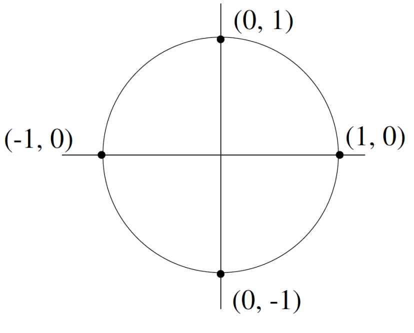
CHAPTER 1

Trigonometry

In this chapter, we review the basic rules and procedures of trigonometry so that you can be successful in calculus. In machine learning, you will find that many algorithms, e.g., clustering algorithms, use distance equations to define groups of data points or clusters. Therefore a basic understanding of trigonometry is necessary.

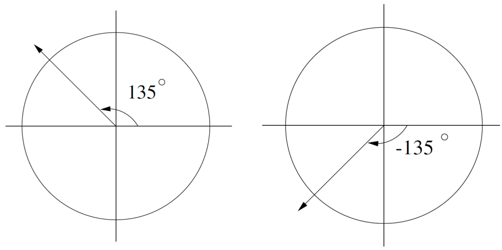
1.1 The Unit Circle

The first key to understanding trigonometry is to know the unit circle. The **unit circle** is the circle centered at $(0,0)$ with radius 1.

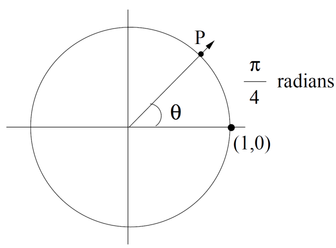


Consider an angle θ in the unit circle. The angle is positive if it is measured counterclockwise from the positive x-axis and negative if it is measured clockwise.

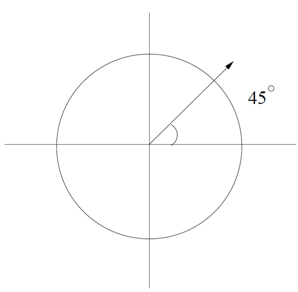
The above angles are measured using **degrees**. An angle θ may also be measured using **radians**. The radian measurement corresponds to a distance around the unit circle's circumference, C, where $C = 2\pi$.



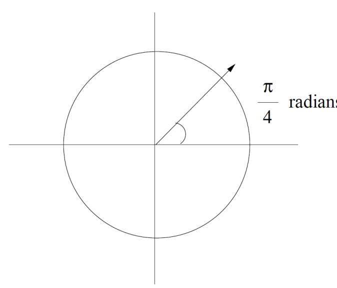
Let us measure an arc on the unit circle starting at $(1, 0)$ of length $\frac{\pi}{4}$ and ending at a point P. If we draw a ray from the origin through point P, we have formed an angle θ, where $\theta = \frac{\pi}{4}$ radians. The following two angles



are the same



or,



Definition 1.1 (Radians and Degrees). *To convert radians to degrees and vice versa, use the following equation*

$$\pi \ \textit{radians} = 180^\circ$$

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Example 1.1. Convert 30° to radians.

$$\begin{aligned}
\pi \text{ radians} &= 180^\circ \\
\frac{\pi}{180} \text{ radians} &= 1^\circ \\
30 \cdot \frac{\pi}{180} \text{ radians} &= 30 \cdot 1^\circ \\
\frac{\pi}{6} \text{ radians} &= 30^\circ
\end{aligned}$$

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Example 1.2. Convert $\frac{8\pi}{5}$ radians to degrees.

$$\begin{aligned}
\pi \text{ radians} &= 180^\circ \\
\frac{8}{5}\pi \text{ radians} &= \frac{8}{5} \cdot 180^\circ \\
\frac{8\pi}{5} \text{ radians} &= 288^\circ
\end{aligned}$$

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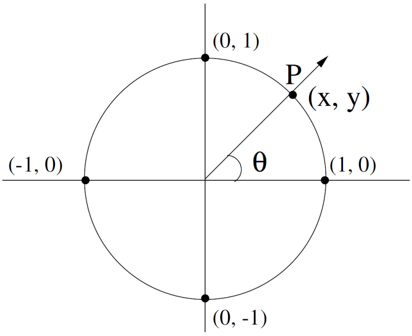
1.2 Trigonometric Functions

Consider the point $P = (x, y)$ where the angle of measure θ intersects the unit circle. We use the coordinates x and y of this point to define **six trigonometric** functions of θ.

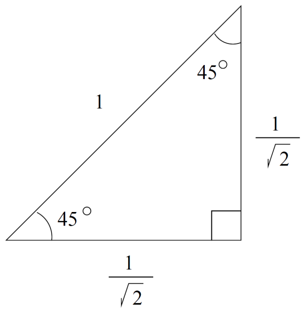


We define the **cosine** of θ to be the x-coordinate of this point and the **sine** of θ to be the y-coordinate. We use the abbreviations "cos" for cosine and "sin" for sine. Thus

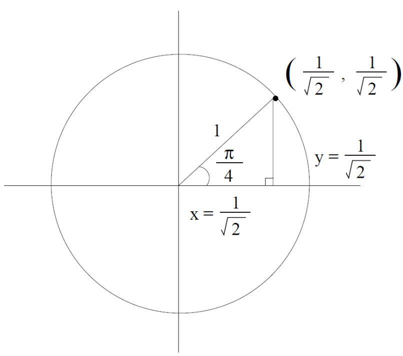
$$-1 \leq \cos\theta \leq 1 \quad \text{and} \quad -1 \leq \sin\theta \leq 1$$

no matter what θ is.

We can use geometry to determine $\sin\frac{\pi}{4}$ and $\cos\frac{\pi}{4}$. Since $\frac{\pi}{4}$ radians equals 45°, consider a 45° – 45° – 90° triangle with a hypotenuse of length 1. Such a triangle must have legs each of length $\frac{1}{\sqrt{2}}$.



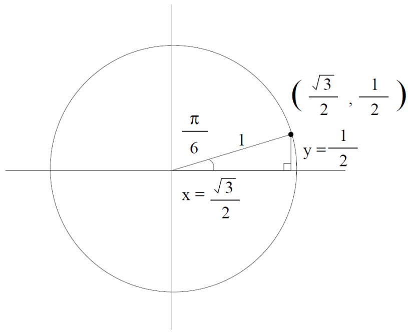
By moving this triangle into the unit circle and remembering that 45° equals $\frac{\pi}{4}$ radians, we find that the corresponding point on the unit circle has (x, y)-coordinates $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

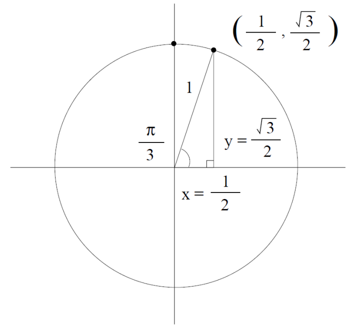


Thus

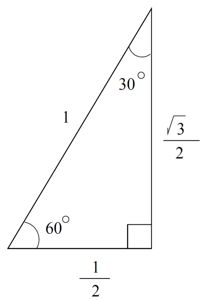
$$\sin\frac{\pi}{4} = y = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos\frac{\pi}{4} = y = \frac{1}{\sqrt{2}}$$

We can find $\sin\frac{\pi}{3}$, $\cos\frac{\pi}{3}$, $\sin\frac{\pi}{6}$, and $\cos\frac{\pi}{6}$ in a similar fashion by noting that $\frac{\pi}{3}$ equals 60°, and $\frac{\pi}{6}$ radians equals 30°.





Notice, a $30° - 60° - 90°$ triangle with a hypotenuse of length 1 gives us the information we need.



Thus

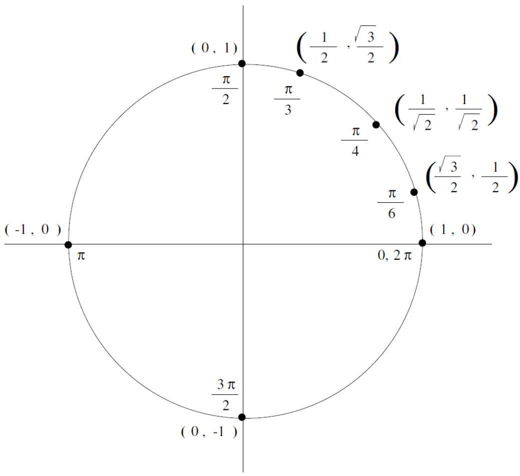
$$\sin\frac{\pi}{6} = \frac{1}{2} \quad \text{and} \quad \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2},$$

and

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\frac{\pi}{3} = \frac{1}{2}.$$

In calculus, we measure angles in *radians*, and we often use the trig values we just found, so it will be helpful to memorize the "enhanced" unit circle below.

Four other trigonometric functions are defined using sine and cosine. The are the **secant** ("sec"), **cosecant** ("csc"), **tangent** ("tan"), and



cotangent ("cot") trigonometric functions, defined as follows:

$$\sec\theta = \frac{1}{\cos\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Since $\cos\theta$ and $\sin\theta$ are 0 for some values of θ, the trig functions $\sec\theta$, $\csc\theta$, $\tan\theta$ and $\cot\theta$ are undefined for some values of θ. For more information, see the graphs of the trig functions in Section 1.6.

Example 1.3. Find $\cot\frac{\pi}{6}$.

$$\cot\frac{\pi}{6} = \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

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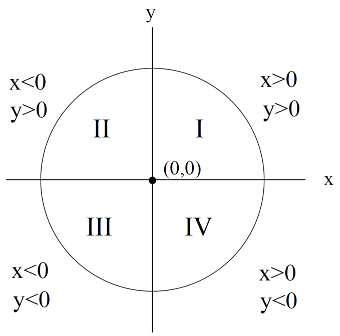
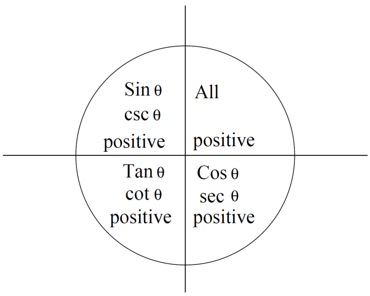
The unit circle has four quadrants:

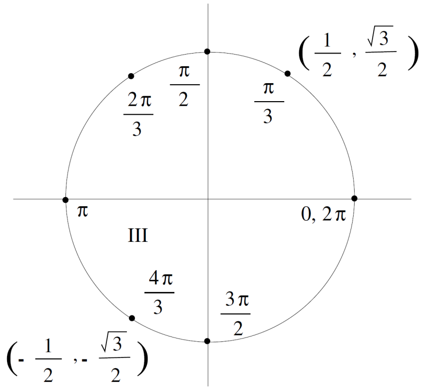
Note that a point in Quad II will have a negative x-value and a positive y-value. Thus, cosine is negative, and sine is positive in Quad II. To remember which trig functions are positive in which quadrant, use the mnemonic, "<u>A</u>ll <u>S</u>tudents <u>T</u>ake <u>C</u>alculus" working counter-clockwise from Quad I.





Example 1.4. Find $\cos\frac{4\pi}{3}$.

We will first find in which quadrant $\frac{4\pi}{3}$ lies.



Due to symmetry, $\frac{\pi}{3}$ and $\frac{4\pi}{3}$ have the same coordinates except for the negative signs. Thus

$$\cos\frac{4\pi}{3} = -\frac{1}{2}.$$

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1.3 Trigonometric Identities

The following trigonometric identities will be helpful in calculus.

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Even/Odd Formulas

$$\begin{aligned} \sin(-\theta) &= -\sin\theta & \csc(-\theta) &= -\csc\theta \\ \cos(-\theta) &= \cos\theta & \sec(-\theta) &= \sec\theta \\ \tan(-\theta) &= -\tan\theta & \cot(-\theta) &= -\cot\theta \end{aligned}$$

Double Angle Formulas

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ \tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} \end{aligned}$$

Sum and Difference Formulas

$$\begin{aligned} \sin(\alpha\pm\beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha\pm\beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\ \tan(\alpha\pm\beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \end{aligned}$$

Product to Sum Formulas

$$\begin{aligned} \sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \\ \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \\ \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \\ \cos\alpha\sin\beta &= \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)] \end{aligned}$$

Sum to Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

It is not necessary to memorize all of the above identities!

Knowing how an identity is derived, one can reduce the amount of memorization necessary.

The equation of the unit circle is $x^2+y^2=1$. The Pythagorean theorem also gives us the equation $x^2+y^2=1$. This equation resuts

$$\cos^2\theta + \sin^2\theta = 1.$$

Dividing this equation by $\cos^2\theta$, we obtain

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad \text{or} \quad 1+\tan^2\theta = \sec^2\theta.$$

Dividing $\cos^2\theta + \sin^2\theta = 1$ by $\sin^2\theta$, we obtain

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \quad \text{or} \quad \cot^2\theta + 1 = \csc^2\theta.$$

Using

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$

and

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

with $a=b=\theta$, we obtain the formulas

$$\sin 2\theta = 2\sin\theta\cos\theta \quad \text{and} \quad \cos 2\theta = \cos^2\theta - \sin^2\theta.$$

Also

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1-\cos^2\theta) \\ &= 2\cos^2\theta - 1\end{aligned}$$

and

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= (1-\sin^2\theta) - \sin^2\theta \\ &= 1 - 2\sin^2\theta\end{aligned}$$

using the identity $\cos^2\theta + \sin^2\theta = 1$. We can then solve for $\cos^2\theta$ in $\cos 2\theta = 2\cos^2\theta - 1$ to get

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

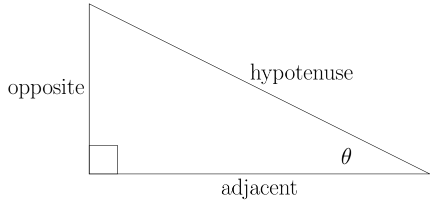
Solving for $\sin^2\theta$ in $\cos 2\theta = 1 - 2\sin^2\theta$ gives us

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}.$$

1.4 Finding the Values of Trigonometric Functions

Given the values for one trig function, we can find the values for the other five trig functions. There are two ways to do this.

1. Use identities.
2. Use a right triangle with 0 radians $< \theta < \frac{\pi}{2}$ radians.



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

One can remember these equations using the acronym " SOH CAH TOA" where: S=sine, O=opposite, H=hypotenuse, C=cosine, and T=tangent.

We assumed θ was in Quad I in the previous triangle. However, using symmetry, we can assume θ is in any quadrant. But be careful of the signs of the trig functions when θ is in Quad II, III, of IV!

Example 1.5. (**Using Method 1**) Given θ is in Quad III, and $\cot\theta = 2$, find the values of the remaining trig functions.

Solution. First, cotangent and tangent are reciprocals, so

$$\tan\theta - \frac{1}{\cot\theta} = \frac{1}{2}.$$

Next use the identity $\cot^2\theta + 1 = \csc^2\theta$ to get $\csc\theta$:

$$\begin{aligned} \cot^2\theta + 1 &= \csc^2\theta \\ 2^2 + 1 &= \csc^2\theta \\ \csc\theta &= \pm\sqrt{5} \\ \csc\theta &= -\sqrt{5} \end{aligned}$$

since θ is in Quad III.
Cosecant and sine are reciprocals, so

$$\sin\theta = \frac{1}{\csc\theta} = -\frac{1}{\sqrt{5}}$$

Using $\sin^2\theta + \cos^2\theta = 1$, we get

$$\begin{aligned} \cos^2\theta + \left(-\frac{1}{\sqrt{5}}\right)^2 &= 1 \\ \cos^2\theta &= \frac{4}{5} \\ \cos\theta &= \pm\frac{2}{\sqrt{5}} \\ \cos\theta &= -\frac{2}{\sqrt{5}} \end{aligned}$$

since θ is in Quad II.
Cosine and secant are reciprocals, so

$$\sec\theta = \frac{1}{\cos\theta} = -\frac{\sqrt{5}}{2}.$$

This gives us all six trig functions.

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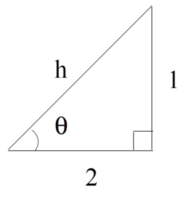
Example 1.6. (**Using Method 2**) Given θ is in Quad III, and $\cot\theta = 2$, find the values of the remaining trig functions.

Solution. First,

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adj}}{\text{opp}} = \frac{2}{1}$$

We then have the following triangle



Where $1^2 + 2^2 = h^2$ and $h = \sqrt{5}$.

Next note that θ is in Quad III. Thus,

$$\begin{aligned}
\cos\theta &= -\frac{\text{adj}}{\text{hyp}} = -\frac{2}{\sqrt{5}} \\
\sin\theta &= -\frac{\text{opp}}{\text{hyp}} = -\frac{1}{\sqrt{5}} \\
\sec\theta &= \frac{1}{\cos\theta} = -\frac{\sqrt{5}}{2} \\
\csc\theta &= \frac{1}{\sin\theta} = -\sqrt{5} \\
\sin\theta &= \frac{\text{opp}}{\text{adj}} = \frac{1}{2}
\end{aligned}$$

This gives us all six trig functions.

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1.5 Solving Equations Involving Trig Functions

We will begin by considering equations with one term involving a trigonometric function.

Example 1.7. Solve $2\sin x = 1$.

We want to find which values of x make this equation true. We will not rewrite this equation as $x = \cdots$. We will isolate the trig function instead:

$$\sin x = \frac{1}{2}$$

To find the solutions to this equation, we find the radian value(s) that will give us sine equal to $\frac{1}{2}$.

For x in $[0, 2\pi]$, we have 2 solutions: $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. However, if x can be any number, note that $\frac{\pi}{6} + 2\pi$, $\frac{\pi}{6} + 4\pi$, $\frac{\pi}{6} - 2\pi$, and $\frac{5\pi}{6} + 2\pi$ are also solutions. In fact, for x in $(-\infty, \infty)$, we have an infinite number of solutions that can be represented as

$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi$$

where n is any integer.

$\diamondsuit$

Example 1.8. Solve $2 \sin 2x = -1$ for x in $[0, 2\pi]$.

We begin by solving for $\sin 2x$.

$$\sin 2x = -\frac{1}{2}$$

We want solutions x for which

$$0 \leq x \leq 2\pi \quad \text{or} \quad 0 \leq 2x \leq 4\pi.$$

For which radian values θ between 0 and 4π does $\sin \theta = -\frac{1}{2}$?

Between 0 and 2π, $\sin \theta = -\frac{1}{2}$ for

$$\theta = \frac{7\pi}{6} \quad \text{and} \quad \theta = \frac{11\pi}{6}.$$

Between 2π and 4π, $\sin \theta = -\frac{1}{2}$ for

$$\theta = \frac{7\pi}{6} + 2\pi = \frac{19\pi}{6} \quad \text{and} \quad \theta = \frac{11\pi}{6} + 2\pi = \frac{23\pi}{6}.$$

To compensate for the $2x$ in $\sin 2x$, we will set $\theta = 2x$.

$$\sin \frac{7\pi}{6} = -\frac{1}{2} = \sin 2x \implies 2x = \frac{7\pi}{6} \implies x = \frac{7\pi}{12}$$

Similarly,

$$\sin \frac{11\pi}{6} = -\frac{1}{2} = \sin 2x \implies x = \frac{11\pi}{12},$$
$$\sin \frac{19\pi}{6} = -\frac{1}{2} = \sin 2x \implies x = \frac{19\pi}{12},$$

and

$$\sin \frac{23\pi}{6} = -\frac{1}{2} = \sin 2x \implies x = \frac{23\pi}{12}.$$

Thus the solutions for x in $[0, 2\pi]$ are

$$x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}.$$

$\diamondsuit$

We will now consider equations with more than one term involving a trigonometric function. The key concept involved in these types of equations is factoring.

Example 1.9. Solve $2\cos^2 x \tan x - \tan x = 0$ for $x = 0$ in $[0, 2\pi]$.

We begin by factoring $\tan x$ out of each term.

$$\tan x(2\cos^2 x - 1) = 0.$$

Thus either

$$\tan x = 0 \quad \text{or} \quad 2\cos^2 x - 1 = 0.$$

- If $\tan x = 0$, then $x = 0, \pi, 2\pi$.
- If $2\cos^2 x - 1 = 0$, then $\cos^2 x = \frac{1}{2}$ and $\cos x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$. So for x in $[0, 2\pi]$, $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

The solution set is

$$\left\{0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}.$$

$\Diamond$

In some problems, we will first use identities and then factor.

Example 1.10. Solve $\cos x = \cos 2x$ for $x = 0$ in $[0, 2\pi]$.

We will first use the identity $\cos 2x = 2\cos^2 x - 1$.

$$\begin{aligned} \cos x &= 2\cos^2 x - 1 \\ 2\cos^2 x - 1 - \cos x &= 0 \end{aligned}$$

Next, factor.

$$\begin{aligned} 2\cos^2 x - \cos x - 1 &= 0 \\ (2\cos x + 1)(\cos x - 1) &= 0 \end{aligned}$$

Thus either

$$2\cos x + 1 = 0 \quad \text{or} \quad \cos x - 1 = 0.$$

- If $2\cos x + 1 = 0$, then $\cos x = -\frac{1}{2}$ and $x = \frac{2\pi}{3}, \frac{4\pi}{3}$.
- If $\cos x - 1 = 0$, then $\cos x = 1$ and $x = 0, 2\pi$.

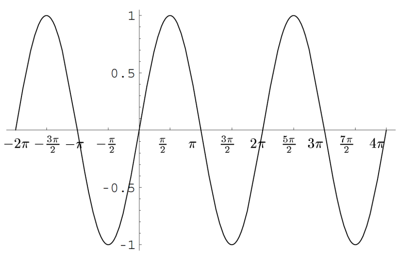
The solution set is

$$\left\{0, \pi, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}.$$

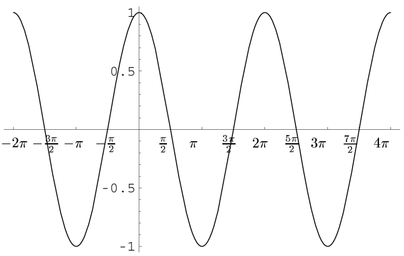
$\Diamond$

1.6 Graphs of Trig Functions

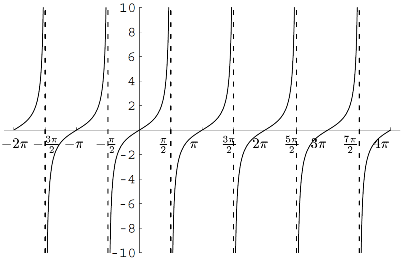
The graphs of the six trig functions are shown below. The trig functions are all periodic; a function is periodic with period p if $f(x+p) = f(x)$ for all real numbers x. Such functions repeat every p unit along the x-axis. Sine and cosine have periods of 2π. Tangent, cotangent, secant, and cosecant have periods of π.



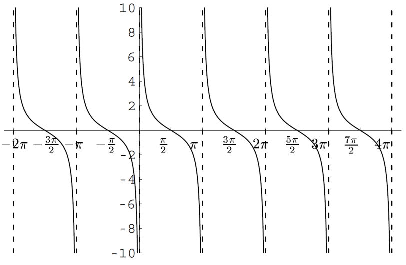
$y = \sin x$



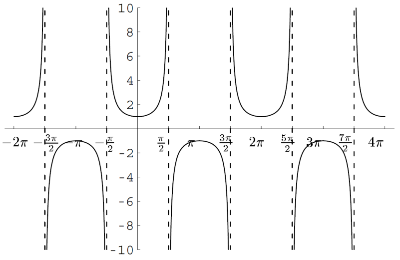
$y = \cos x$



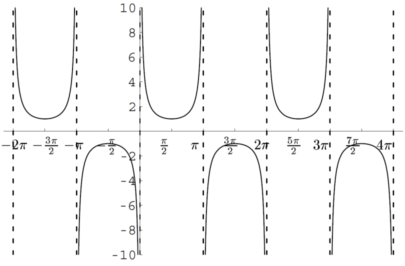
$y = \tan x$



$y = \cot x$



$y = \sec x$



$y = \csc x$

1.7 Common Trigonometry Errors

1.7.1 Degrees & Radians

While degrees is easy to visualize, nearly all calculations are performed in *radians*. For example, evaluate $\cos(x)$ at $x = 10$,

$$\cos(10) = -0.839071529076 \quad \text{in radians}$$
$$\cos(10) = 0.984807753012 \quad \text{in degrees}$$

Not only are the results different, but they are also of a different sign!

1.7.2 $\cos(x)$ is NOT Multiplication

Trig functions are *functions*, e.g. $f(x)$, not multiplication. For example,

1. $\cos(x + y) \neq \cos(x) + \cos(y)$
2. $\cos(3x) \neq 3\cos(x)$.

To convince yourself, consider the following where $x = \pi$ and $y = 2\pi$.
Example 1:

$$\begin{aligned}\cos(\pi + 2\pi) &\neq \cos(\pi) + \cos(2\pi)\\ \cos(3\pi) &\neq -1 + 1\\ -1 &\neq 0,\end{aligned}$$

Example 2:

$$\begin{aligned}\cos(3\pi) &\neq 3\cos(\pi)\\ -1 &\neq 3(-1)\\ -1 &\neq -3.\end{aligned}$$

1.7.3 Powers of Trig Functions

Remember that if n is a *positive integer*, then

$$\sin^n x = (\sin x)^n.$$

Also, note the difference in the following,

$$\sin^n x \quad \text{vs.} \quad \sin x^n.$$

which we clarify as $\sin(x^2)$. In the first case, we are taking the sin *then* raising the result to the n-th power, and in the second, we are raising x to the n-th power, *then* taking the sin. We can clarify the second using parenthesis, $\sin(x^n)$.

1.7.4 Inverse Trig Notation

Inverse trig notation is awkward; the -1 in $\cos^{-1} x$ is NOT an exponent like discussed above. Nor is it

$$\cos^{-1} x \neq \frac{1}{\cos x}.$$

Instead, the more clear notation is,

$$\cos^{-1} x = \arccos x.$$

1.8 Exercises

Exercise 1.1 (Conversions)*.* Change to radian measure:

1. $50°$
2. $375°$

$\lozenge$

Exercise 1.2 (Conversions)*.* Change to degree measure:

1. $-\frac{5\pi}{6}$
2. $\frac{7\pi}{8}$

$\lozenge$

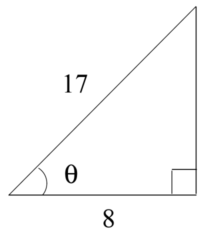
Exercise 1.3 (Function values)*.* Find the function values:

1. $\sin \frac{5\pi}{3}$
2. $\csc \frac{11\pi}{4}$
3. $\sec \frac{11\pi}{6}$
4. $\cot \frac{5\pi}{4}$

$\lozenge$

Exercise 1.4 (Function values)*.* Function values and a quadrant are specified. Find the other five function values.

1. $\sin \theta = \frac{1}{3}$, II
2. $\tan \theta = 5$, III



$\diamond$

Exercise 1.5 (Six Trig Functions). Find the six trigonometric function values for the following θ:

$\diamond$

Exercise 1.6 (Function Solutions). Solve, finding all solutions in the range specified.

1. All solutions: $2\cos^2 x = 1$
2. Range of $[0, 2\pi]$: $\sec^2 x - 4 = 0$
3. Range of $[0, 2\pi]$: $\cos 2x \sin x + \sin x = 0$
4. Range of $[0, 2\pi]$: $\cos 2x - \sin x = 1$

$\diamond$