

Mathematics Refresher Course

First Two Sessions

Paul Dubois

September 2023

Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. We will try to cover most of the prerequisites of the courses in the master's, i.e. basic algebra/analysis and basic applications.

1 Presentation

- Paul Dubois
- 3rd year PhD @ Centrale / TheraPanacea
- Research topic: AI applied to radiotherapy
- Email: `b00795695@essec.edu` (for any question)

Course structure

- 8*3h arranged as 1h20min lecture - 1/3h break - 1h20min lecture
- ~~No pb class planned, but lectures will have integrated live exercises~~
- ~~Interrupt if needed (do not wait for the end of the lecture)~~
- In this document, you will find the content of the first two sessions, with the small exercises we did "live".
- The remaining six sessions will be problem solving.
In case a session is spent on a topic you already, you can skip it
on the condition that you submit all compulsory exercises corresponding to that session.

- Examination

- The course is pass/fail
 - Spoiler: All of you will pass
 - ~~Home exercises, you will need 80+% to pass~~
 - ~~to complete exercises, it should take 30min to 1h~~
 - ~~2-4 exercises~~
 - ~~Hand in paper of PDF~~
 - ~~In the unlikely event of not passing, you will be able to do some extra work to pass~~
 - To pass, I will ask you, for each session, to either be in class, or submit the compulsory exercises.
 - The submission deadlines for the exercises set is exactly one week after the corresponding class.
- Submitting
 1. Solve exercises
 2. Export you work to a single PDF file (e.g. using a scanning smart-phone app)
 3. Rename your file "submission_nb_family_name.pdf" where:
 - "nb" is "2" for exercises set 2, "3" for exercise set 3, etc...
 - "family_name" is your family name in latin alphabet, capital letters

*Example: if I wanted to submit exercise set 1, the name of my file should have been "**submission_1_DUBOIS.pdf**"*
 4. Send me one new email per submission, please do not use the "reply" button, create a new email;
For the subject, you can just put the name of the file (or anything else that makes sense).

2 Sets

- sets of numbers (\mathbb{N} , \mathbb{Z} , \mathbb{R} , \mathbb{Q} , \mathbb{P})
- complex sets (with $\{\}$)
- examples (draw them):
 - $\{n \mid 4 < n < 10, n \in \mathbb{N}\}$
 - $\{2n - 1 \mid 4 < n < 10, n \in \mathbb{N}\}$
 - $\{x \mid 4 < x < 10, x \in \mathbb{R}\}$
 - $\{x \mid 4 < x^2 < 10\}$
 - $\{(x, y) \mid 0 < x < 2, 1 < y < 3, x \in \mathbb{R}, y \in \mathbb{R}\}$

- live exercises: draw set + define set from drawing
- intervals ($[a, b]$ & (a, b)); example: $[-2, 3)$
- sets unions & intersections
- examples:
 - $[0, 1) \cup (2, 3]$
 - $(0, 1) \cap [0.5, 2]$
 - $[-2, 5) \cap \mathbb{N}$
 - $[-2, 5) \cap \mathbb{Z}$
- live exercises:
 - compute and plot the intersection and union of $A = (1, 5)$ and $B = (3, 7]$.
 - compute and plot the intersection and union of $C = (-\infty, 2]$ and $D = [0, +\infty)$.
- quantifiers: \forall, \exists
- simple example: $S = \{1, 3, 5, 7, 8\}$; $\forall s \in S, s \leq 10$
- example (combined): "for any number, there is a (natural) number greater" ($\forall x \in \mathbb{R}, \exists n \in \mathbb{N} s.t. n > x$)
- live exercises:
 - $S = \{5, 6, 3, 1\}$ "all elements of S are positive"
 - $S = \{5, 6, 3, 1\}$ "there is an odd element in S "
 - $S = \{5, 6, 3, 1\}$ "there is an even element in S that is not a multiple of 4"
- implications $\implies, \impliedby, \iff$
- examples:
 - $x > 1 \implies x \text{ positive}$
 - $k \in \mathbb{Z} \impliedby k \in \mathbb{N}$
 - $k \in \mathbb{Z} \text{ and } k \geq 0 \iff k \in \mathbb{N}$
- live exercises:
 - "if x is positive, then it is the square of another number"
 - " n is pair is equivalent to $n = 2m$ for some integer m "
- extreme values (min,max vs inf,sup)
- live exercises:
 - find the extreme values of the set $A = \{x \in \mathbb{R} \mid x > 0\}$.
 - find the extreme values of the set $B = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$.

3 Boolean Algebra

- principle (only 0 and 1)
- + and * for booleans: \vee and \wedge
- *not* (\neg)
- tables
- De Morgan's law ($\neg(a \wedge b) = \neg a \vee \neg b$ and $\neg(a \vee b) = \neg a \wedge \neg b$)
- *implications* operators (\implies , \impliedby , \iff); *xor* operator ($\underline{\vee}$)
- live exercise:
 - express $\underline{\vee}$ in terms of \vee, \wedge, \neg
 - express \implies in terms of \vee, \wedge, \neg
 - express \wedge in terms of \vee, \neg
 - express \vee in terms of \wedge, \neg

4 Modular arithmetic

- Euclidean division of a by b ($a = bk + r$ with $0 \leq r < b$)
- example with $a = 35$, $b = 2, 3, 4, 5, 6, 7, 8$
- modular classes ($12 \equiv 7 \equiv 22 \equiv 102 \equiv -3 \equiv -103 \pmod{5}$ i.e. $\{2+5k \mid k \in \mathbb{Z}\}$)
- live exercises:
 - give 3 numbers that are congruent to 3 mod 7
 - give a test in terms of modular arithmetic that is equivalent to " n is odd"
 - give a test in terms of modular arithmetic that is equivalent to " n is a multiple of k " (for k a natural number greater than two)
 - what does it mean for n to say that $n \equiv 5 \pmod{10}$?
 - find the least positive value of x such that $71 \equiv x \pmod{8}$
- modular operations ($+, -, *$ mod n)
- GCD and $\square^{-1} \pmod{p}$
- example:
 - compute the GCD of 270 and 192 (answer: 6)
 - compute $5^{-1} \pmod{11}$
- live exercises:
 - find the least positive value of x such that $89 \equiv (x + 3) \pmod{4}$
 - what is $x \pmod{10}$ if $96 \equiv x/7 \pmod{5}$
 - find an x such that $5x \equiv 4 \pmod{11}$

- if x is congruent to $13 \pmod{17}$ then $7x - 3$ is congruent to which number $\pmod{17}$?

5 Functions

- functions def
- image vs pre-image
- injective vs surjective
- example of a function injective + proof it is
- example of a function surjective + proof it is
- example of a function not injective + proof it is not
- example of a function not surjective + proof it is not

6 Counting Arrangements

People in a company In a company, there are 800 employees. 300 are men, 352 are union members, 424 are married, 188 are union men, 166 are married men, 208 are union members and married, 144 are married union men.

How many single, non-union women are there?

The padlock A padlock has a 4-digit code, each number being a number from 0 to 9.

1. How many possible codes are there?
2. How many possible codes are there with 4 different digits?
3. How many codes ending in an even number are there?
4. How many codes are there ending with an even number and with 4 different numbers?

In each of the situations:

- How many codes are there containing at least one digit 4?
- How many codes are there containing exactly one digit 4?

Airplanes Company A company has 4 airplanes, 4 pilots and 8 flight attendants. How many different ways are there to assign pilots and flight attendants to helicopters so that each helicopter has one pilot and two flight attendants?

Explanations on the binomial coefficient.

Bookshelf We want to store 4 mathematics books (separate), 6 physics books, and 3 chemistry books on a shelf. How many ways can this storage be done:

If books are to be grouped by subjects.

If only math books should be grouped.

ESSEC's Stairs At ESSEC, the staircase between the ground floor and the 1st floor is made up of 16 steps. To climb this staircase, I can go up one step for each step, or up two steps. How many ways are there to climb this staircase?