TDT4136 Introduction to Artificial Intelligence

Lecture 8: First Order Logic

Chapter 8 in the textbook

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Outline

- Syntax of FOL (Predicate) logic
- Semantics of FOL
- Quantifiers

Next week: Inference in FOL (chapter 9)

Pros and Cons of propositional logic

PROs:

- Propositional logic is declarative: pieces of syntax correspond to facts
- Propositional logic allows partial information (unlike most data structures and databases)
- Propositional logic is compositional: meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- Meaning in propositional logic is context-independent (unlike natural language, where meaning depends on context)

CON:

- Propositional logic has very limited expressive power (unlike natural language)

Limitations of propositional logic

- Some knowledge is hard or impossible to encode in the propositional logic.
- Two cases that are specially hard to represent:
  - Statements about similar objects, relations
  - Statements referring to groups of objects.

Limitations of propositional logic - Example 1

- **Statements referring to groups of objects required to be enumerated**
- Example: Assume we want to express *Every student likes vacation*:

  John likes vacation $\wedge$
  Mary likes vacation $\wedge$
  Ann likes vacation $\wedge$
  ....

  Problem: KB grows large
  Possible solution: ??
- *All students like vacation.*

Limitations of propositional logic - Example 2: Seniority domain

- Statements about similar objects and relations need to be enumerated
- Assume we have:
  Stig is older than Sissel
  Sissel is older than Paul
  Stig is older than Sissel $\wedge$ Sissel is older than Paul $\implies$ Stig is older than Paul
- We can derive *Stig is older than Paul*

Example on Limitations of propositional logic - Example 2 cont.

- Assume we add *Hanne is older than Sissel* into the KB
- The current KB now:
  Stig is older than Sissel
  Sissel is older than Paul
  Stig is older than Sissel $\wedge$ *Sissel is older than Paul* $\implies$ *Stig is older than Paul*
  Hanne is older than Sissel
- What else do we need to have in the KB in order to derive *Hanne is older than Paul*?

Example on Limitations of propositional logic - Example 2 cont.

- The current KB:
  Stig is older than Sissel
  Sissel is older than Paul
  Stig is older than Sissel $\wedge$ *Sissel is older than Paul* $\implies$ *Stig is older than Paul*
  Hanne is older than Sissel
- What else do we need to have in the KB in order to derive *Hanne is older than Paul*?

  We need:
  Hanne is older than Sissel $\wedge$ *Sissel is older than Paul* $\implies$ *Hanne is older than Paul*

What is the problem?

Limitations of propositional logic -Example 2 cont.

Problem? KB grows large
Possible solution: ??
PersA is older than PersB $\wedge$ PersB is older than PersC $\implies$
PersA is older than PersC

Limitations of Propositional Logic -example 3: Wumpus

- Consider the statement "If there is breeze in a square,there must be pit in an adjacent square"
- In propositional logic we need 16 sentences (one for each square) to represent this statement (for 4x4 grid):

  - B1,1 $\implies$ P1,2 $\vee$ P2,1

  - B1,2 $\implies$ P1,2 $\vee$ P1,3 $\vee$ P2,2

  - ...

  - ...

- **We want to be able to say this in one single sentence.**

How to say it in one sentence

- Our statement above refers to 2 types objects (pit and square). The square has the property to be breezy. The relationship between a square and pit is adjacency, i.e., neighbourhood.

- In FOL, this statement is represented by means of the following formula - instead of 16 sentences in propositional logic,:

  $\forall$ square, adjacent(square,pit) $\implies$ breezy(square)

First Order Logic (FOL)

- More expressive than propositional logic. While PL assumes that the world contains facts, FOL assumes the world contains
  - Objects: trees, people, numbers, movies,Trump, maps, colours, hypotheses, Wumpus....
  - Relations: square, smelly, brother of, older than, owns, has colour, adjacent to....
  - Functions: brother-of, colour-of, adjacent-to,....
- Representing objects, their properties, relations and statements about them.

Syntax of FOL - elements

- Constants: NTNU, KingHarald, 5, ...
- Predicates: Brother, $>$, $=$, ...
- Functions: Sqrt, LeftLegOf
- Variables: x, y, a, b, ...
- Connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- Quantifiers: $\forall, \exists$

- First order logic or "predicate calculus" introduces **variables** and **quantifiers** to refer to objects in the world, their relations, group of objects, and to express general rules.

Functions versus Relations

- Functions are a way of referring to individuals indirectly, e.g.,
  - brother-of (Janne) and Edvard would refer to the same object/individual if Janne's brother is the person named Edvard.

- Relations hold among objects
  - Brother(Janne,Edvard) is true if Edvard is Janne's brother
  - Unary, binary, n-ary relations

Syntax

- Atomic sentence: predicate(term$_1$,...,term$_n$)

  Term: constant, or variable or function(term$_1$,...,term$_m$)

- Complex sentences: Composed of atomic sentences using connectives

  $\neg S,\ S_1 \vee S_2\ ,\ S_1 \wedge S_2\ ,\ S_1 \implies S_2,\ S_1 \Leftrightarrow S_2$

  Examples:

  - Brother(JonSnow, AryaStark) $\implies$ Sister(AryaStark, JonSnow)
  - Childish(Trump) $\vee$ BestStudents(Students(NTNU), Norway)

Universal Quantifiers

- Quantifications express properties of collections of objects.
- $\forall$: "For all"
- $\forall \langle \textit{variable} \rangle \ \langle \textit{sentence} \rangle$
  We can state the following: $\forall$ x P(x)
  English translation: "for all values of x, P(x) is true"
  Example: P(x) : x+1 $\geq$ x
  English translation: "for all values of x, x+1$\geq$ x is true"

Universal quantification

- Everyone at NTNU is smart:
  $\forall x \;\; At(x, NTNU) \implies Smart(x)$
- $\forall x \;\; P$ is true in a model m iff P is true with x being each possible object in the model
- Equivalent to the conjunction of instantiations of P

$$
\begin{aligned}
& (At(KingJohn, NTNU) \implies Smart(KingJohn)) \\
\wedge \quad & (At(Richard, NTNU) \implies Smart(Richard)) \\
\wedge \quad & (At(NTNU, NTNU) \implies Smart(NTNU)) \\
\wedge \quad & \ldots
\end{aligned}
$$

Existential quantification

- $\exists$: "There exist a/some"
- $\exists \langle variables \rangle \ \langle sentence \rangle$
- Someone at NTNU is smart:
  $\exists x \ \ At(x, NTNU) \wedge Smart(x)$
- $\exists x \ \ P$ is true in a model m iff P is true with x being **some** possible object in the model
- Equivalent to the disjunction of instantiations of P

$$\begin{array}{ll} & (At(KingJohn, NTNU) \wedge Smart(KingJohn)) \\ \vee & (At(Richard, NTNU) \wedge Smart(Richard)) \\ \vee & (At(NTNU, NTNU) \wedge Smart(NTNU)) \\ \vee & \ldots \end{array}$$

Example for comparison of Propositional Logic and FOL

Primitives in Propositional Logic. Ski-race example.[1]

Objects

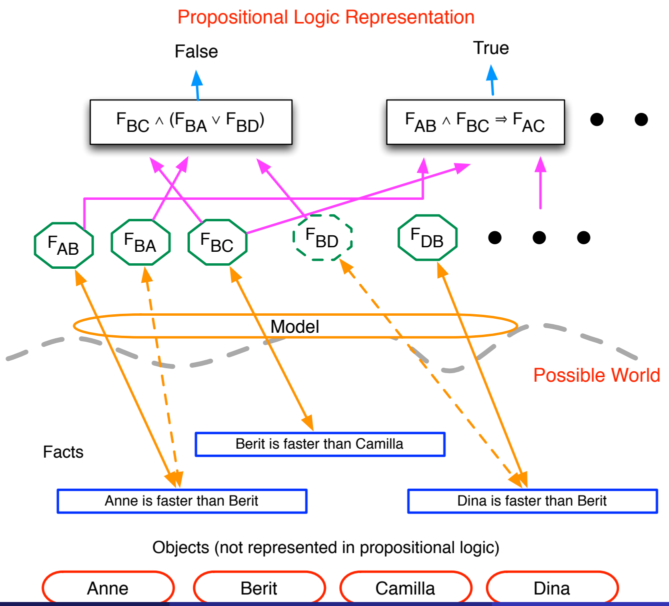
- Anne (A), Berit (B), Camilla (C), Dina (D)
- These are not actually represented in propositional logic.
- Only True-or-False facts **about** them are represented.
- Objects alone do not have a truth value, whereas all primitives in propositional logic do.

Propositional Symbols

- F_{AB}(Anne is faster than Berit) F_{BA} (Berit is faster than Anne), .. F_{AC}
- These have truth values and are the primitive terms.
- Their logical combinations into sentences (representing specific facts or general rules) also have truth values.

[1] Thanks to Keith Downing for this example

Example- Ski-race in Propositional Logic



Example- Ski Race in FOL

Objects

- Anne, Berit, Camilla, Dina
- These are now represented in the logic, even though they still have no truth value.

Functions

- best10K(person) $\rightarrow$ time. Mapping from athlete to their best 10K time.
- rank(person) $\rightarrow$ integer. Mapping from athlete to their seeding in the competition.
- start(person) $\rightarrow$ integer. Mapping from athlete to start order in the race, where slowest start first.
- These have no truth value and map one primitive object (person) to another (number).

Example- Ski Race in FOL

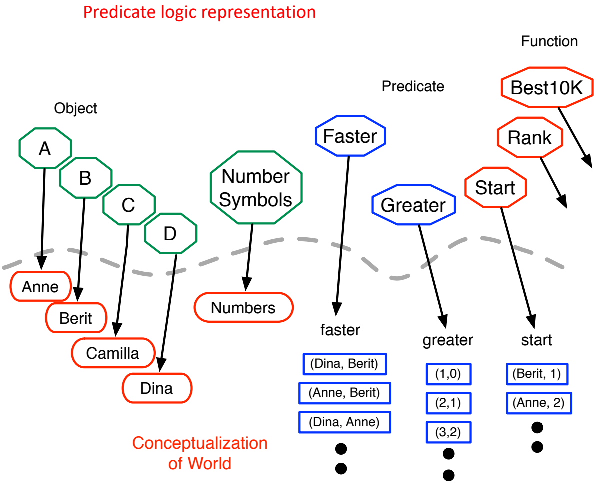
Relations

- greater(X,Y) $\rightarrow$ {True, False }. Is number X greater than number Y?
- faster(X,Y) $\rightarrow$ {True, False }. Is athlete X faster than athlete Y?
- These always have a truth value.
- These are often viewed as explicit lists of tuples, one list for each TRUE relation. So in one possible world, faster is represented by:

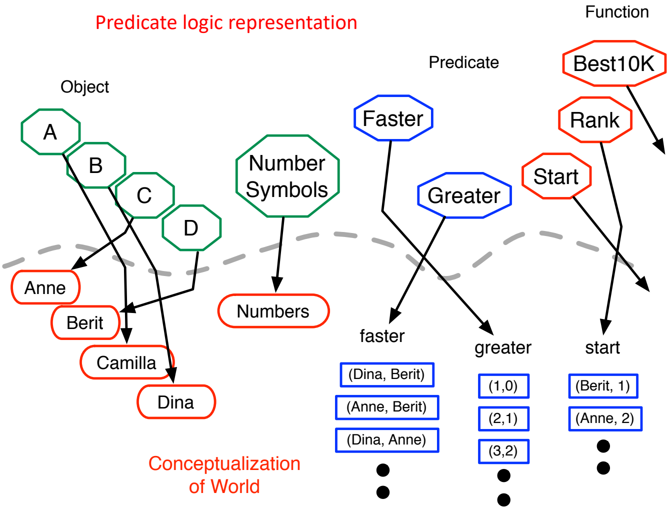
  { (anne, berit), (anne, camilla), (dina, anne), (dina, camilla), (camilla, berit), (dina, berit) }

Example- Interpretations in First-Order Logic

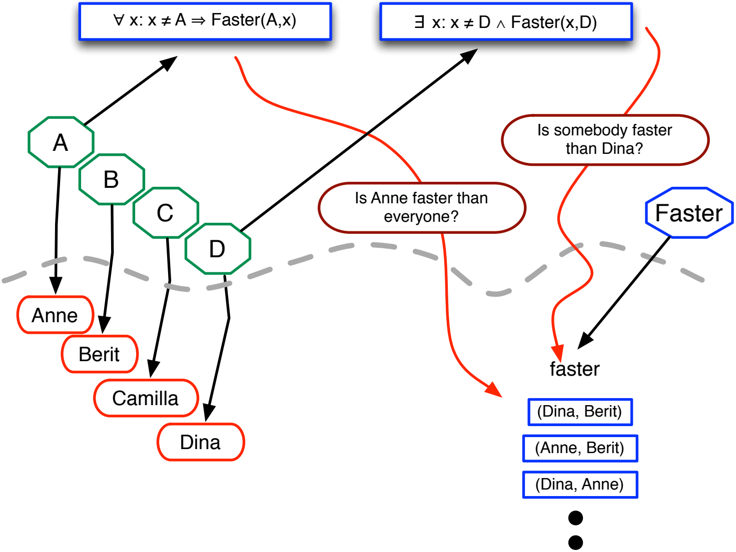
Interpretation = Mapping from constant, function and predicate symbols of the representation to the conceptualization.



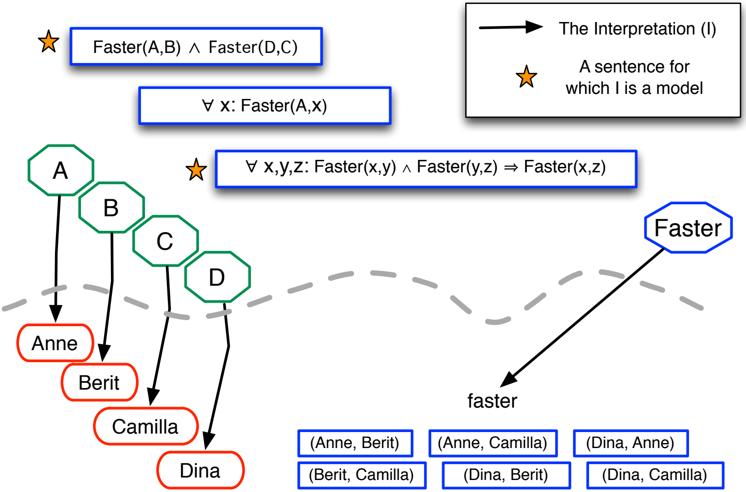
Example- Another Legal Interpretation



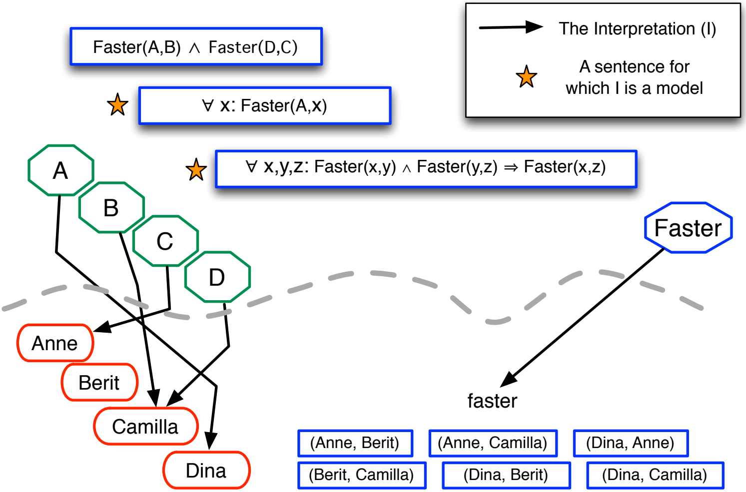
Example - Evaluating Sentences with Quantified Variables



Example - Ski Models, Interpretation 1



Example - Ski Models for Interpretation 2



Common mistakes with Quantifiers

- Typically, $\implies$ is the main connective with $\forall$
- Common mistake: using $\wedge$ as the main connective with $\forall$:
  Let us represent this sentence: "Everybody at NTNU is smart"

$$\forall x \;\; At(x, NTNU) \wedge Smart(x)$$

– Correct? No, it means "Everyone is at NTNU and everyone is smart"

Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\implies$ as the main connective with $\exists$:
  Let us represent this sentence:"There is a smart person at NTNU"–
  Correct?

  $$\exists x \;\; At(x, NTNU) \implies Smart(x)$$

  - No, because it becomes true if there is anyone who is not at NTNU!

Connections between $\forall$ and $\exists$

- All statements made with one quantifier can be converted into equivalent statements with the other quantifier by using negation.
- Remember De Morgan's Rule-
  - $P \wedge Q \equiv (\neg(\neg P \vee \neg Q))$
  - $P \vee Q \equiv (\neg(\neg P \wedge \neg Q))$
  - $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
  - $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- Generalized De Morgan's rules
  - $\forall x \, P(x) \equiv \neg\exists x(\neg P(x))$
  - $\exists x \, P(x) \equiv \neg\forall x(\neg P(x))$
  - $\neg\forall x \, P(x) \equiv \exists x(\neg P(x))$
  - $\neg\exists x \, P(x) \equiv \forall x(\neg P(x))$

Connections between $\forall$ and $\exists$

- Conversion/negation of a quantifier means to represent the quantified statement in terms of the "other" quantifier.
- Negation rules/laws:
  - $\neg\exists \equiv \forall\neg$
  - $\neg\forall \equiv \exists\neg$
  - $\neg\forall\neg \equiv \exists$
  - $\neg\exists\neg \equiv \forall$

Multiple Quantifiers

- more complex sentences can be formulated by multiple variables and by nesting quantifiers
- variables must be introduced by quantifiers, and belong to the innermost quantifier that mention them
- examples
  $\forall x, y\ Parent(x, y) \implies Child(y, x)$
  $\forall x\ Human(x)\, \forall y\ Loves(x, y)$
  $\exists x\ Human(x)\, \forall y\ Loves(x, y)$
  $\exists x\ Human(x)\, \forall y\ Loves(y, x)$

Multiple Quantifiers -2

- "For all x, there exists a y such that P(x,y)"
  - $\forall x \, \exists y P(x, y)$
  - Example: $\forall x \, \exists y (x + y = 0)$
- "There exists an x such that for all y P(x,y) is true"
  - $\exists x \, \forall y P(x, y)$
  - Example: $\exists x \, \forall y (x * y = 0)$

Order of Quantifiers

- Reversing the order of the same type of quantifiers does not change the truth value of a sentence

  $\forall x\, \forall y P(x, y)$ is the same as $\forall y\, \forall x P(x, y)$

- $\exists x\, \forall y$ *and* $\forall x\, \exists y$ are not equivalent!

  $\forall x\, \exists y\ Loves(x, y)$
  $\exists y\, \forall x\ Loves(x, y)$

Example -Order of Quantifiers

- $\forall x\, \exists y\; Loves(x, y)$
  - Everyone in the world loves someone.
  - With parantehеses: $\forall x\, (\exists y\; Loves(x, y)\,)$
  - y is inside the **scope** of x
- $\exists x\, \forall y\; Loves(x, y)$
  - There is a person who loves everyone n the world.
  - The same person x loves everybody.
  - y is inside the scope of x

More examples- Order of Quantifiers

- $\exists y \, \forall x Loves(x, y)$
  - There is someone whom everybody likes
  - Everybody likes the same y.
  - x is inside the scope of y.
- What about "There is a puppy that likes every woman."

More examples - Order of Quantifiers

- What about "There is a puppy that loves every woman."
- $\exists p Puppy(p) \land (\forall w Woman(w) \implies Loves(p, w))$

Negating multiple Quantifiers

- Recall negation rules for single quantifiers:

$\neg\forall x\, P(x) = \exists x \neg P(x)$
$\neg\exists\, x P(x) = \forall x \neg P(x)$

- You change the quantifier(s), and negate what it's quantifying:

$\neg(\forall x\, \exists y\, P(x, y)) \equiv \exists x\, \neg\, \exists y P(x, y) \equiv \exists x\, \forall y\, \neg P(x, y)$

Higher-Order Logic

- FOL is called first-order because it allows quantifiers to range over objects (terms) but not properties, relations, or functions applied to those objects.
- Second-order logic allows quantifiers to range over predicates and functions as well:

- $\forall x \, \forall y [(x = y) \leftrightarrow (\forall P \, P(x) \leftrightarrow P(y))]$
  Means that two objects are equal if and only if they have exactly the same properties.
- $\forall F \, \forall G [(F = G) \leftrightarrow (\forall x \, F(x) = G(x))]$
  Means that two functions are equal if and only if they have the same value for all possible arguments.

Wumpus in first-order logic

- Perception

$\forall$ s, b,g,u,c,t Percept([Stench, b, g, u, c], t) $\implies$ Stench(t)
$\forall$ s, b,g,u,c,t Percept([s, Breeze, g, u, c], t) $\implies$ Breez(t)
$\forall$ s, b,g,u,c,t Percept([s, b, Glitter, u, c], t) $\implies$ Glitter(t)
$\forall$ s, b,g,u,c,t Percept([s, b, g, Bump, c], t) $\implies$ Bump(t)
$\forall$ s, b,g,u,c,t Percept([s, b, g, u, Scream], t) $\implies$ Scream(t)

where s is square, t is timestep.

Wumpus in first-order logic

- Represent a square as an object *s* with coordinate functions x(s) and y(s).
- Represent square adjacency with a predicate

$$
\begin{aligned}
\forall s, q \; Adjacent(s, q) \Leftrightarrow\ & x(s) = x(q) \wedge y(s) = y(q) + 1 \\
& \vee\, x(s) = x(q) \wedge y(s) + 1 = y(q) \\
& \vee\, y(s) = y(q) \wedge x(s) = x(q) + 1 \\
& \vee\, y(s) = y(q) \wedge x(s) + 1 = x(q)
\end{aligned}
$$

Reflex Agent in the Wumpus World

- rules that directly connect percepts to actions
  $\forall$ s, b,g,u,c,t Percept([s, b, Glitter, u,c], t) $\implies$ Action(Grab, t)
- requires a rule for each different combination of percepts - every possible state
- can be simplified by intermediate predicates
  $\forall$ s, b,g,u,c,t Percept([Stench, b, g, u, c], t) $\implies$ Smelly(t)
  $\forall$ s, b,g,u,c,t Percept([s, Breeze, g, u, c], t) $\implies$ Breezy(t)
  $\forall$ s, b,g,u,c,t Percept([s, b, Glitter, u, c], t) $\implies$ AtGold(t)
  $\forall$ s, b,g,u,c,t Percept([s, b, g, Bump, c], t) $\implies$ Bump(t)
  $\forall$ s, b,g,u,c,t Percept([s, b, g, u, Scream], t) $\implies$ Scream(t)
  $\forall$ t AtGold(t) $\implies$ Action(Grab, t)
  ...

Reflex Wumpus Agent

- Assume the agent has the following percept statement at time 5: Percept [noe , none , glitter , none , none , 5]
- We want to query for an appropriate action. Find an action (*ask* the KB): ASK ($\exists a\, Action(a, 5)$)
- 1. $\forall s, b, g, u, c, t[Percept(stench, b, g, u, c, t) \implies Smelly(t)]$
  $\forall s, b, g, u, c, t[Percept(s, b, glitter, u, c, t) \implies AtGold(t)]$
  ...
- 2. Step: Choice of action
  $\forall t[AtGold(t) \implies Action(grab, t)]$
  $AtGold(5) \implies Action(grab, 5)$
- Our reflex agent does not know when it should climb out of the cave (with gold) and cannot avoid an infinite loop.

Deducing hidden properties

- Properties of locations :
  - $\forall x, t \;\; At(Agent, x, t) \wedge Smell(t) \implies Smelly(x)$
  - $\forall x, t \;\; At(Agent, x, t) \wedge Breeze(t) \implies Breezy(x)$

  where x is square, t time.
- Squares are breezy near a pit. Definition for the *Breezy* predicate *(x and y are squares)*:
  $$\forall y \;\; Breezy(y) \Leftrightarrow \exists x \;\; Pit(x) \wedge Adjacent(x, y)$$
- Propositional logic would require an axiom for each square!
- Diagnostic rule - infer cause from effect
  $$\forall y \;\; Breezy(y) \implies \exists x \;\; Pit(x) \wedge Adjacent(x, y)$$
- Causal rule - infer effect from cause
  $$\forall x, y \;\; Pit(x) \wedge Adjacent(x, y) \implies Breezy(y)$$