Exercises Set 6

Solutions

1- Lagrangian Multiplier Technique

1.1 - Unconstrained optimization

$$f(x,y) = 2(x-3)^2 + 4(y+1)^2 + \dots$$

$$\Rightarrow \quad f(x,y) \geq f(3,-1)$$

$$(x^*, y^*) = (3,-1)$$

$$\overline{\nabla}.f = \begin{pmatrix} 4x - 12 \\ 8y + 8 \end{pmatrix}$$

$$\nabla.f = 0 \iff \begin{cases} 4x - 12 = 0 & \iff x = 3 \\ 8y - 8 = 0 & \iff y = -1 \end{cases}$$

Hence, extremum of f is reached for $x=3$, $y=-1$.

f is bounded below & unbounded above, so $(3,-1)$ must be min of f.

1.2 - (Equality) Constrained optimization

$$3x + 5y = 2 \quad \Rightarrow \quad y = \frac{2}{5} - \frac{3}{5}x$$

let $g(x) = f\left(x, \frac{2}{5} - \frac{3}{5}x\right)$

$$= 2x^2 - 12x + 4\left(\frac{2}{5} - \frac{3}{5}x\right)^2 + 8\left(\frac{2}{5} - \frac{3}{5}x\right) + 20$$

$$g'(x) = 4x - 12 + 8\left(\frac{2}{5} - \frac{3}{5}x\right)\left(-\frac{3}{5}\right) - \frac{24}{5}$$

$$= \frac{100}{25}x - \frac{300}{25} - \frac{48}{25} + \frac{72}{25}x - \frac{120}{25}$$

$$= \frac{172}{25}x - \frac{468}{25}$$

$$g'(x) = 0 \iff x = \frac{468}{172} = \frac{234}{86} = \frac{117}{43}$$

$$y = \frac{2}{5} - \frac{3}{5} \cdot \frac{117}{43} = \frac{86 - 351}{215} = -\frac{265}{215} = -\frac{53}{43}$$

hence, min is reached for $x = \frac{117}{43}$, $y = -\frac{53}{43}$

1.3 - Lagrange Multiplier

$$\mathcal{L}(x,y,\lambda) = 2x^2 - 12x + 4y^2 + 8y + 20 - 3\lambda x - 5\lambda y + 2\lambda$$

$$\nabla.\mathcal{L} = \begin{pmatrix} 4x - 12 - 3\lambda \\ 8y + 8 - 5\lambda \\ -3x - 5y + 2 \end{pmatrix}$$

$$\nabla.\mathcal{L} = 0 \iff \begin{cases} 4x - 12 - 3\lambda = 0 \\ 8y + 8 - 5\lambda = 0 \\ -3x - 5y + 2 = 0 \end{cases}$$

$$\iff \begin{cases} 4x \quad -3\lambda = 12 \\ 8y \quad -5\lambda = -8 \\ -3x \quad -5y = -2 \end{cases}$$

let $A = \begin{pmatrix} 4 & 0 & -3 \\ 0 & 8 & -5 \\ -3 & -5 & 0 \end{pmatrix}$, $u = \begin{pmatrix} 12 \\ -8 \\ -2 \end{pmatrix}$

need $w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}$ s.t. $A.w = u$

$\iff w = A^{-1}.u$

$$A^{-1} = \frac{1}{172}\begin{pmatrix} 25 & -15 & -24 \\ -15 & 9 & -20 \\ -24 & -20 & -32 \end{pmatrix}$$

so $w = \frac{1}{172}\begin{pmatrix} 468 \\ -212 \\ -64 \end{pmatrix} = \begin{pmatrix} 117/43 \\ -53/43 \\ -16/43 \end{pmatrix}$

so $x = \frac{117}{43}$, $y = -\frac{53}{43}$, $\lambda = \frac{-16}{43}$

$\Rightarrow$ we find the same x, y as before

1.4 - (Inequality) Constrained Optimization

$f(x) = (x-1)^2 - 1$

$\Rightarrow f(x) \geq f(1)$

$3.1 > 2$ $\cancel{\checkmark}$ so we should acheive equality constr.

hence, $3x^* = 2 \iff x^* = \frac{2}{3}$

$f(x) = (x+1)^2 - 1$

$\Rightarrow f(x) \geq f(-1)$

$3 . -1 \leq 2$ so $x^* = -1$

1.5 - Lagrange Multipliers

$f(x) = x^2 - 2x$

$\mathcal{L}(x, \lambda) = x^2 - 2x - 3\lambda x + 2\lambda$

$$\nabla . \mathcal{L} = \begin{pmatrix} 2x - 2 - 3\lambda \\ -3x + 2 \end{pmatrix}$$

$$\nabla . \mathcal{L} = 0 \iff \begin{cases} -3x + 2 = 0 & \iff x = 2/3 \\ 2 . \frac{2}{3} - 2 - 3\lambda = 0 & \iff \lambda = -\frac{2}{9} \end{cases}$$

$\Rightarrow$ find $x^* = \frac{2}{3}$ & constraint is met as $\lambda = -\frac{2}{9} < 0$

$f(x) = x^2 + 2x$

$\mathcal{L}(x, \lambda) = x^2 + 2x - 3\lambda x + 2\lambda$

$$\nabla . \mathcal{L} = \begin{pmatrix} 2x + 2 - 3\lambda \\ -3x + 2 \end{pmatrix}$$

$$\nabla . \mathcal{L} = 0 \iff \begin{cases} -3x + 2 = 0 & \iff x = 2/3 \\ 2 . \frac{2}{3} + 2 - 3\lambda = 0 & \iff \lambda = \frac{10}{9} \end{cases}$$

$\lambda > 0$ so no need of bdry condi° : $f'(x) = 0$

$\iff 2x + 2 = 0$

$\iff x = -1 \in$ is in domain

$\Rightarrow$ find $x^* = -1$ & constraint is not met on its boundary as $\lambda \geq 0$.

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2 - S.V.M. & 3 - Logistic Regression were done live-style in class.