CHAPTER 1

Linear Algebra

1.1 Working with Vectors

1.1.1 Vector

Definition 1.1 (**Vector**). *A vector in n-space is an ordered list of n numbers.* ♠

Numbers can be represented as either a row vector or a column vector:

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

We can also think of a vector as defining a point in n-dimensional space, usually $\mathbb{R}^n$; each element of the vector defines the coordinate of the point in a particular direction.

For example the vector $\mathbf{A} = \begin{bmatrix} 2 & 1 \end{bmatrix}$ represents the x, y point on a 2-dimensional plane, where $x = 2$ and $y = 1$.

1.1.2 Vector Addition and Subtraction

If two vectors, $\mathbf{u}$ and $\mathbf{v}$, have the same length (i.e. have the same number of elements), they can be added (subtracted) together:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 & u_2 + v_2 & \cdots & u_k + v_n \end{bmatrix}$$

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} u_1 - v_1 & u_2 - v_2 & \cdots & u_k - v_n \end{bmatrix}$$

1.1.3 Scalar Multiplication

The product of a scalar c (i.e. a constant) and vector $\mathbf{v}$ is:

$$c\mathbf{v} = \begin{bmatrix} cv_1 & cv_2 & \dots & cv_n \end{bmatrix}$$

1.1.4 Vector Inner Product

The inner product (also called the dot product or scalar product) of two vectors $\mathbf{u}$ and $\mathbf{v}$ is again defined if and only if (iff) they have the same number of elements:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

If

$$\mathbf{u} \cdot \mathbf{v} = 0,$$

the two vectors are orthogonal (perpendicular).

1.1.5 Vector Norm

The norm of a vector is a measure of its length. There are many different ways to calculate the norm, but the most common is the Euclidean norm (which corresponds to our usual conception of distance in three-dimensional space):

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1 v_1 + v_2 v_2 + \dots + v_n v_n}$$

Example 1.1 (Vector Algebra). Let $\mathbf{a} = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$. Calculate the following:

1. $\mathbf{a} - \mathbf{b}$
2. $\mathbf{a} \cdot \mathbf{b}$

Solution. The solution requires basic arithmetic:

1. $\mathbf{a} - \mathbf{b} = \begin{bmatrix} (2-3) & (1-4) & (2-5) \end{bmatrix} = \begin{bmatrix} -1 & -3 & -3 \end{bmatrix}$
2. $\mathbf{a} \cdot \mathbf{b} = (2 \times 3) + (1 \times 4) + (2 \times 5) = 6 + 4 + 10 = 20$

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Exercise 1.1 (Vector Algebra). Let $\mathbf{u} = \begin{bmatrix} 7 & 1 & -5 & 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 9 & -3 & 2 & 8 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 & 13 & -7 & 2 & 15 \end{bmatrix}$, and $c = 2$. Calculate the following:

1. $\mathbf{u} - \mathbf{v}$
2. $c\mathbf{w}$

3. $\mathbf{u} \cdot \mathbf{v}$

4. $\mathbf{w} \cdot \mathbf{v}$

$\diamondsuit$

1.2 Linear Independence

1.2.1 Linear combinations

Definition 1.2 (Linear combinations). *The vector* $\mathbf{u}$ *is a* ***linear combination*** *of the vectors* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *if*

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}.$$

$\spadesuit$

For example, $\begin{bmatrix} 9 & 13 & 17 \end{bmatrix}$ is a linear combination of the following three vectors: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$, and $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$. This is because $\begin{bmatrix} 9 & 13 & 17 \end{bmatrix} = (2)\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + (-1)\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} + (3)\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$.

1.2.2 Linear independence

Definition 1.3 (Linearly Independent Vectors). *A set of vectors* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *is linearly* independent *if there exist scalars* $c_1 \cdots c_k$*, not all zeros, such that*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

where $\mathbf{0}$ *is the zeros vector. If another solution exists, the set of vectors is linearly* dependent*.* $\spadesuit$

A set S of vectors is linearly dependent iff at least one of the vectors in S can be written as a linear combination of the other vectors in S. Linear independence is only defined for sets of vectors with the same number of elements; any linearly independent set of vectors in n-space contains at most n vectors.

Example 1.2 (Linear Independence)*.* Are the following sets of vectors linearly independent?

1. $\mathbf{a} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 & 6 & 1 \end{bmatrix}$.

2. $\mathbf{a} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 10 & 10 & 0 \end{bmatrix}$.

Solution. Applying Definition 1.3:

1. Yes. For example, if $c_1 = 2$ and $c_2 = -1$, then

$$(2)\begin{bmatrix}2\\3\\1\end{bmatrix} + (-1)\begin{bmatrix}4\\6\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

2. No. There are no constants c_1, c_2, c_3, that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ can be multiplied by to obtain $\mathbf{0}$.

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Exercise 1.2 (Linear Independence). Are the following sets of vectors linearly independent?

1. $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

2. $\mathbf{v}_1 = \begin{bmatrix}2\\2\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-4\\6\\5\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}-2\\8\\6\end{bmatrix}$

$\Diamond$

1.3 Basics of Matrix Algebra

1.3.1 Matrix

Definition 1.4 (Matrix). *A matrix is an array of real numbers arranged in m rows by n columns. The dimensionality of the matrix is defined as the number of rows by the number of columns, $m \times n$.*

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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Note that you can think of vectors as special cases of matrices; a column vector of length k is a $k \times 1$ matrix, while a row vector of the same length is a $1 \times k$ matrix. It's also useful to think of matrices as being made up of a collection of row or column vectors. For example,

$$\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m\end{bmatrix}.$$

1.3.2 Matrix Addition

Let $\mathbf{A}$ and $\mathbf{B}$ be two $m \times n$ matrices.

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

Note that matrices $\mathbf{A}$ and $\mathbf{B}$ must have the same dimensionality.

Example 1.3 (Matrix Addition). Solve $\mathbf{A}+\mathbf{B}$, given:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

Solution. Using simple addition:

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} (1+1) & (2+2) & (3+1) \\ (4+2) & (5+1) & (6+2) \end{bmatrix} = \begin{bmatrix} 2 & 4 & 4 \\ 6 & 6 & 8 \end{bmatrix}.$$

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1.3.3 Scalar Multiplication

Given the scalar s, the scalar multiplication of $s\mathbf{A}$ is

$$s\mathbf{A} = s \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} sa_{11} & sa_{12} & \cdots & sa_{1n} \\ sa_{21} & sa_{22} & \cdots & sa_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ sa_{m1} & sa_{m2} & \cdots & sa_{mn} \end{bmatrix}.$$

Example 1.4 (Scalar Multiplication). Solve $s\mathbf{A}$ given:

$$s = 2, \qquad \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Solution. Using simple multiplication:

$$s\mathbf{A} = 2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2\times 1 & 2\times 2 & 2\times 3 \\ 2\times 4 & 2\times 5 & 2\times 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

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1.3.4 Matrix Multiplication

If $\mathbf{A}$ is an $m \times k$ matrix and $\mathbf{B}$ is a $k \times n$ matrix, then their product $\mathbf{C} = \mathbf{AB}$ is the $m \times n$ matrix where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

We can restate the above equation for any combination of $n \times n$ (i.e. square) matricies, by simply multiplying the rows and columns.

Note that the number of columns of the first matrix must equal the number of rows of the second matrix. The sizes of the matrices (including the resulting product) must be

$$(m \times k)(k \times n) = (m \times n)$$

For example: Let the matrices $\mathbf{A}$ and $\mathbf{B}$ be defined as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}.$$

Solve

$$\mathbf{AB} = \mathbf{C}.$$

First, check the dimensions of $\mathbf{A}$ and $\mathbf{B}$, which are (2×3) and (3×2), respectively. Using the rule above, we use the outer indices $((\boxed{2} \times 3)$ and $(3 \times \boxed{2}))$ and find that $\mathbf{C}$ will be dimension (2×2), e.g.

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

To solve for $\mathbf{C}$, matrix multiplication states that we use the dot product of the first *row* of $\mathbf{A}$ by the first *column* of $\mathbf{B}$, etc., to get the result. We can use the notation

$$c_{11} = \mathbf{A}_{\text{row}_1} \cdot \mathbf{B}_{\text{col}_1}$$

$$c_{12} = \mathbf{A}_{\text{row}_1} \cdot \mathbf{B}_{\text{col}_2}$$

$$c_{21} = \mathbf{A}_{\text{row}_2} \cdot \mathbf{B}_{\text{col}_1}$$

$$c_{22} = \mathbf{A}_{\text{row}_2} \cdot \mathbf{B}_{\text{col}_2}$$

Or, more explicitly

$$c_{11} = a_{11} \times b_{11} + a_{12} \times b_{21} + a_{13} \times b_{31}$$

$$c_{12} = a_{11} \times b_{12} + a_{12} \times b_{22} + a_{13} \times b_{32}$$

$$c_{21} = a_{21} \times b_{11} + a_{22} \times b_{21} + a_{23} \times b_{31}$$

$$c_{22} = a_{21} \times b_{12} + a_{22} \times b_{22} + a_{13} \times b_{32}$$

The same steps used in the example above can be applied to any $n \times m$ or $n \times n$ matrix, provided that the rows and columns have an equal number of elements, e.g. the dimension of $\mathbf{A}_{\text{row}_i}$ is the same as $\mathbf{B}_{\text{col}_j}$.

Note that the matrix vector multiplication is just a simplification of the above example where $\mathbf{B}$ would be a single column, replaced with the vector notation $\mathbf{B}$, such that

$$\mathbf{AB} = \mathbf{C}$$

since $\mathbf{C}$ is a vector.

Also note that if $\mathbf{AB}$ exists, $\mathbf{BA}$ exists only if $\dim(\mathbf{A}) = m \times n$ and $\dim(\mathbf{B}) = n \times m$. Generally $\mathbf{AB} \neq \mathbf{BA}$. Only in special circumstances is $\mathbf{AB} = \mathbf{BA}$ true, e.g. when $\mathbf{A}$ or $\mathbf{B}$ is the identity matrix $\mathbf{I}$, or $\mathbf{A} = \mathbf{B}^{-1}$.

Example 1.5 (Matrix Multiplication). Solve:

1. $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} =$

2. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix} =$

Solution. Applying the example above:

1. $\begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \\ ew + fy & ex + fz \end{bmatrix}.$

2. $\begin{bmatrix} 1(-2) + 2(4) - 1(2), & 1(5) + 2(-3) - 1(1) \\ 3(-2) + 1(4) + 4(2), & 3(5) + 1(-3) + 4(1) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}.$

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1.3.5 Laws of Matrix Algebra

1. Associative: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
   $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
2. Commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
3. Distributive: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
   $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

Note the order of multiplication matters:

$$\mathbf{AB} \neq \mathbf{BA}$$

For example,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix}, \qquad \mathbf{BA} = \begin{bmatrix} 1 & 7 \\ -1 & 3 \end{bmatrix}$$

1.3.6 Transpose

Definition 1.5 (**Transpose**). *The transpose of the $m \times n$ matrix $\mathbf{A}$ is the $n \times m$ matrix $\mathbf{A}^\intercal$ (also written $\mathbf{A}'$ obtained by interchanging the rows and columns of $\mathbf{A}$.* ♠

Example 1.6 (Transpose).

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}, \qquad \mathbf{A}^\intercal = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \qquad \mathbf{v}^\intercal = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}$$

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1.3.7 Properties of the transpose

The following rules apply for transposed matrices:

1. $(\mathbf{A} + \mathbf{B})^\intercal = \mathbf{A}^\intercal + \mathbf{B}^\intercal$
2. $(\mathbf{A}^\intercal)^\intercal = \mathbf{A}$
3. $(s\mathbf{A})^\intercal = s\mathbf{A}^\intercal$
4. $(\mathbf{AB})^\intercal = \mathbf{B}^\intercal\mathbf{A}^\intercal$; and by induction $(\mathbf{ABC})^\intercal = \mathbf{C}^\intercal\mathbf{B}^\intercal\mathbf{A}^\intercal$

Example 1.7 (Matrix Multiplication). Given:

$$(\mathbf{AB})^\intercal = \mathbf{B}^\intercal\mathbf{A}^\intercal$$

It can be shown that

$$(\mathbf{AB})^\intercal = \left[\begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}\right]^\intercal = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}$$

and

$$\mathbf{B}^\intercal\mathbf{A}^\intercal = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}.$$

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Exercise 1.3 (Matrix Multiplication)*.* Given:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 1 & 5 & -7 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & 6 \end{bmatrix}$$

Calculate the following:

1. $\mathbf{Ab}$
2. $\mathbf{bA}$
3. $(\mathbf{bC})^\intercal$
4. $\mathbf{bC}^\intercal$

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1.4 Systems of Linear Equations

1.4.1 Linear Equation

The following equation is linear because there is only one variable per term and is at most degree 1

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where a_i are parameters or coefficients, x_i are variables or unknowns.

We are often interested in solving linear systems like

$$\begin{array}{rcrcr} x & - & 3y & = & -3 \\ 2x & + & y & = & 8 \end{array}$$

More generally, we might have a system of m equations in n unknowns

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & \vdots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

A **solution** to a linear system of m equations in n unknowns is a set of n numbers $x_1, x_2, \cdots, x_n$ that satisfy each of the m equations.

Example 1.8. $x = 3$ and $y = 2$ is the solution to the above 2×2 linear system. If you graph the two lines, you will find that they intersect at $(3, 2)$. $\Diamond$

Does a linear system have one, no, or multiple solutions? For a system of 2 equations with 2 unknowns (i.e., two lines):

- **One solution:** The lines intersect at exactly one point.
- **No solution:** The lines are parallel.
- **Infinite solutions:** The lines coincide.

Methods to solve linear systems:

1. Substitution
2. Elimination of variables
3. Matrix methods

Exercise 1.4 (Linear Equations). Provide a system of 2 equations with 2 unknowns that has

1. one solution
2. no solution
3. infinite solutions

$\Diamond$

1.5 Systems of Equations as Matrices

Matrices provide an easy and efficient way to represent linear systems such as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & \vdots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

as

$$\mathbf{A}\boldsymbol{x} = \mathbf{b}.$$

1.5.1 Coefficient matrix

The $m \times n$ matrix $\mathbf{A}$ is an array of m, n real numbers arranged in m rows by n columns:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The unknown quantities are represented by the vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

The right hand side of the linear system is represented by the vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

1.5.2 Augmented Matrix

When we append $\mathbf{b}$ to the coefficient matrix $\mathbf{A}$, we get the augmented matrix $\bar{\mathbf{A}} = [\mathbf{A}|\mathbf{b}]$:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Exercise 1.5 (Augmented Matrix). Create an augmented matrix that represent the following system of equations:

$$\begin{aligned} 2x_1 - 7x_2 + 9x_3 - 4x_4 &= 8 \\ 41x_2 + 9x_3 - 5x_6 &= 11 \\ x_1 - 15x_2 - 11x_5 &= 9. \end{aligned}$$

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1.6 Solutions to Augmented Matrices & Systems of Equations

1.6.1 Row Echelon Form

Our goal is to translate our augmented matrix or system of equations into row echelon form. This will provide us with the values of the vector $\boldsymbol{x}$ that solve the system. We use the row operations to change coefficients in the lower triangle of the augmented matrix to 0's. An augmented matrix of the form

$$\left[\begin{array}{cccccc|c} \boxed{a'_{11}} & a'_{12} & a'_{13} & \cdots & a'_{1n} & & b'_1 \\ 0 & \boxed{a'_{22}} & a'_{23} & \cdots & a'_{2n} & & b'_2 \\ 0 & 0 & \boxed{a'_{33}} & \cdots & a'_{3n} & & b'_3 \\ 0 & 0 & 0 & \ddots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \boxed{a'_{mn}} & & b'_m \end{array}\right]$$

is said to be in *row echelon form*: each row has more leading zeros than the row preceding it.

1.6.2 Reduced Row Echelon Form

We can go one step further and put the matrix into *reduced row echelon form*. Reduced row echelon form makes the value of $\boldsymbol{x}$ which solves the system very obvious. For a system of m equations in m unknowns, with no all-zero rows, the reduced row echelon form is

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 0 & 0 & b^*_1 \\ 0 & \boxed{1} & 0 & 0 & 0 & b^*_2 \\ 0 & 0 & \boxed{1} & 0 & 0 & b^*_3 \\ 0 & 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 0 & \boxed{1} & b^*_m \end{array}\right]$$

1.6.3 Gaussian and Gauss-Jordan elimination

We can conduct elementary row operations to get our augmented matrix into row echelon or reduced row echelon form. The methods of transforming a matrix, or system, into row echelon and reduced row echelon form are referred to as Gaussian elimination and Gauss-Jordan elimination, respectively.

1.6.3.1 Elementary Row Operations

To do Gaussian and Gauss-Jordan elimination, we use three basic operations to transform the augmented matrix into another augmented matrix that represents an equivalent linear system: equivalent in the sense that the same values of x_j solve both the original and transformed matrix or system:

1.6.3.2 Interchanging Rows

Suppose we have the augmented matrix

$$\bar{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & | & b_1 \\ a_{21} & a_{22} & | & b_2 \end{bmatrix}$$

If we interchange the two rows, we get the augmented matrix

$$\begin{bmatrix} a_{21} & a_{22} & | & b_2 \\ a_{11} & a_{12} & | & b_1 \end{bmatrix}$$

which represents a linear system equivalent to that represented by matrix $\bar{\mathbf{A}}$.

1.6.3.3 Multiplying by a Constant

If we multiply the second row of matrix $\bar{\mathbf{A}}$ by a constant c, we get the augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & | & b_1 \\ ca_{21} & ca_{22} & | & cb_2 \end{bmatrix}$$

which represents a linear system equivalent to that represented by matrix $\bar{\mathbf{A}}$.

1.6.3.4 Adding (subtracting) Rows

If we add (subtract) the first row of matrix $\bar{\mathbf{A}}$ to (from) the second, we obtain the augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & | & b_1 \\ a_{11}+a_{21} & a_{12}+a_{22} & | & b_1+b_2 \end{bmatrix}$$

which represents a linear system equivalent to that represented by matrix $\bar{\mathbf{A}}$.

Example 1.9. Solve the following system of equations by using elementary row operations:

$$\begin{array}{rcrcr} x & - & 3y & = & -3 \\ 2x & + & y & = & 8 \end{array}$$

Solution.

$$\begin{array}{rcrcr} x & - & 3y & = & -3 \\ 2x & + & y & = & 8 \end{array}$$

$$\begin{array}{rcrcr} x & - & 3y & = & -3 \\ & & 7y & = & 14 \end{array}$$

$$\begin{array}{rcrcr} x & - & 3y & = & -3 \\ & & y & = & 2 \end{array}$$

$$\begin{array}{rcr} x & = & 3 \\ y & = & 2 \end{array}$$

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Exercise 1.6 (Solving Systems of Equations). Put the following system of equations into augmented matrix form. Then, using Gaussian or Gauss-Jordan elimination, solve the system of equations by putting the matrix into row echelon or reduced row echelon form.

1. $$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 2 \\ 3x & - & 2y & + & z & = & 1 \\ & & y & - & z & = & 3 \end{array}$$

2. $$\begin{array}{rcrcrcr} 2x & + & 3y & - & z & = & -8 \\ x & + & 2y & - & z & = & 12 \\ -x & - & 4y & + & z & = & -6 \end{array}$$

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1.7 Rank & Number of Solutions

To determine how many solutions exist, we can use information about (1) the number of equations m, (2) the number of unknowns n, and (3) the **rank** of the matrix representing the linear system.

Definition 1.6 (**Rank**). *The maximum number of linearly independent row or column vectors in the matrix.* ♠

This is equivalent to the number of nonzero rows of a matrix in row echelon form. For any matrix $\mathbf{A}$, the row rank always equals column rank, and we refer to this number as the rank of $\mathbf{A}$. For example $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ has Rank $= 3$, and $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ has Rank $= 2$.

Exercise 1.7 (Rank of Matrices). Find the rank of each matrix below: (Hint: transform the matrices into row echelon form. Remember that the number of nonzero rows of a matrix in row echelon form is the rank of that matrix)

1. $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 3 & 3 & -3 & 3 \\ 1 & 3 & 1 & 1 & 3 \\ 1 & 3 & 2 & -1 & -2 \\ 1 & 3 & 0 & 3 & -2 \end{bmatrix}$

$\diamondsuit$

1.8 The Inverse of a Matrix

Definition 1.7 (Identity Matrix). *The $n \times n$ identity matrix $\mathbf{I}_n$ is the matrix whose diagonal elements are* 1 *and all off-diagonal elements are* 0*.* ♠

Examples:

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 1.8 (Inverse Matrix). *An $n \times n$ matrix $\mathbf{A}$ is* nonsingular *or* invertible *if there exists an $n \times n$ matrix $\mathbf{A}^{-1}$ such that*

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. If there is no such $\mathbf{A}^{-1}$, then $\mathbf{A}$ is singular or not invertible. ♠

Example 1.10. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}.$$

Since

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

we conclude that $\mathbf{B}$ is the inverse, $\mathbf{A}^{-1}$, of $\mathbf{A}$ and that $\mathbf{A}$ is nonsingular.

$\diamondsuit$

1.8.1 Properties of the Inverse

- If the inverse exists, it is unique.
- If $\mathbf{A}$ is nonsingular, then $\mathbf{A}^{-1}$ is nonsingular.
- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular, then $\mathbf{Ab}$ is nonsingular
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- If $\mathbf{A}$ is nonsingular, then $(\mathbf{A}^\intercal)^{-1} = (\mathbf{A}^{-1})^\intercal$

1.8.2 Procedure to Find the Inverse

Given $\mathbf{A}^{-1}$; we know that if $\mathbf{B}$ is the inverse of $\mathbf{A}$, then

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n.$$

Looking only at the first and last parts of this

$$\mathbf{AB} = \mathbf{I}_n.$$

Solving for $\mathbf{B}$ is equivalent to solving for n linear systems, where each column of $\mathbf{B}$ is solved for the corresponding column in $\mathbf{I}_n$. We can solve the systems simultaneously by augmenting $\mathbf{A}$ with $\mathbf{I}_n$ and performing Gauss-Jordan elimination on $\mathbf{A}$. If Gauss-Jordan elimination on $[\mathbf{A}|\mathbf{I}_n]$ results in $[\mathbf{I}_n|\mathbf{B}]$, then $\mathbf{B}$ is the inverse of $\mathbf{A}$. Otherwise, $\mathbf{A}$ is singular.

Therefore, to calculate the inverse of $\mathbf{A}$:

1. Form the augmented matrix $[\mathbf{A}|\mathbf{I}_n]$
2. Using elementary row operations, transform the augmented matrix to reduced row echelon form.
3. The result of step 2 is an augmented matrix $[\mathbf{C}|\mathbf{B}]$.
   (a) If $\mathbf{C} = \mathbf{I}_n$, then $\mathbf{B} = \mathbf{A}^{-1}$.
   (b) If $\mathbf{C} \neq \mathbf{I}_n$, then $\mathbf{C}$ has a row of zeros. This means $\mathbf{A}$ is singular and $\mathbf{A}^{-1}$ does not exist.

Example 1.11. Find the inverse of the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$$

Solution. Solve using the following steps:

$$\mathbf{A}^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right]$$
$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$
$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/4 & 0 & 1/4 \\ 0 & 2 & 0 & -15/4 & 1 & 3/4 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$
$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/4 & 0 & 1/4 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$
$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13/8 & -1/2 & -1/8 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$
$$= \left[\begin{array}{ccc} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & -1/4 \end{array}\right]$$

◊

Exercise 1.8. Find the inverse of the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

◊

1.9 Linear Systems and Inverses

Let's return to the matrix representation of a linear system

$$\mathbf{A}\boldsymbol{x} = \mathbf{b}$$

If $\mathbf{A}$ is an $n \times n$ matrix, then $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ is a system of n equations in n unknowns. Note that $\boldsymbol{x}$ and $\mathbf{b}$ are $n \times 1$. Suppose $\mathbf{A}$ is nonsingular. Then

$\mathbf{A}^{-1}$ exists. To solve this system, we can multiply each side by $\mathbf{A}^{-1}$ and reduce it as follows:

$$\begin{aligned}
\mathbf{A}^{-1}(\mathbf{A}\boldsymbol{x}) &= \mathbf{A}^{-1}\mathbf{b} \\
(\mathbf{A}^{-1}\mathbf{A})\boldsymbol{x} &= \mathbf{A}^{-1}\mathbf{b} \\
\mathbf{I}_n\boldsymbol{x} &= \mathbf{A}^{-1}\mathbf{b} \\
\boldsymbol{x} &= \mathbf{A}^{-1}\mathbf{b}
\end{aligned}$$

Hence, given $\mathbf{A}$ and $\mathbf{b}$, and given that $\mathbf{A}$ is nonsingular, then $\boldsymbol{x} = \mathbf{A}^{-1}\mathbf{b}$ is a unique solution to this system.

Exercise 1.9 (Solve linear system using inverses). Use the inverse matrix to solve the following linear system:

$$\begin{aligned}
-3x + 4y &= 5 \\
2x - y &= -10
\end{aligned}$$

Hint: the linear system above can be written in the matrix form $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ given
$\mathbf{A} = \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$. ◊

1.10 Determinants

Definition 1.9 (Nonsingular). *A* square *matrix is nonsingular* iff *its* determinant is not zero. ♠

Determinants can be used to determine whether a square matrix is nonsingular. The determinant of a 1×1 matrix, $\mathbf{A}$, equals a_{11}. The determinant of a 2×2 matrix,

$$\mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

is

$$\begin{aligned}
\det(\mathbf{A}) &= |\mathbf{A}| \\
&= a_{11}|a_{22}| - a_{12}|a_{21}| \\
&= a_{11}a_{22} - a_{12}a_{21}
\end{aligned}$$

We can extend the second to last equation above to get the definition of the determinant of a 3×3 matrix:

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\
&\quad - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31})
\end{aligned}$$

Let's extend this now to any $n \times n$ matrix. Let's define $\mathbf{A}_{ij}$ as the $(n-1) \times (n-1)$ sub-matrix of $\mathbf{A}$ obtained by deleting row i and column j. Let the (i, j)-th **minor** of $\mathbf{A}$ be the determinant of$\mathbf{A}_{ij}$:

$$M_{ij} = |\mathbf{A}_{ij}|$$

Then for any $n \times n$ matrix $\mathbf{A}$

$$|\mathbf{A}| = a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{n+1}a_{1n}M_{1n}$$

Example 1.12 (Determinants)*.* Does the following matrix have an inverse?

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$$

Solution. We find the solution by:

1. Calculate its determinant.

$$\begin{aligned}
&= 1(2-15) - 1(0-15) + 1(0-10) \\
&= -13 + 15 - 10 \\
&= -8
\end{aligned}$$

2. Since $|\mathbf{A}| \neq 0$, we conclude that $\mathbf{A}$ has an inverse.

$\diamondsuit$

Exercise 1.10 (Determinants and Inverses)*.* Determine whether the following matrices are nonsingular:

1. $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 4 & 1 & 4 \end{bmatrix}$

$\diamondsuit$

1.11 Matrix Inverse using the Determinant

Thus far, we have algorithms to

1. Find the solution of a linear system,

2. Find the inverse of a matrix.

At this point, we have no way of telling how the solutions x_j change as the parameters a_{ij} and b_i change, except by changing the values and "re-solving" the algorithms.

With determinants, we have an explicit formula for the inverse, and therefore an explicit formula for the solution of an $n \times n$ linear system. Hence, we can examine how changes in the parameters and b_i affect the solutions x_j.

Definition 1.10 (**Determinant Formula for the Inverse**). *The determinant of a* 2×2 *matrix* $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is defined as:*

$$\frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

♠

Example 1.13 (Determinants and Inverses)*.* Calculate the inverse of matrix $\mathbf{A}$ from Exercise 1.9 using the determinant formula.

Solution. Recall,

$$\mathbf{A} = \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

then:

$$\det(\mathbf{A}) = (-3)(-1) - (4)(2) = 3 - 8 = -5$$

$$\frac{1}{\det(\mathbf{A})} \begin{bmatrix} -1 & -4 \\ -2 & -3 \end{bmatrix}$$

$$\frac{1}{-5} \begin{bmatrix} -1 & -4 \\ -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}.$$

◊

Exercise 1.11 (Calculate Inverse using Determinant Formula). Calculate the inverse of $\mathbf{A}$, where

$$\mathbf{A} = \begin{bmatrix} 3 & 5 \\ -7 & 2 \end{bmatrix}.$$

◊