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Answer 1

Since p is prime and $gcd(x, p) = 1$, by using Fermat's Little Theorem,

$$x^{p-1} \equiv 1 \ (mod \ p)$$

Since p and y are positive integers, by the definition,

$$p - 1 = qy + r \quad \exists q, r \in \mathbb{Z} \quad 0 \leq r < y$$

Then,

$$x^{p-1} = x^{qy+r} = (x^y)^q x^r$$

If we prove that $r = 0$ then we can prove $y \mid (p - 1)$.

Since $x^y \equiv 1 \ (mod \ p)$ then $(x^y)^q \equiv 1 \ (mod \ p)$. Moreover, since both $(x^y)^q \equiv 1 \ (mod \ p)$ and $x^{p-1} \equiv 1 \ (mod \ p)$ holds then we can conclude that $x^r \equiv 1 \ (mod \ p)$ from the equality we obtained. Since y is the smallest positive integer that satisfies $x^y \equiv 1 \ (mod \ p)$ and according to the definition of division $0 \leq r < y$, we can conclude that $r = 0$.
Thus, by the definition, $y \mid (p - 1)$.

Answer 2

Let's assume that $169 \mid (2n^2 + 10n - 7), \ \exists n \in \mathbb{Z}^+$, then by the definition;

$$2n^2 + 10n - 7 = 169k \qquad \exists n \in \mathbb{Z}^+ \quad \exists k \in \mathbb{Z} \tag{1}$$

If we pass $169k$ to the left side of the (1),

$$2n^2 + 10n - 7 - 169k = 0 \qquad \exists n \in \mathbb{Z}^+ \quad \exists k \in \mathbb{Z} \tag{2}$$

To find the $n \in \mathbb{Z}^+$ values that satisfy (2), we can use discriminant method since the equation in quadratic form,

$$\begin{aligned} \Delta = b^2 - 4ac = 10^2 + 8 * (7 + 169k) = 13 * (12 + 8 * 13k) \quad k \in \mathbb{Z} \\ n = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-10 \pm \sqrt{13 * (12 + 8 * 13k)}}{4} \end{aligned} \tag{3}$$

Since $k \in \mathbb{Z}$, we can conclude that $\Delta = 13*(12+8*13k) \in \mathbb{Z}$. Moreover, for $\sqrt{\Delta}$ to be reel $\Delta \geq 0$ and for $\Delta = 0$ $\ n = -5/2 \notin \mathbb{Z}^+$, hence we can conclude that $\Delta = 13*(12+8*13k) \in \mathbb{Z}^+$.
Then, as can be seen from (3), for n to be an integer, $\Delta = 13*(12+8*13k)$ must be perfect square.
According to The Fundamental Theorem Of Arithmetic, since $\Delta = 13*(12+8*13k) \in \mathbb{Z}^+$, Δ can be written uniquely in the form of $p_1 * p_2 * ...p_i$ where p_i is a prime number and $i \in \mathbb{Z}^+$.
Moreover, by the definition, since $\Delta = 13*(12+8*13k)$ must be perfect square then it can be written uniquely as $p_1^{2s_1} * p_2^{2s_2} * ...p_i^{2s_i}$ where p_i is a prime number, $i \in \mathbb{Z}^+$ and $s_i \in \mathbb{Z}^+$. Since 13 is a multiplier of Δ and is a prime number, then $13 \mid (12+8*13k)$ must be satisfied. As can be seen that $13 \mid (8*13k)$, however since $12 \not\equiv 0 \ (mod\ 13)$, $13 \nmid 12$. Hence, $13 \nmid (12+8*13k)$. Since there exist no $k \in \mathbb{Z}$ that makes Δ perfect square, there exist no arbitrarily chosen $n \in \mathbb{Z}^+$ that satisfies $169 \mid (2n^2+10n-7)$. Thus, by using proof by contradiction, $169 \nmid (2n^2+10n-7)$, $\forall n \in \mathbb{Z}^+$.

Answer 3

Since $a \equiv b \ (mod\ m)$ and $a \equiv b \ (mod\ n)$, by the definition, $m \mid (a-b)$ and $n \mid (a-b)$. Then, by the definition,

$$a - b = mk \qquad \exists k \in \mathbb{Z} \tag{4}$$

Since $n \mid (a-b)$, then $n \mid mk$. Also since given that $gcd(m,n) = 1$, $n \nmid m$ so that $n \mid k$. Thus, by the definition,

$$k = nt \qquad \exists t \in \mathbb{Z} \tag{5}$$

Therefore, by using (4) and (5) we can conclude that,

$$a - b = mnt \qquad m, n \in \mathbb{Z}^+ \quad t \in \mathbb{Z} \tag{6}$$

Thus, by the definition, we can conclude from (6),

$$a \equiv b \ (mod\ m \times n)$$

Answer 4

To prove the given argument, we can use Mathematical Induction Method.
Let the given argument be P(n,k).

Basis Step: Let us show that P(1,k) is true for $\forall k \in \mathbb{Z}^+$. For arbitrarily chosen $k \in \mathbb{Z}^+$;

$$\begin{aligned}\sum_{j=1}^{n=1} j(j+1)(j+2)...(j+k-1) &= 1*(1+1)*(1+2)...*(1+k-1) \\ &= 1*2*3*...*k\end{aligned} \tag{7}$$

Moreover,

$$\frac{n(n+1)(n+2)...(n+k)}{(k+1)} = \frac{1*2*3*..*(k+1)}{k+1} = 1*2*3*...*k \tag{8}$$

From (7) and (8);

$$\sum_{j=1}^{n=1} j(j+1)(j+2)...(j+k-1) = \frac{n(n+1)(n+2)...(n+k)}{(k+1)} = 1*2*3*...*k$$

Thus we proved that P(1,k) is true $\forall k \in \mathbb{Z}^+$ since we proved it for an arbitrary $k \in \mathbb{Z}^+$.

Inductive Step: Assume that P(n,k) is true where $n, k \in \mathbb{Z}^+$. Then, let us show that P(n+1,k) is also true.

$$\begin{aligned} &\sum_{j=1}^{n+1} j(j+1)(j+2)...(j+k-1) \\ &= \sum_{j=1}^{n} j(j+1)(j+2)...(j+k-1) + (n+1)*(n+2)*...*(n+1+k-1) \end{aligned} \tag{9}$$

We assumed that P(n,k) is true, therefore we can write (9) as,

$$\sum_{j=1}^{n+1} j(j+1)(j+2)...(j+k-1) = \frac{n(n+1)(n+2)...(n+k)}{(k+1)} + (n+1)*(n+2)*...*(n+1+k-1) \tag{10}$$

If we rearrange the right side of (10);

$$\begin{aligned} \sum_{j=1}^{n+1} j(j+1)(j+2)...(j+k-1) &= \frac{n(n+1)(n+2)...(n+k) + (k+1)(n+1)(n+2)...(n+k)}{(k+1)} \\ &= \frac{(n+1)(n+2)...(n+k)(n+k+1)}{(k+1)} \end{aligned} \tag{11}$$

From (11),

$$\frac{(n+1)(n+2)...(n+k)(n+k+1)}{(k+1)} = \frac{(n+1)((n+1)+1)((n+1)+2)...((n+1)+k)}{(k+1)} \tag{12}$$

Thus, we proved that P(n+1,k) is also true if P(n,k) is true, $\forall k \in \mathbb{Z}^+$ since we proved it for an arbitrary $k \in \mathbb{Z}^+$.

In conclusion, we proved that the given statement is true for all positive integers k and n, by using mathematical induction method.

Answer 5

Basis Step: Since for $H_0 = 1$, $H_1 = 3$, $H_2 = 5$,

$$H_0 \leq 7^0, \qquad H_1 \leq 7^1, \qquad H_2 \leq 7^2$$

we can take them as our base cases.

Inductive Step: Let $n \geq 3$, assume that $H_m \leq 7^m$ for all integer m's where $0 \leq m < n$. Then for H_n by our inductive hypothesis,

$$\begin{aligned} H_n &= 5H_{n-1} + 5H_{n-2} + 63H_{n-3} \\ H_n &\leq 5 * 7^{n-1} + 5 * 7^{n-2} + 63 * 7^{n-3} \\ H_n &\leq 343 * 7^{n-3} \\ H_n &\leq 7^n \end{aligned}$$

Thus, we have proved that $H_n \leq 7^n$ holds for all $n \geq 0$ by using strong induction method.