Practice

Time Complexity

$f(n)$	$g(n)$	$O/\Omega/\Theta$
$n-100$	$n-200$	Θ
$n^{1/2}$	$n^{2/3}$	O
$100n+\log n$	$n+(\log n)^2$	$\Theta\ (a)$
$\log 2n$	$\log 3n$	$\Theta\ (b)$
$10\log n$	$\log n^2$	$\Theta\ (c)$
$n^{1/2}$	$5^{log_2 n}$	$O\ (d)$
2^n	2^{n+1}	$\Theta\ (e)$

(a): $n\ dominates\ (\log n)^c \rightarrow n+(\log n)^2 = \Theta(n)$

(b): $\log ab = \log a + \log b$

(c): $\log a^b = b\log a$

(d): $5 = 2^{2x}\ where\ x > 0 \rightarrow 5^{log_2 n} = (2^{2x})^{log_2 n} = (2^{log_2 n})^{2x} = \Omega(n^{1/2})$

(e): $2^{n+1} = 2 \times 2^n$

Fibonacci 1

$$F_n = \begin{cases} 0, if\ n = 0 \\ 1, if\ n = 1 \\ F_{n-1} + F_{n-2}, otherwise \end{cases}$$

Prove: $F_n = \Omega(\sqrt{2^n})$.

By trial and error: It appears that $F(n) \geq 2^{n/2}$ for all $n \geq 7$

To prove: For all positive integers $n \geq 7 \rightarrow F_n \geq 2^{n/2}$

By induction on n. Base case: $n = 7$

Step, assume: Indeed, true that for all $i = 7, 8, \ldots, k \rightarrow F_i \geq 2^{i/2}$

To prove: $F_{k+1} \geq 2^{k+1/2}$

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LHS: $F_{k+1} = F_k + F_{k-1} \geq 2^{k/2} + 2^{(k-1)/2}$

Suffices to prove: $2^{k/2} + 2^{(k-1)/2} \geq 2^{(k+1)/2}$

$2^{1/2} + 1 \geq 2^{2/2}$ (by dividing the above by $2^{(k-1)/2}$)

It is indeed true that $2^{1/2} + 1 \geq 2^{2/2} = 1$

Multiplication

Figure 1.1 Multiplication à la Français.

```
function multiply(x, y)
Input: Two n-bit integers x and y, where y ≥ 0
Output: Their product

if y = 0: return 0
z = multiply(x, ⌊y/2⌋)
if y is even:
    return 2z
else:
    return x + 2z
```

Suppose instead of both x and y being n-bit, x is n-bit and y is m-bit. What is the worst-case time efficiency of *multiply*?

Proposed: $O(nm)$

Time Efficiency:

- $\#\ recursive\ calls\ x\ time/call$
- $\#\ worst\ case\ recursive\ calls = O(m)$
- $Worst\ case\ time/call =$
    - $2z\ is\ at\ worst\ O(n+m) \rightarrow because\ very\ last\ addition\ is\ 2z = xy - x$
    - $x\ is\ n\ bits$
    - $So, addition's\ time: O(max\ \{n, n+m\}) = O(max\ \{n, m\})$

So, final answer: $O(m \times \max\{n, m\})$

Fibonacci 2

Practice

Let F_n be the n[th] Fibonacci number, Prove $F_n = O(2^n)$.

- Somewhere, we have shown: $F_n = \Omega(\sqrt{2}^n)$
- But here, seek to show: There exists positive real $F_n \leq c \cdot 2^n$, for all n in N
- Natural proof strategy for "there exists" – construction (i.e., propose some concrete c, and show that it works)
- Try some small values for n, and see what c would work
    - $n = 0, F_0 = 0, 2^0 = 1 \rightarrow c = 1\ works$
    - $n = 1, F_1 = 1, 2^1 = 2 \rightarrow c = 1\ works$
    - $n = 2, F_2 = 1, 2^2 = 4 \rightarrow c = 1\ works$
    - $n = 3, F_3 = 2, 2^3 = 8 \rightarrow c = 1\ works$
    - $n = 4, F_4 = 3, 2^4 = 16 \rightarrow c = 1\ works$
- Appears that $c = 1$works. Adopt it and check if proof goes through. Now, proof by induction with $c = 1$
- Base case, $n = 1, F_1 = 1, 2^1, 1 \leq 2 \rightarrow True$
- Step: Seek to show $F_n \leq 2^n$ given that $F_k \leq 2^k$ for all $k = 1, 2, \dots, n-1$
- $F_n = F_{n-1} + F_{n-2} \leq 2^{n-1} + 2^{n-2}$ by induction assumption
- $F_n = 2^{n-2}\ (2+1) = 3 \times 2^{n-2} \leq 2^n = 2^2 \times 2^{n-2} = 4 \times 2^{n-2} \rightarrow Done$

Fibonacci 3

Let F_n be the n[th] Fibonacci number, Prove $F_n \neq O(n^2)$.

- Recall from logic: not (there exists an egg-laying mammal) = for all mammals m, m is not egg-laying
- Here, $f = O(g)$: There exists positive real c, for all natural n, $f(n) \leq c \cdot g(n)$
- So here, need to prove: Given any positive real c, it is true that there exists n such that $F_n > c \cdot n^2$
- By contradiction: Suppose that there exists positive real c, such that, for all natural n, $F_n \leq c \cdot n^2$
- Then: $F_n = F_{n-1} + F_{n-2} \leq c(n-1)^2 + c(n-2)^2 = c(n^2 - 2n + 1 + n^2 - 4n + 4) = c(2n^2 - 6n + 5) \leq cn^2$
- $2n^2 - 6n + 5 \leq n^2$

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Practice

- $2 - \frac{1}{n^2}(6n - 5) \leq 1$
- This is true only if $\frac{1}{n^2}(6n - 5)$ is "large" compared to $2n^2$
- What is large? We need $\frac{1}{n^2}(6n - 5) \geq 1 \rightarrow true\ for\ n = 1$
- Try $n = 2$: $\frac{1}{4}(12 - 5) = \frac{7}{4} \geq 1$
- Try $n = 3$: $\frac{1}{8}(18 - 5) = \frac{13}{8} \geq 1$
- Try $n = 4$: $\frac{1}{16}(24 - 5) = \frac{19}{16} \geq 1$
- Try $n = 5$: $\frac{1}{25}(30 - 5) = 1$
- Try $n = 6$: $\frac{1}{36}(36 - 5) < 1$
- Try $n = 7$: $\frac{1}{49}(42 - 5) < 1$
- Prove by induction: $6n - 5 < n^2$ for all natural $n > 5$
- Base case $n = 6$: See above
- Step: $6(n - 1) - 5 \leq (n - 1)^2 \rightarrow from\ induction\ assumption$
- $6n - 5 - 6 < n^2 - 2n + 1$
- $6n - 5 \leq n^2 - (2n - 7) \leq n^2\ whenever\ 2n - 7 \geq 0 \rightarrow which\ it\ is\ for\ n \geq 6$
- So far: We have shown that indeed, for $n \geq 6$, $F_n < cn^2 \rightarrow Done$

Selection Sort

$SELECTIONSORT(A[1, \dots, n])$
$foreach\ i\ from\ 1\ to\ n\ do$
$m \leftarrow i - 1 + INDEXOFMIN(A[i, \dots, n])$
$if\ i \neq m\ then\ swap\ A[i], A[m]$

$INDEXOFMIN(B[1, \dots, m])$
$min \leftarrow B[1], idx \leftarrow 1$
$foreach\ j\ from\ 2\ to\ m\ do$
$if\ B[j] <$ min $then$
min $\leftarrow B[j], idx \leftarrow j$
$return\ idx$

What is a meaningful characterization of the time efficiency of $SELECTIONSORT$?

Practice

- Suppose we invoke $INDEXOFMIN(A[5, \dots, 13])$. In $INDEXOFMIN$: $B[1, \dots, 9]$. Suppose now, min is at index 3 in $B[1, \dots, 9]$. This → index of a min in $A[5, \dots, 13]$ is at index $(5 - 1) + 3 = 7$
- Suppose on input: $A[1, \dots, 5] = [13, -23, 45, -23, 1]$. Then A evolves in $SELECTIONSORT$ as follows:
  - $i = 1, m = 2, [-23, 13, 45, -23, 1]$
  - $i = 2, m = 4, [-23, -23, 45, 13, 1]$
  - $i = 3, m = 4, [-23, -23, 13, 45, 1]$
- For time efficiency: Need to make meaningful assumption(s)
- Customary Assumptions: (1) n is unbounded, (ii) each $A[i]$ is bounded
- What should we count? Suppose we all agree that counting # swaps is a meaningful measure for time efficiency
- Then: $Worst\ case\ \#\ swaps = n - 1 = \Theta(n)$
- Now, let's say we want to get a bit more fine-grained. Incorporate (worst case) time for each swap x # swaps
- So now, time efficiency: $(n - 1) + (n - 2) + \cdots + 1 = \Theta(n^2)$

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