

ETC3550/ETC5550

Applied forecasting

Week 9: ARIMA models



AR(1) model

$$y_t = c + \phi_1 y_{t-1} + \varepsilon_t$$
$$(1 - \phi_1 B)y_t = c + \varepsilon_t$$

- When $\phi_1 = 0$, y_t is **equivalent to WN** (with mean c)
- When $\phi_1 = 1$ and $c = 0$, y_t is **equivalent to a RW**
- When $\phi_1 = 1$ and $c \neq 0$, y_t is **equivalent to a RW with drift**
- When $\phi_1 < 0$, y_t tends to **oscillate between positive and negative values**.

Autoregressive models

A multiple regression with **lagged values** of y_t as predictors.

$$\begin{aligned}y_t &= c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t \\ &= c + (\phi_1 B + \phi_2 B^2 + \cdots + \phi_p B^p) y_t + \varepsilon_t\end{aligned}$$

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$$\begin{aligned}(1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p) y_t &= c + \varepsilon_t \\ \phi(B) y_t &= c + \varepsilon_t\end{aligned}$$

- ε_t is white noise.
- $\phi(B) = (1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p)$

Stationarity conditions

We normally restrict autoregressive models to stationary data, and then some constraints on the parameters are needed.

General condition for stationarity

Complex roots of $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$ lie outside the unit circle on the complex plane.

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- For $p = 1$: $-1 < \phi_1 < 1$.
- For $p = 2$: $-1 < \phi_2 < 1$ $\phi_2 + \phi_1 < 1$ $\phi_2 - \phi_1 < 1$.
- More complicated conditions hold for $p \geq 3$.
- fable takes care of this.

Moving Average (MA) models

A multiple regression with **past errors** as predictors.

$$\begin{aligned}y_t &= c + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q} \\&= c + (1 + \theta_1B + \theta_2B^2 + \cdots + \theta_qB^q)\varepsilon_t \\&= c + \theta(B)\varepsilon_t\end{aligned}$$

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- For $q = 1$: $-1 < \theta_1 < 1$.
- For $q = 2$: $-1 < \theta_2 < 1$ $\theta_2 + \theta_1 > -1$ $\theta_1 - \theta_2 < 1$.
- More complicated conditions hold for $q \geq 3$.
- fable takes care of this.

Partial autocorrelations

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= equal to the estimate of ϕ_k in regression:

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- Varying number of terms on RHS gives α_k for different values of k .
- $\alpha_1 = \rho_1$
- same critical values of $\pm 1.96/\sqrt{T}$ as for ACF.
- Last significant α_k indicates the order of an AR model.

ACF and PACF interpretation

AR(p)

- ACF dies out in an exponential or damped sine-wave manner
- PACF has all zero spikes beyond the p th spike

So we have an AR(p) model when

- the ACF is exponentially decaying or sinusoidal
- there is a significant spike at lag p in PACF, but none beyond p

ACF and PACF interpretation

MA(q)

- PACF dies out in an exponential or damped sine-wave manner
- ACF has all zero spikes beyond the q th spike

So we have an MA(q) model when

- the PACF is exponentially decaying or sinusoidal
- there is a significant spike at lag q in ACF, but none beyond q

ARIMA models

ARIMA(p, d, q) model: $\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$

AR: p = order of the autoregressive part

I: d = degree of first differencing involved

MA: q = order of the moving average part.

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- Conditions on AR coefficients ensure stationarity.
- Conditions on MA coefficients ensure invertibility.
- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR(p): ARIMA($p,0,0$)
- MA(q): ARIMA(0,0, q)

Intercept form

$$(1 - \phi_1 B - \dots - \phi_p B^p) y'_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

Mean form

$$(1 - \phi_1 B - \dots - \phi_p B^p)(y'_t - \mu) = (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

- $y'_t = (1 - B)^d y_t$
- μ is the mean of y'_t .
- $c = \mu(1 - \phi_1 - \dots - \phi_p)$.
- fable uses intercept form

Understanding ARIMA models

- If $c = 0$ and $d = 0$, the long-term forecasts will go to zero.
- If $c = 0$ and $d = 1$, the long-term forecasts will go to a non-zero constant.
- If $c = 0$ and $d = 2$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 0$, the long-term forecasts will go to the mean of the data.
- If $c \neq 0$ and $d = 1$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 2$, the long-term forecasts will follow a quadratic trend.

Understanding ARIMA models

Forecast variance and d

- The higher the value of d , the more rapidly the prediction intervals increase in size.
- For $d = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

Cyclic behaviour

- For cyclic forecasts, $p \geq 2$ and some restrictions on coefficients are required.
- If $p = 2$, we need $\phi_1^2 + 4\phi_2 < 0$. Then average cycle of length

$$(2\pi) / \left[\arccos(-\phi_1(1 - \phi_2)/(4\phi_2)) \right]$$