

# ETC3550/ETC5550

## Applied forecasting

Week 11: Regression models



# Multiple regression and forecasting

$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t.$$

- $y_t$  is the variable we want to predict: the “response” variable
- Each  $x_{j,t}$  is numerical and is called a “predictor”. They are usually assumed to be known for all past and future times.
- The coefficients  $\beta_1, \dots, \beta_k$  measure the effect of each predictor *after taking account of the effect of all other predictors in the model*.
- $\varepsilon_t$  is a white noise error term

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**NOT RECOMMENDED!**

# Uses of dummy variables

## Seasonal dummies

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## Public holidays

- For daily data: if it is a public holiday,  $\text{dummy}=1$ , otherwise  $\text{dummy}=0$ .

# Holidays

## For monthly data

- Christmas: always in December so part of monthly seasonal effect
- Easter: use a dummy variable  $v_t = 1$  if any part of Easter is in that month,  $v_t = 0$  otherwise.
- Ramadan and Chinese New Year similar.

# Distributed lags

Lagged values of a predictor.

Example:  $x$  is advertising which has a delayed effect

$x_1$  = advertising for previous month;

$x_2$  = advertising for two months previously;

$\vdots$

$x_m$  = advertising for  $m$  months previously.

# Fourier series

Periodic seasonality can be handled using pairs of Fourier terms:

$$s_k(t) = \sin\left(\frac{2\pi kt}{m}\right) \quad c_k(t) = \cos\left(\frac{2\pi kt}{m}\right)$$

$$y_t = a + bt + \sum_{k=1}^K [\alpha_k s_k(t) + \beta_k c_k(t)] + \varepsilon_t$$

- Every periodic function can be approximated by sums of sin and cos terms for large enough  $K$ .
- Choose  $K$  by minimizing AICc or CV.
- Called “harmonic regression”

# Your turn

- 1 Fit a regression model with a piecewise linear trend with Fourier terms for the US leisure employment data.

```
leisure <- us_employment |>
  filter(
    Title == "Leisure and Hospitality",
    year(Month) > 2001
  ) |>
  mutate(Employed = Employed / 1000) |>
  select(Month, Employed)
```

- 2 Does the model fit well? What are the implications for forecasting?

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**Maximizing  $\bar{R}^2$  is equivalent to minimizing  $\hat{\sigma}^2$ .**

$$\hat{\sigma}^2 = \frac{1}{T - k - 1} \sum_{t=1}^T \varepsilon_t^2$$



# Akaike's Information Criterion

$$\text{AIC} = -2 \log(L) + 2(k + 2)$$

- $L$  = likelihood
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$$\text{AIC}_C = \text{AIC} + \frac{2(k+2)(k+3)}{T-k-3}$$

- Minimizing the AIC or AICc is asymptotically equivalent to minimizing MSE via **leave-one-out cross-validation** (for any linear regression).

# Leave-one-out cross-validation

For regression, leave-one-out cross-validation is faster and more efficient than time-series cross-validation.

- Select one observation for test set, and use *remaining* observations in training set. Compute error on test observation.
- Repeat using each possible observation as the test set.
- Compute accuracy measure over all errors.

# Cross-validation

## Traditional evaluation



# Cross-validation

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## Time series cross-validation



# Cross-validation

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## Leave-one-out cross-validation



# Cross-validation

## Traditional evaluation



## Leave-one-out cross-validation



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- BIC penalizes terms more heavily than AIC
- Also called SBIC and SC.
- Minimizing BIC is asymptotically equivalent to leave- $v$ -out cross-validation when  $v = T[1 - 1/(\log(T) - 1)]$ .

# Choosing regression variables

## Best subsets regression

- Fit all possible regression models using one or more of the predictors.
- Choose the best model based on one of the measures of predictive ability (CV, AIC, AICc).

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## Backwards stepwise regression

- Start with a model containing all variables.
- Subtract one variable at a time. Keep model if lower CV.
- Iterate until no further improvement.
- Not guaranteed to lead to best model.

# Ex-ante versus ex-post forecasts

- *Ex ante* forecasts are made using only information available in advance.
  - ▶ require forecasts of predictors
- *Ex post* forecasts are made using later information on the predictors.
  - ▶ useful for studying behaviour of forecasting models.
- trend, seasonal and calendar variables are all known in advance, so these don't need to be forecast.

# Your turn

- 3 Produce forecasts of US leisure employment using your best regression model.
- 4 Why don't you need to forecast the predictors?