Mathematical Derivations and Proofs

(with Dimensions and Variable Properties)

Mixed Integer Linear Programming - b

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Contents

1 Mathematical Derivations & Proofs

1.1 Introduction

Mixed-Integer Linear Programs (MILPs) augment Linear Programs (LPs) with *integrality* requirements on a subset of variables. The feasible set is the intersection of a rational polyhedron with a lattice, yielding a finite (but exponentially large) set of candidate integer assignments. MILPs are NP-hard in general, yet *exact* algorithms—notably *branch-and-bound* and *branch-and-cut*—solve large instances by combining linear relaxations (for tight *bounds*) with combinatorial search (for *integrality*). From first principles we derive: standard MILP form; LP relaxations and bounds; valid inequalities and classical cuts (Gomory mixed-integer, MIR, covers); correctness of branching; and linearization/decomposition tools. We use bold symbols and explicit dimensions as below.

1.2 Data and Notation

Let

$$\mathbf{A} \in \mathbb{R}^{m\times n}, \qquad \mathbf{b} \in \mathbb{R}^m, \qquad \mathbf{c} \in \mathbb{R}^n,$$

and decision vector $\mathbf{x} \in \mathbb{R}^n$ with componentwise bounds $\ell \le \mathbf{x} \le \mathbf{u}$, where $\ell, \mathbf{u} \in (\mathbb{R} \cup \{\pm\infty\})^n$. Partition the indices $\{1, \ldots, n\}$ into disjoint sets:

$$\mathcal{C} \text{ (continuous)}, \qquad \mathcal{I} \text{ (general integer)}, \qquad \mathcal{B} \text{ (binary)}, \qquad \mathcal{C} \cup \mathcal{I} \cup \mathcal{B} = \{1, \ldots, n\}.$$

Throughout, vectors are columns and inequalities are componentwise.

1.3 MILP Problem Formulation

Consider the following standard MILP in canonical form:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}} \quad & \mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \le \mathbf{b}, \\ & \mathbf{x} \ge \mathbf{0}, \quad \mathbf{y} \in \mathbb{Z}^m, \end{aligned}$$

where:

- $\mathbf{x} \in \mathbb{R}^n$ is the vector of continuous decision variables (column vector, $n \times 1$).
- $\mathbf{y} \in \mathbb{Z}^m$ is the vector of integer decision variables (column vector, $m \times 1$). In many applications, $\mathbf{y}$ is binary, i.e., $\mathbf{y} \in \{0, 1\}^m$.
- $\mathbf{c} \in \mathbb{R}^n$ is the cost coefficient vector for the continuous variables (column vector, $n \times 1$).
- $\mathbf{d} \in \mathbb{R}^m$ is the cost coefficient vector for the integer variables (column vector, $m \times 1$).
- $\mathbf{A} \in \mathbb{R}^{p\times n}$ is the constraint matrix associated with the continuous variables.
- $\mathbf{B} \in \mathbb{R}^{p\times m}$ is the constraint matrix associated with the integer variables.
- $\mathbf{b} \in \mathbb{R}^p$ is the right-hand side vector of constraints (column vector, $p \times 1$).

1.4 LP Relaxation and Dual Derivation

A common approach to solving an MILP is to first relax the integrality constraints on $\mathbf{y}$, allowing $\mathbf{y} \in \mathbb{R}^m$. The resulting LP relaxation is:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n,\, \mathbf{y}\in\mathbb{R}^m} \quad & \mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \le \mathbf{b}, \\ & \mathbf{x} \ge \mathbf{0}. \end{aligned}$$

Let the feasible region of the MILP be $\mathcal{F}_{MILP}$ and that of the LP relaxation be $\mathcal{F}_{LP}$. Notice that $\mathcal{F}_{MILP} \subseteq \mathcal{F}_{LP}$.

Lagrangian for the LP Relaxation

We form the Lagrangian by introducing dual variables $\boldsymbol{\lambda} \in \mathbb{R}^p_{\ge 0}$ associated with the inequality constraints:

$$\mathcal{L}(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}) = \mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} + \boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{b}).$$

Rewriting,

$$\mathcal{L}(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}) = \big(\mathbf{c}^\top + \boldsymbol{\lambda}^\top \mathbf{A}\big)\mathbf{x} + \big(\mathbf{d}^\top + \boldsymbol{\lambda}^\top \mathbf{B}\big)\mathbf{y} - \boldsymbol{\lambda}^\top \mathbf{b}.$$

Dual Function

The dual function $g(\boldsymbol{\lambda})$ is defined as the infimum of the Lagrangian over all $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{y} \in \mathbb{R}^m$:

$$g(\boldsymbol{\lambda}) = \inf_{\substack{\mathbf{x} \geq \mathbf{0} \\ \mathbf{y} \in \mathbb{R}^m}} \mathcal{L}(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}).$$

For the function $\mathcal{L}(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda})$ to be bounded below, we require:

$$\mathbf{c}^\top + \boldsymbol{\lambda}^\top \mathbf{A} \geq \mathbf{0} \quad \text{(componentwise)} \quad \Longleftrightarrow \quad \mathbf{A}^\top \boldsymbol{\lambda} \geq -\mathbf{c},$$

and

$$\mathbf{d}^\top + \boldsymbol{\lambda}^\top \mathbf{B} = \mathbf{0}^\top \quad \Longleftrightarrow \quad \mathbf{B}^\top \boldsymbol{\lambda} = -\mathbf{d}.$$

Assuming these conditions hold, the infimum is achieved at $\mathbf{x} = \mathbf{0}$ and $\mathbf{y} = \mathbf{0}$, giving:

$$g(\boldsymbol{\lambda}) = -\boldsymbol{\lambda}^\top \mathbf{b}.$$

Dual Problem

Thus, the dual problem is:

$$\begin{aligned} \max_{\boldsymbol{\lambda} \in \mathbb{R}^p_{\geq \mathbf{0}}} \quad & -\boldsymbol{\lambda}^\top \mathbf{b} \\ \text{subject to} \quad & \mathbf{A}^\top \boldsymbol{\lambda} \geq -\mathbf{c}, \\ & \mathbf{B}^\top \boldsymbol{\lambda} = -\mathbf{d}. \end{aligned}$$

For many MILP problems, the dual of the LP relaxation is used to obtain a lower bound on the MILP objective.

Weak Duality

Since the feasible region of the MILP is contained in the feasible region of the LP relaxation, and by the weak duality theorem for LP, we have:

$$z_{LP} \leq z_{MILP},$$

where z_{LP} is the optimal value of the LP relaxation and z_{MILP} is the optimal value of the MILP.

1.5 Reparameterized Formulation

Reparameterized the MILP formulation we have

$$\min_{\mathbf{x} \in \mathbb{R}^n} \ \mathbf{c}^\top \mathbf{x} \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \quad \boldsymbol{\ell} \leq \mathbf{x} \leq \mathbf{u}, \quad x_j \in \mathbb{Z} \ \ (j \in \mathcal{I}), \quad x_j \in \{0,1\} \ \ (j \in \mathcal{B}). \tag{1}$$

Inequalities can be converted to equalities with slacks, and free variables split as in LPs. The feasible set is $\mathcal{F} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \boldsymbol{\ell} \leq \mathbf{x} \leq \mathbf{u}\} \cap \big(\mathbb{R}^{\mathcal{C}} \times \mathbb{Z}^{\mathcal{I}} \times \{0,1\}^{\mathcal{B}}\big)$.

1.6 Convex Hull, and Bounds

Let $\mathcal{P} = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \boldsymbol{\ell} \leq \mathbf{x} \leq \mathbf{u}\}$ be the underlying polyhedron and $\mathcal{S} = \mathcal{F} \cap \mathbb{Z}^{\mathcal{I}} \times \{0,1\}^{\mathcal{B}}$ the set of integral feasible points projected onto all variables. The *LP relaxation* drops integrality:

$$z^\star_{\text{LP}} \ = \ \min \ \{ \ \mathbf{c}^\top \mathbf{x} \ : \ \mathbf{x} \in \mathcal{P} \ \}. \tag{2}$$

Lower bound (minimization). Since $\mathcal{S} \subseteq \mathcal{P}$,

$$z^\star_{\text{LP}} \ \leq \ z^\star_{\text{MILP}} \ = \ \min\{\mathbf{c}^\top \mathbf{x} : \mathbf{x} \in \mathcal{S}\}. \tag{3}$$

Proof. Minimizing over a superset cannot exceed the minimum over a subset. ■

Convex hull ideal. Let $\mathrm{conv}(\mathcal{S})$ be the convex hull of integer-feasible points. Then

$$\min\{\mathbf{c}^\top \mathbf{x} : \mathbf{x} \in \mathrm{conv}(\mathcal{S})\} \; = \; z^\star_{\mathrm{MILP}},$$

and $\mathrm{conv}(\mathcal{S})$ is a polyhedron describable by (possibly exponentially many) *valid inequalities.* Cutting-plane methods seek tractable subsets of these inequalities to strengthen Eqn. (2).

1.7 Valid Inequalities and Classical Cuts

An inequality $\boldsymbol{\pi}^\top \mathbf{x} \le \pi_0$ is *valid* for $\mathcal{S}$ if every $\mathbf{x} \in \mathcal{S}$ satisfies it. Adding valid inequalities can only increase $z^\star_{\mathrm{LP}}$, thereby tightening the bound Eqn. (3).

1.7.1 Gomory Mixed-Integer (GMI) Cuts

Consider a basic optimal tableau row of the LP relaxation in standard form with $\mathbf{x} \ge \mathbf{0}$:

$$x_i \; = \; \bar{b}_i \; - \; \sum_{j \in \mathcal{N}} \bar{a}_{ij}\, x_j, \qquad i \in \mathcal{BI} \tag{4}$$

where x_i is a *basic* variable constrained to be integer ($i \in \mathcal{BI}$), and $\mathcal{N}$ indexes nonbasic variables at that basis. Let $f_i \triangleq \{\bar{b}_i\} \in (0,1)$ denote the fractional part and decompose coefficients by sign:

$$\mathcal{N}^+ = \{j : \bar{a}_{ij} \ge 0\}, \qquad \mathcal{N}^- = \{j : \bar{a}_{ij} < 0\}.$$

The *GMI cut* derived from Eqn. (4) is

$$\sum_{j \in \mathcal{N}^+} \frac{\bar{a}_{ij}}{1 - f_i}\, x_j \; + \; \sum_{j \in \mathcal{N}^-} \frac{-\bar{a}_{ij}}{f_i}\, x_j \; \ge \; 1. \tag{5}$$

Validity Proof. Since x_i must be integer, rewrite Eqn. (4) mod 1 and apply the disjunction $x_i \le \lfloor \bar{b}_i \rfloor$ *or* $x_i \ge \lceil \bar{b}_i \rceil$. Aggregating and rescaling the two disjunctive linear inequalities yields Eqn. (5). Any LP basic solution with $f_i > 0$ violates Eqn. (5). ■

1.7.2 Mixed-Integer Rounding (MIR) Cuts

For a single constraint

$$\sum_{j \in \mathcal{I} \cup \mathcal{B}} a_j x_j \; + \sum_{j \in \mathcal{C}} a_j x_j \; \le \; b, \qquad 0 \le x_j \le u_j, \tag{6}$$

let $f = \{b\} \in [0,1)$ and set $\tilde{a}_j = \{a_j\}$ for integer-restricted j and $\tilde{a}_j = \max\{0, a_j\}$ for continuous j. The *MIR* inequality[1] is valid for $\mathcal{S}$ and cuts the fractional point whenever the LP row is violated in the “fractional part”. Derivations follow from mixed-integer rounding of Eqn. (6) and are special cases of Chvtal–Gomory cuts.

1.7.3 Cover Inequalities for 0–1 Knapsacks

For a binary knapsack $\sum_{j=1}^n w_j x_j \le W$, $x_j \in \{0,1\}$, a set C is a *cover* if $\sum_{j \in C} w_j > W$. Then the inequality

$$\sum_{j \in C} x_j \; \le \; |C| - 1 \tag{7}$$

is valid: any integer-feasible solution cannot select all items in C. Lifting adds variables not in C to strengthen Eqn. (7).

[1]One canonical presentation: $\sum_{j \in \mathcal{I} \cup \mathcal{B}} \min\{\tilde{a}_j, \frac{a_j}{1-f}\}\, x_j \; + \; \sum_{j \in \mathcal{C}} a_j^+ x_j \; \le \; \lfloor b \rfloor$; equivalent lifted forms exist.

1.8 Branch-and-Bound: Correctness and Fathoming

Tree structure. Branch-and-bound (B&B) explores a search tree whose nodes correspond to subproblems obtained by fixing or bounding some integer/binary variables. At node ν with additional bounds $\mathbf{L}^\nu \le \mathbf{x} \le \mathbf{U}^\nu$, solve its LP relaxation to obtain:

$$z_{\mathrm{LP}}^\nu \;=\; \min\{\mathbf{c}^\top\mathbf{x} : \mathbf{A}\mathbf{x} \le \mathbf{b},\; \ell \vee \mathbf{L}^\nu \le \mathbf{x} \le \mathbf{u} \wedge \mathbf{U}^\nu\}.$$

Bounding and fathoming rules (minimization). Maintain an incumbent (best known integer-feasible) value $z^\star$. Node ν is *fathomed* if any holds:

1. **Infeasible relaxation:** the node LP is infeasible $\Rightarrow$ no feasible integer descendant exists.
2. **Bounded out:** $z_{\mathrm{LP}}^\nu \ge z^\star$ (by Eqn. (3), no descendant can improve).
3. **Integer feasible:** LP optimum $\mathbf{x}^\nu$ is integer (on $\mathcal{I} \cup \mathcal{B}$), so update $z^\star \leftarrow \min\{z^\star, \mathbf{c}^\top\mathbf{x}^\nu\}$ and fathom.

Branching (correctness). *Proof.* If $\mathbf{x}^\nu$ violates integrality, pick an index $k \in \mathcal{I} \cup \mathcal{B}$ with fractional x_k^ν and create two child nodes

$$\text{left: } x_k \le \lfloor x_k^\nu \rfloor, \qquad \text{right: } x_k \ge \lceil x_k^\nu \rceil.$$

Every integer vector satisfies one branch; thus, the union of child feasible sets equals the parent feasible set restricted to integrality. Recursively applying bounding/fathoming yields an exact algorithm that terminates because a finite number of integral assignments exist under finite bounds. ■

Node and variable selection. Depth-first (memory light), best-bound (bound-driven), and hybrid policies are common. Strong branching scores candidate variables by estimated bound improvement before committing.

1.9 Branch-and-Cut: Integrating Cuts into B&B

At a node ν, after solving the LP relaxation, *separate* the current fractional solution $\mathbf{x}^\nu$ from $\mathrm{conv}(\mathcal{S}^\nu)$ by adding valid inequalities (e.g., GMI, MIR, covers). Re-solve, repeat until no violated cuts are found, then branch if still fractional. **Correctness:** Only valid inequalities for $\mathcal{S}^\nu$ are added, so no integer-feasible point is removed; bounds remain valid, preserving exactness.

1.10 Classical Linearizations (Common Modeling Primitives)

1.10.1 Binary-Continuous Product

Let $y \in \{0,1\}$, $0 \le x \le U$, and $z = yx$. The convex hull of feasible (x, y, z) is enforced by

$$0 \le z \le Uy, \qquad z \le x, \qquad z \ge x - U(1-y), \qquad 0 \le x \le U,\; y \in \{0,1\}. \tag{8}$$

Correctness. *Proof.*If $y = 0$, Eqn. (8) forces $z = 0$ and $0 \le x \le U$. If $y = 1$, it forces $z = x$. The four inequalities describe $\mathrm{conv}\{(x, y, z) : z = yx\}$ with these bounds. ■

1.10.2 Binary–Binary Product

For $w = y_1y_2$, $y_1, y_2 \in \{0,1\}$, the convex hull is

$$w \le y_1, \quad w \le y_2, \quad w \ge y_1 + y_2 - 1, \quad w \in \{0,1\}. \tag{9}$$

1.10.3 General Bilinear with Bounds (McCormick Envelopes)

For bounded continuous $x \in [\ell_x, u_x]$, $u \in [\ell_u, u_u]$, and $z = xu$ (leading to a MILP after embedding with binaries), the convex envelope over the rectangle is

$$z \geq \ell_x u + \ell_u x - \ell_x \ell_u, \qquad z \geq u_x u + u_u x - u_x u_u, \tag{10}$$

$$z \leq u_x u + \ell_u x - u_x \ell_u, \qquad z \leq \ell_x u + u_u x - \ell_x u_u. \tag{11}$$

1.11 Decomposition for Structure

1.11.1 Lagrangian Relaxation

Partition constraints $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ into *easy* and *complicating* sets $\mathbf{A}_1\mathbf{x} \leq \mathbf{b}_1$ and $\mathbf{A}_2\mathbf{x} \leq \mathbf{b}_2$. Relax the second with multiplier $\boldsymbol{\lambda} \geq 0$:

$$\min_{\mathbf{x} \in X} \mathbf{c}^\top \mathbf{x} + \boldsymbol{\lambda}^\top (\mathbf{A}_2\mathbf{x} - \mathbf{b}_2), \quad X = \{\mathbf{x} : \mathbf{A}_1\mathbf{x} \leq \mathbf{b}_1,\ \boldsymbol{\ell} \leq \mathbf{x} \leq \mathbf{u},\ x_j \in \mathbb{Z} \text{ on } \mathcal{I} \cup \mathcal{B}\}.$$

Maximizing over $\boldsymbol{\lambda} \geq 0$ gives a *dual bound* (never exceeding $z^\star_{\text{MILP}}$). Subgradients are $\mathbf{A}_2\mathbf{x} - \mathbf{b}_2$.

1.11.2 Benders Decomposition (LP Subproblem)

With variables split $(\mathbf{x}, \mathbf{y})$ and structure $\min\{\mathbf{c}_x^\top \mathbf{x} + \mathbf{c}_y^\top \mathbf{y} :\ \mathbf{E}\mathbf{x} + \mathbf{F}\mathbf{y} \leq \mathbf{g},\ \mathbf{x} \in \{0,1\}^p,\ \mathbf{y} \in \mathbb{R}^q\}$, fix $\mathbf{x} = \bar{\mathbf{x}}$ to get an LP in $\mathbf{y}$; its dual provides *optimality* and *feasibility* cuts in $(\mathbf{x}, \eta)$ for a master MILP. Iterating converges finitely for rational data.

1.12 Algorithmic Summary (Branch-and-Cut MILP Solver)

1. **Presolve & root LP:** tighten bounds, eliminate redundancies; solve the root LP relaxation to get z_{LP} and $\mathbf{x}^{\text{LP}}$.
2. **Cut separation:** while a separator finds violated valid inequalities (GMI/MIR/covers/problem-specific), add them and re-solve.
3. **Integrality check:** if $\mathbf{x}^{\text{LP}}$ integral, set incumbent and **stop**.
4. **Branching:** select a fractional variable k and create children with $x_k \leq \lfloor x_k^{\text{LP}} \rfloor$ and $x_k \geq \lceil x_k^{\text{LP}} \rceil$.
5. **Node processing:** for each open node: solve LP, add local cuts, apply fathoming rules; update incumbent and global bound.
6. **Selection:** choose next node (best-bound/depth-first/hybrid) until the global lower bound $\geq$ incumbent (within tolerance).

1.13 Complexity and Optimality Guarantees

MILP is NP-hard in general, yet the algorithm above is *exact*:

- **Finite termination:** With finite bounds and integral domains, the search tree has finitely many leaves; fathoming prevents infinite descent.
- **Global optimality:** At termination, the incumbent is feasible and the global bound $\geq$ incumbent; by Eqn. (3), no unprocessed node can improve, so optimality holds.

1.14 Summary of Variables and Their Dimensions

- $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$: constraint matrix, RHS, and costs.
- $\mathbf{x} \in \mathbb{R}^n$: decision variables with bounds $\ell \le \mathbf{x} \le \mathbf{u}$.
- Index sets: $\mathcal{C}$ (continuous), $\mathcal{I}$ (general integer), $\mathcal{B}$ (binary).
- $\mathcal{P}$: LP feasible polyhedron; $\mathcal{S}$: integer-feasible set; $\text{conv}(\mathcal{S})$: convex hull.
- $z^\star_{\text{LP}}, z^\star_{\text{MILP}} \in \mathbb{R}$: optimal objective values of LP relaxation and MILP.
- Node data: bounds $(\mathbf{L}^\nu, \mathbf{U}^\nu)$, relaxation value z^ν_{LP}, incumbent $z^\star$.
- Cut coefficients: $\bar{a}_{ij}, \bar{b}_i$ (tableau), $f_i = \{\bar{b}_i\}$ (fractional part), weights w_j, capacity W for covers.

1.15 Summary

From first principles, a MILP minimizes a linear objective over the intersection of a polyhedron and an integer lattice. The LP relaxation supplies rigorous lower bounds; the *ideal* relaxation is the convex hull $\text{conv}(\mathcal{S})$, approximated in practice via *valid inequalities* (GMI, MIR, cover cuts). *Branch-and-bound* is correct because branching partitions the integer domain while LP bounds prune subtrees; *branch-and-cut* tightens relaxations with cuts at each node without excluding any integer-feasible point. Standard linearizations (binary–continuous, binary–binary, McCormick) embed common nonlinear primitives into MILP form, and decomposition (Lagrangian/Benders) exploits structure for stronger bounds. Together, these ingredients yield exact, scalable algorithms with provable optimality certificates for mixed-integer linear optimization.