Mathematics Refresher Course
First Two Sessions

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Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. We will try to cover most of the prerequisites of the courses in the master's, i.e. basic algebra/analysis and basic applications.

1 Presentation

- Paul Dubois
- 3rd year PhD @ Centrale / TheraPanacea
- Research topic: AI applied to radiotherapy
- Email:`b00795695@essec.edu` (for any question)

Course structure

  - 8*3h arranged as 1h20min lecture - 1/3h break - 1h20min lecture
  - ~~No pb class planned, but lectures will have integrated live exercises~~
  - ~~Interrupt if needed (do *not* wait for the end of the lecture)~~
  - In this document, you will find the content of the first two sessions, with the small exercises we did "live".
  - The remaining six sessions will be problem solving.
    In case a session is spent on a topic you already, you can skip it **on the condition that you submit all compulsory exercises corresponding to that session**.

- Examination

  - The course is pass/fail
  - Spoiler: All of you will pass
  - ~~Home exercises, you will need 80+% to pass~~
  - ~~to complete exercises, it should take 30min to 1h~~
  - ~~2-4 exercises~~
  - ~~Hand in paper of PDF~~
  - ~~In the unlikely event of not passing, you will be able to do some extra work to pass~~
  - To pass, I will ask you, for each session, to either be in class, or submit the compulsory exercises.
  - The submission deadlines for the exercises set is exactly one week after the corresponding class.

- Submitting
  1. Solve exercises
  2. Export you work to a single PDF file (e.g. using a scanning smartphone app)
  3. Rename your file "submission_nb_family_name.pdf" where:
     - "nb" is "2" for exercises set 2, "3" for exercise set 3, etc...
     - "family_name" is your family name in latin alphabet, capital letters

     *Example: if I wanted to submit exercise set 1, the name of my file should have been "**submission_1_DUBOIS.pdf**"*
  4. Send me one new email per submission, please do not use the "reply" button, create a new email;
     For the subject, you can just put the name of the file (or anything else that makes sense).

2 Sets

- sets of numbers ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{P}$)
- complex sets (with {})
- examples (draw them):
  - $\{n \mid 4 < n < 10, n \in \mathbb{N}\}$
  - $\{2n - 1 \mid 4 < n < 10, n \in \mathbb{N}\}$
  - $\{x \mid 4 < x < 10, x \in \mathbb{R}\}$
  - $\{x \mid 4 < x^2 < 10\}$
  - $\{(x, y) \mid 0 < x < 2, 1 < y < 3, x \in \mathbb{R}, y \in \mathbb{R}\}$

- live exercises: draw set + define set from drawing
- intervals ($[a, b]$ & (a, b)); example: $[-2, 3)$
- sets unions & intersections
- examples:
  - $[0, 1) \cup (2, 3]$
  - $(0, 1) \cap [0.5, 2]$
  - $[-2, 5) \cap \mathbb{N}$
  - $[-2, 5) \cap \mathbb{Z}$
- live exercises:
  - compute and plot the inersection and union of $A = (1, 5)$ and $B = (3, 7]$.
  - compute and plot the inersection and union of $C = (-\infty, 2]$ and $D = [0, +\infty)$.
- quantifiers: $\forall$, $\exists$
- simple example: $S = \{1, 3, 5, 7, 8\}$ *;* $\forall s \in S, s \leq 10$
- example (combined): "for any number, there is a (natural) number greater" $(\forall x \in \mathbb{R}, \exists n \in \mathbb{N} s.t. n > x)$
- live exercises:
  - $S = \{5, 6, 3, 1\}$ "all elements of S are positive"
  - $S = \{5, 6, 3, 1\}$ "there is an odd element in S"
  - $S = \{5, 6, 3, 1\}$ "there is an even element in S that is not a multiple of 4"
- implications $\implies$, $\impliedby$, $\iff$
- examples:
  - $x > 1 \implies x$positive
  - $k \in \mathbb{Z} \impliedby k \in \mathbb{N}$
  - $k \in \mathbb{Z}$ and $k \geq 0 \iff k \in \mathbb{N}$
- live exercises:
  - "if x is positive, then it is the square of another number"
  - "n is pair is equivalent to $n = 2m$ for some integer m"
- extreme values (min,max vs inf,sup)
- live exercises:
  - find the extreme values of the set $A = \{x \in \mathbb{R} \mid x > 0\}$.
  - find the extreme values of the set $B = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$.

3 Boolean Algebra

- principle (only 0 and 1)
- $+$ and $*$ for booleans: $\vee$ and $\wedge$
- *not* ($\neg$)
- tables
- De Morgan's law ($\neg(a \wedge b) = \neg a \vee \neg b$ and $\neg(a \vee b) = \neg a \wedge \neg b$)
- *implications* operators ($\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$); x*or* operator ($\veebar$)
- live exercise:
  - express $\veebar$ in terms of $\vee, \wedge, \neg$
  - express $\Longrightarrow$ in terms of $\vee, \wedge, \neg$
  - express $\wedge$ in terms of $\vee, \neg$
  - express $\vee$ in terms of $\wedge, \neg$

4 Modular arithmetic

- Euclidean division of a by b ($a = bk + r$ with $0 \leq r < b$)
- example with $a = 35$, $b = 2, 3, 4, 5, 6, 7, 8$
- modular classes ($12 \equiv 7 \equiv 22 \equiv 102 \equiv -3 \equiv -103 \mod 5$ i.e. $\{2+5k \mid k \in \mathbb{Z}\}$)
- live exercises:
  - give 3 numbers that are congruent to 3 mod 7
  - give a test in terms of modular arithmetic that is equivalent to "n is odd"
  - give a test in terms of modular arithmetic that is equivalent to "n is a nultiple of k" (for k a natural number greater than two)
  - what does it mean for n to say that $n \equiv 5 \mod 10$?
  - find the least positive value of x such that $71 \equiv x \mod 8$
- modular operations (+,-,* $\mod n$)
- GCD and $\square^{-1} \mod p$
- example:
  - compute the GCD of 270 and 192 (answer: 6)
  - compute $5^{-1} \mod 11$
- live exercises:
  - find the least positive value of x such that $89 \equiv (x + 3) \mod 4$
  - what is $x \mod 10$ if $96 \equiv x/7 \mod 5$
  - find an x such that $5x \equiv 4 \mod 11$

- if x is congruent to 13 mod 17 then $7x - 3$ is congruent to which number mod 17?

5 Functions

- functions def
- image vs pre-image
- injective vs surjective
- example of a function injective + proof it is
- example of a function surjective + proof it is
- example of a function not injective + proof it is not
- example of a function not surjective + proof it is not