Anderson-like localization in real-world non-normal networks

Joseph O'Brien, Kleber Oliveira, Timoteo Carletti,
James Gleeson and Malbor Asllani

21 - Sep - 2020 | NetSci 20

UNIVERSITY OF LIMERICK
OLLSCOIL LUIMNIGH

Fondúireacht Eolaíochta Éireann
Dá bhfuil romhainn
Science Foundation Ireland
For what's next

Anderson-like localization

- In 1958 Philip Anderson demonstrated how in apparently **disordered systems** (lattices) the corresponding **eigenvectors are localized** about a central point.

PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. Anderson
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

I. INTRODUCTION

reasonably well, and to prove a theorem about the

- Hatano & Nelson showed this phenomena can also be described through the **non-Hermitian** nature of the **Hamiltonian** operator.

Volume 77, Number 3 PHYSICAL REVIEW LETTERS 15 July 1996

Localization Transitions in Non-Hermitian Quantum Mechanics

Naomichi Hatano* and David R. Nelson
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 15 March 1996)

We study the localization transitions which arise in both one and two dimensions when quantum mechanical particles described by a random Schrödinger equation are subjected to a constant imaginary vector potential. A path-integral formulation relates the transition to flux lines depinned from columnar defects by a transverse magnetic field in superconductors. The theory predicts that, close to the depinning transition, the transverse Meissner effect is accompanied by stretched exponential relaxation of the field into the bulk and a diverging penetration depth. [S0031-9007(96)00677-1]

PACS numbers: 74.60.Ge, 05.30.Jp, 72.15.Rn

Anderson, P.W., Absence of diffusion in certain random lattices. *Phys. Rev.* (1958)
N. Hatano, D.R. Nelson. Localization Transitions in Non-Hermitian Quantum Mechanics. *PRL*. (1996)

Localization on classical systems?

- In **classic synthetic models** used in the complex systems community such **localization has not occurred** in general.
- As such, specific process such as the **maximal entropy random walk,** which requires an a priori knowledge of the network structure, have been introduced in order to demonstrate these features.
- Notable exceptions are **degree-based processes** where the dynamics become **localized** on the **hubs** – ideally the localization would be observed in general structures (lattices, ER, …) and be independent of the process being observed.

Non-normal networks

- Has recently been shown that real systems are generally directed and highly asymmetric ($A^T A \neq AA^T$).
- Generally quantified through two measures

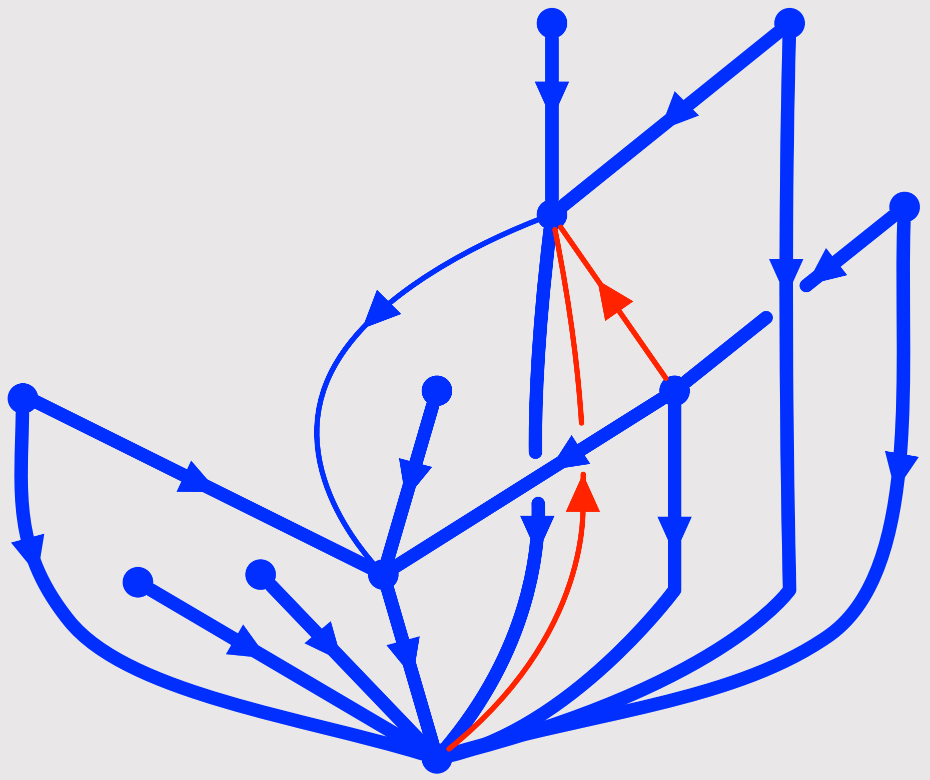
$$d_F(\mathbf{A}) = \sqrt{||\mathbf{A}||_F^2 - \sum_{i=1}^{N} |\lambda_i|^2}$$

and

$$\Delta = |K^< - K^>|/(K^< + K^>),$$

where

$$K^< = \sum_{i<j} \tilde{\mathbf{A}},\ K^> = \sum_{j<i} \tilde{\mathbf{A}}$$



M. Asllani, R. Lambiotte, and T. Carletti. Structure and dynamical behavior of non-normal networks. Sci. Adv. (2018)

Non-normal networks

- Has recently been shown that real systems are generally directed and highly asymmetric ($A^T A \neq AA^T$).
- Generally quantified through two measures

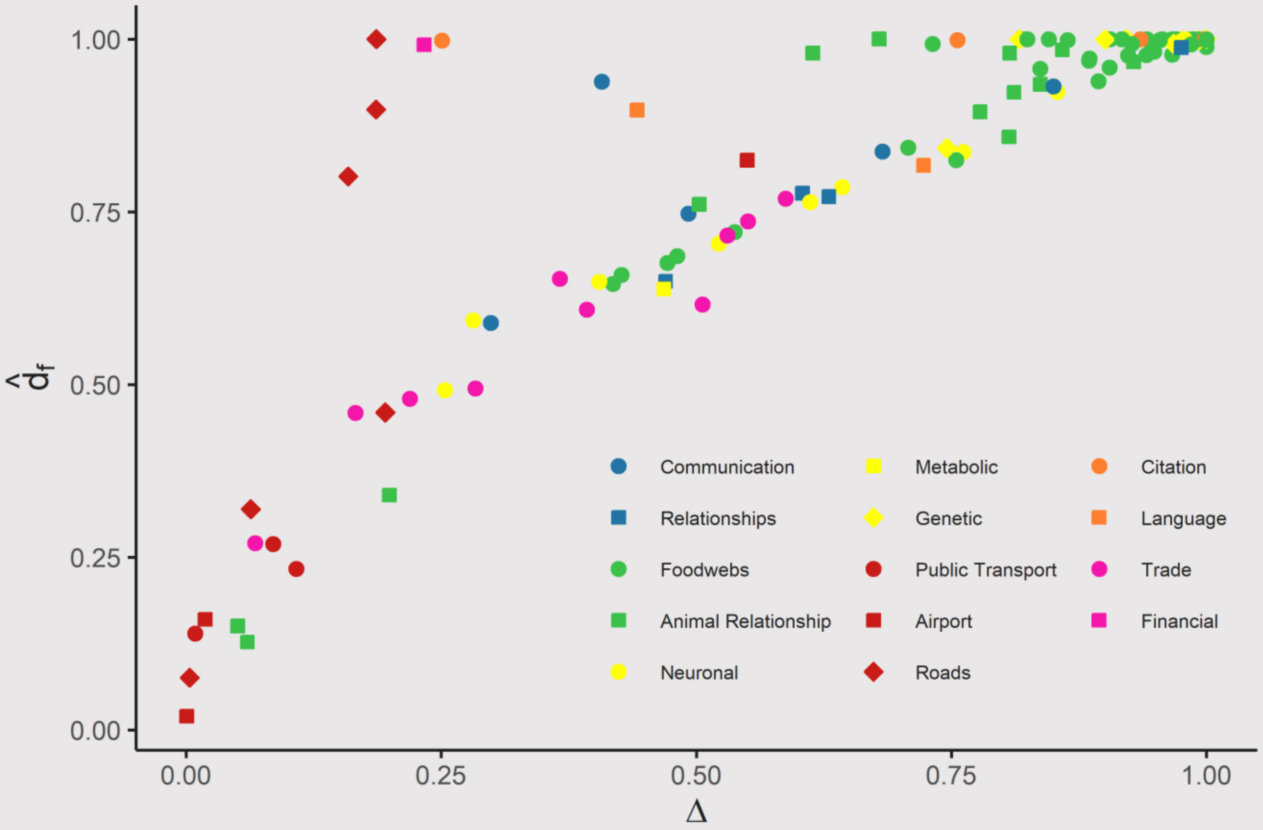
$$d_F(\mathbf{A}) = \sqrt{||\mathbf{A}||_F^2 - \sum_{i=1}^N |\lambda_i|^2}$$

and

$$\Delta = |K^< - K^>|/(K^< + K^>),$$

where

$$K^< = \sum_{i<j} \tilde{\mathbf{A}},\ K^> = \sum_{j<i} \tilde{\mathbf{A}}$$

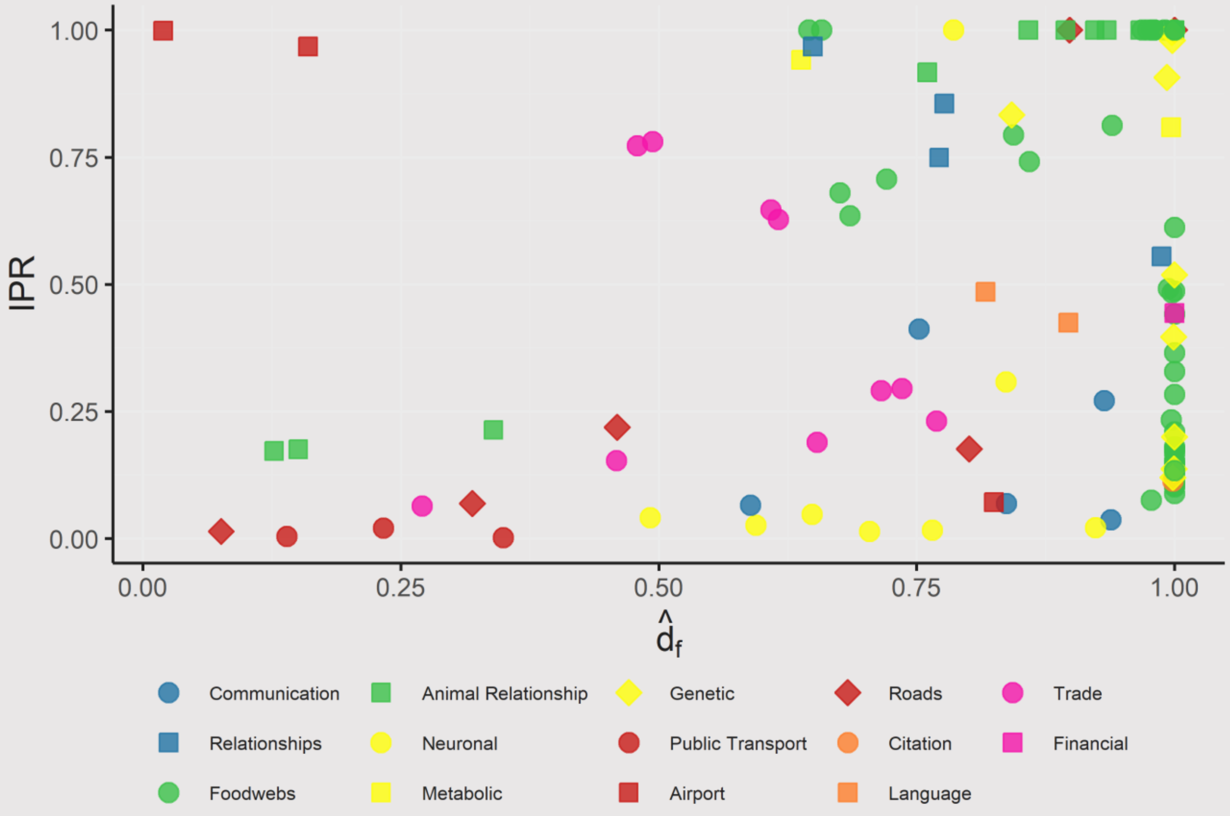


M. Asllani, R. Lambiotte, and T. Carletti. Structure and dynamical behavior of non-normal networks. Sci. Adv. (2018)

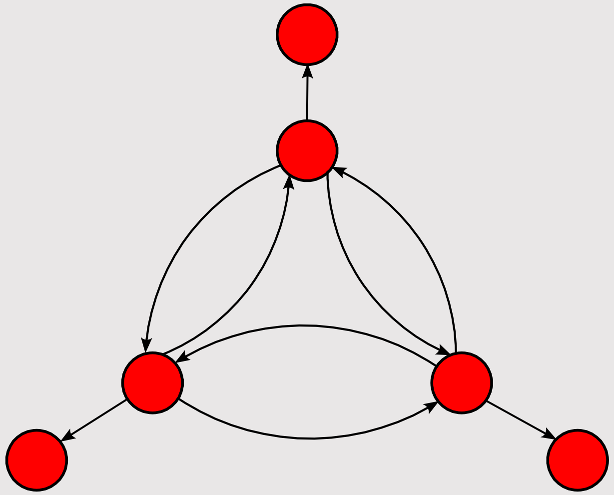
Dynamics localized on the networks?

- Firstly, how do we **measure the localization** of dynamics on a given network?
- Classically the **inverse-participation ratio (IPR)** has been used which considers the eigenvector associated with a given eigenvalue λ

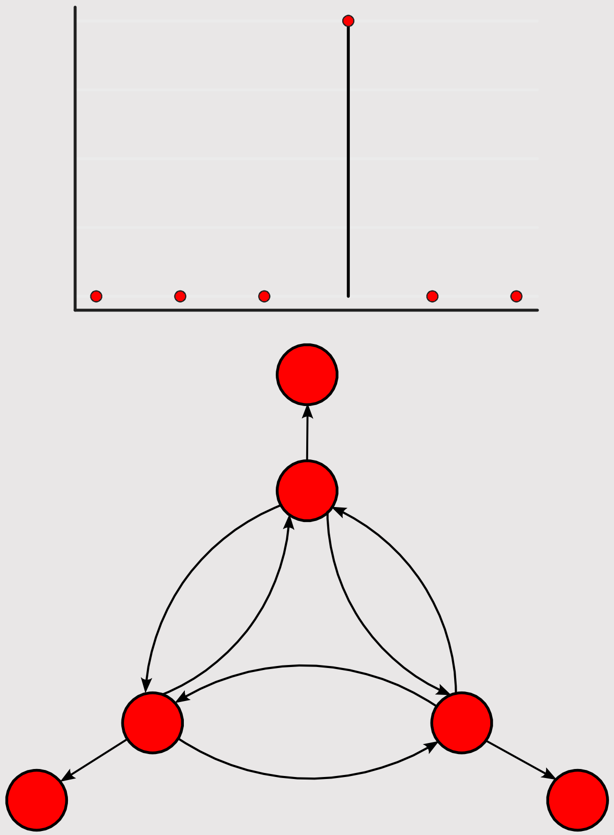
$$\mathrm{IPR}(\lambda) = \sum_i \Phi_i^4(\lambda)$$



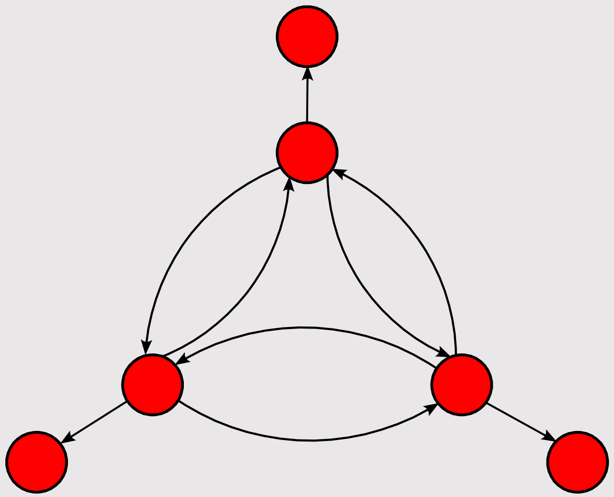
Issues with the IPR

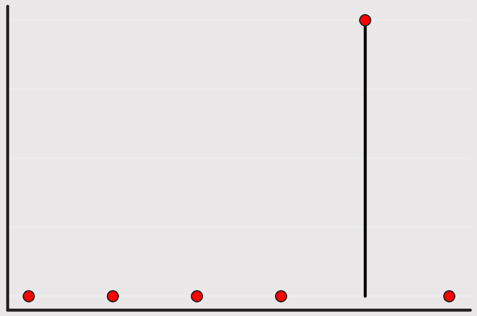


Issues with the IPR

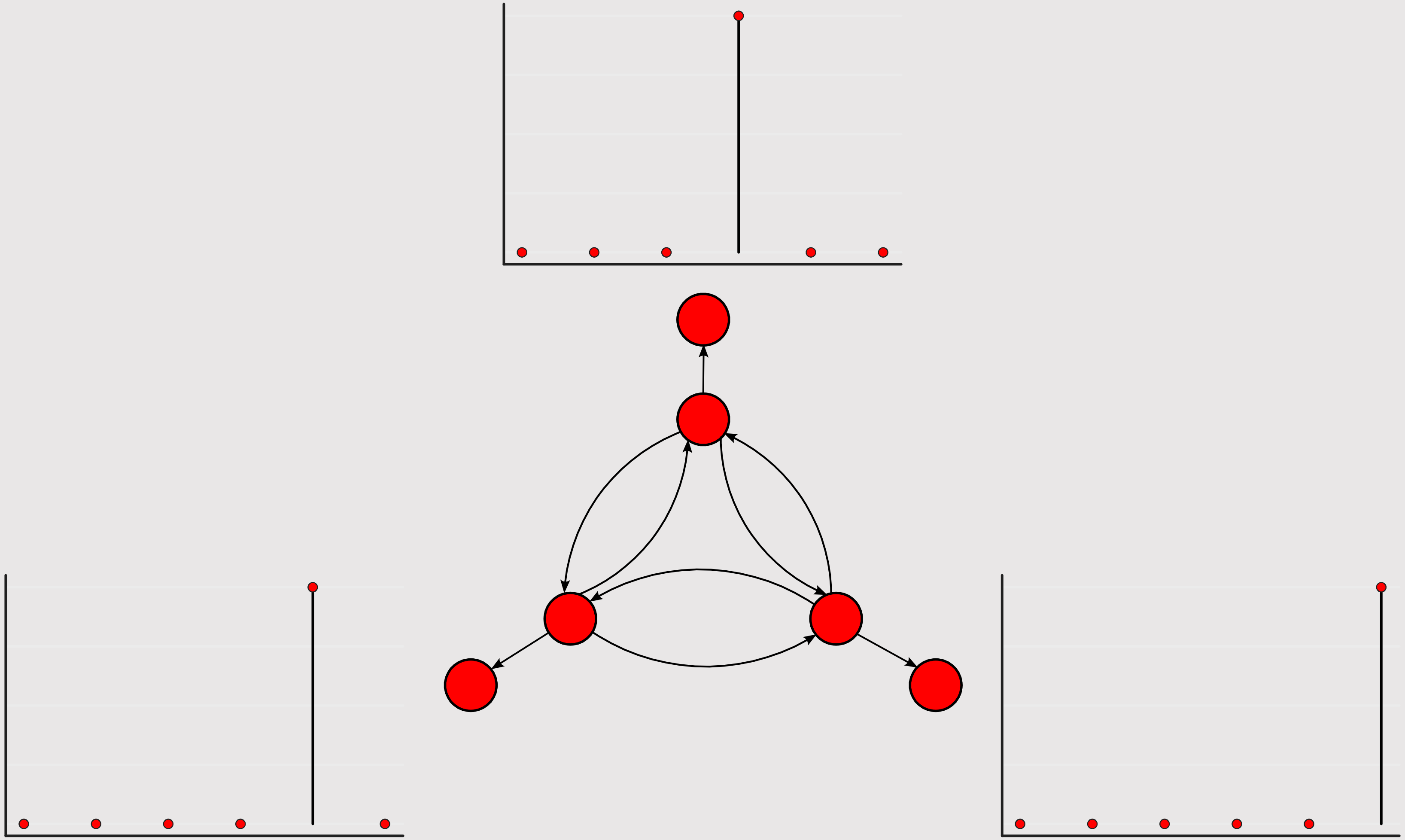


Issues with the IPR



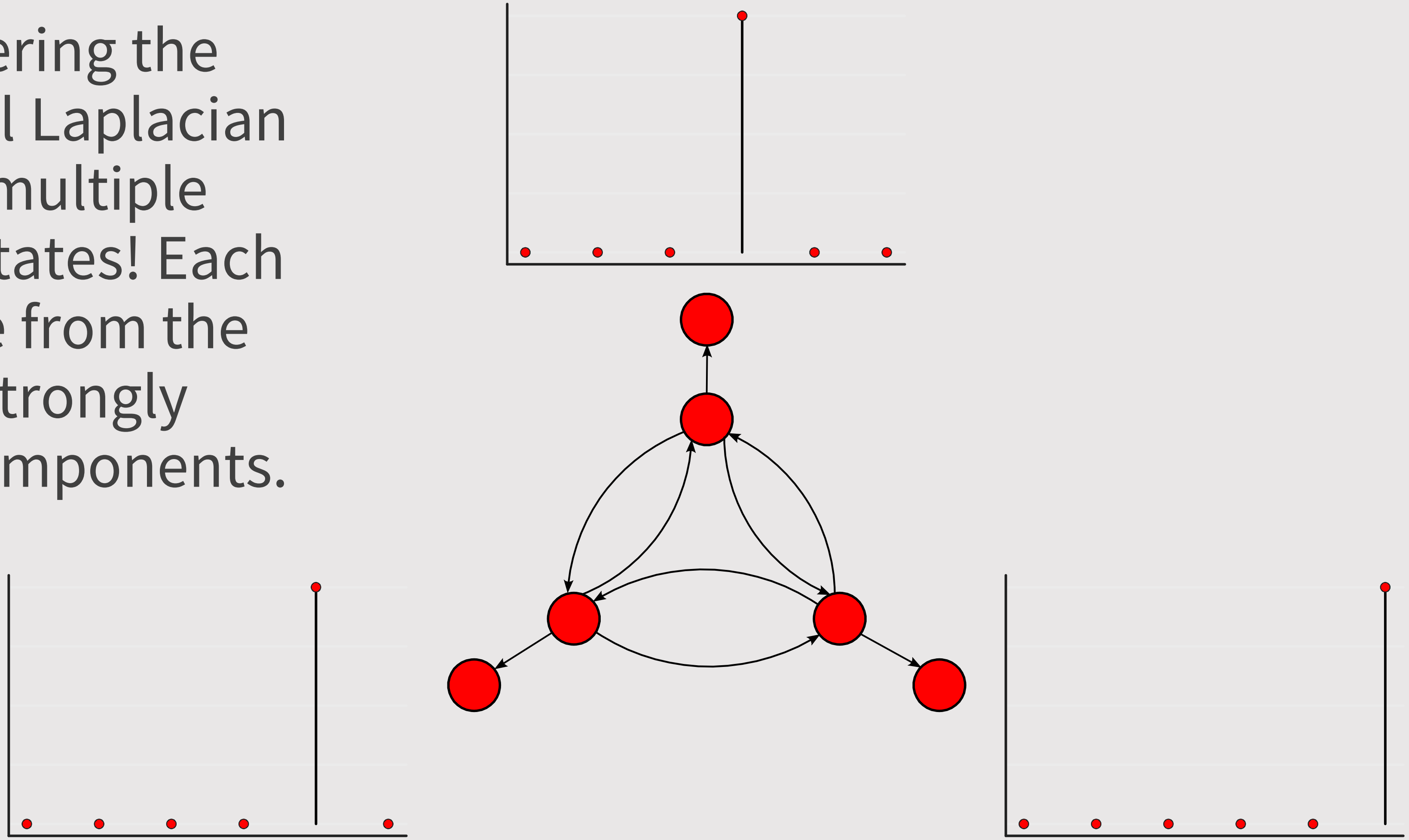


Issues with the IPR



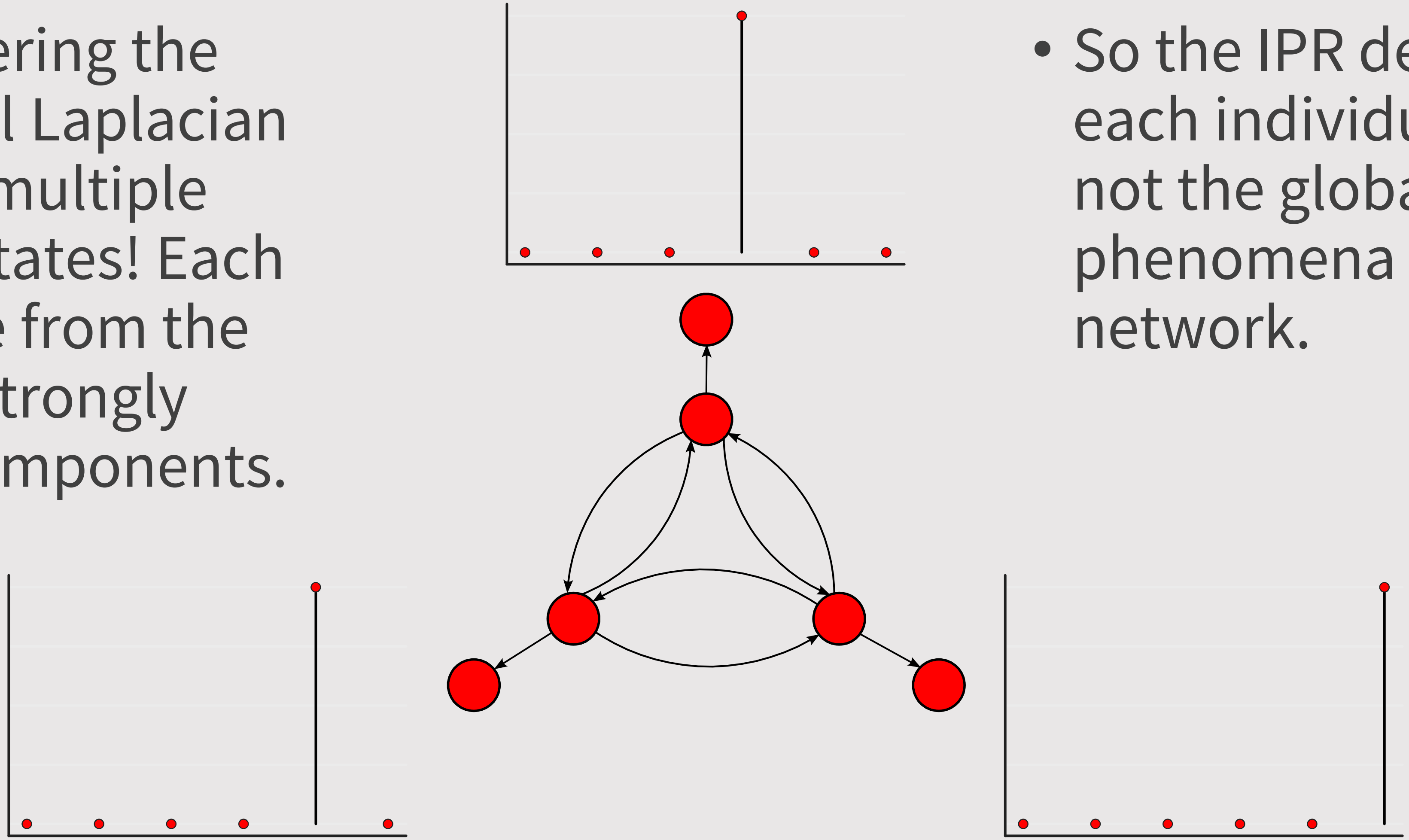
Issues with the IPR

- When considering the combinatorial Laplacian there can be multiple equilibrium states! Each of which arise from the existence of strongly connected components.



Issues with the IPR

- When considering the combinatorial Laplacian there can be multiple equilibrium states! Each of which arise from the existence of strongly connected components.



- So the IPR describes each individually but not the global phenomena on the network.

An alternative approach

- The equilibrium state of a network for a given **dynamical process** on a network may be quantified through other measures based on the network structure rather than the equilibrium eigenvectors.
- In particular we can consider the **generic random walk** which describes random walkers moving between nodes.
- We quantify the inherent randomness (localization) in the process using the **entropy rate**.

Transition matrix

$$h = -\sum_{i,j} T_{ij} \times q_j^{\star} \ln(T_{ij})$$

Stationary distribution

J. Gómez-Gardeñes and V. Latora. Entropy rate of diffusion processes on complex networks. PRE (2008)

Entropy rate of RW on NN networks

- The ER does not have an inherent scale and as such we propose the following measure

$$\hat{h} = \frac{h_{\mathbf{A}}}{h_{\mathrm{H}(\mathbf{A})}}$$

- Where

$$H(\mathbf{A}) = \left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right)/2$$

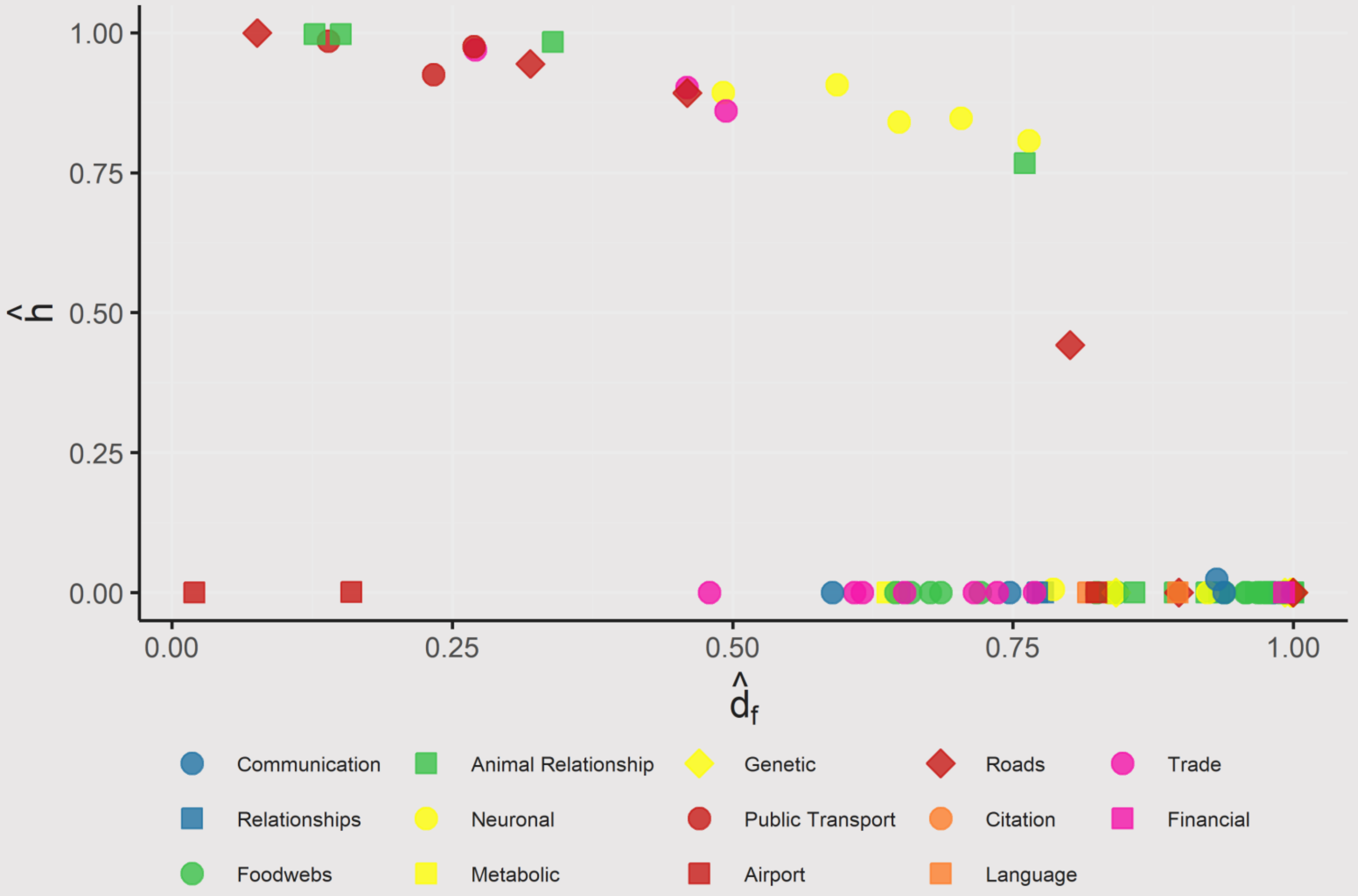
Entropy rate of RW on NN networks

- The ER does not have an inherent scale and as such we propose the following measure

$$\hat{h} = \frac{h_{\mathbf{A}}}{h_{\mathrm{H}(\mathbf{A})}}$$

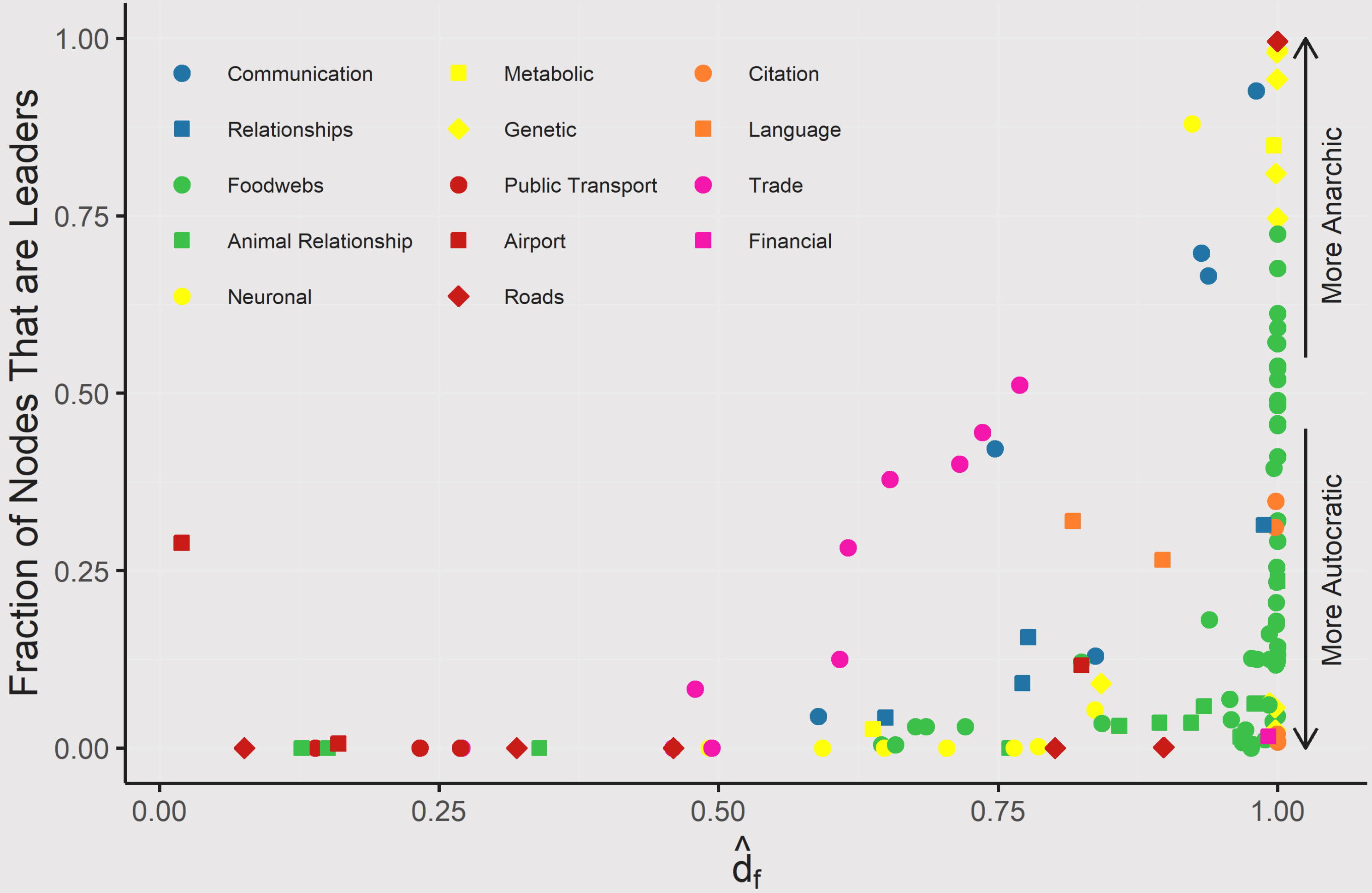
- Where

$$H(\mathbf{A}) = \left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right)/2$$



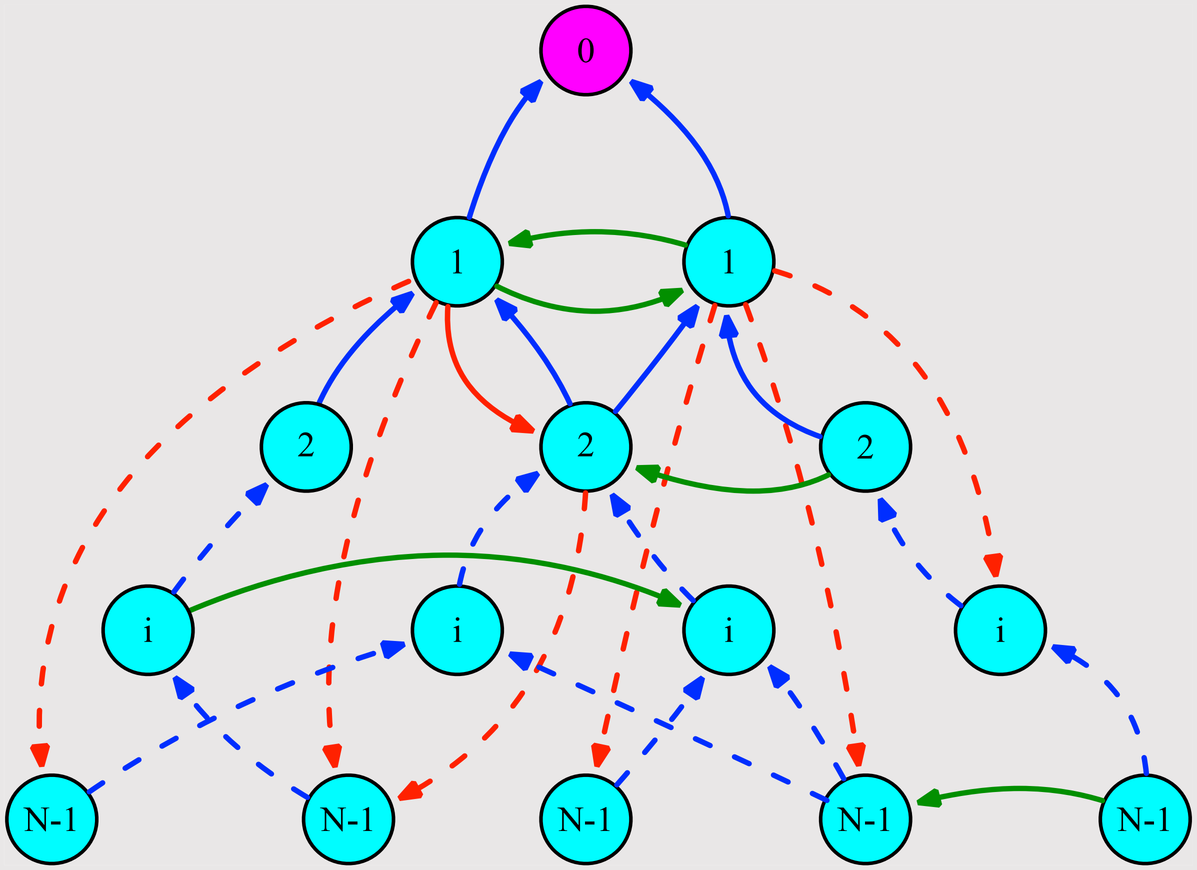
Relationship between ER & Localization

- The collapse in the ER with increasing normality is related to the emergence of **leader nodes** who don't provide to others in the network.
- Where the **level of disorder** in the network can be captured through the number of such nodes.

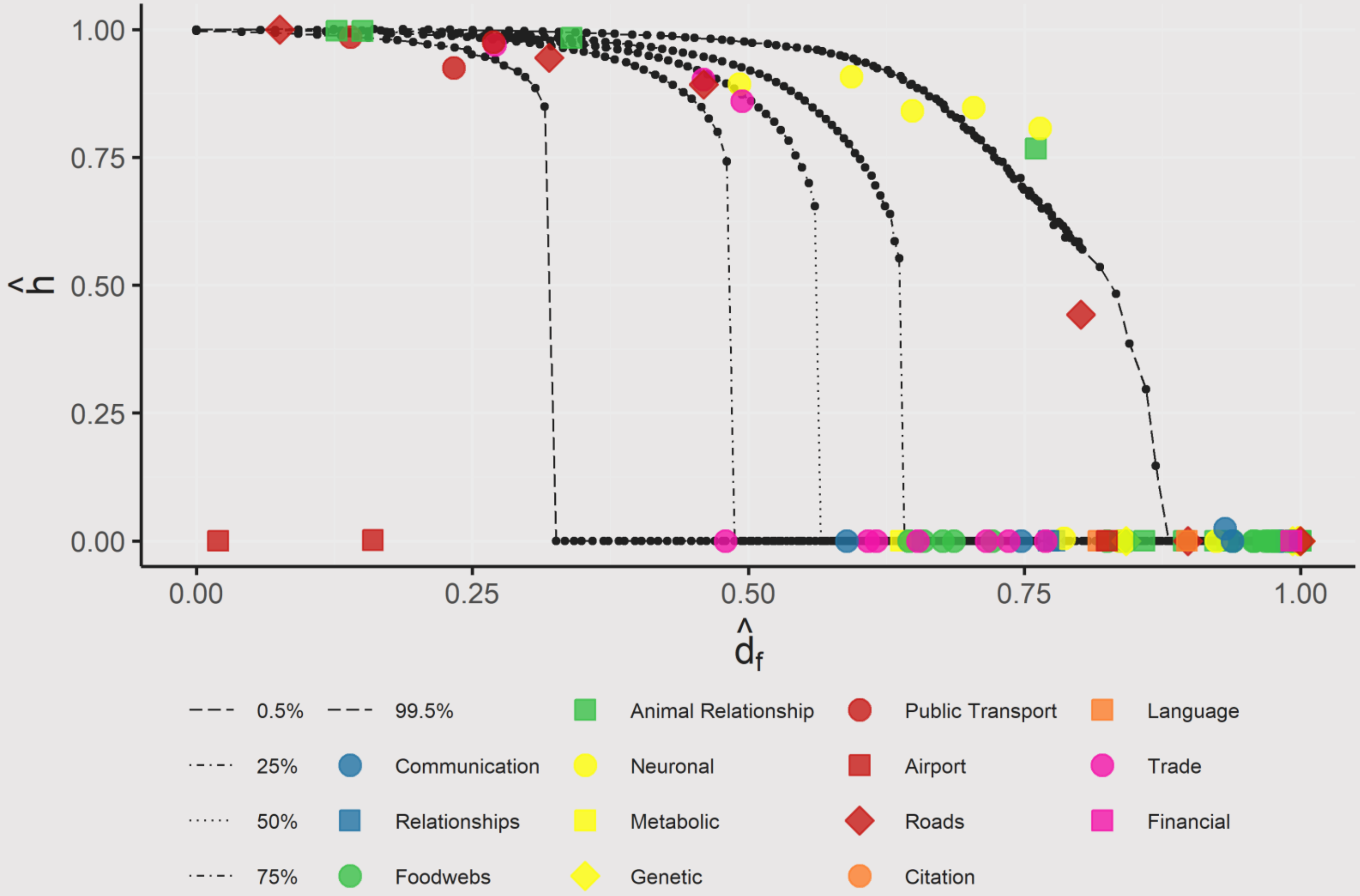


Mechanistic model

- Based upon a **preferential attachment** network with backward edges drawn probabilistically.
- As such offers possibility of **leader nodes** and also a **tuneable** level of **non-normality**.
- Importantly can capture the **localization** present in the **equilibrium** state of empirical networks.



Mechanistic model



Example in empirical systems

- Can a network's non-normality and the leader nodes influence the corresponding dynamics which take place?
- Consider the following diffusively coupled set of equations describing a node's mass at time $x_i(t)$.

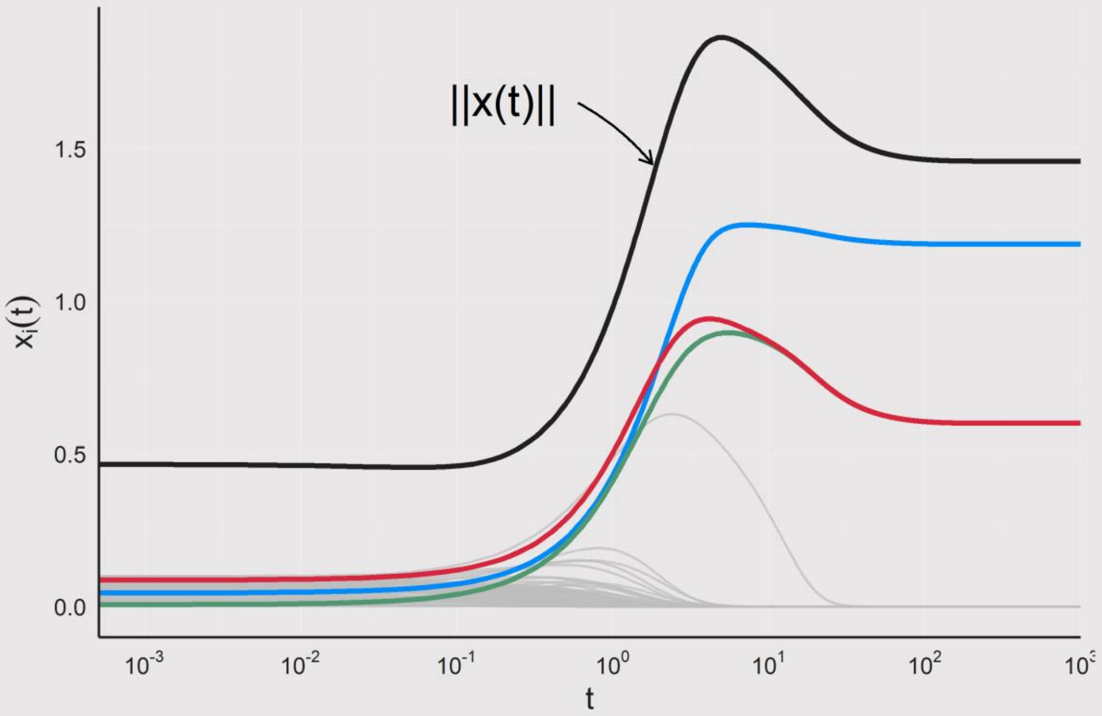
$$\frac{dx_i}{dt} = rx_i(1 - x_i)\left(\frac{x_i}{A} - 1\right) + D\sum_{j=1}^{N} L_{ij}x_j,$$

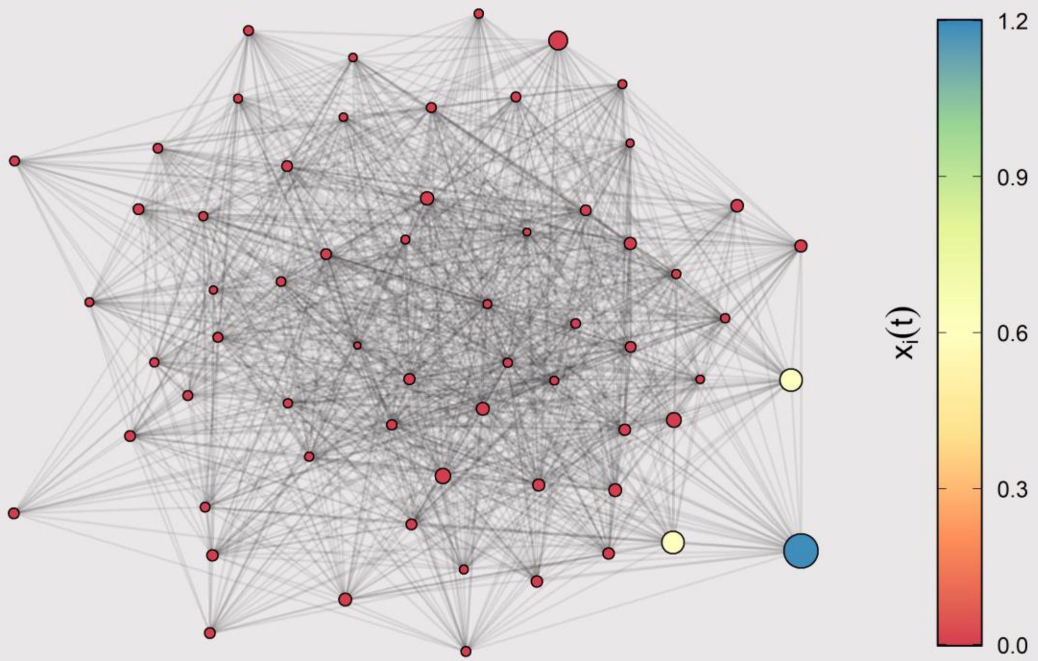
- For symmetric (normal) networks if $x_i(0) < A \;\forall\; i$, then the stable fixed point corresponds to $x_i^* = 0$, i.e., all nodes become extinct.
- In empirical networks however…

M. Asllani and T. Carletti. Topological resilience in non-normal networked systems. PRE (2018)

Example in empirical systems

- … the leader (and two of their neighbours!) will survive.





Conclusions

- The **non-normal** nature of empirical networks, unlike the synthetic models generally used, results in **localization** of the **equilibrium**.
- Classical measures (IPR) aren't satisfactory in describing the localization in **empirical networks** so we use the **entropy rate**.
- The entropy rate demonstrates an interesting relationship with non-normality, specifically an **apparent collapse** in entropy with **increasing NN,** as a consequence of leader nodes emerging.
- **Mechanistic model** introduced which can capture this behaviour.
- Offers **benefits** to empirical dynamical systems beyond the generally studied **symmetric networks**.

Collaborators & Thanks

Kleber Oliveira

Timoteo Carletti

James Gleeson

Malbor Asllani

Thank you for listening!

@obrienj_

joseph.obrien@ul.ie