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THE OPTIMAL STRATEGY FOR RUNNING A RACE (A MATHEMATICAL MODEL FOR WORLD RECORDS FROM 50 m TO 275 km)

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Abstract—The problem of maximizing the distance run in a given time is considered as a problem in optimal control. The parameters involved are the resistive force, the maximal propulsive force available, the energy supply rate, the initial energy level and a fatigue constant. The latter is the proportionality constant relating the additional rate of energy loss due to fatigue at time t to the energy already spent in running up to time t. The constrained optimization problem is reduced to a single variable problem in the first switching time, the time at which the runner changes from maximal thrust to constant cruising speed. An algorithm is presented for the determination of the maximum distance. Short and long race approximations are also presented. Methods are described for determining the parameters.

Predictions of the model are compared with current world track records for both men and women for distances up to 274.48 km (the maximum distance run by a man in 24 hours). The mean absolute error for 24 men's records is 2.35% and for 16 women's records is 1.61%.

1. INTRODUCTION

Scientists interested in man's highest physical achievements have long recognized the existence of valuable physiological data in world running records [1–5]. Most of these early studies were of an empirical rather than a theoretical nature. The extraction of useful data from world records had to await the formulation of a tractable mathematical model of competitive running. In 1974, J. B. Keller [6,7] produced such a model based on simple dynamics and the calculus of variations. Keller determined the theoretical relationship between the shortest time T in which a given distance D can be run in terms of four physiological parameters. By fitting the theoretical curve to four world records he determined the parameters and could then predict record times for distances up to 10 km. The predictions agreed with observations to within a remarkable ± 3 percent. The theory indicated that a runner should run at maximum acceleration for all races of distance D less than a critical distance D_c of 291 m; for longer races he should run at maximum acceleration for one or two seconds, then constant speed for the major portion of the race until the last second or two when there is a slight deceleration.

In this paper, the equivalent problem of how to run the maximum distance in a given time is solved more directly by treating it as a problem in optimal control, without the physical assumptions needed in Keller's solution. The problem is ultimately reduced to a single variable optimization problem in the first switching time. The solution is expressed in dimensionless form and an algorithm is presented for the determination of the maximum distance. Short and long race approximations are presented which show excellent agreement with the exact solutions, so that the relatively complicated exact theory is only needed for intermediate-distance races (300 to 600 m). The reason for the restriction of the model to races under 10 km is elucidated, and the model is extended to longer races by the introduction of a fatigue factor. This factor

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represents an additional variable drain on the runner's energy due to fatigue, which at time t is assumed to be proportional to the energy already spent in running up to time t. A long race approximation incorporating the fatigue factor is developed. The effect of fatigue is negligible for races under 2 or 3 km.

Simple methods for the determination of the five physiological parameters, using the extended model, are presented and predictions of the theory are compared with current world records for both men and women for distances up to 274,480 m (the maximum distance run by a man in 24 hours). The mean absolute error for 24 men's races is 2.35% and for 16 women's races up to 214.9 km is 1.61%.

2. KELLER'S MODEL

How should a runner vary his/her speed $v(t)$ during a race of distance D in order to run it in the shortest time? The time T is given by

$$D = \int_0^T v(t)\,dt. \tag{1}$$

Assume that the total resistive force (both internal and external) per unit mass is proportional to the speed, so that the runner's equation of motion is

$$\frac{dv}{dt} + \frac{1}{\tau} v = f(t), \tag{2}$$

where $f(t)$ is the propulsive force per unit mass and τ is a constant. $f(t)$ is the control variable which is at the runner's disposal, subject to two restrictions. First, it cannot exceed a maximum value F;

$$f(t) \le F. \tag{3}$$

Second, the runner's rate of doing work $f(t)v(t)$ is constrained by his energy supply. Denote by $E(t)$ the energy equivalent of the available oxygen in his muscles per unit mass at time t. Energy is being used at the rate $f(t)v(t)$. Assume energy is being supplied by breathing and circulation at a constant rate σ per unit mass in excess of that supplied in the rest state (as first proposed by Hill [1]). Therefore

$$\frac{dE}{dt} = \sigma - f(t)v(t) \qquad \text{with } E(0) = E_0. \tag{4}$$

Since $E(t)$ cannot be negative,

$$E(t) \ge 0. \tag{5}$$

Given the physiological constants τ, F, σ and E_0, and the distance D, the runner's problem is to choose his speed $v(t)$, consistent with the above equations and inequalities and $v(0) = 0$, so that T is minimal. Instead of minimizing T for given D, we shall consider the reciprocal problem of maximizing D for given T. These two problems clearly yield the same relation between D and T. (Instead of how fast can you run a mile, we ask how far can you run in 4 minutes.) Using (2) to eliminate $f(t)$ and integrating (4), the problem can be reformulated as follows:

Problem: Given positive constants T, τ, F, σ and E_0, find $v \in C[0,T]$ to maximize the area under the velocity curve

$$D = \int_0^T v(t)\,dt$$

subject to the constraints $v(0) = 0$,

$$\dot{v} + \frac{1}{\tau} v \le F, \tag{6}$$

and

$$E(t) = E_0 + \sigma t - \int_0^t v\left(\dot{v} + \frac{1}{\tau} v\right)\,dt \ge 0 \tag{7}$$

for all $t \in [0,T]$ (Note that the minimizing v is $v(t) \equiv 0$).

3. SOLUTION

Case 1. 1st Constraint Tight

Suppose the first constraint (6) is tight, i.e., suppose the runner uses maximal thrust F. The solution of the initial-value problem

$$\dot{v} + \frac{1}{\tau} v = F; \quad v(0) = 0$$

is

$$v(t) = F\tau(1 - e^{-t/\tau}) \tag{8}$$

and then

$$D = F\tau^2 \left(\frac{T}{\tau} - 1 + e^{-T/\tau} \right) \tag{9}$$

provided the second constraint (7) is satisfied, i.e., provided

$$E_0 + \sigma t - \int_0^t v(t) F \, dt \geq 0 \quad \text{for all } t \in [0, T].$$

Substituting (8) for $v(t)$ and integrating, this becomes, in dimensionless form,

$$e^{-t/\tau} - 1 \leq \frac{E_0}{F^2\tau^2} - \left(1 - \frac{\sigma}{F^2\tau}\right) \frac{t}{\tau}, \qquad \text{for all } \frac{t}{\tau} \in \left[0, \frac{T}{\tau}\right]. \tag{10}$$

Now if $\sigma/F^2\tau \geq 1$, i.e., if the energy supply rate σ and/or the resistance coefficient $1/\tau$ are large enough relative to the maximal thrust F per unit mass which the runner can exert, then this inequality is satisfied for all $t/\tau \in (0, \infty)$. In other words, the runner can run with maximum thrust indefinitely; energy considerations are irrelevant! The solution to the problem is given by (8) and (9). On the other hand, if (more realistically) $\sigma/F^2\tau < 1$, then there exists a critical time T_c such that the inequality (10) is satisfied for all $t/\tau \leq T_c/\tau$, that is for all races for which $T \leq T_c$; see Figure 1. Denote the value of D corresponding to $T = T_c$ in (9) by D_c. Thus for short races (sprints or dashes for which $D < D_c$), there is slack in the energy constraint and the runner finishes the race with energy to spare. His optimal strategy is simply to run 'flat out' with maximum thrust F and increasing velocity given by (8) throughout the race. This clearly maximizes the area under the velocity curve. The relation between D and T for such races is given by (9). Note that by (8), $F\tau$ provides an upper bound for a runner's speed.

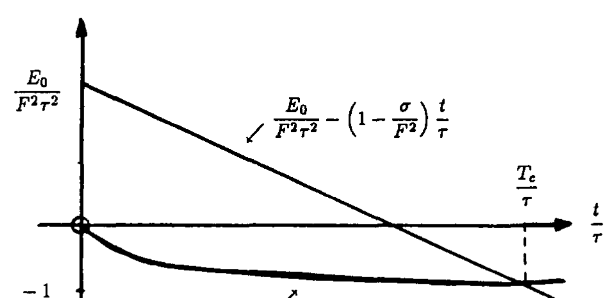

Figure 1.

In the course of running a race of distance $D < D_c$ optimally, a runner runs all races of distances less than D optimally as well (shades of Bellman's principle of optimality). Thus, with facilities such as on-screen electronic timing and freeze-frame techniques, by examining a video of a single record run in the 200 m race, it should be possible to obtain the optimal D vs. T relation for all races under 200 m.

Case 2. Neither Constraint Tight

In this case, the problem is a simple unconstrained problem in the calculus of variations: choose $x(t)$ to maximize

$$\int_0^T G(t, x(t), \dot{x}(t))\, dt,$$

where $G(t, x(t), \dot{x}(t)) = \dot{x}(t) = v(t)$. For such an integrand, the Euler-Lagrange equation implies that the extremals are straight lines, i.e., $x(t) = a + Vt$ or $v(t) = \dot{x}(t) = V$, a constant. Thus, when neither constraint is operative, the runner should run at a constant 'cruising' speed V.

Case 3. 2nd Constraint Tight

Suppose now that the energy constraint (7) is tight throughout some interval, i.e., $E(t) \equiv 0$, or equivalently

$$\begin{aligned} E_0 + \sigma t &= \int_0^t v(\dot{v} + \frac{1}{\tau} v)\, dt \\ &= \frac{1}{2} v^2(t) + \frac{1}{\tau} \int_0^t v^2(t)\, dt \end{aligned} \tag{11}$$

for all time t in this interval. Differentiating with respect to t we have $\dot{E}(t) \equiv 0$ or

$$\frac{d}{dt} v^2(t) + \frac{2}{\tau} v^2(t) = 2\sigma, \tag{12}$$

a simple differential equation for $v^2(t)$, with solution

$$v(t) = \left(\sigma\tau + Ce^{-2t/\tau}\right)^{\frac{1}{2}}, \tag{13}$$

where the arbitrary constant C is determined by the velocity at the beginning of the interval. Note that $v(t)$ is increasing if $C < 0$, constant if $C = 0$ and decreasing if $C > 0$. We will see later that $C > 0$. Thus, in any interval where the energy constraint is tight, the runner's velocity is given by (13), provided the first constraint is satisfied, i.e., provided $\dot{v} + \frac{1}{\tau} v \leq F$. However, since $\dot{v} < 0$ and $v < F\tau$, this is certainly true.

4. SYNTHESIS

An optimal solution consists of a finite number of velocity arcs each of the type described in the above three cases: acceleration arcs during which the runner exerts maximal thrust, constant 'cruising' velocity arcs, and deceleration arcs in intervals during which $E(t) \equiv 0$. These must be combined in such a way that the runner's velocity is continuous throughout the race and the total area under the velocity curve is maximal. Clearly, there is a unique optimal solution consisting in general of three arcs: an acceleration arc given by (9) (the only arc in the case of sprints) during an initial interval $[0, t_1]$, followed by a constant velocity arc during an interval $[t_1, t_2]$, followed by a deceleration arc during the final interval $[t_2, T]$. No other feasible combination of arcs can provide as large an area! The first two intervals are fueled by E_0 and σ, the third only by σ. There are two switching times t_1 and t_2 separating these three intervals; see Figure 2. The continuity requirement at t_1 and t_2 gives

$$V = F\tau \left(1 - e^{-t_1/\tau}\right) \tag{14}$$

and

$$C = \left(V^2 - \sigma\tau\right) e^{2t_2/\tau}. \tag{15}$$

Thus,

$$v(t) = \begin{cases} F\tau(1 - e^{-t/\tau}), & 0 \le t \le t_1 \\ V, & t_1 \le t \le t_2 \, . \\ \left(\sigma\tau + \left(V^2 - \sigma\tau\right) e^{-2(t-t_2)/\tau}\right)^{\frac{1}{2}}, & t_2 \le t \le T \end{cases} \tag{16}$$

Denoting the distances run during the three intervals by D_1, D_2 and D_3 respectively, we have

$$D_1 = \int_0^{t_1} v(t)\, dt = F\tau^2 \left(\frac{t_1}{\tau} - 1 + e^{-t_1/\tau}\right),$$

$$D_2 = \int_{t_1}^{t_2} v(t)\, dt = V\,(t_2 - t_1)$$

and

$$D_3 = \int_{t_2}^{T} v(t)\, dt = \int_{t_2}^{T} \left(\sigma\tau + Ce^{-2t/\tau}\right)^{1/2} dt.$$

Evaluating the integral for D_3 yields

$$D_3 = F\tau^2\left(1 - e^{-t_1/\tau}\right) + \frac{\tau(\sigma\tau)^{1/2}}{2}\left[\ln\frac{V - \sqrt{\sigma\tau}}{V + \sqrt{\sigma\tau}} - 2K - \ln\frac{K-1}{K+1}\right], \tag{17}$$

where

$$K = \left[1 + \left(\frac{V^2}{\sigma\tau} - 1\right) e^{2(t_2 - T)/\tau}\right]^{1/2}.$$

Note that, since $C > 0$, $V^2 > \sigma\tau$. The total distance D is then a function of t_1 and t_2 given by $D = D_1 + D_2 + D_3$. Expressing the result in dimensionless form we find

$$\frac{D}{F\tau^2} = \frac{t_1}{\tau} + \left(1 - e^{-t_1/\tau}\right)\frac{t_2 - t_1}{\tau} + \frac{1}{2}\left(\frac{\sigma}{F^2\tau}\right)^{1/2}\left[\ln\frac{V - \sqrt{\sigma\tau}}{V + \sqrt{\sigma\tau}} - 2K - \ln\frac{K-1}{K+1}\right]. \tag{18}$$

(Keller's equation 3.17 in [7], the counterpart of equation (18), is in error in the D_3 component, since $\tanh^{-1}$ is not defined for the values of the argument shown; e.g., for $\tanh^{-1}(1/\lambda)(\tau/\sigma)^{1/2}$ to be defined, λ must exceed $(\tau/\sigma)^{1/2}$ whereas, by 3.15, $\lambda < (\tau/\sigma)^{1/2}$).

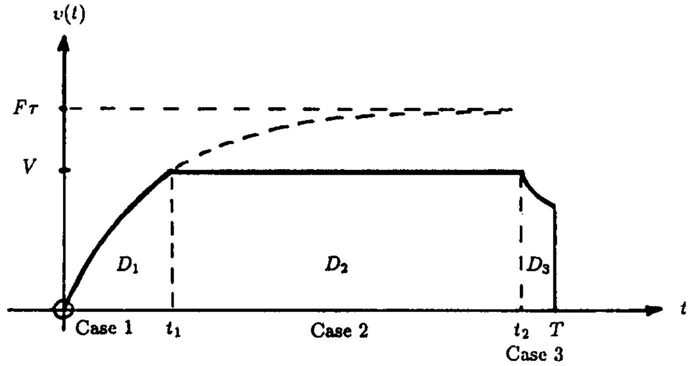

Figure 2.

The switching time t_2 is the time at which $E(t)$ first equals zero, the first time at which (11) is satisfied. Thus

$$\begin{aligned} E_0 + \sigma t_2 &= \frac{1}{2}v^2(t_2) + \frac{1}{\tau}\int_0^{t_2} v^2(t)\, dt \\ &= \frac{1}{2}V^2 + \frac{1}{\tau}\int_0^{t_1} F^2\tau^2(1 - e^{-t/\tau})^2\, dt + \frac{1}{\tau}(t_2 - t_1)V^2, \end{aligned} \tag{19}$$

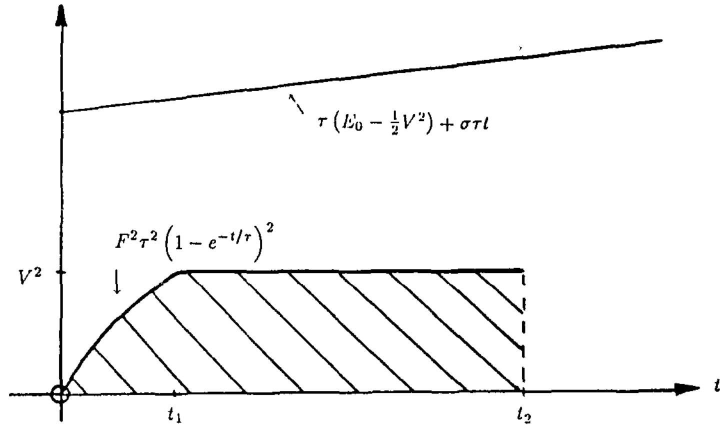

Figure 3. The switching time t_2 is the time at which the shaded area equals $\tau(E_0 - \frac{1}{2}V^2) + \sigma\tau t$.

where V is given by (14). See Figure 3 for a graphical interpretation. This can be solved for t_2, yielding

$$\frac{t_2}{\tau} = \frac{E_0 + F\tau V + \left(F^2\tau^2 - V^2\right)\ln\left(1 - \frac{V}{F\tau}\right)}{V^2 - \sigma\tau}. \tag{20}$$

Hence D is ultimately a function of V only, or equivalently via (14), of t_1 only. The key question for the runner is, "What is my optimal constant speed V?" V is determined by t_1, the time at which he switches from full acceleration to constant speed. Physically, if t_1 is too small he will (inefficiently) finish the race with energy to spare; if t_1 is too large he will use up his energy and enter the deceleration phase too early. So, t_1 must be chosen to maximize D. We do this by setting

$$\frac{dD}{dt_1} = \frac{\partial D}{\partial t_1} + \frac{\partial D}{\partial t_2}\frac{dt_2}{dt_1} = 0,$$

where $\frac{\partial D}{\partial t_1}$ and $\frac{\partial D}{\partial t_2}$ are determined from the expression for D with the D_3 component still in integral form and $\frac{dt_2}{dt_1}$ is obtained from differentiating (20). This yields the equation

$$\frac{E_0 + F\tau V + \left(F^2\tau^2 - V^2\right)\ln\left(1 - \frac{V}{F\tau}\right)}{V^2 - \sigma\tau} = \frac{T}{\tau} + \frac{1}{2}\ln\frac{1}{4}\left(1 - \frac{\sigma\tau}{V^2}\right). \tag{21}$$

ALGORITHM. *To determine the maximum distance D which can be run in time T:*

if $T \le T_c$,		*calculate D from (9);*
if $T > T_c$,	*(i)*	*solve (21) iteratively for V,*
	(ii)	*solve (14) for t_1,*
	(iii)	*calculate t_2 from (20) (Note that t_2/τ is the leftside of (21))*
and	*(iv)*	*calculate D from (18).*

5. DETERMINATION OF THE PARAMETERS FROM WORLD RECORDS

To run a race in record time a runner must be in peak condition and use an optimal or near-optimal strategy. The parameters can therefore be estimated by comparing the predictions of the model with world records. By assuming that races of 220 yds. and less are sprints, Keller determined the best values of F and τ by minimizing the sum of the squares of the relative errors for races of 50 yds., 50 m, 60 yds., 60 m, 100 yds., 100 m, 200 m, and 220 yds. He then determined the best values of E_0 and σ by minimizing the sum of squares of relative errors for 14 races from 400 to 10,000 m. His values are:

$$\begin{aligned} F &= 12.2\,\mathrm{m/sec^2}, \quad \tau = 0.892\,\mathrm{sec}, \\ \sigma &= 9.93\,\mathrm{cal/kg\ sec} = 41.56\,\mathrm{m^2/sec^3} \\ T_c &= 27.7\,\mathrm{sec} \ \text{ and } D_c = 291\,\mathrm{m} \\ E_o &= 575\,\mathrm{cal/kg} = 2409\,\mathrm{m^2/sec^3}. \end{aligned}$$

With these values the model predictions agreed with observations to within $\pm 3\%$. Values of t_1 ranged from 1.78 sec for the 400 m race to 0.75 sec for 10,000 m; values of $T - t_2$ ranged from 0.86 sec at 400 m to 2.12 sec at 10,000 m.

6. APPROXIMATIONS

Short Races

For races with $D < D_c$ the relation between T and D is given by (9). Since $e^{-T/\tau} \ll 1$ for any race of reasonable length, the relation is approximately linear: $D \simeq F\tau(T - \tau)$ or

$$T \simeq \frac{D}{F\tau} + \tau \tag{22}$$

and a graph of D vs. T is a straight line (except near the origin) with T-intercept τ and slope $F\tau$. This approximation is very good, as can be seen by comparing Keller's exact values with those calculated from (22) in Table 1:

Table 1. Comparison of approximate times from (22) with Keller's exact values for sprints.

D	50 yd	50 m	60 yd	60 m	100 yd	100 m	200 m	220 yd
T (exact)(sec)	5.09	5.48	5.93	6.40	9.29	10.07	19.25	19.36
T (approx.)(sec)	5.09	5.49	5.93	6.41	9.30	10.08	19.27	19.38

Long Races

Since t_1 and $T - t_2$ are both small compared with $t_2 - t_1$, the middle constant-velocity interval is the dominant one, especially for longer races. We can therefore approximate (16) by $v(t) = V$, $0 < t < T$, with V determined by the energy constraint (11):

$$E_0 + \sigma T = \frac{1}{2}V^2 + \frac{1}{\tau}TV^2.$$

Neglecting the first term on the right side, in comparison with the second, we find

$$V^2 \simeq \frac{E_0\tau}{T} + \sigma\tau. \tag{23}$$

(This result can also be obtained from (21) by neglecting the ln terms and neglecting $F\tau V$ in comparison with E_0; in fact, the $F\tau V$ term and the ln term on the left side of (21) tend to cancel.) Replacing V in (23) by D/T, solving the resulting quadratic equation for T and discarding the inadmissible root yields

$$T \simeq \frac{1}{2\sigma}\left[\left(E_0^2 + \frac{4\sigma D^2}{\tau}\right)^{\frac{1}{2}} - E_0\right]. \tag{24}$$

Values of T computed from (24) are compared with Keller's exact values in Table 2. The approximation is again very good, even for the 400 m race.

Table 2. Comparison of approximate times from (24) with Keller's exact values.

D (metres)	400	800	1500	3000	5000	10000
T (exact)(sec)	43.27	105.95	219.4	464.96	793.11	1614.1
T (approx.) (sec)	42.83	105.58	219.1	464.63	792.80	1613.8

Note that (23) suggests that a plot of V^2 versus $1/T$ should be linear with V^2-intercept $\sigma\tau$ and slope $E_0\tau$.

7. RESTRICTION OF THE MODEL TO RACES UNDER 10 KM

We return to energy considerations for races with $D > D_c$. Energy is being supplied to the runner at a net marginal rate of σ; while running at a constant velocity V, it is being consumed at a marginal rate of $(\dot{V} + \frac{1}{\tau}V)V$ or V^2/τ. Therefore, the supply rate σ allows the runner to maintain a constant velocity V given by $V^2/\tau = \sigma$ or $V = (\sigma\tau)^{1/2}$. Even though the initial energy E_0 may have been exhausted ('running on empty') the supply rate σ is sufficient to fuel a constant velocity of $(\sigma\tau)^{1/2}$ indefinitely! Using Keller's values, $(\sigma\tau)^{1/2} = 6.09\,\text{m/sec}$. This serves as a lower bound for $v(t)$. However, the average velocity observed for the 10 km race in Keller [6] is 6.03 m/sec and, of course, longer races have even lower average velocities. For example, the current world record for the marathon (26 miles, 385 yds.) corresponds to an average velocity of 5.54 m/sec and the record distance of 274.48 km run in 24 hours corresponds to an average velocity of 3.18 m/sec. This is the reason that Keller's model fails for races of 10 km and longer. It also explains why $V^2 - \sigma\tau > 0$ in (16), (17), (18) and (20), and why the constant C in (13) and (15) is positive, thus verifying that the runner's velocity is indeed decreasing during the final interval (t_2, T). Runners often finish with a burst of acceleration rather than the deceleration of the model; this may be due to their desire to beat their competitors rather than beat the clock. However, the model indicates that the energy needed to produce this final burst would be better used to fuel a larger constant velocity V during the middle interval.

8. EXTENSION OF THE MODEL FOR LONGER RACES

Since runners clearly cannot sustain a velocity of $(\sigma\tau)^{1/2}$ indefinitely, a new factor must be introduced, and a reasonable candidate is fatigue. Fatigue saps the runner's energy and is cumulative. As a first approximation we might assume that the rate of energy loss is simply proportional to the length of time the runner has been running. However, running at a higher velocity inflicts a higher toll. So let us suppose that in (4) there is an additional rate of energy loss due to fatigue, which at time t is proportional to the energy already spent in running up to time t. In other words, the rate of energy loss due to fatigue at time t

$$
\begin{aligned}
&= k \int_0^t f(t)v(t)\,dt \\
&= k \int_0^t \left(\dot{v} + \frac{1}{\tau}v\right) v\,dt \\
&= \frac{kV^2}{\tau} t,
\end{aligned}
$$

since $v(t) = V$ and $\dot{v} = 0$ during most of $(0, t)$, where k is a constant (dimension sec^{-1}). Equation (4) is then replaced by

$$\frac{dE}{dt} = \sigma - \left(\dot{v} + \frac{1}{\tau}v\right) v(t) - \frac{kV^2 t}{\tau},$$

so that

$$E(t) = E_0 + \sigma t - \frac{1}{2}v^2(t) - \frac{1}{\tau}\int_0^t v^2(t)\,dt - \frac{kV^2t^2}{2\tau}. \tag{25}$$

Using the long race approximation, that $v(t) \simeq V$ throughout the race and $E(T) = 0$, gives

$$E_0 + \sigma T = \frac{1}{2}V^2 + \frac{T}{\tau}V^2 + \frac{kV^2T^2}{2\tau}.$$

Solving for V^2, and neglecting 1/2 in comparison with T/τ, we find

$$V^2 \simeq \frac{\frac{E_0\tau}{T} + \sigma\tau}{1 + \gamma T}, \tag{26}$$

where $\gamma = k/2$. If $\gamma = 0$, i.e., with the fatigue factor $1 + \gamma T$ equal to 1, this specializes to the earlier long race approximation (23). Since $V \simeq D/T$ for long races, we have

$$D^2 \simeq \frac{\tau T(E_0 + \sigma T)}{1 + \gamma T}. \tag{27}$$

Solving (27) for T in terms of D we find

$$T \simeq \frac{1}{2\sigma\tau}\left[(A^2 + 4\sigma\tau D^2)^{1/2} - A\right], \tag{28}$$

where $A = E_0\tau - \gamma D^2$. With $\gamma = 0$, (28) specializes to (24).

During the final interval (t_2, T) when the energy constraint is tight, i.e., $E(t) \equiv 0$, so that $\dot{E}(t) = 0$, equation (25) yields the differential equation

$$\frac{d}{dt}v^2(t) + \frac{2}{\tau}v^2(t) = 2\sigma - \frac{2kV^2}{\tau}t$$

in place of (12). The solution is

$$v^2(t) = Ce^{-2t/\tau} + \sigma\tau + \frac{1}{2}kV^2\tau - kV^2t, \quad t \in [t_2, T],$$

and using the condition $v(t_2) = V$ to determine the constant C gives

$$v^2(t) = (V^2 + kV^2t_2 - \sigma\tau - \frac{1}{2}kV^2\tau)e^{-2(t-t_2)/\tau} + \sigma\tau + \frac{1}{2}kV^2(\tau - 2t)$$

for $t \in [t_2, T]$. It is easily shown that $dv^2(t)/dt$ is negative, so that the velocity is decreasing during (t_2, T), again confirming a deceleration during the final interval in an optimally run race.

9. RESULTS

Men's Records

A least squares analysis of current world records [8]* for men for races up to and including 200 m yielded $\tau = 0.739$ and $F\tau = 10.61$. Results calculated using these values in the short race approximation (22) are shown in Table 3. The mean absolute error is under 2%.

Table 3. Comparison of current world record times with those calculated from the model for short races.

D (m)	45.7	50	54.9	60	91.4	100	200
T (record)(sec)	5.15	5.61	5.90	6.50	9.11	9.92	19.72
T (model) (sec)	5.05	5.45	5.91	6.39	9.35	10.16	19.59

Data for races from 800 m to 100,000 m were plotted on a graph of (average velocity)^2 versus time^{-1}. The long race approximation without fatigue (23) suggests that such a plot should be linear. It is approximately linear for races from 800 m to 5000 m but curves away beyond 5000 m as predicted by the long race approximation with fatigue (26). $\sigma\tau$ was estimated from the V^2-intercept of the linear portion. For very long races, when $E_0\tau/T$ is small in comparison with $\sigma\tau$, the approximation (26) becomes $V^2 \simeq \sigma\tau/(1 + \gamma T)$. γ was estimated from the data for the 100 mile race using this approximation. Finally, having values for $\sigma\tau$ and γ, $E_0\tau$ was estimated from (26) using data for the 800 m race. The values obtained are

$$E_0\tau = 2180, \quad \sigma\tau = 40.6, \quad \gamma = 4.08 \times 10^{-5}.$$

*Excluding Ben Johnson's records for 50, 60 and 100 m, which were disallowed in 1990.

Table 4. Comparison of current world record times for long races with those predicted by the extended model.

D (m)	800	1000	1500	1609	2000	3000	5000
T (record)(sec)	101.73	132.18	209.46	226.32	290.81	452.1	778.39
T (model)(sec)	101.8	132.8	211.1	228.3	290.0	449.0	770.6

D (m)	10000	20000	25000	30000	42195	50000	100000	160934
T (record)(sec)	1633.8	3444.2	4435.8	5358.8	7609	10086	22220	41451
T (model)(sec)	1593	3318	4221	5153	7547	9172	21470	41390

Results calculated from (28) are compared with the world records in Table 4. The record distance run in 24 hrs. is 274,480 m; the model predicts 258,900 from (27). Aspiring world record holders might consider concentrating on the 25,000 and 50,000 m races.

For races of intermediate length ($D_c < D < 800\,\text{m}$) the long race approximation is not accurate enough and the exact equations must be used. The only standard race in this range is the 400 m for which the world record time is 43.29 sec. Using the algorithm described earlier and using the value of $V \simeq 9.54$ computed from (23) as a first approximation, the following results are obtained:

$$V = 9.627\,\text{m/sec},\ t_1 = 1.68\,\text{sec},\ t_2 = 42.60\,\text{sec},\ K = 1.0856,$$
$$D_1 = 11.05\,\text{m},\ D_2 = 393.97\,\text{m},\ D_3 = 5.47\,\text{m},$$

yielding 410.5 as the maximum distance which can be run in 43.29 sec. The effect of fatigue at this distance is negligible (0.18%). Values of the fatigue factor $1 + \gamma T$ at various distances are shown in Table 5.

Table 5. Values of the fatigue factor for various races.

Distance D(m)	400	800	1500	5000	10000	50000
Fatigue Factor $1 + \gamma$ T	1.0018	1.0042	1.0086	1.0314	1.0650	1.3742

Women's Records

The ratio of men's record time to women's record time is relatively constant for standard races up to the marathon 42,195 m, ranging from 0.882 for the mile to 0.937 at 100 m, and averaging 0.902 with a standard deviation of only 0.017. (There appear to be no data for women for races between 10,000 m and the marathon.) Thus the best women's performances are about 90% of the best men's.

The procedure outlined above was repeated with the women's data, with the following results: $\tau = 0.72$ and $F\tau = 9.76$ yielding Table 6.

Table 6. Comparison of women's world record times with the model predictions for short races.

D (m)	50	60	100	200
T (record)(sec)	6.06	7.00	10.49	21.34
T (model)(sec)	5.84	6.87	10.97	21.21

The parameters $\sigma\tau$, γ and $E_0\tau$ were estimated as for men, except that γ was obtained from the 100 km rather than the 100 mile record. The values obtained were:

$$E_0\tau = 1955, \quad \sigma\tau = 32.9, \quad \gamma = 5.11 \times 10^{-5},$$

from which the predicted times in Table 7 were calculated.

Table 7. Comparison of women's world record times with those calculated from the extended model for long races.

D (m)	800	1000	1500	1609	2000	3000	5000	10000	42195	100000
T (record)(sec)	113.28	150.6	232.47	256.71	328.69	502.62	877.33	1813.74	8466	26842
T (model)(sec)	113.3	147.8	235.0	254.2	323.1	500.8	871.8	1792	8833	26810

The maximum distance run by a woman in 24 hrs. is 214,901 m. The model predicts 213,040 m. For the 400 m race, the record is 47.60 sec. Using the algorithm yields $V = 8.67\,\mathrm{m/sec}$, $t_1 = 1.55\,\mathrm{sec}$, $t_2 \doteq 46.91\,\mathrm{sec}$, $D_1 = 10.26\,\mathrm{m}$, $D_2 = 393.31\,\mathrm{m}$ and $D_3 = 4.79\,\mathrm{m}$, giving $D = 408.4\,\mathrm{m}$.

10. CONCLUSION

The maximum distance D which can be run in a given time T depends on the parameters F, τ, E_0, σ and γ. F and τ are estimated from short race data; $E_0\tau$, $\sigma\tau$ and γ are estimated from long race data.

The minimum time T to run a race of distance D is computed as follows:

if $D \leq D_c$,	use the short race approximation (22),
if $D \geq 2D_c$,	use the long race approximation (28),
if $D_c < D < 2D_c$,	use the algorithm presented in Section 4.

Note that for intermediate races our algorithm involves the solution of only one nonlinear equation (21) rather than the three nonlinear equations which must be solved simultaneously in Keller's analysis.

The extended model provides results which agree well with those observed over a very wide range of distances from 50 yds. to 274 km and a corresponding range of times from 5 sec to 24 hours. The mean absolute error for 24 men's races is 2.3% and for 16 women's races is 1.6%.

A straightforward dimensional analysis of the problem leads to the conclusion that D must be given by an equation of the form

$$\frac{D}{F\tau^2} = \Phi\left(\frac{T}{\tau}, \frac{E_0}{F^2\tau^2}, \frac{\sigma}{F^2\tau}, \gamma\tau\right), \tag{29}$$

in which all the ratios appearing are dimensionless. Values of the parameters estimated from the data, together with the dimensionless ratios in (29), are presented in Table 8.

Table 8. Values of the parameters estimated from the data.

	Women	Men	Men (Keller [6])
$F\,\mathrm{m/sec^2}$	13.50	14.36	12.2
τ sec	0.723	0.739	0.892
$E_0\,\mathrm{m^2/sec^2}$	2793	3114	2409
E_0 cal/kg	667	743	575
$\sigma\,\mathrm{m^2/sec^3}$	47.0	58.0	41.6
σ cal/kg sec	11.2	13.9	9.93
E_0/σ sec	59.4	53.7	58.0
$\gamma\,\mathrm{sec^{-1}}$	5.11×10^{-5}	4.08×10^{-5}	
$E_0/F^2\tau^2$	29.3	27.7	26.3
$\sigma/F^2\tau$	0.357	0.380	0.313
T_c sec	32.7	31.7	27.6
D_c m	312	328	291

The difference in the estimated values of τ for men and women is only 2%. The estimated values of $E_0/F^2\tau^2$ (27.7 and 29.3), and of $\sigma/F^2\tau$ (0.380 and 0.357) are also close enough to warrant consideration of the possibility that these dimensionless ratios are the same for men and women. The dimensionless ratio $\gamma\tau$, which involves the fatigue factor, appears to be significantly different, 3.02×10^{-5} for men and 3.69×10^{-5} for women. However, if we focus on races in which fatigue does not play a large role, and assume that $E_0/F^2\tau^2$ and $\sigma/F^2\tau$ are the same for men and women, then for a given time T the ratio of the maximum distance which can be run by a woman to that run by a man is, from (29),

$$\frac{D_{\text{women}}}{D_{\text{men}}} = \frac{(F\tau^2)\text{women}}{(F\tau^2)\text{men}} = \frac{F_{\text{women}}}{F_{\text{men}}} = 0.900.$$

The observed average value of the ratio of men's record times to women's record times for 13 standard races up to 10,000 m is 0.904 (with standard deviation 0.019). Even for the marathon the observed ratio is 0.899. For longer races where fatigue becomes more important the ratio decreases to 0.829 at 100 km and to 0.782 for the ratio of distances run in 24 hours.

These speculations raise the question of the reliability of the parameter estimates. Keller's estimates, based on data from 1972 and earlier, differ significantly from ours, yet world records have improved only by about 2% on average since then. This may not be the best way to determine these physiological parameters. On the other hand, the model is robust in the sense that significantly different parameter sets may yield predictions of comparable accuracy.

Finally, the model presented here should apply, with appropriate adjustments, to other kinds of races such as cycling, swimming, ice-skating and horse-racing. The dimensionless ratios appearing in (29) will, of course, have values which are characteristic of the type of race.

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