Exercises Set 5

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September 27, 2023

Abstract

Only the questions with a * are compulsory (but do all of them!).

1 Change of Basis

Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard canonical basis for $\mathbb{R}^3$.

Suppose we have another basis $\mathcal{B}' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ and let Q be the matrix whose columns are the coordinates of

$$\mathbf{u}_1 = \begin{pmatrix} 0.5 \\ -1 \\ 1 \end{pmatrix}_{\mathcal{B}}, \ \mathbf{u}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}_{\mathcal{B}}, \text{ and } \mathbf{u}_3 = \begin{pmatrix} -0.25 \\ 0.5 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

with respect to the standard basis. That is, $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$.

Let $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}'}$. Express $\mathbf{v}$ in the standard basis $\mathcal{B}$.

Let $\mathbf{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}}$. Express $\mathbf{w}$ in the basis $\mathcal{B}'$.

2 Variance and Covariance

Calculate the variance of the following set:

$$\mathcal{S}_1 = \{1.5, 3, 5, 7.5, 8, 9\}$$

Calculate the variance of the following set:

$$\mathcal{S}_2 = \{2, 4, 6, 8, 10\}$$

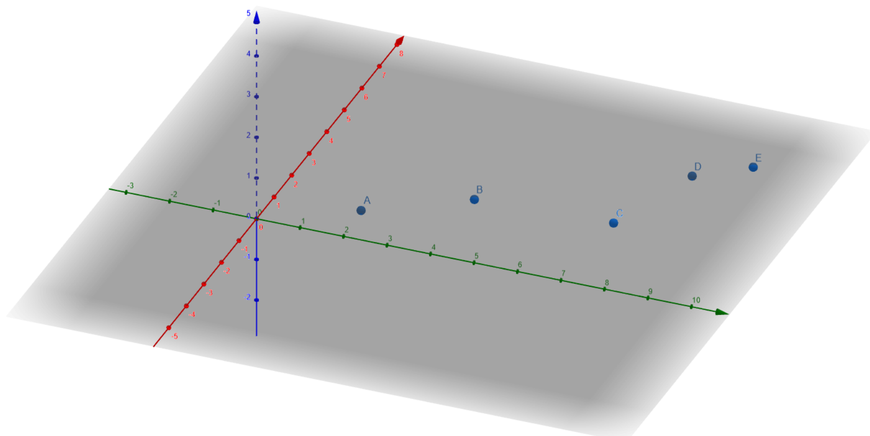
Calculate the covariance of $\mathcal{S}_1$ and $\mathcal{S}_2$.

Compute $\hat{\mathcal{S}_1}$ and $\hat{\mathcal{S}_2}$, the standardized version of $\mathcal{S}_1$ and $\mathcal{S}_2$ (shifted to mean 0 and scaled to have a variance of 1).

Calculate the covariance of $\hat{\mathcal{S}_1}$ and $\hat{\mathcal{S}_2}$. What do you remark?

3 Principal Component Analysis

Let $\mathcal{S} = \{A, B, C, D, E, F\}$ be a set of 5 points in $\mathbb{R}^3$.
$A = \begin{pmatrix} 2 \\ -0.4 \\ 0.1 \end{pmatrix}$, $B = \begin{pmatrix} 4 \\ -0.8 \\ -0.1 \end{pmatrix}$, $C = \begin{pmatrix} 12 \\ -2.4 \\ -0.5 \end{pmatrix}$, $D = \begin{pmatrix} 12 \\ -2.4 \\ 0.5 \end{pmatrix}$, $E = \begin{pmatrix} 14 \\ -2.8 \\ -0.1 \end{pmatrix}$, and $F = \begin{pmatrix} 16 \\ -3.2 \\ 0.1 \end{pmatrix}$.



3.1 Standardization

Calculate $\hat{\mathcal{S}}$, the standardized version of $\mathcal{S}$ (shifted to mean 0 and scaled to have a variance of 1).

3.2 Covariance matrix

Compute the covariance of each pair of features. Compute also the variance of each feature. Arrange the values in a 3×3 matrix (variance is covariance of a feature with itself).

3.3 Eigenvalues of the covariance matrix

Calculate the eigenvalues of the covariance matrix. Use the characteristic polynomial.

The variance explained by each feature is $\frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3}$. Order the features by decreasing importance.

3.4 Feature vectors (the ”principal components”)

For each eigenvalue, calculate the corresponding eigenvectors of the covariance matrix. These are the principal components, also called ”feature vectors”.

3.5 Recasting data on principal components axes

Project each item of data on the first two components, and plot them in a 2D graph.