

# Exercises Set 5

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September 27, 2023

## Abstract

Only the questions with a \* are compulsory (but do all of them!).

## 1 Change of Basis

Let  $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  be the standard canonical basis for  $\mathbb{R}^3$ .

Suppose we have another basis  $\mathcal{B}' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  for  $\mathbb{R}^3$  and let  $Q$  be the matrix whose columns are the coordinates of

$$\mathbf{u}_1 = \begin{pmatrix} 0.5 \\ -1 \\ 1 \end{pmatrix}_{\mathcal{B}}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}_{\mathcal{B}}, \text{ and } \mathbf{u}_3 = \begin{pmatrix} -0.25 \\ 0.5 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

with respect to the standard basis. That is,  $Q = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$ .

Let  $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}'}$ . Express  $\mathbf{v}$  in the standard basis  $\mathcal{B}$ .

Let  $\mathbf{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}}$ . Express  $\mathbf{w}$  in the basis  $\mathcal{B}'$ .

## 2 Variance and Covariance

Calculate the variance of the following set:

$$\mathcal{S}_1 = \{1.5, 3, 5, 7.5, 8, 9\}$$

Calculate the variance of the following set:

$$\mathcal{S}_2 = \{2, 4, 6, 8, 10\}$$

Calculate the covariance of  $\mathcal{S}_1$  and  $\mathcal{S}_2$ .

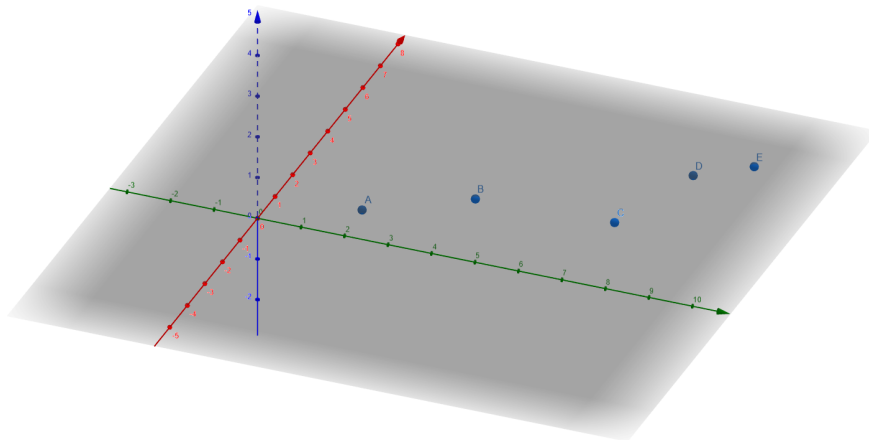
Compute  $\hat{\mathcal{S}}_1$  and  $\hat{\mathcal{S}}_2$ , the standardized version of  $\mathcal{S}_1$  and  $\mathcal{S}_2$  (shifted to mean 0 and scaled to have a variance of 1).

Calculate the covariance of  $\hat{\mathcal{S}}_1$  and  $\hat{\mathcal{S}}_2$ . What do you remark?

### 3 Principal Component Analysis

Let  $\mathcal{S} = \{A, B, C, D, E\}$  be a set of 5 points in  $\mathbb{R}^3$ .

$$A = \begin{pmatrix} 1 \\ 2 \\ 0.1 \end{pmatrix}, B = \begin{pmatrix} 2.5 \\ 4 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 3 \\ 7 \\ -0.2 \end{pmatrix}, D = \begin{pmatrix} 5 \\ 8 \\ 0.1 \end{pmatrix}, \text{ and } E = \begin{pmatrix} 6 \\ 9 \\ 0 \end{pmatrix}.$$



#### 3.1 Standardization

Calculate  $\hat{\mathcal{S}}$ , the standardized version of  $\mathcal{S}$  (shifted to mean 0 and scaled to have a variance of 1).

#### 3.2 Covariance matrix

Compute the covariance of each pair of features. Compute also the variance of each feature. Arrange the values in a  $3 \times 3$  matrix (variance is covariance of a feature with itself).

#### 3.3 Eigenvalues of the covariance matrix

Calculate the eigenvalues of the covariance matrix. Use the characteristic polynomial.

#### 3.4 Feature vectors (the "principal components")

#### 3.5 Recasting data on principal components axes