Practice

Time Complexity

$\boldsymbol{f(n)}$	$\boldsymbol{g(n)}$	$\boldsymbol{O/\Omega/\Theta}$
$n-100$	$n-200$	Θ
$n^{1/2}$	$n^{2/3}$	O
$100n+\log n$	$n+(\log n)^2$	$\Theta\ (a)$
$\log 2n$	$\log 3n$	$\Theta\ (b)$
$10\log n$	$\log n^2$	$\Theta\ (c)$
$n^{1/2}$	$5^{log_2 n}$	$O\ (d)$
2^n	2^{n+1}	$\Theta\ (e)$

(a): $n\ dominates\ (\log n)^c \rightarrow n+(\log n)^2=\Theta(n)$

(b): $\log ab=\log a+\log b$

(c): $\log a^b=b\log a$

(d): $5=2^{2x}\ where\ x>0 \rightarrow 5^{log_2 n}=(2^{2x})^{log_2 n}=(2^{log_2 n})^{2x}=\Omega(n^{1/2})$

(e): $2^{n+1}=2\times 2^n$

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Practice

Fibonacci 1

$$F_n = \begin{cases} 0, if\ n = 0 \\ 1, if\ n = 1 \\ F_{n-1} + F_{n-2}, otherwise \end{cases}$$

Prove: $F_n = \Omega(\sqrt{2}^n)$.

By trial and error: It appears that $F(n) \geq 2^{n/2}$ for all $n \geq 7$

To prove: For all positive integers $n \geq 7 \rightarrow F_n \geq 2^{n/2}$

By induction on n. Base case: $n = 7$

Step, assume: Indeed, true that for all $i = 7, 8, \dots, k \rightarrow F_i \geq 2^{i/2}$

To prove: $F_{k+1} \geq 2^{k+1/2}$

LHS: $F_{k+1} = F_k + F_{k-1} \geq 2^{k/2} + 2^{(k-1)/2}$

Suffices to prove: $2^{k/2} + 2^{(k-1)/2} \geq 2^{(k+1)/2}$

$2^{1/2} + 1 \geq 2^{2/2}$ (by dividing the above by $2^{(k-1)/2}$)

It is indeed true that $2^{1/2} + 1 \geq 2^{2/2} = 1$

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Practice

Multiplication

Figure 1.1 Multiplication à la Français.

```
function multiply(x, y)
Input: Two n-bit integers x and y, where y ≥ 0
Output: Their product

if y = 0: return 0
z = multiply(x, ⌊y/2⌋)
if y is even:
   return 2z
else:
   return x + 2z
```

Suppose instead of both x and y being n-bit, x is n-bit and y is m-bit. What is the worst-case time efficiency of $multiply$?

Proposed: $O(nm)$

Time Efficiency:

- $\#\, recursive\ calls\ x\ time/call$
- $\#\, worst\ case\ recursive\ calls = O(m)$
- $Worst\ case\ time/call =$
    - $2z\ is\ at\ worst\ O(n+m) \rightarrow because\ very\ last\ addition\ is\ 2z = xy - x$
    - $x\ is\ n\ bits$
    - $So, addition's\ time: O(\max\{n, n+m\}) = O(\max\{n, m\})$

So, final answer: $O(m \times \max\{n, m\})$

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Practice

Fibonacci 2

Let F_n be the n[th] Fibonacci number, Prove $F_n = O(2^n)$.

- Somewhere, we have shown: $F_n = \Omega(\sqrt{2}^n)$
- But here, seek to show: There exists positive real $F_n \leq c \cdot 2^n$, for all n in N
- Natural proof strategy for "there exists" – construction (i.e., propose some concrete c, and show that it works)
- Try some small values for n, and see what c would work
  - $n = 0, F_0 = 0, 2^0 = 1 \rightarrow c = 1\ works$
  - $n = 1, F_1 = 1, 2^1 = 2 \rightarrow c = 1\ works$
  - $n = 2, F_2 = 1, 2^2 = 4 \rightarrow c = 1\ works$
  - $n = 3, F_3 = 2, 2^3 = 8 \rightarrow c = 1\ works$
  - $n = 4, F_4 = 3, 2^4 = 16 \rightarrow c = 1\ works$
- Appears that $c = 1$works. Adopt it and check if proof goes through. Now, proof by induction with $c = 1$
- Base case, $n = 1, F_1 = 1, 2^1, 1 \leq 2 \rightarrow True$
- Step: Seek to show $F_n \leq 2^n$ given that $F_k \leq 2^k$ for all $k = 1, 2, \ldots, n-1$
- $F_n = F_{n-1} + F_{n-2} \leq 2^{n-1} + 2^{n-2}$ by induction assumption
- $F_n = 2^{n-2}\,(2 + 1) = 3 \times 2^{n-2} \leq 2^n = 2^2 \times 2^{n-2} = 4 \times 2^{n-2} \rightarrow Done$

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Practice

Fibonacci 3

Let F_n be the n^th^ Fibonacci number, Prove $F_n \neq O(n^2)$.

- Recall from logic: not (there exists an egg-laying mammal) = for all mammals m, m is not egg-laying
- Here, $f = O(g)$: There exists positive real c, for all natural n, $f(n) \leq c \cdot g(n)$
- So here, need to prove: Given any positive real c, it is true that there exists n such that $F_n > c \cdot n^2$
- By contradiction: Suppose that there exists positive real c, such that, for all natural n, $F_n \leq c \cdot n^2$
- Then: $F_n = F_{n-1} + F_{n-2} \leq c(n-1)^2 + c(n-2)^2 = c(n^2 - 2n + 1 + n^2 - 4n + 4) = c(2n^2 - 6n + 5) \leq cn^2$
- $2n^2 - 6n + 5 \leq n^2$
- $2 - \frac{1}{n^2}(6n - 5) \leq 1$
- This is true only if $\frac{1}{n^2}(6n-5)$ is "large" compared to $2n^2$
- What is large? We need $\frac{1}{n^2}(6n-5) \geq 1 \rightarrow true\ for\ n = 1$
- Try $n = 2$: $\frac{1}{4}(12-5) = \frac{7}{4} \geq 1$
- Try $n = 3$: $\frac{1}{8}(18-5) = \frac{13}{8} \geq 1$
- Try $n = 4$: $\frac{1}{16}(24-5) = \frac{19}{16} \geq 1$
- Try $n = 5$: $\frac{1}{25}(30-5) = 1$
- Try $n = 6$: $\frac{1}{36}(36-5) < 1$
- Try $n = 7$: $\frac{1}{49}(42-5) < 1$
- Prove by induction: $6n - 5 < n^2$ for all natural $n > 5$
- Base case $n = 6$: See above
- Step: $6(n-1) - 5 \leq (n-1)^2 \rightarrow from\ induction\ assumption$
- $6n - 5 - 6 < n^2 - 2n + 1$
- $6n - 5 \leq n^2 - (2n - 7) \leq n^2\ whenever\ 2n - 7 \geq 0 \rightarrow which\ it\ is\ for\ n \geq 6$
- So far: We have shown that indeed, for $n \geq 6$, $F_n < cn^2 \rightarrow Done$

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Practice

Selection Sort

$$SELECTIONSORT(A[1, \dots, n])$$
$$foreach\ i\ from\ 1\ to\ n\ do$$
$$m \leftarrow i - 1 + INDEXOFMIN(A[i, \dots, n])$$
$$if\ \ i \neq m\ then\ swap\ A[i], A[m]$$

$$INDEXOFMIN(B[1, \dots, m])$$
$$min \leftarrow B[1], idx \leftarrow 1$$
$$foreach\ j\ from\ 2\ to\ m\ do$$
$$if\ \ B[j] < \min then$$
$$\min \leftarrow B[j], idx \leftarrow j$$
$$return\ idx$$

What is a meaningful characterization of the time efficiency of $SELECTIONSORT$?

- Suppose we invoke $INDEXOFMIN(A[5, \dots, 13])$. In $INDEXOFMIN$: $B[1, \dots, 9]$. Suppose now, min is at index 3 in $B[1, \dots, 9]$. This → index of a min in $A[5, \dots, 13]$ is at index $(5 - 1) + 3 = 7$
- Suppose on input: $A[1, \dots, 5] = [13, -23, 45, -23, 1]$. Then A evolves in $SELECTIONSORT$ as follows:
    - $i = 1, m = 2, [-23, 13, 45, -23, 1]$
    - $i = 2, m = 4, [-23, -23, 45, 13, 1]$
    - $i = 3, m = 4, [-23, -23, 13, 45, 1]$
- For time efficiency: Need to make meaningful assumption(s)
- Customary Assumptions: (1) n is unbounded, (ii) each $A[i]$ is bounded
- What should we count? Suppose we all agree that counting # swaps is a meaningful measure for time efficiency
- Then: $Worst\ case\ \#\ swaps = n - 1 = \Theta(n)$
- Now, let's say we want to get a bit more fine-grained. Incorporate (worst case) time for each swap x # swaps
- So now, time efficiency: $(n - 1) + (n - 2) + \cdots + 1 = \Theta(n^2)$

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Practice

Modular Simplification

1. Is $6^6 \equiv 5^3$ $(mod\ 31)$?

$$6 \times 6 = 36 \equiv 5\ (mod\ 31)$$

So: $(6^2)^3 \equiv (5)^3$ $(mod\ 31)$

2. $2^{125} \equiv ?\,(mod\ 127)$

$$2^7 = 128 = 127 + 1$$

So: $128\ mod\ 127 = 1$

Now: $125/7\ = 17 + 6/7$

So: $2^{125} = 2^{17\times7+6} = 2^{17\times7} \times 2^6$

So: $2^{125} \equiv 2^{17\times7} \times 2^6 \equiv (2^7)^{17} \times 2^6 \equiv 1^{17} \times 2^6 \equiv 64\ (mod\ 127)$

3. Is $4^{1536} - 9^{4824}$ divisible by 35?

$$4^{1536} \equiv 9^{4824}\ (mod\ 35)$$

Trick: Keep exponentiating until numbers start to repeat.

Suppose we repeatedly exponentiate 4:

$$4$$

$$\rightarrow 16$$

$$\rightarrow 64 \equiv 29\ (mod\ 35)$$

$$\rightarrow 116 = 35 \times 3 + 11 = 11\ (mod\ 35)$$

$$\rightarrow 9\ (mod\ 35)$$

$$\rightarrow 36 \equiv 1\ (mod\ 35)$$

So: $4^6 \equiv 1\ (mod\ 35)$. And $1536 = 6 \times 256$. So $4^{1536} \equiv 1\ (mod\ 35)$

Now check whether 1536 is divisible by 4. Indeed: $1536 = 4 \times 384$

Repeat with 9. Repeated exponentiation of 9:

$$9$$

$$\rightarrow 81 \equiv 11 \ (mod\ 35)$$

$$\rightarrow 99 \equiv 29 \ (mod\ 35)$$

$$\rightarrow 261 = 7 \times 35 + 16 \equiv 16 \ (mod\ 35)$$

$$\rightarrow 144 \equiv 4 \times 35 + 4 \equiv 4 \ (mod\ 35)$$

$$\rightarrow 36 \equiv 1 \ (mod\ 35)$$

So: $9^6 \equiv 1 \ (mod\ 35)$

Now: $9^{4824} = 9^{804 \times 6} \equiv 1 \ (mod\ 35)$.

$\therefore It\ is\ divisible\ by\ 35.$

4. $2^{2^{2006}} \ (mod\ 3) = ?$

$$2^{2^{2006}} = (2^2)^{2^{2005}} = 4^{2^{2005}} \equiv 1 \ (mod\ 3)$$

5. Is $5^{30000} - 6^{123456}$ a multiple of 31?

31 is prime. And $5^{30000} = (5^{30})^{1000} \equiv 1 \ (mod\ 31)$.

Compare with $6^{123456} = 6^{123450} \times 6^6$:

$$1 \times 6^6 \equiv 5^3 \equiv 125 \equiv 31 \times 4 + 1 \ (mod\ 31) \equiv 1 \ (mod\ 31)$$

$\therefore It\ is\ a\ multiple\ of\ 31.$

Practice

Proving Multiplicative Inverse

Show that if a has a multiplicative inverse modulo N, then this inverse is unique (modulo N).

Let's assume $a \epsilon \{1, \dots, N-1\}$.

Suppose $b, c \epsilon \{1, \dots, N-1\}$ are both multiplicative inverses of $a\ modulo\ N$. Then:

$$ab \equiv 1\ (mod\ N)$$

$$ac \equiv 1\ (mod\ N)$$

$$ab \equiv ac\ (mod\ N)$$

$$ab \cdot b \equiv ac \cdot b\ (mod\ N)\ (1)$$

(1): Substitution Rule:

$$x \equiv x', y \equiv y' (mod\ N)$$

$$xy \equiv x'y' (mod\ N)$$

Then:

$$(ab) \cdot b \equiv (ab) \cdot c\ (mod\ N)\ (2)$$

(2): Commutativity

$$1 \cdot b \equiv 1 \cdot c\ (mod\ N)$$

$$b \equiv c\ (mod\ N)$$

$$b = c$$

Suppose $p \equiv 3\ (mod\ 4)$. Show that $(p+1)/4$ is an integer.

$$p \equiv 3\ (mod\ 4)$$

$$p = 4k + 3\ for\ some\ k\ \epsilon\ \mathbb{Z}$$

So: $p + 1 = 4k + 4$, which is divisible by 4.

We say that x is a square root of y modulo a prime p if $y \equiv x^2\ (mod\ p)$. Show that if $(i)\ p \equiv 3\ (mod\ 4)$ and $(ii)\ y$ has a square root modulo p, then $y^{(p+1)/4}$ is such a square root.

Let x be the square root of $y\ modulo\ p$. Then: $y \equiv x^2\ (mod\ p)$.

Practice

Write $p = 4k + 3$. Then, $\left(y^{\frac{p+1}{4}}\right)^2 = y^{2(p+1)/4} = y^{2(4k+3+1)/4} = y^{2k+2}$

Keep in mind: $(p + 1)/4 = k + 1$.

Try plugging in x in the last expression:

Is $y^{2k+2} = x^{4k+4} \equiv x^2$?

So, we're asking: Is $x^{4k+4} - x^2 \equiv 0\ (mod\ p)$?

$$x^{4k+4} - x^2 = (x^{2k+2} - x)(x^{2k+2} + x)$$

So at least one of: $x^{2k+2} - x$ or $x^{2k+2} + x$ must be $\equiv 0\ (mod\ p)$.

- $\frac{(p+1)}{4} = \frac{(4k+3+1)}{4} = k + 1$
- $2 \cdot \frac{(p+1)}{4} = 2k + 2$
- $p - 1 = 4k + 2$

We know: There exists $x \epsilon \{1, \dots, p - 1\}$ such that $y \equiv x^2\ (mod\ p)$.

We seek to prove: $\left(y^{\frac{(p+1)}{4}}\right)^2 \equiv y\ (mod\ p)$. Sufficient condition for that to be true:

$\left(y^{\frac{(p+1)}{4}}\right)^2 \cdot y^{-1} \equiv 1\ (mod\ p) \rightarrow$ is okay, because y is invertible modulo p

$$\Rightarrow (y^{2k+2}) \cdot y^{-1} \equiv 1\ (mod\ p)$$
$$\Rightarrow y^{2k+1} \equiv 1\ (mod\ p)$$
$$\Rightarrow (x^2)^{2k+1} \equiv 1\ (mod\ p)$$
$$\Rightarrow x^{4k+2} \equiv 1\ (mod\ p)$$
$$\Rightarrow x^{p-1} \equiv 1\ (mod\ p)$$
$$\Rightarrow True\ (Fermat's\ little\ theorem)$$

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Practice

Proving Recurrence 1

Suppose $x \in \mathbb{Z}^+, y \in \mathbb{Z}_0^+$. Prove recurrence correctness.

$$x^y = \begin{cases} 1, if\ y = 0 \\ (x^2)^{\lfloor y/2 \rfloor}, if\ y\ is\ even \\ x \cdot (x^2)^{\lfloor y/2 \rfloor}, otherwise \end{cases}$$

Case Analysis:

1. If $y = 0$, then $x^y = x^0$. So, the recurrence is correct for the case where $y = 0$
2. If $y \neq 0, y\ is\ even$: then $\lfloor y/2 \rfloor = y/2$. So $x^y = x^{2 \times y/2} = (x^2)^{y/2} = (x^2)^{\lfloor y/2 \rfloor}$
3. If $y \neq 0, y\ is\ odd$: then $\lfloor y/2 \rfloor = (y - 1)/2$. So now:

$$x^y = x^{(2 \times (y-1)/2)+1} = x^{(2 \times \lfloor y/2 \rfloor)+1} = x \cdot x^{2 \times \lfloor y/2 \rfloor}$$

Practice

Proving Recurrence 2

Let $\langle q, r \rangle$ be the quotient and remainder of x/y and $\langle q', r' \rangle$ be the quotient and remainder of $(\lfloor x/2 \rfloor)/y$. Prove recurrence correctness.

$$\langle q, r \rangle = \begin{cases} \langle 0, 0 \rangle, if\ x = 0 \\ \langle 2q', 2r' \rangle, if\ x\ even\ and\ 2r' < y \\ \langle 2q', 2r' + 1 \rangle, if\ x\ odd\ and\ 2r' + 1 < y \\ \langle 2q' + 1, 2r' - y \rangle, if\ x\ even\ and\ 2r' \geq y \\ \langle 2q' + 1, 2r' + 1 - y \rangle, otherwise \end{cases}$$

To be absolutely clear, what are the quotient and remainder of x/y?

We call q the quotient, and r the remainder if and only if q and r are non-negative integers that satisfy:

$$x = q \cdot y + r, where\ r \in \{0, 1, \dots, y - 1\}$$

Proof by case analysis:

1. If $x = 0$, then $x = 0 = 0 \cdot y + 0$. So, recurrence is correct for this case.
2. If x is even and $2r' < y$: then $\lfloor x/2 \rfloor = x/2$. So:

$$\lfloor x/2 \rfloor = x/2 = q' \cdot y + r'$$

$$x = (2q') \cdot y + 2r'$$

$$q = 2q', r = 2r'$$

Where we infer the last line from the facts that: (i) equation is of the form from definition for quotient and remainder, (ii) $r' \geq 0 \rightarrow 2r' \geq 0$, and (iii) we are given $2r' \leq y - 1$.

3. If x is odd and $2r' + 1 < y$: $\lfloor x/2 \rfloor = (x - 1)/2$

$$\lfloor x/2 \rfloor = (x - 1)/2 = q' \cdot y + r'$$

$$x - 1 = (2q') \cdot y + 2r'$$

$$x = (2q') \cdot y + (2r' + 1)$$

4. x is even, $2r' \geq y$: $\lfloor x/2 \rfloor = x/2$. So:

$$\lfloor x/2 \rfloor = x/2 = q' \cdot y + r'$$

$$x = (2q') \cdot y + 2r'$$

Practice

This is of the form of the definition of quotient and remainder, except that we need to confirm that $2r'$ indeed lies between 0 and $y - 1$. Which it does not necessarily. Actually, we are given that $2r' \geq y$ and therefore not between 0 and $y - 1$. Now we observe:

$$x = (2q') \cdot y + 2r'$$
$$x = (2q' + 1) \cdot y + (2r' - y)$$

Now only question that remains: is it the case that $2r' - y \in \{0, 1, \dots, y - 1\}$?

- Is $2r' - y \geq 0$? Yes, because $2r' \geq y$
- Is $2r' - y \leq y - 1$? Yes, because:

$$r' \leq y - 1$$
$$2r' \leq 2y - 2$$
$$2r' - y \leq y - 2 \leq y - 1$$

5. x odd, $2r' + 1 \geq y$:

$$\lfloor x/2 \rfloor = (x - 1)/2 = q' \cdot y + r'$$

$$x = (2q') \cdot y + (2r' + 1)$$
$$x = (2q' + 1) \cdot y + (2r' + 1 - y)$$

Now:

- $2r' + 1 - y \geq 0$ because $2r' + 1 \geq y$.
- $2r' + 1 - y \leq y - 1$ because:

$$r' \leq y - 1$$
$$2r' + 1 \leq 2y - 1$$
$$2r' + 1 - y \leq y - 1$$

Practice

Proving Recurrence 3

Prove that $BinSearch$ is correct.

$BinSearch(A[1, \dots, n], lo, hi, i)$

1. $\boldsymbol{if}\ lo \leq hi\ \boldsymbol{then}$
2. $\quad mid \leftarrow \lfloor (lo + hi)/2 \rfloor$
3. $\quad \boldsymbol{if}\ A[mid] = i\ \boldsymbol{then}\ \boldsymbol{return}\ true$
4. $\quad \boldsymbol{if}\ A[mid] < i\ \boldsymbol{then}\ \boldsymbol{return}\ BinSearch(A, mid + 1, hi, i)$
5. $\quad \boldsymbol{else}\ \boldsymbol{return}\ BinSearch(A, lo, mid - 1, i)$
6. $\boldsymbol{else}\ \boldsymbol{return}\ false$

Above is recursive version of binary search. Iterative version:

$BinSearch(A[1, \dots, n], lo, hi, i)$

1. $\boldsymbol{while}\ lo \leq hi\ \boldsymbol{do}$
2. $\quad mid \leftarrow \lfloor (lo + hi)/2 \rfloor$
3. $\quad \boldsymbol{if}\ A[mid] = i\ \boldsymbol{then}\ \boldsymbol{return}\ true$
4. $\quad \boldsymbol{if}\ A[mid] < i\ \boldsymbol{then}\ lo \leftarrow mid + 1$
5. $\quad \boldsymbol{else}\ hi \leftarrow mid - 1$
6. $\boldsymbol{else}\ \boldsymbol{return}\ false$

Typically, for iterative algorithms, towards correctness, we articulate a *loop invariant*:

Let $lo^{(in)}$ and $hi^{(in)}$ be the values of lo and hi respectively on input. Just before we successfully enter an iteration of the $\boldsymbol{while}$ loop of Line (1), it is true that:

$$i \in A\left[lo^{(in)}, \dots, hi^{(in)}\right] \rightarrow i \in A[lo, \dots, hi]$$

Going back to the recursive version, what is a correctness property?

Given $A[1, \dots, n]$ an array that is sorted, non-decreasing, lo, hi are each $\in \{1, \dots, n\}$ on input, $BinSearch(A, lo, hi, i)$ returns:

- $True \rightarrow (lo \leq hi)\ and\ (i \in A[lo, \dots, hi])$

Practice

- $False \rightarrow either\ (lo > hi)\ or\ (i\ is\ not \in A[lo, \ldots, hi])$

Proof by case analysis:

Case 1: $lo > hi$ on input: then $\boldsymbol{if}$ condition of Line (1) evaluates to **false**, and we correctly return **false** in Line (6). Then, this is either from (a) Line (6) without making any recursive calls, or (b) as the return value from a recursive call from one of Lines (4) or (5).

For (b), we first observe that $lo \leq hi$ because the only recursive calls are within the $\boldsymbol{if}$ block of Line (1). So, all that remains to be proven is that indeed: $i \notin A[lo, \ldots, hi]$.

We prove that by induction on $hi - lo + 1$. Base case: $hi - lo + 1 = 1$. We claim we return $\boldsymbol{false}$ within the first recursive invocation. That is, we claim: (i) $mid + 1 > hi$ and $lo > mid - 1$, (ii) $mid = lo = hi$, and (iii) $i \neq A[mid]$.

(ii) easy to prove:

$$hi - lo + 1 = 1$$
$$\Rightarrow lo = hi$$
$$\Rightarrow mid = \left\lfloor \frac{(lo + hi)}{2} \right\rfloor = \left\lfloor \frac{(lo + lo)}{2} \right\rfloor = \left\lfloor \frac{(2 \cdot lo)}{2} \right\rfloor = \frac{2 \cdot lo}{2} = lo = hi$$

(iii) is $\boldsymbol{true}$, because then we would have returned $\boldsymbol{true}$ in Line (3).

To prove (i): we simply exploit: $mid = hi = lo$

$$mid = hi \Rightarrow mid + 1 > hi$$
$$mid = lo \Rightarrow mid - 1 < lo$$

So, the algorithm is correct if it returns $\boldsymbol{false}$, and $hi - lo + 1 = 1$.

For the step, we know that on input $lo < hi$. So, we returned $\boldsymbol{false}$ in some recursive call. So, all we have to prove to appeal to induction assumption: $hi - (mid + 1) < hi - lo$ and $(mid - 1) - lo < hi - lo$.

Practice

Proving Master Theorem

Give a closed form solution for the following recurrence. Assume: $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is non-decreasing, $a > 0, b > 1, d \geq 0$.

$$f(n) = \begin{cases} \Theta(1), if n \leq 1 \\ a \cdot f\left(\frac{n}{b}\right) + \Theta(n^d), otherwise \end{cases}$$

Proposed approach: Inductive "rewriting" of the function f. But first: adopt concrete functions wherever we have $\Theta(\cdot)$, $O(\cdot)$ or $\Omega(\cdot)$. In this case: adopt 1 for $\Theta(1)$, and n^d for $\Theta(n^d)$. Now onto the rewriting:

$$f(n) = a \cdot f\left(\frac{n}{b}\right) + n^d$$

$$= a \cdot \left(a \cdot f\left(\frac{n}{b^2}\right) + \left(\frac{n}{b}\right)^d\right) + n^d$$

$$= a^2 \cdot f\left(\frac{n}{b^2}\right) + a \cdot \left(\frac{n}{b}\right)^d + n^d$$

$$= a^2\left(a \cdot f\left(\frac{n}{b^3}\right) + \left(\frac{n}{b^2}\right)^d\right) + a \cdot \left(\frac{n}{b}\right)^d + n^d$$

$$= a^3 \cdot f\left(\frac{n}{b^3}\right) + a^2 \cdot \left(\frac{n}{b^2}\right)^d + a \cdot \left(\frac{n}{b}\right)^d + n^d$$

$$= a^3 \cdot f\left(\frac{n}{b^3}\right) + n^d\left(\left(\frac{a}{b^d}\right)^2 + \left(\frac{a}{b^d}\right)^1 + \left(\frac{a}{b^d}\right)^0\right)$$

$$= a^4 \cdot f\left(\frac{n}{b^4}\right) + n^d\left(\left(\frac{a}{b^d}\right)^3 + \left(\frac{a}{b^d}\right)^2 + \left(\frac{a}{b^d}\right)^1 + \left(\frac{a}{b^d}\right)^0\right)$$

...

$$= a^{\log_b n} \cdot f(1) + n^d \cdot \left(\left(\frac{a}{b^d}\right)^{(\log_b n)-1} + \left(\frac{a}{b^d}\right)^{(\log_b n)-2} + \cdots + \left(\frac{a}{b^d}\right)^0\right)$$

$$= a^{\log_b n} + n^d \cdot \left(\left(\frac{a}{b^d}\right)^{(\log_b n)-1} + \left(\frac{a}{b^d}\right)^{(\log_b n)-2} + \cdots + \left(\frac{a}{b^d}\right)^0\right)$$

To figure out the power of a in that last term:

> Power of a is the same as the power of b inside the $f\left(\frac{n}{b^x}\right)$. In other words: what is the power of b, i.e., x for which $\frac{n}{b^x} = 1$? Answer: $\frac{n}{b^x} = 1 \Leftrightarrow n = b^x \Leftrightarrow x = \log_b n$.

Practice

Our next step: Simplify/figure out:

$$S = \left(\frac{a}{b^d}\right)^{(\log_b n)-1} + \left(\frac{a}{b^d}\right)^{(\log_b n)-2} + \cdots + \left(\frac{a}{b^d}\right)^0$$

Suppose:

$$T = r^{q-1} + r^{q-2} + \cdots + r^0$$
$$\Longrightarrow r \cdot T = r^q + r^{q-1} + \cdots + r$$

Now subtract one from the other:

$$\Longrightarrow T - r \cdot T = r^0 - r^q$$
$$\Longrightarrow (1-r) \cdot T = 1 - r^q$$
$$\Longrightarrow T = \frac{1-r^q}{1-r}, provided\ r \neq 1$$

When $r = 1$, how do we figure out what T is? Answer: then, T is:

$$T = 1^{q-1} + 1^{q-2} + \cdots + 1^0$$
$$= 1 + 1 + \cdots + 1 \rightarrow q\ instances\ of\ 1$$
$$= q$$

So, going back to our S:

$$S = \left(\frac{a}{b^d}\right)^{(\log_b n)-1} + \left(\frac{a}{b^d}\right)^{(\log_b n)-2} + \cdots + \left(\frac{a}{b^d}\right)^0$$
$$\Longrightarrow S = \frac{1 - \left(\frac{a}{b^d}\right)^{\log_b n}}{1 - \left(\frac{a}{b^d}\right)}, provided \frac{a}{b^d} \neq 1$$

And:

$$S = \log_b n\, , when \frac{a}{b^d} = 1$$

When is $\frac{a}{b^d} = 1$? Answer: $d = \log_b a$.

So, going back to our $f(n)$: first, the case that $d = \log_b a$.

But even before that: rewrite $a^{\log_b n} = n^{\log_b a}$. Because:

$$x = a^{\log_b n} \Leftrightarrow \log_b x = \log_b a \cdot \log_b n \Leftrightarrow x = n^{\log_b a}$$

Practice

$$f(n) = n^{\log_b a} + n^d \cdot S$$

So, when $d = \log_b a$, $S = \log_b n$. So, in this case:

$$f(n) = n^d + n^d \cdot \log_b n$$
$$= \Theta(n^d \cdot \log n)$$

Onto the other two cases: $d \neq \log_b a$.

$$f(n) = n^{\log_b a} + \cdots + n^d \cdot S$$

Before we continue: a closer look at $\left(\frac{a}{b^d}\right)^{\log_b n}$:

$$\left(\frac{a}{b^d}\right)^{\log_b n} = \frac{a^{\log_b n}}{(b^d)^{\log_b n}}$$
$$= \frac{n^{\log_b a}}{(b^{\log_b n})^d}$$
$$= \frac{n^{\log_b a}}{n^d}$$

So: when $d \neq \log_b a$

$$S = \frac{1 - \frac{n^{\log_b a}}{n^d}}{1 - \left(\frac{a}{b^d}\right)}$$

So, going back to $f(n)$:

$$f(n) = n^{\log_b a} + n^d \cdot S$$
$$= n^{\log_b a} + \frac{1}{1 - \left(\frac{a}{b^d}\right)} \cdot \left(n^d - n^{\log_b a}\right)$$
$$= c \cdot n^{\log_b a} + c' \cdot n^d, for\ positive\ constants\ c, c'$$

So, if $d > \log_b a$: $f(n) = \Theta(n^d)$

And if $d < \log_b a$: $f(n) = \Theta\left(n^{\log_b a}\right)$

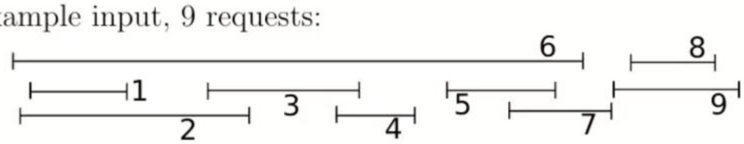
Practice

Proving Greediness

Given as input n meeting requests, $\langle s_1, f_1\rangle, \langle s_2, f_2\rangle, \ldots, \langle s_n, f_n\rangle$, where each $s_i, f_i \in \mathbb{Z}^+$ is a start- and finish-time and $s_i < f_i$. We want a subset of those requests that is of maximum size that are pairwise conflict-free.

Two requests $\langle s_i, f_i\rangle$, $\langle s_j, f_j\rangle$ are in conflict if $s_i \leq f_j$, and $s_j \leq f_i$, or vice versa.

Example input, 9 requests:



Request 5 is in conflict with each Request 6 and 7. But is conflict-free with Request 2.

An optimal (maximum-sized) conflict-free set: $\{1, 3, 5, 9\}$. Another: $\{1, 4, 7, 8\}$.

Prove: this problem possesses a greedy choice.

Candidate greedy choice: request with earliest finish time.

Proof strategy: "cut and paste."

For this problem, we prove two claims in order:

Claim 1: *Suppose for some input of n requests, $O = \{o_1, \ldots, o_k\}$ is an optimal (maximum-sized) set of requests which are pairwise conflict-free ordered in increasing finish time. Suppose our greedy algorithm outputs $G = \{g_1, \ldots, g_l\}$, ordered in increasing finish time. Then, it is true that: for every $i = 1, 2, \ldots, l, f(g_i) \leq f(o_i)$.*

Proof. Note: it must be the case that $l \leq k$. And therefore, $k = l$, i.e., greedy is optimal.

Proof by induction on i. Base case: $i = 1$. In our greedy algorithm, we first pick exactly a meeting that finishes earliest amongst all requests. Therefore, immaterial of what o_1 is, $f(g_1) \leq f(o_1)$.

Induction assumption: for $i = j - 1$, it is true that $f(g_i) \leq f(o_i)$.

Step: to prove that $f(g_j) \leq f(o_j)$. We observe:

- $f(o_{j-1}) \leq s(o_j)$ – because the set O is conflict-free requests, ordered in increasing finish, and therefore, start times.
- $f(g_{j-1}) \leq f(o_{j-1})$ – induction assumption.

Practice

- Therefore, $f(g_{j-1}) \leq s(o_j)$. Therefore $f(g_j) \leq f(o_j)$ – because after we greedily choose g_{j-1} and eliminate all requests that are in conflict, o_j still remains. And our greedy choice is exactly to pick a request that remains that finishes earliest, and we happened to pick g_j.

Claim 2: *Given sets O, G as in Claim 1, o_{l+1} cannot exist in O.*

Proof. By Claim 1, $f(g_l) \leq f(o_l)$. And because the O set is all conflict-free, $f(o_l) \leq s(o_{l+1})$. Therefore, $f(g_l) \leq s(o_{l+1})$. So, o_{l+1} not in conflict with g_l, and so was available to be chosen after g_l was chosen and all conflicts were eliminated.

Contradiction to the assumption that greedy algorithm terminates only when no more requests available to choose from.

Practice

Graph Algorithm 1

Given an undirected graph $G = \langle V, E \rangle$ encoded as an adjacency list, define an array $\mathsf{snd}[\cdot]$ as: for each $u \in V, \mathsf{snd}[u]$ is the sum of the degrees of the neighbours of u.

Devise an algorithm that given input G, computes and outputs an array snd.

$SNDStraightForward(G = \langle V, E \rangle)$

1. $snd \leftarrow new\ array\ of\ |V|\ entries$
2. $\boldsymbol{foreach}\ u \in V\ \boldsymbol{do}\ snd[u] \leftarrow 0$
3. $\boldsymbol{foreach}\ u \in V\ \boldsymbol{do}$
4. $\quad \boldsymbol{foreach}\ v \in Adj[u]\ \boldsymbol{do}$
5. $\quad\quad degreev \leftarrow 0$
6. $\quad\quad \boldsymbol{foreach}\ w \in Adj[v]\ \boldsymbol{do}\ degreev \leftarrow degreev + 1$
7. $\quad\quad snd[u] \leftarrow snd[u] + degreev$
8. $\boldsymbol{return}\ snd$

Time efficiency of $SNDStraightForward$: $O(|V| \cdot (|E|)^2)$

Perhaps a better (more efficiency) approach:

- Visit each vertex as though it is someone's neighbor.
- Measure its degree.
- Walk its adj list again and inform each neighbor of the degree so they can update their snd.

$SNDLinearTime(G = \langle V, E \rangle)$

1. $snd \leftarrow new\ array\ of\ |V|\ entries$
2. $\boldsymbol{foreach}\ u \in V\ \boldsymbol{do}\ snd[u] \leftarrow 0$
3. $deg \leftarrow new\ array\ of\ |V|\ entries$
4. $\boldsymbol{foreach}\ u \in V\ \boldsymbol{do}\ \mathrm{deg}[u] \leftarrow 0$
5. $\boldsymbol{foreach}\ u \in V\ \boldsymbol{do}$

5. $\quad \mathrm{deg}[u] \leftarrow 0$

6. $\quad \boldsymbol{foreach}\ v \in Adj[u]\ \boldsymbol{do}\ \mathrm{deg}[u] \leftarrow \mathrm{deg}[u] + 1$
7. $\quad \boldsymbol{foreach}\ v \in Adj[u]\ \boldsymbol{do}\ snd[v] \leftarrow snd[v] + \mathrm{deg}[u]$

8. $\boldsymbol{return}$ snd

Time efficiency:

- We visit each vertex once – Line (4) $foreach$ loop.
- We visit each edge four times – Line (6) and Line (7), we walk each adj list twice.
- So total time: $O(|V| + |E|)$.

Practice

Graph Algorithm 2

Given an undirected graph G as an adjacency list and an edge e in it, devise a linear-time algorithm to determine whether there is a cycle in G that contains e.

“Go-to” linear time algorithms for graphs: DFS and BFS.

Idea:

- DFS, check if back edge results in DFS tree.
- In fact, edit the explore routine as follows:
    - Keep track of parent in DFS tree.
    - Every time we hit a vertex, check if edge to root of DFS tree, and root is not parent in DFS tree.
    - If yes, immediately output **true**.

$HasCycle(G = \langle V, E \rangle, e = \langle u, v \rangle)$

1. $\boldsymbol{foreach}\ u \in V\ \boldsymbol{do}$
2. $\quad visited(u) \leftarrow false$
3. $\quad \pi(u) \leftarrow NIL$
4. $\boldsymbol{return}\ ExploreModified(\langle V, E \rangle, u, u)$

$ExploreModified(\langle V, E \rangle, \langle u, v \rangle, x)$

1. $visited(x) \leftarrow true$
2. $\boldsymbol{foreach}\ y \in Adj[x]\ \boldsymbol{do}$
3. $\quad \boldsymbol{if}\ visited(y) = false\ \boldsymbol{then}$
4. $\quad\quad \boldsymbol{if}\ (x \neq u)\ or\ (x = u\ and\ y = v)\ \boldsymbol{then}$
5. $\quad\quad\quad \pi(y) \leftarrow x$
6. $\quad\quad\quad ret \leftarrow ExploreModified(\langle V, E \rangle, \langle u, v \rangle, y)$
7. $\quad\quad\quad \boldsymbol{if}\ ret = true\ \boldsymbol{then}$
8. $\quad\quad\quad\quad \boldsymbol{return}\ true$
9. $\quad \boldsymbol{else}$
10. $\quad\quad \boldsymbol{if}\ y = r\ and\ \pi(x) \neq u\ \boldsymbol{then}$
11. $\quad\quad\quad \boldsymbol{return}\ true$
12. $\boldsymbol{return}\ false$

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Practice

Proving DAG

Show that the following algorithm to linearize a DAG can be realized in linear time.

> *Find a source, output it, and delete it from the graph.*
>
> *Repeat until the graph is empty.*

We assume adjacency list representation of the input DAG.

Suppose we first create a new array, call it ni of size $|V|$, where $ni[u]$ is the number of edges incident in $u \in V$ at the start. Can do this in one pass of entire adj list of the graph.

From ni, we can identify all sources. Suppose we create a list of source vertices, call it $srclist$. Then, we remove a vertex from $srclist$ and proceed…

```
ni ← new array of size |V|
foreach u ∈ V do ni[u] ← 0
foreach u ∈ V do
    foreach v ∈ Adj[u] do
        ni[v] ← ni[v] + 1
srclist ← new empty linked list
foreach u ∈ V do
    if ni[u] = 0 then Insert u at head of srclist
while srclist is not empty do
    u ← remove vertex from head of srclist
    foreach v ∈ Adj[u] do
        ni[v] ← ni[v] − 1
        if ni[v] = 0 then
            Add v to head of srclist
    Output u
```

Practice

Proving Depth First Search (DFS)

Prove that DFS on an undirected graph can result in no cross edges.

An edge $\langle u, v \rangle$ is a cross edge if and only if: $pre[v] < post[v] < pre[u] < post[u]$.

Suppose a cross edge, $\langle u, v \rangle$ exists after a run of DFS on an undirected graph G.

At the time $post[v]$ and at all times prior since initialization, $visited[u] = false$.

But that means that in the for loop that immediately precedes $postvisit(v)$, we would have invoked $explore(u)$, thereby setting $visited[u]$ to **true** before the time $post[v]$.

Therefore, we have a contradiction.

Practice

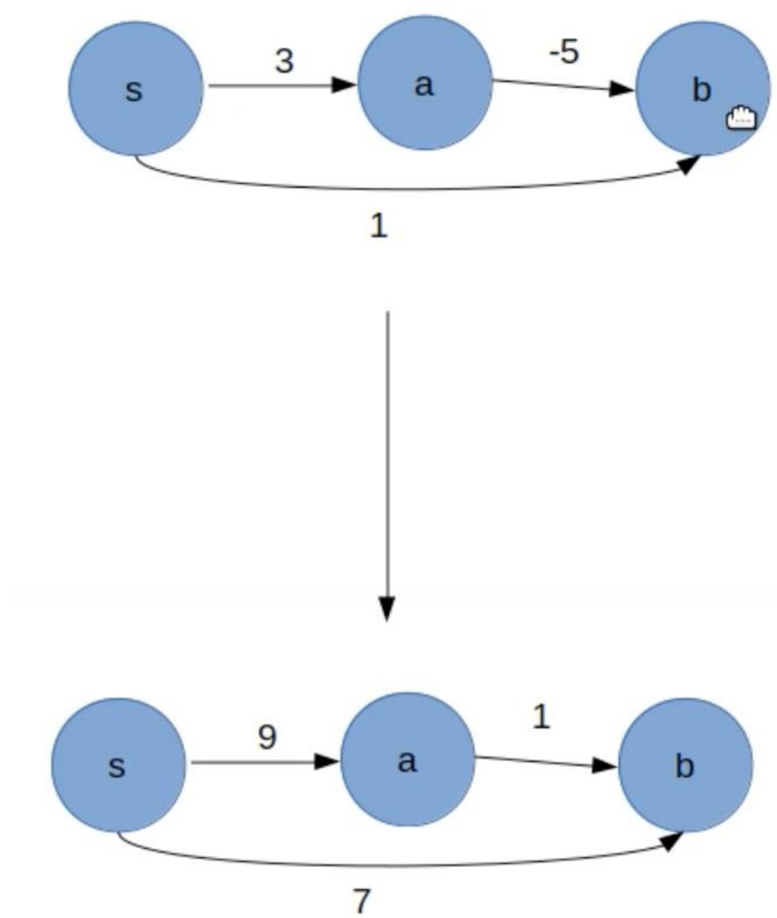
Proving Shortest Path

Professor F. Lake suggests the following algorithm for finding the shortest path from node s to a node t in a directed graph with some negative weight edges: add a large constant to each edge weight so all the weights become positive, then run Dijkstra's algorithm starting at node s, and return the shortest path found to node t.

Is this a valid method? Either prove that it works correctly or give a counterexample.

Directed graph with weights on edges is: $G = \langle V, E, l \rangle$, where $E \subseteq V \times V$, and $l: E \to \mathbb{R}$.

Counterexample, add a constant of 6 to the graph below:



In the unmodified graph, the shortest path is $s \to a \to b$ (-2), but in the modified graph, the shortest path becomes $s \to b$ (7). Since the shortest path changes, this is not a valid method.

Practice

Proving Dijkstra's

Prove: if we initialize $dist(u)$ to ∞, and at the end of a run of Dijkstra's algorithm on $G = \langle V, E, l\rangle$ with source $s \in V$ it is the case that $dist(u) \neq \infty$, then there exists a path $s \rightsquigarrow u$ in G.

Contrapositive: if there exists no path $s \rightsquigarrow u$ in G, then at the end of any run of Dijkstra, $dist(u) = \infty$.

We first observe: the only way $dist(u)$ can change after initialization is via a call $update(e)$ where $e \in E$ is incident on u, i.e., some $\langle v, u\rangle \in E$.

So proof strategy: induction on number of invocations to $update(\cdot)$ that the run of Dijkstra does. Call this number k.

If $k = 0$, then this can only be because $E = \emptyset$. Then, there is no path $s \rightsquigarrow u$. And as we have not changed $dist(u)$ from its initial value, at the end of the run of Dijkstra, $dist(u) = \infty$ as desired.

For the step, we consider two cases.
(i) No edge is incident on u. Then, we know that no $update(\cdot)$ affects $dist(u)$, and therefore $dist(u) = \infty$ as desired.
(ii) There exists some $\langle v, u\rangle \in E$. If the last $update(\cdot)$ we performed is not on any edge incident on u, then $dist(u)$ is the same as it was after $k - 1$ invocations to $update(\cdot)$, and by the induction assumption $dist(u)$ in that case $= \infty$.
The final (sub-)case: the k^{th} update was on some $\langle v, u\rangle$, i.e., edge incident on u. Then there is no path $s \rightsquigarrow v$. Why not? Because if there was, there would be a path to u: $s \rightsquigarrow v \rightarrow u$. And therefore, $dist(v)$ is whatever value it is after $k - 1$ invocations to $update(\cdot)$. And by the induction assumption $dist(v) = \infty$ before $update(v, u)$. Also, again by the induction assumption, $dist(u) = \infty$ before the k^{th} invocation to $update(\cdot)$. Therefore, after the k^{th} invocation, which is $update(v, u)$, $dist(u) = \infty$.

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Practice

Proving Bellman-Ford

Prove: suppose we run Bellman-Ford on $\langle G = \langle V, E, l \rangle, s \in V \rangle$ where we do not know whether G has a negative weight cycle. Also suppose that at the end of that run of Bellman-Ford, we carry out one more $update(e)$ on every $e \in E$. Then: some $dist(u)$ changes in this additional round of updates for some u that is reachable from s if and only if there is a negative weight cycle in G that is reachable from s.

"Only if": we seek to prove: if $dist(u)$ changes, this implies that there is a negative weight cycle.

By Claim (2) of Lecture 5(b): if there exists a shortest path from s to u that is simple, then $|V| - 1$ invocations to $update(\cdot)$ on all edges, as Bellman-Ford does, is sufficient for $dist(u)$ to converge to $\delta(s, u)$. Given that $|V| - 1$ invocations to $update(\cdot)$ on all edges is not sufficient, this can only be because there is a shortest path $s \rightsquigarrow u$ that is not simple. And this in turn is true only if there is a negative cycle reachable from s.

"If": we seek to prove: if there is a negative weight cycle reachable from s, then there exists some u that is reachable from s for which the additional round of $update(\cdot)$ changes $dist(u)$.

An observation: a change to $dist(u)$ has to be a decrease. Because (repeated) invocation(s) to $update(\cdot)$ can only decrease $dist(\cdot)$ value(s).

Suppose $\langle u_o, u_1, \ldots, u_{k-1}, u_o \rangle$ is a negative weight cycle that is reachable from s.

Proof idea: suppose we have a path $\langle u_o, u_1, \ldots, u_{k-1} \rangle$. And we start with some $dist(\cdot)$ value for each u_i. Now suppose the edges in that path have $update(\cdot)$ invoked on them in order. Then, at the end of that round of invocations to $update(\cdot)$, $dist^{(new)}(u_k) \leq \min\{dist(u_k), dist(u_0) + l(u_0, u_1) + \cdots + l(u_{k-2}, u_{k-1})\}$.