Calculus in Brief with Python, Colab, GitHub, and LaTeX CiB - Version 0.46

December 13, 2023

Available on GitHub at:
https://GitHub.com/nicholaskarlson/CiB

Contents

Preface

This text, *Calculus in Brief - CiB* aspires to be more than just another math book. This book strives to foster collaborative math writing. Note that this book has very few references. The reader is encouraged to use resources available on the Web to fact-check. This book's view on "causation" and facts is heavily influenced by Mosteller and Tukey [MT77].

Redefining the Role of the Reader

Calculus in Brief (CiB) is an endeavor to reshape how math is written, understood, and studied. It's not just a passive read but an open-source approach to math, aiming to encourage students to become proactive learners.

This project strives to break the traditional mold of math education and invites readers and professional mathematicians to participate actively.

A Dynamic Relationship with Math

Calculus in Brief is not just a book but a movement and methodology, heralding a new era in how we approach, consume, and interact with math. By positioning the reader as an integral part of the math-book process, CiB fosters a dynamic relationship with math, making mathematics more accessible, proactive, and relevant. In this shifting paradigm, we are all potential mathematicians, creators of interesting and relevant ways to learn and study math.

Please fork the LaTeX source code for CiB (available on GitHub) and create your own book that chooses the facts and exercises most relevant to you! Also, starring the CiB project on GitHub would be greatly appreciated! Thanks for reading CiB!

Chapter 1

Introduction to CiB

Welcome to CiB on GitHub

Calculus in Brief, abbreviated CiB, isn't merely a passive read. It's an endeavor to reshape how math is written, studied, and taught. By presenting an open-source approach to math, the goal is to encourage everyone to become proactive readers and writers of math.

Fostering a Proactive Engagement with Math

Calculus in Brief is a call for a renewed engagement with mathematics. CiB is an endeavor to reshape how math is written, understood, and studied. It's not just a passive read but an open-source approach to math, aiming to encourage students to become proactive learners.

This project strives to break the traditional mold of math education and encourages readers and professional mathematicians to participate actively.

Please fork the LaTeX source code for CiB (available on GitHub) and create your own book on Calculus that chooses the content most relevant to you! Also, starring the CiB project on GitHub would be greatly appreciated! Thanks for reading CiB!

Chapter 2

Open-Source Ethos

The Spirit of Shared Knowledge and Collaboration

Math, like software, is better when it's open. CiB draws inspiration from the open-source software movement; this section elucidates how a collaborative, transparent, and shared approach can enhance our understanding of math. Here, we look at the philosophy behind open-source and how it beautifully combines with the study of mathematics.

Open-Source Math: Preserving Tradition Through Collaborative Exploration

Mathematics, like software, thrives when it embraces openness and transparency. CiB takes a leaf from the proven benefits of the open-source software model; this section highlights how a collaborative and transparent method can improve and deepen our grasp of math and its texts. Here, we explore the principles of open source and how these principles align with the development of mathematics and its texts.

Understanding the Open-Source Ethos

The open-source paradigm revolves around shared ownership, collaboration, and the free exchange of knowledge. In the software realm, this approach has led to groundbreaking innovations built and enhanced by a global community of skilled contributors. United by a mutual objective, these individuals pool their diverse talents and insights to improve and share software solutions for broader public benefit.

Advantages of the Open-Source Framework in Math

Collective Insight

Mirroring the collaborative essence of open-source software, many individuals can offer their perspectives and knowledge, making math texts more robust and varied.

Enhancement and Accuracy

Open platforms foster an environment of constructive criticism, ensuring prompt identification and correction of inaccuracies. This meticulous peer review can help provide a credible and current mathematical text.

Universal Access

Much as open-source software promotes free access and modification, open-source math prioritizes universal accessibility. This ensures mathematics knowledge isn't restricted to a select few but is available to all curious minds.

Potential Challenges

Despite its advantages, melding open-source with math has potential pitfalls. The volume of contributions can complicate accuracy verification processes.

However, the very community championing this open-source approach to math can serve as its vigilant protector. They can ensure that contributions undergo rigorous evaluation and referencing, akin to the meticulous checks within the open-source software community.

Conclusion: Reinvigorating Our Experience with Math

Adopting an open-source perspective to the approach of math signifies a refreshed approach. It beckons a worldwide community to collaborate and forge a comprehensive and exciting math text. In this refreshed approach, every individual can play a part, both as a contributor and a learner. Math texts, through this lens, evolve and flourish, reflecting the collective input of active participants.

Chapter 3

Introduction to GitHub

The Hub for Modern Collaboration

Harnessing GitHub: A New Frontier in Collaborative Math Writing

At the heart of our collaborative math endeavor lies GitHub, a platform traditionally associated with code but now repurposed for our endeavor. This section provides a primer on GitHub, laying the foundation for those unfamiliar and offering insights into its transformative potential for collective math writing, learning, and teaching.

A Brief Introduction to GitHub

Originally conceptualized as a platform for developers, GitHub is a repository hosting service that facilitates version control using Git. At its core, it allows multiple users to work on a project simultaneously, tracking changes and ensuring that the latest version of a project is always accessible. Over the years, GitHub has grown beyond its initial software-centric confines, becoming a hub for all kinds of collaborative projects, from writing to data science and now to math.

Repurposing GitHub for Math Texts

Version Control

Math writing, like software, is dynamic and constantly evolving. As new sources or perspectives emerge, math texts may need revisions. GitHub's version control ensures that every change made to a document is tracked, enabling mathematicians to see how math texts evolve over time.

Collaborative Writing

Multiple contributors can work on a single math text simultaneously. This multi-user capability ensures diverse viewpoints can be seamlessly integrated, making the math text richer and more comprehensive.

Review and Feedback

Just as developers review and comment on code, mathematicians can provide feedback on written content. This feature encourages rigorous peer review, ensuring accuracy and credibility.

Open Access

Math texts on GitHub can be made public, granting anyone access to read, contribute, or fork the text into their own versions. This workflow democratizes math texts, making the creation process a collective endeavor rather than the domain of a select few.

Transparency

All changes and contributions are logged, providing a clear trail of the evolution of a mathematical text. This transparency bolsters the credibility of the text hosted on the platform.

Community Building

Beyond just writing, GitHub fosters a community of mathematicians, enthusiasts, and readers who can discuss, debate, and engage in meaningful dialogues about math and available math texts on GitHub.

Conclusion: Envisioning a Collaborative Mathematical Landscape

Embracing GitHub as a tool for collaborative math signifies more than just a shift in approach; it heralds a new era of inclusivity, transparency, and dynamism in writing, learning, and math teaching.

Chapter 4

Encouragement to Fork

Invitation to Dive Deep and Make It Your Own

CiB isn't a static entity. It thrives on evolution, adaptation, and diversification, much like math itself. We encourage readers to "fork" (a term soon to be discussed) and create their own versions of this book. Read this section to understand the essence of "forking" and how it can be the starting point of your unique math journey.

The Concept of Forking: A Brief Overview

In the realm of software development, particularly in platforms like GitHub, "forking" refers to the act of creating a copy of a project, allowing one to make changes independently of the original. In this context, forking CiB enables readers to take the base content and adapt, modify, and expand upon it, tailoring the narrative to resonate with their perspectives, insights, and understanding.

How to Begin Your Forking Journey

Start Small: You don't need to rewrite entire chapters. Begin by adding annotations, insights, or even footnotes to existing content. As you grow more confident, you can expand and modify larger sections.

Engage with the Community: Share your forked version with other readers. This encourages discourse, debate, and constructive feedback, allowing your text to be refined and enhanced.

Celebrate input: Encourage others around you to fork and create their own versions. The more in-depth the input, the deeper our collective understanding of math potentially becomes.

Conclusion: The Power of Collective Math

The invitation to fork CiB isn't just about creating different versions of a book. It's a call to embrace collective writing, learning, and teaching. By embracing the essence of forking, math is not just something we read but something we actively shape, share, and pass on.

Chapter 5

More About GitHub

Discovering the Power of Collaborative Tools

Diving deeper into the world of GitHub, this chapter provides a comprehensive overview. Beyond its technicalities, we explore how GitHub emerged as a revolutionary platform for collaboration and how it can be leveraged for those interested in writing, teaching, and learning about math.

The Genesis of GitHub

GitHub began as a platform designed for software developers to manage and track changes to their codebase. Launched in 2008, it swiftly gained traction due to its user-friendly interface and efficient version control system powered by Git. Over the years, it evolved from a mere repository hosting service to a dynamic hub of collaboration, housing millions of projects and engaging tens of millions of users worldwide.

GitHub: More than Just Code

While GitHub's origins are rooted in code collaboration, its adaptable nature has made it a favored platform for various non-code projects. Writers, designers, educators, and researchers have discovered the potential of GitHub as a tool for:

Document Collaboration

With its built-in version control, contributors can track changes, revert to previous versions, and seamlessly merge updates.

Project Management

With features like "issues" and "milestones," teams can organize tasks, set goals, and monitor progress.

Open Access & Transparency

Public repositories allow for open contributions, ensuring transparency and fostering a sense of collective ownership.

Collaborative Writing

Multiple contributors can simultaneously work on a single document, with every change being tracked and attributed, facilitating teamwork on extensive projects like books or research papers.

Engaging the Public

With the platform's inherent transparency, researchers can make their work-in-progress accessible to the public, inviting insights, corrections, and contributions.

Case Study: CiB's Use of GitHub

CiB's journey on GitHub is a testament to the platform's potential in mathematical endeavors. By hosting the book on GitHub, the following is possible:

Feedback Loop

Readers can raise "issues," pointing out inaccuracies, suggesting enhancements, or even recommending new sections or topics.

Forking

As previously discussed, readers can "fork" the repository, creating their unique versions of the book while staying connected to the original.

Regular Updates

With math being dynamic, the book can be regularly updated, with new versions being released as and when significant changes are incorporated.

Challenges and Considerations

While GitHub offers many advantages, it's essential to understand its limitations:

Learning Curve

For those unfamiliar with Git or version control, there can be an initial learning curve.

Data Overwhelm

With vast amounts of data and contributions, ensuring quality and accuracy can be challenging.

Diverse Audience Management

Catering to both tech-savvy and non-tech audiences might require creating additional resources or tutorials to ensure inclusivity.

Conclusion: GitHub – A Paradigm Shift in Collaboration

The rise of GitHub marks a significant shift in how we perceive and participate in collaborative projects. Its adaptability, transparency, and user-centric design make it a powerful tool, not just for coders but for anyone passionate about collective endeavors. In the realm of mathematics, GitHub promises a future where texts are continually refined, expanded, and enriched by a global community.

Chapter 6

Forking Process

The Heart of Collaboration on GitHub

The beauty of open-source lies in its democratization of content creation. In this section, we demystify the process of "forking" on GitHub, guiding you step-by-step on how to take CiB and create a version uniquely yours.

Understanding Forking

Before diving into the specifics, it's crucial to understand what "forking" means in the context of GitHub. In the simplest terms, to "fork" a project means to create a personal copy of someone else's project. Forking allows you to freely experiment with changes without affecting the original project. Forking is akin to taking a book you admire and making a copy to write your notes, edits, or additional chapters without altering the original book.

Why Fork?

Experimentation

It provides a safe space where you can test out ideas, make changes, or introduce new content.

Personalization

For projects like CiB, it allows readers to customize the content, tailor it to their perspectives, or even localize it for specific audiences.

Collaboration

If you believe your changes have broad appeal, you can propose that they be incorporated back into the original project, enriching it with your unique contributions.

Step-by-Step Forking Guide

Set Up Your GitHub Account

If you don't have an account on GitHub, you'll need to create one. Visit GitHub's official site and sign up.

Navigate to the CiB Repository

Once logged in, search for the CiB project or navigate to its URL directly.

Click the 'Fork' Button

The fork button is located at the top right corner of the repository page; this button will create a copy of CiB in your account.

Clone Your Forked Repository

Forking allows you to have a local copy on your computer, making editing and experimentation easier. Use the command: `git clone [URL of your forked repo]`.

Make Your Changes

Using your preferred tools, introduce the edits, additions, or modifications you desire.

Commit and Push Changes

Once satisfied, save these changes (known as a "commit") and then "push" them to your forked repository on GitHub.

Optional – Create a Pull Request

If you believe your changes should be incorporated into the original CiB repository, you can create a "pull request." A pull request notifies the original authors of your suggestions.

Things to Keep in Mind

Stay Updated

The original CiB project may undergo updates. It's a good practice to regularly "pull" from the original repo to keep your fork up-to-date.

Engage with the Community

Open-source thrives on community interactions. Engage in discussions, seek feedback, and please remain open to constructive criticism.

Conclusion: Embracing the Forking Culture

Forking is more than just a technical process; it symbolizes the ethos of open-source — a world where knowledge is not hoarded but shared, refined, and built upon collectively. By forking CiB or any other project, you're not just creating a personal copy; you're becoming a part of a global movement that values collaboration, innovation, and the shared pursuit of knowledge. So, embark on this journey, make your unique mark, and contribute to the ever-evolving corpus of collective wisdom.

Chapter 7

Editing and Customizing

Tailoring Repositories to Suit Your Needs

Now, let's build upon the forking process; this segment delves into the next steps. How can you edit and customize your version of CiB? What tools and techniques are available at your disposal? Embark on this informative journey as we guide you through the intricacies of editing on GitHub.

Understanding the GitHub Workspace

Before diving into the specifics of editing, it's essential to familiarize yourself with the GitHub workspace. Think of it as a digital toolshed where each tool serves a unique function:

- **Repository (Repo)**: This is the project's main folder where all your project's files are stored and where you track all changes.
- **Branches**: These are parallel versions of a repository, allowing you to work on features or edits without altering the main project.
- **Commits**: This is a saved change in the repository, akin to saving a file after making edits.
- **Pull Requests**: This is how you notify the main project of desired changes, proposing that your edits be merged with the original.

Editing Files Directly on GitHub

For minor changes, you might opt to edit directly on GitHub:

1. Navigate to the File: Within your forked CiB repository, find the file you want to edit.

2. Click the Pencil Icon: This button allows you to edit the file.

3. Make Your Edits: Modify the content as needed.

4. Save and Commit: Below the editing pane, you'll see a "commit changes" section. Add a brief note summarizing your changes and click 'Commit.'

Editing Files Locally

For extensive customization:

1. Clone Your Repository: Use a tool like Git to clone (download) your forked repo to your local computer.

2. Edit Using Your Preferred Tools: This could range from text editors to specialized software, depending on the file type.

3. Commit and Push: After making your changes, save them (commit) and then upload (push) them to your GitHub repository.

Utilizing Branches for Extensive Customization

Branches are especially useful for significant overhauls or when working on different versions:

1. Create a New Branch: From your main project page, use the branch dropdown to type in a new branch name and create it.

2. Switch to Your Branch: Ensure you're working in this new parallel environment.

3. Make and Commit Changes: As you would in the main project.

4. Merging: Once satisfied with your edits in the branch, you can merge these changes back into the main project or keep them separate as a different version.

Exploring Additional Tools and Extensions

GitHub's ecosystem is rich with tools and extensions to enhance your editing experience:

- **GitHub Desktop**: An application that simplifies the process of managing your repositories without using command-line tools.
- **Markdown Editors**: Since many GitHub files (like READMEs) are written in Markdown, tools like StackEdit or Dillinger can be invaluable.
- **Extensions for Browsers**: Tools like Octotree can help in navigating repositories more effortlessly.

Conclusion: The Art of Tailored Content

Editing and customizing on GitHub might seem daunting initially, but with practice, it transforms into a manageable workflow. Many people find that the ability to take a project like CiB and mold it into something uniquely theirs is empowering. It's a testament to the open-source community's ethos, where shared knowledge becomes the canvas and our collective edits, the brushstrokes, crafting an ever-evolving masterpiece. As you embark on your customization journey, remember that every edit, no matter how small, contributes to the project potentially in significant ways.

Chapter 8

Engaging with the Community

Joining the Global Conversation

The Significance of the GitHub Community

The digital age has bestowed upon us the gift of connectivity. On platforms like GitHub, this connectivity transcends borders, disciplines, and ideologies, culminating in a melting pot of diverse ideas and knowledge. For mathematicians and math enthusiasts, GitHub offers a space not only to store and manage content but also to engage with an audience that is passionate, informed, and eager to contribute.

1. Discussions and Debates

One of the most enriching aspects of the GitHub community is the plethora of discussions that unfold:

- **Issues**: A core feature of GitHub, "issues" allow users to raise questions, report problems, or propose enhancements.
- **GitHub Discussions**: A newer feature, Discussions, acts like a community forum. It's an excellent place for extended conversations, brainstorming, and sharing ideas or resources.

2. Collaborative Content Creation

Beyond solitary endeavors, GitHub shines in its collaborative capabilities:

- **Pull Requests**: If you have made an alteration to a math text or added a new perspective, pull requests are the way to propose these changes to the original repository owner. Pull requests foster a collaborative spirit, where content isn't static but continually evolving with community input.

- **Fork and Merge**: As you've learned, forking allows you to create your version of a repository. Engaging with the Community means you can merge changes from others into your fork, blending a mixture of diverse insights.

3. Building and Nurturing Networks

Connections made on GitHub often spill over into lasting professional relationships:

- **Following and Followers**: Like on social media platforms, you can follow contributors whose work resonates with you. Following contributors creates a curated feed of updates and also allows you to be part of a more extensive network.
- **GitHub Stars**: If a particular project or repository impresses you, give it a star! Starring not only bookmarks the project for you but also shows appreciation to the creator.

4. Learning and Growing Through Feedback

The Community's feedback is an invaluable asset:

- **Code Reviews**: Although traditionally for software, text writers can use this feature to receive feedback on their methodologies or approaches, refining their work.
- **Community Insights**: The "insights" tab on a repository provides analytics. For text writers, this can give a sense of which topics garner more attention and interest.

5. Participating in Community Events

GitHub often hosts and sponsors events:

- **Hackathons**: While traditionally for coders, these events can be repurposed for text writer content creation, where participants collaboratively tackle projects or themes.
- **Webinars and Workshops**: These events can range from mastering GitHub's technical side to thematic discussions on math topics.

A Project of Collective Wisdom

Math, in many ways, is a collective endeavor. GitHub can provide a dynamic Community. By engaging with this Community you can become an active participant in the creation of mathematical texts.

Chapter 9

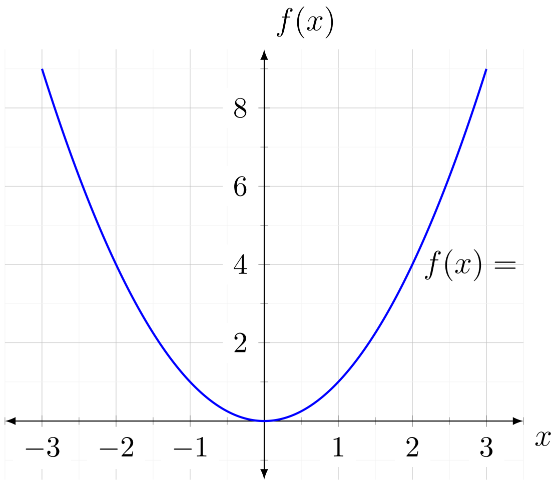
Functions

9.1 What is a Function?

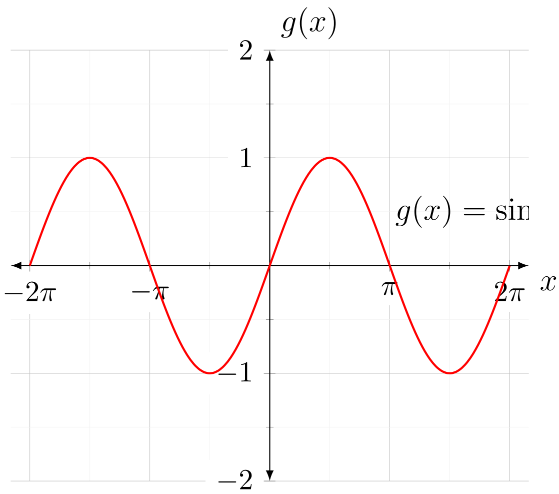
9.1.1 Definition and Examples

Definition 1 *A function is a relation between a set of inputs (domain) and a set of permissible outputs (range) with the property that each input is related to exactly one output.*

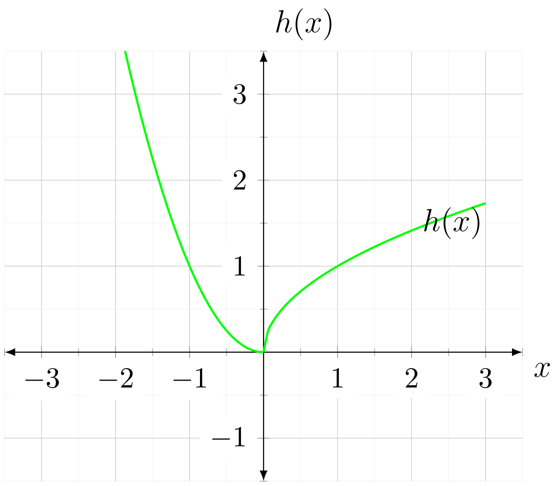
Example 1: Consider the function $f(x) = x^2$, which maps each real number to its square.



Example 2: Consider the function $g(x) = \sin(x)$, which maps real numbers to their sine values.



Example 3: Consider the piecewise function $h(x)$ defined as $h(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$.



9.1.2 Exercises

Exercise Graph the function $f(x) = 3x - 2$. Identify its slope and y-intercept.

Exercise Consider the function $g(x) = \frac{1}{x}$. For what values of x is $g(x)$ undefined?

Exercise Find the domain and range of the function $h(x) = \sqrt{x+4}$.

Exercise Determine if the function $f(x) = x^3 - x$ is even, odd, or neither. Justify your answer.

Exercise Sketch the graph of the piecewise function $p(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x+1 & \text{if } x \geq 0 \end{cases}$. Indicate any points of discontinuity.

Exercise Given the function $f(x) = 2x^2 - 5x + 3$, find $f(-1)$ and $f(2)$.

Exercise For the function $f(x) = |x - 3|$, find the x-coordinate where $f(x) = 0$.

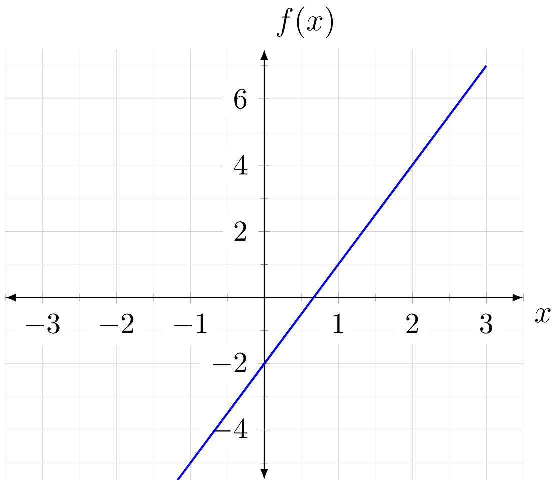
Exercise Create a function $f(x)$ that has a domain of all real numbers except $x = 2$ and a range of $y \geq 0$.

Exercise Determine the intervals on which the function $f(x) = -x^2 + 4x - 3$ is increasing and decreasing.

Exercise For the function $g(x) = \cos(x)$, evaluate $g(\pi/2)$ and $g(\pi)$.

9.1.3 Solutions to Exercises

Solution: To graph $f(x) = 3x - 2$, we note that the slope (m) is 3 and the y-intercept (b) is -2. The graph is a straight line with these characteristics.

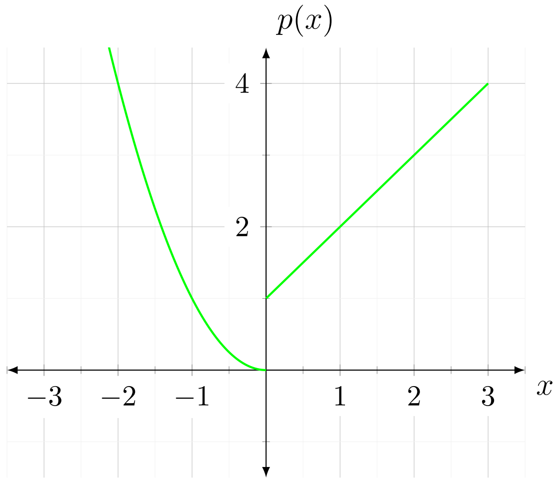


Solution: The function $g(x) = \frac{1}{x}$ is undefined when $x = 0$ because division by zero is not allowed.

Solution: For $h(x) = \sqrt{x+4}$, the domain is $x \geq -4$ (since the expression inside the square root must be non-negative), and the range is $y \geq 0$ (since the square root produces non-negative results).

Solution: The function $f(x) = x^3 - x$ is odd because $f(-x) = (-x)^3 - (-x) = -x^3 + x = -f(x)$.

Solution: The piecewise function $p(x)$ is sketched below. It is discontinuous at $x = 0$ since the left-hand limit (0) does not equal the right-hand limit (1).



Solution: For $f(x) = 2x^2 - 5x + 3$, we find $f(-1) = 2(-1)^2 - 5(-1) + 3 = 10$ and $f(2) = 2(2)^2 - 5(2) + 3 = -1$.

Solution: The function $f(x) = |x - 3|$ equals 0 when $x - 3 = 0$, i.e., $x = 3$.

Solution: One such function is $f(x) = \frac{1}{(x-2)^2}$. The denominator ensures that $x \neq 2$ (domain), and the square ensures all output values are positive (range).

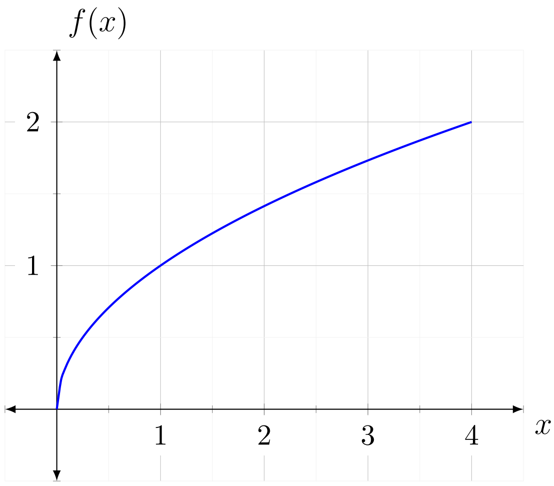
Solution: For $f(x) = -x^2 + 4x - 3$, the function is increasing where its derivative $f'(x) = -2x + 4$ is positive, i.e., for $x < 2$, and decreasing where $f'(x)$ is negative, i.e., for $x > 2$.

Solution: For $g(x) = \cos(x)$, we have $g(\pi/2) = \cos(\pi/2) = 0$ and $g(\pi) = \cos(\pi) = -1$.

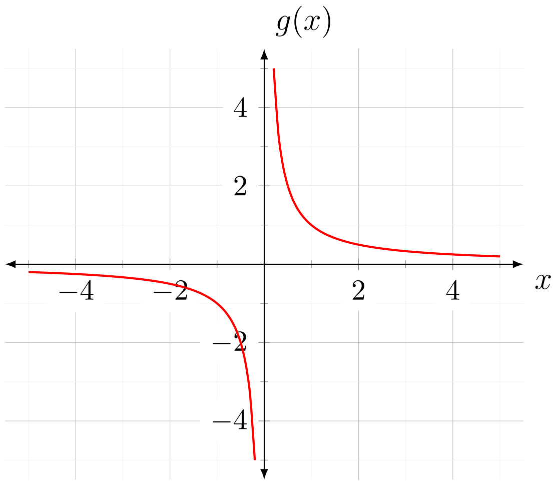
9.2 Domain and Range

Definition 2 *The domain of a function is the set of all possible inputs, while the range is the set of all possible outputs.*

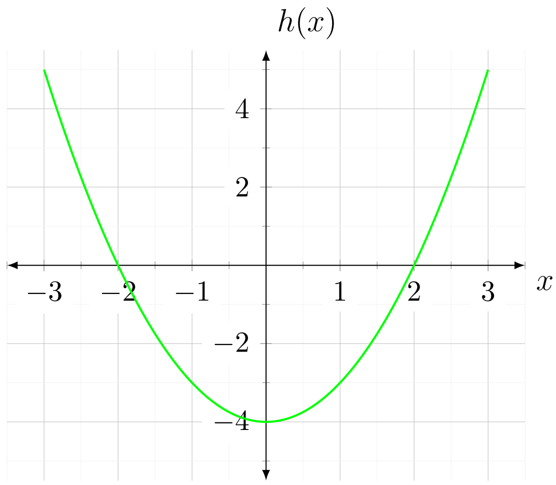
Example 4: For $f(x) = \sqrt{x}$, the domain is $x \geq 0$ and the range is $y \geq 0$. The graph of this function is as follows:



Example 5: Consider the function $g(x) = \frac{1}{x}$. Its domain is $x \neq 0$ and its range is $y \neq 0$. The graph is given by:



Example 6: For the function $h(x) = x^2 - 4$, the domain is all real numbers, but the range is $y \geq -4$. Its graph is:



9.2.1 Exercises on Domain and Range

Exercise Determine the domain and range of the function $f(x) = 2x + 3$.

Exercise Find the domain and range of the function $g(x) = \frac{1}{x-2}$.

Exercise Identify the domain and range for the quadratic function $h(x) = x^2 - 6x + 9$.

Exercise For the function $k(x) = \ln(x-1)$, determine its domain and range.

Exercise Consider the function $m(x) = \frac{x}{x^2-4}$. What are its domain and range?

Exercise Find the domain and range for the trigonometric function $n(x) = \sin(x)$.

Exercise Determine the domain and range of the absolute value function $p(x) = |x + 5|$.

Exercise For the piecewise function $q(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ 2x + 3 & \text{if } x > 0 \end{cases}$, identify the domain and range.

Exercise Consider the exponential function $r(x) = 2^x$. What is its domain and range?

Exercise Identify the domain and range for the cubic function $s(x) = x^3 - 3x$.

9.2.2 Solutions to Exercises on Domain and Range

Solution: For $f(x) = 2x + 3$, the domain is all real numbers since there are no restrictions on x. The range is also all real numbers because as x takes any real value, $2x + 3$ also covers all real numbers.

Solution: The function $g(x) = \frac{1}{x-2}$ is undefined when $x = 2$, so the domain is $x \neq 2$. The range is all real numbers except $y \neq 0$ because the function never touches the y-axis.

Solution: For the quadratic function $h(x) = x^2 - 6x + 9$, the domain is all real numbers. Since the vertex of this parabola is at $x = 3$ and it opens upwards, the minimum value is $h(3) = 0$, thus the range is $y \geq 0$.

Solution: The function $k(x) = \ln(x - 1)$ requires $x - 1 > 0$, so the domain is $x > 1$. The range of a natural logarithm function is all real numbers.

Solution: For $m(x) = \frac{x}{x^2-4}$, the function is undefined for $x = \pm 2$, so the domain is $x \neq \pm 2$. The range is all real numbers, as the function can take any y-value.

Solution: The domain of the trigonometric function $n(x) = \sin(x)$ is all real numbers. The range of the sine function is between -1 and 1, inclusive, so $y \in [-1, 1]$.

Solution: For the absolute value function $p(x) = |x+5|$, the domain is all real numbers. The range is $y \geq 0$ since absolute values are always non-negative.

Solution: The piecewise function $q(x)$ is defined for all real numbers, so the domain is all real numbers. To find the range, consider each piece: x^2 and $2x + 3$. The range is all non-negative values for x^2 and all real numbers for $2x + 3$, thus the range is all real numbers.

Solution: For the exponential function $r(x) = 2^x$, the domain is all real numbers, as exponents can take any real value. The range is $y > 0$ since exponential functions are always positive.

Solution: The cubic function $s(x) = x^3 - 3x$ has a domain of all real numbers. The range is also all real numbers because as a cubic function, it extends infinitely in both the positive and negative y-directions.

9.3 Types of Functions

9.3.1 Linear Functions

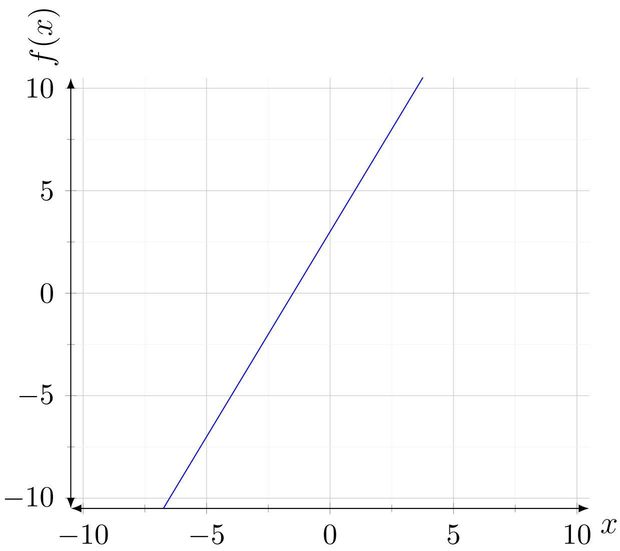
Definition 3 *A linear function has the form $f(x) = mx + b$, where m is the slope of the line and b is the y-intercept.*

Linear functions are fundamental in calculus and represent the simplest form of polynomial functions. Their graph is a straight line, and they have constant rates of change.

Properties of Linear Functions:

- The slope m determines the steepness of the line and its direction (increasing or decreasing).
- The y-intercept b is the point where the line crosses the y-axis.
- Linear functions have a constant rate of change, which is equal to the slope m.

Graphing a Linear Function: Consider the function $f(x) = 2x + 3$. Here, $m = 2$ and $b = 3$. The graph can be plotted as follows:



This line intersects the y-axis at (0, 3) and has a slope of 2, indicating it rises 2 units for every 1 unit it moves to the right.

Slope-Intercept Form: The equation $f(x) = mx + b$ is known as the slope-intercept form of a linear function, where the slope m and y-intercept b are easily identifiable.

Applications of Linear Functions: Linear functions model relationships with a constant rate of change and are widely used in various fields such as economics, physics, and social sciences.

Exercise Find the slope and y-intercept of the linear function $g(x) = -3x+7$ and sketch its graph.

Exercise If a linear function passes through the points (1, 2) and (3, 6), determine its equation.

Exercises on Linear Functions

Exercise Find the slope and y-intercept of the linear function $g(x) = -3x+7$ and sketch its graph.

Exercise If a linear function passes through the points (1, 2) and (3, 6), determine its equation.

Exercise Determine the equation of the line that is parallel to $f(x) = 4x-5$ and passes through the point (2, 3).

Exercise Find the point of intersection of the linear functions $f(x) = 2x+1$ and $g(x) = -x+5$.

Exercise A rental car company charges a base fee of \$50 and an additional \$20 per day. Write a linear function representing the total cost C as a function of the number of days d rented.

Exercise Graph the linear function $h(x) = -\frac{1}{2}x + 4$ and identify where it intersects the x-axis and y-axis.

Exercise A phone company offers a monthly plan for \$30 with an additional charge of \$0.05 per minute of calls. Write a linear function representing the monthly cost M as a function of the number of minutes m used.

Exercise Determine whether the points (2, 4), (3, 6), and (5, 10) lie on the same linear function. If so, find the equation of the line.

Exercise Find the slope of the line that passes through the points (-1, -2) and (3, 4).

Exercise A company's profit P in thousands of dollars is linearly related to the number of units n sold, with a profit of \$4,000 for 500 units and a loss of \$2,000 for 200 units. Determine the linear function that models the profit.

Solutions to Exercises on Linear Functions

Solution: For $g(x) = -3x + 7$, the slope (m) is -3, and the y-intercept (b) is 7. The graph is a straight line with a downward slope, crossing the y-axis at (0, 7).

Solution: To find the equation of a line passing through (1, 2) and (3, 6), calculate the slope $m = \frac{6-2}{3-1} = 2$. Using the point-slope form: $y-2 = 2(x-1)$, the equation is $y = 2x$.

Solution: A line parallel to $f(x) = 4x-5$ has the same slope, m $= 4$. Passing through (2, 3), use point-slope form: $y - 3 = 4(x - 2)$, yielding $y = 4x - 5$.

Solution: To find the intersection of $f(x) = 2x + 1$ and $g(x) = -x + 5$, set them equal: $2x + 1 = -x + 5$. Solving gives $x = \frac{4}{3}$, and substituting back, $y = \frac{11}{3}$. Thus, they intersect at $(\frac{4}{3}, \frac{11}{3})$.

Solution: The total cost C is given by $C = 50 + 20d$, where d is the number of days rented.

Solution: For $h(x) = -\frac{1}{2}x + 4$, the graph intersects the y-axis at (0, 4) and intersects the x-axis at x where $h(x) = 0$. Setting the equation to 0 gives $x = 8$. So, it intersects the x-axis at (8, 0).

Solution: The monthly cost M is $M = 30 + 0.05m$, where m is the number of minutes used.

Solution: The slope between (2, 4) and (3, 6) is 2, and between (3, 6) and (5, 10) is also 2. Since the slope is consistent, they lie on the same line. The line equation, using point-slope form, is $y = 2x$.

Solution: The slope is $\frac{4-(-2)}{3-(-1)} = \frac{6}{4} = \frac{3}{2}$.

Solution: Let $P = mn + b$. Using given points, form equations: $4000 = 500m + b$ and $-2000 = 200m + b$. Solving these simultaneously gives $m = 20$ and $b = -6000$. So, $P = 20n - 6000$.

9.3.2 Polynomial Functions

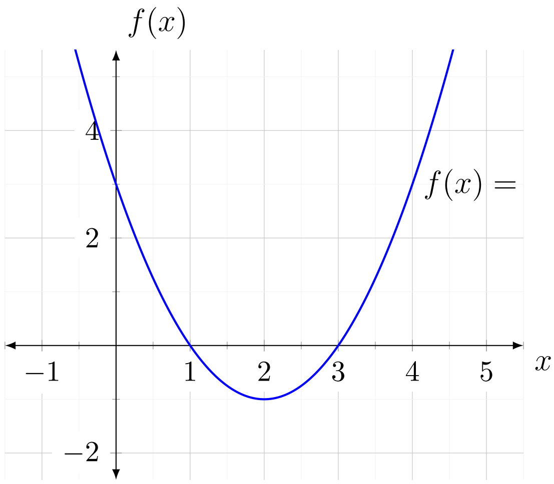
Definition 4 *A polynomial function is of the form* $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$*, where* n *is a non-negative integer, and* $a_n, a_{n-1}, \ldots, a_0$ *are constants.*

Polynomial functions are fundamental in calculus and algebra. They can range from simple linear functions to complex equations with high degrees. The highest power of x in the function (the degree of the polynomial) greatly influences the shape and behavior of its graph.

Characteristics of Polynomial Functions:

- The degree of the polynomial determines its basic shape and the number of roots.
- Polynomials are continuous and smooth functions.
- The leading coefficient a_n affects the end behavior of the polynomial.

Graphing Polynomial Functions: Consider the quadratic function $f(x) = x^2 - 4x + 3$. Its graph can be plotted as follows:



This graph represents a parabola opening upwards with its vertex and intercepts easily identifiable.

Real-World Applications: Polynomial functions model numerous real-world phenomena, including projectile motion, profit and loss calculations, population growth, and much more.

Exercise Sketch the graph of the cubic function $g(x) = x^3 - 6x^2 + 11x - 6$. Identify its roots and turning points.

Exercise Find a polynomial of degree 3 that has roots at $x = 1, 2$, and 3.

Exercises on Polynomial Functions

Exercise Sketch the graph of the cubic function $g(x) = x^3 - 6x^2 + 11x - 6$. Identify its roots and turning points.

Exercise Find a polynomial of degree 3 that has roots at $x = 1, 2$, and 3.

Exercise Determine the degree and leading coefficient of the polynomial $h(x) = 4x^4 - 3x^3 + 2x^2 - x + 7$.

Exercise For the quadratic function $f(x) = 2x^2 - 8x + 6$, find the vertex and axis of symmetry. Also, determine whether it opens upwards or downwards.

Exercise Write the polynomial function $p(x) = x^3 - 4x$ in factored form and identify its zeros.

Exercise A polynomial function has zeros at $x = -2, 0$, and 4 with a leading coefficient of 1. Write the equation of this polynomial.

Exercise Given the polynomial $q(x) = x^4 - 10x^2 + 9$, find the y-intercept and x-intercepts (if any).

Exercise Graph the polynomial $r(x) = -x^3 + 3x^2 - 2x$ and determine the intervals where the function is increasing and decreasing.

Exercise For the polynomial function $s(x) = 3x^3 - 5x^2 + x - 2$, use synthetic division to determine whether $x = 1$ is a zero of s.

Exercise Consider the polynomial $t(x) = x^4 - 2x^2 - 3$. Find all the real zeros of t and sketch the graph of the function.

Solutions to Exercises on Polynomial Functions

Solution: For $g(x) = x^3 - 6x^2 + 11x - 6$, the roots are found by solving $g(x) = 0$. The roots are $x = 1, 2, 3$. Turning points occur where the derivative changes sign. Calculating $g'(x) = 3x^2 - 12x + 11$ and finding its roots gives the turning points at approximately $x \approx 1.57$ and $x \approx 2.43$.

Solution: A polynomial of degree 3 with roots at $x = 1, 2, 3$ can be written as $f(x) = (x - 1)(x - 2)(x - 3)$.

Solution: The degree of the polynomial $h(x) = 4x^4 - 3x^3 + 2x^2 - x + 7$ is 4, and the leading coefficient is 4.

Solution: For $f(x) = 2x^2 - 8x + 6$, the vertex form is $f(x) = 2(x - 2)^2 + 2$. The vertex is at (2, 2), and the axis of symmetry is $x = 2$. The parabola opens upwards since the coefficient of x^2 is positive.

Solution: The polynomial $p(x) = x^3 - 4x$ can be factored as $p(x) = x(x^2 - 4) = x(x - 2)(x + 2)$. The zeros are at $x = 0, 2, -2$.

Solution: A polynomial with zeros at $x = -2, 0$, and 4 and a leading coefficient of 1 is $f(x) = (x+2)x(x-4) = x^3 - 4x^2 - 2x$.

Solution: For $q(x) = x^4 - 10x^2 + 9$, the y-intercept is $q(0) = 9$. To find the x-intercepts, solve $x^4 - 10x^2 + 9 = 0$, which gives $x \approx \pm 3.16, \pm 0.84$.

Solution: Graphing $r(x) = -x^3 + 3x^2 - 2x$, the function increases where $r'(x) > 0$ and decreases where $r'(x) < 0$. Calculating $r'(x) = -3x^2 + 6x - 2$ and finding its roots gives the intervals of increase and decrease.

Solution: Using synthetic division to divide $3x^3 - 5x^2 + x - 2$ by $x - 1$, we find that the remainder is not zero. Hence, $x = 1$ is not a zero of $s(x)$.

Solution: To find the real zeros of $t(x) = x^4 - 2x^2 - 3$, solve $x^4 - 2x^2 - 3 = 0$. The real zeros are approximately $x \approx \pm 1.73, \pm 1.00$. Graphing $t(x)$ would show a polynomial crossing the x-axis at these points.

9.3.3 Rational Functions

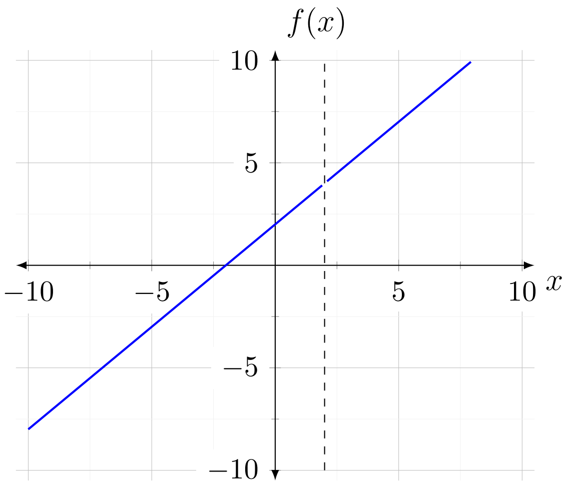
Definition 5 *A rational function is the ratio of two polynomial functions, typically expressed in the form* $f(x) = \frac{p(x)}{q(x)}$, *where both* $p(x)$ *and* $q(x)$ *are polynomial functions and* $q(x) \neq 0$.

Rational functions can exhibit a variety of behaviors, including asymptotes and discontinuities, making them interesting subjects of study in calculus.

Characteristics of Rational Functions:

- The vertical asymptotes occur at values of x for which $q(x) = 0$, provided these points are not cancelled out by the numerator.
- Horizontal asymptotes are determined by the degrees of $p(x)$ and $q(x)$.
- Rational functions may have holes if a factor in the denominator is cancelled out by the numerator.

Graphing Rational Functions: Consider the function $f(x) = \frac{x^2-4}{x-2}$. Here, we have a hole at $x = 2$ because the factor $(x - 2)$ is cancelled. The graph can be plotted as follows:



Applications of Rational Functions: Rational functions are used in various fields, including engineering, economics, and physics, to model scenarios where variables inversely affect each other.

Exercise Graph the rational function $g(x) = \frac{1}{x-1}$ and determine its vertical and horizontal asymptotes.

Exercise Consider the function $h(x) = \frac{x^2-1}{x^2+x-2}$. Identify its vertical and horizontal asymptotes, and sketch the graph.

Exercises on Rational Functions

Exercise Graph the rational function $g(x) = \frac{1}{x-1}$ and determine its vertical and horizontal asymptotes.

Exercise Consider the function $h(x) = \frac{x^2-1}{x^2+x-2}$. Identify its vertical and horizontal asymptotes, and sketch the graph.

Exercise Find the domain of the rational function $f(x) = \frac{2x-3}{x^2-4}$.

Exercise Determine the vertical and horizontal asymptotes for the function $p(x) = \frac{x^3-4x}{x^2-5x+6}$ and sketch its graph.

Exercise For the rational function $q(x) = \frac{x^2+2x+1}{x^2-1}$, identify any holes or asymptotes, and graph the function.

Exercise Given $r(x) = \frac{3x-5}{2x+4}$, calculate $r(2)$ and determine the behavior of $r(x)$ as x approaches infinity.

Exercise Sketch the graph of the function $s(x) = \frac{x}{x^2-9}$ and find the x- and y-intercepts.

Exercise Analyze the rational function $t(x) = \frac{5x-4}{x-3}$ for any asymptotes or discontinuities and provide its graph.

Exercise Find the range of the function $u(x) = \frac{2}{x-1} + 3$.

Exercise Determine the inverse of the function $v(x) = \frac{1}{2-x}$.

Solutions to Exercises on Rational Functions

Solution: For $g(x) = \frac{1}{x-1}$, the vertical asymptote is at $x = 1$ (where the denominator equals zero), and the horizontal asymptote is $y = 0$ (as x approaches infinity, $g(x)$ approaches 0).

Solution: In $h(x) = \frac{x^2-1}{x^2+x-2}$, the vertical asymptotes are at the roots of the denominator, $x = 1$ and $x = -2$. The horizontal asymptote is $y = 1$ as the degrees of the numerator and denominator are the same.

Solution: The domain of $f(x) = \frac{2x-3}{x^2-4}$ is all real numbers except $x = \pm 2$, where the denominator equals zero.

Solution: For $p(x) = \frac{x^3-4x}{x^2-5x+6}$, vertical asymptotes are at $x = 2$ and $x = 3$, the roots of the denominator. There is no horizontal asymptote as the degree of the numerator is greater than that of the denominator.

Solution: In $q(x) = \frac{x^2+2x+1}{x^2-1}$, there is a hole at $x = -1$ (factor cancelled in numerator and denominator), vertical asymptotes at $x = \pm 1$, and no horizontal asymptote.

Solution: For $r(x) = \frac{3x-5}{2x+4}$, $r(2) = \frac{1}{8}$. As x approaches infinity, $r(x)$ approaches $\frac{3}{2}$.

Solution: In $s(x) = \frac{x}{x^2-9}$, the x-intercepts are at $x = 0$, y-intercept at $y = 0$, and vertical asymptotes at $x = \pm 3$.

Solution: The function $t(x) = \frac{5x-4}{x-3}$ has a vertical asymptote at $x = 3$ and no horizontal asymptote. Discontinuity occurs at $x = 3$.

Solution: To find the range of $u(x) = \frac{2}{x-1} + 3$, consider the behavior as x approaches 1 and infinity. The range is all real numbers except $y = 3$.

Solution: The inverse of $v(x) = \frac{1}{2-x}$ can be found by swapping x and y and solving for y: $x = \frac{1}{2-y}$ leads to $y = 2 - \frac{1}{x}$.

9.3.4 Trigonometric Functions

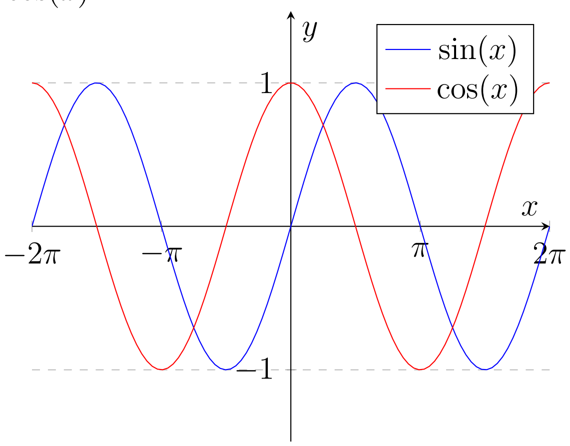
Definition 6 *Trigonometric functions include* $\sin(x)$*,* $\cos(x)$*, and* $\tan(x)$*, relating angles to ratios of sides in right triangles.*

Trigonometric functions are fundamental in calculus, physics, and engineering. They are periodic functions and have various properties that are essential in the study of waves, oscillations, and circular motion.

Properties of Trigonometric Functions:

- The functions $\sin(x)$ and $\cos(x)$ are periodic with a period of 2π.
- The function $\tan(x)$ is periodic with a period of π.
- These functions have specific symmetries: $\sin(x)$ is odd, $\cos(x)$ is even, and $\tan(x)$ is odd.

Graphing Trigonometric Functions: Using 'pgfplots' to graph $\sin(x)$ and $\cos(x)$:



Applications of Trigonometric Functions: Trigonometric functions model phenomena such as sound waves, light waves, and describe the motion of pendulums and other periodic phenomena.

Exercise Graph the function $f(x) = \tan(x)$ and identify its asymptotes.

Exercise Find the amplitude and period of the function $g(x) = 3\sin(2x)$.

Exercise Determine the phase shift and vertical shift for the function $h(x) = \cos(x - \pi/2) + 1$.

Exercises on Trigonometric Functions

Exercise Graph the function $f(x) = \tan(x)$ and identify its asymptotes.

Exercise Find the amplitude and period of the function $g(x) = 3\sin(2x)$.

Exercise Determine the phase shift and vertical shift for the function $h(x) = \cos(x - \pi/2) + 1$.

Exercise Prove that $\sin^2(x) + \cos^2(x) = 1$ using a unit circle.

Exercise Evaluate $\sin(\pi/6)$, $\cos(\pi/3)$, and $\tan(\pi/4)$.

Exercise Sketch the graph of $f(x) = \sin(x) - \cos(x)$ over the interval $[0, 2\pi]$.

Exercise Find the values of x where $\sin(x) = -\sqrt{3}/2$ in the interval $[-\pi, \pi]$.

Exercise For the function $g(x) = 2\cos(3x - \pi)$, identify the amplitude, period, phase shift, and sketch its graph.

Exercise Determine the exact value of $\tan(\pi/6) + \cos(\pi/3)$.

Exercise Solve for x in the equation $3\sin(x) + 4\cos(x) = 0$ over the interval $[0, 2\pi]$.

Solutions to Exercises on Trigonometric Functions

Solution: For $f(x) = \tan(x)$, the graph has vertical asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ where the cosine function in the denominator equals zero.

Solution: The function $g(x) = 3\sin(2x)$ has an amplitude of 3 (the coefficient of the sine function) and a period of $\frac{\pi}{2}$ (computed as $\frac{2\pi}{2} = \pi$).

Solution: For $h(x) = \cos(x - \pi/2) + 1$, the phase shift is $\frac{\pi}{2}$ to the right, and the vertical shift is 1 unit upwards.

Solution: Using the unit circle, we know that $\sin(x)$ and $\cos(x)$ represent the y-coordinate and x-coordinate, respectively, of a point on the circle. Since the radius of the unit circle is 1, we have $\sin^2(x) + \cos^2(x) = 1^2 = 1$.

Solution: Evaluate $\sin(\pi/6) = 1/2$, $\cos(\pi/3) = 1/2$, and $\tan(\pi/4) = 1$.

Solution: To graph $f(x) = \sin(x) - \cos(x)$, plot the sine and cosine functions separately and then subtract their values for each x in the interval $[0, 2\pi]$.

Solution: The values of x where $\sin(x) = -\sqrt{3}/2$ in the interval $[-\pi, \pi]$ are $x = -\pi/3$ and $x = -2\pi/3$.

Solution: For $g(x) = 2\cos(3x - \pi)$, the amplitude is 2, the period is $\frac{2\pi}{3}$, and the phase shift is $\frac{\pi}{3}$ to the right. Sketch the graph accordingly.

Solution: The exact value of $\tan(\pi/6) + \cos(\pi/3)$ is $\frac{\sqrt{3}}{3} + \frac{1}{2}$.

Solution: Solve $3\sin(x) + 4\cos(x) = 0$ over the interval $[0, 2\pi]$. Dividing by $\cos(x)$ and applying trigonometric identities, we find $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

9.3.5 Exponential and Logarithmic Functions

Definition 7 *Exponential functions have the form $f(x) = a^x$ where a is a positive constant. Logarithmic functions are the inverses of exponential functions and are usually expressed as $g(x) = \log_a(x)$ for a logarithm to the base a.*

Exponential and logarithmic functions are widely used in many fields, including physics, engineering, and economics, due to their properties of describing growth and decay processes.

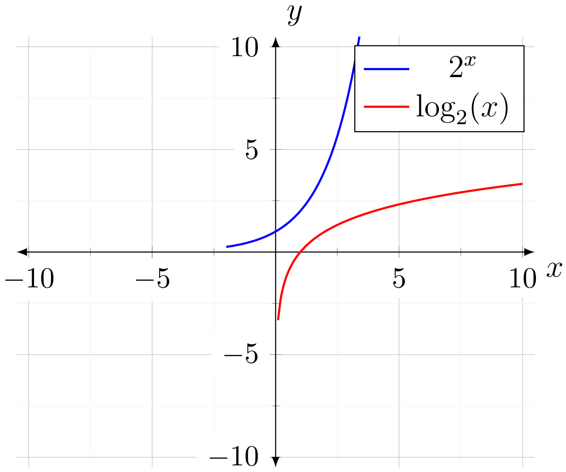
Characteristics of Exponential Functions:

- Exponential growth occurs when $a > 1$, and exponential decay happens when $0 < a < 1$.
- The graph of an exponential function always lies above the x-axis and increases or decreases rapidly.
- Exponential functions have a horizontal asymptote at $y = 0$.

Characteristics of Logarithmic Functions:

- The logarithmic function $\log_a(x)$ is undefined for $x \leq 0$.
- The graph of a logarithmic function passes through the point $(1, 0)$ and has a vertical asymptote at $x = 0$.
- Logarithmic functions increase slowly and are used to model the inverse of exponential growth or decay.

Graphing Exponential and Logarithmic Functions: Here we graph $f(x) = 2^x$ and its inverse $g(x) = \log_2(x)$ using 'pgfplots':



Exercise Graph the exponential function $h(x) = e^x$ and identify its key characteristics.

Exercise Sketch the graph of $k(x) = \log_e(x)$ (also known as $\ln(x)$) and discuss its properties.

Exercises on Exponential and Logarithmic Functions

Exercise Graph the exponential function $h(x) = e^x$ and identify its key characteristics.

Exercise Sketch the graph of $k(x) = \log_e(x)$ (also known as $\ln(x)$) and discuss its properties.

Exercise Evaluate the expression $e^{\ln(5)}$ and explain the relationship between the exponential and logarithmic functions.

Exercise Find the solution to the equation $3^x = 27$.

Exercise Determine the x-intercept of the logarithmic function $f(x) = \log_2(x - 1)$.

Exercise Solve for x in the equation $\ln(x) + \ln(2x) = 3$.

Exercise Graph the function $g(x) = 2^x + 3$ and describe how it differs from the basic exponential function $f(x) = 2^x$.

Exercise Find the domain and range of the function $h(x) = \ln(x-2)$.

Exercise Calculate the derivative of the function $f(x) = e^{2x}$ using the rules of differentiation for exponential functions.

Exercise Given the function $g(x) = \log_{10}(x^2 - 1)$, find the points of discontinuity.

Solutions to Exercises on Exponential and Logarithmic Functions

Solution: For $h(x) = e^x$, the graph is an increasing exponential function. It passes through the point (0, 1) and has a horizontal asymptote at $y = 0$. The function is always positive and increases rapidly as x increases.

Solution: The function $k(x) = \log_e(x)$, or $\ln(x)$, has a graph that passes through the point (1, 0) and has a vertical asymptote at $x = 0$. The function is defined for $x > 0$ and increases slowly as x increases.

Solution: Evaluate $e^{\ln(5)}$. Since the exponential and logarithmic functions are inverses of each other, $e^{\ln(5)} = 5$.

Solution: Solve $3^x = 27$. Recognizing that $27 = 3^3$, we find $x = 3$.

Solution: To find the x-intercept of $f(x) = \log_2(x-1)$, set $f(x) = 0$. This yields $\log_2(x-1) = 0$, so $x - 1 = 1$, and thus $x = 2$.

Solution: Solve $\ln(x) + \ln(2x) = 3$. Combine the logarithms: $\ln(2x^2) = 3$. Exponentiating both sides gives $2x^2 = e^3$. Solving for x yields $x = \pm\sqrt{\frac{e^3}{2}}$, but only the positive solution is valid since $\ln(x)$ is undefined for negative x.

Solution: Graph $g(x) = 2^x + 3$. This graph is a vertical shift of the function $f(x) = 2^x$ upwards by 3 units. It increases exponentially and passes through the point (0, 4).

Solution: The domain of $h(x) = \ln(x-2)$ is $x > 2$, and the range is all real numbers since the logarithmic function can take any real value.

Solution: The derivative of $f(x) = e^{2x}$ is $f'(x) = 2e^{2x}$, using the chain rule for differentiation.

Solution: For $g(x) = \log_{10}(x^2 - 1)$, the function is undefined at points where $x^2 - 1 \leq 0$, which occurs at $x = -1$ and $x = 1$. These are the points of discontinuity.

9.4 Function Properties

9.4.1 Continuity

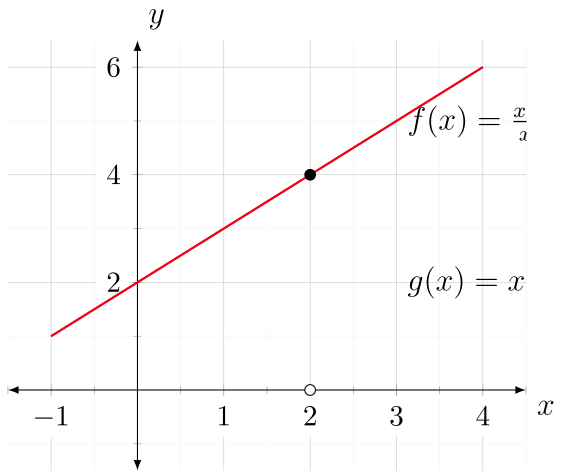
Definition 8 *A function is continuous at a point if the limit of the function as it approaches the point is equal to the function value at that point. Formally, a function $f(x)$ is continuous at a point c if $\lim_{x \to c} f(x) = f(c)$.*

Continuity is a fundamental concept in calculus, essential for understanding limits, derivatives, and integrals.

Types of Discontinuities:

- A function has a *removable discontinuity* at a point if the limit exists but is not equal to the function value at that point.
- A *jump discontinuity* occurs when the left-hand and right-hand limits exist but are not equal to each other.
- An *infinite discontinuity* occurs when the function approaches infinity near the point of discontinuity.

Graphing Continuous Functions: To illustrate continuity, consider the function $f(x) = \frac{x^2-4}{x-2}$ and its simplified form $g(x) = x+2$. The function $f(x)$ has a removable discontinuity at $x = 2$.



Applications of Continuity: Continuous functions are used in various applications, including physics (to model continuous motion), economics (to describe continuous growth), and engineering (in signal processing).

Exercise Determine if the function $h(x) = \frac{x^2-1}{x-1}$ is continuous at $x = 1$.

Exercise Evaluate the continuity of the piecewise function $k(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ x+2 & \text{if } x > 2 \end{cases}$ at $x = 2$.

Exercises on Continuity

Exercise Determine if the function $h(x) = \frac{x^2-1}{x-1}$ is continuous at $x = 1$.

Exercise Evaluate the continuity of the piecewise function $k(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ x+2 & \text{if } x > 2 \end{cases}$ at $x = 2$.

Exercise Is the function $f(x) = \frac{1}{x}$ continuous at $x = 0$? Justify your answer.

Exercise Consider the function $g(x) = \ln(x)$. Determine the points of discontinuity, if any.

Exercise Prove that the function $p(x) = x^3 - 3x + 1$ is continuous for all real numbers.

Exercise Find the value of a for which the function $q(x) = \begin{cases} 2x+a & \text{if } x \leq 3 \\ x^2 - 1 & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$.

Exercise Determine whether the function $r(x) = \frac{\sin(x)}{x}$ is continuous at $x = 0$.

Exercise For the function $s(x) = \sqrt{x}$, identify the intervals where it is continuous.

Exercise Show that the function $t(x) = e^{-x^2}$ is continuous everywhere.

Exercise Is the function $u(x) = \frac{|x|}{x}$ continuous at $x = 0$? Explain your reasoning.

Solutions to Exercises on Continuity

Solution: For $h(x) = \frac{x^2-1}{x-1}$, factorize the numerator to get $h(x) = \frac{(x-1)(x+1)}{x-1}$. At $x = 1$, the function simplifies to $h(1) = 2$, and the limit as x approaches 1 is also 2. Hence, $h(x)$ is continuous at $x = 1$.

Solution: For $k(x)$, at $x < 2$, $k(x) = x^2$; at $x = 2$, $k(x) = 4$; and at $x > 2$, $k(x) = x + 2$. The left-hand limit as x approaches 2 is 4, and the right-hand limit is also 4. Since the function value at $x = 2$ is 4, $k(x)$ is continuous at $x = 2$.

Solution: The function $f(x) = \frac{1}{x}$ is not continuous at $x = 0$ because it is not defined there (the denominator becomes zero), and the limits as x approaches 0 are $\pm\infty$.

Solution: The function $g(x) = \ln(x)$ is continuous for all $x > 0$. It has a discontinuity at $x = 0$ because it is not defined for $x \leq 0$.

Solution: Since $p(x) = x^3 - 3x + 1$ is a polynomial function, it is continuous for all real numbers (polynomials are continuous everywhere).

Solution: For $q(x)$ to be continuous at $x = 3$, the values from the left and right must be equal at $x = 3$. Setting $2(3) + a = 3^2 - 1$ gives $a = 5$.

Solution: The function $r(x) = \frac{\sin(x)}{x}$ is continuous at $x = 0$ by applying L'Hôpital's Rule or recognizing that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

Solution: The function $s(x) = \sqrt{x}$ is continuous for all $x \geq 0$ since it is defined and smooth over this interval.

Solution: The function $t(x) = e^{-x^2}$ is continuous everywhere as the composition of continuous functions (exponential and polynomial) is continuous.

Solution: The function $u(x) = \frac{|x|}{x}$ is not continuous at $x = 0$ as it is not defined there (the denominator becomes zero).

9.4.2 Differentiability

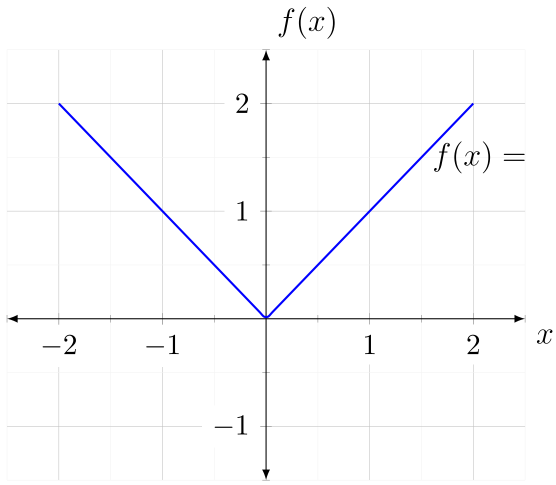
Definition 9 *A function is differentiable at a point if it has a defined derivative at that point. Formally, a function $f(x)$ is differentiable at c if the limit $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists.*

Differentiability is a key concept in calculus, relating closely to the continuity of a function and its smoothness.

Characteristics of Differentiable Functions:

- If a function is differentiable at a point, it is also continuous at that point, but the converse is not necessarily true.
- A function may fail to be differentiable at a point if it has a corner, cusp, or vertical tangent, or is discontinuous at that point.

Graphing and Analyzing Differentiability: Consider a function $f(x)$ with a cusp at $x = 0$. For example, $f(x) = |x|$ is not differentiable at $x = 0$ due to the cusp.



Applications of Differentiability: Differentiability is used in physics to analyze motion (velocity and acceleration) and in economics to find rates of change (marginal costs and revenues).

Exercise Determine whether the function $g(x) = x^2$ is differentiable at $x = 0$ and explain why.

Exercise Analyze the differentiability of the piecewise function $h(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$ at $x = 1$.

Exercises on Differentiability

Exercise Determine whether the function $g(x) = x^2$ is differentiable at $x = 0$ and explain why.

Exercise Analyze the differentiability of the piecewise function $h(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$ at $x = 1$.

Exercise Check if the function $f(x) = \frac{1}{x}$ is differentiable at $x = 0$ and provide a justification for your answer.

Exercise Determine whether the function $k(x) = |x|$ is differentiable at $x = 0$. Give reasons for your answer.

Exercise For the function $p(x) = x^3$, find the derivative at $x = 2$ and discuss its differentiability at this point.

Exercise Consider the function $q(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Is $q(x)$ differentiable at $x = 0$?

Exercise Evaluate the differentiability of the function $r(x) = \sqrt{x}$ at $x = 0$.

Exercise Find the points of non-differentiability, if any, for the function $s(x) = \frac{|x|}{x}$.

Exercise Given the function $t(x) = \ln(x)$, determine its differentiability over its domain.

Exercise Prove that the function $u(x) = e^x$ is differentiable for all real numbers x.

Solutions to Exercises on Differentiability

Solution: The function $g(x) = x^2$ is differentiable at $x = 0$. This is because its derivative $g'(x) = 2x$ is defined at $x = 0$ (with $g'(0) = 0$), showing that the function is smooth and has no sharp corners or cusps at this point.

Solution: The piecewise function $h(x)$ is differentiable at $x = 1$ as both pieces of the function have the same derivative at this point. For $x \leq 1$, the derivative is $2x$, and for $x > 1$, it is 2. Since both derivatives equal 2 at $x = 1$, the function is differentiable there.

Solution: The function $f(x) = \frac{1}{x}$ is not differentiable at $x = 0$ because it is not defined at this point. The function has a discontinuity (infinite discontinuity) at $x = 0$, which precludes differentiability.

Solution: The function $k(x) = |x|$ is not differentiable at $x = 0$. Although the function is continuous at $x = 0$, it has a sharp corner at this point, which means that the limit defining the derivative does not exist.

Solution: For $p(x) = x^3$, the derivative is $p'(x) = 3x^2$. At $x = 2$, $p'(2) = 12$. The function is differentiable at this point as the derivative is defined and finite.

Solution: The function $q(x)$ is differentiable at $x = 0$. Although the expression $x^2 \sin(\frac{1}{x})$ seems complex, its derivative at $x = 0$ is 0, making the function differentiable at this point.

Solution: The function $r(x) = \sqrt{x}$ is not differentiable at $x = 0$. The derivative, $\frac{1}{2\sqrt{x}}$, becomes infinite as x approaches 0, indicating a vertical tangent line at this point.

Solution: The function $s(x) = \frac{|x|}{x}$ is not differentiable at $x = 0$ as it is not continuous there. The function has a jump discontinuity at this point.

Solution: The function $t(x) = \ln(x)$ is differentiable over its domain $(0, \infty)$. Its derivative, $\frac{1}{x}$, is defined for all $x > 0$.

Solution: The function $u(x) = e^x$ is differentiable for all real numbers x because its derivative, e^x, is defined and continuous for all x.

9.4.3 Asymptotic Behavior

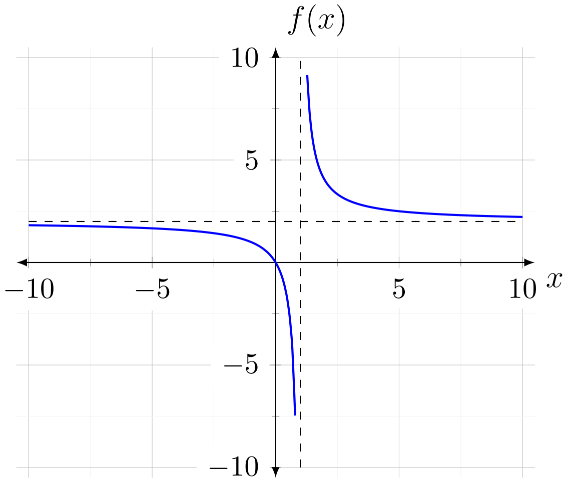
Definition 10 *Asymptotic behavior refers to the behavior of a function as it approaches infinity or a particular value. A function may approach a finite value (horizontal asymptote), become infinitely large (vertical asymptote), or approach a certain path (slant asymptote).*

Understanding asymptotes is crucial in calculus for analyzing function limits and behavior at extreme values.

Types of Asymptotes:

- *Horizontal asymptotes* occur when a function approaches a horizontal line as x goes to infinity or minus infinity.
- *Vertical asymptotes* occur at values of x where the function tends towards infinity or minus infinity.
- *Slant (or oblique) asymptotes* occur when the function approaches a line that is not horizontal as x goes to infinity or minus infinity.

Graphing Asymptotic Behavior: Consider the function $f(x) = \frac{2x}{x-1}$. It has a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 2$.



Applications of Asymptotic Analysis: Asymptotic analysis is used in various fields, such as physics for studying motion at high velocities, in economics for long-term growth models, and in probability for extreme value theory.

Exercise Determine the horizontal and vertical asymptotes of the function $g(x) = \frac{3x^2-x}{x^2+2}$.

Exercise Analyze the asymptotic behavior of the function $h(x) = \frac{x^2+3x}{x+1}$.

Exercises on Asymptotic Behavior

Exercise Determine the horizontal and vertical asymptotes of the function $g(x) = \frac{3x^2-x}{x^2+2}$.

Exercise Analyze the asymptotic behavior of the function $h(x) = \frac{x^2+3x}{x+1}$.

Exercise Find all asymptotes of the function $f(x) = \frac{4x^3-2x+1}{2x^2-3x+5}$.

Exercise Determine the horizontal asymptote, if any, for the function $p(x) = e^x$.

Exercise Identify any vertical asymptotes for the function $q(x) = \ln(x-2)$.

Exercise For the function $r(x) = \frac{x}{x^2-4}$, describe its asymptotic behavior as x approaches 2 and -2.

Exercise Calculate the slant asymptote for the function $s(x) = \frac{x^2-x+1}{x-3}$.

Exercise Examine the function $t(x) = \frac{1}{\sqrt{x^2+1}}$ for horizontal asymptotes.

Exercise Does the function $u(x) = \frac{\sin(x)}{x}$ have any horizontal asymptotes as x approaches infinity?

Exercise Analyze the function $v(x) = x\sin(\frac{1}{x})$ for x approaching 0 and determine if there are any asymptotes.

Solutions to Exercises on Asymptotic Behavior

Solution: For $g(x) = \frac{3x^2-x}{x^2+2}$, the horizontal asymptote is found by comparing the degrees of the numerator and denominator. Since they are the same, the horizontal asymptote is $y = \frac{3}{1} = 3$. There are no vertical asymptotes as the denominator is never zero.

Solution: The function $h(x) = \frac{x^2+3x}{x+1}$ has a slant asymptote since the degree of the numerator is one more than the degree of the denominator. Dividing the numerator by the denominator gives the slant asymptote $y = x + 2$. There is no vertical asymptote as $x + 1$ does not equal zero for any real x.

Solution: For $f(x) = \frac{4x^3-2x+1}{2x^2-3x+5}$, the degree of the numerator is higher than the degree of the denominator, so there is no horizontal asymptote. For vertical asymptotes, solve $2x^2 - 3x + 5 = 0$, which has no real solutions, so there are no vertical asymptotes.

Solution: The function $p(x) = e^x$ does not have a horizontal asymptote as it grows exponentially and does not approach a finite value as x approaches infinity.

Solution: The function $q(x) = \ln(x - 2)$ has a vertical asymptote at $x = 2$ since the logarithm becomes undefined at this point.

Solution: For $r(x) = \frac{x}{x^2-4}$, the function has vertical asymptotes at $x = 2$ and $x = -2$ where the denominator equals zero.

Solution: The function $s(x) = \frac{x^2-x+1}{x-3}$ has a slant asymptote since the degree of the numerator is one more than the denominator. Divide $x^2 - x + 1$ by $x - 3$ to find the slant asymptote equation.

Solution: The function $t(x) = \frac{1}{\sqrt{x^2+1}}$ has a horizontal asymptote at $y = 0$ as the value approaches zero when x approaches infinity.

Solution: The function $u(x) = \frac{\sin(x)}{x}$ has a horizontal asymptote at $y = 0$ as x approaches infinity since the sine function oscillates between -1 and 1 while x grows without bound.

Solution: The function $v(x) = x\sin(\frac{1}{x})$ for x approaching 0 does not have any asymptotes. As x approaches 0, $\sin(\frac{1}{x})$ oscillates rapidly, but the product with x ensures that the function remains bounded and approaches 0.

9.4.4 Periodicity and Symmetry

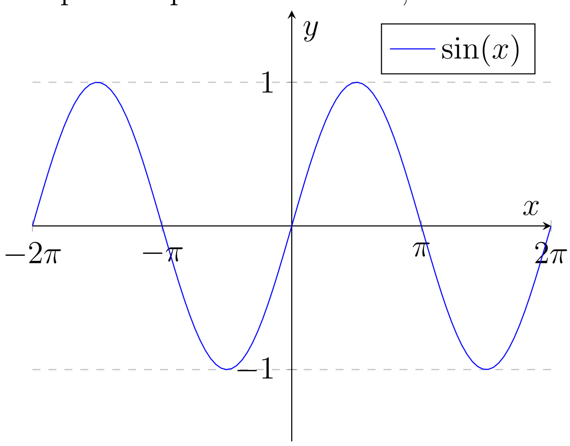
Definition 11 *A function is periodic if it repeats its values at regular intervals, and it is symmetric if it exhibits mirror symmetry about an axis or point.*

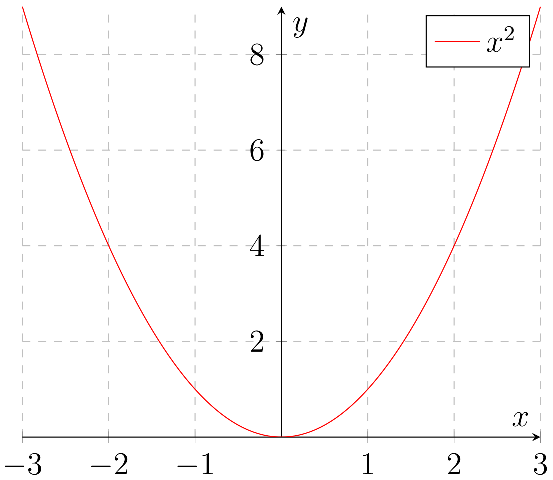
Periodicity and symmetry play important roles in understanding the behavior and properties of functions, especially in the fields of trigonometry and wave analysis.

Types of Symmetry:

- A function is *even* if $f(-x) = f(x)$, showing symmetry about the y-axis.
- A function is *odd* if $f(-x) = -f(x)$, showing rotational symmetry about the origin.

Graphing Periodic and Symmetric Functions: For instance, $\sin(x)$ is an example of a periodic function, and x^2 is an example of an even function.





Applications of Periodicity and Symmetry: These concepts are frequently applied in physics for analyzing wave patterns, in engineering for signal processing, and in mathematics for solving trigonometric equations.

Exercise Determine if the function $f(x) = \cos(x)$ is periodic, and if so, find its period.

Exercise Analyze the symmetry of the function $g(x) = x^3 - x$. Is it even, odd, or neither?

Exercises on Periodicity and Symmetry

Exercise Determine if the function $f(x) = \cos(x)$ is periodic, and if so, find its period.

Exercise Analyze the symmetry of the function $g(x) = x^3 - x$. Is it even, odd, or neither?

Exercise Identify the periodicity of the function $h(x) = \sin(2x)$ and specify the period.

Exercise Check whether the function $p(x) = x^4 - 6x^2 + 8$ is even, odd, or neither.

Exercise For the function $q(x) = \tan(x)$, determine if it is periodic and find the period.

Exercise Investigate the symmetry of the function $r(x) = \frac{1}{x}$. Is it symmetric about the y-axis, the origin, or neither?

Exercise Examine the function $s(x) = e^x$ for periodicity. Is this function periodic?

Exercise Determine the period of the function $t(x) = \cos(3x - \pi)$.

Exercise Analyze the function $u(x) = x^2 - 2x + 1$ for symmetry. Is it symmetric about the y-axis?

Exercise Check whether the function $v(x) = \sin(x) + \cos(x)$ is periodic, and if so, identify its period.

Solutions to Exercises on Periodicity and Symmetry

Solution: The function $f(x) = \cos(x)$ is periodic. Its period is 2π since $\cos(x + 2\pi) = \cos(x)$ for all x.

Solution: The function $g(x) = x^3 - x$ is odd. This is because $g(-x) = (-x)^3 - (-x) = -x^3 + x = -g(x)$.

Solution: The function $h(x) = \sin(2x)$ is periodic with a period of π. This is because $\sin(2(x + \pi)) = \sin(2x + 2\pi) = \sin(2x)$.

Solution: The function $p(x) = x^4 - 6x^2 + 8$ is even since $p(-x) = (-x)^4 - 6(-x)^2 + 8 = x^4 - 6x^2 + 8 = p(x)$.

Solution: The function $q(x) = \tan(x)$ is periodic with a period of π because $\tan(x + \pi) = \tan(x)$ for all x.

Solution: The function $r(x) = \frac{1}{x}$ is neither even nor odd, and it does not exhibit symmetry about the y-axis or the origin.

Solution: The function $s(x) = e^x$ is not periodic. It does not repeat its values at regular intervals.

Solution: The function $t(x) = \cos(3x - \pi)$ is periodic with a period of $\frac{2\pi}{3}$. This period is derived from the coefficient of x in the argument of the cosine function.

Solution: The function $u(x) = x^2 - 2x + 1$ is not symmetric about the y-axis. It is a parabola that opens upwards and does not have symmetry about the y-axis.

Solution: The function $v(x) = \sin(x) + \cos(x)$ is periodic with a period of 2π since both sine and cosine functions have a period of 2π.

9.5 Function Transformations

9.5.1 Translation and Scaling

Definition 12 *Translation shifts a function horizontally or vertically, while scaling changes its size.*

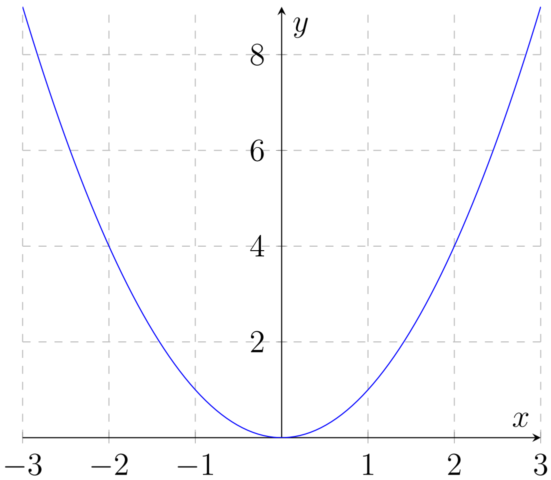
Translation and scaling are fundamental operations in function manipulation, allowing for adjustments in position and size of the graph of a function.

Types of Transformations:

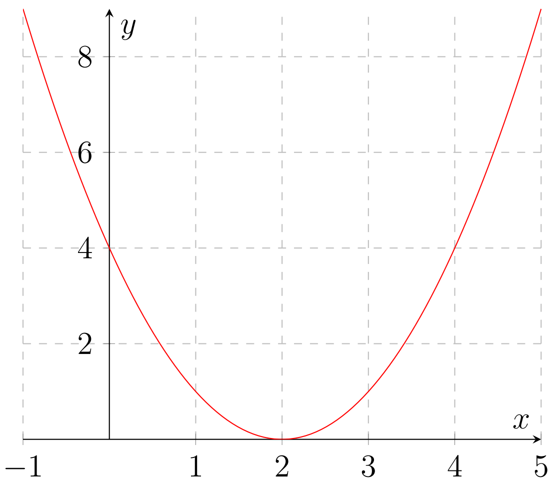
- *Horizontal Translation:* Shifting a function left or right. For instance, $f(x)$ to $f(x-h)$ shifts the function right by h.
- *Vertical Translation:* Moving a function up or down. For example, $f(x)$ to $f(x)+k$ raises the function by k.
- *Horizontal Scaling:* Stretching or compressing a function horizontally. $f(x)$ to $f(cx)$ scales the function horizontally.
- *Vertical Scaling:* Altering the function's height. $f(x)$ to $cf(x)$ scales the function vertically.

Graphing Translated and Scaled Functions: Let's consider the function $f(x) = x^2$, its translation, and scaling.

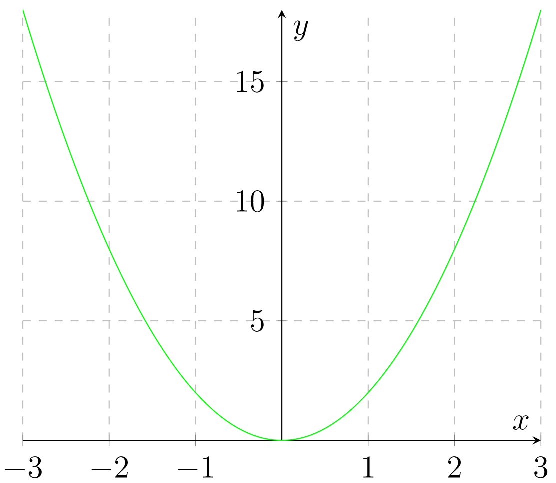
Original Function $f(x) = x^2$



Translated Function $f(x-2) = (x-2)^2$



Scaled Function $2f(x) = 2x^2$



Applications of Translation and Scaling: These transformations are widely used in data analysis for adjusting graphs, in physics to model motion and waves, and in other mathematical contexts for function analysis.

Exercise Graph the function $g(x) = (x-3)^2+2$ and describe its translation and scaling relative to $f(x) = x^2$.

Exercise For the function $h(x) = \frac{1}{2}\sin(2x+\pi)$, identify the translations and scalings applied to the basic sine function.

Exercises on Translation and Scaling

Exercise Graph the function $g(x) = (x-3)^2+2$ and describe its translation and scaling relative to $f(x) = x^2$.

Exercise For the function $h(x) = \frac{1}{2}\sin(2x + \pi)$, identify the translations and scalings applied to the basic sine function.

Exercise Consider the function $p(x) = -3\cos(x-\pi/2)$. Describe its vertical and horizontal translations, and vertical scaling.

Exercise Determine the translation and scaling transformations applied to the function $f(x) = x^2$ to obtain $q(x) = 4(x + 1)^2 - 5$.

Exercise Sketch the graph of $r(x) = \frac{1}{3}(x - 2)^3 + 4$ and explain the transformations applied to the basic cubic function $f(x) = x^3$.

Exercise Graph the function $s(x) = 2\sin(\pi x - \pi/2) + 1$ and describe its periodicity, translation, and scaling.

Exercise Analyze the function $t(x) = \frac{x+2}{x-1}$ for vertical and horizontal translations and any scaling factors.

Exercise Identify the transformations applied to the exponential function $f(x) = e^x$ to produce $u(x) = 2e^{-x+3} - 1$.

Exercise Consider the function $v(x) = \sqrt{x + 4} - 2$. Determine the transformations applied to the basic square root function $f(x) = \sqrt{x}$.

Exercise Graph the function $w(x) = \ln(x-3)+2$ and describe its translation and scaling relative to the natural logarithm function $f(x) = \ln(x)$.

Solutions to Exercises on Translation and Scaling

Solution: The function $g(x) = (x - 3)^2 + 2$ is a translation of $f(x) = x^2$. It is translated 3 units to the right and 2 units up.

Solution: For $h(x) = \frac{1}{2}\sin(2x + \pi)$, the function is scaled vertically by a factor of $\frac{1}{2}$, horizontally by a factor of $\frac{1}{2}$ (period is π), and translated $\frac{\pi}{2}$ units to the left.

Solution: The function $p(x) = -3\cos(x - \pi/2)$ is vertically scaled by a factor of 3, inverted (due to the negative sign), and translated $\frac{\pi}{2}$ units to the right.

Solution: In $q(x) = 4(x+1)^2 - 5$, the function $f(x) = x^2$ is horizontally translated 1 unit to the left, vertically scaled by a factor of 4, and translated 5 units down.

Solution: The function $r(x) = \frac{1}{3}(x-2)^3 + 4$ is a transformation of $f(x) = x^3$ with a horizontal translation 2 units to the right, vertical scaling by $\frac{1}{3}$, and vertical translation 4 units up.

Solution: For $s(x) = 2\sin(\pi x - \pi/2) + 1$, the function is vertically scaled by 2, translated $\frac{1}{2}$ unit to the right, and translated 1 unit up. The period is $\frac{2\pi}{\pi} = 2$.

Solution: The function $t(x) = \frac{x+2}{x-1}$ involves a horizontal translation 1 unit to the right and 2 units to the left in the numerator. There's no clear vertical or horizontal scaling.

Solution: The function $u(x) = 2e^{-x+3} - 1$ is translated 3 units to the right, reflected over the y-axis, vertically scaled by 2, and translated 1 unit down from $f(x) = e^x$.

Solution: In $v(x) = \sqrt{x+4} - 2$, the square root function $f(x) = \sqrt{x}$ is translated 4 units to the left and 2 units down.

Solution: The function $w(x) = \ln(x-3) + 2$ is a translation of the natural logarithm function $f(x) = \ln(x)$, shifted 3 units to the right and 2 units up.

9.5.2 Reflection and Rotation

Definition 13 *Reflection inverts a function across a line, and rotation turns it around a point.*

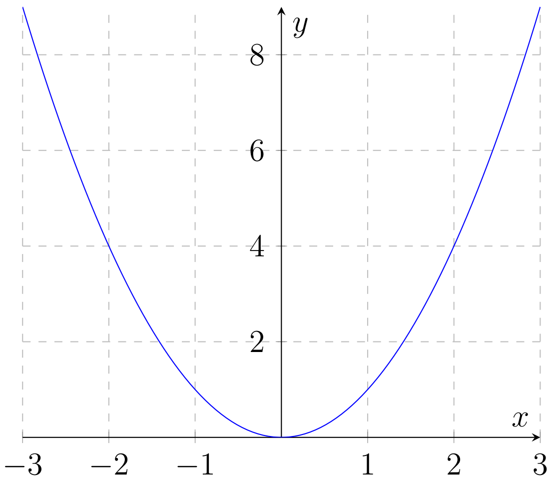
Reflection and rotation are key transformations in function manipulation, changing the orientation of the graph of a function.

Types of Transformations:

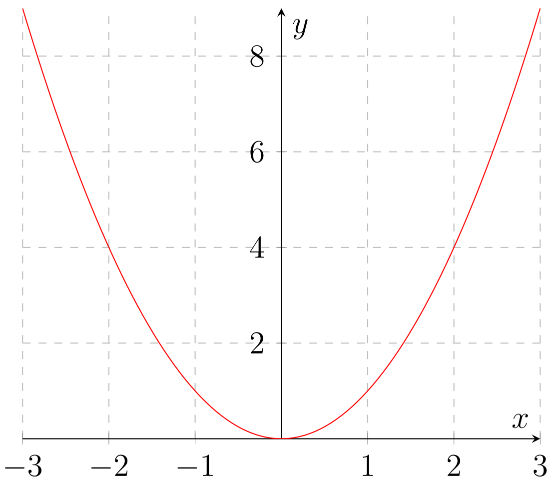
- *Reflection Across the y-axis:* Reflecting a function $f(x)$ to $f(-x)$ inverts it across the y-axis.
- *Reflection Across the x-axis:* Reflecting a function $f(x)$ to $-f(x)$ inverts it across the x-axis.
- *Rotation:* Rotation involves turning the graph of a function around a specific point, often the origin.

Graphing Reflected and Rotated Functions: Consider the function $f(x) = x^2$ and its reflection across the y-axis.

Original Function $f(x) = x^2$



Reflected Function $f(-x) = (-x)^2$



Applications of Reflection and Rotation: These transformations are used in various applications, such as in physics for understanding wave properties, in engineering for signal processing, and in computer graphics for image manipulation.

Exercise Graph the function $g(x) = -\frac{1}{x}$ and describe its reflection relative to the basic function $f(x) = \frac{1}{x}$.

Exercise Consider the function $h(x) = \sqrt{-x}$. Describe the transformation applied to the basic function $f(x) = \sqrt{x}$.

Exercises on Reflection and Rotation

Exercise Graph the function $g(x) = -\frac{1}{x}$ and describe its reflection relative to the basic function $f(x) = \frac{1}{x}$.

Exercise Consider the function $h(x) = \sqrt{-x}$. Describe the transformation applied to the basic function $f(x) = \sqrt{x}$.

Exercise Reflect the function $p(x) = x^3$ across the x-axis and sketch the resulting graph.

Exercise Graph the function $q(x) = -\sin(x)$ and explain how it is transformed from the basic sine function.

Exercise Analyze the rotation of the function $r(x) = \cos(-x)$ and describe how it differs from $f(x) = \cos(x)$.

Exercise Reflect the function $s(x) = e^x$ across the y-axis and describe the changes in its graph.

Exercise Graph $t(x) = -\ln(x)$ and discuss its reflection in relation to the function $f(x) = \ln(x)$.

Exercise Consider the function $u(x) = \tan(-x)$. Is it a reflection or a rotation of the basic tangent function? Explain.

Exercise Sketch the function $v(x) = \sqrt{1-x}$ and describe the transformation from $f(x) = \sqrt{x}$.

Exercise Reflect the function $w(x) = x^2 + x$ across the y-axis and sketch the resulting graph.

Solutions to Exercises on Reflection and Rotation

Solution: The function $g(x) = -\frac{1}{x}$ is a reflection of $f(x) = \frac{1}{x}$ across the x-axis. The negative sign in front of the function inverts it vertically.

Solution: The function $h(x) = \sqrt{-x}$ is a reflection of $f(x) = \sqrt{x}$ across the y-axis. The negative sign inside the square root reflects the function horizontally.

Solution: Reflecting $p(x) = x^3$ across the x-axis results in $-x^3$. This inversion changes the direction of the cubic curve, flipping it vertically.

Solution: The function $q(x) = -\sin(x)$ is the basic sine function reflected across the x-axis. The negative sign reverses the peaks and troughs of the sine wave.

Solution: The function $r(x) = \cos(-x)$ is the same as $f(x) = \cos(x)$ due to the even property of the cosine function. It represents a rotation of the cosine function by π radians, but the graph remains unchanged.

Solution: Reflecting $s(x) = e^x$ across the y-axis results in e^{-x}. The graph changes from increasing exponentially to decreasing exponentially as x increases.

Solution: Graphing $t(x) = -\ln(x)$ shows it is a reflection of $f(x) = \ln(x)$ across the x-axis. The logarithmic curve is inverted vertically.

Solution: The function $u(x) = \tan(-x)$ is a reflection of the basic tangent function across the y-axis due to the odd property of the tangent function.

Solution: The function $v(x) = \sqrt{1-x}$ is transformed from $f(x) = \sqrt{x}$ by a horizontal reflection across the y-axis and a horizontal translation 1 unit to the right.

Solution: Reflecting $w(x) = x^2 + x$ across the y-axis results in $(-x)^2 - x = x^2 - x$. The quadratic curve is flipped horizontally.

Chapter 10

Limits and Continuity - Detail

10.1 Introduction to Limits - Detail

10.1.1 Definition of a Limit - Detail

Definition 14 *The limit of $f(x)$ as x approaches a is L if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.*

This definition, known as the $\epsilon - \delta$ definition of a limit, is fundamental in calculus. It formalizes the concept of approaching a value.

Understanding $\epsilon - \delta$ Criteria:

- ϵ (epsilon) represents how close $f(x)$ is to L.
- δ (delta) represents how close x is to a.
- The statement means that $f(x)$ gets arbitrarily close to L as x approaches a.

Applications of Limits: Limits are used in calculus to define derivatives, integrals, and continuity. They are essential in understanding instantaneous rates of change and areas under curves.

Exercise Using the $\epsilon - \delta$ definition, show that the limit of $f(x) = x^2$ as x approaches 2 is 4.

Exercise Consider $f(x) = \frac{1}{x}$. Explain using the $\epsilon - \delta$ definition why the limit as x approaches 0 does not exist.

Exercises on the Definition of a Limit

Exercise Using the $\epsilon - \delta$ definition, show that the limit of $f(x) = x^2$ as x approaches 2 is 4.

Exercise Consider $f(x) = \frac{1}{x}$. Explain using the $\epsilon - \delta$ definition why the limit as x approaches 0 does not exist.

Exercise Prove that the limit of $g(x) = 3x+2$ as x approaches 4 is 14 using the $\epsilon - \delta$ approach.

Exercise Use the $\epsilon-\delta$ definition to demonstrate that the limit of $h(x) = \sqrt{x}$ as x approaches 9 is 3.

Exercise Verify that the limit of $p(x) = \frac{x^2-1}{x-1}$ as x approaches 1 is 2 using the $\epsilon - \delta$ criteria.

Exercise Determine whether the limit of $q(x) = \frac{\sin(x)}{x}$ as x approaches 0 exists, and if so, find the limit using the $\epsilon - \delta$ definition.

Exercise Prove that the limit of $r(x) = \frac{1}{x^2}$ as x approaches -3 is $\frac{1}{9}$, employing the $\epsilon - \delta$ method.

Exercise Using the $\epsilon - \delta$ definition, establish that the limit of $s(x) = \frac{x-2}{x^2-4}$ as x approaches 2 does not exist.

Exercise Determine the limit of $t(x) = x^3$ as x approaches -1 using the $\epsilon - \delta$ definition, and verify your result.

Exercise Use the $\epsilon - \delta$ approach to show that the limit of $u(x) = \frac{1}{\sqrt{x+4}}$ as x approaches 0 is $\frac{1}{2}$.

Solutions to Exercises on the Definition of a Limit

Solution: To show the limit of $f(x) = x^2$ as x approaches 2 is 4, choose $\delta = \min\left(1, \frac{\epsilon}{5}\right)$. For $0 < |x-2| < \delta$, we have $|f(x) - 4| = |x^2 - 4| = |(x-2)(x+2)| < 5|x-2| < \epsilon$.

Solution: For $f(x) = \frac{1}{x}$, as x approaches 0, the function values increase without bound. Therefore, no matter how small δ is chosen, $|f(x) - L|$ cannot be made less than any $\epsilon > 0$. Hence, the limit does not exist.

Solution: To show the limit of $g(x) = 3x+2$ as x approaches 4 is 14, choose $\delta = \frac{\epsilon}{3}$. Then for $0 < |x-4| < \delta$, we have $|g(x) - 14| = |3x + 2 - 14| = 3|x-4| < \epsilon$.

Solution: For $h(x) = \sqrt{x}$ and x approaching 9, choose $\delta = \min(1, \epsilon^2)$. Then $0 < |x-9| < \delta$ implies $|\sqrt{x} - 3| < \epsilon$.

Solution: For $p(x) = \frac{x^2-1}{x-1}$, factorize the numerator as $(x-1)(x+1)$. The limit as x approaches 1 is 2, as $p(x)$ simplifies to $x+1$ and $p(1) = 2$. To show this, choose $\delta = \epsilon$.

Solution: For $q(x) = \frac{\sin(x)}{x}$ as x approaches 0, use the squeeze theorem. Since $-1 \leq \sin(x) \leq 1$, then $-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$. As x approaches 0, both bounds approach 0, so the limit is 0.

Solution: To show the limit of $r(x) = \frac{1}{x^2}$ as x approaches -3 is $\frac{1}{9}$, choose $\delta = \min(1, 3\epsilon)$. Then $0 < |x+3| < \delta$ implies $|r(x) - \frac{1}{9}| < \epsilon$.

Solution: For $s(x) = \frac{x-2}{x^2-4}$, the function is undefined at $x = 2$, and the denominator approaches 0 as x approaches 2. Thus, the function values become unbounded, and the limit does not exist.

Solution: The limit of $t(x) = x^3$ as x approaches -1 is -1. Choose $\delta = \sqrt[3]{\epsilon}$. Then $0 < |x+1| < \delta$ implies $|x^3 + 1| < \epsilon$.

Solution: To show the limit of $u(x) = \frac{1}{\sqrt{x+4}}$ as x approaches 0 is $\frac{1}{2}$, choose $\delta = \min(1, 4\epsilon^2)$. Then $0 < |x| < \delta$ implies $|u(x) - \frac{1}{2}| < \epsilon$.

10.1.2 One-sided Limits

In calculus, the concept of limits is essential for understanding the behavior of functions as they approach specific values. One-sided limits are a special case of limits that help us analyze the behavior of a function as it approaches a particular point from either the left or the right.

Definition 15 *The one-sided limits of $f(x)$ as x approaches a from the left (denoted as $x \to a^-$) and as x approaches a from the right (denoted as $x \to a^+$) are the values the function approaches as x gets arbitrarily close to a from the left or the right, respectively. Mathematically, we write:*

$$\lim_{x \to a^-} f(x) = L^- \quad \textit{and} \quad \lim_{x \to a^+} f(x) = L^+$$

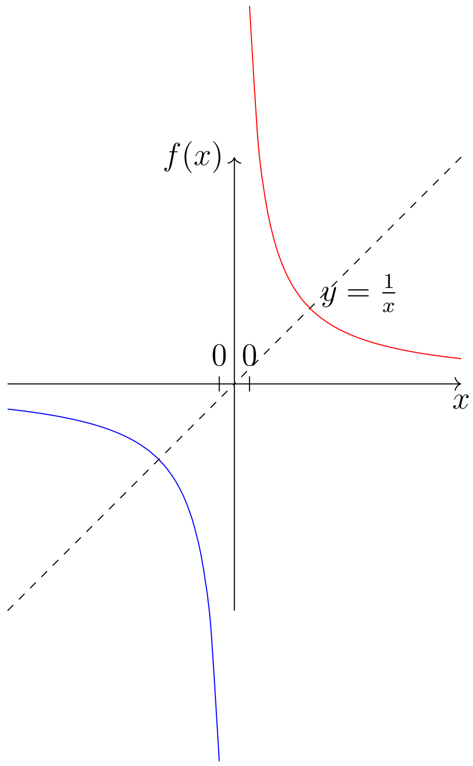
where L^- and L^+ are the one-sided limits from the left and right, respectively.

To visualize one-sided limits, consider the following example:

Example 7: Let's examine the function $f(x) = \frac{1}{x}$. We want to find the one-sided limits of this function as x approaches 0.

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{x} = +\infty$$

As x approaches 0 from the left, $\frac{1}{x}$ goes to negative infinity, and as x approaches 0 from the right, $\frac{1}{x}$ goes to positive infinity.



In the graph, the function approaches negative infinity as x approaches 0 from the left and positive infinity as x approaches 0 from the right, indicating the one-sided limits.

Understanding one-sided limits is crucial for analyzing the behavior of functions near specific points, especially when dealing with discontinuities or asymptotes.

Practice Problems

Let's work on some practice problems to reinforce our understanding of one-sided limits.

Problem 1: Find the one-sided limits of the following function as x approaches the given values:

(a) $\lim_{x\to 2^-}(x^2 - 4x + 4)$

(b) $\lim_{x\to 2^+}(x^2 - 4x + 4)$

(c) $\lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{|x|}\right)$

(d) $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{|x|}\right)$

Problem 2: Consider the function $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$ Find the following one-sided limits:

(a) $\lim_{x\to 0^-} f(x)$

(b) $\lim_{x\to 0^+} f(x)$

Problem 3: Determine if the following statements are true or false:

(a) If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ exists.

(b) If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist and are finite, then $\lim_{x\to a} f(x)$ exists and is finite.

(c) If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist but are not equal, then $\lim_{x\to a} f(x)$ does not exist.

Solution:

(a) True. If the left-hand and right-hand limits are equal, then the two-sided limit exists and is equal to that common value.

(b) True. If both one-sided limits exist and are finite, then the two-sided limit exists and is also finite.

(c) True. If the one-sided limits are not equal, it indicates a discontinuity at $x = a$, so the two-sided limit does not exist.

Now, you have a set of practice problems for students to work on to enhance their understanding of one-sided limits. The solutions to these problems are also provided to help students check their work.

Practice Problems - Solutions

Let's solve the practice problems related to one-sided limits.

Problem 4: Find the one-sided limits of the following function as x approaches the given values:

(a) $\lim_{x\to 2^-}(x^2 - 4x + 4)$

(b) $\lim_{x\to 2^+}(x^2 - 4x + 4)$

(c) $\lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{|x|}\right)$

(d) $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{|x|}\right)$

Solution:

(a) To find $\lim_{x\to 2^-}(x^2 - 4x + 4)$, we substitute $x = 2$ into the expression:

$$\lim_{x\to 2^-}(x^2 - 4x + 4) = (2^2 - 4\cdot 2 + 4) = 0$$

The limit exists, and its value is 0.

(b) To find $\lim_{x\to 2^+}(x^2 - 4x + 4)$, we substitute $x = 2$ into the expression:

$$\lim_{x\to 2^+}(x^2 - 4x + 4) = (2^2 - 4\cdot 2 + 4) = 0$$

The limit exists, and its value is 0.

(c) To find $\lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{|x|}\right)$, we substitute $x = 0$ from the left side:

$$\lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{|x|}\right) = \left(\frac{1}{0} - \frac{1}{|0|}\right) = -\infty$$

The limit is $-\infty$.

(d) To find $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{|x|}\right)$, we substitute $x = 0$ from the right side:

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{|x|}\right) = \left(\frac{1}{0} - \frac{1}{|0|}\right) = +\infty$$

The limit is $+\infty$.

Problem 5: Consider the function $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$ Find the following one-sided limits:

(a) $\lim_{x\to 0^-} f(x)$

(b) $\lim_{x\to 0^+} f(x)$

Solution:

(a) To find $\lim_{x\to 0^-} f(x)$, we approach 0 from the left side:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1) = 1$$

The left-hand limit is 1.

(b) To find $\lim_{x\to 0^+} f(x)$, we approach 0 from the right side:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2x) = 0$$

The right-hand limit is 0.

Problem 6: Determine if the following statements are true or false:

(a) If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ exists.

(b) If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist and are finite, then $\lim_{x\to a} f(x)$ exists and is finite.

(c) If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist but are not equal, then $\lim_{x\to a} f(x)$ does not exist.

Solution:

(a) True. If the left-hand and right-hand limits are equal, then the two-sided limit exists and is equal to that common value.

(b) True. If both one-sided limits exist and are finite, then the two-sided limit exists and is also finite.

(c) True. If the one-sided limits are not equal, it indicates a discontinuity at $x = a$, so the two-sided limit does not exist.

10.1.3 Limits Involving Infinity

Limits as x Approaches Infinity

When dealing with limits involving infinity, we are interested in understanding the behavior of a function as x approaches positive or negative infinity. Let's start by discussing limits as x approaches positive infinity (∞).

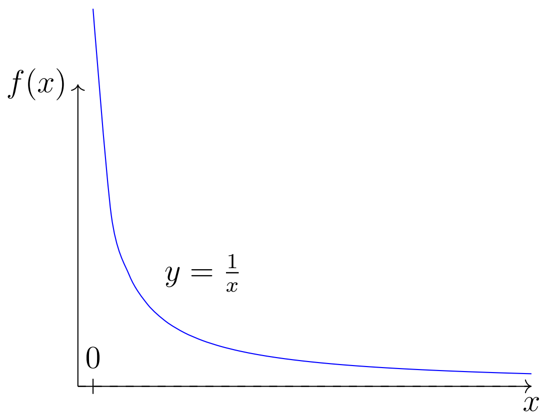
Definition 16 *A limit involving infinity is the value that $f(x)$ approaches as x approaches positive infinity ($x \to \infty$), or as $f(x)$ approaches infinity for some finite x.*

To visualize this concept, consider the following example:

Example 8: Let's examine the function $f(x) = \frac{1}{x}$. We want to find the limit of this function as x approaches positive infinity ($x \to \infty$).

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

As x becomes larger and larger, $\frac{1}{x}$ approaches zero.



In the graph, as x goes to infinity, $\frac{1}{x}$ approaches 0, which is the limit.

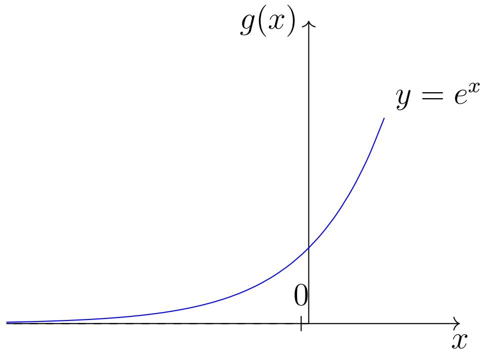
Limits as x Approaches Negative Infinity

Similarly, we can analyze limits as x approaches negative infinity ($x \to -\infty$).

Example 9: Consider the function $g(x) = e^x$. We want to find the limit of this function as x approaches negative infinity ($x \to -\infty$).

$$\lim_{x\to-\infty} e^x = 0$$

As x becomes more and more negative, e^x approaches zero.



In the graph, as x goes to negative infinity, e^x approaches 0, which is the limit.

Understanding limits involving infinity is crucial for analyzing the long-term behavior of functions as x becomes infinitely large or small.

Practice Problems

Let's work on some practice problems to reinforce our understanding of limits involving infinity.

Problem 7: Find the limit as x approaches positive infinity for the following functions:

(a) $\lim_{x\to\infty} \frac{3x+2}{x-1}$

(b) $\lim_{x\to\infty} \sqrt{x^2+1}$

(c) $\lim_{x\to\infty} \frac{2x^3+4x^2-3}{x^4+5x^2+1}$

Solution:

(a) To find $\lim_{x\to\infty} \frac{3x+2}{x-1}$, we divide both the numerator and the denominator by the highest power of x:

$$\lim_{x\to\infty} \frac{3x+2}{x-1} = \lim_{x\to\infty} \frac{\frac{3x}{x}+\frac{2}{x}}{\frac{x}{x}-\frac{1}{x}} = \lim_{x\to\infty} \frac{3+0}{1-0} = 3$$

The limit is 3.

(b) To find $\lim_{x\to\infty} \sqrt{x^2+1}$, we can use the fact that $\sqrt{x^2} = x$ for all positive x:

$$\lim_{x\to\infty}\sqrt{x^2+1}=\lim_{x\to\infty}\sqrt{x^2(1+\frac{1}{x^2})}=\lim_{x\to\infty}\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}=\lim_{x\to\infty}x\cdot 1=\infty$$

The limit is ∞.

(c) To find $\lim_{x\to\infty}\frac{2x^3+4x^2-3}{x^4+5x^2+1}$, we divide both the numerator and the denominator by the highest power of x in the denominator:

$$\lim_{x\to\infty}\frac{2x^3+4x^2-3}{x^4+5x^2+1}=\lim_{x\to\infty}\frac{x^3(2+\frac{4}{x}-\frac{3}{x^3})}{x^4(1+\frac{5}{x^2}+\frac{1}{x^4})}=\lim_{x\to\infty}\frac{2+0-0}{1+0+0}=2$$

The limit is 2.

Problem 8: Find the limit as x approaches negative infinity for the following functions:

(a) $\lim_{x\to-\infty}\frac{2x^2-5x+1}{3x^2+2}$

(b) $\lim_{x\to-\infty}\frac{4x}{x^2+1}$

(c) $\lim_{x\to-\infty}e^x$

Solution:

(a) To find $\lim_{x\to-\infty}\frac{2x^2-5x+1}{3x^2+2}$, we divide both the numerator and the denominator by the highest power of x:

$$\lim_{x\to-\infty}\frac{2x^2-5x+1}{3x^2+2}=\lim_{x\to-\infty}\frac{x^2(2-\frac{5}{x}+\frac{1}{x^2})}{x^2(3+\frac{2}{x^2})}=\lim_{x\to-\infty}\frac{2-0+0}{3+0}=\frac{2}{3}$$

The limit is $\frac{2}{3}$.

(b) To find $\lim_{x\to-\infty}\frac{4x}{x^2+1}$, we divide both the numerator and the denominator by the highest power of x:

$$\lim_{x\to-\infty}\frac{4x}{x^2+1}=\lim_{x\to-\infty}\frac{x(4)}{x^2(1+\frac{1}{x^2})}=\lim_{x\to-\infty}\frac{4}{1+0}=4$$

The limit is 4.

(c) To find $\lim_{x\to-\infty} e^x$, we know that as x approaches negative infinity, e^x approaches 0.

$$\lim_{x\to-\infty} e^x = 0$$

The limit is 0.

Practice Problems - Solutions

Let's solve the practice problems related to limits involving infinity.

Problem 9: Find the limit as x approaches positive infinity for the following functions:

(a) $\lim_{x\to\infty} \frac{3x+2}{x-1}$

(b) $\lim_{x\to\infty} \sqrt{x^2+1}$

(c) $\lim_{x\to\infty} \frac{2x^3+4x^2-3}{x^4+5x^2+1}$

Solution:

(a) To find $\lim_{x\to\infty} \frac{3x+2}{x-1}$, we divide both the numerator and the denominator by the highest power of x:

$$\lim_{x\to\infty} \frac{3x+2}{x-1} = \lim_{x\to\infty} \frac{\frac{3x}{x}+\frac{2}{x}}{\frac{x}{x}-\frac{1}{x}} = \lim_{x\to\infty} \frac{3+0}{1-0} = 3$$

The limit is 3.

(b) To find $\lim_{x\to\infty} \sqrt{x^2+1}$, we can use the fact that $\sqrt{x^2} = x$ for all positive x:

$$\lim_{x\to\infty} \sqrt{x^2+1} = \lim_{x\to\infty} \sqrt{x^2(1+\frac{1}{x^2})} = \lim_{x\to\infty} \sqrt{x^2}\sqrt{1+\frac{1}{x^2}} = \lim_{x\to\infty} x\cdot 1 = \infty$$

The limit is ∞.

(c) To find $\lim_{x\to\infty} \frac{2x^3+4x^2-3}{x^4+5x^2+1}$, we divide both the numerator and the denominator by the highest power of x in the denominator:

$$\lim_{x\to\infty} \frac{2x^3+4x^2-3}{x^4+5x^2+1} = \lim_{x\to\infty} \frac{x^3(2+\frac{4}{x}-\frac{3}{x^3})}{x^4(1+\frac{5}{x^2}+\frac{1}{x^4})} = \lim_{x\to\infty} \frac{2+0-0}{1+0+0} = 2$$

The limit is 2.

Problem 10: Find the limit as x approaches negative infinity for the following functions:

(a) $\lim_{x\to-\infty} \frac{2x^2-5x+1}{3x^2+2}$

(b) $\lim_{x\to-\infty} \frac{4x}{x^2+1}$

(c) $\lim_{x\to-\infty} e^x$

Solution:

(a) To find $\lim_{x\to-\infty} \frac{2x^2-5x+1}{3x^2+2}$, we divide both the numerator and the denominator by the highest power of x:

$$\lim_{x\to-\infty} \frac{2x^2-5x+1}{3x^2+2} = \lim_{x\to-\infty} \frac{x^2(2-\frac{5}{x}+\frac{1}{x^2})}{x^2(3+\frac{2}{x^2})} = \lim_{x\to-\infty} \frac{2-0+0}{3+0} = \frac{2}{3}$$

The limit is $\frac{2}{3}$.

(b) To find $\lim_{x\to-\infty} \frac{4x}{x^2+1}$, we divide both the numerator and the denominator by the highest power of x:

$$\lim_{x\to-\infty} \frac{4x}{x^2+1} = \lim_{x\to-\infty} \frac{x(4)}{x^2(1+\frac{1}{x^2})} = \lim_{x\to-\infty} \frac{4}{1+0} = 4$$

The limit is 4.

(c) To find $\lim_{x\to-\infty} e^x$, we know that as x approaches negative infinity, e^x approaches 0.

$$\lim_{x\to-\infty} e^x = 0$$

The limit is 0.

10.1.4 Properties of Limits

Limit Laws

In calculus, we often encounter limits of functions, and there are several standard limit laws that help us simplify the evaluation of limits. These laws include the sum law, product law, quotient law, and power law. Let's explore each of these limit laws and provide graphical interpretations.

Theorem 1 *Standard limit laws include the sum law, product law, quotient law, and power law.*

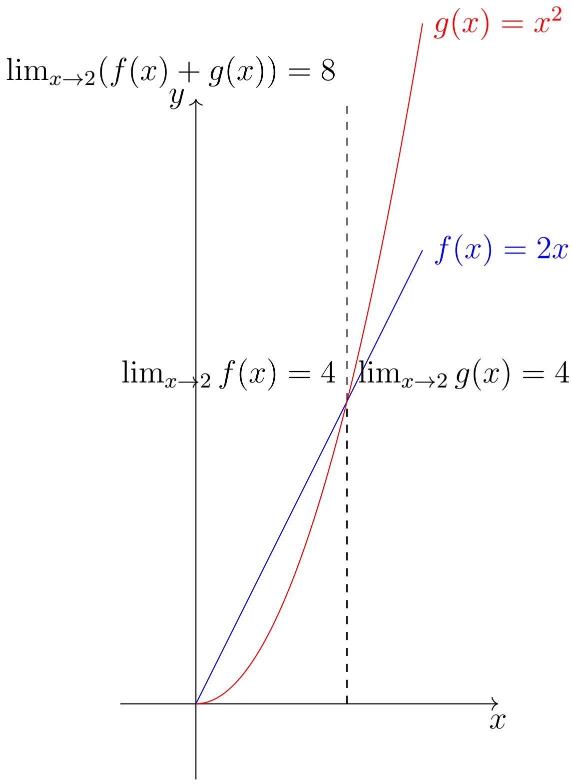
Sum Law

The sum law states that if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then:

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

In other words, the limit of the sum of two functions is equal to the sum of their limits. Let's illustrate this with a graphical example.

Example 10: Consider the functions $f(x) = 2x$ and $g(x) = x^2$. We want to find $\lim_{x\to 2}(2x + x^2)$.



As x approaches 2, both $f(x)$ and $g(x)$ approach 4. Therefore, by the sum law:

$$\lim_{x\to 2}(2x + x^2) = \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x) = 4 + 4 = 8$$

The sum law allows us to find limits of complex functions by breaking them down into simpler parts.

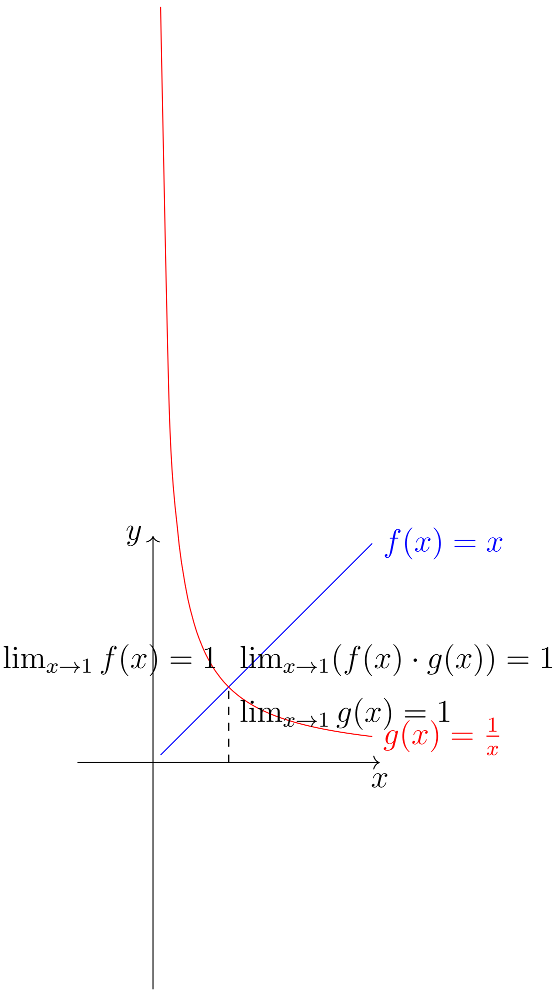
Product Law

The product law states that if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then:

$$\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

In other words, the limit of the product of two functions is equal to the product of their limits. Let's illustrate this with a graphical example.

Example 11: Consider the functions $f(x) = x$ and $g(x) = \frac{1}{x}$. We want to find $\lim_{x\to 1}(x \cdot \frac{1}{x})$.



As x approaches 1, both $f(x)$ and $g(x)$ approach 1. Therefore, by the product law:

$$\lim_{x \to 1}(x \cdot \frac{1}{x}) = \lim_{x \to 1} f(x) \cdot \lim_{x \to 1} g(x) = 1 \cdot 1 = 1$$

The product law allows us to find limits of products of functions without directly evaluating the product.

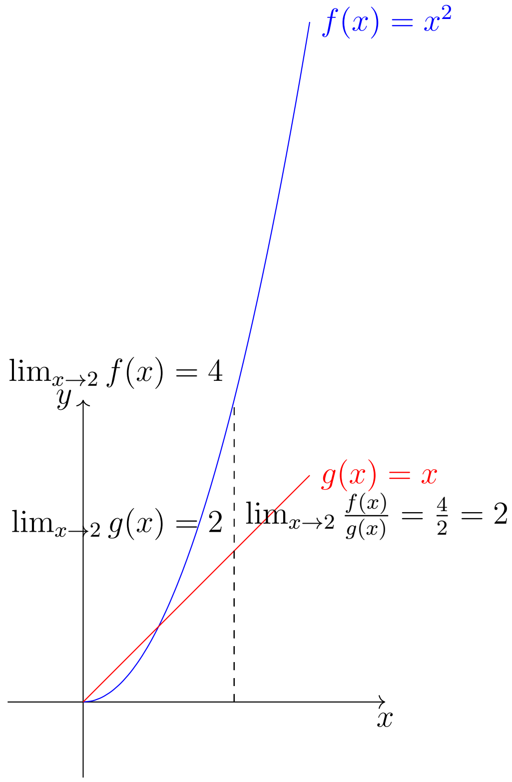
Quotient Law

The quotient law states that if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and $\lim_{x \to a} g(x) \neq 0$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

In other words, the limit of the quotient of two functions is equal to the quotient of their limits, provided that the limit of the denominator is not zero. Let's illustrate this with a graphical example.

Example 12: Consider the functions $f(x) = x^2$ and $g(x) = x$. We want to find $\lim_{x\to 2} \frac{x^2}{x}$.



As x approaches 2, both $f(x)$ and $g(x)$ approach 4 and 2, respectively. Therefore, by the quotient law:

$$\lim_{x\to 2} \frac{x^2}{x} = \frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)} = \frac{4}{2} = 2$$

The quotient law helps us find limits involving fractions of functions by simplifying the process.

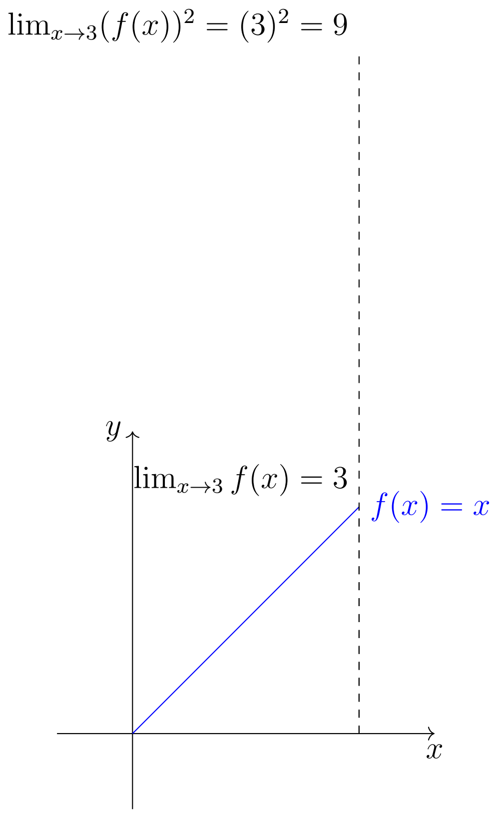
Power Law

The power law states that if $\lim_{x\to a} f(x)$ exists and n is a positive integer, then:

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$$

In other words, the limit of a function raised to a power is equal to the function's limit raised to that power. Let's illustrate this with a graphical example.

Example 13: Consider the function $f(x) = x$. We want to find $\lim_{x \to 3}(x^2)$.



As x approaches 3, $f(x)$ approaches 3. Therefore, by the power law:

$$\lim_{x \to 3}(x^2) = [\lim_{x \to 3} f(x)]^2 = (3)^2 = 9$$

The power law simplifies the evaluation of limits involving powers of functions.

In summary, these standard limit laws (sum law, product law, quotient law, and power law) provide essential tools for evaluating limits and understanding the behavior of functions as they approach certain points.

Practice Problems

Let's practice applying the limit laws we discussed in the previous section.

Problem 11: Find the following limits using the sum law, product law, quotient law, or power law as appropriate:

(a) $\lim_{x\to 0}(3x + 2x^2)$

(b) $\lim_{x\to 1}(x^2 + x)$

(c) $\lim_{x\to 2}\left(\frac{x^2}{x+1}\right)$

(d) $\lim_{x\to 2}(x - 1)^3$

Solution:

(a) Using the sum law:

$$\lim_{x\to 0}(3x + 2x^2) = \lim_{x\to 0} 3x + \lim_{x\to 0} 2x^2 = 0 + 0 = 0$$

(b) Using the sum law:

$$\lim_{x\to 1}(x^2 + x) = \lim_{x\to 1} x^2 + \lim_{x\to 1} x = 1 + 1 = 2$$

(c) Using the quotient law:

$$\lim_{x\to 2}\left(\frac{x^2}{x+1}\right) = \frac{\lim_{x\to 2} x^2}{\lim_{x\to 2}(x + 1)} = \frac{4}{3} = \frac{4}{3}$$

(d) Using the power law:

$$\lim_{x\to 2}(x - 1)^3 = (\lim_{x\to 2}(x - 1))^3 = (2 - 1)^3 = 1$$

Problem 12: Evaluate the following limits. You may use any of the limit laws:

(a) $\lim_{x\to 0}(2x^3 - 3x^2 + 4x - 1)$

(b) $\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$

(c) $\lim_{x\to -1}\left(\frac{x^2+2x-3}{x+1}\right)$

(d) $\lim_{x\to 4}\left(\frac{x^3-64}{x-4}\right)$

Solution:

(a) Using the sum law and power law:

$$\lim_{x\to 0}(2x^3-3x^2+4x-1) = \lim_{x\to 0}(2x^3)-\lim_{x\to 0}(3x^2)+\lim_{x\to 0}(4x)-\lim_{x\to 0}(1) = 0-0+0-1 = -1$$

(b) Using the quotient law:

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right) = \frac{\lim_{x\to 3}(x^2-9)}{\lim_{x\to 3}(x-3)} = \frac{0}{0} \text{ (indeterminate)}$$

We can simplify further using factoring:

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right) = \lim_{x\to 3}\left(\frac{(x+3)(x-3)}{x-3}\right) = \lim_{x\to 3}(x+3) = 6$$

(c) Using the quotient law:

$$\lim_{x\to -1}\left(\frac{x^2+2x-3}{x+1}\right) = \frac{\lim_{x\to -1}(x^2+2x-3)}{\lim_{x\to -1}(x+1)} = \frac{0}{0} \text{ (indeterminate)}$$

We can simplify further using factoring:

$$\lim_{x\to -1}\left(\frac{x^2+2x-3}{x+1}\right) = \lim_{x\to -1}\left(\frac{(x+3)(x-1)}{x+1}\right) = \lim_{x\to -1}(x-1) = -2$$

(d) Using the quotient law:

$$\lim_{x\to 4}\left(\frac{x^3-64}{x-4}\right) = \frac{\lim_{x\to 4}(x^3-64)}{\lim_{x\to 4}(x-4)} = \frac{0}{0} \text{ (indeterminate)}$$

We can simplify further using factoring:

$$\lim_{x\to 4}\left(\frac{x^3-64}{x-4}\right) = \lim_{x\to 4}\left(\frac{(x-4)(x^2+4x+16)}{x-4}\right) = \lim_{x\to 4}(x^2+4x+16) = 68$$

Practice Problems - Solutions

Let's go through the solutions and explanations for the practice problems.

Problem 13: Find the following limits using the sum law, product law, quotient law, or power law as appropriate:

(a) $\lim_{x\to 0}(3x+2x^2)$

(b) $\lim_{x\to 1}(x^2+x)$

(c) $\lim_{x\to 2}\left(\frac{x^2}{x+1}\right)$

(d) $\lim_{x\to 2}(x-1)^3$

Solution:

(a) Using the sum law:

$$\lim_{x\to 0}(3x+2x^2) = \lim_{x\to 0} 3x + \lim_{x\to 0} 2x^2 = 0+0 = 0$$

(b) Using the sum law:

$$\lim_{x\to 1}(x^2+x) = \lim_{x\to 1} x^2 + \lim_{x\to 1} x = 1+1 = 2$$

(c) Using the quotient law:

$$\lim_{x\to 2}\left(\frac{x^2}{x+1}\right) = \frac{\lim_{x\to 2} x^2}{\lim_{x\to 2}(x+1)} = \frac{4}{3} = \frac{4}{3}$$

(d) Using the power law:

$$\lim_{x\to 2}(x-1)^3 = (\lim_{x\to 2}(x-1))^3 = (2-1)^3 = 1$$

Problem 14: Evaluate the following limits. You may use any of the limit laws:

(a) $\lim_{x\to 0}(2x^3-3x^2+4x-1)$

(b) $\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$

(c) $\lim_{x\to -1}\left(\frac{x^2+2x-3}{x+1}\right)$

(d) $\lim_{x\to 4}\left(\frac{x^3-64}{x-4}\right)$

Solution:

(a) Using the sum law and power law:

$$\lim_{x\to 0}(2x^3-3x^2+4x-1) = \lim_{x\to 0}(2x^3)-\lim_{x\to 0}(3x^2)+\lim_{x\to 0}(4x)-\lim_{x\to 0}(1) = 0-0+0-1 = -1$$

(b) Using the quotient law:

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right) = \frac{\lim_{x\to 3}(x^2-9)}{\lim_{x\to 3}(x-3)} = \frac{0}{0} \text{ (indeterminate)}$$

We can simplify further using factoring:

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right) = \lim_{x\to 3}\left(\frac{(x+3)(x-3)}{x-3}\right) = \lim_{x\to 3}(x+3) = 6$$

(c) Using the quotient law:

$$\lim_{x\to -1}\left(\frac{x^2+2x-3}{x+1}\right) = \frac{\lim_{x\to -1}(x^2+2x-3)}{\lim_{x\to -1}(x+1)} = \frac{0}{0} \text{ (indeterminate)}$$

We can simplify further using factoring:

$$\lim_{x\to -1}\left(\frac{x^2+2x-3}{x+1}\right) = \lim_{x\to -1}\left(\frac{(x+3)(x-1)}{x+1}\right) = \lim_{x\to -1}(x-1) = -2$$

(d) Using the quotient law:

$$\lim_{x\to 4}\left(\frac{x^3-64}{x-4}\right) = \frac{\lim_{x\to 4}(x^3-64)}{\lim_{x\to 4}(x-4)} = \frac{0}{0} \text{ (indeterminate)}$$

We can simplify further using factoring:

$$\lim_{x\to 4}\left(\frac{x^3-64}{x-4}\right) = \lim_{x\to 4}\left(\frac{(x-4)(x^2+4x+16)}{x-4}\right) = \lim_{x\to 4}(x^2+4x+16) = 68$$

10.1.5 Squeeze Theorem

The Squeeze Theorem is a powerful tool for finding the limit of a function when it is bounded between two other functions whose limits are known. This theorem is particularly useful when direct substitution or algebraic simplification of a limit expression is difficult. The theorem can be stated as follows:

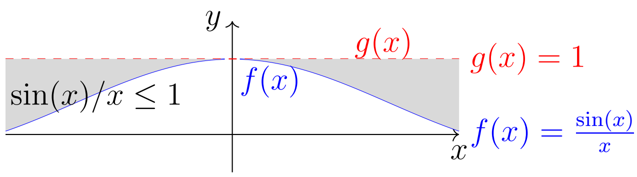
Theorem 2 (Squeeze Theorem) *If $f(x) \leq g(x) \leq h(x)$ for all x near a, and the limits of $f(x)$ and $h(x)$ as x approaches a are equal, then the limit of $g(x)$ as x approaches a is the same:*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \implies \lim_{x \to a} g(x) = L$$

This theorem essentially states that if $g(x)$ is "squeezed" between two functions $f(x)$ and $h(x)$, and both $f(x)$ and $h(x)$ approach the same limit L as x approaches a, then $g(x)$ also approaches L as x approaches a.

Let's illustrate the Squeeze Theorem with a graphical example.

Example 14: Consider the functions $f(x) = \frac{\sin(x)}{x}$ and $g(x) = 1$ for $x \neq 0$. We want to find the limit of $g(x)$ as x approaches 0.



As shown in the graph, for all $x \neq 0$, $f(x) = \frac{\sin(x)}{x}$ is bounded between -1 and 1. Additionally, $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ as it is a well-known limit.

Therefore, by the Squeeze Theorem, since $-1 \leq \frac{\sin(x)}{x} \leq 1$ for all $x \neq 0$ and $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$, we conclude that $\lim_{x \to 0} 1 = 1$.

So, $\lim_{x \to 0} g(x) = 1$.

The Squeeze Theorem provides a powerful method to evaluate limits by establishing upper and lower bounds for a function and leveraging known limits of the bounding functions.

Practice Problems

Let's practice using the Squeeze Theorem to evaluate limits.

Problem 15: Evaluate the following limit using the Squeeze Theorem:

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Problem 16: Find the limit as x approaches 0 for the function $f(x)$ defined as follows:

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Problem 17: Evaluate the following limit:

$$\lim_{x \to 0} \left(\cos^2(x) + \sin^2(x) \right)$$

Problem 18: Determine the limit as x approaches 0 for the function $g(x)$ defined as follows:

$$g(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Practice Problems - Solutions

Let's go through the solutions and explanations for the practice problems.

Problem 19: Evaluate the following limit using the Squeeze Theorem:

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Solution: To evaluate this limit, let's use the Squeeze Theorem. First, notice that for all $x \neq 0$, we have $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$. Therefore, we can use the following inequalities:

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Next, consider the limits of the bounds as x approaches 0:

$$\lim_{x \to 0}(-x^2) = 0 \quad \text{and} \quad \lim_{x \to 0}(x^2) = 0$$

Since both x^2 and $-x^2$ approach 0 as x approaches 0, we have:

$$0 \leq \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) \leq 0$$

By the Squeeze Theorem, the limit of $x^2 \sin\left(\frac{1}{x}\right)$ as x approaches 0 is 0.

Problem 20: Find the limit as x approaches 0 for the function $f(x)$ defined as follows:

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution: To find this limit, we can use the Squeeze Theorem. Consider the function $f(x) = x$. We know that $\lim_{x\to 0} x = 0$.

Now, let's analyze the function $g(x)$ defined as $g(x) = 0$ for all x. Clearly, $\lim_{x\to 0} g(x) = 0$.

Since $0 \leq f(x) \leq |x|$ for all x, and both $f(x)$ and $|x|$ approach 0 as x approaches 0, we can apply the Squeeze Theorem to conclude that:

$$\lim_{x\to 0} f(x) = 0$$

Problem 21: Evaluate the following limit:

$$\lim_{x\to 0} \left(\cos^2(x) + \sin^2(x)\right)$$

Solution: This limit is straightforward to evaluate. Notice that $\cos^2(x) + \sin^2(x)$ is the identity for all x. Therefore:

$$\lim_{x\to 0} \left(\cos^2(x) + \sin^2(x)\right) = \lim_{x\to 0} 1 = 1$$

Problem 22: Determine the limit as x approaches 0 for the function $g(x)$ defined as follows:

$$g(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Solution: To find this limit, we can use the Squeeze Theorem. Consider the function $f(x) = \frac{\sin(x)}{x}$. We know that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ as it is a well-known limit.

Now, let's analyze the function $h(x)$ defined as $h(x) = 1$ for all x. Clearly, $\lim_{x\to 0} h(x) = 1$.

Since $1 \leq f(x) \leq h(x)$ for all x, and both $f(x)$ and $h(x)$ approach 1 as x approaches 0, we can apply the Squeeze Theorem to conclude that:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

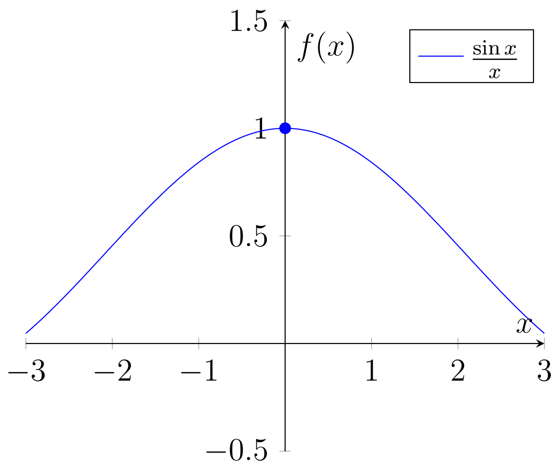
10.1.6 Limits of Trigonometric Functions

In calculus, we often encounter limits involving trigonometric functions. These limits are essential for understanding the behavior of functions near specific points and are used in various mathematical and scientific applications. Let's explore some important limits of trigonometric functions.

Theorem 3 *One of the fundamental limits involving trigonometric functions is:*

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

The limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$ is a crucial result and has significant implications in calculus. It's often referred to as the Sine Limit or Squeeze Theorem. To understand why this limit is equal to 1, let's consider a graphical representation.



As x approaches 0, the graph of $y = \frac{\sin x}{x}$ approaches the horizontal line $y = 1$. This visual representation helps illustrate why $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

The significance of this limit is profound, as it forms the basis for understanding the derivatives of trigonometric functions and plays a crucial role in calculus and mathematical analysis. It is often used as a building block for solving more complex limits involving trigonometric functions.

In addition to the limit mentioned above, there are other important limits involving trigonometric functions that are used in calculus, such as limits involving $\sin x$, $\cos x$, and $\tan x$ as x approaches specific values. These limits provide insights into the behavior of trigonometric functions near certain points and are essential tools in calculus and mathematics.

In practice, these limits are applied to solve various real-world problems, including physics, engineering, and mathematical modeling. Understanding the limits of trigonometric functions is a fundamental skill for students and professionals in these fields.

Practice Problems

Let's practice evaluating limits of trigonometric functions.

Problem 23: Evaluate the following limit:

$$\lim_{x \to 0} \frac{\sin 2x}{x}$$

Problem 24: Find the limit as x approaches 0 for the function $f(x)$ defined as:

$$f(x) = \begin{cases} \frac{\sin 3x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Problem 25: Determine the limit as x approaches $\frac{\pi}{2}$ for the function $g(x)$ given by:

$$g(x) = \cos x$$

Problem 26: Evaluate the following limit:

$$\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$

Practice Problems - Solutions

Let's go through the solutions and explanations for the practice problems.

Problem 27: Evaluate the following limit:

$$\lim_{x \to 0} \frac{\sin 2x}{x}$$

Solution: To evaluate this limit, we can use the Squeeze Theorem. Notice that for all $x \neq 0$, we have $2x \leq \sin 2x \leq 2x$. Therefore, we can use the following inequalities:

$$2x \leq \frac{\sin 2x}{x} \leq 2x$$

Next, consider the limits of the bounds as x approaches 0:

$$\lim_{x \to 0}(2x) = 0$$

Since both $2x$ and $-2x$ approach 0 as x approaches 0, we have:

$$0 \leq \lim_{x \to 0} \frac{\sin 2x}{x} \leq 0$$

By the Squeeze Theorem, the limit of $\frac{\sin 2x}{x}$ as x approaches 0 is 0.

Problem 28: Find the limit as x approaches 0 for the function $f(x)$ defined as:

$$f(x) = \begin{cases} \frac{\sin 3x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution: To find this limit, we can directly apply the Sine Limit:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

In this case, $3x$ is used instead of x, but the form remains the same. Therefore:

$$\lim_{x \to 0} \frac{\sin 3x}{3x} = \frac{1}{3} \cdot 3 = 1$$

So, the limit of $f(x)$ as x approaches 0 is 1.

Problem 29: Determine the limit as x approaches $\frac{\pi}{2}$ for the function $g(x)$ given by:

$$g(x) = \cos x$$

Solution: To find this limit, we can directly substitute $\frac{\pi}{2}$ into the function:

$$\lim_{x \to \frac{\pi}{2}} \cos x = \cos\left(\frac{\pi}{2}\right) = 0$$

So, the limit of $g(x)$ as x approaches $\frac{\pi}{2}$ is 0.

Problem 30: Evaluate the following limit:

$$\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$

Solution: To evaluate this limit, we can apply L'Hôpital's Rule, as the limit has the form $\frac{0}{0}$. Taking the derivatives of the numerator and denominator:

$$\lim_{x\to\frac{\pi}{4}} \frac{\frac{d}{dx}(\sin x - \cos x)}{\frac{d}{dx}(x - \frac{\pi}{4})} = \lim_{x\to\frac{\pi}{4}} \frac{(\cos x + \sin x)'}{1} = \frac{(\cos\frac{\pi}{4} + \sin\frac{\pi}{4})'}{1} = \frac{\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)}{1} = 1$$

So, the limit of $\frac{\sin x - \cos x}{x - \frac{\pi}{4}}$ as x approaches $\frac{\pi}{4}$ is 1.

10.2 Continuity - Detail

10.2.1 Continuity Definition and Examples

In calculus, the concept of continuity is fundamental in understanding the behavior of functions. A function is considered continuous at a point if it satisfies the following criteria:

Definition 17 *A function is continuous at a point $x = a$ if the following conditions hold:*

1. *The function is defined at $x = a$, meaning that $f(a)$ is defined.*
2. *The limit of the function as x approaches a exists, denoted as $\lim_{x\to a} f(x)$.*
3. *The limit $\lim_{x\to a} f(x)$ is equal to the function value at $x = a$, i.e., $\lim_{x\to a} f(x) = f(a)$.*

The concept of continuity is crucial because it helps us determine when a function behaves smoothly without abrupt jumps or breaks. Let's explore some examples to illustrate continuity:

Example 15: Consider the function $f(x) = x^2$. Is this function continuous at $x = 2$?

Solution: To check for continuity, we need to verify the three conditions of continuity.

1. $f(2) = 2^2 = 4$, so the function is defined at $x = 2$.

2. Next, we find the limit as x approaches 2:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 2^2 = 4$$

   The limit exists and is equal to 4.

3. Since $f(2) = 4$ and $\lim_{x \to 2} f(x) = 4$, the limit equals the function value at $x = 2$, and therefore, the function $f(x) = x^2$ is continuous at $x = 2$.

Example 16: Now, let's consider the function $g(x) = \frac{1}{x}$. Is this function continuous at $x = 0$?

Solution: To check for continuity, we again examine the three conditions.

1. $g(0)$ is undefined since division by zero is undefined. Therefore, the function is not defined at $x = 0$.

2. We cannot compute the limit as x approaches 0 for $g(x)$ since it's undefined at that point.

3. Since the function is not defined at $x = 0$, we cannot compare the limit and the function value. Therefore, the function $g(x) = \frac{1}{x}$ is not continuous at $x = 0$.

Example 17: Let's consider a piecewise function:

$$h(x) = \begin{cases} x + 2 & \text{if } x < 1 \\ 3x - 1 & \text{if } x \geq 1 \end{cases}$$

Is this function continuous at $x = 1$?

Solution: To determine continuity, we examine the three conditions for both branches of the piecewise function separately.

For $x < 1$:

1. $h(x) = x + 2$, so it is defined at $x = 1$.

2. The limit as x approaches 1 for this branch:

$$\lim_{x \to 1^-} (x + 2) = 1 + 2 = 3$$

   The limit exists and is equal to 3.

3. $h(1) = 1 + 2 = 3$.

For $x \geq 1$:

1. $h(x) = 3x - 1$, so it is defined at $x = 1$.
2. The limit as x approaches 1 for this branch:

$$\lim_{x \to 1^+} (3x - 1) = 3(1) - 1 = 2$$

   The limit exists and is equal to 2.

3. $h(1) = 3(1) - 1 = 2$.

Since both branches are defined and the limits equal the function values at $x = 1$, the function $h(x)$ is continuous at $x = 1$.

Continuity is a fundamental concept in calculus, and understanding when a function is continuous at a point helps us analyze and work with functions in various applications.

Practice Problems

Let's practice applying the concept of continuity with some example problems:

Problem 31: Determine if the following function is continuous at $x = 3$:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 3 \\ x^2 - 1 & \text{if } x \geq 3 \end{cases}$$

Problem 32: Find the values of a for which the function $g(x) = ax^2 + 1$ is continuous at $x = 2$.

Problem 33: Consider the function $h(x) = \frac{1}{x}$. Determine all the values of x for which $h(x)$ is continuous.

Problem 34: Investigate the continuity of the function $k(x)$ defined as:

$$k(x) = \begin{cases} \sin x & \text{if } x \neq \pi \\ 0 & \text{if } x = \pi \end{cases}$$

Practice Problems - Solutions

Let's go through the solutions and explanations for the practice problems on continuity.

Problem 35: Determine if the following function is continuous at $x = 3$:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 3 \\ x^2 - 1 & \text{if } x \geq 3 \end{cases}$$

Solution: To check for continuity at $x = 3$, we need to examine both branches of the function separately.

For $x < 3$:

1. $f(x) = 2x + 1$, so it is defined at $x = 3$.
2. The limit as x approaches 3 for this branch:

$$\lim_{x \to 3^-} (2x + 1) = 2(3) + 1 = 7$$

   The limit exists and is equal to 7.

3. $f(3) = 2(3) + 1 = 7$.

For $x \geq 3$:

1. $f(x) = x^2 - 1$, so it is defined at $x = 3$.
2. The limit as x approaches 3 for this branch:

$$\lim_{x \to 3^+} (x^2 - 1) = 3^2 - 1 = 8$$

   The limit exists and is equal to 8.

3. $f(3) = 3^2 - 1 = 8$.

Since both branches are defined and the limits equal the function values at $x = 3$, the function $f(x)$ is continuous at $x = 3$.

Problem 36: Find the values of a for which the function $g(x) = ax^2 + 1$ is continuous at $x = 2$.

Solution: For the function $g(x) = ax^2 + 1$ to be continuous at $x = 2$, it must satisfy the three conditions of continuity:

1. $g(2) = a(2)^2 + 1 = 4a + 1$ must be defined.

2. The limit as x approaches 2, denoted as $\lim_{x\to 2} g(x)$, must exist.

3. $\lim_{x\to 2} g(x)$ must be equal to $g(2)$.

First, we consider the limit:

$$\lim_{x\to 2} g(x) = \lim_{x\to 2}(ax^2 + 1) = 4a + 1$$

For the function to be continuous at $x = 2$, the limit must exist, and it must be equal to $g(2)$. Therefore, we have:

$$4a + 1 = 4a + 1$$

No matter the value of a, the function $g(x)$ will be continuous at $x = 2$ since the limit always exists and is equal to $4a+1$. So, there are no restrictions on the value of a for continuity at $x = 2$.

Problem 37: Consider the function $h(x) = \frac{1}{x}$. Determine all the values of x for which $h(x)$ is continuous.

Solution: To find the values of x for which $h(x)$ is continuous, we need to consider where the function is defined and check if the limit as x approaches each point exists.

The function $h(x) = \frac{1}{x}$ is defined for all x except when $x = 0$ since division by zero is undefined. Therefore, $h(x)$ is not continuous at $x = 0$ because it's not even defined at that point.

For all other values of x, $h(x)$ is continuous since it's defined, and the limit as x approaches any point $x = a$ (where $a \neq 0$) exists and is equal to $\frac{1}{a}$.

In summary, $h(x)$ is continuous for all x except $x = 0$.

Problem 38: Investigate the continuity of the function $k(x)$ defined as:

$$k(x) = \begin{cases} \sin x & \text{if } x \neq \pi \\ 0 & \text{if } x = \pi \end{cases}$$

Solution: To investigate the continuity of $k(x)$, we need to check the conditions of continuity at $x = \pi$.

1. $k(\pi) = 0$ is defined.

2. The limit as x approaches π:

$$\lim_{x\to\pi} k(x) = \lim_{x\to\pi} \sin x = \sin(\pi) = 0$$

   The limit exists and is equal to 0.

3. $k(\pi) = 0$.

Therefore, $k(x)$ is continuous at $x = \pi$.

For all other values of x, $k(x)$ is the sine function, which is known to be continuous everywhere.

In conclusion, $k(x)$ is continuous for all x.

10.2.2 Types of Discontinuities

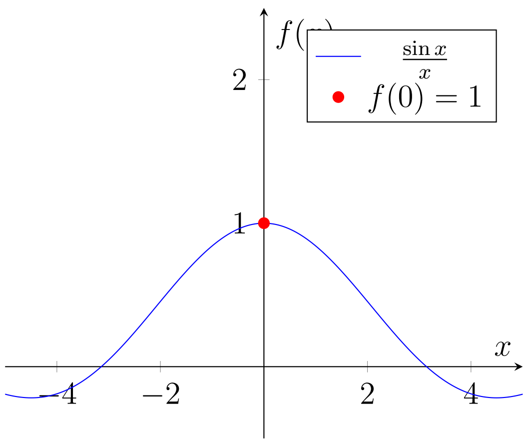
In calculus, discontinuities are points where a function behaves in a way that is not continuous, typically due to a sudden change or a point of undefined behavior. Discontinuities can be classified into three main types: removable, jump, and infinite discontinuities.

Definition 18 ***Removable Discontinuity:*** *A removable discontinuity occurs at a point where the function is undefined or has a jump but can be made continuous by redefining the function value at that point.*

Consider the function $f(x)$ defined as:

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

The graph of $f(x)$ is shown below:



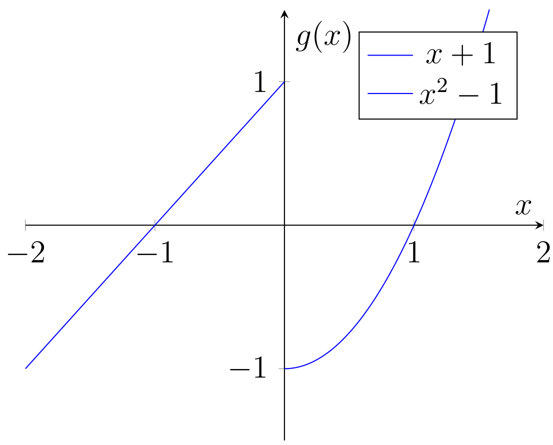
At $x = 0$, the function $f(x)$ is undefined (division by zero). However, we can redefine $f(0)$ as 2 to make it continuous at $x = 0$. This is an example of a removable discontinuity.

Definition 19 ***Jump Discontinuity:*** *A jump discontinuity occurs at a point where there is a sudden jump in the function's values from one side to the other.*

Consider the function $g(x)$ defined as:

$$g(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ x^2 - 1 & \text{if } x \geq 0 \end{cases}$$

The graph of $g(x)$ is shown below:



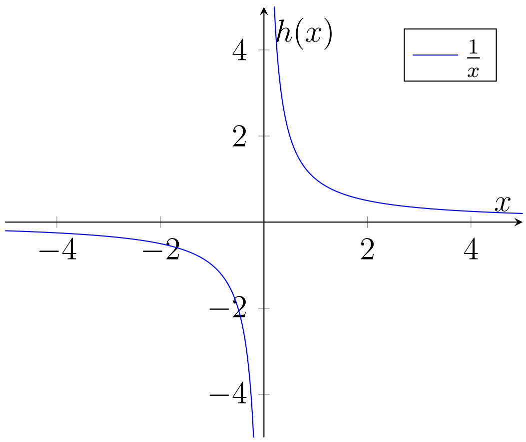
At $x = 0$, there is a sudden jump in the function's values from $g(0^-) = 1$ to $g(0^+) = -1$. This is an example of a jump discontinuity.

Definition 20 ***Infinite Discontinuity:*** *An infinite discontinuity occurs at a point where the function approaches positive or negative infinity as x approaches that point.*

Consider the function $h(x)$ defined as:

$$h(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The graph of $h(x)$ is shown below:



As x approaches 0 from the left ($x \to 0^-$), $h(x)$ approaches negative infinity ($h(x) \to -\infty$), and as x approaches 0 from the right ($x \to 0^+$), $h(x)$ approaches positive infinity ($h(x) \to +\infty$). This is an example of an infinite discontinuity.

Example Problems

Example 18: Consider the function $f(x)$ defined as:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases}$$

Determine the value of k that makes $f(x)$ continuous at $x = 2$.

Solution: For $f(x)$ to be continuous at $x = 2$, we need to ensure that the limit of $f(x)$ as x approaches 2 exists, and it must be equal to the value of $f(x)$ at $x = 2$.

First, let's calculate the limit as x approaches 2:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

We can simplify this limit using algebraic manipulation:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2}$$

Now, we can cancel the common factor $(x - 2)$ from the numerator and denominator:

$$\lim_{x \to 2}(x + 2) = 2 + 2 = 4$$

So, the limit of $f(x)$ as x approaches 2 is 4. To make $f(x)$ continuous at $x = 2$, we set k to be equal to this limit:

$$k = 4$$

Therefore, $f(x)$ is continuous at $x = 2$ when $k = 4$.

Example 19: Consider the function $g(x)$ defined as:

$$g(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

Determine the type(s) of discontinuity, if any, at $x = 0$.

Solution: To determine the type(s) of discontinuity at $x = 0$, we need to analyze the behavior of $g(x)$ as x approaches 0 from both the left ($x \to 0^-$) and the right ($x \to 0^+$).

As x approaches 0 from the left ($x \to 0^-$), $g(x)$ approaches:

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-\sqrt{-x}) = -\sqrt{0} = 0$$

As x approaches 0 from the right ($x \to 0^+$), $g(x)$ approaches:

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (\sqrt{x}) = \sqrt{0} = 0$$

Both the left and right limits are equal and finite (0). Therefore, there is no jump or infinite discontinuity at $x = 0$. However, $g(x)$ is defined differently for $x \geq 0$ and $x < 0$, which means there is a removable discontinuity at $x = 0$.

To make $g(x)$ continuous at $x = 0$, we can redefine the function value at $x = 0$ as $g(0) = 0$. This removes the discontinuity, making $g(x)$ continuous at $x = 0$.

Example 20: Consider the function $h(x)$ defined as:

$$h(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$$

Determine the values of k that make $h(x)$ continuous at $x = 0$.

Solution: For $h(x)$ to be continuous at $x = 0$, we need to ensure that the limit of $h(x)$ as x approaches 0 exists, and it must be equal to the value of $h(x)$ at $x = 0$.

First, let's calculate the limit as x approaches 0:

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \frac{1}{x^2}$$

As x approaches 0, the denominator x^2 approaches 0, and the fraction $\frac{1}{x^2}$ approaches positive infinity ($+\infty$). Therefore, the limit of $h(x)$ as x approaches 0 is $+\infty$.

To make $h(x)$ continuous at $x = 0$, we set k to be equal to this limit:

$$k = +\infty$$

Therefore, $h(x)$ is continuous at $x = 0$ when $k = +\infty$.

Solutions to Example Problems

Example 21: Consider the function $f(x)$ defined as:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases}$$

Determine the value of k that makes $f(x)$ continuous at $x = 2$.

Solution: For $f(x)$ to be continuous at $x = 2$, we need to ensure that the limit of $f(x)$ as x approaches 2 exists, and it must be equal to the value of $f(x)$ at $x = 2$.

First, let's calculate the limit as x approaches 2:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - 4}{x - 2}$$

We can simplify this limit using algebraic manipulation:

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} \frac{(x + 2)(x - 2)}{x - 2}$$

Now, we can cancel the common factor $(x - 2)$ from the numerator and denominator:

$$\lim_{x\to 2}(x + 2) = 2 + 2 = 4$$

So, the limit of $f(x)$ as x approaches 2 is 4. To make $f(x)$ continuous at $x = 2$, we set k to be equal to this limit:

$$k = 4$$

Therefore, $f(x)$ is continuous at $x = 2$ when $k = 4$.

Example 22: Consider the function $g(x)$ defined as:

$$g(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

Determine the type(s) of discontinuity, if any, at $x = 0$.

Solution: To determine the type(s) of discontinuity at $x = 0$, we need to analyze the behavior of $g(x)$ as x approaches 0 from both the left ($x \to 0^-$) and the right ($x \to 0^+$).

As x approaches 0 from the left ($x \to 0^-$), $g(x)$ approaches:

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-\sqrt{-x}) = -\sqrt{0} = 0$$

As x approaches 0 from the right ($x \to 0^+$), $g(x)$ approaches:

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (\sqrt{x}) = \sqrt{0} = 0$$

Both the left and right limits are equal and finite (0). Therefore, there is no jump or infinite discontinuity at $x = 0$. However, $g(x)$ is defined differently for $x \geq 0$ and $x < 0$, which means there is a removable discontinuity at $x = 0$.

To make $g(x)$ continuous at $x = 0$, we can redefine the function value at $x = 0$ as $g(0) = 0$. This removes the discontinuity, making $g(x)$ continuous at $x = 0$.

Example 23: Consider the function $h(x)$ defined as:

$$h(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$$

Determine the values of k that make $h(x)$ continuous at $x = 0$.

Solution: For $h(x)$ to be continuous at $x = 0$, we need to ensure that the limit of $h(x)$ as x approaches 0 exists, and it must be equal to the value of $h(x)$ at $x = 0$.

First, let's calculate the limit as x approaches 0:

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} \frac{1}{x^2}$$

As x approaches 0, the denominator x^2 approaches 0, and the fraction $\frac{1}{x^2}$ approaches positive infinity ($+\infty$). Therefore, the limit of $h(x)$ as x approaches 0 is $+\infty$.

To make $h(x)$ continuous at $x = 0$, we set k to be equal to this limit:

$$k = +\infty$$

Therefore, $h(x)$ is continuous at $x = 0$ when $k = +\infty$.

10.2.3 Intermediate Value Theorem

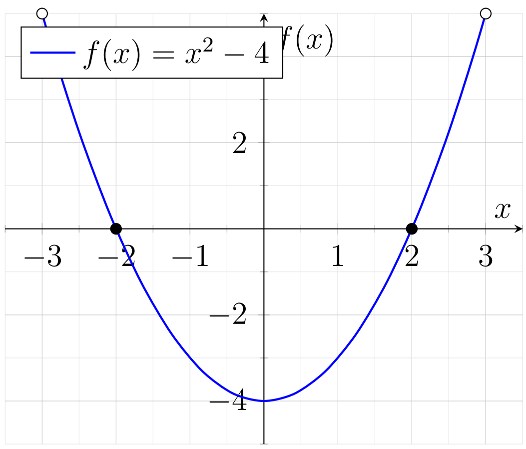
Theorem 4 *If a function is continuous on a closed interval $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then there exists a number c in the interval such that $f(c) = N$.*

Definition 21 *A function f is said to be continuous on a closed interval $[a, b]$ if it is continuous at every point in the interval.*

Discussion 1 *The Intermediate Value Theorem is a fundamental result in calculus that asserts the existence of a point in a continuous function where the function takes on a given value. This theorem is particularly useful in proving the existence of roots.*

Example 24: Consider the function $f(x) = x^2 - 4$ on the interval $[-3, 3]$. Since $f(-3) = 5$ and $f(3) = 5$, and 0 is between 5 and 5, by the Intermediate Value Theorem, there exists a c in $[-3, 3]$ such that $f(c) = 0$.

Graph of $f(x) = x^2 - 4$



Exercise Show using the Intermediate Value Theorem that the function $g(x) = x^3 - x - 1$ has a root in the interval $[1, 2]$.

Practice Problems on Intermediate Value Theorem

Problem 39: Show that the function $h(x) = 3x^3 - 2x + 1$ has at least one root in the interval $[-2, 2]$.

Solution:

Problem 40: Consider the function $k(x) = \cos(x) - x$. Use the Intermediate Value Theorem to prove that there is a root of k in the interval $[0, \frac{\pi}{2}]$.

Solution:

Problem 41: Let $f(x) = e^x - 4x$. Demonstrate using the IVT that f has a root between 1 and 2.

Solution:

Problem 42: Given the function $m(x) = x^5 - 5x + 4$, use the Intermediate Value Theorem to find an interval of length 1 where m must have a root.

Solution:

Problem 43: Assume the function $p(x) = \ln(x) - x^2 + 3$ is continuous on the interval $[1, 4]$. Apply the IVT to prove that p has at least one root in this interval.

Solution:

Solutions to Practice Problems on Intermediate Value Theorem

Solution: To show that $h(x) = 3x^3 - 2x + 1$ has at least one root in the interval $[-2, 2]$, we need to check the values of h at the endpoints and apply the IVT.

At $x = -2$, $h(-2) = 3(-2)^3 - 2(-2) + 1 = -24 + 4 + 1 = -19$.

At $x = 2$, $h(2) = 3(2)^3 - 2(2) + 1 = 24 - 4 + 1 = 21$.

Since $h(-2) < 0$ and $h(2) > 0$, and h is a polynomial (hence continuous), by the IVT, there is at least one root of h in the interval $[-2, 2]$.

Solution: For $k(x) = \cos(x) - x$, we evaluate k at 0 and $\frac{\pi}{2}$:

At $x = 0$, $k(0) = \cos(0) - 0 = 1$.

At $x = \frac{\pi}{2}$, $k(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) - \frac{\pi}{2} = -\frac{\pi}{2}$.

Since $k(0) > 0$ and $k(\frac{\pi}{2}) < 0$, and k is continuous, there must be a root of k in the interval $[0, \frac{\pi}{2}]$ by the IVT.

Solution: To apply the IVT to $f(x) = e^x - 4x$ in the interval [1, 2], we first evaluate f at the endpoints:

At $x = 1$, $f(1) = e^1 - 4(1) = e - 4$.
At $x = 2$, $f(2) = e^2 - 4(2) = e^2 - 8$.
Since $e < 4$ and $e^2 > 8$, it follows that $f(1) < 0$ and $f(2) > 0$. Hence, by the IVT, there is a root of f between 1 and 2.

Solution: For the function $m(x) = x^5 - 5x + 4$, we need to find an interval of length 1 where m has a root.

Consider the intervals $[-1, 0]$ and $[0, 1]$. We compute $m(-1)$, $m(0)$, and $m(1)$:
$m(-1) = (-1)^5 - 5(-1) + 4 = -1 + 5 + 4 = 8$.
$m(0) = 0^5 - 5(0) + 4 = 4$.
$m(1) = 1^5 - 5(1) + 4 = 0$.
Since $m(-1) > 0$ and $m(0) < 0$, there is at least one root in $[-1, 0]$ by the IVT. Similarly, since $m(0) < 0$ and $m(1) = 0$, there is a root at $x = 1$ in $[0, 1]$.

Solution: For $p(x) = \ln(x) - x^2 + 3$, evaluate p at $x = 1$ and $x = 4$:
$p(1) = \ln(1) - 1^2 + 3 = 0 - 1 + 3 = 2$.
$p(4) = \ln(4) - 4^2 + 3 = \ln(4) - 16 + 3 = \ln(4) - 13$.
Since $\ln(4) < 13$, it follows that $p(4) < 0$. Given $p(1) > 0$ and $p(4) < 0$, and p is continuous on $[1, 4]$, by the IVT, there is at least one root of p in this interval.

Chapter 11

The Derivative - Detail

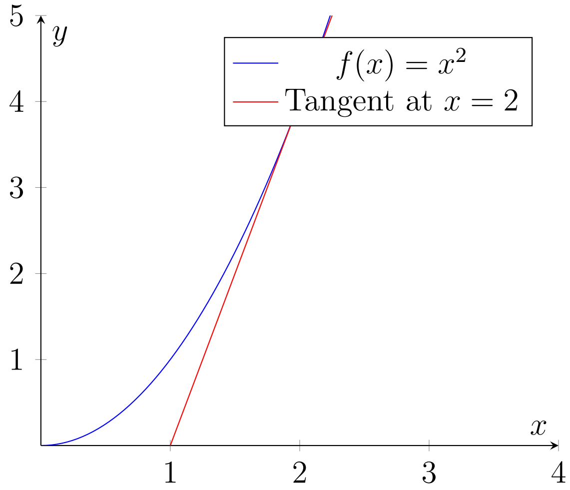
11.1 Understanding Derivatives

11.1.1 Derivative Definition and Notation

Definition 22 *The derivative of a function f at a point a is defined as $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, if this limit exists. This derivative is denoted as $f'(a)$ or $\frac{df}{dx}(a)$.*

The derivative of a function at a point is a fundamental concept in calculus. It represents the rate at which the function's value changes at that point. Geometrically, the derivative corresponds to the slope of the tangent line to the function's graph at that point.

Example Graphical Representation of Derivative: Consider the function $f(x) = x^2$ and the point $a = 2$. The derivative at this point is the slope of the tangent line to the graph of f at $x = 2$.



In this case, the slope of the tangent line (the derivative) at $x = 2$ can be calculated as:

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2 - 4}{h} = 4.$$

Thus, the derivative of $f(x) = x^2$ at $x = 2$ is 4.

The concept of the derivative is widely used in various fields of mathematics, physics, engineering, and beyond. It provides a powerful tool for analyzing and understanding how functions behave and change.

Trigonometric Identities in Calculus

Trigonometric identities are foundational elements in the study of calculus. They facilitate the simplification, integration, and differentiation of trigonometric functions, which are pervasive in various calculus problems. This essay outlines the most common uses of trigonometric identities in calculus, illustrating their importance through practical examples.

11.2 Simplification of Expressions

One of the primary uses of trigonometric identities in calculus is to simplify complex expressions, making them more tractable for further operations like integration or differentiation.

Example

Consider the identity $\sin^2 x + \cos^2 x = 1$. This identity is crucial in simplifying expressions. For instance, to integrate $\int (\sin^2 x + \cos^2 x)\, dx$, we can use this identity to simplify the integrand to 1, making the integral trivial.

11.3 Integration and Differentiation

Trigonometric identities are indispensable tools in solving integrals and derivatives involving trigonometric functions.

Example: Integration

To integrate $\int \sin x \cos x\, dx$, we can use the identity $\sin 2x = 2 \sin x \cos x$. This simplifies the integral to $\frac{1}{2} \int \sin 2x\, dx$, which is straightforward to solve.

Example: Differentiation

When differentiating a function like $\tan x$, knowing that $\tan x = \frac{\sin x}{\cos x}$ helps. The derivative, using the quotient rule, becomes $\sec^2 x$.

11.4 Solving Trigonometric Equations

Trigonometric identities are crucial in solving trigonometric equations, which is a common problem in calculus.

Example

Consider solving $\tan x = \frac{\sin x}{\cos x} = 1$. Knowing the trigonometric identities helps to understand that this equation is true when $x = \frac{\pi}{4} + n\pi$, where n is an integer.

11.5 Substitution in Integrals

Trigonometric identities are often used for substitution in integrals, particularly in cases involving square roots or other complex expressions.

Example

In the integral $\int \sqrt{1-x^2}\,dx$, a common approach is to use a trigonometric substitution such as $x = \sin\theta$, then applying the identity $\cos^2\theta = 1 - \sin^2\theta$ to simplify the integrand.

11.6 Conclusion - Trigonometric Identities in Calculus

Trigonometric identities are not just mathematical curiosities; they are essential tools in calculus. They streamline the process of integration and differentiation, simplify complex expressions, and enable the solving of challenging trigonometric equations. Their utility in various calculus problems underscores the interconnectedness of different branches of mathematics and highlights the elegance and efficiency of mathematical concepts in problem-solving.

Answer Key with Explanations for Trigonometric Examples in Calculus

This answer key provides detailed solutions and explanations for the example problems presented in the previous essay on the use of trigonometric identities in calculus.

11.7 Simplification of Expressions

Example: Simplify and integrate $\int(\sin^2 x + \cos^2 x)\,dx$.

Solution: Using the trigonometric identity $\sin^2 x + \cos^2 x = 1$, the integral simplifies to $\int 1\,dx$. The solution is then:

$$\int 1\,dx = x + C,$$

where C is the constant of integration.

11.8 Integration and Differentiation

11.8.1 Integration

Example: Integrate $\int \sin x \cos x\,dx$.

Solution: Utilizing the identity $\sin 2x = 2\sin x\cos x$, we can rewrite the integral as $\frac{1}{2}\int \sin 2x\,dx$. Integrating $\sin 2x$ gives us:

$$\frac{1}{2}\int \sin 2x\,dx = -\frac{1}{4}\cos 2x + C.$$

11.8.2 Differentiation

Example: Differentiate $\tan x$.

Solution: Knowing that $\tan x = \frac{\sin x}{\cos x}$, we apply the quotient rule for differentiation:

$$\begin{aligned}\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$

11.9 Solving Trigonometric Equations

Example: Solve $\tan x = 1$.

Solution: The equation $\tan x = \frac{\sin x}{\cos x} = 1$ implies that $\sin x = \cos x$. This is true when $x = \frac{\pi}{4} + n\pi$, where n is an integer.

11.10 Substitution in Integrals

Example: Solve $\int \sqrt{1-x^2}\,dx$ using trigonometric substitution.

Solution: Using the substitution $x = \sin\theta$, we have $dx = \cos\theta\,d\theta$. The integral becomes:

$$\begin{aligned}\int \sqrt{1-\sin^2\theta}\cos\theta\,d\theta &= \int \cos^2\theta\,d\theta\\ &= \int \frac{1+\cos 2\theta}{2}\,d\theta\\ &= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C\\ &= \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{4} + C.\end{aligned}$$

Conclusion - Explanations for Trig Examples

These solutions demonstrate the application of trigonometric identities in various calculus problems, emphasizing the importance of understanding and utilizing these identities in solving integrals, derivatives, and trigonometric equations.

Exercises on Derivatives

The following exercises will help solidify your understanding of calculating derivatives. Try to solve them and then check your answers against the provided solutions.

Exercise Find the derivative of the function $f(x) = 3x^3 - 2x^2 + 5x - 7$ at $x = 1$.

Solution: Using the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for $f(x) = 3x^3 - 2x^2 + 5x - 7$, we find

$$f'(1) = \lim_{h \to 0} \frac{3(1+h)^3 - 2(1+h)^2 + 5(1+h) - 7 - (3 - 2 + 5 - 7)}{h}.$$

Simplifying and evaluating this limit gives the derivative at $x = 1$.

Exercise Determine the slope of the tangent line to the curve $g(x) = \sqrt{x}$ at $x = 4$.

Solution: The slope of the tangent line is given by the derivative of $g(x)$ at $x = 4$. The derivative is

$$g'(x) = \frac{1}{2\sqrt{x}}.$$

Therefore, the slope at $x = 4$ is $g'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$.

Exercise Calculate the derivative of $h(x) = \frac{1}{x^2}$ at any point x.

Solution: The derivative of $h(x) = \frac{1}{x^2}$ is

$$h'(x) = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}.$$

Simplifying this expression will give the general formula for the derivative of $h(x)$.

Exercise What is the derivative of the constant function $k(x) = 17$?

Solution: For any constant function $k(x) = c$, where c is a constant, the derivative is zero. Hence, $k'(x) = 0$.

These exercises cover a range of fundamental concepts in differentiation. By working through these problems, you'll gain a better understanding of how to compute derivatives and interpret their geometric meaning.

Solutions to Exercises on Derivatives

Here are the detailed solutions and explanations for the exercises on derivatives.

Solution: To find the derivative of the function $f(x) = 3x^3 - 2x^2 + 5x - 7$ at $x = 1$, we first calculate the derivative of $f(x)$ and then evaluate it at $x = 1$.

The derivative $f'(x)$ is calculated using standard differentiation rules:

$$f'(x) = \frac{d}{dx}(3x^3) - \frac{d}{dx}(2x^2) + \frac{d}{dx}(5x) - \frac{d}{dx}(7) = 9x^2 - 4x + 5.$$

Evaluating this at $x = 1$ gives:

$$f'(1) = 9(1)^2 - 4(1) + 5 = 9 - 4 + 5 = 10.$$

Therefore, the derivative of $f(x)$ at $x = 1$ is 10.

Solution: The slope of the tangent line to the curve $g(x) = \sqrt{x}$ at $x = 4$ is given by $g'(4)$.

First, find the derivative $g'(x)$:

$$g'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$$

Now, evaluate $g'(x)$ at $x = 4$:

$$g'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The slope of the tangent line at $x = 4$ is $\frac{1}{4}$.

Solution: To calculate the derivative of $h(x) = \frac{1}{x^2}$ at any point x, we use the power rule for differentiation:

$$h'(x) = \frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}(x^{-2}) = -2x^{-3} = -\frac{2}{x^3}.$$

So, the derivative of $h(x)$ is $-\frac{2}{x^3}$.

Solution: The derivative of a constant function $k(x) = 17$ is zero. This is because the rate of change of a constant function is always zero. Hence, $k'(x) = 0$.

These solutions provide a step-by-step approach to understanding and solving basic derivative problems. They demonstrate the application of differentiation rules and the interpretation of derivatives in various contexts.

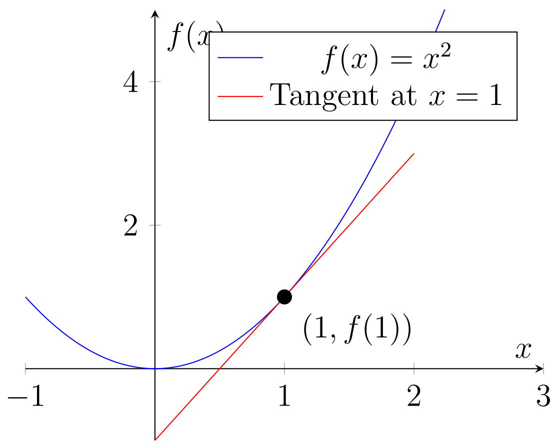
11.10.1 Interpreting Derivatives

Discussion 2 *The derivative of a function at a point provides critical information about the behavior of the function at that point. It gives the slope of the tangent line to the function's graph and represents the rate of change of the function with respect to its variable. This can be interpreted in various ways:*

- ***Slope of Tangent:*** *The derivative at a point a gives the slope of the tangent line to the graph of the function at that point. A positive derivative indicates an increasing function, while a negative derivative indicates a decreasing function.*
- ***Rate of Change:*** *The derivative is a measure of how fast the function's value is changing at that point. It is especially important in physics for understanding velocity and acceleration.*

11.10.2 Graphical Interpretation

Consider the function $f(x) = x^2$. The derivative of f at a point $x = a$ is given by $f'(a) = 2a$. This derivative represents the slope of the tangent line at $x = a$.



In this graph, the tangent line at $x = 1$ has a slope of $f'(1) = 2$, illustrating how the derivative provides the slope at a specific point on the curve.

11.10.3 Practical Implications

Understanding derivatives is crucial in many scientific and engineering fields. For instance, in physics, the derivative of position with respect to time gives the velocity, and the derivative of velocity gives acceleration. In economics, the derivative can represent the rate of change in cost or revenue.

Interpreting derivatives thus provides a powerful tool for analyzing and predicting the behavior of various phenomena described mathematically.

Exercises on Interpreting Derivatives

The following exercises are designed to help you apply and deepen your understanding of derivatives.

Exercise Given the function $f(x) = x^3 - 3x^2 + 5$, find the slope of the tangent line to the graph of f at $x = 2$.

Exercise Consider the function $g(x) = \frac{1}{x}$. Determine where the function is increasing and decreasing.

Exercise A particle moves along a line with its position at time t given by $s(t) = 2t^2 - 3t + 1$. Find the velocity of the particle at $t = 4$.

Exercise Find the points on the curve $y = x^4 - 4x^3$ where the tangent is horizontal.

11.10.4 Solutions to Exercises on Interpreting Derivatives

Solution: First, find the derivative $f'(x)$ of $f(x) = x^3 - 3x^2 + 5$:

$$f'(x) = 3x^2 - 6x.$$

Then, find the slope of the tangent line at $x = 2$:

$$f'(2) = 3(2)^2 - 6(2) = 12 - 12 = 0.$$

The slope of the tangent line at $x = 2$ is 0.

Solution: Find the derivative of $g(x) = \frac{1}{x}$:

$$g'(x) = -\frac{1}{x^2}.$$

The function is increasing where $g'(x) > 0$ and decreasing where $g'(x) < 0$. Since $g'(x)$ is always negative, $g(x)$ is always decreasing.

Solution: The velocity of the particle is the derivative of $s(t)$:

$$s'(t) = 4t - 3.$$

The velocity at $t = 4$ is:

$$s'(4) = 4(4) - 3 = 16 - 3 = 13.$$

The velocity at $t = 4$ is 13 units per time interval.

Solution: Find y' and set it equal to 0 to find points where the tangent is horizontal:

$$y' = 4x^3 - 12x^2,$$

$$0 = 4x^3 - 12x^2.$$

Solving this gives the points where the tangent is horizontal.

These exercises and solutions explore various aspects of derivatives, from the slope of tangent lines to real-world applications such as motion analysis.

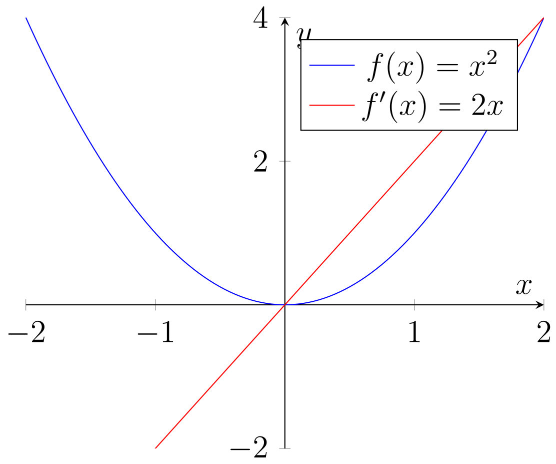
11.10.5 Derivative as a Function

Definition 23 *The derivative of a function f, denoted as $f'(x)$ or $\frac{df}{dx}$, is a function that assigns to each point x in the domain of f the slope of the tangent line to the graph of f at x. This function provides a means to understand how f changes at each point.*

11.10.6 A Derivative's Graphical Interpretation

Consider a function $f(x)$ and its derivative $f'(x)$. The graph of $f'(x)$ can be understood as a plot of the slopes of the tangents to the graph of $f(x)$ at each point.

Example 26: Let $f(x) = x^2$. The derivative function is $f'(x) = 2x$.



In this graph, the blue curve represents the function $f(x) = x^2$, and the red curve represents its derivative $f'(x) = 2x$. The derivative graph shows the slope of the tangent to $f(x)$ at each point x.

11.10.7 Significance of the Derivative Function

The derivative function $f'(x)$ provides valuable information about the behavior of $f(x)$:

- Where $f'(x) > 0$, $f(x)$ is increasing.
- Where $f'(x) < 0$, $f(x)$ is decreasing.
- Where $f'(x) = 0$, $f(x)$ may have a local maximum, minimum, or inflection point.

Understanding the derivative as a function itself allows for a deeper analysis of the original function, particularly in understanding its growth, decay, and points of extrema.

Exercises on Derivative as a Function

The following exercises are designed to help you apply and deepen your understanding of the derivative as a function.

Exercise Find the derivative function $f'(x)$ for $f(x) = 3x^3 - 2x + 1$. Then, determine the intervals where $f(x)$ is increasing and decreasing.

Exercise Given the function $g(x) = \cos(x)$, sketch the graph of g and its derivative $g'(x)$ on the same axes. Indicate points of maximum, minimum, and points where the tangent is horizontal.

Exercise For the function $h(x) = \frac{1}{x^2}$, compute the derivative $h'(x)$. Use the derivative to discuss the concavity of the function.

Exercise Consider $f(x) = e^x$. Find $f'(x)$ and describe how the behavior of $f'(x)$ relates to the behavior of $f(x)$.

11.10.8 Solutions to Exercises on Derivative as a Function

Solution: The derivative of $f(x) = 3x^3 - 2x + 1$ is $f'(x) = 9x^2 - 2$. To find where $f(x)$ is increasing or decreasing, determine the sign of $f'(x)$ for different intervals. $f(x)$ is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$.

Solution: The derivative of $g(x) = \cos(x)$ is $g'(x) = -\sin(x)$. Sketching g and g' together, you'll notice that $g'(x)$ is zero at the maxima and minima of $g(x)$, indicating points where the tangent to $g(x)$ is horizontal.

Solution: The derivative of $h(x) = \frac{1}{x^2}$ is $h'(x) = -\frac{2}{x^3}$. The concavity of $h(x)$ is determined by the sign of $h'(x)$. Since $h'(x)$ is always negative, $h(x)$ is concave down for its entire domain.

Solution: The derivative of $f(x) = e^x$ is $f'(x) = e^x$, which is the same as $f(x)$. This means that the rate of change of e^x is equal to its value at any point, showing an exponential rate of increase.

These exercises and solutions provide a practical application of the concept of the derivative as a function, enhancing understanding through specific examples.

11.11 Techniques of Differentiation

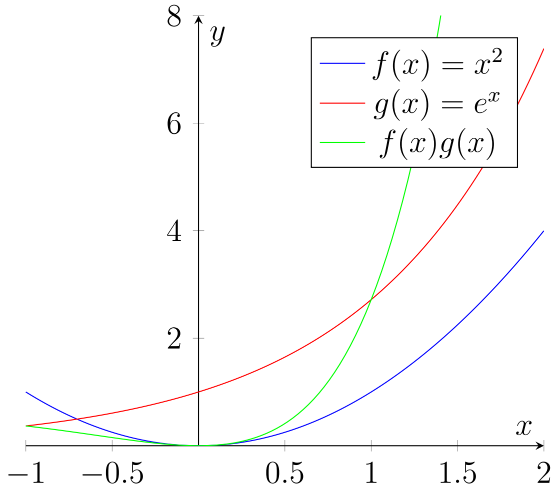
11.11.1 Product Rule and Quotient Rule

Theorem 5 *The Product Rule states that for two differentiable functions f and g, the derivative of their product fg is given by $(fg)' = f'g + fg'$. The Quotient Rule states that for two differentiable functions f and g (with g not equal to zero), the derivative of their quotient $\frac{f}{g}$ is given by $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.*

11.11.2 Graphical Representation of the Product Rule

Consider two functions $f(x)$ and $g(x)$. The graph below illustrates the product rule by showing $f(x)$, $g(x)$, and their product $f(x)g(x)$ along with their derivatives.

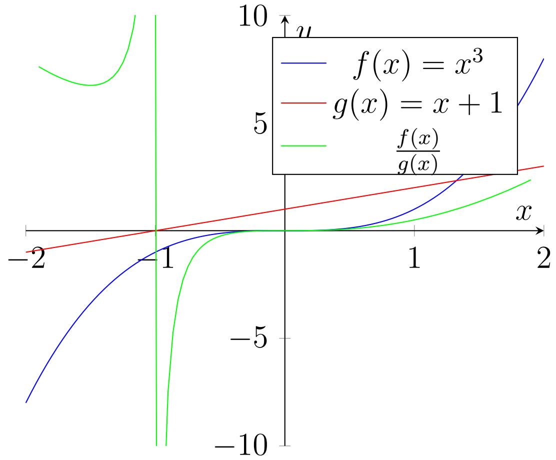
Example 27: Let $f(x) = x^2$ and $g(x) = e^x$.



11.11.3 Graphical Representation of the Quotient Rule

Similarly, for the quotient rule, consider functions $f(x)$ and $g(x)$. The graph below illustrates the quotient rule by showing $f(x)$, $g(x)$, and their quotient $\frac{f(x)}{g(x)}$ along with their derivatives.

Example 28: Let $f(x) = x^3$ and $g(x) = x + 1$.



These graphical representations help in visualizing how the derivatives of the product and quotient of two functions are related to the derivatives of the individual functions. They also highlight the importance of understanding these rules for effective differentiation of more complex functions.

Exercises on Product Rule and Quotient Rule

The following exercises aim to help you practice and understand the Product Rule and Quotient Rule for differentiation.

Exercise Using the Product Rule, find the derivative of $h(x) = (x^3 + 2x)(\sin x)$.

Exercise Apply the Quotient Rule to differentiate $f(x) = \frac{x^2-1}{x+2}$.

Exercise Given $g(x) = e^x \cdot \ln x$, use the Product Rule to find $g'(x)$.

Exercise Differentiate the function $k(x) = \frac{\tan x}{x^2+1}$ using the Quotient Rule.

Solutions to Exercises on Product Rule and Quotient Rule

Solution: For $h(x) = (x^3 + 2x)(\sin x)$, we apply the Product Rule:

$h'(x) = (x^3 + 2x)' \sin x + (x^3 + 2x)(\sin x)' = (3x^2 + 2) \sin x + (x^3 + 2x) \cos x.$

Solution: For $f(x) = \frac{x^2-1}{x+2}$, apply the Quotient Rule:

$$f'(x) = \frac{(x^2-1)'(x+2) - (x^2-1)(x+2)'}{(x+2)^2} = \frac{2x(x+2) - (x^2-1)}{(x+2)^2}.$$

Solution: For $g(x) = e^x \cdot \ln x$, the Product Rule gives:

$$g'(x) = (e^x)' \ln x + e^x (\ln x)' = e^x \ln x + \frac{e^x}{x}.$$

Solution: For $k(x) = \frac{\tan x}{x^2+1}$, use the Quotient Rule:

$$k'(x) = \frac{(\tan x)'(x^2+1) - \tan x (x^2+1)'}{(x^2+1)^2} = \frac{\sec^2 x (x^2+1) - 2x \tan x}{(x^2+1)^2}.$$

These exercises provide a practical application of the Product Rule and Quotient Rule in different scenarios, enhancing the understanding of these fundamental differentiation techniques.

11.11.4 Chain Rule

Theorem 6 *If a function y is a function of u, which is in turn a function of x, then the derivative of y with respect to x is given by the product of the derivatives of y with respect to u and u with respect to x, that is,* $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Application of the Chain Rule

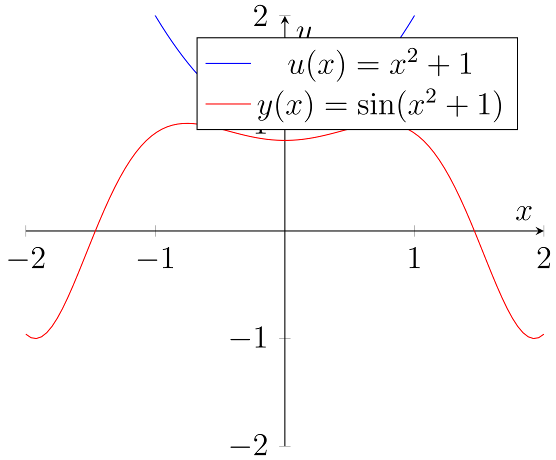
The Chain Rule is particularly useful when dealing with composite functions, where one function is nested within another.

Example 29: Consider $y = \sin(u)$ and $u = x^2 + 1$. We want to find $\frac{dy}{dx}$.
Applying the Chain Rule, we have:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u) \cdot 2x = \cos(x^2 + 1) \cdot 2x.$$

11.11.5 Chain Rule Graphical Representation

To visualize the Chain Rule, we can plot both the inner function $u(x) = x^2 + 1$ and the composite function $y(x) = \sin(x^2 + 1)$, along with their derivatives.



This graph shows how the Chain Rule applies to the composition of functions, allowing us to differentiate complex functions by breaking them down into simpler parts.

Examples and Exercises

The following examples and exercises provide practice in applying the Chain Rule to different types of functions.

Exercise Find the derivative of $f(x) = e^{3x^2+2x}$ using the Chain Rule.

Exercise If $g(x) = \sqrt{1 + \sin x}$, compute $g'(x)$ using the Chain Rule.

These exercises are designed to enhance understanding and proficiency in applying the Chain Rule to a variety of functions.

Additional Exercises on the Chain Rule

The following exercises aim to help students practice the application of the Chain Rule in differentiating composite functions.

Exercise Differentiate $h(x) = \cos(2x^3 - 5x)$.

Exercise Find the derivative of the function $f(x) = \ln(x^2 + 3x + 1)$.

Exercise Compute the derivative of $g(x) = \sqrt{e^{x^2}}$.

Exercise Determine $\frac{d}{dx}\left(\tan(\sqrt{x+1})\right)$.

Solutions to Additional Exercises on the Chain Rule

Solution: For $h(x) = \cos(2x^3 - 5x)$, we have:

$$h'(x) = -\sin(2x^3 - 5x) \cdot \frac{d}{dx}(2x^3 - 5x) = -\sin(2x^3 - 5x) \cdot (6x^2 - 5).$$

Solution: For $f(x) = \ln(x^2 + 3x + 1)$, apply the Chain Rule:

$$f'(x) = \frac{1}{x^2 + 3x + 1} \cdot \frac{d}{dx}(x^2 + 3x + 1) = \frac{2x + 3}{x^2 + 3x + 1}.$$

Solution: For $g(x) = \sqrt{e^{x^2}}$, we find:

$$g'(x) = \frac{1}{2\sqrt{e^{x^2}}} \cdot \frac{d}{dx}(e^{x^2}) = \frac{e^{x^2} \cdot 2x}{2\sqrt{e^{x^2}}} = x\sqrt{e^{x^2}}.$$

Solution: For $\frac{d}{dx}\left(\tan(\sqrt{x+1})\right)$, use the Chain Rule:

$$\frac{d}{dx}\left(\tan(\sqrt{x+1})\right) = \sec^2(\sqrt{x+1}) \cdot \frac{1}{2\sqrt{x+1}}.$$

These exercises cover a range of functions, providing students with the opportunity to practice the Chain Rule in various contexts and enhance their understanding of its application in calculus.

11.11.6 Higher-Order Derivatives

Definition 24 *The n-th order derivative of a function f, denoted $f^{(n)}$, is the derivative of $f^{(n-1)}$. It represents the rate of change of the $(n-1)$-th derivative of f.*

11.11.7 Understanding Higher-Order Derivatives

Higher-order derivatives can be seen as successive rates of change. The second derivative, $f''(x)$, often represents the curvature or concavity of the function $f(x)$. Higher-order derivatives have applications in physics, engineering, and other fields.

Example 30: Consider the function $f(x) = x^3 - 3x^2 + 2x$. Find the first, second, and third derivatives.
Solution: The first derivative $f'(x)$ represents the rate of change of f with respect to x:

$$f'(x) = 3x^2 - 6x + 2.$$

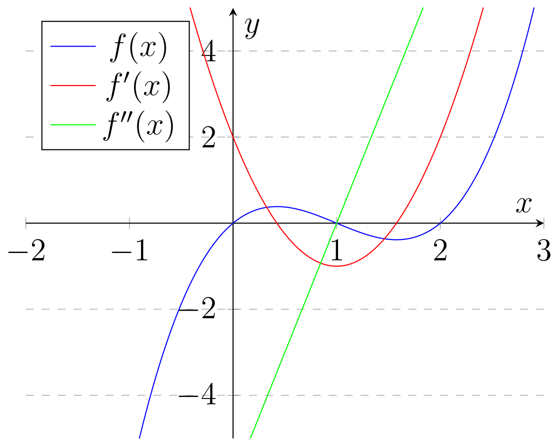
The second derivative $f''(x)$ shows how the slope of $f(x)$ changes:

$$f''(x) = 6x - 6.$$

The third derivative $f'''(x)$ gives the rate of change of the curvature of $f(x)$:

$$f'''(x) = 6.$$

11.11.8 Graphical Illustration of Higher-Order Derivatives



This graph illustrates the original function $f(x)$ and its first and second derivatives, providing a visual representation of how each successive derivative relates to the original function.

Exercises on Higher-Order Derivatives

Exercise Find the first four derivatives of the function $g(x) = \sin(x)$.

Exercise If $h(x) = e^{2x}$, compute $h'''(x)$.

These exercises allow students to practice calculating higher-order derivatives, which is a crucial skill in advanced calculus and its applications.

Additional Exercises on Higher-Order Derivatives

The following exercises aim to further reinforce the concept of higher-order derivatives through practical applications.

Exercise Find the first three derivatives of the function $p(t) = \ln(t^2 + 1)$.

Exercise Given $f(x) = \frac{1}{x^2+1}$, compute $f''(x)$ and $f'''(x)$.

Exercise For the function $g(x) = e^{-x}\cos(x)$, determine $g'(x)$, $g''(x)$, and $g'''(x)$.

Exercise Calculate the fourth derivative of $h(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$.

Solutions to Additional Exercises on Higher-Order Derivatives

Solution: For $p(t) = \ln(t^2 + 1)$, we find:

$$p'(t) = \frac{2t}{t^2+1}, \quad p''(t) = \frac{2-2t^2}{(t^2+1)^2}, \quad p'''(t) = \frac{-8t(t^2-3)}{(t^2+1)^3}.$$

Solution: For $f(x) = \frac{1}{x^2+1}$, we calculate:

$$f''(x) = \frac{6x^2-2}{(x^2+1)^3}, \quad f'''(x) = \frac{-24x(x^2-1)}{(x^2+1)^4}.$$

Solution: For $g(x) = e^{-x}\cos(x)$:

$$g'(x) = -e^{-x}(\cos(x) + \sin(x)),$$

$$g''(x) = e^{-x}(2\sin(x)),$$
$$g'''(x) = e^{-x}(2\cos(x) - 2\sin(x)).$$

Solution: For $h(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$:

$$h''''(x) = 24.$$

These exercises provide a comprehensive range of functions to practice differentiating, helping students to deepen their understanding of higher-order derivatives and their applications in various contexts.

11.11.9 L'Hôpital's Rule for Indeterminate Forms

Theorem 7 (L'Hôpital's Rule) *If the functions $f(x)$ and $g(x)$ are differentiable and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$, and if $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists or equals $\pm\infty$, then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Graphical Illustration

To understand L'Hôpital's Rule, let's consider the indeterminate form $\frac{0}{0}$. Suppose $f(x)$ and $g(x)$ approach 0 as x approaches c, making $\frac{f(x)}{g(x)}$ indeterminate at $x = c$.

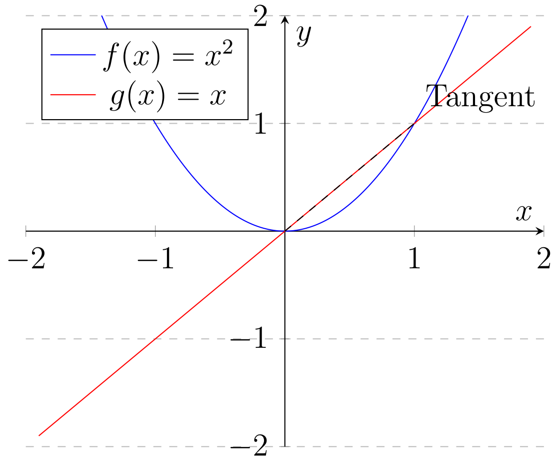


Figure 11.1: Graphical interpretation of L'Hôpital's Rule

Applications

L'Hôpital's Rule is particularly useful in calculus for resolving limits that result in indeterminate forms like $\frac{0}{0}$ or $\frac{\infty}{\infty}$. It simplifies complex limit evaluations, especially when direct substitution in the limit leads to an indeterminate form.

Examples

Example 31: Evaluate $\lim_{x \to 0} \frac{\sin(x)}{x}$.
Solution: Direct substitution gives $\frac{0}{0}$, an indeterminate form. Applying L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = \cos(0) = 1.$$

Example 32: Find $\lim_{x \to \infty} \frac{e^x}{x^2}$.
Solution: Both numerator and denominator approach infinity as $x \to \infty$. Applying L'Hôpital's Rule twice:

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty.$$

Conclusion

L'Hôpital's Rule streamlines the process of evaluating limits that lead to indeterminate forms, which are common in calculus. Its application requires understanding of derivatives and their properties.

Practice Problems

To further strengthen your understanding of L'Hôpital's Rule, try solving the following problems:

Problem 44: Evaluate the limit: $\lim_{x \to 0} \frac{e^x - 1}{x}$.

Problem 45: Find the limit: $\lim_{x \to \infty} \frac{\ln(x)}{x}$.

Problem 46: Determine $\lim_{x \to 0} \frac{\sin(2x)}{3x}$.

Problem 47: Compute the limit: $\lim_{x \to 0^+} x \ln(x)$.

Problem 48: Evaluate $\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$.

Hints

- For problems that result in the $\frac{0}{0}$ indeterminate form, apply L'Hôpital's Rule by differentiating the numerator and denominator separately.
- If the indeterminate form persists after the first application of the rule, you may need to apply L'Hôpital's Rule multiple times.
- Remember that understanding the behavior of functions as they approach a point is key to correctly applying L'Hôpital's Rule.

These problems are designed to cover various scenarios where L'Hôpital's Rule can be applied. They will help you develop a deeper understanding of how to deal with indeterminate forms in calculus.

Solutions to Practice Problems

Solution: Evaluate $\lim_{x\to 0} \frac{e^x-1}{x}$.

Solution: Applying L'Hôpital's Rule, we differentiate the numerator and denominator:

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = e^0 = 1.$$

Solution: Find $\lim_{x\to\infty} \frac{\ln(x)}{x}$.

Solution: Using L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Solution: Determine $\lim_{x\to 0} \frac{\sin(2x)}{3x}$.

Solution: Applying L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sin(2x)}{3x} = \lim_{x\to 0} \frac{2\cos(2x)}{3} = \frac{2}{3}.$$

Solution: Compute $\lim_{x\to 0^+} x\ln(x)$.

Solution: This is an indeterminate form of type $0 \cdot (-\infty)$. Rewriting the expression:

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x}.$$

Now applying L'Hôpital's Rule:

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

Solution: Evaluate $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$.

Solution: Using L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin(x)}{2x} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2}.$$

These solutions provide step-by-step application of L'Hôpital's Rule to solve the given limits. They illustrate the procedure of differentiating the numerator and denominator to resolve indeterminate forms.

11.11.10 Implicit Differentiation

Method 1 *Implicit differentiation is used when a function is given in an implicit form $F(x, y) = 0$ rather than the explicit form $y = f(x)$. It involves differentiating both sides of the equation with respect to x and solving for $\frac{dy}{dx}$.*

11.11.11 Principle of Implicit Differentiation

In implicit differentiation, each term involving y is differentiated using the chain rule, as y is a function of x.

Example 33: Differentiate the equation $x^2 + y^2 = 25$ implicitly.
Solution: Differentiating both sides with respect to x:

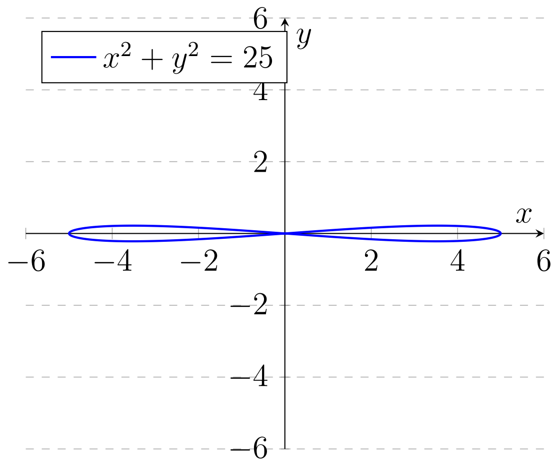
$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25),$$

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x}{y}.$$

11.11.12 Graphical Representation of Implicit Functions



This graph shows the circle defined by the equation $x^2+y^2 = 25$. Implicit differentiation helps find the slope of the tangent at any point on this curve.

11.11.13 Practical Engineering Example: Pressure and Volume in Thermodynamics

Example 34: In thermodynamics, the relationship between the pressure P, volume V, and temperature T of a gas is often described by the equation $PV = nRT$, where n is the number of moles of the gas and R is the universal gas constant. If the temperature is held constant, find the rate at which pressure changes with respect to volume.

Solution: Given the equation $PV = nRT$ with constant T, we differentiate implicitly with respect to V:

$$P\frac{dV}{dV} + V\frac{dP}{dV} = 0,$$

$$P + V\frac{dP}{dV} = 0,$$

$$\frac{dP}{dV} = -\frac{P}{V}.$$

This result indicates that the rate at which pressure changes with respect to volume is inversely proportional to the volume itself, assuming temperature remains constant. This principle is crucial in understanding how gases behave under compression or expansion in various engineering applications.

Discussion 3 *This example demonstrates the application of implicit differentiation in the field of thermodynamics, specifically in understanding how the properties of gases change under different conditions. Such principles are fundamental in engineering disciplines, especially in mechanical and chemical engineering.*

Exercises on Implicit Differentiation

Exercise Use implicit differentiation to find $\frac{dy}{dx}$ for the curve $x^3+y^3-3xy = 0$.

Exercise Find the slope of the tangent line to the ellipse $4x^2 + 9y^2 = 36$ at the point $(2, \sqrt{2})$.

These exercises aim to provide hands-on experience with implicit differentiation, enhancing the understanding of this essential calculus technique.

More Exercises on Implicit Differentiation

Exercise Find $\frac{dy}{dx}$ for the curve given by the equation $\sin(xy) = x + y$.

Exercise Differentiate implicitly to find the slope of the curve $e^{x+y} = x^2+y^2$ at any point.

Exercise For the curve defined by $\ln(x) + \ln(y) = 1$, find $\frac{d^2y}{dx^2}$.

Exercise Use implicit differentiation to find the slope of the tangent line to the curve $x^4 + y^4 = 16$ at the point $(1, \sqrt[4]{15})$.

Exercise Find $\frac{dy}{dx}$ for the ellipse $4x^2 + y^2 = 16$ using implicit differentiation.

Solutions to More Exercises on Implicit Differentiation

Solution: Differentiating $\sin(xy) = x + y$ implicitly:

$$\cos(xy)(y + x\frac{dy}{dx}) = 1 + \frac{dy}{dx},$$

$$\frac{dy}{dx}(x\cos(xy) - 1) = 1 - y\cos(xy),$$

$$\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) - 1}.$$

Solution: Differentiating $e^{x+y} = x^2 + y^2$:

$$e^{x+y}(1 + \frac{dy}{dx}) = 2x + 2y\frac{dy}{dx},$$

$$\frac{dy}{dx}(e^{x+y} - 2y) = 2x - e^{x+y},$$

$$\frac{dy}{dx} = \frac{2x - e^{x+y}}{e^{x+y} - 2y}.$$

Solution: Differentiating $\ln(x) + \ln(y) = 1$:

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = -\frac{y}{x}.$$

To find $\frac{d^2y}{dx^2}$, differentiate $\frac{dy}{dx}$ again:

$$\frac{d^2y}{dx^2} = -\frac{x\frac{dy}{dx} - y}{x^2} = -\frac{-y^2/x - y}{x^2} = \frac{y(y+x)}{x^3}.$$

Solution: For $x^4 + y^4 = 16$ at $(1, \sqrt[4]{15})$:

$$4x^3 + 4y^3\frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = -\frac{x^3}{y^3} = -\frac{1}{(\sqrt[4]{15})^3}.$$

Solution: For $4x^2 + y^2 = 16$:

$$8x + 2y\frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = -\frac{8x}{2y} = -\frac{4x}{y}.$$

These exercises provide a diverse range of problems to practice implicit differentiation and help students understand its applications in different scenarios.

11.12 Applications of Derivatives

11.12.1 Tangent Lines and Normal Lines

Application 1 *The equation of the tangent line to the curve* $y = f(x)$ *at the point* $(a, f(a))$ *is given by* $y - f(a) = f'(a)(x - a)$*. The normal line is perpendicular to the tangent line at the point of tangency.*

Example 35: Consider the function $f(x) = x^2$. Find the equation of the tangent and normal lines to the curve at the point where $x = 1$.

Solution: First, we find the derivative of $f(x) = x^2$, which is $f'(x) = 2x$. At $x = 1$, $f'(1) = 2 \times 1 = 2$. The equation of the tangent line at $x = 1$ is:

$$y - 1 = 2(x - 1)$$

or

$$y = 2x - 1.$$

To find the equation of the normal line, we use the fact that the slope of the normal line is the negative reciprocal of the tangent line's slope. Thus, the slope of the normal line is $-\frac{1}{2}$, and its equation is:

$$y - 1 = -\frac{1}{2}(x - 1)$$

or

$$y = -\frac{1}{2}x + \frac{3}{2}.$$

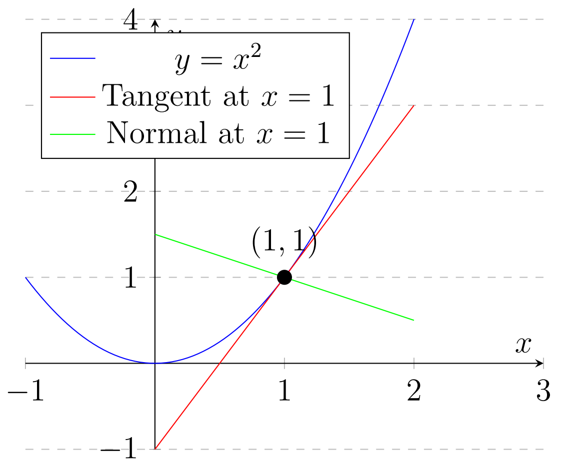


Figure 11.2: Graph of the function $y = x^2$ with its tangent and normal lines at $x = 1$.

Discussion 4 *This example illustrates how to determine the equations of tangent and normal lines to a given curve at a specific point. The graphical representation helps in visualizing these concepts, which are fundamental in differential calculus and have various applications in fields such as engineering and physics.*

Practice Problems on Tangent and Normal Lines

Problem 49: Find the equations of the tangent and normal lines to the curve $y = \sqrt{x}$ at the point where $x = 4$.

Problem 50: Consider the function $f(x) = \frac{1}{x}$. Determine the equation of the tangent line at the point $x = 2$ and the equation of the normal line at the same point.

Problem 51: Given the function $f(x) = 3x^3 - 2x + 1$, compute the equations of the tangent and normal lines at the point where $x = -1$.

Problem 52: For the function $f(x) = \sin(x)$, find the equations of the tangent and normal lines at $x = \frac{\pi}{4}$.

Problem 53: Determine the equations of the tangent and normal lines to the curve given by $y = \ln(x)$ at the point where $x = e$ (where e is the base of the natural logarithm).

Problem 54: Consider the curve defined by the equation $x^2 + y^2 = 25$. Find the equations of the tangent and normal lines at the point $(3, 4)$ on the curve.

Discussion 5 *These problems involve different types of functions, including polynomial, trigonometric, logarithmic, and even a circle equation. Solving these problems will give students practice in applying the concepts of derivatives to find tangent and normal lines in various contexts.*

Solutions to Practice Problems on Tangent and Normal Lines

Solution: The function is $y = \sqrt{x}$. Its derivative is $\frac{1}{2\sqrt{x}}$. At $x = 4$, the slope of the tangent line is $\frac{1}{2\sqrt{4}} = \frac{1}{4}$. The equation of the tangent line is given by $y - f(a) = f'(a)(x - a)$, which simplifies to $y - 2 = \frac{1}{4}(x - 4)$. The normal line is perpendicular to the tangent line, so its slope is the negative reciprocal, -4. The equation of the normal line is $y - 2 = -4(x - 4)$.

Solution: For $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$. At $x = 2$, $f'(2) = -\frac{1}{4}$, so the tangent line is $y - \frac{1}{2} = -\frac{1}{4}(x - 2)$. The slope of the normal line is 4, so its equation is $y - \frac{1}{2} = 4(x - 2)$.

Solution: The derivative of $f(x) = 3x^3 - 2x + 1$ is $f'(x) = 9x^2 - 2$. At $x = -1$, $f'(-1) = 7$. The equation of the tangent line is $y - 2 = 7(x + 1)$. The normal line has a slope of $-\frac{1}{7}$, so its equation is $y - 2 = -\frac{1}{7}(x + 1)$.

Solution: For $f(x) = \sin(x)$, $f'(x) = \cos(x)$. At $x = \frac{\pi}{4}$, $f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. The tangent line equation is $y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$. The normal line has a slope of $-\frac{\sqrt{2}}{2}$, with its equation being $y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$.

Solution: For $y = \ln(x)$, $\frac{dy}{dx} = \frac{1}{x}$. At $x = e$, the derivative is $\frac{1}{e}$. The tangent line equation is $y - 1 = \frac{1}{e}(x - e)$. The normal line's slope is $-e$, so its equation is $y - 1 = -e(x - e)$.

Solution: For the circle $x^2 + y^2 = 25$, implicit differentiation gives $2x + 2yy' = 0$. At $(3, 4)$, $y' = -\frac{3}{4}$. The equation of the tangent line is $y - 4 = -\frac{3}{4}(x - 3)$. The normal line, with a slope of $\frac{4}{3}$, has the equation $y - 4 = \frac{4}{3}(x - 3)$.

Discussion 6 *These solutions demonstrate the application of derivatives to find equations of tangent and normal lines. They involve direct computation of derivatives, using implicit differentiation, and understanding the relationship between slopes of tangent and normal lines.*

11.12.2 Increasing and Decreasing Functions

Theorem 8 *A function f is increasing on an interval if, for any two numbers x_1 and x_2 in the interval with $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$. Similarly, f is decreasing on an interval if $f(x_1) \geq f(x_2)$ for $x_1 < x_2$ in the interval. This can often be determined by analyzing the sign of its derivative f'.*

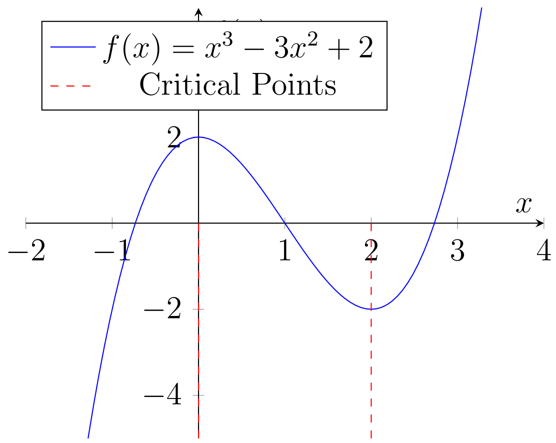
Example 36: Consider the function $f(x) = x^3 - 3x^2 + 2$. To determine where the function is increasing or decreasing, we first find its derivative: $f'(x) = 3x^2 - 6x$. Setting $f'(x) = 0$, we find critical points at $x = 0$ and $x = 2$.

We then test intervals around these critical points to determine the sign of $f'(x)$:

- For $x < 0$, take $x = -1$: $f'(-1) = -3 - 6 = -9$ (negative, so f is decreasing).

- For $0 < x < 2$, take $x = 1$: $f'(1) = 3 - 6 = -3$ (negative, so f is still decreasing).
- For $x > 2$, take $x = 3$: $f'(3) = 27 - 18 = 9$ (positive, so f is increasing).

Thus, $f(x)$ is decreasing on $(-\infty, 2]$ and increasing on $[2, +\infty)$.



Discussion 7 *This example illustrates how to find intervals where a function is increasing or decreasing. By examining the derivative, we can identify critical points and the behavior of the function around these points. The graph visually confirms these findings.*

Practice Problems on Increasing and Decreasing Functions

Problem 55: Determine the intervals where the function $f(x) = x^2 - 4x + 3$ is increasing and decreasing.

Problem 56: Given the function $g(x) = \frac{1}{x}$, find the intervals where the function is increasing and where it is decreasing.

Problem 57: For the function $h(x) = e^{-x}$, identify the intervals of increase and decrease.

Problem 58: Consider the function $p(x) = x^3 - 9x^2 + 24x - 15$. Analyze the function's increasing and decreasing behavior.

Problem 59: Let $q(x) = \sin(x)$ on the interval $[0, 2\pi]$. Determine where the function is increasing and decreasing.

Discussion 8 *These problems involve different types of functions, including polynomial, rational, exponential, and trigonometric functions. Students should first find the derivative of the function, determine critical points, and then analyze the sign of the derivative on intervals around these points to conclude where the function is increasing or decreasing.*

Solutions to Practice Problems

Solution: To find where $f(x) = x^2 - 4x + 3$ is increasing and decreasing, we first find its derivative:

$$f'(x) = 2x - 4.$$

Setting $f'(x) = 0$, we find the critical point at $x = 2$.

Examining the sign of $f'(x)$ around this point:

- For $x < 2$, say $x = 1$, $f'(1) = -2$, so $f(x)$ is decreasing.
- For $x > 2$, say $x = 3$, $f'(3) = 2$, so $f(x)$ is increasing.

Thus, $f(x)$ is decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$.

Solution: For $g(x) = \frac{1}{x}$, the derivative is

$$g'(x) = -\frac{1}{x^2}.$$

Since $g'(x)$ is always negative for $x \neq 0$, $g(x)$ is always decreasing on its domain.

Solution: For $h(x) = e^{-x}$, the derivative is

$$h'(x) = -e^{-x}.$$

Since $h'(x) < 0$ for all x, $h(x)$ is always decreasing.

Solution: Consider $p(x) = x^3 - 9x^2 + 24x - 15$. The derivative is

$$p'(x) = 3x^2 - 18x + 24.$$

Setting $p'(x) = 0$ yields $x = 2$ and $x = 4$ as critical points.

Checking intervals around these points:

- For $x < 2$, $p'(x) > 0$, so $p(x)$ is increasing.
- For $2 < x < 4$, $p'(x) < 0$, so $p(x)$ is decreasing.
- For $x > 4$, $p'(x) > 0$, so $p(x)$ is increasing.

Therefore, $p(x)$ is increasing on $(-\infty, 2) \cup (4, \infty)$ and decreasing on $(2, 4)$.

Solution: For $q(x) = \sin(x)$ on $[0, 2\pi]$, the derivative is

$$q'(x) = \cos(x).$$

We find critical points where $\cos(x) = 0$, which are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

Examining the sign of $q'(x)$:

- For $0 < x < \frac{\pi}{2}$, $q'(x) > 0$, so $q(x)$ is increasing.
- For $\frac{\pi}{2} < x < \frac{3\pi}{2}$, $q'(x) < 0$, so $q(x)$ is decreasing.
- For $\frac{3\pi}{2} < x < 2\pi$, $q'(x) > 0$, so $q(x)$ is increasing.

Thus, $q(x)$ is increasing on $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$ and decreasing on $(\frac{\pi}{2}, \frac{3\pi}{2})$.

11.12.3 Maxima and Minima

Method 2 *Critical points, where $f'(x) = 0$ or $f'(x)$ does not exist, are potential locations of local maxima or minima. These points are found by solving the equation $f'(x) = 0$ or determining where $f'(x)$ is undefined.*

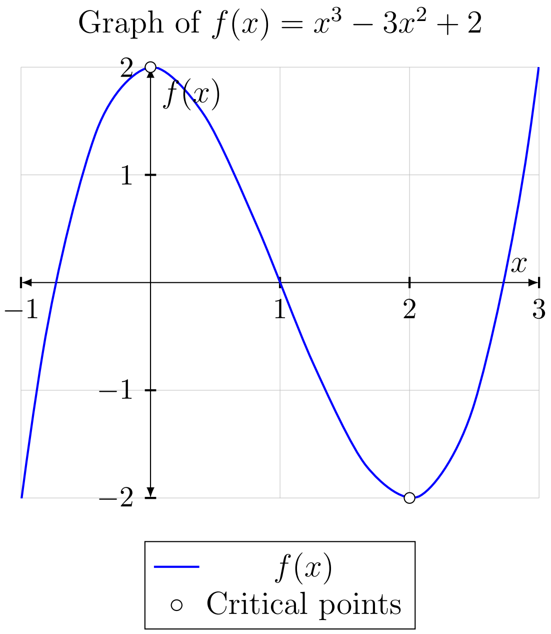
Example 37: Consider the function $f(x) = x^3 - 3x^2 + 2$. To find its critical points, we compute its derivative:

$$f'(x) = 3x^2 - 6x.$$

Setting $f'(x) = 0$, we find:

$$3x^2 - 6x = 0 \Rightarrow x(x - 2) = 0.$$

Therefore, the critical points are $x = 0$ and $x = 2$.



Discussion 9 *To determine whether these critical points are maxima or minima, we can use the first or second derivative test. For the first derivative test, we check the sign of $f'(x)$ before and after the critical point. For the second derivative test, we check whether $f''(x)$ is positive or negative at the critical point.*

Exercise Find the maxima and minima of the function $g(x) = x^4 - 4x^3 + 6x^2$ using the first and second derivative tests.

Practice Problems on Maxima and Minima

Problem 60: Find the critical points of the function $h(x) = x^4 - 8x^2 + 16$ and determine whether they are maxima, minima, or neither.

Problem 61: Consider the function $p(x) = 3x^3 - 12x + 9$. Identify all the critical points and classify them using the second derivative test.

Problem 62: Determine the local maxima and minima of the function $q(x) = x^5 - 5x^4 + 10x^3 - 10$ using the first derivative test.

Problem 63: For the function $r(x) = \sin(x) + \cos(2x)$ on the interval $[0, 2\pi]$, find the critical points and use the second derivative test to classify them.

Problem 64: Analyze the function $s(t) = e^{-t} \cdot \sin(t)$ to find its local maxima and minima. Consider the interval $[0, 2\pi]$.

Exercise Sketch the graph of one of the above functions, marking the critical points and indicating the regions where the function is increasing or decreasing.

Exercise For one of the functions above, calculate the inflection points and discuss how these points relate to the function's concavity and the locations of maxima and minima.

Solutions to Practice Problems

Solution: For the function $h(x) = x^4 - 8x^2 + 16$, we find the critical points by first computing the derivative:

$$h'(x) = 4x^3 - 16x.$$

Setting $h'(x) = 0$ gives:

$$4x(x^2 - 4) = 0 \Rightarrow x = 0, \pm 2.$$

Using the second derivative test, we find:

$$h''(x) = 12x^2 - 16.$$

At $x = 0$, $h''(0) = -16$ (negative), so $x = 0$ is a local maximum. At $x = \pm 2$, $h''(\pm 2) = 16$ (positive), so $x = \pm 2$ are local minima.

Solution: The critical points of $p(x) = 3x^3 - 12x + 9$ are found by setting the derivative $p'(x) = 9x^2 - 12$ equal to zero:

$$9x^2 - 12 = 0 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm\sqrt{\frac{4}{3}}.$$

Second derivative $p''(x) = 18x$ at $x = \sqrt{\frac{4}{3}}$ is positive, indicating a local minimum, and at $x = -\sqrt{\frac{4}{3}}$ is negative, indicating a local maximum.

Solution: For $q(x) = x^5 - 5x^4 + 10x^3 - 10$, the first derivative is:

$$q'(x) = 5x^4 - 20x^3 + 30x^2.$$

Setting $q'(x) = 0$ and factoring gives:

$$x^2(5x^2 - 20x + 30) = 0.$$

The critical points are found by solving this equation. Use the first derivative test to determine whether these points are maxima or minima.

Solution: For $r(x) = \sin(x) + \cos(2x)$, the derivative is:

$$r'(x) = \cos(x) - 2\sin(2x).$$

Setting $r'(x) = 0$ and solving on the interval $[0, 2\pi]$ gives the critical points. Use the second derivative test to classify these points.

Solution: To find the critical points of $s(t) = e^{-t} \cdot \sin(t)$ on $[0, 2\pi]$, compute the derivative and set it to zero. Then, use the first or second derivative test to classify the critical points.

11.12.4 Maxima and Minima - Again

In calculus, finding the maximum and minimum values of a function is a fundamental task. Critical points are potential locations of local maxima or minima. These critical points occur where the derivative of the function either equals zero or does not exist.

To understand this concept visually, let's consider some examples.

Example 38: Consider the function $f(x) = x^3 - 3x^2 + 2x + 1$. To find its critical points, we need to find where $f'(x) = 0$ or $f'(x)$ does not exist.

To find the critical points, we first find $f'(x)$:

$$f'(x) = 3x^2 - 6x + 2$$

Now, we set $f'(x) = 0$ to find the values of x where the slope is zero:

$$3x^2 - 6x + 2 = 0$$

Solving this quadratic equation, we find two critical points at $x = 1$ and $x = 2$.

Next, we check if there are any critical points where $f'(x)$ does not exist. In this case, there are no such points.

Therefore, the critical points of the function $f(x)$ are $x = 1$ and $x = 2$, which are potential locations of local maxima or minima.

To determine whether these points correspond to maxima or minima, we can use the second derivative test or analyze the behavior of the function around these points.

In this example, we discuss how to find critical points using the derivative of a function. These critical points are essential for identifying potential

local maxima or minima, which are crucial in optimization and real-world problem-solving.
Solution: In calculus, finding the maximum and minimum values of a function is a fundamental concept. Critical points play a crucial role in identifying these extrema. Critical points are points on the graph of a function where either the derivative is zero or the derivative does not exist.

Example 39: Consider the function $f(x) = x^3 - 3x^2 + 2x + 1$. We want to find its critical points, which are potential locations of local maxima or minima.

First, let's calculate the derivative of $f(x)$, denoted as $f'(x)$:

$$f'(x) = \frac{d}{dx}(x^3 - 3x^2 + 2x + 1) = 3x^2 - 6x + 2$$

Next, we set $f'(x)$ equal to zero to find where the slope of the function is zero:

$$3x^2 - 6x + 2 = 0$$

To solve this quadratic equation, we can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case, $a = 3$, $b = -6$, and $c = 2$. Applying the quadratic formula:

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3}$$

Simplifying:

$$x = \frac{6 \pm \sqrt{12 - 24}}{6} = \frac{6 \pm \sqrt{-12}}{6} = \frac{6 \pm 2i\sqrt{3}}{6} = 1 \pm \frac{\sqrt{3}}{3}i$$

So, the critical points are $x = 1 + \frac{\sqrt{3}}{3}i$ and $x = 1 - \frac{\sqrt{3}}{3}i$. However, these points are complex numbers, and we are typically interested in real-valued critical points. In this case, there are no real critical points where $f'(x)$ equals zero.

Now, let's consider whether there are any critical points where $f'(x)$ does not exist. In this example, the derivative $f'(x) = 3x^2 - 6x + 2$ is a polynomial, which is defined for all real numbers. Therefore, there are no critical points where $f'(x)$ does not exist.

In conclusion, for the function $f(x) = x^3 - 3x^2 + 2x + 1$, there are no real critical points where $f'(x)$ equals zero or where $f'(x)$ does not exist. Thus, there are no potential local maxima or minima for this function among its critical points.

Practical Engineering Example: Optimizing Beam Design

In engineering, the concept of maxima and minima is crucial for optimizing various designs. Let's consider a practical example involving the design of a cantilever beam.

Suppose you are an engineer tasked with designing a cantilever beam that can support a maximum load while minimizing the material used. The beam's length is fixed at L meters, and you can adjust its width w and height h in meters.

Problem 65: Determine the dimensions of the cantilever beam that will maximize its load-bearing capacity while minimizing the amount of material used.

Solution: To solve this problem, we need to establish a mathematical relationship between the load-bearing capacity and the amount of material used.

Let's denote the load-bearing capacity as P (in Newtons) and the amount of material used as A (in square meters). We'll start by defining these quantities:

$$P = k \cdot w \cdot h^2$$

The load-bearing capacity, P, is directly proportional to the product of width, w, and the square of the height, h. Here, k is a constant that depends on the material properties and other factors.

The amount of material used, A, is given by the surface area of the beam:

$$A = 2 \cdot w \cdot h + L \cdot w$$

We need to find the dimensions (w, h) that maximize P while minimizing A. This is an optimization problem. To solve it, we can use the method of critical points.

First, we'll find the critical points by setting the derivatives of P and A with respect to w and h equal to zero:

$$\begin{cases} \frac{dP}{dw} = k \cdot h^2 = 0 \\ \frac{dP}{dh} = 2k \cdot w \cdot h = 0 \\ \frac{dA}{dw} = 2h + L = 0 \\ \frac{dA}{dh} = 2w = 0 \end{cases}$$

From the first two equations, we find that either $h = 0$ or $w = 0$ to maximize P. However, a beam with zero width or zero height is not practical. Therefore, we exclude these possibilities.

From the last two equations, we find that $w = -L/2$ and $h = 0$. However, these values are also impractical.

Thus, there are no critical points for this problem.

This result means that we need to consider the boundary conditions. Since w and h are positive values, we must choose values of w and h that satisfy the constraints $w > 0$, $h > 0$, and A and L are fixed constants.

Without loss of generality, let's set $L = 1$ meter (the length of the beam) and $A = 1$ square meter (to simplify the problem).

Given the constraints, we can use various numerical optimization techniques to find the dimensions (w, h) that maximize P. This could involve using software tools like MATLAB or Python with libraries such as SciPy.

In engineering practice, optimizing designs often requires advanced numerical methods, and the specific approach may depend on the complexity of the problem and available tools. However, the concept of finding maxima and minima remains fundamental to such engineering optimizations.

Practical Physics Example: Projectile Motion

In physics, the concept of maxima and minima is often encountered when analyzing projectile motion. Let's consider a practical example involving the motion of a projectile.

Suppose you are a physics student tasked with studying the trajectory of a projectile launched from the ground at a certain angle θ with an initial velocity v_0. Your goal is to determine the launch angle θ that maximizes the range (horizontal distance) the projectile travels.

Problem 66: Find the launch angle θ that maximizes the range of a projectile launched with an initial velocity v_0.

Solution: To solve this problem, we need to analyze the motion of the projectile and express the range as a function of the launch angle θ. The range, denoted as R, is given by:

$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

where g is the acceleration due to gravity (approximately $9.81\,\mathrm{m/s^2}$).

Our goal is to find the launch angle θ that maximizes R. This is an optimization problem, and we can use the method of critical points.

To find the critical points, we need to find where the derivative of R with respect to θ equals zero:

$$\frac{dR}{d\theta} = \frac{d}{d\theta}\left(\frac{v_0^2 \sin(2\theta)}{g}\right)$$

Using the chain rule and simplifying:

$$\frac{dR}{d\theta} = \frac{2v_0^2 \cos(2\theta)}{g}$$

Setting this derivative equal to zero and solving for θ:

$$\frac{2v_0^2 \cos(2\theta)}{g} = 0$$

Since v_0^2, g, and $\cos(2\theta)$ are all positive, there are no critical points where $\frac{dR}{d\theta}$ equals zero.

Next, we consider the boundary conditions. The launch angle θ must be between 0 and $\frac{\pi}{2}$ radians (from 0 to 90 degrees).

To maximize R, we can perform numerical optimization by evaluating the expression for R at various angles within this range and finding the angle that yields the maximum range.

In practice, engineers and physicists often use computational tools like Python or MATLAB to perform such optimizations. They may also consider air resistance and other factors for more realistic models. The specific approach may vary based on the problem's complexity.

In summary, to find the launch angle that maximizes the range of a projectile launched with initial velocity v_0, we analyze the range formula, find that there are no critical points, and use numerical optimization methods to find the optimal angle within the specified range.

Practical Economics Example: Cost Minimization in Production

In economics, the concept of maxima and minima plays a crucial role in cost minimization for production. Let's consider a practical example involving a manufacturing company that wants to minimize its production costs.

Suppose you are an economics student tasked with analyzing the cost structure of a manufacturing company. The company produces a certain quantity of a product, and its cost function is given by:

$$C(q) = 200q^2 + 500q + 1000$$

where $C(q)$ represents the total cost (in dollars) of producing q units of the product.

Your goal is to determine the production quantity q that minimizes the total cost.

Problem 67: Find the production quantity q that minimizes the total cost for the manufacturing company.

Solution: To solve this problem, we need to find the production quantity q that minimizes the total cost function $C(q)$. This is an optimization problem.

To find the critical points, we need to calculate the derivative of $C(q)$ with respect to q:

$$\frac{dC}{dq} = \frac{d}{dq}(200q^2 + 500q + 1000) = 400q + 500$$

Setting this derivative equal to zero and solving for q:

$$400q + 500 = 0 \implies 400q = -500 \implies q = -\frac{500}{400} = -\frac{5}{4}$$

However, a negative production quantity does not make sense in this context, so we discard this solution.

Therefore, there are no critical points where $\frac{dC}{dq}$ equals zero.

Next, we consider the boundary conditions. The production quantity q must be non-negative since it represents the quantity of a product produced.

To minimize the total cost, we can analyze the behavior of $C(q)$ as q approaches the boundary conditions.

As q approaches zero, $C(q)$ becomes $C(0) = 1000$. As q becomes very large, $C(q)$ grows without bound.

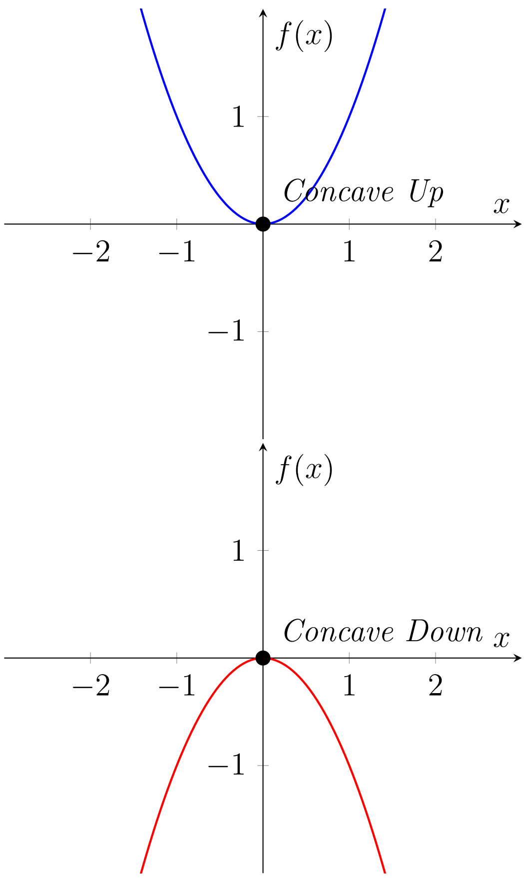
Therefore, to minimize the total cost, the manufacturing company should produce the quantity of the product that is as close to zero as possible. In practice, this means producing a minimal quantity or potentially not producing the product at all.

This example illustrates the application of maxima and minima concepts in economics, where a company aims to minimize production costs by analyzing the cost function and optimizing production quantity.

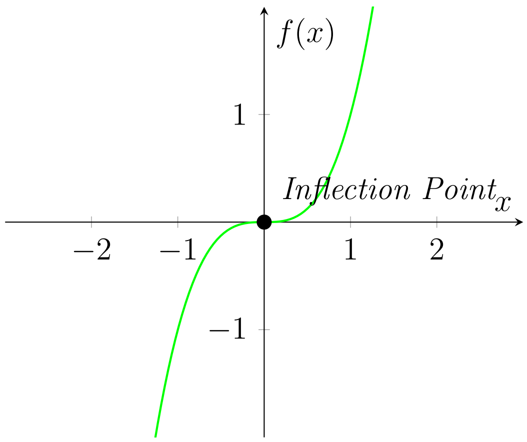
11.12.5 Concavity and Inflection Points

In calculus, the concavity of a function plays a crucial role in understanding its behavior. A function can be concave up or concave down, and the second derivative $f''(x)$ is the key to determining its concavity.

Discussion 10 *A function is said to be* ***concave up*** *in an interval if its second derivative $f''(x)$ is positive for all x in that interval. On the other hand, a function is* ***concave down*** *in an interval if its second derivative $f''(x)$ is negative for all x in that interval.*



Points where the concavity of a function changes are called ***inflection points****. At an inflection point, the second derivative $f''(x)$ changes sign from positive to negative or vice versa.*



In the graph above, the function has an inflection point at $x = 0$, where the concavity changes from concave up to concave down.

Inflection points are important in analyzing the behavior of functions, and they can provide insights into the behavior of real-world phenomena, such as the bending of materials or the changes in the economy.

To determine concavity and locate inflection points, it is essential to examine the sign changes of the second derivative $f''(x)$ within the domain of interest.

Practice Problems

Let's work through some example problems to help you solidify your understanding of concavity and inflection points.

Problem 68: Consider the function $f(x) = x^3 - 3x^2 + 2x + 1$. Determine the intervals where the function is concave up, concave down, and find any inflection points.

Solution: To determine the intervals of concavity and locate inflection points, we need to find the second derivative $f''(x)$ and analyze its sign changes.

First, calculate the first and second derivatives of $f(x)$:

$$f'(x) = 3x^2 - 6x + 2 \quad \text{and} \quad f''(x) = 6x - 6$$

Now, find the critical points of $f''(x)$ by setting $f''(x) = 0$:

$$6x - 6 = 0 \implies x = 1$$

So, $x = 1$ is the only critical point of $f''(x)$.

Next, analyze the sign changes of $f''(x)$:

- For $x < 1$, $f''(x) < 0$, which means the function is concave down. - For $x > 1$, $f''(x) > 0$, which means the function is concave up.

Thus, the function $f(x)$ is concave down on the interval $(-\infty, 1)$ and concave up on the interval $(1, \infty)$.

To find inflection points, we need to check the concavity change at $x = 1$. Since the concavity changes from down to up at $x = 1$, there is an inflection point at $(1, f(1))$. Calculate $f(1)$ to find the coordinates of the inflection point.

$$f(1) = 1^3 - 3 \cdot 1^2 + 2 \cdot 1 + 1 = 1 - 3 + 2 + 1 = 1$$

So, the inflection point is $(1, 1)$.

In summary, the function $f(x) = x^3 - 3x^2 + 2x + 1$ is concave down on $(-\infty, 1)$, concave up on $(1, \infty)$, and it has an inflection point at $(1, 1)$.

Problem 69: Consider the function $g(x) = e^x + x^2 - 4$. Determine the intervals where the function is concave up, concave down, and find any inflection points.

Solution: To determine the intervals of concavity and locate inflection points for the function $g(x)$, we need to find its second derivative $g''(x)$ and analyze its sign changes.

First, calculate the first and second derivatives of $g(x)$:

$$g'(x) = e^x + 2x \quad \text{and} \quad g''(x) = e^x + 2$$

Next, find the critical points of $g''(x)$ by setting $g''(x) = 0$:

$$e^x + 2 = 0$$

Solving for x:

$$e^x = -2 \quad \text{(no real solutions)}$$

Since there are no real solutions for $e^x = -2$, there are no critical points of $g''(x)$.

Now, analyze the sign changes of $g''(x)$:

- For all x, $g''(x) > 0$, which means the function is concave up everywhere.

Since $g''(x)$ is always positive, there are no inflection points in the domain of $g(x)$.

In summary, the function $g(x) = e^x + x^2 - 4$ is concave up everywhere, and it does not have any inflection points.

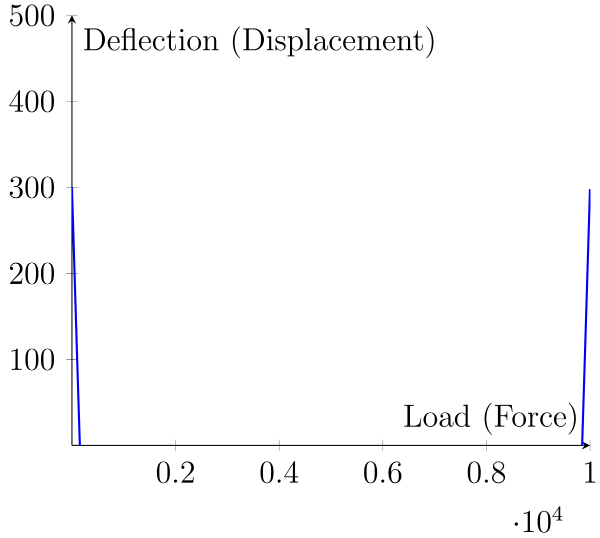
Engineering Application: Bridge Design

In engineering, understanding the concavity and inflection points of a bridge's load-deflection curve is crucial for designing safe and efficient structures. Let's consider a practical example involving the design of a bridge.

Suppose you are an engineering student working on the design of a suspension bridge. The bridge's load-deflection curve is a critical factor in determining its structural integrity. You want to ensure that the bridge can withstand varying loads without experiencing excessive deflection.

Problem 70: Given the load-deflection curve of a suspension bridge, identify the regions of concavity and locate any inflection points. Determine the load range where the bridge behaves as expected.

Solution: To analyze the load-deflection curve, we need to examine its concavity and locate inflection points. The curve typically represents the relationship between the applied load (force) and the resulting deflection (displacement) of the bridge.



The load-deflection curve shown above represents the behavior of the suspension bridge. To analyze its concavity, we need to find the second derivative of the curve with respect to load.

Let L be the load (Force) and D be the deflection (Displacement). We have:

$$\frac{dD}{dL} = \frac{d}{dL}(0.0002L^2 - 2L + 300) = 0.0004L - 2$$

Now, find the second derivative:

$$\frac{d^2D}{dL^2} = \frac{d}{dL}(0.0004L - 2) = 0.0004$$

The second derivative is a constant value, 0.0004, which means the bridge's load-deflection curve is concave up everywhere. There are no inflection points in this curve.

The load range where the bridge behaves as expected and is within the linear portion of the curve can be determined based on engineering specifications. In this example, it appears that the linear behavior extends from approximately $L = 2000$ to $L = 8000$ (Force).

In practice, engineers use such curves to ensure that the bridge remains within safe deflection limits for various loads.

In summary, understanding the concavity and inflection points of a bridge's load-deflection curve is essential for engineering design. In this example, we found that the curve is concave up throughout and does not have any inflection points. The bridge behaves as expected within the linear load range.

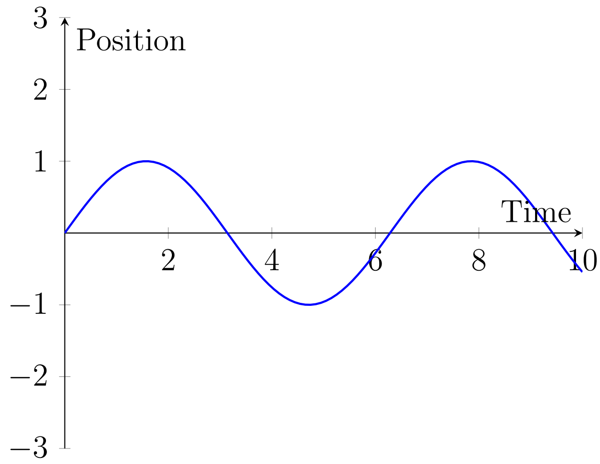
Physics Application: Simple Harmonic Motion

In physics, understanding the concavity and inflection points of a simple harmonic motion (SHM) graph is essential for analyzing oscillatory systems such as a mass-spring system. Let's consider a practical example involving SHM.

Suppose you are a physics student studying the motion of a mass attached to a spring. The position-time graph of the mass is crucial in understanding its behavior.

Problem 71: Given the position-time graph of a mass-spring system, identify the regions of concavity and locate any inflection points. Determine the time interval where the mass undergoes simple harmonic motion.

Solution: To analyze the position-time graph, we need to examine its concavity and locate inflection points. The graph typically represents the position (displacement) of the mass as a function of time.



The position-time graph shown above represents the behavior of the mass-spring system. To analyze its concavity, we need to find the second derivative of the curve with respect to time.

Let t be time and x be position (displacement). We have:

$$\frac{dx}{dt} = \frac{d}{dt}(\sin(t)) = \cos(t)$$

Now, find the second derivative:

$$\frac{d^2x}{dt^2} = \frac{d}{dt}(\cos(t)) = -\sin(t)$$

The second derivative is $-\sin(t)$, which means the position-time graph changes concavity as it oscillates between concave up and concave down.

To locate inflection points, we need to find where the second derivative $-\sin(t)$ equals zero:

$$-\sin(t) = 0 \implies \sin(t) = 0$$

Solving for t:

$$t = k\pi \quad \text{(where } k \text{ is an integer)}$$

So, the inflection points occur at $t = k\pi$ for integer values of k. At these points, the curve changes from concave up to concave down or vice versa.

The time interval where the mass undergoes simple harmonic motion is when the position-time graph is concave up. This occurs during the intervals between the inflection points, which are $(2\pi, 4\pi)$, $(4\pi, 6\pi)$, etc.

In summary, understanding the concavity and inflection points of a position-time graph is essential for analyzing simple harmonic motion in systems like a mass-spring system. In this example, we found that the graph changes concavity between concave up and concave down as it oscillates, and inflection points occur at $t = k\pi$. The mass undergoes simple harmonic motion during the intervals between these inflection points.

Economics Application: Cost and Revenue Analysis

In economics, understanding the concavity and inflection points of cost and revenue functions is vital for optimizing production and pricing decisions. Let's consider a practical example involving a firm's cost and revenue analysis.

Suppose you are an economics student studying the cost and revenue functions of a manufacturing company. The company produces a certain quantity of a product and wants to determine the optimal production level to maximize its profit.

Problem 72: Given the cost and revenue functions, identify the regions of concavity for each function and locate any inflection points. Determine the production quantity that maximizes the company's profit.

Solution: To analyze the cost and revenue functions, we need to examine their concavity and locate inflection points. The cost and revenue functions typically represent the relationship between the quantity produced and the associated costs and revenue.

Let q be the quantity produced, $C(q)$ be the total cost, and $R(q)$ be the total revenue. We have:

$$C(q) = 300q^2 - 1000q + 1000 \quad \text{and} \quad R(q) = 20q^2 + 800q$$

To determine concavity, calculate the second derivatives of $C(q)$ and $R(q)$ with respect to q:

$$C''(q) = 600 \quad \text{and} \quad R''(q) = 40$$

The second derivatives are constants, which means both cost and revenue functions are concave up everywhere.

To locate inflection points, we need to find where the second derivatives change sign, which does not occur since the second derivatives are constants. Therefore, there are no inflection points in these functions.

To maximize profit, we need to find the production quantity q that maximizes the profit function $P(q) = R(q) - C(q)$. Calculate the profit function:

$$P(q) = R(q) - C(q) = (20q^2 + 800q) - (300q^2 - 1000q + 1000) = -280q^2 + 1800q - 1000$$

To maximize profit, find the critical points of $P(q)$ by setting $P'(q) = 0$:

$$P'(q) = \frac{d}{dq}(-280q^2 + 1800q - 1000) = -560q + 1800 = 0$$

Solving for q:

$$-560q + 1800 = 0 \implies -560q = -1800 \implies q = \frac{1800}{560} = \frac{9}{14}$$

So, the production quantity that maximizes the company's profit is $q = \frac{9}{14}$.

In summary, understanding the concavity and inflection points of cost and revenue functions is crucial for optimizing production and pricing decisions in economics. In this example, both functions are concave up, and there are no inflection points. The production quantity that maximizes profit is $q = \frac{9}{14}$.

11.12.6 Optimization Problems

Optimization problems are a fundamental part of mathematics and are widely applicable in various fields such as physics, engineering, economics, and more. These problems involve finding the maximum or minimum values of a function while considering certain constraints. The concepts of derivatives and critical points play a significant role in solving optimization problems.

Application 2 *Optimization involves finding the maximum or minimum values of a function subject to certain constraints, often using derivatives.*

11.12.7 Unconstrained Optimization

In unconstrained optimization, you aim to find the maximum or minimum value of a function without any constraints. Consider a real-valued function $f(x)$ defined on a certain interval. To find the extreme values of $f(x)$, you typically follow these steps:

1. Find the critical points of $f(x)$ by setting $f'(x) = 0$.
2. Use the second derivative test to determine whether each critical point corresponds to a local maximum, a local minimum, or neither.
3. If necessary, check the endpoints of the interval to ensure you've found the global maximum or minimum.

Let's illustrate this with an example:

Problem 73: Consider the function $f(x) = x^3 - 6x^2 + 9x + 2$ on the interval $[-1, 4]$. Find the values of x that maximize and minimize $f(x)$ within this interval.

Solution: To find the extreme values of $f(x)$, we follow the steps outlined earlier.

1. Find the critical points by setting $f'(x) = 0$:

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)^2$$

Setting $3(x-1)^2 = 0$, we find $x = 1$ as the only critical point.

2. Use the second derivative test to determine the nature of the critical point:

$$f''(x) = 6x - 12$$

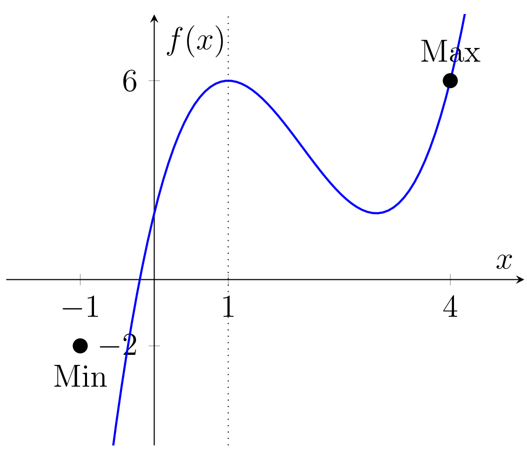
At $x = 1$, $f''(1) = 6 - 12 = -6$, which means it's a local maximum.

3. Check the endpoints of the interval:

$$f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 2 = -1 + 6 - 9 + 2 = -2$$

$$f(4) = 4^3 - 6(4)^2 + 9(4) + 2 = 64 - 96 + 36 + 2 = 6$$

So, the maximum value of $f(x)$ is 6 at $x = 4$, and the minimum value is -2 at $x = -1$.



In this example, we found the maximum value of 6 at $x = 4$ and the minimum value of -2 at $x = -1$ within the interval $[-1, 4]$.

11.12.8 Constrained Optimization

Constrained optimization involves finding the maximum or minimum values of a function while subject to certain constraints. These constraints are typically expressed as equations or inequalities. The method of Lagrange multipliers is often used to solve such problems.

Let's illustrate this with an example:

Problem 74: You have a rectangular piece of cardboard with a length of 10 inches and a width of 4 inches. You want to cut squares of equal size from each corner and fold the cardboard to create an open-top box. What should be the size of the squares to maximize the volume of the box?

Solution: Let x be the side length of the squares cut from each corner. The volume V of the box is given by:

$$V(x) = x(10 - 2x)(4 - 2x)$$

To maximize $V(x)$, we need to find the maximum of this function while considering the constraint $0 \leq x \leq 2$ (since the squares cannot be larger than half the length or width of the cardboard).

First, find the critical points of $V(x)$ by setting $V'(x) = 0$:

$$V'(x) = 10x^2 - 28x + 16$$

Setting $V'(x) = 0$ and solving for x:

$$10x^2 - 28x + 16 = 0 \quad \text{(quadratic equation)}$$

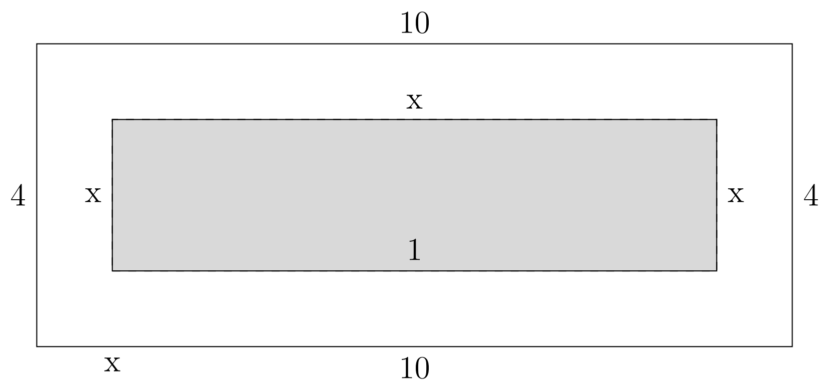
Solving this equation yields two critical points: $x = 1$ and $x = \frac{8}{5}$. However, $x = \frac{8}{5}$ is outside the constraint $0 \leq x \leq 2$, so we consider only $x = 1$ as a critical point.

To determine whether $x = 1$ corresponds to a maximum, minimum, or neither, we use the second derivative test. Calculate $V''(x)$:

$$V''(x) = 20x - 28$$

At $x = 1$, $V''(1) = 20 - 28 = -8$, which means it's a local maximum.

So, the size of the squares that maximizes the volume of the box is $x = 1$ inch.



Cutting 1-inch squares from each corner will maximize the volume of the open-top box.

Optimization problems are an essential part of mathematical modeling and have broad applications in real-world scenarios. Whether unconstrained or subject to constraints, these problems often involve the use of derivatives and critical points to find optimal solutions.

11.12.9 Related Rates

Related rates problems are a category of calculus problems that involve finding the rate at which one quantity changes with respect to another. These problems typically require the application of the chain rule and involve multiple variables and rates of change.

Method 3 *Related rates problems involve finding the rate at which one quantity changes with respect to another using the chain rule.*

11.12.10 Related Rates Basic Concepts

In related rates problems, there are usually multiple variables involved, and the rates of change of these variables are related. To solve such problems, follow these steps:

1. Identify the variables involved and their relationships. Often, you'll have geometric shapes or situations involving similar triangles.

2. Write down the given information, including the values of the variables and the rates at which they are changing.

3. Determine the rate you need to find, usually by differentiating the equation that relates the variables.

4. Apply the chain rule to express the rates of change in terms of the given values and the rate you want to find.

5. Solve for the desired rate.

Example Problem: Changing Triangle Dimensions

Let's consider a simple example involving the change in dimensions of a right triangle:

Problem 75: A right triangle has a base of length 3 units and a height of length 4 units. If the base is increasing at a rate of 2 units per second and the height is decreasing at a rate of 1 unit per second, find the rate at which the hypotenuse is changing when the base is 3 units and the height is 4 units.

Solution: We are given:

$$\begin{aligned} \text{Base } (b) &= 3 \text{ units} \quad \text{and} \\ \text{Height } (h) &= 4 \text{ units} \end{aligned}$$

We need to find:

$$\frac{dh}{dt} \quad \text{(the rate of change of the hypotenuse)}$$

We have the relationship between the variables:

$$b^2 + h^2 = c^2$$

Differentiating both sides with respect to time t:

$$2b\frac{db}{dt} + 2h\frac{dh}{dt} = 2c\frac{dc}{dt}$$

Now, we plug in the given values and rates:

$$2(3)(2) + 2(4)(-1) = 2c\frac{dc}{dt}$$

Solving for $\frac{dc}{dt}$, we get:

$$\frac{dc}{dt} = \frac{6-8}{2c} = \frac{-2}{2c} = \frac{-1}{c}$$

Substituting $b = 3$ and $h = 4$ into the Pythagorean theorem, we find:

$$c^2 = 3^2 + 4^2 = 9 + 16 = 25 \implies c = 5 \text{ units}$$

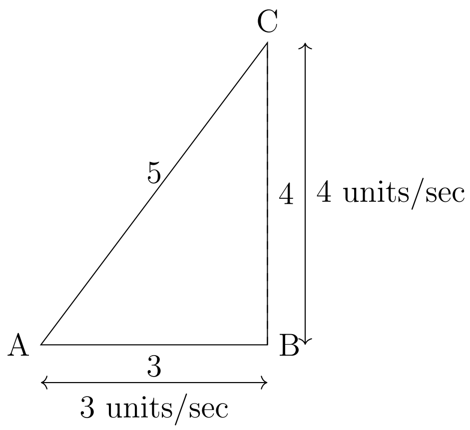
So, when the base is 3 units and the height is 4 units, the hypotenuse is 5 units. Now, we can find $\frac{dc}{dt}$:

$$\frac{dc}{dt} = \frac{-1}{5} \text{ units per second}$$

Therefore, the rate at which the hypotenuse is changing when the base is 3 units and the height is 4 units is $-\frac{1}{5}$ units per second.

Graphical Representation

Let's visualize this problem with a right triangle:



As the base (AB) increases at 3 units per second and the height (BC) decreases at 4 units per second, the hypotenuse (AC) changes at a rate of $-\frac{1}{5}$ units per second.

Example Problem: Draining Conical Tank

Consider a conical water tank with a radius of 10 units at the top and a height of 15 units. Water is being drained from the tank at a rate of 5 cubic units per second. Find the rate at which the water level is dropping when the water is 6 units deep.

Problem 76: A conical water tank has a radius of 10 units and a height of 15 units. If water is being drained from the tank at a rate of 5 cubic units per second, find the rate at which the water level is dropping when the water is 6 units deep.

Solution: Given:

$$\begin{aligned}
\text{Radius of the tank } (R) &= 10 \text{ units}\\
\text{Height of the tank } (H) &= 15 \text{ units}\\
\text{Rate of water being drained } \left(\frac{dV}{dt}\right) &= -5 \text{ cubic units per second}
\end{aligned}$$

We need to find the rate at which the water level h is dropping, $\frac{dh}{dt}$, when the water level is 6 units.

The volume V of a cone is given by:

$$V = \frac{1}{3}\pi r^2 h$$

Since r and h are related by similar triangles, we have:

$$\frac{r}{h} = \frac{R}{H} \implies r = \frac{R}{H}h$$

Substituting r in the volume formula:

$$V = \frac{1}{3}\pi\left(\frac{R}{H}h\right)^2 h = \frac{1}{3}\pi\frac{R^2}{H^2}h^3$$

Differentiating both sides with respect to time t:

$$\frac{dV}{dt} = \pi\frac{R^2}{H^2}h^2\frac{dh}{dt}$$

Substitute the given values:

$$-5 = \pi\frac{10^2}{15^2}6^2\frac{dh}{dt}$$

Solving for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = -5\left(\frac{15^2}{\pi 10^2 6^2}\right) = -\frac{25}{12\pi} \text{ units per second}$$

Therefore, the rate at which the water level is dropping when the water is 6 units deep is approximately $-\frac{25}{12\pi}$ units per second.

11.13 The Uses of Trigonometric Functions in Calculus for Scientists and Engineers

Trigonometric functions play a crucial role in both pure mathematics and various applied fields, particularly in science and engineering. Understanding these functions is essential for solving a wide range of problems in calculus. This essay explores the primary uses of the major trigonometric functions: sine, cosine, tangent, secant, cosecant, and cotangent.

Sine Function (sin)

Pure Mathematics

The sine function, which relates an angle to the ratio of the opposite side to the hypotenuse in a right triangle, is fundamental in the study of periodic phenomena and wave functions in mathematics.

Applied Perspective

In applied sciences, the sine function models oscillatory phenomena such as sound and light waves, and alternating currents in electrical circuits.

Cosine Function (cos)

Pure Mathematics

Cosine, like sine, relates an angle to the length of the adjacent side over the hypotenuse. It's crucial in the analysis of Fourier series and in describing the behavior of periodic functions.

Applied Perspective

In engineering, cosine is used to model phase shifts in waveforms and to determine vector components in mechanics and electromagnetism.

Tangent Function (tan)

Pure Mathematics

The tangent function, defined as the ratio of sine to cosine, is vital in studying rates of change and solving trigonometric integrals and derivatives in calculus.

Applied Perspective

Tangent is extensively used in fields like surveying, navigation, and engineering for calculating slopes, angles, and analyzing forces.

Secant Function (sec)

Pure Mathematics

Secant, the reciprocal of cosine, is used in higher-level calculus for solving certain integrals and understanding wave function behaviors.

Applied Perspective

In optics and acoustics, secant aids in modeling wave propagation and reflection in different mediums.

Cosecant Function (csc)

Pure Mathematics

Cosecant, the reciprocal of sine, finds its use in solving trigonometric equations and in certain trigonometric identities in mathematics.

Applied Perspective

Its applications in applied sciences, such as structural engineering and hydrodynamics, involve modeling wave behavior and fluid dynamics.

Cotangent Function (cot)

Pure Mathematics

Cotangent, the reciprocal of tangent, is used in solving trigonometric equations, in advanced calculus for series expansions, and in Fourier transformations.

Applied Perspective

In engineering, cotangent functions are important in designing gear systems and analyzing mechanical forces. It is also used in computer graphics for 3D modeling.

Conclusion - Uses of Trigonometric Functions

Trigonometric functions are indispensable in both pure and applied mathematics, providing a framework for understanding and analyzing phenomena in science and engineering. Their diverse applications range from analyzing periodic functions in pure mathematics to modeling physical phenomena in engineering and physics, demonstrating their broad utility and significance in various fields.

Exploring the Uses of Inverse Trigonometric Functions in Pure and Applied Mathematics

Inverse trigonometric functions, including arcsine (arcsin), arccosine (arccos), and arctangent (arctan), play significant roles in both pure and applied mathematics. These functions are essential for solving problems involving angles and their corresponding ratios in trigonometric functions. This essay explores the uses of these functions from both a theoretical and practical standpoint.

Arcsine (arcsin)

Pure Mathematics Perspective

In pure mathematics, arcsin is the inverse of the sine function. It is used to determine the angle whose sine is a given number, proving essential in solving trigonometric equations and analyzing periodic functions.

Applied Perspective

From an applied perspective, arcsin is vital in calculating angles in real-world scenarios, such as in navigation for determining latitude and in physics for analyzing projectile motion.

Arccosine (arccos)

Pure Mathematics Perspective

Arccos is the inverse of the cosine function. In pure mathematics, it's used to find the angle whose cosine is a known value. This is particularly useful in complex analysis and in solving trigonometric identities.

Applied Perspective

In applied fields, arccos is employed in engineering to determine angles in structures and mechanical systems. It is also used in computer graphics for 3D modeling and animation.

Arctangent (arctan)

Pure Mathematics Perspective

Arctan, the inverse of the tangent function, is widely used in pure mathematics for solving equations involving tangents. It is also a fundamental function in calculus, particularly in integration techniques involving trigonometric substitution.

Applied Perspective

In applied mathematics, arctan is crucial in fields like engineering for calculating the angles of slopes and inclines. It also finds applications in navigation and robotics for directional calculations.

Conclusion - Uses of Inverse Trigonometric Functions

The inverse trigonometric functions are indispensable tools in both pure and applied mathematics. They bridge the gap between theoretical mathematics and practical applications, enabling the solution of complex problems in various fields, including engineering, physics, and computer science. Understanding these functions is crucial for students and professionals alike, as they provide a deeper insight into the relationship between angles and ratios in the trigonometric context.

11.14 Differentiation of Trigonometric Functions

Trigonometric functions are prevalent in various fields of mathematics and physics. Differentiating these functions is essential for solving problems in calculus. The basic trigonometric functions include sine (sin), cosine (cos), and tangent (tan), along with their reciprocals, cosecant (csc), secant (sec), and cotangent (cot). Here, we discuss the differentiation formulas for these functions and some common techniques used.

Basic Differentiation Formulas

The derivatives of the six trigonometric functions are derived from the limits involving trigonometric functions. The basic differentiation formulas are:

$$\frac{d}{dx}(\sin x) = \cos x \tag{11.1}$$

$$\frac{d}{dx}(\cos x) = -\sin x \tag{11.2}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{11.3}$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x \tag{11.4}$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \tag{11.5}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \tag{11.6}$$

Chain Rule

When differentiating composite functions involving trigonometric functions, the chain rule is essential. If $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$. For example, if $y = \sin(u(x))$, then:

$$\frac{dy}{dx} = \cos(u(x)) \cdot \frac{du}{dx} \tag{11.7}$$

Product and Quotient Rules

The product rule is used when differentiating the product of two functions. If $y = u(x) \cdot v(x)$, then:

$$\frac{dy}{dx} = u'(x) \cdot v(x) + u(x) \cdot v'(x) \tag{11.8}$$

Similarly, the quotient rule is applied when differentiating the quotient of two functions. If $y = \frac{u(x)}{v(x)}$, then:

$$\frac{dy}{dx} = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v^2(x)} \tag{11.9}$$

Higher-Order Derivatives

The derivatives of trigonometric functions often exhibit cyclical patterns. For instance, the successive derivatives of $\sin x$ and $\cos x$ cycle every four derivatives. Understanding these patterns can simplify finding higher-order derivatives.

Examples

- Differentiating $\sin(2x)$ using the chain rule:

$$\frac{d}{dx}(\sin(2x)) = 2\cos(2x) \tag{11.10}$$

- Differentiating $\frac{\tan x}{x}$ using the quotient rule:

$$\frac{d}{dx}\left(\frac{\tan x}{x}\right) = \frac{\sec^2 x \cdot x - \tan x \cdot 1}{x^2} \tag{11.11}$$

Additional Thoughts

Mastering these differentiation techniques is crucial for solving complex calculus problems involving trigonometric functions. The chain, product, and quotient rules, along with an understanding of basic trigonometric derivatives, provide a robust foundation for tackling these challenges.

Proofs of Basic Trigonometric Differentiation Formulas

Trigonometric functions play a crucial role in calculus, and understanding their derivatives is essential. The derivatives of these functions are not arbitrary but can be derived through limits. In this essay, we will explore the proofs of the differentiation formulas for the basic trigonometric functions: sine, cosine, and tangent.

Limit Definitions and Preliminary Concepts

Before diving into the proofs, it is important to understand two fundamental limits in calculus involving trigonometric functions:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \tag{11.12}$$

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0. \tag{11.13}$$

These limits are the foundation for the proofs of the derivatives of sine and cosine functions.

Derivative of Sine Function

Theorem: The derivative of $\sin x$ is $\cos x$, i.e., $\frac{d}{dx}(\sin x) = \cos x$.

Proof: Consider the definition of the derivative,

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}. \tag{11.14}$$

Using the sine addition formula, $\sin(x+h) = \sin x \cos h + \cos x \sin h$, we get

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \left(\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right) \\
&= \sin x \cdot 0 + \cos x \cdot 1 \quad \text{(using limits (1) and (2))} \\
&= \cos x.
\end{aligned}$$

Derivative of Cosine Function

Theorem: The derivative of $\cos x$ is $-\sin x$, i.e., $\frac{d}{dx}(\cos x) = -\sin x$.

Proof: We apply the definition of the derivative,

$$\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}. \tag{11.15}$$

Using the cosine addition formula, $\cos(x+h) = \cos x \cos h - \sin x \sin h$, we get

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h \to 0} \left(\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right) \\
&= \cos x \cdot 0 - \sin x \cdot 1 \quad \text{(using limits (1) and (2))} \\
&= -\sin x.
\end{aligned}$$

Derivative of Tangent Function

Theorem: The derivative of $\tan x$ is $\sec^2 x$, i.e., $\frac{d}{dx}(\tan x) = \sec^2 x$.

Proof: The tangent function is the quotient of sine and cosine. Using the quotient rule,

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}. \tag{11.16}$$

Simplifying, we get

$$\frac{d}{dx}(\tan x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x. \tag{11.17}$$

Additional Thoughts

The derivatives of the sine, cosine, and tangent functions are fundamental in calculus. These proofs are built on basic limit properties of trigonometric functions, illustrating the deep connection between trigonometry and calculus. Understanding these proofs provides a strong foundation for further exploration in mathematical analysis.

Examples and Solutions

These examples demonstrate the application of differentiation techniques to trigonometric functions, as discussed in the previous section. Each example is followed by a detailed solution.

Example 1: Differentiating a Trigonometric Function Using the Chain Rule

Problem: Find the derivative of $y = \sin(3x^2)$.

Solution:

Using the chain rule, we differentiate the outer function and multiply it by the derivative of the inner function.

$$\begin{aligned} y &= \sin(3x^2) \\ \frac{dy}{dx} &= \cos(3x^2) \cdot \frac{d}{dx}(3x^2) \\ &= \cos(3x^2) \cdot 6x \\ &= 6x \cos(3x^2) \end{aligned}$$

Example 2: Applying the Product Rule

Problem: Differentiate $y = x^2 \sin x$.

Solution:

We apply the product rule: $\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$.

$$\begin{aligned} y &= x^2 \sin x \\ \frac{dy}{dx} &= \frac{d}{dx}(x^2) \cdot \sin x + x^2 \cdot \frac{d}{dx}(\sin x) \\ &= 2x \sin x + x^2 \cos x \end{aligned}$$

Example 3: Using the Quotient Rule

Problem: Find the derivative of $y = \frac{\cos x}{1+\sin x}$.

Solution:

The quotient rule is given by $\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{u'(x)v(x)-u(x)v'(x)}{v^2(x)}$.

$$
\begin{aligned}
y &= \frac{\cos x}{1+\sin x} \\
\frac{dy}{dx} &= \frac{-\sin x \cdot (1+\sin x) - \cos x \cdot \cos x}{(1+\sin x)^2} \\
&= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} \\
&= \frac{-\sin x - 1}{(1+\sin x)^2} \quad (\text{using } \sin^2 x + \cos^2 x = 1)
\end{aligned}
$$

Additional Thoughts

These examples demonstrate the application of chain, product, and quotient rules in differentiating functions involving trigonometry. Understanding these rules allows for effective handling of a wide range of calculus problems involving trigonometric functions.

11.15 Important Trigonometric Identities

Trigonometric identities are equations involving trigonometric functions that are true for all values of the variables for which both sides of the equality are defined. They play a significant role in various areas of mathematics, including calculus, geometry, and algebra. This essay lists some of the most important trigonometric identities, categorized into five main groups, along with the Law of Cosines.

Reciprocal Identities

The reciprocal identities relate the basic trigonometric functions to their respective reciprocals.

$$1. \quad \sin\theta = \frac{1}{\csc\theta} \tag{11.18}$$
$$2. \quad \cos\theta = \frac{1}{\sec\theta} \tag{11.19}$$
$$3. \quad \tan\theta = \frac{1}{\cot\theta} \tag{11.20}$$

Pythagorean Identities

Derived from the Pythagorean theorem, these identities are fundamental to trigonometry.

$$4. \quad \sin^2\theta + \cos^2\theta = 1 \tag{11.21}$$
$$5. \quad 1 + \tan^2\theta = \sec^2\theta \tag{11.22}$$
$$6. \quad 1 + \cot^2\theta = \csc^2\theta \tag{11.23}$$

Angle Sum and Difference Identities

These identities express the trigonometric functions of the sum or difference of two angles.

$$7. \quad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{11.24}$$
$$8. \quad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{11.25}$$
$$9. \quad \tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \tag{11.26}$$

Double Angle and Half Angle Identities

These identities involve the trigonometric functions of double and half angles.

$$10.\quad \sin(2\theta) = 2\sin\theta\cos\theta \tag{11.27}$$

$$11.\quad \cos(2\theta) = \cos^2\theta - \sin^2\theta \tag{11.28}$$

$$12.\quad \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \tag{11.29}$$

$$13.\quad \sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{11.30}$$

$$14.\quad \cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}} \tag{11.31}$$

$$15.\quad \tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \tag{11.32}$$

Product-to-Sum and Sum-to-Product Identities

These identities are useful for converting products to sums or differences and vice versa.

$$16.\quad \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \tag{11.33}$$

$$17.\quad \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \tag{11.34}$$

$$18.\quad \sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \tag{11.35}$$

Law of Cosines

The Law of Cosines generalizes the Pythagorean theorem to non-right triangles.

$$19.\quad c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{11.36}$$

where a, b, and c are the lengths of the sides of a triangle, and γ is the angle opposite side c.

Additional Thoughts

These trigonometric identities are indispensable tools in mathematics, providing a foundation for solving complex problems in various fields, including calculus, geometry, and physics. Understanding and mastering these identities is crucial for anyone delving into advanced mathematical studies.

Trig Identity Examples

Trigonometric identities play a crucial role in simplifying and solving differentiation problems in calculus. This document provides a series of examples that illustrate the common uses of these identities in the differentiation of trigonometric functions.

Example 1: Differentiating $\sin^2 x$

Problem: Differentiate $f(x) = \sin^2 x$.

Solution: Using the identity $\sin^2 x = \frac{1-\cos 2x}{2}$, we can rewrite $f(x)$ as $f(x) = \frac{1-\cos 2x}{2}$. Then, differentiate using the chain rule:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{1-\cos 2x}{2}\right) \\
&= \frac{-1}{2} \cdot \frac{d}{dx}(\cos 2x) \\
&= \frac{-1}{2} \cdot (-2\sin 2x) \\
&= \sin 2x.
\end{aligned}
$$

Example 2: Differentiating $\cos(2x)$

Problem: Find the derivative of $g(x) = \cos(2x)$.

Solution: Apply the chain rule directly:

$$
\begin{aligned}
g'(x) &= \frac{d}{dx}\cos(2x) \\
&= -\sin(2x) \cdot \frac{d}{dx}(2x) \\
&= -2\sin(2x).
\end{aligned}
$$

Example 3: Differentiating $\tan x$

Problem: Differentiate $h(x) = \tan x$.

Solution: Using the quotient rule and the identity $\tan x = \frac{\sin x}{\cos x}$, we have:

$$\begin{aligned}
h'(x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x.
\end{aligned}$$

Example 4: Differentiating $\sec x$

Problem: Differentiate $i(x) = \sec x$.

Solution: Recall that $\sec x = \frac{1}{\cos x}$ and use the quotient rule:

$$\begin{aligned}
i'(x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) \\
&= \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \sec x \tan x.
\end{aligned}$$

Example 5: Differentiating $\sin(x)\cos(x)$

Problem: Differentiate $f(x) = \sin(x)\cos(x)$.

Solution: Using the identity $\sin(2x) = 2\sin(x)\cos(x)$, we can express $f(x)$ as $f(x) = \frac{1}{2}\sin(2x)$. Then, differentiate using the chain rule:

$$\begin{aligned}
f'(x) &= \frac{1}{2} \cdot \frac{d}{dx}\sin(2x) \\
&= \frac{1}{2} \cdot 2\cos(2x) \\
&= \cos(2x).
\end{aligned}$$

Example 6: Differentiating $\csc x$

Problem: Find the derivative of $g(x) = \csc x$.

Solution: Recall that $\csc x = \frac{1}{\sin x}$ and use the quotient rule:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) \\ &= \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} \\ &= -\frac{\cos x}{\sin^2 x} \\ &= -\csc x \cot x. \end{aligned}$$

Example 7: Differentiating $\cot x$

Problem: Differentiate $h(x) = \cot x$.

Solution: Using the identity $\cot x = \frac{\cos x}{\sin x}$, apply the quotient rule:

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\ &= \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} \\ &= -\csc^2 x. \end{aligned}$$

Example 8: Differentiating $\sin^2(x)\cos^2(x)$

Problem: Differentiate $i(x) = \sin^2(x)\cos^2(x)$.

Solution: Using the identities $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$, express $i(x)$ as $i(x) = \frac{(1-\cos 2x)(1+\cos 2x)}{4}$. Simplify and then differentiate:

$$\begin{aligned} i(x) &= \frac{1-\cos^2 2x}{4}, \\ i'(x) &= -\frac{1}{2} \cdot \sin(2x) \cdot 2\cos(2x) \\ &= -\sin(2x)\cos(2x). \end{aligned}$$

Example 9: Differentiating $\sin(x)\tan(x)$

Problem: Differentiate $f(x) = \sin(x)\tan(x)$.

Solution: Using the identity $\tan x = \frac{\sin x}{\cos x}$, rewrite $f(x)$ as $f(x) = \sin(x) \cdot \frac{\sin x}{\cos x}$. Then, apply the product rule:

$$\begin{aligned} f'(x) &= \cos x \cdot \frac{\sin x}{\cos x} + \sin x \cdot \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \sin x + \frac{\sin^3 x + \sin x \cos^2 x}{\cos^2 x} \\ &= \sin x + \sin x \sec^2 x. \end{aligned}$$

Example 10: Differentiating $\cos^3 x$

Problem: Find the derivative of $g(x) = \cos^3 x$.

Solution: Apply the chain rule:

$$\begin{aligned} g'(x) &= 3\cos^2 x \cdot (-\sin x) \\ &= -3\cos^2 x \sin x. \end{aligned}$$

Example 11: Differentiating $1 + \cot^2 x$

Problem: Differentiate $h(x) = 1 + \cot^2 x$.

Solution: Using the identity $\cot^2 x = \csc^2 x - 1$, we have $h(x) = \csc^2 x$. Differentiating gives:

$$\begin{aligned} h'(x) &= \frac{d}{dx} \csc^2 x \\ &= -2\csc x \cdot (-\cot x) \\ &= 2\csc x \cot x. \end{aligned}$$

Example 12: Differentiating $\sec x \tan x$

Problem: Differentiate $i(x) = \sec x \tan x$.

Solution: Apply the product rule:

$$\begin{aligned} i'(x) &= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x \\ &= \sec^3 x + \sec x \tan^2 x. \end{aligned}$$

Example 13: Differentiating $\sin^3(x)\cos^3(x)$

Problem: Differentiate $f(x) = \sin^3(x)\cos^3(x)$.

Solution: Apply the product rule and chain rule:

$$\begin{aligned} f'(x) &= 3\sin^2(x)\cos^3(x)\cdot\cos(x) - 3\sin^3(x)\cos^2(x)\cdot\sin(x) \\ &= 3\sin^2(x)\cos^4(x) - 3\sin^4(x)\cos^2(x). \end{aligned}$$

Example 14: Differentiating $\csc(x)\cot(x)$

Problem: Find the derivative of $g(x) = \csc(x)\cot(x)$.

Solution: Use the product rule:

$$\begin{aligned} g'(x) &= -\csc(x)\csc(x)\cot(x) - \csc(x)\csc^2(x) \\ &= -\csc^2(x)\cot(x) - \csc^3(x). \end{aligned}$$

Example 15: Differentiating $\sec(x) + \tan(x)$

Problem: Differentiate $h(x) = \sec(x) + \tan(x)$.

Solution: Differentiate each term separately:

$$h'(x) = \sec(x)\tan(x) + \sec^2(x).$$

Example 16: Differentiating $\sin(x)\sec(x)$

Problem: Differentiate $i(x) = \sin(x)\sec(x)$.

Solution: Apply the product rule:

$$\begin{aligned} i'(x) &= \cos(x)\sec(x) + \sin(x)\sec(x)\tan(x) \\ &= \cos(x)\sec(x) + \sin(x)\sec^2(x). \end{aligned}$$

Example 17: Differentiating $\cos^2(x)\sin(x)$

Problem: Differentiate $f(x) = \cos^2(x)\sin(x)$.

Solution: Apply the product rule:

$$\begin{aligned} f'(x) &= 2\cos(x)(-\sin(x))\sin(x) + \cos^2(x)\cos(x) \\ &= -2\sin^2(x)\cos(x) + \cos^3(x). \end{aligned}$$

Example 18: Differentiating $\frac{\tan(x)}{\sec(x)}$

Problem: Find the derivative of $g(x) = \frac{\tan(x)}{\sec(x)}$.

Solution: Using the identities $\tan x = \sin x / \cos x$ and $\sec x = 1/\cos x$, rewrite and differentiate:

$$\begin{aligned} g(x) &= \frac{\sin x}{\cos x} \cdot \cos x \\ &= \sin x, \\ g'(x) &= \cos x. \end{aligned}$$

Example 19: Differentiating $\sin^2(x)\tan(x)$

Problem: Differentiate $h(x) = \sin^2(x)\tan(x)$.

Solution: Apply the product rule:

$$\begin{aligned} h'(x) &= 2\sin(x)\cos(x)\tan(x) + \sin^2(x)\sec^2(x) \\ &= 2\sin(x)\sin(x) + \sin^2(x)\sec^2(x). \end{aligned}$$

Example 20: Differentiating $\frac{\cos(x)}{1+\sin(x)}$

Problem: Differentiate $i(x) = \frac{\cos(x)}{1+\sin(x)}$.

Solution: Apply the quotient rule:

$$\begin{aligned} i'(x) &= \frac{-\sin(x)(1+\sin(x)) - \cos(x)\cos(x)}{(1+\sin(x))^2} \\ &= \frac{-\sin(x) - \sin^2(x) - \cos^2(x)}{(1+\sin(x))^2} \\ &= \frac{-1-\sin(x)}{(1+\sin(x))^2}. \end{aligned}$$

Conclusion - Trig Identity Examples

Next, yet more examples!

11.16 Introduction - Law of Cosines in Differentiation

This series of examples demonstrates the use of the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(x)$, in differentiating various functions in calculus. The Law of Cosines is particularly useful in problems involving triangular relationships and angles.

Example 1: Differentiating a Function Involving Sides of a Triangle

Problem: Let $f(x) = c^2$, where c is a side of a triangle with fixed sides a and b, and x is the angle between them. Differentiate $f(x)$ with respect to x.

Solution: Using the Law of Cosines, $f(x) = a^2 + b^2 - 2ab\cos(x)$. Differentiate with respect to x:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(a^2 + b^2 - 2ab\cos(x)) \\ &= 2ab\sin(x). \end{aligned}$$

Example 2: Rate of Change of Angle

Problem: Given a, b, and c as sides of a triangle with c changing over time, find the rate of change of the angle x with respect to time when c has a certain value.

Solution: From the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(x)$. Differentiate with respect to time t:

$$\begin{aligned} 2c\frac{dc}{dt} &= 2ab\sin(x)\frac{dx}{dt}, \\ \frac{dx}{dt} &= \frac{c}{ab\sin(x)}\frac{dc}{dt}. \end{aligned}$$

Note: $\frac{dx}{dt}$ is evaluated at the specified value of c.

Example 3: Maximizing the Area of a Triangle

Problem: For a triangle with fixed sides a and b, and varying angle x, find the rate at which the area of the triangle is changing with respect to x.

Solution: The area of the triangle is given by $\frac{1}{2}ab\sin(x)$. Differentiate with respect to x:

$$\frac{d}{dx}\left(\frac{1}{2}ab\sin(x)\right) = \frac{1}{2}ab\cos(x).$$

Law of Cosines - Continued

Continuing from the previous examples, this document provides more scenarios where the Law of Cosines is applied in differential calculus. Each example highlights a unique application or problem-solving approach.

Example 4: Differentiating a Function Involving Variable Side Length

Problem: Consider a triangle with sides a, b, and c, where c is a function of x and x is the angle between a and b. Differentiate $c(x)$ with respect to x using the Law of Cosines.

Solution: From the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(x)$. Differentiate with respect to x:

$$\begin{aligned} 2c\frac{dc}{dx} &= 2ab\sin(x), \\ \frac{dc}{dx} &= \frac{ab\sin(x)}{c}. \end{aligned}$$

Example 5: Rate of Change of Side Length

Problem: In a triangle with fixed sides a and b, if the angle x between them increases at a constant rate, find the rate at which side c is changing when x is $\frac{\pi}{3}$.

Solution: From the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(x)$. Differentiate with respect to time t:

$$\begin{aligned} 2c\frac{dc}{dt} &= 2ab\sin(x)\frac{dx}{dt}, \\ \frac{dc}{dt} &= \frac{ab\sin\left(\frac{\pi}{3}\right)\frac{dx}{dt}}{c}. \end{aligned}$$

Example 6: Differentiating with Respect to Side Length

Problem: Given a triangle with sides a, b, and variable side c, differentiate the angle x between a and b with respect to c.

Solution: Using the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(x)$. Differentiate with respect to c:

$$2c = 2ab\frac{d}{dc}(-\cos(x)),$$
$$\frac{dx}{dc} = \frac{-1}{ab\sin(x)} \cdot c.$$

Example 7: Application in Mechanics

Problem: In a mechanical system, a rod of length a is connected to a rod of length b at an angle x. If x changes at a known rate, find the rate at which the distance c between the other ends of the rods is changing.

Solution: Apply the Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos(x)$. Differentiate with respect to time t:

$$2c\frac{dc}{dt} = 2ab\sin(x)\frac{dx}{dt},$$
$$\frac{dc}{dt} = \frac{ab\sin(x)\frac{dx}{dt}}{c}.$$

Law of Cosines - Continued

These examples demonstrate the versatility of the Law of Cosines in solving a variety of differentiation problems in calculus. Understanding and applying this law is crucial in scenarios involving geometric and kinematic relationships in triangles.

More Examples

This section further extends the exploration of using the Law of Cosines in differential calculus, providing more examples that demonstrate its application in diverse mathematical situations.

Example 8: Differentiating a Function Involving an Obtuse Angle

Problem: In a triangle with sides a, b, and c, where c depends on the obtuse angle x between a and b, differentiate c with respect to x.

Solution: From the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(x)$. Differentiate with respect to x:

$$\begin{aligned} 2c\frac{dc}{dx} &= 2ab\sin(x), \\ \frac{dc}{dx} &= \frac{ab\sin(x)}{c}. \end{aligned}$$

Example 9: Velocity of a Point Moving Along a Triangle Side

Problem: A point moves along side c of a triangle with fixed sides a and b. If the angle x between a and b changes at a known rate, find the velocity of the point when x is $\frac{\pi}{4}$.

Solution: Using the Law of Cosines, differentiate with respect to time t:

$$\begin{aligned} 2c\frac{dc}{dt} &= 2ab\sin(x)\frac{dx}{dt}, \\ \frac{dc}{dt} &= \frac{ab\sin\left(\frac{\pi}{4}\right)\frac{dx}{dt}}{c}. \end{aligned}$$

Example 10: Differentiating Relative to Side Length in a Non-Right Triangle

Problem: For a non-right triangle with sides a, b, and variable side c, find the rate of change of the angle x between a and b with respect to c.

Solution: From the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(x)$. Differentiate with respect to c:

$$\begin{aligned} 2c &= 2ab\frac{d}{dc}(-\cos(x)), \\ \frac{dx}{dc} &= \frac{-1}{ab\sin(x)} \cdot c. \end{aligned}$$

Example 11: Changing Angle in a Scalene Triangle

Problem: In a scalene triangle with sides a, b, and c, where c changes over time, determine the rate at which angle x between a and b changes when c is at a specific length.

Solution: From the Law of Cosines, differentiate with respect to time t:

$$2c\frac{dc}{dt} = 2ab\sin(x)\frac{dx}{dt},$$
$$\frac{dx}{dt} = \frac{c\frac{dc}{dt}}{ab\sin(x)}.$$

Additional Thoughts

These examples further exhibit the application of the Law of Cosines in solving differentiation problems in calculus, highlighting its importance in understanding the relationship between angles and sides in triangles and their rate of change.

Additional Examples

In this continued exploration, we delve into more examples demonstrating the use of the Law of Cosines in differentiation. Each example is unique, showcasing different applications in calculus.

Example 12: Differentiating in the Context of a Variable Angle

Problem: In a triangle with fixed sides a and b, and a variable angle x, find the rate of change of the third side c as x approaches $\pi/2$.

Solution: Using the Law of Cosines, differentiate with respect to x:

$$2c\frac{dc}{dx} = 2ab\sin(x),$$
$$\frac{dc}{dx} = \frac{ab\sin(x)}{c}.$$

Evaluate this derivative as x approaches $\pi/2$.

Example 13: Kinematics in a Triangle

Problem: In a triangle with constant sides a and b, if the angle x increases at a constant rate k, determine how quickly the length of side c is changing when $x = \pi/6$.

Solution: Apply the Law of Cosines and differentiate with respect to time t:

$$\begin{aligned} 2c\frac{dc}{dt} &= 2ab\sin(x)\cdot k, \\ \frac{dc}{dt} &= \frac{abk\sin(\pi/6)}{c}. \end{aligned}$$

Example 14: Angle Change in a Dynamic Triangle

Problem: Given a dynamic triangle where sides a and b remain constant, but side c varies with time, find the rate of change of the angle x between a and b when c is at its maximum.

Solution: From the Law of Cosines, differentiate x with respect to time t:

$$\begin{aligned} 2c\frac{dc}{dt} &= 2ab\sin(x)\frac{dx}{dt}, \\ \frac{dx}{dt} &= \frac{c\frac{dc}{dt}}{ab\sin(x)}. \end{aligned}$$

At the maximum of c, $\frac{dc}{dt} = 0$, thus $\frac{dx}{dt} = 0$.

Example 15: Velocity of the Apex Angle in a Varying Triangle

Problem: In a triangle with a fixed base b and varying sides a and c, if a increases at a rate v_a, find the rate of change of the apex angle x opposite to b.

Solution: Using the Law of Cosines, $b^2 = a^2 + c^2 - 2ac\cos(x)$. Differentiate with respect to time t:

$$\begin{aligned} 0 &= 2a\frac{da}{dt} + 2c\frac{dc}{dt} - 2\left(c\frac{da}{dt} + a\frac{dc}{dt}\right)\cos(x) - 2ac\sin(x)\frac{dx}{dt}, \\ \frac{dx}{dt} &= \frac{a\frac{da}{dt} + c\frac{dc}{dt} - \left(c\frac{da}{dt} + a\frac{dc}{dt}\right)\cos(x)}{ac\sin(x)}. \end{aligned}$$

Substitute $\frac{da}{dt} = v_a$ and solve for $\frac{dx}{dt}$.

Additional Thoughts

These examples illustrate the broad range of applications for the Law of Cosines in differential calculus, from kinematics in triangles to dynamic geometrical situations. Mastery of these techniques offers deeper insight into the behavior of varying systems in mathematical and physical contexts.

Additional Advanced Examples

This document presents further advanced examples of using the Law of Cosines in differentiation, exploring new scenarios and challenges in calculus.

Example 16: Rate of Change of the Side in a Growing Triangle

Problem: In a triangle with sides a, b, and c, where a is increasing at a constant rate and b and c are fixed, find how quickly the angle x opposite to c is changing.

Solution: Using the Law of Cosines, differentiate x with respect to time t:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos(x), \\ 0 &= 2a\frac{da}{dt} - 2b\frac{da}{dt}\cos(x) - ab\sin(x)\frac{dx}{dt}, \\ \frac{dx}{dt} &= \frac{2(a - b\cos(x))\frac{da}{dt}}{ab\sin(x)}. \end{aligned}$$

Example 17: Differentiation in a Triangle with a Moving Side

Problem: Consider a triangle where side c is changing over time while sides a and b are constant. If the rate of change of c is known, find the rate of change of angle x between a and b.

Solution: Apply the Law of Cosines and differentiate with respect to time t:

$$\begin{aligned} 2c\frac{dc}{dt} &= 2ab\sin(x)\frac{dx}{dt}, \\ \frac{dx}{dt} &= \frac{c\frac{dc}{dt}}{ab\sin(x)}. \end{aligned}$$

Example 18: Differentiating the Angle Between Two Varying Sides

Problem: In a triangle with two sides a and b changing over time, determine the rate of change of the angle x opposite the fixed side c.

Solution: From the Law of Cosines, differentiate with respect to time t:

$$\begin{aligned} 0 &= 2a\frac{da}{dt} + 2b\frac{db}{dt} - 2(\frac{da}{dt}b + a\frac{db}{dt})\cos(x) - 2ab\sin(x)\frac{dx}{dt}, \\ \frac{dx}{dt} &= \frac{a\frac{da}{dt} + b\frac{db}{dt} - (\frac{da}{dt}b + a\frac{db}{dt})\cos(x)}{ab\sin(x)}. \end{aligned}$$

Example 19: Calculating the Rate of Decrease of an Angle

Problem: In a decreasing triangle with sides a, b, and c, where a decreases at a known rate, find how quickly the angle x between b and c is decreasing.

Solution: Differentiate the angle x with respect to time t using the Law of Cosines:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos(x), \\ 0 &= 2a\frac{da}{dt} - 2b\frac{da}{dt}\cos(x) - ab\sin(x)\frac{dx}{dt}, \\ \frac{dx}{dt} &= \frac{2(a - b\cos(x))\frac{da}{dt}}{ab\sin(x)}. \end{aligned}$$

Additional Thoughts

These advanced examples further illustrate the diverse applications of the Law of Cosines in differentiation, offering insights into complex scenarios involving changing angles and sides in triangles. Mastery of these concepts provides a deeper understanding of dynamic relationships in geometry and their calculus applications.

Differentiation Examples Using the Trigonometric Identity $2\sin^2(x) = 1 - \cos(2x)$

This collection of examples demonstrates the use of the trigonometric identity $2\sin^2(x) = 1 - \cos(2x)$ in differentiation. This identity is useful in simplifying expressions involving sine and cosine functions, thereby facilitating the differentiation process.

Example 1: Differentiating a Function Involving $\sin^2(x)$

Problem: Differentiate $f(x) = 2\sin^2(x)$.

Solution: Using the identity, $f(x) = 1 - \cos(2x)$. Differentiate with respect to x:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(1 - \cos(2x)) \\ &= 2\sin(2x). \end{aligned}$$

Example 2: Finding the Derivative of a Composite Function

Problem: Find the derivative of $g(x) = 2\sin^2(ax)$, where a is a constant.

Solution: Using the identity, $g(x) = 1 - \cos(2ax)$. Differentiate with respect to x:

$$\begin{aligned} g'(x) &= \frac{d}{dx}(1 - \cos(2ax)) \\ &= 2a\sin(2ax). \end{aligned}$$

Example 3: Differentiating a Product Involving $\sin^2(x)$

Problem: Differentiate $h(x) = x^2 \cdot 2\sin^2(x)$.

Solution: Use the identity and the product rule:

$$\begin{aligned} h(x) &= x^2 \cdot (1 - \cos(2x)), \\ h'(x) &= 2x \cdot (1 - \cos(2x)) + x^2 \cdot 2\sin(2x). \end{aligned}$$

Example 4: Differentiating an Expression with Multiple Trig Functions

Problem: Differentiate $i(x) = \cos(x) \cdot 2\sin^2(x)$.

Solution: Apply the identity and the product rule:

$$i(x) = \cos(x) \cdot (1 - \cos(2x)),$$
$$i'(x) = -\sin(x) \cdot (1 - \cos(2x)) + \cos(x) \cdot 2\sin(2x).$$

Additional Thoughts

These examples illustrate the practical application of the trigonometric identity $2\sin^2(x) = 1 - \cos(2x)$ in differentiating functions in calculus. This identity simplifies expressions involving squares of sine functions, making the differentiation process more efficient.

Continued Differentiation Examples Using $2\sin^2(x) = 1 - \cos(2x)$

Building on the previous examples, this document presents more scenarios where the trigonometric identity $2\sin^2(x) = 1 - \cos(2x)$ is applied in differentiation. Each example explores a unique application of this identity in calculus.

Example 5: Differentiating a Trigonometric Function Raised to a Power

Problem: Differentiate $f(x) = (2\sin^2(x))^3$.

Solution: Apply the identity and the chain rule:

$$f(x) = (1 - \cos(2x))^3,$$
$$f'(x) = 3(1 - \cos(2x))^2 \cdot (-2\sin(2x)).$$

Example 6: Finding the Derivative of a Function Involving Sine and Cosine

Problem: Find the derivative of $g(x) = \tan(x) \cdot 2\sin^2(x)$.

Solution: Use the identity and the product rule:

$$g(x) = \tan(x) \cdot (1 - \cos(2x)),$$
$$g'(x) = \sec^2(x) \cdot (1 - \cos(2x)) + \tan(x) \cdot 2\sin(2x).$$

Example 7: Differentiating an Expression with an Angle Multiple

Problem: Differentiate $h(x) = \sin(3x) \cdot 2\sin^2(2x)$.

Solution: Apply the identity and the product rule:

$$h(x) = \sin(3x) \cdot (1 - \cos(4x)),$$
$$h'(x) = 3\cos(3x) \cdot (1 - \cos(4x)) - \sin(3x) \cdot 4\sin(4x).$$

Example 8: Rate of Change of a Combined Trigonometric Function

Problem: Determine the rate of change of $i(x) = x \cdot 2\sin^2(x) + \cos(x)$ at $x = \pi/4$.

Solution: Use the identity and differentiate:

$$i(x) = x \cdot (1 - \cos(2x)) + \cos(x),$$
$$i'(x) = (1 - \cos(2x)) + x \cdot 2\sin(2x) - \sin(x),$$
$$i'(\pi/4) = \left(1 - \cos\left(\frac{\pi}{2}\right)\right) + \frac{\pi}{4} \cdot 2\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right).$$

Additional Thoughts

These examples further demonstrate the utility of the trigonometric identity $2\sin^2(x) = 1 - \cos(2x)$ in solving diverse differentiation problems in calculus. Mastery of this identity allows for the simplification and efficient resolution of complex trigonometric expressions.

Extended Differentiation Examples Using $2\sin^2(x) = 1 - \cos(2x)$

This document presents additional examples that apply the trigonometric identity $2\sin^2(x) = 1 - \cos(2x)$ in differentiation, showcasing unique and varied applications in calculus.

Example 9: Differentiating a Function Involving a Reciprocal

Problem: Differentiate $f(x) = \frac{1}{2\sin^2(x)}$.

Solution: Use the identity and the quotient rule:

$$f(x) = \frac{1}{1 - \cos(2x)},$$
$$f'(x) = \frac{2\sin(2x)}{(1 - \cos(2x))^2}.$$

Example 10: Differentiating a Function with Multiple Trigonometric Terms

Problem: Find the derivative of $g(x) = 2\sin^2(x) + 3\cos^2(x)$.

Solution: Apply the identity and differentiate:

$$g(x) = (1 - \cos(2x)) + 3(1 - \sin^2(x)),$$
$$g'(x) = 2\sin(2x) - 6\sin(x)\cos(x).$$

Example 11: Rate of Change in a Trigonometric Polynomial

Problem: Determine the rate of change of $h(x) = 2x^3\sin^2(x) - \cos(x)$.

Solution: Use the identity and the product rule:

$$h(x) = 2x^3(1 - \cos(2x)) - \cos(x),$$
$$h'(x) = 6x^2(1 - \cos(2x)) + 2x^3 \cdot 2\sin(2x) + \sin(x).$$

Example 12: Differentiating a Complex Trigonometric Function

Problem: Differentiate $i(x) = e^{2\sin^2(x)}$.

Solution: Apply the identity and the chain rule:

$$i(x) = e^{1-\cos(2x)},$$
$$i'(x) = e^{1-\cos(2x)} \cdot 2\sin(2x).$$

Additional Thoughts

These additional examples further illustrate the versatility and utility of the trigonometric identity $2\sin^2(x) = 1 - \cos(2x)$ in the context of differentiation in calculus. This identity is a powerful tool in simplifying and solving complex trigonometric expressions, enhancing the efficiency and effectiveness of mathematical analysis.

Advanced Differentiation Examples Using $2\sin^2(x) = 1 - \cos(2x)$

This document further expands on the use of the trigonometric identity $2\sin^2(x) = 1 - \cos(2x)$ in advanced differentiation scenarios. Each example is designed to explore complex applications and challenges in calculus.

Example 13: Differentiating a Composite Trigonometric Function

Problem: Differentiate $f(x) = \ln(2\sin^2(x))$.

Solution: Apply the identity and the chain rule:

$$\begin{aligned} f(x) &= \ln(1 - \cos(2x)), \\ f'(x) &= \frac{1}{1 - \cos(2x)} \cdot 2\sin(2x). \end{aligned}$$

Example 14: Rate of Change in a Harmonic Function

Problem: Find the rate of change of $g(x) = \cos(x) \cdot 2\sin^2(x)$ when $x = \frac{\pi}{3}$.

Solution: Use the identity and differentiate:

$$\begin{aligned} g(x) &= \cos(x) \cdot (1 - \cos(2x)), \\ g'(x) &= -\sin(x) \cdot (1 - \cos(2x)) + \cos(x) \cdot 2\sin(2x), \\ g'\left(\frac{\pi}{3}\right) &= -\sin\left(\frac{\pi}{3}\right) \cdot \left(1 - \cos\left(\frac{2\pi}{3}\right)\right) + \cos\left(\frac{\pi}{3}\right) \cdot 2\sin\left(\frac{2\pi}{3}\right). \end{aligned}$$

Example 15: Differentiating an Inverse Trigonometric Function

Problem: Differentiate $h(x) = \arctan(2\sin^2(x))$.

Solution: Apply the identity and the chain rule:

$$
\begin{aligned}
h(x) &= \arctan(1 - \cos(2x)), \\
h'(x) &= \frac{1}{1 + (1 - \cos(2x))^2} \cdot 2\sin(2x).
\end{aligned}
$$

Example 16: Differentiating a Product with Exponential Function

Problem: Find the derivative of $i(x) = e^x \cdot 2\sin^2(x)$.

Solution: Use the identity and the product rule:

$$
\begin{aligned}
i(x) &= e^x \cdot (1 - \cos(2x)), \\
i'(x) &= e^x \cdot (1 - \cos(2x)) + e^x \cdot 2\sin(2x).
\end{aligned}
$$

Additional Thoughts

These advanced examples continue to demonstrate the significant role of the trigonometric identity $2\sin^2(x) = 1 - \cos(2x)$ in solving complex differentiation problems. Understanding and applying this identity is crucial for efficiently tackling challenging scenarios in calculus.

Differentiation Examples Using the Trigonometric Identity $2\cos^2(x) = 1 + \cos(2x)$

This series of examples demonstrates the application of the trigonometric identity $2\cos^2(x) = 1+\cos(2x)$ in the context of differentiation. This identity is instrumental in simplifying expressions involving cosine functions, making the differentiation process more streamlined.

Example 1: Differentiating a Function Involving $\cos^2(x)$

Problem: Differentiate $f(x) = 2\cos^2(x)$.

Solution: Using the identity, $f(x) = 1 + \cos(2x)$. Differentiate with respect to x:

$$f'(x) = -2\sin(2x).$$

Example 2: Finding the Derivative of a Composite Function

Problem: Find the derivative of $g(x) = 2\cos^2(ax)$, where a is a constant.

Solution: Using the identity, $g(x) = 1 + \cos(2ax)$. Differentiate with respect to x:

$$g'(x) = -2a\sin(2ax).$$

Example 3: Differentiating a Product Involving $\cos^2(x)$

Problem: Differentiate $h(x) = x^2 \cdot 2\cos^2(x)$.

Solution: Use the identity and the product rule:

$$\begin{aligned} h(x) &= x^2 \cdot (1 + \cos(2x)), \\ h'(x) &= 2x \cdot (1 + \cos(2x)) - x^2 \cdot 2\sin(2x). \end{aligned}$$

Example 4: Differentiating an Expression with Multiple Trig Functions

Problem: Differentiate $i(x) = \sin(x) \cdot 2\cos^2(x)$.

Solution: Apply the identity and the product rule:

$$\begin{aligned} i(x) &= \sin(x) \cdot (1 + \cos(2x)), \\ i'(x) &= \cos(x) \cdot (1 + \cos(2x)) - \sin(x) \cdot 2\sin(2x). \end{aligned}$$

Additional Thoughts

These examples illustrate the practical use of the trigonometric identity $2\cos^2(x) = 1 + \cos(2x)$ in differentiating functions in calculus. This identity simplifies expressions involving squares of cosine functions, enhancing the efficiency of the differentiation process.

Further Differentiation Examples Using $2\cos^2(x) = 1 + \cos(2x)$

Expanding on the previous examples, this document provides more scenarios illustrating the use of the trigonometric identity $2\cos^2(x) = 1 + \cos(2x)$ in differentiation. Each example is crafted to explore different aspects of this identity in calculus.

Example 5: Differentiating a Function Involving a Negative Cosine Term

Problem: Differentiate $f(x) = 2\cos^2(x) - x^2$.

Solution: Apply the identity and differentiate:

$$f(x) = 1 + \cos(2x) - x^2,$$
$$f'(x) = -2\sin(2x) - 2x.$$

Example 6: Rate of Change in a Trigonometric Ratio

Problem: Find the derivative of $g(x) = \frac{1}{2\cos^2(x)}$.

Solution: Use the identity and the quotient rule:

$$g(x) = \frac{1}{1 + \cos(2x)},$$
$$g'(x) = \frac{-2\sin(2x)}{(1 + \cos(2x))^2}.$$

Example 7: Differentiating a Trigonometric Polynomial

Problem: Differentiate $h(x) = 3x^3 + 2\cos^2(x)$.

Solution: Apply the identity and differentiate:

$$h(x) = 3x^3 + 1 + \cos(2x),$$
$$h'(x) = 9x^2 - 2\sin(2x).$$

Example 8: Differentiating a Composite Trigonometric Function

Problem: Find the derivative of $i(x) = \tan(x) \cdot 2\cos^2(x)$.

Solution: Use the identity and the product rule:

$$\begin{aligned} i(x) &= \tan(x) \cdot (1 + \cos(2x)), \\ i'(x) &= \sec^2(x) \cdot (1 + \cos(2x)) + \tan(x) \cdot (-2\sin(2x)). \end{aligned}$$

Additional Thoughts

These additional examples continue to demonstrate the application of the trigonometric identity $2\cos^2(x) = 1 + \cos(2x)$ in solving diverse differentiation problems in calculus. Mastery of this identity enables efficient tackling of trigonometric expressions, facilitating a deeper understanding of mathematical analysis.

Advanced Differentiation Examples Using $2\cos^2(x) = 1 + \cos(2x)$

Continuing our exploration, this document presents more advanced examples applying the trigonometric identity $2\cos^2(x) = 1 + \cos(2x)$ in differentiation, showcasing a range of complex applications in calculus.

Example 9: Differentiating an Exponential Trigonometric Function

Problem: Differentiate $f(x) = e^{2\cos^2(x)}$.

Solution: Apply the identity and the chain rule:

$$\begin{aligned} f(x) &= e^{1+\cos(2x)}, \\ f'(x) &= e^{1+\cos(2x)} \cdot (-2\sin(2x)). \end{aligned}$$

Example 10: Differentiating a Function with Trigonometric Coefficients

Problem: Find the derivative of $g(x) = (2\cos^2(x)) \cdot \ln(x)$.

Solution: Use the identity and the product rule:

$$\begin{aligned} g(x) &= (1+\cos(2x)) \cdot \ln(x), \\ g'(x) &= \ln(x) \cdot (-2\sin(2x)) + \frac{1+\cos(2x)}{x}. \end{aligned}$$

Example 11: Rate of Change in a Harmonic Oscillator

Problem: Determine the derivative of $h(x) = \cos(x) \cdot 2\cos^2(x)$ at $x = \pi/4$.
Solution: Apply the identity and differentiate:

$$\begin{aligned} h(x) &= \cos(x) \cdot (1+\cos(2x)), \\ h'(x) &= -\sin(x) \cdot (1+\cos(2x)) + \cos(x) \cdot (-2\sin(2x)), \\ h'(\pi/4) &= -\sin(\pi/4) \cdot (1+\cos(\pi/2)) + \cos(\pi/4) \cdot (-2\sin(\pi/2)). \end{aligned}$$

Example 12: Differentiating a Composite Function with Angle Multiplication

Problem: Find the derivative of $i(x) = 2\cos^2(3x)$.
Solution: Use the identity and differentiate:

$$\begin{aligned} i(x) &= 1+\cos(6x), \\ i'(x) &= -6\sin(6x). \end{aligned}$$

Additional Thoughts

These advanced examples further illustrate the significant role of the trigonometric identity $2\cos^2(x) = 1 + \cos(2x)$ in solving complex differentiation problems in calculus. This identity is a valuable tool in simplifying and solving intricate trigonometric expressions, enhancing the depth of mathematical analysis.

Continued Advanced Differentiation Examples Using $2\cos^2(x) = 1 + \cos(2x)$

This document adds more advanced examples showcasing the use of the trigonometric identity $2\cos^2(x) = 1 + \cos(2x)$ in the context of differentiation. Each example is designed to explore complex and diverse applications in calculus.

Example 13: Differentiating a Function Involving Angular Frequency

Problem: Differentiate $f(x) = 2\cos^2(\omega x)$, where ω is a constant angular frequency.

Solution: Use the identity and differentiate:

$$f(x) = 1 + \cos(2\omega x),$$
$$f'(x) = -2\omega \sin(2\omega x).$$

Example 14: Finding the Derivative of a Complex Trigonometric Expression

Problem: Determine the derivative of $g(x) = \sqrt{2\cos^2(x) + x^2}$.

Solution: Apply the identity and the chain rule:

$$g(x) = \sqrt{1 + \cos(2x) + x^2},$$
$$g'(x) = \frac{1}{2\sqrt{1 + \cos(2x) + x^2}} \cdot (-2\sin(2x) + 2x).$$

Example 15: Rate of Change in a Function with Inverse Trigonometry

Problem: Find the derivative of $h(x) = \arccos(2\cos^2(x))$.

Solution: Use the identity and the chain rule:

$$h(x) = \arccos(1 + \cos(2x)),$$
$$h'(x) = -\frac{1}{\sqrt{1 - (1 + \cos(2x))^2}} \cdot (-2\sin(2x)).$$

Example 16: Differentiating a Product of Trigonometric and Algebraic Functions

Problem: Differentiate $i(x) = x^3 \cdot 2\cos^2(x)$.

Solution: Apply the identity and the product rule:

$$i(x) = x^3 \cdot (1 + \cos(2x)),$$
$$i'(x) = 3x^2 \cdot (1 + \cos(2x)) + x^3 \cdot (-2\sin(2x)).$$

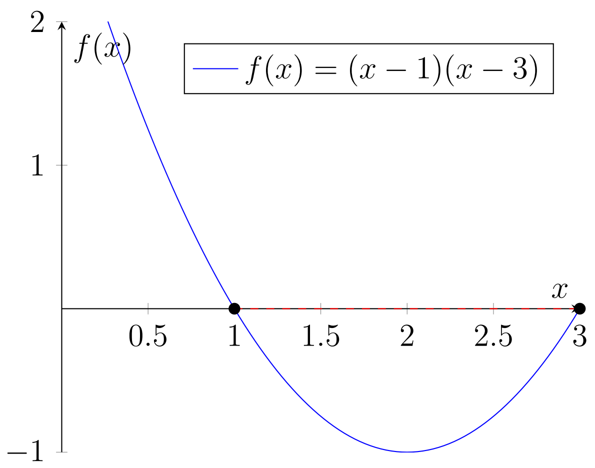
Additional Thoughts

These additional advanced examples further demonstrate the versatility and utility of the trigonometric identity $2\cos^2(x) = 1 + \cos(2x)$ in solving complex differentiation problems in calculus. Mastery of this identity is crucial for efficiently handling intricate trigonometric expressions in mathematical analysis.

11.17 Rolle's, Mean Value, and Extreme Value Theorems

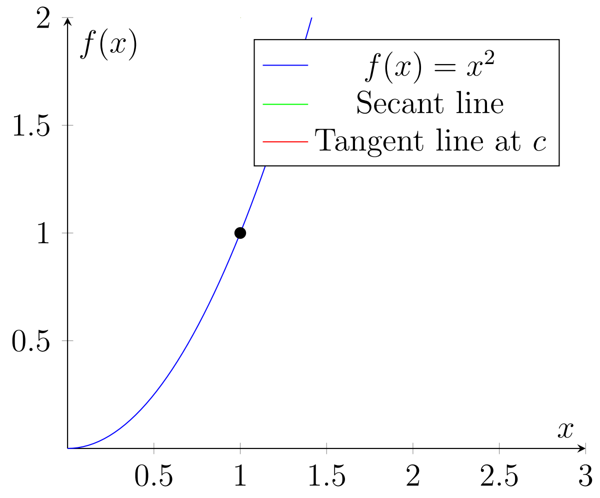
11.17.1 Rolle's Theorem

Theorem 9 (Rolle's Theorem) *If a function f is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists at least one c in the interval (a, b) such that $f'(c) = 0$.*



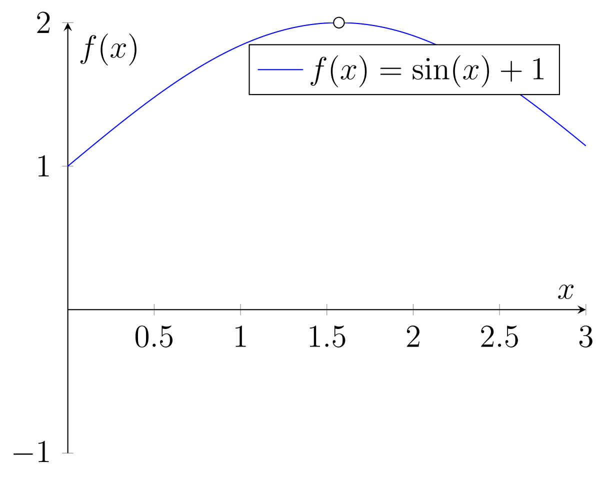
11.17.2 Mean Value Theorem

Theorem 10 (Mean Value Theorem) *If a function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists at least one c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*



11.17.3 Extreme Value Theorem

Theorem 11 (Extreme Value Theorem) *If a function f is continuous on a closed interval $[a, b]$, then f must attain a maximum and a minimum, each at least once on that interval.*



These theorems are fundamental in calculus and provide critical insights into the behavior of differentiable functions. Rolle's Theorem and the Mean Value Theorem connect the properties of derivatives to the shape of a function's graph, while the Extreme Value Theorem guarantees the existence of global maximum and minimum values under certain conditions.

Exercises on Rolle's, Mean Value, and Extreme Value Theorems

The following exercises will help you apply and understand the concepts of Rolle's Theorem, Mean Value Theorem, and Extreme Value Theorem.

Exercise Given the function $f(x) = x^3 - 6x^2 + 9x$ on the interval $[0, 3]$, verify if Rolle's Theorem can be applied. If so, find the value of c in the interval where $f'(c) = 0$.

Exercise Use the Mean Value Theorem to find a value c in the interval $[1, 4]$ such that the tangent line at c is parallel to the secant line joining the points $(1, f(1))$ and $(4, f(4))$ for the function $f(x) = x^2$.

Exercise Consider the function $g(x) = \sin(x)$ on the interval $[0, \pi]$. Use the Extreme Value Theorem to discuss the existence of global maximum and minimum values of $g(x)$ on this interval.

Exercise For the function $h(x) = \frac{1}{x}$ on the interval $[1, 3]$, determine if the Mean Value Theorem applies. If it does, find the value of c that satisfies the theorem.

Solutions to Exercises on Rolle's, Mean Value, and Extreme Value Theorems

Solution: First, we check if $f(a) = f(b)$ for $a = 0$ and $b = 3$:

$$f(0) = 0^3 - 6 \times 0^2 + 9 \times 0 = 0,$$

$$f(3) = 3^3 - 6 \times 3^2 + 9 \times 3 = 0.$$

Since $f(0) = f(3)$ and $f(x)$ is continuous and differentiable on $[0, 3]$, Rolle's Theorem applies.

To find c, we solve $f'(x) = 0$:

$$f'(x) = 3x^2 - 12x + 9,$$

$$0 = 3x^2 - 12x + 9.$$

Solving this quadratic equation gives the value of c in the interval $[0, 3]$.

Solution: For $f(x) = x^2$, the Mean Value Theorem states there exists a c in $[1, 4]$ such that:

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

Since $f'(x) = 2x$, we find c by solving:

$$2c = \frac{4^2 - 1^2}{4 - 1}.$$

Solution: The function $g(x) = \sin(x)$ is continuous on $[0, \pi]$. By the Extreme Value Theorem, it attains a maximum and minimum on this interval. The maximum is at $x = \frac{\pi}{2}$ and the minimum at $x = 0$ or $x = \pi$.

Solution: Since $h(x) = \frac{1}{x}$ is continuous and differentiable on $[1, 3]$, the Mean Value Theorem applies. To find c, solve:

$$h'(c) = \frac{h(3) - h(1)}{3 - 1},$$

where $h'(x) = -\frac{1}{x^2}$. Solving this will give the value of c.

These exercises and solutions provide practical applications of Rolle's Theorem, Mean Value Theorem, and Extreme Value Theorem, reinforcing their concepts through specific examples.

Chapter 12

Integration

12.1 Antiderivatives

12.1.1 Definition and Basic Techniques

Definition 25 *An antiderivative of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.*

Example 40: An antiderivative of $f(x) = 2x$ is $F(x) = x^2 + C$, where C is a constant.

12.1.2 Indefinite Integrals

Definition 26 *The indefinite integral of $f(x)$, denoted $\int f(x)\,dx$, represents the collection of all its antiderivatives.*

12.1.3 Initial Value Problems

Problem 77: Given a differential equation $f'(x) = g(x)$ and an initial condition $f(x_0) = y_0$, find the function $f(x)$.

12.2 Definite Integrals

12.2.1 Definition and Interpretation

Definition 27 *The definite integral of $f(x)$ from a to b, denoted $\int_a^b f(x)\,dx$, is the limit of the sum of areas of rectangles under the curve of $f(x)$ as the width of the rectangles approaches zero.*

12.2.2 Properties of Definite Integrals

Theorem 12 *Properties include linearity, additivity, and the fact that the integral from a to a is zero.*

12.2.3 Fundamental Theorem of Calculus

Theorem 13 *If F is an antiderivative of f on an interval, then $\int_a^b f(x)\,dx = F(b) - F(a)$.*

12.2.4 Numerical Integration Methods

Method 4 *Methods include the Trapezoidal Rule and Simpson's Rule, which provide approximations of definite integrals.*

12.3 Applications of Integrals

12.3.1 Area Between Curves

Application 3 *The area between curves $f(x)$ and $g(x)$ from a to b is $\int_a^b |f(x) - g(x)|\,dx$.*

12.3.2 Volumes of Solids of Revolution

Application 4 *The volume of a solid obtained by rotating a region about an axis can be found using the Disk or Washer methods.*

12.3.3 Arc Length and Surface Area

Method 5 *Formulas for the arc length of a curve and the surface area of a solid of revolution involve definite integrals.*

12.3.4 Work and Fluid Forces

Application 5 *Work done in moving an object and the force exerted by a fluid can be calculated using integration.*

Chapter 13

Advanced Topics in Calculus

13.1 Sequences and Series

13.1.1 Convergence and Divergence

Definition 28 *A sequence $\{a_n\}$ converges if it approaches a limit as n goes to infinity. A series $\sum a_n$ converges if the sequence of its partial sums converges.*

13.1.2 Tests for Convergence

Method 6 *Tests include the Integral Test, Comparison Test, Ratio Test, and Root Test.*

13.1.3 Power Series and Taylor Series

Definition 29 *A power series is a series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$. A Taylor series is a power series that represents a function.*

13.2 Multivariable Calculus

13.2.1 Functions of Several Variables

Definition 30 *A function of several variables is a function that takes several inputs and produces a single output.*

13.2.2 Partial Derivatives

Definition 31 *The partial derivative of a function with respect to one of its variables is its derivative with respect to that variable, holding the other variables constant.*

13.2.3 Multiple Integrals

Definition 32 *Multiple integrals involve integration over more than one variable, such as double and triple integrals.*

13.2.4 Vector Calculus

Discussion 11 *Vector calculus involves differentiation and integration of vector fields, including topics like gradient, divergence, and curl.*

13.3 Differential Equations

13.3.1 First Order Differential Equations

Method 7 *Methods for solving include separation of variables, integrating factors, and exact equations.*

13.3.2 Second Order Linear Differential Equations

Method 8 *Includes homogeneous and non-homogeneous equations, characteristic equations, and particular solutions.*

13.3.3 Systems of Differential Equations

Discussion 12 *Systems of differential equations involve several interrelated differential equations and can be solved using matrix methods and eigenvalues.*

13.4 Conclusion

This concludes the math currently included in CiB. Please fork the LaTeX source code for CiB (available on GitHub) and create your own book that chooses the facts and exercises most relevant to you! Also, starring the CiB project on GitHub would be greatly appreciated! Thanks for reading CiB!

Appendix I

Basic GitHub Guide

A Quick Start to Your GitHub Journey

Welcome to the fascinating world of GitHub, a platform that has revolutionized the way we collaborate on projects, share code, and build software together. Whether you are a programmer, a writer, or a mathematician, GitHub provides a set of powerful tools to help you collaborate with others, manage your projects, and contribute to the vast world of open-source software. In this guide, we will walk you through the foundational steps to get started with GitHub, helping you to navigate, contribute, and make the most out of this incredible platform.

Creating Your GitHub Account

The first step to joining the GitHub community is to create an account. Here's how you can do it:

1. Visit the GitHub website.
2. Click on the "Sign up" button.
3. Fill in the required information, including your username, email address, and password.
4. Verify your account and complete the sign-up process.

Once you have created your account, take a moment to explore your new GitHub dashboard. Here, you will find a variety of tools and features that will help you manage your projects, collaborate with others, and discover new and interesting repositories.

Creating Your First Repository

A repository (or "repo") is a digital directory where you can store your project files. Here's how you can create your first repository:

1. From your GitHub dashboard, click on the "New" button to create a new repository.
2. Give your repository a name and provide a brief description.
3. Initialize this repository with a README file. (This is an optional step, but it's a good practice to include a README file in every repository to explain what your project is about.)
4. Click "Create repository."

Congratulations! You have just created your first GitHub repository. You can now start adding files, collaborating with others, and managing your project right from GitHub.

Making Changes and Commits

GitHub uses Git, a version control system, to keep track of changes made to your project. Here's a quick guide on how to make changes and commits:

1. Navigate to your repository on GitHub.
2. Find the file you want to edit, and click on it.
3. Click the pencil icon to start editing.
4. Make your changes and then scroll down to the "Commit changes" section.
5. Provide a commit message that explains the changes you made.
6. Choose whether you want to commit directly to the main branch or create a new branch for your changes.
7. Click "Commit changes."

Your changes are now saved, and a new commit is created. Every commit has a unique ID, making it easy to track changes, revert to previous versions, and collaborate with others.

Collaborating with Others

One of the biggest strengths of GitHub is its collaborative nature. Here are some ways you can collaborate with others:

- **Forking:** You can fork a repository, create your own copy, make changes, and then propose those changes back to the original project.
- **Issues:** Use issues to report bugs, request new features, or start a discussion with the community.
- **Pull Requests:** Propose changes to a project by creating a pull request. This allows others to review your changes, discuss them, and eventually merge them into the project.

Conclusion: Embarking on Your GitHub Adventure

Now that you have a basic understanding of GitHub and how it works, you are ready to embark on your GitHub adventure. Explore repositories, contribute to open-source projects, collaborate with others, and build amazing things together. Remember, the GitHub community is vast and supportive, and there is a wealth of knowledge and resources available to help you along the way. Happy coding!

Appendix II

Basic Python and Colab Guide

Introduction to Python for Calculus

Python's versatility in mathematics, science, engineering, and data analysis stems from its simplicity and extensive library ecosystem. This guide will delve into Python packages essential for math and calculus, showcasing their utility in Google Colab notebooks.

Setting Up Python and Google Colab

Google Colab is a free, cloud-based platform enabling Python programming in a browser. It offers free computational resources, ideal for Python learning and experimentation.

Visit Google Colab to start. Create a new notebook, and use code cells to write and execute Python code with Shift+Enter.

Important Python Packages for Calculus

NumPy

NumPy, fundamental for scientific computing, offers support for large, multi-dimensional arrays and matrices, along with various mathematical functions.

SymPy

SymPy, a library for symbolic mathematics, allows algebraic manipulations and equation solving symbolically, perfect for calculus operations like differentiation and integration.

Matplotlib

Matplotlib, a Python plotting library, creates static, interactive, and animated visualizations, useful for graphing functions and data in calculus.

Pandas

Pandas provide high-performance, easy-to-use data structures, and data analysis tools, invaluable for handling and analyzing mathematical data.

SciPy

SciPy extends NumPy by adding a collection of algorithms and high-level commands for data manipulation and visualization.

Examples and Applications

Calculating Derivatives and Integrals with SymPy

Illustrate using SymPy to compute derivatives and integrals of functions.

Data Visualization with Matplotlib and Pandas

Demonstrate graphing functions and datasets, highlighting calculus concepts.

Solving Equations with SciPy

Show how to solve equations that commonly appear in calculus.

Numerical Methods in Python

Discuss Python's capabilities in numerical differentiation and integration, useful in calculus.

Using Python for Limits and Continuity

Examples showcasing how Python can help in understanding limits and continuity in functions.

Interactive Learning with Google Colab

Highlight the benefits of using Colab notebooks for interactive calculus learning, including real-time collaboration and easy sharing.

Creating a Colab Notebook for Practice Problems

In this section, we will guide you through creating a Google Colab notebook to solve calculus practice problems using Python.

Setting Up Your Colab Notebook

To start solving calculus problems with Python:

1. Open Google Colab.
2. Choose 'New Notebook' to create a blank notebook.
3. Rename the notebook to something descriptive, like 'Calculus Practice'.

Installing Necessary Libraries

At the beginning of your notebook, install any necessary Python libraries. For these exercises, ensure NumPy, SymPy, and Matplotlib are available:

```
Remove # if the following packages are not installed:
# !pip install numpy sympy matplotlib
```

Solving Exercise 1: Graphing a Linear Function

Let's solve the first exercise, which involves graphing a linear function.

```
import numpy as np
import matplotlib.pyplot as plt

# Define the function
def f(x):
    return 3*x - 2

# Generate x values
x = np.linspace(-10, 10, 400)

# Plot the function
plt.plot(x, f(x))
plt.xlabel('x')
plt.ylabel('f(x)')
plt.title('Graph of f(x) = 3x - 2')
plt.grid(True)
```

```
plt.show()

# Slope and y-intercept
print("Slope: 3")
print("Y-intercept: -2")
```

Solving Exercise 2: Identifying Undefined Points in a Function

Now, let's address the second exercise, which requires identifying for what values the function $g(x) = \frac{1}{x}$ is undefined.

```
import sympy as sp

x = sp.symbols('x')
g = 1/x

# Display the function
sp.plot(g, (x, -10, 10), title='Graph of g(x) = 1/x',
        xlabel='x', ylabel='g(x)', ylim=(-10, 10))

# Identifying undefined points
print("g(x) is undefined for x = 0")
```

Accessing the Completed Colab Notebook

The Colab notebook we've discussed is available for viewing and interaction. You can access it by clicking on the following link: Finished Colab Notebook on Graphing and Analysis. This link will take you directly to the notebook hosted on Google Colab, where you can view the complete code and run it yourself.

https://colab.research.google.com/drive/1HF-cmwqfIZ803i1i-CFyR7ssWsDt7WvV

Adding the Notebook to Your GitHub Repository

If you have downloaded the Colab notebook to your local machine and want to add it to your Git repository, follow these terminal commands on your Ubuntu machine:

```
# Navigate to your local Git repository directory
cd path/to/your/repo
```

```
# Add the Colab notebook file to the repository
git add name_of_the_notebook.ipynb

# Commit the addition with a descriptive message
git commit -m "Add Colab notebook for calculus exercises"

# Push the changes to the remote GitHub repository
git push origin main
```

Using Colab Notebooks for Problem Solving

These examples demonstrate how you can use Google Colab and Python to solve and visualize calculus problems. You can use similar steps to tackle other exercises, explore different functions, and deepen your understanding of calculus concepts.

Conclusion: Interactive Learning with Colab Notebooks

Google Colab notebooks offer an interactive and accessible way to explore calculus using Python. By integrating theoretical concepts with computational examples, students can gain a deeper understanding of calculus. We encourage you to use these notebooks to solve exercises, visualize mathematical concepts, and explore the vast possibilities that Python and Colab offer.

Conclusion: Python and Colab in Calculus

Python, with its comprehensive libraries, offers a powerful toolset for calculus exploration. Combined with Google Colab, it provides an accessible, interactive platform for learning and experimentation. Embrace Python and Colab to enhance your understanding of calculus and to explore mathematical problems creatively and efficiently.

Appendix III

Basic LaTeX Guide

A Quick Start to Your LaTeX Journey

Welcome to the immersive world of LaTeX, a typesetting system widely used for creating scientific and professional documents due to its powerful handling of formulas and bibliographies. This guide is designed to offer you the foundational steps to grasp the basics of LaTeX, enabling you to craft documents of high typographic quality akin to this book.

Setting Up Your LaTeX Environment

Before you can start creating documents with LaTeX, you need to set up a working LaTeX environment on your computer. Here's how you can do it:

1. Download and install a TeX distribution, which includes LaTeX. For Windows, MiKTeX is a popular choice, while Mac users might prefer MacTeX, and TeX Live is widely used on Linux.

2. Install a LaTeX editor. Some popular options include TeXShop (for Mac), TeXworks (cross-platform), and Overleaf (an online LaTeX editor).

3. Ensure that your TeX distribution and LaTeX editor are properly configured and integrated.

Creating Your First LaTeX Document

Once your LaTeX environment is set up, you are ready to create your first LaTeX document. Follow these steps:

1. Open your LATEX editor and create a new document.
2. Insert the following code to set up a basic LATEX document:

```
\documentclass{article}
\begin{document}
Hello, \LaTeX\ world!
\end{document}
```

3. Save your document with a .tex file extension.
4. Compile your document using your LATEX editor. This process converts your .tex file into a PDF document.
5. View the output PDF and admire your first LATEX creation.

Understanding LATEX Commands and Environments

LATEX documents are created using a series of commands and environments. Commands typically start with a backslash \ and are used to format text, insert special characters, or execute functions. Environments are used to define specific sections of your document that require special formatting.

- **Commands:** For example, \{italics} will render the word "italics" in italic font.
- **Environments:** To create a bulleted list, you would use the *itemize* environment:

```
\begin{itemize}
    \item First item
    \item Second item
\end{itemize}
```

Adding Structure to Your Document

LATEX makes it easy to structure your documents with sections, subsections, and chapters. Here's how you can add structure:

```
\section{Introduction}
This is the introduction of your document.
\subsection{Background}
This subsection provides background information.
\subsubsection{Details}
This is a subsubsection for more detailed information.
```

Including Mathematical Formulas

LaTeX excels at typesetting mathematical formulas. Use the *equation* environment or the $ sign for inline formulas. For example:

```
The quadratic formula is \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).
```

Adding Images and Tables

You can also include images and tables in your LaTeX documents:

- **Images:** Use the *graphicx* package and the *includegraphics* command.
- **Tables:** Use the *tabular* environment to create tables.

Compiling Your Document

LaTeX documents need to be compiled to produce a PDF. This can be done through your LaTeX editor. If your document includes bibliographies or cross-references, you may need to compile multiple times.

Conclusion: Embracing the Power of LaTeX

Congratulations! You have taken your first steps into the world of LaTeX. With practice, you will discover that LaTeX is a powerful tool for creating professional-quality documents, from simple articles to complex books. Embrace the learning curve, explore the vast array of packages available, and join the community of LaTeX users who are ready to help you on your journey. Happy typesetting!

Bibliography

[MT77] F. Mosteller and J. W. Tukey. *Data Analysis and Regression: A Second Course in Statistics*. Addison-Wesley Pub Co, Reading, MA, 1977.