

# ETC3550/ETC5550

## Applied forecasting

Week 9: ARIMA models



# Random walk model

If differenced series is white noise with zero mean:

$$y_t - y_{t-1} = \varepsilon_t \quad \text{or} \quad y_t = y_{t-1} + \varepsilon_t$$

where  $\varepsilon_t \sim NID(0, \sigma^2)$ .

- Model behind the **naïve method**.
- Forecast are equal to the last observation (future movements up or down are equally likely).

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$$\begin{aligned} y_{T+h} &= y_{T+h-1} + \varepsilon_{T+h} \\ &= y_{T+h-2} + \varepsilon_{T+h-1} + \varepsilon_{T+h} \\ &= \dots \\ &= y_T + \varepsilon_{T+1} + \dots + \varepsilon_{T+h} \end{aligned}$$

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$$\begin{aligned} \text{So} \quad & E(y_{T+h} | y_1, \dots, y_T) = y_T \\ \text{and} \quad & \text{Var}(y_{T+h} | y_1, \dots, y_T) = h\sigma^2 \end{aligned}$$

# Random walk with drift model

If differenced series is white noise with non-zero mean:

$$y_t - y_{t-1} = c + \varepsilon_t \quad \text{or} \quad y_t = c + y_{t-1} + \varepsilon_t$$

where  $\varepsilon_t \sim NID(0, \sigma^2)$ .

- $c$  is the **average change** between consecutive observations.
- Model behind the **drift method**.

# Backshift operator notation

- $B$  shifts the data back one period.  $By_t = y_{t-1}$
- $B^2$  shifts the data back two periods:  $B(By_t) = B^2y_t = y_{t-2}$
- A difference can be written as  $(1 - B)y_t$
- A  $d$ th-order difference can be written as  $(1 - B)^d y_t$
- A seasonal difference followed by a first difference can be written as  $(1 - B)(1 - B^m)y_t$

# AR(1) model

$$y_t = c + \phi_1 y_{t-1} + \varepsilon_t$$
$$(1 - \phi_1 B)y_t = c + \varepsilon_t$$

- When  $\phi_1 = 0$ ,  $y_t$  is **equivalent to WN** (with mean  $c$ )
- When  $\phi_1 = 1$  and  $c = 0$ ,  $y_t$  is **equivalent to a RW**
- When  $\phi_1 = 1$  and  $c \neq 0$ ,  $y_t$  is **equivalent to a RW with drift**
- When  $\phi_1 < 0$ ,  $y_t$  tends to **oscillate between positive and negative values**.



# Autoregressive models

A multiple regression with **lagged values** of  $y_t$  as predictors.

$$\begin{aligned} y_t &= c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t \\ &= c + (\phi_1 B + \phi_2 B^2 + \cdots + \phi_p B^p) y_t + \varepsilon_t \end{aligned}$$

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$$\begin{aligned}(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) y_t &= c + \varepsilon_t \\ \phi(B) y_t &= c + \varepsilon_t\end{aligned}$$

- $\varepsilon_t$  is white noise.
- $\phi(B) = (1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p)$

# Stationarity conditions

We normally restrict autoregressive models to stationary data, and then some constraints on the parameters are needed.

## General condition for stationarity

Complex roots of  $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$  lie outside the unit circle on the complex plane.

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- For  $p = 1$ :  $-1 < \phi_1 < 1$ .
- For  $p = 2$ :  $-1 < \phi_2 < 1$        $\phi_2 + \phi_1 < 1$        $\phi_2 - \phi_1 < 1$ .
- More complicated conditions hold for  $p \geq 3$ .
- fable takes care of this.

# Moving Average (MA) models

A multiple regression with **past errors** as predictors.

$$\begin{aligned}y_t &= c + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q} \\&= c + (1 + \theta_1B + \theta_2B^2 + \cdots + \theta_qB^q)\varepsilon_t \\&= c + \theta(B)\varepsilon_t\end{aligned}$$

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- $\varepsilon_t$  is white noise.
- $\theta(B) = (1 + \theta_1B + \theta_2B^2 + \dots + \theta_qB^q)$

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## General condition for invertibility

Complex roots of  $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$  lie outside the unit circle on the complex plane.

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- For  $q = 1$ :  $-1 < \theta_1 < 1$ .
- For  $q = 2$ :  $-1 < \theta_2 < 1$        $\theta_2 + \theta_1 > -1$        $\theta_1 - \theta_2 < 1$ .
- More complicated conditions hold for  $q \geq 3$ .
- fable takes care of this.



# ARIMA models

**ARIMA( $p, d, q$ ) model:**  $\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$

AR:  $p$  = order of the autoregressive part

I:  $d$  = degree of first differencing involved

MA:  $q$  = order of the moving average part.

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- Conditions on AR coefficients ensure stationarity.
- Conditions on MA coefficients ensure invertibility.
- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR( $p$ ): ARIMA( $p,0,0$ )
- MA( $q$ ): ARIMA(0,0, $q$ )

## Intercept form

$$(1 - \phi_1 B - \dots - \phi_p B^p) y'_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

## Mean form

$$(1 - \phi_1 B - \dots - \phi_p B^p)(y'_t - \mu) = (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

- $y'_t = (1 - B)^d y_t$
- $\mu$  is the mean of  $y'_t$ .
- $c = \mu(1 - \phi_1 - \dots - \phi_p)$ .
- fable uses intercept form

# Understanding ARIMA models

- If  $c = 0$  and  $d = 0$ , the long-term forecasts will go to zero.
- If  $c = 0$  and  $d = 1$ , the long-term forecasts will go to a non-zero constant.
- If  $c = 0$  and  $d = 2$ , the long-term forecasts will follow a straight line.
- If  $c \neq 0$  and  $d = 0$ , the long-term forecasts will go to the mean of the data.
- If  $c \neq 0$  and  $d = 1$ , the long-term forecasts will follow a straight line.
- If  $c \neq 0$  and  $d = 2$ , the long-term forecasts will follow a quadratic trend.

# Understanding ARIMA models

## Forecast variance and $d$

- The higher the value of  $d$ , the more rapidly the prediction intervals increase in size.
- For  $d = 0$ , the long-term forecast standard deviation will go to the standard deviation of the historical data.

## Cyclic behaviour

- For cyclic forecasts,  $p \geq 2$  and some restrictions on coefficients are required.
- If  $p = 2$ , we need  $\phi_1^2 + 4\phi_2 < 0$ . Then average cycle of length  $(2\pi) / [\arccos(-\phi_1(1 - \phi_2)/(4\phi_2))]$ .

# Exercise

- Find an ARIMA model for the `pelt::Lynx` data