Complex Networks
percolation

2017.12.21(Thu)

Goal

metrics	algorithms
models	processes

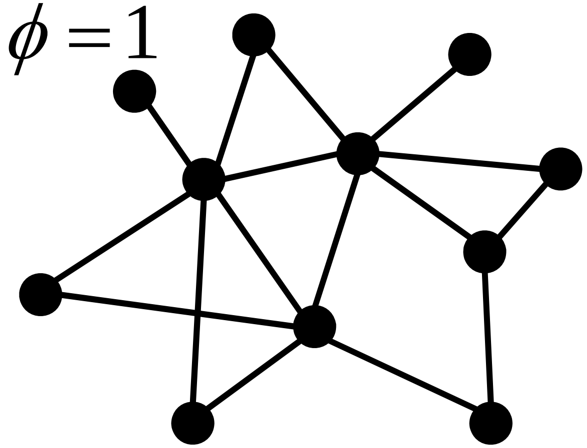
contents of this chapter

- Percolation : one of the simplest processes taking place on networks

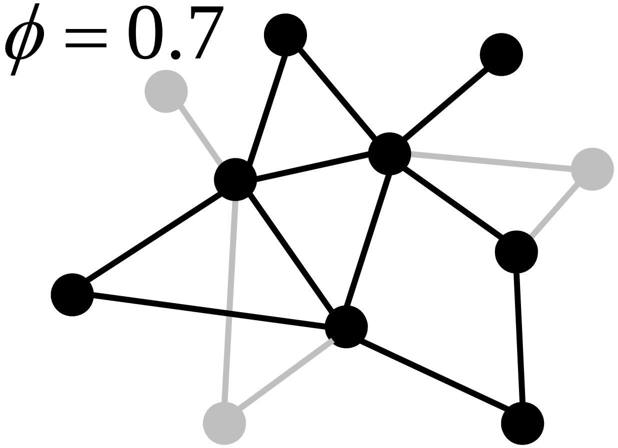
Processes on networks

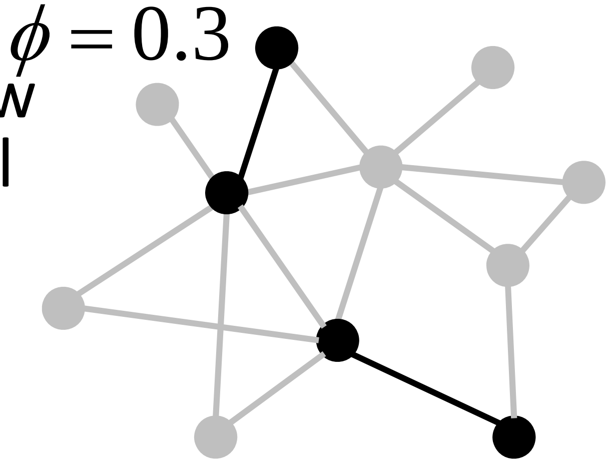
- To make the connection between network structure and function
  - Network failure and resilience
  - Dynamic systems on networks
  - Epidemic and spreading processes

percolation



- Removing some vertices in a network
  - failure of routers on the Internet
  - vaccination / immunization against the spread of disease
- “knock-on” effect
  - vaccination of small fraction of the population can effectively prevent the spread of disease
  - “herd immunity”
- Percolation theory : to understand how the knock-on effects of vertex removal affect the network as a whole





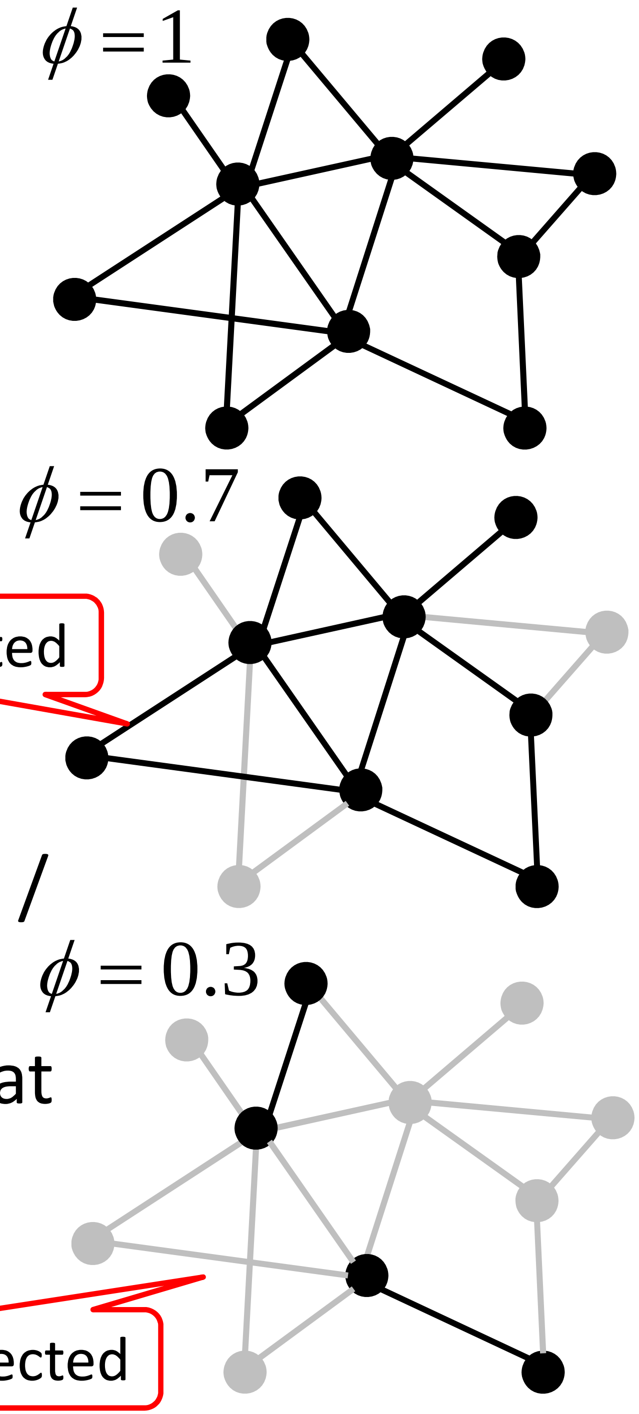
vertex / edge removal

- site percolation : vertex removal ⇦
- bond percolation : edge removal

- random removal ⇦
- removal from highest degrees
- removal of highest betweenness

Uniform random removal of vertices

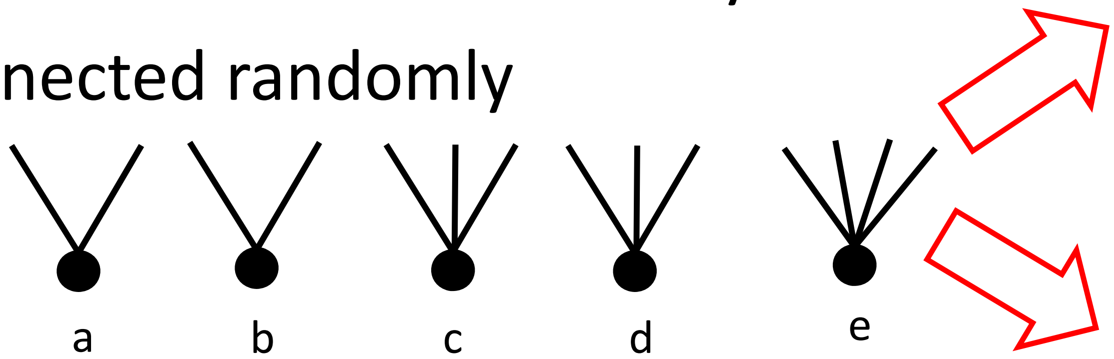
- ϕ: probability that a vertex is present (occupation probability)
  - $\phi = 1$: no vertices have been removed
  - $\phi = 0$: all vertices have been removed
- Percolation transition : formation / dissolution of a giant component
- Percolation threshold : the point at which the transition occurs

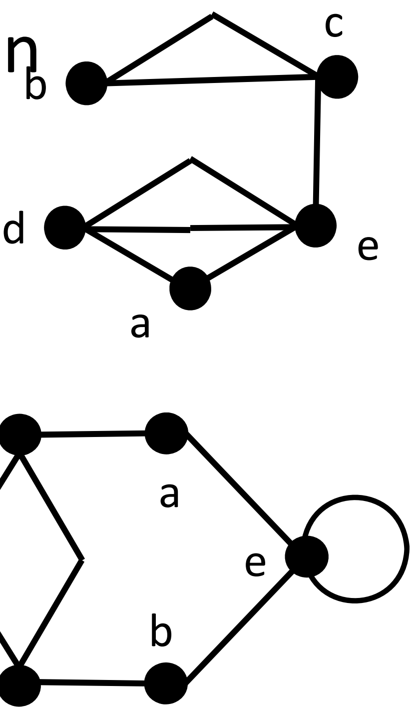


configuration model

- Degree distribution of a naïve random graph is Poisson distribution
- Configuration model: a method for generating networks of arbitrary degree distribution
  - "stumps" are prepared according to given degree distribution and they are connected randomly

v	deg
a	2
b	2
c	3
d	3
e	4



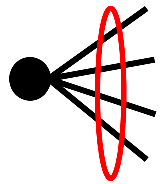


Uniform removal in the configuration model (1)

- Site percolation process on networks generated using the configuration model
- Configuration model network with degree distribution : p_k
- Occupation probability : ϕ
- Size of the giant cluster S

removed $(1-\phi)$ or present but not connected to GC via its neighbors

  - k: degree
  - u : average probability that the vertex is not connected to the giant cluster via a particular neighbor
  - u^k: probability of its not belonging to the giant cluster
  - $\sum_k p_k u^k (= g_0(u))$: average probability of not being in the giant cluster
  - $1 - g_0(u)$: average probability of being in the giant cluster
  - $S = \phi[1 - g_0(u)]$: total fraction of the giant cluster (because ϕ is the probability of remaining nodes



k

$$g_0(z) = \sum_{k=0}^{n} p_k z^k$$:generating function

Uniform removal of the configuration model (2)

All neighbors of vertex A are not connected to GC

- Total probability of not connecting to the giant component via vertex A is $1 - \phi + \phi u^k$

vertex A is removed

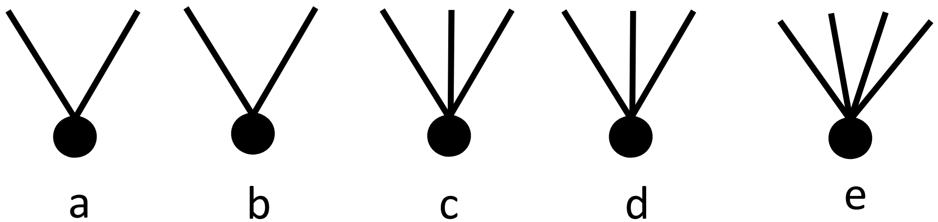
- Excess degree distribution
  - Probability of an edge reaches to a vertex of degree k (other than the edge we arrived along)
  - $q_k = \frac{(k+1)p_{k+1}}{\langle k \rangle}$
- Average probability that a vertex is not connected to the giant component :
  $u = \sum_{k=0}^{\infty} q_k \left(1 - \phi + \phi u^k\right) = 1 - \phi + \phi \sum_{k=0}^{\infty} q_k u^k$
- $= 1 - \phi + \phi g_1(u)$

$$g_1(z) = \sum_{k=0}^{\infty} q_k z^k$$:generating function for the excess degree distribution

Excess degree distribution (p.445)

- Configuration model with degree distribution p_k when vertices are chosen randomly
- If we take a vertex and follow one of its edges to the vertex at the other end, what is the probability that this vertex will have degree k?
  - It is not p_k(because a vertex of degree zero will not be reached)
  - Probability of selecting an edge ending at any particular vertex of degree k : $k/(2m-1) \approx k/2m$
  - There are np_k such vertices → probability of an edge attaching to any vertex with degree k is $\frac{k}{2m} \times np_k = \frac{kp_k}{\langle k \rangle}$
  - The probability is proportional not to p_k but to kp_k

high degree is more likely



2m edges

$$\langle k \rangle = \frac{2m}{n}$$

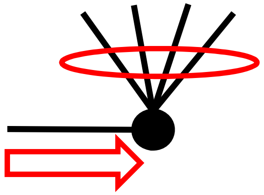
Your friends have more friends than you do

- Average degree of a neighbor :$\sum_k k \frac{k p_k}{\langle k \rangle} = \frac{\langle k^2 \rangle}{\langle k \rangle}$
- Average degree : $\langle k \rangle = \frac{2m}{n}$
- $\frac{\langle k^2 \rangle}{\langle k \rangle} - \langle k \rangle = \frac{1}{\langle k \rangle}(\langle k^2 \rangle - \langle k \rangle^2) = \frac{\sigma_k^2}{\langle k \rangle} > 0$

Network	n	Average degree	Average neighbor degree	$\frac{\langle k^2 \rangle}{\langle k \rangle}$
Biologists	1520252	15.5	68.4	130.2
Mathematicians	253339	3.9	9.5	13.2
Internet	22963	4.2	224.3	261.5

a vertex of degree k appears as one of the neighbors of exactly k other vertices -> overrepresented

Excess degree



- The number of edges attached to a vertex other than the edge we arrived along
- One less than the total degree
- $q_k = \frac{(k+1)p_{k+1}}{\langle k \rangle}$: excess degree distribution

denominator is still just <k> ($\sum_{k=0}^{\infty} q_k = 1$)

Generating functions for degree distributions (p.450)

- $g_0(z) = \sum_{k=0}^{\infty} p_k z^k$
- $g_1(z) = \sum_{k=0}^{\infty} q_k z^k$

These are not independent

- $q_k = \frac{(k+1)p_{k+1}}{\langle k \rangle}$
- $g_1(z) = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} (k+1) p_{k+1} z^k = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} k p_k z^{k-1} = \frac{1}{\langle k \rangle} \frac{dg_0}{dz}$
- $\langle k \rangle = g_0'(1)$
- $g_1(z) = \frac{g_0'(z)}{g_0'(1)}$

Graphical solution of function u

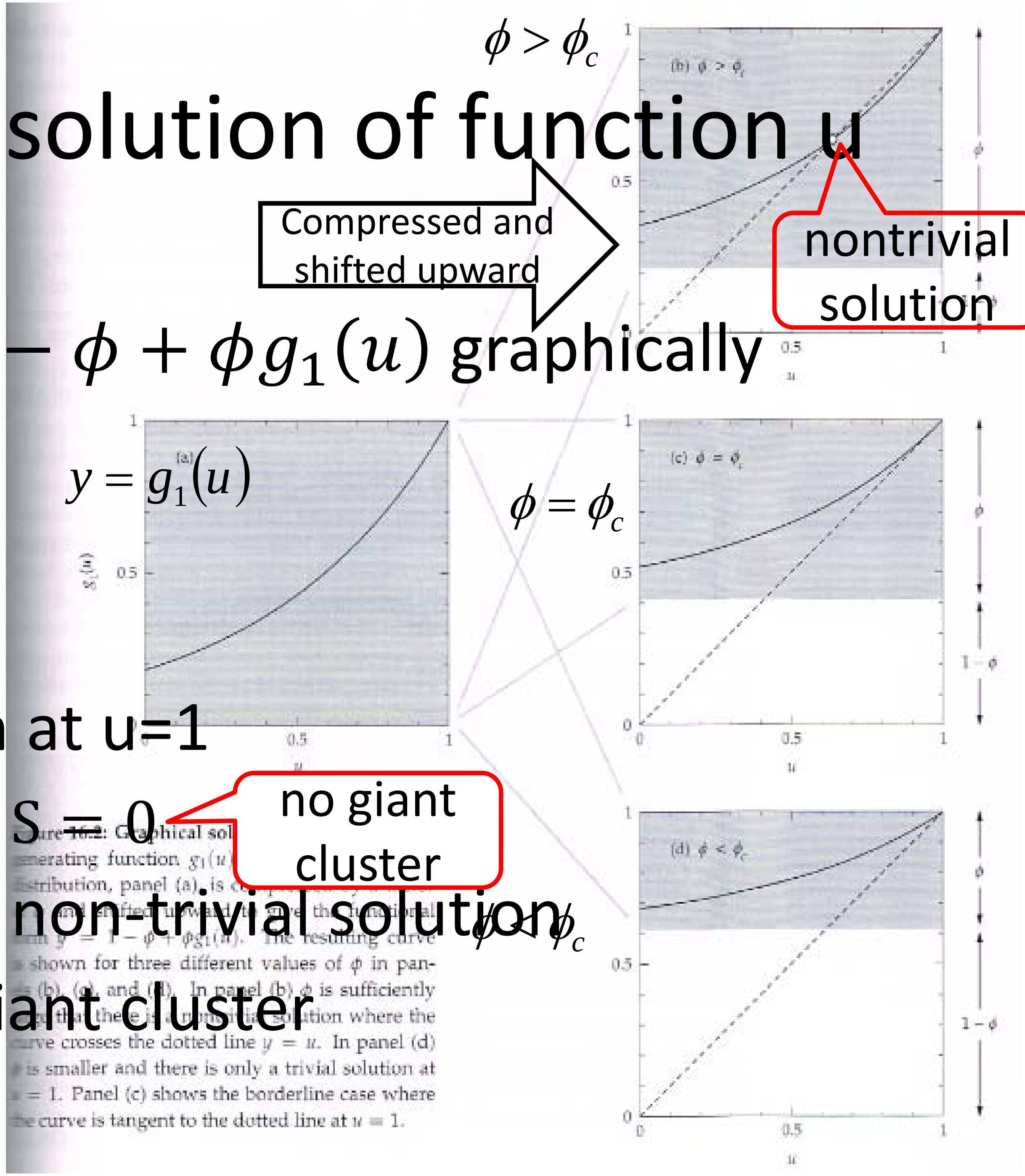
- $S = \phi[1 - g_0(u)]$ size of giant component
- $g_0(z) = \sum_{k=0}^{\infty} p_k z^k$ generating function for degree distribution
- $u = 1 - \phi + \phi g_1(u)$ total probability of not connecting to the giant component via vertex A
- $g_1(u) = \sum_{k=0}^{\infty} q_k u^k$ generating function for excess degree distribution
- $q_k = \frac{(k+1)p_{k+1}}{\langle k \rangle}$ excess degree distribution

Graphical solution of function u

- Solving $u = 1 - \phi + \phi g_1(u)$ graphically

$$\begin{cases} y = 1 - \phi + \phi g_1(u) \\ y = u \end{cases}$$

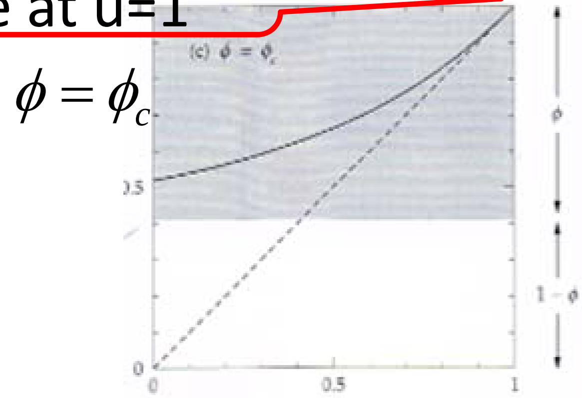
- Trivial solution at u=1
  - $g_1(1) = 1 \rightarrow S = 0$
- When there is non-trivial solution there can be a giant cluster



Critical occupation probability

curve is tangent to the dotted line at u=1

- Percolation threshold



- $\left[\frac{d}{du}\left(1-\phi+\phi g_1(u)\right)\right]_{u=1} = 1$
  - $\phi_c = \frac{1}{g_1'(1)}$
  - $g_1'(1) = \frac{1}{\langle k\rangle}\sum_{k=0}^{\infty} k(k+1)p_{k+1} = \frac{1}{\langle k\rangle}\sum_{k=0}^{\infty} k(k-1)p_k$

$$= \frac{\langle k^2\rangle - \langle k\rangle}{\langle k\rangle}$$

- $\phi_c = \frac{\langle k\rangle}{\langle k^2\rangle - \langle k\rangle}$: minimum fraction of vertices that must be present in the configuration model network for a giant component to exist

if $\langle k^2\rangle \gg \langle k\rangle$, then ϕ_c is low and the network will have a giant cluster

Giant cluster of Poisson/power-law distribution

- $\phi_c = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$
- Poisson distribution :$p_k = e^{-c} \frac{c^k}{k!} \rightarrow \phi_c = \frac{1}{c}$
  - ¾ of the vertices have to be removed before the giant cluster is destroyed
- Power-law distribution : $\langle k^2 \rangle$ diverges for large networks $\rightarrow \phi_c = 0$
  - No matter how many vertices are removed, there will always be a giant cluster

robust against random failure

disease spread out of control

Size of the giant cluster

- As well as percolation threshold, the size of the giant cluster also plays a role for assessing robustness of a network
- We can solve for u for some special cases
- A network with an exponential degree distribution
  - $p_k = \left(1 - e^{-\lambda}\right)e^{-\lambda k}, \lambda > 0$
  - $g_0(z) = \frac{e^{\lambda}-1}{e^{\lambda}-z}, g_1(z) = \left(\frac{e^{\lambda}-1}{e^{\lambda}-z}\right)^2$ (p.468)
  - $u = 1 - \phi + \phi g_1(u)$
  - $u = 1 - \phi + \phi\left(\frac{e^{\lambda}-1}{e^{\lambda}-u}\right)^2$
  - $u\left(e^{\lambda} - u\right)^2 - (1 - \phi)\left(e^{\lambda} - u\right)^2 - \phi\left(e^{\lambda} - 1\right)^2 = 0$

Size of giant cluster (exponential degree distribution)

- $u(e^{\lambda}-u)^2-(1-\phi)(e^{\lambda}-u)^2-\phi(e^{\lambda}-1)^2=0$
- $u=1$ is always a solution →it must contain a factor of $u-1$
- $(u-1)\left[u^2+(\phi-2e^{\lambda})u+\phi-2\phi e^{\lambda}+e^{2\lambda}\right]=0$
- $u=e^{\lambda}-\frac{1}{2}\phi-\sqrt{\frac{1}{4}\phi^2+\phi(e^{\lambda}-1)}$

This can became greater than 1 for small ϕ

  – the other solution is greater than one so it cannot be probability u
- $S=\phi[1-g_0(u)]=\phi\left[1-\frac{2(e^{\lambda}-1)}{\phi+\sqrt{\phi^2+4\phi(e^{\lambda}-1)}}\right]$

$=\phi\left[1-2(e^{\lambda}-1)\frac{\phi-\sqrt{\phi^2+4\phi(e^{\lambda}-1)}}{\phi^2-(\phi^2+4\phi(e^{\lambda}-1))}\right]=$

$\frac{3}{2}\phi-\sqrt{\frac{1}{4}\phi^2+\phi(e^{\lambda}-1)}$

Size of giant cluster (exponential degree distribution)

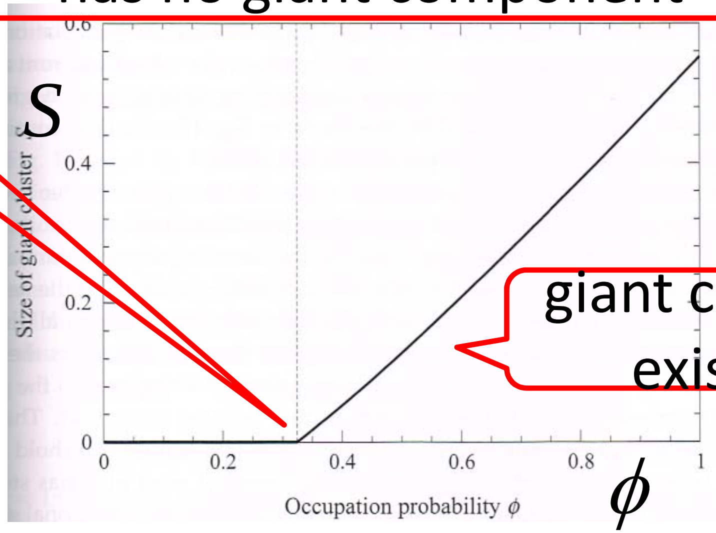
- When $u = 1$, $u = e^{\lambda} - \frac{1}{2}\phi - \sqrt{\frac{1}{4}\phi^2 + \phi(e^{\lambda} - 1)} = 1$
- $e^{\lambda} - 1 - \frac{1}{2}\phi = \sqrt{\frac{1}{4}\phi^2 + \phi(e^{\lambda} - 1)}$
- $\phi_c = \frac{1}{2}(e^{\lambda} - 1)$

percolation threshold

- If λ becomes large, ϕ_c can become greater than one
  - $\frac{1}{2}(e^{\lambda} - 1) = 1 \rightarrow \lambda = ln3$

When $\lambda > \overset{1.09}{ln3}$, the network has no giant component

- When $\lambda = 1/2$, $\boxed{\phi_c\text{=0.324…}}$
  - Phase transition
  - Sharp transition is true in an infinite networks



giant cluster exists

Non uniform removal of vertices

- Vertices are removed randomly (previous discussion)
- High-degree vertices are removed $\rightarrow$ percolation will be quite different