ETC3550/ETC5550 Applied forecasting

Week 7: Seasonal exponential smoothing

Historical perspective

- Developed in the 1950s and 1960s as methods (algorithms) to produce point forecasts.
- Combine a "level", "trend" (slope) and "seasonal" component to describe a time series.
- The rate of change of the components are controlled by "smoothing parameters": α, β and γ respectively.
- Need to choose best values for the smoothing parameters (and initial states).
- Equivalent ETS state space models developed in the 1990s and 2000s.

ETS models

General notation   E T S : **E**ponen**T**ial **S**moothing

Error **T**rend **S**eason

Error: Additive (`"A"`) or multiplicative (`"M"`)

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Error: Additive (`"A"`) or multiplicative (`"M"`)

Trend: None (`"N"`), additive (`"A"`), multiplicative (`"M"`), or damped (`"Ad"` or `"Md"`).

ETS models

General notation   E T S : **E**xponen**T**ial **S**moothing

Error **T**rend **S**eason

Error: Additive (`"A"`) or multiplicative (`"M"`)

Trend: None (`"N"`), additive (`"A"`), multiplicative (`"M"`), or damped (`"Ad"` or `"Md"`).

Seasonality: None (`"N"`), additive (`"A"`) or multiplicative (`"M"`)

ETS(A,A,A): Holt-Winters additive method

State equations	$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$
	$b_t = b_{t-1} + \beta\varepsilon_t$
	$s_t = s_{t-m} + \gamma\varepsilon_t$
Observation equation	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$
Forecast equation	$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$

- Forecast errors: $\varepsilon_t = y_t - \hat{y}_{t|t-1} \sim \text{NID}(0, \sigma^2)$
- k is integer part of $(h-1)/m$.

ETS(A,N,A): No trend, additive seasonal model

State equations	$\ell_t = \ell_{t-1} + \alpha\varepsilon_t$
	$s_t = s_{t-m} + \gamma\varepsilon_t$
Observation equation	$y_t = \ell_{t-1} + s_{t-m} + \varepsilon_t$
Forecast equation	$\hat{y}_{t+h|t} = \ell_t + s_{t+h-m(k+1)}$

- Forecast errors: $\varepsilon_t = y_t - \hat{y}_{t|t-1} \sim \text{NID}(0, \sigma^2)$
- k is integer part of $(h-1)/m$.

Models so far

Additive Error	Seasonal Component	
Trend Component	N (None)	A (Additive)
N (None)	A,N,N	A,N,A
A (Additive)	A,A,N	A,A,A

All ETS models we will use

Additive Error	Seasonal Component		
Trend Component	N (None)	A (Additive)	M (Multiplicative)
N (None)	A,N,N	A,N,A	A,N,M
A (Additive)	A,A,N	A,A,A	A,A,M
A_d (Additive damped)	A,A_d,N	A,A_d,A	A,A_d,M

All ETS models we will use

Additive Error		Seasonal Component		
Trend Component		N (None)	A (Additive)	M (Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
A	(Additive)	A,A,N	A,A,A	A,A,M
A_d	(Additive damped)	A,A_d,N	A,A_d,A	A,A_d,M

Multiplicative Error		Seasonal Component		
Trend Component		N (None)	A (Additive)	M (Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A_d	(Additive damped)	M,A_d,N	M,A_d,A	M,A_d,M

ETS(M,N,N): SES with multiplicative errors.

State equation	$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$
Observation equation	$y_t = \ell_{t-1}(1 + \varepsilon_t)$
Forecast equation	$\hat{y}_{t+h\mid t} = \ell_t$

- Relative forecast errors: $\varepsilon_t = \frac{y_t - \hat{y}_{t\mid t-1}}{\hat{y}_{t\mid t-1}} \sim \text{NID}(0, \sigma^2)$
- Same point forecasts as ETS(A,N,N) with additive errors and same α.
- Different prediction intervals from ETS(A,N,N).

ETS(M,A,N): Holt's method with multiplicative errors.

State equations	$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$
	$b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$
Observation equation	$y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t)$
Forecast equation	$\hat{y}_{t+h|t} = \ell_t + hb_t$

- Relative forecast errors: $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Same point forecasts as ETS(A,A,N) with additive errors and same α and β
- Different prediction intervals from ETS(A,A,N).

ETS(M,A,M): Holt-Winters multiplicative method

State equations	$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$
	$b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$
	$s_t = s_{t-m}(1 + \gamma\varepsilon_t)$
Observation equation	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$
Forecast equation	$\hat{y}_{t+h|t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$

- Relative forecast errors: $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- k is integer part of $(h-1)/m$.

ETS models

Additive Error	Seasonal Component		
Trend Component	N (None)	A (Additive)	M (Multiplicative)
N (None)	A,N,N	A,N,A	A,N,M
A (Additive)	A,A,N	A,A,A	A,A,M
A_d (Additive damped)	A,A_d,N	A,A_d,A	A,A_d,M

Multiplicative Error	Seasonal Component		
Trend Component	N (None)	A (Additive)	M (Multiplicative)
N (None)	M,N,N	M,N,A	M,N,M
A (Additive)	M,A,N	M,A,A	M,A,M
A_d (Additive damped)	M,A_d,N	M,A_d,A	M,A_d,M

Additive error models

Trend	Seasonal		
	N	A	M
N	$y_t = \ell_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t$	$y_t = \ell_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = \ell_{t-1} s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/\ell_{t-1}$
A	$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$ $b_t = b_{t-1} + \beta\varepsilon_t$	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$ $b_t = b_{t-1} + \beta\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $b_t = b_{t-1} + \beta\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1} + b_{t-1})$
A_d	$y_t = \ell_{t-1} + \phi b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t$ $b_t = \phi b_{t-1} + \beta\varepsilon_t$	$y_t = \ell_{t-1} + \phi b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t$ $b_t = \phi b_{t-1} + \beta\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1})s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $b_t = \phi b_{t-1} + \beta\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1} + \phi b_{t-1})$

Multiplicative error models

Trend	Seasonal		
	N	**A**	**M**
N	$y_t = \ell_{t-1}(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$	$y_t = (\ell_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + \alpha(\ell_{t-1} + s_{t-m})\varepsilon_t$ $s = s_{t-m} + \gamma(\ell_{t-1} + s_{t-m})\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m}(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$
A	$y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$
A$_\mathbf{d}$	$y_t = (\ell_{t-1} + \phi b_{t-1})(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + \phi b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1})s_{t-m}(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + \phi b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

ETS models

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Trend		N	A	M
Component		(None)	(Additive)	(Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
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Additive Error

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Multiplicative Error

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Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

Automatic forecasting

From Hyndman et al. (IJF, 2002):

- Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE (or some other criterion).
- Select best method using AICc:
- Produce forecasts using best method.
- Obtain forecast intervals using underlying state space model.

Method performed very well in M3 competition.

Residuals

Response residuals

$$\hat{e}_t = y_t - \hat{y}_{t|t-1}$$

Innovation residuals

Additive error model:

$$\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1}$$

Multiplicative error model:

$$\hat{\varepsilon}_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}}$$