Tutorial & Homework #3 Due Date: 30 August 2018

1. In a magnetic levitation experiment, as illustrated in figure 1, a metallic object of mass m is held up in the air suspended under an electromagnet. The vertical displacement of the object can be described by the following nonlinear differential equation:

$$m\frac{d^2H}{dt^2} = mg - k\frac{I^2}{H^2}$$

where m- mass of the metallic object, g- gravity of acceleration constant, k-a positive constant, H

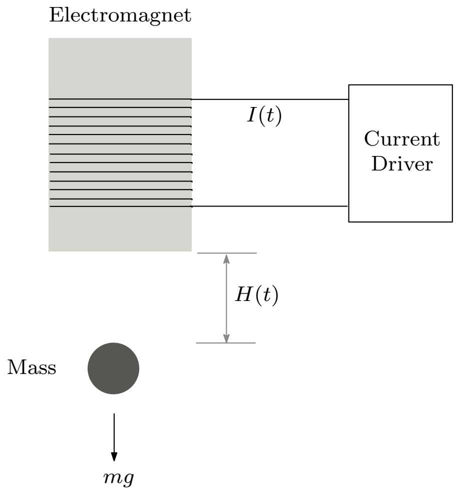

Figure 1: Magnetic levitation

-distance between the electromagnet and the metallic object (output signal), I - electromagnet's current (input signal).

   a) Show that a system's equilibrium will be achieved when $H_0 = I_0\frac{k}{mg}$.

   b) Linearlize the equation about the euquilibrium point in part (a) and show that the resulting transfer function $\frac{\delta H(s)}{\delta I(s)}$. (Hint: The set point is the equilibrium point.)

2. Let us explore simple state space models for population growth called the Lotka-Volterra models.

   a) Let $N(t)$ be the population of a species at time t. A logistic growth model was proposed which assumed that the growth rate k linearly decreased with increase in population and hence read as

$$\frac{dN}{dt} = k_0(1 - \frac{N}{C})N$$

   For this system with one state, find the equilibria in $N \geq 0$ and linearize about each of them and solve the linearized equation. Based on the solution, which equilibrium is stable and which is unstable?

   b) Now, consider two species in an ecosystem that are competing against each other (e.g, rabbits and deer). The growth model for them with $x(t)$ and $y(t)$ being the populations of rabbits and deer respectively are

$$\frac{dx}{dt} = k_R(1 - \frac{x}{C_R}) - \alpha xy, \qquad \frac{dy}{dt} = k_D(1 - \frac{x}{C_D}) - \beta xy.$$

Now, identify the equilibria of this system in the region $x, y \geq 0$. Solve the linearized equations and conclude stability. Which equilibria are stable?

(Hint: Analyse the eigenvalues of matrix A after linearlization as discussed in the class lectures.)

(BONUS problem:) Consider the case when $k_R = 2$, $k_D = 1$ (rabbits reproduce faster than deer), $C_R = 2$, $C_D = 1$ (more rabbits can be supported than deer), $\alpha = 3$, $\beta = 2$ (rabbits being more timid are more affected due to competition). With these values, find the equilibrium points in the region $x, y \geq 0$. Classify them into both species becoming extinct, one species becoming extinct and coexistence.

3. (Lotka Volterra predator-prey model) Now consider a two species ecosystem again but this time being a predator and a prey. Forgetting the logistics, assume that without the prey, predators die exponentially and without the predators, the prey grow exponentially with some rates, and due to the interaction, predator population increases and the prey population decreases. The dynamic equation with $x(t)$ and $y(t)$ being the prey and the predator populations respectively, we have

$$\frac{dx}{dt} = k_X x - \alpha x y, \qquad \frac{dy}{dt} = -k_Y y + \beta x y.$$

Identify the equilibria in $x, y \geq 0$

a) Linearize the system about the equilibria, solve the linearized equation and conclude stability.

b) This model was proposed by Lotka and Volterra to explain the oscillatory behaviour in the catch yield of a certain species of edible fish among the fishermen of the Adriatic sea. Can you now explain the oscillatory behaviour?

4. An inverted pendulum[1] mounted on a motor-driven cart is shown in figure 2. We assume the rod connecting the mass m and the cart is of length l and weightless. The equations of motion are given as follows:

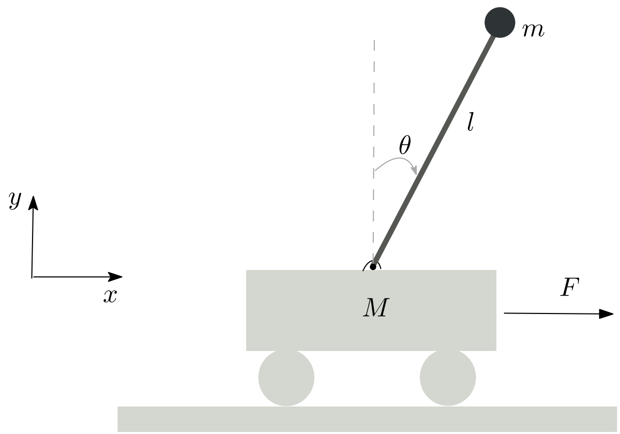

Figure 2: Inverted pendulum on a moving cart

$$(M + m)\ddot{x}(t) + ml\ddot{\theta}(t)\cos(\theta(t)) - ml\dot{\theta}^2(t)\ \sin(\theta(t)) = F(t)$$

[1]The numerous practical applications of the inverted pendulum make its study even more interesting and important. In robotics, balancing systems are developed using inverted pendulums. These find application in transport machines that need to balance objects, in systems that support walking for patients, in robots that are used in domestic and industrial use and in object transport using drones. Even large scale constructions such as buildings are modeled as inverted pendulums. Some of the most famous applications of the inverted pendulum are the personal transporter Segway and vertical landing of SpaceX re-entry rocket.

$$m\ddot{x}(t)\ \cos(\theta(t)) + ml\ddot{\theta}(t) - mg\sin(\theta(t)) = 0$$

a) Write the state space representation of the inverted pendulum system taking input as $F(t)$ and outputs as $x(t)$ and $\theta(t)$

b) Compute the equilibrium point and linearize the system around the equilibrium point.

Hint: In (a), remember the number of state variables are equal to the number of differentiations appearing in the physial laws. In (b), for small values of θ assume $\sin\theta \approx \theta$, $\cos(\theta) \approx 1$ and $\dot{\theta}^2 \approx 0$.

5. In class lectures, we studied state-space formulation of the RLC circuit; see figure 3, taking $V_R(t)$ and $V_C(t)$ as the state variables. Is it possible to formulate state space equations taking $V_L(t)$ and $I_C(t)$ as state variables? If no, argue why? If yes, write the state space formulation.

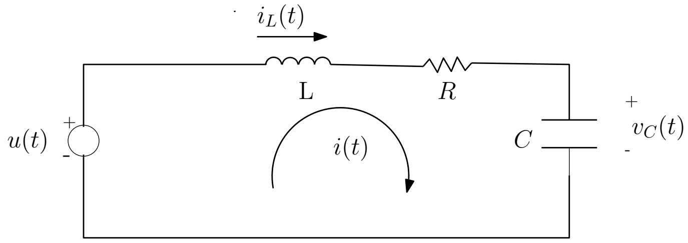

Figure 3: RLC circuit

6. Find the state-space representation of the electrical circuit shown in the figure 4, with $v(t)$ as the input and $v_L(t)$ as the output, $i_2(t) = i_L(t)$.

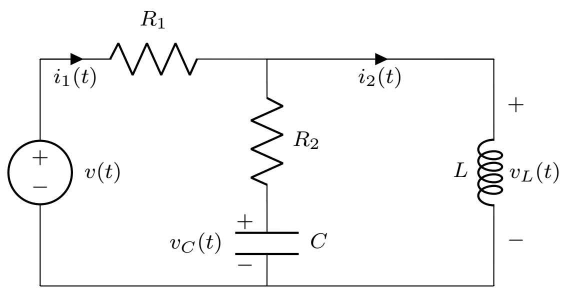

Figure 4: Electrical circuit

7. Let $P = \{p_1, p_2, \cdots, p_k\}$ be the set of distinct poles of an LTI system. Let the state space representation of the LTI is given as follows:

$$\dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t) + Du(t).$$

Here, A is a $n \times n$ matrix. Let $\Lambda := \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ denote set of distinct eigen values of A.

(a) Why it is true that $m \leq n$?

(b) Show that $P \subseteq \Lambda$, i.e., $k \leq m$.

(c) When do you think the condition $P \subset \Lambda$ holds, i.e., when not all the eigen values of A are poles of the LTI system.

Hint: If $\{1, 1, 3, 4, 3\}$ are eigen values of a matrix then distinct eigen values are $\{1, 3, 4\}$. For problem (c) you can provide the answer using a simple example.

8. State space representations are, in general, not unique. One system can be represented in several possible ways. For example, consider the following systems:

(a) $\dot{x} = -5x + 3u, \quad y = 7x.$

(b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u, \qquad y = [7 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(c)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u, \qquad y = [7 \;\; 3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Show that these systems will result in the same transfer function. Provide a justification.

9. Consider the systems described by the transfer functions below. Write two state space representations of the systems that would yield the given transfer function

a) $\frac{1}{(s+1)(s+3)}$

b) $\frac{s+1}{(s+2)(s+3)}$

10. Given the system in state variable form, find the transfer function, $T(s) = Y(s)/U(s)$, where $U(s)$ is the input and $Y(s)$ is the output, and develop equivalent block diagram showing phase variables.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$