Exercises Set 3

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Abstract

Only the questions with a * are compulsory (but do all of them!).

1 Fundamental Theorem of Calculus

Statement

Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all $x \in [a, b]$, by $F(x) = \int_a^b f(t)dt$.

Then F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b), and $F'(x) = f(x)$ for all $xin\,(a, b)$ so F is an anti-derivative of f.

Generalization / Corollary

Let $f(x)$ be a continuous function on the closed interval $[a, b]$, and let $F(x)$ be an anti-derivative of $f(x)$. Prove that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Application

Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_0^{\pi/2} \sin(x)\,dx$$

Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_1^4 \frac{1}{x^2}\,dx$$

2 Integration Techniques

Reminder

$$\int f(u)\,du = \int f(g(x)) \cdot g'(x)\,dx \quad \text{or} \quad \int f(u)\,du = \int f(x) \cdot \frac{du}{dx}\,dx$$

$$\int u\,dv = uv - \int v\,du$$

Substitution / Change of Variable

Exercise 1: Evaluate the following integral using the method of substitution:

$$\int e^{2x}\cos(2x)\,dx$$

Hint: Let $u = e^{2x}$ and then find du to perform the substitution.

Exercise 2: Evaluate the following integral using the method of substitution:

$$\int \frac{2x}{(x^2+1)^2}\,dx$$

Hint: Let $u = x^2 + 1$ and then find du to perform the substitution.

Exercise 3: Evaluate the following integral using the method of substitution:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

Hint: Let $u = 1 - x^2$ and then find du to perform the substitution.

Exercise 4: (*) Evaluate the following integral using a trigonometric substitution:

$$\int \frac{1}{\sqrt{4-x^2}}\,dx$$

Hint: Use the substitution $x = 2\sin(\theta)$ to simplify the integral.

Integration by Parts

Exercise A: Compute the following integral using integration by parts:

$$\int x\ln(x)\,dx$$

Hint: Choose $u = \ln(x)$ and $dv = x\,dx$, and then use the integration by parts formula.

Exercise B: Find the value of the integral using integration by parts:

$$\int x^2 e^x\,dx$$

Hint: Choose $u = x^2$ and $dv = e^x\,dx$, and then use the integration by parts formula.

Exercise C: (*) Compute the following integral using integration by parts:

$$\int x\cos(x)\,dx$$

Hint: Choose $u = x$ and $dv = \cos(x)\,dx$, and then use the integration by parts formula.

Further integration techniques

Exercise α: Perform partial fraction decomposition on the following rational expression:

$$\frac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1}$$

Hint: Factor the denominator and express the given expression as a sum of simpler fractions.

Exercise β: (*) Evaluate the following improper integral by decomposition:

$$\int_0^{+\infty} e^{-x}\, dx$$

Hint: Evaluate the improper integral by considering the limits is a, and let a approach infinity.

Exercise γ: Approximate the value of the integral

$$\int_0^{\pi/2} \sin(x)\, dx$$

using the Trapezoidal Rule with $n = 4$ sub-intervals.

Exercise δ: Estimate the value of the integral

$$\int_0^{\pi/2} \sin(x)\, dx$$

using Simpson's Rule with $n = 3$ sub-intervals.

3 Applications

Areas between curves

Determine the area of the region enclosed by the curves $y = \sin(x)$ and $y = -\sin(x)$ over the interval $[0, \pi]$.

Hint: Begin by finding the points of intersection between the two curves within the given interval. Then, set up the integral to calculate the area between the curves.

Volumes of revolution (Disk Method)

Find the volume of the solid generated by revolving the region bounded by $y = x^2$ and the x-axis, over the interval $[0, 1]$, about the x-axis using the disk method.

Hint: Determine the limits of integration, the radius of the disks, and set up the integral to find the volume.

Arc length of curves (*)

Find the arc length of the curve defined by $y = \sqrt{x}$ over the interval $[1, 4]$.

Hint: Use the formula for arc length $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$ to calculate the arc length of the curve.

Surface area of revolution
Determine the surface area of the solid generated by revolving the curve $y = x^2$ over the interval $[0, 1]$ about the x-axis.
Hint: Use the formula for surface area of revolution $\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$ to calculate the surface area.

4 First Order Differential Equations

Basics

Exercise 0: Solve the following first-order differential equation:

$$\frac{dy}{dx} = 3x^2 - 2x + 1$$

Hint: Integrate both sides with respect to x to find the solution.

Exercise 1: Solve the following first-order differential equation:

$$\frac{dy}{dx} = 5y$$

Hint: Calculate the derivative of λe^x and adjust λ.

Separable

Exercise 2: Solve the following separable differential equation:

$$\frac{dy}{dx} = \frac{x}{y}$$

Hint: Separate the variables x and y, and then integrate both sides to find the solution.

Exercise 3: Find the solution to the separable differential equation:

$$\frac{dy}{dx} = 2x^2 e^y$$

Hint: Separate the variables x and y, and then integrate both sides to determine the solution.

Integrating Factor

Exercise 4: Solve the following linear first-order differential equation:

$$\frac{dy}{dx} + 2y = 4x$$

Hint: Use an integrating factor.

Exercise 5: (*) Find the solution to the linear first-order differential equation:

$$\frac{dy}{dx} - \frac{1}{x}y = x^3$$

Hint: Use an integrating factor.

5 Second Order Differential Equations

Basics

Exercise 0: Solve the following first-order differential equation:

$$\frac{d^2y}{dx^2} = e^x + 4\sin(2x) - 5x$$

Hint: Integrate twice both sides with respect to x to find the solution.

Exercise 1: Find the general solution to the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

Hint: Assume a solution of the form $y(x) = e^{rx}$ and find the values of r that satisfy the equation.

Exercise 2: Find the general solution to the differential equation:

$$\frac{d^2y}{dx^2} + 4y = 0$$

Hint: Assume a solution of the form $y(x) = e^{rx}$ and find the values of r that satisfy the equation (r may be complex... what is exponential of a complex number?).

Separable

Exercise 3: Solve the following separable differential equation:

$$y'' = (y')^2$$

Hint: Separate the variables.

Non-homogeneous

Exercise 4: Consider the non-homogeneous linear second-order differential equation with constant coefficients:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 6x^2 + 10x + 2$$

Hint: Begin by finding the general solution to the associated homogeneous equation. Then, use the method of undetermined coefficients to find a particular solution for the non-homogeneous part.

Exercise 5: Solve the following non-homogeneous linear second-order differential equation with constant coefficients:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 3e^{2t}$$

Hint: First, find the general solution to the homogeneous equation. Then, use the method of undetermined coefficients to find a particular solution for the non-homogeneous part.

Boundary Conditions

Exercise 6: Consider the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

Find the particular solution of this differential equation that satisfies the boundary conditions $y(0) = 1$ and $y(2) = 5$. *Hint:* First, solve the homogeneous equation, and then find a particular solution that satisfies the given boundary conditions.

Exercise 7: (*) Given the differential equation:

$$\frac{d^2y}{dx^2} + 4y = 12x$$

Find the particular solution of this differential equation subject to the boundary conditions $y(0) = 0$ and $y'(0) = 2$. *Hint:* Solve the homogeneous equation, find a particular solution for the non-homogeneous part, and apply the given boundary conditions.