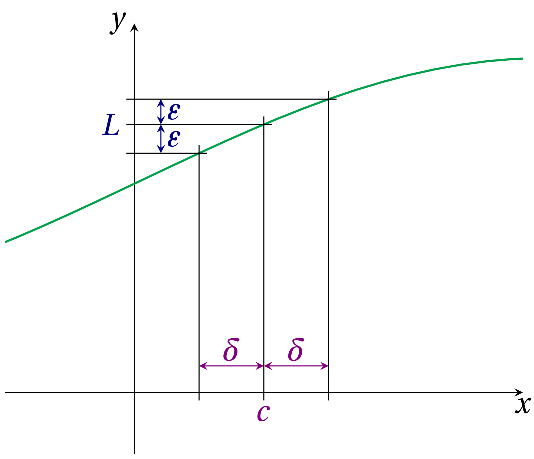
Chapter 1

Introduction to Topology

1.1 Continuity

Definition 1.1.1. The function $f:\mathbb{R}\to\mathbb{R}$ is continuous at a point $x\in\mathbb{R}$ if x is sufficiently closed to x, and $f(x)$ is sufficiently closed to $f(a)$, that is

$$\text{If } x\in(a-\delta,\, a+\delta), \quad \text{then } f(x)\in\big(f(a)-\varepsilon, f(a)+\varepsilon\big)$$



Theorem 1.1.1. The function $f:\mathbb{R}$ is continuous (you can check by $\delta-\varepsilon$ definition) if and only if the subset $S\subset\mathbb{R}$ and its preimage $f^{-1}(S)$ are both open.

Example 1.1.2. Let $f:\mathbb{R}\to\mathbb{R}$ be the function $f(x)=4x^2-4x$. Compute the following preimages

$$f^{-1}\{5\},\ f^{-1}(8,24),$$

For $f^{-1}\{5\}$ (this is a fixed point), we need to solve the equation $4x^2 - 4x = 5 \Rightarrow 4x^2 - 4x - 5 = 0$.

$$\begin{aligned} 4x^2 - 4x - 5 &= 0 \\ x &= \frac{-(-4) \pm \sqrt{16 + 4(4 \times 5)}}{2(4)} \\ &= \frac{1 \pm \sqrt{6}}{2} \end{aligned}$$

the results yield to $f^{-1}\{5\} = \left\{ \frac{1-\sqrt{6}}{2}, \frac{1+\sqrt{6}}{2} \right\}$.

$$\begin{gathered} 8 < 4x^2 - 4x < 24 \\ 4x^2 - 4x > 8 \quad \text{and} \quad 4x^2 - 4x < 24 \\ x^2 - x - 2 > 0 \qquad x^2 - x - 6 < 0 \\ (x-2)(x+1) > 0 \qquad (x-3)(x+2) < 0 \end{gathered}$$

1.2 Homeomorphism

A homeomorphism is a map between two topological spaces that preserves some properties.

Definition 1.2.1. The two topological spaces (X, τ_X), (Y, τ_Y) are homeomorphic if they are continuous mapping $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$, where 1_X and 1_Y are both indentity functions such that $1_X : X \to X$ and $1_Y : Y \to Y$.

Example 1.2.2. The surface of a cube is homeomorphic to a sphere on the same dimension.

Example 1.2.3. $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \to \mathbb{R}$ is a homeomorphism.

Example 1.2.4. Let S^n denotes the n-dimensional unit sphere, in mathematical way, it can be written as

$$S^n = \left\{ (x_0, x_1, \cdots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=0}^{n} x_i^2 = x_0^2 + x_1^2 + \cdots + x_n^2 = 1 \right\}$$

Let $N = (1, 0, \cdots, 0)$ denote the north pole of the n-dimensional unit sphere, then $S^n \setminus \{N\}$ will be homeomorphic to $\mathbb{R}^n$.

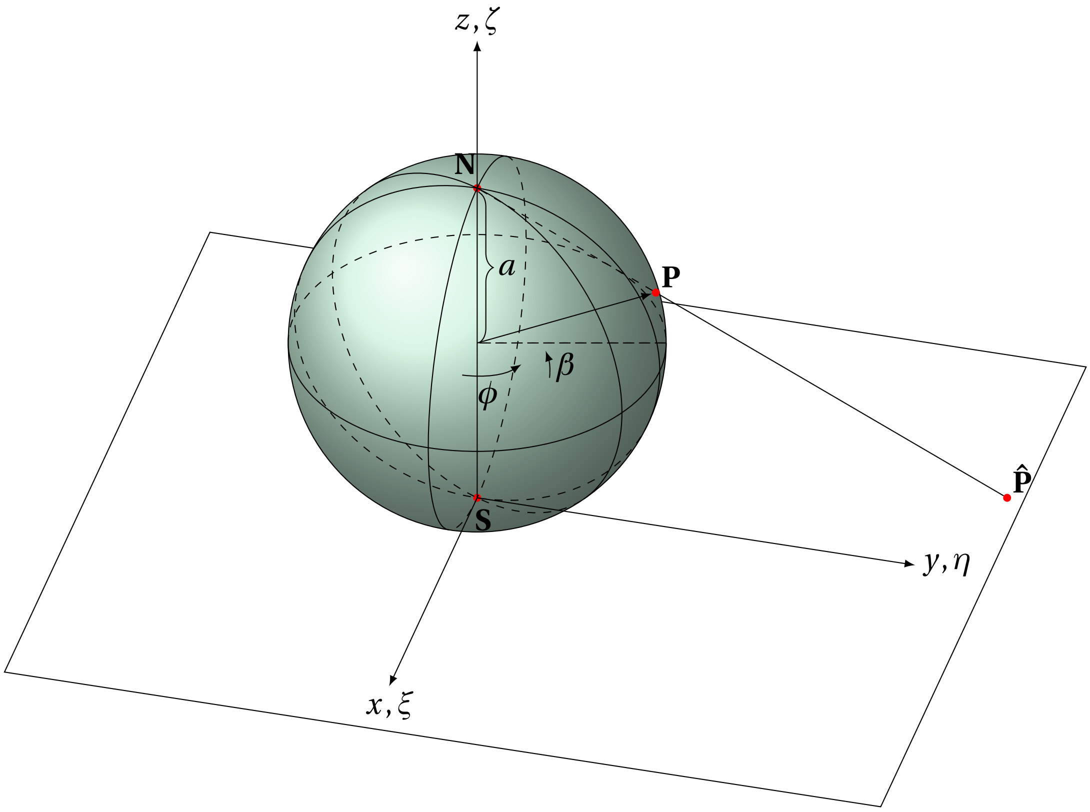


Figure 1.1: Source: http://gallery.bridgesmathart.org/exhibitions/2016-joint-mathematics-meetings/henrys

By defining a mapping $f: S^n \setminus \{N\} \to \mathbb{R}^n$, and there has another preimage function $g: \mathbb{R}^n \to S^n \setminus \{N\}$ which are given by

$$f(x_0, x_1, \cdots, x_n) = \frac{1}{1 - x_0}(x_1, x_2, \cdots, x_n)$$

and

$$g(y_1, y_2, \cdots, y_n) = \frac{1}{1+|y|^2}(|y|^2 - 1, 2y_1, \cdots, 2y_n)$$

where $|y|^2 = y_1^2 + y_2^2 + \cdots + y_n^2$.

Certainly, since $x_0 \neq 1$ (you had already taken away the north pole), for all $(x_0, x_1, \cdots, x_n) \in S^n \setminus \{N\}$, f is well-defined and continuous function. For all $(y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, we have

$$\begin{aligned}
(|y|^2 - 1)^2 + 4y_1^2 + \cdots + 4y_n^2 &= |y|^4 - 2|y|^2 + 1 + 4(y_1^2 + \cdots + y_n^2) \\
&= |y|^4 - 2|y|^2 + 1 + 4|y|^2 \\
&= |y|^4 + 2|y|^2 + 1 \\
&= (|y|^2 + 1)^2
\end{aligned}$$

implies that g is also well-defined and continuous function.

Now, in order to identify whether $S^n \setminus \{N\}$ is homeomorphic to $\mathbb{R}^n$, we need to check if $f \circ g = 1_X$ and $g \circ f = 1_Y$ (the $\circ$ here is function composition, not multiplication). In another words, this is equivalent to show that the functions f and g are mutually inverse.

$$\begin{aligned}
g\big(f(x_0, x_1, \cdots, x_n)\big) &= g\left(\frac{1}{1-x_0}(x_1, x_2, \cdots, x_n)\right) \\
&= \frac{1}{1 + \dfrac{x_0^2 + x_1^2 + \cdots + x_n^2}{(1-x_0)^2}} \left(\frac{x_1^2 + \cdots x_n^2}{(1-x_0)^2} - 1, \frac{2x_1}{1-x_0}, \cdots, \frac{2x_n}{1-x_0}\right) \\
&= \frac{(1-x_0)^2}{2-2x_0}\left(\frac{1-x_0^2}{(1-x_0)^2} - 1, \frac{2x_1}{1-x_0}, \cdots, \frac{2x_n}{1-x_0}\right) \\
&= \frac{1}{2}(1 + x_0 - 1 + x_0, 2x_1, \cdots, 2x_n) \\
&= (x_0, x_1, \cdots, x_n) \\
&= 1_X
\end{aligned}$$

where $1_X : (x_0, x_1, \cdots, x_n) \to (x_0, x_1, \cdots, x_n)$ is an identity mapping.

On the other hand, for any $(y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, we need to compute

$$\begin{aligned}
f\big(g(y_1, y_2, \cdots, y_n)\big) &= f\left(\frac{1}{1+|y|^2}(|y|^2 - 1, 2y_1, \cdots, 2y_n)\right) \\
&= \frac{1}{1 - \dfrac{|y|^2 - 1}{|y|^2 + 1}}\left(\frac{2y_1}{|y|^2 + 1}, \cdots, \frac{2y_n}{|y|^2 + 1}\right) \\
&= (y_1, \cdots, y_n) \\
&= 1_Y
\end{aligned}$$

Non-example 1.2.5. The cartesian plane $\mathbb{R}^2$ is not homeomorphic to the real line $\mathbb{R}$.

Theorem 1.2.1. There is no continuous surjective map $\mathbb{R} \to S^0$.

1.3 Hausdorff Property

Definition 1.3.1. A topological space X is Hausdorff if for any distinct points $x, y \in X$, there are two open subsets $U, V \subset X$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.

Theorem 1.3.1 (Weierstrass Intermediate Value Theorem)**.** *content...*

Theorem 1.3.2 (Intermediate Value Theorem)**.** If $f : [a, b] \to \mathbb{R}$ is a continuous function and $f(a) < 0$, $f(b) > 0$, then there is some value $x \in [a, b]$ such that $f(x) = 0$.