ETC3550/ETC5550 Applied forecasting

Week 11: Regression models

Multiple regression and forecasting

$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t.$$

- y_t is the variable we want to predict: the "response" variable
- Each $x_{j,t}$ is numerical and is called a "predictor". They are usually assumed to be known for all past and future times.
- The coefficients $\beta_1, \ldots, \beta_k$ measure the effect of each predictor *after taking account of the effect of all other predictors in the model.*
- ε_t is a white noise error term

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NOT RECOMMENDED!

Uses of dummy variables

Seasonal dummies

- For quarterly data: use 3 dummies
- For monthly data: use 11 dummies
- For daily data: use 6 dummies
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Public holidays

- For daily data: if it is a public holiday, dummy=1, otherwise dummy=0.

Holidays

For monthly data

- Christmas: always in December so part of monthly seasonal effect
- Easter: use a dummy variable $v_t = 1$ if any part of Easter is in that month, $v_t = 0$ otherwise.
- Ramadan and Chinese New Year similar.

Distributed lags

Lagged values of a predictor.

Example: x is advertising which has a delayed effect

x_1 = advertising for previous month;

x_2 = advertising for two months previously;

$\vdots$

x_m = advertising for m months previously.

Fourier series

Periodic seasonality can be handled using pairs of Fourier terms:

$$s_k(t) = \sin\left(\frac{2\pi k t}{m}\right) \qquad c_k(t) = \cos\left(\frac{2\pi k t}{m}\right)$$

$$y_t = a + bt + \sum_{k=1}^{K} \left[\alpha_k s_k(t) + \beta_k c_k(t)\right] + \varepsilon_t$$

- Every periodic function can be approximated by sums of sin and cos terms for large enough K.
- Choose K by minimizing AICc or CV.
- Called "harmonic regression"

1. Fit a regression model with a piecewise linear trend with Fourier terms for the US leisure employment data.

```
leisure <- us_employment |>
  filter(
    Title == "Leisure and Hospitality",
    year(Month) > 2001
  ) |>
  mutate(Employed = Employed / 1000) |>
  select(Month, Employed)
```

2. Does the model fit well? What are the implications for forecasting?

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Maximizing $\bar{R}^2$ is equivalent to minimizing $\hat{\sigma}^2$.

$$\hat{\sigma}^2 = \frac{1}{T - k - 1}\sum_{t=1}^{T}\varepsilon_t^2$$

Akaike's Information Criterion

$$\text{AIC} = -2\log(L) + 2(k + 2)$$

- L = likelihood
- k = # predictors in model.
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$$AIC_C = AIC + \frac{2(k+2)(k+3)}{T-k-3}$$

- Minimizing the AIC or AICc is asymptotically equivalent to minimizing MSE via **leave-one-out cross-validation** (for any linear regression).

Leave-one-out cross-validation

For regression, leave-one-out cross-validation is faster and more efficient than time-series cross-validation.

- Select one observation for test set, and use *remaining* observations in training set. Compute error on test observation.
- Repeat using each possible observation as the test set.
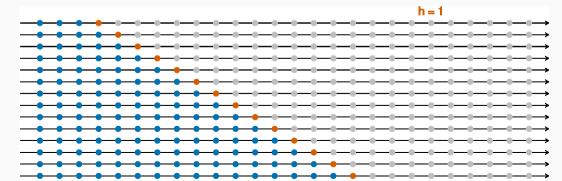- Compute accuracy measure over all errors.
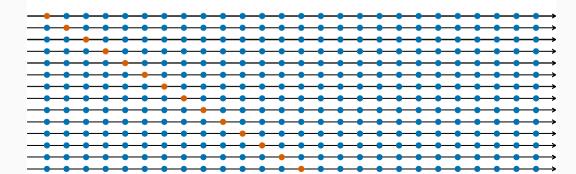
Cross-validation

Traditional evaluation
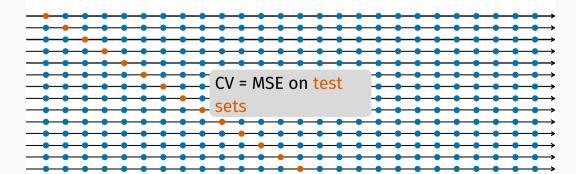
Cross-validation

Traditional evaluation

Time series cross-validation

Cross-validation

Traditional evaluation

Leave-one-out cross-validation

Cross-validation

Traditional evaluation

Training data   Test data

time

Leave-one-out cross-validation

CV = MSE on test sets

Bayesian Information Criterion

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- BIC penalizes terms more heavily than AIC
- Also called SBIC and SC.
- Minimizing BIC is asymptotically equivalent to leave-v-out cross-validation when $v = T[1 - 1/(log(T) - 1)]$.

Choosing regression variables

Best subsets regression

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- Choose the best model based on one of the measures of predictive ability (CV, AIC, AICc).

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Backwards stepwise regression

- Start with a model containing all variables.
- Subtract one variable at a time. Keep model if lower CV.
- Iterate until no further improvement.
- Not guaranteed to lead to best model.

Ex-ante versus ex-post forecasts

- *Ex ante forecasts* are made using only information available in advance.
  - ▸ require forecasts of predictors
- *Ex post forecasts* are made using later information on the predictors.
  - ▸ useful for studying behaviour of forecasting models.
- trend, seasonal and calendar variables are all known in advance, so these don't need to be forecast.

3 Produce forecasts of US leisure employment using your best regression model.

4 Why don't you need to forecast the predictors?