

# ETC3550/ETC5550

## Applied forecasting

Week 10: ARIMA models



# Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	↑	↑
	Non-seasonal part of the model	Seasonal part of of the model

where  $m$  = number of observations per year.

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Diagram illustrating the components of the ARIMA(1, 1, 1)(1, 1, 1)<sub>4</sub> model:

- $(1 - \phi_1 B)$ : Non-seasonal AR(1)
- $(1 - \Phi_1 B^4)$ : Seasonal AR(1)
- $(1 - B)$ : Non-seasonal difference
- $(1 - B^4)$ : Seasonal difference
- $(1 + \theta_1 B)$ : Non-seasonal MA(1)
- $(1 + \Theta_1 B^4)$ : Seasonal MA(1)

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All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t = & (1 + \phi_1)y_{t-1} - \phi_1 y_{t-2} + (1 + \Phi_1)y_{t-4} \\ & - (1 + \phi_1 + \Phi_1 + \phi_1 \Phi_1)y_{t-5} + (\phi_1 + \phi_1 \Phi_1)y_{t-6} \\ & - \Phi_1 y_{t-8} + (\Phi_1 + \phi_1 \Phi_1)y_{t-9} - \phi_1 \Phi_1 y_{t-10} \\ & + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \Theta_1 \varepsilon_{t-4} + \theta_1 \Theta_1 \varepsilon_{t-5}. \end{aligned}$$

# Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

**ARIMA(0,0,0)(0,0,1)<sub>12</sub> will show:**

- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ....

**ARIMA(0,0,0)(1,0,0)<sub>12</sub> will show:**

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

# Point forecasts

- 1 Rearrange ARIMA equation so  $y_t$  is on LHS.
- 2 Rewrite equation by replacing  $t$  by  $T + h$ .
- 3 On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with  $h = 1$ . Repeat for  $h = 2, 3, \dots$



# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

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- $v_{T+1|T} = \hat{\sigma}^2$  for all ARIMA models regardless of parameters and orders.
- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[ 1 + \sum_{i=1}^{h-1} \theta_i^2 \right], \quad \text{for } h = 2, 3, \dots$$

# Prediction intervals

- Prediction intervals **increase in size with forecast horizon.**
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed.**
- Prediction intervals tend to be too narrow.
  - ▶ the uncertainty in the parameter estimates has not been accounted for.
  - ▶ the ARIMA model assumes historical patterns will not change during the forecast period.
  - ▶ the ARIMA model assumes uncorrelated future errors

# ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

# ARIMA vs ETS

## ETS models

Combination  
of components

9 ETS models with  
multiplicative errors

3 ETS models with  
additive errors and  
multiplicative  
seasonality

## ARIMA models

Modelling  
autocorrelations

Potentially  $\infty$  models

All stationary models  
Many large models

6 fully additive  
ETS models

# Equivalences

ETS model	ARIMA model	Parameters
ETS(A,N,N)	ARIMA(0,1,1)	$\theta_1 = \alpha - 1$
ETS(A,A,N)	ARIMA(0,2,2)	$\theta_1 = \alpha + \beta - 2$ $\theta_2 = 1 - \alpha$
ETS(A,A <sub>d</sub> ,N)	ARIMA(1,1,2)	$\phi_1 = \phi$ $\theta_1 = \alpha + \phi\beta - 1 - \phi$ $\theta_2 = (1 - \alpha)\phi$
ETS(A,N,A)	ARIMA(0,0,m)(0,1,0) <sub>m</sub>	
ETS(A,A,A)	ARIMA(0,1,m + 1)(0,1,0) <sub>m</sub>	
ETS(A,A <sub>d</sub> ,A)	ARIMA(1,0,m + 1)(0,1,0) <sub>m</sub>	