# Exercises Set 3

## Solutions

### 1 - Fundamental Theorem of Calculus

$$F(x) = \int_0^x f(t)\,dt \implies F'(x) = f(x)$$

$$\int_a^b f(t)\,dt = \underbrace{\int_0^b f(t)\,dt}_{F(b)} - \underbrace{\int_0^a f(t)\,dt}_{F(a)} = F(b) - F(a)$$

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$$\int_0^{\pi/2} \sin(x)\,dx = \Big[-\cos(x)\Big]_0^{\pi/2} = -\underbrace{\cos(\pi/2)}_{0} - (-\underbrace{\cos(0)}_{1}) = 1$$

$$\int_1^4 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}$$

### 2 - Integration Thechniques

$$\int f(u)\,du = \int f(t)\frac{du}{dt}\,dt \quad \text{where } \frac{du}{dt} \text{ is derivative of } u \text{ w.r.t. } t.$$

$$\int uv' = uv - \int vu'$$

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Ex. 1: $(x^2+1)' = 2x$

so $\left(\ln(x^2+1)\right)' = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$

Hence, $\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1) + c$

Ex. 2: $\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin(u)^2}} \cdot \cos(u)\,du$ with $x = \sin(u)$, $u = \arcsin(x)$, $\frac{dx}{du} = \cos(u)$

$= \int \frac{1}{\sqrt{\cos(u)^2}} \cdot \cos(u)\,du$ as $\cos(t)^2 + \sin(t)^2 = 1$

$= \int 1\,du = u + c = \arcsin(x) + c$

Ex.3 ; $\int \frac{1}{4+x^2}\,dx = \int \frac{1}{4+4u^2}\cdot 2\,du$ with $u = x/2$, $x = 2u$, $\frac{dx}{du} = 2$

$$= \int \frac{2}{4}\,\frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\cdot \arctan(u) + c$$

$$= \frac{1}{2}\cdot \arctan\left(\frac{x}{2}\right) + c$$

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Ex. A; $\int x\,.\ln(x)\,dx = \frac{1}{2}x^2.\ln(x) - \int \frac{1}{2}x^2.\frac{1}{x}\,dx$

$$= \frac{1}{2}x^2.\ln(x) - \int \frac{1}{2}x\,dx$$

$$= \frac{1}{2}x^2.\ln(x) - \frac{1}{4}x^2 + c$$

Ex. B; $\int x^2.e^x\,dx = x^2.e^x - \int 2x.e^x\,dx$

$$= x^2.e^x - 2xe^x + \int 2e^x\,dx$$

$$= (x^2 - 2x + 2).e^x + c$$

Ex. C; $\int x.\cos(x)\,dx = x.\sin(x) - \int \sin(x)\,dx$

$$= x.\sin(x) + \cos(x) + c$$

Ex. D: $\int e^{2x}\cos(2x)\,dx = \frac{1}{2}e^{2x}\cos(2x) - \int \cancel{\tfrac{1}{2}}e^{2x}.\sin(2x).\cancel{2}\,dx$

$$= \frac{1}{2}e^{2x}\cos(2x) - \frac{1}{2}e^{2x}\sin(2x) - \int e^{2x}\cos(2x)\,dx$$

$$\Leftrightarrow 2\int e^{2x}.\cos(2x)\,dx = \frac{1}{2}e^{2x}\left(\cos(2x) - \sin(2x)\right) + 2c$$

$$\Leftrightarrow \int e^{2x}.\cos(2x)\,dx = \frac{1}{4}e^{2x}\left(\cos(2x) - \sin(2x)\right) + c$$

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Ex d; $(x^3 - x^2 + x - 1)' = 3x^2 - 2x + 1$

Hence, $\left(\ln(x^3 - x^2 + x - 1)\right)' = \frac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1}$

so $\int \frac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1}\,dx = \ln(x^3 - x^2 + x - 1) + c$

Ex. α (bis):

$$3x^2 - 2x - 1 = (x-1)(3x+1)$$

$$x^3 - x^2 + x - 1 = (x-1)(x^2+1)$$

so $$\frac{3x^2-2x-1}{x^3-x^2+x-1} = \frac{3x+1}{x^2+1} = \frac{3}{2}\left(\frac{2x}{x^2+1}\right) + \frac{1}{x^2+1}$$

Finally, $$\int \frac{3x^2-2x-1}{x^3-x^2+x-1}\,dx = \frac{3}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \frac{3}{2}\ln(x^2+1) + \arctan(x) + C$$

Ex. β: $$\int_0^{+\infty} e^{-x}\,dx = \lim_{a\to+\infty} \int_0^a e^{-x}\,dx$$

$$= \lim_{a\to+\infty} \left[-e^{-x}\right]_0^a$$

$$= \lim_{a\to+\infty} \underbrace{-e^{-a}}_{\to 0} + \underbrace{e^0}_{1} = 1$$

Ex. γ: 4 intervals: $\left[0, \frac{\pi}{8}\right]$ $\left[\frac{\pi}{8}, \frac{\pi}{4}\right]$ $\left[\frac{\pi}{4}, \frac{3\pi}{8}\right]$ $\left[\frac{3\pi}{8}, \frac{\pi}{2}\right]$

$\sin(0) = 0$

$\sin\left(\frac{\pi}{8}\right) \simeq 0.383$

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \simeq 0.707$

$\sin\left(\frac{3\pi}{8}\right) \simeq 0.924$

$\sin\left(\frac{\pi}{2}\right) = 1$

Trapezoidal approx.: $$\int_0^{\pi/2} \sin(x)\,dx \simeq \left(\frac{\pi}{2} - 0\right)\left(\frac{1}{8}\sin(0) + \frac{1}{4}\sin\left(\frac{\pi}{8}\right) + \frac{1}{4}\sin\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin\left(\frac{3\pi}{4}\right) + \frac{1}{8}\sin\left(\frac{\pi}{2}\right)\right) \simeq 0.987$$

Real value is 1, so with only 4 interval, the approximation is <u>very</u> good!

Ex. 8: 3 intervals: $\left[0, \frac{\pi}{6}\right]$ $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

$$\sin(0) = 0$$
$$\sin\left(\frac{\pi}{12}\right) \simeq 0.259$$
$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = 0.5$$
$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \simeq 0.707$$
$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \simeq 0.866$$
$$\sin\left(\frac{5\pi}{12}\right) \simeq 0.966$$
$$\sin\left(\frac{\pi}{2}\right) = 1$$

Simpson's rule:
$$\int_0^{\pi/6} \sin(x)\,dx \simeq \left(\frac{\pi}{6} - 0\right) \frac{1}{6}\left(\sin(0) + 4.\sin\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{6}\right)\right)$$
$$\simeq 0.134$$
$$\int_{\pi/6}^{\pi/3} \sin(x)\,dx \simeq \left(\frac{\pi}{3} - \frac{\pi}{6}\right) \frac{1}{6}\left(\sin\left(\frac{\pi}{6}\right) + 4.\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\right)$$
$$\simeq 0.366$$
$$\int_{\pi/3}^{\pi/2} \sin(x)\,dx \simeq \left(\frac{\pi}{2} - \frac{\pi}{3}\right) \frac{1}{6}\left(\sin\left(\frac{\pi}{3}\right) + 4.\sin\left(\frac{5\pi}{12}\right) + \sin\left(\frac{\pi}{2}\right)\right)$$
$$\simeq 0.500$$

$$\Rightarrow \int_0^{\pi/2} \sin(x)\,dx \simeq 1.000$$

Approximating calculations to $10^{-3}$, we get the correct answer with only 3 intervals!

Simpson's rule is a very powerful tool to approximate integrals of continuous smooth functions.

Trapezoid rule is preferred for non-smooth / non-continuous functions.

## 3- Applications

$$A = \int_0^{\pi} \sin(x) - (-\sin(x))\, dx$$
$$= \int_0^{\pi} 2\sin(x)\, dx = \left[-2\cos(x)\right]_0^{\pi} = -2\cos(\pi) + 2\cos(0) = 4$$

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$$V = \int_0^1 \pi (x^2)^2\, dx = \pi \int_0^1 x^4\, dx = \pi \left[\tfrac{1}{5}x^5\right]_0^1 = \pi/5$$

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Using symmetry by the $y=x$ line, the arc length of $y=\sqrt{x}$ from $x=1$ to $x=4$ is the same as the arclength of $y=x^2$ from $x=1$ to $x=2$.

$$\mathcal{L} = \int_1^2 \sqrt{1+(2x)^2}\, dx$$
$$= \int_1^2 \sqrt{1+4x^2}\, dx$$
$$= \left[x\sqrt{1+4x^2}\right]_1^2 - \int_1^2 \frac{x}{2\sqrt{1+4x^2}}\, dx$$
$$= \left[x\sqrt{1+4x^2}\right]_1^2 - \left[\frac{1}{8}\ln\left(\sqrt{4x^2+1}\right)\right]_1^2$$

$$\left(\ln\left(\sqrt{4x^2+1}\right)\right)' = \frac{1}{\sqrt{4x^2+1}} \cdot \frac{1}{2\sqrt{4x^2+1}} \cdot 8x$$

$$\left(\ln\left(\sqrt{4x^2+1} + 2x\right)\right)'$$
$$= \frac{2}{\sqrt{4x^2+1}+2x}\left(\frac{4\cancel{8}x}{\cancel{2}\sqrt{4x^2+1}} + 2\right)$$
$$= \frac{4}{\sqrt{4x^2+1}+2x} + \frac{8x}{(4x^2+1)+2x\sqrt{4x^2+1}}$$
$$= \frac{16x^2+4+8x\sqrt{4x^2+1}+8x\sqrt{4x^2+1}+16x^2}{(4x^2+1)^{3/2}+2x(4x^2+1)+2x(4x^2+1)+4x^2\sqrt{4x^2+1}}$$
$$= \frac{x}{\sqrt{4x^2+1}} \cdot \frac{4\left(2x+\sqrt{4x^2+1}\right)}{\left(2x+\sqrt{4x^2+1}\right)}$$

$$\boxed{\left(2x+\sqrt{4x^2+1}\right)^2 = 4x^2 + 4x\sqrt{4x^2+1} + 4x^2+1}$$

$$= \frac{4x}{\sqrt{4x^2+1}}$$

$$= 2\sqrt{1+4\cdot 2^2} - \sqrt{1+4\cdot 1^2} - \left(\frac{1}{8}\ln\left(\sqrt{4\cdot 2^2+1}\right) - \frac{1}{8}\ln\left(\sqrt{4\cdot 1^2+1}\right)\right)$$
$$= 2\sqrt{17} - \sqrt{5} - \frac{1}{8}\ln\left(\sqrt{17}\right) + \frac{1}{8}\ln\left(\sqrt{5}\right) = 2\sqrt{17} - \sqrt{5} - \frac{1}{16}\ln(17) + \frac{1}{16}\ln(5)$$

$$\mathcal{S} = \int_0^1 2\pi\, x^2 \sqrt{1+(2x)^2}\, dx$$

$$= \int_0^1 2\pi\, x^2 \sqrt{1+4x^2}\, dx$$

... (use Wolfram Alpha) ...

$$= \left[ \frac{\pi}{32} \left( 2x\sqrt{4x^2+1}\,(8x^2+1) - \sinh^{-1}(2x) \right] \right._0^1$$

$$= \frac{\pi}{32} \left( 2\sqrt{5} \cdot 9 - \sinh^{-1}(2) \right)$$

$$= \frac{9\pi}{16}\sqrt{5} - \frac{\pi \sinh^{-1}(2)}{32} \simeq 3.8097$$