

# ETC3550/ETC5550

## Applied forecasting

Week 7: Seasonal exponential  
smoothing



# ETS models

**General notation**    E T S : ExponenTial S moothing



Error   T r e n d   S e a s o n

**Error:** Additive ("A") or multiplicative ("M")

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**General notation**    E T S : ExponenTial S moothing



Diagram illustrating the components of the ETS notation: Error, Trend, and Season.

**Error**   **Trend**   **Season**

**Error:** Additive ("A") or multiplicative ("M")

**Trend:** None ("N"), additive ("A"), or damped ("Ad").

# ETS models

**General notation**    E T S : ExponenTial Smoothing



The diagram consists of three arrows pointing upwards from the words 'Error', 'Trend', and 'Season' to the letters 'E', 'T', and 'S' respectively in the notation 'ETS : ExponenTial Smoothing'. The 'T' in 'Trend' and 'T' in 'ExponenTial' are bolded in the original image.

Error **T**rend **S**eason

**Error:** Additive ("A") or multiplicative ("M")

**Trend:** None ("N"), additive ("A"), or damped ("Ad").

**Seasonality:** None ("N"), additive ("A") or multiplicative ("M")

# ETS(A,N,A): No trend, additive seasonal model

State equations

$$\ell_t = \ell_{t-1} + \alpha \varepsilon_t$$

$$S_t = S_{t-m} + \gamma \varepsilon_t$$

Observation equation

$$y_t = \ell_{t-1} + S_{t-m} + \varepsilon_t$$

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t + S_{t+h-m(k+1)}$$

- Forecast errors:  $\varepsilon_t = y_t - \hat{y}_{t|t-1} \sim \text{NID}(0, \sigma^2)$
- $k$  is integer part of  $(h-1)/m$ .

# ETS(A,A,A): Holt-Winters additive method

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$$

$$b_t = b_{t-1} + \beta \varepsilon_t$$

$$s_t = s_{t-m} + \gamma \varepsilon_t$$

Observation equation

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$$

- Forecast errors:  $\varepsilon_t = y_t - \hat{y}_{t|t-1} \sim \text{NID}(0, \sigma^2)$
- $k$  is integer part of  $(h - 1)/m$ .

# ETS(M,A,M): Holt-Winters multiplicative method

State equations

$$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$$

$$b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$$

$$s_t = s_{t-m}(1 + \gamma\varepsilon_t)$$

Observation equation

$$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$$

Forecast equation

$$\hat{y}_{t+h|t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$$

- Relative forecast errors:  $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- $k$  is integer part of  $(h - 1)/m$ .

# All ETS models

## Additive Error

## Seasonal Component

Trend Component		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
A	(Additive)	A,A,N	A,A,A	A,A,M
A <sub>d</sub>	(Additive damped)	A,A <sub>d</sub> ,N	A,A <sub>d</sub> ,A	A,A <sub>d</sub> ,M



# All ETS models

## Additive Error

		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
	Trend Component			
	N (None)	A,N,N	A,N,A	A,N,M
	A (Additive)	A,A,N	A,A,A	A,A,M
	A <sub>d</sub> (Additive damped)	A,A <sub>d</sub> ,N	A,A <sub>d</sub> ,A	A,A <sub>d</sub> ,M

## Multiplicative Error

		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
	Trend Component			
	N (None)	M,N,N	M,N,A	M,N,M
	A (Additive)	M,A,N	M,A,A	M,A,M
	A <sub>d</sub> (Additive damped)	M,A <sub>d</sub> ,N	M,A <sub>d</sub> ,A	M,A <sub>d</sub> ,M

# Additive error models

Trend	Seasonal		
	N	A	M
N	$y_t = \ell_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t$	$y_t = \ell_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = \ell_{t-1} s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / \ell_{t-1}$
A	$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$ $b_t = b_{t-1} + \beta \varepsilon_t$	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$ $b_t = b_{t-1} + \beta \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $b_t = b_{t-1} + \beta \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / (\ell_{t-1} + b_{t-1})$
A <sub>d</sub>	$y_t = \ell_{t-1} + \phi b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t$ $b_t = \phi b_{t-1} + \beta \varepsilon_t$	$y_t = \ell_{t-1} + \phi b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t$ $b_t = \phi b_{t-1} + \beta \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1}) s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $b_t = \phi b_{t-1} + \beta \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / (\ell_{t-1} + \phi b_{t-1})$

# Multiplicative error models

Trend	Seasonal		
	N	A	M
N	$y_t = \ell_{t-1}(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$	$y_t = (\ell_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + \alpha(\ell_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + s_{t-m})\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m}(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$
A	$y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$
A <sub>d</sub>	$y_t = (\ell_{t-1} + \phi b_{t-1})(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + \phi b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1})s_{t-m}(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + \phi b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

# Estimating ETS models

- Smoothing parameters  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\phi$ , and the initial states  $\ell_0$ ,  $b_0$ ,  $s_0$ ,  $s_{-1}, \dots, s_{-m+1}$  are estimated by maximising the “likelihood” = the probability of the data arising from the specified model.
- For models with additive errors equivalent to minimising SSE.
- For models with multiplicative errors, **not** equivalent to minimising SSE.

# Innovations state space models

Let  $\mathbf{x}_t = (\ell_t, b_t, s_t, s_{t-1}, \dots, s_{t-m+1})$  and  $\varepsilon_t \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$ .

$$y_t = \underbrace{h(\mathbf{x}_{t-1})}_{\mu_t} + \underbrace{k(\mathbf{x}_{t-1})\varepsilon_t}_{e_t}$$

$$\mathbf{x}_t = f(\mathbf{x}_{t-1}) + g(\mathbf{x}_{t-1})\varepsilon_t$$

## Additive errors

$$k(\mathbf{x}_{t-1}) = 1. \quad y_t = \mu_t + \varepsilon_t.$$

## Multiplicative errors

$$k(\mathbf{x}_{t-1}) = \mu_t. \quad y_t = \mu_t(1 + \varepsilon_t).$$

$$\varepsilon_t = (y_t - \mu_t)/\mu_t \text{ is relative error.}$$

# Innovations state space models

## Estimation

$$\begin{aligned} L^*(\theta, \mathbf{x}_0) &= T \log \left( \sum_{t=1}^T \varepsilon_t^2 \right) + 2 \sum_{t=1}^T \log |k(\mathbf{x}_{t-1})| \\ &= -2 \log(\text{Likelihood}) + \text{constant} \end{aligned}$$

- Estimate parameters  $\theta = (\alpha, \beta, \gamma, \phi)$  and initial states  $\mathbf{x}_0 = (\ell_0, b_0, s_0, s_{-1}, \dots, s_{-m+1})$  by minimizing  $L^*$ .

# Parameter restrictions

## Traditional region

- $0 < \alpha < 1$
- $0 < \beta < \alpha$
- $0 < \gamma < 1 - \alpha$
- $0.8 < \phi < 0.98$  — to prevent numerical difficulties.

## Admissible region

- To prevent observations in the distant past having a continuing effect on current forecasts
- Usually (but not always) less restrictive than *traditional* region
- e.g., ETS(A,N,N): *traditional*  $0 < \alpha < 1$   
*admissible*  $0 < \alpha < 2$

# Parameter restrictions

**table default:**  
intersection of both regions

## *Traditional* region

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- $0 < \beta < \alpha$
- $0 < \gamma < 1 - \alpha$
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- Usually (but not always) less restrictive than *traditional* region
- e.g., ETS(A,N,N): *traditional*  $0 < \alpha < 1$   
*admissible*  $0 < \alpha < 2$



# Model selection

## Akaike's Information Criterion

$$\text{AIC} = -2 \log(L) + 2k$$

where  $L$  is the likelihood and  $k$  is the number of parameters & initial states estimated in the model.

# Model selection

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## Corrected AIC

$$AIC_c = AIC + \frac{2k(k+1)}{T - k - 1}$$

which is the AIC corrected (for small sample bias).

# Model selection

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## Bayesian Information Criterion

$$BIC = AIC + k[\log(T) - 2].$$

# AIC and cross-validation

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

# Automatic forecasting

## From Hyndman et al. (IJF, 2002):

- Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE (or some other criterion).
- Select best method using AICc:
- Produce forecasts using best method.
- Obtain forecast intervals using underlying state space model.

Method performed very well in M3 competition.

## Some unstable models

- Some of the combinations of (Error, Trend, Seasonal) can lead to numerical difficulties; see equations with **division by a state**.
- These are:  $ETS(A,N,M)$ ,  $ETS(A,A,M)$ ,  $ETS(A,A_d,M)$ .

# ETS models

## Additive Error

		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
Trend Component	N (None)	A,N,N	A,N,A	<del>A,N,M</del>
	A (Additive)	A,A,N	A,A,A	<del>A,A,M</del>
	A <sub>d</sub> (Additive damped)	A,A <sub>d</sub> ,N	A,A <sub>d</sub> ,A	<del>A,A<sub>d</sub>,M</del>

## Multiplicative Error

		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
Trend Component	N (None)	M,N,N	M,N,A	M,N,M
	A (Additive)	M,A,N	M,A,A	M,A,M
	A <sub>d</sub> (Additive damped)	M,A <sub>d</sub> ,N	M,A <sub>d</sub> ,A	M,A <sub>d</sub> ,M

# Residuals

## Response residuals

$$\hat{e}_t = y_t - \hat{y}_{t|t-1}$$

## Innovation residuals

Additive error model:

$$\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1}$$

Multiplicative error model:

$$\hat{\varepsilon}_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}}$$