Exercises Set 5

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September 27, 2023

Abstract

Only the questions with a * are compulsory (but do all of them!).

1 Change of Basis

Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard canonical basis for $\mathbb{R}^3$.

Suppose we have another basis $\mathcal{B}' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ and let Q be the matrix whose columns are the coordinates of

$$\mathbf{u}_1 = \begin{pmatrix} 0.5 \\ -1 \\ 1 \end{pmatrix}_{\mathcal{B}}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}_{\mathcal{B}}, \text{ and } \mathbf{u}_3 = \begin{pmatrix} -0.25 \\ 0.5 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

with respect to the standard basis. That is, $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$.

Let $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}'}$. Express $\mathbf{v}$ in the standard basis $\mathcal{B}$.

Let $\mathbf{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{B}}$. Express $\mathbf{w}$ in the basis $\mathcal{B}'$.

2 Variance and Covariance

Calculate the variance of the following set:

$$\mathcal{S}_1 = \{1.5, 3, 5, 7.5, 8, 9\}$$

Calculate the variance of the following set:

$$\mathcal{S}_2 = \{2, 4, 6, 8, 10\}$$

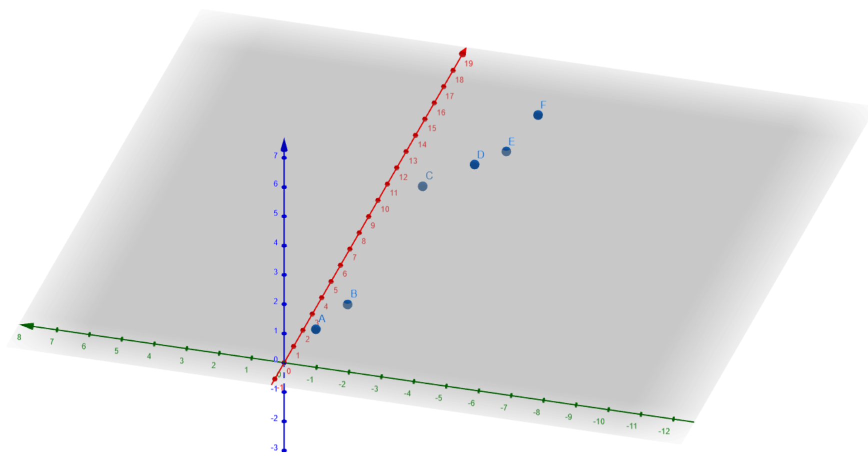
Calculate the covariance of $\mathcal{S}_1$ and $\mathcal{S}_2$.

Compute $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$, the standardized version of $\mathcal{S}_1$ and $\mathcal{S}_2$ (shifted to mean 0 and scaled to have a variance of 1).

Calculate the covariance of $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$. What do you remark?

3 Principal Component Analysis

Let $\mathcal{S} = \{A, B, C, D, E, F\}$ be a set of 5 points in $\mathbb{R}^3$.
$A = \begin{pmatrix} 2 \\ -0.4 \\ 0.1 \end{pmatrix}$, $B = \begin{pmatrix} 4 \\ -0.8 \\ -0.1 \end{pmatrix}$, $C = \begin{pmatrix} 12 \\ -2.4 \\ -0.5 \end{pmatrix}$, $D = \begin{pmatrix} 12 \\ -2.4 \\ 0.5 \end{pmatrix}$, $E = \begin{pmatrix} 14 \\ -2.8 \\ -0.1 \end{pmatrix}$, and $F = \begin{pmatrix} 16 \\ -3.2 \\ 0.1 \end{pmatrix}$.



3.1 Standardization *

Calculate $\hat{\mathcal{S}}$, the standardized version of $\mathcal{S}$ (shifted to mean 0 and scaled to have a variance of 1).

3.2 Covariance matrix *

Compute the covariance of each pair of features. Compute also the variance of each feature. Arrange the values in a 3×3 matrix (variance is covariance of a feature with itself).

3.3 Eigenvalues of the covariance matrix *

Calculate the eigenvalues of the covariance matrix. Use the characteristic polynomial.

The variance explained by each feature is $\frac{\lambda_i}{\lambda_1+\lambda_2+\lambda_3}$[1]. Order the features by decreasing importance.

[1] Where λ_i are the eigenvalues.

3.4 Feature vectors (the ”principal components”) *

For each eigenvalue, calculate the corresponding eigenvectors of the covariance matrix. These are the principal components, also called ”feature vectors”.

3.5 Recasting data on principal components axes *

Project each item of data on the first two components, and plot them in a 2D graph.

3.6 Importance of standardization

Redo this exercise without standardizing your data to variance of one.

4 Principal Component Analysis with Python

(see notebook)