Exercises Set 2

Solutions

1 - System of Linear Equations

Reduced Row Echelon Form

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 2 & 3 \\ 0 & 0 & 0 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1/3 & 1/2 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad L_2 = L_2/6$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 7/3 & 3 \\ 0 & 1 & 1/3 & 1/2 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad L_1 = L_1 - 2.L_2$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 7/3 & 3 \\ 0 & 1 & 1/3 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad L_3 = L_3/10$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 7/3 & 3 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad L_2 = L_2 - 1/2.L_3$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 7/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad L_1 = L_1 - 3.L_3$$

Let $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $v = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$ $A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & -1 & 3 \end{pmatrix}$

Then, the system of linear equations is just $A.u = v$

$\iff u = A^{-1}.v$

The augmented matrix $(A \vdots Id_3)$ is:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 3 & 0 & 0 & 1 \end{array}\right)$$

[...]

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/11 & -2/11 & 3/11 \\ 0 & 1 & 0 & -8/11 & 7/11 & -5/11 \\ 0 & 0 & 1 & -5/11 & 3/11 & 1/11 \end{array}\right)$$

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 7 & -2 & 3 \\ -8 & 7 & -5 \\ -5 & 3 & 1 \end{pmatrix}$$

so $u = \frac{1}{11}\begin{pmatrix} 39 \\ -32 \\ -12 \end{pmatrix} = \begin{pmatrix} 39/11 \\ -32/11 \\ -12/11 \end{pmatrix}$

2 - Vector Spaces

Suppose $x.v_1 + y.v_2 + z v_3 = 0$

$$\iff \begin{cases} x + 3z = 0 \\ 2x + y + 5z = 0 \\ y + 2z = 0 \end{cases}$$

$L_2 - 2L_1 - L_3 \implies -3z = 0 \implies z = 0$

so $L_1 \implies x = 0$ & $L_3 \implies y = 0$

Thus, $x = y = z = 0$, so $\{v_1, v_2, v_3\}$ are linearly indep.

Suppose $f(x) = ax^2 + bx + c$

Then $f(x) = \frac{a}{3} p_3(x) + \frac{b}{2} p_2(x) + \left(\frac{a}{3} + c\right) p_1(x)$

Hence, $\{p_1, p_2, p_3\}$ spans $\mathbb{P}_2$

3 - Matrix Inverses

$\det(A) = 2^2 - 1.3 = 1$

$A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ -> as if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$A A^{-1} = Id_2$ & $A^{-1} A = Id_2$

$\det(B) = 2.2.1 + 1.0.(-1) + 3.1.3$
$- (-1).2.3 - 3.0.2 - 1.1.1$
$= 4 + 0 + 9 + 6 - 0 - 1 = 18$

The augmented matrix $(B \,\vdots\, Id_3)$ is:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \to [\ldots] \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/9 & 4/9 & -1/3 \\ 0 & 1 & 0 & -1/18 & 5/18 & 1/6 \\ 0 & 0 & 1 & 5/18 & -7/18 & 1/6 \end{array}\right)$$

So $B^{-1} = \frac{1}{18} \begin{pmatrix} 2 & 8 & -6 \\ -1 & 5 & 3 \\ 5 & -7 & 3 \end{pmatrix}$

$B B^{-1} = Id_3$ $B^{-1} B = Id_3$

4 - Eigenvalues & Eigenvectors

$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ $\quad \det(A - \lambda I_2) = (3-\lambda)^2 - 1$

$= 8 - 6\lambda + \lambda^2$

$\Delta = 6^2 - 4.8.1 = 36 - 32 = 4$

$\lambda = \frac{6 \pm 2}{2} = 3 \pm 1 = 2, 4$

Supp $u = \begin{pmatrix} x \\ y \end{pmatrix}$ is s.t. $Au = 2u$

$\Leftrightarrow \begin{cases} 3x + y = 2x \\ x + 3y = 2y \end{cases}$

$\Leftrightarrow x + y = 0$

let $x = 1 \Rightarrow y = -1$ so $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$Au = 4u$

$\Leftrightarrow \begin{cases} 3x + y = 4x \\ x + 3y = 4y \end{cases}$

$\Leftrightarrow x = y$

let $x = 1 \Rightarrow y = 1$ so $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ $\quad \det(B - \lambda I_2) = (1-\lambda)^2$

$\Rightarrow \lambda = 1$

Supp $u = \begin{pmatrix} x \\ y \end{pmatrix}$ s.t. $Au = u$

$\Leftrightarrow \begin{cases} x - y = x \\ y = y \end{cases}$

$\Rightarrow y = 0$

let $x = 1$ so $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \qquad \det(C - \lambda I_3) = (1-\lambda)(2-\lambda)(3-\lambda)$$

$$\Rightarrow \lambda = 1, 2, 3$$

Supp. $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is s.t. $Cu = u$

$$\begin{cases} x + 2y = x \\ 2y = y \\ y + 3z = z \end{cases}$$

$\Rightarrow$ $y = 0$ & $z = 0$

let $x = 1$ so $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$Cu = 2u$

$$\begin{cases} x + 2y = 2x \\ 2y = 2y \\ y + 3z = 2z \end{cases}$$

$$\begin{cases} x = 2y \\ z = -y \end{cases}$$

set $y = 1$ so $u = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$Cu = 3u$

$$\begin{cases} x + 2y = 3x \\ 2y = 3y \\ y + 3z = 3z \end{cases}$$

$\Rightarrow$ $y = 0$ & $x = 0$

let $z = 1$ so $u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

5 - Diagonalization

$A = PDP^{-1}$ with $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $\qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$

$$P^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

B : Only one eigenvalue $\Rightarrow$ diagonalization is not possible (check out "Jordan normal form" if interested)

$$C = PDP^{-1} \quad \text{with} \quad P = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1/6 & 1/3 \end{pmatrix}$$

6 - Orthogonal Vectors

$\underline{u} \cdot \underline{v} = 2.1 + 2.(-1) + 0.1 = 0$ hence, $\underline{u} \perp \underline{v}$

$\underline{u} \cdot \underline{w} = 2.0 + 1(-1) + 0.(-2) = -1$ hence, $\underline{u} \not\perp \underline{w}$

$\underline{v} \cdot \underline{w} = 1.0 + 2.1 + 1.(-2) = 0$ hence, $\underline{v} \perp \underline{w}$

$\|\underline{v}_1\| = \sqrt{5}$

$$\underline{\tilde{v}}_1 = \frac{1}{\|\underline{v}_1\|} v_1 \simeq \begin{pmatrix} 0.447 \\ 0.894 \\ 0 \end{pmatrix}$$

$$\underline{v}_2' = \underline{v}_2 - (\underline{\tilde{v}}_1 \cdot \underline{v}_2)\, \underline{\tilde{v}}_1$$

$$\underline{\tilde{v}}_2 = \frac{1}{\|\underline{v}_2'\|} \underline{v}_2' \simeq \begin{pmatrix} 0.365 \\ -0.183 \\ 0.913 \end{pmatrix}$$

$$\underline{v}_3' = \underline{v}_3 - (\underline{\tilde{v}}_1, \underline{v}_3)\, \underline{\tilde{v}}_1 - (\underline{\tilde{v}}_2, \underline{v}_3)\, \underline{\tilde{v}}_2$$

$$\underline{\tilde{v}}_3 = \frac{1}{\|\underline{v}_3'\|} \underline{v}_3' \simeq \begin{pmatrix} -0.816 \\ 0.408 \\ 0.408 \end{pmatrix}$$