Time Integration Methods

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1 Introduction

In many cases, physical systems will be in an initial equilibrium state and then be subjected to time-varying influences on the system. Based upon the initial state and the time-varying influences such systems will show a specific response that needs to be predicted for engineering purposes.

Mathematical models are most often used to model the behaviour of physical systems. These are usually defined by partial differential equations derived from fundamental theories of physics (e.g. preservation of energy, equilibrium of forces). Systems with time-varying influences and behaviour are referred to as “initial-value problems”.

The solution of mathematical models in engineering is often accomplished using the Finite Element Method (FEM). Using FEM to treat the spatial aspects of such problems the partial differential equations can be converted to a (generally large) set of ordinary differential equations.

Initial value problems can be solved in two separate ways. Either a transformation is performed into a different, orthogonal system of so-called eigensolutions. Or a time integration (time-stepping) scheme is used to solve the system in a sequence of discrete steps tracing the response of the system from the initial state step-by-step in (generally a large) number of small steps to its final state.

Finding the eigensolutions of large systems is a very complex task often beyond the limits of computation. Performing a time integration solution is therefore an appropriate choice for many problems in engineering.

In particular, problems in instationary heat transfer may be solved using this approach.

Another important field of application are problems in structural dynamics, e.g. earthquake excitation and response.

This script will cover aspects of time integration schemes often used for these purposes in engineering.

2 Heat Transfer Analysis

The differential form of the **system equation** governing certain forms of physical behaviour of a three-dimensional continuum is given as follows:

$$\frac{\partial f_0(x_0)}{\partial x_0} + \frac{\partial f_1(x_1)}{\partial x_1} + \frac{\partial f_2(x_2)}{\partial x_2} = q \qquad \text{for all } \mathbf{x} \in K$$

The application of this general form to the problem of heat transfer in a 2-dimensional body results in (see for example: [1]):

$$\frac{\partial}{\partial x_0}(k\frac{\partial T}{\partial x_0}) + \frac{\partial}{\partial x_1}(k\frac{\partial T}{\partial x_1}) = \rho c \frac{\partial T}{\partial t} \qquad \text{for all } \mathbf{x} \in K$$

with: k = thermal conductivity
c = specific heat
ρ = density of the body

the boundary conditions are:

$T(x,t)$ = f(x, t)	on that part of the boundary where the temperature is specified
$k\frac{\partial T}{\partial n}(x,t)$ = g(x, t)	on that part of the boundary where the temperature stream in the direction of the outward normal vector is specified

where f(x,t) and g(x,t) are known functions that must be predefined.

In case, the problem to be solved is an initial value problem, additional initial conditions are given by:

$$T(x,0) = T_0(x) \qquad \text{for all } \mathbf{x} \in K$$

Applying the Finite Element Method on the spatial aspects of the problem, the partial differential equation can be converted to a set of ordinary differential equations of the form:

$$\mathbf{C}\dot{\mathbf{T}}(t) + \mathbf{K}\mathbf{T}(t) = \mathbf{F}(t) \qquad \text{and} \quad \mathbf{T}(0) = \mathbf{T}_0 \qquad (1)$$

where: $\mathbf{C}$ is the specific heat matrix

$\dot{\mathbf{T}}$ is the vector of temperature gradients
$\mathbf{K}$ is the thermal conductivity matrix
$\mathbf{T}$ is the temperature vector
$\mathbf{F}$ is forcing function vector of induced temperatures
$\mathbf{T}_0$ is the initial temperature vector at time=0

Eigenfunction Solution

Substitution of the eigenfunction solution for the homogeneous case with

$$\mathbf{T} = e^{-\beta t}\,\mathbf{q} \qquad \text{and} \qquad \dot{\mathbf{T}} = \beta\, e^{-\beta t}\,\mathbf{q}$$

results in: $e^{-\beta t}(\mathbf{K} - \beta\,\mathbf{C})\,*\mathbf{q} = \mathbf{0}$

where: β is one of n-eigenvalues for which this equation is true
q is the corresponding eigenvector

The solution for eigenvalues is given by:

$$\det\,(\mathbf{K} - \beta\,\mathbf{C}) = 0$$

This, however, is only a viable solution for small values of n.

The general solution of this eigenproblem yields a number of n-eigenpairs $(\beta_i, \mathbf{q}_i)$ with an eigenvalue and eigenvector each. The eigenvectors are both $\mathbf{K}\,and\,\mathbf{C}$ orthogonal.

The solution for all eigenpairs can be written as:

$$\mathbf{T} = \mathbf{Q}\,\mathbf{y}$$

with: $\mathbf{Q} = [\mathbf{q_0} \quad \mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_{n-1}]$

and: $\mathbf{y} = [y_0(t) \quad y_1(t) \quad y_2(t) \quad \ldots \quad y_{n-1}(t)]$, the modal participation

Substituting this into equation (1), and premultiplying by $\mathbf{Q}^T$ results in:

$$\mathbf{Q}^T\mathbf{C}\,\mathbf{Q}\,\dot{\mathbf{y}} + \mathbf{Q}^T\,\mathbf{K}\,\mathbf{Q}\,\mathbf{y} = \mathbf{Q}^T\,\mathbf{F}$$

since $\mathbf{Q}$ is both $\mathbf{K}\,and\,\mathbf{C}$ orthogonal, this will result in the following set of equations:

$$c_i\,\dot{y} + k_i\,y = f_i \qquad \text{with: } \beta_i = \frac{k_i}{c_i}$$

where: $c_i = \mathbf{q}_i^T\,\mathbf{C}\,\mathbf{q_i}$, $\quad k_i = \mathbf{q}_i^T\,\mathbf{K}\,\mathbf{q_i}$, $\quad f_i = \mathbf{q}_i^T\,\mathbf{F}$

Time Integration Solution

Solving equations of this type in time, i.e. at a set of discrete times t_k , can be accomplished by a time stepping scheme such as (Ralston)

$$y_{k+1} = \sum_{i=0}^{p} a_i \, y_{k-i} + \Delta t \sum_{i=-1}^{p} b_i \, \dot{y}_{k-i} \; + \text{R (truncation error)}$$

For p>0 this is called a multi-step method and for p= 0, it is called a one-step method:

$$y_{k+1} = a_0 \, y_k + \Delta t \, (b_{-1} \, \dot{y}_{k+1} + b_0 \, \dot{y}_k)$$

Application of time stepping schemes in the solution of initial value problems are approximations to the exact solution and must therefore address the issues of

consistency, **stability** and **order of method**

Consistency and **stability** are both necessary and sufficient conditions for a method to be convergent, i.e. for the discrete solution to converge to the solution as the time step approaches zero ($\Delta t \rightarrow 0$).

These conditions result in the requirements that:

$$a_0 = 1, \;\; and \;\; b_{-1} + b_0 = 1$$

which can be reduced to a single variable parameter α that can be used for controlling stability resulting in the following equation that is consistent for all α:

$$y_{k+1} = y_k + \Delta t \, (\alpha \, \dot{y}_{k+1} + (1-\alpha) \, \dot{y}_k)$$

Stability considerations are controlled by the parameter α and lead to the following definition of the critical time step Δt_{cr}:

$$\Delta t_{cr} = \frac{2}{\beta_{\max}(1-2\alpha)}$$

where β_{max} is the largest eigenvalue of problem to be solved.

The **order** of the method can be shown to be of first order unless α=½ when it becomes second order. This gives an indication of the rate of convergence as the time step approaches zero ($\Delta t \rightarrow 0$).

For α=0, the method is called **explicit method** as the Right-Hand-Side only contains information at the current time step k. The critical time step is:

$$\Delta t_{cr} = \frac{2}{\beta_{\max}}$$

For nonzero values of α, the method is called **implicit method** as the Right-Hand-Side also contains information at the next time step (k+1).

For ($0 \le \alpha < ½$) the method is called **conditionally stable** with $\Delta t \le \Delta t_{cr}$. For ($½ \le \alpha \le 1$) the method is called **unconditionally stable** and any time step can be used with regard to stability but must be chosen sufficiently small for accuracy.

Generalization of this method to multi-degree-of-freedom systems in heat transfer leads to:

$$\mathbf{T}_{k+1} = \mathbf{T}_k + \Delta t\ (\alpha \dot{\mathbf{T}}_{k+1} + (1-\alpha)\dot{\mathbf{T}}_k)$$

or $$\mathbf{T}_k = \mathbf{T}_{k-1} + \Delta t\ (\alpha \dot{\mathbf{T}}_k + (1-\alpha)\dot{\mathbf{T}}_{k-1})$$

The solution algorithm can either be defined in $\dot{\mathbf{T}}_k$ as the primary dependent variables or in $\mathbf{T}_k$. Using $\dot{\mathbf{T}}_k$ for these purposes has the advantage of being able to consider all values of α and Δt.

Substitution into equation (1) $\mathbf{C}\dot{\mathbf{T}}_k + \mathbf{K}\ \mathbf{T}_k = \mathbf{F}_k$ results in:

$$(\mathbf{C} + \alpha\, \Delta t\, \mathbf{K})\dot{\mathbf{T}}_k = \mathbf{F}_k - \mathbf{K}\,(\mathbf{T}_{k-1} + \Delta t\,(1-\alpha)\dot{\mathbf{T}}_{k-1})$$

The solution algorithm is defined by the following steps:

<table>
<tr><td colspan="2">at time =0: calculate initial temperature gradients $\dot{\mathbf{T}}_0$
$\mathbf{C}\ \dot{\mathbf{T}}_0 = \mathbf{F}_0 - \mathbf{K}\ \mathbf{T}_0$</td></tr>
<tr><td colspan="2">evaluate constant coefficient matrix $\mathbf{CM} = (\mathbf{C} + \alpha\, \Delta t\, \mathbf{K})$</td></tr>
<tr><td colspan="2">decompose constant coefficient matrix only once</td></tr>
<tr><td>for k=1 and</td><td>k < timeSteps</td></tr>
<tr><td></td><td>time = time+Δt</td></tr>
<tr><td></td><td>calculate temperature gradients at restrained nodes
$\dot{\mathbf{T}}_k = (\mathbf{T}_k - \mathbf{T}_{k-1})/\Delta t$</td></tr>
<tr><td></td><td>calculate $\hat{\mathbf{T}} = \mathbf{T}_{k-1} + \Delta t\,(1-\alpha)\dot{\mathbf{T}}_{k-1}$</td></tr>
<tr><td></td><td>modification of RHS: $\mathbf{R} = \mathbf{F}_k - \mathbf{K}\ \hat{\mathbf{T}}$</td></tr>
<tr><td></td><td>backsubstitution of coefficient matrix with new RHS
$(\mathbf{C} + \alpha\, \Delta t\, \mathbf{K})\dot{\mathbf{T}}_k = \mathbf{R}$, solve for temperature gradient $\dot{\mathbf{T}}_k$</td></tr>
<tr><td></td><td>compute temperature at next time step $\mathbf{T}_k = \hat{\mathbf{T}} + \alpha\, \Delta t\ \dot{\mathbf{T}}_k$</td></tr>
</table>

It should be noted that in case α=0: $\mathbf{T}_k = \hat{\mathbf{T}}$. If $\mathbf{C}$ is diagonal, no equations need to be solved.

2.1 Example: Two Degree-of-Freedom System

Consider the following two degree-of-freedom problem:

$$\dot{\mathbf{x}}_1 + \mathbf{x}_1 - \mathbf{x}_2 = f(t)$$
$$\dot{\mathbf{x}}_2 - \mathbf{x}_1 + 1.01\,\mathbf{x}_2 = 0$$

In matrix formulation:

$$\begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{Bmatrix} 1 & -1 \\ -1 & 1.01 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f(t) \\ 0 \end{Bmatrix}$$

Compute the eigenvalues.

$$\begin{vmatrix} 1-\beta & -1 \\ -1 & 1.01-\beta \end{vmatrix} = (1-\beta)(1.01-\beta) - 1 = 0$$

$$\beta^2 - 2.01\beta + 0.01 = 0$$
$$\beta_1 = 2.0050125$$
$$\beta_1 = 0.005$$

Write a program to solve this system by one of the algorithms discussed. Solve for the two load cases

Problem 1: $\mathbf{x}_1(0) = \mathbf{x}_2(0) = 1;$ $\quad f(t) \equiv 0$
Problem 2: $\mathbf{x}_1(0) = \mathbf{x}_2(0) = 0;$ $\quad f(t) = \sin(.03t)$
for $0 \le t \le 400$.

For each problem consider the following α and Δt values:

1. $\alpha = 0$; Δt for stability; $\Delta t = 2$
2. $\alpha = ½$ $\Delta t = 5, 10, 20, 40$
3. $\alpha = ⅔$ $\Delta t = 5, 10, 20, 40$
4. $\alpha = 1$ $\Delta t = 5, 10, 20, 40$

3 Dynamic Motion Analysis

problem: $\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{f}(t)$ and $\mathbf{u}(0)=\mathbf{d_0}$, $\dot{\mathbf{u}}(0)=\mathbf{v_0}$

where: $\mathbf{M}$ is the structural mass matrix

$\ddot{\mathbf{u}}$ is the acceleration vector

$\mathbf{C}$ is the structural damping matrix

$\dot{\mathbf{u}}$ is the velocity vector

$\mathbf{K}$ is the structural stiffness matrix

$\mathbf{u}$ is the displacement vector

$\mathbf{f}$ is forcing function vector of predefined deformations

$\mathbf{d_0}$ and $\mathbf{v_0}$ are the initial deformation and velocity vectors

Eigenfunction Solution

Substitution of the eigenfunction solution for the homogeneous case with

$$\mathbf{u}=e^{i\omega t}\,\mathbf{q} \qquad \dot{\mathbf{u}}=i\,\omega e^{i\omega t}\,\mathbf{q} \qquad \ddot{\mathbf{u}}=-\omega^2\,e^{i\omega t}\,\mathbf{q}$$

results in: $e^{i\omega t}(\mathbf{K}-\omega^2\,\mathbf{M})\,*\mathbf{q}=\mathbf{0}$

where: ω is one of n-eigenvalues for which this equation is true

q is the corresponding eigenvector

The solution for eigenvalues is given by:

$$\det\,(\mathbf{K}-\omega^2\,\mathbf{M})=0$$

This, however, is only a viable solution for small values of n.

The general solution of this eigenproblem yields a number of n-eigenpairs $(\omega_i, \mathbf{q}_i)$) with an eigenvalue and eigenvector each. The eigenvectors are both $\mathbf{K}\,and\,\mathbf{M}$ orthogonal.

The solution for all eigenpairs can be written as:

$$\mathbf{u}=\mathbf{Q}\,\mathbf{y}$$

with: $\mathbf{Q}=[\mathbf{q_0} \quad \mathbf{q}_1 \quad \mathbf{q}_2 \quad ... \quad \mathbf{q}_{n-1}]$ matrix of eigenvectors for free vibration, undamped set of equations

and: $\mathbf{y}=[y_0(t) \quad y_1(t) \quad y_2(t) \quad ... \quad y_{n-1}(t)]$, vector of modal participations

Substituting this into equation (1), and premultiplying by $\mathbf{Q}^T$ results in:

$$\mathbf{Q}^T\mathbf{M}\mathbf{Q}\,\ddot{\mathbf{y}} + \mathbf{Q}^T\mathbf{C}\mathbf{Q}\,\dot{\mathbf{y}} + \mathbf{Q}^T\,\mathbf{K}\mathbf{Q}\,\mathbf{y} = \mathbf{Q}^T\,\mathbf{f}$$

$\mathbf{Q}$ is both $\mathbf{K}$ *and* $\mathbf{M}$ orthogonal. Q, however, is not generally C orthogonal. It will only be C orthogonal if damping is assumed to be either mass or stiffness proportional modal damping. This will result in the following equations for each mode:

$$m_i\,\ddot{y} + c_i\,\dot{y} + k_i\,y = f_i \qquad \text{with:} \quad \omega_i^{\,2} = \frac{k_i}{m_i}$$

frequency of each modal response

where:
$$m_i = \mathbf{q}_i^T\,\mathbf{M}\,\mathbf{q_i}$$
$$c_i = \mathbf{q}_i^T\,\mathbf{C}\,\mathbf{q_i}$$
$$k_i = \mathbf{q}_i^T\,\mathbf{K}\,\mathbf{q_i}$$
$$f_i = \mathbf{q}_i^T\,\mathbf{f}$$

Often, these equations are written in the form:

$$\ddot{y} + 2\xi\omega\,\dot{y} + \omega^2\,y = f/m \qquad \text{with:} \quad \xi = \frac{c}{2\sqrt{\frac{k}{m}}}$$

initial conditions are defined by:

$$y(0) = d_0 \quad \text{and} \quad \dot{y}(0) = v_0$$

A distinction is made between

$(0 < \xi < 1)$	underdamped systems
$\xi = 1$	critical damping
$\xi > 1$	overdamped systems

Time Integration Solution

Various numerical methods were developed for Numerical Time Integration Methods in Structural Dynamics. Reference [3] presents an overview over the design and evaluation of these methods.

1. **Newmark's method [4]**:

solve: $\mathbf{M}\,\mathbf{a}_{k+1} + \mathbf{C}\,\mathbf{v}_{k+1} + \mathbf{K}\,\mathbf{d}_{k+1} = \mathbf{F}_{k+1}$

Newmark's method is defined by the following formulas:

$$\mathbf{d}_{k+1} = \mathbf{d}_k + \Delta t\,\mathbf{v}_k + (\tfrac{1}{2} - \beta)\,\Delta t^2\,\mathbf{a}_k + \beta\,\Delta t^2\,\mathbf{a}_{k+1}$$

$$\mathbf{v}_{k+1} = \mathbf{v}_k + (1-\gamma)\,\Delta t\,\mathbf{a}_k + \gamma\,\Delta t\,\mathbf{a}_{k+1}$$

The algorithm can be implemented in terms of the unknown variables for $\mathbf{a}_{k+1}$, $\mathbf{v}_{k+1}$ or $\mathbf{d}_{k+1}$. Using $\mathbf{a}_{k+1}$ for these purposes simply requires substitution of Newmark's formulas into the governing equations resulting in:

$$(\mathbf{M} + \gamma\,\Delta\mathbf{t}\,\mathbf{C} + \beta\,\Delta\mathbf{t}^2\,\mathbf{K})\,\mathbf{a}_{k+1} = \mathbf{F}_{k+1} - \mathbf{C}\,\hat{\mathbf{v}}_{k+1} - \mathbf{K}\,\hat{\mathbf{d}}_{k+1}$$

where:
$$\hat{\mathbf{d}}_{k+1} = \mathbf{d}_k + \Delta t\,\mathbf{v}_k + \Delta t^2\,(\tfrac{1}{2} - \beta)\,\mathbf{a}_k$$
$$\hat{\mathbf{v}}_{k+1} = \mathbf{v}_k + \Delta t\,(1-\gamma)\,\mathbf{a}_k \qquad \text{are called the predictors}$$

and:
$$\mathbf{d}_{k+1} = \hat{\mathbf{d}}_{k+1} + \beta\,\Delta t^2\,\mathbf{a}_{k+1}$$
$$\mathbf{v}_{k+1} = \hat{\mathbf{v}}_{k+1} + \gamma\,\Delta t\,\mathbf{a}_{k+1} \qquad \text{are called the correctors}$$

The method is consistent for all values of γ and β. It is unconditionally stable for $(\gamma \geq \tfrac{1}{2})$ and $(\beta \geq \gamma/2)$.

It is conditionally stable for $(\gamma \geq \tfrac{1}{2})$ and $(\beta < \gamma/2)$. The critical time step is:

$$\omega\,\Delta t_{cr} = \frac{\xi(\gamma - \tfrac{1}{2}) + [\gamma/2 - \beta + \xi^2(\gamma - \tfrac{1}{2})^2]^{1/2}}{(\gamma/2 - \beta)}$$

2. **Wilson's Θ-method [5]**:

Wilson suggested in 1973 a solution approach as a special case of equilibrium collocation. The general family of equilibrium collocation methods is defined by two parameters (β,γ) and the time step $(t+\Delta t)$ where equilibrium is satisfied. Wilson suggested equilibrium to be satisfied a time step of $(t+\theta\Delta t)$ or $(t+\tau)$ where $(1\le\theta<2)$.

This approach results in the following modifications to Newmark's scheme:

$$\mathbf{d}_{k+\theta}=\mathbf{d}_k+\tau\,\mathbf{v}_k+(\tfrac{1}{2}-\beta)\tau^2\,\mathbf{a}_k+\beta\tau^2\,\mathbf{a}_{k+\theta}$$
$$\mathbf{v}_{k+\theta}=\mathbf{v}_k+(1-\gamma)\tau\,\mathbf{a}_k+\gamma\tau\,\mathbf{a}_{k+\theta}$$

The algorithm will again be implemented in terms of the unknown variables for $\mathbf{a}_{k+\theta}$. This simply requires substitution of the above formulas into the governing equations resulting in:

$$\mathbf{M}\,\mathbf{a}_{k+\theta}+\mathbf{C}\,\mathbf{v}_{k+\theta}+\mathbf{K}\,\mathbf{d}_{k+\theta}=\mathbf{F}_{k+\theta}$$
$$(\mathbf{M}+\gamma\tau\,\mathbf{C}+\beta\tau^2\,\mathbf{K})\,\mathbf{a}_{k+\theta}=\mathbf{F}_{k+\theta}-\mathbf{C}\,\hat{\mathbf{v}}_{k+\theta}-\mathbf{K}\,\hat{\mathbf{d}}_{k+\theta}\qquad\text{with: }\tau=\theta\,\Delta t$$

where: $\mathbf{F}_{k+\theta}=(1-\theta)\mathbf{F}_k+\theta\mathbf{F}_{k+1}$ collocation of force

$$\hat{\mathbf{d}}_{k+\theta}=\mathbf{d}_k+\tau\,\mathbf{v}_k+\tau^2\,(\tfrac{1}{2}-\beta)\,\mathbf{a}_k$$
$$\hat{\mathbf{v}}_{k+\theta}=\mathbf{v}_k+\tau\,(1-\gamma)\,\mathbf{a}_k$$

the system is solved for $\mathbf{a}_{k+\theta}$ and from collocation of acceleration $\mathbf{a}_{k+\theta}=(1-\theta)\,\mathbf{a}_k+\theta\,\mathbf{a}_{k+1}$ we get

$$\mathbf{a}_{k+1}=\mathbf{a}_k+\frac{1}{\theta}(\mathbf{a}_{k+\theta}-\mathbf{a}_k)$$

with this, Newmark's algorithm can proceed as defined above

$$\mathbf{d}_{k+1}=\mathbf{d}_k+\Delta t\,\mathbf{v}_k+\Delta t^2\,(\tfrac{1}{2}-\beta)\,\mathbf{a}_k+\beta\,\Delta t^2\,\mathbf{a}_{k+1}$$
$$\mathbf{v}_{k+1}=\mathbf{v}_k+\Delta t\,(1-\gamma)\,\mathbf{a}_k+\gamma\,\Delta t\,\mathbf{a}_{k+1}$$

The method is generally "started" by determining $\mathbf{a}_0$ from the initial conditions which is not consistent with the general approach ($\mathbf{a}_{0+\theta}$). This effect is documented in the literature and may lead to "overshoot" in early parts of an analysis.

For values of $\beta=1/6$ and $\gamma=1/2$ the method is referred to as Wilson's Θ-method. The value of $(\theta=1.420815)$ is recommended for this method with the additional requirement for $(\theta>1.366025)$.

3. **Taylor's α-method [6]**:

It is well documented in the literature that solution algorithms for time integration methods may also induce "algorithmic damping" into the system which is not part of the physical problem. In an attempt to improve the algorithmic damping of Newmark's method, Taylor suggested in 1977 a modification of Newmark's method which is referred to in the literature as the α-method.

In this method equilibrium is modified by including some stiffness proportional damping into the equilibrium equations:

$$\mathbf{M}\,\mathbf{a}_{k+1} + \mathbf{C}\,\mathbf{v}_{k+1} + (1+\alpha)\mathbf{K}\,\mathbf{d}_{k+1} - \alpha\,\mathbf{K}\,\mathbf{d}_k = \mathbf{F}_{k+1}$$

$$\mathbf{d}_{k+1} = \mathbf{d}_k + \Delta t\,\mathbf{v}_k + (\tfrac{1}{2} - \beta)\,\Delta t^2\,\mathbf{a}_k + \beta\,\Delta t^2\,\mathbf{a}_{k+1}$$
$$\mathbf{v}_{k+1} = \mathbf{v}_k + (1-\gamma)\,\Delta t\,\mathbf{a}_k + \gamma\,\Delta t\,\mathbf{a}_{k+1}$$

The algorithm will again be implemented in terms of the unknown variables for $\mathbf{a}_{k+1}$. This simply requires substitution of the above formulas into the governing equations resulting in:

$$(\mathbf{M} + \gamma\,\Delta\mathbf{t}\,\mathbf{C} + (1+\alpha)\beta\,\Delta\mathbf{t}^2\,\mathbf{K})\,\mathbf{a}_{k+1} = \mathbf{F}_{k+1} - \mathbf{C}\,\hat{\mathbf{v}}_{k+1} - \mathbf{K}[(1+\alpha)\,\hat{\mathbf{d}}_{k+1} - \alpha\,\mathbf{d}_k]$$

where:
$$\hat{\mathbf{d}}_{k+1} = \mathbf{d}_k + \Delta t\,\mathbf{v}_k + \Delta t^2\,(\tfrac{1}{2} - \beta)\,\mathbf{a}_k$$
$$\hat{\mathbf{v}}_{k+1} = \mathbf{v}_k + \Delta t\,(1-\gamma)\,\mathbf{a}_k$$

and:
$$\mathbf{d}_{k+1} = \hat{\mathbf{d}}_{k+1} + \beta\,\Delta t^2\,\mathbf{a}_{k+1}$$
$$\mathbf{v}_{k+1} = \hat{\mathbf{v}}_{k+1} + \gamma\,\Delta t\,\mathbf{a}_{k+1}$$

A one-parameter model can be defined by constraining two parameters comparable to Wilson's Θ-method. Taylor's α-method is defined by

$$\gamma = \tfrac{1}{2} - \alpha \qquad \text{and} \qquad \beta = \tfrac{1}{4}(1-\alpha)^2$$

with: $(-\tfrac{1}{3} \le \alpha \le 0)$

A discussion on consistency, stability and accuracy can be found in [3]. From this results:

- the minimum order of accuracy for all α, β, γ and ξ (real damping) is one.
- if $(\alpha + \gamma = \tfrac{1}{2})$ and either $\xi = 0$ or $(\gamma = \tfrac{1}{2})$ the method is second order accurate
- if $\xi = 0$, $(\alpha + \gamma = \tfrac{1}{2})$ and $(\beta = \tfrac{1}{12} + \alpha^2)$ the method is third order accurate
- the methods are unconditionally stable for $(-\tfrac{1}{2} \le \alpha \le 0)$, $(\beta \ge \tfrac{1}{4} - \frac{\alpha}{2})$ and $(\alpha + \gamma = \tfrac{1}{2})$

Implementation

All three methods may be combined into a single algorithm:

$$[\mathbf{M}+\gamma\tau\mathbf{C}+(1+\alpha)\beta\tau^2\mathbf{K}]\mathbf{a}_{k+\theta}=(1-\theta)\mathbf{F}_k+\theta\mathbf{F}_{k+1}-\mathbf{C}\hat{\mathbf{v}}_{k+\theta}-\mathbf{K}[(\mathbf{1}+\alpha)\hat{\mathbf{d}}_{k+\theta}-\alpha\mathbf{d}_k]$$

with:

$$\hat{\mathbf{d}}_{k+\theta}=\mathbf{d}_k+\tau\mathbf{v}_k+\tau^2(\tfrac{1}{2}-\beta)\mathbf{a}_k$$

$$\hat{\mathbf{v}}_{k+\theta}=\mathbf{v}_k+\tau(1-\gamma)\mathbf{a}_k$$

and:

$$\mathbf{a}_{k+1}=\mathbf{a}_k+\frac{1}{\theta}(\mathbf{a}_{k+\theta}-\mathbf{a}_k)$$

$$\mathbf{d}_{k+1}=\mathbf{d}_k+\Delta t\,\mathbf{v}_k+\Delta t^2(\tfrac{1}{2}-\beta)\mathbf{a}_k+\beta\Delta t^2\mathbf{a}_{k+1}$$

$$\mathbf{v}_{k+1}=\mathbf{v}_k+\Delta t(1-\gamma)\mathbf{a}_k+\gamma\Delta t\mathbf{a}_{k+1}$$

In Newmark's and Taylor's methods backsubstitution of the equation will yield the acceleration at the next time step $\mathbf{a}_{k+1}$ and the predictors $\hat{\mathbf{d}}_{k+1}$ and $\hat{\mathbf{v}}_{k+1}$ can be used to compute the displacement $\mathbf{d}_{k+1}$ and velocity $\mathbf{v}_{k+1}$ at the next time step. In Wilson's method this is not possible as backsubstitution of the equation will yield the acceleration $\mathbf{a}_{k+\theta}$ at time step ($k+\theta$) and the predictors $\hat{\mathbf{d}}_{k+\theta}$ and $\hat{\mathbf{v}}_{k+\theta}$ are also evaluated at time step ($k+\theta$). Therefore, acceleration, displacement and velocity must be recomputed separately.

Integration parameters are set depending on the method chosen:

Newmark's method: $\alpha=0,\ \theta=1$ and β,γ will have to be defined

Wilson's Θ-method $\alpha=0,\ \beta=1/6,\ \gamma=1/2$ and θ will have to be defined

Taylor's α-method $\beta=\frac{1}{4}(1-\alpha)^2,\ \gamma=1/2,\ \theta=1$ and α will have to be defined

The solution algorithm is defined by the following steps:

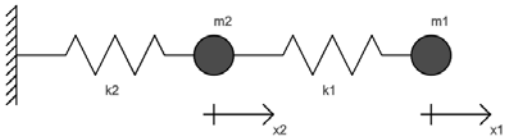
at time step k=0: $\dot{\mathbf{u}}_0 = \mathbf{v_0}$ and $\mathbf{u}_0 = \mathbf{d_0}$ are set to initial conditions calculate initial accelerations $\mathbf{a_0}$ by solving $\mathbf{M}\mathbf{a}_0 = \mathbf{F}_0 - \mathbf{C}\mathbf{v}_0 - \mathbf{K}\mathbf{d}_0$	
compute and decompose constant coefficient matrix $\mathbf{M} + \gamma\tau\mathbf{C} + (1+\alpha)\beta\tau^2\mathbf{K}$ only once	
for $k \le k_{max}$	
	k = k+1, time = time+dt
	calculate predictors $\hat{\mathbf{d}}_{k+\theta} = \mathbf{d}_k + \tau\mathbf{v}_k + \tau^2(½ - \beta)\mathbf{a}_k$ $\hat{\mathbf{v}}_{k+\theta} = \mathbf{v}_k + \tau(1-\gamma)\mathbf{a}_k$
	calculate new RHS $\mathbf{R} = (1-\theta)\mathbf{F}_k + \theta\mathbf{F}_{k+1} - \mathbf{C}\hat{\mathbf{v}}_{k+\theta} - \mathbf{K}[(1+\alpha)\hat{\mathbf{d}}_{k+\theta} - \alpha\mathbf{d}_k]$
	backsubstitution of coefficient matrix with new RHS $\mathbf{M} + \gamma\tau\mathbf{C} + (1+\alpha)\beta\tau^2\mathbf{K} = \mathbf{R}$ for acceleration $\mathbf{a}_{k+\theta}$
	compute accelerations, displacements and velocities at next time step $\mathbf{a}_{k+1} = \mathbf{a}_k + \frac{1}{\theta}(\mathbf{a}_{k+\theta} - \mathbf{a}_k)$ $\mathbf{d}_{k+1} = \mathbf{d}_k + \Delta t\,\mathbf{v}_k + \Delta t^2(½ - \beta)\mathbf{a}_k + \beta\Delta t^2\mathbf{a}_{k+1}$ $\mathbf{v}_{k+1} = \mathbf{v}_k + \Delta t(1-\gamma)\mathbf{a}_k + \gamma\Delta t\,\mathbf{a}_{k+1}$

3.1 Example 1: Two Degree-of-Freedom System

Consider the following two degree-of-freedom problem:

$$m_1 \ddot{x}_1 + k_{11} x_1 + k_{12} x_2 = f_1(t)$$

$$m_2 \ddot{x}_2 + k_{12} x_1 + k_{22} x_2 = f_2(t)$$



where: $k_1 = 1.$; $k_2 = 0.01$; $m_1 = 1.$; $m_2 = 1.$;

a) Write a program to solve this system using the Newmark algorithm. Solve for the following initial value problem:

$$x_1(0)=1; \quad x_2(0)=0.9; \qquad f(t) \equiv 0$$

$$\dot{x}_1(0)=\dot{x}_2(0)=0.;$$

b) Write a program to solve this system using the 3 algorithms discussed in class. Solve for the following initial value problem:

$$x_1(0)=1; \quad x_2(0)=0.9; \qquad f(t) \equiv 0$$

$$\dot{x}_1(0)=\dot{x}_2(0)=0.;$$

and solve for the following forcing function problem:

$$x_1(0)=x_2(0)=0; \qquad f(t)=\sin(.03t)$$

for $0 \le t \le 400$.

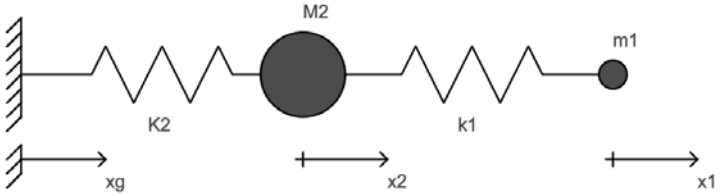
For each problem consider the following α and Δt values:

5. $\alpha = 0$; Δt for stability; $\Delta t = 2$
6. $\alpha = ½$ $\Delta t = 5, 10, 20, 40$
7. $\alpha = ⅔$ $\Delta t = 5, 10, 20, 40$
8. $\alpha = 1$ $\Delta t = 5, 10, 20, 40$

3.2 Example 2: Two Degree-of-Freedom System with Ground Acceleration

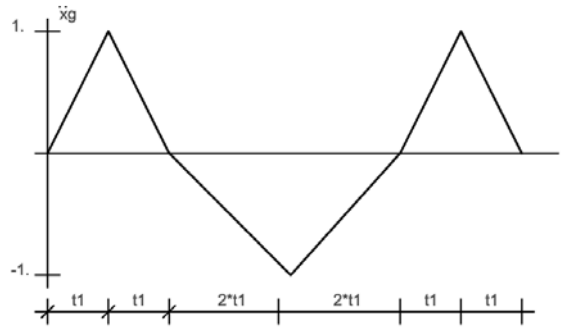
Consider the following two degree-of-freedom problem:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = -\mathbf{M}\ddot{\mathbf{x}}_g = \mathbf{F}(t)$$



where: $k_1 = 0.01$; $K_2 = 1.$; $m_1 = 0.01$; $M_2 = 1.$; and x_1, x_2 are defined relative to x_g

with the following definition for the ground acceleration:



where: $t_1 = 0.8$ and $0 \le t \le 125$.

solve the system for the following integration parameters:

1. Newmark: β = 0.25; γ=0.5 Δt = 0.4
2. Wilson-θ: θ = 1.420815 Δt = 0.4
3. α-method: α = -0.1 Δt = 0.4

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