# Student Information

Name : Onat Özdemir

ID : 2310399

# Answer 1

## a)

Let K be an experiment which can be described as selecting a box and picking a random ball from the selected box. Additionally, let's we represent the balls in X as,

$$X = \{X_{red1}, X_{red2}, X_{green1}, X_{green2}, X_{blue1}, X_{blue2}\}$$

and in Y as

$$Y = \{Y_{red}, Y_{green1}, Y_{green2}, Y_{blue1}, Y_{blue2}\}$$

Then the sample space of K,

$$\Omega = \{(X, X_{red1}), (X, X_{red2}), (X, X_{green1}), (X, X_{green2}), (X, X_{blue1}), (X, X_{blue2}), (Y, Y_{red}), (Y, Y_{green1}), (Y, Y_{green2}), (Y, Y_{blue1}), (Y, Y_{blue2})\}$$

Given condition which is knowing that the selected box is X, limits the sample space to,

$$\Omega' = \{(X, X_{red1}), (X, X_{red2}), (X, X_{green1}), (X, X_{green2}), (X, X_{blue1}), (X, X_{blue2})\}$$

Since we are randomly choosing the ball, all outcomes in $\Omega'$ are equally likely. Therefore,

$$P(A|B) = \frac{\textit{number of favourable outcomes in } \Omega'}{\textit{number of total outcomes in } \Omega'}$$

where $P(A|B)$ represents probability that we pick a green ball given that the selected box is X, and our favourable outcomes in this case are $(X, X_{red1}), (X, X_{red2})$. And the number of total outcomes in $\Omega'$ equals to 6. So if we apply the above formula,

$$P(A|B) = \frac{2}{6}$$

## b)

We can pick a red ball in 2 different ways: either we can pick the X box and pick a red ball from it or we can pick the Y box and then pick a red ball from it. Before starting the calculation, let's denote some events:
Choosing the Y box as $Y$

Choosing the X box as $X$
Choosing a red ball as $R$
Since $Y$ and $X$ are both exhaustive and mutually exclusive events, by the law of Total Probability,

$$P(R) = P(R|Y) * P(Y) + P(R|X) * P(X) \tag{1}$$

Since the ball is chosen randomly, then probability of picking a ball from the selected box is as equally likely as choosing the another one. So, probability of picking a ball with x color from the selected box can be calculated as

$$P(x) = \frac{number\ of\ balls\ with\ color\ x}{number\ of\ total\ balls} \tag{2}$$

Since choosing the X box and choosing the Y box are exhaustive events, then $P(Y) = 1 - P(X) = 0.6$. Additionally, $P(R|Y)$ can be derived from (2), $P(R|Y) = 1/5 = 0.2$.Hence,

$$P(R|Y) * P(Y) = (0.6) * (0.2) = 0.12 \tag{3}$$

We can apply the same procedure to calculate $P(R|X) * P(X)$.By using (2), $P(R|X) = 1/3$.Hence,

$$P(R|X) * P(X) = (0.4) * (1/3) = 2/15 \tag{4}$$

By using (1), (3) and (4) ;

$$P(R) = 0.12 + 2/15 = 38/150 \tag{5}$$

## c)

Since we have got previously known information, to calculate the probability that we had chosen the Y box when we know that picked ball is blue, we can use Bayes' Theorem:

$$P(Y|B) = \frac{P(B|Y) * P(Y)}{P(B)} \tag{6}$$

where $Y$ represents choosing the Y box and $B$ represents picking a blue ball.Firstly, let's calculate $P(B)$.We can pick a blue ball in 2 different ways: either we can pick the X box and pick a blue ball from it or we can pick the Y box and then pick a blue ball from it.Since $Y$ and $X$ are both exhaustive and mutually exclusive events, by the law of Total Probability,

$$P(B) = P(B|Y) * P(Y) + P(B|X) * P(X) \tag{7}$$

Since the ball is chosen randomly, then probability of picking a ball from the selected box is as equally likely as choosing the another one. So we can use (2) for this question too, to calculate the probability of choosing a blue ball from the selected box:

$$\begin{aligned} P(B|Y) &= 2/5 \\ P(B|X) &= 2/6 = 1/3 \end{aligned} \tag{8}$$

Since $Y$ and $X$ are both exhaustive events, $P(Y) = 1 - P(X) = 0.6$. Then,

$$\begin{aligned} P(B|Y) * P(Y) &= (2/5) * (0.6) = 0.24 \\ P(B|X) * P(X) &= (1/3) * (0.4) = 2/15 \end{aligned} \tag{9}$$

Hence from (7),

$$P(B) = P(B|Y) * P(Y) + P(B|X) * P(X) = 112/300 \tag{10}$$

We have already found $P(B|Y) * P(Y)$ from (9). By using (6),(9) and (10) :

$$P(Y|B) = \frac{(24/100)}{(112/300)} = 72/112 \tag{11}$$

Thus, probability of we had chosen the Y box when we know that picked ball is blue is 72/112

# Answer 2

## a)

In order to prove given argument, we should be able to prove both: "$A$ and $B$ are mutually exclusive" implies that "$\overline{A}$ and $\overline{B}$ are exhaustive", and "$\overline{A}$ and $\overline{B}$ are exhaustive" implies that "$A$ and $B$ are mutually exclusive".

**1-Prove if "$A$ and $B$ are mutually exclusive" then "$\overline{A}$ and $\overline{B}$ are exhaustive":** Assume that $A$ and $B$ are mutually exclusive events.Then, by the definition,

$$A \cap B = \emptyset \tag{12}$$

If we take the complement of (12), by the De Morgan's Rule:

$$\overline{A} \cup \overline{B} = \Omega \tag{13}$$

Since (13) satisfies, by the definition of exhaustive events, we showed that $\overline{A}$ and $\overline{B}$ are exhaustive events. Since we assumed "$A$ and $B$ are mutually exclusive events" and reached (13), we proved that if "$A$ and $B$ are mutually exclusive" then "$\overline{A}$ and $\overline{B}$ are exhaustive events".

**2-Prove that "if $\overline{A}$ and $\overline{B}$ are exhaustive events" then "$A$ and $B$ are mutually exclusive events":** Assumed that $\overline{A}$ and $\overline{B}$ are exhaustive events. Then, by the definition,

$$\overline{A} \cup \overline{B} = \Omega \tag{14}$$

If we take the complement of (14), by the De Morgan's Rule:

$$A \cap B = \emptyset \tag{15}$$

Since (15) satisfies, by the definition of mutually exclusive events, we showed that $A$ and $B$ are mutually exclusive events. Since we assumed "$\overline{A}$ and $\overline{B}$ are exhaustive events" and reached (15), we proved that if "$\overline{A}$ and $\overline{B}$ are exhaustive events" then "$A$ and $B$ are mutually exclusive events".

We were able to prove both if "$A$ and $B$ are mutually exclusive events" then "$\overline{A}$ and $\overline{B}$ are exhaustive events", and if "$\overline{A}$ and $\overline{B}$ are exhaustive events" then "$A$ and $B$ are mutually exclusive events". Thus, we proved that $A$ and $B$ are mutually exclusive events if and only if $\overline{A}$ and $\overline{B}$ are exhaustive events.

### b)

Assume that given statement is true. Then,
1) If "$A$, $B$ and $C$ are mutually exclusive events" then "$\overline{A}$, $\overline{B}$ and $\overline{C}$ are exhaustive events"
2) If "$\overline{A}$, $\overline{B}$ and $\overline{C}$ are exhaustive events" then "$A$, $B$ and $C$ are mutually exclusive events"
must satisfy. Let $\overline{B} = A$, $\overline{C} = \overline{A}$ and $A \neq \emptyset$ then,

$$\begin{aligned}(\overline{A} \cup \overline{B}) \cup \overline{C} &= (\overline{A} \cup A) \cup \overline{C} \\ &= \Omega \cup \overline{C} \quad \textit{Complement Law} \\ &= \Omega \quad \textit{Dominance Law}\end{aligned}$$

Since $(\overline{A} \cup \overline{B}) \cup \overline{C} = \Omega$ by the definition of exhaustive events, $\overline{A}, \overline{B}$ and $\overline{C}$ are exhaustive events. Since we assumed that given statement is true and $\overline{A}, \overline{B}$ and $\overline{C}$ are exhaustive events, by the second condition, A, B and C must be mutually exclusive events.

$$\begin{aligned}A \cap C &= A \cap \overline{\overline{A}} \\ &= A \cap A \quad \textit{Involution Law} \\ &= A \quad \textit{Idempotent Law}\end{aligned}$$

Since $A \cap C = A$ and $A \neq \emptyset$, A and C don't satisfy the definition of mutually exclusive events. So, $A$ and $C$ are not mutally exclusive events. Hence, we have found one case that doesn't satisfy given statement. Thus, by using proof by contradiction, we disproved given statement.

## Answer 3

### a)

Since for each die roll we have either a successful die or unsuccessful die, rolling a single die is a Bernoulli Experiment and rolling five dice is 5 times repeated Bernoulli Experiment. As we know, distribution of n times repeated Bernoulli Experiment is a binomial distribution. Let's define a binomial random variable X as number of successful dice in rolling 5 dice. So we should find $P\{X = 2\}$ where $P\{X = x\}$ is probability mass function of X. As we know from the Binomial distribution,

$$P\{X = k\} = C(n, k) * P(s)^k * (1 - P(s))^{n-k} \tag{16}$$

where n is number of repeated Bernoulli Experiment, k is the number of successful dice and $P(s)$ is the probability of a single successful die. In our case, since 2 of the faces of the die which are

5 and 6 out of 6 faces are successful and probabilities of each face is equally likely since dice are fair, $P(s) = \frac{\text{Number of succesful outcomes}}{\text{Number of total outcomes}} = 2/6 = 1/3$ and from the question $n = 5$ and $k = 2$.
If we put the obtained parameters to (16),

$$P\{X = 2\} = C(5, 2) * (1/3)^2 * (2/3)^3 = 80/243 \tag{17}$$

Hence, probability that we have exactly 2 successful dice ($P\{X = 2\}$) equals to 80/243.

### b)

For this question let's use X as the way it is defined in (a) part. So we are trying to find $P\{1 < X \leq 5\}$. Let $F_X(x)$ be the cumulative distribution function of X. By using the definition of F(x) and the given formula ("Probability and Statistics For Computer Scientists" page 41):

$$P\{1 < X \leq 5\} = F_X(5) - F_X(1) \tag{18}$$

By using the definition of F(x)

$$F_X(5) = \sum_{y=0}^{5} P_X(y) = 1 \tag{19}$$

$$F_X(1) = \sum_{y=0}^{1} P_X(y) = P_X(0) + P_X(1) \tag{20}$$

By using (16),

$$P_X(0) = C(5, 0) * (2/3)^5 = 32/243 \quad P_X(1) = C(5, 1) * (1/3) * (2/3)^4 = 80/243 \tag{21}$$

Hence, $F_X(1) = 112/243$.

Combining (18),(19),(20) and (21),

$$P\{1 < X \leq 5\} = 1 - 112/243 = 131/243 \tag{22}$$

Hence, probability that we have at least 2 successful dice equals to 131/243.

## Answer 4

### a)

Since $\{B = 0\}$ and $\{B = 1\}$ are exhaustive and mutually exclusive events, we can use Addition Rule to calculate $P(A = 1, C = 0)$:

$$P(A = 1, C = 0) = \sum_{b} P(A = 1, B = b, C = 0) = P(A = 1, B = 0, C = 0) + P(A = 1, B = 1, C = 0) \tag{23}$$

From the given table, from row 5, $P(A = 1, B = 0, C = 0) = 0.06$ and from row 7, $P(A = 1, B = 1, C = 0) = 0.09$. Hence, from (23), $P(A = 1, C = 0) = 0.06 + 0.09 = 0.15$

### b)

Since (A,C) = (a,c) are mutually exclusive and exhaustive events for the different pairs of (a,c), we can use Addition Rule to calculate $P(B=1)$:

$$P(B=1) = \sum_a \sum_c P(A=a, B=1, C=c) \tag{24}$$

Then, from the table:

$$\begin{aligned} P(A=0, B=1, C=0) &= 0.21 \ (Row\ 3) \\ P(A=0, B=1, C=1) &= 0.02 \ (Row\ 4) \\ P(A=1, B=1, C=0) &= 0.09 \ (Row\ 7) \\ P(A=1, B=1, C=1) &= 0.08 \ (Row\ 8) \end{aligned} \tag{25}$$

Hence, by using (24) and (25):

$$P(B=1) = 0.21 + 0.02 + 0.09 + 0.08 = 0.4$$

### c)

Let's assume that A and B independent random variables, then by the definition, for all (a,b) pairs:

$$P(A=a, B=b) = P(A=a) * P(B=b) \tag{26}$$

must be satisfied. Let's observe $P(A=1, B=1)$ case. Since we assumed A and B are independent then by (26):

$$P(A=1, B=1) = P(A=1) * P(B=1) \tag{27}$$

must be satisfied. Since $\{C=1\}$ and $\{C=0\}$ are mutually exclusive and exhaustive events by the Addition Rule:

$$P(A=1, B=1) = P(A=1, B=1, C=0) + P(A=1, B=1, C=1)$$

Then, from the given table:

$$P(A=1, B=1) = 0.09 + 0.08 = 0.17$$

Since (B,C) = (b,c) are mutually exclusive and exhaustive events for the different pairs of (b,c), we can use Addition Rule to calculate $P(A=1)$:

$$P(A=1) = \sum_b \sum_c P(A=1, B=b, C=c) \tag{28}$$

Then,

$$\begin{aligned}
P(A=1,B=0,C=0) &= 0.06\ (Row\ 5) \\
P(A=1,B=0,C=1) &= 0.32\ (Row\ 6) \\
P(A=1,B=1,C=0) &= 0.09\ (Row\ 7) \\
P(A=1,B=1,C=1) &= 0.08\ (Row\ 8)
\end{aligned} \tag{29}$$

Hence, by using (28) and (29), $P(A=1) = 0.06+0.32+0.09+0.08 = 0.55$. Also we have already found $P(B) = 0.4$ from the b section. Since $P(A=1,B=1) = 0.17 \neq P(A=1) * P(B=1) = 0.4 * 0.55 = 0.22$, (27) is not satisfied. Thus,by using proof by contradiction, we've proved that A and B are not independent random variables.

## d)

By the definition, A and B random variables are conditionally independent under the condition $C = 1$ if:

$$P(A=a \cap B=b|C=1) = P(A=a,B=b|C=1) = P(A=a|C=1) * P(B=b|C=1) \tag{30}$$

for all (a,b) pairs. By the definition of conditional probability:

$$\begin{aligned}
P(A=a|C=1) &= \frac{P(A=a,C=1)}{P(C=1)} \\
P(B=b|C=1) &= \frac{P(B=b,C=1)}{P(C=1)} \\
P(A=a,B=b|C=1) &= \frac{P(A=a,B=b,C=1)}{P(C=1)}
\end{aligned} \tag{31}$$

Since (A,B) = (a,b) are mutually exclusive and exhaustive events for the different pairs of (a,b), we can use Addition Rule to calculate $P(C=1)$:

$$P(C=1) = \sum_a \sum_b P(A=a,B=b,C=1) \tag{32}$$

Then,

$$\begin{aligned}
P(A=0,B=0,C=1) &= 0.08\ (Row\ 2) \\
P(A=0,B=1,C=1) &= 0.02\ (Row\ 4) \\
P(A=1,B=0,C=1) &= 0.32\ (Row\ 6) \\
P(A=1,B=1,C=1) &= 0.08\ (Row\ 8)
\end{aligned} \tag{33}$$

By using (32), $P(C=1) = 0.08+0.02+0.32+0.8 = 0.5$ If we can show that (30) satisfies for all (a,b) pairs, then we can prove that A and B are conditional independent random variables under the condition $C = 1$.

Since $\{B = 1\}$ and $\{B = 0\}$ are mutually exclusive and exhaustive events we can use Addition Rule:

$$\begin{aligned} P(A=1, C=1) &= P(A=1, B=0, C=1) + P(A=1, B=1, C=1) \\ P(A=1, C=1) &= 0.32 + 0.08 = 0.4 \\ P(A=0, C=1) &= P(A=0, B=0, C=1) + P(A=0, B=1, C=1) \\ P(A=0, C=1) &= 0.08 + 0.02 = 0.1 \end{aligned} \tag{34}$$

Since $\{A = 1\}$ and $\{A = 0\}$ are mutually exclusive and exhaustive events we can use Addition Rule:

$$\begin{aligned} P(B=1, C=1) &= P(A=1, B=1, C=1) + P(A=0, B=1, C=1) \\ P(B=1, C=1) &= 0.08 + 0.02 = 0.1 \\ P(B=0, C=1) &= P(A=0, B=0, C=1) + P(A=1, B=0, C=1) \\ P(B=0, C=1) &= 0.08 + 0.32 = 0.4 \end{aligned} \tag{35}$$

Additionally, from the table,

$$\begin{aligned} P(A=0, B=0, C=1) &= 0.08 \\ P(A=0, B=1, C=1) &= 0.02 \\ P(A=1, B=0, C=1) &= 0.32 \\ P(A=1, B=1, C=1) &= 0.08 \end{aligned} \tag{36}$$

Using (31)

$$\begin{aligned} P(A=0|C=1) &= \frac{P(A=0, C=1)}{P(C=1)} = 0.1/0.5 = 0.2 \\ P(A=1|C=1) &= \frac{P(A=1, C=1)}{P(C=1)} = 0.4/0.5 = 0.8 \\ P(B=0|C=1) &= \frac{P(B=0, C=1)}{P(C=1)} = 0.4/0.5 = 0.8 \\ P(B=1|C=1) &= \frac{P(B=1, C=1)}{P(C=1)} = 0.1/0.5 = 0.2 \\ P(A=0, B=0|C=1) &= \frac{P(A=0, B=0, C=1)}{P(C=1)} = 0.08/0.5 = 0.16 \\ P(A=0, B=1|C=1) &= \frac{P(A=0, B=1, C=1)}{P(C=1)} = 0.02/0.5 = 0.04 \\ P(A=1, B=0|C=1) &= \frac{P(A=1, B=0, C=1)}{P(C=1)} = 0.32/0.5 = 0.64 \\ P(A=1, B=1|C=1) &= \frac{P(A=1, B=1, C=1)}{P(C=1)} = 0.08/0.5 = 0.16 \end{aligned} \tag{37}$$

Let's observe that whether for all (a,b) (30) holds or not:

$$\begin{aligned}
P(A=0,B=0|C=1) = 0.16 = P(A=0|C=1) * P(B=0|C=1) = 0.16 \\
P(A=0,B=1|C=1) = 0.04 = P(A=0|C=1) * P(B=1|C=1) = 0.04 \\
P(A=1,B=0|C=1) = 0.64 = P(A=1|C=1) * P(B=0|C=1) = 0.64 \\
P(A=1,B=1|C=1) = 0.16 = P(A=1|C=1) * P(B=1|C=1) = 0.16
\end{aligned} \tag{38}$$

In (38), we are able to show that for all (a,b) pairs (30) holds. Thus, we have proved that A and B are conditionally independent random variables under the condition $C = 1$