# Exercises Set 3

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**Abstract**

Only the questions with a * are compulsory (but do all of them!).

## 1 Fundamental Theorem of Calculus

**Statement**   Let $f$ be a continuous real-valued function defined on a closed interval $[a, b]$. Let $F$ be the function defined, for all $x \in [a, b]$, by $F(x) = \int_a^b f(t)dt$.

Then $F$ is uniformly continuous on $[a, b]$ and differentiable on the open interval $(a, b)$, and $F'(x) = f(x)$ for all $x in (a, b)$ so $F$ is an anti-derivative of $f$.

**Generalization / Corollary**   Let $f(x)$ be a continuous function on the closed interval $[a, b]$, and let $F(x)$ be an anti-derivative of $f(x)$. Prove that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

**Application**   Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_0^{\pi/2} \sin(x)\,dx$$

Evaluate the following definite integral using the Fundamental Theorem of Calculus:

$$\int_1^4 \frac{1}{x^2}\,dx$$

## 2 Integration techniques

**Reminder**

$$\int f(u)\,du = \int f(g(x)) \cdot g'(x)\,dx \quad \text{or} \quad \int f(u)\,du = \int f(x) \cdot \frac{du}{dx}\,dx$$

$$\int u\,dv = uv - \int v\,du$$

**Substitution / Change of Variable**
**Exercise 1:** Evaluate the following integral using the method of substitution:

$$\int e^{2x} \cos(2x)\, dx$$

*Hint:* Let $u = e^{2x}$ and then find $du$ to perform the substitution.

**Exercise 2:** Evaluate the following integral using the method of substitution:

$$\int \frac{2x}{(x^2 + 1)^2}\, dx$$

*Hint:* Let $u = x^2 + 1$ and then find $du$ to perform the substitution.

**Exercise 3:** Evaluate the following integral using the method of substitution:

$$\int \frac{1}{\sqrt{1 - x^2}}\, dx$$

*Hint:* Let $u = 1 - x^2$ and then find $du$ to perform the substitution.

**Exercise 4:** Evaluate the following integral using a trigonometric substitution:

$$\int \frac{1}{\sqrt{4 - x^2}}\, dx$$

*Hint:* Use the substitution $x = 2\sin(\theta)$ to simplify the integral.

**Integration by Parts**
**Exercise A:** Compute the following integral using integration by parts:

$$\int x \ln(x)\, dx$$

*Hint:* Choose $u = \ln(x)$ and $dv = x\, dx$, and then use the integration by parts formula.

**Exercise B:** Find the value of the integral using integration by parts:

$$\int x^2 e^x\, dx$$

*Hint:* Choose $u = x^2$ and $dv = e^x\, dx$, and then use the integration by parts formula.

**Exercise C:** Compute the following integral using integration by parts:

$$\int x \cos(x)\, dx$$

*Hint:* Choose $u = x$ and $dv = \cos(x)\, dx$, and then use the integration by parts formula.

# 3 Applications

**Areas between curves**
Determine the area of the region enclosed by the curves $y = \sin(x)$ and $y = -\sin(x)$ over the interval $[0, \pi]$.
*Hint:* Begin by finding the points of intersection between the two curves within the given interval. Then, set up the integral to calculate the area between the curves.

**Volumes of revolution**   (Disk Method)
Find the volume of the solid generated by revolving the region bounded by $y = x^2$ and the x-axis, over the interval $[0, 1]$, about the x-axis using the disk method.
*Hint:* Determine the limits of integration, the radius of the disks, and set up the integral to find the volume.

**Arc length of curves**
Find the arc length of the curve defined by $y = \sqrt{x}$ over the interval $[1, 4]$.
*Hint:* Use the formula for arc length $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$ to calculate the arc length of the curve.

**Surface area of revolution**
Determine the surface area of the solid generated by revolving the curve $y = x^2$ over the interval $[0, 1]$ about the x-axis.
*Hint:* Use the formula for surface area of revolution $\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$ to calculate the surface area.