Exercises Set 2

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Abstract

Only the questions with a * are compulsory (but do all of them!).

1 Systems of Linear Equations

Reduced Row Echelon Form Find the Reduced Row Echelon Form of the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 2 & 3 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

Gaussian Elimination (*) Solve the following linear system using Gaussian Elimination:

$$\begin{aligned} 2x + y - z &= 4 \\ 3x + 2y + z &= 5 \\ x - y + 3z &= 7 \end{aligned}$$

Start by writing the augmented matrix for the system and perform the necessary row operations to find the solution.

2 Vector Spaces

Linear Independence Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

Show that $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ are linearly independent.

Space of Polynomials Let $\mathbb{P}_2$ be the space of polynomials of degree at most 2. Consider the following polynomials:

$$p_1(x) = 1, \quad p_2(x) = 2x, \quad p_3(x) = 3x^2 - 1$$

Show that the polynomials $p_1(x)$, $p_2(x)$, and $p_3(x)$ form a spanning set for $\mathbb{P}_2$. Express an arbitrary polynomial $q(x) \in \mathbb{P}_2$ as a linear combination of $p_1(x)$, $p_2(x)$, and $p_3(x)$.

3 Matrix Inverses

2x2 Matrices Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Determine whether matrix A is invertible. If it is, find its inverse A^{-1}. Verify your result by multiplying A by its inverse A^{-1} and showing that you get the identity matrix.

3x3 Matrices (*) Let

$$B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

Determine whether matrix B is invertible. If it is, find its inverse B^{-1}. Verify your result by multiplying B by its inverse B^{-1} and showing that you get the identity matrix.

4 Eigenvalues and Eigenvectors

Basic 2x2 Case (*) Consider the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Find the eigenvalues of matrix A.
For each eigenvalue, find the corresponding eigenvector.

Repeated Eigenvalues Consider the matrix

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Find the eigenvalues of matrix B.
For each eigenvalue, find the corresponding eigenvector.

Basic 3x3 Case Consider the matrix

$$C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Find the eigenvalues of matrix C.
For each eigenvalue, find the corresponding eigenvector.

5 Diagonalization

For each matrix from the "Eigenvalues and Eigenvectors" section, determine whether matrix is diagonalizable. If it is, diagonalize it by finding a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.