Conjugate Calculus Rules

$g(\mathbf{x})$	$g^*(\mathbf{y})$
$\sum_{i=1}^m f_i(\mathbf{x}_i)$	$\sum_{i=1}^m f_i^*(\mathbf{y}_i)$
$\alpha f(\mathbf{x})\ \ (\alpha > 0)$	$\alpha f^*(\mathbf{y}/\alpha)$
$\alpha f(\mathbf{x}/\alpha)\ \ (\alpha > 0)$	$\alpha f^*(\mathbf{y})$
$f(\mathcal{A}(\mathbf{x}-\mathbf{a})) + \langle \mathbf{b}, \mathbf{x}\rangle + c$	$f^*\left((\mathcal{A}^T)^{-1}(\mathbf{y}-\mathbf{b})\right) + \langle \mathbf{a}, \mathbf{y}\rangle - c - \langle \mathbf{a}, \mathbf{b}\rangle$

Conjugate Functions

f	$\mathrm{dom}(f)$	f^*	assumptions
e^x	$\mathbb{R}$	$y\log y - y\ (\mathrm{dom}(f^*) = \mathbb{R}_+)$	
$-\log x$	$\mathbb{R}_{++}$	$-1-\log(-y)\ (\mathrm{dom}(f^*) = \mathbb{R}_{--})$	
$\max\{1-x, 0\}$	$\mathbb{R}$	$y + \delta_{[-1,0]}(y)$	
$\frac{1}{p}\|x\|^p$	$\mathbb{R}$	$\frac{1}{q}\|y\|^q$	$p > 1, \frac{1}{p} + \frac{1}{q} = 1$
$-\frac{x^p}{p}$	$\mathbb{R}_+$	$-\frac{(-y)^q}{q}\ (\mathrm{dom}(f^*) = \mathbb{R}_{--})$	$0 < p < 1,\ \frac{1}{p} + \frac{1}{q} = 1$
$\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$	$\mathbb{R}^n$	$\frac{1}{2}(\mathbf{y}-\mathbf{b})^T\mathbf{A}^{-1}(\mathbf{y}-\mathbf{b}) - c$	$\mathbf{A} \in \mathbb{S}^n_{++},\ \mathbf{b} \in \mathbb{R}^n, c \in \mathbb{R}$
$\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$	$\mathbb{R}^n$	$\frac{1}{2}(\mathbf{y}-\mathbf{b})^T\mathbf{A}^{\dagger}(\mathbf{y}-\mathbf{b}) - c$ $(\mathrm{dom}(f^*) = \mathbf{b} + \mathrm{Range}(\mathbf{A}))$	$\mathbf{A} \in \mathbb{S}^n_{+},\ \mathbf{b} \in \mathbb{R}^n, c \in \mathbb{R}$
$\sum_{i=1}^n x_i \log x_i$	$\mathbb{R}^n_+$	$\sum_{i=1}^n e^{y_i - 1}$	
$\sum_{i=1}^n x_i \log x_i$	Δ_n	$\log\left(\sum_{i=1}^n e^{y_i}\right)$	
$-\sum_{i=1}^n \log x_i$	$\mathbb{R}^n_{++}$	$-n - \sum_{i=1}^n \log(-y_i)$	
$\log\left(\sum_{i=1}^n e^{x_i}\right)$	$\mathbb{R}^n$	$\sum_{i=1}^n y_i \log y_i\ (\mathrm{dom}(f^*) = \Delta_n)$	
$\max_i\{x_i\}$	$\mathbb{R}^n$	$\delta_{\Delta_n}(\mathbf{y})$	
$\delta_C(\mathbf{x})$	C	$\sigma_C(\mathbf{y})$	$C \subseteq \mathbb{E}$
$\sigma_C(\mathbf{x})$	$\mathbb{E}$	$\delta_{\mathrm{cl}(\mathrm{conv}(C))}(\mathbf{y})$	$C \subseteq \mathbb{E}$
$\\|\mathbf{x}\\|$	$\mathbb{E}$	$\delta_{B_{\\|\cdot\\|_*}[\mathbf{0},1]}(\mathbf{y})$	
$-\sqrt{\alpha^2 - \\|\mathbf{x}\\|^2}$	$B[\mathbf{0}, \alpha]$	$\alpha\sqrt{\\|\mathbf{y}\\|_*^2 + 1}$	$\alpha > 0$
$\sqrt{\alpha^2 + \\|\mathbf{x}\\|^2}$	$\mathbb{E}$	$-\alpha\sqrt{1 - \\|\mathbf{y}\\|_*^2}$ $(\mathrm{dom}\ f^* = B_{\\|\cdot\\|_*}[\mathbf{0}, 1])$	$\alpha > 0$
$\frac{1}{2}\\|\mathbf{x}\\|^2$	$\mathbb{E}$	$\frac{1}{2}\\|\mathbf{y}\\|_*^2$	
$\frac{1}{2}\\|\mathbf{x}\\|^2 + \delta_C(\mathbf{x})$	C	$\frac{1}{2}\\|\mathbf{y}\\|^2 - \frac{1}{2}d_C^2(\mathbf{y})$	$\emptyset \neq C \subseteq \mathbb{E}$, $\mathbb{E}$ Euclidean
$\frac{1}{2}\\|\mathbf{x}\\|^2 - \frac{1}{2}d_C^2(\mathbf{x})$	$\mathbb{E}$	$\frac{1}{2}\\|\mathbf{y}\\|^2 + \delta_C(\mathbf{y})$	$\emptyset \neq C \subseteq \mathbb{E}$ closed convex

Smooth Functions

$f(\mathbf{x})$	$\text{dom}(f)$	parameter	norm
$\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$ $(\mathbf{A} \in \mathbb{S}^n, \mathbf{b} \in \mathbb{R}^n, c \in \mathbb{R})$	$\mathbb{R}^n$	$\|\mathbf{A}\|_{p,q}$	l_p
$\langle \mathbf{b}, \mathbf{x} \rangle + c$ $(\mathbf{b} \in \mathbb{E}^*, c \in \mathbb{R})$	$\mathbb{E}$	0	any norm
$\frac{1}{2}\|\mathbf{x}\|_p^2,\ p \in [2, \infty)$	$\mathbb{R}^n$	$p-1$	l_p
$\sqrt{1 + \|\mathbf{x}\|_2^2}$	$\mathbb{R}^n$	1	l_2
$\log(\sum_{i=1}^n e^{x_i})$	$\mathbb{R}^n$	1	l_2, l_∞
$\frac{1}{2}d_C^2(\mathbf{x})$ $(\emptyset \neq C \subseteq \mathbb{E}$ closed convex$)$	$\mathbb{E}$	1	Euclidean
$\frac{1}{2}\|\mathbf{x}\|^2 - \frac{1}{2}d_C^2(\mathbf{x})$ $(\emptyset \neq C \subseteq \mathbb{E}$ closed convex$)$	$\mathbb{E}$	1	Euclidean
$H_\mu(\mathbf{x})\ (\mu > 0)$	$\mathbb{E}$	$\frac{1}{\mu}$	Euclidean

Strongly Convex Functions

$f(\mathbf{x})$	$\text{dom}(f)$	s.c. parameter	norm
$\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$ $(\mathbf{A} \in \mathbb{S}^n_{++}, \mathbf{b} \in \mathbb{R}^n, c \in \mathbb{R})$	$\mathbb{R}^n$	$\lambda_{\min}(\mathbf{A})$	l_2
$\frac{1}{2}\|\mathbf{x}\|^2 + \delta_C(\mathbf{x})$ $(\emptyset \neq C \subseteq \mathbb{E}$ convex$)$	C	1	Euclidean
$-\sqrt{1 - \|\mathbf{x}\|_2^2}$	$B_{\|\cdot\|_2}[\mathbf{0}, 1]$	1	l_2
$\frac{1}{2}\|\mathbf{x}\|_p^2\ \ (p \in (1, 2])$	$\mathbb{R}^n$	$p-1$	l_p
$\sum_{i=1}^n x_i \log x_i$	Δ_n	1	l_2 or l_1

Orthogonal Projections

set (C)	$P_C(\mathbf{x})$	assumptions
$\mathbb{R}^n_+$	$[\mathbf{x}]_+$	$-$
$\text{Box}[\boldsymbol{\ell},\mathbf{u}]$	$P_C(\mathbf{x})_i = \min\{\max\{x_i,\ell_i\},u_i\}$	$\ell_i \le u_i$
$B_{\|\cdot\|_2}[\mathbf{c},r]$	$\mathbf{c} + \frac{r}{\max\{\|\mathbf{x}-\mathbf{c}\|_2,r\}}(\mathbf{x}-\mathbf{c})$	$\mathbf{c}\in\mathbb{R}^n, r>0$
$\{\mathbf{x}:\mathbf{A}\mathbf{x}=\mathbf{b}\}$	$\mathbf{x}-\mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{x}-\mathbf{b})$	$\mathbf{A}\in\mathbb{R}^{m\times n}$, $\mathbf{b}\in\mathbb{R}^m$, $\mathbf{A}$ full row rank
$\{\mathbf{x}:\mathbf{a}^T\mathbf{x}\le b\}$	$\mathbf{x}-\frac{[\mathbf{a}^T\mathbf{x}-b]_+}{\|\mathbf{a}\|^2}\mathbf{a}$	$\mathbf{0}\ne\mathbf{a}\in\mathbb{R}^n, b\in\mathbb{R}$
Δ_n	$[\mathbf{x}-\mu^*\mathbf{e}]_+$ where $\mu^*\in\mathbb{R}$ satisfies $\mathbf{e}^T[\mathbf{x}-\mu^*\mathbf{e}]_+=1$	
$H_{\mathbf{a},b}\cap\text{Box}[\boldsymbol{\ell},\mathbf{u}]$	$P_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x}-\mu^*\mathbf{a})$ where $\mu^*\in\mathbb{R}$ satisfies $\mathbf{a}^TP_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x}-\mu^*\mathbf{a})=b$	$\mathbf{a}\in\mathbb{R}^n\setminus\{\mathbf{0}\}$, $b\in\mathbb{R}$
$H^-_{\mathbf{a},b}\cap\text{Box}[\boldsymbol{\ell},\mathbf{u}]$	$\begin{cases} P_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x}), & \mathbf{a}^T\mathbf{v}_\mathbf{x}\le b,\\ P_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x}-\lambda^*\mathbf{a}), & \mathbf{a}^T\mathbf{v}_\mathbf{x}> b,\end{cases}$ $\mathbf{v}_\mathbf{x}=P_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x})$, $\mathbf{a}^TP_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x}-\lambda^*\mathbf{a})=b, \lambda^*>0$	$\mathbf{a}\in\mathbb{R}^n\setminus\{\mathbf{0}\}$, $b\in\mathbb{R}$
$B_{\|\cdot\|_1}[\mathbf{0},\alpha]$	$\begin{cases} \mathbf{x}, & \|\mathbf{x}\|_1\le\alpha,\\ \mathcal{T}_{\lambda^*}(\mathbf{x}), & \|\mathbf{x}\|_1>\alpha,\end{cases}$ $\|\mathcal{T}_{\lambda^*}(\mathbf{x})\|_1=\alpha$, $\lambda^*>0$	$\alpha>0$
$\{\mathbf{x}:\boldsymbol{\omega}^T\lvert\mathbf{x}\rvert\le\beta,$ $-\boldsymbol{\alpha}\le\mathbf{x}\le\boldsymbol{\alpha}\}$	$\begin{cases} \mathbf{v}_\mathbf{x}, & \boldsymbol{\omega}^T\lvert\mathbf{v}_\mathbf{x}\rvert\le\beta,\\ \mathcal{S}_{\lambda^*\boldsymbol{\omega},\boldsymbol{\alpha}}(\mathbf{x}), & \boldsymbol{\omega}^T\lvert\mathbf{v}_\mathbf{x}\rvert>\beta,\end{cases}$ $\mathbf{v}_\mathbf{x}=P_{\text{Box}[-\boldsymbol{\alpha},\boldsymbol{\alpha}]}(\mathbf{x})$, $\boldsymbol{\omega}^T\lvert\mathcal{S}_{\lambda^*\boldsymbol{\omega},\boldsymbol{\alpha}}(\mathbf{x})\rvert=\beta,\lambda^*>0$	$\boldsymbol{\omega}\in\mathbb{R}^n_{++}$, $\boldsymbol{\alpha}\in[0,\infty]^n$, $\beta\in\mathbb{R}_{++}$
$\{\mathbf{x}>\mathbf{0}:\Pi x_i\ge\alpha\}$	$\begin{cases} \mathbf{x}, & \mathbf{x}\in C,\\ \left(\frac{x_j+\sqrt{x_j^2+4\lambda^*}}{2}\right)_{j=1}^n, & \mathbf{x}\notin C,\end{cases}$ $\Pi_{j=1}^n\left((x_j+\sqrt{x_j^2+4\lambda^*})/2\right)=\alpha,\lambda^*>0$	$\alpha>0$
$\{(\mathbf{x},s):\|\mathbf{x}\|_2\le s\}$	$\left(\frac{\|\mathbf{x}\|_2+s}{2\|\mathbf{x}\|_2}\mathbf{x},\frac{\|\mathbf{x}\|_2+s}{2}\right)$ if $\|\mathbf{x}\|_2\ge\lvert s\rvert$ $(\mathbf{0},0)$ if $s<\|\mathbf{x}\|_2<-s$, $(\mathbf{x},s)$ if $\|\mathbf{x}\|_2\le s$.	
$\{(\mathbf{x},s):\|\mathbf{x}\|_1\le s\}$	$\begin{cases} (\mathbf{x},s), & \|\mathbf{x}\|_1\le s,\\ (\mathcal{T}_{\lambda^*}(\mathbf{x}),s+\lambda^*), & \|\mathbf{x}\|_1>s,\end{cases}$ $\|\mathcal{T}_{\lambda^*}(\mathbf{x})\|_1-\lambda^*-s=0$, $\lambda^*>0$	

Prox Calculus Rules

$f(\mathbf{x})$	$\text{prox}_f(\mathbf{x})$	assumptions
$\sum_{i=1}^m f_i(\mathbf{x}_i)$	$\text{prox}_{f_1}(\mathbf{x}_1) \times \cdots \times \text{prox}_{f_m}(\mathbf{x}_m)$	
$g(\lambda \mathbf{x} + \mathbf{a})$	$\frac{1}{\lambda}\left[\text{prox}_{\lambda^2 g}(\mathbf{a} + \lambda \mathbf{x}) - \mathbf{a}\right]$	$\lambda \neq 0, \mathbf{a} \in \mathbb{E}$, g proper
$\lambda g(\mathbf{x}/\lambda)$	$\lambda \text{prox}_{g/\lambda}(\mathbf{x}/\lambda)$	$\lambda \neq 0$, g proper
$g(\mathbf{x}) + \frac{c}{2}\|\mathbf{x}\|^2 + \langle \mathbf{a}, \mathbf{x} \rangle + \gamma$	$\text{prox}_{\frac{1}{c+1}g}(\frac{\mathbf{x}-\mathbf{a}}{c+1})$	$\mathbf{a} \in \mathbb{E}, c > 0, \gamma \in \mathbb{R}$, g proper
$g(\mathcal{A}(\mathbf{x}) + \mathbf{b})$	$\mathbf{x} + \frac{1}{\alpha}\mathcal{A}^T(\text{prox}_{\alpha g}(\mathcal{A}(\mathbf{x}) + \mathbf{b}) - \mathcal{A}(\mathbf{x}) - \mathbf{b})$	$\mathbf{b} \in \mathbb{R}^m$, $\mathcal{A} : \mathbb{V} \to \mathbb{R}^m$, g proper closed convex, $\mathcal{A} \circ \mathcal{A}^T = \alpha I$, $\alpha > 0$
$g(\|\mathbf{x}\|)$	$\text{prox}_g(\|\mathbf{x}\|)\frac{\mathbf{x}}{\|\mathbf{x}\|}, \quad \mathbf{x} \neq \mathbf{0}$ $\{\mathbf{u} : \|\mathbf{u}\| = \text{prox}_g(0)\}, \quad \mathbf{x} = \mathbf{0}$	g proper closed convex, $\text{dom}(g) \subseteq [0, \infty)$

Prox Computations

$f(\mathbf{x})$	$\mathrm{dom}(f)$	$\mathrm{prox}_f(\mathbf{x})$	assumptions
$\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{b}^T\mathbf{x}+c$	$\mathbb{R}^n$	$(\mathbf{A}+\mathbf{I})^{-1}(\mathbf{x}-\mathbf{b})$	$\mathbf{A}\in\mathbb{S}^n_+$, $\mathbf{b}\in\mathbb{R}^n$, $c\in\mathbb{R}$
λx^3	$\mathbb{R}_+$	$\frac{-1+\sqrt{1+12\lambda[x]_+}}{6\lambda}$	$\lambda>0$
μx	$[0,\alpha]\cap\mathbb{R}$	$\min\{\max\{x-\mu,0\},\alpha\}$	$\mu\in\mathbb{R}$, $\alpha\in[0,\infty]$
$\lambda\\|\mathbf{x}\\|$	$\mathbb{E}$	$\left(1-\frac{\lambda}{\max\{\\|\mathbf{x}\\|,\lambda\}}\right)\mathbf{x}$	$\\|\cdot\\|$ - Euclidean, $\lambda>0$
$-\lambda\\|\mathbf{x}\\|$	$\mathbb{E}$	$\left(1+\frac{\lambda}{\\|\mathbf{x}\\|}\right)\mathbf{x}$, $\mathbf{x}\neq\mathbf{0}$, $\{\mathbf{u}:\\|\mathbf{u}\\|=\lambda\}$, $\mathbf{x}=\mathbf{0}$.	$\\|\cdot\\|$ - Euclidean, $\lambda>0$
$\lambda\\|\mathbf{x}\\|_1$	$\mathbb{R}^n$	$\mathcal{T}_\lambda(\mathbf{x})=[\lvert\mathbf{x}\rvert-\lambda\mathbf{e}]_+\odot\mathrm{sgn}(\mathbf{x})$	$\lambda>0$
$\\|\boldsymbol{\omega}\odot\mathbf{x}\\|_1$	$\mathrm{Box}[-\boldsymbol{\alpha},\boldsymbol{\alpha}]$	$\mathcal{S}_{\boldsymbol{\omega},\boldsymbol{\alpha}}(\mathbf{x})$	$\boldsymbol{\alpha}\in[0,\infty]^n$,$\boldsymbol{\omega}\in\mathbb{R}^n_{++}$
$\lambda\\|\mathbf{x}\\|_\infty$	$\mathbb{R}^n$	$\mathbf{x}-\lambda P_{B_{\\|\cdot\\|_1}[\mathbf{0},1]}(\mathbf{x}/\lambda)$	$\lambda>0$
$\lambda\\|\mathbf{x}\\|_a$	$\mathbb{E}$	$\mathbf{x}-\lambda P_{B_{\\|\cdot\\|_{a,*}}[\mathbf{0},1]}(\mathbf{x}/\lambda)$	$\\|\mathbf{x}\\|_a$ – norm, $\lambda>0$
$\lambda\\|\mathbf{x}\\|_0$	$\mathbb{R}^n$	$H_{\sqrt{2\lambda}}(x_1)\times\cdots\times H_{\sqrt{2\lambda}}(x_n)$	$\lambda>0$
$\lambda\\|\mathbf{x}\\|^3$	$\mathbb{E}$	$\frac{2}{1+\sqrt{1+12\lambda\\|\mathbf{x}\\|}}\mathbf{x}$	$\\|\cdot\\|$ - Euclidean, $\lambda>0$
$-\lambda\sum_{j=1}^n\log x_j$	$\mathbb{R}^n_{++}$	$\left(\frac{x_j+\sqrt{x_j^2+4\lambda}}{2}\right)_{j=1}^n$	$\lambda>0$
$\delta_C(\mathbf{x})$	$\mathbb{E}$	$P_C(\mathbf{x})$	$\emptyset\neq C\subseteq\mathbb{E}$
$\lambda\sigma_C(\mathbf{x})$	$\mathbb{E}$	$\mathbf{x}-\lambda P_C(\mathbf{x}/\lambda)$	$\lambda>0$, $C\neq\emptyset$ closed convex
$\lambda\max\{x_i\}$	$\mathbb{R}^n$	$\mathbf{x}-P_{\Delta_n}(\mathbf{x}/\lambda)$	$\lambda>0$
$\lambda\sum_{i=1}^k x_{[i]}$	$\mathbb{R}^n$	$\mathbf{x}-\lambda P_C(\mathbf{x}/\lambda)$, $C=H_{\mathbf{e},k}\cap\mathrm{Box}[\mathbf{0},\mathbf{e}]$	$\lambda>0$
$\lambda\sum_{i=1}^k \lvert x_{\langle i\rangle}\rvert$	$\mathbb{R}^n$	$\mathbf{x}-\lambda P_C(\mathbf{x}/\lambda)$, $C=B_{\\|\cdot\\|_1}[\mathbf{0},k]\cap\mathrm{Box}[-\mathbf{e},\mathbf{e}]$	$\lambda>0$
$\lambda M_f^\mu(\mathbf{x})$	$\mathbb{E}$	$\mathbf{x}+\frac{\lambda}{\mu+\lambda}\left(\mathrm{prox}_{(\mu+\lambda)f}(\mathbf{x})-\mathbf{x}\right)$	$\lambda,\mu>0$, f proper closed convex
$\lambda d_C(\mathbf{x})$	$\mathbb{E}$	$\mathbf{x}+\min\left\{\frac{\lambda}{d_C(\mathbf{x})},1\right\}(P_C(\mathbf{x})-\mathbf{x})$	$\emptyset\neq C$ closed convex, $\lambda>0$
$\frac{\lambda}{2}d_C^2(\mathbf{x})$	$\mathbb{E}$	$\frac{\lambda}{\lambda+1}P_C(\mathbf{x})+\frac{1}{\lambda+1}\mathbf{x}$	$\emptyset\neq C$ closed convex, $\lambda>0$
$\lambda H_\mu(\mathbf{x})$	$\mathbb{E}$	$\left(1-\frac{\lambda}{\max\{\\|\mathbf{x}\\|,\mu+\lambda\}}\right)\mathbf{x}$	$\lambda,\mu>0$
$\rho\\|\mathbf{x}\\|_1^2$	$\mathbb{R}^n$	$\left(\frac{v_ix_i}{v_i+2\rho}\right)_{i=1}^n$, $\mathbf{v}=\left[\sqrt{\frac{\rho}{\mu}}\lvert\mathbf{x}\rvert-2\rho\right]_+$,$\mathbf{e}^T\mathbf{v}=1$ ($\mathbf{0}$ when $\mathbf{x}=\mathbf{0}$)	$\rho>0$
$\\|\mathbf{A}\mathbf{x}\\|_2$	$\mathbb{R}^n$	$\mathbf{x}-\mathbf{A}^T(\mathbf{A}\mathbf{A}^T+\alpha^*\mathbf{I})^{-1}\mathbf{A}\mathbf{x}$, $\alpha^*=0$ if $\\|\mathbf{v}_0\\|_2\le\lambda$; otherwise, $\\|\mathbf{v}_{\alpha^*}\\|_2=\lambda$; $\mathbf{v}_\alpha\equiv(\mathbf{A}\mathbf{A}^T+\alpha\mathbf{I})^{-1}\mathbf{A}\mathbf{x}$	$\mathbf{A}\in\mathbb{R}^{m\times n}$ with full row rank