Summer course 2024

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Part I

Practice

Chapter 1

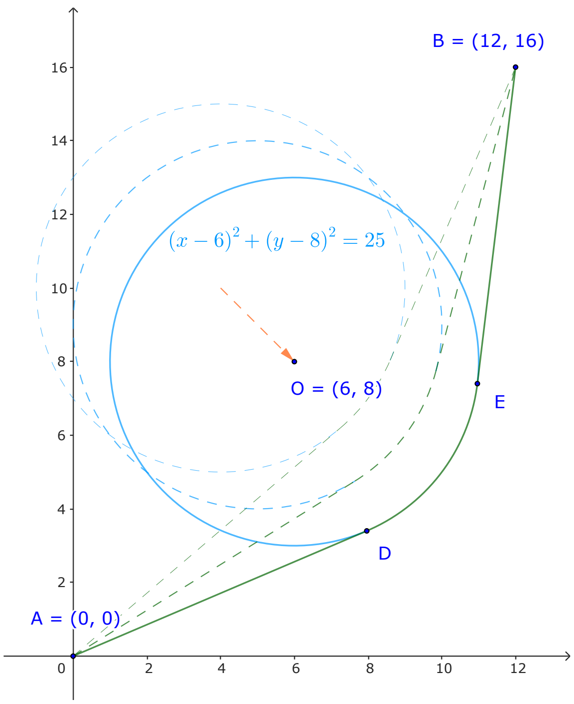
Getting Started

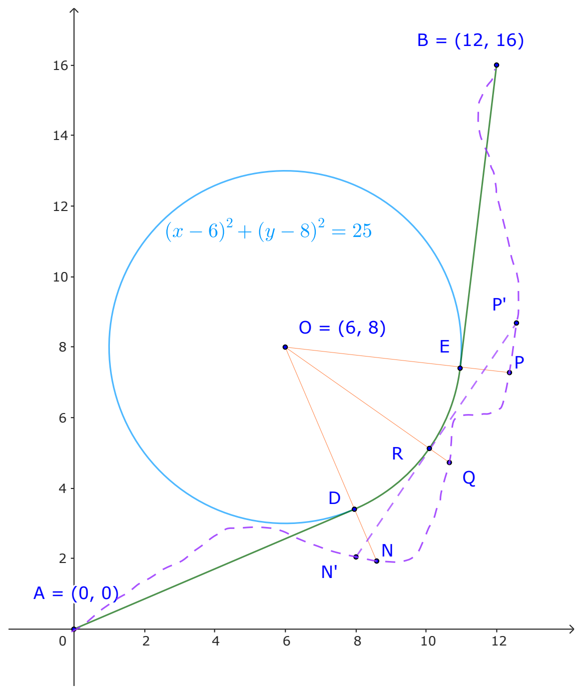
1.1 Problems and Exercises

Problem 1.1.1 (1.3.4 AHSME 1996)**.** In the $xy-$plane, what is the length of the shortest path from $(0,0)$ to $(12,16)$ that does not go inside the circle $(x-6)^2+(y-8)^2=25$?

Remark. Let A, B, and O be $(0,0), (12,16)$, and $(6,8)$. Note that O is the midpoint of AB and the tangent lengths from A and B to the circle $(x-6)^2+(y-8)^2=25$ are the same, or $AD = BE$.

To give us a hint, or rather a feeling, consider the 5-radius circle moves from the top left of the diagram in the direction of a vector through O perpendicular to AB A string with two fixed ends at A and B is bent by the circle's force, thus it dilates and become the path $ADEB$ when the circle center reaches O. See below in the left diagram. It seems that the $ADEB$ path containing segment AD, arc $\overset{\frown}{DE}$, and segment EB, is the shortest path from $(0,0)$ to $(12,16)$ that does not go inside the circle.





Solution. Assume that the shortest path from A to B is not $ADEB$. Assume that there is shortest curve above shown in the right diagram. Let the extensions of OD and OE meet the curve at N and P. Let Q be a point on the curve NP. Assume that Q is not on the perimeter of the circle. Let R be the intersection of OQ and the circle. The tangent line through R shall intersect the curve at N' and P'. It is easy to see that $AN'RP'B$ is a shorter than $AN'QP'B$ (because $Q \neq R$.) The contradiction means that every point on the curve NP must be on the circle. Thus $NP \equiv DE$.

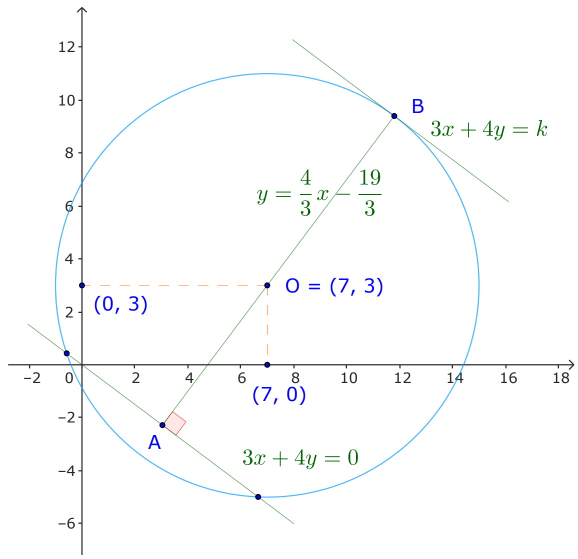
Now AD and BE are the shortest distances from A, B to the circle, respectively. Thus $ADEB$ is the shortest path. Note that $\triangle AOD$ is right at D, $AD = \sqrt{AO^2 - OD^2} = \sqrt{10^2-5^2} = 5\sqrt{3}$. Similarly $BE = 5\sqrt{3}$. Furthermore $AO = 2OD$, thus $\angle AOD = 60°$, similarly $\angle BOE = 60°$, so $\angle DOE = 60°$. Thus $\overset{\frown}{DE}$ is one-sixth of the perimeter of the circle, or $\overset{\frown}{DE} = \frac{1}{6}2\pi(5) = \frac{5\pi}{3}$.

Therefore $ADEB = \boxed{10\sqrt{3}+\frac{5\pi}{3}.}$ ◻

Problem 1.1.2 (1.3.5 AHSME 1996)**.** Given that $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value of $3x + 4y$?

Remark (Geometric properties)**.** The general strategy to solve problems created using on AM-GM inequality such as $x^2 + y^2 \geq 2xy$ or square-is-non-negative $(ax - by)^2 \geq 0$ is to transform a quadratic expression containing both first and second degrees of variables x and y into some sum of squares and then determine the condition where the equality stands.

$$x^2 - 14x = x^2 - 2(x)(7) + 7^2 - 7^2 = (x-7)^2 - 49$$
$$y^2 - 6y = y^2 - 2(y)(3) + 3^2 - 3^2 = (y-3)^2 - 9$$
$$\Rightarrow x^2 + y^2 - (14x + 6y) - 6 = (x-7)^2 + (y-3)^2 - 64$$



Solution. [Geometric properties] Note that $x^2 + y^2 = 14x + 6y + 6 \Leftrightarrow (x-7)^2 + (y-3)^2 - 64$. Thus $(x-7)^2 + (y-3)^2 = 64$. This means that (x, y) are points on the circle radius 8, centred at $(7, 3)$. A is the midpoint of the chord which is the intersection of the circle with the line $3x + 4y = 0$, or $y = -\frac{3}{4}x$. A is on a line through O, slop $\frac{4}{3}$, which is $y - 3 = \frac{4}{3}(x - 7)$, or $y = \frac{4}{3}x - \frac{19}{3}$. This line intersects the circle at $B(x, y)$, where line $3x + 4 = k$ tangent with the circle and k has the maximal value.

$$\left.\begin{aligned} (x-7)^2 + (y-3)^2 = 64 \\ y = \frac{4}{3}x - \frac{19}{3} \end{aligned}\right\} \Rightarrow (x-7)^2 + \left(\frac{4}{3}x - \frac{19}{3} - 3\right)^2 = 64 \Rightarrow \frac{1}{9}(25x^2 - 350x + 1225) = 64 \Rightarrow x \in \{\frac{11}{5}, \frac{59}{5}\}$$

It is easy to see that for B, $x = \frac{59}{5}$, $y = \frac{47}{5}$. Then $3x + 4y = \boxed{73.}$ □

Solution. [Algebraic manipulation + Cauchy-Schwarz] Note that $x^2+y^2 = 14x+6y+6 \Leftrightarrow (x-7)^2+(y-3)^2-64$. By Cauchy-Schwarz inequality

$$(3(x-7) + 4(y-3))^2 \leq ((x-7)^2 + (y-3)^2)(3^2 + 4^2) \Rightarrow 3x + 4y - 33 \leq= 40 \Rightarrow 3x + 4y \leq \boxed{73.}$$

□

Remark ((Quadratic) Discriminant properties)**.** The second strategy is to maximize $k = 3x + 4y$, knowing that $x^2 + y^2 = 14x + 6y + 6$. If we substitute y with k and x then the equality becomes a quadratic equation for x, which by the existence of x enforces a condition of non-negativity on its discriminant.

Solution. [(Quadratic) Discriminant properties] Let $k = 3x + 4y$, then $y = \frac{3x-k}{4}$, by substitution:

$$x^2 + \left(\frac{3x-k}{4}\right)^2 + 14x + 6\left(\frac{3x-k}{4}\right) + 6 \Rightarrow 25x^2 - (6k+152)x + (k^2 - 24k - 96) = 0$$
$$\Delta = (6k+152)^2 - 4(25)(k^2 - 24k - 96) \geq 0 \Rightarrow -64(k+7)(k-73) \geq 0$$

It is easy to see that the largest value of such k is $\boxed{73.}$ □

Remark (Polar coordinate + Cauchy-Schwarz)**.** The third strategy is coming from the idea of parameterize the circle $(x-7)^2 + (y-3)^2 = 64$ using polar coordinate. This would lead to expressing x and y as linear combinations of $\cos\theta$ and $\sin\theta$, then applying Cauchy-Schwarz inequality knowing that $\cos^2\theta + \sin^2\theta = 1$.

Solution. [Polar coordinate + Cauchy-Schwarz] Note that

$$a\cos\theta + b\sin\theta \leq \sqrt{(a^2+b^2)(\cos^2\theta + \sin^2\theta)} = \sqrt{(a^2+b^2)}$$
$$(x-7)^2 + (y-7)^2 = 64 \Rightarrow \left(\frac{x-7}{8}\right)^2 + \left(\frac{y-3}{8}\right)^2 = 1$$
$$\Rightarrow \exists\theta : \ \frac{x-7}{8} = \cos\theta, \ \frac{y-3}{8} = \sin\theta$$
$$\Rightarrow 3x + 4y = 33 + 24\cos\theta + 32\sin\theta \leq 33 + \sqrt{24^2 + 32^2} = \boxed{73.}$$

□

Remark (Substitution + Function inspection)**.** The fourth strategy is based on the idea of direct substitution, i.e. solving the equation $(x-7)^2 + (y-3)^2 = 64$ for y and then investigate $3x + 4y$ as a single-variable function of x.

Solution. [Substitution + Function inspection] Note that $x^2 + y^2 = 14x + 6y + 6 \Leftrightarrow (x-7)^2 + (y-3)^2 - 64$, thus

$$(y-3)^2 = 64 - (x-7)^2 \Rightarrow y = 3 \pm \sqrt{64 - (x-7)^2} \Rightarrow \max 3x + 4y = 3x + 4(3 + \sqrt{64 - (x-7)^2})$$
$$(x-7)^2 + (y-3)^2 = 64 \Rightarrow |x-7| \leq 8 \Rightarrow -1 \leq x \leq 15$$

Thus, the maximal value of $3x+4y$ is the maximal value of the function $f(x) = 3x+4(3+\sqrt{64-(x-7)^2})$, where $x \in [-1, 15]$. On the $I = (-1, 15)$ interval, $f(x)$ is a continuous and twice differentiable function:

$$f'(x) = 3 - \frac{4(x-7)}{\sqrt{64-(x-7)^2}}, \ f''(x) = -\frac{256}{(-64-(x-7)^2)^{\frac{3}{2}}} < 0, \ \forall x \in I.$$

Therefore f' is a strictly decreasing function on I, and since:

$$f'(x) = 0 \Leftrightarrow 3 - \frac{4(x-7)}{\sqrt{64-(x-7)^2}} = 0 \Leftrightarrow 9(64 - (x-7)^2) = 16(x-7)^2, x \geq 7 \Leftrightarrow x = \frac{59}{5}.$$

It means that $\frac{59}{5}$ is a zero of $f'(x)$. It is easy to see that $f'(x) > 0$ when $x < 7$,

$$f'(x) > 0, \ \forall x \in \left(-1, \frac{59}{5}\right), \ f'(x) < 0, \ \forall x \in x\left(\frac{59}{5}, 7\right)$$

Hence, $f(x)$ attains maximal value at $x = \frac{59}{5}$, which is $3\left(\frac{59}{5}\right) + 4(3 + \sqrt{64 - \left(\frac{59}{5} - 7\right)^2}) = \boxed{73.}$ □

Problem 1.1.3 (1.3.6 AHSME 1994)**.** When n standard six-sided dice are rolled, the probability of obtaining a sum of 1994 is greater than zero and is the same as the probability of obtaining a sum of S. What is the smallest possible value of S?

> **Remark** (Identifying the case based on the existence of the probability)**.** The unknown n seems to be problematic. We don't know what it is and it is difficult to calculate the probability if the sum of n dice is 1994. Let's say that it is p. Note that on a standard dice, 1 and 6, 2 and 5, 3 and 4 are on opposite sides, so the probability of S - the sum of the opposite sides is also p. Now, S is minimal if both the number of terms is S is as small as possible and the terms also as small as possible. It then means that we can actually determine n because of the existence of a fixed value (1994) sum of n numbers.

Solution. [Identifying the case based on the existence of the probability] On a standard dice, 1 and 6, 2 and 5, 3 and 4 are on opposite sides. To obtain a sum of 1994 with the most six sides on the top faces of the dice require that 332 sixes and 1 two face up. Then 332 ones and 1 five will face down and $332 + 5 = \boxed{337.}$ □

> **Remark.** In the below solution, the equivalence of the two cases is based on counting.

Solution. [Identifying the case based on counting] When n dice are rolled, the sum is between n and $6n$ (inclusive). The sum $n + k$ can be obtain in the same number of ways as the sum of $6n - k$ and this number of ways increases as k increases from 0 to $\frac{5n}{2}$. Thus $S = n + k$ is minimized by *choosing n and k* such that $6n - k = 1994$ (in other words there exists n and k such that the probability of obtaining a sum of $6n - k = 1994$ is greater than zero.) It is easy to see that the least multiple of 6 that *does not exceed 1994* is $1998 = 6 \cdot 333$, thus $n = 333, k = 4$. Therefore $S = 333 + 4 = \boxed{337.}$ □

> **Remark** (Identifying the case based on lower bound estimation)**.** In the below solution, the reasoning is more detailed in order to find a property of S (having remainder 1 when divided by 7) and then a lower bound for S (which obviously at least n.)

Solution. [Identifying the case based on lower bound estimation] Each roll of the n dice can be represented as a tuple $(a_1, a_2, \ldots, a_n)$. There is a one-on-one correspondence between the rolls $(a_1, a_2, \ldots, a_n)$ and the rolls $(7 - a_1, 7 - a_2, \ldots, 7 - a_n)$. Thus for all x, the probability to obtain a sum x is the same as the probability to obtain a sum of $7n - x$. Now $7n - S = 1994$, thus $S = 1994 - 7n \equiv 1 \pmod 7$. In order for the probability for obtaining a sum of 1994 is greater than zero,

$$n \geq \frac{1994}{6} = 332 + \frac{1}{3} \Rightarrow n \geq 333,\ S \geq n, S \geq 333, S \equiv 1 \pmod 7 \Rightarrow S \geq \boxed{337.}$$

□

> **Remark** (Generating function)**.** Generating function can be the best way *to express a generic sum.*

Solution. [Generating function] When n dice are rolled, the sum can be any integer from n to $6n$. For any k between n and $6n$, *the number of ways the sum k can appear is the coefficient of x^k* in the generating function:

$$P(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^n.$$

By symmetry, this coefficient is the same as the coefficient of $x^7 n - k$. For $n > 1$, the coefficients in $P(x)$ are strictly increasing for $k = n, n + 1, \ldots, \lfloor \frac{7n}{2}. \rfloor$ (why?) Therefore the only value of $S\ ne1994$ for which the coefficients of x^S and $x^1 994$ are equal. In this case:

$$\left.\begin{aligned} 7n - S = 1994 \Rightarrow S = 7(n - 285) + 1 \\ 6n \geq 1994 \Rightarrow n \geq 333 \end{aligned}\right\} \Rightarrow S\ ge7(333 - 285) + 1 = \boxed{337.}$$

□

Problem 1.1.4 (1.3.7 AIME 1994)**.** Find the positive integer n for which

$$\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \cdots + \lfloor \log_2 n \rfloor = 1994.$$

(For real x, $\lfloor x \rfloor$ is the greatest integer $\leq x$.)

Remark (Distribution of numbers into intervals)**.** The key idea is to investigate the terms of the sum:

$$\begin{aligned}
&\lfloor \log_2 1 \rfloor = 0\\
&\lfloor \log_2 2 \rfloor = \lfloor \log_2 3 \rfloor = 1\\
&\lfloor \log_2 4 \rfloor = \lfloor \log_2 5 \rfloor = \ldots = \lfloor \log_2 7 \rfloor = 2\\
&\lfloor \log_2 8 \rfloor = \lfloor \log_2 9 \rfloor = \ldots = \lfloor \log_2 15 \rfloor = 3
\end{aligned}$$

All the numbers under the $\log_2$ with in an interval of $\left[2^m, 2^{m+1} - 1\right]$ have the same value.

Solution. [Distribution of numbers into intervals $\left[2^m, 2^{m+1} - 1\right]$]

$$\begin{aligned}
&\lfloor \log_2 k \rfloor = m \Leftrightarrow m \leq \log_2 k < m + 1 \Leftrightarrow 2^m \leq k \geq 2^{m+1} - 1\\
&\lfloor \log_2 2^m \rfloor + \lfloor \log_2 2^m + 1 \rfloor + \cdots + \left\lfloor \log_2 2^{m+1} - 1 \right\rfloor = m(2^{m+1} - 1 - 2^m + 1) = m2^m
\end{aligned}$$

Now $8 \cdot 2^8 = 2048$, and

$$\begin{aligned}
&1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + 5 \cdot 2^5 +, 6 \cdot 2^6 + 7 \cdot 2^7 = 1538 \Rightarrow 2^8 \leq n < 2^9 - 1\\
&1994 = \lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \cdots + \lfloor \log_2 n \rfloor = 1538 + 8 \cdot (n - 2^8 + 1) \Rightarrow n = \boxed{312.}
\end{aligned}$$

$\square$

Problem 1.1.5 (1.3.8)**.** For any sequence of real numbers $A = (a_1, a_2, a_3, \ldots)$, define ΔA to be the sequence $(a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots)$, whose n^{th} term is $a_{n+1} - a_n$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1, and that $a_{19} = a_{94} = 0$. Find a_1.

> **Remark** (Generic formula + Solving system of equations)**.** The first approach is pretty standard: let (b_n) be the delta sequence (the differences). We use their differences to find a formula of b_n and b_1, then a formula of a_n and b_1. By substitution of a_n with the values of a_{19} and a_{94} we obtain a system of two equations with b_1, which then can be replaced by the unknowns a_2 and a_1.

Solution. [Generic formula + Solving system of equations] Let $(b_1, b_2, b_3, \ldots)$ be the ΔA sequence, then for all $n \geq 1 : b_n = a_{n+1} - a_n$. Now, for $n \geq 1$:

$$b_{n+1} - b_n = 1 \Rightarrow b_n = b_{n-1} + 1 = b_{n-2} + 2 = \cdots = b_1 + (n-1)$$

$$a_{n+1} = a_n + b_1 + (n-1) = a_{n-1} + 2b_1 + [(n-1) + (n-2)] = \cdots = a_1 + nb_1 + \frac{n(n-1)}{2}$$

$$\left.\begin{aligned} a_{19} = 0 \Rightarrow a_1 + 18(a_2 - a_1) + \frac{18(17)}{2} = 0 \\ a_{94} = 0 \Rightarrow a_1 + 93(a_2 - a_1) + \frac{93(92)}{2} = 0 \end{aligned}\right\} \Rightarrow a_1 = \boxed{837}, a_2 = 782.$$

□

> **Remark** (Telescopic sums twice)**.** The second approach is based on telescoping sums.

Solution. [Telescopic sums twice] Note that $(a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) = a_{n+2} - 2a_{n+1} + a_n$, thus:

$$\left.\begin{aligned} a_3 - 2a_2 + a_1 &= 1 \\ a_4 - 2a_3 + a_2 &= 1 \\ a_5 - 2a_4 + a_3 &= 1 \\ &\ldots \\ a_{n-1} - 2a_{n-2} + a_{n-3} &= 1 \\ a_n - 2a_{n-1} + a_{n-2} &= 1 \\ a_{n+1} - 2a_n + a_{n-1} &= 1 \end{aligned}\right\} \Rightarrow a_{n+1} - a_n = (n-1) + (a_2 - a_1),\ \forall n \geq 1,$$

$$\left.\begin{aligned} a_{n+1} - a_n &= (n-1) + (a_2 - a_1) \\ a_n - a_{n-1} &= (n-2) + (a_2 - a_1) \\ &\ldots \\ a_3 - a_2 &= (1) + (a_2 - a_1) \\ a_2 - a_1 &= (0) + (a_2 - a_1) \end{aligned}\right\} \Rightarrow a_{n+1} - a_1 = n(a_2 - a_1) + \frac{n(n-1)}{2}$$

The rest can be done similarly to the first solution. □

> **Remark** (Sequence reconstruction)**.** The third approach is based on a simple idea of*working backward.*

Solution. [Sequence reconstruction] Since $\Delta(\Delta A) = 1$, thus there exist k such that ΔA

$$\Delta A = (k, k+1, k+2, \ldots) \Rightarrow A = (a_1, a_1 + k, a_1 + 2k + 1, a_1 + 3k + 3, \ldots)$$

$$a_n = a_1 + k(n-1) + \frac{(n-1)(n-2)}{2}.$$

The rest can be done similarly to the first solution. □

Remark (Quadratic representation of a sequence whose secondary difference is a constant)**.** The fourth approach is based on some understanding of *representation of generic terms of sequences.*

Solution. [Quadratic representation of a sequence whose secondary difference is a constant] Since the result of two finite differences of a sequence is a constant sequence, thus that sequence is a quadratic. Furthermore, we know that $f(19) = f(94) = 0$ so the quadratic is

$$f(x) = a(x-19)(x-94) \text{ for some constant } a.$$

Now,

$$\begin{aligned} & f(19) = 0, f(20) = -74a, f(18) = 76a \Rightarrow (f(20) - f(19)) + (f(19) - f(18)) = 1 \\ & \Rightarrow a = \frac{1}{2} \Rightarrow f(1) = \frac{1}{2}(-18)(-93) = \boxed{837.} \end{aligned}$$

□

Remark (Generic formula by induction principle)**.** The fifth approach is to find a *recurrence relation* for a_n based on preceding terms.

Solution. [Generic formula by induction principle] Let $\Delta^1 = \Delta A$, and $\Delta^n A = \Delta(\Delta^{n-1} A)$. By induction principle:

Claim — $a_n = \binom{n-1}{1}\Delta a_n + \binom{n-1}{2}\Delta^2 a_n + \binom{n-1}{3}\Delta^3 a_n + \cdots, \forall n \geq 1.$

Then,

$$\begin{aligned} & a_n = a_1 + \binom{n-1}{1}\Delta a_1 + \binom{n-1}{2}\Delta^2 a_1 + \binom{n-1}{3}\Delta^3 a_1 + \cdots \\ & \Rightarrow a_n = a_1 + \binom{n-1}{1}(a_2 - a_1) + \binom{n-1}{2} \cdot 1 = a_1 + n(a_2 - a_1) + \binom{n-1}{2} \end{aligned}$$

The rest can be done similarly to the first solution. □

Remark (Generic formula using initial terms and pattern recognition)**.** The six approach is to find a pattern for a *generic formula* of a_n based on n, initial terms a_1, a_2.

Solution. [Generic formula using initial terms and pattern recognition] Let $a_1 = a, a_2 = b$, then $a_{n+1} = 2a_n + 1 - a_{n-1}$, we compute the first few terms to find a pattern

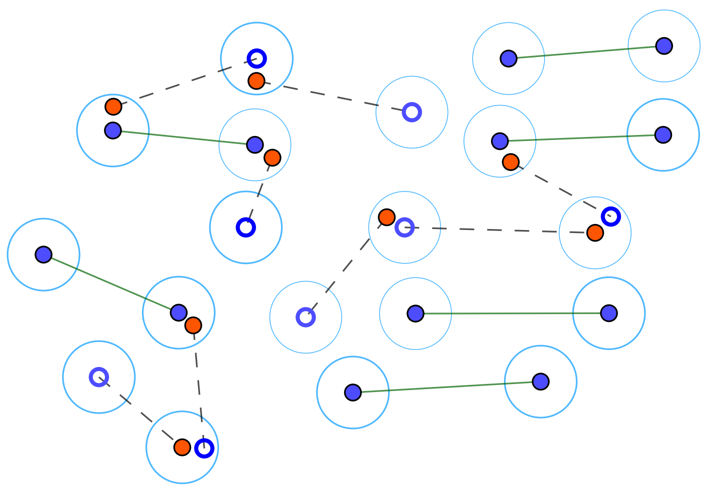
$$\left.\begin{aligned} a_3 &= 2b + 1 - a \\ a_4 &= 3b + (1+2) - 2a \\ a_5 &= 4b + (1+2+3) - 3a \end{aligned}\right\} \Rightarrow a_n = (n-1)b + \frac{(n-2)(n-1)}{2} - (n-2)a.$$

The rest can be done similarly to the first solution. □

Problem 1.1.6 (1.3.9 USAMO 1989)**.** The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

> **Remark** (Construction method)**.** In the first approach, the *construction method* is used to identify the 6 games and 12 players.

Solution. [Construction method] 14 two-person games means $14 \cdot 2 = 28$ places to participate. By condition, if each of the players plays one game, each occupies one place, we have 20 places. so there are 8 places to put some players who already has at least a game. For those players who in these positions, there are 8 games them plays. So, if we delete these games, there are $14 - 8 = 6$ games in where each of the players plays not more than once. This is the set of $\boxed{6}$ games with $6 \cdot 2 = \boxed{12}$ distinct players.



For illustration, in the diagram above, 20 large circles represent the players, the blue dots represent 20 places for each of the players the red dots represent 8 extra places. We removed 8 dotted segment connecting 8 pairs of red - (circled) blue dots, leaving 8 green segments connecting 8 pairs of (full) blue-blue dots. □

> **Remark** (Pigeonhole Principle + Graph representation)**.** In this second approach with a graph representation, the *Pigeonhole Principle* is used.

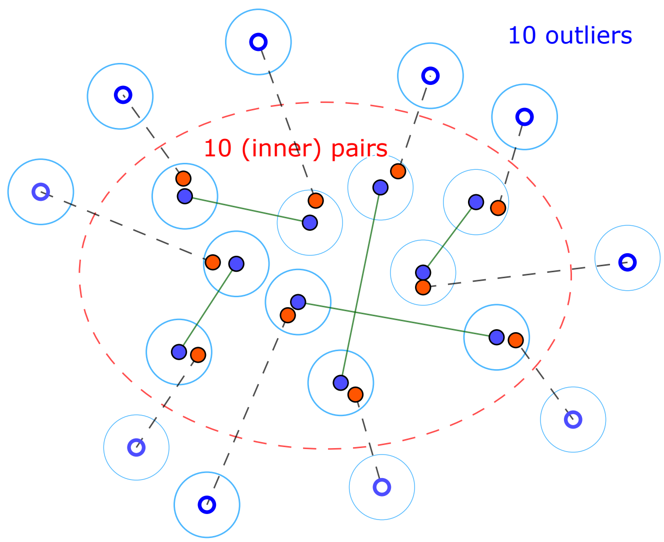
Solution. [Pigeonhole Principle + Graph representation] Consider a graph representation of the problem: a vertex represents a player, an edge connecting two players represents the game which plays by them. The degree of a vertex is the number of edges connected to the vertex. The sum of the degrees of 20 vertices is 28, by the Pigeonhole Principle, at most 8 vertices have degrees greater than 1. Now by removing at most 8 edges connected to the vertices with degree greater than 1, we are left with at least 6 edges connecting vertices of degrees 1. This is the set of $\boxed{6}$ games with $6 \cdot 2 = \boxed{12}$ distinct players. □

> **Remark** (Extremal Principle based on Maximum Matching strategy)**.** In this third approach, the *Extremal Principle* is used together with the Proof by Contradiction. Note that this approach proves the *the existence of a required set of games* but does not identify those games.

Solution. [Extremal Principle based on Maximum Matching strategy] A set of games is called *matching set* if they are played by distinct players. By the Extremal Principle, there is a *maximum matching set* S.

Assume that the cardinality of S *is at most 5*, so there are at most 10 so-call *inner* players (who playing the games in S) Thus there are at least $20 - 10 = 10$ so-called *outlier* players (who do not play any games in S). By the given condition each of them played at least one game, if there is a match played by any two outlier players, then this game can be added to S, thus increases the cardinality of S, which is a contradiction.

Therefore, these 10 outlier players have to play 10 games with 10 inner players. Thus, the number of games is at least $5 + 10 = 15 > 14$, a contradiction.



For illustration, in the diagram above, 10 pair of players as large circles in the region with the red dotted border. They play 5 games between each other. 10 outliers, who cannot play any games between each other, must be forced to play at least one game with one of the inner players. □

Problem 1.1.7 (1.3.10 USAMO 1995)**.** A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows O. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

> **Remark** (Construction method)**.** For each q, the problem is how to construct a sequence of numbers starting from 0,= and ending by q, using one the the given six trigonometric functions. For example $q = 1$, can easily be produce since $\cos 0 = 1$. Thus is is worth to investigate how we can produce different numbers from an arbitrary x. Now for any rational q, there exist m, n relatively prime integers such that $q = \frac{m}{n} = \frac{nk+r}{n} = k + \frac{r}{n}, 0 \leq r < n$. Since 1 can be produced from 0, then k from 0 and somehow $\frac{1}{n}$ from n seems to be necessary as well.

Solution. [Construction method] We prove the following claims:

Claim —

$$\begin{cases} \tan(\cos^{-1}(\sin(\tan^{-1}(x)))) = \frac{1}{x} & (1) \\ \cos(\tan^{-1}(\sqrt{x})) = \frac{1}{\sqrt{1+x}} & (2) \end{cases}$$

Proof.

(1) $\forall\theta:\ \cos\left(\frac{\pi}{2} - x\right) = \sin(x) \Rightarrow \cos^{-1}(\sin(\theta)) = \frac{\pi}{2} - \theta,\ \tan\left(\frac{\pi}{2} - \theta\right) = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} = \frac{1}{\tan(\theta)}$

$$\Rightarrow \cos^{-1}(\sin(\tan^{-1}(x))) = \frac{\pi}{2} - \tan^{-1}(x) \Rightarrow \tan(\cos^{-1}(\sin(\tan^{-1}(x)))) = \tan\left(\frac{\pi}{2} - \tan^{-1}(x)\right) = \frac{1}{x}$$

(2) $\forall\theta : \cos(\theta) = \dfrac{\cos(\theta)}{\cos^2(\theta) + \sin^2(\theta)} = \dfrac{1}{1 + \tan^2(\theta)}$. Let $\theta = \tan^{-1}(\sqrt{x})$:

$$\Rightarrow \cos(\tan^{-1}(\sqrt{x})) = \frac{1}{1 + \tan^2(\tan^{-1}(\sqrt{x}))} = \frac{1}{\sqrt{1+x}}.$$

■

Using (2) we can produce $\frac{1}{n+1}$ from $\dfrac{1}{n}$:

$$\frac{1}{n} = \frac{1}{\sqrt{n^2}} \to \frac{1}{\sqrt{n^2+1}} \to \cdots \to \frac{1}{\sqrt{n^2+2n+1}} = \frac{1}{n+1} \qquad (3)$$

Thus by (3) and (1) we can produce any positive integer $n+1$ from n :

$$n \to \frac{1}{n} \to \frac{1}{n+1} \to n+1$$

WLOG assume that $q = \frac{m}{n}$, where m, n are positive relatively prime numbers, $m = nk + r$, $0 \leq r < n$:

$$q = k + \frac{r}{n} \leftarrow \cdots \leftarrow 1 + \frac{r}{n} \leftarrow \frac{r}{n}.$$

Thus, producing $q = (m, n)$ means to produce (n, r), where r is the remainder in the division of m by n :

$$(m, n) \to (n, r) \to (r, r_1) \to \cdots \to (1, 0),\ \text{which is the Euclidean algorithm.}$$

Therefore it is possible to produce any rational number q. □

> **Remark** (Hypothesis on a stronger result + Induction method)**.** As we have seen, the key idea derives from the equality $\cos(\tan^{-1}(\sqrt{x})) = \dfrac{1}{\sqrt{1+x}}$, thus for $x = \frac{n-m}{m}$, then $\dfrac{1}{\sqrt{1+x}} = \sqrt{\frac{m}{n}}$. This gives the idea that perhaps all elements of set of $S = \{\sqrt{\frac{m}{n}} | m, n \in \mathbb{Z}^+\}$ are producible, there for all elements of $\mathbb{Q}$ (rational numbers) are producible too, since $\mathbb{Q} \subset S$.

Solution. [Hypothesis on a stronger result + Induction method] First, we prove the following claims:

> **Claim —**
> $$\begin{cases} \tan(\cos^{-1}(\sin(\tan^{-1}(x)))) = \frac{1}{x} & (1) \\ \cos(\tan^{-1}(\sqrt{x})) = \frac{1}{\sqrt{1+x}} & (2) \end{cases}$$

Now, we prove the stronger hypothesis by using the induction principle on $m + n$:

> **Claim —** If m and n are relatively prime nonnegative integers such that $n > 0$, then the some finite sequence of buttons will yield $\sqrt{\dfrac{m}{n}}$.

For the base case: $m + n = 1, m$ then $n = 1, m = 0$, thus $\sqrt{\dfrac{m}{n}} = 0$, which is the initial value.

For the inductive step: Let assume that for all k and ℓ relatively prime nonnegative integers such that $\ell > 0$, $k + \ell < m + n$, $\sqrt{\dfrac{k}{\ell}}$ is producible after a finite number of steps. Consider two sub-cases.

Case 1: $0 < m \leq n$. $\sqrt{\frac{n-m}{m}}$ can be obtained by the inductive hypothesis, thus

$$\cos\left(\tan^{-1}\left(\sqrt{\frac{n-m}{m}}\right)\right) = \frac{1}{\sqrt{1+\frac{n-m}{m}}} = \sqrt{\frac{m}{n}} \text{ also can be obtained.}$$

Case 2: $n < m$, since $\sqrt{\frac{m}{n}}$ can be obtained by the previous case:

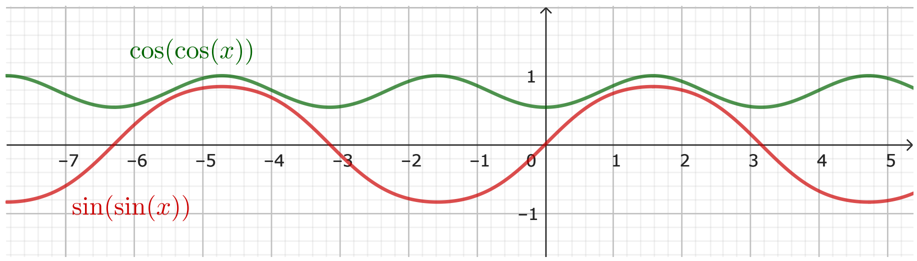
$$\cos\left(\tan^{-1}\left(\sqrt{\frac{n}{m}}\right)\right) = \sin\left(\tan^{-1}\left(\sqrt{\frac{m}{n}}\right)\right) \Rightarrow \tan\left(\cos^{-1}\left(\sin\left(\tan^{-1}\left(\sqrt{\frac{n}{m}}\right)\right)\right)\right) = \sqrt{\frac{m}{n}}.$$

$\square$

Problem 1.1.8 (1.3.11 Russia 1995)**.** Solve the equation

$$\cos(\cos(\cos(\cos x))) = \sin(\sin(\sin(\sin x)))$$

Lemma
$\cos(\cos(x)) > \sin(\sin(x)),\ \forall x.$



Proof 1. We use the following claim:

Claim — $|\sin(x)| < |x|, \forall x \in \left(0, \frac{\pi}{2}\right)$.

It is sufficient to prove for $\forall x \in \left(0, \frac{\pi}{2}\right)$ (why?), thus:

$$\cos^2(\cos(x)) = 1 - \sin^2(\cos(x)) > 1 - \cos^2(x) = sin^2(x) > sin^2(sin(x)) \Rightarrow \cos(\cos(x)) > \sin(\sin(x)).$$

□

Proof 2.

$$\left.\begin{aligned} \sin(x) + \cos(x) &= \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) \\ \sin(x) - \cos(x) &= \sqrt{2}\cos\left(x - \frac{3\pi}{4}\right) \end{aligned}\right\} \Rightarrow |\sin(x) + \cos(x)| \leq \sqrt{2} < \frac{\pi}{2}$$

$$\Rightarrow 0 < \frac{\frac{\pi}{2} - \sin x \pm \cos x}{2} < \frac{\frac{\pi}{2} + \sqrt{2}}{2} < \frac{\pi}{2}.$$

Therefore

$$\begin{aligned} \cos(\cos(x)) - \sin(\sin(x)) &= \cos(\cos(x)) - \cos\left(\frac{\pi}{2} - \sin(x)\right) \\ &= 2\sin\left(\frac{\frac{\pi}{2} - \sin x - \cos x}{2}\right)\sin\left(\frac{\frac{\pi}{2} - \sin x + \cos x}{2}\right) > 0 \end{aligned}$$

□

Solution. Applying the lemma

$$\cos(\cos(\cos(\cos x))) > \cos(\cos(\sin(\sin x))) > \sin(\sin(\sin(\sin x))).$$

□

Problem 1.1.9 (1.3.12 IMO 1976)**.** Determine, with proof, the largest number that is the product of positive integers whose sum is 1976.

> **Remark** (Examine multiple cases to find efficient factors)**.** We investigate a few smaller cases: $4 = 2+2$, $2 \cdot 3 > 5$, $3 \cdot 3 > 2 \cdot 2 \cdot 2 > 6$, $3 \cdot 4 > 7$, $3 \cdot 5 > 8, \ldots$

Solution. [Examine multiple cases to find efficient factors] There cannot be any integers larger than 4 in the maximal product, because for $n > 4$, we can replace n by 3 and $n - 3$ to get a larger product.

There cannot be any 1s, because there must be an integer $r > 1$ (otherwise the product would be 1) and $r + 1 > 1 \cdot r$.

We can also replace any 4s by two 2s leaving the product unchanged.

Finally, there cannot be more than two 2s, because we can replace three 2s by two 3s to get a larger product.

Thus the product must consist of 3s, and either zero, one or two 2s. The number of 2s is determined by the remainder on dividing the number 1976 by 3.

$1976 = 3 \cdot 658 + 2$, so there must be just one 2, giving the product $\boxed{2 \cdot 3^{658}.}$ □

> **Remark** (Proving and using a generic inequality)**.** Similar idea to arrive at the inequality: $n^3 < 3^n$. This inequality is the essence of the problem.

Solution. [Proving and using a generic inequality] Since $3 \cdot 3 = 2 \cdot 2 \cdot 2 + 1$, 3's are more efficient than 2's. We to prove that 3's are more efficient than anything.

Let there be a positive integer n. If 3 is more efficient than n, then

$$n^3 < 3^n.$$

We prove that all integers greater than 3 are less efficient than 3 as follow. When n increases by 1, then the RHS of (*) is multiplied by 3. The other side is multiplied by $\frac{(n+1)^3}{n^3}$, and we prove that this is less than 3 for all $n > 3$:

$$\frac{(n+1)^3}{n^3} < 3 \Leftrightarrow \frac{n+1}{n} < \sqrt[3]{3} \Leftrightarrow 1 < (\sqrt[3]{3} - 1)n \Leftrightarrow \frac{1}{\sqrt[3]{3} - 1} < n.$$

It is enough to show that $\frac{1}{\sqrt[3]{3}-1} < 4$. Simplifying, we get $5 < 4\sqrt[3]{3} \Rightarrow 125 < 64 * 3 = 192$, which is true.

Now, we see that all n greater than 3 are less efficient than 3, so we never use anything greater than 3. Therefore we use as many 3s in groups of 2 as possible, and since the remainder when 1976 is divided by 6 is 2, we can use 658 3s and one 2. So the greatest product is $\boxed{3^{658} \cdot 2}$. □

Problem 1.1.10 (1.3.13 Putnam 1978)**.** Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

> **Remark** (Pigeonhole Principle)**.** What are the pairs of integers in A whose sum is 104?
>
> $$4 + 100 = 7 + 97 = 10 + 94 = \cdots = 49 + 55, \text{ a total of 16 pairs.}$$
>
> If both members of a pair is chosen, then those are the desired ones. Two numbers are not in any pairs: 1, 52. These two and 16 pairs make 18 holes. We need to choose 20 pigeons.

Solution. [Pigeonhole Principle] Let

$$A_1 = \{1\}, A_2 = \{52\}, B_1 = \{4, 100\}, B_2 = \{7, 97\}, \ldots, B_{16} = \{49, 55\}.$$

If we choose $19 < \boxed{20}$ numbers from these 18 sets, there would be at least a B_i $(1 \leq i \leq 16)$ set where we select two distinct integers who sum is 104. □

Problem 1.1.11 (1.3.14 Putnam 1994)**.** Let (a_n) be a sequence of positive reals such that, for all n,

$$a_n \leq a_{2n} + a_{2n+1}.$$

Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

> **Remark** (Partial sums)**.** The objective here, basically, is to prove this sum $\sum_{n=1}^{\infty} a_n$ is larger or smaller than anything. Note that from this given condition $a_n \leq a_{2n} + a_{2n+1}$, it is safe to assume that the sequence diverges to positive infinity. It is sufficient if we can prove that the limit of the partial sums does not exist:
>
> $$\lim_{n \to \infty} \sum_{n=1}^{m} a_n = \infty$$

Solution. [Partial sums + Induction Principle] From the given conditions:

$$\left.\begin{array}{l} a_2 + a_3 \geq a_1 \\ (a_4 + a_5) + (a_6 + a_7) \geq a_2 + a_3 \geq a_1 \\ (a_8 + a_9) + \cdots + (a_{14} + a_{15}) \geq (a_4 + a_5) + (a_6 + a_7) \geq a_2 + a_3 \geq a_1 \\ \ldots \end{array}\right\}$$

It is easy to prove by the induction principle:

> **Claim —** $a_{2^n} + \cdots + a_{2^{n+1}-1} \geq a_1$.

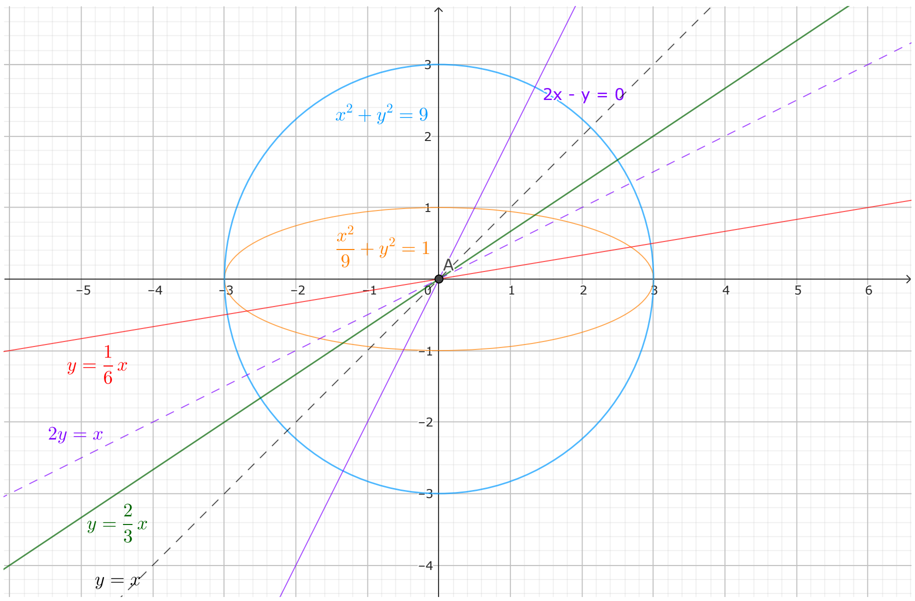
Therefore

$$\sum_{k=1}^{2^n-1} a_k = \sum_{k=0}^{n-1} (a_{2^k} + \cdots + a_{2^{k+1}-1}) \geq n a_1.$$

Since $a_1 > 0$, thus for any $r > 0$, $\exists N > \frac{r}{a_1}$, such that for all $n > N$, $\sum_{k=1}^{2^n-1} a_k = n a_1 > N a_1 > r$. □

Problem 1.1.12 (1.3.15 Putnam 1994)**.** Find the positive value of m such that the area in the first quadrant enclosed by the ellipse $x^2/9 + y^2 = 1$, the $x-$axis, and the line $y = 2x/3$ is equal to the area in the first quadrant enclosed by the ellipse $x^2/9 + y^2 = 1$, the $y-$axis, and the line $y = mx$.

> **Remark.** The key idea here is a transformation of an eclipse can make it becoming a circle. The transformation that retains ration likely can be reversed and the ratios still valid.



Solution. [Transformation of an eclipse] Stretch the eclipse $x^2/9 + y^2 = 1$, by a factor 3 along the $y-$axis, so that the point (x, y) goes to $(x, 3y)$. Then the ellipse becomes the circle $x^2 + y^2 = 9$, and the line $2y = x$, becomes the line $y = 2x$.

Obviously, the required line in the stretched figure is the reflection of the line $y = 2x$ over $y = x$, which is the line $2y = x$. Now, shrinking back to the original preserves the ratio of the areas and gives the line $6y = x$, whose slope is $\boxed{\frac{1}{6}}$. □

Problem 1.1.13 (1.3.16 Putnam 1990)**.** Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely *all the points whose distance from the center is irrational.* How many punches are needed to remove every point?

> **Remark** (Proof of existence)**.** Assume we have two points at random, what would be their distance: rational, irrational? How about 3 points? In this proof, we prove the existence of the smallest number of punches.

Solution. [Proof of existence] Two punches are not enough since given any two points P and Q we can find a point R whose distance from each of P and Q is rational, which is one of the intersections of two circles centred at P and Q with rational radius larger then half of the length of PQ.

Three punches will be enough.

First, take two punches at two distinct points P and Q. Since each punch leaves *countably many* circles, and any two distinct circles intersection at most two points, thus two punches leaves a set S of *countably many* points.

Now, the circles centred at each point of S intersect line PQ at *countably many* points. Let R be a point on line PQ other than any of these intersections. It is obvious that the distance from R to any point on the circles is irrational. Thus a punch at R will remove all the circles. □

> **Remark** (Construction method)**.** In this approach, we show that there points can be chosen so that three punches can be done to remove all the points. The key here is to show that from any point there is always an *irrational distance* to one of the three selected points.

Solution. [Construction method] Let $\alpha = \sqrt[3]{2}$, and let $A = (-\alpha, 0), B = (0.0), C = (\alpha, 0)$, then if $P = (x, y)$ is an arbitrary point, then:

$$AP^2 - 2BP^2 + CP^2 = (x + \alpha)^2 + y^2 - 2(x^2 + y^2) + (x - \alpha)^2 + y^2 = 2\alpha^2 \text{ is irrational.}$$

Thus all of them cannot be rational. Therefore P is removed. □

Part II

Tests