# Technical Note: INS Temporal Propagation

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***Abstract*— Due to space limitations in [1] and [2], this Technical Note is supplied to explain an approach for the time propagation of the state vector estimate $\hat{x}$ by the function $\boldsymbol{f}(\boldsymbol{x}(t), \mathbf{u}(t))$ in an inertial navigation system (INS).**

## I. Introduction

While many reference frames may be used, the Earth-centered Earth-fixed (ECEF) reference frame is a convenient reference frame for GPS-aided INS. One reason for this choice is that satellite navigation solutions are resolved in the ECEF reference frame. Another is that the ECEF frame works globally, not yielding singularities in the polar regions.

In this article we define the $e$-frame as the ECEF-frame (or Earth-frame), and the $i$-frame as the Earth-Centered-Inertial (ECI) frame (or inertial-frame). The direction cosine matrix (DCM), or rotation matrix, from body-frame to Earth-frame is $\mathbf{R}_b^e$. The position of the body $b$-frame with respect to the Earth $e$-frame resolved in the $e$-frame is $\mathbf{r}_{eb}^e$. Similarly the velocity of the $b$-frame with respect to the $e$-frame resolved in the $e$-frame is $\mathbf{v}_{eb}^e$. The rotation matrix from $b$-frame to $e$-frame is $\mathbf{R}_b^e$.

Expanding eqn. (1) in both [1] and [2], the INS kinematic equations defining $\boldsymbol{f}(\boldsymbol{x}(t), \mathbf{u}(t))$ in the ECEF frame (see Section 11.2.2 of [3]) are

$$\dot{\mathbf{r}}_{eb}^e = \mathbf{v}_{eb}^e \tag{1}$$

$$\dot{\mathbf{v}}_{eb}^e = \mathbf{R}_b^e \mathbf{f}_{ib}^b + \mathbf{g}_b^e - 2\boldsymbol{\Omega}_{ie}^e \mathbf{v}_{eb}^e \tag{2}$$

$$\dot{\mathbf{R}}_b^e = \mathbf{R}_b^e (\boldsymbol{\Omega}_{ib}^e - \boldsymbol{\Omega}_{ie}^e). \tag{3}$$

The inputs $\mathbf{u}$, are specific force $\mathbf{f}_{ib}^b$ and angular rate $[\boldsymbol{\omega}_{ib}^b \times] = \boldsymbol{\Omega}_{ib}^b$, with respect to the inertial-frame $i$. The local gravity vector is $\mathbf{g}^e$, and the Earth-rotation rate is $[\boldsymbol{\omega}_{ie}^e \times] = \boldsymbol{\Omega}_{ie}^e$. The notation $[\mathbf{a}\times]$ represents the skew-symmetric matrix corresponding to the vector $\mathbf{a}$.

Fig. 1 is a block diagram showing how the angular-rate and specific-force measurements, of an Inertial Measurement Unit (IMU), are used to update the Earth-referenced attitude, velocity, and position states at $t_i$.

## II. INS Time Propagation Equations

### A. Attitude Update

Using eqn. (2.54) in [3], the time derivative of the rotation matrix $\mathbf{R}_b^e$ is

$$\dot{\mathbf{R}}_b^e = \mathbf{R}_b^e \boldsymbol{\Omega}_{eb}^b, \tag{4}$$

where $\boldsymbol{\Omega}_{eb}^b = [\boldsymbol{\omega}_{eb}^b \times]$.

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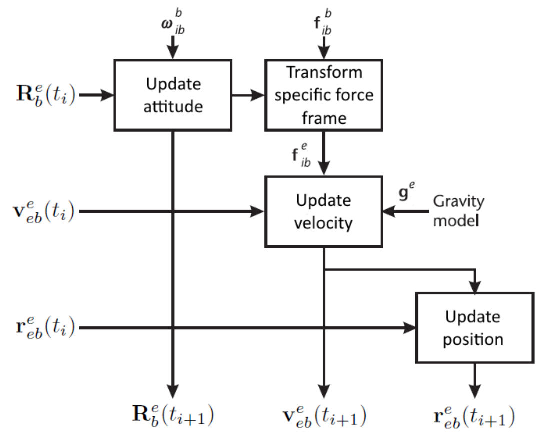


Fig. 1. Block diagram of the ECEF referenced INS equations.

Let the attitude increment over the IMU measurement interval $\tau_i$ be defined as $\boldsymbol{\alpha}_{ib}^b = \boldsymbol{\omega}_{ib}^b \tau_i$. Appendix I shows that the discrete-time rotation matrix update that is equivalent to eqn. (4) can be computed as

$$\mathbf{R}_b^e(t_{i+1}) = \mathbf{R}_b^e(t_i)\mathbf{R}_{b(t)}^{b(t+\tau_i)} - \boldsymbol{\Omega}_{ie}^b \mathbf{R}_b^e(t_i)\tau_i + \mathbf{R}_b^e(t_i). \tag{5}$$

The integrating factor $\mathbf{R}_{b(t)}^{b(t+\tau_i)}$ represents the effect of the angle rotation corresponding to $\boldsymbol{\Omega}_{ib}^b$ over the interval $[t, t+\tau_i]$, and $\mathbf{R}_b^e(t_i)$ and $\mathbf{R}_b^e(t_{i+1})$ are the prior and updated rotation matrix, respectively.

**Attitude Increment**

The IMU angular rate has two components $\boldsymbol{\omega}_{ib}^b$ and $\boldsymbol{\omega}_{ie}^b$. The body rate portion $\boldsymbol{\omega}_{ib}^b$ can be large and change rapidly, which is the reason its portion of the attitude increment receives special treatment.

By eqn. (2.62) of [3],

$$\mathbf{R}_{b(t)}^{b(t+\tau_i)} = \mathbf{R}_{b-}^{b+} = \exp[\boldsymbol{\alpha}_{ib}^b \times]. \tag{6}$$

The power-series expansion of the matrix exponential is

$$\exp[\boldsymbol{\alpha}_{ib}^b \times] = \sum_{r=0}^{\infty} \frac{[\boldsymbol{\alpha}_{ib}^b \times]^r}{r!}. \tag{7}$$

As shown in eqns. (2.66) and (2.67) of [3], the odd and even powers of the attitude increment skew-symmetric matrix are,

$$[\boldsymbol{\alpha}_{ib}^b \times]^{2r+1} = (-1)^r ||\boldsymbol{\alpha}_{ib}^b||^r [\boldsymbol{\alpha}_{ib}^b \times] \tag{8}$$

$$[\boldsymbol{\alpha}_{ib}^b \times]^{2r} = (-1)^r ||\boldsymbol{\alpha}_{ib}^b||^{2r} [\boldsymbol{\alpha}_{ib}^b \times]^2, \tag{9}$$

where $r = 1, 2, 3, \cdots$. Expanding eqn. (6) using eqns. (7)-(9),

$$\mathbf{R}_{b-}^{b+} = \mathbf{I}_3 + \left(\sum_{r=0}^{\infty}(-1)^r \frac{||\boldsymbol{\alpha}_{ib}^b||^{2r}}{(2r+1)!}\right)[\boldsymbol{\alpha}_{ib}^b \times] + \left(\sum_{r=0}^{\infty}(-1)^r \frac{||\boldsymbol{\alpha}_{ib}^b||^{2r}}{(2r+2)!}\right)[\boldsymbol{\alpha}_{ib}^b \times]^2. \quad (10)$$

Eqn. (10) is equivalent to (see eqn. (2.69) of [3], and [4], [5]),

$$\mathbf{R}_{b-}^{b+} = \mathbf{I}_3 + \frac{\sin(||\boldsymbol{\alpha}_{ib}^b||)}{||\boldsymbol{\alpha}_{ib}^b||}[\boldsymbol{\alpha}_{ib}^b \times] + \frac{1-\cos(||\boldsymbol{\alpha}_{ib}^b||)}{||\boldsymbol{\alpha}_{ib}^b||^2}[\boldsymbol{\alpha}_{ib}^b \times]^2. \quad (11)$$

When the CPU requires tradeoffs related to implementation, computation of trigonometric functions can be prohibitive. In these cases, the Taylor series expansions may be truncated. For example, the fourth-order approximation of eqn. (10) is

$$\mathbf{R}_{b-}^{b+} = \mathbf{I}_3 + \left(1 - \frac{||\boldsymbol{\alpha}_{ib}^b||^2}{6}\right)[\boldsymbol{\alpha}_{ib}^b \times] + \left(\frac{1}{2} - \frac{||\boldsymbol{\alpha}_{ib}^b||^2}{24}\right)[\boldsymbol{\alpha}_{ib}^b \times]^2. \quad (12)$$

Such tradeoffs should be thoroughly evaluated in simulation.

### B. Velocity Update

Neglecting acceleration in the ECEF frame, assume the ECI frame instantaneously coincides with the ECEF frame, such that $\ddot{\mathbf{r}}_{ib}^e = \ddot{\mathbf{r}}_{eb}^e$, $\dot{\mathbf{r}}_{ib}^e = \dot{\mathbf{r}}_{eb}^e$, and $\mathbf{r}_{ib}^e = \mathbf{r}_{eb}^e$. The velocity update in the ECEF frame is

$$\mathbf{v}_{eb}^e(t_{i+1}) = \mathbf{v}_{eb}^e(t_i) + \left(\mathbf{f}_{ib}^e + \mathbf{g}_b^e(\mathbf{r}_{eb}^e(t_i)) - 2\boldsymbol{\Omega}_{ie}^e \mathbf{v}_{eb}^e(t_i)\right)\tau_i \quad (13)$$

where $\mathbf{r}_{eb}^e(t_i)$ and $\mathbf{v}_{eb}^e(t_i)$ are the prior position and velocity, respectively, and $\mathbf{v}_{eb}^e(t_{i+1})$ is the updated velocity for the IMU interval $\tau_i$.

**Specific-Force Frame Transformation**

The frame transformation of specific-force takes the form

$$\mathbf{f}_{ib}^e = \mathbf{R}_b^e \mathbf{f}_{ib}^b. \quad (14)$$

Incorporating the updated rotation matrix $\mathbf{R}_b^e(t_{i+1})$ from eqn. (5), the specific force transformation is

$$\mathbf{f}_{ib}^e = \mathbf{R}_b^e(t_{i+1})\mathbf{f}_{ib}^b. \quad (15)$$

### C. Position Update

Using eqn. (1), the position update is derived as follows:

$$\begin{aligned}\mathbf{r}_{eb}^e(t_{i+1}) &= \mathbf{r}_{eb}^e(t_i) + (\mathbf{v}_{eb}^e(t_i) + \mathbf{v}_{eb}^e(t_{i+1}))\frac{\tau_i}{2}, \quad (16)\\ &= \mathbf{r}_{eb}^e(t_i) + \mathbf{v}_{eb}^e(t_i)\tau_i \\ &\quad + \left(\mathbf{f}_{ib}^e + \mathbf{g}^e(\mathbf{r}_{eb}^e(t_i)) - 2\boldsymbol{\Omega}_{ie}^e \mathbf{v}_{eb}^e(t_i)\right)\frac{\tau_i^2}{2}. \quad (17)\end{aligned}$$

where $\mathbf{r}_{eb}^e(t_i)$ and $\mathbf{r}_{eb}^e(t_{i+1})$ are the prior and updated positions, respectively.

### D. Gravity Model

A precise gravity model [6] formulated in the ECEF frame, is defined as

$$\gamma_{ib}^e = -\frac{\mu}{||\mathbf{r}_{eb}^e||^3} \times \left\{\mathbf{r}_{eb}^e + \frac{3}{2}J_2\frac{R_0^2}{||\mathbf{r}_{eb}^e||^2}\begin{bmatrix}(1-5(\mathbf{r}_{eb,z}^e)^2/||\mathbf{r}_{eb}^e||)\mathbf{r}_{eb,x}^e \\ (1-5(\mathbf{r}_{eb,z}^e)^2/||\mathbf{r}_{eb}^e||)\mathbf{r}_{eb,y}^e \\ (3-5(\mathbf{r}_{eb,z}^e)^2/||\mathbf{r}_{eb}^e||)\mathbf{r}_{eb,z}^e\end{bmatrix}\right\}, \quad (18)$$

where the Earth's second gravitational constant $J_2 = 1.082627E^{-3}$ $(m^3/s^2)$, the Equatorial radius is $R_0 = 6.378137E^{+6}$ $(m)$, the gravitational constant $\mu = 3.986004418E^{+14}$ $(m^3/s^2)$, and the Earth-rotation rate is $\omega_{ie} = 7.292115E^{-5}$ $(rad/s)$.

## Appendix I
## Attitude Update Derivation

The goal of this appendix is to derive eqn. (5) from eqn. (4). The starting point of this derivation is eqn. (4) and the related definitions:

$$\begin{aligned}\dot{\mathbf{R}}_b^e &= \mathbf{R}_b^e \boldsymbol{\Omega}_{eb}^b \\ \boldsymbol{\Omega}_{eb}^b &\triangleq [\boldsymbol{\omega}_{eb}^b \times] = \boldsymbol{\Omega}_{ib}^b - \boldsymbol{\Omega}_{ie}^b \\ \boldsymbol{\alpha}_{ib}^b &\triangleq \boldsymbol{\omega}_{ib}^b \delta t.\end{aligned}$$

The definition of the derivative is

$$\dot{\mathbf{R}}_b^e = \lim_{\delta t \mapsto 0}\frac{\mathbf{R}_b^e(t+\delta t) - \mathbf{R}_b^e(t)}{\delta t}. \quad (19)$$

Assuming that the time increment $\delta t$ is sufficiently small:

$$\begin{aligned}\mathbf{R}_b^e(t+\delta t) &= \dot{\mathbf{R}}_b^e(t)\delta t + \mathbf{R}_b^e(t) \\ &= \mathbf{R}_b^e(t)\boldsymbol{\Omega}_{eb}^b \delta t + \mathbf{R}_b^e(t) \\ &= \mathbf{R}_b^e(t)(\boldsymbol{\Omega}_{ib}^b - \boldsymbol{\Omega}_{ie}^b)\delta t + \mathbf{R}_b^e(t) \\ &= \mathbf{R}_b^e(t)\boldsymbol{\Omega}_{ib}^b \delta t - \mathbf{R}_b^e(t)\boldsymbol{\Omega}_{ie}^b \delta t + \mathbf{R}_b^e(t) \\ &= \mathbf{R}_b^e(t)[\boldsymbol{\alpha}_{ib}^b \times] - \mathbf{R}_b^e(t)\mathbf{R}_e^b(t)\boldsymbol{\Omega}_{ie}^b \mathbf{R}_b^e(t)\delta t + \mathbf{R}_b^e(t) \\ &= \mathbf{R}_b^e(t)\mathbf{R}_{b(t)}^{b(t+\delta t)} - \boldsymbol{\Omega}_{ie}^b \mathbf{R}_b^e(t)\delta t + \mathbf{R}_b^e(t) \quad (20)\end{aligned}$$

Note that $\mathbf{R}_{b(t)}^{b(t+\delta t)}$ is defined in eqn. (6) as the exponential of $[\boldsymbol{\alpha}_{ib}^b \times]$. The last step in deriving eqn. (20) is not obvious, see Section 2.71 in [3]. The only derivation in the assumption is that the IMU angular-rate measurement is constant over the integration interval.

## References


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