

Random Forests™

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RF motivation

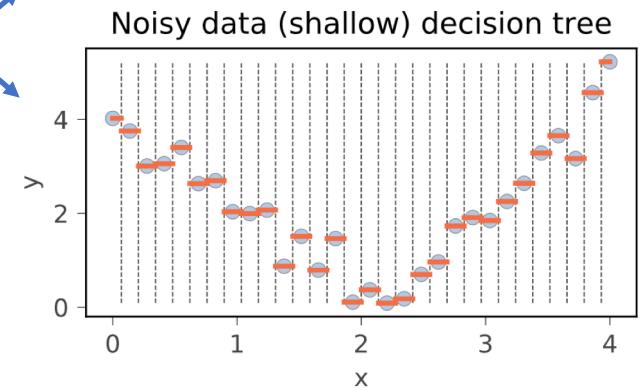
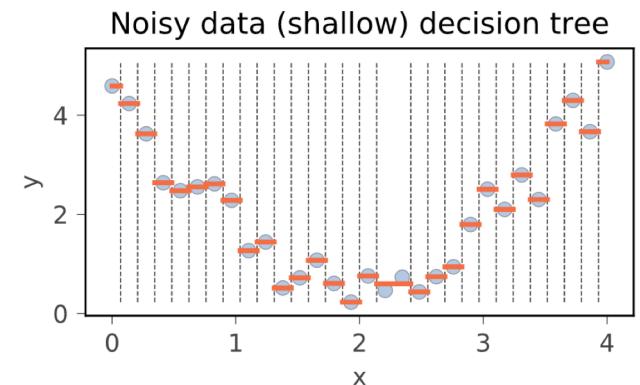
Leo Breiman (1996) introduced bagging
then Random Forests (2001)

<https://www.stat.berkeley.edu/~breiman/randomforest2001.pdf>

- Decision trees can often get training errors close to zero because we can grow very large trees to partition the feature space into tiny regions with 1 or just a few observations/samples; trees have very **low bias**
- The downside is that decision trees overfit or, using stats nerds terminology, decision trees have **high variance** and don't generalize well
- Goal: keep the low bias, but increase the generality (reduce the variance)

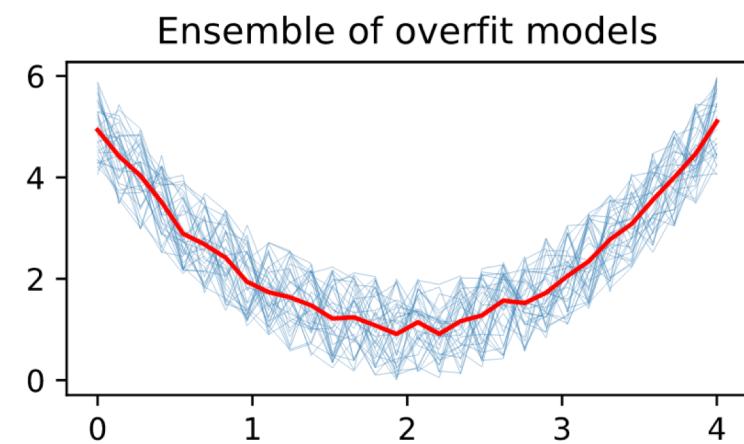
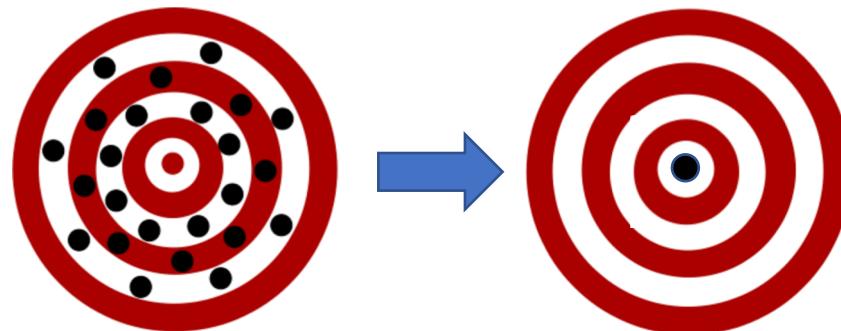
Variance is the key word here

- Decision trees are sensitive to quirks in the data and so similar but different training sets drawn from the same distribution can yield very different models (parameters)
- (That's the definition of model variance)
- Ex: two i.i.d. X training sets fit by regr tree
- The predictions made by decision trees trained on i.i.d. sets bounce around, but their average approaches the correct value (same for i.d. sets)



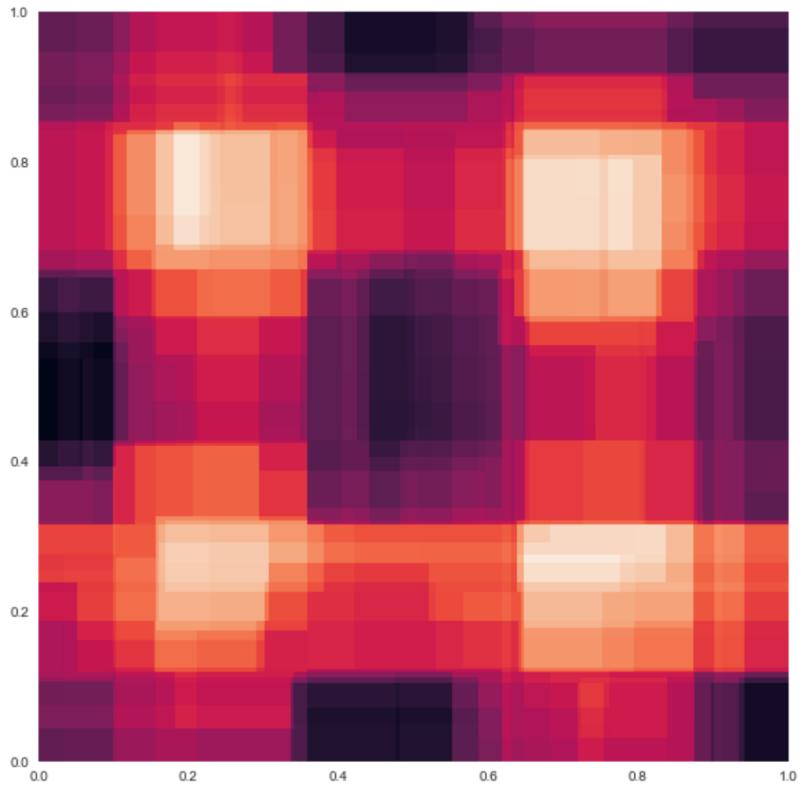
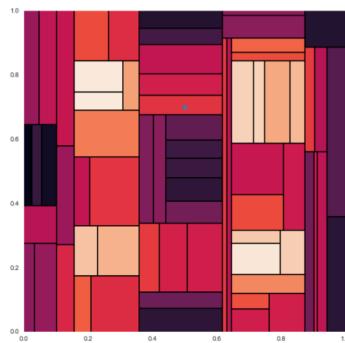
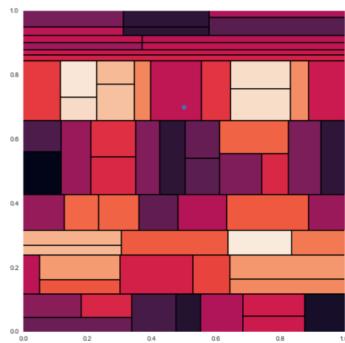
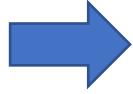
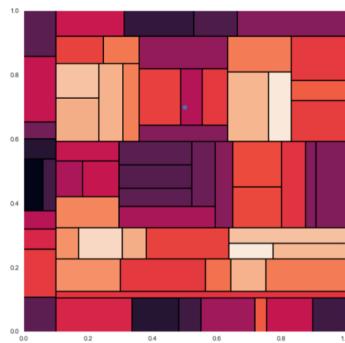
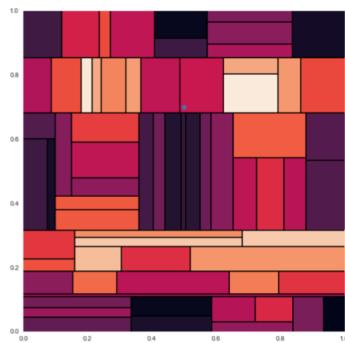
Ensemble of high-variance decision trees

- Decision trees are inconsistent but are accurate on average across multiple identically distributed (i.d.) training sets
- Pretend we do have multiple i.d. X's and multiple fit models
- Then, we can average predictions from multiple models to get an accurate prediction:



Target image credit: <http://ficscience.blogspot.com/2011/02/technicalities-of-technical-terms.html>

RF: Overlapping feature space partitions



Images generated by USF MSDS alumnus Tyler White
<https://gist.github.com/tylerwx51/fc8b316337833c877785222d463a45b0>

Ensemble's effect on bias and variance

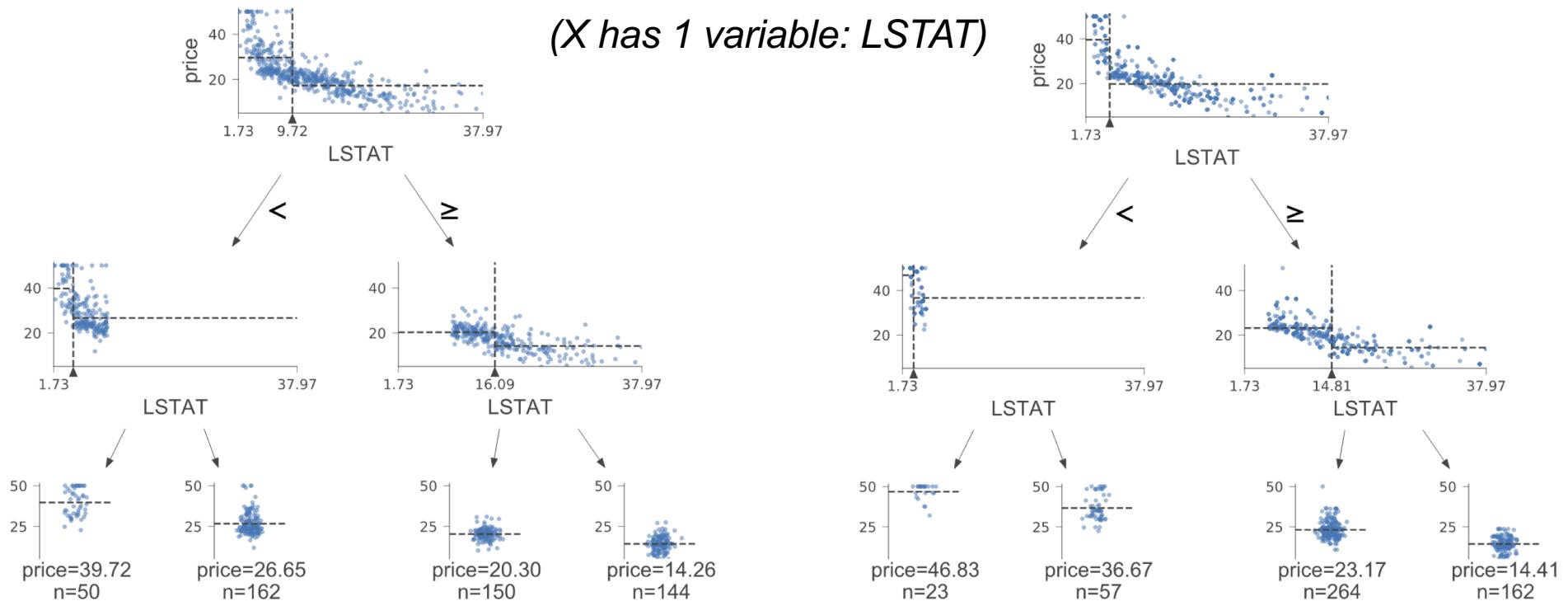
- Train T trees on T identically distributed X data sets (doesn't have to be independent)
- The average of the tree predictions is the same as the expected prediction from any tree trained on one of the X
- Central limit theorem says that if variance of T i.i.d. random variables is σ^2 , the variance of the average is $\frac{\sigma^2}{T}$
- So, as we add trees, the variance of the prediction (average from ensemble trees) tightens towards true prediction!
- Problem: we only have one X training data set

Bagging (Bootstrap aggregation)

- We only have one training set, X , but can *bootstrap* to get as many as we want from the same distribution
- These bootstrapped sets are i.d. but not independent (some overlap)
- *Bootstrap*: from n records, randomly select n with replacement
- Each bootstrapped data set will have about 2/3 of the unique overall records
- *Aggregation*: average the ensemble predictions
(majority vote for classification)
- Analogy: real estate agents sent out to sample NYC apartments/prices; there will be some overlap but average of all agents' predictions is more accurate than an individual's

Bootstrapping gives slightly different trees

```
df = df.sample(len(df), replace=True) # boston data set  
t = DecisionTreeRegressor(max_depth=2)  
t.fit(X,y)
```



Problem: bootstrapping is i.d. not i.i.d.

- If just identically distributed and not necessarily independent, with positive pairwise correlation ρ between trees, then variance of the average for random vars is: (Eqn 15.1 ESLII Hastie *et al*):

$$\rho\sigma^2 + \frac{1 - \rho}{T}\sigma^2$$

- As T , (num trees) grows, 2nd term disappears but correlation of trees would still have variance proportional to ρ (proportional to σ^2 , variance of any single tree predictor)
- ESLII: "...the size of the correlation of pairs of bag trees limits the benefits of averaging."
- So, bagging helps to reduce variance but still has correlation $\rho > 0$

What does “correlated trees” mean?

- Model parameters are correlated; “not independent” better term
- **Analogy.** Real estate agents sampling bias: if they visit apartments they observe other agents visit, variance might not shrink with more agents; worst-case: they all visit the same apartments; best case is they choose all apts independently
- **Worst-case.** Imagine replicating single decision tree T times, variance of average prediction would be same as variance of single tree (i.e., bad strategy); if $\rho=1$, variance of avg = σ^2
- **Best-case.** if $\rho=0$, trees are uncorrelated and variance of average converges to 0 as T grows

$$\rho\sigma^2 + \frac{1-\rho}{T}\sigma^2$$

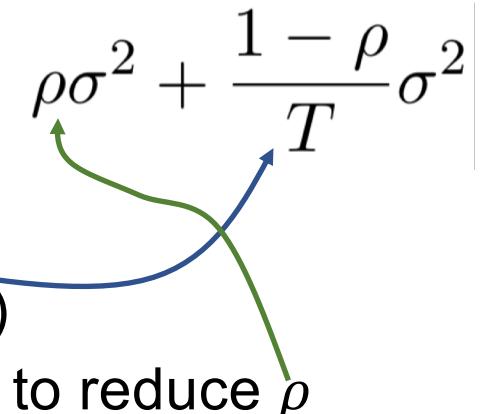
Bagging summary

- Adding bagged trees does not increase **bias**; the avg of tree predictions is same as the expected prediction from any single bootstrapped tree
- Keep in mind, however, that a tree fit to bootstrapped data is only using 2/3 of the data and so each bagged tree will have higher bias than single decision tree fit to entire data set
- Bagging does reduce **variance**, it is averaging predictions from lots of non-identical models afterall
- Degree of tree similarity, ρ , influences ability to reduce variance to 0
- If $\rho=1$, no reduction in variance, but $\rho=0$ means adding trees reduces variance to 0 asymptotically

$$\rho\sigma^2 + \frac{1-\rho}{T}\sigma^2$$

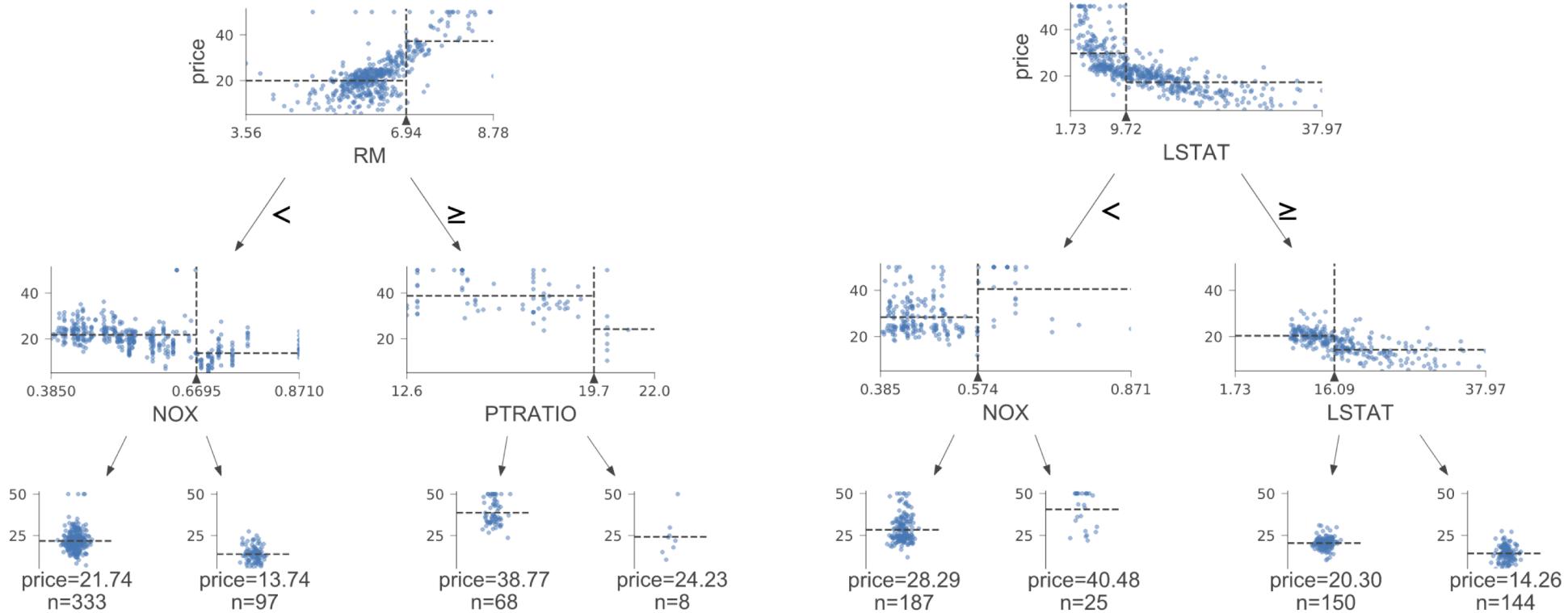
De-correlating trees to get Random Forests (RFs)

- Bagging overcomes most of the overfitting (Hastie *et al* p589 say ρ usually small like 0.05)
- Can do a little better de-correlation of the trees to reduce ρ
- Restrict the available features at each split (random selection of m)
- Choose max features per split, $m \leq p$, such as $\text{sqrt}(p)$
- Imagine one strongly predictive var out of p , then all trees would be similar; initial root splits, and many others, will likely be same
- Make sure chance of selecting predictive variables (m/p) is high enough to find them (See ESLII p596)
- Let validation error be your guide to m (and other hyperparameters)

$$\rho\sigma^2 + \frac{1 - \rho}{T}\sigma^2$$


(Boston data)

Max features = 5 (of 13) gets diff trees

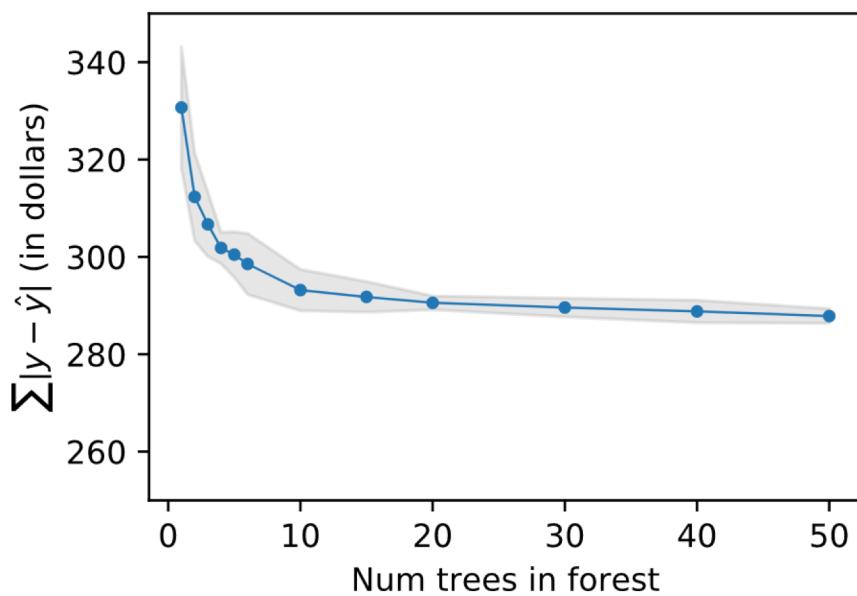


Choose 5 randomly selected features during EACH split

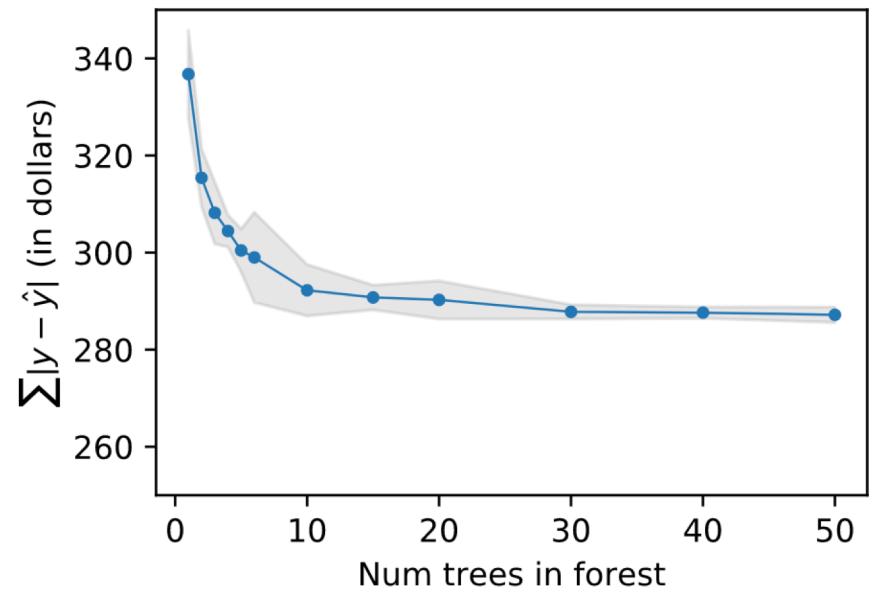
Compare bagged trees vs RF (Rent)

Very little difference with just p=4 features

Rent bagged forest error vs num trees
5 trials, 41055 training obs., 7245 test
4 max features



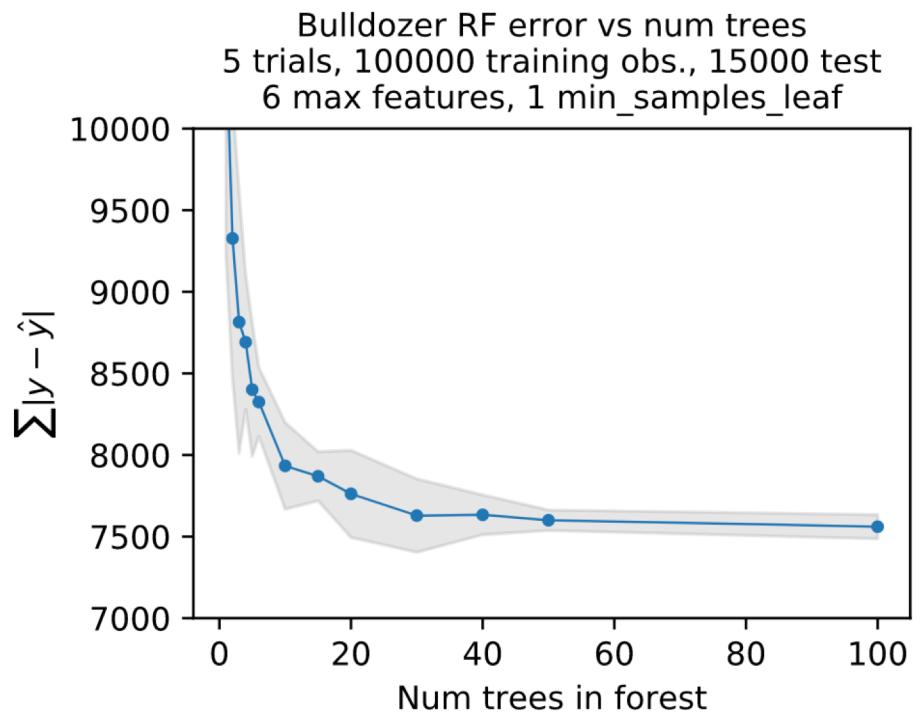
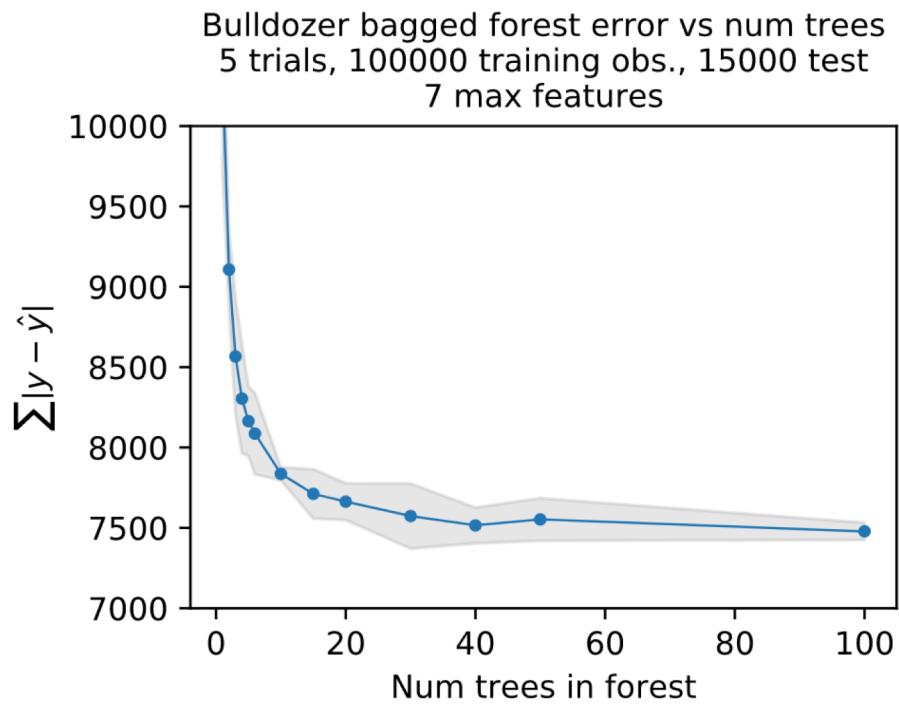
Rent RF error vs num trees
5 trials, 41055 training obs., 7245 test
1 max features



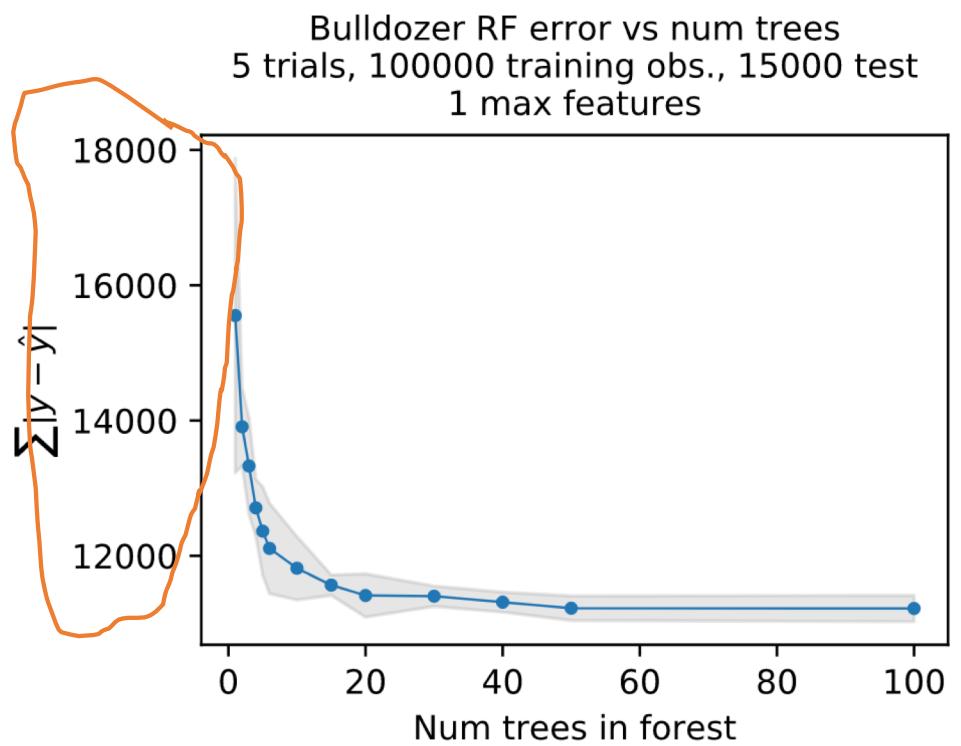
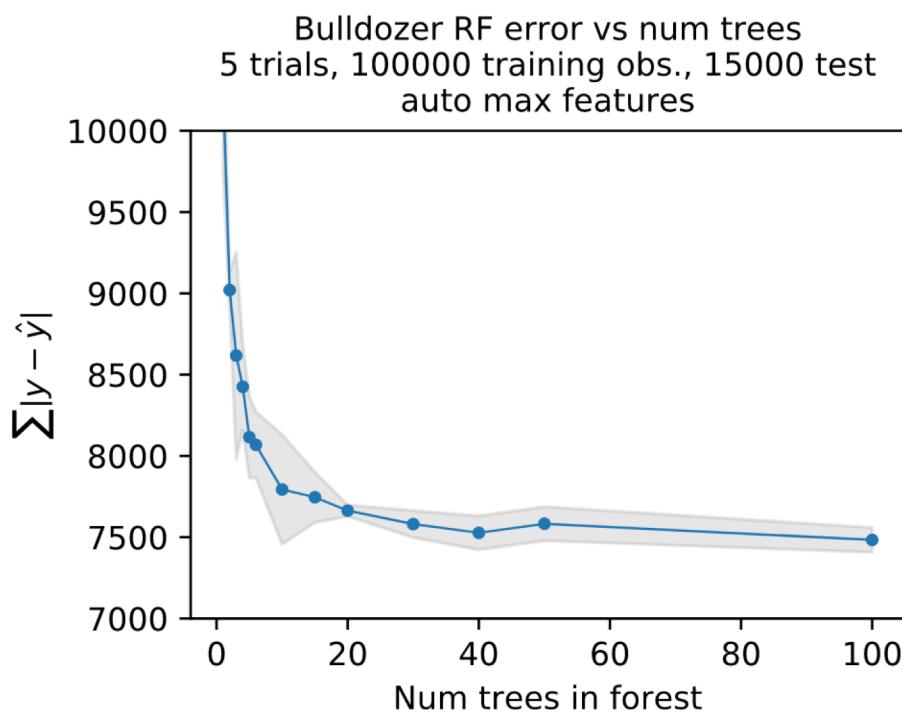
Some data sets will be more sensitive to max features than others

Compare bagged trees vs RF (bulldozer)

For this data, using all p=7 is best

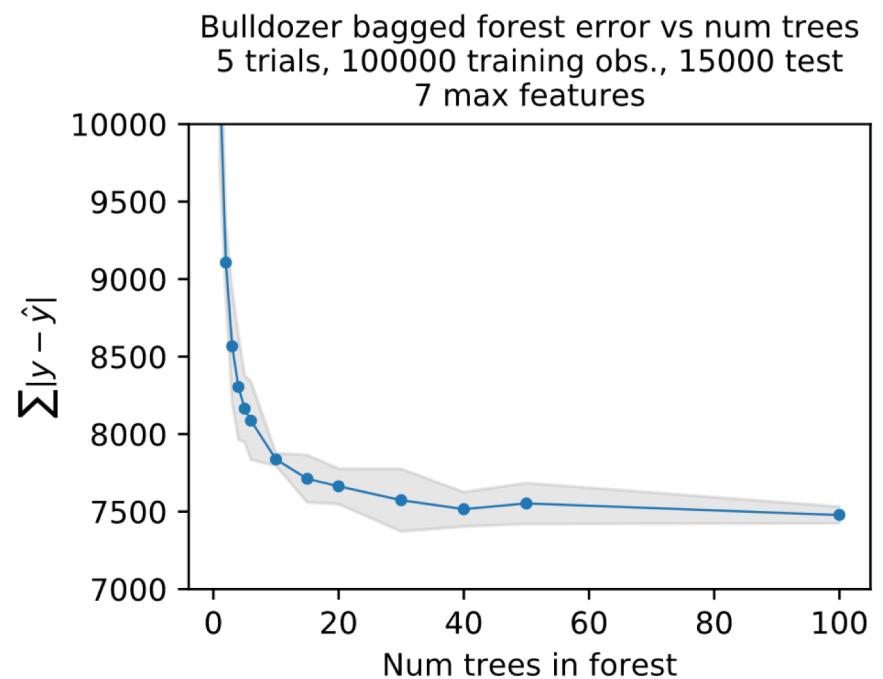


If max_features too low, high bias

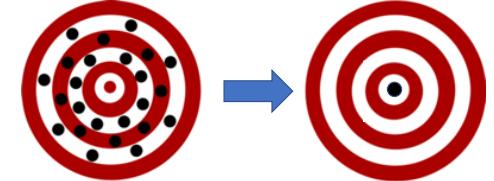


Effect of forest size on bias

- Why does accuracy improve greatly (initially) as we add trees?
 - Each tree sees only 2/3 of data so adding bootstrapped trees increases use of training data
- Why does accuracy asymptotically approach a minimum instead of continual improvement?
 - With enough trees, ensemble sees 100% of the training data; it's approaching the bias of single decision tree in ideal world where it's not overfit



Properties (Breiman 2001)



- p4 “*Random forests do not overfit as more trees are added*” **Why?**
 - Adding more trees actually REDUCES variance, and overfitting==variance
 - New trees get averaged in so each new tree has less individual effect
 - New trees balance each other out, one might be too high, another too low
- p7 “*It's relatively robust to y outliers and X noise*” **Why?**
 - y outliers get shunted to their own leaf since doing so reducing loss function, particularly if squared-error is used
 - Noise vars aren't predictive so not chosen as split vars
- p10 Bagging helps more the more unstable the model. **Why?**
 - Averaging is a smoothing operator, squeezing predictions to centroid
 - If model is low variance already, there is no point in bagging

Properties continued

- RFs are scale and range insensitive in features and target y **Why?**
 - Comparing feature values in decision nodes not doing math on them
 - Computing mean or mode of y to predict
- ESLII p596 “*Classifiers are less sensitive to variance [than regressors]*” **Why?**
 - (not sure haha) I believe it has something to do with mode vs mean (mode is same until a threshold whereas mean is influenced by any value added, unless it is also the mean)

Bootstrapping vs subsampling

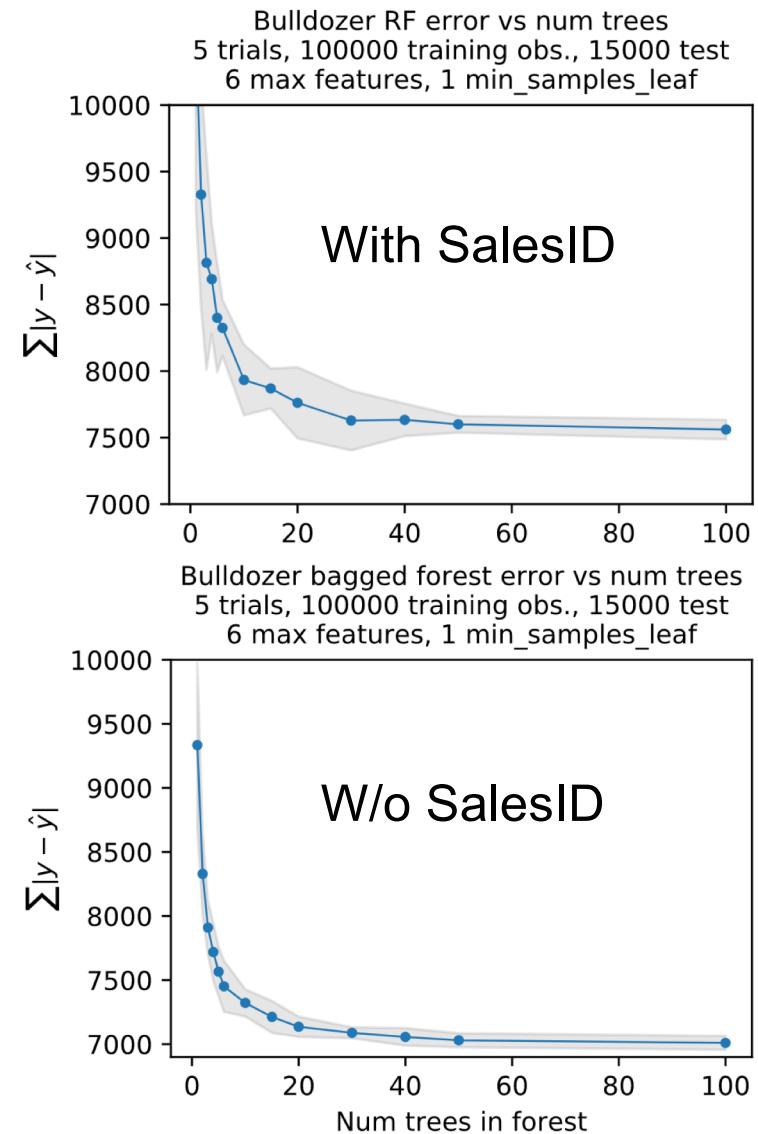
- Bootstrapping is sampling with replacement vs subsampling w/o replacement
-  Friedman and Hall (2000): subsampling also works, showing that training trees with $n/2$ subsamples is similar in bias/variance to bagging <http://statweb.stanford.edu/~jhf/ftp/bag.pdf>
- Smaller training set is a big win in terms of speed
- Using even smaller fractions of n improve generality (reduce variance) because trees are less correlated (they work on different data chunks); note that each tree would become more biased as n subsample size decreases

RF Tuning strategy

- Good news: very little tuning needed
- Start with maybe 20 trees and work upwards til validation error stops getting better; or just pick 100
- Sklearn uses `max_features= sqrt(p)` by default; try dropping this using `log(p)`, or similar; ESLII suggests $p/3$ for regression and \sqrt{p} for classification
- Try adjusting `min samples per leaf`: 1, 3, 5, 10, 25, 100
- Goal: minimize validation error
- Can also try grid search, but I never bother;
Start with num trees, then tune the others

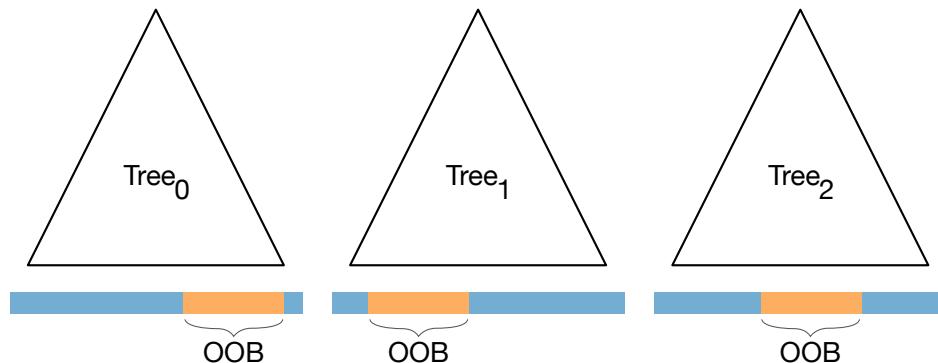
Feature engineering beats model tuning

- SalesID: unique record ID, and is never seen again in future predictions
- Is that useful for prediction? No
- Does the model think it's useful? Yes
- Model is overfit not on noise but on falsely-predictive feature
 - Could be that sales ID correlates with inflation or change in type of models sold in auction creates “trend” in sale prices
- A case where using LESS data improves the model a lot (\$500 diff)
- Dropping useless features also often gives a small bump



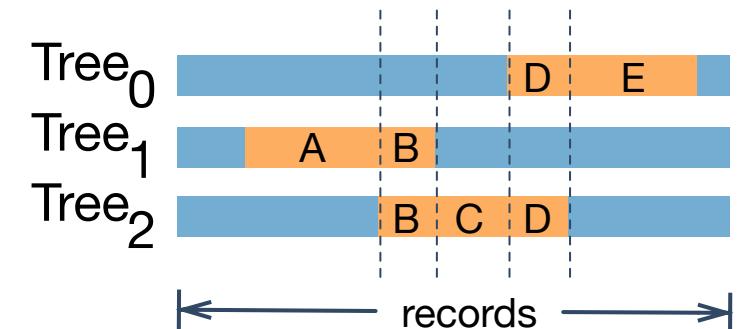
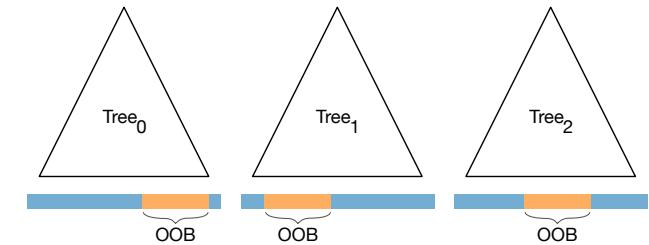
RF Out-of-bag (OOB)

- RFs have a major advantage over other models: OOB metrics
- Each tree is trained on 63% of data, leaving 27% OOB
- OOB subsamples available to the trees are different
- It's an excellent estimate of the validation error
- Stick with OOB unless default sklearn score() is not suitable
- Not having to process training and validation sets separately is a huge productivity win (assuming significant feature engineering)



Computing OOB predictions

- Get \hat{y}_i by averaging estimates from trees not trained using (x_i, y_i)
 - Image to right; blue is training set, OOB orange
 - Trees from same labeled region of $x^{(i)}$ give \hat{y}_i
 - Must find all trees not trained on $x^{(i)}$
 - E.g., compute \hat{y}_i for B region using Trees 1, 2
 - No OOB error estimate is possible for unlabeled regions
- Do not make predictions for OOB and compute error per tree!
- Each tree has high bias, so OOB scores from 1 tree'd be very high
- Must compute metric on predicted \hat{y} vector as usual (MSE,accuracy)
- Algorithms for regression and classification near end of lecture



OOB continued

- OOB error will slightly underestimate test set error. Why?
 - At least one of the trees in the forest is trained on the OOB samples
- Some research suggests OOB overestimates for binary classification <https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0201904>
- OOB metrics don't affect training, just gives metric
- Compare OOB with validation set:
 - If we add (predictive) feature and OOB is still ok, but the validation set is worse, the validation set is not good; e.g., different distribution or time sensitive
- OOB not to be used with time-sensitive data sets. Why not?

Extremely randomized trees (Geurts *et al* 2006)

- The variable/value pair is highly sensitive to the training set, and responsible much of error rate
- “*The optimal cut-point was shown to depend very strongly on the particular learning sample used...this cut-point variance appeared to be responsible for a significant part of the error rates of tree-based methods.*” <https://link.springer.com/article/10.1007/s10994-006-6226-1>
- Geurts wondered if more randomness could reduce variance further
- Pick random split value in $\min(X[:,j]) \dots \max(X[:,j])$, ignoring y!
- Like RF, select $m \leq p$ variables and choose var/value with lowest loss
- Fits using entire X training set, not bootstrap and not subsample (trying to reduce bias)
- Our use of just 11 (not n) X candidate values in the project is similar (an effort to reduce variance and increase speed)

The RF algorithms

Fitting RFs

Algorithm: $RFfit(X, y, loss, ntrees, max_features, min_samples_leaf)$

for $i = 1..ntrees$ **do**

$X', y' = bootstrap(X, y, size = |X|)$

$T_i = RFdtreefit(X', y', loss, max_features, min_samples_leaf)$

end

For regression, pass in loss = MSE or stddev

For classifier, pass in loss = gini

Fitting a single tree in RF

Same as decision tree except
we pass max_features to
RFbestsplit()

Algorithm: $RFdtreefit(X, y, loss, max_features, min_samples_leaf)$

if $|X| < min_samples_leaf$ **then** return Leaf(y)

$col, split = RFbestsplit(X, y, loss, max_features)$

if $col = -1$ **then** return Leaf(y)

$lchild = RFdtreefit(X[X_{col} \leq split], y[X_{col} \leq split], \dots)$

$rchild = RFdtreefit(X[X_{col} > split], y[X_{col} > split], \dots)$

return $DecisionNode(col, split, lchild, rchild)$

Finding best split in decision node in RF

Algorithm: $RFbestsplit(X, y, loss, max_features)$

$best = (col = -1, split = -1, loss = loss(y))$

$vars = \text{pick } max_features \text{ variables from all } p$

Only diff with decision tree

for $col \in vars$ **do**

$candidates = \text{randomly pick } k \ll n \text{ values from } X_{col}$

foreach $split \in candidates$ **do**

$yl = y[X_{col} \leq split]$

$yr = y[X_{col} > split]$

Pick, say, 11 not all possible X values. We get better generality and code is much faster!

if $|yl| = 0$ or $|yr| = 0$ **then continue**

$l = \frac{loss(yl) + loss(yr)}{2}$

Should pick midpoint between split value and next smallest X

if $l = 0$ **then return** $col, split$

if $l < best[loss]$ **then** $best = (col, split, l)$

end

end

return $best[col], best[split]$



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Simplest RF prediction (ESLII p588)

- But doesn't use all information to make best prediction
- Should use weighted averages / votes

Regression: $\hat{f}_{\text{rf}}^B(x) = \frac{1}{B} \sum_{b=1}^B T_b(x).$

Classification: Let $\hat{C}_b(x)$ be the class prediction of the b th random-forest tree. Then $\hat{C}_{\text{rf}}^B(x) = \text{majority vote } \{\hat{C}_b(x)\}_1^B.$

RF prediction

Algorithm: $RFpredict_{regr}(\{T_1..T_{ntrees}\}, x)$

Let $leaves = \{leaf(T_t, x) \forall t = 1..ntrees\}$

$nobs = \sum_{t=1}^{ntrees} |leaves_t|$

$ysum = \sum_{t=1}^{ntrees} \sum_{y \in leaves_t} y$

return $\frac{1}{nobs} ysum$

Count all y votes among the leaves
reached by running x down each tree

Weighted average of y values
among the leaves reached by
running x down each tree

Algorithm: $RFpredict_{class}(\{T_1..T_{ntrees}\}, x)$

$counts[k] = 0 \forall \text{classes } k$

foreach $t = 1..ntrees$ **do**

$leaf = leaf(T_t, x)$ *(leaf reached by x)*

foreach $y \in leaf$ **do**

$counts[y] += 1$ *(track count of leaf modes)*

end

end

return $\text{argmax}(counts)$

OOB regression scoring

Algorithm: $oob_score_{regr}(RF, X, y)$

Let $oob_counts[i] = 0 \forall$ trees $i = 1..|RF|$

Let $oob_preds[i] = 0 \forall$ trees $i = 1..|RF|$

foreach $t \in RF$ **do**

$leafsizes = |t.leaf(X[t.oob, :])|$ (*Num samples in leaf reached by each X*)

$oob_preds[t.oob] += leafsizes \otimes t.predict(X[t.oob, :])$

$oob_counts[t.oob] += leafsizes$

end

$oob_avg_preds = \frac{oob_preds[oob_counts > 0]}{oob_counts[oob_counts > 0]}$

return R^2 score for $(y[oob_counts > 0], oob_avg_preds)$

Assumes each tree collects OOB sample indexes during fit()



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OOB classification scoring

Algorithm: $oob_score_{class}(RF, X, y)$

Let $oob_counts[i] = 0 \forall \text{ trees } i = 1..|RF|$

Let $oob_preds[i] = 0 \forall \text{ trees } i = 1..|RF|$

foreach $t \in RF$ **do**

$leafsizes = |t.leaf(X[t.oob, :])|$ (*Num samples in leaf reached by each X*)

$tpred = t.predict(X[t.oob, :])$

$oob_preds[t.oob, tpred] += leafsizes$ (*count weighted class votes*)

$oob_counts[t.oob] += 1$ (*track num trees used for each X*)

end

$oob_votes[i] = \arg \max_j oob_preds[i, j] \forall i : oob_counts[i] > 0$

return *accuracy of* $y[oob_counts > 0] = oob_votes$

Assumes each tree collects OOB sample indexes during fit()