Ficha 3 - Integração dupla

1 a) $\int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int_0^1\int_0^1 x^2y + xy^2\,dx\,dy =$

$= \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^1 dy = \int_0^1 \left(\frac{y}{3}+\frac{y^2}{2}\right)dy = \left[\frac{y^2}{6}+\frac{y^3}{6}\right]_0^1 = \frac{1}{6}+\frac{1}{6}=\frac{2}{6}=\frac{1}{3}$

b) $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x+y)\,dx\,dy = \int_0^{\frac{\pi}{2}}\left[-\cos(x+y)\right]_0^{\frac{\pi}{2}} dy = \int_0^{\frac{\pi}{2}} -\cos\left(\frac{\pi}{2}+y\right)+\cos(y)\,dy =$

$= \int_0^{\frac{\pi}{2}} \sin(y)+\cos(y)\,dy = \left[-\cos(y)+\sin(y)\right]_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right)+\cos(0)=1+1=2$

c) $\int_1^3\int_0^1 (\sqrt{y}+x-3xy^2)\,dx\,dy = \int_1^3 \left[x\sqrt{y}+\frac{x^2}{2}-\frac{3x^2y^2}{2}\right]_0^1 dy =$

$=\int_1^3 \left(\sqrt{y}+\frac{1}{2}-\frac{3y^2}{2}\right)dy = \left[\frac{2\sqrt{y^3}}{3}+\frac{y}{2}-\frac{3y^3}{6}\right]_1^3 = \frac{2\sqrt{27}}{3}+\frac{3}{2}-\frac{3\times 27}{6}-\frac{2}{3}-\frac{1}{2}+\frac{1}{2}=$

$= 2\sqrt{3}-\frac{38}{3}$

$\begin{cases} u = x^2+y \\ du = 2x\,dx \end{cases}$

d) $\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{\frac{\pi}{2}}} x\sin(x^2+y)\,dx\,dy = \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{\frac{\pi}{2}}} \sin(u)\,du\,dy =$

$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[-\cos(u)\right]_0^{\sqrt{\frac{\pi}{2}}}dy = \frac{1}{2}\int_0^{\frac{\pi}{2}}\left[-\cos(x^2+y)\right]_0^{\sqrt{\frac{\pi}{2}}}dy = \frac{1}{2}\int_0^{\frac{\pi}{2}} -\cos\left(\frac{\pi}{2}+y\right)+\cos(y)\,dy=$

$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(y)+\cos(y)\,dy = \frac{1}{2}\left[-\cos(y)+\sin(y)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\sin\left(\frac{\pi}{2}\right)+\cos(0)\right) = \frac{1}{2}(1+1)=\frac{2}{2}=1$

e) $\int_0^{\pi}\int_0^{\pi} \sin^2(x)\sin^2(y)\,dx\,dy = \int_0^{\pi} \sin^2(y)\int_0^{\pi} \frac{1}{2}-\frac{\cos(2x)}{2}\,dx\,dy=$

$=\int_0^{\pi} \sin^2(y)\times\left[\frac{x}{2}-\frac{\sin(2x)}{4}\right]_0^{\pi} dy = \int_0^{\pi} \sin^2(y)\times\left[\frac{\pi}{2}\right]dy = \frac{\pi}{2}\int_0^{\pi} \frac{1}{2}-\frac{\cos(2y)}{2}\,dy=$

$=\frac{\pi}{2}\times\left[\frac{y}{2}-\frac{\sin(2y)}{4}\right]_0^{\pi} = \frac{\pi}{2}\times\frac{\pi}{2}=\frac{\pi^2}{4}$

f) $\int_1^2\int_0^2 y^{-3}e^{\frac{2x}{y}}\,dx\,dy = \int_1^2 \frac{1}{y^3}\int_0^2 e^{\frac{2x}{y}}\,dx\,dy = \int_1^2\frac{1}{y^3}\left[\frac{e^{\frac{2x}{y}}}{\frac{2}{y}}\right]_0^2 dy=$

$\begin{cases} u = \frac{4}{y} \\ du = -\frac{4}{y^2}dy \end{cases}$

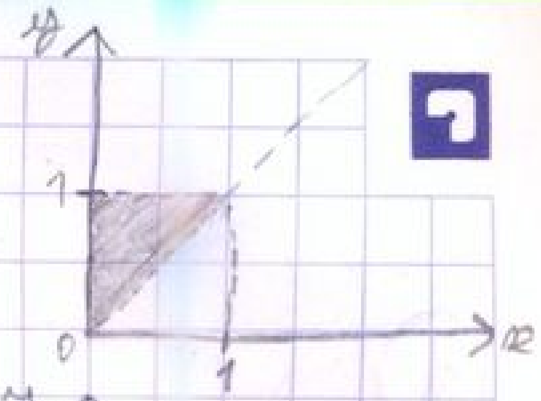
$=\int_1^2 \frac{1}{y^3}\times\left[\frac{e^{\frac{4}{y}}}{\frac{2}{y}}-\frac{y}{2}\right]dy = \int_1^2 \frac{e^{\frac{4}{y}}}{2y^2}-\frac{1}{2y^2}\,dy = \int_1^2\frac{e^{\frac{4}{y}}}{2y^2}\,dy-\int_1^2\frac{1}{2y^2}\,dy=$

$=-\frac{1}{8}\int_1^2 e^{\frac{4}{y}}\times\left(-\frac{4}{y^2}\right)dy - \frac{1}{2}\int_1^2\frac{1}{y^2}\,dy = -\frac{1}{8}\int_1^2 e^{u}\,du - \frac{1}{2}\times\left[-\frac{1}{y}\right]_1^2 =$

$=-\frac{1}{8}e^{u}\Big|_1^2 - \frac{1}{2}\times\left[-\frac{1}{2}+1\right] = -\frac{1}{8}\times e^{\frac{4}{y}}\Big|_1^2 - \frac{1}{2}\times\frac{1}{2} = -\frac{1}{8}\times\left[e^2-e^4\right]-\frac{1}{4}=$

$=\frac{e^2-e^4}{-8}-\frac{2}{8} = -\frac{e^2-e^4+2}{8} = \frac{e^4-e^2-2}{8}$

② a) $\int_0^1\left[\int_0^y f(x,y)\,dx\right]dy = \int_0^1\int_x^1 f(x,y)\,dy\,dx$

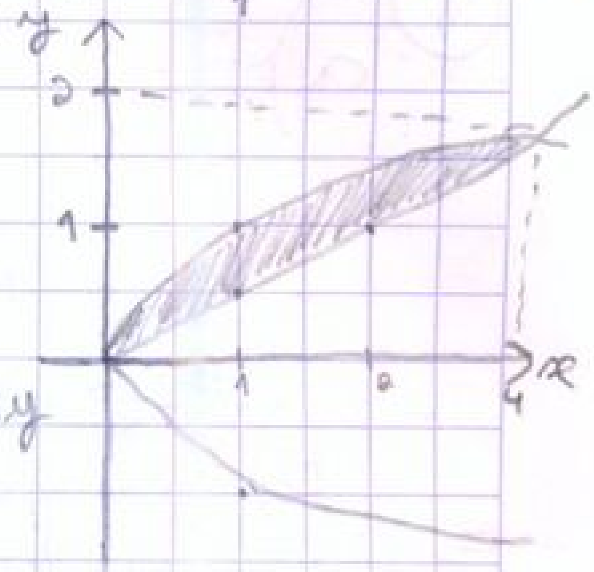


b) $\int_0^2\left[\int_{y^2}^{2y} f(x,y)\,dx\right]dy = \int_0^4\int_{\frac{x}{2}}^{\sqrt{x}} f(x,y)\,dy\,dx$

$x = y^2 \Leftrightarrow y = \pm\sqrt{x}$

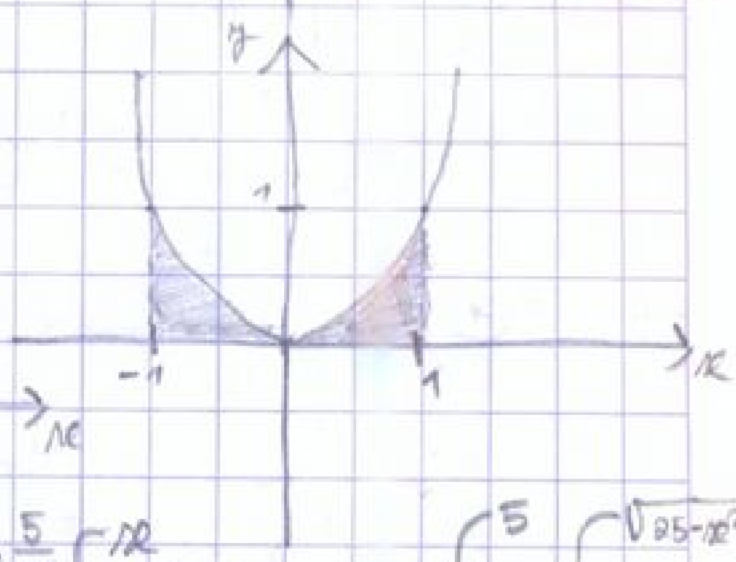
$x = 2y \Leftrightarrow y = \frac{x}{2}$

$\frac{x^2}{4} = x \Leftrightarrow x\left(\frac{x}{4}-1\right)=0 \Leftrightarrow x=0 \vee x=4$



c) $\int_{-1}^1\left[\int_0^{x^2} f(x,y)\,dy\right]dx = \int_0^1\int_{-1}^{-\sqrt{y}} f(x,y)\,dx\,dy + \int_0^1\int_{\sqrt{y}}^1 f(x,y)\,dx\,dy$

$y = x^2 \Rightarrow x = \pm\sqrt{y}$

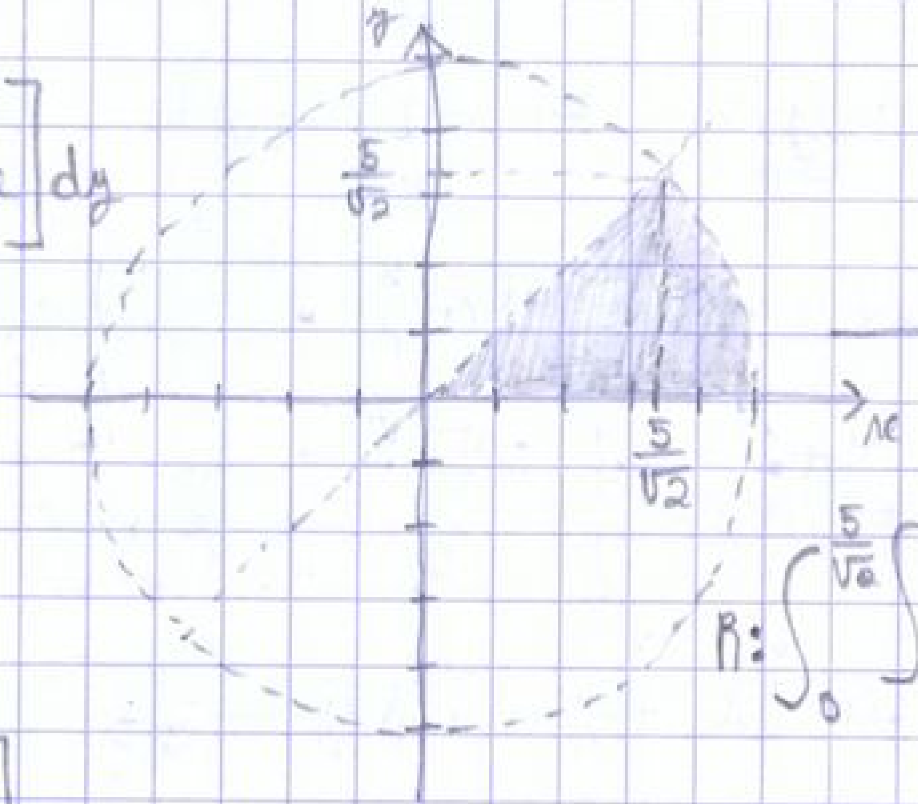


d) $\int_0^{\frac{5}{\sqrt2}}\left[\int_y^{\sqrt{25-y^2}} f(x,y)\,dx\right]dy$

$x = y \Rightarrow y = x$ $\quad y = \sqrt{25-x^2}$ $\Uparrow$

$x = \sqrt{25-y^2} \Rightarrow x^2+y^2 = 25$

$2x^2 = 25 \Leftrightarrow x = \pm\sqrt{\frac{25}{2}} = \pm\frac{5}{\sqrt2}$

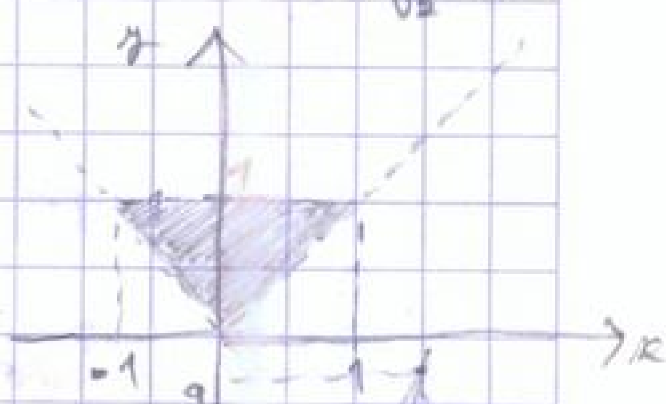


R: $\int_0^{\frac{5}{\sqrt2}}\int_0^x f(x,y)\,dy\,dx + \int_{\frac{5}{\sqrt2}}^5\int_0^{\sqrt{25-x^2}} f(x,y)\,dy\,dx$

e) $\int_0^1\left[\int_{-y}^y f(x,y)\,dx\right]dy =$

CA:
$x = y \Rightarrow y = x$
$x = -y \Rightarrow y = -x$

$= \int_{-1}^0\int_{-x}^1 f(x,y)\,dy\,dx + \int_0^1\int_x^1 f(x,y)\,dy\,dx$



f) $\int_1^3\left[\int_x^{x^2} f(x,y)\,dy\right]dx =$

CA:
$y = x^2 \Rightarrow x = \pm\sqrt{y}$
$y = x \Rightarrow x = y$

$= \int_1^3\int_{\sqrt{y}}^y f(x,y)\,dx\,dy + \int_3^9\int_{\sqrt{y}}^3 f(x,y)\,dx\,dy$

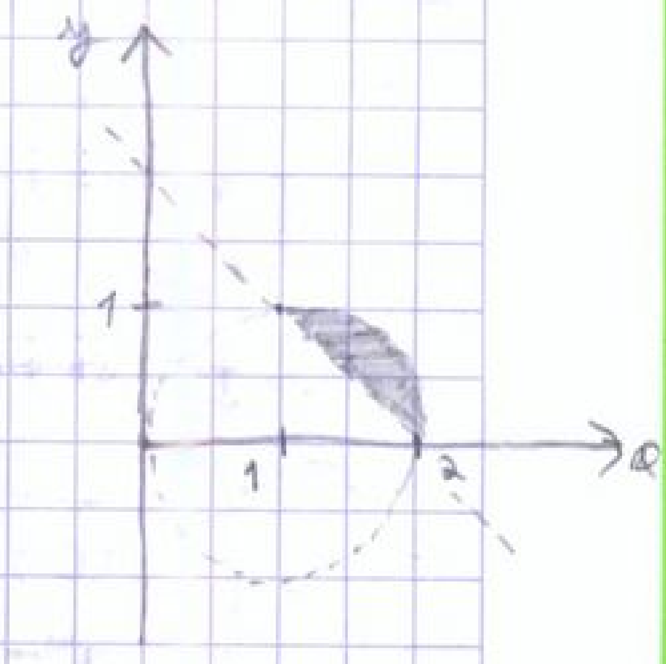
g) $\int_1^2\left[\int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,dy\right]dx$

$y = \sqrt{2x-x^2} \Rightarrow y^2+x^2-2x=0 \Leftrightarrow y^2+(x-1)^2=1 \Leftrightarrow$

$\Leftrightarrow x-1 = \sqrt{1-y^2} \Leftrightarrow x = 1+\sqrt{1-y^2}$

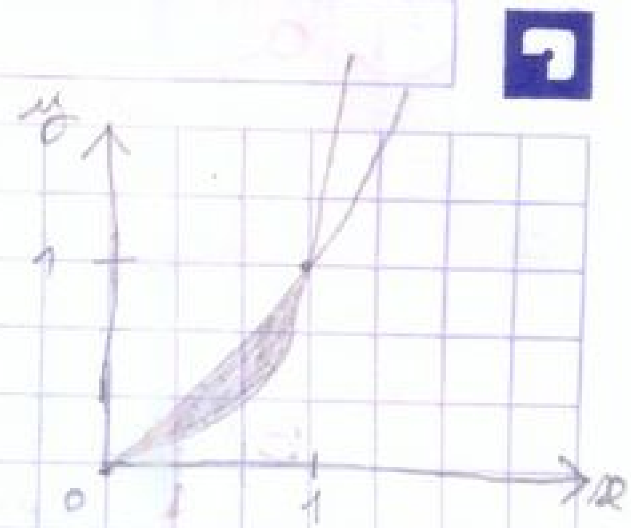
$y = 2-x \Rightarrow x = 2-y$

R: $\int_0^1\int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx\,dy$



3 a) $A = \int_0^1 \int_{x^3}^{x^2} 1 \, dy \, dx = \int_0^1 x^2 - x^3 \, dx =$

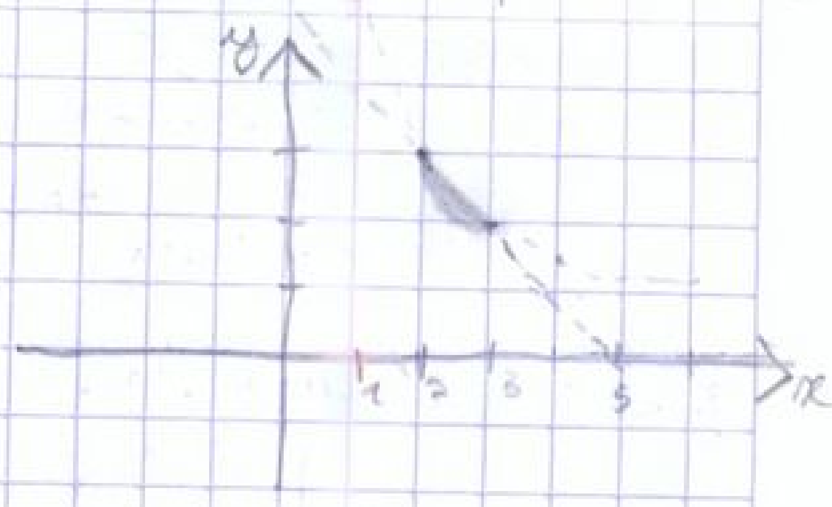
$= \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$ u.a.



b) $x + y = 5 \Leftrightarrow y = 5 - x \qquad xy = 6 \Leftrightarrow y = \frac{6}{x}$

$5 - x = \frac{6}{x} \Leftrightarrow 5x - x^2 - 6 = 0 \Leftrightarrow x = \frac{-5 \pm \sqrt{25-24}}{-2} \Leftrightarrow$

$\Leftrightarrow x = \frac{-5 \pm 1}{-2} \Leftrightarrow x = 2 \vee x = 3$



$A = \int_2^3 \int_{\frac{6}{x}}^{5-x} 1 \, dy \, dx = \int_2^3 5 - x - \frac{6}{x} \, dx =$

$= \left[5x - \frac{x^2}{2} - 6 \times \ln(x) \right]_2^3 = 15 - \frac{9}{2} - 6\ln(3) - (10 - 2 - 6\ln(2)) = \frac{5}{2} + 6 \ln\left(\frac{2}{3}\right)$

c) $4x - x^2 = -3x + 6 \Leftrightarrow x^2 - 7x + 6 = 0 \Leftrightarrow$

C.A.: $4x - x^2 = 0 \Leftrightarrow x(4-x) = 0 \Leftrightarrow x = 0 \vee x = 4$

$\Leftrightarrow x = \frac{7 \pm \sqrt{49-24}}{2} \Leftrightarrow x = \frac{7 \pm 5}{2} \Leftrightarrow x = 6 \vee x = 1$

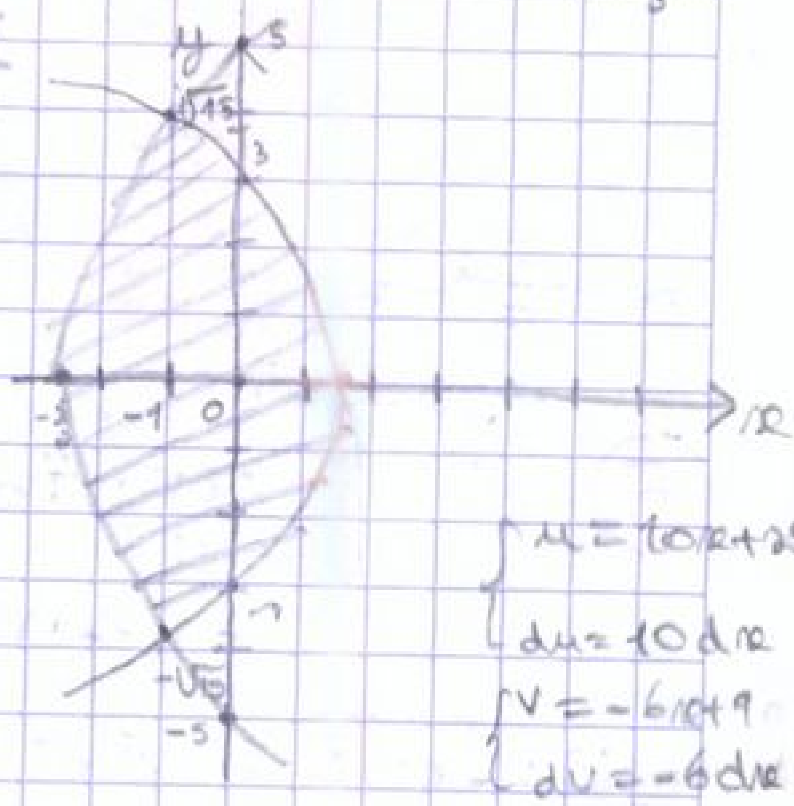
$A = \int_1^2 \int_{-3x+6}^{4x-x^2} 1 \, dy \, dx + \int_2^4 \int_0^{4x-x^2} 1 \, dy \, dx =$

$= \int_1^2 4x - x^2 + 3x - 6 \, dx + \int_2^4 4x - x^2 \, dx = \int_1^2 -x^2 + 7x - 6 \, dx + \int_2^4 4x - x^2 \, dx =$

$= \left[-\frac{x^3}{3} + \frac{7x^2}{2} - 6x \right]_1^2 + \left[\frac{4x^2}{2} - \frac{x^3}{3} \right]_2^4 = -\frac{8}{3} + 14 - 12 + \frac{1}{3} - \frac{7}{2} + 6 + 32 - \frac{64}{3} - 8 + \frac{8}{3} =$

$= -\frac{63}{3} + 32 - \frac{7}{2} = -21 + 32 - \frac{7}{2} = \frac{22}{2} - \frac{7}{2} = \frac{15}{2}$

d) $10x + 25 = -6x + 9 \Leftrightarrow 16x = -16 \Leftrightarrow x = -1$



$y^2 = 25 \Leftrightarrow y = \pm 5 \qquad y^2 = 9 \Rightarrow y = \pm 3$

$10x = -25 \Leftrightarrow x = -\frac{5}{2} \qquad 6x = 9 \Leftrightarrow x = \frac{3}{2}$

$A = \int_{-\frac{5}{2}}^{-1} \int_{-\sqrt{10x+25}}^{\sqrt{10x+25}} 1 \, dy \, dx + \int_{-1}^{\frac{3}{2}} \int_{-\sqrt{-6x+9}}^{\sqrt{-6x+9}} 1 \, dy \, dx =$

$\begin{cases} u = 10x + 25 \\ du = 10 \, dx \end{cases} \quad \begin{cases} v = -6x + 9 \\ dv = -6 \, dx \end{cases}$

$= \int_{-\frac{5}{2}}^{-1} 2\sqrt{10x+25} \, dx + \int_{-1}^{\frac{3}{2}} 2\sqrt{-6x+9} \, dx = \frac{1}{5} \int_{-\frac{5}{2}}^{-1} \sqrt{u} \, du - \frac{1}{3} \int_{-1}^{\frac{3}{2}} \sqrt{v} \, dv =$

$= \frac{1}{5} \times \left[\frac{2u^{\frac{3}{2}}}{3} \right]_{-\frac{5}{2}}^{-1} - \frac{1}{3} \times \left[\frac{2v^{\frac{3}{2}}}{3} \right]_{-1}^{\frac{3}{2}} = \frac{1}{5} \times \left[\frac{2(10x+25)^{\frac{3}{2}}}{3} \right]_{-\frac{5}{2}}^{-1} - \frac{1}{3} \left[\frac{2(-6x+9)^{\frac{3}{2}}}{3} \right]_{-1}^{\frac{3}{2}} =$

$= \frac{1}{5} \times \left[\frac{30\sqrt{15}}{3} \right] - \frac{1}{3} \times \left[-\frac{30\sqrt{15}}{3} \right] = \frac{10\sqrt{15}}{5} + \frac{10\sqrt{15}}{3} = 2\sqrt{15} + \frac{10\sqrt{15}}{3} =$

$= \frac{16\sqrt{15}}{3}$

4) a) $\int_1^2 \left[\int_0^{y^2} e^{\frac{x}{y^2}}\, dx\right] dy = \int_1^2 \left[\frac{e^{\frac{x}{y^2}}}{\frac{1}{y^2}}\right]_0^{y^2} dy = \int_1^2 y^2 e - y^2\, dy =$

$= \left[\frac{y^3 e}{3} - \frac{y^3}{3}\right]_1^2 = \frac{8e}{3} - \frac{8}{3} - \frac{e}{3} + \frac{1}{3} = \frac{7e}{3} - \frac{7}{3} = \frac{7}{3}(e-1)$

$\begin{cases} u = x^2 - y^2 \\ du = -2y\, dy \end{cases}$

b) $\int_0^1 \left[\int_0^x y\sqrt{x^2-y^2}\,dy\right]dx = \int_0^1 -\frac12\left[\int_0^x \sqrt{u}\,du\right]dx = -\frac12\int_0^1\left[\frac{2u^{\frac32}}{3}\right]_0^x dx =$

$= -\frac12\int_0^1 \left[\frac{2(x^2-y^2)^{\frac32}}{3}\right]_0^x dx = -\frac12\int_0^1 -\frac{2x^3}{3}\,dx = \frac12 \times \left[\frac{2x^4}{12}\right]_0^1 = \frac12\times\frac{2}{12} = \frac{1}{12}$

$\begin{cases} u = xy+1 \\ du = x\,dy \end{cases}$

c) $\int_0^1\left[\int_0^1 \frac{x}{(xy+1)^2}\,dy\right]dx = \int_0^1\int_0^1 \frac{1}{u^2}\,du\,dx = \int_0^1\left[-\frac1u\right]_0^1 dx =$

$=\int_0^1 -\frac{1}{xy+1}\Big|_0^1 dx = \int_0^1 -\frac{1}{x+1}+1\,dx = \left[-\ln(x+1)+x\right]_0^1 = 1-\ln(2)$

$\begin{cases} u = \frac{x}{y} \\ du = -\frac{x}{y^2}\,dy \end{cases}$

d) $\int_{\frac14}^1\left[\int_{x^2}^x \sqrt{\frac{x}{y}}\,dy\right]dx = \int_{\frac14}^1\int_{x^2}^x \sqrt{u}\times\left(-\frac{y^2}{x}\right)du\,dx = \int_{\frac14}^1\int_{x^2}^x -\frac{y^2\sqrt u}{x}\,du\,dx=$

$=\int_{\frac14}^1\int_{x^2}^x \frac{-\frac{x^2}{u^2}\sqrt u}{x}\,du\,dx = \int_{\frac14}^1\int_{x^2}^x -\frac{x}{u^{\frac32}}\,du\,dx = \int_{\frac14}^1 -x\int_{x^2}^x u^{-\frac32}\,du\,dx=$

$=\int_{\frac14}^1 -x\left[\frac{u^{-\frac12}}{-\frac12}\right]_{x^2}^x dx = \int_{\frac14}^1 -x\left[-\frac{2}{\sqrt u}\right]_{x^2}^x dx = \int_{\frac14}^1 -x\left[-\frac{2}{\sqrt{\frac xy}}\right]_{x^2}^x dx=$

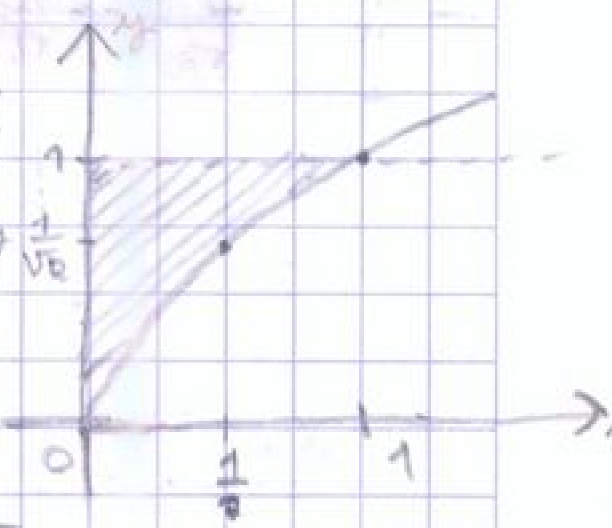
$=\int_{\frac14}^1 -x\times\left[-2+\frac{2}{\sqrt{\frac1x}}\right]dx = \int_{\frac14}^1 -x\times\left[-2+2\sqrt x\right]dx = \int_{\frac14}^1 2x-2x\sqrt x\,dx=$

$=\left[x^2-2\times\frac{2x^{\frac52}}{5}\right]_{\frac14}^1 = 1-\frac45-\left[\frac1{16}-\frac{4\times\sqrt{\left(\frac14\right)^5}}{5}\right] = \frac15-\left(\frac1{16}-\frac1{40}\right)=$

$=\frac15-\left(\frac{40}{640}-\frac{16}{640}\right)=\frac15-\frac{24}{640}=\frac{640}{3200}-\frac{120}{3200}=\frac{520}{3200}=\frac{13}{80}$

5) a) $\int_0^1\int_{\sqrt x}^1 \sin\left(\frac{y^3+1}{2}\right)dy\,dx =$ $\boxed{\text{CA: } y=\sqrt x \quad y=1}$

$\begin{cases} u = \frac{y^3}{2}+\frac12 \\ du = \frac32 y^2\,dy \end{cases}$

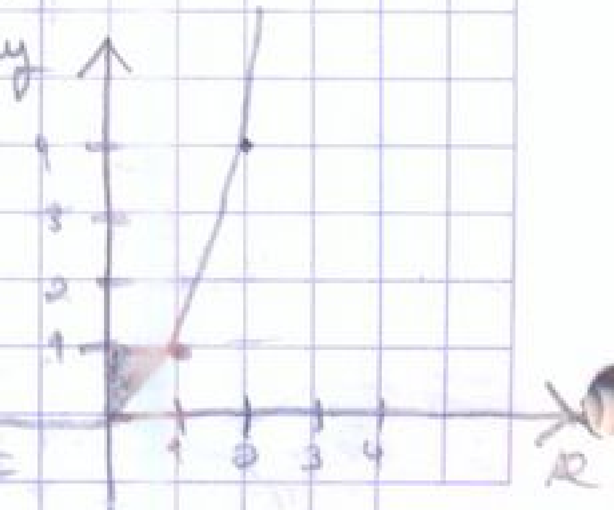


$=\int_0^1\int_0^{y^2}\sin\left(\frac{y^3+1}{2}\right)dx\,dy = \int_0^1 y^2\sin\left(\frac{y^3+1}{2}\right)dy=$

$=\frac23\int_0^1 \sin(u)\,du = \frac23\left(-\cos(u)\right)\Big|_0^1 = \frac23\left(-\cos\left(\frac{y^3+1}{2}\right)\right)\Big|_0^1=$

$=\frac23\left(\cos\left(\frac12\right)-\cos(1)\right)$

$\begin{cases} u = x^4+y^2 \\ du = 4x^3\,dx \end{cases}$

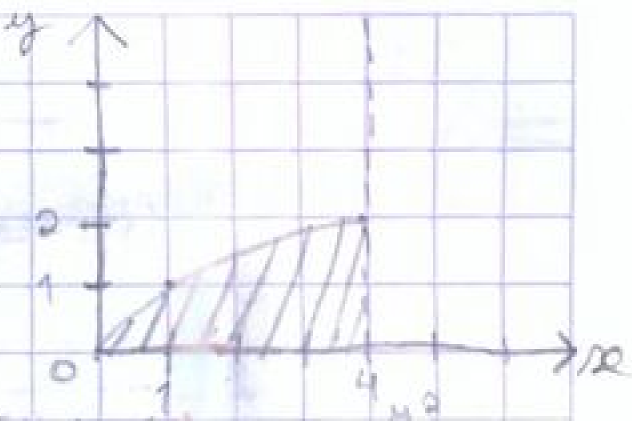


b) $\int_0^1\int_{x^2}^1 \frac{x^3}{\sqrt{x^4+y^2}}\,dy\,dx = \int_0^1\int_0^{\sqrt y}\frac{x^3}{\sqrt{x^4+y^2}}\,dx\,dy=$

$=\int_0^1\frac14\int_0^{\sqrt y}\frac{1}{\sqrt u}\,du\,dy = \frac14\int_0^1\left[2\sqrt u\right]_0^{\sqrt y} = \frac14\int_0^1\left[2\sqrt{x^4+y^2}\right]_0^{\sqrt y}dy=$

$=\frac14\int_0^1 2\sqrt2\, y-2y\,dy = \frac14\left[\sqrt2\, y^2-y^2\right]_0^1 = \frac{\sqrt2-1}{4}$

6) $\int_0^2\int_{y^2}^4 y(1+x^2)^{-1}\,dx\,dy = \int_0^2 y\int_{y^2}^4 \frac{1}{1+x^2}\,dx\,dy =$



$= \int_0^2 y\left[\arctan(x)\right]_{y^2}^4 = \int_0^2 y(\arctan(4)-\arctan(y^2))\,dy =$

$= \int_0^2 y\arctan(4)\,dy - \int_0^2 y(\arctan(y^2))\,dy = \begin{cases}u=\arctan(y^2)\\ du=\frac{2y}{y^4+1}\end{cases} \begin{cases}v=\frac{y^2}{2}\\ dv=y\end{cases}$

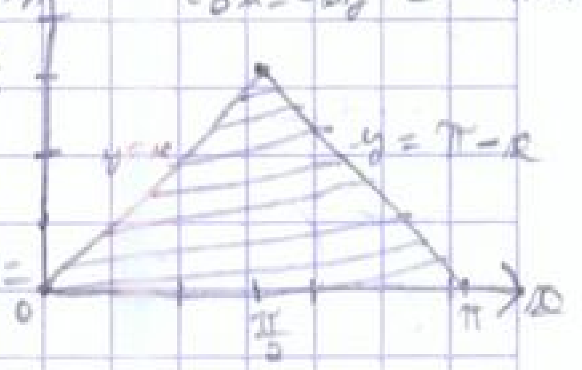
$= \left[\frac{y^2}{2}\arctan(4)\right]_0^2 - \left[\frac{y^2}{2}\arctan(y^2)\Big|_0^2 - \int_0^2 \frac{y^3}{y^4+1}\,dy\right] =$

$= 2\arctan(4) - 2\arctan(4) + \frac{1}{4}\int_0^2 \frac{4y^3}{y^4+1}\,dy = \frac{1}{4}\times\ln(y^4+1)\Big|_0^2 =$

$= \frac{1}{4}\times\ln(2^4+1) - \frac{1}{4}\times\cancel{\ln(1)}^{0} = \frac{1}{4}\ln(17)$

$\begin{cases}u=x\\ du=dx\end{cases} \begin{cases}v=\sin(x+y)\\ dv=\cos(x+y)\,dx\end{cases} \quad \begin{cases}u=-y\\ du=-dy\end{cases} \begin{cases}v=\frac{-\cos(2y)}{2}\\ dv=\sin(2y)\end{cases}$

7) $\int_0^{\frac{\pi}{2}}\int_y^{\pi-y} x\cos(x+y)\,dx\,dy = \int_0^{\frac{\pi}{2}} x\sin(x+y)\Big|_y^{\pi-y} - \int_y^{\pi-y}\sin(x+y)\,dx\,dy =$

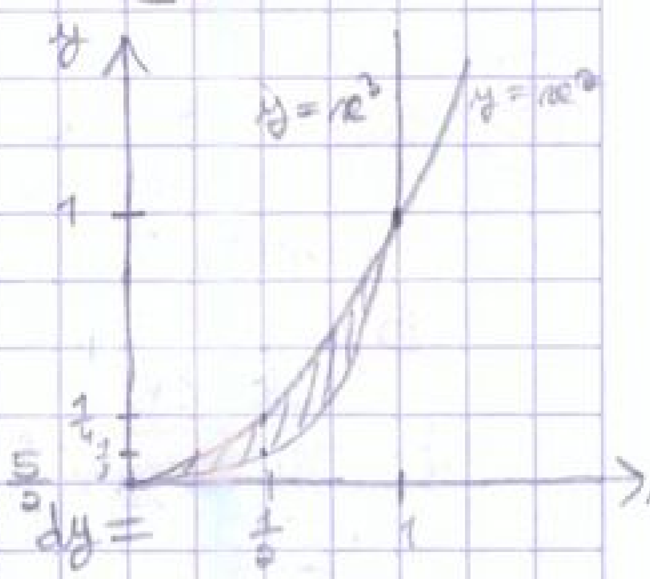


$= \int_0^{\frac{\pi}{2}} -y\sin(2y) - \left(-\cos(x+y)\right)\Big|_y^{\pi-y}\,dy = \int_0^{\frac{\pi}{2}} -y\sin(2y) + (-1-\cos(2y))\,dy =$

$= \int_0^{\frac{\pi}{2}} -y\sin(2y) - \cos(2y) - 1\,dy = \int_0^{\frac{\pi}{2}} -y\sin(2y)\,dy + \left[-\frac{\sin(2y)}{2} - y\right]_0^{\frac{\pi}{2}} =$

$= \frac{y\cos(2y)}{2}\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\cos(2y)}{2}\,dy + \left[-\frac{\pi}{2}\right] = -\frac{\pi}{4} - \left[\frac{\sin(2y)}{4}\right]_0^{\frac{\pi}{2}} - \frac{\pi}{2} =$

$= -\frac{3\pi}{4} - 0 = -\frac{3\pi}{4}$



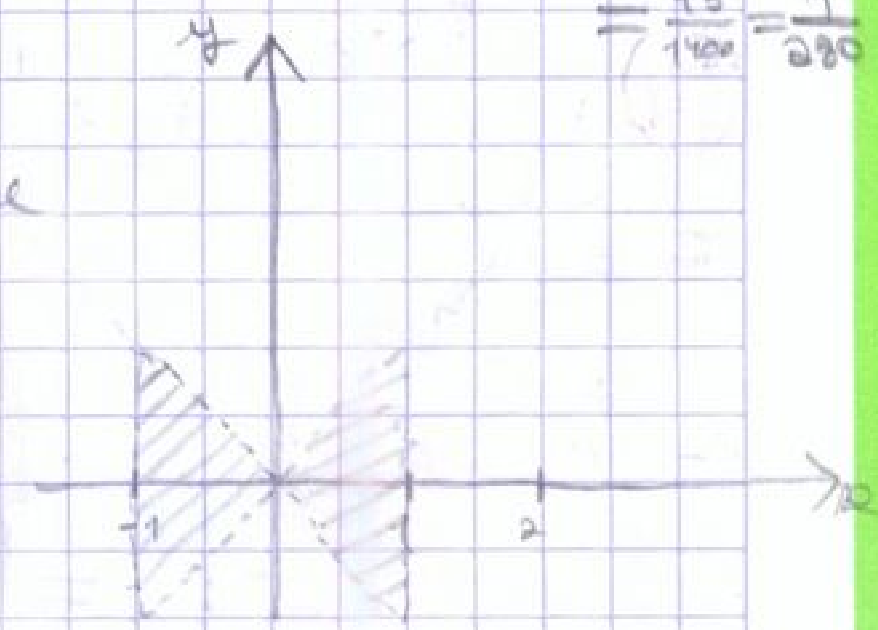
8) $\int_0^1\int_{\sqrt{y}}^{\sqrt[3]{y}} (x^4+y^2)\,dx\,dy = \int_0^1 \left[\frac{x^5}{5} + xy^2\right]_{\sqrt{y}}^{\sqrt[3]{y}}\,dy =$

$= \int_0^1 \frac{y^{\frac{5}{3}}}{5} + y^{\frac{7}{3}} - \left(\frac{y^{\frac{5}{2}}}{5} + y^{\frac{5}{2}}\right)dy = \int_0^1 \frac{y^{\frac{5}{3}}}{5} + y^{\frac{7}{3}} - \frac{y^{\frac{5}{2}}}{5} - y^{\frac{5}{2}}\,dy =$

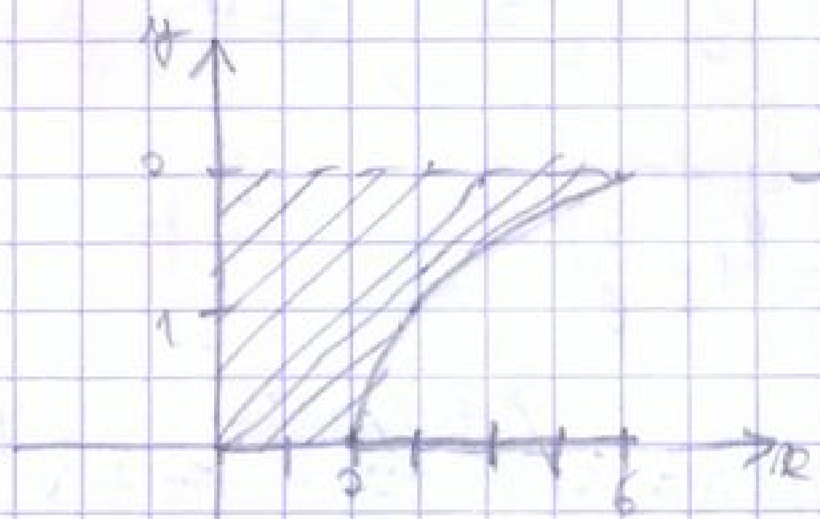
$= \left[\frac{3y^{\frac{8}{3}}}{40} + \frac{3y^{\frac{10}{3}}}{10} - \frac{2y^{\frac{7}{2}}}{35} - \frac{2y^{\frac{7}{2}}}{7}\right]_0^1 = \frac{3}{40} + \frac{3}{10} - \frac{2}{35} - \frac{2}{7} = \frac{15}{40} - \frac{12}{35} = \frac{525}{1400} - \frac{480}{1400} =$

$= \frac{45}{1400} = \frac{9}{280}$

9) Pelas propriedades dos integrais, conclui-se que $\iint_\Omega (3xy^2 - y)\,dx\,dy = 0$

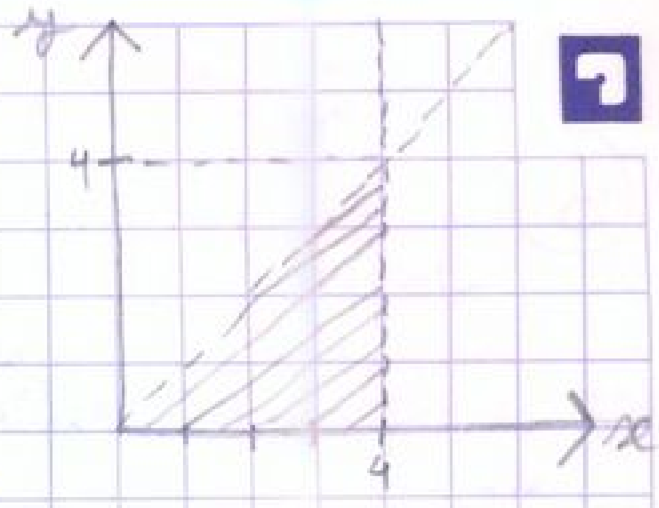


10) $\int_0^2\int_0^{y^2+2} \frac{1}{y^2+2}\,dx\,dy =$



$= \int_0^2 1\,dy = [y]_0^2 = 2$

(11) $\int_0^4\int_0^x 2(x^2+3x+2)^{-1}\,dy\,dx = \int_0^4 \frac{2x}{x^2+3x+2}\,dx =$



$= \int_0^4 \frac{2x}{(x+1)(x+2)}\,dx =$

CA:

$2x = A(x+2)+B(x+1)$

$2x = Ax+2A+Bx+B$

$\begin{cases}A+B=2\\2A+B=0\end{cases} \Leftrightarrow \begin{cases}B=4\\A+2=0 \Leftrightarrow A=-2\end{cases}$

$= \int_0^4 -\frac{2}{x+1}+\frac{4}{x+2} =$

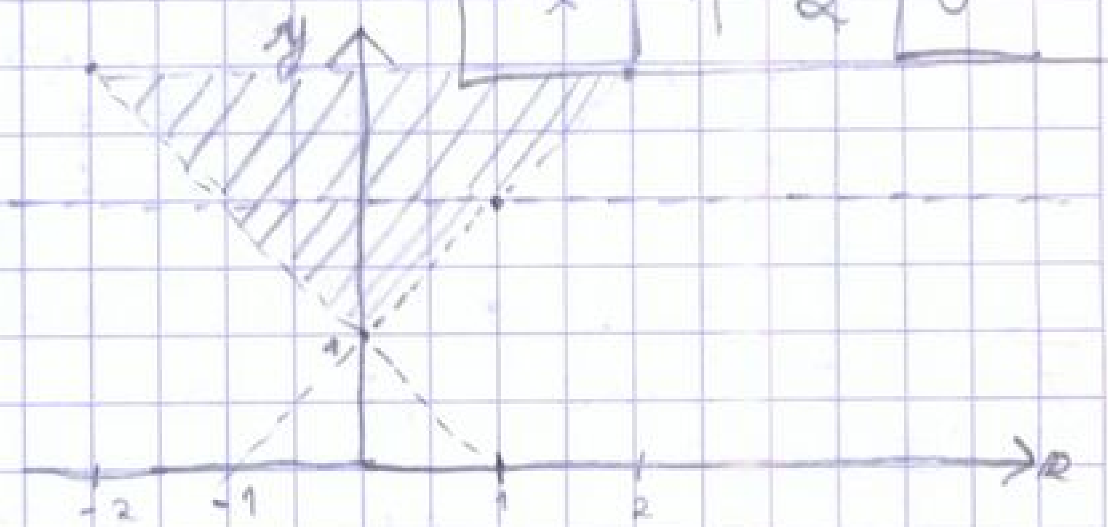
$2x =$

	1	3	2
-1		-1	-2
x	1	2	0

$= -2\ln(x+1)+4\ln(x+2)\Big|_0^4 = -2\ln(5)+4\ln(6)-4\ln(2) =$

$= -2\ln(5)+4\ln(3)$

(12) $\int_1^3\int_{1-y}^{y-1}(2x-y^2)\,dx\,dy =$



$= \int_1^3 \left[x^2-xy^2\right]_{1-y}^{y-1} =$

$= \int_1^3 (y-1)^2-(y-1)y^2-\left((1-y)^2-(1-y)y^2\right)dy =$

$= \int_1^3 y^2-2y+1-y^3+y^2-(1-2y+y^2-y^2+y^3)\,dy = \int_1^3 -2y^3+2y^2\,dy =$

$= \left[-\frac{y^4}{2}+\frac{2y^3}{3}\right]_1^3 = -\frac{81}{2}+18-\left(-\frac{1}{2}+\frac{2}{3}\right) = -\frac{81}{2}+18+\frac{1}{2}-\frac{2}{3} = -22-\frac{2}{3} = -\frac{68}{3}$

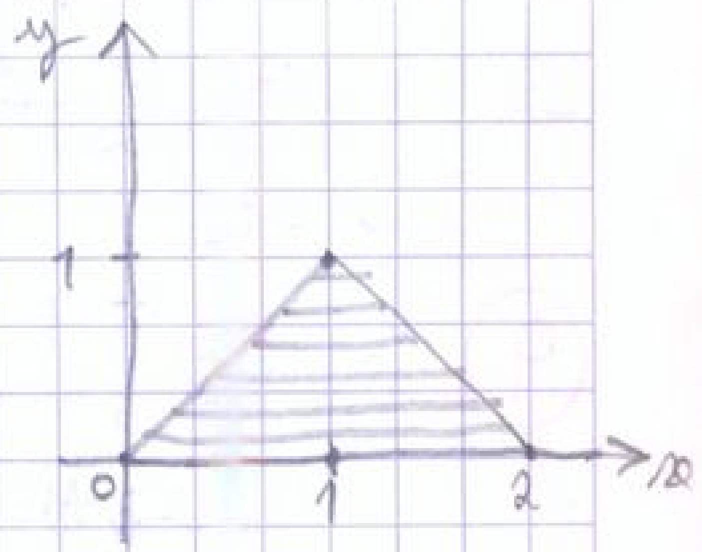
(13) a) $\int_0^1\int_0^x e^{\frac{y}{x}}\,dy\,dx = \int_0^1 \left[x\,e^{\frac{y}{x}}\right]_0^x dx = \int_0^1 xe-x\,dx = \left[\frac{x^2e}{2}-\frac{x^2}{2}\right]_0^1 = \frac{e-1}{2}$

b) $\int_0^1\int_0^y x^2e^{y^4}\,dx\,dy = \int_0^1\left[\frac{x^3e^{y^4}}{3}\right]_0^y dy = \int_0^1 \frac{y^3e^{y^4}}{3}\,dy =$ $\begin{cases}u=y^4\\du=4y^3\end{cases}$

$= \frac{1}{3}\int_0^1 y^3e^{y^4}\,dy = \frac{1}{12}\int_0^1 4y^3e^{y^4}\,dy = \frac{1}{12}\int_0^1 e^u\,du = \frac{1}{12}\times e^u\Big|_0^1 =$

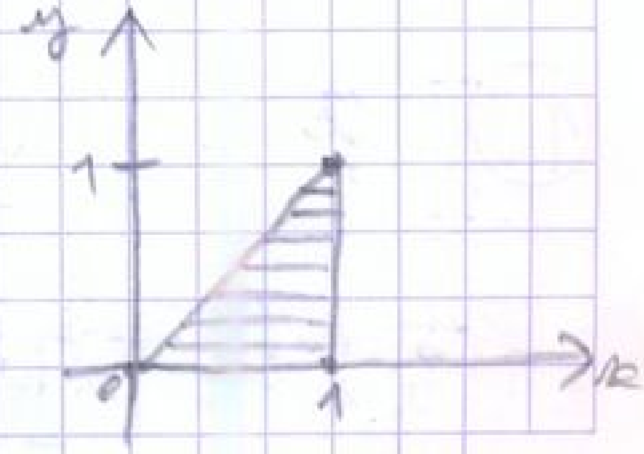
$= \frac{1}{12}\times e^{y^4}\Big|_0^1 = \frac{1}{12}\times e - \frac{1}{12}\times 1 = \frac{e-1}{12}$

(14) $\int_0^1\int_y^{2-y} 2x\,dy = \int_0^1\left[x^2\right]_y^{2-y}dy = \int_0^1 (2-y)^2-y^2\,dy =$



$= \int_0^1 4-4y+y^2-y^2\,dy = \int_0^1 4-4y\,dy =$

$= \left[4y-2y^2\right]_0^1 = 4-2 = 2$

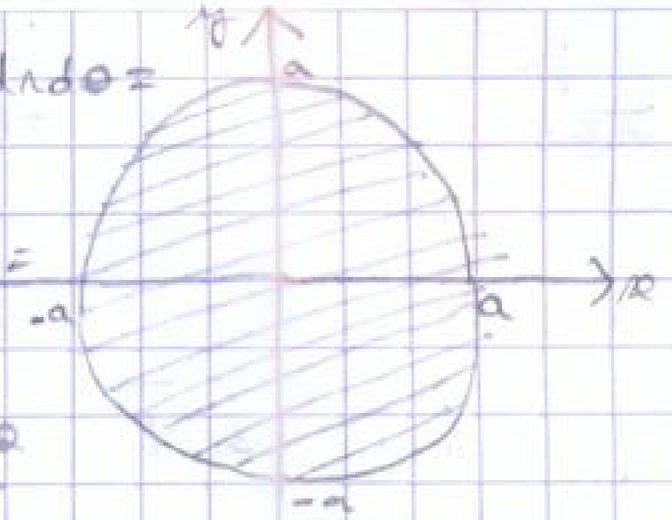


(15) a) $\int_0^1\int_y^1 (x+y+1)\,dx\,dy = \int_0^1\left[\frac{x^2}{2}+xy+x\right]_y^1 dy =$

$= \int_0^1 \frac{1}{2}+y+1-\frac{y^2}{2}-y^2-y\,dy = \int_0^1 \frac{3}{2}-\frac{3y^2}{2}\,dy = \left[\frac{3}{2}y-\frac{3y^3}{6}\right]_0^1 =$

$= \frac{3}{2}-\frac{3}{6} = 1$

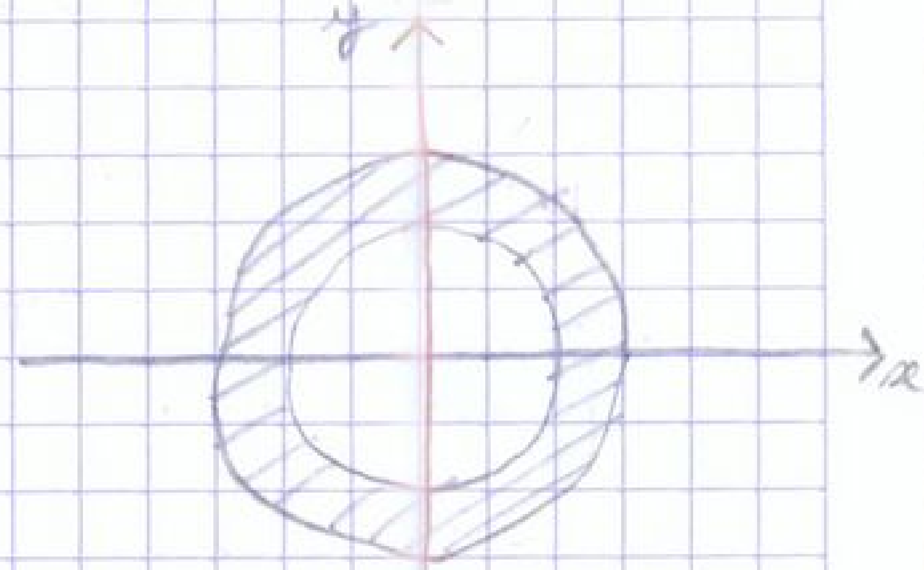
b) $\int_0^{2\pi}\int_0^a (r\cos\theta + r\sin\theta + 1)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^a (r^2\cos\theta + r^2\sin\theta + r)\,dr\,d\theta =$



$= \int_0^{2\pi}\left[\frac{r^3\cos\theta}{3}+\frac{r^3\sin\theta}{3}+\frac{r^2}{2}\right]_0^a = \int_0^{2\pi}\frac{a^3\cos\theta}{3}+\frac{a^3\sin\theta}{3}+\frac{a^2}{2}\,d\theta =$

$= \left[\frac{a^3\sin\theta}{3}-\frac{a^3\cos\theta}{3}+\frac{a^2}{2}\theta\right]_0^{2\pi} = -\frac{a^3}{3}+\frac{2\pi a^2}{2}+\frac{a^3}{3}=\pi a^2$

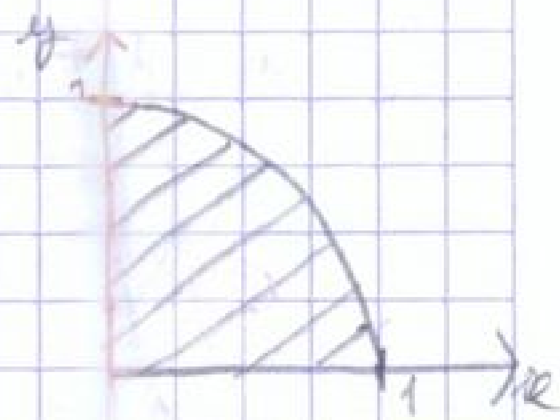
16) $\int_0^{2\pi}\int_2^3 \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\; r\,dr\,d\theta =$



$=\int_0^{2\pi}\int_2^3 r^2\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^3}{3}\right]_2^3 d\theta=$

$=\int_0^{2\pi}\left(9-\frac{8}{3}\right)d\theta = \int_0^{2\pi}\frac{19}{3}\,d\theta = \frac{19}{3}\times 2\pi = \frac{38\pi}{3}$

17) $f_m = \frac{\iint_D f(x,y)\,dx\,dy}{\iint_D dx\,dy} = \frac{4}{\pi}\iint_D xy\,dx\,dy =$



$= \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_0^1 r^3\cos\theta\sin\theta\,dr\,d\theta = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\left[\frac{r^4\cos\theta\sin\theta}{4}\right]_0^1 d\theta =$

$= \frac{4}{\pi}\times\frac{1}{8}\int_0^{\frac{\pi}{2}}\sin(2\theta)\,d\theta = \frac{1}{2\pi}\times\left(\frac{-\cos(2\theta)}{2}\right)\Big|_0^{\frac{\pi}{2}} = \frac{1}{2\pi}\times\left(\frac{1}{2}+\frac{1}{2}\right)=\frac{1}{2\pi}$

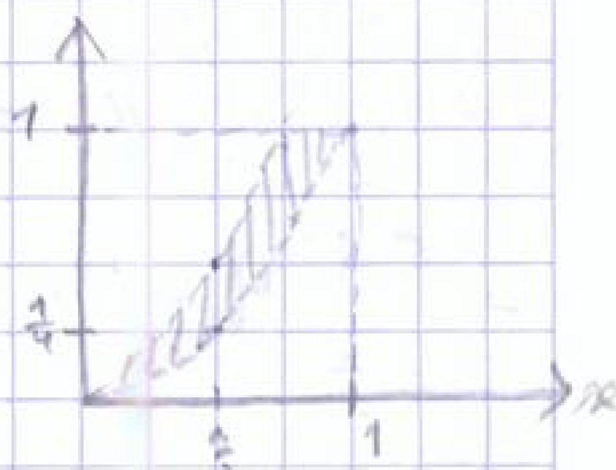
18) $x+y\le 1 \Rightarrow y \le 1-x$

$\int_0^1\int_0^{1-x} 1-x-y\,dx\,dy = \int_0^1\left[y-xy-\frac{y^2}{2}\right]_0^{1-x}dx =$

$=\int_0^1 1-x-x(1-x)-\frac{(1-x)^2}{2}\,dx = \int_0^1 1-x-x+x^2-\frac{x^2-2x+1}{2}\,dx$

$=\int_0^1\frac{x^2-2x+1}{2}\,dx = \frac{1}{2}\left[\frac{x^3}{3}-x^2+x\right]_0^1 = \frac{1}{2}\left(\frac{1}{3}-1+1\right)=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$

19) $\int_0^1\int_{\sqrt{\frac{y}{2}}}^{\sqrt{y}} x+y\,dx\,dy = \int_0^1\left[\frac{x^2}{2}+yx\right]_{\sqrt{\frac{y}{2}}}^{\sqrt{y}}dy=$



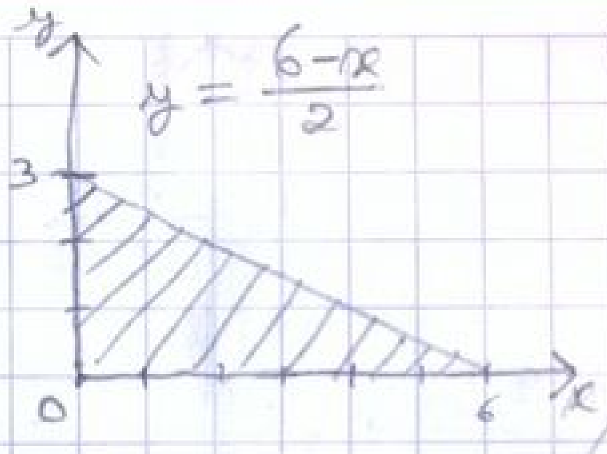
$=\int_0^1 \frac{y}{2}+y\sqrt{y}-\left(\frac{y}{4}+\frac{y\sqrt{y}}{\sqrt{2}}\right)dy=$

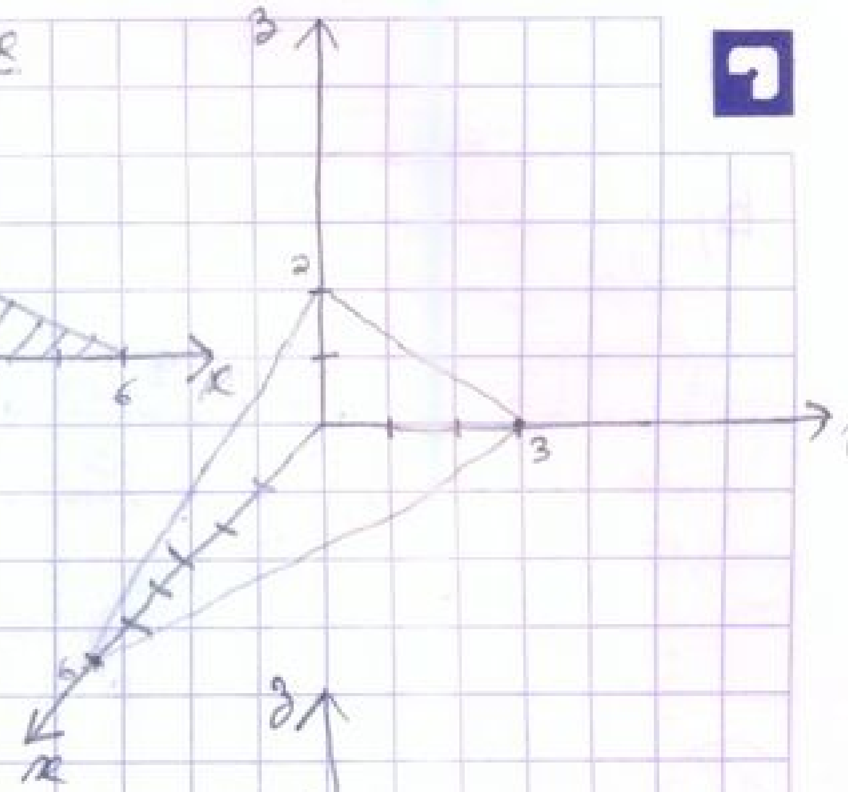
$=\int_0^1\frac{y}{4}+y^{\frac{3}{2}}\left(1-\frac{1}{\sqrt{2}}\right)dy = \left[\frac{y^2}{8}+\left(1-\frac{1}{\sqrt{2}}\right)\frac{2y^{\frac{5}{2}}}{5}\right]_0^1=$

$=\frac{1}{8}+\left(1-\frac{1}{\sqrt{2}}\right)\times\frac{2}{5} = \frac{1}{8}+\frac{2}{5}-\frac{2}{5\sqrt{2}}=\frac{5}{40}+\frac{16}{40}-\frac{2\sqrt{2}}{5\times2}=$

$=\frac{21}{40}-\frac{\sqrt{2}}{5}$

20 a) $x=0 \wedge y=0 \Rightarrow 3z=6 \Leftrightarrow z=2$
$x=0 \wedge z=0 \Rightarrow 2y=6 \Leftrightarrow y=3$
$y=0 \wedge z=0 \Rightarrow x=6$



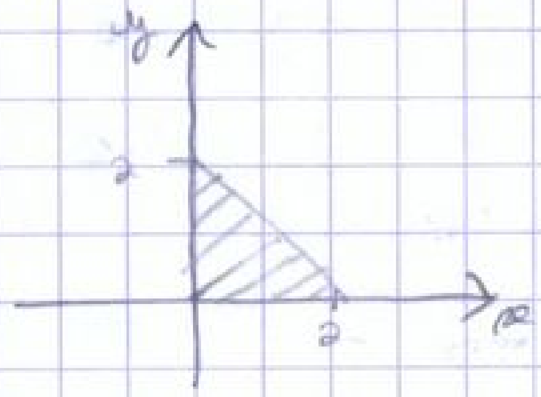


$$\int_0^6\int_0^{\frac{6-x}{2}} 1\,dy\,dx = \int_0^6 \frac{6-x}{2}\,dx =$$

$$= \frac{1}{2}\left[6x - \frac{x^2}{2}\right]_0^6 = \frac{1}{2}\left(36 - \frac{36}{2}\right) = \frac{36}{4} = 18$$

$$V_{\text{pirâmide}} = \frac{1}{3} V_{\text{prisma}} = \frac{1}{3} \times 18 = 6 \text{ u.v.}$$

b) $$\int_0^2\int_0^{2-x} 4-x^2\,dy\,dx =$$

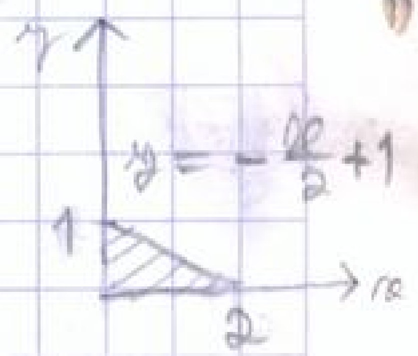


$$= \int_0^2 \left[4y - yx^2\right]_0^{2-x} dx =$$

$$= \int_0^2 8 - 4x - 2x^2 + x^3\,dx = \left[8x - 2x^2 - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^2 =$$

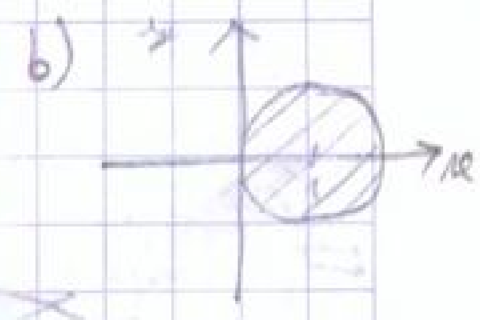
$$= 16 - 8 - \frac{16}{3} + 4 = \frac{36}{3} - \frac{16}{3} = \frac{20}{3} \text{ u.v.}$$

c) $$\int_0^2\int_0^{-\frac{x}{2}+1} x^3 y\,dy\,dx = \int_0^2 \left[\frac{x^3y^2}{2}\right]_0^{-\frac{x}{2}+1} dx =$$

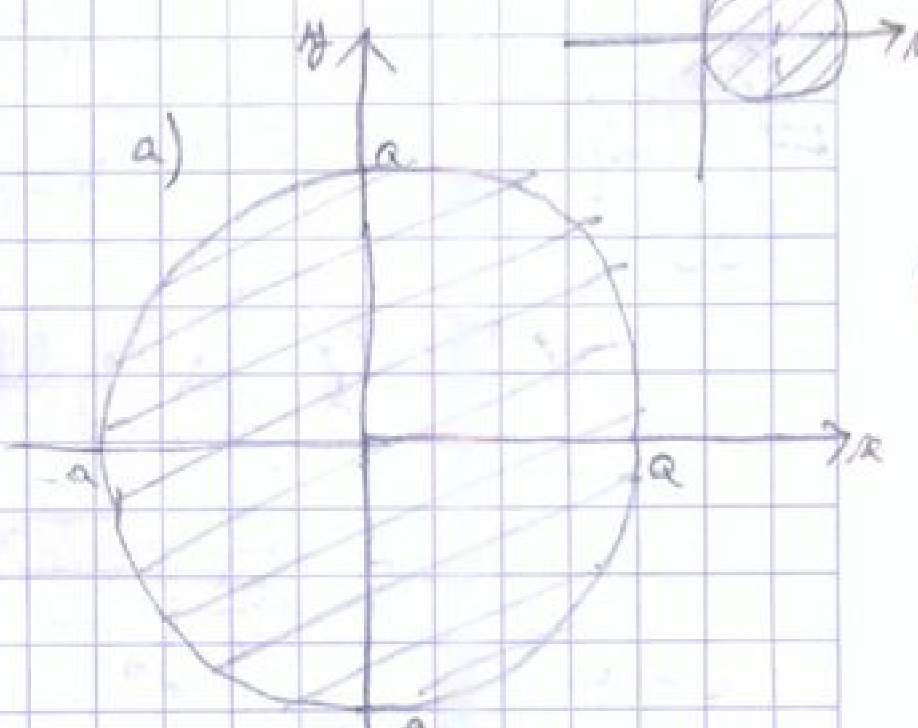


$$= \int_0^2 \frac{x^3\left(\frac{x^2}{4} - x + 1\right)}{2}\,dx = \frac{1}{2}\int_0^2 \frac{x^5}{4} - x^4 + x^3\,dx =$$

$$= \frac{1}{2}\left[\frac{x^6}{24} - \frac{x^5}{5} + \frac{x^4}{4}\right]_0^2 = \frac{1}{2}\left(\frac{64}{24} - \frac{32}{5} + 4\right) = \frac{1}{2} \times \frac{4}{15} = \frac{2}{15}$$



21 a) $$\int_0^{2\pi}\int_0^a f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$



b) $x^2+y^2 \le 2x \Leftrightarrow (x-1)^2 + y^2 \le 1$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

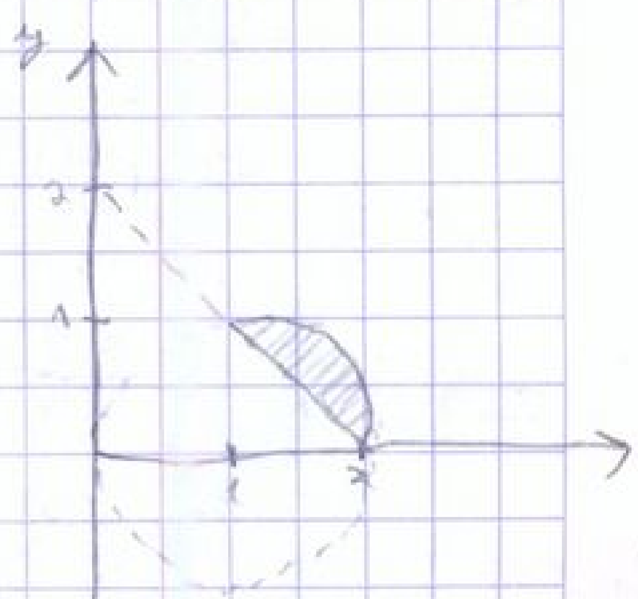
22 $y = \sqrt{2x - x^2} \Leftrightarrow y^2 + (x-1)^2 = 1$

$2 - r\cos\theta = r\sin\theta \Leftrightarrow r = \frac{2}{\sin\theta + \cos\theta}$ ← limite inferior

limite superior

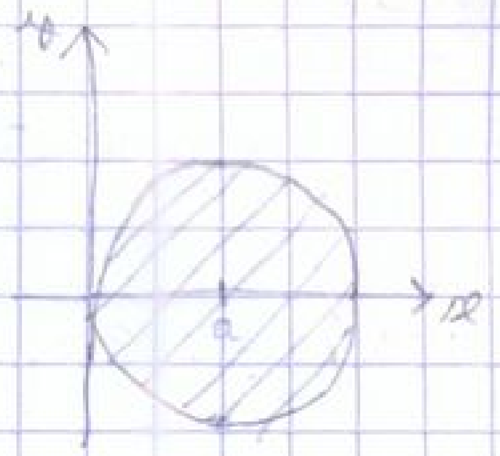
$\sqrt{2r\cos\theta - r^2\cos^2\theta} = r\sin\theta \Leftrightarrow r^2 = 2r\cos\theta \Leftrightarrow r = 2\cos\theta$

$$\int_0^{\frac{\pi}{4}}\int_{\frac{2}{\sin\theta+\cos\theta}}^{2\cos\theta} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$



23) $y=\sqrt{2ax-x^2} \Rightarrow r\sin\theta = \sqrt{2ar\cos\theta - r^2\cos^2\theta} \Leftrightarrow r^2\sin^2\theta = 2ar\cos\theta - r^2\cos^2\theta \Leftrightarrow$

$\Leftrightarrow r^2 = 2ar\cos\theta \Leftrightarrow r = 2a\cos\theta$



$$\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta}(r^2\cos^2\theta + r^2\sin^2\theta)\, r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^3\, dr\, d\theta =$$

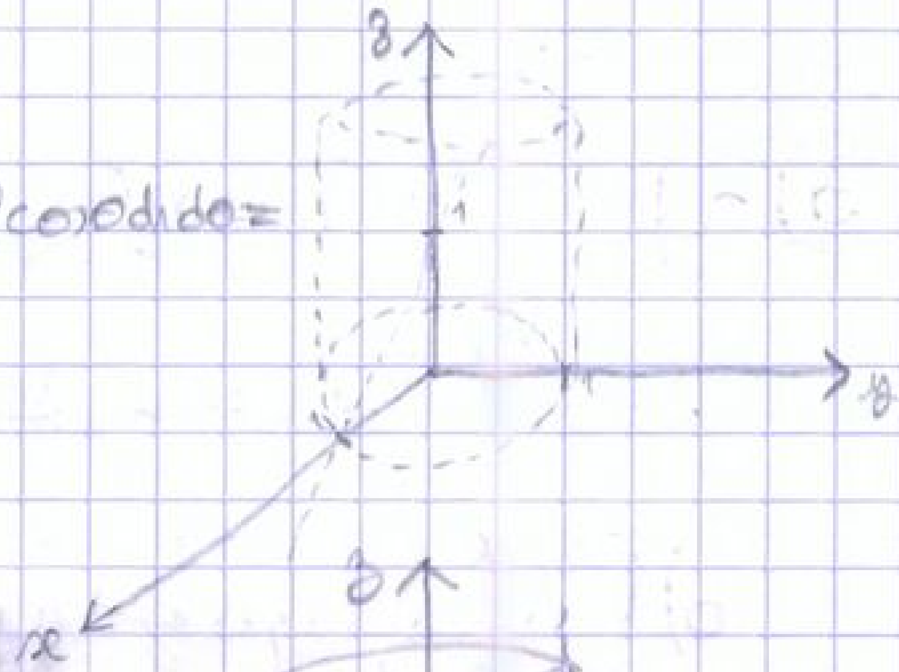
$$= \int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta = \int_0^{\frac{\pi}{2}}\frac{16a^4\cos^4\theta}{4}\,d\theta = \int_0^{\frac{\pi}{2}} 4a^4\cos^4\theta\, d\theta =$$

$$= 4a^4\int_0^{\frac{\pi}{2}}\left[\frac{1+\cos(2\theta)}{2}\right]^2 d\theta = 4a^4\int_0^{\frac{\pi}{2}}\frac{\cos^2(2\theta)+2\cos(2\theta)+1}{4}\,d\theta =$$

$$= a^4\int_0^{\frac{\pi}{2}}\frac{1+\cos(4\theta)}{2}+2\cos(2\theta)+1\, d\theta = a^4\left[\frac{\theta}{2}+\frac{\sin(4\theta)}{8}+\sin(2\theta)+\theta\right]_0^{\frac{\pi}{2}} =$$

$$= a^4\frac{\pi}{4}+a^4\frac{\pi}{2} = \frac{3\pi a^4}{4}$$

24) a) $\int_0^{2\pi}\int_0^1 (1-r\cos\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r - r^2\cos\theta\, dr\, d\theta =$



$$=\int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^3\cos\theta}{3}\right]_0^1 = \int_0^{2\pi}\frac{1}{2} - \frac{\cos\theta}{3}\,d\theta =$$

$$=\left[\frac{1}{2}\theta - \frac{\sin\theta}{3}\right]_0^{2\pi} = \pi$$

b) $\int_0^{2\pi}\int_0^1 r^3\, dr\, d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 d\theta =$

$$=\int_0^{2\pi}\frac{1}{4}\, d\theta = \frac{\theta}{4}\Big|_0^{2\pi} = \frac{2\pi}{4}=\frac{\pi}{2}$$

c) $\int_0^{2\pi}\int_0^2 (4-r^2)\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 4r - r^3\, dr\, d\theta =$ $\quad z=0 \Rightarrow x^2+y^2=4$

$$=\int_0^{2\pi}\left[2r^2-\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 8-4\, d\theta = \int_0^{2\pi}4\,d\theta = 4\theta\Big|_0^{2\pi} = 8\pi$$

$\begin{cases} u = 4-r^2 \\ du = -2r \end{cases}$

d) $\int_0^{2\pi}\int_0^1\sqrt{4-r^2}\,r\,dr\,d\theta = \int_0^{2\pi} -\frac{1}{2}\int_0^1 -2r\sqrt{4-r^2}\,dr\,d\theta =$

$$= -\frac{1}{2}\int_0^{2\pi}\int_0^1\sqrt{u}\,du\,d\theta = -\frac{1}{2}\int_0^{2\pi}\left[\frac{2u^{\frac{3}{2}}}{3}\right]_0 d\theta = -\frac{1}{2}\int_0^{2\pi}\frac{2(4-r^2)^{\frac{3}{2}}}{3}\Big|_0^1 d\theta =$$

$$= -\frac{1}{2}\times\int_0^{2\pi} 2\sqrt{3} - \frac{16}{3}\,d\theta = -\frac{1}{2}\times\left[2\pi\times2\sqrt{3} - 2\pi\times\frac{16}{3}\right] = \frac{16\pi}{3} - 2\pi\sqrt{3} = \frac{2\pi}{3}(8-3\sqrt{3})$$

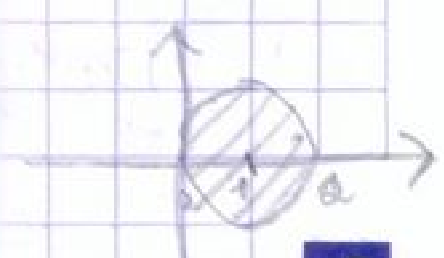
$x^2+y^2=2x$

$(x-1)^2+y^2=1$

$r^2=2r\cos\theta$

$r=2\cos\theta$

e) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}(2r\cos\theta+1)\,r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{2r^3\cos\theta}{3}+\frac{r^2}{2}\right]_0^{2\cos\theta}d\theta =$



$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{16\cos^4\theta}{3}+\frac{4\cos^2\theta}{2}\,d\theta = 3\pi$$