

# Exercises Set 6

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## Abstract

As this is the last session, there will be no compulsory questions this time.

## 1 Lagrangian multiplier technique



### 1.1 Unconstrained optimization

Let  $f(x, y) = 2x^2 - 12x + 4y^2 + 8y + 20$ .

Find  $(x^*, y^*) \in \mathbb{R}^2$  such that  $f$  reaches its minimum (i.e.  $f(x^*, y^*) \leq f(x, y) \quad \forall (x, y) \in \mathbb{R}^2$ ).

### 1.2 (Equality) Constrained optimization

Let  $f(x, y) = 2x^2 - 12x + 4y^2 + 8y + 20$ .

Suppose further that we want  $3x + 5y = 2$ .

Find  $(x^*, y^*) \in \mathbb{R}^2$  such that  $3x^* + 5y^* = 2$  and  $f$  reaches its minimum (i.e.  $f(x^*, y^*) \leq f(x, y) \quad \forall (x, y) \in \mathbb{R}^2, 3x + 5y = 2$ ).

### 1.3 Lagrange multiplier

Let  $f(x, y) = 2x^2 - 12x + 4y^2 + 8y + 20$ .

Suppose further that we want  $3x + 5y = 2$ .

Let  $\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(3x + 5y - 2)$ .

Find the point where  $\nabla \cdot \mathcal{L} = 0$

## 1.4 (Inequality) Constrained optimization

Let  $f(x) = x^2 - 2x$ .

Suppose further that we want  $3x \leq 2$ .

Find  $x^* \in \mathbb{R}$  such that  $3x^* \leq 2$  and  $f$  reaches its minimum (i.e.  $f(x^*) \leq f(x) \quad \forall x \in \mathbb{R}, 3x \leq 2$ ).

Let  $f(x) = x^2 + 2x$ .

Suppose further that we want  $3x \leq 2$ .

Find  $x^* \in \mathbb{R}$  such that  $3x^* \leq 2$  and  $f$  reaches its minimum (i.e.  $f(x^*) \leq f(x) \quad \forall x \in \mathbb{R}, 3x \leq 2$ ).

## 1.5 Lagrange multiplier

Let  $f(x) = x^2 - 2x$ .

Suppose further that we want  $3x \leq 2$ .

Let  $\mathcal{L}(x, \lambda) = f(x) - \lambda(3x - 2)$ .

Find the point where  $\nabla \cdot \mathcal{L} = 0$  and  $\lambda \geq 0$ .

Let  $f(x) = x^2 + 2x$ .

Suppose further that we want  $3x \leq 2$ .

Let  $\mathcal{L}(x, \lambda) = f(x) - \lambda(3x - 2)$ .

Find the point where  $\nabla \cdot \mathcal{L} = 0$  and  $\lambda \geq 0$ .

## 2 Support Vector Machines

### 2.1 Theory

Define a line in  $\mathbb{R}^2$  with parameters  $\mathbf{w}$  and  $b$  defined by  $\mathbf{w} \cdot \mathbf{x} = b$  (or  $\mathbf{w} \cdot \mathbf{x} - b = 0$ ) for  $\mathbf{x} \in \mathbb{R}^2$ . This line cut the plane in 2 regions:

- $\mathcal{R}_{-1}$ :  $\mathbf{x} \in \mathbb{R}^2$  such that  $\mathbf{w} \cdot \mathbf{x} - b < 0$
- $\mathcal{R}_{+1}$ :  $\mathbf{x} \in \mathbb{R}^2$  such that  $\mathbf{w} \cdot \mathbf{x} - b > 0$

The goal is to find  $\mathbf{w}$  and  $b$  such that all points of the first class are in the first region, and all points of the second class are in the second region.

Let our data be  $\{(\mathbf{x}_k, y_k)\}_{k=1}^N$  where  $\mathbf{x}_k$  is the coordinate of the point in  $\mathbb{R}^2$  and  $y_k$  is the label (+1 or -1).

The best line is not only separating the data in two sets, but also maximizing the distance between the line and the points.

Calculate the distance between the line and a point  $\mathbf{x} \in \mathbb{R}^2$ .

Let  $h(x) = \mathbf{w}^T \cdot \mathbf{x} - b$ . We will predict first class -1 if  $h(x) < 0$  (as this means  $x \in \mathcal{R}_{-1}$ ); and second class +1 if  $h(x) > 0$  (as this means  $x \in \mathcal{R}_{+1}$ ). To have each item of data classified correctly, we need:

- if  $y_k = +1$  then  $h(\mathbf{x}_k) > 0$
- if  $y_k = -1$  then  $h(\mathbf{x}_k) < 0$

Hence, we need  $y_k \cdot h(\mathbf{x}_k) > 0 \forall k$ .

The margin is defined as the minimal distance between a point of data and the line.

Formulate the constrained optimization problem to find the best (i.e. maximizing the margin) parameters  $\mathbf{w}$  and  $b$ .

## 2.2 Practice

The training dataset consists of the following data points:

Positive class (" +1 "):

- (2, 2)
- (1, 1)

Negative class (" -1 "):

- (0, 1)
- (1, 0)

The SVM model's objective function for a linearly separable dataset is as follows:

$$\text{Minimize } \frac{1}{2} \|\mathbf{w}\|^2$$

Subject to the constraints:

$$y_i(\mathbf{w} \cdot \mathbf{x}_i + b) \geq 1 \text{ for all } i$$

Where:

$y_i$  is the class label (+1 or -1) of the  $i$ -th data point.

$\mathbf{w}$  is the weight vector normal to the hyperplane.

$\mathbf{x}_i$  is the  $i$ -th data point.

$b$  is the bias term.

1. Calculate the optimal values of  $\mathbf{w}$  and  $b$  to separate the data points while maximizing the margin.
2. Determine the equation of the optimal hyperplane in the form  $\mathbf{w} \cdot \mathbf{x} + b = 0$ .
3. Identify the support vectors in the dataset.
4. Calculate the margin, which is the perpendicular distance from the hyperplane to the nearest support vector.
5. Classify a new data point, (3, 2), based on the learned SVM model.

### 3 Logistic Regression

You are working on a binary classification problem where you are using logistic regression to predict whether a student will pass (1) or fail (0) an exam based on the number of hours they have studied. You have collected the data in the provided table.

Hours Studied ( $X$ )	Pass ( $Y$ )
3	0
4	0
5	1
6	0
7	1
8	1
9	1

You want to fit a logistic regression model to this data and find the best-fitting sigmoid function, which is represented as:

$$P(Y = 1|X) = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X)}}$$

Where  $P(Y = 1|X)$  is the probability of passing the exam given the hours studied.

1. Calculate the cost function (log loss) for the given data and the predicted probabilities. The log loss for a single data point is given as:

$$\text{Log Loss} = -(Y \cdot \log(P(Y = 1|X)) + (1 - Y) \cdot \log(1 - P(Y = 1|X)))$$

2. Calculate the total cost (log loss) summing for all data points.
3. Use gradient descent or any other optimization method to find the values of  $\beta_0$  and  $\beta_1$  that minimize the total cost; you may use a computer to perform gradient descent.
4. Calculate the probability  $P(Y = 1|X = 10)$  using the logistic regression model<sup>1</sup>.
5. Calculate the probability  $P(Y = 1|X = 1)$  using the logistic regression model<sup>2</sup>.
6. Calculate the probability  $P(Y = 1|X = 0)$  using the logistic regression model<sup>3</sup>.

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<sup>1</sup>That is, the estimated probability of passing given the fact that the student have been studying for ten hours.

<sup>2</sup>That is, the estimated probability of passing given the fact that the student have been studying for one hour.

<sup>3</sup>That is, the estimated probability of passing given the fact that the student have not studied at all.