Ficha 2 - Funções a várias variáveis, gradiente

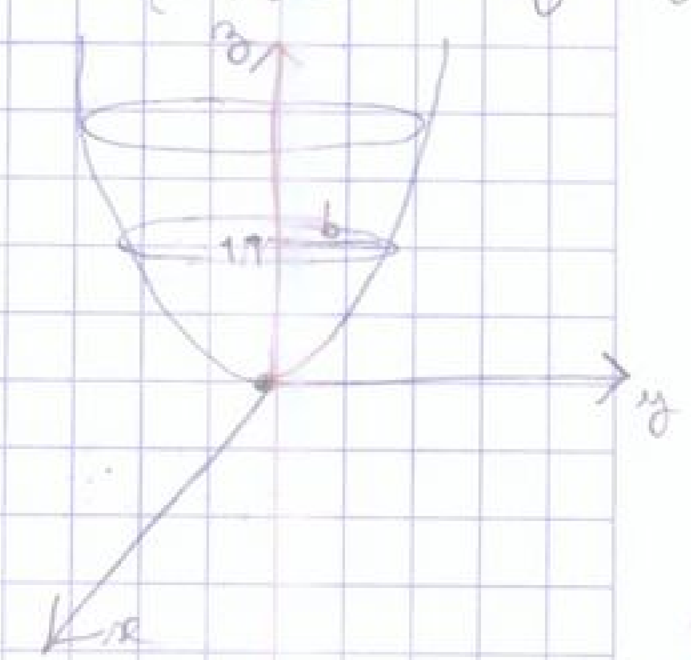
1 a) $f(x,y,z) = |xy| + 2|xz| + 2|yz|$. $D_f = \{(x,y,z) \in \mathbb{R}^3 : x \neq 0 \wedge y \neq 0 \wedge z \neq 0\}$

b) $(1,1,0) \cdot (x,y,z) = \sqrt{2} \times \sqrt{x^2+y^2+z^2} \times \cos\theta \Leftrightarrow$

$\Leftrightarrow \theta = \arccos \dfrac{x+y}{\sqrt{2(x^2+y^2+z^2)}}$, $D_f = \mathbb{R}^3 \setminus \{(0,0,0)\}$

c) $f(x,y,z) = A_{base} \times h = 1 \times 1 \times h = h = |z|$, $D_f = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$

2 a) Paraboloide elíptico



b) $b \to +\infty \Rightarrow z = x^2$

Transforma-se num cilindro parabólico.

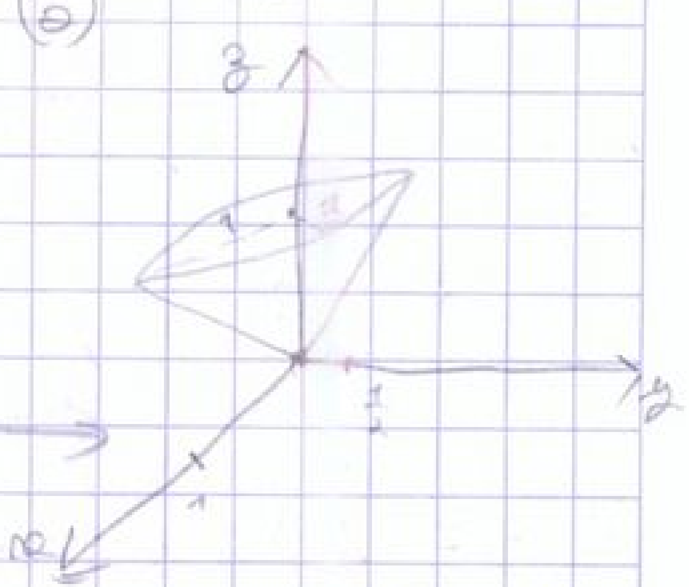
c) É uma elipse no plano $z=1$

d) $b \to +\infty \Rightarrow x^2 = 1 \Leftrightarrow x = 1 \vee x = -1$

Transforma-se em duas retas paralelas no plano $z=1$: $x=-1$ e $x=1$

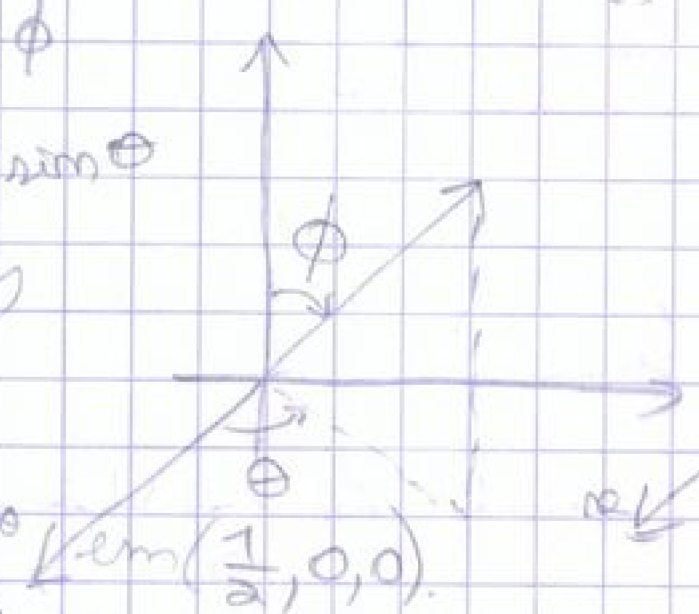
3 a) $z = \sqrt{x^2+4y^2} \Leftrightarrow x^2+4y^2 = z^2 \Leftrightarrow \dfrac{x^2}{1^2} + \dfrac{y^2}{\left(\frac{1}{2}\right)^2} = z^2$

Cone elíptico



$z = \rho\cos\phi$

b) $x = \rho\sin\phi\cos\theta$ $\quad y = \rho\sin\phi\sin\theta$



$x^2+y^2+z^2 = \rho$ $\quad x^2+y^2+z^2 = x \Leftrightarrow$

$\Leftrightarrow \left(x - \frac{1}{2}\right)^2 + y^2 + z^2 = \frac{1}{4}$

Sup. esférica de raio $\frac{1}{2}$ com centro em $\left(\frac{1}{2}, 0, 0\right)$.

4 a) $\lim\limits_{x\to 0} f(x,0) = \lim\limits_{x\to 0} 0 = 0$

b) $\lim\limits_{y\to 0} f(0,y) = \lim\limits_{y\to 0} 0 = 0$

c) $\lim\limits_{x\to 0} f(x,mx) = \lim\limits_{x\to 0} \dfrac{m x^2}{x^2+m^2x^2} = \lim\limits_{x\to 0} \dfrac{m}{m^2+1} = \dfrac{m}{m^2+1}$

d) $x = \theta\cos\theta$ $\quad y = \theta\sin\theta$

$f(\theta\cos\theta, \theta\sin\theta) = \dfrac{\theta^2\sin\theta\cos\theta}{\theta^2\cos^2\theta + \theta^2\sin^2\theta} = \dfrac{\theta^2\sin\theta\cos\theta}{\theta^2} = \sin\theta\cos\theta$

$\lim\limits_{\theta\to 0^+} \sin\theta\cos\theta = 0 \times 1 = 0$

e) $x = \sin(3\theta)\cos(\theta)$ $\quad y = \sin(3\theta)\sin(\theta)$

$f(\sin(3\theta)\cos\theta, \sin(3\theta)\sin\theta) = \dfrac{\sin^2(3\theta)\sin\theta\cos\theta}{\sin^2(3\theta)\cos^2(\theta) + \sin^2(3\theta)\sin^2(\theta)} = \sin\theta\cos\theta$

$\lim\limits_{\theta\to \frac{\pi}{3}} \sin\theta\cos\theta = \sin\left(\frac{\pi}{3}\right) \times \cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{4}$

f) $x = \frac{1}{t}$ $\quad y = \frac{\sin(t)}{t}$

$f\left(\frac{1}{t}, \frac{\sin(t)}{t}\right) = \dfrac{\frac{\sin(t)}{t^2}}{\frac{1+\sin^2(t)}{t^2}} = \dfrac{\sin(t)}{1+\sin^2(t)}$ $\quad \lim\limits_{t\to +\infty} \dfrac{\sin(t)}{1+\sin^2(t)}$ não existe.

Logo, não existe $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ ao longo de C.

⑤ a) $\frac{\partial \rho}{\partial \theta} = -\sin\phi \sin\theta \qquad \frac{\partial \rho}{\partial \phi} = \cos\phi \cos\theta$

b) $\frac{\partial g}{\partial x} = \frac{2x}{2\sqrt{x^2+4y^2}} = \frac{x}{\sqrt{x^2+4y^2}} \qquad \frac{\partial g}{\partial y} = \frac{8y}{2\sqrt{x^2+4y^2}} = \frac{4y}{\sqrt{x^2+4y^2}}$

c) $\frac{\partial h}{\partial x} = \frac{2}{1+(2x+y)^2} \qquad \frac{\partial h}{\partial y} = \frac{1}{1+(2x+y)^2}$

d) $\frac{\partial u}{\partial x} = -\frac{e^z}{x^2y^2} \qquad \frac{\partial u}{\partial y} = -\frac{2e^z}{xy^3} \qquad \frac{\partial u}{\partial z} = \frac{e^z}{xy^2}$

e) $\frac{\partial w}{\partial x} = \frac{z}{3x+3y} \qquad \frac{\partial w}{\partial y} = \frac{3}{3x+3y} \qquad \frac{\partial w}{\partial z} = \frac{x}{3x+3y}$

f) $\frac{\partial v}{\partial x} = y^z x^{y^z-1} \qquad \frac{\partial v}{\partial y} = zy^{z-1} \times x^{y^z} \times \ln(x) \qquad \frac{\partial v}{\partial z} = y^z \ln(y) x^{y^z} \ln(x)$

g) $\frac{\partial f}{\partial x} = \frac{2x + \frac{3x^2}{2\sqrt{x^3+y^2}}}{x^2+\sqrt{x^3+y^2}} = \frac{4x\sqrt{x^3+y^2}+3x^2}{2(x^3+y^2+x^2\sqrt{x^3+y^2})}$

$\frac{\partial f}{\partial y} = \frac{\frac{2y}{2\sqrt{x^3+y^2}}}{x^2+\sqrt{x^3+y^2}} = \frac{y}{(x^3+y^2+x^2\sqrt{x^3+y^2})}$

⑥ a) $\frac{\partial f}{\partial x} = e^y \sin(z+x) + x e^y \cos(z+x) = e^y(\sin(z+x) + x\cos(z+x))$

$\frac{\partial f}{\partial y} = e^y x \sin(z+x) \qquad \frac{\partial f}{\partial z} = x e^y \cos(z+x)$

$\vec{\nabla} f = e^y(\sin(z+x) + x\cos(z+x), x\sin(z+x), x\cos(z+x))$

b) $\frac{\partial g}{\partial x} = -5(-x+2y)^4 \qquad \frac{\partial g}{\partial y} = 10(-x+2y)^4 \qquad \frac{\partial g}{\partial z} = -\frac{2}{z^2}$

$\vec{\nabla} g = \left(-5(-x+2y)^4, 10(-x+2y)^4, -\frac{2}{z^2}\right)$

⑦ a) $\frac{\partial f}{\partial x} = 4-y^2 = 4-4\sin^2(t) = 4(1-\sin^2(t)) = 4\cos^2(t)$

$\frac{\partial f}{\partial y} = -2yx = -2(2\cos(t) \times 2\sin(t)) = -8\sin(t)\cos(t)$

$\vec{\alpha}'(t) = (-2\sin(t), 2\cos(t))$

$f'(t) = \vec{\nabla} f(\alpha(t)) \cdot \vec{\alpha}'(t) = (4\cos^2(t), -8\sin(t)\cos(t)) \cdot (-2\sin(t), 2\cos(t)) =$

$= -8\sin(t)\cos^2(t) - 16\sin(t)\cos^2(t) = -24\sin(t)\cos^2(t)$

b) $f(\alpha(t)) = 2\cos(t)(4-4\sin^2(t)) = 8\cos(t) - 8\cos(t)\sin^2(t) =$

$= 8\cos(t)(1-\sin^2(t)) = 8\cos^3(t)$

$f'(\alpha(t)) = -24\sin(t)\cos^2(t)$

(8) $\frac{\partial f}{\partial x} = 3 \times \frac{\frac{1}{y}}{\frac{x}{y}} = \frac{3}{x}$ $\frac{\partial f}{\partial y} = 3 \times \frac{-\frac{x}{y^2}}{\frac{x}{y}} = -\frac{3}{y}$

$\frac{\partial f}{\partial z} = \ln\left(\frac{x}{y}\right)$ $\vec{\nabla} f = \left(\frac{3}{x}, -\frac{3}{y}, \ln\left(\frac{x}{y}\right)\right)$

$\vec{PQ} = Q - P = (1,0,3)$ $\|\vec{PQ}\| = \sqrt{10}$ $\hat{n} = \frac{1}{\sqrt{10}}(1,0,3)$

No ponto P: $\vec{\nabla} f(1,2,-2) = (-2, 1, -\ln(2))$

$f'_{(P,\hat{n})} = \vec{\nabla} f(P) . \hat{n} = (-2,1,-\ln(2)).(1,0,3) \times \frac{1}{\sqrt{10}} = \frac{-2-3\ln(2)}{\sqrt{10}} =$

$= -\frac{\sqrt{10}}{5} - \frac{3\sqrt{10}\ln(2)}{10}$

(9) $\frac{\partial f}{\partial x} = e^{y^2-z^2}$ $\frac{\partial f}{\partial y} = 2xy\,e^{y^2-z^2}$ $\frac{\partial f}{\partial z} = -2xz\,e^{y^2-z^2}$ $P(1,0,-2) \Rightarrow t=1$

$\vec{\nabla} f = \left(e^{y^2-z^2}, 2xy\,e^{y^2-z^2}, -2xz\,e^{y^2-z^2}\right)$ $\vec{\nabla} f(P) = (1,4,4)$

$\hat{n} = \hat{T}(1) = \frac{\vec{r}'(1)}{\|\vec{r}'(1)\|}$ $\vec{r}'(t) = (1, -2\sin(t-1), -2e^{t-1})$

$\vec{r}'(1) = (1,0,-2)$ $\|\vec{r}'(1)\| = \sqrt{1+4} = \sqrt{5}$ $\hat{T}(1) = \frac{1}{\sqrt{5}}(1,0,-2)$

$f'_{(P,\hat{n})} = \vec{\nabla} f(P) . \hat{n} = (1,4,4).(1,0,-2) \times \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}(1-8) = -\frac{7}{\sqrt{5}} = -\frac{7\sqrt{5}}{5}$

(10) $\frac{\partial f}{\partial x} = 2(x+y^2+z^3)$ $\frac{\partial f}{\partial y} = 4y(x+y^2+z^3)$ $\frac{\partial f}{\partial z} = 6z^2(x+y^2+z^3)$

$\vec{\nabla} f = \left(2(x+y^2+z^3), 4y(x+y^2+z^3), 6z^2(x+y^2+z^3)\right)$ $\vec{\nabla} f(P) = (6,-12,18)$

$\vec{a} = (1,1,0)$ $\|\vec{a}\| = \sqrt{2}$ $\hat{n} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{\sqrt{2}}(1,1,0)$

$f'_{(P,\hat{n})} = \vec{\nabla} f(P) . \hat{n} = \frac{1}{\sqrt{2}} \times ((6,-12,18).(1,1,0)) = \frac{1}{\sqrt{2}} \times (6-12) = -\frac{6\sqrt{2}}{2} = -3\sqrt{2}$

(11) $\frac{\partial f}{\partial x} = 2y^2 e^{2x}$ $\frac{\partial f}{\partial y} = 2y\,e^{2x}$ $\vec{\nabla} f = (2y^2 e^{2x}, 2y\,e^{2x})$

$\vec{\nabla} f(P) = (2,2)$ $f'_{(P,\hat{n})} = (2,2).\hat{n}$ $f'_{(P,\hat{n})}$ tem valor máximo quando $(2,2)$ e $\hat{n}$ fazem um ângulo de $0°$ entre si.

Logo, $\hat{n}$ tem de resultar da normalização do vetor $(2,2)$, isto é: $\|(2,2)\| = \sqrt{8}$

$\hat{n} = \frac{1}{\sqrt{8}}(2,2) = \frac{(2,2)}{2\sqrt{2}} = \frac{1}{\sqrt{2}}(1,1)$ R: Segundo direção e sentido definidos pelo versor $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$.

(12) $\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$ $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$ $\frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$\vec{\nabla} f = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ $\vec{\nabla} f(P) = \frac{1}{\sqrt{6}}(1,-2,1)$

$f'_{(P,\hat{n})} = \frac{1}{\sqrt{6}}(1,-2,1).\hat{n}$ $f'_{(P,\hat{n})}$ tem valor máximo quando $(1,-2,1)$ e $\hat{n}$ fazem um ângulo de $0°$ entre si.

Logo, $\hat{n}$ tem de resultar da normalização de $(1,-2,1)$: $\hat{n} = \frac{1}{\sqrt{6}}(1,-2,1)$

R: Segundo a direção e sentido definidos por $\frac{1}{\sqrt{6}}(\hat{i} - 2\hat{j} + \hat{k})$

(13) $\frac{\partial f}{\partial x} = \frac{\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$ $\quad \frac{\partial f}{\partial y} = \frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \frac{y}{x^2+y^2}$

$\vec{\nabla} f = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ $\quad \vec{XO} = O - X = (-x,-y)$ $\quad \hat{n} = \frac{(-x,-y)}{\sqrt{x^2+y^2}}$

$f'(X,\hat{n}) = \vec{\nabla} f \cdot \hat{n} = \left(\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)\cdot(-x,-y)\right) \times \frac{1}{\sqrt{x^2+y^2}} =$

$= \left(-\frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2}\right) \times \frac{1}{\sqrt{x^2+y^2}} = -\left(\frac{x^2+y^2}{x^2+y^2}\right) \times \frac{1}{\sqrt{x^2+y^2}} = -\frac{1}{\sqrt{x^2+y^2}}$

(14) a) $\frac{\partial f}{\partial x} = 2x+y$ $\quad \frac{\partial f}{\partial y} = x+z$ $\quad \frac{\partial f}{\partial z} = y$ $\quad \vec{\nabla} f = (2x+y, x+z, y)$

$\vec{\nabla} f(P) = (2,3,0)$ $\quad z = 3 - x^2 - y^2 + 6y \Leftrightarrow \underbrace{x^2+y^2-6y+z-3}_{g(x,y,z)} = 0$

$\frac{\partial g}{\partial x} = 2x$ $\quad \frac{\partial g}{\partial y} = 2y-6$ $\quad \frac{\partial g}{\partial z} = 1$ $\quad \nabla g = (2x, 2y-6, 1)$ $\quad \nabla g(P) = (2,-6,1)$

$\|\vec{\nabla} g(P)\| = \sqrt{4+36+1} = \sqrt{41}$ $\quad \vec{u} = \pm\frac{1}{\sqrt{41}}(2,-6,1)$ $\quad \vec{u}$ normal à superfície.

$f'(P,\hat{u}) = \vec{\nabla} f(P)\cdot\vec{u} = (2,3,0)\cdot(2,-6,1)\times\left(\pm\frac{1}{\sqrt{41}}\right) = \pm\frac{4-18}{\sqrt{41}} = \pm\frac{14}{\sqrt{41}}$

b) $\frac{\partial f}{\partial x} = 2x$ $\quad \frac{\partial f}{\partial y} = 2y$ $\quad \frac{\partial f}{\partial z} = -2z$ $\quad \vec{\nabla} f = (2x, 2y, -2z)$ $\quad \vec{\nabla} f(Q) = (6,8,-10)$

$z^2 = x^2+y^2$

$2x^2+2y^2-z^2 = 25 \Leftrightarrow 2x^2+2y^2-(x^2+y^2) = 25 \Leftrightarrow x^2+y^2 = 25 \wedge z = 5$

$\frac{\partial s}{\partial x} = 2x$ $\quad \frac{\partial s}{\partial y} = 2y$ $\quad \frac{\partial s}{\partial z} = 0$ $\quad \vec{\nabla} s = (2x, 2y, 0)$

$\vec{\nabla} s(Q) = (6,8,0)$ $\quad$ vetor perpendicular: $(-8,6,0) = (-4,3,0)$

$\hat{n} = \frac{(-4,3,0)}{\sqrt{16+9}} = \frac{1}{5}(-4,3,0)$

$f'(Q,\hat{n}) = \vec{\nabla} f(Q)\cdot\hat{n} = ((6,8,-10)\cdot(-4,3,0))\times\frac{1}{5} = \frac{-24+24}{5} = 0$

(15) Parametrização de $[AB]$: $\vec{r}(t) = A + t(B-A)$, $t \in [0,1]$
em que $\vec{r}(0) = A$ e $\vec{r}(1) = B$

Considere-se a função composta $g(t) = f(\vec{r}(t))$, $t \in [0,1]$

$g(0) = f(A)$ e $g(1) = f(B)$. Se $g(0) = g(1)$, então, pelo Teorema de Rolle, $\exists t \in]0,1[: g'(t) = 0$

Sabendo que $g'(t) = \nabla f(\vec{r}(t))\cdot\vec{r}'(t) = \nabla f(\vec{r}(t))\cdot(B-A)$

Então:

$\exists c \in [AB] : 0 = \vec{\nabla} f(c)\cdot(B-A)$

16) $B - A = (1,3,2) - (0,1,1) = (1,2,1)$

Parametrização de $[AB]$: $\vec{r}(t) = (0,1,1) + t(1,2,1),\ t \in [0,1] \Leftrightarrow$

$\Leftrightarrow \vec{r}(t) = (t, 1+2t, 1+t),\ t \in [0,1]$

$f(B) = 8 - 9 + 4 = 3 \qquad f(A) = -1 + 1 = 0 \qquad \vec{\nabla} f(x,y,z) = (4z, -2y, 4x+2z)$

$C \in [AB] \Rightarrow C = (t, 1+2t, 1+t),\ t \in [0,1]$

$\nabla f(C) = (4+4t, -2-4t, 4t+2+2t) = (4+4t, -2-4t, 6t+2)$

$\vec{\nabla} f(C) \cdot (1,2,1) = 4 + 4t - 4 - 8t + 6t + 2 = 2t + 2$

$2t + 2 = 3 - 0 \Leftrightarrow 2t = 1 \Leftrightarrow t = \frac{1}{2} \qquad C\left(\frac{1}{2}, 2, \frac{3}{2}\right)$

17) $\overbrace{x^3 + y^2 + 2x - 6}^{f(x,y)} = 0 \qquad \frac{\partial f}{\partial x} = 3x^2 + 2 \qquad \frac{\partial f}{\partial y} = 2y \qquad \vec{\nabla} f = (3x^2+2, 2y)$

$\vec{\nabla} f(-1,3) = (5,6)$ Vetor Normal: $5\hat{i} + 6\hat{j}$ Vetor tangente: $6\hat{i} - 5\hat{j}$

18) Se a trajetória segue a direção de maior taxa de variação:

$\nabla T(x,y) = K\vec{r}\,'(t)$ (isto é, $\nabla T(x,y)$ tem a mesma direção do vetor tangente à trajetória).

$\nabla T(x,y) = \left(-\sqrt{2}\, e^{-y} \sin(x), -\sqrt{2}\, e^{-y} \cos(x)\right) \qquad \vec{r}\,'(t) = (x'(t), y'(t))$

$\begin{cases} -\sqrt{2}\, e^{-y} \sin(x) = K x'(t) \\ -\sqrt{2}\, e^{-y} \cos(x) = K y'(t) \end{cases} \Leftrightarrow \begin{cases} K = \dfrac{-\sqrt{2}\, e^{-y} \sin(x)}{x'(t)} \\ -\sqrt{2}\, e^{-y} \cos(x) = \dfrac{-\sqrt{2}\, e^{-y} \sin(x)}{x'(t)}\, y'(t) \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \text{———} \\ x'(t) \cos(x) = \sin(x)\, y'(t) \end{cases} \Leftrightarrow y'(t) = \frac{\cos(x)}{\sin(x)}\, x'(t) \Leftrightarrow \qquad \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{y'(t)}{x'(t)}$

$\Leftrightarrow y = \int \frac{\cos(x)}{\sin(x)}\, dx \Leftrightarrow y = \ln(\sin(x)) + C$ Como $C \in$ curva:

$0 = \ln\left(\sin\left(\frac{\pi}{4}\right)\right) + C \Leftrightarrow C = -\ln\left(\frac{\sqrt{2}}{2}\right) \Leftrightarrow C = -\ln\left(\frac{1}{\sqrt{2}}\right) \Leftrightarrow C = \ln(\sqrt{2})$

Logo, $y = \ln(\sin(x)) + \ln(\sqrt{2}) = \ln(\sqrt{2}\sin(x))$

19) $\overbrace{z - xy}^{s_1(x,y,z)} = 0 \qquad \frac{\partial s_1}{\partial x} = -y \qquad \frac{\partial s_1}{\partial y} = -x \qquad \frac{\partial s_1}{\partial z} = 1 \qquad \vec{\nabla} s_1(x,y,z) = (-y, -x, 1)$

Se o plano é horizontal, $\vec{\nabla} s_1 = \vec{0} \Leftrightarrow \begin{cases} -y = 0 \\ -x = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \Rightarrow z = 0$

Verifica-se no ponto $O = (0,0,0)$.

$\overbrace{4x + 2y - x^2 + xy - y^2 - z}^{s_2(x,y,z)} = 0 \qquad \frac{\partial s_2}{\partial x} = 4 - 2x + y \qquad \frac{\partial s_2}{\partial y} = 2 + x - 2y \qquad \frac{\partial s_2}{\partial z} = -1$

$\vec{\nabla} s_2(x,y,z) = (4 - 2x + y, 2 + x - 2y, -1)$

Se o plano é horizontal, $\vec{\nabla} s_2 = \vec{0} \Leftrightarrow \begin{cases} 4 - 2x + y = 0 \\ 2 + x - 2y = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x - 4 \\ 2 + x - 4x + 8 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \text{———} \\ 3x = 10 \end{cases} \Leftrightarrow \begin{cases} y = \frac{8}{3} \\ x = \frac{10}{3} \end{cases} \qquad z = \frac{40}{3} + \frac{16}{3} - \frac{100}{9} + \frac{80}{9} - \frac{64}{9} = \frac{28}{3}$

Verifica-se no ponto $P\left(\frac{10}{3}, \frac{8}{3}, \frac{28}{3}\right)$.

(20) $x^2+y^2+z^2=3 \Leftrightarrow \overbrace{x^2+y^2+z^2-3}^{s(x,y,z)}=0$ vetor normal

$\frac{\partial s}{\partial x}=2x \quad \frac{\partial s}{\partial y}=2y \quad \frac{\partial s}{\partial z}=2z \quad \vec{\nabla} s(x,y,z)=(2x,2y,2z) \quad \vec{\nabla} s(P)=(2,2,2)$

$2x+2y+2z=d$ Como $P \in$ plano: $2\times1+2\times1+2\times1=d \Leftrightarrow d=6$

Equação cartesiana $\rightarrow 2x+2y+2z=6 \Leftrightarrow x+y+z=3$

(21) $xy+yz+xz=11 \Leftrightarrow \overbrace{xy+yz+xz-11}^{s(x,y,z)}=0$ $\quad \vec{\nabla} s(P)=(5,4,3)$

$\frac{\partial s}{\partial x}=y+z \quad \frac{\partial s}{\partial y}=x+z \quad \frac{\partial s}{\partial z}=y+x \quad \vec{\nabla} s(x,y,z)=(y+z,x+z,y+x)$

reta normal: $r: (x,y,z)=(1,2,3)+t(5,4,3), t\in\mathbb{R}$

$5x+4y+3z=d$ Como $P\in$ plano: $5\times1+4\times2+3\times3=d=22$

eq. do plano tangente: $5x+4y+3z=22$

(22) $\overbrace{x^2+y^2+z^2-8x-8y-6z+24}^{s(x,y,z)}=0 \quad \vec{\nabla} s(2,1,1)=(-4,-6,-4)$

$\frac{\partial s}{\partial x}=2x-8 \quad \frac{\partial s}{\partial y}=2y-8 \quad \frac{\partial s}{\partial z}=2z-6 \quad \vec{\nabla} s(x,y,z)=(2x-8,2y-8,2z-6)$

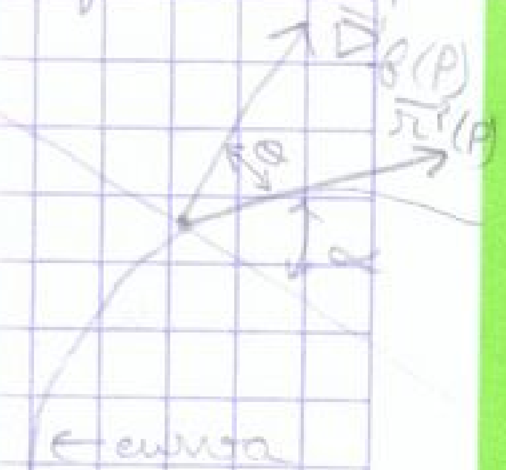
$\overbrace{x^2+3y^2+2z^2-9}^{e(x,y,z)}=0 \quad \vec{\nabla} e(2,1,1)=(4,6,4)$

$\frac{\partial e}{\partial x}=2x \quad \frac{\partial e}{\partial y}=6y \quad \frac{\partial e}{\partial z}=4z \quad \vec{\nabla} e(x,y,z)=(2x,6y,4z)$

$\vec{\nabla} s(2,1,1)=-\vec{\nabla} e(2,1,1)$, logo as duas superfícies são tangentes nesse ponto.

(23) $\vec{r}'(t)=\left(2,-\frac{3}{t^2},-4t\right) \quad P=(2,3,-2) \Rightarrow t=1$ plano tangente à superfície

$\vec{r}'(1)=(2,-3,-4) \quad \vec{\nabla} g(x,y,z)=(2x,2y,6z)$



$\vec{\nabla} g(2,3,-2)=(4,6,-12)$

$\vec{\nabla} g(P)\cdot\vec{r}'(1)=\|\vec{\nabla} g(P)\|\times\|\vec{r}'(1)\|\times\cos\theta \Leftrightarrow$

$\Leftrightarrow (4,6,-12)\cdot(2,-3,-4)=\sqrt{16+36+144}\times\sqrt{4+9+16}\times\cos\theta \Leftrightarrow$

$\Leftrightarrow \cos\theta=\frac{8-18+48}{\sqrt{196}\times\sqrt{29}} \Leftrightarrow \theta=\arccos\left(\frac{19}{7\sqrt{29}}\right)=\arccos\left(\frac{19\sqrt{29}}{203}\right)$

$\alpha=\frac{\pi}{2}-\theta=\frac{\pi}{2}-\arccos\left(\frac{19\sqrt{29}}{203}\right)$

$P(x_0,y_0,z_0) \leftarrow$ ponto de intersecção

(24) $\underbrace{x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}}_{g(x,y,z)}=w^{\frac{1}{2}} \quad \vec{\nabla} g(x,y,z)=\frac{1}{2}\left(x^{-\frac{1}{2}},y^{-\frac{1}{2}},z^{-\frac{1}{2}}\right)$

plano tangente à superfície: $(X-P)\cdot\vec{\nabla} g(x,y,z)=0 \Leftrightarrow (x-x_0,y-y_0,z-z_0)\cdot\left(\frac{1}{\sqrt{x_0}},\frac{1}{\sqrt{y_0}},\frac{1}{\sqrt{z_0}}\right)=$

$\Leftrightarrow \frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\frac{x_0}{\sqrt{x_0}}+\frac{y_0}{\sqrt{y_0}}+\frac{z_0}{\sqrt{z_0}} \Leftrightarrow \frac{1}{\sqrt{x_0}}x+\frac{1}{\sqrt{y_0}}y+\frac{1}{\sqrt{z_0}}z=w^{\frac{1}{2}}$

Ponto com os eixos: $x=x_0^{\frac{1}{2}}w^{\frac{1}{2}}$; $y=y_0^{\frac{1}{2}}w^{\frac{1}{2}}$; $z=z_0^{\frac{1}{2}}w^{\frac{1}{2}}$

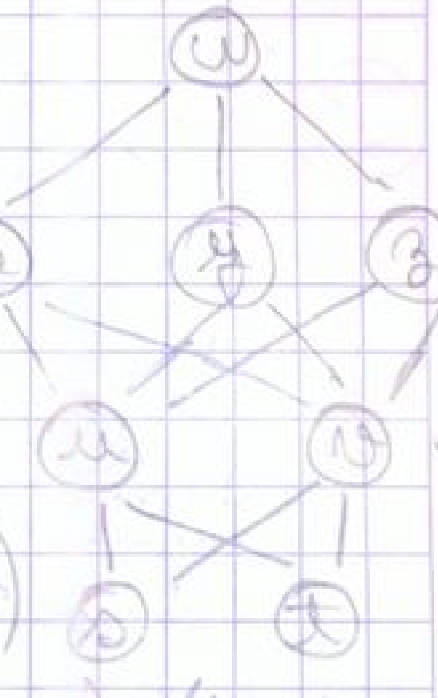
$x+y+z=w^{\frac{1}{2}}\times w^{\frac{1}{2}}=w$ c.q.p.

(25) $\frac{\partial y}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{\cos(xy) - yx\sin(xy) - y\sin(x)}{-x^2\sin(xy) + \cos(x)} =$

$= \frac{xy\sin(xy) + y\sin(x) - \cos(xy)}{\cos(x) - x^2\sin(xy)}$

(26) $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2x+y}{4z^3+3z^2}$ $\quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{2y+x}{4z^3+3z^2}$

(27) $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial u} \times \frac{\partial u}{\partial s} + \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial v} \times \frac{\partial v}{\partial s} +$



$+ \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial u} \times \frac{\partial u}{\partial s} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial v} \times \frac{\partial v}{\partial s} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial s} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial v} \times \frac{\partial v}{\partial s} =$

$= \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial u} \times \frac{\partial u}{\partial s} + \frac{\partial x}{\partial v} \times \frac{\partial v}{\partial s}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial u} \times \frac{\partial u}{\partial s} + \frac{\partial y}{\partial v} \times \frac{\partial v}{\partial s}\right) +$
$\frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial u} \times \frac{\partial u}{\partial s} + \frac{\partial z}{\partial v} \times \frac{\partial v}{\partial s}\right)$

$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial u} \times \frac{\partial u}{\partial t} + \frac{\partial x}{\partial v} \times \frac{\partial v}{\partial t}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial u} \times \frac{\partial u}{\partial t} + \frac{\partial y}{\partial v} \times \frac{\partial v}{\partial t}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial u} \times \frac{\partial u}{\partial t} + \frac{\partial z}{\partial v} \times \frac{\partial v}{\partial t}\right)$

(28) $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{1}{1+2(y+z)} = -\frac{1}{1+2y+2z}$

$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{2y+2z}{1+2y+2z}$ $\quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{2y+2z}{1+2y+2z}\right) =$

$= -\frac{2\frac{\partial z}{\partial x} \times (1+2y+2z) - (2y+2z) \times 2\frac{\partial z}{\partial x}}{(1+2y+2z)^2} = -\frac{-\frac{2}{1+2y+2z}(1+2y+2z) - (2y+2z) \times \left(-\frac{2}{1+2y+2z}\right)}{(1+2y+2z)^2}$

$= \frac{2 - \frac{2(2y+2z)}{1+2y+2z}}{(1+2y+2z)^2} = \frac{\frac{2+4y+4z-4y-4z}{1+2y+2z}}{(1+2y+2z)^2} = \frac{2}{(1+2y+2z)^3}$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{1}{1+2y+2z}\right) = -\frac{-(2+2\frac{\partial z}{\partial y})}{(1+2y+2z)^2} = \frac{2 - \frac{4y+4z}{1+2y+2z}}{(1+2y+2z)^2} =$

$= \frac{\frac{2+4y+4z-4y-4z}{1+2y+2z}}{(1+2y+2z)^2} = \frac{2}{(1+2y+2z)^3}$

(29) $\underbrace{e^{\cos(z)}\ln(z+1) - \arctan(2x+y)}_{F(x,y,z)} = 0$

$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{-\frac{2}{1+(2x+y)^2}}{-\sin(z)e^{\cos(z)}\ln(z+1) + e^{\cos(z)} \times \frac{1}{z+1}}$

$\left.\frac{\partial z}{\partial x}\right|_P = -\frac{-\frac{2}{1+0}}{-\sin(0)e^{\cos(0)}\ln(1) + e^{\cos(0)} \times \frac{1}{1}} = \frac{2}{e}$

$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{-\frac{1}{1+(2x+y)^2}}{-\sin(z)e^{\cos(z)}\ln(z+1) + e^{\cos(z)} \times \frac{1}{z+1}}$

$\left.\frac{\partial z}{\partial y}\right|_P = -\frac{-\frac{1}{1+0}}{-\sin(0)e^{\cos(0)}\ln(1) + e^{\cos(0)} \times \frac{1}{1}} = \frac{1}{e}$

30
$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{\ln(y)}{y^2+2z}\qquad \left.\frac{\partial z}{\partial x}\right|_P=-\frac{\ln(1)}{1^2+4}=0$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}=-\frac{\frac{x}{y}+2yz}{y^2+2z}\qquad \left.\frac{\partial z}{\partial y}\right|_P=-\frac{\frac{1}{1}+2\times1\times2}{1^2+4}=-\frac{5}{5}=-1$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(-\frac{\frac{x}{y}+2yz}{y^2+2z}\right)=-\frac{\left(\frac{1}{y}+2y\frac{\partial z}{\partial x}\right)(y^2+2z)-\left(\frac{x}{y}+2yz\right)\times 2\frac{\partial z}{\partial x}}{(y^2+2z)^2}$$

$$\left.\frac{\partial^2 z}{\partial x\partial y}\right|_P=-\frac{(1+0)(1+4)-(1+4)\times2\times0}{(1+4)^2}=-\frac{5}{25}=-\frac{1}{5}$$

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(-\frac{\ln(y)}{y^2+2z}\right)=-\frac{\frac{y^2+2z}{y}-\ln(y)\left(2y+2\frac{\partial z}{\partial y}\right)}{(y^2+2z)^2}$$

$$\left.\frac{\partial^2 z}{\partial y\partial x}\right|_P=-\frac{5-\ln(1)\,(2\times1+2\times(-1))}{(1+4)^2}=-\frac{5}{25}=-\frac{1}{5}$$

31
$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\times\frac{\partial r}{\partial x}+\frac{\partial z}{\partial s}\times\frac{\partial s}{\partial x}+\frac{\partial z}{\partial v}\times\frac{\partial v}{\partial x}=\frac{1}{v}\times\cos(y)+\frac{1}{v}\times y\cos(x)-\frac{(r+s)}{v^2}\times2=$$
$$=\frac{\cos(y)+y\cos(x)}{2x-y}-2\,\frac{x\cos(y)+y\sin(x)}{(2x-y)^2}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial r}\times\frac{\partial r}{\partial y}+\frac{\partial z}{\partial s}\times\frac{\partial s}{\partial y}+\frac{\partial z}{\partial v}\times\frac{\partial v}{\partial y}=\frac{1}{v}\times(-x\sin(y))+\frac{1}{v}\times\sin(x)-\frac{r+s}{v^2}\times(-1)=$$
$$=\frac{\sin(x)-x\sin(y)}{2x-y}+\frac{x\cos(y)+y\sin(x)}{(2x-y)^2}$$

$P=\left(2,0,\arccos\left(\frac{1}{\sqrt{2}}\right)\right)$

32
$$\sqrt{2}\times\cos(-2\times0+z)=1\Leftrightarrow\cos(z)=\frac{1}{\sqrt{2}}\Leftrightarrow z=\arccos\left(\frac{1}{\sqrt{2}}\right)$$

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{\frac{\cos(-2y+z)}{2}x^{-\frac{1}{2}}}{-\sqrt{x}\sin(-2y+z)}\qquad\left.\frac{\partial z}{\partial x}\right|_P=-\frac{\cos\left(\arccos\left(\frac{1}{\sqrt{2}}\right)\right)\times2^{-\frac{1}{2}}}{-\sqrt{2}\sin\left(\arccos\left(\frac{1}{\sqrt{2}}\right)\right)}$$

$$=\frac{\frac{1}{4}}{\sqrt{2}\sin\left(\arccos\left(\frac{1}{\sqrt{2}}\right)\right)}=\frac{1}{4}$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}=-\frac{2\sqrt{x}\sin(-2y+z)}{-\sqrt{2}\sin\left(\arccos\left(\frac{1}{\sqrt{2}}\right)\right)}\qquad\left.\frac{\partial z}{\partial y}\right|_P=\frac{2\sqrt{2}\sin\left(\arccos\left(\frac{1}{\sqrt{2}}\right)\right)}{-\sqrt{2}\sin\left(\arccos\left(\frac{1}{\sqrt{2}}\right)\right)}=2$$

$P_1\left(3,1,-\frac{5}{2}\right)\quad P_2(3,1,1)$

33
$$3z^2-3^2+3z-5=0\Leftrightarrow2z^2+3z-5=0\Leftrightarrow z=\frac{-3\pm\sqrt{9+40}}{4}\Leftrightarrow z=\frac{-3\pm7}{4}\Leftrightarrow z=-\frac{5}{2}\vee z=1$$

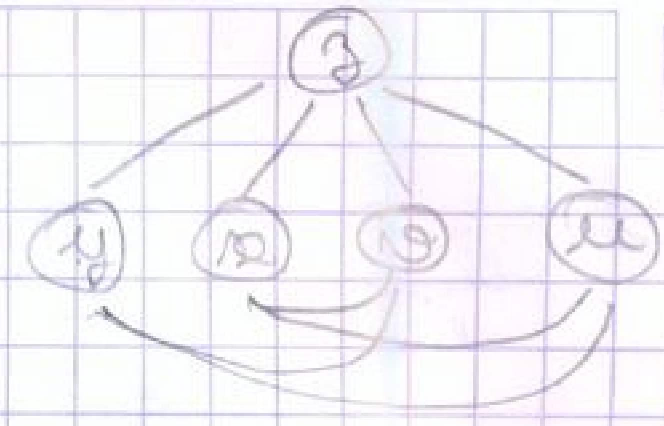
$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{z^2+y^2z}{2xz-2yz+xy^2}\qquad\left.\frac{\partial z}{\partial x}\right|_{P_1}=-\frac{\frac{25}{4}-\frac{10}{4}}{-15+5+3}=-\frac{\frac{15}{4}}{-7}=\frac{15}{28}$$

$$\left.\frac{\partial z}{\partial x}\right|_{P_2}=-\frac{2}{6-2+3}=-\frac{2}{7}$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}=-\frac{-z^2+2xyz}{2xz-2yz+xy^2}$$

$$\left.\frac{\partial z}{\partial y}\right|_{P_1}=-\frac{-\frac{25}{4}-15}{-7}=-\frac{85}{28}\qquad\left.\frac{\partial z}{\partial y}\right|_{P_2}=-\frac{-1+6}{7}=-\frac{5}{7}$$

(34) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u} + \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial u} =$

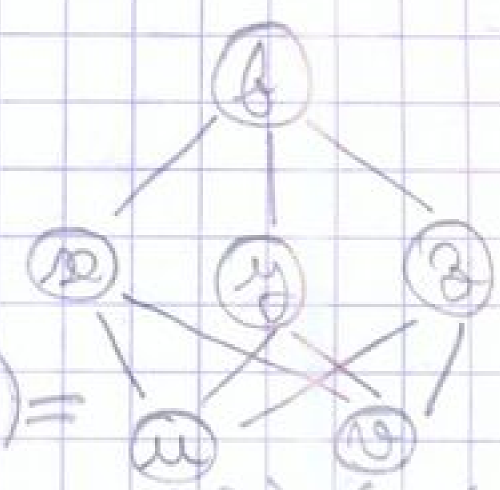


$= (2y + 2v) \times (-\sin(u)) + 1 \times 2 + \frac{1}{u} =$

$= \frac{1}{u} + 2 - 2\sin(u)\left(\cos(u) + \sin(v) + v\right)$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v} + \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial v} = (2y + 2v) \times \cos(v) + 1 \times 3 + (2y + 2v) =$

$= 3 + 2\left(\cos(u) + \sin(v) + v\right)\left(1 + \cos(v)\right)$



(35) $\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \times \frac{\partial z}{\partial u} =$

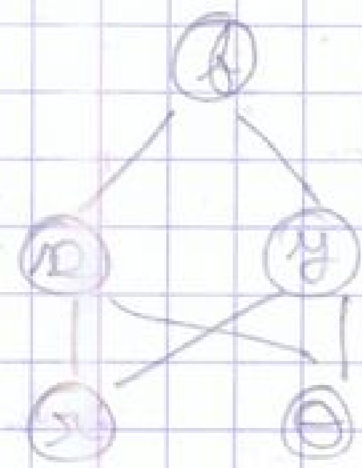
$= \frac{y}{z} \times \sec^2(u-1) + \frac{x}{z} \times 2u - \frac{xy}{z^2} \times \left(-2uv\sin(u^2v)\right) =$

$= \frac{(u^2 - v^2)\sec^2(u-1)}{\cos(u^2v)} + \frac{2u\left(\tan(u-1) - e^v\right)}{\cos(u^2v)} + \frac{2uv\sin(u^2v) \times \left(\tan(u-1) - e^v\right)(u^2 - v^2)}{\cos^2(u^2v)} =$

$= \frac{u^2 - v^2}{\cos(u^2v)\cos^2(u-1)} + \frac{2u\left(\tan(u-1) - e^v\right)}{\cos(u^2v)} + \frac{2uv\sin(u^2v)\left(\tan(u-1) - e^v\right)(u^2 - v^2)}{\cos^2(u^2v)}$

$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \times \frac{\partial z}{\partial v} = \frac{y}{z} \times (-e^v) + \frac{x}{z} \times (-2v) - \frac{xy}{z^2}\left(-u^2\sin(u^2v)\right) =$

$= \frac{-e^v(u^2 - v^2)}{\cos(u^2v)} - \frac{2v\left(\tan(u-1) - e^v\right)}{\cos(u^2v)} + \frac{u^2\sin(u^2v)\left(\tan(u-1) - e^v\right)(u^2 - v^2)}{\cos^2(u^2v)}$



(36) a) $\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta)$

$\frac{\partial u}{\partial \theta} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \theta} = -r\sin(\theta)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} r\cos(\theta)$

b) $\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(-r\sin(\theta)\frac{\partial f}{\partial x} + r\cos(\theta)\frac{\partial f}{\partial y}\right) =$

$=$