

ETC3550/ETC5550

Applied forecasting

Week 10: ARIMA models



Seasonal ARIMA models

| ARIMA | $\underbrace{(p, d, q)}$ | $\underbrace{(P, D, Q)_m}$ |
|-------|-----------------------------------|----------------------------------|
| | ↑ | ↑ |
| | Non-seasonal part of the model | Seasonal part of of the model |

where m = number of observations per year.

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Diagram illustrating the components of the ARIMA(1, 1, 1)(1, 1, 1)₄ model:

- $(1 - \phi_1 B)$: Non-seasonal AR(1)
- $(1 - \Phi_1 B^4)$: Seasonal AR(1)
- $(1 - B)$: Non-seasonal difference
- $(1 - B^4)$: Seasonal difference
- $(1 + \theta_1 B)$: Non-seasonal MA(1)
- $(1 + \Theta_1 B^4)$: Seasonal MA(1)

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All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t = & (1 + \phi_1)y_{t-1} - \phi_1 y_{t-2} + (1 + \Phi_1)y_{t-4} \\ & - (1 + \phi_1 + \Phi_1 + \phi_1 \Phi_1)y_{t-5} + (\phi_1 + \phi_1 \Phi_1)y_{t-6} \\ & - \Phi_1 y_{t-8} + (\Phi_1 + \phi_1 \Phi_1)y_{t-9} - \phi_1 \Phi_1 y_{t-10} \\ & + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \Theta_1 \varepsilon_{t-4} + \theta_1 \Theta_1 \varepsilon_{t-5}. \end{aligned}$$

Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

ARIMA(0,0,0)(0,0,1)₁₂ will show:

- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36,

ARIMA(0,0,0)(1,0,0)₁₂ will show:

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

Point forecasts

- 1 Rearrange ARIMA equation so y_t is on LHS.
- 2 Rewrite equation by replacing t by $T + h$.
- 3 On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with $h = 1$. Repeat for $h = 2, 3, \dots$

Prediction intervals

95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

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- $v_{T+1|T} = \hat{\sigma}^2$ for all ARIMA models regardless of parameters and orders.
- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2 \right], \quad \text{for } h = 2, 3, \dots$$

Prediction intervals

- Prediction intervals **increase in size with forecast horizon.**
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed.**
- Prediction intervals tend to be too narrow.
 - ▶ the uncertainty in the parameter estimates has not been accounted for.
 - ▶ the ARIMA model assumes historical patterns will not change during the forecast period.
 - ▶ the ARIMA model assumes uncorrelated future errors

ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

ARIMA vs ETS

ETS models

Combination
of components

9 ETS models with
multiplicative errors

3 ETS models with
additive errors and
multiplicative
seasonality

ARIMA models

Modelling
autocorrelations

Potentially ∞ models

All stationary models
Many large models

6 fully additive
ETS models

Equivalences

| ETS model | ARIMA model | Parameters |
|--------------------------|--------------------------------------|--|
| ETS(A,N,N) | ARIMA(0,1,1) | $\theta_1 = \alpha - 1$ |
| ETS(A,A,N) | ARIMA(0,2,2) | $\theta_1 = \alpha + \beta - 2$ $\theta_2 = 1 - \alpha$ |
| ETS(A,A _d ,N) | ARIMA(1,1,2) | $\phi_1 = \phi$ $\theta_1 = \alpha + \phi\beta - 1 - \phi$ $\theta_2 = (1 - \alpha)\phi$ |
| ETS(A,N,A) | ARIMA(0,0,m)(0,1,0) _m | |
| ETS(A,A,A) | ARIMA(0,1,m + 1)(0,1,0) _m | |
| ETS(A,A _d ,A) | ARIMA(1,0,m + 1)(0,1,0) _m | |