Getting a grip on recursion

"To understand recursion, one must first understand recursion"

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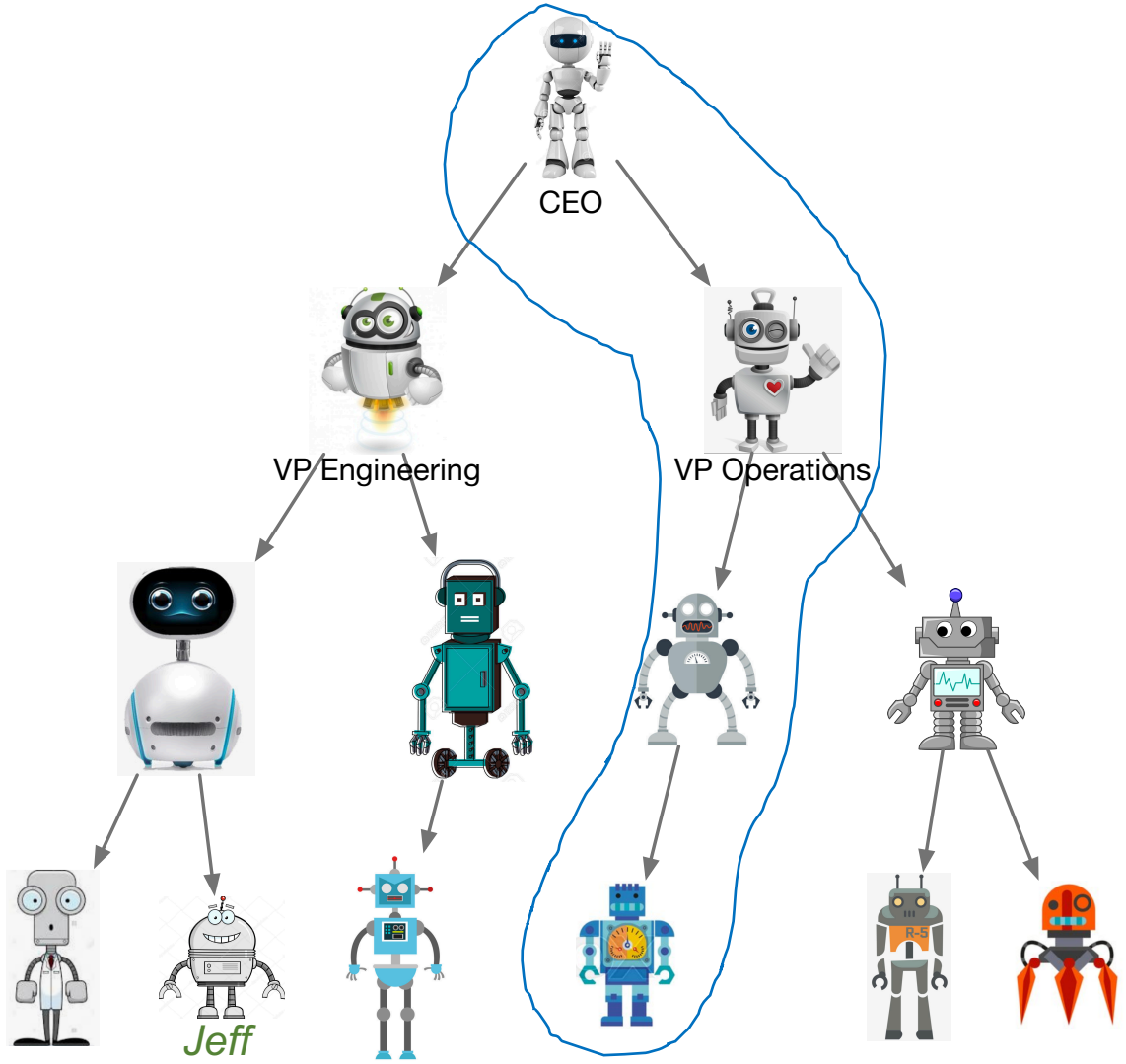
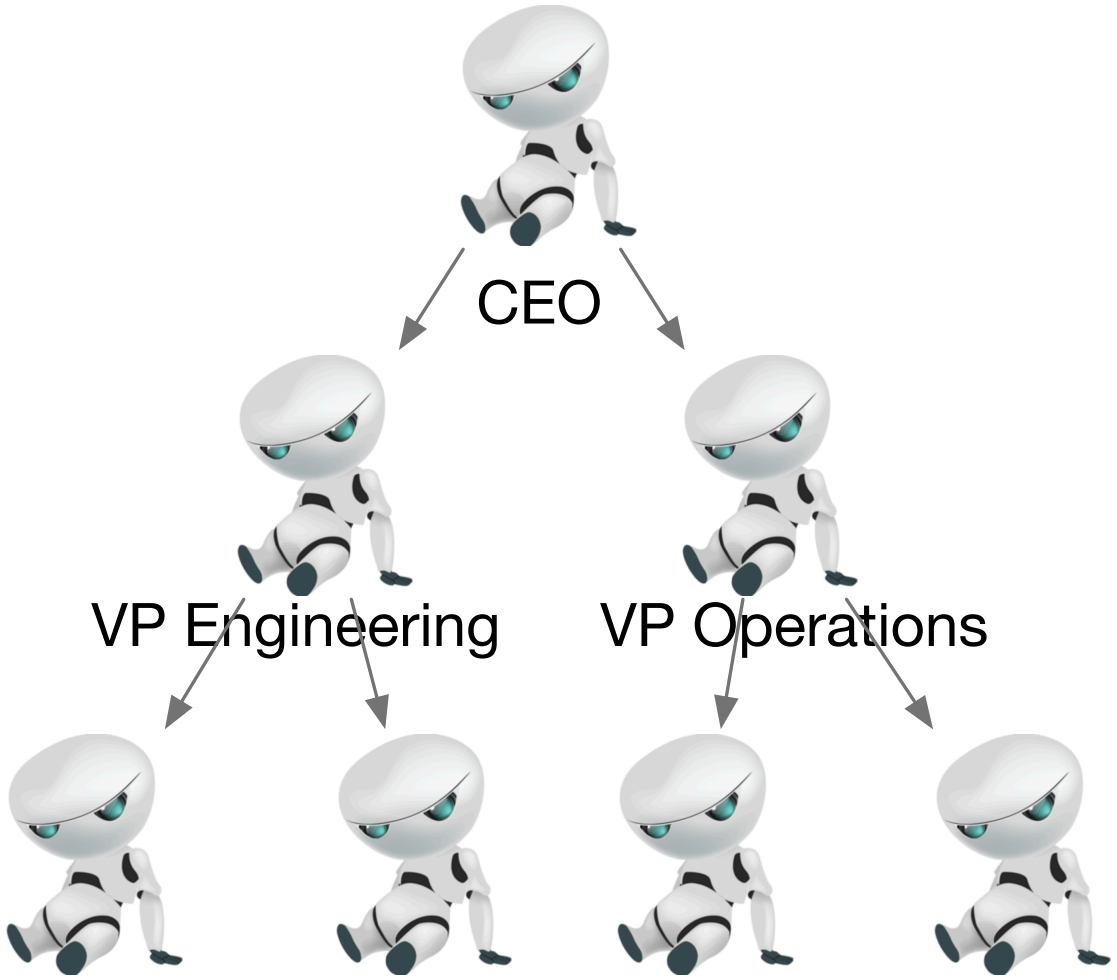
Recursion is just delegation

- Consider an org chart
- Bosses delegate to subordinates
- CEO wants work done:
  - sends emails to 2 VPs
  - who launch 2 other workers
  - those workers launch emails too
  - until potentially all *n* contacted
- Max work is *n* workers times work each worker performs
- Trace from CEO to mailroom cost is just height of org chart: *log(n)* in typical case
- Example: phone tree



Recursion is more like startup org chart

- One person has to play every role, cloning themselves
- Person ”pauses” current work, performs subtask(s), then continues where they left off, possibly using subtask results
- Each call or email to worker is analogous to a function call
- “Recursive leap of faith”
- (A large company could launch multiple core/employees)



Let's start with math recurrence relations

- Factorial definition:
  - Let 0! = 1
  - Define n! = n * (n-1)! for n>=1
- Recurrent math function calls become recursive function call in Python
- Non-recursive version is harder to understand and less natural
- This has linear time *O(n)*; T(n) = k + T(n-1) = kn; more clear in non-recursive version

```
def fact(n):
    if n==0: return 1
    return n * fact(n-1)
```

```
def factloop(n):
    r = 1
    for i in range(1,n+1):
        r *= i
    return r
```

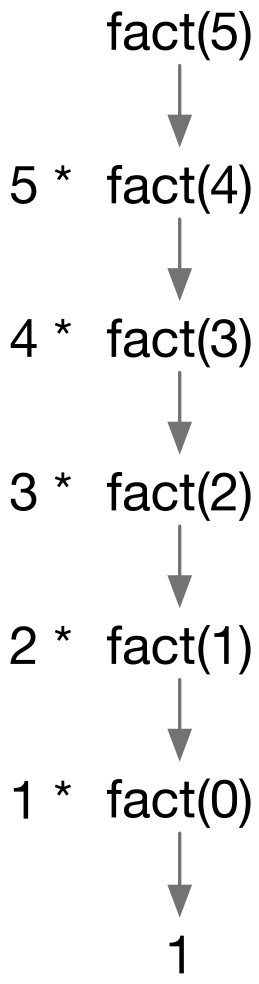
Fibonacci sequence

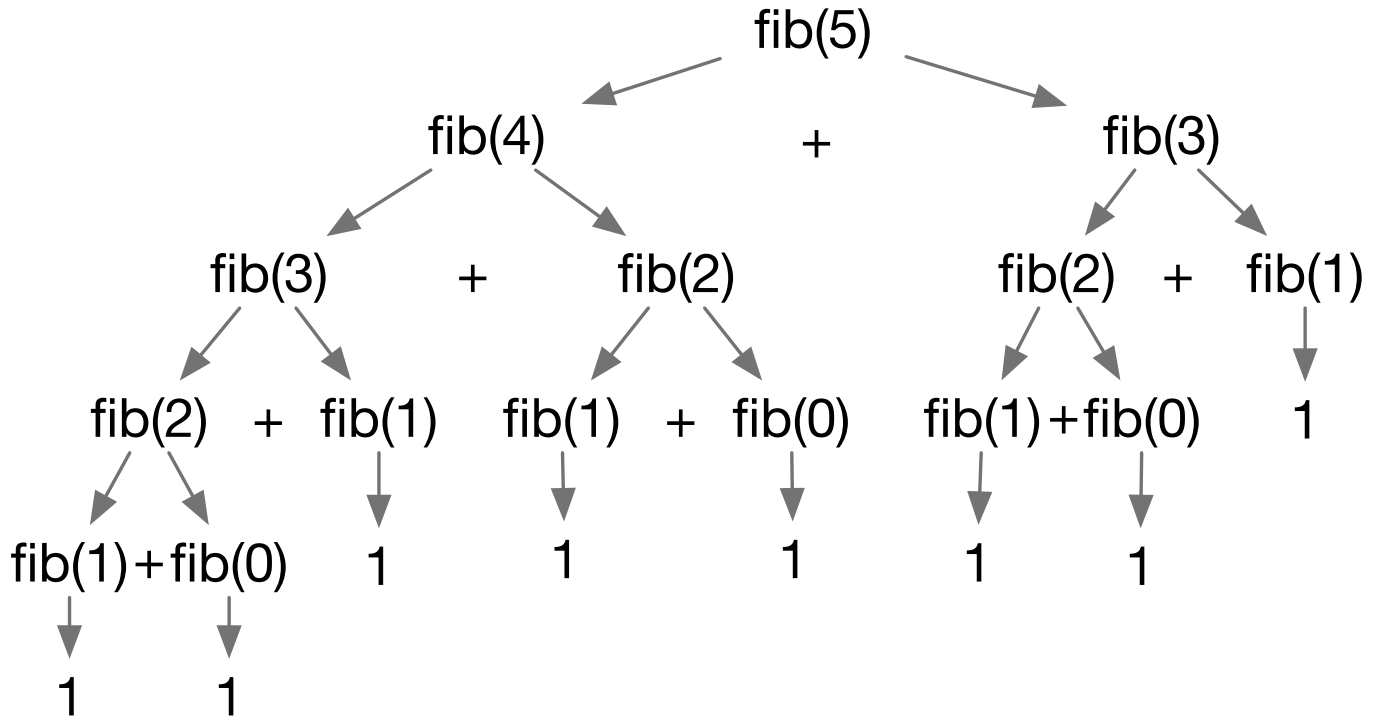
```
def fib(n):
    if n==0 or n==1: return 1
    return fib(n-1) + fib(n-2)
```

- Let F(0) = F(1) = 1
- Define F(n) = F(n-1) + F(n-2) for n>=2
- Recursive implementation is very natural but mucho inefficient!
- We do small bit of work to combine results of two subproblems, and each subproblem is only 1 and 2 elements smaller
- Compare this to fact(n) where we do small bit of work using result of a **single** subproblem, which has linear time

Compare fact, fib call trees visually

Difference comes down to two versus one subproblem!
Key idea: recursion traces out a tree of function calls
(implies recursion is most natural way to walk trees)





How fast is fib(n)?

```
def fib(n):
    if n==0 or n==1: return 1
    return fib(n-1) + fib(n-2)
```

- Seems like it should linear, right?
- Computation of fib(n-1) and fib(n-2) overlap, repeating same redundant computations
- Solving and combining results from two similarly-sized subproblems yields exponential complexity, O(k^n)
- fib(30) takes 0.5s on my fast mac
- fib(36) takes 9.0s
- fib(37) takes 16.7s

Exercise

```
def fib(n):
    if n==0 or n==1: return 1
    return fib(n-1) + fib(n-2)
```

- What is T(n) for fib()? Try to show it's exponential.
- T(n) = k + T(n-1) + T(n-2)
- T(n) = k + (k + T(n-2) + T(n-3)) + (k + T(n-3) + T(n-4))
- T(n) = k + (k + (k + T(n-3) + T(n-4)) + (k + T(n-4) + T(n-5))) + (k + (k + T(n-4) + T(n-5)) + (k + T(n-5) + T(n-6)))
- T(n) = 7k + T(n-3) + 3T(n-4) + 3T(n-5) + T(n-6)
- Yikes! I don't see the pattern, do you?
- Intuition: recursion tree is bushy; height n => 2^n nodes

Proof by “complete induction”

- Try T(n) = 2^n; it’s a good bet given the explosion of computations then prove by induction
- Complete induction: Assume T(i)=2^i holds for all i<n
- Observe that T(n+2) = k + T(n+1) + T(n)
- Complete induction assumption lets us directly substitute:
- T(n+2) = k + 2^(n+1) + 2^n = k + 2^1*2^n + 2^n = k + 2^n(2 + 1)
- So T(n) = k + 3*2^n, which is O(2^n)

Summary: formula for recursive functions

def f(input):
1. check termination condition
2. process the active input region / current node, etc…
3. invoke f on subregion(s)
4. combine and return results

Steps 2 and 4 are optional

```
def fact(n):
    if n==0: return 1
    return n * fact(n-1)
```

```
def fib(n):
    if n==0 or n==1: return 1
    return fib(n-1) + fib(n-2)
```

Terminology: *currently-active region* or *element* is what function is currently trying to process. Here, that is argument n (the “region” is the numbers 0..n)

Recursion at its finest: Divide and conquer

- The big idea: break a big problem down into smaller subproblems via recursion until subproblems are so small we can solve in constant time; then, merge partial results in linear time as you climb back up the recursive calls
- Examples: merge sort, quick sort, decision tree construction
- Recursion is easiest way to describe algorithms that break problems into multiple subproblems
- In contrast, algorithms that use a single recursive call are easy to convert to loops, and loops are usually more efficient

The nature of divide and conquer alg's

- Divide and conquer algorithms make 2 or more recursive calls where each subproblem is a **fraction** of the size of the currently active problem
- (Binary search splits problem in half each step but it descends into just one half with one recursive call not two and it doesn't merge results; technically, not divide and conquer…just divide)
- The cost of any recursive algorithms depends on:
  - the number of recursive subproblem calls per active region
  - the size of the subproblems; for example, n-1 or n/2 for active size n?
  - the work required for each active region

Merge sort
O(n log n)

- The idea is to split currently active region in half, sorting both the left and right subregions, then merge two sorted subregions
- Eventually, the regions are so small we can sort in constant time; i.e., sorting 2 nums is easy
- Merging two sorted lists can be done in linear time

https://youtu.be/XaqR3G_NVoo

Algorithm is simple but efficient (nlogn)

```
def msort(A):
    if len(A)==1:
        return A
    if len(A)==2:
        return A if A[0]<A[1] else [A[1],A[0]]
    mid = int(len(A)/2)
    left = msort(A[:mid])
    right = msort(A[mid:])
    return sorted(left + right)
```

- “sorted(left + right)” is cheating but used for simplicity here; merging two sorted lists is O(n)
- (Note: constant on this is huge)

Exercise: write pow2(n)

- Compute 2^n recursively by multiplying 2 together n times

```
def pow2(n):
    if n==0: return 1
    return 2 * pow2(n-1)
```

- Now do iterative version

```
def pow2(n):
    v = 1
    for i in range(n): v *= 2
    return v
```

Can we do pow2 more efficiently?

- Divide and conquer; hint 2^n = 2^(n/2) * 2^(n/2) if n is even
  If n odd, it's 2 times 2^floor(n/2)

```
def pow2(n):
    if n==0: return 1
    half = pow2(int(n/2))
    if n % 2==0: # if even
        return half * half
    return 2 * half * half
```

- A non-recursive version would be awkward and hard to write!

Recursion summary

- Use recursion when implementing a recurrence relation (not always for efficiency reasons, such as gradient descent)
- Use recursion when you can break problem down into subproblems whose results can be merged to solve overall problem; divide and conquer
- Use recursion when the same operation and navigation procedure applies to any element of a data structure; binary search, walking trees and graphs; self-similar structures
- Recursion traces out a function call tree

An Aside: Caching partial results with dynamic programming

Dynamic programming trades memory for speed

- We can often can decompose a problem into subproblems and combine the partial results to form the overall result
- If the same computation is repeated, it's a candidate for dynamic programming
- Dynamic programming is a terrible, meaningless name that simply means caching or *memoizing* partial results and using those to avoid redundant computations
- Idea: compute "1+1+1+1". That's 4. If I append "+1", what is sum?
- You reused the 4 result hopefully; that's dynamic programming

https://www.geeksforgeeks.org/dynamic-programming

Example: Use *dynamic programming* to solve Fibonacci in O(n) not O(k^n)

- Make cache F of size n, working upwards, fill in F

```
def cachefib(n):
    F = [0 for i in range(n+1)]
    F[0] = F[1] = 1          # init cache
    for i in range(2,n+1): # work up not down
        F[i] = F[i-1] + F[i-2]
    return F[n]
```

- cachefib(1000) take 0.3s compared to fib(37) at 16.7s (wow)
- Algorithmic changes matter much more than code optimizations!!!

Example: use cache with recursive fib()

- The recursive version is more natural; to make efficient record partial results and look in cache before computing anything

```
memo = {}
def memofib(n):
    if n==0 or n==1: return 1
    # return if already computed
    if n in memo: return memo[n]
    f = memofib(n-1) + memofib(n-2)
    memo[n] = f # memoize result
    return f
```

Memoized recursive function template

def f(input):

1. check termination condition as usual
2. return memoized result if available
3. process the active input region / current node, etc…
4. invoke f on subregion(s)
5. combine results and memoize
6. return results

Blue regions differ from normal recursive function template

Check out the recursion notebook

https://github.com/parrt/msds689/blob/master/notes/recursion-notebook.ipynb