Exercises Set 4

Paul Dubois

September 20, 2023

Abstract

Only the questions with a * are compulsory (but do all of them!).

1 First Order Differential Equations

Basics

Exercise 0: Solve the following first-order differential equation:

$$\frac{dy}{dx} = 3x^2 - 2x + 1$$

Hint: Integrate both sides with respect to x to find the solution.

Exercise 1: Solve the following first-order differential equation:

$$\frac{dy}{dx} = 5y$$

Hint: Calculate the derivative of $\alpha e^{\lambda x}$ and adjust α and β.
Suppose that we also want $y(0) = 2$. Adjust further α and β.

Separable

Exercise 2: Solve the following separable differential equation:

$$\frac{dy}{dx} = \frac{x}{y}$$

Hint: Separate the variables x and y, and then integrate both sides to find the solution.

Exercise 3: Find the solution to the separable differential equation:

$$\frac{dy}{dx} = 2x^2 e^y$$

Hint: Separate the variables x and y, and then integrate both sides to determine the solution.

Integrating Factor

Exercise 4: Solve the following linear first-order differential equation:

$$\frac{dy}{dx} + 2y = 4x$$

Hint: Use an integrating factor.

Exercise 5: (*) Find the solution to the linear first-order differential equation:

$$\frac{dy}{dx} - \frac{1}{x}y = x^3$$

Hint: Use an integrating factor.

2 Second Order Differential Equations

Basics

Exercise 0: Solve the following second-order differential equation:

$$\frac{d^2y}{dx^2} = e^x + 4\sin(2x) - 5x$$

Hint: Integrate twice both sides with respect to x to find the solution.

Exercise 1: Find the general solution to the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

Hint: Assume a solution of the form $y(x) = e^{rx}$ and find the values of r that satisfy the equation.

Exercise 2: Find the general solution to the differential equation:

$$\frac{d^2y}{dx^2} + 4y = 0$$

Hint: Assume a solution of the form $y(x) = e^{rx}$ and find the values of r that satisfy the equation (r may be complex... what is exponential of a complex number?).

Separable

Exercise 3: Solve the following separable differential equation:

$$y'' = (y')^2$$

Hint: Separate the variables.

Non-homogeneous

Exercise 4: Consider the non-homogeneous linear second-order differential equation with constant coefficients:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 6x^2 + 10x + 2$$

Hint: Begin by finding the general solution to the associated homogeneous equation. Then, use the method of undetermined coefficients to find a particular solution for the non-homogeneous part.

Exercise 5: Solve the following non-homogeneous linear second-order differential equation with constant coefficients:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 3e^{2t}$$

Hint: First, find the general solution to the homogeneous equation. Then, use the method of undetermined coefficients to find a particular solution for the non-homogeneous part.

Boundary Conditions

Exercise 6: Consider the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

Find the particular solution of this differential equation that satisfies the boundary conditions $y(0) = 1$ and $y(2) = 5$. *Hint:* First, solve the homogeneous equation, and then find a particular solution that satisfies the given boundary conditions.

Exercise 7: (*) Given the differential equation:

$$\frac{d^2y}{dx^2} + 4y = 12x$$

Find the particular solution of this differential equation subject to the boundary conditions $y(0) = 0$ and $y'(0) = 2$. *Hint:* Solve the homogeneous equation, find a particular solution for the non-homogeneous part, and apply the given boundary conditions.