

Exercises Set 5

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Abstract

Only the questions with a * are compulsory (but do all of them!).

1 Change of Basis

Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard canonical basis for \mathbb{R}^3 .

Suppose we have another basis $\mathcal{B}' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for \mathbb{R}^3 and let Q be the matrix whose columns are the coordinates of

$$\mathbf{u}_1 = \begin{pmatrix} 0.5 \\ -1 \\ 1 \end{pmatrix}_{\mathcal{B}}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}_{\mathcal{B}}, \text{ and } \mathbf{u}_3 = \begin{pmatrix} -0.25 \\ 0.5 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

with respect to the standard basis. That is, $Q = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$.

Let $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}'}$. Express \mathbf{v} in the standard basis \mathcal{B} .

Let $\mathbf{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}}$. Express \mathbf{w} in the basis \mathcal{B}' .

2 Variance and Covariance

Calculate the variance of the following set:

$$\mathcal{S}_1 = \{1.5, 3, 5, 7.5, 8, 9\}$$

Calculate the variance of the following set:

$$\mathcal{S}_2 = \{2, 4, 6, 8, 10\}$$

Calculate the covariance of \mathcal{S}_1 and \mathcal{S}_2 .

Compute $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$, the standardized version of \mathcal{S}_1 and \mathcal{S}_2 (shifted to mean 0 and scaled to have a variance of 1).

Calculate the covariance of $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$. What do you remark?

3 Principal Component Analysis

Let $\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ be a set of 5 points in \mathbb{R}^3 .

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \mathbf{x}_4 = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}, \text{ and } \mathbf{x}_5 = \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix}.$$