Mathematics Refresher Course
First Two Sessions

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Abstract

This course teaches basic mathematical methodologies for proofs. It is intended for students with a lack of mathematical background, or with a lack of confidence in mathematics. We will try to cover most of the prerequisites of the courses in the master's, i.e. basic algebra/analysis and basic applications.

1 Presentation

- Paul Dubois
- 3rd year PhD @ Centrale / TheraPanacea
- Research topic: AI applied to radiotherapy
- Email:`b00795695@essec.edu` (for any question)

Course structure

  - 8*3h arranged as 1h20min lecture - 1/3h break - 1h20min lecture
  - ~~No pb class planned, but lectures will have integrated live exercises~~
  - ~~Interrupt if needed (do *not* wait for the end of the lecture)~~
  - In this document, you will find the content of the first two sessions, with the small exercises we did "live".
  - The remaining six sessions will be problem solving.
    In case a session is spent on a topic you already, you can skip it **on the condition that you submit all compulsory exercises corresponding to that session**.

- Examination

  - The course is pass/fail
  - Spoiler: All of you will pass
  - ~~Home exercises, you will need 80+% to pass~~
  - ~~to complete exercises, it should take 30min to 1h~~
  - ~~2-4 exercises~~
  - ~~Hand in paper of PDF~~
  - ~~In the unlikely event of not passing, you will be able to do some extra work to pass~~
  - To pass, I will ask you, for each session, to either be in class, or submit the compulsory exercises.
  - The submission deadlines for the exercises set is exactly one week after the corresponding class.

- Submitting
  1. Solve exercises
  2. Export you work to a single PDF file (e.g. using a scanning smartphone app)
  3. Rename your file "submission_nb_family_name.pdf" where:
     - "nb" is "2" for exercises set 2, "3" for exercise set 3, etc...
     - "family_name" is your family name in latin alphabet, capital letters

     *Example: if I wanted to submit exercise set 1, the name of my file should have been "**submission_1_DUBOIS.pdf**"*
  4. Send me one new email per submission, please do not use the "reply" button, create a new email;
     For the subject, you can just put the name of the file (or anything else that makes sense).

2 Sets

- sets of numbers ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{P}$)
- complex sets (with {})
- examples (draw them):
  - $\{n \mid 4 < n < 10, n \in \mathbb{N}\}$
  - $\{2n - 1 \mid 4 < n < 10, n \in \mathbb{N}\}$
  - $\{x \mid 4 < x < 10, x \in \mathbb{R}\}$
  - $\{x \mid 4 < x^2 < 10\}$
  - $\{(x, y) \mid 0 < x < 2, 1 < y < 3, x \in \mathbb{R}, y \in \mathbb{R}\}$

- live exercises: draw set + define set from drawing
- intervals ($[a, b]$ & (a, b)); example: $[-2, 3)$
- sets unions & intersections
- examples:
  - $[0, 1) \cup (2, 3]$
  - $(0, 1) \cap [0.5, 2]$
  - $[-2, 5) \cap \mathbb{N}$
  - $[-2, 5) \cap \mathbb{Z}$
- live exercises:
  - compute and plot the inersection and union of $A = (1, 5)$ and $B = (3, 7]$.
  - compute and plot the inersection and union of $C = (-\infty, 2]$ and $D = [0, +\infty)$.
- quantifiers: $\forall$, $\exists$
- simple example: $S = \{1, 3, 5, 7, 8\}$ *;* $\forall s \in S, s \leq 10$
- example (combined): "for any number, there is a (natural) number greater" $(\forall x \in \mathbb{R}, \exists n \in \mathbb{N} s.t. n > x)$
- live exercises:
  - $S = \{5, 6, 3, 1\}$ "all elements of S are positive"
  - $S = \{5, 6, 3, 1\}$ "there is an odd element in S"
  - $S = \{5, 6, 3, 1\}$ "there is an even element in S that is not a multiple of 4"
- implications $\implies$, $\impliedby$, $\iff$
- examples:
  - $x > 1 \implies x$positive
  - $k \in \mathbb{Z} \impliedby k \in \mathbb{N}$
  - $k \in \mathbb{Z}$ and $k \geq 0 \iff k \in \mathbb{N}$
- live exercises:
  - "if x is positive, then it is the square of another number"
  - "n is pair is equivalent to $n = 2m$ for some integer m"
- extreme values (min,max vs inf,sup)
- live exercises:
  - find the extreme values of the set $A = \{x \in \mathbb{R} \mid x > 0\}$.
  - find the extreme values of the set $B = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$.

3 Boolean Algebra

- principle (only 0 and 1)
- $+$ and $*$ for booleans: $\vee$ and $\wedge$
- *not* ($\neg$)
- tables
- De Morgan's law ($\neg(a \wedge b) = \neg a \vee \neg b$ and $\neg(a \vee b) = \neg a \wedge \neg b$)
- *implications* operators ($\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$); x*or* operator ($\veebar$)
- live exercise:
  - express $\veebar$ in terms of $\vee, \wedge, \neg$
  - express $\Longrightarrow$ in terms of $\vee, \wedge, \neg$
  - express $\wedge$ in terms of $\vee, \neg$
  - express $\vee$ in terms of $\wedge, \neg$

4 Modular arithmetic

- Euclidean division of a by b ($a = bk + r$ with $0 \leq r < b$)
- example with $a = 35$, $b = 2, 3, 4, 5, 6, 7, 8$
- modular classes ($12 \equiv 7 \equiv 22 \equiv 102 \equiv -3 \equiv -103 \mod 5$ i.e. $\{2+5k \mid k \in \mathbb{Z}\}$)
- live exercises:
  - give 3 numbers that are congruent to 3 mod 7
  - give a test in terms of modular arithmetic that is equivalent to "n is odd"
  - give a test in terms of modular arithmetic that is equivalent to "n is a nultiple of k" (for k a natural number greater than two)
  - what does it mean for n to say that $n \equiv 5 \mod 10$?
  - find the least positive value of x such that $71 \equiv x \mod 8$
- modular operations (+,-,* $\mod n$)
- GCD and $\square^{-1} \mod p$
- example:
  - compute the GCD of 270 and 192 (answer: 6)
  - compute $5^{-1} \mod 11$
- live exercises:
  - find the least positive value of x such that $89 \equiv (x + 3) \mod 4$
  - what is $x \mod 10$ if $96 \equiv x/7 \mod 5$
  - find an x such that $5x \equiv 4 \mod 11$

- if x is congruent to 13 mod 17 then $7x - 3$ is congruent to which number mod 17?

5 Functions

- functions def
- image vs pre-image
- injective vs surjective
- example of a function injective + proof it is
- example of a function surjective + proof it is
- example of a function not injective + proof it is not
- example of a function not surjective + proof it is not

6 Counting Arrangements

People in a company In a company, there are 800 employees.
300 are men, 352 are union members, 424 are married, 188 are union men, 166 are married men, 208 are union members and married, 144 are married union men.
How many single, non-union women are there?

The padlock A padlock has a 4-digit code, each number being a number from 0 to 9.

1. How many possible codes are there?
2. How many possible codes are there with 4 different digits?
3. How many codes ending in an even number are there?
4. How many codes are there ending with an even number and with 4 different numbers?

In each of the situations:

- How many codes are there containing at least one digit 4?
- How many codes are there containing exactly one digit 4?