ECE 406 COURSE NOTES
ALGORITHM DESIGN AND ANALYSIS

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1 INTRODUCTION AND BASIC ARITHMETIC

1.1 Algorithms, Correctness, Termination, Efficiency

1.1.1 Algorithms

Given the specification for a function, an algorithm is the procedure to compute it.

Example:

$$F: Z_o^+ \to Z_o^+, \text{ where } F(n) = \begin{cases} 0, if\ n = 0 \\ 1, if\ n = 1 \\ F(n-1) + F(n-2), otherwise \end{cases}$$

Commonly used sets: $N, Z(all\ ints), Z^+(positive\ ints), Z_o^+(non-negative\ ints), R, \ldots$

Fibonacci Sequence:

$FIB_1(n)$

$1\ if\ n = 0\ then\ return\ 0$

$2\ if\ n = 1\ then\ return\ 1$

$3\ return\ FIB_1(n-1) + FIB_1(n-2)$

Important aspects:

- Function has been specified as a recurrence, so a recursive algorithm seems natural
- Imperative (procedural) specification of an algorithm has consequences:
    - Intuiting correctness can be a challenge
    - Intuiting time and space efficiency may be easier
- No mundane error checking, can focus on core logic
- Input value n is unbounded but finite

1.1.2 Correctness

Correctness refers to an algorithm's ability to guarantee expected termination. In the case of FIB_1, it is a direct encoding of the recurrence.

1.1.3 Termination

The end of an algorithm. It can be proven that FIB_1 terminates on every input $n \in Z_o^+$ by induction on n.

1.1.4 Time Efficiency

Can be calculated by counting the number of: (i) comparisons – these happen on Lines (1) and (2), and (ii) number of additions – this happens on Line (3).

Suppose $T(n)$ represents the time efficiency of FIB_1:

$$T(n) = \begin{cases} 1, if\ n = 0 \\ 2, if\ n = 1 \\ 3 + T(n-1) + T(n-2), otherwise \end{cases}$$

How bad is $T(n)$? Is it exponential in n?

For all n, $T(n) \geq F(n)$.

1.1.5 Claim 1

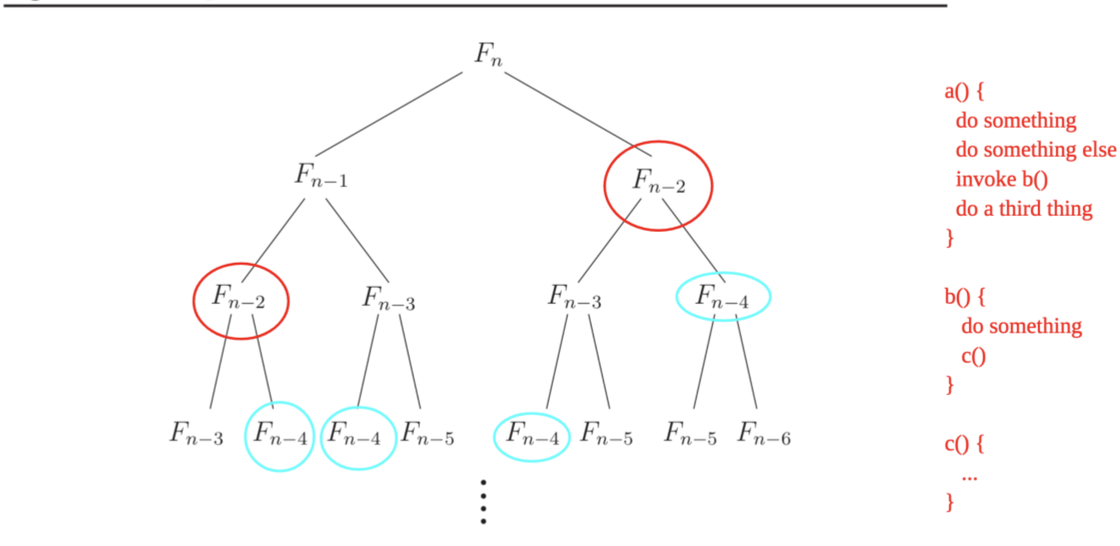
For all $n \epsilon Z_o^+, F(n) \geq \left(\sqrt{2}\right)^n$.

If this claim is true, then $T(n) \geq \left(\sqrt{2}\right)^n$, and because $\sqrt{2} > 1$, $T(n)$ is exponential in n.

Proof for the claim: by induction on n.

Does a better algorithm exist from the standpoint of time efficiency?

Figure 0.1 The proliferation of recursive calls in `fib1`.



Recall how subroutine (recursive, in this case) invocation works:

- Every node in the tree corresponds to an invocation of the algorithm
- Sequence of invocations corresponds to a pre-order traversal
- Maximum depth of the call stack at any moment: n

Main point in this case: Redundancy, F_i, appears more than once.

1.1.6 More Efficient Algorithm

$FIB_2(n)$

$1\ if\ n = 0\ then\ return\ 0$

$2\ create\ an\ array\ f[0, \dots, n]$

$3\ f[0] \leftarrow 0, f[1] \leftarrow 1$

$4\ foreach\ i\ from\ 2\ to\ n\ do$

$5\ f[i] \leftarrow f[i-1] + f[i-2]$

$6\ return\ f[n]$

Let $U(n)$ be the # of comparisons plus additions on input n:

$$U(n) = \begin{cases} 1, if\ n = 0 \\ n, otherwise \end{cases}$$

Linear in n for $n \geq 1$, more efficient than FIB_1.

1.1.7 Note on Measuring Time Efficiency

Need to pick the right level of abstraction, meaning picking some kind of “hot spot” or “hot operation,” then count. For example, number of additions, comparisons, recursive calls, etc.

1.2 Big-O Notation

1.2.1 Definition 1 (O)

Let $f: N \rightarrow R^+$, and $g: N \rightarrow R^+$ be functions. Define $f = O(g)$ if there exists a constant $c \epsilon R^+$ such that $f(n) \leq c \cdot g(n)$.

- $N = Z^+: \{1,2,3, \dots\}, R^+: set\ of\ positive\ real\ numbers$
- Typically consider non-decreasing functions only

1.2.2 Definition 2 (Ω)

Define $f = \Omega(g)$ if $g = O(f)$

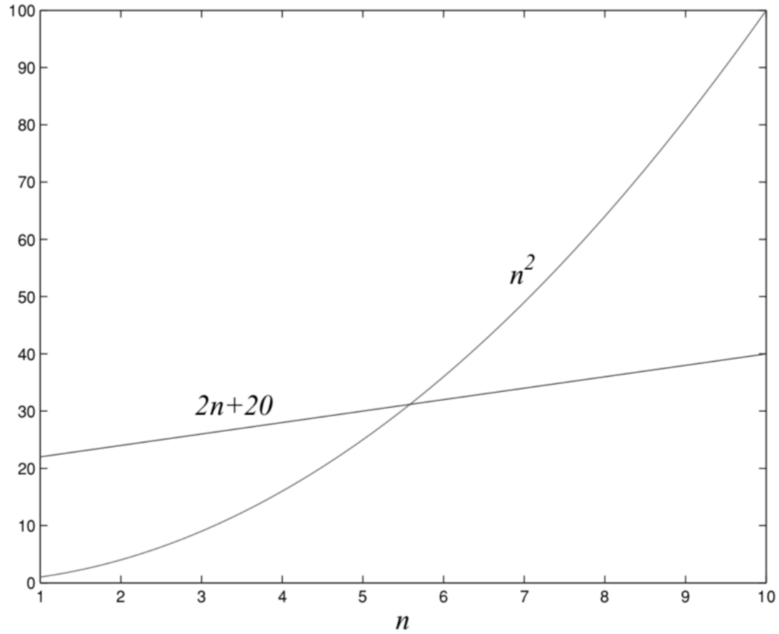
1.2.3 Definition 3 (Θ)

Define $f = \Theta(\text{g})$ if $f = O(g)$ and $f = \Omega(g)$

- $f = O(g)$ analogous to $f \leq g$
- $f = \Omega(g)$ analogous to $f \geq g$
- $f = \Theta(g)$ analogous to $f = g$

1.2.4 Example

Figure 0.2 Which running time is better?



Precise answer to this question: depends on n.

But in big-O notation:

- $2n + 20 = O(n^2)$. Proof: Adopt as the constant $c \epsilon R$ for any $c > 22$
- $2n + 2 \neq \Omega(n^2) \therefore 2n + 2 \neq \Theta(n^2)$

1.2.5 Big-O Explanation

Suppose algorithm A runs in $2n + 20$ time, B in n^2, and C in 2^n. Now suppose the speed of the computer doubles, which algorithm gives the best payoff?

For a given time period t, what is the largest input n each algorithm can handle? Set $runtime = t$ and $runtime = 2t$, solve for n:

Algorithm	**Old Computer**	**New Computer**
A	${}^{t}/_{2} - 10$	$t - 10$
B	$\sqrt{t}$	$\sqrt{2} \cdot \sqrt{t}$
C	$\log_2 t$	$1 + \log_2 t$

So, payoff with algorithm A is approximately $2x$, B is $1.4x$, and C is $1 +$.

1.2.6 Big-O Simplifications

- Multiplicative and additive constants can be omitted
- n^a dominates n^b for $a > b \geq 0$
- Any exponential dominates any polynomial, any polynomial dominates any logarithm
- Big-O simplifications should be used prudently, not applicable in all settings

1.3 Arithmetic

1.3.1 Addition

Hypothesize access to a function $T : \{0, 1\} \times \{0, \dots, 9\} \times \{0, \dots, 9\} \rightarrow \{0, 1\} \times \{0, \dots, 9\}$:

Carry	One Digit	Other Digit	Result Carry	Result Sum
0	0	0	0	0
1	0	0	0	1
0	0	1	0	1
…				
1	8	9	1	8
0	9	9	1	8
1	9	9	1	9

To add 7,814 and 93,404:

$$C \rightarrow 1\ 1\ 1\ 0\ 0\ 0$$

$$1 \rightarrow 0\ 0\ 7\ 8\ 1\ 4$$

$$2 \rightarrow 0\ 9\ 3\ 4\ 0\ 4$$

$$T \rightarrow 1\ 0\ 1\ 2\ 1\ 8$$

\# digits needed to encode $x \in Z^+ = \lfloor \log_{10} x \rfloor + 1$

\# bits needed to encode $x \in Z^+ = \lfloor \log_2 x \rfloor + 1$

So, time efficiency of an algorithm to add $x, y \in Z^+$ as measured by number of lookups to T:

- $1 + \lfloor \log_{10}(\max\{x, y\}) \rfloor$ in the best case
- $2 + \lfloor \log_{10}(\max\{x, y\}) \rfloor$ in the worst case
- So, either way, $\Theta(n)$, or linear time, where n is the size of the input

1.3.2 Multiplication

For $x, y \in Z_o^+$, encoded in binary:

$$x \times y = \begin{cases} 0, if\ y = 0 \\ 2\left(x \times {}^{y}/_{2}\right), if\ y\ even, y > 0 \\ x + 2\left(x \times \lfloor {}^{y}/_{2} \rfloor\right), otherwise \end{cases}$$

Straightforward encoding as recursive algorithm $MULTIPLY(x, y)$.

Figure 1.1 Multiplication à la Français.

```
function multiply(x, y)
Input: Two n-bit integers x and y, where y ≥ 0
Output: Their product

if y = 0: return 0
z = multiply(x, ⌊y/2⌋)
if y is even:
   return 2z
else:
   return x + 2z
```

Worst case running time:

- $Let\ \#\ bits\ to\ encode\ each\ of\ x\ and\ y = n$
- $\#\ recursive\ calls = \Theta(\text{n})$
- $In\ each\ call$:
    - One comparison to 0, one division by 2 (right bit shift), one assignment to z, one check for evenness (check LSB), one multiplication by 2 (left bit shift), one addition of $O(n)$-bit numbers
- So, $O(n^2)$ in the worst case

1.3.3 Division

Definition 1: Given $x \in Z_o^+, y \in Z^+$, the pair $\langle q, r \rangle$ where $q \in Z_o^+$, $r \in \{0, 1, \dots, y-1\}$ of x divided by y are those that satisfy:

$$x = q \cdot y + r$$

Claim 1: For every $x \in Z_o^+$, $y \in Z^+$, $\langle q, r \rangle$ as defined above (i) exists, and (ii) is unique.

To specify a recurrence for $\langle q, r \rangle$, denote as $\langle q', r' \rangle$, the result of $\lfloor {}^{x}/_{2} \rfloor$ divided by y. Now:

$$\langle q, r \rangle = \begin{cases} \langle 0, 0 \rangle, if\ x = 0 \\ \langle 2q', 2r' \rangle, if\ x\ even\ and\ 2r' < y \\ \langle 2q', 2r' + 1 \rangle, if\ x\ odd\ and\ 2r' + 1 < y \\ \langle 2q' + 1, 2r' - y \rangle, if\ x\ even\ and\ 2r' \geq y \\ \langle 2q' + 1, 2r' + 1 - y \rangle, otherwise \end{cases}$$

Claim 2: The above recurrence is correct.

Proof: Cases are exhaustive. Proof by case-analysis and induction on # bits to encode x.

By induction assumption: $0 \leq r' \leq y - 1 \therefore 0 \leq 2r' \leq 2y - 2$.

Figure 1.2 Division.

```
function divide(x, y)
Input: Two n-bit integers x and y, where y ≥ 1
Output: The quotient and remainder of x divided by y

if x = 0: return (q, r) = (0, 0)
(q, r) = divide(⌊x/2⌋, y)
q = 2 · q,  r = 2 · r
if x is odd: r = r + 1
if r ≥ y: r = r − y,  q = q + 1
return (q, r)
```

Running time: $O(n^2)$.

2 ALGORITHMS WITH NUMBERS

2.1 Modular Arithmetic

- **Definition 1:** $For\ x \in Z, N \in Z^+, x\ modulo\ N\ is\ the\ remainder\ of\ x\ divided\ by\ N$
- **Definition 2:** $x \equiv y\ (mod\ N)\ if\ N\ divides\ x - y.$ "≡" read as "$congruent\ to$"
  - Example: $373 \equiv 13\ (mod\ 60), 59 \equiv -1\ (mod\ 60)$

2.1.1 Example Application: Two's Complement Arithmetic

Suppose we want to represent, using n bits, positive and negative integers, and 0. Could reserve 1 bit for sign. This would allow us to represent integers in the interval $[-(2^{n-1} - 1), 2^{n-1} - 1]$, with a "positive zero" and a "negative zero."

In two's complement arithmetic, we have exactly one bit-string for 0, and represent integers in the interval $[-2^{n-1}, 2^{n-1}-1]$. How? Represent any $x \epsilon [-2^{n-1}, 2^{n-1}-1] \cap Z$ as the non-negative integer modulo 2^n. So:

$$0 \leq x \leq 2^{n-1} - 1 \rightarrow x \text{ is represented as } x$$

- Ex: $For\ n = 5, (9)_{10}\ written\ as\ (01001)_2$

$$-2^{n-1} \leq x < 0 \rightarrow x \text{ is represented as } 2^n + x$$

- Ex: $For\ n = 5, (-9)_{10} \equiv 32 + (-9) \equiv 23\ (mod\ 32), written\ as\ (10111)_2$

All arithmetic performed modulo 2^n:

- Ex: $For\ n = 5, 13 + (-7) \equiv 01101 + 11001 \equiv 100110 \equiv 00110 \equiv 6\ (mod\ 32)$

Claim 1: $x \equiv x', y \equiv y' (mod\ N)\ implies: x + y \equiv x' + y', xy \equiv x'y' (mod\ N)$

Claim 2:

$$\begin{aligned} x + (y + z) &\equiv (x + y) + z && \text{(mod N)} \\ xy &\equiv yx && \text{(mod N)} \\ x(y + z) &\equiv xy + xz && \text{(mod N)} \end{aligned}$$

Example: $2^{3045} \equiv (2^5)^{609} \equiv (1)^{609} \equiv 1\ (mod\ 31)$

2.1.2 Modular Addition, Subtraction

$$x + y \equiv \begin{cases} x + y\ (mod\ N), if\ 0 \leq x + y < N \\ x + y - N\ (mod\ N), otherwise \end{cases}$$

$$x - y \equiv \begin{cases} x - y\ (mod\ N), if\ 0 \leq x - y < N \\ x - y + N\ (mod\ N), otherwise \end{cases}$$

- Any intermediate result is between $-(N-1)$ and $2(N-1)$
- So, time efficiency is $O(n)$, where $n = \lceil \log N \rceil$

2.1.3 Modular Multiplication

Let $MULT$ and DIV be our algorithms for non-modular multiplication and division:

$$x \times y \equiv r\ (mod\ N), where\ \langle q, r \rangle = DIV(MULT(x, y), N)$$

- Any intermediate result (specifically, result of $MULT$) is between 0 and $(N-1)^2$
- So, time efficiency is $O(n^2)$, where $n = \lceil \log N \rceil$

2.1.4 Modular Exponentiation

- Recall: We used "repeated doubling" for non-modular multiplication and "repeated halving" for non-modular division
- Similarly, here, use "repeated squaring":

$$x^y = \begin{cases} 1, if\ y = 0 \\ (x^2)^{\lfloor y/2 \rfloor}, if\ y\ is\ even \\ x \cdot (x^2)^{\lfloor y/2 \rfloor}, otherwise \end{cases}$$

Figure 1.4 Modular exponentiation.

```
function modexp(x, y, N)
Input: Two n-bit integers x and N, an integer exponent y
Output: x^y mod N

if y = 0: return 1
z = modexp(x, ⌊y/2⌋, N)
if y is even:
   return z^2 mod N
else:
   return x · z^2 mod N
```

$Time\ Efficiency: O(n^3)$

2.1.5 Towards Modular Division: GCD Using Euclid

- In a non-modular world, $a/b = a \times b^{-1}$. Only case for which b^{-1} doesn't exist: $b = 0$
- In a modular world, $b^{-1}\ (mod\ N)$ may not exist even if $b \not\equiv 0\ (mod\ N)$

Crucial building block: GCD. Example: What is $\gcd(1035, 759)$?

$1035 = 3^2 \cdot 5 \cdot 23 \rightarrow 759 = 3 \cdot 11 \cdot 23 \rightarrow \gcd(1035, 759) = 3 \cdot 23$

But factoring into prime factors conjectured to be computationally hard in the worse case

Claim 3: $x, y \in \mathbb{Z}^+, x \geq y \rightarrow \gcd(x, y) = \gcd(x\ mod\ y, y)$

Proof: Suffices to prove: $\gcd(x, y) = \gcd(x - y, y)$. Now prove $\leq$ and $\geq$.

Figure 1.5 Euclid's algorithm for finding the greatest common divisor of two numbers.

```
function Euclid(a, b)
Input: Two integers a and b with a ≥ b ≥ 0
Output: gcd(a, b)

if b = 0: return a
return Euclid(b, a mod b)
```

How fast does *Euclid* converge?

Claim 4: $a \geq b \geq 0 \rightarrow a \bmod b < a/2$

So: Guaranteed to lose at least 1 bit for every recursive call → time efficiency is $O(n^3)$

2.1.6 Towards Modular Division: Extended Euclid

Claim 5: d *divides* a *and* b, *and* $d = ax + by$ *for some* $x, y \in \mathbb{Z} \rightarrow \mathrm{d} = \gcd(\mathrm{a}, \mathrm{b})$

Claim 6: *Let* $d = \gcd(a, b)$, $d = ax + by$ *and* $d = bx' + (a \bmod b)y'$ *for some* $x, y, x', y' \in \mathbb{Z}$. *Then*:

$$\langle x, y \rangle = \begin{cases} \langle 1, 0 \rangle, if\ b = 0 \\ \langle y', x' - \lfloor a/b \rfloor y' \rangle, otherwise \end{cases}$$

Figure 1.6 A simple extension of Euclid's algorithm.

```
function extended-Euclid(a, b)
Input: Two positive integers a and b with a ≥ b ≥ 0
Output: Integers x, y, d such that d = gcd(a, b) and ax + by = d

if b = 0: return (1, 0, a)
(x', y', d) = extended-Euclid(b, a mod b)
return (y', x' − ⌊a/b⌋y', d)
```

Example run on input $\langle 359, 82 \rangle$:

Arguments	**Return Value**
$\langle 359, 82 \rangle$	$\langle -37, 162, 1 \rangle$
$\langle 82, 31 \rangle$	$\langle 14, -37, 1 \rangle$
$\langle 31, 20 \rangle$	$\langle -9, 14, 1 \rangle$

$\langle 20, 11\rangle$	$\langle 5, -9, 1\rangle$
$\langle 11, 9\rangle$	$\langle -4, 5, 1\rangle$
$\langle 9, 2\rangle$	$\langle 1, -4, 1\rangle$
$\langle 2, 1\rangle$	$\langle 0, 1, 1\rangle$
$\langle 1, 0\rangle$	$\langle 1, 0, 1\rangle$

A good example to segue to our final step in modular division.

We figured out: $\gcd(359,82) = 1$, which implies:

- $82 \times 162 \equiv 1\ (mod\ 359)$. So: 162 is multiplicative inverse of 82 $modulo$ 359
- So, for example: 116 $divided\ by$ 82 $modulo$ 359 $\equiv 116 \times 162 = 124\ (mod\ 359)$

2.1.7 Modular Division

Definition 3: $x\ is\ the\ multiplicative\ inverse\ of\ a\ modulo\ N\ if\ ax \equiv 1\ (mod\ N)$

Claim 7: $For\ every\ a\ \epsilon\ \{0, \dots, N-1\}, there\ exists\ at\ most\ a^{-1}\ (mod\ N)$

Claim 8: $Given\ \langle a, N\rangle, where\ a\ \epsilon\ \{0, \dots, N-1\}, a^{-1}\ (mod\ N)\ may\ not\ exist$

Definition 4: $If\ \gcd(x, y) = 1, then\ we\ say\ that\ x\ is\ relatively\ prime\ to\ y$

Claim 9: $a^{-1}\ (mod\ N)\ exists\ if\ and\ only\ if\ a\ and\ N\ are\ relatively\ prime$

So, to compute $a/b\ (mod\ N)$:

(i) Determine whether $\gcd(b, N) = 1$

(ii) If yes to (i) determine $b^{-1}\ (mod\ N)$, and

(iii) If yes to (i) compute $a \times b^{-1}\ (mod\ N)$

(i) and (ii) are done simultaneously by $extended - Euclid$

Running Time: $(i) + (ii) = O(n^3), (iii) = O(n^2). So\ (i) + (ii) + (iii) = O(n^3)$

2.2 Primality Testing

Given $n\ \epsilon\ \mathbb{Z}^+$, is n prime?

For a decision problem, i.e., co-domain of function to be computed as $\{true, false\}$, a randomized algorithm:

- Has access to an unbiased coin
- Is deemed to be correct if:
  - $\Pr\{Algorithm\ outputs\ false\ |\ input\ instance\ is\ false\} = 1$
  - $\Pr\{Algorithm\ outputs\ true |\ input\ instance\ is\ true\} \geq {}^{1}/_{2}$

Suppose:

- We run such an algorithm k times, pairwise independently
- We return $true$ if and only if every run returns $true$
- Then, $\Pr\{we\ return\ true\ incorrectly\} \leq 2^{-k}$

2.2.1 Fermat's Little Theorem

Claim 1: $p\ prime \rightarrow for\ all\ a \in [1, p) \cap \mathbb{Z}, a^{p-1} \equiv 1\ (mod\ p)$

To prove Fermat's little theorem, leverage the following:

Claim 2: $p\ prime, a, i, j \in \{1, 2, \ldots, p-1\}\ and\ i \neq j \rightarrow a \cdot i \not\equiv a \cdot j\ (mod\ p)$

Proof for Claim 2: We know that a, p are relatively prime, so $a^{-1}\ (mod\ p)$ exists.

$$a \cdot i \equiv a \cdot j\ (mod\ p) \rightarrow a \cdot i \cdot a^{-1} \equiv a \cdot j \cdot a^{-1}\ (mod\ p) \rightarrow i \equiv j\ (mod\ p) \rightarrow i = j$$

Proof for Claim 1: From Claim 2:

$$\{1, 2, \ldots, p-1\} = \{a \cdot 1\ mod\ p, a \cdot 2\ mod\ p, \ldots, a \cdot (p-1)\ mod\ p\}$$

Also, $(p-1)!^{-1}\ (mod\ p)$ exists.

So, $(p-1)! \equiv a^{p-1} \cdot (p-1)!\ (mod\ p) \rightarrow a^{p-1} \equiv 1\ (mod\ p)$

Figure 1.7 An algorithm for testing primality.

```
function primality(N)
Input: Positive integer N
Output: yes/no

Pick a positive integer a < N at random
if a^(N-1) ≡ 1 (mod N):
   return yes
else:
   return no
```

Issues with the algorithm:

1. Fermat's little theorem is not an "if and only if": Carmichael numbers.
2. Suppose N is not prime/Carmichael. For a chosen, say, uniformly from $\{1, \ldots, N-1\}$, what is $\Pr\{a^{N-1} \not\equiv 1\ (mod\ N)\}$?
   - We know such an a exists, but how likely is it that we will pick it?

We do not deal with (1) – cop out: Carmichael numbers are rare.

Claim 3: If $a^{N-1} \not\equiv 1\ (mod\ N)$ for a, N relatively prime, then it must hold for at least half the choices $a \in \{1, \ldots, N-1\}$.

Proof: If there exists no $b \in \{1, \ldots, N-1\}$ with $b^{N-1} \equiv 1\ (mod\ N)$, then we are done.

If such a b exists, then $(b \cdot a)^{N-1} \not\equiv 1\ (mod\ N)$.

Also, if b, c exist with $b \neq c$, $b^{N-1} \equiv c^{N-1} \equiv 1\ (mod\ N)$, then:

$$b \cdot a \not\equiv c \cdot a\ (mod\ N)$$

So, at least as many $\not\equiv 1\ (mod\ N)$ as there are $\equiv 1\ (mod\ N)$.

So:

$$\Pr\{Algorithm\ 1.7\ returns\ yes\ when\ N\ is\ prime/Carmichael\} = 1$$

$$\Pr\{Algorithm\ 1.7\ returns\ yes\ when\ N\ is\ not\ prime/Carmichael\}, < 1/2$$

For k runs of $Algorithm$ 1.7 on uniform, independent choices of a:

$$\Pr\{Algorithm\ 1.7\ returns\ yes\ on\ all\ k\ runs\ when\ N\ is\ not\ prime/Carmichael\} \leq 2^{-k}$$

2.3 Generating an n-Bit Prime

Claim 4: $\Pr\{uniformly\ chosen\ n-bit\ number\ is\ prime\} \approx 1/n.$

So, algorithm for generating a prime:

1. Randomly generate n-bit number, r.
2. Check whether r is prime.
3. If not, go to Step (1).

Guaranteed return in Step (2) if r is indeed prime.

Each trial in the above algorithm is a Bernoulli trial:

- Only one of two outcomes: success or failure

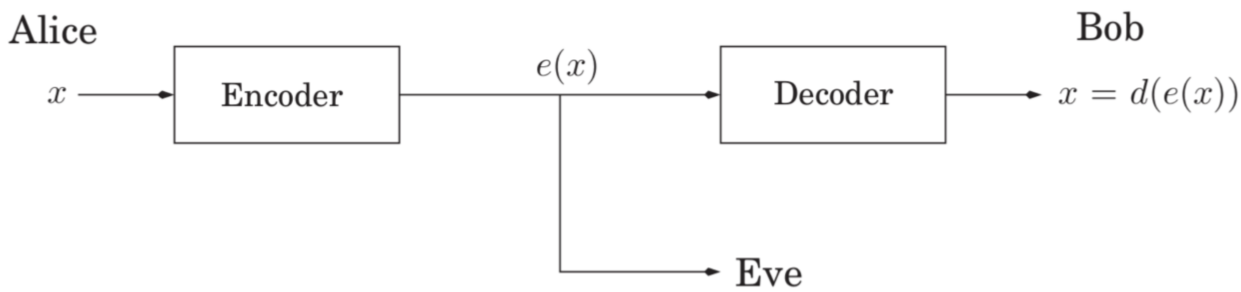
In a Bernoulli trial if $\Pr\{success\} = p$, then $expected\ \#\ trials\ to\ see\ a\ success = 1/p$

- Example: $Expected\ \#\ tosses\ of\ fair\ coin\ to\ see\ a\ heads = 2$
- Example: $Expected\ \#\ tosses\ of\ fair\ die\ to\ see, say, a\ 3 = 6$

So, we expect the above algorithm to halt in n iterations.

2.4 Cryptography

RSA: Exploits presumed computational hardness of factoring vs. computational ease of GCD, primality testing and modular exponentiation.



Symmetric Key Cryptography:

- $e(\cdot) = d(\cdot) \rightarrow Alice\ and\ Bob\ both\ know\ e(\cdot)\ and\ d(\cdot)$
    - Example: $e(x) = x \oplus r, d(y) = y \oplus r$
- They keep these secrets from everyone else
- Bootstrapping Problem: How do Alice and Bob share $e(\cdot)$, $d(\cdot)$?

Public Key Cryptography:

- Bob publishes $e(\cdot)$ to the whole world
- Bob keeps $d(\cdot)$ to himself
- RSA is an example of a public key cryptography scheme

2.4.1 Claim 1

Let p, q be primes and $N = pq$. For any e relatively prime to $(p-1)(q-1)$:

1. The function $f: \{0, 1, \dots, N-1\} \rightarrow \{0, 1, \dots, N-1\}$ where $f(x) = x^e\ mod\ N$ is a bijection
2. Let $d = e^{-1}\ (mod\ \ (p-1)(q-1))$. Then, $(x^e)^d \equiv x\ (mod\ N)$

Bob publishes the pair $\langle N, e \rangle$. Alice encodes message x as $f(x)$ from the claim.

Bob keeps the d in the claim secret to themselves.

2.4.2 Example

$p = 11, q = 17$. Then, $N = pq = 187$. The only messages that can be sent: $\{0, 1, \dots, 186\}$.

We could pick $e = 7$: $\gcd(10 \times 16, 7) = 1$

Then, $d = 23$: $7^{-1} \equiv 23\ (mod\ 160)$.

To send the message 98, Alice would send $98^7\ mod\ 187 = 21$.

Bob would decode the message as $21^{23}\ mod\ 187 = 98$.

2.4.3 Attacks

Attacker knows: (i) $\langle N, e\rangle$, (ii) $x^e \ mod\ N$.

- Attack 1: Attacker determines x given (i) and (ii).
- Attack 2: Even more devastating – attacker factors $N = p \cdot q$.
  - They can then compute $d = e^{-1}\ (mod\ (p-1)(q-1))$.

2.4.4 Proof for Claim 1

Property #2 implies Property #1.

To prove Property #2:

Because $ed \equiv 1\ \big(mod\ (p-1)(q-1)\big), ed = 1 + k(p-1)(q-1)$ for some $k \in \mathbb{Z}$.

We seek to show: $x^{ed} - x = x^{1+k(p-1)(q-1)} - x$ is divisible by N, and is therefore $\equiv 0\ mod\ N$.

Now, by Fermat's little theorem: $x \cdot (x^{p-1})^{k(q-1)} - x \equiv 0\ mod\ p$.

And again, by Fermat's little theorem: $x \cdot (x^{q-1})^{k(p-1)} - x \equiv 0\ mod\ q$.

So: $x^{ed} - x$ is divisible by the product of the two primes $pq = N$.

Figure 1.9 RSA.

Bob chooses his public and secret keys.

- He starts by picking two large (n-bit) random primes p and q.
- His public key is (N, e) where $N = pq$ and e is a $2n$-bit number relatively prime to $(p-1)(q-1)$. A common choice is $e = 3$ because it permits fast encoding.
- His secret key is d, the inverse of e modulo $(p-1)(q-1)$, computed using the extended Euclid algorithm.

Alice wishes to send message x to Bob.

- She looks up his public key (N, e) and sends him $y = (x^e \bmod N)$, computed using an efficient modular exponentiation algorithm.
- He decodes the message by computing $y^d \bmod N$.

Example:

```
$ ssh-keygen -t rsa
...
$ cat id_rsa.pub
ssh-rsa AAAAB3NzaC1yc2EAAAADAQABAAABAQDJAs5HIayjHG
LdvEeiaRI2R3TG8+chfGYrjEWc82bV3ndC87+dYAFVXyVDDc2C
OvDHY6cNcN4vpjc0fZbieeJWCOwjFV8qt5VZDTdvtLJSBilH1j
lJI6FoGBjwMjqoDsXR0n3e7rundqr0xLsk6RoIVunhluloj2Ss
L2fwU7/pbhrvWBBx1jP6aaCkW5sAEu143xM71C2bAMqzoS47WY
+xH91sgm8hwji/KUEoHeeVMrc54bTsozPQp4t+3QwjDfqEMeTy
BvJ93ZTZFHJiQVORTnw3x8HyzNgYTDZVnFJi6kWx8suggcokgz
ffAM+7xNJlzDmJlbY3N+WHuRDsvFpf me@localhost
```