

# Exercises Set 2

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## Abstract

Only the questions with a \* are compulsory (but do all of them!).

## 1 Systems of Linear Equations

**Reduced Row Echelon Form** Find the Reduced Row Echelon Form of the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 2 & 3 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

**Gaussian Elimination** (\*) Solve the following linear system using Gaussian Elimination:

$$2x + y - z = 4$$

$$3x + 2y + z = 5$$

$$x - y + 3z = 7$$

Start by writing the augmented matrix for the system and perform the necessary row operations to find the solution.

## 2 Vector Spaces

**Linear Independence** Consider the following vectors in  $\mathbb{R}^3$ :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

Show that  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$  are linearly independent.

**Space of Polynomials** Let  $\mathbb{P}_2$  be the space of polynomials of degree at most 2. Consider the following polynomials:

$$p_1(x) = 1, \quad p_2(x) = 2x, \quad p_3(x) = 3x^2 - 1$$

Show that the polynomials  $p_1(x)$ ,  $p_2(x)$ , and  $p_3(x)$  form a spanning set for  $\mathbb{P}_2$ . Express an arbitrary polynomial  $q(x) \in \mathbb{P}_2$  as a linear combination of  $p_1(x)$ ,  $p_2(x)$ , and  $p_3(x)$ .

### 3 Matrix Inverses

**2x2 Matrices** Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Determine whether matrix  $A$  is invertible. If it is, find its inverse  $A^{-1}$ . Verify your result by multiplying  $A$  by its inverse  $A^{-1}$  and showing that you get the identity matrix.

**3x3 Matrices** (\*) Let

$$B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

Determine whether matrix  $B$  is invertible. If it is, find its inverse  $B^{-1}$ . Verify your result by multiplying  $B$  by its inverse  $B^{-1}$  and showing that you get the identity matrix.

### 4 Eigenvalues and Eigenvectors

**Basic 2x2 Case** (\*) Consider the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Find the eigenvalues of matrix  $A$ .  
For each eigenvalue, find the corresponding eigenvector.

**Repeated Eigenvalues** Consider the matrix

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Find the eigenvalues of matrix  $B$ .  
For each eigenvalue, find the corresponding eigenvector.

**Basic 3x3 Case** Consider the matrix

$$C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Find the eigenvalues of matrix  $C$ .

For each eigenvalue, find the corresponding eigenvector.

## 5 Diagonalization