Application of Complex Networks: Control of Networked Multi-Agent Systems (2)

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Complex Networks, Jan 11th, 2018

Introduction

Control of multi-agent systems

- Very active research area in systems control (2000~)

Consensus problem

- One of the basic problems for multi-agent systems
- Initiated the research trend in this area
- Systems control approach: Theory-based with applications

In this lecture

- Basics of multi-agent consensus

What is consensus?

Flocks of fish/birds

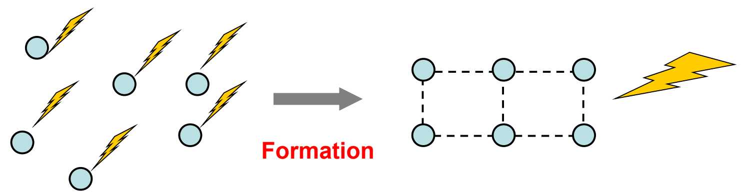
Load balancing among servers

Formation of autonomous robots

Sensor networks

Example 1: Autonomous robots

- Cluster of small robots for planetary exploration
  - High flexibility and reliability at low cost
  - Communication is limited by on-board power
- Array antenna
  - Multiple antennas coupled for directed transmission
  - Formation of robots based on distributed control laws

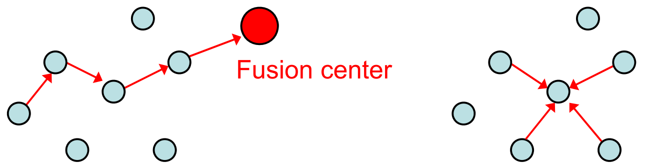


Example 2: Sensor networks

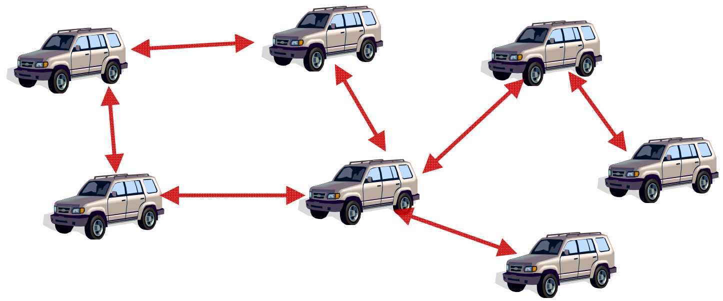
- Spatially distributed autonomous sensors with wireless communication capability
- Problem: When each sensor measures unknown parameter + noise, want to find the average of all measurements.

Centralized scheme

Distributed scheme



Consensus problem

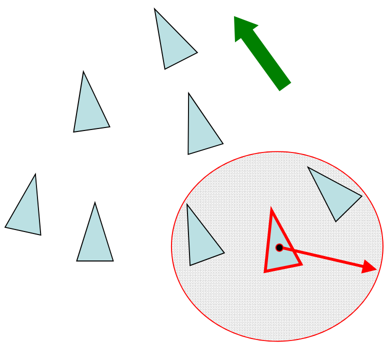


- Network of agents without a leader
- Each agent communicates with others and updates its state
- All agents should arrive at the same (unspecified) state

Achieve global objectives through local interaction!

Some history (1): Boids

- Flocking of birds: Formation flying without a leader
- What are the simple control laws for each bird?
- Simulation-based study by Raynolds
- Three rules
  - Separation
  - Alignment
  - Cohesion



Raynolds (1987)

Some history (2): Model by Vicek et al.

- Proposed a mathematical model of agents' dynamics
  - Each agent moves on a plane at constant speed
  - Align with the directions of neighboring agents
- Flocking behavior was observed by simulation

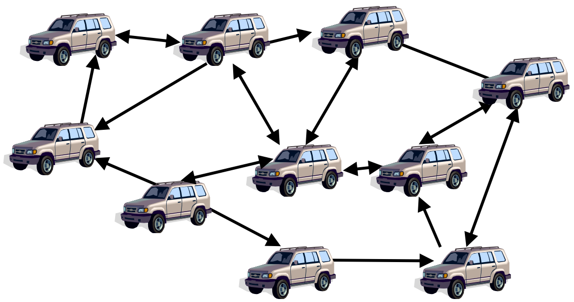
Vicek et al. (1995)

Analytic results by Jadbabaie et al.

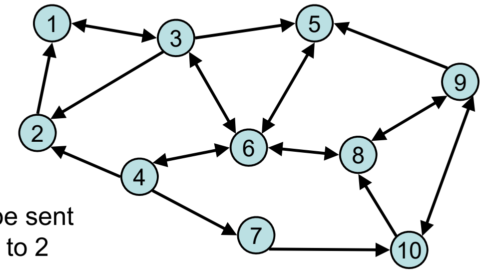
- Proved that all agents converge to the same direction if there is sufficient connectivity structure
  - Motivated control researchers to study multi-robot problems

Jadbabaie, Lin, & Morse (2003), Tsitsiklis & Bertsekas (1989)

Network of agents

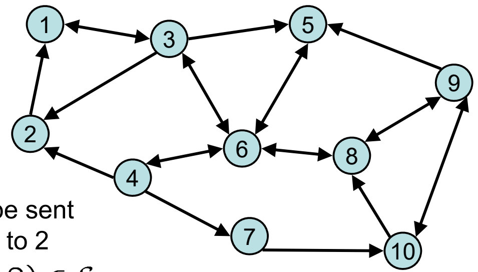


Network of agents



Info can be sent
from 4 to 2

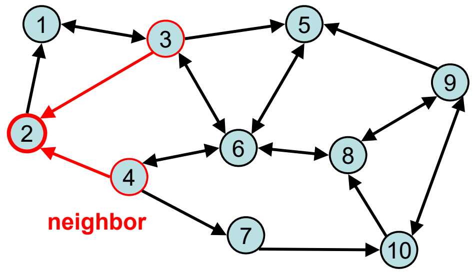
Connectivity in multi-agent systems



Info can be sent
from 4 to 2
⇔ $(4,2) \in \mathcal{E}$

- Represented as a graph
  - Node set $\mathcal{V} = \{1, 2, \ldots, N\}$ ⇒ Indices for the agents
  - Edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ ⇒ Communication among the agents

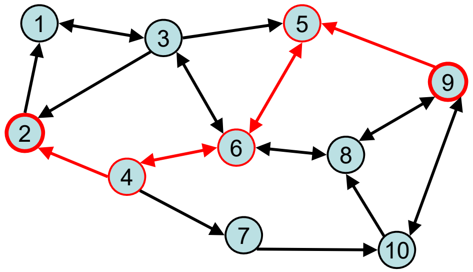
Connectivity in multi-agent systems



- Neighbor set $\mathcal{N}_i$ ⇒ Indices of agents that can send info to agent ***i***
  - Example: For agent 2

$$\mathcal{N}_2 = \{3, 4\} = \{j \in \mathcal{V} : \ (j, 2) \in \mathcal{E}\}$$

Basics of graphs

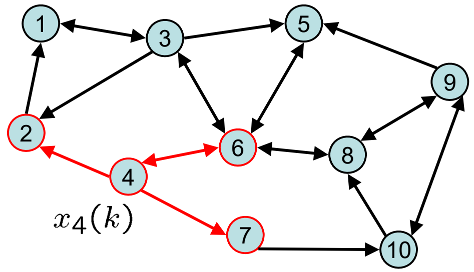


Agents 2 and 9
The whole graph
⇒ Connected!

- Types: Directed/Undirected
- Nodes i and j are connected
  ⇔ Agent j is reachable from i by following edges
- Graph is (strongly) connected
  ⇔ Any two nodes are connected

Protocol for distributed algorithms

- At time k, agent i does the following:
  1. Sends its value $x_i(k)$ to the neighbor agents
  2. Updates its value based on the received info and obtains $x_i(k+1)$



Average consensus

- **Problem:** Find a distributed algorithm satisfying the two conditions:
  1. All agents converge to the same value.
     $$|x_i(k) - x_j(k)| \to 0, \quad k \to \infty, \ \forall i, j = 1, 2, \dots, N$$
  2. The value is the average of the initial values.

Algorithms in this lecture

Two classes of consensus problems
1. Real-valued
2. Integer-valued (Quantized)

- Algorithms may be deterministic or probabilistic
- Graph structure: Undirected and connected

Average consensus (1)

Real-valued case

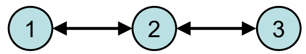
Real-valued average consensus

- Each agent has a real value $x_i(k)$
- Average consensus

$$x_i(k) \to \underbrace{\frac{1}{N}\sum_{j=1}^{N} x_j(0)}_{\text{Average of initial values}}, \quad k \to \infty, \ \forall i = 1, 2, \ldots, N$$

- Example

Initial values 1 2 2



Ave$=$1.666

Distributed algorithm

- Update scheme for agent ***i***:

$$x_i(k+1) = W_{ii}x_i(k) + \textstyle\sum_{j\in\mathcal{N}_i} W_{ij}x_j(k)$$

where

$$W_{ij} = \begin{cases} \frac{1}{1+\max\{d_i,d_j\}} & \text{if } j \in \mathcal{N}_i \\ 1 - \sum_{\ell\in\mathcal{N}_i} W_{i\ell} & \text{if } i = j \\ 0 & \text{Otherwise} \end{cases}$$

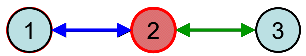
$d_i = |\mathcal{N}_i|$ Number of neighbors for agent ***i***

- Can be implemented in a distributed manner

Xiao, Boyd, Lall (2005)

Example

Init. values 1 2 2



Ave$=$1.666

- Update scheme for agent 1:

$$x_1(k+1) = \underbrace{\frac{2}{3}}_{=1-\frac{1}{3}}x_1(k) + \underbrace{\frac{1}{3}}_{=\frac{1}{1+\max\{1,2\}}}x_2(k)$$

- Update scheme for agent 2:

$$x_2(k+1) = \underbrace{\frac{1}{3}}_{=1-\frac{1}{3}-\frac{1}{3}}x_2(k) + \underbrace{\frac{1}{3}}_{=\frac{1}{1+\max\{1,2\}}}x_1(k) + \underbrace{\frac{1}{3}}_{=\frac{1}{1+\max\{1,2\}}}x_3(k)$$

Example

Init. values 1 2 2

$$1 \leftrightarrow 2 \leftrightarrow 3$$

Ave=1.666

- Distributed algorithm:

$$x_1(k+1) = \frac{2}{3}x_1(k) + \underline{\frac{1}{3}}x_2(k)$$

$$x_2(k+1) = \frac{1}{3}x_2(k) + \underline{\frac{1}{3}}x_1(k) + \underline{\frac{1}{3}}x_3(k)$$

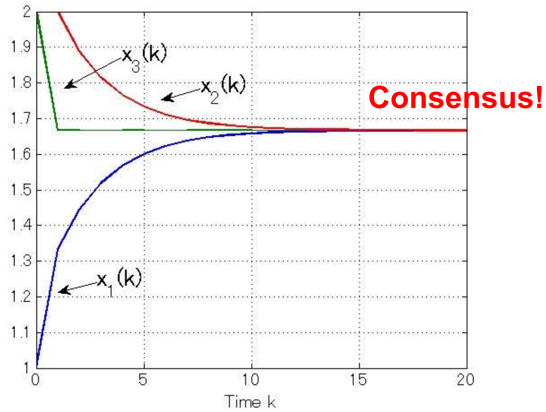
$$x_3(k+1) = \frac{2}{3}x_3(k) + \underline{\frac{1}{3}}x_2(k)$$

Example

Init. values 1 2 2

$$1 \leftrightarrow 2 \leftrightarrow 3$$

Ave=1.666



Example

Init. values 1 2 2

$$1 \leftrightarrow 2 \leftrightarrow 3$$

Ave=1.666

- Distributed algorithm in vector form:

$$x(k+1) = \underline{\begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix}} x(k), \quad x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Each element is nonnegative, and

- Sum of elements in each row = 1 ⇒ Row stochastic
- Sum of elements in each column = 1 ⇒ Column stochastic

General form of the algorithm

$$x(k+1) = \underline{W}x(k) \qquad x(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_N(k) \end{bmatrix}$$

Stochastic matrix
(Row and column)

- Property 1
  - Because $\boldsymbol{W}$ is <u>row</u> stochastic,

$$W\mathbf{1} = \mathbf{1} \qquad \text{where } \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

  - The matrix has eigenvalue 1
  - Corresponding eigenvector is a (scalar multiple of) vector **1**: $c\mathbf{1}$

General form of the algorithm

$$x(k+1) = \underline{W} x(k) \qquad x(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_N(k) \end{bmatrix}$$

Stochastic matrix
(Row and column)

- Property 2
  - Because $\boldsymbol{W}$ is column stochastic,

$$\mathbf{1}^T W = \mathbf{1}^T \quad \text{where } \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

  - Thus

$$\sum_{i=1}^{N} x_i(k+1) = \mathbf{1}^T x(k+1) = \mathbf{1}^T W x(k)$$

$$= \mathbf{1}^T x(k) = \sum_{i=1}^{N} x_i(k)$$

$\boxed{\text{Sum of all elements is invariant!}}$

Average vector

- By properties 1 and 2,

For eigenvalue 1, the eigenvector is in the form $x^* = c\mathbf{1}$

and satisfies

$$\sum_{i=1}^{N} x_i^* = \sum_{i=1}^{N} x_i(0)$$

Hence

The desired average!

$$x_i^* = \frac{1}{N} \sum_{i=1}^{N} x_i(0) \qquad \forall i = 1, \ldots, N$$

- However, there may be other vectors as the eigenvector.
- If the graph is <u>connected</u>, then it is <u>unique</u>.

(by the Perron-Frobenius theorem)

Convergence of the algorithm

$$x(k+1) = W x(k) \qquad x(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_N(k) \end{bmatrix}$$

- Computation via power method
  - The state $x(k)$ converges to the eigenvector x^*

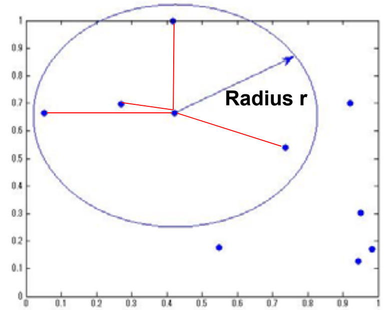
- **Result:** If the network of agents forms a connected graph,

then average consensus is achieved:

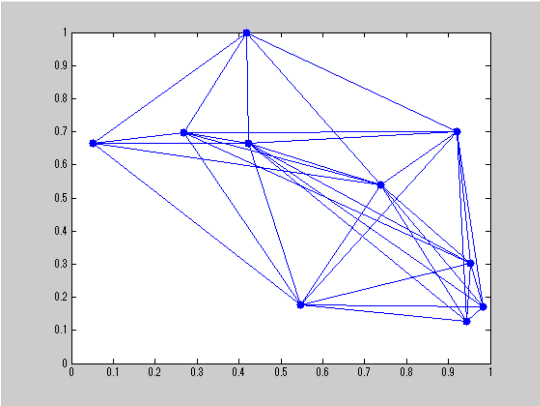
$$x_i(k) \to \frac{1}{N} \sum_{j=1}^{N} x_j(0), \quad k \to \infty, \ \forall i = 1, 2, \ldots, N$$

Autonomous mobile robots: Randezvous

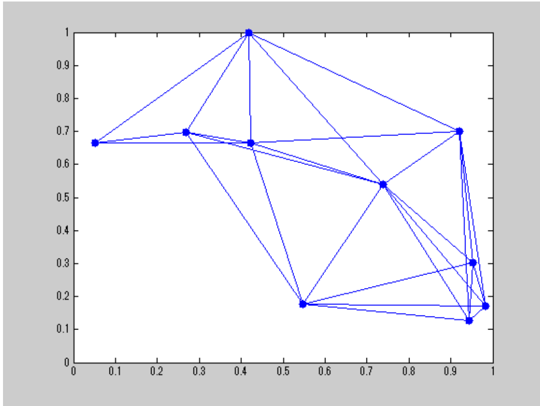
- 10 agents
- Random graph:
  - Initial positions are uniformly distributed
  - Neigbors are agents within radius r



- Radius r=0.8   # of edges 35

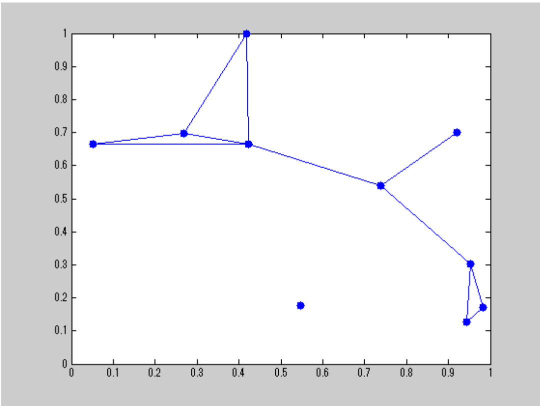


- Radius r=0.6   # of edges 27
  - The graph is a subgraph of the previous one.



- Radius r=0.38   # of edges 11

  Disconnected !



Recap

Average consensus: Real-valued case

- True average can be obtained.
- Connected graph must be used
- Matrix theory

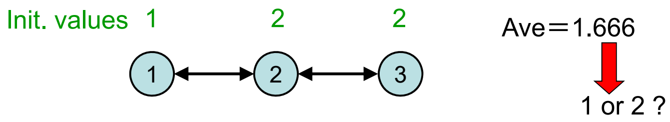
Average consensus (2)

Integer-valued (quantized) case

Quantized average consensus

- Each agent's value $x_i(k)$ is an integer

- **What's different:**
  - True average of N integers ≠ integer
  - Approximation of the average is not unique
  - Convergence in finite time is possible (i.e., not asymptotic)



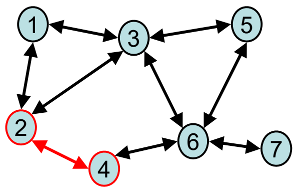
Kashap, Basar, Srikant (2007)

Probabilistic communication

Gossip algorithm

- Agents decide to communicate at a random time with randomly chosen neighbor.
- To each edge, assign a probability to be chosen.

- No need of a common clock.
  (asynchronous communication)



Boyd, Ghosh, Prabhakar, Shah (2006)

Quantized average consensus

- Problem:

Find a distributed algorithm such that

1. Each agent's value is always an integer
2. Sum of all agents' value is constant
3. For sufficiently large k, the agents achieve average consensus, that is,

$$x_i(k) = \left\lceil \frac{1}{N}\sum_{j=1}^{N} x_j(0) \right\rceil \quad \text{or} \quad \left\lfloor \frac{1}{N}\sum_{j=1}^{N} x_j(0) \right\rfloor, \quad i = 1, \ldots, N$$

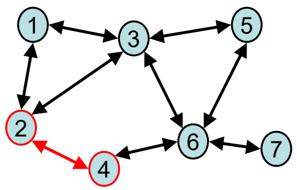
Kashap, Basar, Srikant (2007)

Quantized gossip algorithm

- At time $\boldsymbol{k}$, one edge (i, j) is randomly chosen.
- Agents i, j update their values to $x_i(k+1),\ x_j(k+1)$ by
  - If $x_i(k) = x_j(k)$, then the values stay the same.
  - If $|x_i(k) - x_j(k)| = 1$, then exchange the values (Swapping)
  - Otherwise, if $x_i(k) < x_j(k)$, then let

$$x_i(k+1) = x_i(k) + 1$$
$$x_j(k+1) = x_j(k) - 1$$

1. Sum of both values remains the same
2. Their difference is reduced



Quantized gossip algorithm

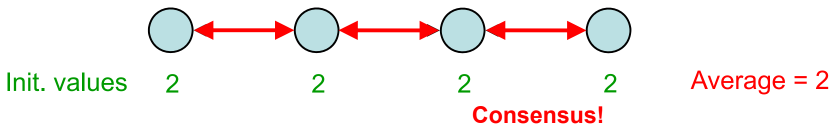
Result:

The algorithm achieves quantized average consensus with probability 1 in finite time.

Two important properties:

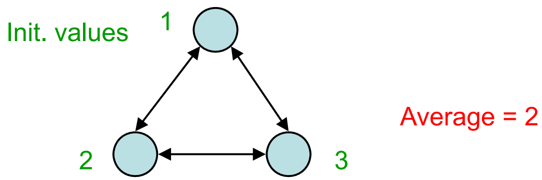
- Swapping
- Probabilistic algorithm

Example 1 (Swapping)

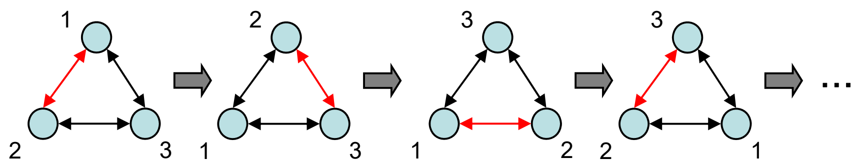


- For each edge, the difference in values is at most 1.
- The average is unknown from local info.
- By swapping, consensus is possible.
  - Agents with values 1 and 3 become neighbors (with prob. 1).

Example 2 (Probabilistic algorithm)



- Example of a deterministic algorithm: Periodic comm.



- Only swapping occurs, thus no consensus.
- Under probabilistic comm., convergence in a few steps.

Recap

Average consensus: Quantized-valued case

- Approximate average
- Gossip algorithm – Probabilistic but always correct
- Theory of Markov chain
- Performance at the order of $O(N^2)$

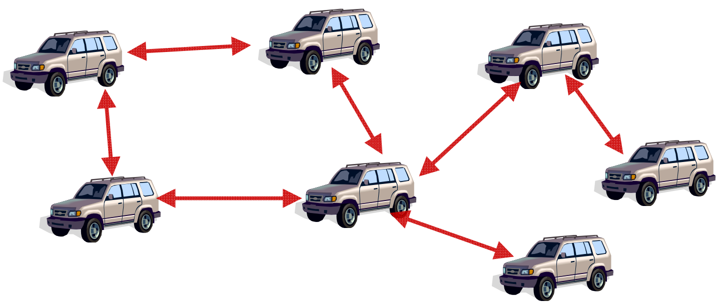
Summary

- Multi-agent systems and consensus problems
- Graph representation of network structures
- Distributed algorithms: Deterministic vs Probabilistic
- Update schemes for different agent values
  (real, quantized, and binary)

New challenges

- Performance
- Communication (time delay, data rate, graph,…)
- Dynamics of the agents (high dim., nonlinear,…)

Consensus problem



- Network of agents without a leader
- Each agent communicates with others and updates its state
- All agents should arrive at the same (unspecified) state

Achieve global objectives through local interaction!

Assignment

- Regarding the topics of the lectures of Dec 25th and Jan 11th, make a summary in 2 to 4 pages of A4 paper that reports your findings (through books and papers) and/or your thoughts on the topics in view of what you learned so far about Complex Networks.
- Deadline: Jan 22th (Mon) Noon
- Submit through OCW-i in Tokyo Tech Portal
- On the first page, write the name of this lecture, your name and student ID.