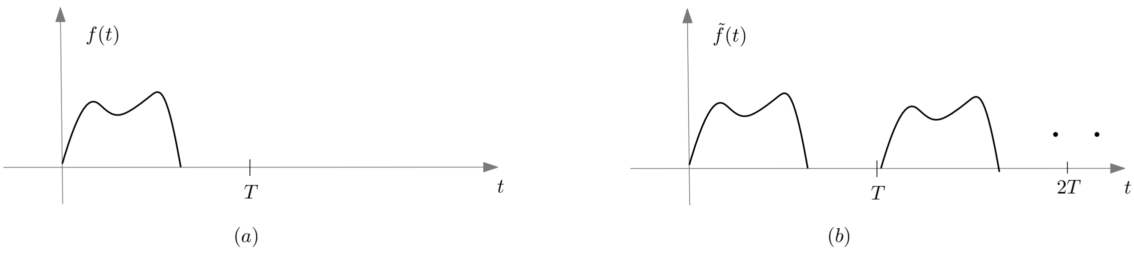
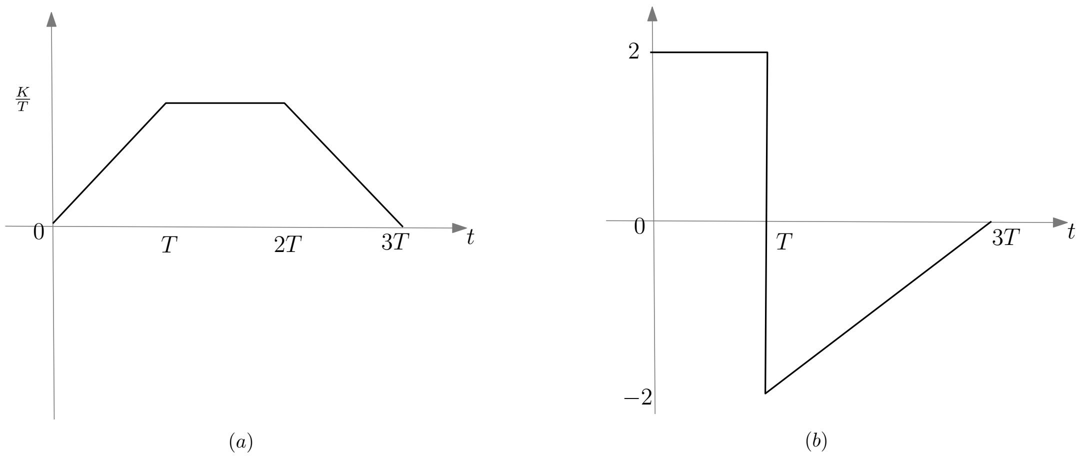
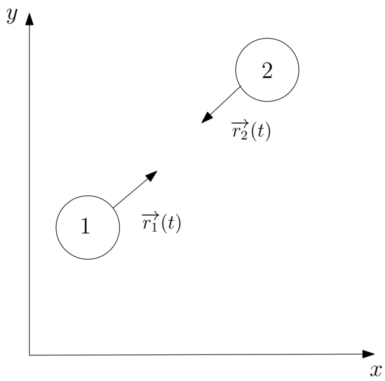
1. A signal $f(t)$ is such that it vanishes after a time T. i.e. $f(t) = 0 \ \forall \ t > T$. It's Laplace transform is given by $F(s)$. Now, consider the periodic repetition of $f(t)$ with period T given by $\tilde{f}(t)$, as illustrated below. Prove that: $L[\tilde{f}(t)] = \frac{1}{1-e^{-Ts}}F(s)$



2. Compute the Laplace Transform of the time signals given below



3. Consider two robots moving in a plane attempting to approach each other (rendezvous problem).The robots do not have a global coordinate system for reference. They can measure only relative positions.



The position of robot (1) is given by $\overrightarrow{r_1}(t) = [x_1(t) \quad y_1(t)]^T$ and position of robot (2) is given by $\overrightarrow{r_2}(t) = [x_2(t) \quad y_2(t)]^T$. The two agents decide to rendezvous by controlling their velocities such that they point along the line joining them and towards each other as shown in Figure ??. The steering law (depending only on the relative coordinates) is given by,

$$\frac{d\overrightarrow{r_1}}{dt} = K_1(\overrightarrow{r_2} - \overrightarrow{r_1}); \quad \frac{d\overrightarrow{r_2}}{dt} = K_2(\overrightarrow{r_1} - \overrightarrow{r_2})$$

where $K_1,\ K_2\ > 0$ are the positive gains.

a) Solve the above system of coupled linear differential equations using Laplace transform.

b) Prove that the robots achieve rendezvous asymptotically. i.e.

$$\lim_{t\to\infty} \vec{r_1}(t) = \lim_{t\to\infty} \vec{r_2}(t)$$

c) Try to prove the above result without inverting the Laplace transform and solving for differential equation.

d) Where do the robots finally rendezvous? How does that meeting point depend on K_1 and K_2? Interpret the result geometrically.

4. An oversimplified model of a car is shown below, where $p(t)$ is the pedal position on the accelerator,

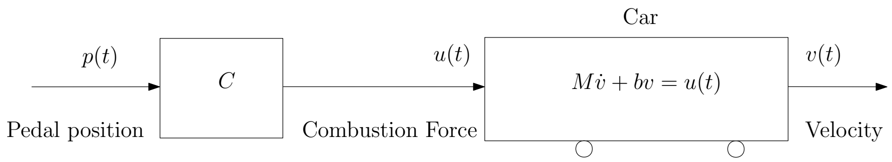


Figure 1

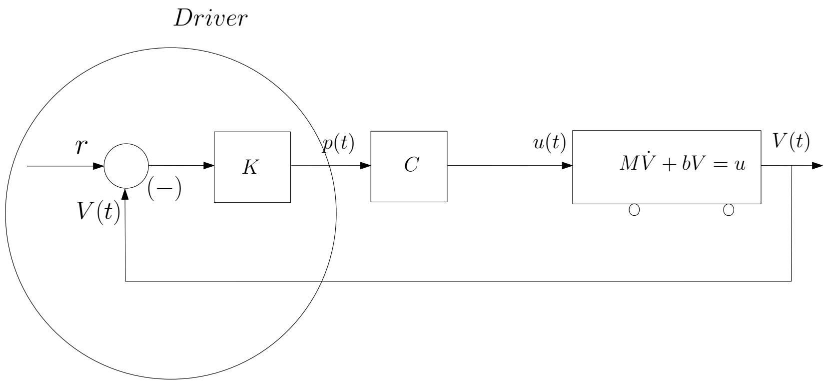
$u(t)$ is the combustion force, $v(t)$ is the car velocity, M is the mass of the car, b is the friction coefficient from the road. Assume that the force from the engine depends linearly on the pedal position through a gain C (negecting the dynamics of ignition-combustion).
A driver decides to control the speed of the car to a desired reference speed 'r' by adopting the following scheme.

$$p(t) = K(r - v(t))$$

He adjusts the pedal position proportional to the error in speed (this is called proportional controller).

$$\text{If} \quad v(t) < r, \quad p(t) > 0, \ \text{Speed up}$$

$$\text{If} \quad v(t) > r, \quad p(t) < 0, \ \text{Speed down}$$



a) Write and solve the differential equation of the controlled system. Evaluate $\lim_{t\to\infty} v(t)$. Is it r? Why does $\lim_{t\to\infty} v(t) \neq r$ when $b \neq 0$.(Think physically)

5. Consider the signal $f(t) = e^{3t}$. Find its Laplace transform and apply final value theorem to get $\lim_{t\to\infty} f(t)$. Is it in expectation with the obvious answer by looking at $f(t)$? Explain the reason for the apparent contradiction seen in the result. Now with this understanding, state the final value theorem with a caveaut.

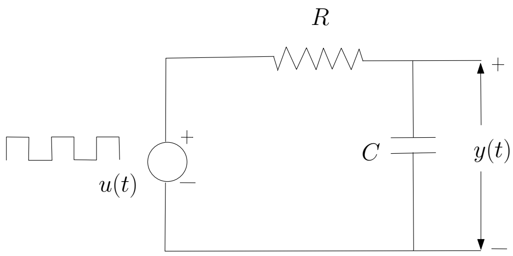
6. a) Give an example of a function $f(t)$ that is not Laplace transformable. i.e. $L[f(t)]$ does not exist for any value of s. (i.e. The integral for evaluating the transform never converges for any s)

   b) If $f(t)$ is Laplace transformable, is it true that it's derivative $\dot{f}(t)$ is also Laplace transformable? If yes, prove. If no, give a counter example of a function $f(t)$ that is Laplace transformable but it's derivative is not.

   c) If $f(t)$ is Fourier transformable, then is it also Laplace transformable? What about the converse? Give counter examples/proofs to support your answer.
   (Recall that $f(t)$ is Laplace transformable if and only if there exist constants K,c such that $|f(t)| < Ke^{ct}$. i.e. $f(t)$ does not grow faster than some exponential)

7. Consider an RC circuit as shown below.

Important manipulations



   a) Consider the case when $u(t)$ is a periodic unit rectangular pulse waveform with duty cycle D as illustrated below.

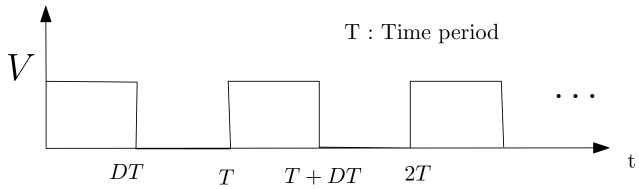


Figure 2

   Model the system and write the equations in Laplace domain assuring $y(0) = 0$ (Capacitor initially uncharged).

   b) Obtain an expression for $y(t)$ and physically interpret each of the terms.

   c) Evaluate $\lim_{T \to 0} y(t)$ (high frequency limit) and $\lim_{T \to \infty} y(t)$ (low frequency limit) and check that they agree with your intuition of what to expect in the circuit.

8. Calculate the transfer functions for the mechanical and electrical systems below. Also obtain the electrical equivalents for the given mechanical systems and vice versa.

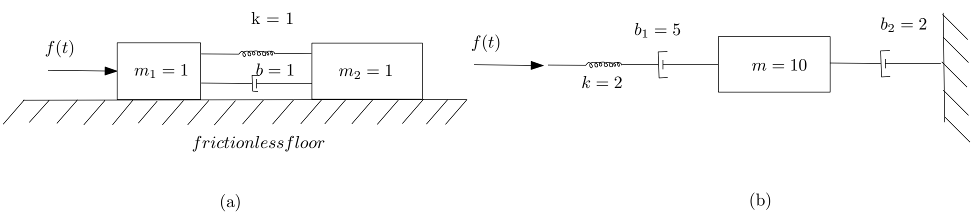
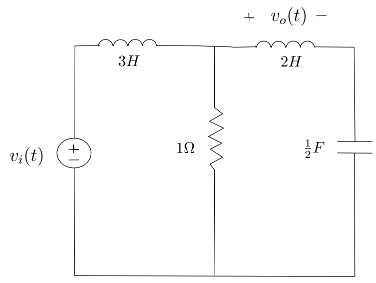
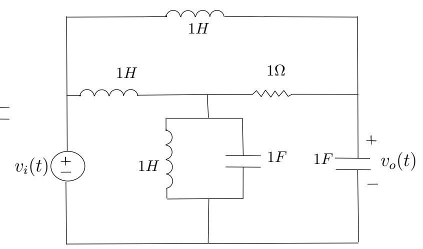


Figure 3: Mechanical systems



(a)



(b)

Figure 4: Electrical systems

9. Solve the following linear time invariant differential equations with impulsive inputs both classically and through Laplace transform techniques. Observe that the Laplace transform method is more elegant for handling such types of inputs.

   a) $\dot{x}(t) + x(t) = \delta(t),\ t > 0^-,\ x(0^-) = 1$

   b) $\ddot{x} + 2\dot{x} + x = \delta(t) + \dot{\delta}(t),\quad t > 0^-,\ x(0^-) = 1,\ \dot{x}(0^-) = 1$

   Note that $\delta(t)$ is the Dirac delta function defined as:

$$\delta(t) = 0, \forall t \neq 0$$

   and

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$