Exercises Set 3

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Abstract

Only the questions with a * are compulsory (but do all of them!).

1 Fundamental Theorem of Calculus

Statement Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all $x \in [a, b]$, by $F(x) = \int_a^b f(t)dt$.

Then F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b), and $F'(x) = f(x)$ for all $xin\,(a, b)$ so F is an anti-derivative of f.

Generalization / Corollary Let $f(x)$ be a continuous function on the closed interval $[a, b]$, and let $F(x)$ be an anti-derivative of $f(x)$. Prove that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Application Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_0^{\pi/2} \sin(x)\,dx$$

Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_1^4 \frac{1}{x^2}\,dx$$

2 Integration Techniques

Reminder

$$\int f(u)\,du = \int f(g(x)) \cdot g'(x)\,dx \quad \text{or} \quad \int f(u)\,du = \int f(x) \cdot \frac{du}{dx}\,dx$$

$$\int u\,dv = uv - \int v\,du$$

Substitution / Change of Variable

Exercise 1: Evaluate the following integral using the method of substitution:

$$\int e^{2x} \cos(2x)\, dx$$

Hint: Let $u = e^{2x}$ and then find du to perform the substitution.

Exercise 2: Evaluate the following integral using the method of substitution:

$$\int \frac{2x}{(x^2+1)^2}\, dx$$

Hint: Let $u = x^2 + 1$ and then find du to perform the substitution.

Exercise 3: Evaluate the following integral using the method of substitution:

$$\int \frac{1}{\sqrt{1-x^2}}\, dx$$

Hint: Let $u = 1 - x^2$ and then find du to perform the substitution.

Exercise 4: Evaluate the following integral using a trigonometric substitution:

$$\int \frac{1}{\sqrt{4-x^2}}\, dx$$

Hint: Use the substitution $x = 2\sin(\theta)$ to simplify the integral.

Integration by Parts

Exercise A: Compute the following integral using integration by parts:

$$\int x \ln(x)\, dx$$

Hint: Choose $u = \ln(x)$ and $dv = x\, dx$, and then use the integration by parts formula.

Exercise B: Find the value of the integral using integration by parts:

$$\int x^2 e^x\, dx$$

Hint: Choose $u = x^2$ and $dv = e^x\, dx$, and then use the integration by parts formula.

Exercise C: Compute the following integral using integration by parts:

$$\int x \cos(x)\, dx$$

Hint: Choose $u = x$ and $dv = \cos(x)\, dx$, and then use the integration by parts formula.

Further integration techniques
Exercise α: Perform partial fraction decomposition on the following rational expression:

$$\frac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1}$$

Hint: Factor the denominator and express the given expression as a sum of simpler fractions.

Exercise β: Evaluate the following improper integral by decomposition:

$$\int_0^{+\infty} e^{-x}\, dx$$

Hint: Evaluate the improper integral by considering the limits is a, and let a approach infinity.

Exercise γ: Approximate the value of the integral

$$\int_0^{\pi/2} \sin(x)\, dx$$

using the Trapezoidal Rule with $n = 4$ sub-intervals.

Exercise δ: Estimate the value of the integral

$$\int_0^{\pi/2} \sin(x)\, dx$$

using Simpson's Rule with $n = 3$ sub-intervals.

3 Applications

Areas between curves
Determine the area of the region enclosed by the curves $y = \sin(x)$ and $y = -\sin(x)$ over the interval $[0, \pi]$.
Hint: Begin by finding the points of intersection between the two curves within the given interval. Then, set up the integral to calculate the area between the curves.

Volumes of revolution (Disk Method)
Find the volume of the solid generated by revolving the region bounded by $y = x^2$ and the x-axis, over the interval $[0, 1]$, about the x-axis using the disk method.
Hint: Determine the limits of integration, the radius of the disks, and set up the integral to find the volume.

Arc length of curves
Find the arc length of the curve defined by $y = \sqrt{x}$ over the interval $[1, 4]$.
Hint: Use the formula for arc length $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$ to calculate the arc length of the curve.

Surface area of revolution
Determine the surface area of the solid generated by revolving the curve $y = x^2$ over the interval $[0, 1]$ about the x-axis.
Hint: Use the formula for surface area of revolution $\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$ to calculate the surface area.

4 First Order Differential Equations