Mathematics Refresher Course

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September 2023

Contents

Math Refresher 2023

This course teaches basic mathematical metho dologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. We will try to cover most of the prerequisites of the courses in the master's, i.e. basic algebra/analysis and basic applications.

1 Presentation

- Paul Dubois
- 3rd year PhD @ Centrale / TheraPanacea
- Research topic: AI applied to radiotherapy
- Email: `b00795695@essec.edu` (for any question)
- Course structure
  - 8*3h arranged as 1h20min lecture - 1/3h break - 1h20min lecture
  - No pb class planned, but lectures will have integrated live exercises
  - Interrupt if needed (do *not* wait for the end of the lecture)
- Examination
  - The course is pass/fail
  - Spoiler: All of you will pass
  - Home exercises, you will need 80+% to pass
  - How long do you need to complete exercises (should take 30min to 1h)?
  - How many exercises do you want? (2-4?)
  - Hand in paper of PDF? (vote)
  - In the unlikely event of not passing, you will be able to do some extra work to pass
- Course notes are still under construction (as I will adjust according to the speed of the class); I will give it to you at the end of the course.
- Final questions before we start?

2 Assumed to be known

- 4 operations (+,-,*,/)
- integer vs rational vs decimal
- what is a prime number
- basic (linear) equations solving

3 Sets

- sets of numbers ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{P}$)
- complex sets (with $\{\}$)
- examples (draw them):
  - $\{n \mid 4 < n < 10, n \in \mathbb{N}\}$
  - $\{2n - 1 \mid 4 < n < 10, n \in \mathbb{N}\}$
  - $\{x \mid 4 < x < 10, x \in \mathbb{R}\}$
  - $\{x \mid 4 < x^2 < 10\}$
  - $\{(x, y) \mid 0 < x < 2, 1 < y < 3, x \in \mathbb{R}, y \in \mathbb{R}\}$
- live exercises: draw set + define set from drawing
- intervals ($[a, b]$ & (a, b)); example: $[-2, 3)$
- sets unions & intersections
- examples:
  - $[0, 1) \cup (2, 3]$
  - $(0, 1) \cap [0.5, 2]$
  - $[-2, 5) \cap \mathbb{N}$
  - $[-2, 5) \cap \mathbb{Z}$
- live exercises:
  - compute and plot the inersection and union of $A = (1, 5)$ and $B = (3, 7]$.
  - compute and plot the inersection and union of $C = (-\infty, 2]$ and $D = [0, +\infty)$.
- quantifiers: $\forall$, $\exists$
- exmaple (simple):
  - $S = \{1, 3, 5, 7, 8\}$: $\forall s \in S$, s.t. ≤ 10
  - $S = \{1, 3, 5, 7, 8\}$: $\exists s \in \S$ s.t. s is pair
- example (combined): "for any number, there is a (natural) number greater" ($\forall x \in \mathbb{R}, \exists n \in \mathbb{N} s.t. n > x$)
- live exercises:
  - $S = \{5, 6, 3, 1\}$ "all elements of S are positive"
  - $S = \{5, 6, 3, 1\}$ "there is an odd element in S"
  - $S = \{5, 6, 3, 1\}$ "there is an even element in S that is not a multiple of 4"
- implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$
- examples:

- $x > 1 \implies x$positive
- $k \in \mathbb{Z} \impliedby k \in \mathbb{N}$
- $k \in \mathbb{Z}$ and $k \geq 0 \iff k \in \mathbb{N}$

- live exercises:
  - "if x is positive, then it is the square of another number"
  - "n is pair is equivalent to $n = 2m$ for some integer m"
- extreme values (min,max vs inf,sup)
- live exercises:
  - find the extreme values of the set $A = \{x \in \mathbb{R} \mid x > 0\}$.
  - find the extreme values of the set $B = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$.

4 Boolean Algebra

- principle (only 0 and 1)
- $+$ and $*$ for booleans: $\vee$ and $\wedge$
- *not* ($\neg$)
- tables
- De Morgan's law ($\neg(a \wedge b) = \neg a \vee \neg b$ and $\neg(a \vee b) = \neg a \wedge \neg b$)
- *implications* operators ($\implies$, $\impliedby$, $\iff$); x*or* operator ($\veebar$)
- live exercise:
  - express $\veebar$ in terms of $\vee, \wedge, \neg$
  - express $\implies$ in terms of $\vee, \wedge, \neg$
  - express $\wedge$ in terms of $\vee, \neg$
  - express $\vee$ in terms of $\wedge, \neg$

5 Modular arithmetic

- Euclidean division of a by b ($a = bk + r$ with $0 \leq r < b$)
- example with $a = 35$, $b = 2, 3, 4, 5, 6, 7, 8$
- modular classes ($12 \equiv 7 \equiv 22 \equiv 102 \equiv -3 \equiv -103 \mod 5$ i.e. $\{2+5k \mid k \in \mathbb{Z}\}$)
- live exercises:
  - give 3 numbers that are congruent to 3 mod 7
  - give a test in terms of modular arithmetic that is equivalent to "n is odd"
  - give a test in terms of modular arithmetic that is equivalent to "n is a nultiple of k" (for k a natural number greater than two)

- what does it mean for n to say that $n \equiv 5 \mod 10$?
- find the least positive value of x such that $71 \equiv x \mod 8$

- modular operations (+,-,* $\mod n$)
- GCD and $\square^{-1} \mod p$
- example:
  - compute the GCD of 270 and 192 (answer: 6)
  - compute $5^{-1} \mod 11$
- live exercises:
  - find the least positive value of x such that $89 \equiv (x+3) \mod 4$
  - what is $x \mod 10$ if $96 \equiv x/7 \mod 5$
  - find an x such that $5x \equiv 4 \mod 11$
  - if x is congruent to $13 \mod 17$ then $7x - 3$ is congruent to which number $\mod 17$?

6 Functions

- functions def
- image vs pre-image
- span vs kernel
- examples:
  - $f : x \to 3x + 1$
  - $g : x \to x^2 - 1$
  - $h : x \to 8$
- live exercises:
  - compute the image of 2 by $f(x) = \frac{(x+1)^2 - x}{x-3}$
  - compute the preimage(s) of 5 by $f(x) = 2x - 3$
  - compute the kernel of $f(x) = -3x + 2$
  - compute the span of $f(x) = 5 - (2x)^4$
- typical plotting of functions: set of points (x, y) s.t. $y = f(x)$

7 Sequences

- sequences def: general formula
- example: $u_n = n^3 - 5n^2$
- sequences def: recursive formula

- example: $u_0 = 5, u_{n+1} = u_n^2 - u_n + 2$
- live exercises:
  - consider the (arithmetic) sequence $\{a_n\}$ defined by $a_{n+1} = a_n + 2$ and $a_0 = -1$:
    - find the first five terms of the sequence
    - find the common difference between consecutive terms
    - find a formula for a_n (without using a_{n-1})
  - consider the (geometric) sequence $\{b_n\}$ defined by $b_n = 3 * 2^n$
    - find the first five terms of the sequence
    - find the common ratio between consecutive terms
    - find a formula for b_{n+1} (using only b_n, no n)

8 Essence of proofs

- proof: assumption => conclusion
- direct with $n \geq 0 \implies 2n \geq 4n$
- cases split with $n \equiv n^2 \mod 2$
- contradiction with $\sqrt{2} \notin \mathbb{Q}$
- induction with $u_0 = 2, u_{n+1} = \frac{u_n+1}{2} \implies u_n > 1$
- live exercises:
  - prove that for all real numbers x, if x is positive, then x^3 is also positive
  - prove that the square root of 3 is irrational, i.e., it cannot be expressed as a fraction of two integers.
  - prove by mathematical induction that for all non-negative integers n, $3^n - 1$ is divisible by 2.
  - use mathematical induction to prove that for all positive integers n, the sum of the first n odd integers is given by the formula: $1 + 3 + 5 + ... + (2n - 1)$ is n^2.

9 Asymptotic analysis

- definition (ε, δ)
- examples / live exercises:
  - prove that limit of $u_n = \frac{n^2+1}{n^2}$ as $n \to +\infty$ is 1
  - prove that limit of $f(x) = \frac{2x-1}{x}$ as $x \to -\infty$ is 2
  - prove that limit of $u_n = \frac{1}{\sqrt{n}}$ as $n \to +\infty$ is 0

- prove that $u_n = 2n^3$ diverges to $+\infty$ as $n \to +\infty$
- prove that limit of $f(x) = \frac{1}{x^2}$ as $x \to 0$ is $+\infty$
- prove that limit of $f(x) = \frac{1}{x}$ as $x \to 0^-$ is $-\infty$

- operations on limits ($+$, $-$, $*$, and $/$)
- live exercises:
  - calculate $\lim_{n\to\infty} \left(2 + \frac{-1}{2n}\right)\left(3 - \frac{4}{-n^2}\right) + 5$
  - calculate $\lim_{n\to\infty} \frac{-2n+1}{8n}$
  - calculate $\lim_{x\to\infty} \frac{3x^2+2x}{4x^2-1}$
  - determine the behaviour of $u_n = (-2)^n$ as $n \to +\infty$

10 Large operators

- $\sum$, $\prod$, $\bigcup$, $\bigcap$
- examples:
  - "product of numbers from 10 to 20"
  - "sum of squares up to 10"
  - $\bigcup_{x\in\{1,4,10.5,21.75\}} [x - 0.5, x + 0.5]$
  - $\bigcap_{n\in\mathbb{N}^*} \left[-\frac{1}{n}, \frac{1}{n}\right]$
- live exercises:
  - what set does the last example corresponds to?
  - define the factorial
  - give an expression for the sum of inverses from 1 to 1000
  - give an expression for the product of all prime numbers smaller than 10000
  - give an expression for the sum of factorials from 100 to 200

11 Series

- definition: sum of a sequence
- partial sums: $S_n = \sum_{k=0}^{n} u_k$
- examples:
  - $S_n = \sum_{k=0}^{n} k^2$
  - $S_n = \sum_{k=0}^{n} \frac{1}{k!}$
  - $S_n = \sum_{k=0}^{n} \frac{1}{2^k}$
- popular series:

- geometric series
- harmonic series
- alternating series

- convergence: if the sequence of partial sums converges
- convergence tests:
  - comparison test
  - integral test (see later)
  - ratio test
  - root test
  - alternating series test
- live exercises:
  - prove that the series $\sum_{k\in\mathbb{N}} \frac{1}{k} - \frac{1}{k+1}$ converges
  - prove that the series $\sum_{k\in\mathbb{N}} \frac{1}{k!}$ converges
  - prove that the series $\sum_{k\in\mathbb{N}} \frac{1}{2^k}$ converges
  - prove that the series $\sum_{k\in\mathbb{N}} \frac{1}{k}$ diverges
  - prove that the series $\sum_{k\in\mathbb{N}} \frac{1}{k^2}$ converges
  - prove that the series $\sum_{k\in\mathbb{N}} \frac{k^10}{2^k}$ converges

12 Affine functions

- definition: $f(x) = ax + b$ (a is the slope, b is the intercept)
- examples:
  - $f(x) = 2x + 1$
  - $f(x) = -3x + 2$
  - $f(x) = 5$
- live exercises:
  - plot the function $f(x) = 2x + 1$
  - plot the function $f(x) = -3x + 2$
  - find the affine function that passes through the points $(1, 2)$ and $(3, 4)$
- parallel (same slope) and orthogonal lines (negative reciprocal slope)
- live exercises:
  - find the equation of the line parallel to $y = 2x + 1$ that passes through $(5, 3)$
  - find the equation of the line orthogonal to $y = 2x + 1$ that passes through $(8, 7)$

13 Quadratic functions / equations

- definition: $f(x) = ax^2 + bx + c$ (a is the quadratic coefficient, b is the linear coefficient, c is the constant)
- example: $f(x) = x^2 + 3$ (plot it)
- solving quadratic equations (do demo)
- 3 forms of quadratic functions:
  - $f(x) = a(x - x_1)(x - x_2)$
  - $f(x) = ax^2 + bx + c$
  - $f(x) = a(x - x_0)^2 + y_0$

TODO:

- Graph of usual functions
- Derivatives
- Usual functions (sin, cos, tan, exp, log)
- Integration
- Complex numbers
- Vectors (concept, sum, scalar product)
- Equations for lines (2D, 3D) and planes (3D)
- Matrices (concept, sum, product)
- Mutli-dimensional functions
- Inversing matrices (+ row reduction; span)
- Linear regression
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