, c, d are positive real numbers. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

Solution. Note that the denominator of the fraction on the right is the sum of the denominators of the fractions on the left. So by T2 Lemma for the $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 4$, and $b_1 = a, b_2 = b, b_3 = c, b_4 = d$, sequences

$$\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d} \geq \frac{(1+1+2+4)^2}{a+b+c+d} = \frac{64}{a+b+c+d}.$$

□

Example 3.6.8 (OT-22-23-S3-P8)

a, b, c are positive real numbers. Prove that

$$\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1} \geq \frac{3(a+b+c)}{3+a+b+c}.$$

Solution. Note that by multiplying both the nominator and denominator of each fraction on the left with a, b, c, respectively, we have

$$\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1} = \frac{a^2}{a(b+1)}+\frac{b^2}{b(c+1)}+\frac{c^2}{c(a+1)}$$

Now by T2 Lemma for the $a_1 = a, a_2 = b, a_3 = c$, and $b_1 = a(b+1), b_2 = b(c+1), b_3 = c(a+1)$, sequences

$$\frac{a^2}{a(b+1)}+\frac{b^2}{b(c+1)}+\frac{c^2}{c(a+1)} \geq \frac{(a+b+c)^2}{ab+bc+ca+a+b+c}.$$

By Cauchy-Schwarz Inequality

$$(ab+bc+ca)^2 = (a\cdot b+b\cdot c+c\cdot a)^2 \leq (a^2+b^2+c^2)(b^2+c^2+a^2) \Rightarrow ab+bc+ca \leq a^2+b^2+c^2$$
$$\Rightarrow 3(ab+bc+ca) \leq a^2+b^2+c^2+2(ab+bc+ca) = (a+b+c)^2$$

Thus

$$\frac{(a+b+c)^2}{ab+bc+ca+a+b+c} \geq \frac{(a+b+c)^2}{\frac{1}{3}(a+b+c)^2+a+b+c} = \frac{3(a+b+c)}{3+a+b+c}.$$

□

Example 3.6.9 (OT-22-23-S3-P9)

a, b, c are positive real numbers. Prove that

$$a^5+b^5+c^5 \geq a^4b+b^4c+c^4a.$$

Solution. Lets assume that $a \geq b \geq c$, then $a^4 \geq b^4 \geq c^4$, by rearrangement inequality,

$$a^4\cdot a+b^4\cdot b+c^4\cdot c \geq a^4\cdot b+b^4\cdot c+c^4\cdot a.$$

Similarly for the case $a \geq c \geq b$. □

Example 3.6.10 (OT-22-23-S3-P10)

a, b, c are positive real numbers. Prove that

$$\frac{a^2+c^2}{b} + \frac{b^2+a^2}{c} + \frac{c^2+b^2}{a} \geq 2(a+b+c).$$

Solution. Lets assume that $a \geq b \geq c$, then $a^2 \geq b^2 \geq c^2$, and $\frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c}$, thus by rearrangement inequality,

$$\begin{aligned} \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} &\geq \frac{a^2}{a} + \frac{b^2}{b} + \frac{c^2}{c} \\ \frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b} &\geq \frac{a^2}{a} + \frac{b^2}{b} + \frac{c^2}{c} \\ \Rightarrow \frac{a^2+c^2}{b} + \frac{b^2+a^2}{c} + \frac{c^2+b^2}{a} &\geq 2(a+b+c). \end{aligned}$$

□

Chapter 4

Two olympiad geometry problems

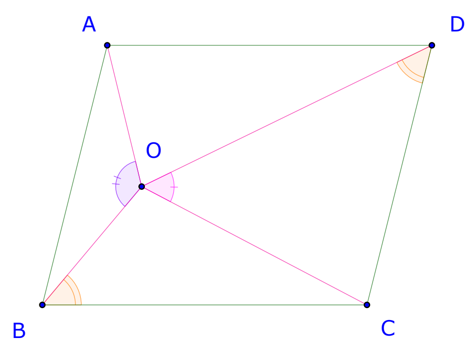
4.1 Examples

Example 4.1.1 (Canada MO 1997/P4)

The point O is situated inside the parallelogram $ABCD$ such that

$$\angle AOB + \angle COD = 180^\circ.$$

Prove that $\angle OBC = \angle ODC$.



Definition (Geometric transformations)**.** A transformation is an operation that moves, flips, or otherwise changes a figure to create a new figure. A rigid transformation (also known as an isometry or congruence transformation) is a transformation that does not change the size or shape of a figure.

The rigid transformations are *translations*, *reflections*, and *rotations.* The new figure created by a transformation is called the image. The original figure is called the preimage.

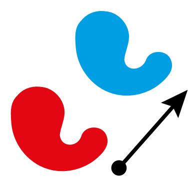


Figure 4.1: Translation

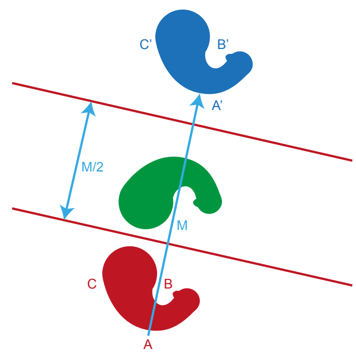


Figure 4.2: Reflections

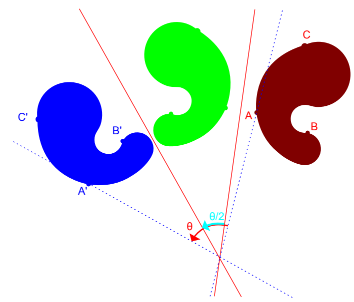
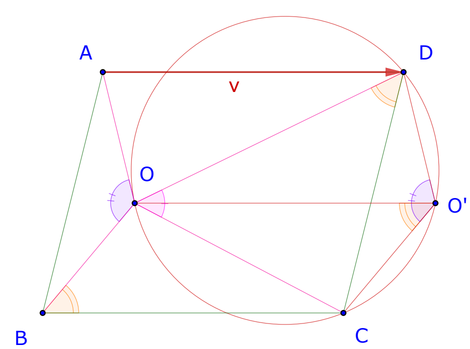


Figure 4.3: Rotations

> **Remark.** In this proof, we look at the sum $\angle AOB + \angle COD = 180^\circ$. Instead to have the two angles sharing a same vertex to have their sum of measures to be 180°, we *translate* - the angle $\angle AOB$ to $\angle DO'C$ so that they form a quadrilateral $ODO'C$. The main reason of doing so is because in the quadrilateral $ODO'C$ two opposite angles having a sum of 180°, which makes the quadrilateral to be cyclic.

1^{st} ***proof based on translation to cyclic quadrilateral****.* The translation by $\overline{v} = \overrightarrow{AD}$ maps A to D, B to C, and O to O'. Therefore $CO' \parallel BO$, $CO' = BO$, so $OBCO'$ is a parallelogram. Thus $\angle OO'C = \angle OBC$.



Furthermore $\angle DO'C = \angle AOB$, so $\angle DO'C + \angle COD = 180^\circ$, thus $CODO'$ is a cyclic quadrilateral. Therefore $\angle ODC = \angle OO'C = \angle OBC$. $\square$

Theorem (Law of Sines)

In $\triangle ABC$, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$.

Remark. In this proof, we extend DO to meet AB at Q. By doing so, we have

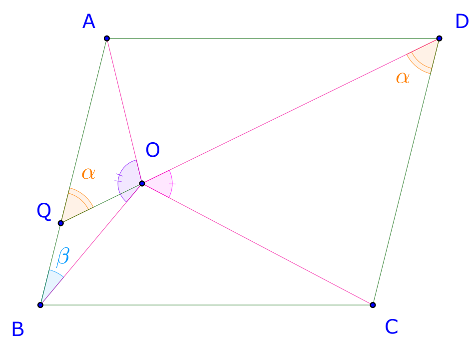
$$\angle AQO = \angle ODC \text{ and } \angle AOQ = \angle AOB - \angle QOB = 180^\circ - \angle DOC - \angle QOB = \angle BOC.$$

Thus if $\angle OBC = \angle ODC$, then $\triangle AOQ \sim \triangle COB$. We prove this similarity by using the Law of Sines.

2^{nd} ***proof based on similarity by the Law of Sines.*** Let Q be the intersection of DO and AB.

We have, $\angle AQO = \angle ODC$ and $\angle AOQ = \angle AOB - \angle QOB = 180^\circ - \angle DOC - \angle QOB = \angle BOC$.

Thus, if (*) $\frac{CO}{BO} = \frac{AO}{QO}$ then $\triangle AOQ \sim \triangle COB$, hence, $\angle OBC = \angle OQA = \angle ODC$.



Now, let $\alpha = \angle ODC$, $\beta = \angle OBA$.

By the Law of Sines for $\triangle CDO$, and note that $\angle COD = 180^\circ - \angle AOB$, $CD = AB$,

$$\frac{CD}{CO} = \frac{\sin\angle COD}{\sin\alpha} \Rightarrow \frac{AB}{CO} = \frac{\sin\angle AOB}{\sin\alpha} \Rightarrow \frac{CO}{\sin\alpha} = \frac{AB}{\sin\angle AOB} \quad (1).$$

By the Law of Sines for $\triangle AOB$, and by (1),

$$\frac{AB}{\sin\angle AOB} = \frac{AO}{\sin\beta} \Rightarrow \frac{CO}{\sin\alpha} = \frac{AO}{\sin\beta} \Rightarrow \frac{CO}{AO} = \frac{\sin\alpha}{\sin\beta}.$$

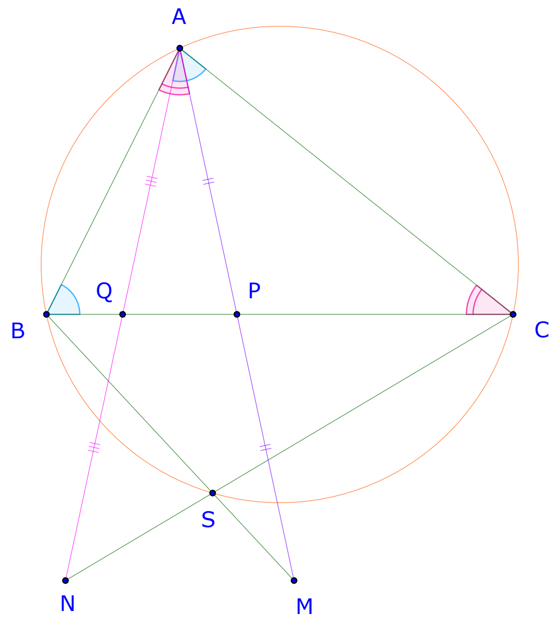
By the Law of Sines for $\triangle BQO$ and note that $\angle BQO = 180^\circ - \alpha$, thus

$$\frac{BO}{QO} = \frac{\sin\angle BQO}{\sin\beta} = \frac{\sin\alpha}{\sin\beta} \Rightarrow \frac{CO}{AO} = \frac{BO}{QO} \Rightarrow \frac{CO}{BO} = \frac{AO}{QO}.$$

The last ratio equation is what we need for (*). $\square$

Example 4.1.2 (IMO 2014/4)

Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA, \angle CAQ = \angle ABC$. Let M and N be points on lines AP and AQ, respectively, such that P and Q are midpoints of AM and AN, respectively. Prove that the intersection S of BM and CN is on the circumference of $\triangle ABC$.



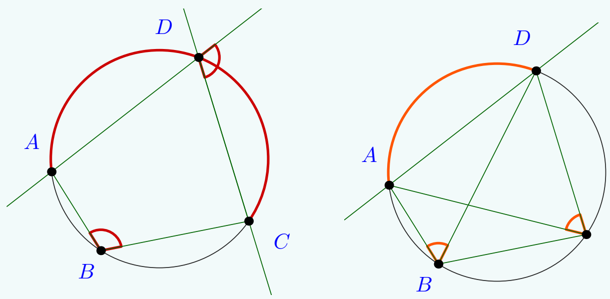
Theorem (Cyclic Quadrilaterals)

Let $ABCD$ be a convex quadrilateral. Then the following are **equivalent**:

(i) $ABCD$ is cyclic.

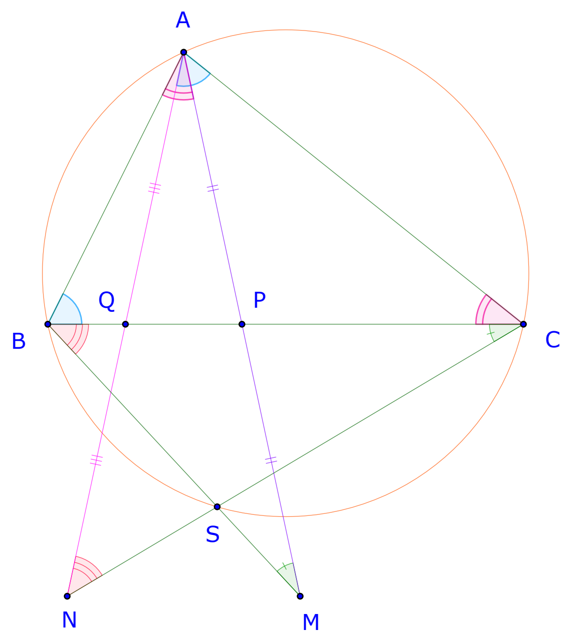
(ii) $\angle ABC + \angle CDA = 180°$.

(iii) $\angle ABD = \angle ACD$.



Remark. First, we chase the angles to prove that $\triangle PBA \sim \triangle QAC \sim \triangle ABC$. Second, if $\triangle BPM \sim \triangle NQC$ is true, then by proving that $\angle BSC = 180° - \angle BAC$. the quadrilateral $ABSC$ cyclic.

1^{st} ***proof based on Cyclic Quadrilaterals.*** First, $\angle PAB = \angle BCA = \angle C$, $\angle CAQ = \angle ABC = \angle B$. Thus $\triangle PBA \sim \triangle QAC \sim \triangle ABC$.



Thus, $\frac{PB}{PA} = \frac{QA}{QC}$, or $\frac{PB}{PM} = \frac{QN}{QC}$ (1).

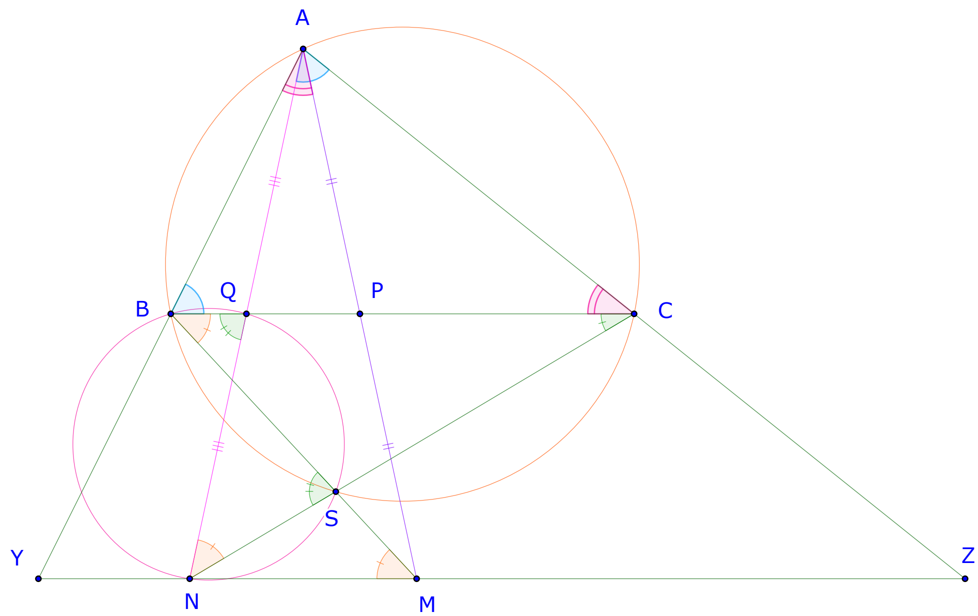
Now, $\angle BPM = \angle PAB + \angle PAB = \angle B + \angle C = \angle QAC + \angle QCA = \angle NQC$ (2).

From (1) and (2) $\triangle BPM \sim \triangle NQC$, so $\angle SBC = \angle MBP$, $\angle SCB = \angle NCQ$. Thus, $\triangle BSC \sim \triangle BPM$, or $\angle BSC = \angle BPM = \angle B + \angle C = 180° - \angle A$. Therefore $ABSC$ is cyclic and S is on the circle (ABC). □

> **Remark.** Here we show another proof based on similarity using median segments and cyclic quadrilateral.

2^{nd} ***proof based on Cyclic Quadrilaterals***. First, $\angle PAB = \angle BCA = \angle C$, $\angle CAQ = \angle ABC = \angle B$. Thus $\triangle PBA \sim \triangle QAC \sim \triangle ABC$.

Now, extend AB and AC to intersect line MN at Y and Z, respectively. Since Q, P are median segment, thus $MN \parallel BC$. Therefore $\triangle AMY \sim \triangle ABP \sim \triangle CAQ \sim \triangle ZAN$. B is the midpoint of AY in $\triangle AMY$, C is the midpoint of ZA in $\triangle ZAN$. By similarity $\angle BNY = \angle CNA$.

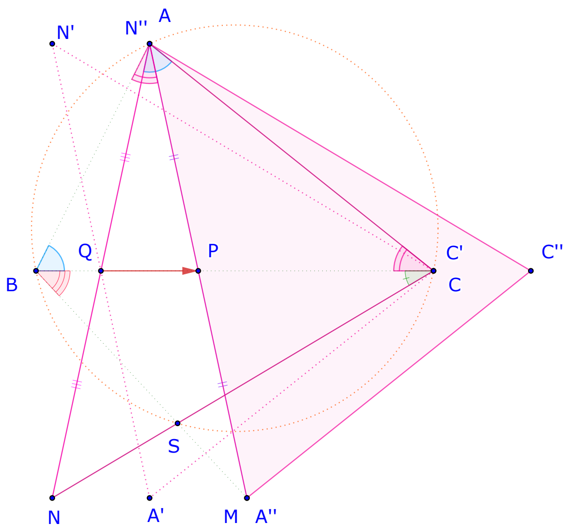


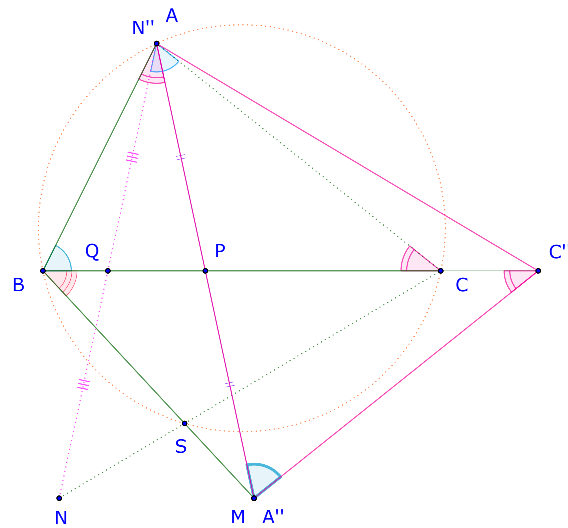
Therefore $\angle SBC = \angle BMN = \angle CNQ$, so $BQSN$ is cyclic, thus $\angle NSB = \angle NQB = \angle CQA = \angle A$. Hence, $\angle BSC = 180° - \angle A$, $ABSC$ is cyclic and S is on the circle (ABC). □

> **Remark.** From the diagram, if we can prove that $\angle ABS + \angle ACS = 180°$. then $ABSC$ is cyclic and obviously we are done. From the construction, it seems that $AM = AN$. If that is the case, then we can flip the $\triangle ANC$ over the line BC then move it along BC so that the bases $AN \equiv MA$. Thus, there is no more point S.

3^{rd} ***proof based on rigid transformations.*** First, $\angle PAB = \angle BCA = \angle C$, $\angle CAQ = \angle ABC = \angle B$. Thus $\triangle PBA \sim \triangle QAC \sim \triangle ABC$,

$$\frac{AP}{AB} = \frac{AC}{BC}, \ \frac{AQ}{AC} = \frac{AB}{BC} \Rightarrow AP = AQ.$$

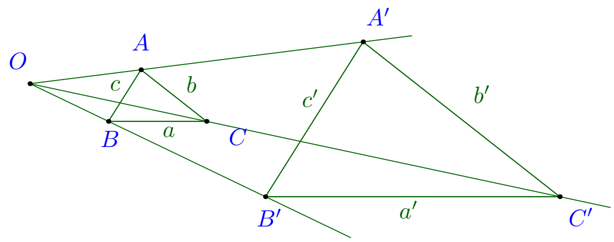




Thus, we can reflect the triangle ANC over the line BC to be $A'N'C'$, then translate it by the vector QP to be $A"N"C"$. Then segment $AM \equiv N"A"$. P is still the midpoint of $AM \equiv N"A"$, and $\angle BPA = \angle CQA = 180° - \angle C"PA$. Thus $C", C, P, Q, B$ arew collinear.

In quadrilateral $AC"A"B$, $\angle C"BA = \angle C"A"A$, thus it is cyclic, therefore $\angle A"BA + \angle A"C"A = 180°$. Thus $\angle ABS + \angle ACS = 180°$. Therefore $ABSC$ is cyclic and S is on the circle (ABC). $\square$

Definition (Homothety)**.** A **homothety** (or homothecy) is a transformation of space which dilates distances *with respect to a fixed point.* A homothety can be an *enlargement* (resulting figure is larger), *identity* transformation (resulting figure is congruent), or a *contraction* (resulting figure is smaller).



A homothety with center O and factor k, denoted by $\mathcal{H}_{(O,k)}$, sends point A to a point A', and $OA' = k \cdot OA$.

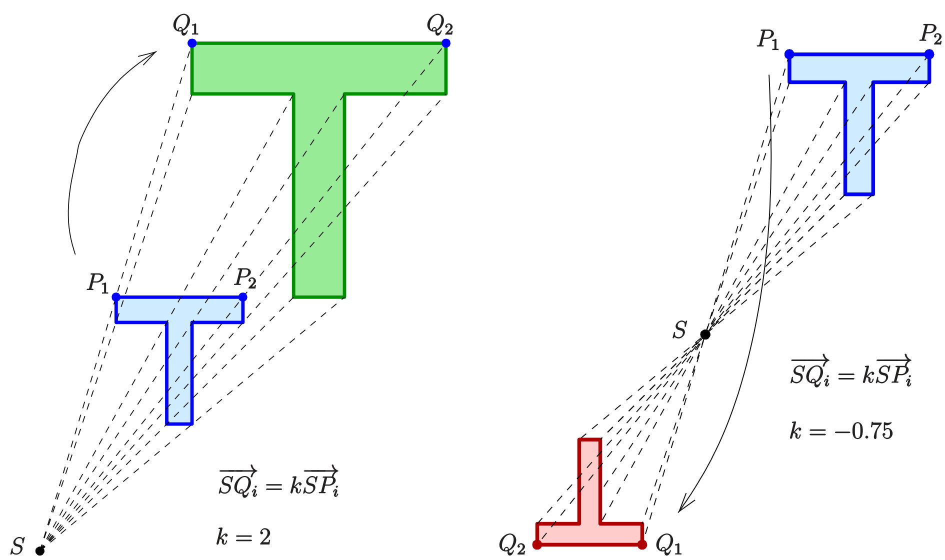
Fact (Homothety Images)**.** Let $\mathcal{H}_{(O,k)}$ be a homothety,

(i) For point A, $\mathcal{H}_{(O,k)}(A) = A' \Rightarrow O, A, A'$ colliner. The lines connecting each point to its images are all concurrent.

(ii) For line ℓ, $\mathcal{H}_{(O,k)}(\ell) = \ell' \Rightarrow \ell \parallel \ell'$. The image of a line is parallen to the line.

(iii) For polygon P, $\mathcal{H}_{(O,k)}(P) = P' \Rightarrow P \sim P'$. The image of a polygon is similar to the polygon.

Fact (Homothety Factor)**.** Let $\mathcal{H}_{(O,k)}$ be a homothety,



(i) If $k > 0$, then the image and the original will be on the same side of the center, they are are scaled and translated similar to one another.

(ii) If $k < 0$, the image and the original will be on different sides of the center, i.e. the center will be between them. They are scaled, $180°-$rotated and translated images of one another.

(iii) If $|k| > 1$, then the homothety is a magnification (enlargement);

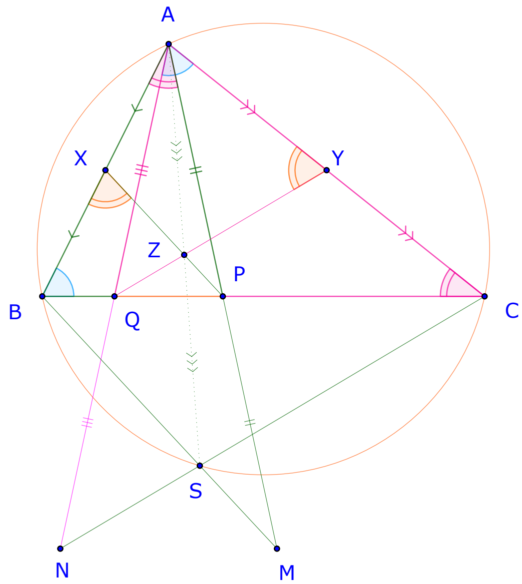
(iv) If $|k| < 1$, then it is a reduction (shrinking).

(v) A homothety with factor $k = -1$ is a $180°$ rotation about the center.

(vi) Circles are geometrically similar to one another and *rotation invariant.* These two homothetic centers lie on the line joining the centers of the two circles.

> **Remark.** It seems that a homothety $\mathcal{H}_{(A,2)}$, with center A and factor 2 sends P to M, and Q to N. Thus it sends some Z point to $.S$ Z would be an intersecion of the originals of MB and NC.

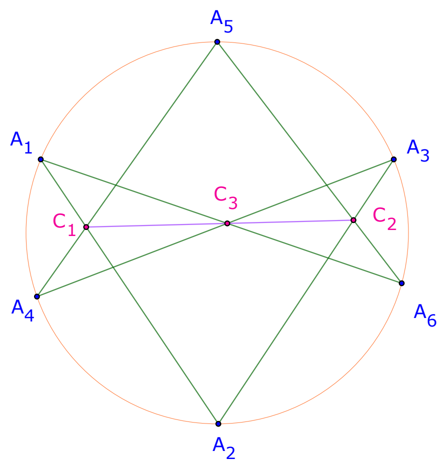
4^{th} ***proof based on Homothety.*** Let X and Y be the midpoint of AB and AC, respectively. Let Z be the intersection of PX and QY. $\angle PAB = \angle BCA = \angle C$, $\angle CAQ = \angle ABC = \angle B$. Thus $\triangle PBA \sim \triangle QAC$, thus $\angle BXP = \angle AYQ$. By Cyclic Quadrilaterals $AXZY$ is cyclic.



The homothety $\mathcal{H}_{(A,2)}$ (center A and factor 2) sends X, Y, P, Q to B, C, M, N, respectively. $S = MB \cap NC$ ($MB \cap NC$ denotes the intersection of MB and NC) thus it is the image of $PX \cap QY = Z$. $AXZY$ is cyclic. $ABSC$ is similar to $AXZY$ and therefore cyclic, too. Hence, S is on the circle (ABC). $\square$

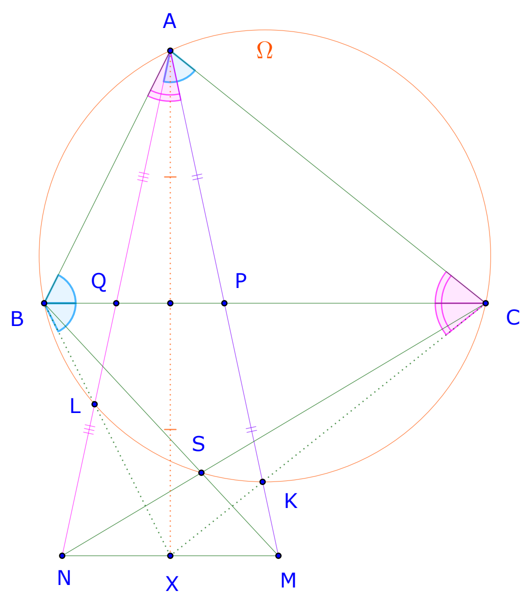
Theorem (Pascal's Theorem)

Let $\mathcal{P} = A_1A_2A_3A_4A_5A_6$ be a hexagon, $C_1 = A_1A_2 \cap A_4A_5$, $C_2 = A_2A_3 \cap A_5A_6$, $C_3 = A_3A_4 \cap A_6A_1$. Then $\mathcal{P}$ is a cyclic hexagon (which is circumscribed by a circle) if and only if C_1, C_2, C_3 are collinear.



Remark. Let Ω be the circle. $L = \Omega \cap AN$, $K = \omega \cap AM$. Is $ALBSCK$ a cyclic hexagon?

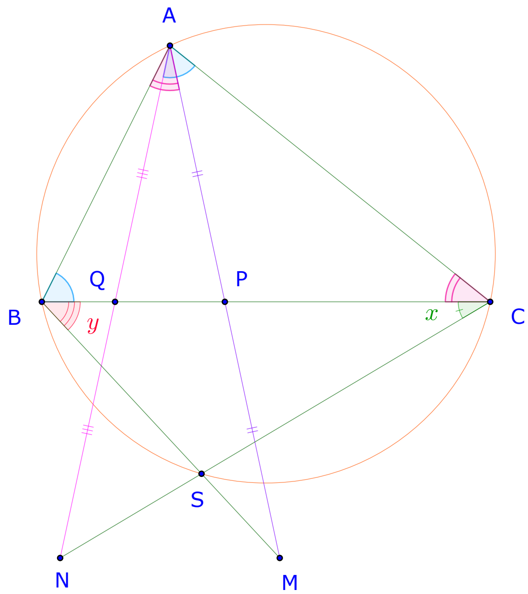
5^{th} ***proof based on Pascal's Theorem.*** Let Ω denote the circle. Let points $L = \Omega \cap AN$, $K = \Omega \cap AM$. Since $\angle LBC = \angle LAC = \angle CBA$ and $\angle KCB = \angle KAB = \angle BCA$, thus $X = BL \cap CK$ is an image of A by the reflection over the line BC.



Since Q and P are midpoints of AN and AM, so X must be on NM. Thus if $\mathcal{P} = A_1A_2A_3A_4A_5A_6$ denotes $ALBSCK$, then since $N = AL \cap SC$, $X = LB \cap CK$, and $M = BS \cap KA$ are collinear thus $ALBSCK$ is cyclic. Therefore S is on the circle (ABC). $\square$

Remark. In this proof, we will let $\angle QCN = x$, $\angle PBM = y$, then by trigonometry, prove that $x + y = 180°$.

7^{th} proof based on the Law of Sines. First $\angle PAB = \angle BCA = \angle C$, $\angle CAQ = \angle ABC = \angle B$. Thus $\triangle PBA \sim \triangle QAC \sim \triangle ABC$.



By the Law of Sines for $\triangle QCN$, and note that $\angle QNC = \angle AQC - x = \angle A - x$,

$$\frac{QN}{QC} = \frac{\sin \angle QCN}{\sin \angle QNC} = \frac{\sin x}{\sin(\angle A - x)},\ QN = QA,\ \frac{QA}{QC} = \frac{AB}{AC} \Rightarrow \frac{AB}{AC} = \frac{QA}{QC} = \frac{QN}{QC} = \frac{\sin x}{\sin(\angle A - x)} \quad (1)$$

Similarly for $\triangle PBM$, $\dfrac{AC}{AB} = \dfrac{\sin y}{\sin(\angle A - y)}$ (2).

From (1) and (2), and note that $\sin\alpha \sin\beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$,

$$1 = \frac{AB}{AC} \cdot \frac{AC}{AB} = \frac{\sin x}{\sin(\angle A - x)} \cdot \frac{\sin y}{\sin(\angle A - y)} \Rightarrow \sin(\angle A - x)\sin(\angle A - y) = \sin x \sin y$$

$$\Rightarrow \cos(y - x) - \cos(x + y - 2\angle A) = \cos(x - y) - \cos(x + y) \Rightarrow \cos(2\angle A - (x + y)) = \cos(x + y) \quad (3)$$

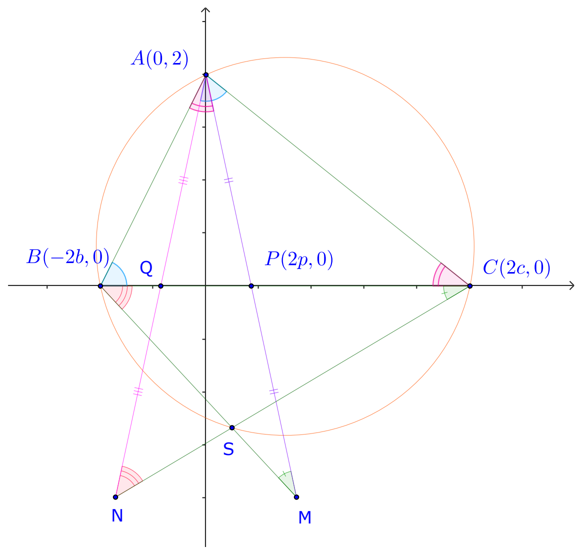
Since $x < \angle A, y < \angle A$ and $\cos\alpha = \cos(-\alpha)$,

$$(3) \Rightarrow 2A - (x + y) = x + y \Rightarrow x + y = A \Rightarrow \angle BSC = 180° - \angle A.$$

Therefore $ABSC$ is cyclic and S is on the circle (ABC). $\square$

> **Remark.** Following is a *more advanced solution* based on analytical geometry. Make sure that you are through the definitions, theorems, and examples of the LPS Vol 3., Geometry Part, Computational Geometry.

8^{th} ***proof based on analytical geometry.*** Let $A(2,0)$, $B(-2b,0)$, $C(2c,0)$, $P(2p,0)$ $(0<b<p<c)$ be the coordinates. Let A, B, C be the measures of the $\angle A$, $\angle B$, $\angle C$, respectively;



$$\cot B = \frac{OB}{OA} = b, \cot C = c, \cot A = \cot\left(180^\circ - B - C\right) = -\cot\left(B+C\right) = \frac{1-bc}{b+c}$$
$$\frac{PO}{AO} = \cot A \Rightarrow p = \cot \angle APB = \frac{1-bc}{b+c}, \Rightarrow P\left(\frac{2(1-bc)}{b+c},0\right),\ Q\left(\frac{-2(1-bc)}{b+c},0\right),$$
$$P,\ Q \text{ are midpoints of } AM,\ AN \Rightarrow M\left(\frac{4(1-bc)}{b+c},2\right),\ N\left(\frac{-4(1-bc)}{b+c},-2\right).$$

The slope of line BM is $\dfrac{2-0}{\dfrac{4(1-bc)}{b+c}-(-2b)} = \dfrac{b+c}{b^2-bc+2}$. The equation of line BM is

$$y-0 = \left(\frac{b+c}{b^2-bc+2}\right)x-(-2b) \Rightarrow y = \left(\frac{b+c}{b^2-bc+2}\right)x + \frac{2b(b+c)}{b^2-bc+2}.$$

Let D be the circumcentre of (ABC). $D_x = \frac{1}{2}(B_x+C_x) = \dfrac{2c+(2b)}{2} = c-b$. Line AC is $y = -\dfrac{1}{c}x+2$ the perpendicular bisector of AC has the slope c and is through $(c,1)$. Thus, its equation of is $y = cx+(1-c^2)$. Therefore $D_y = c(c-b)+(1-c^2) = 1-bc$. Furtheremore, the circumradius of (ABC) is:

$$R = \frac{AB\cdot BC\cdot CA}{4[ABC]} = \frac{2\sqrt{1+b^2}\cdot(2c-2b)\cdot 2\sqrt{1+4^2}}{4\cdot\frac{1}{2}(2)((2c-2b))} = \sqrt{(1+b^2)(1+c^2)}.$$

Thus, the equation of the circumcircle (ABC) is $\left[x-(c-b)\right]^2+\left[y-(1-bc)\right]^2 = (1+b^2)(1+c^2)$. It is easy to verify that the intersection of BM with (ABC) is the same as the intersection of CN with (ABC), which is point S with coordinates symmetric in regards to b and c,

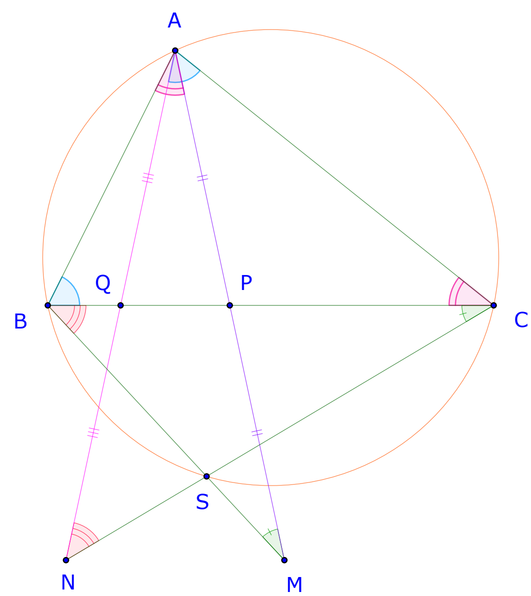
$$S\left(2\frac{(c-b)(2-bc)}{(c-b)^2+4}, -2\frac{(c+b)^2}{(c-b)^2+4}\right) \Rightarrow S = BM\cap CN \in (ABC)$$

$\square$

> **Remark.** Following is a *more advanced solution* based on complex numbers. Make sure that you are through the definitions, theorems, and examples of the LPS Vol 3., Geometry Part, Complex Numbers.

9^{th} ***proof based on complex number***. Let $X = a + ib$ complex number is represented by a point $X(a, b)$ on the complex plane. In short, we write X, and interpret it as both complex number and a number on the complex plane (of course depending on what context).

Note that for points X, Y on the complex plane then $X - Y$ is the *directed* distance of them (It is easy to derive by denote $X(x_1, x_2)$, $Y(y_1, y_2)$ then $(X - Y)(x_1 - x_2, y_1 - y_2)$.) Also note that $\overline{X}$ is the conjugate of X.



Now, $\triangle ABC \sim \triangle PBA$, ABC is anti-clockwise, while PBA is clockwise, thus they have different orientations, therefore

$$\frac{A-P}{B-P} = \overline{\left(\frac{C-A}{B-A}\right)} \quad (1)$$

Similarly

$$\frac{C-Q}{A-Q} = \overline{\left(\frac{C-A}{B-A}\right)} \quad (2)$$

But P and Q are midpoints of AM and AN, respectively, thus

$$(3) \quad \begin{cases} M - P = -(A - P) \\ N - Q = -(A - Q) \end{cases}$$

From (1), (2), and (3)

$$\frac{M-P}{B-P} = \frac{C-Q}{N-Q}, \text{ thus } \triangle MPB \sim \triangle CQN.$$

From here it is easy to see that $\angle BSC + \angle A = 180°$, thus $BM \cap CN \in (ABC)$. $\square$

4.2 Problems

Theorem (Ceva Theorem)

Let ABC be a triangle, and let D, E, F be points on lines BC, CA, AB, respectively. Lines AD, BE, CF are concurrent *if and only if*:

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

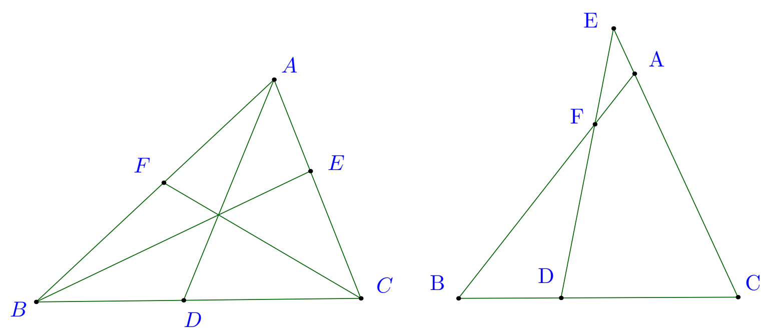


Figure 4.4: Ceva (left) and Menelaus (right) theorems

Theorem (Menelaus Theorem)

Let ABC be a triangle, and let D, F be points on lines BC, AB, respectively. E is on the extension of CA. Points D, E, F are collinear *if and only if*:

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Lemma (Sinus ratio of midpoint)

In $\triangle ABC$, D is midpoint of BC, then $\dfrac{\sin \angle BAD}{\sin \angle DAC} = \dfrac{\sin \angle ABC}{\sin \angle ACB}$.

Attention:

In this problem section, some problem can be solved by different (suggested) approaches. The submission of multiple solutions with significant differences to the same problem will earn more than a single solution.

Problem 4.2.1 (OT-22-23-S4-P1)**.** The diagonals of a convex quadrilateral $ABCD$ intersect at E. Prove that the circumcenters of the triangles ABE, BCE, CDE, and DAE are vertices of a parallelogram.

Problem 4.2.2 (OT-22-23-S4-P2)**.** Let Ω be the circumcircle of triangle ABC. Line ℓ_1 tangent to Ω at C. Line ℓ_2 is parallel to ℓ_1 intersecting BC and AC at points D and E, respectively. Prove that $ABDE$ is a cyclic quadrilateral.

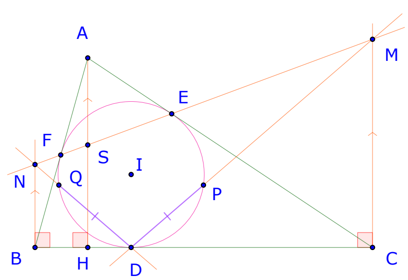
Problem 4.2.3 (OT-22-23-S4-P3)**.** The diagonals of trapezoid $ABCD$ meet at point E. Point P is on CD; points R and S on the bases BC and AD, respectively, such that segments PR and PS are parallel to the diagonals of the trapezoid. Prove that line RS passes through point E.

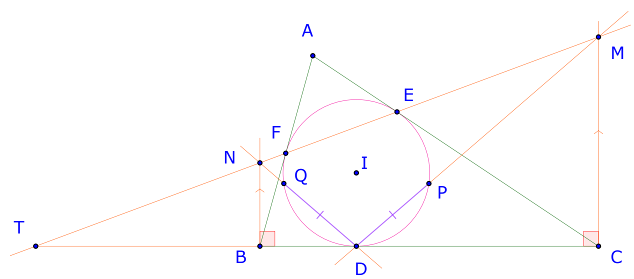
Problem 4.2.4 (OT-22-23-S4-P4)**.** In convex quadrilateral $ABCD$, point F is the intersection of AC and BD. If $\angle CAD = 90°$, $\angle ADB = \frac{1}{2}\angle ACB$, and $\angle CBD + 2\angle ADC = 180°$, prove that $BF = 2AF$.

> **Remark.** Let point E on BF be the foot of the angle bisector of $\angle ACB$. Prove that $ADCE$ is a cyclic quadrilateral.

Problem 4.2.5 (OT-22-23-S4-P5)**.** Let ABC be an acute triangle such that $AB \neq AC$. The incircle (I) touches the sides BC, CA and AB at D, E, and F, respectively. Line through C and perpendicular to BC intersects line through EF at M. Line through B and perpendicular to BC intersects line through EF at N. Line DM and DN intersect the circle (I) again at P and Q, respectively. Prove that $DP = DQ$.

> **Remark.** For a proof based on *triangle similarity*, let S be the intersection point of the altitude from A with the line EF. For $DP = DQ$, we need $\widehat{DP} = \widehat{DQ}$, or $\angle CDM = \angle BDN$. Don't forget the property of incircle (I). See below on the left.

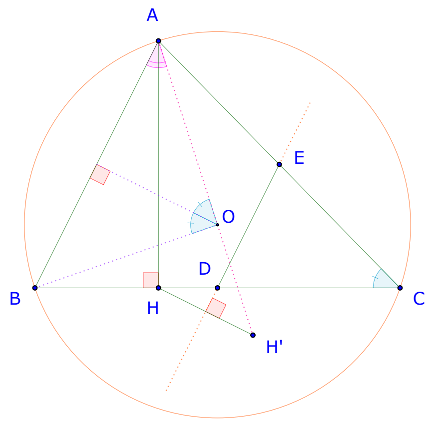




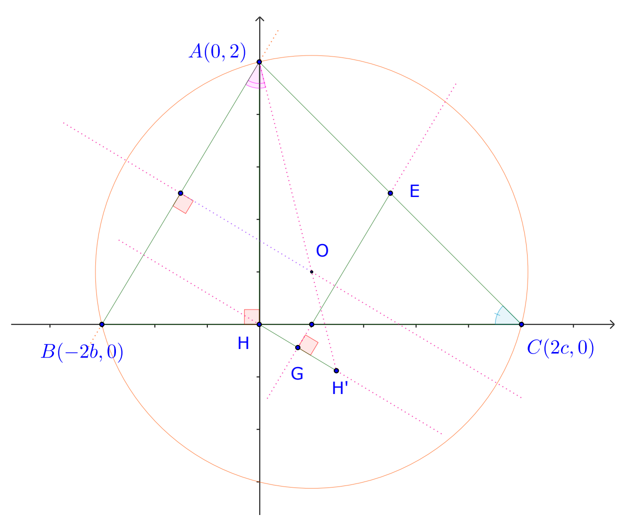
> **Remark.** For a proof based on *collinearity*, Let $T = \overline{EF} \cap \overline{BC}$. $\angle BDN = \angle CDM$ can be proven by the Menelaus Theorem for $\triangle ABC$ and line EFT. Don't forget the property of incircle (I). See above on the right.

Problem 4.2.6 (OT-22-23-S4-P6)**.** ABC is a triangle with only acute angles. Points H is the feet of the altitude from A to BC. Points D and E are midpoints of BC and CA, respectively. Let H' be the image of H by the reflection with respect to the line DE. Prove that the circumcenter O of $\triangle ABC$ is on the line through AH'.

> **Remark.** For a proof based on *cyclic quadrilateral*, you can start by proving $AHH'C$ is cyclic. Then note that if O is on line AH', then $\angle AOB = 2\angle ACB$, so $\angle BAH' = \angle BAO = 90° - \angle ACB$. See below.

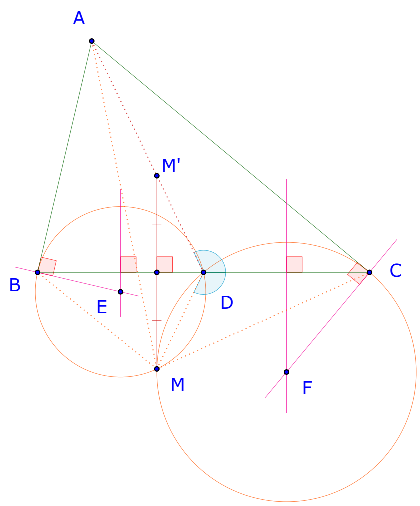


> **Remark.** For a proof based on *analytical geometry*, transform $\triangle ABC$ into $A(0,2)$, $B(-2b,0)$, and $C(2c,0)$ $(b,c>0)$. This way X is fixed but positions of B and C are depending on b and c, thus it comprehends any shape of an acute triangle. The factor 2 will greatly simplify calculation for midpoints. First determine the coordinates of the circumcenter O based on the perpendicular bisectors of line AB and line BC. Then determine H' coordinates based on G, which is the intersecion of the line through H and perpendicular to ED. By comparing the *slopes* of line AO and AH', you can prove that O is on the line AH'. See below.

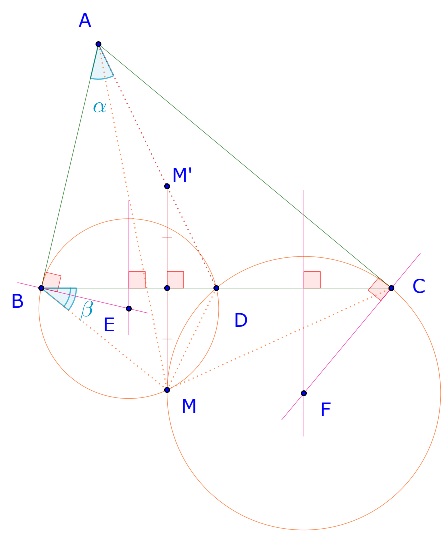


Problem 4.2.7 (OT-22-23-S4-P7)**.** In $\triangle ABC$, D is midpoint of BC. Circle Ω passes through D and tangent to AB at B. Circle Γ passes through D and tangent to AC at C. Ω and Γ meet again at $M \not\equiv D$. N is the image of M by the reflection with respect to BC. Prove that N is on AD.

> **Remark.** For a proof based on *cyclic quadrilateral*, in order to have M' to be on AD, we need line AD to be the reflection of line MD with respect to BC. To reach that you have to, first prove that M is on the circumcircle (ABC), then continue to find similar triangle pair(s) to show $\angle ADC = \angle MDC$. See below on the left.



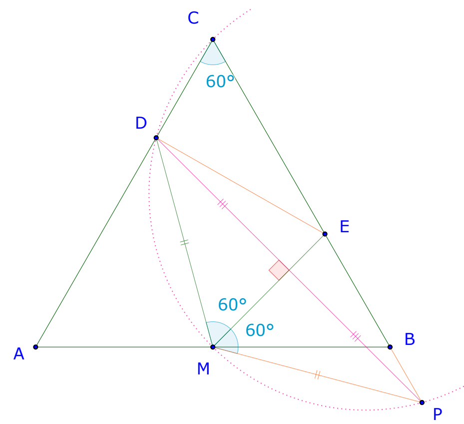
> **Remark.** For a proof based on *trigonometry*, consider using the Law of Sines and then the lemma Sinus ratio of midpoint to prove that $\angle ADB = \angle BDM$.



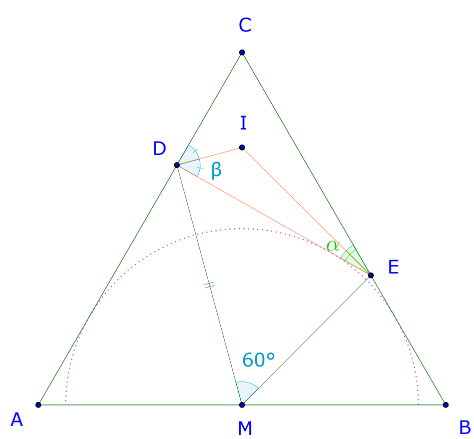
Problem 4.2.8 (OT-22-23-S4-P8)**.** $\triangle ABC$ is equilateral. M is midpoint of AB. Points D and E are on on AC and BC, respectively, such that $\angle DME = 60°$. Prove that, $AD - DE + EB = \frac{1}{2}AB$.

Remark. For a proof based on *cyclic quadrilateral*, the reflection of DE over EM would produce EP, and *if* P *is collinear* with E and B, then the segment DE can be brought onto the perimeter of $\triangle ABC$, thus *in principle* the terms of the equation $AD - DE + EB = \frac{1}{2}AB$ can be computed based on the same types (AB, BC, CA.)

Are E, B, and P collinear? $\angle DMP = 120°$, $\angle C = 60°$, so $CDMP$ is cyclic. Thus, we can try a *synthetic* proof by constructing a circle through C, D, and M meeting EB at P. Then investigate if P is the image of D by the reflection over EM.



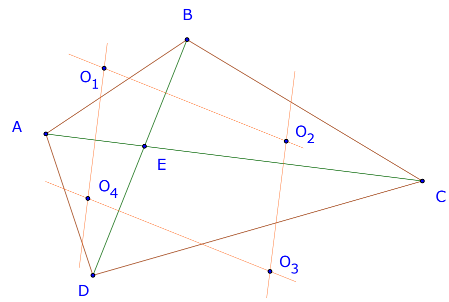
Remark. For another proof based on *cyclic quadrilateral*, the quantity $AD - DE + EB$ reminds similar sums, which are (alternate) sums of sides of a triangles with respect to the tangent points of the incircle. Perhaps there is a circle tangent to all AD, DE, and EB sides? It is worth to consider I the incenter of $\triangle CDE$. $ADEB$ is not likely cyclic, but $DIEM$ could be.



4.3 Solution

Problem 4.3.1 (OT-22-23-S4-P1)**.** The diagonals of a convex quadrilateral $ABCD$ intersect at E. Prove that the circumcenters of the triangles ABE, BCE, CDE, and DAE are vertices of a parallelogram.

Proof. Let O_1, O_2, O_3, and O_4 be the circumcenters of $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively.



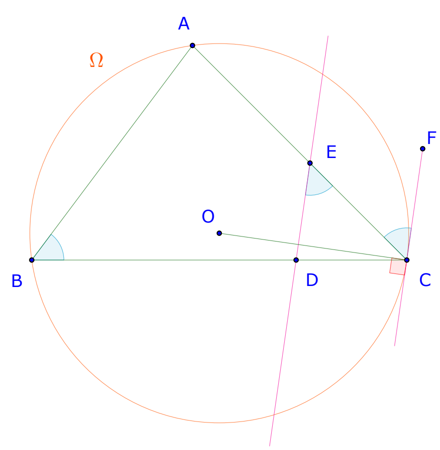
O_1 and O_2 are on the perpendicular bisector of BE, which is parallel to the perpendicular bisector of DE, where O_3 and O_4 are on, thus $O_1O_2 \parallel O_3O_4$. Similarly $O_2O_3 \parallel O_1O_4$. Hence , $\boxed{O_1O_2O_3O_4 \text{ is a parallelogram.}}$ □

Example 4.3.2 (OT-22-23-S4-P2)

Let Ω be the circumcircle of triangle ABC. Line ℓ_1 tangent to Ω at C. Line ℓ_2 is parallel to ℓ_1 intersecting BC and AC at points D and E, respectively. Prove that $ABDE$ is a cyclic quadrilateral.

> **Remark.** Always try to prove by definition first, if you don't have any other more apparent reason. For example, $\angle ABC = \angle DEC$?

JBMO Shortlist 2015. Note that CF is the tangent, so $\angle ABC = \angle ACF$. Since $DE \parallel CF$, so $\angle DEC = \angle ECF$.



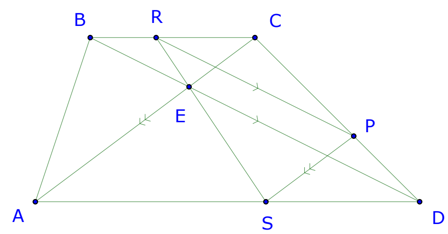
Hence, $\angle ABC = \angle DEC$, so $\boxed{ABDE \text{ is a cyclic quadrilateral.}}$ □

Example 4.3.3 (OT-22-23-S4-P3)

The diagonals of trapezoid $ABCD$ meet at point O. Point M is on CD; points P and Q on the bases BC and AD, respectively, such that segments MP and MQ are parallel to the diagonals of the trapezoid. Prove that line PQ passes through point O.

I.F Sharrygin, 2011, The Correspondence Round. **Proof based on triangle similarity.**

$$PS \parallel CA \Rightarrow \frac{AS}{AD} = \frac{CP}{CD} \text{ and } PR \parallel DB \Rightarrow \frac{CP}{CD} = \frac{CR}{CB} \Rightarrow \frac{AS}{AD} = \frac{CR}{CB} \Rightarrow \frac{AS}{CR} = \frac{AD}{CB} = \frac{AE}{CE}.$$



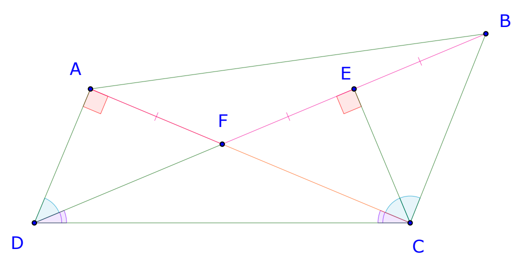
Thus, $\triangle AES \sim \triangle CER \Rightarrow \angle AES = \angle CER$, hence, $\boxed{\mathcal{B}(R, E, S).}$ □

Example 4.3.4 (OT-22-23-S4-P4)

In convex quadrilateral $ABCD$, F is the intersection of AC and BD If $\angle CAD = 90°$, $\angle ADB = \frac{1}{2}\angle ACB$, and $\angle CBD + 2\angle ADC = 180°$, prove that $BF = 2AF$.

> **Remark.** Let point E on BF be the foot of the angle bisector of $\angle ACB$. We prove that $ADCE$ is a cyclic quadrilateral.

Iran Geometry MO 2018/Medium/P2. **Proof based on cyclic quadrilateral** Let E be the foot of the angle bisector of $\angle ACB$.



Since $\angle ADB = \frac{1}{2}\angle ACB$, then $\angle ADE = \angle ACE$, thus $ADCE$ is a cyclic quadrilateral. $\angle CAD = 90°$, thus CD is the diameter of the $(ADCE)$, therefore $\angle CED = 90°$. In $\triangle BCF$, CE is both the angle bisector and the altitude from C, thus, it is isosceles. Therefore CE is also the median, so $BE = EF$ (1).

Now, $180° = \angle CBD + 2\angle ADC = \angle CBD + 2\angle ADB + 2\angle BDC = \angle CBD + \angle BCF + 2\angle BDC$. In $\triangle BCD$, $\angle CBD + \angle BCF + \angle FCD + \angle CDB = 180°$, thus $\angle FDC = \angle FCD$. By that $\triangle ACD \cong \triangle EDC$, so $AD = EC$, thus $\triangle AFD \cong \triangle EFC$, or $AF = EF$ (2).

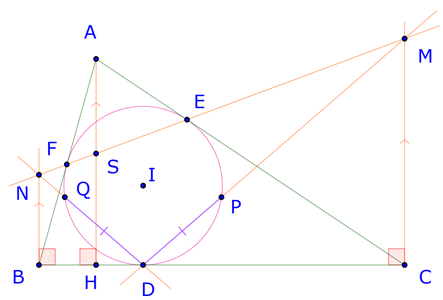
From (1) and (2), $\boxed{BF = 2AF.}$ □

Example 4.3.5 (OT-22-23-S4-P5)

Let ABC be an acute triangle such that $AB \neq AC$. The incircle (I) touches the sides BC, CA and AB at D, E, and F, respectively. Line through C and perpendicular to BC intersects line through EF at M. Line through B and perpendicular to BC intersects line through EF at N. Line DM and DN intersect the circle (I) again at P and Q, respectively. Prove that $DP = DQ$.

Remark. Let S be the intersection point of the altitude from A with the line EF. For $DP = DQ$, we need $\widehat{DP} = \widehat{DQ}$, or $\angle CDM = \angle BDF$.

Romania JBMO 2016/P4. **Proof based on triangle similarity** Let S be the intersection point of the altitude from A with the line EF. $\triangle BNF \sim \triangle ASF, \triangle CME \sim \triangle ASE$.

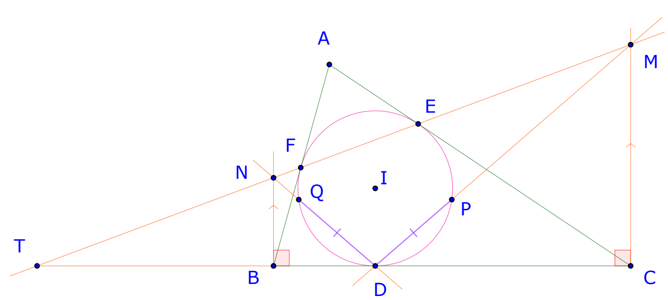


In $\triangle ABC$, $AE = AF$, thus $\dfrac{BN}{BF} = \dfrac{AS}{AF} = \dfrac{AS}{AE} = \dfrac{CM}{CE}$. Now, $BF = BD$, $CE = CD$, so $\dfrac{BN}{BD} = \dfrac{BN}{BF} = \dfrac{CM}{CE} = \dfrac{CM}{CD}$. Then, $\triangle BDN \sim \triangle CDM$, so $\angle BDN = \angle CDM$, $\widehat{DQ} = \widehat{DP}$. Hence, $\boxed{DP = DQ.}$ ☐

Remark. Let $T = \overline{EF} \cap \overline{BC}$. $\angle BDN = \angle CDM$ can be proven by the Menelaus Theorem for $\triangle ABC$ and $\overline{EFT}$.

Romania JBMO 2016/P4. **Proof based on collinearity** Let $T = \overline{EF} \cap \overline{BC}$, by Menelaus Theorem for $\triangle ABC$ and $\overline{EFT}$.

$$1 = \frac{TB}{TC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} \Rightarrow \frac{TB}{TC} = \frac{FB}{EC} (\text{since } AE = AF).$$

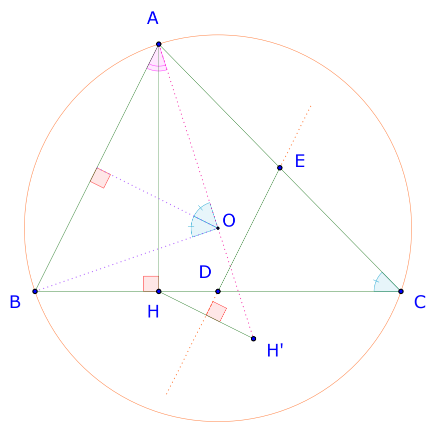


Thus $\triangle TBN \sim \triangle TCN$, or $\dfrac{BN}{CM} = \dfrac{TB}{TC} = \dfrac{FB}{EC}$. Since $FB = BD, EC = CD$ thus $\dfrac{BN}{CM} = \dfrac{BD}{BC}$, so $\triangle BDN \sim \triangle CDM$, so $\angle BDF = \angle CDM$, $\angle BDF = \angle CDM$, and $\widehat{DQ} = \widehat{DP}$. Hence, $\boxed{DP = DQ.}$ ☐

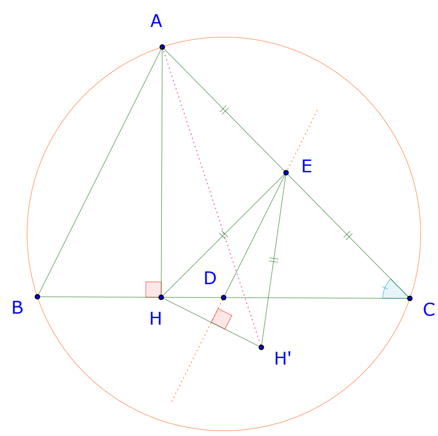
Example 4.3.6 (OT-22-23-S4-P6)

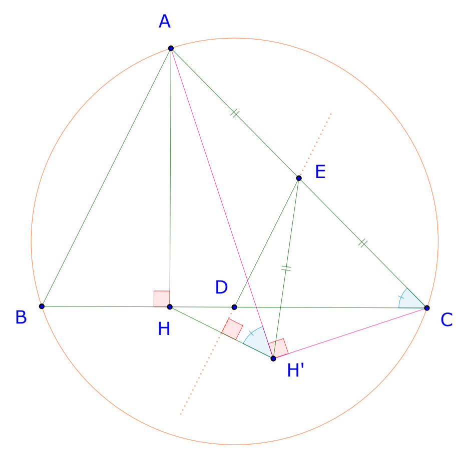
ABC is a triangle with only acute angles. Points H is the feet of the altitude from A to BC. Points D and E are midpoints of BC and CA, respectively. Let H' be the image of H by the reflection with respect to the line DE. Prove that the circumcenter O of $\triangle ABC$ is on the line through AH'.

Remark. Note that if O is on line AH', then $\angle AOB = 2\angle ACB$, so $\angle BAH' = \angle BAO = 90° - \angle ACB$.



Iran Geometry MO 2015/Medium/P2. **Proof based on cyclic quadrilateral** First, $\triangle AHC$ is right, E is midpoint of AC, thus $EA = EC = EH$. Since H' is the image of the reflection of H over ED, thus $EH = EH'$. See the diagram on the left below.



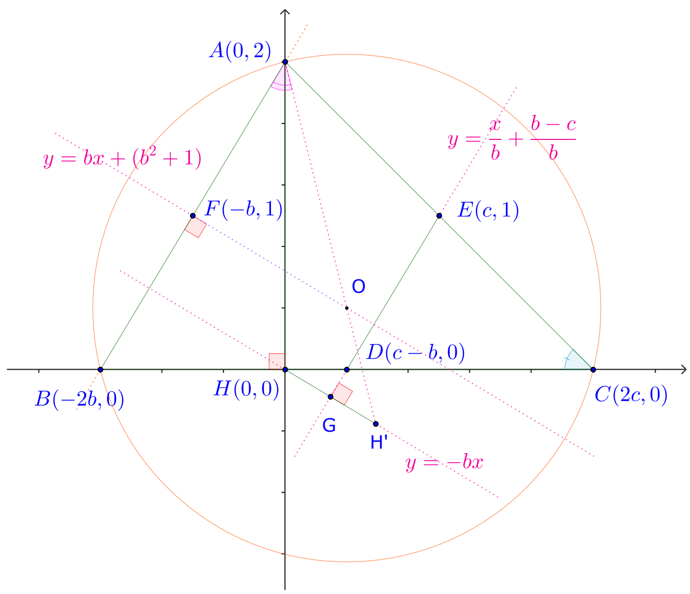


Second, in $\triangle AH'C$, E is the midpoint of AC and $EA = EC = EH'$, thus $\triangle AH'C'$ is right at H'. Since $\angle AHC = \angle AH'C = 90°$, it implies that $AHH'C$ is cyclic.

Now, $\angle HAH' = 180° - \angle H'HA - \angle AH'H = 90° - \angle H'HC - \angle C = \angle EDC - \angle C = \angle B - \angle C$. Therefore $\angle BAH' = \angle BAH + \angle HAH' = 90° - \angle B + \angle B - \angle C = 90° - \angle C$. Because for the circumcenter O, $\angle BAO = 90° - \angle ACB$, thus $\boxed{A,\ O, \text{ and } H' \text{ are collinear.}}$ □

> **Remark.** In this proof, we first use a *homothety* to transform $\triangle ABC$ into $A(0,2)$, $B(-2b,0)$, and $C(2c,0)$ $(b,c>0)$. This way X is fixed but positions of B and C are depending on b and c, thus it comprehends any shape of an acute triangle. The factor 2 will greatly simplify calculation for midpoints.
>
> We first determine the coordinates of the circumcenter O based on the perpendicular bisectors of line AB and line BC. Then we determine H' coordinates based on G, which is the intersection of the line through H and perpendicular to ED. By comparing the *slopes* of line AO and AH', we can prove that O is on the line AH'.

Iran Geometry MO 2015/Medium/P2. **Proof based on analytical geometry** First, line AB is $y-0=\dfrac{0-2}{(-2b)-0}(x-0)$, or $y=\dfrac{1}{b}x$. Thus its perpendicular bisector is $y-1=-b(x+b)$, or $y=-bx-b^2+1 \quad (1)$.



The $x-$ coordinate of the circumcenter O should be the same as of point D, or $c-b$. Its $y-$coordinate is computed based on the perpendicular bisector in (1), $y=-b(c-b)-b^2+1=-bc+1$. Thus $O\,(c-b,1-bc)$.

Second, the line ED is $y-1=\frac{0-1}{(c-b)-c}(x-c)$, or $y=\frac{x}{b}+\frac{b-c}{b}$. Thus, the line through H and perpendicular to ED is $y=-bx$. The coordinates G, the intersection of this line and ED is the solution (x,y) to

$$\begin{cases} y=\frac{x}{b}+\frac{b-c}{b} \\ y=-bx \end{cases} \Rightarrow x\left(\frac{1}{b}+b\right)+\frac{b-c}{b}=0 \Rightarrow x=\frac{c-b}{1+b^2},\ y=\frac{b(b-c)}{1+b^2} \Rightarrow G\left(\frac{(c-b)}{1+b^2},\frac{b(b-c)}{1+b^2}\right)$$

H' is the image of H over ED, in other words G is the midpoint of HH'. $H(0,0)$, thus $H'\left(\frac{2(c-b)}{1+b^2},\frac{2b(b-c)}{1+b^2}\right)$.

Thus, the slopes of line AO and AH' are the same, because

$$(AO):\ \frac{(1-bc)-2}{(c-b)-0}=\frac{1+bc}{b-c} \qquad (AH'):\ \frac{\frac{2b(b-c)}{b^2+1}-2}{\frac{2(c-b)}{b^2+1}-0}=\frac{2b^2-2bc-2b^2-2}{2(c-b)}=\frac{1+bc}{b-c}.$$

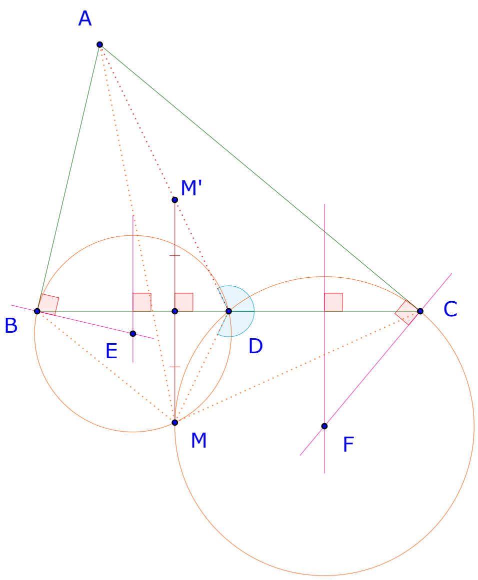
Therefore $\boxed{O \text{ is on the line } AH'.}$ □

Example 4.3.7 (OT-22-23-S4-P7)

In $\triangle ABC$, D is midpoint of BC. Circle Ω passes through D and tangent to AB at B. Circle Γ passes through D and tangent to AC at C. Ω and Γ meet again at $M \not\equiv D$. M' is the image of M by the reflection with respect to BC. Prove that M' is on AD.

Remark. In order to have M' to be on AD, we need line AD to be the reflection of line MD with respect to BC. First we prove that M is on the circumcircle (ABC), then $\triangle AMC \sim \triangle BMD$ and $\triangle ACD \sim triangle CMD$, so that $\angle ADC = \angle MDC$.

Turkey JMO 2015/P4. **Proof based on cyclic quadrilateral** Note that $\angle BMD = \angle ABD$, $\angle DMC = \angle ACD$, thus $\angle BMC = \angle ABD + \angle ACD = 180^\circ - \angle BAC$. Therefore, $ABMC$ is a cyclic quadrilateral (1).



In the cyclic $ABMC$, $\angle MBD = \angle MAC$, and $\angle AMC = \angle ABC = \angle ABD (= \frac{1}{2}\widehat{BD}) = \angle BMD$. Thus $\triangle AMC \sim \triangle BMD$, $\dfrac{MC}{MD} = \dfrac{AC}{BD} = \dfrac{AC}{CD}$ (D is the midpoint of BC.) It implies that $\triangle ACD \sim \triangle CMD$, thus $\angle ADC = \angle MDC$. Thus line AD is the reflection of line MD with respect to BC. Hence, $\boxed{M' \text{ is on } AD.}$ ☐

Lemma

In $\triangle ABC$, D is midpoint of BC, then

$$\frac{\sin \angle BAD}{\sin \angle DAC} = \frac{\sin \angle ABC}{\sin \angle ACB}.$$

Proof. By the Law of Sines for $\triangle ABD$,

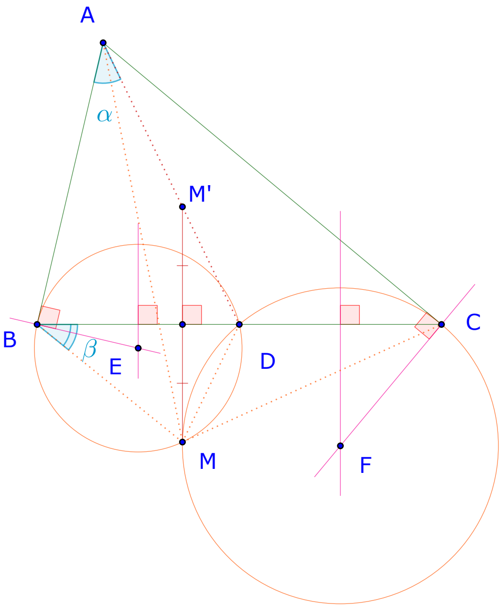
$$\frac{BD}{\sin BAD} = \frac{AD}{\sin ABD} \Rightarrow \frac{\sin \angle BAD}{\sin \angle ABD} = \frac{BD}{AD}, \text{ Similarly } \frac{\sin \angle DAC}{\sin \angle ACD} = \frac{CD}{AD}.$$

$BD = CD$, so the equality follows. ☐

Remark. For a proof based on *trigonometry*, we use the lemma to show that $\angle ADB = \angle BDM$,

Turkey JMO 2015/P4. **Proof based on Law of Sines** By the lemma

$$\frac{\sin \angle BAD}{\sin \angle DAC} = \frac{\sin \angle ABC}{\sin \angle ACB} \text{ and } \frac{\sin \angle BMD}{\sin \angle DMC} = \frac{\sin \angle MBC}{\sin \angle MCB}$$



Note that $\angle BMD = \angle ABC$ and $\angle DMC = \angle ACB$, thus

$$\frac{\sin \angle BAD}{\sin \angle DAC} = \frac{\sin \angle MBC}{\sin \angle MCB} \quad (1)$$

Let $\alpha = \angle BAD, \beta = \angle CBM$, then

$$(2) \quad \begin{cases} \angle DAC = 180° - (\alpha + \angle BAC + \angle ACB) \Rightarrow \sin \angle DAC = \sin(\alpha + \angle BAC + \angle ACB) \\ \angle MCB = 180° - (\beta + \angle BMD + \angle DMC) \Rightarrow \sin \angle DAC = \sin(\beta + \angle BMD + \angle DMC) \end{cases}$$

Substitue (2) into (1), and note that $\angle BMD = \angle ABC$ and $\angle DMC = \angle ACB$, we have

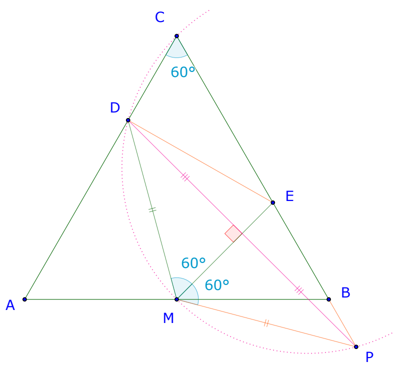
$$\frac{\sin \alpha}{\sin(\alpha + \angle BAC + \angle ACB)} = \frac{\sin \angle \beta}{\sin(\beta + \angle BMD + \angle DMC)} \Rightarrow \sin \alpha = \sin \beta$$

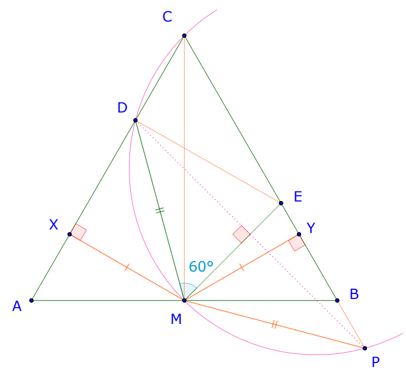
Now in $\triangle ABD$, $\angle ADB = 180° - \alpha - \angle ABC$. Similarly $\angle BDM = 180° - \alpha - \angle BMD$. Thus $\angle ADB = \angle BDM$. Hence, $\boxed{M' \text{ is on } AD.}$ □

Example 4.3.8 (OT-22-23-S4-P8)

$\triangle ABC$ is equilateral. M is midpoint of AB. Points D and E are on on AC and BC, respectively, such that $\angle DME = 60°$. Prove that, $AD - DE + EB = \frac{1}{2}AB$.

Remark. We try a *synthetic* proof by constructing a circle through C, D, and M meeting EB at P. $CDMP$ is cyclic, so $\angle DMP = 120°$, $\angle C = 60°$. We prove that P is the reflection of D over EM, then $DE = EP$.



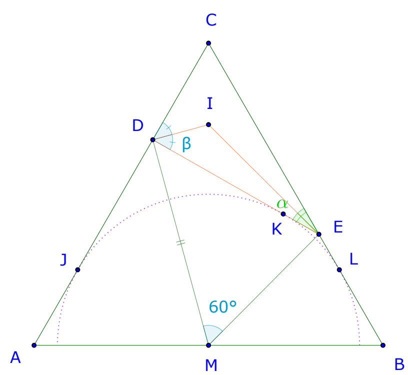


Polish Junior MO Finals 2018/P5. **First proof based on cyclic quadrilateral** Let circle (CDM) meet EB at P. $CDMP$ cyclic, thus $\angle ADM = \angle MPC$. $\angle DCM = \angle MCP$, thus $MP = MD$, by subtending the arcs with the same size. $EMP = 60°$, so P is the reflection of D over ME. Thus $DE = EP$.

Let X and Y the foot of the perpendiculars from M to AB and AC, respectively. $MX = MY$, thus $\triangle XDM \cong \triangle YPM$ (ASA), then $XD = YP$.

$AD - ED + EB = AX + XD - DE + EB = AX + YP - EP + EB = AX + YB = \frac{1}{2}AM + \frac{1}{2}MB = \frac{1}{2}AB$. $\square$

Remark. I is the incenter of $\triangle CDE$. $DIEM$ is cyclic. (M) circle is tangent to all AD, DE, and EB.



Polish Junior MO Finals 2018/P5. **Second proof based on cyclic quadrilateral** Let I the incenter of $\triangle CDE$. It is easy to see that $\angle DIE = 2\angle DCE = 120°$. Thus $DIEM$ is cyclic.

Let denote the angles as show on the diagram. It is easy to see that $\angle MDE = \angle ADM = \angle EMB = \delta$, $\angle DEM = \angle MEB = \angle AMD = \gamma$. Thus DM and EM are the external angle bisectors of $\angle D$ and $\angle E$ of $\triangle CDE$. They meet at M, midpoint of AB. Let J, K, and L be the tangent points of the circle with AD, DE, and EB.

Then $AD - EB + EB = AJ + JD - DK - KE + EL + LB = AJ + LB = \frac{1}{2}AM + \frac{1}{2}MB = \frac{1}{2}AB$. $\square$