

Exercises Set 5

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Abstract

Only the questions with a * are compulsory (but do all of them!).

1 Change of Basis

Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard canonical basis for \mathbb{R}^3 .

Suppose we have another basis $\mathcal{B}' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for \mathbb{R}^3 and let Q be the matrix whose columns are the coordinates of

$$\mathbf{u}_1 = \begin{pmatrix} 0.5 \\ -1 \\ 1 \end{pmatrix}_{\mathcal{B}}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}_{\mathcal{B}}, \text{ and } \mathbf{u}_3 = \begin{pmatrix} -0.25 \\ 0.5 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

with respect to the standard basis. That is, $Q = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$.

Let $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}'}$. Express \mathbf{v} in the standard basis \mathcal{B} .

Let $\mathbf{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}}$. Express \mathbf{w} in the basis \mathcal{B}' .

2 Variance and Covariance

Calculate the variance of the following set:

$$\mathcal{S}_1 = \{1.5, 3, 5, 7.5, 8, 9\}$$

Calculate the variance of the following set:

$$\mathcal{S}_2 = \{2, 4, 6, 8, 10\}$$

Calculate the covariance of \mathcal{S}_1 and \mathcal{S}_2 .

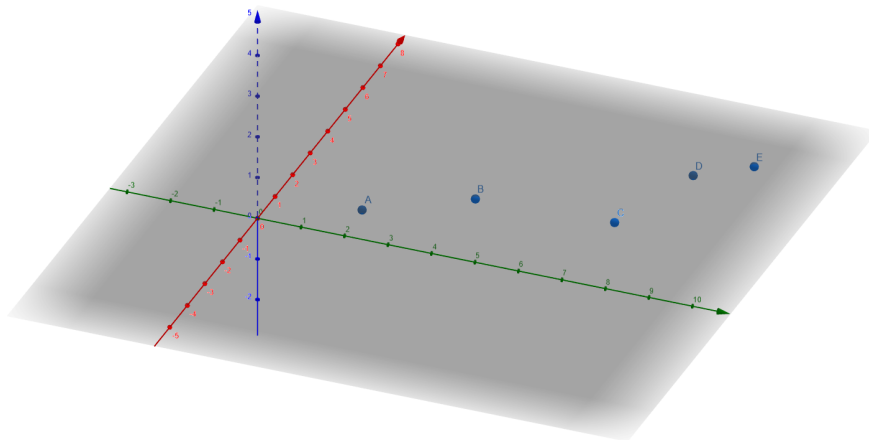
Compute $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$, the standardized version of \mathcal{S}_1 and \mathcal{S}_2 (shifted to mean 0 and scaled to have a variance of 1).

Calculate the covariance of $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$. What do you remark?

3 Principal Component Analysis

Let $\mathcal{S} = \{A, B, C, D, E\}$ be a set of 5 points in \mathbb{R}^3 .

$$A = \begin{pmatrix} 1 \\ 2 \\ 0.1 \end{pmatrix}, B = \begin{pmatrix} 2.5 \\ 4 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 3 \\ 7 \\ -0.2 \end{pmatrix}, D = \begin{pmatrix} 5 \\ 8 \\ 0.1 \end{pmatrix}, \text{ and } E = \begin{pmatrix} 6 \\ 9 \\ 0 \end{pmatrix}.$$



3.1 Standardization

Calculate $\hat{\mathcal{S}}$, the standardized version of \mathcal{S} (shifted to mean 0 and scaled to have a variance of 1).

3.2 Covariance matrix

Compute the covariance of each pair of features. Compute also the variance of each feature. Arrange the values in a 3×3 matrix (variance is covariance of a feature with itself).

3.3 Eigenvalues of the covariance matrix

Calculate the eigenvalues of the covariance matrix. Use the characteristic polynomial.

The variance explained by each feature is $\frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3}$. Order the features by decreasing importance.

3.4 Feature vectors (the "principal components")

3.5 Recasting data on principal components axes