Discrete Haar Wavelet Transforms

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PREP - Wavelet Workshop, 2006

Outline

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Today's Schedule

9:00-10:15	**Lecture One:** Why Wavelets?
10:15-10:30	Coffee Break (OSS 235)
10:30-11:45	**Lecture Two:** Digital Images, Measures, and Huffman Codes
12:00-1:00	Lunch (Cafeteria)
1:30-2:45	**Lecture Three:** Fourier Series, Convolution and Filters
2:45-3:00	Coffee Break (OSS 235)
3:00-4:15	⇒**Lecture Four:** 1D and 2D Haar Transforms
5:30-6:30	Dinner (Cafeteria)

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Building the Haar Matrix

Putting Two Filters Together

- Consider again the filter $\mathbf{h} = (h_0, h_1) = (\frac{1}{2}, \frac{1}{2})$.
- If we compute $\mathbf{y} = \mathbf{h} * \mathbf{x}$, we obtain the components

$$y_n = \frac{1}{2}x_n + \frac{1}{2}x_{n-1}$$

- We could write down the convolution matrix

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Putting Two Filters Together

$$H = \left[\begin{array}{ccccccccccccccc}
\ddots & & \ddots & \ddots & \ddots & & & & & & & & & & \\
 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\cdots & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \\
 & & & & & & & & & \ddots & \ddots & & & & \ddots
\end{array}\right]$$

Building the Haar Matrix

Putting Two Filters Together

- . . . but we can't invert the process.
- What would we need to be able to invert the process?
- We have averages of consecutive numbers - if we had the directed distance between these averages and the consecutive numbers, then we could invert.
- The directed distance is exactly the sequence $\mathbf{x}$ convolved with the filter $\mathbf{g} = (\frac{1}{2}, -\frac{1}{2})$.

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Building the Haar Matrix

Putting Two Filters Together

Indeed if

$$y_n = \frac{1}{2}x_n + \frac{1}{2}x_{n-1} \qquad \text{and} \qquad z_n = \frac{1}{2}x_n - \frac{1}{2}x_{n-1}$$

then

$$x_n = y_n + z_n \qquad \text{and} \qquad x_{n-1} = y_n - z_n$$

Building the Haar Matrix

Putting Two Filters Together

Perhaps we could invert the process if we used both filters. We know that G is

Building the Haar Matrix

Putting Two Filters Together

$$G = \left[\begin{array}{ccccccccccccccc}
\ddots & & \ddots & \ddots & \ddots & & & & & & & & & & \\
 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\cdots & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \\
 & & & & & & & & & \ddots & \ddots & & & & \ddots
\end{array}\right.$$

Building the Haar Matrix
Putting Two Filters Together

So that

$$\begin{bmatrix} H \\ \hline G \end{bmatrix} \cdot \mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \hline \mathbf{z} \end{bmatrix}$$

Building the Haar Matrix

Putting Two Filters Together

If we think about inverting, we can write down:

Building the Haar Matrix

Putting Two Filters Together

$$
\begin{aligned}
\vdots \quad & \quad \vdots \\
y_1 - z_1 &= \frac{x_1 + x_0}{2} - \frac{x_1 - x_0}{2} = x_0 \\
y_1 + z_1 &= \frac{x_1 + x_0}{2} + \frac{x_1 - x_0}{2} = x_1 \\
y_2 - z_2 &= \frac{x_2 + x_1}{2} - \frac{x_2 - x_1}{2} = x_1 \\
y_2 + z_2 &= \frac{x_2 + x_1}{2} + \frac{x_2 - x_1}{2} = x_2 \\
y_3 - z_3 &= \frac{x_3 + x_2}{2} - \frac{x_3 - x_2}{2} = x_2 \\
y_3 + z_3 &= \frac{x_3 + x_2}{2} + \frac{x_3 - x_2}{2} = x_3 \\
\vdots \quad & \quad \vdots
\end{aligned}
$$

Building the Haar Matrix

Putting Two Filters Together

But there is some redundancy here - we do not need all the values of y_n, z_n to recover x_n:

Building the Haar Matrix

Putting Two Filters Together

$$
\begin{aligned}
&\vdots && \vdots \\
y_1 - z_1 &= \frac{x_1 + x_0}{2} - \frac{x_1 - x_0}{2} = x_0 \\
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y_3 + z_3 &= \frac{x_3 + x_2}{2} + \frac{x_3 - x_2}{2} = x_3 \\
&\vdots && \vdots
\end{aligned}
$$

Building the Haar Matrix

Putting Two Filters Together

- So we can omit every other row in H, G and still produce enough to be able to recover $\mathbf{x}$
- This is called *downsampling*.
- We are also now in a position to truncate our matrix. Indeed, if $\mathbf{x} = (x_0, \ldots, x_N)$, then it is natural to truncate the matrix and write:

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Building the Haar Matrix

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Building the Haar Matrix

Putting Two Filters Together

$$\tilde{W}_N = \left[\begin{array}{cccccccccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & & & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & & \cdots & 0 & 0 \\ \vdots & & & & \ddots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & & & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & & \cdots & 0 & 0 \\ \vdots & & & & \ddots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Building the Haar Matrix

Putting Two Filters Together

This matrix is easy to invert if we remember the formulas:

$$x_n = y_n + z_n \qquad \text{and} \qquad x_{n-1} = y_n - z_n$$

We have:

Building the Haar Matrix

Putting Two Filters Together

$$\tilde{W}_N^{-1} = \left[\begin{array}{cccc|cccc} 1 & 0 & & 0 & -1 & 0 & & 0 \\ 1 & 0 & & 0 & 1 & 0 & & 0 \\ 0 & 1 & & 0 & 0 & -1 & & 0 \\ 0 & 1 & & 0 & 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & 0 & 0 & 0 & & 0 \\ 0 & 0 & & 0 & 0 & 0 & & 0 \\ 0 & 0 & & 1 & 0 & 0 & & -1 \\ 0 & 0 & & 1 & 0 & 0 & & 1 \end{array}\right]$$

Building the Haar Matrix

Putting Two Filters Together

- Note we are very close to having $\tilde{W}_N$ an orthogonal matrix.
- We have $\tilde{W}_N^T = \frac{1}{2}\tilde{W}_N^{-1}$.
- If we multiply $\tilde{W}_N$ by $\sqrt{2}$, we will obtain an orthogonal matrix.
- We have:

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$$W_N = \left[\begin{array}{c} H \\ \hline G \end{array}\right] = \left[\begin{array}{cccccc} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & & 0 \quad 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & & 0 \quad 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \\ \hline -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & & 0 \quad 0 \\ 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & & 0 \quad 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \end{array}\right]$$

Building the Haar Matrix

Putting Two Filters Together

- W_N is called the Discrete Haar Wavelet Transform
- The filter

$$\boxed{\mathbf{h} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)}$$

  is called the Haar filter.
- Note that $H(\omega) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}e^{i\omega}$ satisfies $H(\pi) = 0$, but $H(0) = \frac{\sqrt{2}}{2}$. We will still consider this to be a lowpass filter- the $\frac{\sqrt{2}}{2}$ resulted when we made the transform orthogonal.

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Building the Haar Matrix

Why the Word Wavelet?

- The word wavelet comes from the more classical treatment of the topic. Here, we work in $L^2(\mathbb{R})$ and *downsampling* is basically a way to move between nested subspaces V_j that are generated by the translates and dilates of a single scaling function.
- If V_0 is the space of piecewise constants with possible breaks at $\mathbb{Z}$, then the characteristic function $\phi(t) = \chi_{[0,1)}(t)$ and its translates form an orthonormal basis for V_0.

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Building the Haar Matrix

Why the Word Wavelet?

Building the Haar Matrix

Why the Word Wavelet?

If V_1 is the space of piecewise constants with possible breakpoints at $\frac{1}{2}\mathbb{Z}$, then $V_0 \subset V_1$, and the functions $\sqrt{2}\phi(2t-k)$ form an orthornormal basis for V_1.

Building the Haar Matrix

Why the Word Wavelet?

Note that

$$\phi(t) = \sqrt{2}\left(\frac{\sqrt{2}}{2}\phi(2t) + \frac{\sqrt{2}}{2}\phi(2t-1)\right)$$

is called a dilation equation.

Building the Haar Matrix

Why the Word Wavelet?

- We can get the Haar filter coefficients from the dilation equation.
- The word wavelet refers to the function $\psi(t)$ that generates a basis for the orthogonal complement of V_0 in V_1.
- In this case, the wavelet function is

$$\psi(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \end{cases}$$

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Building the Haar Matrix

Why the Word Wavelet?

Note that $\psi(t) \in V_1$ and

$$\psi(t) = \phi(t) = \sqrt{2}\left(\frac{\sqrt{2}}{2}\phi(2t) - \frac{\sqrt{2}}{2}\phi(2t-1)\right)$$

so that the highpass filter $\mathbf{g} = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ can be read from this dilation equation.

Building the Haar Matrix

Why the Word Wavelet?

- I opted to stay away from the classical approach to wavelets.
- It is beautiful theory, but too much for sophomores and juniors.
- I believe it's better to give them a practical introduction to Fourier series and convolution, and then derive the the discrete wavelet transform by using a lowpass/highpass filter pair and downsampling.

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Building the Haar Matrix
Examples

Let's have a look at the Mathematica notebook

`HaarTransforms1D.nb`

for a bit more on Haar Transforms.

Coding the Haar Transform

Implementing W_N

- The natural inclination when coding the DHWT is to simply write a loop and compute the lowpass portion and the highpass portion in the same loop.
- This bogs down in Mathematica and is also difficult to generalize when we consider longer filters.
- If we look at the lowpass portion of the transform, $H\mathbf{v}$, we can see a better way to code things.

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Coding the Haar Transform

Implementing W_N

Consider $H\mathbf{v}$ when $v \in \mathbb{R}^8$. We have

$$H\mathbf{v} = \frac{\sqrt{2}}{2} \begin{bmatrix} v_1 + v_2 \\ v_3 + v_4 \\ v_5 + v_6 \\ v_7 + v_8 \end{bmatrix}$$

Coding the Haar Transform

Implementing W_N

If we rewrite this, we have

$$H\mathbf{v} = \frac{\sqrt{2}}{2}\begin{bmatrix} v_1 + v_2 \\ v_3 + v_4 \\ v_5 + v_6 \\ v_7 + v_8 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \\ v_5 & v_6 \\ v_7 & v_8 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = V\mathbf{h}$$

Coding the Haar Transform

Implementing W_N

In a similar way we see that

$$G\mathbf{v} = V\mathbf{g}$$

So all we need to do to compute $W_n\mathbf{v}$ is to create V, multiply it with $\mathbf{h}$ and $\mathbf{g}$, and join to the two blocks together!

Coding the Haar Transform

Implementing W_N

Here is some Mathematica code to do it:

```
DHWT[v_]:=Module[{V,lp,hp,y},
     V=Partition[v,2,2];
     lp=V.{1,1};
     hp=V.{1,-1};
     y=Join[lp,hp];
     Return[Sqrt[2]*y/2];
];
```

Coding the Haar Transform

Implementing W_N^T

- Writing the code for the inverse transform is a bit trickier.
- Now the computation is

$$W_N^T \mathbf{v} = \left[H^T \middle| G^T \right] \mathbf{v}$$

- Let's again look at a vector $\mathbf{v} \in \mathbb{R}^8$ and consider the product $W_8^T \mathbf{v}$:

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Implementing W_N^T

- Writing the code for the inverse transform is a bit trickier.
- Now the computation is

$$W_N^T \mathbf{v} = \left[H^T \middle| \, G^T \right] \mathbf{v}$$

- Let's again look at a vector $\mathbf{v} \in \mathbb{R}^8$ and consider the product $W_8^T \mathbf{v}$:

Coding the Haar Transform

Implementing W_N^T

$$W_8^T \mathbf{v} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix}$$

Coding the Haar Transform

Implementing W_N^T

or

$$W_8^T \mathbf{v} = \frac{\sqrt{2}}{2} \begin{bmatrix} v_1 - v_5 \\ v_1 + v_5 \\ v_2 - v_6 \\ v_2 + v_6 \\ v_3 - v_7 \\ v_3 + v_7 \\ v_4 - v_8 \\ v_4 + v_8 \end{bmatrix}$$

Coding the Haar Transform

Implementing W_N^T

- The matrix V takes a bit different shape this time.
- Now V is

$$V = \begin{bmatrix} v_1 & v_5 \\ v_2 & v_6 \\ v_3 & v_7 \\ v_4 & v_8 \end{bmatrix}$$

- We need to dot V with both **h** and **g** but then intertwine the results.

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Coding the Haar Transform

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Coding the Haar Transform

Implementing W_N^T

Let's return to the Mathematica notebook

`HaarTransforms1D.nb`

to see how to code the inverse.

2D Haar Transform

Building the 2D Transform

- Let's now assume A is an $N \times N$ image with N even.
- How do we transform A?
- If we compute $W_N A$, we are simply applying the DHWT to each column of A:

2D Haar Transform

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2D Haar Transform

Building the 2D Transform

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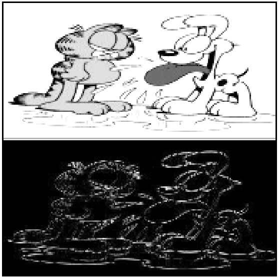

$W_N A$

2D Haar Transform

Building the 2D Transform

- We've processed the columns of A - what should we do to process the rows of A as well?
- Answer: Compute $W_N A W_N^T$.

2D Haar Transform

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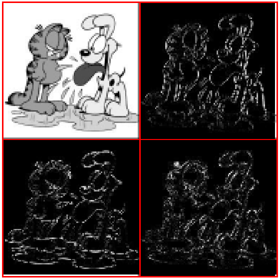

$W_N A W_N^T$

2D Haar Transform

Building the 2D Transform

- If we look at $W_N A W_N^T$ in block format, we can get a better idea what's going on.
- $$\begin{aligned} W_N A W_N^T &= \left[\begin{array}{c} H \\ \hline G \end{array}\right] A \left[\begin{array}{c} H \\ \hline G \end{array}\right]^T = \left[\begin{array}{c} HA \\ \hline GA \end{array}\right] \left[\begin{array}{c|c} H^T & G^T \end{array}\right] \\ &= \left[\begin{array}{c|c} HAH^T & HAG^T \\ \hline GAH^T & GAG^T \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathcal{B} & \mathcal{V} \\ \hline \mathcal{H} & \mathcal{D} \end{array}\right] \end{aligned}$$

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2D Haar Transform

Building the 2D Transform

- HAH^T averages along the columns of A and then along the rows of HA. This will produce an approximation (or blur) $\mathcal{B}$ of A.
- HAG^T averages along the columns of A and then differences along the rows of HA. This will produce vertical differences $\mathcal{V}$ between $\mathcal{B}$ and A.

2D Haar Transform

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2D Haar Transform

Building the 2D Transform

- GAH^T differences along the columns of A and then averages along the rows of GA. This will produce horizontal differences $\mathcal{H}$ between $\mathcal{B}$ and A.
- GAG^T differences along the columns of A and then differences along the rows of GA. This will produce diagonal differences $\mathcal{V}$ between $\mathcal{B}$ and A.

2D Haar Transform

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2D Haar Transform

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2D Haar Transform

Building the 2D Transform

To better understand these block forms, let's look at the Mathematica notebook

`HaarTransforms2D.nb`

2D Haar Transform

Coding the 2D Transform

- Coding the 2D Haar transform is easy - we already have a routine that will apply the DHWT to each column of A,
- so we can easily write a routine to compute $C = W_N A$. Let's call this routine `W`.
- Our goal is to compute $B = W_N A W_N^T = C W_N^T$.
- It turns out that writing code for $C W_N^T$ is a bit tedious, but if we use some linear algebra . . .

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2D Haar Transform

Coding the 2D Transform

- If we transpose both sides of $B = CW_N^T$, we have

$$B^T = W_N C^T$$

- So we can simply apply `W` to C^T and transpose the result.
- One student wasn't so sure about this . . .
- Let's return to `HaarTransforms2D.nb` to write some code for the 2D Haar Wavelet Transform.

2D Haar Transform

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2D Haar Transform

Iterating

- It's time to explain the `NumIterations` directive you have seen in the Mathematica notebooks.
- We can motivate the idea by looking at the cumulative energy of an image A and its wavelet transform.

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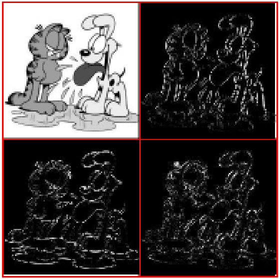
2D Haar Transform

Iterating

Here is a 200×200 image and it's transform:

A

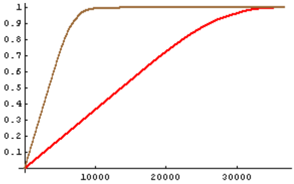


$W_N A W_N^T$

2D Haar Transform

Iterating

Here are the cumulative energies for both A (red) and its transform (brown):



2D Haar Transform

Iterating

- To give you an idea, the largest 10000 elements in A make up about 36.5% of the energy in A while the first 10000 elements in the transform comprise about 99.5% of the energy in the transform.
- The wavelet transform is totally invertible, so if we were to Huffman encode the transform, the bit stream should be markedly smaller.

2D Haar Transform

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2D Haar Transform

Iterating

- We can get even more concentration of the energy if we iterate the wavelet transform. That is, after computing the wavelet transform of A, we extract the blur and compute a wavelet transform of it.
- We could repeat this process p times if the dimensions of A are divisible by 2^p.

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2D Haar Transform

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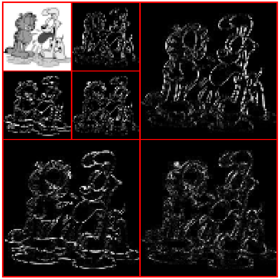
2D Haar Transform

Iterating

Now suppose we iterate 2 times:



2D Haar Transform

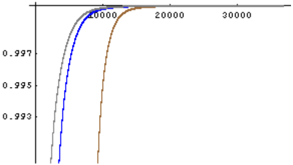
Iterating

or 3 times:

2D Haar Transform

Iterating

Here are the cumulative energy vectors for 1 iteration (brown), 2 iterations (blue), and 3 iterations (gray):



In the Classroom

Teaching Ideas

- The students really enjoy the material in this chapter. It is quite straightforward and ties together everything new we've done to date.
- I have them look at the entropy of particular vectors when processed by the Haar transform. This gives them some idea of the potential for wavelet-based compression.

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- As you might imagine, we do lots of coding in this chapter.
- I let the students work in pairs and they write code for the Haar transform and its inverse (1D and 2D) as well as iterated versions.
- They can get pretty frustrated with Mathematica at this point - it is good to show them some simple debugging techniques.

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- I let them use their own images/audio files (sometimes dangerous).
- To test their iterated 1D inverse, they must download an audio clip from my website that has been transformed p times, guess at p, and then apply their inverse to guess the movie clip.

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Today's Schedule

9:00-10:15	**Lecture One:** Why Wavelets?
10:15-10:30	Coffee Break (OSS 235)
10:30-11:45	**Lecture Two:** Digital Images, Measures, and Huffman Codes
12:00-1:00	Lunch (Cafeteria)
1:30-2:45	**Lecture Three:** Fourier Series, Convolution and Filters
2:45-3:00	Coffee Break (OSS 235)
3:00-4:15	**Lecture Four:** 1D and 2D Haar Transforms
5:30-6:30	$\Rightarrow$Dinner (Cafeteria)

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