

Firn Densification Model

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1 Introduction

The top layer of snow on a glacier or ice sheet increases in density as depth increases. Several models have been created to simulate this process, some based on temperature and others based on enthalpy. I have re-created these models with the finite-element software package FEniCS.

2 Temperature Solution

We begin with the standard heat-transport equation (Patterson, 2001)

$$\rho c_i \frac{\partial T}{\partial t} = k_i \frac{\partial^2 T}{\partial z^2} + \left(\frac{dk}{dt} - \rho c_i w \right) \frac{\partial T}{\partial z}$$

with heat sources from the deformation of ice omitted, ρ density, c_i heat capacity, k_i thermal conductivity, w vertical velocity, and T temperature of firm. To solve the total derivative dk/dt we must apply the chain rule

$$\frac{dk_i}{dz} = \frac{\partial k_i}{\partial \rho} \frac{\partial \rho}{\partial z} + \frac{\partial k_i}{\partial T} \frac{\partial T}{\partial z}.$$

The thermal conductivity of ice is defined by Arthern et. al, 1998 as

$$k_i = 2.1 \left(\frac{\rho}{\rho_i} \right)^2,$$

and gives

$$\frac{\partial k_i}{\partial \rho} = 4.2 \frac{\rho}{\rho_i^2}$$

and

$$\frac{\partial k_i}{\partial T} = \frac{4.2}{\rho_i^2} \left(\frac{\partial \rho}{\partial T} \right).$$

Patterson, 2001 defined ρ in terms of T and from this can be derived

$$\frac{\partial \rho}{\partial T} = 5.6 \times 10^{-2} \exp((-5.7 \times 10^{-3})T).$$

The vertical velocity of ice, w , is directly proportional to the accumulation, b ,

$$w = -\frac{b}{\rho}$$

with b in units of $kg m^{-2} s^{-1}$. The densification process is defined with the material derivative

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + w \frac{\partial \rho}{\partial z}.$$

Arthern et. al, 2010 described the total derivative with the formula

$$\frac{d\rho}{dt} = \begin{cases} c_0(\rho_i - \rho), & \rho \leq 550 \text{ kg m}^{-3} \\ c_1(\rho_i - \rho), & \rho > 550 \text{ kg m}^{-3} \end{cases}$$

Zwally and Li, 2002 defined the multiplying constant c with an arrhenius-type relation

$$c_0 = c_1 = b\rho_i/\rho_w\beta(T)K_{0G}(T)\exp\left(-\frac{E(T)}{RT}\right),$$

with $K_{0G}(T)\exp(-E(T)/(RT)) = 8.36T^{-2.061}$, and $\beta(T)$ a smoothing function to match a desired density rate. Arthern et. al 2010 developed a semi-empirical formula by coupling the rate equations for Nabarro-Herring creep and normal grain-growth :

$$\begin{cases} c_0 = M_0 b g \frac{k_{c0}}{k_g} \exp\left(-\frac{E_c}{RT} + \frac{E_g}{RT_{avg}}\right) \\ c_1 = M_1 b g \frac{k_{c1}}{k_g} \exp\left(-\frac{E_c}{RT} + \frac{E_g}{RT_{avg}}\right) \end{cases},$$

with the creep coefficients defined as

$$\begin{cases} k_{c0} = 9.2 \times 10^{-9} \text{ m}^3 \text{ s kg}^{-1} \\ k_{c1} = 3.7 \times 10^{-9} \text{ m}^3 \text{ s kg}^{-1} \end{cases}$$

and M defined by Ligtenberg et. al 2011 to better fit with observed densification rates in higher-temperature environments :

$$\begin{cases} M_0 = 2.366 - 0.293 \ln(b * 1 \times 10^3) \\ M_1 = 1.435 - 0.151 \ln(b * 1 \times 10^3) \end{cases}.$$

Ligtenberg et. al 2011 developed a firm surface density expression from data :

$$\rho_s = -151.94 + 1.4266(73.6 + 1.06T_s + 0.0669A + 4.77V_a).$$

3 Enthalpy Solution

As stated in Aschwanden et. al 2012, we take 'enthalpy' to be synonymous with 'internal energy' due to the exclusion of work done with changing volume. The equation used here is the shallow-enthalpy :

$$\rho \frac{\partial H}{\partial t} = \frac{\partial}{\partial z} \left(\left\{ \begin{matrix} K_i(H), & \text{Temperate} \\ K_0, & \text{Cold} \end{matrix} \right\} \frac{\partial H}{\partial z} \right) + w\rho \frac{\partial H}{\partial z}.$$

Strain heating has been neglected and the advective term $w\rho \partial H/\partial z$ has been added. The coefficient for temperate ice is

$$K_i(H) = \frac{k_i}{c_i}.$$

The coefficient for cold ice is

$$K_0 = \frac{1}{10} K_i(H).$$

Temperate firm is defined as firm with $H > H_s$ and cold firm $H \leq H_s$, with

$$H_s = \int_{T_0}^{T_m} c_i(T) dT$$

where $T_m = 273.15$ K and $T_0 = 0.0$ K. The enthalpy can be found with a constant heat capacity of $2009 \text{ J kg}^{-1} \text{ K}^{-1}$ with the linear equation

$$H = \begin{cases} c_i(T - T_0), & \text{where } T \leq T_m \\ c_i(T_w - T_0) + \omega L_f, & \text{where } T > T_m \end{cases}$$

where L_f is the latent heat of fusion and ω represents the water content percentage of firn and is given by

$$\omega L_f = H - c_i(T_w - T_0).$$

It is convenient to find the temperature :

$$T = \frac{H}{c}.$$

The density equations remain the same as the temperature solution. The surface density can be described as :

$$\rho_s^{n+1} = \rho_s^n d_p + \rho_s^{n-1}(1 - d_p),$$

where

$$\rho_s^n = \rho_s^{n-1} + d\omega_s \rho_b,$$

$$d\omega_s = \omega_s^n - \omega_s^{n-1},$$

$$d_p = \frac{d}{l_s},$$

$$d = \omega_s dt,$$

l_s is the length of the surface node, and

$$\rho_b = \begin{cases} 300 \text{ kg m}^{-3}, & T > T_w \\ 1000 \text{ kg m}^{-3}, & T \leq T_w \end{cases}.$$

4 Finite Element Method

This section will focus on the enthalpy solution; solving these equations with FENiCS is done with Galerkin's method which requires finding the weak formulation :

$$0 = \int_{\Omega} \left\{ \begin{matrix} K_i(H) \\ K_0 \end{matrix} \right\} \nabla^2 H \psi \, d\Omega + \int_{\Omega} w \rho \nabla H \psi \, d\Omega - \int_{\Omega} \rho \frac{\partial H}{\partial t} \psi \, d\Omega.$$

Here we have integrated over the entire domain, Ω , and multiplied by a test function ψ . After integrating the diffusive term by parts this becomes

$$f_H = - \int_{\Omega} \left\{ \begin{matrix} K_i(H) \\ K_0 \end{matrix} \right\} \nabla H \nabla \psi \, d\Omega + \int_{\Omega} w \rho \nabla H \psi \, d\Omega - \int_{\Omega} \rho \frac{\partial H}{\partial t} \psi \, d\Omega.$$

We can discretize the enthalpy time-differential with the second-order accurate backward-difference formula

$$\frac{\partial H}{\partial t} = \frac{H^{k-2} - 4H^{k-1} + 3H^k}{2\Delta t},$$

with superscripts referring to time index. This equation can be represented in FENiCS as :

$$\begin{aligned} f_H = & (\text{rho} * (H_2 - 4 * H_1 + 3 * H) / (2 * dt) * \text{psi} \\ & + k / c * K\text{coef} * \text{inner}(\text{grad}(H), \text{grad}(\text{psi})) \\ & + \text{rho} * w * \text{grad}(H) * \text{psi}) * dx \end{aligned}$$

The variable $K\text{coef}$ is a coefficient vector which will be updated dynamically depending on the temperature of firn (either 1.0 or 0.1).

The weak form for density is found similarly :

$$f_{\rho} = \int_{\Omega} \frac{\partial \rho}{\partial t} \phi \, d\Omega + \int_{\Omega} w \nabla \rho \phi \, d\Omega - \int_{\Omega} \frac{d\rho}{dt} \phi \, d\Omega.$$

With the partial-time-differential of density defined identically to the enthalpy equation, this can be represented in FENiCS with the Arthern et. al 2010 densification equation as :

$$\begin{aligned} c &= b * g * \text{rhoCoef} / \text{kg} * \\ &\quad \exp(-E_c / (R * T) + E_g / (R * T_{\text{avg}})) \\ \text{drhodt} &= c * (\text{rho}_i - \text{rho}) \\ f_rho &= ((\text{rho}_2 - 4 * \text{rho}_1 + 3 * \text{rho}) / (2 * dt) \\ &\quad - \text{drhodt} + w * \text{grad}(\text{rho})) * \text{phi} * dx \end{aligned}$$

The variable rhoCoef is another dynamically-updated-coefficient vector and is either k_{c0} or k_{c1} depending upon the density at the node.

We can define the function space for the entire non-linear problem as

$$U = \Omega \times \Omega,$$

with corresponding trial and test functions respectively defined as

$$v, u \subset U.$$

The test functions for each function can now be described as

$$\psi, \phi \subset u.$$

In FENiCS these spaces can be defined by this :

$$\begin{aligned} V &= \text{FunctionSpace}(\text{mesh}, 'Lagrange', 1) \\ MV &= V * V \\ h &= \text{Function}(MV) \\ H, \text{rho} &= \text{split}(h) \\ dh &= \text{TrialFunction}(MV) \\ dH, \text{drho} &= \text{split}(dh) \\ j &= \text{TestFunction}(MV) \\ \text{psi}, \text{phi} &= \text{split}(j) \end{aligned}$$

We define the function as

$$f = f_H + f_{\rho}.$$

Solving this system can be accomplished by *Newton's Method* and requires the derivation of the Jacobian :

$$J = \frac{\partial f}{\partial U}$$

In FENiCS this becomes :

```
f = f_H + f_rho
J = derivative(f, h, dh)
```

Now all that is left is to iterate through time and call the `solve` method :

```
solve(f == 0, h, [Hbc, Dbc], J=J)
```

$$l_{n+1} = l_n \frac{\rho_{ini}}{\rho}$$

$$s_n = (z_s - z_b) \frac{s_{n-1} - z_b}{z_s - z_b} + w_s dt$$

5 Verification of Program

Using the values given in the previous two sections results in these figures :

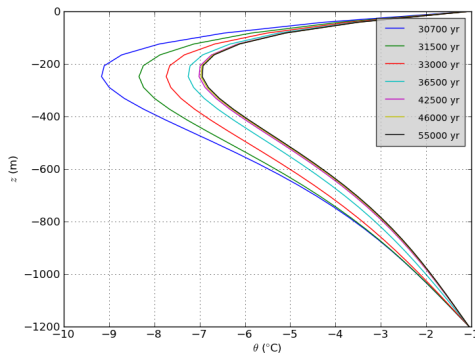


Figure 1: Temperature of Ice after Biasing the Temperature by $+15^{\circ}\text{C}$

6 Interpretation