Exercise Set 4

Solutions

1 - First Order Differential Equations

Basics

Exercise 0: $\frac{dy}{dx} = 3x^2 - 2x + 1$

$\Leftrightarrow \int \frac{dy}{dx} \, dx = \int 3x^2 - 2x + 1 \, dx$

$\Leftrightarrow \int dy = y = x^3 - x^2 + x + c \quad , \quad c \in \mathbb{R}$

Exercise 1: $(e^{\lambda x})' = \lambda . e^{\lambda x}$

so if $y' = 5y$, set $\lambda = 5$;

$(e^{5x})' = 5 e^{5x}$

The general solution is $y = C . e^{5x} \quad , \quad c \in \mathbb{R}$

Separable

Exercise 2: $\frac{dy}{dx} = \frac{x}{y}$

$\Leftrightarrow y \, dy = x \, dx$

$\Leftrightarrow \int y \, dy = \int x \, dx$

$\Leftrightarrow \frac{1}{2} y^2 = \frac{1}{2} x^2 + c$

$\Leftrightarrow y^2 = x^2 + c$

$\Leftrightarrow y = \pm \sqrt{x^2 + c}$

Exercise 3: $\frac{dy}{dx} = 2x^2 e^y$

$\Leftrightarrow e^{-y} dy = 2x^2 dx$

$\Leftrightarrow \int e^{-y} dy = \int 2x^2 dx$

$\Leftrightarrow -e^{-y} = \frac{2}{3} x^3 + c$

$$\iff e^{y} = -\frac{2}{3}x^{3} + c$$

$$\iff y = \ln\left(-\frac{2}{3}x^{3} + c\right)$$

Integrating Factor

Exercise 4 : $y' + 2y = 4x$

$$I.F. = e^{\int 2\,dx} = e^{2x}$$

So $e^{2x}(y' + 2y) = e^{2x}.4x$

integrating: $\int e^{2x}(y'+2y)\,dx = \int e^{2x}.4x\,dx$

$$\iff e^{2x}.y = 2x.e^{2x} - e^{2x} + c$$

$$\iff y = 2x - 1 + c.e^{-2x}$$

Exercise 5 : $y' - \frac{1}{x}y = x^{3}$

$$I.F. = e^{\int -\frac{1}{x}\,dx} = e^{-\ln(x)} = \frac{1}{x}$$

So $\frac{1}{x}y' - \frac{1}{x^{2}}y = x^{2}$

integrating: $\int \frac{1}{x}\left(y' - \frac{1}{x}y\right)dx = \int x^{2}\,dx$

$$\iff \frac{1}{x}y = \frac{1}{3}x^{3} + c$$

$$\iff y = \frac{1}{3}x^{4} + cx$$

2 - Second Order Differential Equation

Basics

Exercise 0: $\frac{d^{2}y}{dx^{2}} = e^{x} + 4\sin(2x) - 5x$

$$\iff \int \frac{d^{2}y}{dx^{2}}\,dx = \int e^{x} + 4\sin(2x) - 5x\,dx$$

$$\iff \frac{dy}{dx} = e^{x} - 2\cos(2x) - \frac{5}{2}x^{2} + c$$

$$\iff \int \frac{dy}{dx}\,dx = \int e^{x} - 2\cos(2x) - \frac{5}{2}x^{2} + c\,dx$$

$$\iff y = e^{x} - \sin(2x) + \frac{5}{6}x^{3} + Cx + D$$

Exercise 1: $y'' + 2y' + y = 0$

$y = e^{rx} \Rightarrow y' = re^{rx} \Rightarrow y'' = r^2 e^{rx}$

so $y'' + 2y' + y$ becomes $r^2 e^{rx} + 2re^{rx} + e^{rx}$
$= e^{rx}[r^2 + 2r + 1]$

so $y'' + 2y' + y = 0$

$\Leftrightarrow e^{rx}(r^2 + 2r + 1) = 0$

$\Rightarrow r^2 + 2r + 1 = 0$

$\Delta = 2^2 - 4.1.1 = 0$

$r_0 = \frac{-2}{2.1} = -1$

let $r = -1$ gives $y = e^{-x}$

because $\Delta = 0$, we also try $y = x.e^{-x}$:

$y' = e^{-x} - x.e^{-x}$

$y'' = -e^{-x} - e^{-x} + x.e^{-x} = x.e^{-x} - 2.e^{-x}$

$y'' + 2y' + y = x.e^{-x} - 2e^{-x} + 2(e^{-x} - xe^{-x}) + xe^{-x}$
$= 0$

$y = x.e^{-x}$ is also a solution

Therefore, the general solution is $y = Ae^{-x} + B\,x.e^{-x}$
$A, B \in \mathbb{R}$

Exercise 2: $y'' + 4y = 0$

$y = e^{rx} \Rightarrow$ need $r^2 + 4 = 0$
$\Leftrightarrow r^2 = -4$
$\Leftrightarrow r = \pm 2i$

so $y = Ae^{2ix} + B\,e^{-2ix}$ $A, B \in \mathbb{C}$
$= A(\cos(2x) + i\sin(2x)) + B(\cos(2x) - i\sin(2x))$

letting $C = A + B$, $D = (A - B)i$:

$y = C.\cos(2x) + D\sin(2x)$

Separable :

Exercise 3 : $y'' = (y')^2$

let $z = y'$; $z' = z^2$

$$\Leftrightarrow \frac{dz}{dx} = z^2$$

$$\Leftrightarrow \frac{dz}{z^2} = dx$$

$$\Leftrightarrow \int \frac{dz}{z^2} = \int dx$$

$$\Leftrightarrow -\frac{1}{z} = x - c \qquad c \in \mathbb{R}$$

$$\Leftrightarrow z = \frac{1}{c-x}$$

$$\Leftrightarrow \frac{dy}{dx} = \frac{1}{c-x}$$

$$\Leftrightarrow \int dy = \int \frac{dx}{c-x}$$

$$\Leftrightarrow y = \ln(c-x) + D \qquad D \in \mathbb{R}$$

Non - Homogeneous

Exercise 4 : $y'' + 4y' + 4y = 6x^2 + 10x + 2$

$y = e^{\lambda x}$: $\lambda^2 + 4\lambda + 4 = 0$ (homogeneous case)

$$\Delta = 4^2 - 4.4.1 = 0$$

$$\lambda_0 = \frac{-4}{2.1} = -2$$

so $y_{hom} = A e^{-2x} + B\, x . e^{-2x}$

$y = ax^2 + bx + c \Rightarrow y' = 2ax + b \Rightarrow y'' = 2a$

$y'' + 4y' + 4y = 2a + 8ax + 4b + 4ax^2 + 4bx + 4c$

so $y'' + 4y' + 4y = 6x^2 + 10x + 2$

$$\Leftrightarrow \begin{cases} 2a + 4b + 4c = 2 \\ 8a + 4b = 10 \\ 4a = 6 \end{cases}$$

- $a = 3/2$
- $12 + 4b = 10 \Leftrightarrow b = -\frac{1}{2}$
- $3 - 2 + 4c = 2 \Leftrightarrow c = \frac{1}{4}$

$$y_{part} = \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{4}$$

$$y = y_{hom} + y_{part} = A e^{-2x} + B x . e^{-2x} + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{4}$$

$$A, B \in \mathbb{R}$$

Exercise 5: $y'' - 2y' + 5y = 3e^{2t}$

$y = e^{\lambda t}$: $\lambda^2 - 2\lambda + 5 = 0$

$\Delta = (-2)^2 - 4.1.5 = 4 - 20 = -16$

$\lambda = \frac{2 \pm 4i}{2.1} = 1 \pm 2i$

$y_{hom} = A e^{(1+2i)t} + B e^{(1-2i)t}$ $\quad A, B \in \mathbb{C}$

$= e^t [C . \cos(2t) + D . \sin(2t)]$ $\quad C = A + B ; \; D = (A - B)i$

let $y = \alpha . e^{2t}$ $\Rightarrow$ $y' = 2\alpha . e^{2t}$, $y'' = 4\alpha . e^{2t}$

$\Rightarrow y'' - 2y' + 5y = e^{2t} (\cancel{4\alpha} - 2 \cancel{2\alpha} + 5.\alpha)$

$= e^{2t} . 5\alpha$

Hence, we want $y'' - 2y' + 5y = 3.e^{2t}$

$\Leftrightarrow e^{2t} . 5\alpha = 3.e^{2t}$

$\Leftrightarrow \alpha = 3/5$

so $y_{part} = \frac{3}{5} e^{2t}$

Thus, $y = e^t (C . \cos(2t) + D . \sin(2t)) + \frac{3}{5} e^{2t}$

Boundary Conditions

Exercise 6: $y'' + 2y' - 3y = 0$

let $y = e^{\lambda x}$ $\Rightarrow$ $\lambda^2 + 2\lambda - 3 = 0$

$\Delta = 2^2 - 4.(-3).1$

$= 4 + 12 = 16$

so $\lambda = \frac{-2 \pm 4}{2.1} = -1 \pm 2 = -3$ or 1

Thus, $y = A e^x + B e^{-3x}$

$y(0) = 1$ so $A + B = 1$

$y(2) = 5$ so $A e^2 + B e^{-6} = 5$

$\Rightarrow A(e^2 - e^{-6}) = 5 - e^{-6}$

$\Rightarrow A = \frac{5 - e^{-6}}{e^2 - e^{-6}}$ $\Rightarrow B = \frac{e^2 - 5}{e^2 - e^{-6}}$

so $y = \frac{5 - e^{-6}}{e^2 - e^{-6}} e^x + \frac{e^2 - 5}{e^2 - e^{-6}} e^{-3x}$

Exercise 7: $y'' + 4y = 12x$

let $y = e^{\lambda x} \Rightarrow \lambda^2 + 4 = 0$

$\Leftrightarrow \lambda = \pm 2i$

So $y_{hom} = C\cos(2x) + D\sin(2x)$

let $y = 3x \Rightarrow y'' = 0$ so $y'' + 4y = 12x$

hence, $y_{part} = 3x$

So $y = C.\cos(2x) + D\sin(2x) + 3x$

& $y' = -2C.\sin(2x) + 2D\cos(2x) + 3$

$y(0) = 0 \Leftrightarrow C.\underbrace{\cos(0)}_{1} + D\underbrace{\sin(0)}_{0} + \underbrace{3.0}_{0} = 0$

$\Leftrightarrow C = 0$

$y'(0) = 0 \Leftrightarrow -2C\underbrace{\sin(0)}_{0} + 2D\underbrace{\cos(0)}_{1} + 3 = 0$

$\Leftrightarrow D = -3/2$

Thus, $y = -\frac{3}{2}\sin(2x) + 3x$