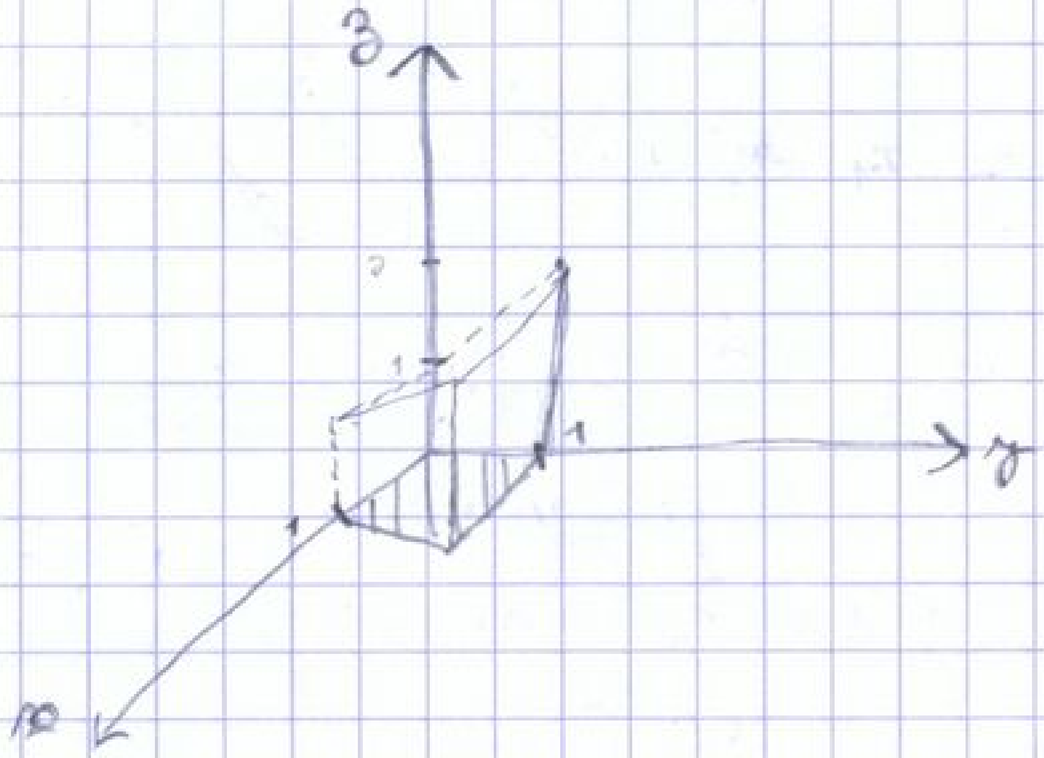
Ficha 4 - Integração tripla

① $\int_0^3\int_0^1\int_0^2 1\,dx\,dy\,dz = \int_0^3\int_0^1 2\,dy\,dz =$

$= \int_0^3 2\,dz = 3\times 2 = 6$ u.v.

② a)



b) $\int_0^1\int_0^1\int_0^{1+y} 1\,dz\,dx\,dy =$

$= \int_0^1\int_0^1 1+y\,dx\,dy =$

$= \int_0^1 1+y\,dy = \left[y+\frac{y^2}{2}\right]_0^1 = 1+\frac{1}{2} = \frac{3}{2}$

③ a) $\int_0^1\int_1^{2y}\int_0^x (x+2z)\,dz\,dx\,dy = \int_0^1\int_1^{2y}\left[xz+z^2\right]_0^x dx\,dy = \int_0^1\int_1^{2y} x^2+x^2\,dx\,dy =$

$= \int_0^1 \left[\frac{2x^3}{3}\right]_1^{2y} dy = \int_0^1 \frac{16y^3}{3}-\frac{2}{3}\,dy = \left[\frac{16y^4}{12}-\frac{2}{3}y\right]_0^1 = \frac{4}{3}-\frac{2}{3} = \frac{2}{3}$

b) $\int_0^1\int_0^x\int_0^y y\,dz\,dy\,dx = \int_0^1\int_0^x y^2\,dy\,dx = \int_0^1\left[\frac{y^3}{3}\right]_0^x dx = \int_0^1 \frac{x^3}{3}\,dx =$

$= \left[\frac{x^4}{12}\right]_0^1 = \frac{1}{12}$

c) $\int_1^2\int_y^{y^2}\int_0^{\ln(x)} y e^z\,dz\,dx\,dy = \int_1^2\int_y^{y^2}\left[y e^z\right]_0^{\ln(x)} dx\,dy = \int_1^2\int_y^{y^2} yx - y\,dx\,dy$

$= \int_1^2 \left[\frac{yx^2}{2}-yx\right]_y^{y^2} dy = \int_1^2 \frac{y^5}{2}-y^3-\frac{y^3}{2}+y^2\,dy = \left[\frac{y^6}{12}-\frac{y^4}{4}-\frac{y^4}{8}+\frac{y^3}{3}\right]_1^2 =$

$= \frac{64}{12}-4-2+\frac{8}{3}-\frac{1}{12}+\frac{1}{4}+\frac{1}{8}-\frac{1}{3} = \frac{47}{24}$

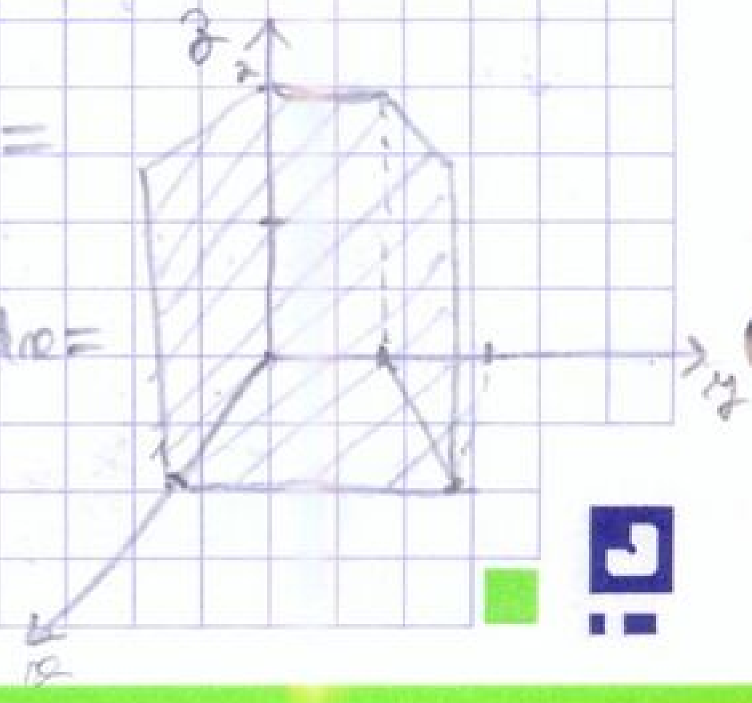
④ $\int_0^1\int_0^1\int_0^1 y e^{-xy}\,dx\,dy\,dz = \int_0^1\int_0^1\left[-e^{-xy}\right]_0^1 dy\,dz = \int_0^1\int_0^1 -e^{-y}+1\,dy\,dz =$

$= \int_0^1\left[e^{-y}+y\right]_0^1 dz = \int_0^1 \frac{1}{e}+1-1\,dz = \left[\frac{1}{e}z\right]_0^1 = \frac{1}{e}$

⑤ $\int_0^1\int_0^2\int_0^{1+x}(x^2+z)\,dy\,dz\,dx = \int_0^1\int_0^2 x^2+z+x^3+zx\,dz\,dx =$

$= \int_0^1\left[x^2z+\frac{z^2}{2}+x^3z+\frac{xz^2}{2}\right]_0^2 = \int_0^1 2x^2+2x^3+2x+2\,dx =$

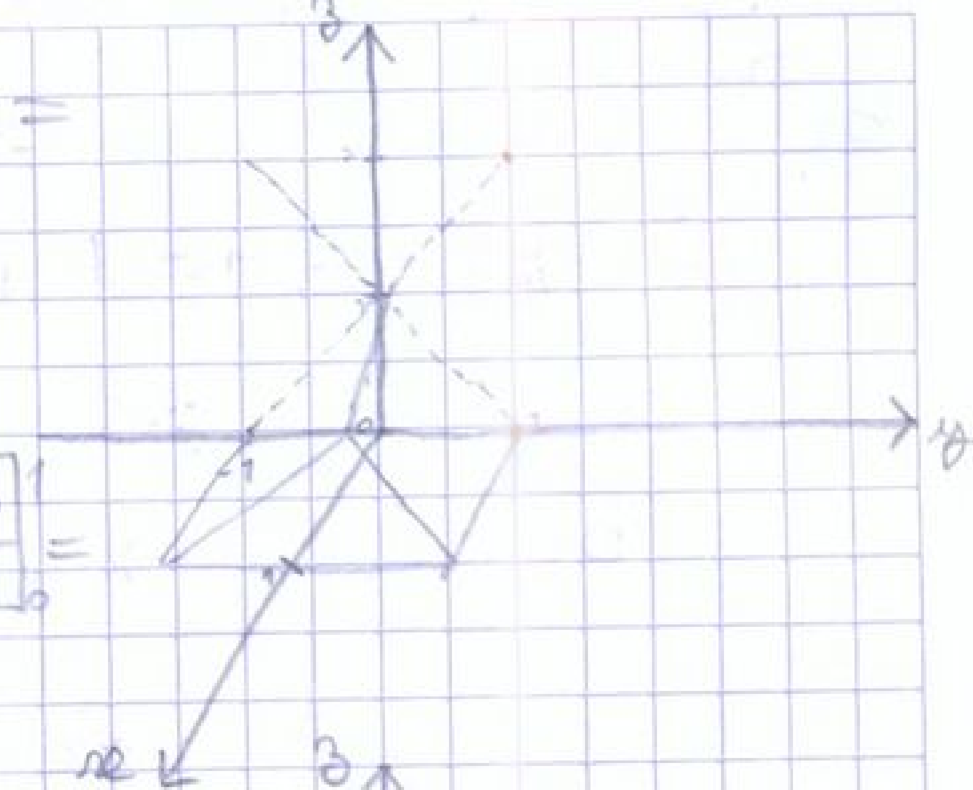
$= \left[\frac{2x^3}{3}+\frac{x^4}{2}+x^2+2x\right]_0^1 = \frac{2}{3}+\frac{1}{2}+1+2 = \frac{25}{6}$



(6) $\int_0^1\int_{-1}^0\int_0^{y+1} z\,dz\,dy\,dx + \int_0^1\int_0^1\int_0^{1-y} z\,dz\,dy\,dx =$

$= \int_0^1\int_{-1}^0 \frac{(y+1)^2}{2}\,dy\,dx + \int_0^1\int_0^1 \frac{(1-y)^2}{2}\,dy\,dx =$

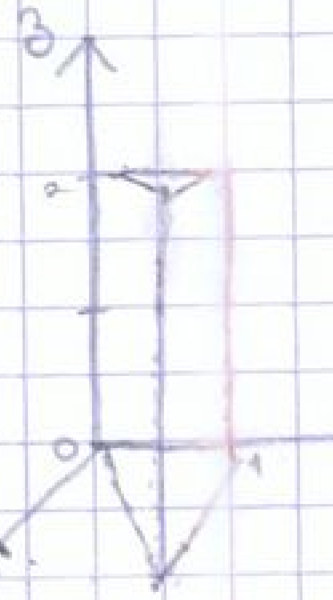
$= \int_0^1 \frac{1}{2}\left[\frac{y^3}{3}+y^2+y\right]_{-1}^0 dx + \int_0^1 \frac{1}{2}\left[y-y^2+\frac{y^3}{3}\right]_0^1 =$



$= \frac{1}{2}\times\int_0^1 \frac{1}{3}-1+1\,dx + \frac{1}{2}\int_0^1 -1+1-\frac{1}{3}\,dx =$

$= \frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\times\left(-\frac{1}{3}\right) = \frac{1}{6}-\frac{1}{6} = 0$

$\begin{cases} u = 2x \\ du = 2dx \end{cases} \begin{cases} v' = e^x \\ v = e^x \end{cases}$ $\begin{cases} u = 4y \\ du = 4dy \end{cases} \begin{cases} v' = e^y \\ v = e^y \end{cases}$



(7) $\int_0^1\int_0^y\int_0^{x+y} 2e^x\,dz\,dx\,dy = \int_0^1\int_0^y 2xe^x + 2ye^x\,dx\,dy =$

$= \int_0^1 \left[2xe^x - 2\int_0^y e^x\,dx + 2ye^x\right]_0^y = \int_0^1 2ye^y - 2e^y + 2 + 2ye^y - 2y\,dy$

$= \int_0^1 4ye^y - 2e^y - 2y + 2\,dy = \left[4ye^y\right]_0^1 - \int_0^1 4e^y\,dy - 2e^y - y^2 + 2y\Big]_0^1$

$= 4e - 4e + 4 - 2e + 2 - 1 + 2 = 7 - 2e$

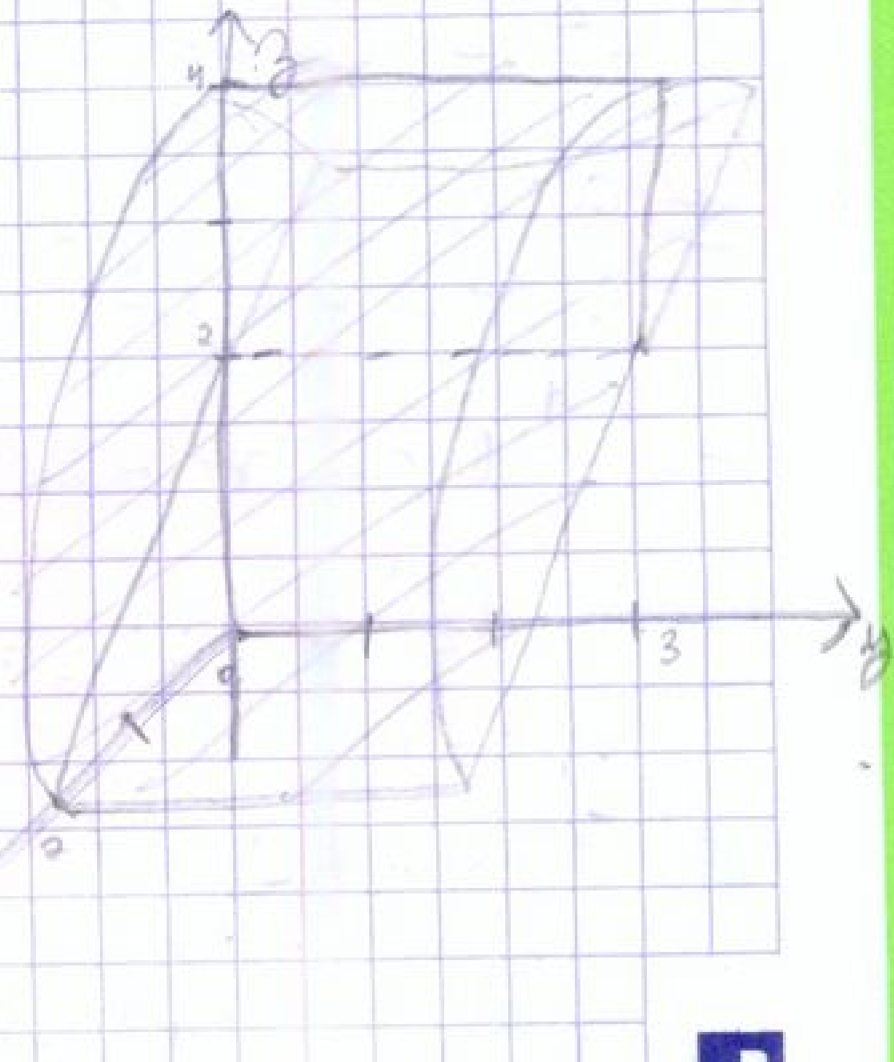
(8) $\int_0^1\int_0^{1-x}\int_0^{x^2+y^2} 1\,dz\,dy\,dx = \int_0^1\int_0^{1-x} x^2+y^2\,dy\,dx =$

$= \int_0^1 \left[yx^2 + \frac{y^3}{3}\right]_0^{1-x} dx = \int_0^1 x^2 - x^3 + \frac{(1-x)^3}{3}\,dx =$

$= \int_0^1 x^2 - x^3 + \frac{(1-x)(1-2x+x^2)}{3}\,dx = \int_0^1 x^2 - x^3 + \frac{1-2x+x^2-x+2x^2-x^3}{3}\,dx = \left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 + \frac{1}{3}\int_0^1 -x^3+3x^2-3x+1\,dx =$

$= \frac{1}{3}-\frac{1}{4}+\frac{1}{3}\left[-\frac{x^4}{4}+x^3-\frac{3x^2}{2}+x\right]_0^1 = \frac{1}{3}-\frac{1}{4}+\frac{1}{3}\left(-\frac{1}{4}+1-\frac{3}{2}+1\right) = \frac{1}{6}$

(9) $2-x = 4-x^2 \Leftrightarrow x^2 - x - 2 = 0 \Leftrightarrow x = -1 \vee x = 2$



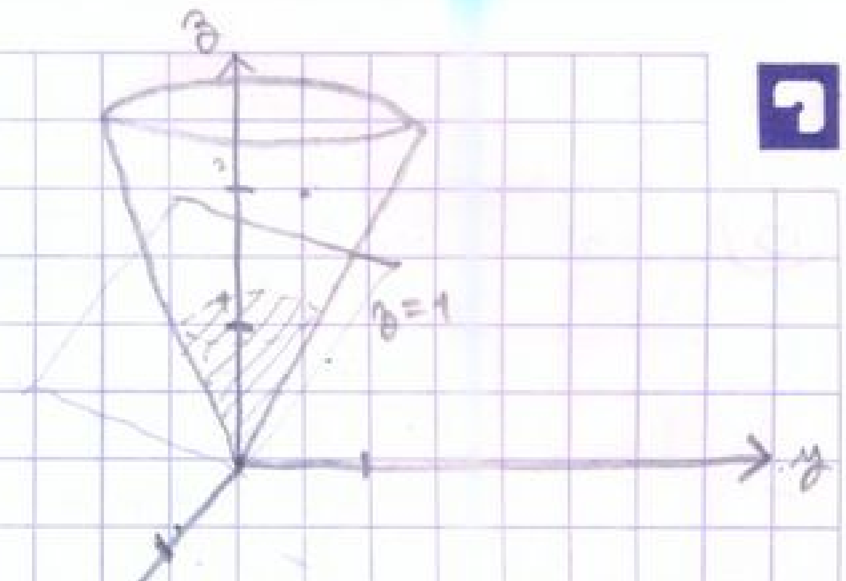
$\int_{-1}^2\int_0^3\int_{2-x}^{4-x^2} dz\,dy\,dx = \int_{-1}^2\int_0^3 4-x^2-2+x\,dy\,dx =$

$= \int_{-1}^2 \left[-x^2y + xy + 2y\right]_0^3 dx = \int_{-1}^2 -3x^2+3x+6\,dx =$

$= \left[-x^3 + \frac{3x^2}{2} + 6x\right]_{-1}^2 = -8+6+12-\left(1+\frac{3}{2}-6\right) =$

$= 10+5-\frac{3}{2} = \frac{30}{2}-\frac{3}{2} = \frac{27}{2}$

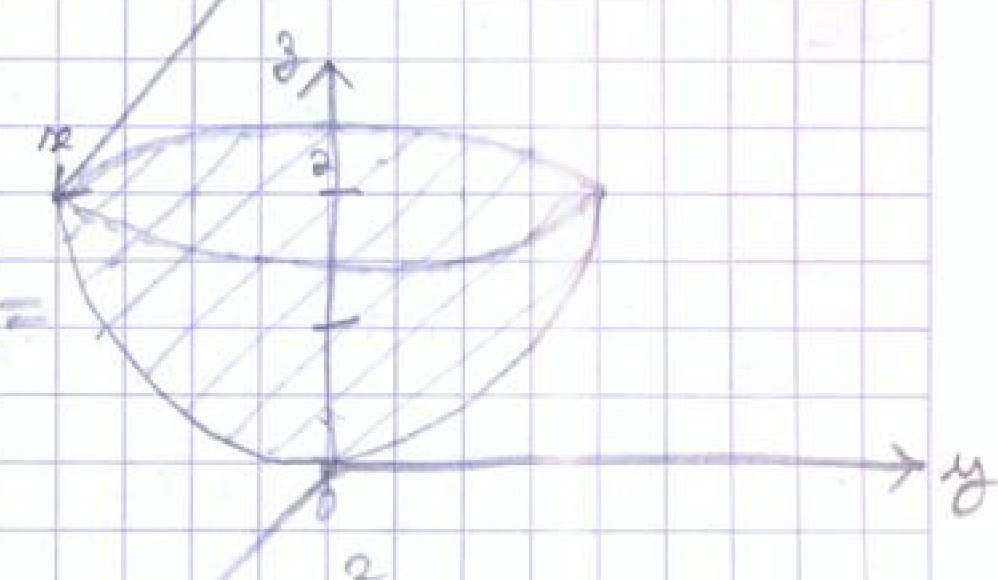
10 a) $\int_0^{2\pi}\int_0^1\int_r^1 r^2\,dz\,dr\,d\theta =$



$=\int_0^{2\pi}\int_0^1 r^2 - r^3\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^3}{3}-\frac{r^4}{4}\right]_0^1 d\theta =$

$=\int_0^{2\pi}\frac{1}{3}-\frac{1}{4}\,d\theta = \left[\frac{\theta}{12}\right]_0^{2\pi} = \frac{2\pi}{12}=\frac{\pi}{6}$

b) $\int_0^{2\pi}\int_0^2\int_{\frac{r^2}{2}}^2 r^2\times r\,dz\,dr\,d\theta =$

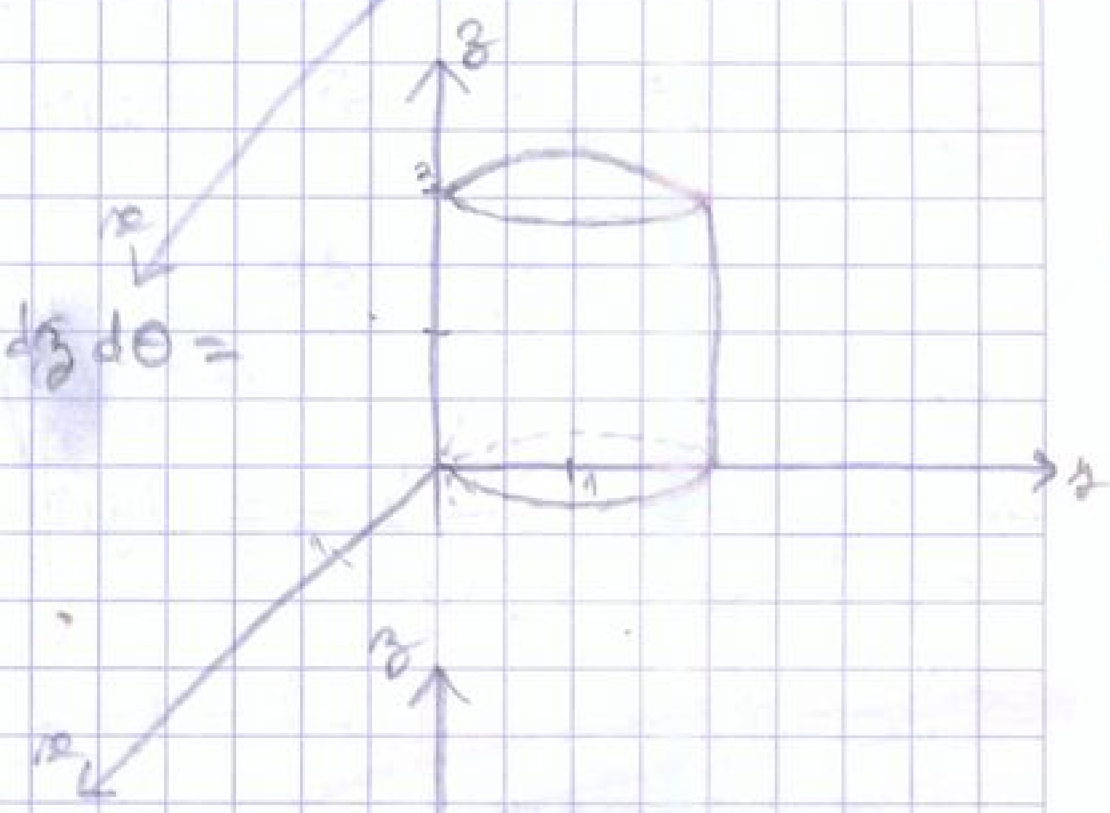


$=\int_0^{2\pi}\int_0^2 2r^3-\frac{r^5}{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{2}-\frac{r^6}{12}\right]_0^2 d\theta=$

$=\int_0^{2\pi} 8-\frac{64}{12}\,d\theta = \left[\frac{8}{3}\theta\right]_0^{2\pi} = \frac{16\pi}{3}$

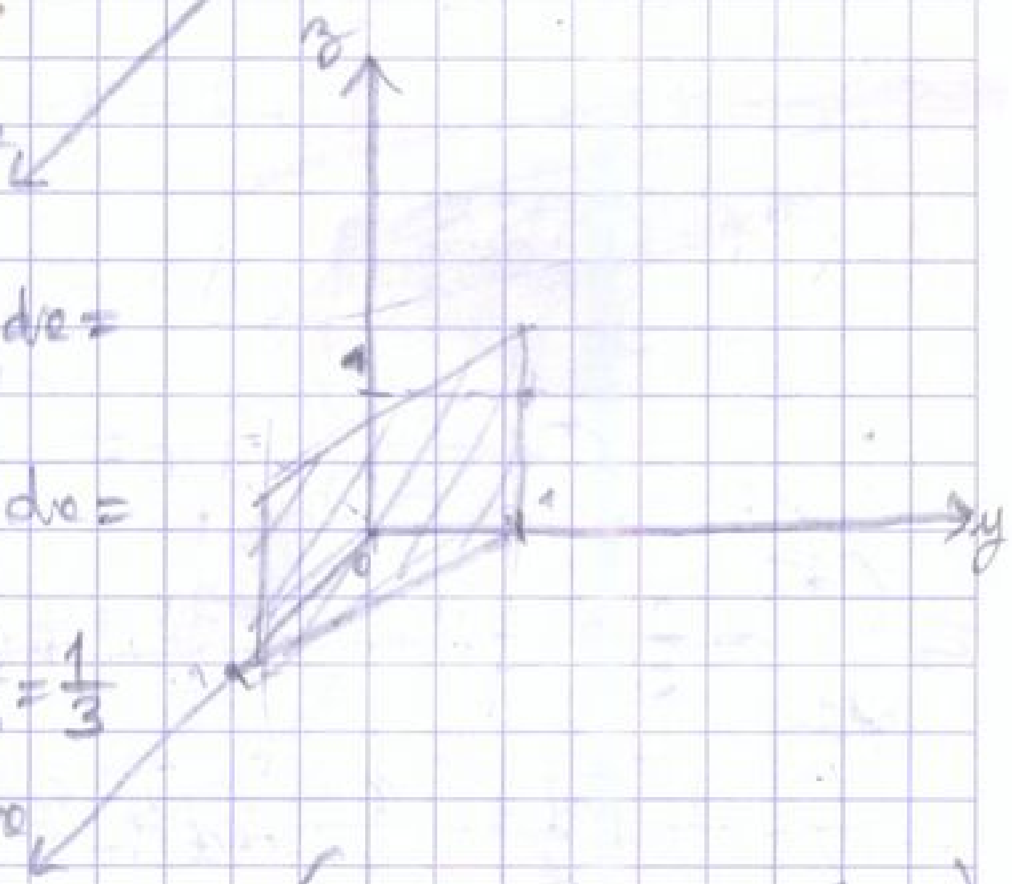
$r^2 = 2r\cos\theta \Leftrightarrow r = 2\cos\theta$

c) $x^2+y^2 = 2x \Leftrightarrow (x-1)^2+y^2=1$



$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2\int_0^{2\cos\theta} r^3\,dr\,dz\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 4\cos^4\theta\,dz\,d\theta =$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^4\theta\,d\theta = 3\pi$



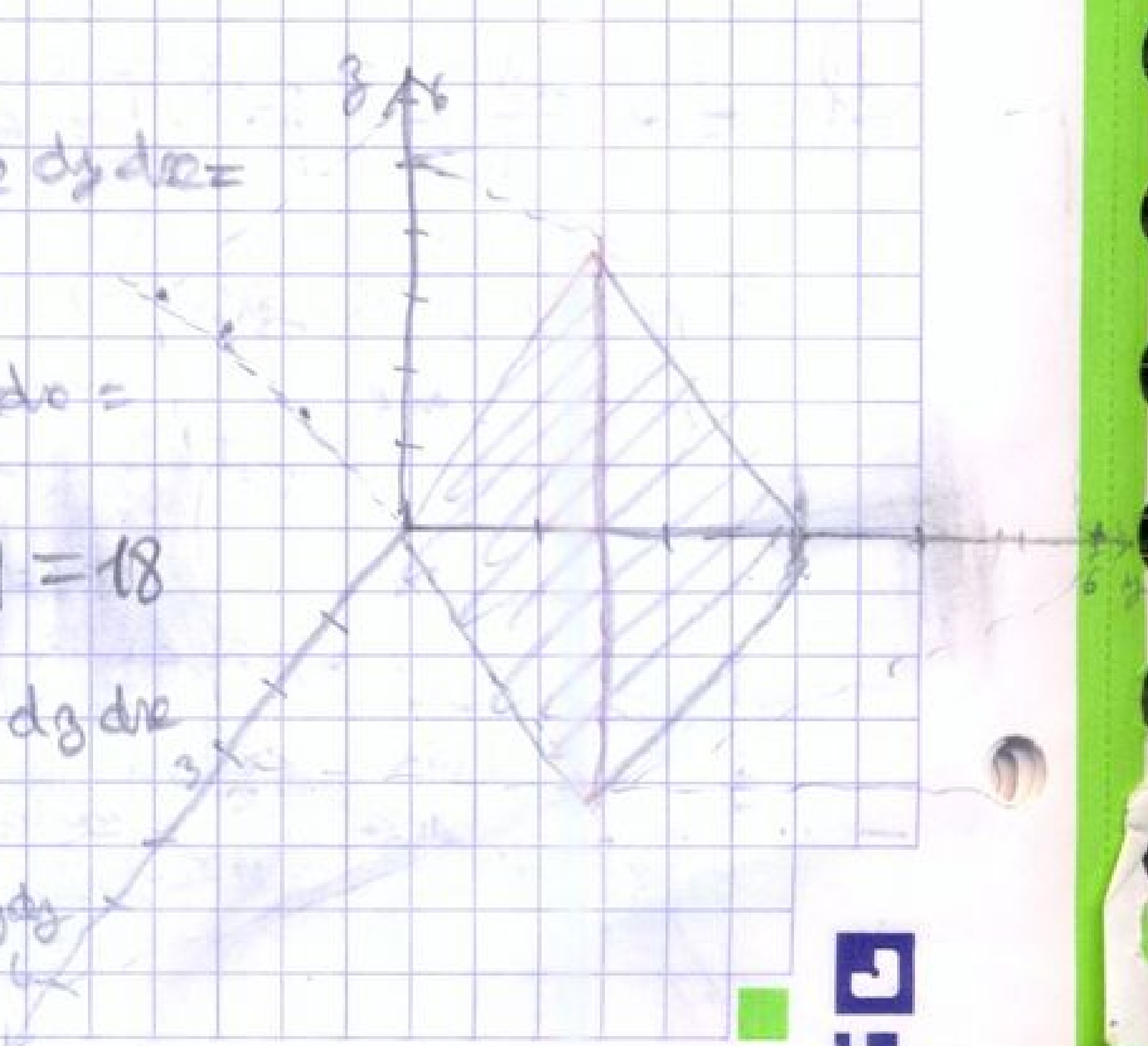
11 a) $\int_0^1\int_0^{1-x}\int_0^{x+y} dz\,dy\,dx = \int_0^1\int_0^{1-x} x+y\,dy\,dx=$

$=\int_0^1 \left[xy+\frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 x-x^2+\frac{1-2x+x^2}{2}\,dx=$

$=\frac{1}{2}\int_0^1 -x^2+1\,dx = \frac{1}{2}\left[-\frac{x^3}{3}+x\right]_0^1 = \frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$

b) $\int_0^1\int_0^x\int_0^{1-x} dy\,dz\,dx + \int_0^1\int_x^1\int_{z-x}^{1-x} dy\,dz\,dx$ (não sei porquê ...)

12 a) $\int_0^3\int_x^{6-x}\int_0^{2x} dz\,dy\,dx = \int_0^3\int_x^{6-x} 2x\,dy\,dx=$



$=\int_0^3 12x-2x^2-2x^2\,dx = \int_0^3 -4x^2+12x\,dx =$

$=\left[-\frac{4x^3}{3}+6x^2\right]_0^3 = -\frac{27\times 4}{3}+6\times 9 = -36+54 = 18$

b) $\int_0^6\int_{\frac{z}{2}}^3\int_x^{6-x} dy\,dx\,dz$; $\int_0^3\int_0^{2x}\int_x^{6-x} dy\,dz\,dx$

xyz

$\int_0^6\int_{\frac{z}{2}}^3\int_{\frac{z}{2}}^{y} dx\,dy\,dz + \int_0^6\int_3^{6-\frac{z}{2}}\int_{\frac{z}{2}}^{6-y} dx\,dy\,dz$

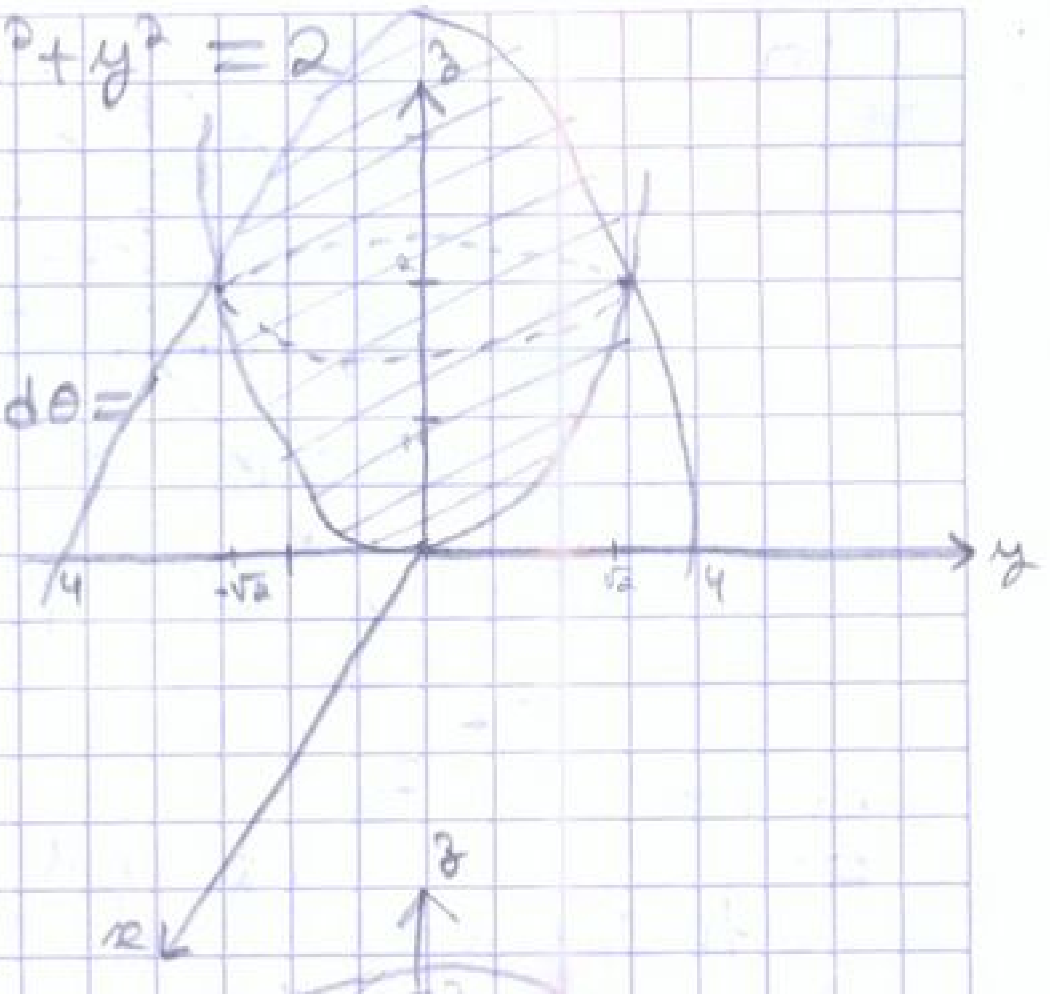
(13) $4-x^2-y^2 = x^2+y^2 \Leftrightarrow 2x^2+2y^2=4 \Leftrightarrow x^2+y^2=2$

$z = 4-(x^2+y^2) = 4-r^2 \quad z = x^2+y^2 = r^2$

$\int_0^{2\pi}\int_0^{\sqrt2}\int_{r^2}^{4-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt2} 4r - r^3 - r^3\,dr\,d\theta =$

$= \int_0^{2\pi}\left[2r^2 - \frac{r^4}{2}\right]_0^{\sqrt2} d\theta = \int_0^{2\pi} 4-2\,d\theta =$

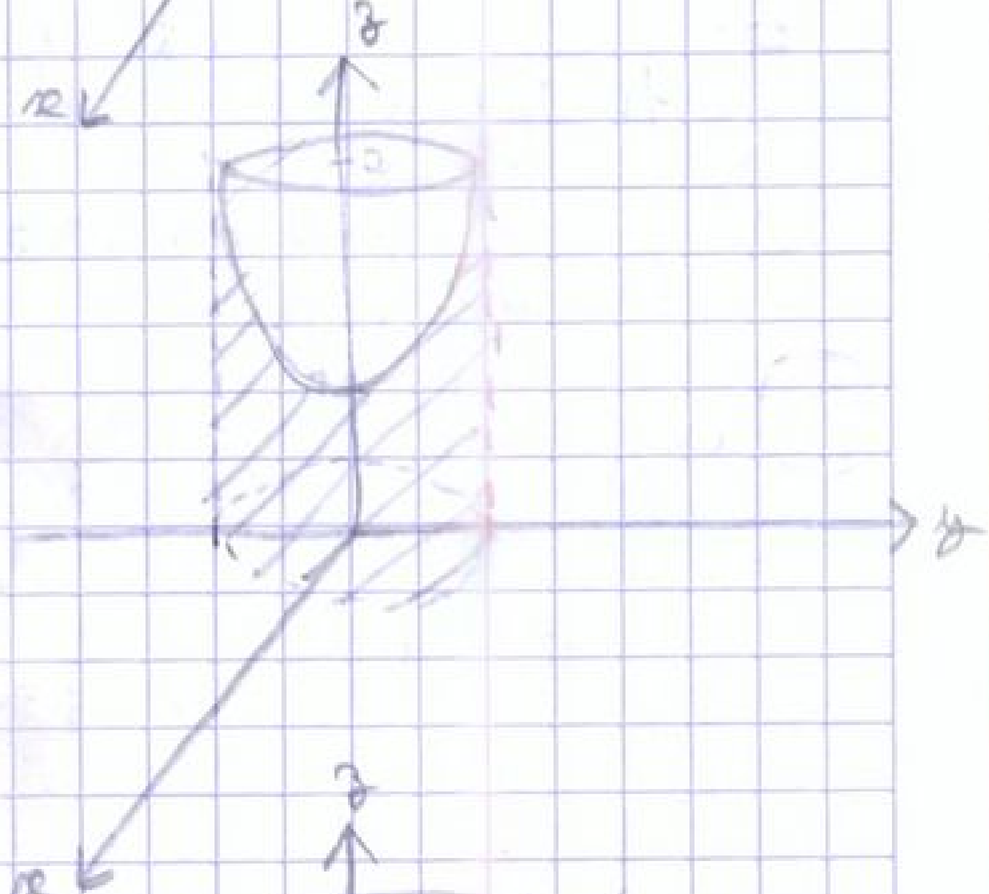
$= [2\theta]_0^{2\pi} = 4\pi$



(14) $z = 1+x^2+y^2 = 1+r^2 \quad x^2+y^2=1 \Rightarrow r=1$

$\int_0^{2\pi}\int_0^1\int_0^{1+r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r+r^3\,dr\,d\theta$

$=\int_0^{2\pi}\left[\frac{r^2}{2}+\frac{r^4}{4}\right]_0^1 = \int_0^{2\pi}\frac12+\frac14 = \frac{6\pi}{4}=\frac{3\pi}{2}$



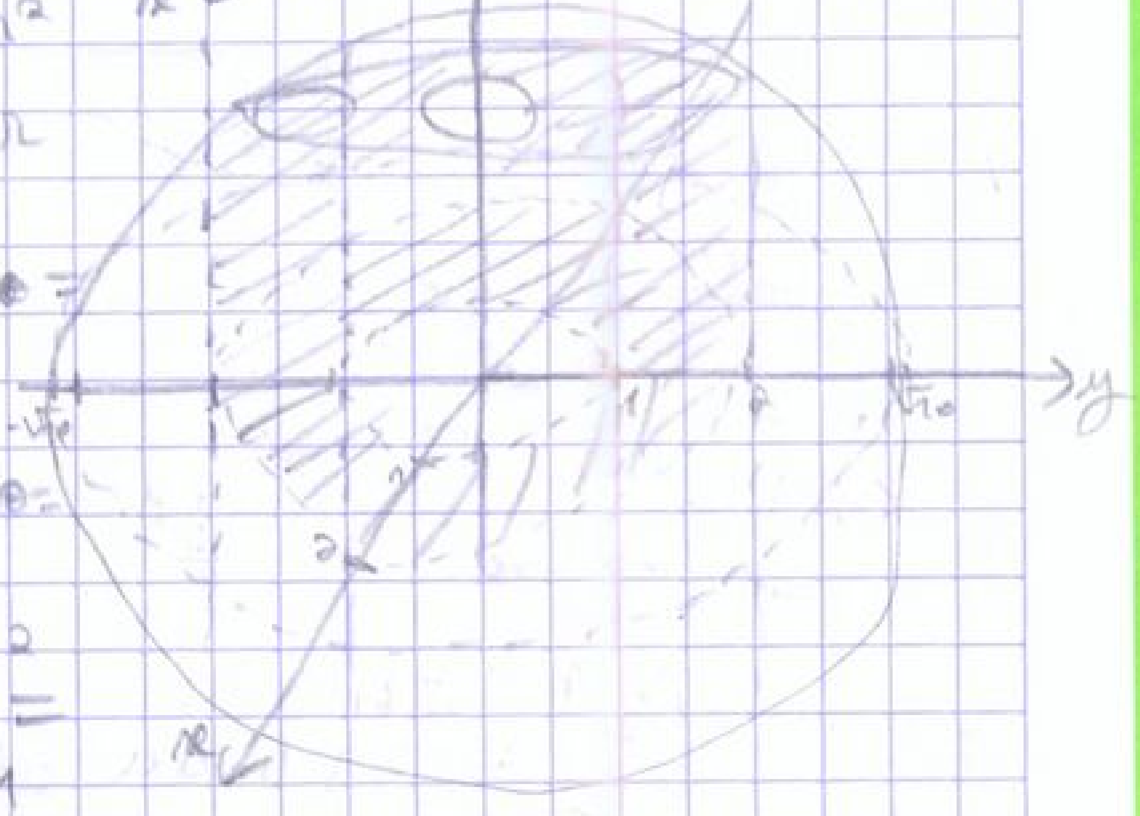
(15) $z^2 = 10-r^2 \Leftrightarrow z=\sqrt{10-r^2}$ $\quad \begin{cases}u=10-r^2\\ du=-2r\end{cases}$

$\int_0^{2\pi}\int_1^2\int_0^{\sqrt{10-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_1^2 r\sqrt{10-r^2}\,dr\,d\theta =$

$=\int_0^{2\pi} -\frac12\int_1^2 -2r\sqrt{10-r^2}\,dr\,d\theta = -\frac12\int_0^{2\pi}\int_1^2\sqrt{u}\,du\,d\theta =$

$=\int_0^{2\pi}-\frac12\times\left[\frac{2u^{\frac32}}{3}\right]_1^2 = -\frac12\int_0^{2\pi}\left[\frac{2(10-r^2)^{\frac32}}{3}\right]_1^2 =$

$=\int_0^{2\pi} -\frac12\times\left(\frac{12\sqrt6}{3} - \frac{18\sqrt3}{3}\right)d\theta = \int_0^{2\pi} -\frac{6\sqrt6}{3}+9 = \pi\left(18-4\sqrt6\right)$



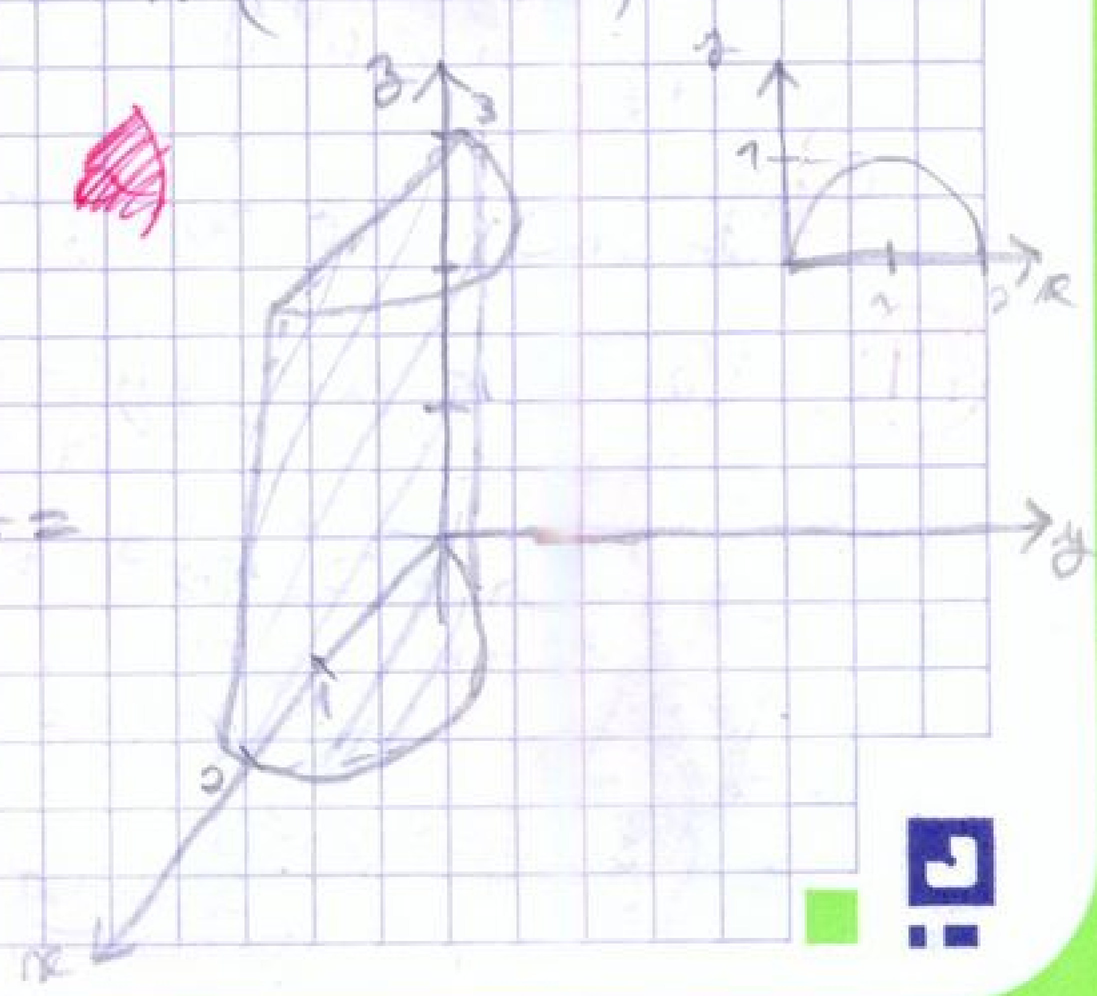
(16) $y^2+x^2 = 2x \Leftrightarrow (x-1)^2+y^2=1$ $\quad r = 2\cos\theta$

a) $\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}\int_0^3 r\, f(r,\theta,z)\,dz\,dr\,d\theta$

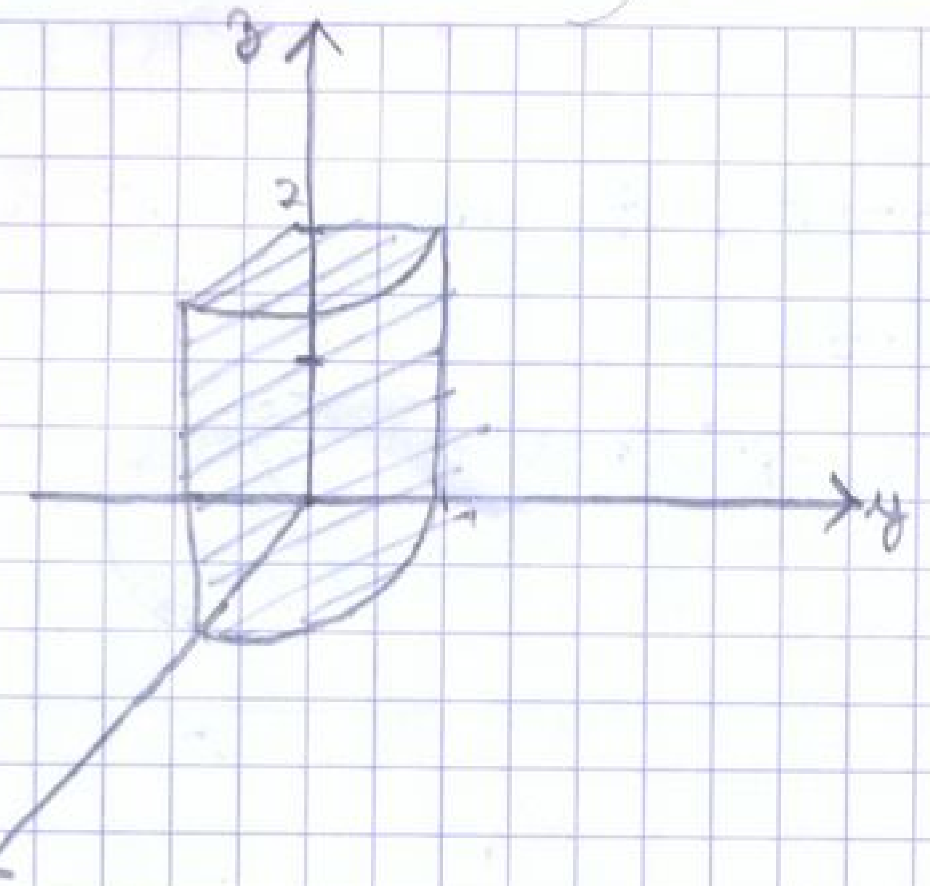
b) $\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}\int_0^3 r\,dz\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} 3r\,dr\,d\theta =$

$=\int_0^{\frac{\pi}{2}}\left[\frac{3r^2}{2}\right]_0^{2\cos\theta}d\theta = \int_0^{\frac{\pi}{2}} 6\cos^2\theta\,d\theta =$

$=\frac{3\pi}{2}$



17 a)



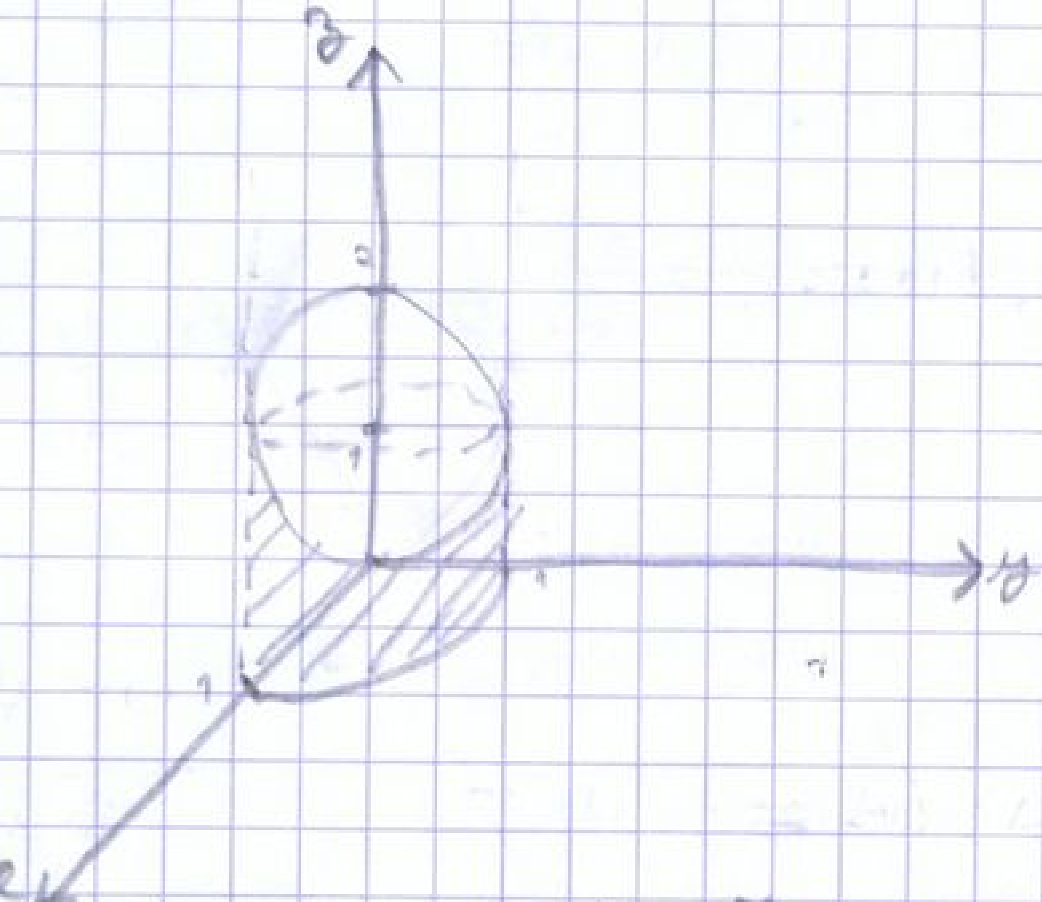
$y=\sqrt{1-x^2} \Leftrightarrow y^2+x^2=1 \Leftrightarrow$

$\Leftrightarrow x=\sqrt{1-y^2}$

b) $\int_0^2\int_0^1\int_0^{\sqrt{1-y^2}} dx\,dy\,dz$

c) $\int_0^{\frac{\pi}{2}}\int_0^1\int_0^2 r\,dz\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^1 2r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\left[r^2\right]_0^1 d\theta = \int_0^{\frac{\pi}{2}} 1\,d\theta = \frac{\pi}{2}$

18 a)



$y=\sqrt{1-x^2} \Leftrightarrow y^2+x^2=1$

$z=1+\sqrt{1-(x^2+y^2)}$

$(z-1)^2+x^2+y^2=1 \Rightarrow$

$\Rightarrow y=\sqrt{1-x^2-(z-1)^2}$

$\begin{cases} u=1-r^2 \\ du=-2r \end{cases}$

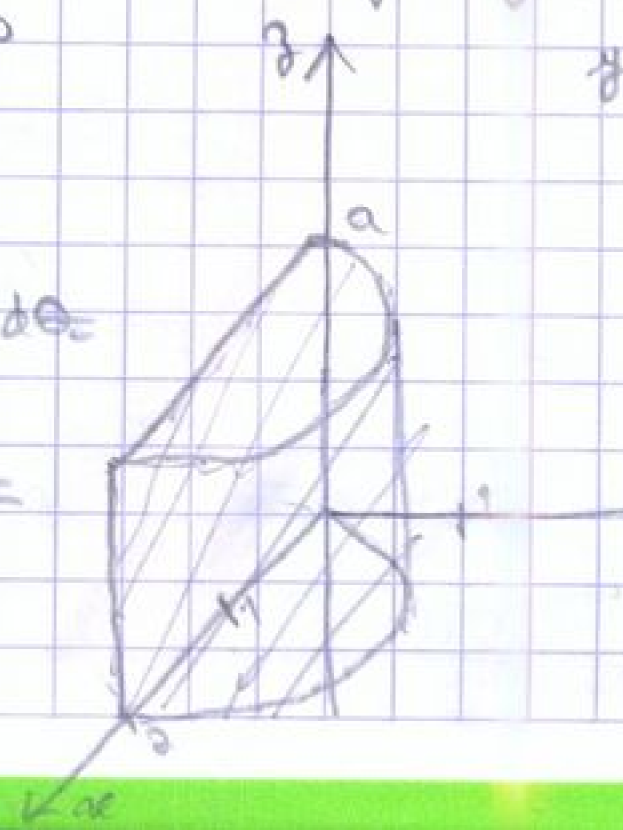
b) $\int_0^{\frac{\pi}{2}}\int_0^1\int_0^{1+\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^1 r + r\sqrt{1-r^2}\,dr\,d\theta =$

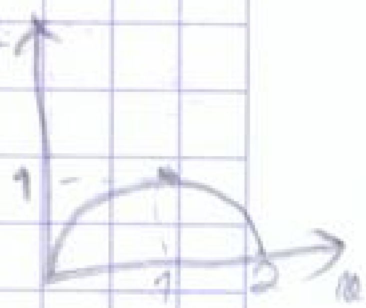
$= \int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\Big|_0^1 - \frac{1}{2}\int_0^1\sqrt{u}\,du\right]d\theta = \int_0^{\frac{\pi}{2}}\left[\frac{1}{2} - \frac{1}{2}\left[\frac{2(1-r^2)^{\frac{3}{2}}}{3}\right]_0^1\right]d\theta =$

$= \int_0^{\frac{\pi}{2}}\left[\frac{1}{2}-\frac{1}{2}\left(-\frac{2}{3}\right)\right]d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}+\frac{2}{6}\,d\theta = \left[\frac{5}{6}\theta\right]_0^{\frac{\pi}{2}} = \frac{5}{6}\times\frac{\pi}{2} = \frac{5\pi}{12}$

c) $\int_0^1\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dz\,dx + \int_0^1\int_1^{1+\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-(z-1)^2}} dy\,dz\,dx$

19 $y=\sqrt{2x-x^2} \Leftrightarrow (x-1)^2+y^2=1$





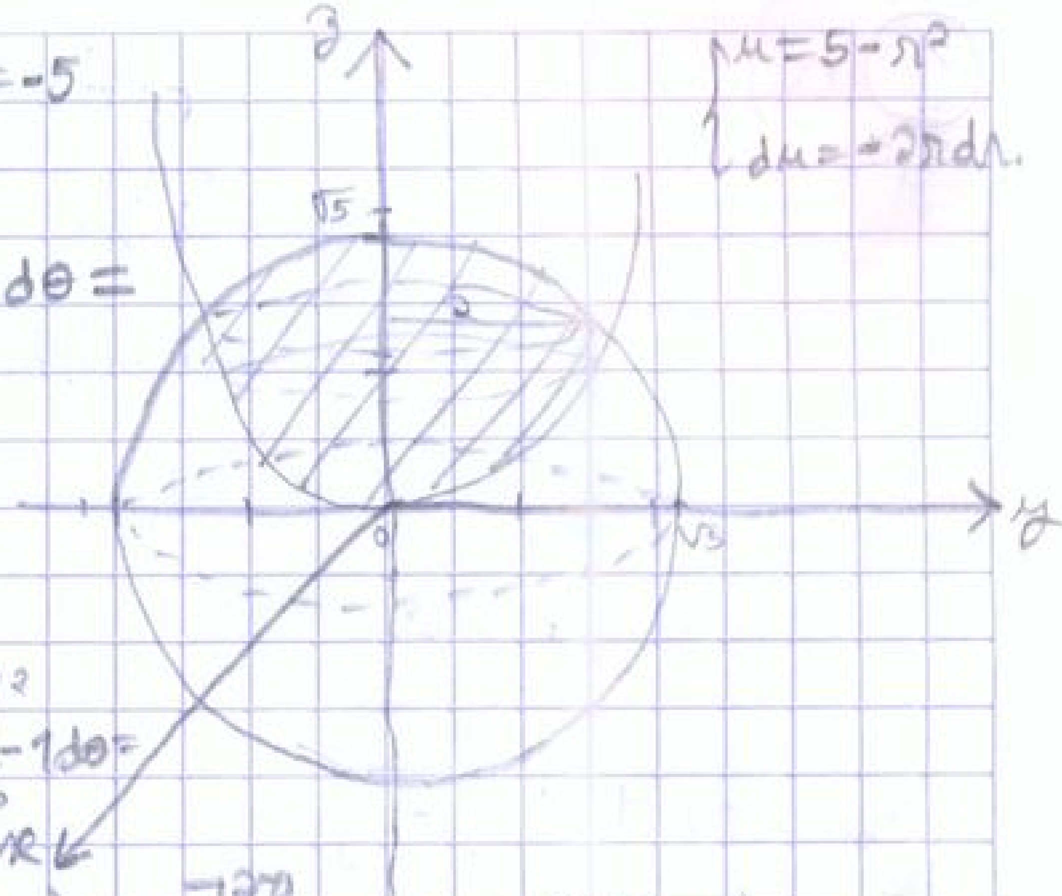
$\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}\int_0^a 3r^2\,dz\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}\frac{z^2r^2}{2}\Big|_0^a dr\,d\theta$

$= \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}\frac{a^2r^2}{2}\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\left[\frac{a^2r^3}{6}\right]_0^{2\cos\theta}d\theta =$

$= \int_0^{\frac{\pi}{2}}\frac{8a^2\cos^3\theta}{6}\,d\theta = \frac{8a^2}{9}$

(20) $z^2+4z-5=0 \Rightarrow z=\frac{-4\pm\sqrt{16+20}}{2} \Leftrightarrow z=1 \vee z=-5$

$\begin{cases} u=5-r^2 \\ du=-2r\,dr \end{cases}$



$$\int_0^{2\pi}\int_0^2\int_{\frac{r^2}{4}}^{\sqrt{5-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r\sqrt{5-r^2}-\frac{r^3}{4}\,dr\,d\theta =$$

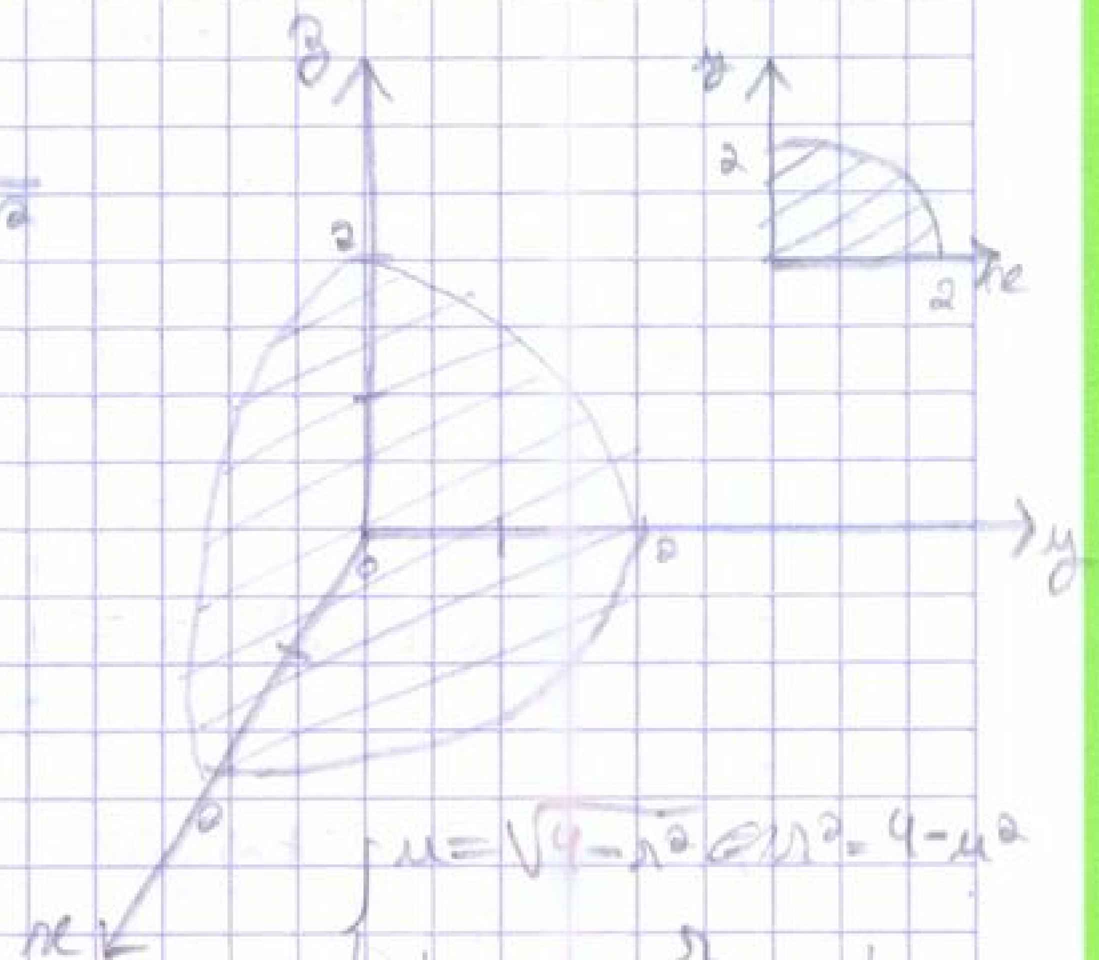
$$= \int_0^{2\pi}\left[-\frac{1}{2}\int_0^2 -2r\sqrt{5-r^2}\,dr - \left[\frac{r^4}{16}\right]_0^2\right]d\theta =$$

$$= \int_0^{2\pi}\left[-\frac{1}{2}\int_0^2 \sqrt{u}\,du - 1\right]d\theta = \int_0^{2\pi}\left[-\frac{1}{2}\times\frac{2(5-r^2)^{\frac{3}{2}}}{3}\right]_0^2 - 1\,d\theta =$$

$$= \int_0^{2\pi} -\frac{1}{3}+\frac{1}{3}(5\sqrt{5})-1\,d\theta = \left[-\frac{\theta}{3}+\frac{\theta}{3}(5\sqrt{5})-\theta\right]_0^{2\pi} = -\frac{2\pi}{3}+\frac{2\pi}{3}(5\sqrt{5})-2\pi =$$

$$= -\frac{8\pi}{3}+\frac{2\pi}{3}(5\sqrt{5}) = \frac{2\pi}{3}(5\sqrt{5}-4)$$

(21) $x^2+y^2+z^2=4 \Leftrightarrow z^2+r^2=4 \Leftrightarrow z=\sqrt{4-r^2}$



$\begin{cases} u=\sqrt{4-r^2} \Leftrightarrow r^2=4-u^2 \\ du=-\frac{r}{\sqrt{4-r^2}}\,dr \end{cases}$

$$\int_0^{\frac{\pi}{2}}\int_0^2\int_0^{\sqrt{4-r^2}} r^3\cos\theta\sin\theta\,dz\,dr\,d\theta =$$

$$= \int_0^{\frac{\pi}{2}}\int_0^2 r^3\sqrt{4-r^2}\cos\theta\sin\theta\,dr\,d\theta =$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(2\theta)\int_0^2 r^3\sqrt{4-r^2}\,dr\,d\theta =$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(2\theta)\int_0^2 r^3 u\left(-\frac{\sqrt{4-r^2}}{r}\right)du\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(2\theta)\int_0^2 -u^2r^2\,du\,d\theta =$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(2\theta)\int_0^2 -u^2(4-u^2)\,du\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(2\theta)\left[-\frac{4u^3}{3}+\frac{u^5}{5}\right]_0^2 d\theta =$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(2\theta)\left[-\frac{4(\sqrt{4-r^2})^3}{3}+\frac{(\sqrt{4-r^2})^5}{5}\right]_0^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin\theta\times\left(\frac{32}{3}-\frac{32}{5}\right)d\theta =$$

$$= \frac{1}{2}\times\left[-\frac{64}{15}\cos\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\times\left(-\frac{64}{15}\cos\left(\frac{\pi}{2}\right)+\frac{64}{15}\cos(0)\right) = \frac{1}{2}\times\frac{64}{15}\times 1 = \frac{64}{30} =$$

$$= \frac{32}{15}$$

R: $\frac{32}{15}$