Exercise Set 5

Solutions

1 Change of Basis

$$v = -1 \cdot u_1 + 3u_2 + 2u_3$$

$$= \begin{pmatrix} -0,5 + 3.2 + 2.(-0,25) \\ 1 + 3.0 + 2.(0,5) \\ -1 + 3(-1) + 2.0 \end{pmatrix}_{\mathcal{B}}$$

$$= \begin{pmatrix} 5 \\ 5 \\ -2 \end{pmatrix}_{\mathcal{B}} = 5.e_1 + 5e_2 - 2e_3$$

$$w = \alpha u_1 + \beta u_2 + \gamma u_3$$

$$(=) \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}} = \alpha \begin{pmatrix} 0,5 \\ -1 \\ 1 \end{pmatrix}_{\mathcal{B}} + \beta \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}_{\mathcal{B}} + \gamma \begin{pmatrix} -0,25 \\ 0,5 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

$$(=) \begin{cases} -1 = \frac{1}{2}\alpha + 2\beta - \frac{1}{4}\gamma \\ 3 = -\alpha + \frac{1}{2}\gamma \\ 2 = \alpha - \beta \end{cases}$$

letting $x = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ & $P = \begin{pmatrix} 1/2 & 2 & -1/4 \\ -1 & 0 & 1/2 \\ 1 & -1 & 0 \end{pmatrix}$

we want $w = Px$

$$(=) \quad P^{-1} w = x$$

$$P^{-1} = \begin{pmatrix} 1/2 & 1/4 & 1 \\ 1/2 & 1/4 & 0 \\ 1 & 5/2 & 2 \end{pmatrix}$$

$$\text{so} \quad x = \begin{pmatrix} -1/2 & +3/4 & +2 \\ -1/2 & +3/4 & \\ -1 & +5/2 & +4 \end{pmatrix} = \begin{pmatrix} 9/4 \\ 1/4 \\ 21/2 \end{pmatrix}$$

$$\alpha = \frac{9}{4} \; ; \; \beta = \frac{1}{4} \; ; \; \gamma = \frac{21}{2}$$

2 - Variance & Covariance

$$\varphi_1 = \{1.5\ ;\ 3\ ;\ 5\ ;\ 7.5\ ;\ 8\ ;\ 9\}$$

$$\varphi_2 = \{0\ ;\ 2\ ;\ 4\ ;\ 6\ ;\ 8\ ;\ 10\}$$

φ_1 : mean : $\frac{17}{3}$
φ_2 : mean : 5
$\Big\}\ cov.(\varphi_1, \varphi_2) = 11$

unbias sample std

φ_1 std : ~ 2.99
φ_2 std : ~ 3.74

$$\hat{\varphi}_1 = \{-1.39\ ;\ -0.89\ ;\ -0.22\ ;\ 0.61\ ;\ 0.78\ ;\ 1.11\}$$

$$\hat{\varphi}_2 = \{-1.34\ ;\ -0.80\ ;\ -0.27\ ;\ 0.27\ ;\ 0.80\ ;\ 1.34\}$$

$$cov.(\hat{\varphi}_1, \hat{\varphi}_2) = \sim 0.9818$$

3 - Principal Component Analysis

$$\varphi = \left\{\begin{pmatrix}2\\-0.4\\0.1\end{pmatrix}, \begin{pmatrix}4\\-0.8\\-0.1\end{pmatrix}, \begin{pmatrix}12\\-2.4\\-0.5\end{pmatrix}, \begin{pmatrix}12\\-2.4\\0.5\end{pmatrix}, \begin{pmatrix}14\\-2.8\\-0.1\end{pmatrix}, \begin{pmatrix}16\\-3.2\\0.1\end{pmatrix}\right\}$$

3.1) $$\hat{\varphi} = \left\{\begin{pmatrix}-1.414\\1.414\\0.304\end{pmatrix}, \begin{pmatrix}-1.067\\1.061\\-0.304\end{pmatrix}, \begin{pmatrix}0.354\\-0.354\\-1.521\end{pmatrix}, \begin{pmatrix}0.354\\-0.354\\1.521\end{pmatrix}, \begin{pmatrix}0.707\\-0.707\\-0.304\end{pmatrix}, \begin{pmatrix}1.061\\-1.061\\0.304\end{pmatrix}\right\}$$

3.2) $$C = \begin{pmatrix}1 & -1 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

3.3) $$\det|C - \lambda I| = (1-\lambda)^3 - (1-\lambda)$$
$$= -2\lambda + 3\lambda^2 - \lambda^3$$
$$= -\lambda(\lambda^2 - 3\lambda + 2)$$
$$\Delta = (-3)^2 - 4.2.1 = 1$$
$$\lambda = \frac{3 \pm 1}{2.1} = 2 \text{ or } 1$$
or $\lambda = 0$

$\lambda = 2$	explains	66.6%
$\lambda = 1$	—	33.3%
$\lambda = 0$	—	0 %

3.4) $\lambda = 2$: $u = \begin{pmatrix}x\\y\\z\end{pmatrix}$ want $Cu = 2u$

$$(=) \begin{pmatrix}x-y\\y-x\\z\end{pmatrix} = \begin{pmatrix}2x\\2y\\2z\end{pmatrix}$$

$$(=) \begin{cases}x+y=0\\z=0\end{cases} \quad \text{set } x=1 \rightarrow u = \begin{pmatrix}1\\-1\\0\end{pmatrix} \quad \text{norm : } u = \begin{pmatrix}0.707\\-0.707\\0\end{pmatrix}$$

$\lambda = 1$:

$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ want $Cu = u$

$\Leftrightarrow \begin{pmatrix} x-y \\ y-x \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ set $z = 1$ $\rightarrow$ $u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 0$:

$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ want $Cu = u$

$\Leftrightarrow \begin{cases} x - y = 0 \Leftrightarrow x = y \\ y - x = 0 \\ z = 0 \end{cases}$ set $x = 1$ $\rightarrow$ $u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

normalized : $u = \begin{pmatrix} 0.707 \\ 0.707 \\ 0 \end{pmatrix}$

3.5) Components of interest : $\begin{pmatrix} 0.707 \\ -0.707 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

↑ 66.6 % ↑ 33.3 %

of the variance explained

$S^* = \left\{ \begin{pmatrix} -2.191 \\ 0.333 \end{pmatrix}, \begin{pmatrix} -1.643 \\ -0.333 \end{pmatrix}, \begin{pmatrix} 0.548 \\ -1.667 \end{pmatrix}, \begin{pmatrix} 0.548 \\ 1.667 \end{pmatrix}, \begin{pmatrix} 1.095 \\ -0.333 \end{pmatrix}, \begin{pmatrix} 1.643 \\ 0.333 \end{pmatrix} \right\}$

3.6) See notebook !