Ficha 5 - Integrais de linha

1) a) $\vec{F}(x,y)=(\overset{M}{2-y},\overset{N}{x})$ $\quad \vec{r}(t)=(t-\sin(t), 1-\cos(t)),\ t\in[0,2\pi]$

$\vec{F}[\vec{r}(t)]=(2-(1-\cos(t)), t-\sin(t))=(1+\cos(t), t-\sin(t))$

$\vec{r}'(t)=(1-\cos(t), \sin(t))$ $\quad \vec{F}[\vec{r}(t)].\vec{r}'(t)=(1+\cos(t), t-\sin(t)).(1-\cos(t),\sin(t))=$

$=1-\cos^2(t)+t\sin(t)-\sin^2(t)=1+t\sin(t)-(\cos^2(t)+\sin^2(t))=t\sin(t)$

$\int_C(2-y)\,dx+x\,dy=\int_0^{2\pi}\vec{F}[\vec{r}(t)].\vec{r}'(t)\,dt=\int_0^{2\pi}t\sin(t)\,dt=$ $\quad \begin{cases}u=t\\ du=dt\end{cases}\begin{cases}v=-\cos(t)\\ dv=\sin(t)\end{cases}$

$=-t\cos(t)\Big|_0^{2\pi}-\int_0^{2\pi}-\cos(t)\,dt=-2\pi+\int_0^{2\pi}\cos(t)\,dt=-2\pi+\sin(t)\Big|_0^{2\pi}=-2\pi$

b) $\vec{F}(x,y)=(2xy, x^2+z, y)$ $\quad \vec{PQ}=Q-P=(5,8,0)-(1,0,2)=(4,8,-2)$

$\vec{r}(t)=(1,0,2)+t(4,8,-2)\Leftrightarrow\vec{r}(t)=(1+4t, 8t, 2-2t), t\in[0,1]$ $\quad \vec{r}'(t)=(4,8,-2)$

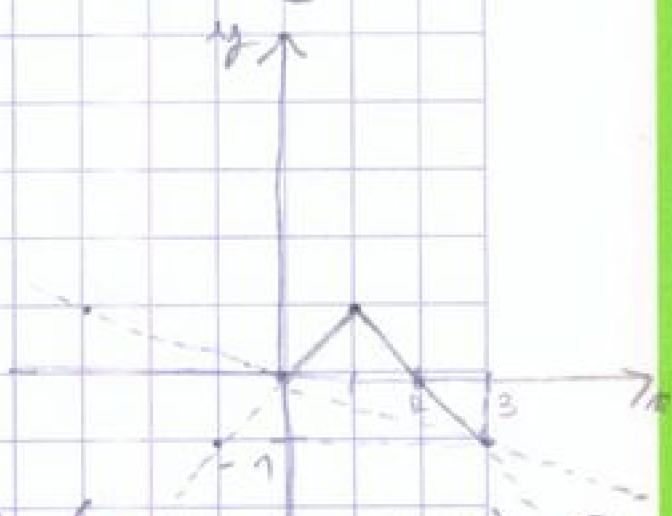
$\vec{F}[\vec{r}(t)]=(2(1+4t)(8t), (1+4t)^2+2-2t, 8t)=(16t+64t^2, 16t^2+6t+3, 8t)$

$\vec{F}[\vec{r}(t)].\vec{r}'(t)=(16t+64t^2, 16t^2+6t+3, 8t).(4,8,-2)=64t+256t^2+128t^2+48t+24-16t=$

$=384t^2+96t+24$ $\quad \int_C 2xy\,dx+(x^2+z)\,dy+y\,dz=\int_0^1\vec{F}[\vec{r}(t)].\vec{r}'(t)\,dt=$

$=\int_0^1 384t^2+96t+24\,dt=\left[\frac{384t^3}{3}+48t^2+24t\right]_0^1=\frac{384}{3}+48+24=\frac{600}{3}=200$

c) $\vec{F}(x,y)=(x^2+y^2, x^2-y^2)$



$\vec{r_1}(t)=(t,t), t\in[0,1]$ $\quad \vec{r_2}(t)=(t, 2-t), t\in[1,3]$

$\vec{r_1}'(t)=(1,1), t\in[0,1]$ $\quad \vec{r_2}'(t)=(1,-1), t\in[1,3]$

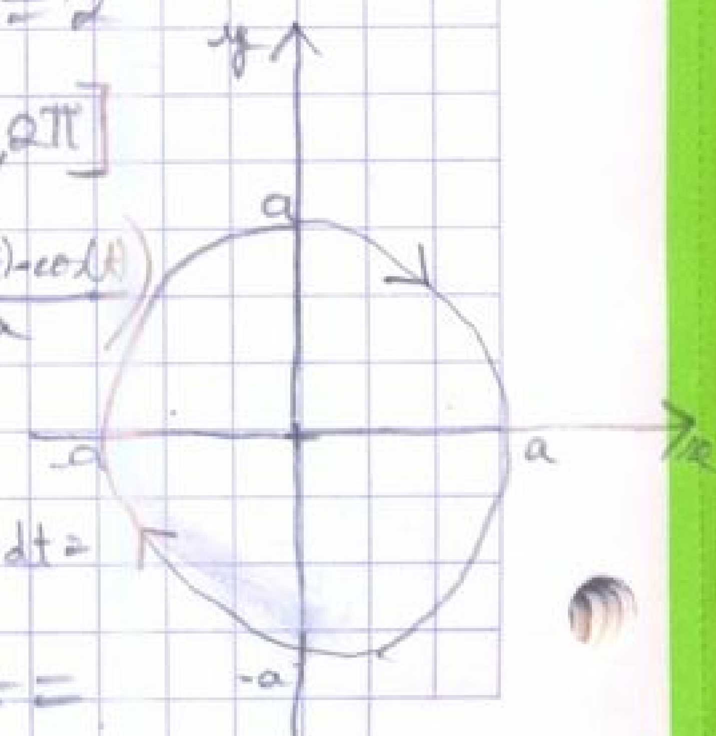
$\vec{F}[\vec{r_1}(t)]=(2t^2,0)$ $\quad \vec{F}[\vec{r_2}(t)]=(t^2+4-4t+t^2, t^2-4+4t-t^2)=(2t^2-4t+4, 4t-4)$

$\vec{F}[\vec{r_1}(t)].\vec{r_1}'(t)=(2t^2,0).(1,1)=2t^2$ $\quad \vec{F}[\vec{r_2}(t)].\vec{r_2}'(t)=(2t^2-4t+4, 4t-4).(1,-1)=$
$=2t^2-4t+4-4t+4=2t^2-8t+8$

$\int_C(x^2+y^2)\,dx+(x^2-y^2)\,dy=\int_0^1 2t^2\,dt+\int_1^3 2t^2-8t+8\,dt=\left[\frac{2t^3}{3}\right]_0^1+\left[\frac{2t^3}{3}-4t^2+8t\right]_1^3=$

$=\frac{2}{3}+\frac{2\times27}{3}-36+24-\frac{2}{3}+4-8=18-36+24+4-8=20-18=2$

d) $\vec{F}=\left(\frac{x+y}{x^2+y^2}, \frac{y-x}{x^2+y^2}\right)$ $\quad \vec{r}(t)=(a\cos(t), a\sin(t)), t\in[0,2\pi]$



$\vec{F}[\vec{r}(t)]=\left(\frac{a\cos(t)+a\sin(t)}{a^2}, \frac{a\sin(t)-a\cos(t)}{a^2}\right)=\left(\frac{\cos(t)+\sin(t)}{a}, \frac{\sin(t)-\cos(t)}{a}\right)$

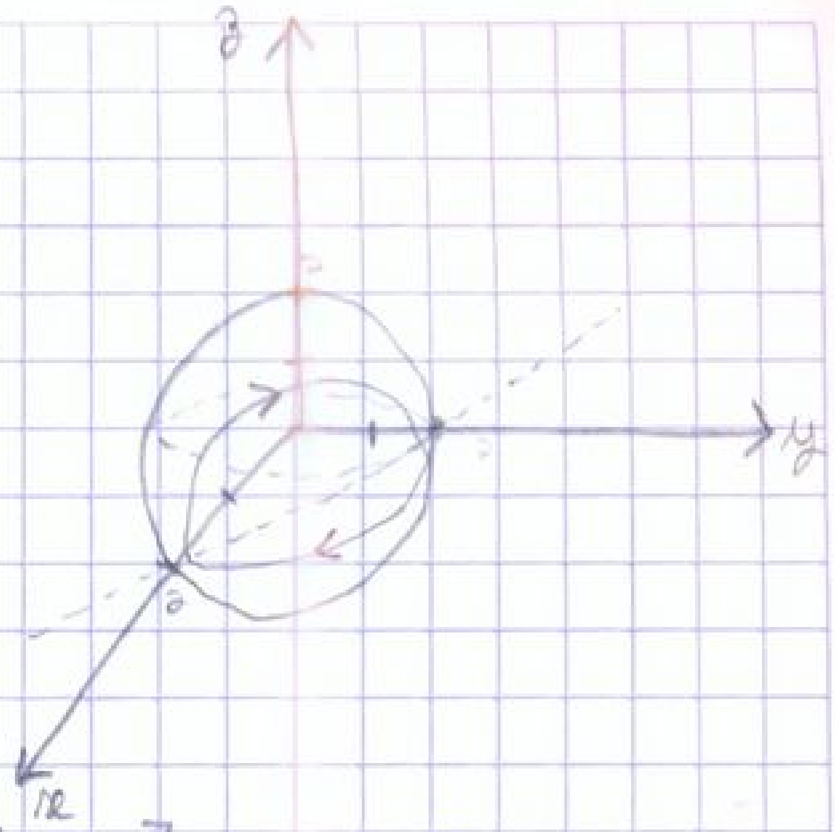
$\vec{r}'(t)=(-a\sin(t), a\cos(t))$

$\int_C\frac{(x+y)\,dx+(y-x)\,dy}{x^2+y^2}=\int_0^{2\pi}\left(\frac{\cos(t)+\sin(t)}{a}, \frac{\sin(t)-\cos(t)}{a}\right).(-a\sin(t), a\cos(t))\,dt=$

$=\int_0^{2\pi}-\sin(t)\cos(t)-\sin^2(t)+\cos(t)\sin(t)-\cos^2(t)\,dt=\int_0^{2\pi}-1\,dt=$

$=-\int_0^{2\pi}1\,dt=-[t]_0^{2\pi}=-2\pi$

e) $\vec{F}=(y,z,x)$ $\quad x+y=2 \Rightarrow y=2-x$



$x^2+(2-x)^2+z^2=4 \Leftrightarrow x^2+4-4x+x^2+z^2=4 \Leftrightarrow$

$\Leftrightarrow 2x^2-4x+z^2=0 \Leftrightarrow x^2-2x+\frac{z^2}{2}=0 \Leftrightarrow$

$\Leftrightarrow (x-1)^2+\frac{z^2}{2}=1 \qquad y=1-\cos(t)$

$\begin{cases} x-1=\cos(t) \\ z=\sqrt{2}\sin(t) \end{cases} \Leftrightarrow \begin{cases} x=1+\cos(t) \\ z=\sqrt{2}\sin(t) \end{cases}$

$\vec{r}(t)=\left(1+\cos(t),1-\cos(t),\sqrt{2}\sin(t)\right),\ t\in[0,2\pi]$

$\vec{F}[\vec{r}(t)]=\left(1-\cos(t),\sqrt{2}\sin(t),1+\cos(t)\right),t\in[0,2\pi]$

$\vec{r}'(t)=\left(-\sin(t),\sin(t),\sqrt{2}\cos(t)\right),\ t\in[0,2\pi]$

$\vec{F}[\vec{r}(t)].\vec{r}'(t)=\left(1-\cos(t),\sqrt{2}\sin(t),1+\cos(t)\right).\left(-\sin(t),\sin(t),\sqrt{2}\cos(t)\right)=$

$=-\sin(t)+\sin(t)\cos(t)+\sqrt{2}\sin^2(t)+\sqrt{2}\cos(t)+\sqrt{2}\cos^2(t)=$

$=\sqrt{2}+\sqrt{2}\cos(t)-\sin(t)+\sin(t)\cos(t)$ (teorema da ortogonalidade)

$\int_C y\,dx+z\,dy+x\,dz=\int_0^{2\pi}\sqrt{2}+\sqrt{2}\cos(t)-\sin(t)+\cancel{\sin t\cos(t)}\,dt=$

$=\left[\sqrt{2}t+\sqrt{2}\sin(t)+\cos(t)\right]_0^{2\pi}=2\pi\sqrt{2}+1-(1)=2\sqrt{2}\pi$

b) $\vec{r}_1(t)=(t,1),t\in[-1,1]$ $\quad \vec{r}_2(t)=(-1,t),t\in[-1,1]$

$\vec{r}_3(t)=(t,-1),t\in[-1,1]$ $\quad \vec{r}_4(t)=(1,t),t\in[-1,1]$