Exercises Set 5

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Abstract

Only the questions with a * are compulsory (but do all of them!).

1 Change of Basis

Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard canonical basis for $\mathbb{R}^3$.

Suppose we have another basis $\mathcal{B}' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ and let Q be the matrix whose columns are the coordinates of

$$\mathbf{u}_1 = \begin{pmatrix} 0.5 \\ -1 \\ 1 \end{pmatrix}_{\mathcal{B}}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}_{\mathcal{B}}, \text{ and } \mathbf{u}_3 = \begin{pmatrix} -0.25 \\ 0.5 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

with respect to the standard basis. That is, $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$.

Let $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}'}$. Express $\mathbf{v}$ in the standard basis $\mathcal{B}$.

Let $\mathbf{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}}$. Express $\mathbf{w}$ in the basis $\mathcal{B}'$.

2 Variance and Covariance

Calculate the variance of the following set:

$$\mathcal{S}_1 = \{1.5, 3, 5, 7.5, 8, 9\}$$

Calculate the variance of the following set:

$$\mathcal{S}_2 = \{2, 4, 6, 8, 10\}$$

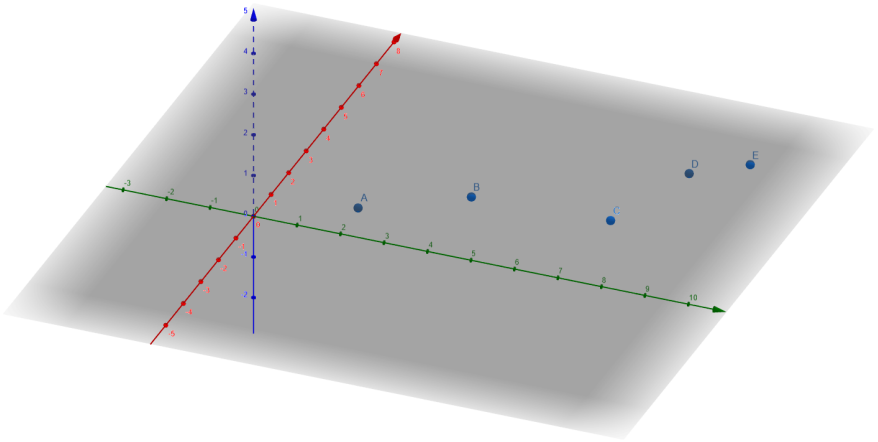
Calculate the covariance of $\mathcal{S}_1$ and $\mathcal{S}_2$.

Compute $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$, the standardized version of $\mathcal{S}_1$ and $\mathcal{S}_2$ (shifted to mean 0 and scaled to have a variance of 1).

Calculate the covariance of $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$. What do you remark?

3 Principal Component Analysis

Let $\mathcal{S} = \{A, B, C, D, E\}$ be a set of 5 points in $\mathbb{R}^3$.
$A = \begin{pmatrix} 1 \\ 2 \\ 0.1 \end{pmatrix}$, $B = \begin{pmatrix} 2.5 \\ 4 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 3 \\ 7 \\ -0.2 \end{pmatrix}$, $D = \begin{pmatrix} 5 \\ 8 \\ 0.1 \end{pmatrix}$, and $E = \begin{pmatrix} 6 \\ 9 \\ 0 \end{pmatrix}$.



3.1 Step 1: Standardization

3.2 Step 2: Covariance Matrix computation