

# Exercises Set 3

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## Abstract

Only the questions with a \* are compulsory (but do all of them!).

## 1 Fundamental Theorem of Calculus

**Statement** Let  $f$  be a continuous real-valued function defined on a closed interval  $[a, b]$ . Let  $F$  be the function defined, for all  $x \in [a, b]$ , by  $F(x) = \int_a^x f(t) dt$ .

Then  $F$  is uniformly continuous on  $[a, b]$  and differentiable on the open interval  $(a, b)$ , and  $F'(x) = f(x)$  for all  $x \in (a, b)$  so  $F$  is an anti-derivative of  $f$ .

**Generalization / Corollary** Let  $f(x)$  be a continuous function on the closed interval  $[a, b]$ , and let  $F(x)$  be an anti-derivative of  $f(x)$ . Prove that

$$\int_a^b f(x) dx = F(b) - F(a).$$

**Application** Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_0^{\pi/2} \sin(x) dx$$

Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_1^4 \frac{1}{x^2} dx$$

## 2 Integration techniques

**Reminder**

$$\int f(u) du = \int f(g(x)) \cdot g'(x) dx \quad \text{or} \quad \int f(u) du = \int f(x) \cdot \frac{du}{dx} dx$$

$$\int u dv = uv - \int v du$$

### Substitution / Change of Variable

**Exercise 1:** Evaluate the following integral using the method of substitution:

$$\int e^{2x} \cos(2x) dx$$

*Hint:* Let  $u = e^{2x}$  and then find  $du$  to perform the substitution.

**Exercise 2:** Evaluate the following integral using the method of substitution:

$$\int \frac{2x}{(x^2 + 1)^2} dx$$

*Hint:* Let  $u = x^2 + 1$  and then find  $du$  to perform the substitution.

**Exercise 3:** Evaluate the following integral using the method of substitution:

$$\int \frac{1}{\sqrt{1 - x^2}} dx$$

*Hint:* Let  $u = 1 - x^2$  and then find  $du$  to perform the substitution.

### Integration by Parts

**Exercise A:** Compute the following integral using integration by parts:

$$\int x \ln(x) dx$$

*Hint:* Choose  $u = \ln(x)$  and  $dv = x dx$ , and then use the integration by parts formula.

**Exercise B:** Find the value of the integral using integration by parts:

$$\int x^2 e^x dx$$

*Hint:* Choose  $u = x^2$  and  $dv = e^x dx$ , and then use the integration by parts formula.

**Exercise C:** Compute the following integral using integration by parts:

$$\int x \cos(x) dx$$

*Hint:* Choose  $u = x$  and  $dv = \cos(x) dx$ , and then use the integration by parts formula.

## 3 Applications

### Areas between curves

Determine the area of the region enclosed by the curves  $y = \sin(x)$  and  $y = -\sin(x)$  over the interval  $[0, \pi]$ .

*Hint:* Begin by finding the points of intersection between the two curves within the given interval. Then, set up the integral to calculate the area between the curves.

**Volumes of revolution** (Disk Method)

Find the volume of the solid generated by revolving the region bounded by  $y = x^2$  and the x-axis, over the interval  $[0, 1]$ , about the x-axis using the disk method.

*Hint:* Determine the limits of integration, the radius of the disks, and set up the integral to find the volume.

**Arc length of curves**

Find the arc length of the curve defined by  $y = \sqrt{x}$  over the interval  $[1, 4]$ .

*Hint:* Use the formula for arc length  $\int_a^b \sqrt{1 + (f'(x))^2} dx$  to calculate the arc length of the curve.