

CHAPTER 1

Algebra

1.1 Basic Properties & Facts

Arithmetic Operations

$$ab + ac = a(b + c)$$

$$a \left(\frac{b}{c} \right) = \frac{ab}{c}$$

$$\frac{\left(\frac{a}{b} \right)}{c} = \frac{a}{bc}$$

$$\frac{a}{\left(\frac{b}{c} \right)} = \frac{ac}{b}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

$$\frac{a - b}{c - d} = \frac{b - a}{d - c}$$

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{ab + ac}{a} = b + c, \quad a \neq 0$$

$$\frac{\left(\frac{a}{b} \right)}{\left(\frac{c}{d} \right)} = \frac{ad}{bc}$$

Exponent Properties

$$a^n a^m = a^{n+m}$$

$$\frac{a^n}{a^m} = a^{n-m} = \frac{1}{a^{m-n}}$$

$$(a^n)^m = a^{nm}$$

$$a^0 = 1, a \neq 0$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$a^{-n} = \frac{1}{a^n}$$

$$\frac{1}{a^{-n}} = a^n$$

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$$

$$a^{\frac{n}{m}} = \left(a^{\frac{1}{m}}\right)^n = (a^n)^{\frac{1}{m}}$$

Properties of Radicals

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

$$\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}$$

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[n]{a^n} = a, \text{ if } n \text{ is odd}$$

$$\sqrt[n]{a^n} = |a|, \text{ if } n \text{ is even}$$

Properties of Inequalities

If $a < b$ then $a + c < b + c$ and $a - c < b - c$.

If $a < b$ and $c > 0$ then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

If $a < b$ and $c < 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

Properties of Absolute Value

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

$$|a| \geq 0$$

$$|-a| = |a|$$

$$|ab| = |a||b|$$

$$\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$$

$$|a + b| \leq |a| + |b| \text{ Triangle Inequality}$$

Distance Formula If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are two points, the distance between them is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Complex Numbers

$$i = \sqrt{-1}$$

$$i^2 = -1$$

$$\sqrt{-a} = i\sqrt{a}, \quad a \geq 0$$

$$(a + bi) + (c + di) = a + c + (b + d)i$$

$$(a + bi) - (c + di) = a - c + (b - d)i$$

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

$$(a + bi)(c - di) = a^2 + b^2$$

$$|a + bi| = \sqrt{a^2 + b^2} \text{ Complex Modulus}$$

$$(a + bi) = a - bi \text{ Complex Conjugate}$$

$$(a + bi)(a + bi) = |a + bi|^2$$

Logarithms and Log Properties

Definition

$$y = \log_b x \quad \equiv \quad x = b^y$$

Special Logarithms

$$\begin{aligned}\ln x &= \log_e x \text{ natural log} \\ \log x &= \log_{10} x \text{ common log} \\ e &= 2.718281828\end{aligned}$$

Logarithm Properties

$$\begin{aligned}\log_b b &= 1 \\ \log_b 1 &= 0 \\ \log_b b^x &= x \\ b^{\log_b x} &= x \\ \log_b(x^r) &= r \log_b x \\ \log_b(xy) &= \log_b x + \log_b y \\ \log_b\left(\frac{x}{y}\right) &= \log_b x - \log_b y\end{aligned}$$

The domain of $\log_b x$ is $x > 0$.

1.2 Factoring & Solving

Factoring Formulas

$$\begin{aligned}x^2 - a^2 &= (x + a)(x - a) \\ x^2 + 2ax + a^2 &= (x + a)^2 \\ x^2 - 2ax + a^2 &= (x - a)^2 \\ x^2 + (a + b)x + ab &= (x + a)(x + b) \\ x^3 + 3ax^2 + 3a^2x + a^3 &= (x + a)^3 \\ x^3 - 3ax^2 + 3a^2x - a^3 &= (x - a)^3 \\ x^3 + a^3 &= (x + a)(x^2 - ax + a^2) \\ x^3 - a^3 &= (x - a)(x^2 - ax + a^2) \\ x^{2n} - a^{2n} &= (x^n - a^n)(x^n + a^n)\end{aligned}$$

If n is odd then

$$\begin{aligned}x^n - a^n &= (x - a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-1}) \\ x^n + a^n &= (x + a)(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \cdots + a^{n-1})\end{aligned}$$

Quadratic Formula

Solve $ax^2 + bx + c = 0$, $a \neq 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac > 0$ Two real unequal solutions.

If $b^2 - 4ac = 0$ Repeated real solution.

If $b^2 - 4ac < 0$ Two complex solutions.

Square Root Property

If $x^2 = p$ then $x = \pm\sqrt{p}$.

Absolute Value Equations/Inequalities

If b is a positive number

$$|p| = b \implies p = -b \text{ or } p = b$$

$$|p| < b \implies -b < p < b$$

$$|p| > b \implies p < -b \text{ or } p > b$$

1.3 Functions & Graphs

Constant Function

Given

$$y = a \quad \text{or} \quad f(x) = a$$

The graph is a horizontal line passing through the point $(0, a)$.

Line/Linear Function

Given

$$y = mx + b \quad \text{or} \quad f(x) = mx + b$$

The graph is a line with point $(0, b)$ and slope m .

Slope

Slope of the line containing the two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

Slope-intercept form

The equation of the line with slope m and y -intercept $(0, b)$ is

$$y = mx + b$$

Point-Slope form

The equation of the line with slope m and passing through the point (x_1, y_1) is

$$y = y_1 + m(x - x_1)$$

Parabola/Quadratic Function

Case 1

$$y = a(x - h)^2 + k \quad \text{or} \quad f(x) = a(x - h)^2 + k$$

The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at (h, k) .

Case 2

$$y = ax^2 + bx + c \quad \text{or} \quad f(x) = ax^2 + bx + c$$

The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Case 3

$$x = ay^2 + by + c \quad \text{or} \quad g(y) = ay^2 + by + c$$

The graph is a parabola that opens right if $a > 0$ or left if $a < 0$ and has a vertex at $\left(g\left(-\frac{b}{2a}\right), -\frac{b}{2a}\right)$.

Circle

Equation:

$$(x - h)^2 + (y - k)^2 = r^2$$

Graph: circle with radius r and center (r, k) .

Ellipse

Equation:

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

Graph: ellipse with center (h, k) with vertices a units right/left from the center and vertices b units up/down from the center.

Hyperbola

Case 1

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

Graph: hyperbola that opens left and right, has a center at (h, k) , vertices a units left/right of center and asymptotes that pass through center with slope $\pm \frac{b}{a}$.

Case 2

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Graph: hyperbola that opens up and down, has a center at (h, k) , vertices b units up/down from the center and asymptotes that pass through center with slope $\pm \frac{b}{a}$.

1.4 Common Algebraic Errors

Error	Reason/Justification/Example
$\frac{2}{0} \neq 0$ and $\frac{2}{0} \neq 2$	Division by zero is undefined!
$-3^2 \neq 9$	$-3^2 = -9, (-3)^2 = 9$. Watch parenthesis!
$(x^2)^3 \neq x^5$	$(x^2)^3 = x^2 x^2 x^2 = x^6$
$\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$	$\frac{1}{2} = \frac{1}{1+1} \neq \frac{1}{1} + \frac{1}{1} = 2$
$\frac{1}{x^2+x^3} \neq x^{-2} + x^{-3}$	This is a more complex version of the previous error.
$\frac{a+bx}{a} \neq 1 + bx$	$\frac{a+bx}{a} = \frac{a}{a} + \frac{bx}{a} = 1 + \frac{bx}{a}$ Beware of incorrect canceling!
$-a(x-1) \neq -ax - a$	$-a(x-1) = -ax + a$ Make sure you distribute the “-”!
$(x+a)^2 \neq x^2 + a^2$	$(x+a)^2 = (x+a)(x+a)$ $= x^2 + 2ax + a^2$
$\sqrt{x^2 + a^2} \neq x + a$	$5 = \sqrt{25} = \sqrt{3^2 + 4^2}$ $\neq \sqrt{3} + \sqrt{4} = 3 + 4 = 7$
$\sqrt{x+a} \neq \sqrt{x} + \sqrt{a}$	See previous error.

Error

$(x + a)^n \neq x^n + a^n$ and,

$$\sqrt[n]{x + a} \neq \sqrt[n]{x} + \sqrt[n]{a}$$

$$2(x + 1)^2 \neq (2x + 2)^2$$

$$(2x + 2) \neq 2(x + 1)^2$$

$$\sqrt{-x^2 + a^2} \neq -\sqrt{x^2 + a^2}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} \neq \frac{ab}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{c} \neq \frac{ac}{b}$$

Reason/Justification/Example

More general versions of previous three errors.

$$\begin{aligned} 2(x + 1)^2 &= 2(x^2 + 2x + 1) \\ &= 2x^2 + 4x + 2, \\ (2x + 2)^2 &= 4x^2 + 8x + 4. \end{aligned}$$

Square first, then distribute!

See previous example. You cannot factor out a constant if there is a power on the parenthesis!

$$\sqrt{-x^2 + a^2} = (-x^2 + a^2)^{\frac{1}{2}}$$

Now see the previous error.

$$\frac{\frac{a}{b}}{\left(\frac{c}{d}\right)} = \frac{\left(\frac{a}{1}\right)}{\left(\frac{b}{c}\right)} = \left(\frac{a}{1}\right) \left(\frac{c}{b}\right) = \frac{ac}{b}$$

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{1}\right)} = \left(\frac{a}{b}\right) \left(\frac{1}{c}\right) = \frac{a}{bc}$$