

# Exercises Set 3

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September 16, 2023

## Abstract

Only the questions with a \* are compulsory (but do all of them!).

## 1 Fundamental Theorem of Calculus

### Statement

Let  $f$  be a continuous real-valued function defined on a closed interval  $[0, x]$ . Let  $F$  be the function defined, for all  $t \in [0, x]$ , by  $F(x) = \int_0^x f(t)dt$ .

Then  $F$  is uniformly continuous on  $[0, x]$  and differentiable on the open interval  $(a, b)$ , and  $F'(x) = f(x)$  for all  $x \in (a, b)$  so  $F$  is an anti-derivative of  $f$ .

### Generalization / Corollary

Let  $f(x)$  be a continuous function on the closed interval  $[a, b]$ , and let  $F$  be an anti-derivative of  $f$ . Prove that

$$\int_a^b f(x) dx = F(b) - F(a).$$

### Application

Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_0^{\pi/2} \sin(x) dx$$

Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_1^4 \frac{1}{x^2} dx$$

## 2 Integration Techniques

### Reminder

$$\int f(u) du = \int f(g(x)) \cdot g'(x) dx \quad \text{or} \quad \int f(u) du = \int f(x) \cdot \frac{du}{dx} dx$$

$$\int uv' = uv - \int vu'$$

### Substitution / Change of Variable

**Exercise 1:** Evaluate the following integral using the method of substitution:

$$\int \frac{2x}{x^2 + 1} dx$$

*Hint:* Let  $u = x^2 + 1$  and then find  $du$  to perform the substitution.

**Exercise 2: (\*)** Evaluate the following integral using the method of substitution:

$$\int \frac{1}{\sqrt{1-x^2}} dx$$

*Hint:* Let  $x = \sin(u)$  and then find  $du$  to perform the substitution.

**Exercise 3:** Evaluate the following integral using a trigonometric substitution:

$$\int \frac{1}{4+x^2} dx$$

*Hint:* Use the substitution  $u = x/2$  to simplify the integral.

### Integration by Parts

**Exercise A:** Compute the following integral using integration by parts:

$$\int x \ln(x) dx$$

**Exercise B:** Find the value of the integral using integration by parts:

$$\int x^2 e^x dx$$

**Exercise C: (\*)** Compute the following integral using integration by parts:

$$\int x \cos(x) dx$$

**Exercise D:** Evaluate the following integral using the method of substitution:

$$\int e^{2x} \cos(2x) dx$$

### Further integration techniques

**Exercise  $\alpha$ :** Perform partial fraction decomposition on the following rational expression:

$$\frac{3x^2 - 2x - 1}{x^3 - x^2 + x - 1}$$

*Hint:* Factor the denominator and express the given expression as a sum of simpler fractions.

**Exercise  $\beta$ : (\*)** Evaluate the following improper integral by decomposition:

$$\int_0^{+\infty} e^{-x} dx$$

*Hint:* Evaluate the improper integral by considering the limits is  $a$ , and let  $a$  approach infinity.

**Exercise  $\gamma$ :** Approximate the value of the integral

$$\int_0^{\pi/2} \sin(x) dx$$

using the Trapezoidal Rule with  $n = 4$  sub-intervals.

**Exercise  $\delta$ :** Estimate the value of the integral

$$\int_0^{\pi/2} \sin(x) dx$$

using Simpson's Rule with  $n = 3$  sub-intervals.

### 3 Applications

#### Areas between curves

Determine the area of the region enclosed by the curves  $y = \sin(x)$  and  $y = -\sin(x)$  over the interval  $[0, \pi]$ .

*Hint:* Begin by finding the points of intersection between the two curves within the given interval. Then, set up the integral to calculate the area between the curves.

#### Volumes of revolution (Disk Method)

Find the volume of the solid generated by revolving the region bounded by  $y = x^2$  and the x-axis, over the interval  $[0, 1]$ , about the x-axis using the disk method.

*Hint:* Determine the limits of integration, the radius of the disks, and set up the integral to find the volume.

#### Arc length of curves (\*)

Find the arc length of the curve defined by  $y = \sqrt{x}$  over the interval  $[1, 4]$ .

*Hint:* Use the formula for arc length  $\int_a^b \sqrt{1 + (f'(x))^2} dx$  to calculate the arc length of the curve.

#### Surface area of revolution

Determine the surface area of the solid generated by revolving the curve  $y = x^2$  over the interval  $[0, 1]$  about the x-axis.

*Hint:* Use the formula for surface area of revolution  $\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$  to calculate the surface area.

## 4 First Order Differential Equations

### Basics

**Exercise 0:** Solve the following first-order differential equation:

$$\frac{dy}{dx} = 3x^2 - 2x + 1$$

*Hint:* Integrate both sides with respect to  $x$  to find the solution.

**Exercise 1:** Solve the following first-order differential equation:

$$\frac{dy}{dx} = 5y$$

*Hint:* Calculate the derivative of  $\lambda e^x$  and adjust  $\lambda$ .

### Separable

**Exercise 2:** Solve the following separable differential equation:

$$\frac{dy}{dx} = \frac{x}{y}$$

*Hint:* Separate the variables  $x$  and  $y$ , and then integrate both sides to find the solution.

**Exercise 3:** Find the solution to the separable differential equation:

$$\frac{dy}{dx} = 2x^2 e^y$$

*Hint:* Separate the variables  $x$  and  $y$ , and then integrate both sides to determine the solution.

### Integrating Factor

**Exercise 4:** Solve the following linear first-order differential equation:

$$\frac{dy}{dx} + 2y = 4x$$

*Hint:* Use an integrating factor.

**Exercise 5: (\*)** Find the solution to the linear first-order differential equation:

$$\frac{dy}{dx} - \frac{1}{x}y = x^3$$

*Hint:* Use an integrating factor.

## 5 Second Order Differential Equations

### Basics

**Exercise 0:** Solve the following first-order differential equation:

$$\frac{d^2y}{dx^2} = e^x + 4\sin(2x) - 5x$$

*Hint:* Integrate twice both sides with respect to  $x$  to find the solution.

**Exercise 1:** Find the general solution to the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

*Hint:* Assume a solution of the form  $y(x) = e^{rx}$  and find the values of  $r$  that satisfy the equation.

**Exercise 2:** Find the general solution to the differential equation:

$$\frac{d^2y}{dx^2} + 4y = 0$$

*Hint:* Assume a solution of the form  $y(x) = e^{rx}$  and find the values of  $r$  that satisfy the equation ( $r$  may be complex... what is exponential of a complex number?).

### Separable

**Exercise 3:** Solve the following separable differential equation:

$$y'' = (y')^2$$

*Hint:* Separate the variables.

### Non-homogeneous

**Exercise 4:** Consider the non-homogeneous linear second-order differential equation with constant coefficients:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 6x^2 + 10x + 2$$

*Hint:* Begin by finding the general solution to the associated homogeneous equation. Then, use the method of undetermined coefficients to find a particular solution for the non-homogeneous part.

**Exercise 5:** Solve the following non-homogeneous linear second-order differential equation with constant coefficients:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 3e^{2t}$$

*Hint:* First, find the general solution to the homogeneous equation. Then, use the method of undetermined coefficients to find a particular solution for the non-homogeneous part.

**Boundary Conditions**

**Exercise 6:** Consider the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

Find the particular solution of this differential equation that satisfies the boundary conditions  $y(0) = 1$  and  $y(2) = 5$ . *Hint:* First, solve the homogeneous equation, and then find a particular solution that satisfies the given boundary conditions.

**Exercise 7: (\*)** Given the differential equation:

$$\frac{d^2y}{dx^2} + 4y = 12x$$

Find the particular solution of this differential equation subject to the boundary conditions  $y(0) = 0$  and  $y'(0) = 2$ . *Hint:* Solve the homogeneous equation, find a particular solution for the non-homogeneous part, and apply the given boundary conditions.