Practice

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TIME EFFICIENCY

$\boldsymbol{f(n)}$	$\boldsymbol{g(n)}$	$\boldsymbol{O/\Omega/\Theta}$
$n - 100$	$n - 200$	Θ
$n^{1/2}$	$n^{2/3}$	O
$100n + \log n$	$n + (\log n)^2$	$\Theta\ (a)$
$\log 2n$	$\log 3n$	$\Theta\ (b)$
$10 \log n$	$\log n^2$	$\Theta\ (c)$
$n^{1/2}$	$5^{log_2 n}$	$O\ (d)$
2^n	2^{n+1}	$\Theta\ (e)$

(a): $n\ dominates\ (\log n)^c \rightarrow n + (\log n)^2 = \Theta(n)$

(b): $\log ab = \log a + \log b$

(c): $\log a^b = b \log a$

(d): $5 = 2^{2x}\ where\ x > 0 \rightarrow 5^{log_2 n} = (2^{2x})^{log_2 n} = (2^{log_2 n})^{2x} = \Omega(n^{1/2})$

(e): $2^{n+1} = 2 \times 2^n$

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TIME EFFICIENCY: FIBONACCI 1

$$F_n = \begin{cases} 0, if\ n = 0 \\ 1, if\ n = 1 \\ F_{n-1} + F_{n-2}, otherwise \end{cases}$$

Prove: $F_n = \Omega(\sqrt{2}^n)$.

By trial and error: It appears that $F(n) \geq 2^{n/2}$ for all $n \geq 7$

To prove: For all positive integers $n \geq 7 \rightarrow F_n \geq 2^{n/2}$

By induction on n. Base case: $n = 7$

Step, assume: Indeed, true that for all $i = 7, 8, \ldots, k \rightarrow F_i \geq 2^{i/2}$

To prove: $F_{k+1} \geq 2^{k+1/2}$

LHS: $F_{k+1} = F_k + F_{k-1} \geq 2^{k/2} + 2^{(k-1)/2}$

Suffices to prove: $2^{k/2} + 2^{(k-1)/2} \geq 2^{(k+1)/2}$

$2^{1/2} + 1 \geq 2^{2/2}$ (by dividing the above by $2^{(k-1)/2}$)

It is indeed true that $2^{1/2} + 1 \geq 2^{2/2} = 1$

Practice

TIME EFFICIENCY: MULTIPLICATION

Figure 1.1 Multiplication à la Français.

```
function multiply(x, y)
Input: Two n−bit integers x and y, where y ≥ 0
Output: Their product

if y = 0: return 0
z = multiply(x, ⌊y/2⌋)
if y is even:
   return 2z
else:
   return x + 2z
```

Suppose instead of both x and y being n-bit, x is n-bit and y is m-bit. What is the worst-case time efficiency of $multiply$?

Proposed: $O(nm)$

Time Efficiency:

- $\# \, recursive\ calls\ x\ time/call$
- $\# \, worst\ case\ recursive\ calls = O(m)$
- $Worst\ case\ time/call =$
    - $2z\ is\ at\ worst\ O(n+m) \rightarrow because\ very\ last\ addition\ is\ 2z = xy - x$
    - $x\ is\ n\ bits$
    - $So, addition's\ time: O(\max\{n, n+m\}) = O(\max\{n, m\})$

So, final answer: $O(m \times \max\{n, m\})$

Practice

TIME EFFICIENCY: FIBONACCI 2

Let F_n be the nth Fibonacci number, Prove $F_n = O(2^n)$.

- Somewhere, we have shown: $F_n = \Omega(\sqrt{2}^n)$
- But here, seek to show: There exists positive real $F_n \leq c \cdot 2^n$, for all n in N

- Natural proof strategy for "there exists" – construction (i.e., propose some concrete c, and show that it works)

- Try some small values for n, and see what c would work
  - $n = 0, F_0 = 0, 2^0 = 1 \rightarrow c = 1\ works$
  - $n = 1, F_1 = 1, 2^1 = 2 \rightarrow c = 1\ works$
  - $n = 2, F_2 = 1, 2^2 = 4 \rightarrow c = 1\ works$
  - $n = 3, F_3 = 2, 2^3 = 8 \rightarrow c = 1\ works$
  - $n = 4, F_4 = 3, 2^4 = 16 \rightarrow c = 1\ works$

- Appears that $c = 1$works. Adopt it and check if proof goes through. Now, proof by induction with $c = 1$

- Base case, $n = 1, F_1 = 1, 2^1, 1 \leq 2 \rightarrow True$

- Step: Seek to show $F_n \leq 2^n$ given that $F_k \leq 2^k$ for all $k = 1, 2, \ldots, n-1$
- $F_n = F_{n-1} + F_{n-2} \leq 2^{n-1} + 2^{n-2}$ by induction assumption
- $F_n = 2^{n-2}\ (2 + 1) = 3 \times 2^{n-2} \leq 2^n = 2^2 \times 2^{n-2} = 4 \times 2^{n-2} \rightarrow Done$

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TIME EFFICIENCY: FIBONACCI 3

Let F_n be the n[th] Fibonacci number, Prove $F_n \neq O(n^2)$.

- Recall from logic: not (there exists an egg-laying mammal) = for all mammals m, m is not egg-laying
- Here, $f = O(g)$: There exists positive real c, for all natural n, $f(n) \leq c \cdot g(n)$
- So here, need to prove: Given any positive real c, it is true that there exists n such that $F_n > c \cdot n^2$
- By contradiction: Suppose that there exists positive real c, such that, for all natural n, $F_n \leq c \cdot n^2$
- Then: $F_n = F_{n-1} + F_{n-2} \leq c(n-1)^2 + c(n-2)^2 = c(n^2 - 2n + 1 + n^2 - 4n + 4) = c(2n^2 - 6n + 5) \leq cn^2$
- $2n^2 - 6n + 5 \leq n^2$
- $2 - \frac{1}{n^2}(6n - 5) \leq 1$
- This is true only if $\frac{1}{n^2}(6n - 5)$ is "large" compared to $2n^2$
- What is large? We need $\frac{1}{n^2}(6n - 5) \geq 1 \rightarrow true\ for\ n = 1$
- Try $n = 2$: $\frac{1}{4}(12 - 5) = \frac{7}{4} \geq 1$
- Try $n = 3$: $\frac{1}{8}(18 - 5) = \frac{13}{8} \geq 1$
- Try $n = 4$: $\frac{1}{16}(24 - 5) = \frac{19}{16} \geq 1$
- Try $n = 5$: $\frac{1}{25}(30 - 5) = 1$
- Try $n = 6$: $\frac{1}{36}(36 - 5) < 1$
- Try $n = 7$: $\frac{1}{49}(42 - 5) < 1$
- Prove by induction: $6n - 5 < n^2$ for all natural $n > 5$
- Base case $n = 6$: See above
- Step: $6(n-1) - 5 \leq (n-1)^2 \rightarrow from\ induction\ assumption$
- $6n - 5 - 6 < n^2 - 2n + 1$
- $6n - 5 \leq n^2 - (2n - 7) \leq n^2\ whenever\ 2n - 7 \geq 0 \rightarrow which\ it\ is\ for\ n \geq 6$
- So far: We have shown that indeed, for $n \geq 6$, $F_n < cn^2 \rightarrow Done$

Practice

TIME EFFICIENCY: SELECTION SORT

$SELECTIONSORT(A[1, \dots, n])$

$foreach\ i\ from\ 1\ to\ n\ do$

$m \leftarrow i - 1 + INDEXOFMIN(A[i, \dots, n])$

$if\ \ i \neq m\ then\ swap\ A[i], A[m]$

$INDEXOFMIN(B[1, \dots, m])$

$min \leftarrow B[1], idx \leftarrow 1$

$foreach\ j\ from\ 2\ to\ m\ do$

$if\ \ B[j] < \min then$

$\min \leftarrow B[j], idx \leftarrow j$

$return\ idx$

What is a meaningful characterization of the time efficiency of $SELECTIONSORT$?

- Suppose we invoke $INDEXOFMIN(A[5, \dots, 13])$. In $INDEXOFMIN$: $B[1, \dots, 9]$. Suppose now, min is at index 3 in $B[1, \dots, 9]$. This → index of a min in $A[5, \dots, 13]$ is at index $(5 - 1) + 3 = 7$
- Suppose on input: $A[1, \dots, 5] = [13, -23, 45, -23, 1]$. Then A evolves in $SELECTIONSORT$ as follows:
  - $i = 1, m = 2, [-23, 13, 45, -23, 1]$
  - $i = 2, m = 4, [-23, -23, 45, 13, 1]$
  - $i = 3, m = 4, [-23, -23, 13, 45, 1]$
- For time efficiency: Need to make meaningful assumption(s)
- Customary Assumptions: (1) n is unbounded, (ii) each $A[i]$ is bounded
- What should we count? Suppose we all agree that counting # swaps is a meaningful measure for time efficiency
- Then: $Worst\ case\ \#\ swaps = n - 1 = \Theta(n)$
- Now, let's say we want to get a bit more fine-grained. Incorporate (worst case) time for each swap x # swaps
- So now, time efficiency: $(n - 1) + (n - 2) + \dots + 1 = \Theta(n^2)$

Practice

MODULAR SIMPLIFICATION

1. Is $6^6 \equiv 5^3\ (mod\ 31)$?

   $6 \times 6 = 36 \equiv 5\ (mod\ 31)$

   So: $(6^2)^3 \equiv (5)^3\ (mod\ 31)$

2. $2^{125} \equiv ?\ (mod\ 127)$

   $2^7 = 128 = 127 + 1$

   So: $128\ mod\ 127 = 1$

   Now: $125/7\ = 17 + 6/7$

   So: $2^{125} = 2^{17\times7+6} = 2^{17\times7} \times 2^6$

   So: $2^{125} \equiv 2^{17\times7} \times 2^6 \equiv (2^7)^{17} \times 2^6 \equiv 1^{17} \times 2^6 \equiv 64\ (mod\ 127)$

3. Is $4^{1536} - 9^{4824}$ divisible by 35?

   $4^{1536} \equiv 9^{4824}\ (mod\ 35)$

   Trick: Keep exponentiating until numbers start to repeat.

   Suppose we repeatedly exponentiate 4:

   4

   $\rightarrow 16$

   $\rightarrow 64 \equiv 29\ (mod\ 35)$

   $\rightarrow 116 = 35 \times 3 + 11 = 11\ (mod\ 35)$

   $\rightarrow 9\ (mod\ 35)$

   $\rightarrow 36 \equiv 1\ (mod\ 35)$

   So: $4^6 \equiv 1\ (mod\ 35)$. And $1536 = 6 \times 256$. So $4^{1536} \equiv 1\ (mod\ 35)$

   Now check whether 1536 is divisible by 4. Indeed: $1536 = 4 \times 384$

   Repeat with 9. Repeated exponentiation of 9:

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$\rightarrow 81 \equiv 11 \ (mod\ 35)$

$\rightarrow 99 \equiv 29 \ (mod\ 35)$

$\rightarrow 261 = 7 \times 35 + 16 \equiv 16 \ (mod\ 35)$

$\rightarrow 144 \equiv 4 \times 35 + 4 \equiv 4 \ (mod\ 35)$

$\rightarrow 36 \equiv 1 \ (mod\ 35)$

So: $9^6 \equiv 1 \ (mod\ 35)$

Now: $9^{4824} = 9^{804\times 6} \equiv 1 \ (mod\ 35)$.

$\therefore It\ is\ divisible\ by\ 35.$

4. $2^{2^{2006}} \ (mod\ 3) = ?$

   $2^{2^{2006}} = (2^2)^{2^{2005}} = 4^{2^{2005}} \equiv 1 \ (mod\ 3)$

5. Is $5^{30000} - 6^{123456}$ a multiple of 31?

   31 is prime. And $5^{30000} = (5^{30})^{1000} \equiv 1 \ (mod\ 31)$.

   Compare with $6^{123456} = 6^{123450} \times 6^6$:

   $1 \times 6^6 \equiv 5^3 \equiv 125 \equiv 31 \times 4 + 1 \ (mod\ 31) \equiv 1 \ (mod\ 31)$

   $\therefore It\ is\ a\ multiple\ of\ 31.$

Practice

PROOF: MULTIPLICATIVE INVERSE

Show that if a has a multiplicative inverse modulo N, then this inverse is unique (modulo N).

Let's assume $a \epsilon \{1, \dots, N-1\}$.

Suppose $b, c \epsilon \{1, \dots, N-1\}$ are both multiplicative inverses of $a\ modulo\ N$. Then:

$$ab \equiv 1\ (mod\ N)$$
$$ac \equiv 1\ (mod\ N)$$

$$ab \equiv ac\ (mod\ N)$$
$$ab \cdot b \equiv ac \cdot b\ (mod\ N)\ (1)$$

(1): Substitution Rule:

$$x \equiv x', y \equiv y' (mod\ N)$$
$$xy \equiv x'y' (mod\ N)$$

Then:

$$(ab) \cdot b \equiv (ab) \cdot c\ (mod\ N)\ (2)$$

(2): Commutativity

$$1 \cdot b \equiv 1 \cdot c\ (mod\ N)$$
$$b \equiv c\ (mod\ N)$$
$$b = c$$

Suppose $p \equiv 3\ (mod\ 4)$. Show that $(p+1)/4$ is an integer.

$$p \equiv 3\ (mod\ 4)$$
$$p = 4k + 3\ for\ some\ k\ \epsilon\ \mathbb{Z}$$

So: $p + 1 = 4k + 4$, which is divisible by 4.

Practice

We say that x is a square root of y modulo a prime p if $y \equiv x^2 \ (mod\ p)$. Show that if $(i)\ p \equiv 3\ (mod\ 4)$ and $(ii)\ y$ has a square root modulo p, then $y^{(p+1)/4}$ is such a square root.

Let x be the square root of $y\ modulo\ p$. Then: $y \equiv x^2\ (mod\ p)$.

Write $p = 4k + 3$. Then, $\left(y^{\frac{p+1}{4}}\right)^2 = y^{2(p+1)/4} = y^{2(4k+3+1)/4} = y^{2k+2}$

Keep in mind: $(p+1)/4 = k + 1$.

Try plugging in x in the last expression:

Is $y^{2k+2} = x^{4k+4} \equiv x^2$?

So, we're asking: Is $x^{4k+4} - x^2 \equiv 0\ (mod\ p)$?

$$x^{4k+4} - x^2 = (x^{2k+2} - x)(x^{2k+2} + x)$$

So at least one of: $x^{2k+2} - x$ or $x^{2k+2} + x$ must be $\equiv 0\ (mod\ p)$.

- $\frac{(p+1)}{4} = \frac{(4k+3+1)}{4} = k + 1$
- $2 \cdot \frac{(p+1)}{4} = 2k + 2$
- $p - 1 = 4k + 2$

We know: There exists $x\ \epsilon\ \{1, \dots, p-1\}$ such that $y \equiv x^2\ (mod\ p)$.

We seek to prove: $\left(y^{\frac{(p+1)}{4}}\right)^2 \equiv y\ (mod\ p)$. Sufficient condition for that to be true:

$\left(y^{\frac{(p+1)}{4}}\right)^2 \cdot y^{-1} \equiv 1\ (mod\ p) \rightarrow$ is okay, because y is invertible modulo p

$$\Rightarrow (y^{2k+2}) \cdot y^{-1} \equiv 1\ (mod\ p)$$
$$\Rightarrow y^{2k+1} \equiv 1\ (mod\ p)$$
$$\Rightarrow (x^2)^{2k+1} \equiv 1\ (mod\ p)$$
$$\Rightarrow x^{4k+2} \equiv 1\ (mod\ p)$$
$$\Rightarrow x^{p-1} \equiv 1\ (mod\ p)$$
$$\Rightarrow True\ (Fermat's\ little\ theorem)$$

Practice

PROOF: RECURRENCE CORRECTNESS 1

Suppose $x \epsilon \mathbb{Z}^+, y \epsilon \mathbb{Z}_0^+$. Prove recurrence correctness.

$$x^y = \begin{cases} 1, if\ y = 0 \\ (x^2)^{\lfloor y/2 \rfloor}, if\ y\ is\ even \\ x \cdot (x^2)^{\lfloor y/2 \rfloor}, otherwise \end{cases}$$

Case Analysis:

1. If $y = 0$, then $x^y = x^0$. So, the recurrence is correct for the case where $y = 0$

2. If $y \neq 0, y\ is\ even$: then $\lfloor y/2 \rfloor = y/2$. So $x^y = x^{2 \times y/2} = (x^2)^{y/2} = (x^2)^{\lfloor y/2 \rfloor}$

3. If $y \neq 0, y\ is\ odd$: then $\lfloor y/2 \rfloor = (y-1)/2$. So now:

$$x^y = x^{(2 \times (y-1)/2)+1} = x^{(2 \times \lfloor y/2 \rfloor)+1} = x \cdot x^{2 \times \lfloor y/2 \rfloor}$$

Practice

PROOF: RECURRENCE CORRECTNESS 2

Let $\langle q, r\rangle$ be the quotient and remainder of x/y and $\langle q', r'\rangle$ be the quotient and remainder of $(\lfloor x/2 \rfloor)/y$. Prove recurrence correctness.

$$\langle q, r\rangle = \begin{cases} \langle 0, 0\rangle, if\ x = 0 \\ \langle 2q', 2r'\rangle, if\ x\ even\ and\ 2r' < y \\ \langle 2q', 2r' + 1\rangle, if\ x\ odd\ and\ 2r' + 1 < y \\ \langle 2q' + 1, 2r' - y\rangle, if\ x\ even\ and\ 2r' \geq y \\ \langle 2q' + 1, 2r' + 1 - y\rangle, otherwise \end{cases}$$

To be absolutely clear, what are the quotient and remainder of x/y?

We call q the quotient, and r the remainder if and only if q and r are non-negative integers that satisfy:

$$x = q \cdot y + r, where\ r\ \epsilon\ \{0, 1, \dots, y - 1\}$$

Proof by case analysis:

1. If $x = 0$, then $x = 0 = 0 \cdot y + 0$. So, recurrence is correct for this case.
2. If x is even and $2r' < y$: then $\lfloor x/2 \rfloor = x/2$. So:

$$\lfloor x/2 \rfloor = x/2 = q' \cdot y + r'$$

$$x = (2q') \cdot y + 2r'$$

$$q = 2q', r = 2r'$$

Where we infer the last line from the facts that: (i) equation is of the form from definition for quotient and remainder, (ii) $r' \geq 0 \rightarrow 2r' \geq 0$, and (iii) we are given $2r' \leq y - 1$.

3. If x is odd and $2r' + 1 < y$: $\lfloor x/2 \rfloor = (x - 1)/2$

$$\lfloor x/2 \rfloor = (x - 1)/2 = q' \cdot y + r'$$

$$x - 1 = (2q') \cdot y + 2r'$$

$$x = (2q') \cdot y + (2r' + 1)$$

4. x is even, $2r' \geq y$: $\lfloor x/2 \rfloor = x/2$. So:

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$$\lfloor x/2 \rfloor = x/2 = q' \cdot y + r'$$

$$x = (2q') \cdot y + 2r'$$

This is of the form of the definition of quotient and remainder, except that we need to confirm that $2r'$ indeed lies between 0 and $y - 1$. Which it does not necessarily. Actually, we are given that $2r' \geq y$ and therefore not between 0 and $y - 1$. Now we observe:

$$x = (2q') \cdot y + 2r'$$
$$x = (2q' + 1) \cdot y + (2r' - y)$$

Now only question that remains: is it the case that $2r' - y \in \{0, 1, \dots, y - 1\}$?

- Is $2r' - y \geq 0$? Yes, because $2r' \geq y$
- Is $2r' - y \leq y - 1$? Yes, because:

$$r' \leq y - 1$$
$$2r' \leq 2y - 2$$
$$2r' - y \leq y - 2 \leq y - 1$$

5. x odd, $2r' + 1 \geq y$:

$$\lfloor x/2 \rfloor = (x - 1)/2 = q' \cdot y + r'$$

$$x = (2q') \cdot y + (2r' + 1)$$
$$x = (2q' + 1) \cdot y + (2r' + 1 - y)$$

Now:

- $2r' + 1 - y \geq 0$ because $2r' + 1 \geq y$.
- $2r' + 1 - y \leq y - 1$ because:

$$r' \leq y - 1$$
$$2r' + 1 \leq 2y - 1$$
$$2r' + 1 - y \leq y - 1$$

Practice

PROOF: RECURRENCE CORRECTNESS 3

Prove that $BinSearch$ is correct.

$BinSearch(A[1, \dots, n], lo, hi, i)$

1. $\boldsymbol{if}\ lo \leq hi\ \boldsymbol{then}$
2. $\quad mid \leftarrow \lfloor (lo + hi)/2 \rfloor$
3. $\quad \boldsymbol{if}\ A[mid] = i\ \boldsymbol{then}\ \boldsymbol{return}\ true$
4. $\quad \boldsymbol{if}\ A[mid] < i\ \boldsymbol{then}\ \boldsymbol{return}\ BinSearch(A, mid + 1, hi, i)$
5. $\quad \boldsymbol{else}\ \boldsymbol{return}\ BinSearch(A, lo, mid - 1, i)$
6. $\boldsymbol{else}\ \boldsymbol{return}\ false$

Above is recursive version of binary search. Iterative version:

$BinSearch(A[1, \dots, n], lo, hi, i)$

1. $\boldsymbol{while}\ lo \leq hi\ \boldsymbol{do}$
2. $\quad mid \leftarrow \lfloor (lo + hi)/2 \rfloor$
3. $\quad \boldsymbol{if}\ A[mid] = i\ \boldsymbol{then}\ \boldsymbol{return}\ true$
4. $\quad \boldsymbol{if}\ A[mid] < i\ \boldsymbol{then}\ lo \leftarrow mid + 1$
5. $\quad \boldsymbol{else}\ hi \leftarrow mid - 1$
6. $\boldsymbol{else}\ \boldsymbol{return}\ false$

Typically, for iterative algorithms, towards correctness, we articulate a *loop invariant*:

Let $lo^{(in)}$ and $hi^{(in)}$ be the values of lo and hi respectively on input. Just before we successfully enter an iteration of the $\boldsymbol{while}$ loop of Line (1), it is true that:

$$i \in A\left[lo^{(in)}, \dots, hi^{(in)}\right] \rightarrow i \in A[lo, \dots, hi]$$

Going back to the recursive version, what is a correctness property?

Practice

Given $A[1, \ldots, n]$ an array that is sorted, non-decreasing, lo, hi are each $\epsilon\ \{1, \ldots, n\}$ on input, $BinSearch(A, lo, hi, i)$ returns:

- $True \rightarrow (lo \leq hi)\ and\ (i\ \epsilon\ A[lo, \ldots, hi])$
- $False \rightarrow either\ (lo > hi)\ or\ (i\ is\ not\ \epsilon\ A[lo, \ldots, hi])$

Proof by case analysis:

Case 1: $lo > hi$ on input: then $\boldsymbol{if}$ condition of Line (1) evaluates to **false**, and we correctly return **false** in Line (6). Then, this is either from (a) Line (6) without making any recursive calls, or (b) as the return value from a recursive call from one of Lines (4) or (5).

For (b), we first observe that $lo \leq hi$ because the only recursive calls are within the $\boldsymbol{if}$ block of Line (1). So, all that remains to be proven is that indeed: $i \notin A[lo, \ldots, hi]$.

We prove that by induction on $hi - lo + 1$. Base case: $hi - lo + 1 = 1$. We claim we return $\boldsymbol{false}$ within the first recursive invocation. That is, we claim: (i) $mid + 1 > hi$ and $lo > mid - 1$, (ii) $mid = lo = hi$, and (iii) $i \neq A[mid]$.

(ii) easy to prove:

$$hi - lo + 1 = 1$$
$$\Rightarrow lo = hi$$
$$\Rightarrow mid = \left\lfloor \frac{(lo + hi)}{2} \right\rfloor = \left\lfloor \frac{(lo + lo)}{2} \right\rfloor = \left\lfloor \frac{(2 \cdot lo)}{2} \right\rfloor = \frac{2 \cdot lo}{2} = lo = hi$$

(iii) is $\boldsymbol{true}$, because then we would have returned $\boldsymbol{true}$ in Line (3).

To prove (i): we simply exploit: $mid = hi = lo$

$$mid = hi \Rightarrow mid + 1 > hi$$
$$mid = lo \Rightarrow mid - 1 < lo$$

So, the algorithm is correct if it returns $\boldsymbol{false}$, and $hi - lo + 1 = 1$.

For the step, we know that on input $lo < hi$. So, we returned $\boldsymbol{false}$ in some recursive call. So, all we have to prove to appeal to induction assumption: $hi - (mid + 1) < hi - lo$ and $(mid - 1) - lo < hi - lo$.

Practice

PROOF: MASTER THEOREM CORRECTNESS

Give a closed form solution for the following recurrence. Assume: $f: \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, $a > 0, b > 1, d \geq 0$.

$$f(n) = \begin{cases} \Theta(1), if n \leq 1 \\ a \cdot f\left(\frac{n}{b}\right) + \Theta(n^d), otherwise \end{cases}$$

Proposed approach: Inductive "rewriting" of the function f. But first: adopt concrete functions wherever we have $\Theta(\cdot)$, $O(\cdot)$ or $\Omega(\cdot)$. In this case: adopt 1 for $\Theta(1)$, and n^d for $\Theta(n^d)$. Now onto the rewriting:

$$f(n) = a \cdot f\left(\frac{n}{b}\right) + n^d$$

$$= a \cdot \left(a \cdot f\left(\frac{n}{b^2}\right) + \left(\frac{n}{b}\right)^d\right) + n^d$$

$$= a^2 \cdot f\left(\frac{n}{b^2}\right) + a \cdot \left(\frac{n}{b}\right)^d + n^d$$

$$= a^2 \left(a \cdot f\left(\frac{n}{b^3}\right) + \left(\frac{n}{b^2}\right)^d\right) + a \cdot \left(\frac{n}{b}\right)^d + n^d$$

$$= a^3 \cdot f\left(\frac{n}{b^3}\right) + a^2 \cdot \left(\frac{n}{b^2}\right)^d + a \cdot \left(\frac{n}{b}\right)^d + n^d$$

$$= a^3 \cdot f\left(\frac{n}{b^3}\right) + n^d \left(\left(\frac{a}{b^d}\right)^2 + \left(\frac{a}{b^d}\right)^1 + \left(\frac{a}{b^d}\right)^0\right)$$

$$= a^4 \cdot f\left(\frac{n}{b^4}\right) + n^d \left(\left(\frac{a}{b^d}\right)^3 + \left(\frac{a}{b^d}\right)^2 + \left(\frac{a}{b^d}\right)^1 + \left(\frac{a}{b^d}\right)^0\right)$$

...

$$= a^{\log_b n} \cdot f(1) + n^d \cdot \left(\left(\frac{a}{b^d}\right)^{(\log_b n)-1} + \left(\frac{a}{b^d}\right)^{(\log_b n)-2} + \cdots + \left(\frac{a}{b^d}\right)^0\right)$$

$$= a^{\log_b n} + n^d \cdot \left(\left(\frac{a}{b^d}\right)^{(\log_b n)-1} + \left(\frac{a}{b^d}\right)^{(\log_b n)-2} + \cdots + \left(\frac{a}{b^d}\right)^0\right)$$

To figure out the power of a in that last term:

> Power of a is the same as the power of b inside the $f\left(\frac{n}{b^x}\right)$. In other words: what is the power of b, i.e., x for which $\frac{n}{b^x} = 1$? Answer: $\frac{n}{b^x} = 1 \Leftrightarrow n = b^x \Leftrightarrow x = \log_b n$.

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Our next step: Simplify/figure out:

$$S = \left(\frac{a}{b^d}\right)^{(\log_b n)-1} + \left(\frac{a}{b^d}\right)^{(\log_b n)-2} + \cdots + \left(\frac{a}{b^d}\right)^0$$

Suppose:

$$T = r^{q-1} + r^{q-2} + \cdots + r^0$$
$$\Longrightarrow r \cdot T = r^q + r^{q-1} + \cdots + r$$

Now subtract one from the other:

$$\Longrightarrow T - r \cdot T = r^0 - r^q$$
$$\Longrightarrow (1 - r) \cdot T = 1 - r^q$$
$$\Longrightarrow T = \frac{1 - r^q}{1 - r}, provided\ r \neq 1$$

When $r = 1$, how do we figure out what T is? Answer: then, T is:

$$T = 1^{q-1} + 1^{q-2} + \cdots + 1^0$$
$$= 1 + 1 + \cdots + 1 \to q\ instances\ of\ 1$$
$$= q$$

So, going back to our S:

$$S = \left(\frac{a}{b^d}\right)^{(\log_b n)-1} + \left(\frac{a}{b^d}\right)^{(\log_b n)-2} + \cdots + \left(\frac{a}{b^d}\right)^0$$
$$\Longrightarrow S = \frac{1 - \left(\frac{a}{b^d}\right)^{\log_b n}}{1 - \left(\frac{a}{b^d}\right)}, provided \frac{a}{b^d} \neq 1$$

And:

$$S = \log_b n, when \frac{a}{b^d} = 1$$

When is $\frac{a}{b^d} = 1$? Answer: $d = \log_b a$.

So, going back to our $f(n)$: first, the case that $d = \log_b a$.

But even before that: rewrite $a^{\log_b n} = n^{\log_b a}$. Because:

$$x = a^{\log_b n} \Longleftrightarrow \log_b x = \log_b a \cdot \log_b n \Longleftrightarrow x = n^{\log_b a}$$

Practice

$$f(n) = n^{\log_b a} + n^d \cdot S$$

So, when $d = \log_b a$, $S = \log_b n$. So, in this case:

$$f(n) = n^d + n^d \cdot \log_b n$$
$$= \Theta(n^d \cdot \log n)$$

Onto the other two cases: $d \neq \log_b a$.

$$f(n) = n^{\log_b a} + \cdots + n^d \cdot S$$

Before we continue: a closer look at $\left(\frac{a}{b^d}\right)^{\log_b n}$:

$$\left(\frac{a}{b^d}\right)^{\log_b n} = \frac{a^{\log_b n}}{(b^d)^{\log_b n}}$$
$$= \frac{n^{\log_b a}}{(b^{\log_b n})^d}$$
$$= \frac{n^{\log_b a}}{n^d}$$

So: when $d \neq \log_b a$

$$S = \frac{1 - \frac{n^{\log_b a}}{n^d}}{1 - \left(\frac{a}{b^d}\right)}$$

So, going back to $f(n)$:

$$f(n) = n^{\log_b a} + n^d \cdot S$$
$$= n^{\log_b a} + \frac{1}{1 - \left(\frac{a}{b^d}\right)} \cdot \left(n^d - n^{\log_b a}\right)$$
$$= c \cdot n^{\log_b a} + c' \cdot n^d, for\ positive\ constants\ c, c'$$

So, if $d > \log_b a$: $f(n) = \Theta(n^d)$

And if $d < \log_b a$: $f(n) = \Theta\left(n^{\log_b a}\right)$

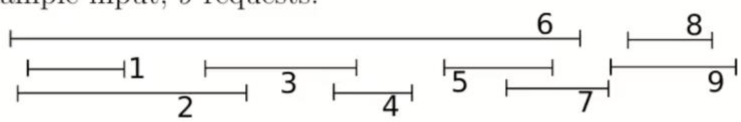
Practice

PROOF: GREEDY CHOICE

Given as input n meeting requests, $\langle s_1, f_1\rangle, \langle s_2, f_2\rangle, \ldots, \langle s_n, f_n\rangle$, where each $s_i, f_i \in \mathbb{Z}^+$ is a start- and finish-time and $s_i < f_i$. We want a subset of those requests that is of maximum size that are pairwise conflict-free.

Two requests $\langle s_i, f_i\rangle$, $\langle s_j, f_j\rangle$ are in conflict if $s_i \leq f_j$, and $s_j \leq f_i$, or vice versa.

Example input, 9 requests:



Request 5 is in conflict with each Request 6 and 7. But is conflict-free with Request 2.

An optimal (maximum-sized) conflict-free set: $\{1, 3, 5, 9\}$. Another: $\{1, 4, 7, 8\}$.

Prove: this problem possesses a greedy choice.

Candidate greedy choice: request with earliest finish time.

Proof strategy: "cut and paste."

For this problem, we prove two claims in order:

Claim 1: *Suppose for some input of n requests, $O = \{o_1, \ldots, o_k\}$ is an optimal (maximum-sized) set of requests which are pairwise conflict-free ordered in increasing finish time. Suppose our greedy algorithm outputs $G = \{g_1, \ldots, g_l\}$, ordered in increasing finish time. Then, it is true that: for every $i = 1, 2, \ldots, l, f(g_i) \leq f(o_i)$.*

Proof. Note: it must be the case that $l \leq k$. And therefore, $k = l$, i.e., greedy is optimal.

Proof by induction on i. Base case: $i = 1$. In our greedy algorithm, we first pick exactly a meeting that finishes earliest amongst all requests. Therefore, immaterial of what o_1 is, $f(g_1) \leq f(o_1)$.

Induction assumption: for $i = j - 1$, it is true that $f(g_i) \leq f(o_i)$.

Practice

Step: to prove that $f(g_j) \leq f(o_j)$. We observe:

- $f(o_{j-1}) \leq s(o_j)$ – because the set O is conflict-free requests, ordered in increasing finish, and therefore, start times.
- $f(g_{j-1}) \leq f(o_{j-1})$ – induction assumption.
- Therefore, $f(g_{j-1}) \leq s(o_j)$. Therefore $f(g_j) \leq f(o_j)$ – because after we greedily choose g_{j-1} and eliminate all requests that are in conflict, o_j still remains. And our greedy choice is exactly to pick a request that remains that finishes earliest, and we happened to pick g_j.

Claim 2: *Given sets O, G as in Claim 1, o_{l+1} cannot exist in O.*

Proof. By Claim 1, $f(g_l) \leq f(o_l)$. And because the O set is all conflict-free, $f(o_l) \leq s(o_{l+1})$. Therefore, $f(g_l) \leq s(o_{l+1})$. So, o_{l+1} not in conflict with g_l, and so was available to be chosen after g_l was chosen and all conflicts were eliminated.

Contradiction to the assumption that greedy algorithm terminates only when no more requests available to choose from.

Practice

GRAPH ALGORITHM 1

Given an undirected graph $G = \langle V, E \rangle$ encoded as an adjacency list, define an array $\mathsf{snd}[\cdot]$ as: for each $u \in V$, $\mathsf{snd}[u]$ is the sum of the degrees of the neighbours of u.

Devise an algorithm that given input G, computes and outputs an array snd.

$SNDStraightForward(G = \langle V, E \rangle)$

$snd \leftarrow new\ array\ of\ |V|\ entries$

$\boldsymbol{foreach}\ u \in V\ \boldsymbol{do}\ snd[u] \leftarrow 0$

$\boldsymbol{foreach}\ u \in V\ \boldsymbol{do}$

$\quad \boldsymbol{foreach}\ v \in Adj[u]\ \boldsymbol{do}$

$\quad\quad degreev \leftarrow 0$

$\quad\quad \boldsymbol{foreach}\ w \in Adj[v]\ \boldsymbol{do}\ degreev \leftarrow degreev + 1$

$\quad\quad snd[u] \leftarrow snd[u] + degreev$

$\boldsymbol{return}\ snd$

Time efficiency of $SNDStraightForward$: $O(|V| \cdot (|E|)^2)$

Perhaps a better (more efficiency) approach:

- Visit each vertex as though it is someone's neighbor.
- Measure its degree.
- Walk its adj list again and inform each neighbor of the degree so they can update their snd.

$SNDLinearTime(G = \langle V, E \rangle)$

$snd \leftarrow new\ array\ of\ |V|\ entries$

$\boldsymbol{foreach}\ u \in V\ \boldsymbol{do}\ snd[u] \leftarrow 0$

$deg \leftarrow new\ array\ of\ |V|\ entries$

Practice

4. $\boldsymbol{foreach}\ u \in V\ \boldsymbol{do}\ \deg[u] \leftarrow 0$

5. $\boldsymbol{foreach}\ u \in V\ \boldsymbol{do}$

5. $\quad \deg[u] \leftarrow 0$

6. $\quad \boldsymbol{foreach}\ v \in Adj[u]\ \boldsymbol{do}\ \deg[u] \leftarrow \deg[u] + 1$

7. $\quad \boldsymbol{foreach}\ v \in Adj[u]\ \boldsymbol{do}\ snd[v] \leftarrow snd[v] + \deg[u]$

8. $\boldsymbol{return}\ snd$

Time efficiency:

- We visit each vertex once – Line (4) $foreach$ loop.
- We visit each edge four times – Line (6) and Line (7), we walk each adj list twice.
- So total time: $O(|V| + |E|)$.

Practice

GRAPH ALGORITHM 2

Given an undirected graph G as an adjacency list and an edge e in it, devise a linear-time algorithm to determine whether there is a cycle in G that contains e.

"Go-to" linear time algorithms for graphs: DFS and BFS.

- DFS, check if back edge results in DFS tree.
- In fact, edit the explore routine as follows:
  - Keep track of parent in DFS tree.
  - Every time we hit a vertex, check if edge to root of DFS tree, and root is not parent in DFS tree.
  - If yes, immediately output **true**.

```
HasCycle(G = ⟨V, E⟩, e = ⟨u, v⟩)
foreach u ∈ V do
    visited(u) ← false
    π(u) ← NIL
return ExploreModified(⟨V, E⟩, u, u)
```

```
ExploreModified(⟨V, E⟩, ⟨u, v⟩, x)
visited(x) ← true
foreach y ∈ Adj[x] do
    if visited(y) = false then
        if (x ≠ u) or (x = u and y = v) then
            π(y) ← x
            ret ← ExploreModified(⟨V, E⟩, ⟨u, v⟩, y)
            if ret = true then
                return true
    else
        if y = r and π(x) ≠ u then
            return true
return false
```

Practice

PROOF: DIRECTED ACYCLIC GRAPH (DAG) ALGORITHM

Show that the following algorithm to linearize a DAG can be realized in linear time.

> *Find a source, output it, and delete it from the graph.*
>
> *Repeat until the graph is empty.*

We assume adjacency list representation of the input DAG.

Suppose we first create a new array, call it ni of size $|V|$, where $ni[u]$ is the number of edges incident in $u \in V$ at the start. Can do this in one pass of entire adj list of the graph.

From ni, we can identify all sources. Suppose we create a list of source vertices, call it $srclist$. Then, we remove a vertex from $srclist$ and proceed…

```
ni ← new array of size |V|
foreach u ∈ V do ni[u] ← 0
foreach u ∈ V do
  foreach v ∈ Adj[u] do
    ni[v] ← ni[v] + 1
srclist ← new empty linked list
foreach u ∈ V do
  if ni[u] = 0 then Insert u at head of srclist
while srclist is not empty do
  u ← remove vertex from head of srclist
  foreach v ∈ Adj[u] do
    ni[v] ← ni[v] − 1
    if ni[v] = 0 then
      Add v to head of srclist
  Output u
```

Practice

PROOF: DEPTH FIRST SEARCH (DFS) ALGORITHM

Prove that DFS on an undirected graph can result in no cross edges.

An edge $\langle u, v \rangle$ is a cross edge if and only if: $pre[v] < post[v] < pre[u] < post[u]$.

Suppose a cross edge, $\langle u, v \rangle$ exists after a run of DFS on an undirected graph G.

At the time $post[v]$ and at all times prior since initialization, $visited[u] = false$.

But that means that in the for loop that immediately precedes $postvisit(v)$, we would have invoked $explore(u)$, thereby setting $visited[u]$ to **true** before the time $post[v]$.

Therefore, we have a contradiction.

Practice

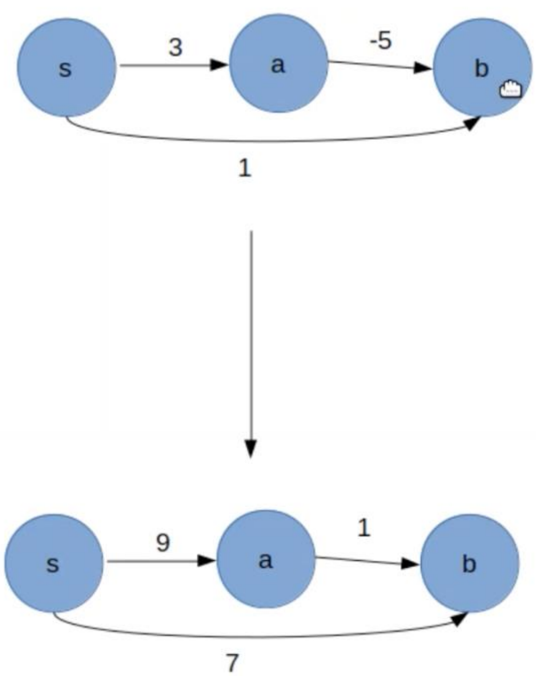
PROOF: SHORTEST PATH ALGORITHM

Professor F. Lake suggests the following algorithm for finding the shortest path from node s to a node t in a directed graph with some negative weight edges: add a large constant to each edge weight so all the weights become positive, then run Dijkstra's algorithm starting at node s, and return the shortest path found to node t.

Is this a valid method? Either prove that it works correctly or give a counterexample.

Directed graph with weights on edges is: $G = \langle V, E, l \rangle$, where $E \subseteq V \times V$, and $l: E \to \mathbb{R}$.

Counterexample, add a constant of 6 to the graph below:



In the unmodified graph, the shortest path is $s \to a \to b$ (-2), but in the modified graph, the shortest path becomes $s \to b$ (7). Since the shortest path changes, this is not a valid method.

Practice

PROOF: DIJKSTRA'S ALGORITHM

Prove: if we initialize $dist(u)$ to ∞, and at the end of a run of Dijkstra's algorithm on $G = \langle V, E, l \rangle$ with source $s \in V$ it is the case that $dist(u) \neq \infty$, then there exists a path $s \rightsquigarrow u$ in G.

Contrapositive: if there exists no path $s \rightsquigarrow u$ in G, then at the end of any run of Dijkstra, $dist(u) = \infty$.

We first observe: the only way $dist(u)$ can change after initialization is via a call $update(e)$ where $e \in E$ is incident on u, i.e., some $\langle v, u \rangle \in E$.

So proof strategy: induction on number of invocations to $update(\cdot)$ that the run of Dijkstra does. Call this number k.

If $k = 0$, then this can only be because $E = \emptyset$. Then, there is no path $s \rightsquigarrow u$. And as we have not changed $dist(u)$ from its initial value, at the end of the run of Dijkstra, $dist(u) = \infty$ as desired.

For the step, we consider two cases.
(i) No edge is incident on u. Then, we know that no $update(\cdot)$ affects $dist(u)$, and therefore $dist(u) = \infty$ as desired.
(ii) There exists some $\langle v, u \rangle \in E$. If the last $update(\cdot)$ we performed is not on any edge incident on u, then $dist(u)$ is the same as it was after $k - 1$ invocations to $update(\cdot)$, and by the induction assumption $dist(u)$ in that case $= \infty$.
The final (sub-)case: the k^{th} update was on some $\langle v, u \rangle$, i.e., edge incident on u. Then there is no path $s \rightsquigarrow v$. Why not? Because if there was, there would be a path to u: $s \rightsquigarrow v \rightarrow u$. And therefore, $dist(v)$ is whatever value it is after $k - 1$ invocations to $update(\cdot)$. And by the induction assumption $dist(v) = \infty$ before $update(v, u)$. Also, again by the induction assumption, $dist(u) = \infty$ before the k^{th} invocation to $update(\cdot)$. Therefore, after the k^{th} invocation, which is $update(v, u)$, $dist(u) = \infty$.

Practice

PROOF: BELLMAN-FORD ALGORITHM

Prove: suppose we run Bellman-Ford on $\langle G = \langle V, E, l \rangle, s \in V \rangle$ where we do not know whether G has a negative weight cycle. Also suppose that at the end of that run of Bellman-Ford, we carry out one more $update(e)$ on every $e \in E$. Then: some $dist(u)$ changes in this additional round of updates for some u that is reachable from s if and only if there is a negative weight cycle in G that is reachable from s.

"Only if": we seek to prove: if $dist(u)$ changes, this implies that there is a negative weight cycle.

By Claim (2) of Lecture 5(b): if there exists a shortest path from s to u that is simple, then $|V| - 1$ invocations to $update(\cdot)$ on all edges, as Bellman-Ford does, is sufficient for $dist(u)$ to converge to $\delta(s, u)$. Given that $|V| - 1$ invocations to $update(\cdot)$ on all edges is not sufficient, this can only be because there is a shortest path $s \rightsquigarrow u$ that is not simple. And this in turn is true only if there is a negative cycle reachable from s.

"If": we seek to prove: if there is a negative weight cycle reachable from s, then there exists some u that is reachable from s for which the additional round of $update(\cdot)$ changes $dist(u)$.

An observation: a change to $dist(u)$ has to be a decrease. Because (repeated) invocation(s) to $update(\cdot)$ can only decrease $dist(\cdot)$ value(s).

Suppose $\langle u_o, u_1, \ldots, u_{k-1}, u_o \rangle$ is a negative weight cycle that is reachable from s, where $u_0 = u_k$.

Proof idea: we know that $\sum_{i=1}^{k} l(u_{i-1}, u_i) < 0$.

Practice

Assume, for the purpose of contradiction, that no $dist(u)$ changed in the additional round of $update(\cdot)$'s, for any $u \in V$. Now consider the vertices $u_0, \dots, u_{k-1}, u_k$ in the negative weight cycle above.

We first observe for all $u_i \in \{u_1, \dots, u_k\}$, it is true that: $dist(u_i) \leq dist(u_{i-1}) + l(u_{i-1}, u_i)$ after the last round of updates.

So now:

$$\sum_{i=1}^{k} dist(u_i) \leq \sum_{i=1}^{k} \big(dist(u_{i-1}) + l(u_{i-1}, u_i)\big)$$

$$\sum_{i=1}^{k} dist(u_i) = \sum_{i=1}^{k} dist(u_{i-1}) + \sum_{i=1}^{k} l(u_{i-1}, u_i)$$

But:

$$\sum_{i=1}^{k} dist(u_i) = \sum_{i=1}^{k} dist(u_{i-1})$$

To see this:

$$dist(u_1) + dist(u_2) + \dots + dist(u_k) = dist(u_0) + dist(u_1) + \dots + dist(u_{k-1})$$

Because $u_0 = u_k$.

So:

$$\sum_{i=1}^{k} l(u_{i-1}, u_i) \geq 0$$

This is the contradiction.

This proof is "constructive" – it is saying that $dist(u)$ must change (decrease) for some vertex u in a negative weight cycle reachable from s in this additional round of calls to $update(\cdot)$.

Practice

MINIMUM SPANNING TREE 1

Give an example of a connected undirected $G = \langle V, E, l \rangle$ such that the set of edges $\{\langle u, v \rangle$: there exists a cut $\langle S, V \setminus S \rangle$ such that $S \subset V$ and $\langle u, v \rangle$ is an edge of smallest weight that crosses $\langle S, V \setminus S \rangle$ does not form an MST.

Let X be that set of edges.

What is the set X for our example graph: complete graph with vertices $V = \{a, b, c\}$.

$l(a, b) = l(b, c) = l(a, c) = 1$.

- Is $\langle a, b \rangle \in X$? Yes. It is a light edge that crosses the cut: $\langle \{a\}, \{b, c\} \rangle$.
- Is $\langle b, c \rangle \in X$? Yes. Consider the cut $\langle \{b\}, \{a, c\} \rangle$.
- Is $\langle a, c \rangle \in X$? Yes. Consider the cut $\langle \{a\}, \{c, b\} \rangle$.

So, we are done. Because $X = \{\langle a, b \rangle, \langle b, c \rangle, \langle a, c \rangle\}$ is not an MST of that G.

Because that is not acyclic, i.e., not a tree.

Practice

MINIMUM SPANNING TREE 2

Professor Sabatier conjectures the following converse of what we say under “now the general approach” on page 3 of Lecture 6a:

> Let $G = \langle V, E, l\rangle$ be a connected undirected graph. Let: (i) $A \subseteq E$ that is included in some MST of G, (ii) $\langle S, V \setminus S\rangle$ be any cut of G that respects A, and (iii) $A \cup \{\langle u, v\rangle\}$ also be included in some MST of G. Then, $\langle u, v\rangle$ is an edge of smallest weight that crosses the cut $\langle S, V \setminus S\rangle$.

Show that the professor’s conjecture is not necessarily true.

Let $V = \{a, b, c\}$, $E = \{\langle a, b\rangle, \langle a, c\rangle\}$, and $l(a, b) = 1$, $l(a, c) = 1000$.

Now let $A = \emptyset$. Then the cut $\langle \{a\}, \{c, b\}\rangle$ respects A.

And $\langle a, c\rangle$ is not a light edge that crosses that cut but is in a (the) MST of the graph.

Practice

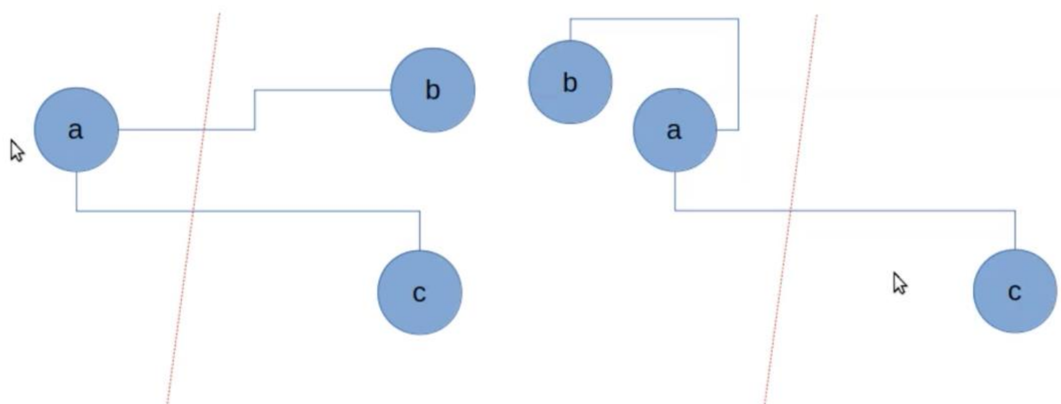
MIN-CUT AND OPTIMAL SUBSTRUCTURE

Consider the min-cut problem: given as input an undirected graph $G = \langle V, E\rangle$, what is the minimum number of edges that cross any cut $\langle S, V \setminus S\rangle$ where $S \subset V$?

Bob claims that the problem has the following optimal substructure. Given a cut $\langle S, V \setminus S\rangle$ that is a min-cut for G, then $\langle S \setminus \{u\}, V \setminus (S \cup \{u\})\rangle$ is a min-cut for G', where we get G' from G by removing $u \in V$ and all edges incident on it, provided both $S \setminus \{u\}$ and $V \setminus (S \cup \{u\})$ are non-empty.

Refute Bob's claim.

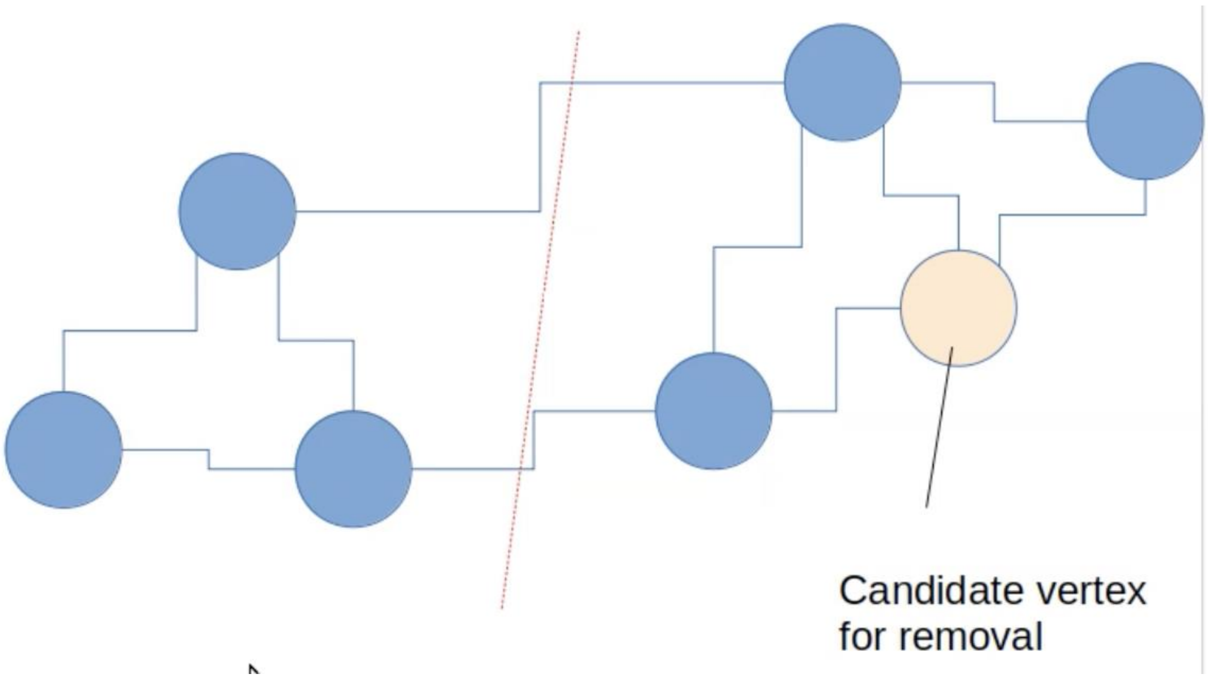
Examples:



Original min-cut $\langle\{a, b\}, \{c\}\rangle$, and indeed, for this min-cut, it turns out:

$\langle\{a\}, \{c\}\rangle$ is indeed a min-cut for G' for this G.

Practice



Counterexample: In this example, the min-cut is 2. However, if the candidate vertex is removed, the graph can be shifted such that the min-cut now becomes 1. Hence, this is a valid counterexample, and Bob's claim is refuted.

Practice

OPTIMAL SUBSTRUCTURE ALGORITHM

The interval-scheduling problem from "Proof: Greedy Choice" possesses optimal substructure. What is it, and how do we exploit it to realize an algorithm?

We are given as input a set of requests $R = \{r_1, \ldots, r_n\}$, where each $r_i = \langle s_i, f_i \rangle$ such that $s_i, f_i \in \mathbb{Z}^+$ and $s_i < f_i$.

For single-source shortest-paths: if $s \rightsquigarrow x \rightsquigarrow y$ is a shortest-path from s to y, then the $s \rightsquigarrow x$ sub-path is a shortest path from s to x.

We could ask: suppose $r_{i_1}, r_{i_2}, \ldots, r_{i_k}$ is an optimal sequence of requests that are non-conflicting such that $f_{i_j} \leq f_{i_{j+1}}$. Is there anything I can say about the optimality of $r_{i_1}, \ldots, r_{i_{k-1}}$? More specifically, is it an optimal solution to a sub-problem?

I think: the answer is yes. A sub-problem for which $r_{i_1}, \ldots, r_{i_{k-1}}$ has to be an optimal solution: suppose in the input, the requests $r_1, \ldots, r_n$ are ordered by non-decreasing finish time. Then: $r_i, \ldots, r_{i_{k-1}}$ has to be an optimal solution to all requests that end at or before s_{i_k}.

In fact: we can "lop off" or "eat into" optimal solution from both directions.

Specifically:

- Assume input set of requests $r_1, \ldots, r_n$ are sorted non-decreasing by finish time. That is: $f_1 \leq f_2 \leq \cdots \leq f_n$.
- Now: suppose $M[i, j]$ is the max # requests I can schedule that start at or after f_i and end at or before s_j.
- Also denote as $R_{i,j}$ the set of requests that start at or after f_i and end at or before s_j.

Then:

$$M[i,j] = \begin{cases} 0, & if\ R_{i,j} = \emptyset \\ 1 + \max\limits_{\substack{i<k<j \\ r_k \in R_{i,j}}} \{M[i,k] + M[k,j]\}, & otherwise \end{cases}$$

Our final solution: $M[0, n+1]$, where we introduce fictitious requests r_0, r_{n+1} with $f_0 < s_1$, $s_{n+1} > f_n$.

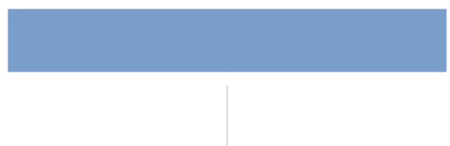
Practice

TIME EFFICIENCY: K-ARY SEARCH

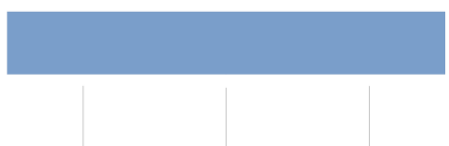
In binary search, we split the input sorted array into two pieces, each of size $n/2$, and recursively search on one of those pieces.

Alice proposes k-ary search, in which we split the array into k pieces, each of size n/k. What is the worst-case time-efficiency of k-ary search as a function of $\langle n, k \rangle$?

Inspired by the eggs-building problem, Alice wonders whether setting $k = \sqrt{n}$ yields a more efficient algorithm than binary search. Does it?



Comparison against middle item



k-ary search with k = 4
In each recursive step, we would
do 3 comparisons in the worst-case.

Recurrence for binary search: $T(n) = T(n/2) + 1$.

For k-ary search, recurrence for worst-case time-efficiency: $T(n) = T(n/k) + (k-1)$. To solve the recurrence:

$$\begin{aligned}
T(n) &= T(n/k) + (k-1) \\
&= T(n/k^2) + 2(k-1) \\
&= T(n/k^3) + 3(k-1) \\
&= \cdots \\
&= T(1) + \log_k n \cdot (k-1) \\
&= \Theta(k \cdot \log_k n)
\end{aligned}$$

Practice

Where we figure the last term as follows: we ask for what x is $n/k^x = 1$? Answer: $n/k^x = 1 \Leftrightarrow n = k^x \Leftrightarrow \log n = x \cdot \log k \Leftrightarrow x = \log n \,/ \log k = \log_k n$

So, if we set $k = \sqrt{n} = n^{1/2}$, then:

$$T(n) = \Theta\left(\sqrt{n} \cdot \log_{\sqrt{n}} n\right) = \Theta\left(2 \cdot \sqrt{n}\right) = \Theta\left(\sqrt{n}\right)$$

And if we do binary search, $T(n) = \Theta(\log n)$. And $\sqrt{n} = \Omega(\log n)$, and $\sqrt{n} \neq O(\log n)$. So, setting $k = \sqrt{n}$ yields a strictly worse performing algorithm than setting $k = 2$.

Practice

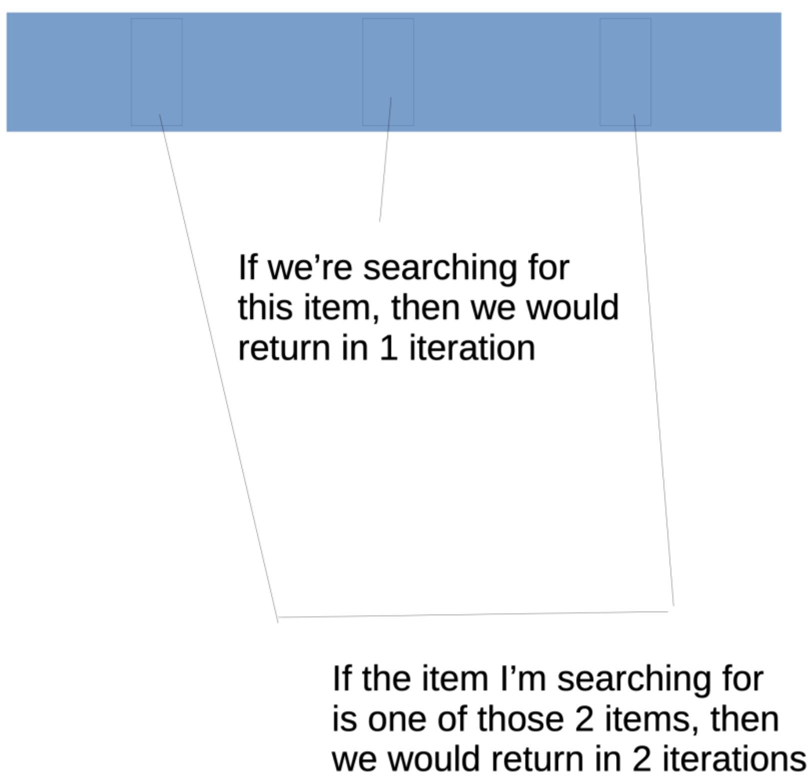
TIME EFFICIENCY: BINARY SEARCH

Carry out an expected- (or average-) case analysis of the time-efficiency of binary search.

First off, we should distinguish a successful (binary) search from an unsuccessful search. Because expected-case time-efficiency of an unsuccessful search is just $\Theta(\log n)$.

For a successful search, the time it takes depends on the item we are looking for. Assume: (i) every item in array is distinct, and (ii) every item in the array is equally likely to be searched for.

Now, if X is a random variable that is the number of comparisons or # iterations or # recursive calls we perform before we find the item we seek is below. As simplification, assume that we have $2^k - 1 = n$ items in the array, for some positive integer k.



Practice

$$E[X] = 1 \times \frac{1}{n} + 2 \times \frac{2}{n} + 3 \times \frac{4}{n} + 4 \times \frac{8}{n} + \cdots$$

$$= 1 \times \frac{2^0}{2^k - 1} + 2 \times \frac{2^1}{2^k - 1} + 3 \times \frac{2^2}{2^k - 1} + \cdots + k \times \frac{2^{k-1}}{2^k - 1}$$

$$= \frac{1}{2^k - 1} \cdot (1 \times 2^0 + 2 \times 2^1 + \cdots + k \times 2^{k-1})$$

$$= \frac{1}{2^k - 1} \cdot \Big((2^0 + 2^1 + \cdots + 2^{k-1}) + (2^1 + \cdots + 2^{k-1}) + (2^2 + \cdots + 2^{k-1}) + \cdots + (2^{k-1})\Big)$$

$$= \frac{1}{2^k - 1} \cdot \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} 2^{j-1}$$

$$= \frac{1}{2^k - 1} \cdot \sum_{i=0}^{k-1} \left(2^i + 2^{i+1} + \cdots + 2^{k-1}\right)$$

$$= \frac{1}{2^k - 1} \cdot \sum_{i=0}^{k-1} \left(2^k - 2^i\right) = \frac{1}{2^k - 1} \cdot \left(\sum_{i=0}^{k-1} 2^k - \sum_{i=0}^{k-1} 2^i\right)$$

$$= \frac{1}{2^k - 1} \cdot \left(k \cdot 2^k - \sum_{i=0}^{k-1} 2^i\right)$$

$$= \frac{1}{2^k - 1} \cdot \Big(k \cdot 2^k - (2^k - 1)\Big)$$

$$= \frac{1}{2^k - 1} \cdot \Big((k - 1) \cdot 2^k + 1\Big) = \Theta(k) = \Theta(\log n)$$

Practice

INTEGER LINEAR PROGRAM: VERTEX COVER

Given an undirected graph $G = \langle V, E \rangle$, a *vertex cover* for it is a set $C \subseteq V$ with the property: $\langle u, v \rangle \in E$ implies at least one of $u, v \in C$.

An optimization problem is: given as input undirected $G = \langle V, E \rangle$, compute the minimum size of a vertex cover. Encode this optimization problem as an Integer Linear Program (ILP).

Adopt as unknowns in our output ILP, $x_1, \ldots, x_{|V|}$, where x_i corresponds to a vertex $i \in V$. More specifically, constrain each $x_i \in \{0, 1\}$, with $x_i = 1$ if vertex i is in the vertex cover, and $x_i = 0$ otherwise.

So immediately, we have the constraints:

- For all $i = 1, \ldots, |V|, x_i \geq 0$.
- For all $i = 1, \ldots, |V|, x_i \leq 1$.

To model the constraints of a vertex cover:

For each $\langle u, v \rangle \in E$, a constraint:

- $x_u + x_v \geq 1$.

And finally, our optimization objective:

- Minimize $\sum_{u \in V} x_u$, or Maximize $-\sum_{u \in V} x_u$.

That's it. What is the size of the output ILP instance, given as input $\langle G \rangle$?

Answer: $\Theta(|V| + |E|)$.

Practice

INTEGER LINEAR PROGRAM: ALGORITHM

Consider the following restricted, decision version of ILP, which is known as ZOE.

Given as input $\boldsymbol{A} \in \{0,1\}^{n\times m}$, does there exist $\boldsymbol{x} \in \{0,1\}^m$ such that $\boldsymbol{Ax} = 1$?

Suppose we have access to an oracle for this decision version. That is, given any such $\boldsymbol{A}$, it outputs **true** if indeed such an $\boldsymbol{x}$ exist, and **false** otherwise, and it does so in constant-time.

Devise a polynomial-time algorithm that given as input such an $\boldsymbol{A}$, outputs a vector $\boldsymbol{x}$ such that $\boldsymbol{Ax} = 1$ if indeed such an $\boldsymbol{x}$ exists, and the string 'no solution' otherwise.

Suppose $\boldsymbol{x} = [x_1, x_2, \dots, x_m]$. And suppose the oracle is denoted $I(\cdot)$.

First invoke $I(A)$. If the output is **false**, then output 'no solution' and halt.

Otherwise, we know that such an $\boldsymbol{x}$ exists, and we need to find it.

First try $x_1 = 0$. Then simplify $\boldsymbol{Ax}$. That is, suppose $\boldsymbol{A} = [a_{i,j}]$. Then, $\boldsymbol{Ax}$ is:

$$\begin{bmatrix} \sum_{j=1}^{m} a_{1,j} \cdot x_j \\ \sum_{j=1}^{m} a_{2,j} \cdot x_j \\ \dots \\ \sum_{j=1}^{m} a_{n,j} \cdot x_j \end{bmatrix}$$

If we adopt $x_1 = 0$, then $\boldsymbol{Ax}$ is:

$$\begin{bmatrix} \sum_{j=2}^{m} a_{1,j} \cdot x_j \\ \sum_{j=2}^{m} a_{2,j} \cdot x_j \\ \dots \\ \sum_{j=2}^{m} a_{n,j} \cdot x_j \end{bmatrix}$$

Practice

So $A[1, \ldots, n; 2, \ldots, m]$, i.e., the original A matrix with the $m - 1$ columns $2, \ldots, m$ only is a new instance of ZOE of size $n \times (m - 1)$. Now invoke $I(A[1, \ldots, n; 2, \ldots, m])$. If it returns **true**, then we know that an $\boldsymbol{x}$ exists to the original instance $\boldsymbol{A}$ of ZOE such that $x_1 = 0$. If the return is **false**, then we know that $x_1 = 1$.

Rewritten more carefully:

Suppose we determine that $x_1 \neq 0$, i.e., $x_1 = 1$. Then, in any row i of $\boldsymbol{A}$ that $a_{i,1} = 1$, suppose for some $j \neq 1$, $a_{i,j} = 1$. Then the corresponding $x_j = 0$. Because that is the only way that $\sum_{j=1}^{m} a_{i,j} \cdot x_j = 1$. Thus, we can go ahead and adopt 0 for all those x_j's.

Once we determine x_1, similarly determine x_2 unless it has already been determined to be by trial-and-error, with at most 2 possibilities for it. Note that the only rows that are useful in determining x_2 are those rows i in which (i) $a_{i,2} = 1$. Also, if x_1 was determined to be 1, then we also should not consider any row in which (ii) $a_{i,1} = 1$. If no such rows exist that satisfy both (i) and (ii), then we know $x_2 = 0$.

So, in the worst case, number of invocations to $I(\cdot)$ is $m + 1$. So, we have constructed a polynomial-time algorithm to determine such an $\boldsymbol{x}$ if on exists.

So, the point is: within a "polynomial factor" finding such an $\boldsymbol{x}$ is no more difficult than determining whether an instance of ZOE is **true**.

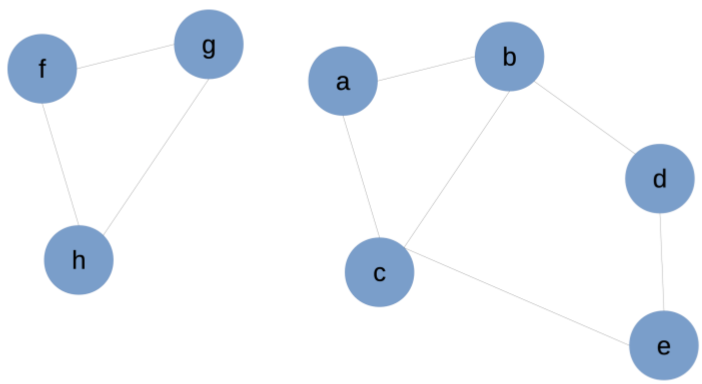
Practice

NP: CLIQUE

A *clique* in an undirected graph $G = \langle V, E\rangle$ is a subset of the vertices $C \subseteq V$ with the property: $u, v \in C$ distinct $\Longrightarrow \langle u, v\rangle \in E$. That is, a clique is a complete subgraph of G.

Is the following decision problem in **NP**?

Given input (i) an undirected graph $G = \langle V, E\rangle$, and (ii) an integer k, does G have more than one clique of size k?



Example cliques in the above graph: $\{a\}, \{a, b\}, \{a, b, c\}, \{d, e\}, \{f, g, h\}$.

Decision question is asking: (1) does G have a clique of size k? (2) If yes, does G have more than one clique of size k?

Answer: yes, it is in **NP**.

A solution/witness/certificate for a **true** instance is two distinct cliques each of size k in G.

A verification algorithm, given as input the instance and a solution for it, would check: (1) each claimed clique in the solution is indeed of size k, (2) that the two claimed cliques in the solution are distinct, and (3) that indeed the two claimed cliques are cliques in the graph, i.e., each pair of distinct vertices in each claimed clique has an edge between them.

Size of the solution above: at worst $2 \times |V|$, i.e., $O(n)$ where n is the size of the instance $\langle G, k\rangle$.

Check (1) can be carried out in time $O(n)$.

Check (2) can be carried out in time $O(n^2)$.

Check (3) can be carried out in time $O(n^3)$.

So, the verification algorithm is polynomial time.

Practice

NP: SIMPLE PATHS AND SUM OF EDGE-WEIGHTS

Is the following decision problem in **NP**?

Given as inputs (i) a graph $G = \langle V, E, l\rangle$ with $l: E \to \mathbb{Z}^+$, (ii) two distinct vertices $a, b \in V$, and (iii) an integer k, does every simple path $a \rightsquigarrow b$ have sum of edge-weights $\leq k$?

It appears unlikely to be in **NP**. Why? A natural solution/witness/certificate for a **true** instance would comprise some evidence that <u>every</u> path $a \rightsquigarrow b$ has weight $\leq k$.

And there may be, in the worst-case, exponentially many simple paths $a \rightsquigarrow b$.

Practice

POLYNOMIAL-TIME ALGORITHM: CLIQUE

Suppose you have a polynomial-time algorithm, that given inputs (i) undirected $G = \langle V, E \rangle$ and (ii) an integer k, correctly returns **true** if G has a clique of size k and **false** otherwise.

Devise a polynomial-time algorithm that given input undirected $G = \langle V, E \rangle$ outputs a clique of maximum size.

A strategy:

- First identify m the size of a clique of maximum size in G.
- Then, with our knowledge of m, go about identifying the vertices in a clique of size m.

To identify the maximum-size for a clique in G:

> Suppose the algorithm for the decision version is denoted $\mathcal{D}$.
>
> Now, an upper-bound for m is $|V|$. A lower-bound for m is 1 if G is non-empty.
>
> To identify m, perform binary search on k between 1 and $|V|$ with repeated calls to $\mathcal{D}(G, k)$.
>
> If the running-time of $\mathcal{D}(G, k)$ is $O(n^c)$ for constant c on input of size n. Then, our binary search has running time $O(\log n \cdot n^c)$, which is polynomial in n.

Now, given that we have identified m, we set about identifying the vertices in a clique of size m. Perform a trial-and-error on each vertex in V with repeated invocations to $\mathcal{D}$. Specifically, for each $u \in V$, we ask: does u have to remain in G for G to have a clique of size m?

So, here's an algorithm:

$G' \leftarrow$ a copy of G
foreach *vertex* u *in* G **do**
    Remove u and all incident edges from G'
    $\mathit{has\_clique} \leftarrow \mathcal{D}(G', m)$
    if *has_clique is* false **then**
        Restore u and all incident edges to G'
The vertices in G' that remain comprise a clique of size m

Running time: if $\mathcal{D}(G, k)$ runs in time $O(n^c)$. Then the above algorithm has time-efficiency $O(n \cdot n^c + n^2) = O(\max\{n^2, n^{c+1}\})$.

Practice

REDUCTION: SUBSET-SUM, KNAPSACK

Consider the following two decision problems.

> SUBSET-SUM: given a set of positive integers S and an integer k, does there exist a subset of S whose members' sum is k?
>
> KNAPSACK: given (i) n value-weight pairs $\langle v_1, w_1 \rangle, \ldots, \langle v_n, w_n \rangle$ of positive integers, and (ii) two positive integers V, W, does there exist a subset of the items whose sum of values $= V$ and sum of weights $= W$?

Show that SUBSET-SUM ≤ KNAPSACK.

An "instance" of the SUBSET-SUM problem is a concrete set S and a concrete integer k.

E.g., an instance of SUBSET-SUM is $\langle \{1, 8, 3, 14, 22, 6\}, 9 \rangle$.

An instance may be either **true** or **false**.

E.g., the above instance is **true**. A solution/witness/certificate is $\{1, 8\}$.

The instance $\langle \{1, 2\}, 5 \rangle$ is **false**.

Customary way to prove SUBSET-SUM ≤ KNAPSACK is by construction.

Meaning: we produce a function $f: I_{SUBSET-SUM} \rightarrow I_{KNAPACK}$ such that f satisfies the following two properties:

1. s is a **true** instance of SUBSET-SUM $\Leftrightarrow f(s)$ is a **true** instance of KNAPSACK, and
2. f is polynomial-time computable.

Consider the following candidate for f:

Given an instance of SUBSET-SUM, which is $\langle \{i_1, \ldots, i_n\}, k \rangle$ map it to the following instance of KNAPSACK: $\langle \{\langle i_1, i_1 \rangle, \ldots, \langle i_n, i_n \rangle\}, k, k \rangle$.

Practice

- Both properties (1) and (2) for f are satisfied by this candidate function.

Note: the above f we proposed that does work as a reduction is <u>not</u> invertible.

Proof for that: problem is the onto-ness, i.e., there exist instances of KNAPSACK that are not produced at all by f for any input instance of SUBSET-SUM.

E.g., $\langle\{\langle 1,2\rangle\},1,2\rangle$. That instance of KNAPSACK is not produced by f as output for any input.

Practice

REDUCTION: HAM-PATH, HAM-PATH-START-END

Consider the following two decision problems.

HAM-PATH: given an undirected graph, does it have a simple path of all vertices?

HAM-PATH-START-END: given an undirected graph G, and two distinct vertices a, b in it, does there exist a simple path of all vertices in G whose first vertex is a and last vertex is b?

Show that:

(1) HAM-PATH $\leq$ HAM-PATH-START-END, and
(2) HAM-PATH-START-END $\leq$ HAM-PATH

Note that an instance of HAM-PATH is a 1-tuple: $\langle G \rangle$ where G is an undirected graph.

E.g., of a **false** instance of HAM-PATH: $\langle\langle\{a, b\}, \emptyset\rangle\rangle$.

E.g., of a **true** instance of HAM-PATH: $\langle\langle\{a\}, \emptyset\rangle\rangle$. Another one: $\langle\langle\{a, b\}, \{\langle a, b\rangle\}\rangle\rangle$.

An instance of HAM-PATH-START-END: $\langle G, a, b\rangle$.

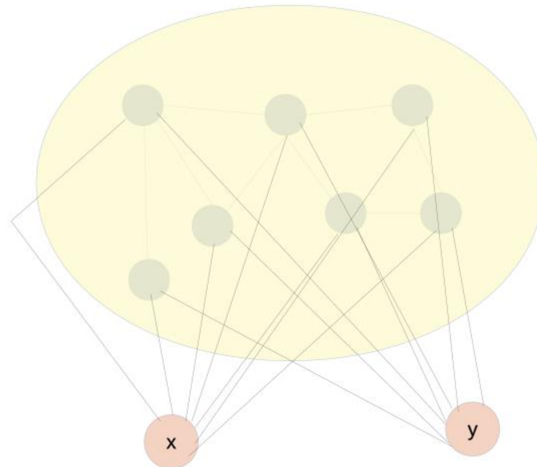
E.g., of a **true** instance of HAM-PATH-START-END: $\langle\langle\{a, b\}, \{\langle a, b\rangle\}\rangle, a, b\rangle$.

E.g., of a **false** instance of HAM-PATH-START-END: certainly, any **false** instance of HAM-PATH.

But also, some **true** instances of HAM-PATH. E.g., $\langle\langle\{a, b, c\}, \{\langle a, b\rangle, \langle b, c\rangle\}\rangle, a, b\rangle$.

For reduction (1): given an instance $\langle G \rangle$ of HAM-PATH, produce a new graph G' as follows.

Practice



Introduce two <u>new</u> vertices, call them x, y. By "new" we mean two vertices that don't exist in G.

Then introduce edges $\langle x, u\rangle, \langle y, u\rangle$ for every vertex u in G. (But no edge $\langle x, y\rangle$).

The output instance of HAM-PATH-START-END is $\langle G', x, y\rangle$.

Does the above function have the two properties we seek?

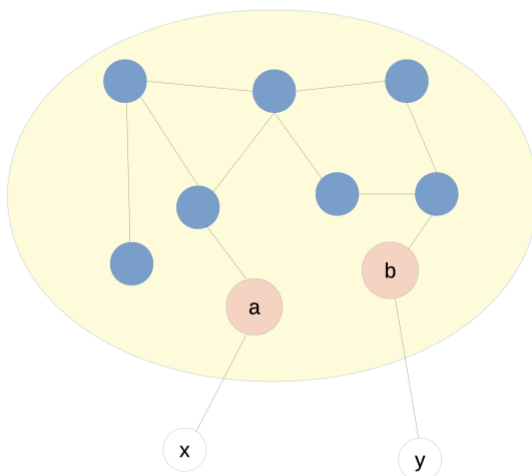
- It is indeed polynomial-time (specifically, linear-time) computable in the size of the instance of HAM-PATH, i.e., $\langle G\rangle$.
- For the "only if" direction of the "if and only if" property: if $\langle G\rangle$ is a **true** instance of HAM-PATH, then there exists a simple path $\langle u_1, u_2, \dots, u_{|V|}\rangle$ in G. Then, $\langle G', x, y\rangle$ produced as output by our function above is a **true** instance of HAM-PATH-START-END because in G', $\langle x, u_1, \dots, u_{|V|}, y\rangle$ is a simple path.
- For the "if" direction let's do it "directly" instead of proving the contrapositive just as an exercise.
  Suppose $\langle G', x, y\rangle$ is a **true** instance of HAM-PATH-START-END <u>that is produced as output by our function</u>. For the purpose of contradiction, for such a particular **true** instance of HAM-PATH-START-END, assume that the input instance $\langle G\rangle$ of HAM-PATH is **false**.
  The fact that $\langle G', x, y\rangle$ is a **true** instance of HAM-PATH-START-END means that there exists a path $\langle x, u_1, \dots, u_{|V|}, y\rangle$ where V is the set of vertices in the input graph G, and each $V = \{u_1, \dots, u_{|V|}\}$ in G. And the edges $u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_{|V|}$ exist in G.

Practice

Thus, the Hamiltonian path $\langle u_1, \dots, u_{|V|} \rangle$ exists in G, which contradicts the assumption that $\langle G \rangle$ is a **false** instance of HAM-PATH.

(2) In this direction, we observe that if $\langle G, a, b \rangle$ is a **true** instance of HAM-PATH-START-END, then $\langle G \rangle$ is a **true** instance of HAM-PATH. The problem is when $\langle G, a, b \rangle$ is a **false** instance of HAM-PATH-START-END. Because that does not necessarily imply that then $\langle G \rangle$ is a **false** instance of HAM-PATH – there may be a Hamiltonian path in G whose end points are not a, b.

A reduction is to alter G so that we "force" any Hamiltonian path that exists in it to be only one with end points a, b.



Given an instance $\langle G, a, b \rangle$ of HAM-PATH-START-END, we create a new graph G' from G as follows. We add two new vertices, x, y. We then add two edges, $\langle x, a \rangle, \langle y, b \rangle$. That's it. We now claim that there is a Hamiltonian path $a \rightsquigarrow b$ in G if and only if there is a Hamiltonian path in G'.

For the "only if" – let p be a Hamiltonian path $a \rightsquigarrow b$ in G. Then $x - p - y$ is a Hamiltonian path in G'. For the "if" direction, we prove the contrapositive. Let p be any path in G whose endpoints are a, b. Then we know that p is not Hamiltonian. Then neither is the path $x - p - y$ in G. Furthermore, any Hamiltonian path in G <u>must</u> have x, y as its endpoints.

Practice

REDUCTION: HAM-PATH, HAM-CYCLE

Let HAM-CYCLE be the following decision problem. Given an undirected graph, does it have a simple cycle of all vertices?

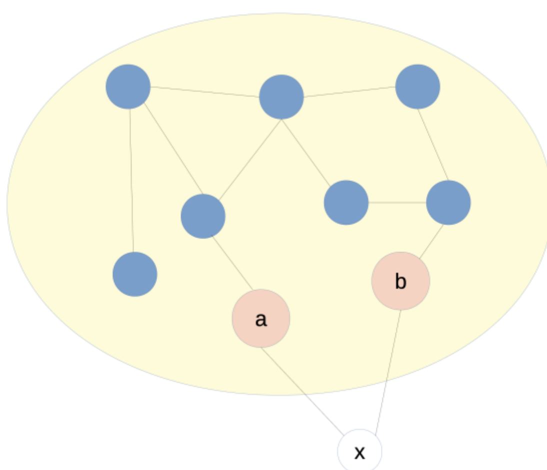
Show that:

(1) HAM-PATH from the previous problem $\leq$ HAM-CYCLE.

(2) HAM-CYCLE $\leq$ HAM-PATH.

(1) While it's certainly possible to reduce directly, we can also leverage the transitivity of $\leq$, and reduce via HAM-PATH-START-END.

Suppose $\langle G, a, b \rangle$ is an instance of HAM-PATH-START-END. Then, a reduction to HAM-CYCLE is as follows.



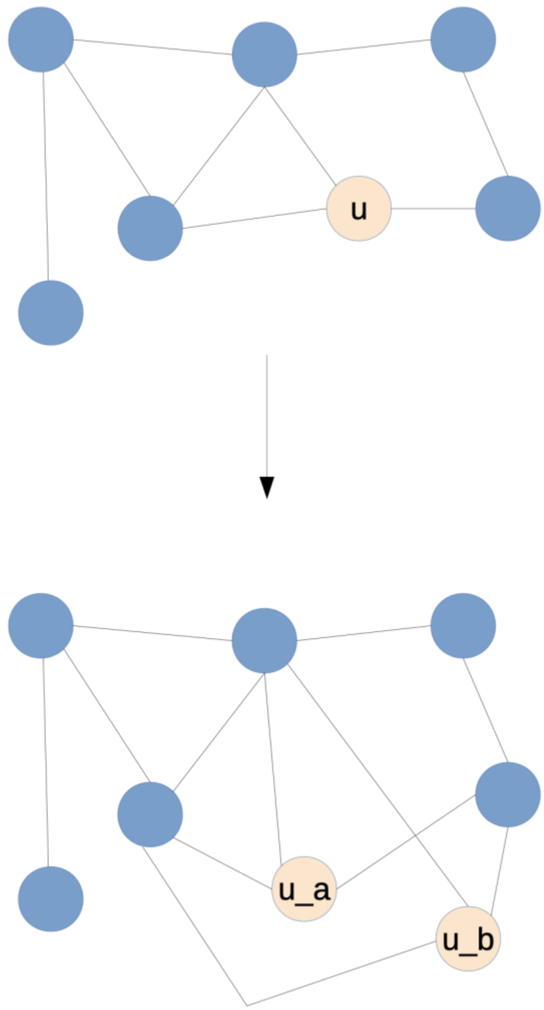
Create a new graph G' from G as follows. Add a new vertex x. Add new edges $\langle x, a \rangle, \langle x, b \rangle$. That's it.

Now we claim that $\langle G, a, b \rangle$ is a **true** instance of HAM-PATH-START-END if and only if $\langle G' \rangle$ is a **true** instance of HAM-CYCLE. For the "only if" direction, if p is a Hamiltonian path in G with endpoints a, b, then $x - p - x$ is a Hamiltonian cycle in G'. For the "if" direction, suppose there exists a Hamiltonian cycle in G'. Write it as a path $x \rightsquigarrow x$; this is possible because we could start at any vertex in the graph and write a Hamiltonian cycle as a simple cycle of all vertices that start and end at that vertex.

Practice

Then, such a cycle must be $x - a \overset{h}{\rightsquigarrow} b - x$ where h is a Hamiltonian path with end points a, b. But as those edges are present in G, we know that the Hamiltonian path h exists in G.

(2) For a reduction in the other direction, given any instance $\langle G \rangle$ of HAM-CYCLE, pick any vertex u and "split" it into two vertices, call them u_a, u_b.



By "split it," we mean, transform G as follows to a new graph G'.

Introduce two new vertices u_a, u_b. For every edge $\langle u, v \rangle$, add edges $\langle u_a, v \rangle, \langle u_b, v \rangle$. Finally, remove u and all edges incident on it from the graph.

Now, we can prove that $\langle G \rangle$ is a **true** instance of HAM-CYCLE if and only if $\langle G', u_a, u_b \rangle$ is a **true** instance of HAM-PATH-START-END.