

**“All things being equal, the simplest solution tends to be the best one.”**

**William of Ockham**

# Lecture I: Dimensionality Reduction

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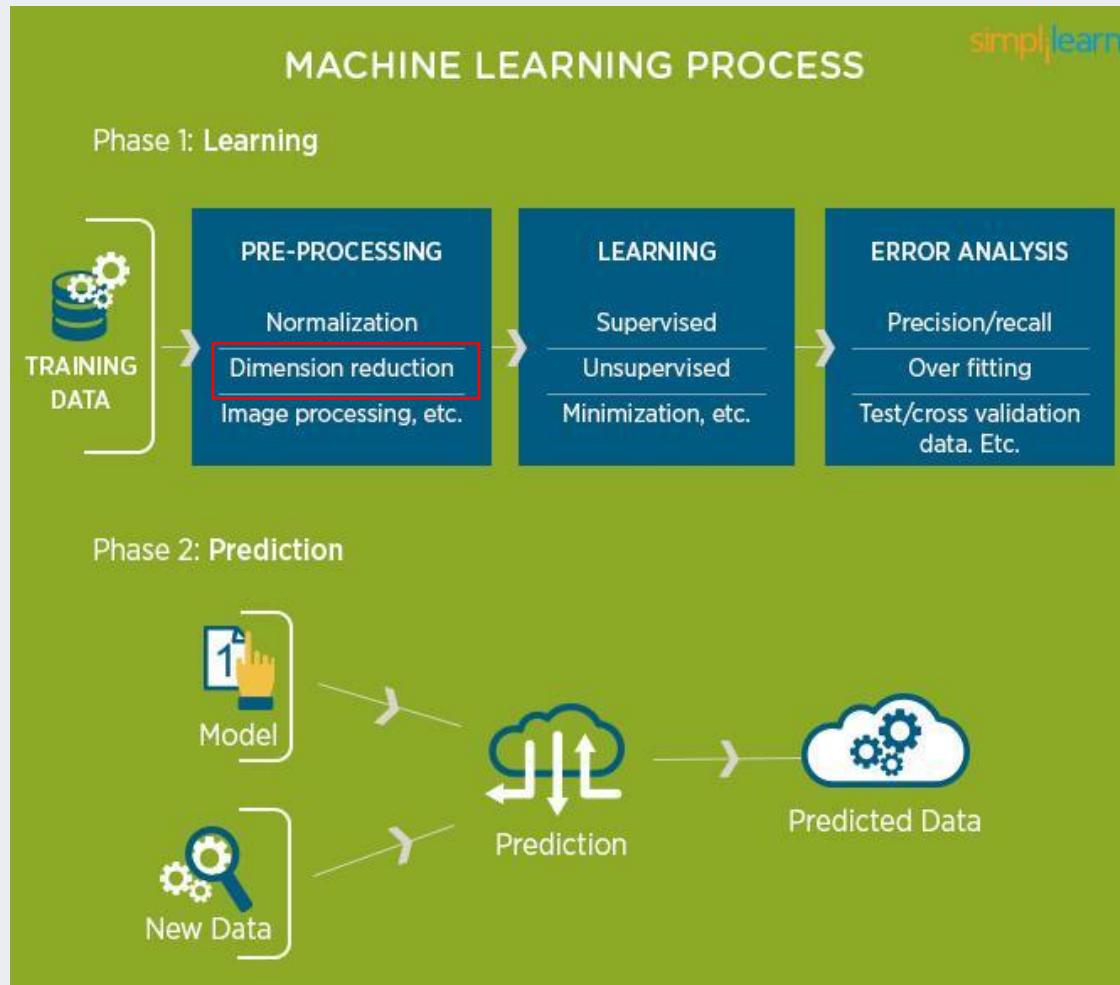
Korea University

# AGENDA

- 01 Dimensionality Reduction: Overview
- 02 Supervised Methods
- 03 Unsupervised Methods: Linear Mapping
- 04 Unsupervised Methods: Non-linear Mapping

# Data Analytics Process

- Process of Business Analytics with Machine Learning



# High-dimensional Data

- Examples of high dimensional data

Document classification:

Billions of documents x Thousands/  
Millions of words/bigrams matrix



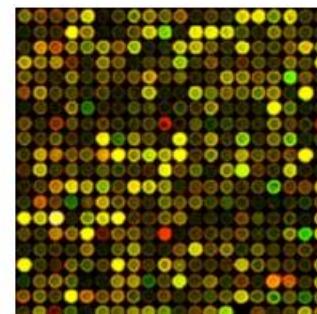
Recommendation systems:

480,189 users x 17,770 movies matrix



Clustering gene expression profiles:

10,000 genes x 1,000 conditions

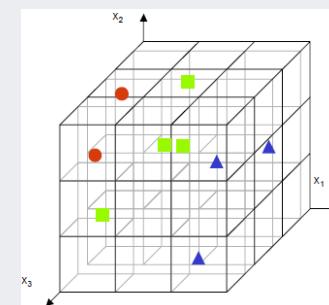
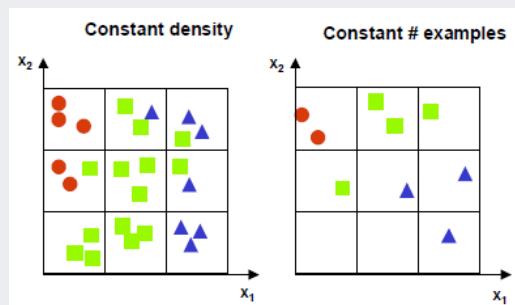
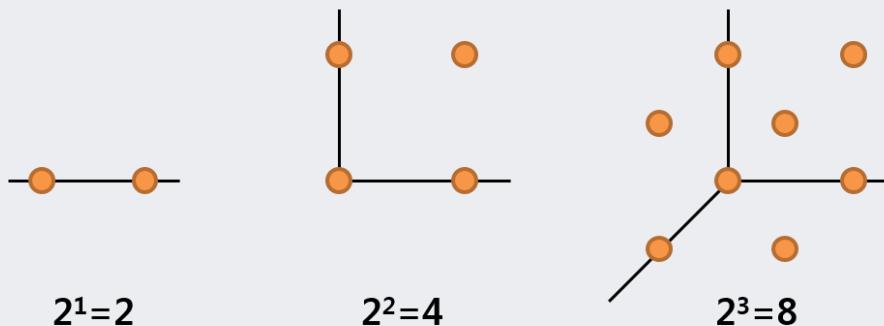


# Dimensionality Reduction: Overview

- Curse of dimensionality

- ✓ The number of instances increases exponentially to achieve the same explanation ability when the number of variables increases

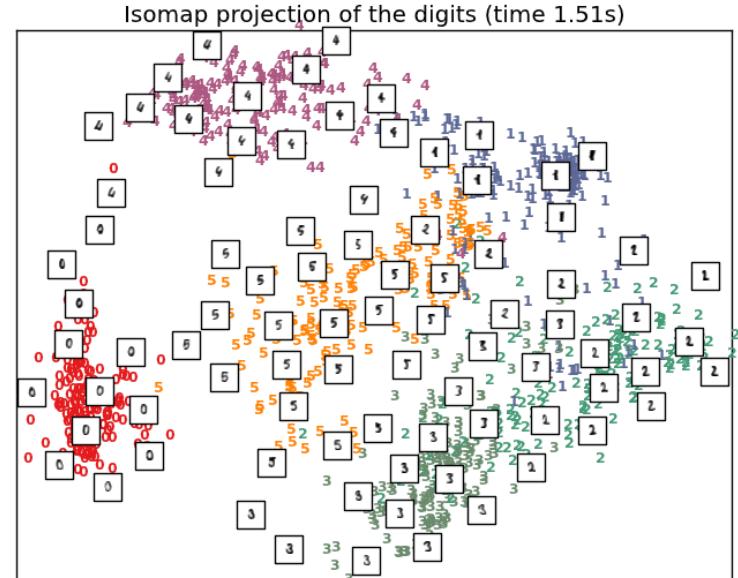
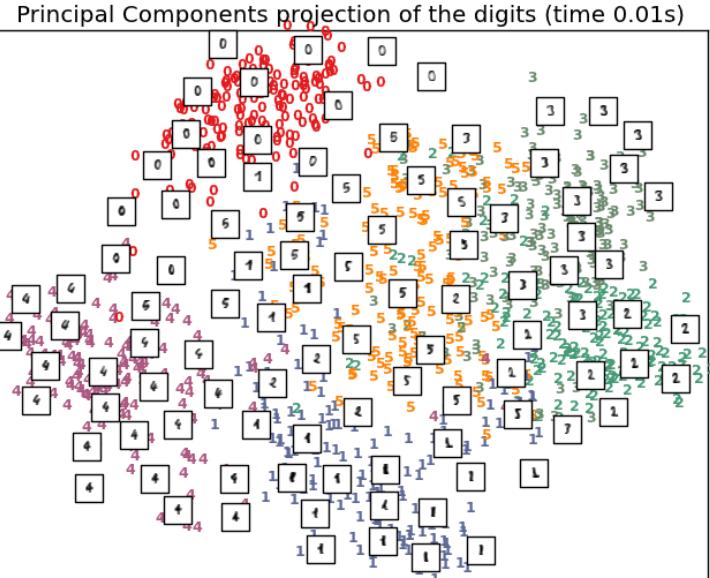
“If there are various logical ways to explain a certain phenomenon, the simplest is the best” - Occam’s Razor



# Dimensionality Reduction: Overview

- Curse of dimensionality

- ✓ Sometimes, an intrinsic dimension is relatively low compared to the original dimension.
  - Ex: handwritten digits in a 16 by 16 pixel (256 dimensions)
  - Reduced to two dimensions by PCA and ISOMAP



# Dimensionality Reduction: Overview

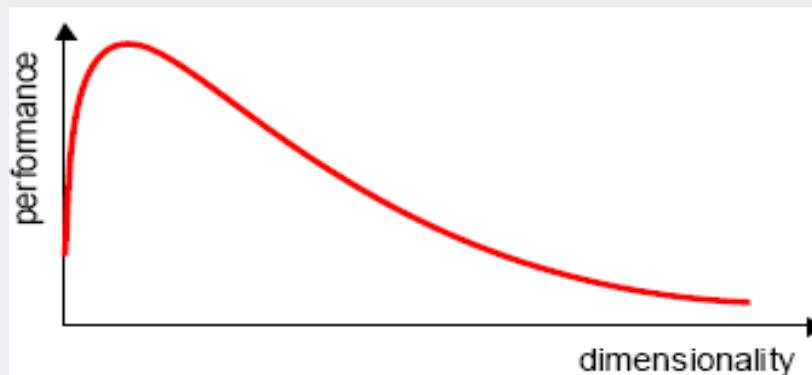
- Curse of dimensionality

- ✓ Problems caused by high-dimensionality

- Increase the probability of having noise in data → degenerate the prediction performance
    - Increase computational burden for training/applying prediction models
    - Require more number of examples to secure generalization ability of prediction model

- ✓ To resolve the curse of dimensionality

- Utilize domain knowledge
    - Use a regularization term in objective function
    - Employ a quantitative reduction technique



# Dimensionality Reduction: Overview

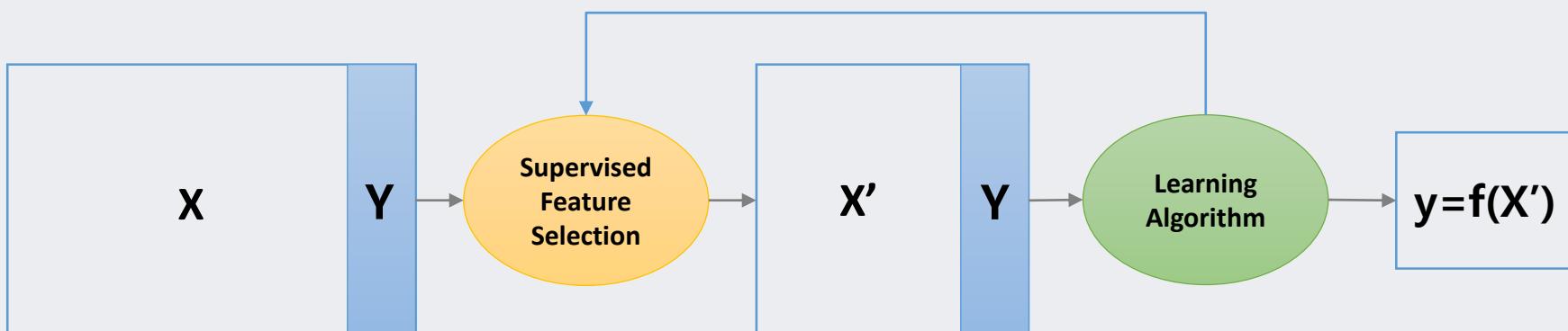
- Backgrounds
  - ✓ Theoretically, model performance improves when the number of variables increases  
**(Under variable independence condition)**
  - ✓ In reality, model performance degenerates due to variable dependence, existence of noise, etc.
- Purpose
  - ✓ Identify a subset of variables that best fit the model
- Effect
  - ✓ Remove correlations between variables
  - ✓ Simplified post-processing
  - ✓ Remove redundant or unnecessary variables while keeping relevant information
  - ✓ Visualization can be possible

# Dimensionality Reduction: Overview

- Supervised vs. Unsupervised Dimensionality Reduction

- ✓ Supervised dimensionality reduction

- Use data mining models to verify the reduced dimensions
    - Dimensionality reduction results can be different according to the data mining algorithms employed

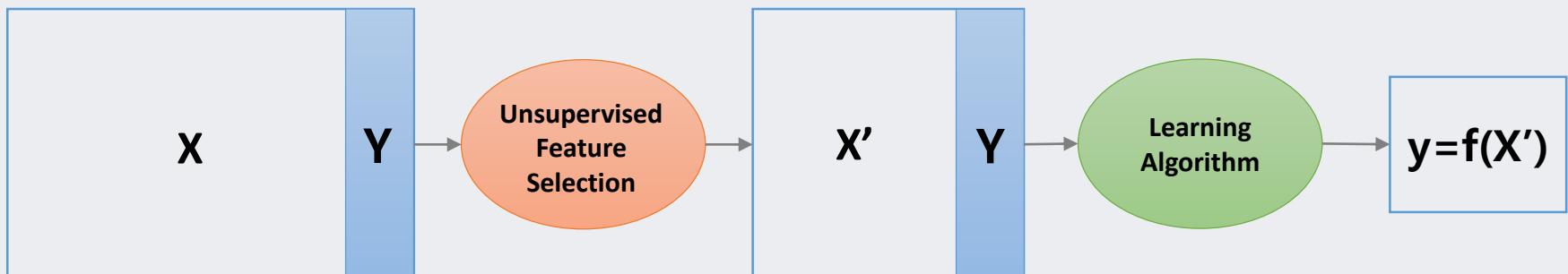


# Dimensionality Reduction: Overview

- Supervised vs. Unsupervised Dimensionality Reduction

- ✓ Unsupervised dimensionality reduction

- Find a set of coordinate systems in a lower dimension that preserve the information (e.g., variance, distance, etc.) in the original input space as much as possible
    - Do not use data mining models during the process
    - Dimensionality reduction results are identical if the data and method is same



# Dimensionality Reduction: Overview

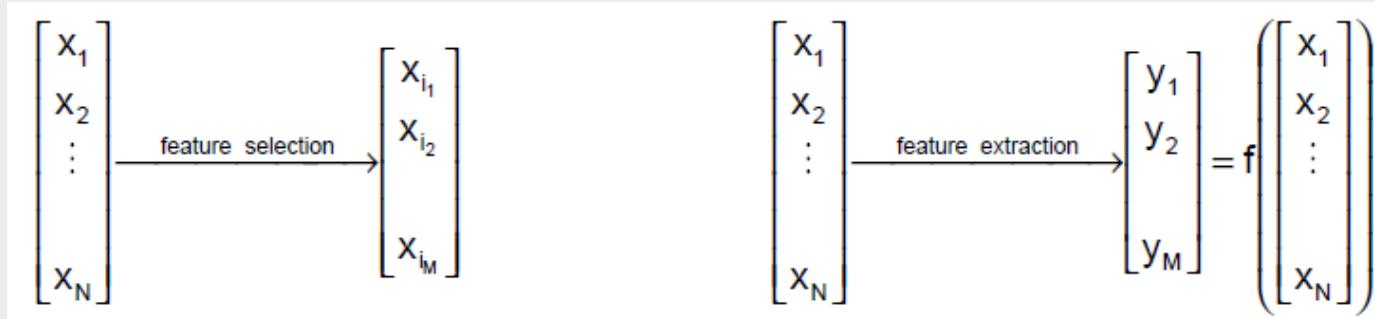
- Dimensionality reduction techniques

- ✓ Variable/feature selection

- Select a subset of variables from the original variable set
  - Filter – Variable selection and model training are independent
  - Wrapper – Variable selection is done to optimizes the result of the considered data mining model

- ✓ Variable/feature extraction

- Extract a new smaller set of variables that preserve the characteristics of the original data
  - Performance metric that is independent from data mining models is used



# Dimensionality Reduction: Overview

- Selection vs. Extraction

- ✓ Conceptual difference between variable selection and variable extraction

$X_1$	$X_2$	$X_3$	...	$X_n$
...	...	...	...	...
...	...	...	...	...
...	...	...	...	...
...	...	...	...	...
...	...	...	...	...

Variable selection

$X_1$	$X_5$	$X_8$
...	...	...
...	...	...
...	...	...
...	...	...
...	...	...

Variable extraction

$Z_1$	$Z_2$	$Z_3$
...	...	...
...	...	...
...	...	...
...	...	...
...	...	...

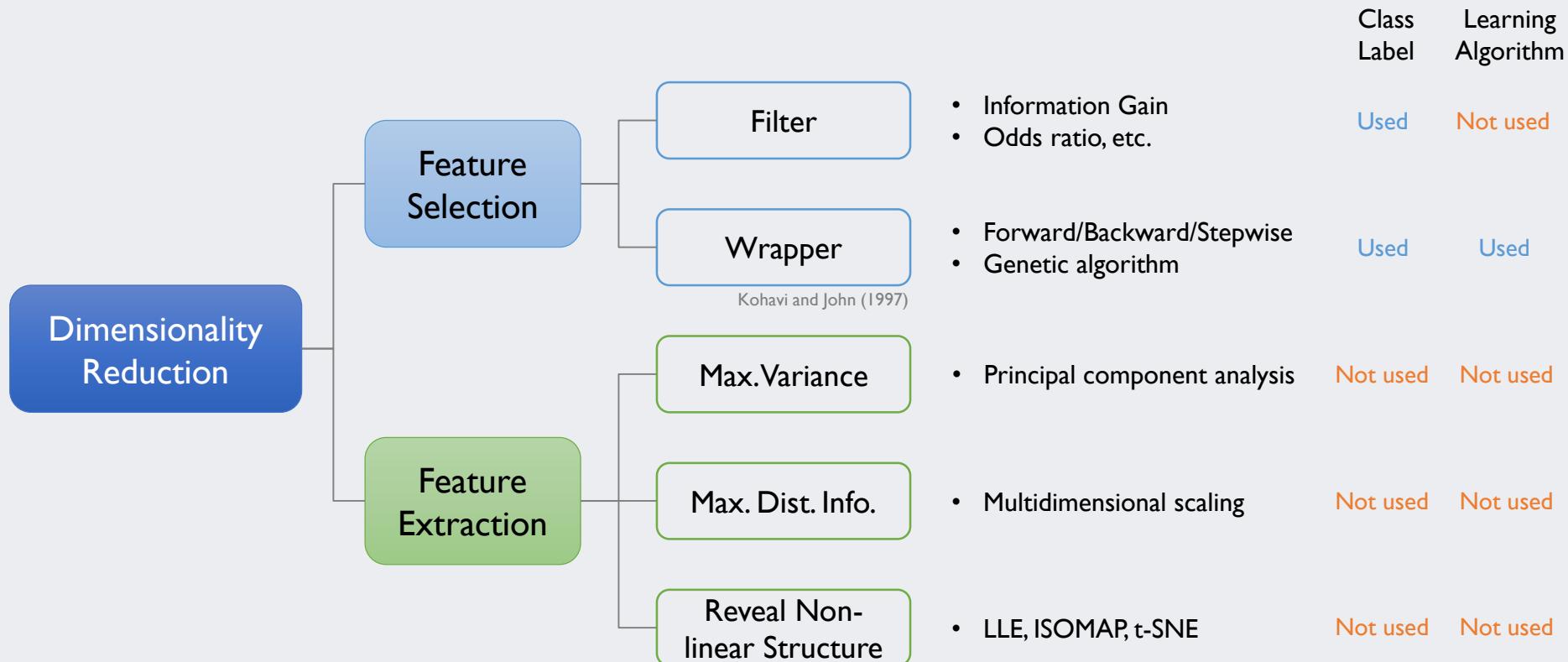
$$Z_1 = X_1 + 0.2*X_2$$

$$Z_2 = X_3 - 2*X_5$$

$$Z_3 = X_4 + X_6 - X_9$$

# Dimensionality Reduction: Overview

- A simplified taxonomy of dimensionality reduction techniques



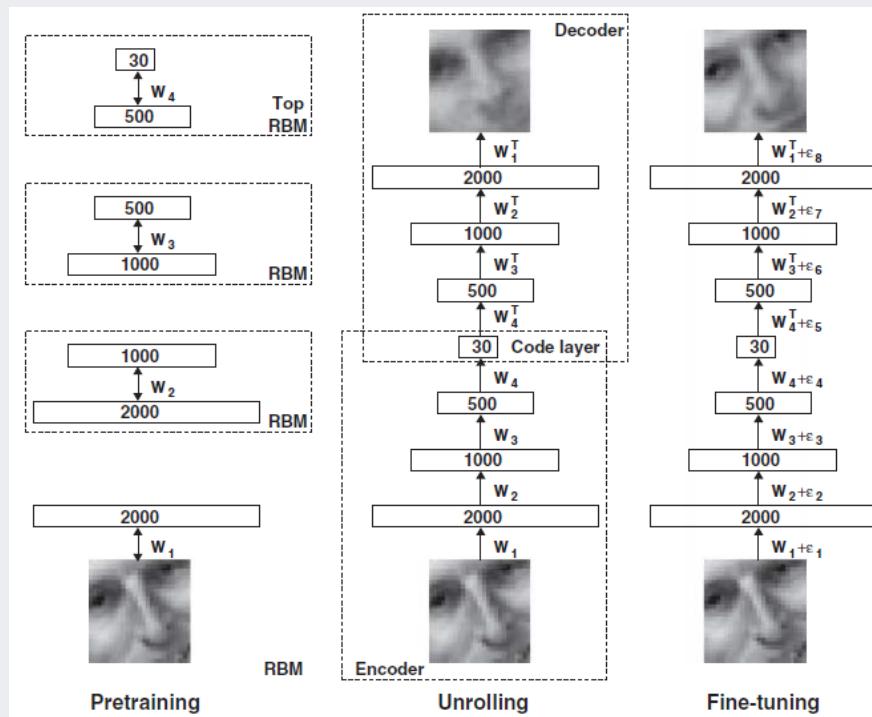
# Dimensionality Reduction: Recent Trends

- Representation learning: Deep auto-encoder

- ✓ Try to extract (learn) features from very low-level components (e.g., image pixels, text words, etc.)

- Article recommendation

- Bengio, Y., Courville, A., & Vincent, P. (2013). Representation learning: A review and new perspectives. *IEEE transactions on pattern analysis and machine intelligence*, 35(8), 1798-1828.

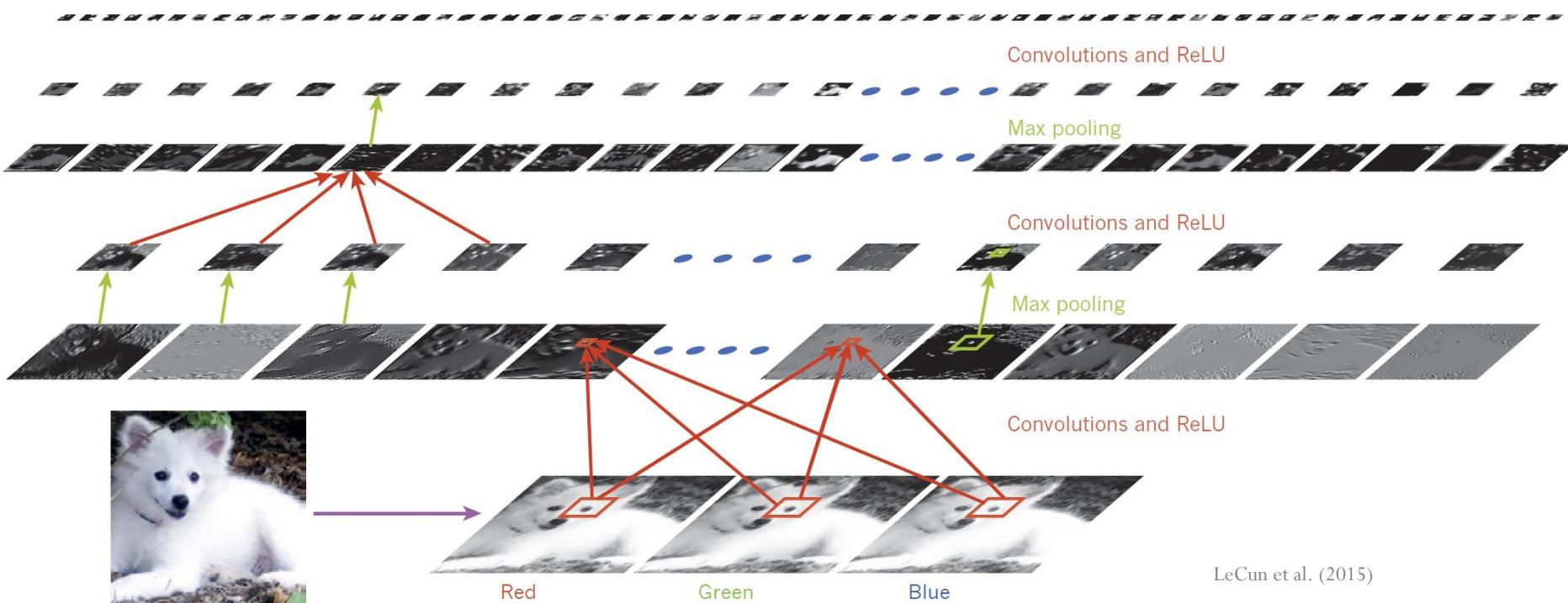


Hinton and Salakhutdinov (2006)

# Dimensionality Reduction: Recent Trends

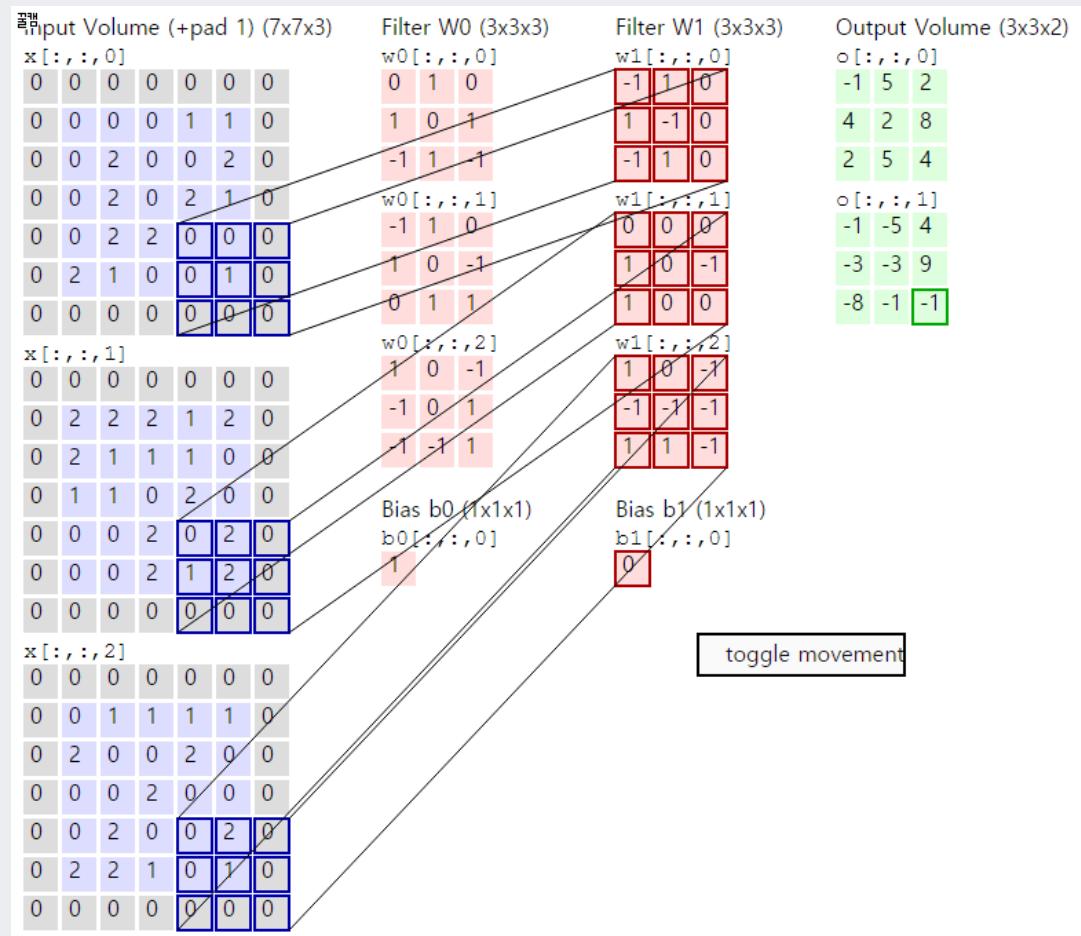
- Representation learning: Convolutional neural network
  - ✓ Try to extract (learn) features from very low-level components (e.g., image pixels, text words, etc.)

Samoyed (16); Papillon (5.7); Pomeranian (2.7); Arctic fox (1.0); Eskimo dog (0.6); white wolf (0.4); Siberian husky (0.4)



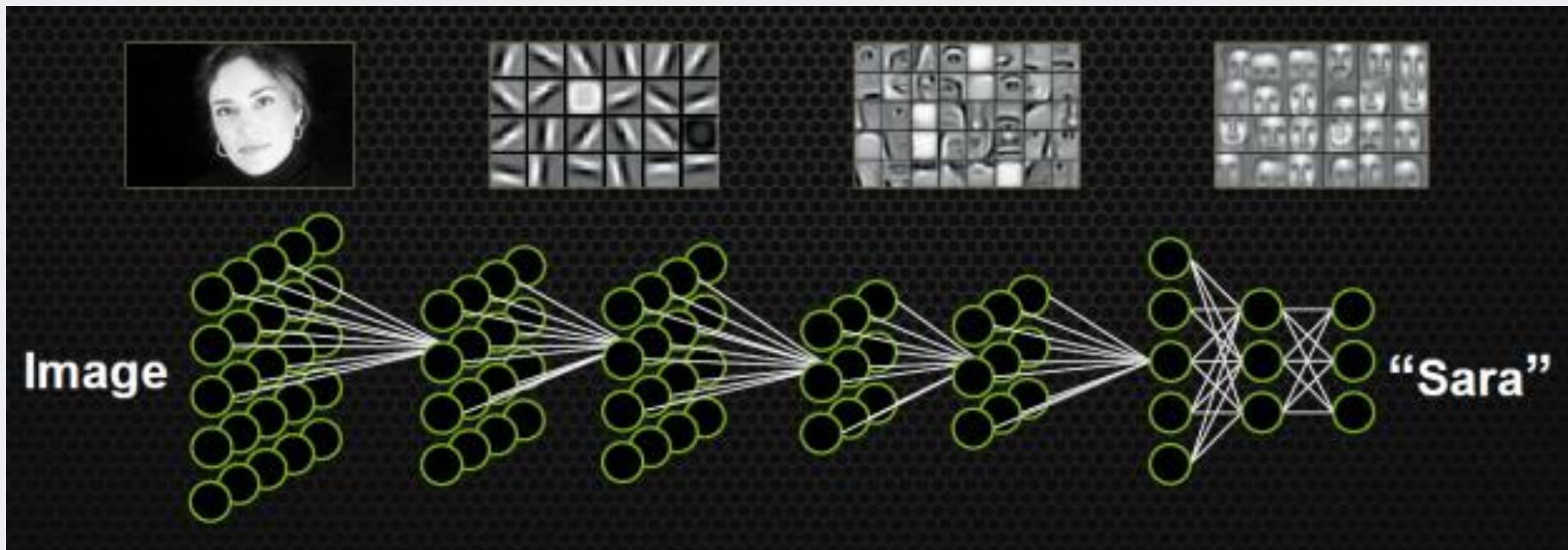
# Dimensionality Reduction: Recent Trends

- Representation learning: Convolutional neural network
  - ✓ Try to extract (learn) features from very low-level components (e.g., image pixels, text words, etc.)



# Dimensionality Reduction: Recent Trends

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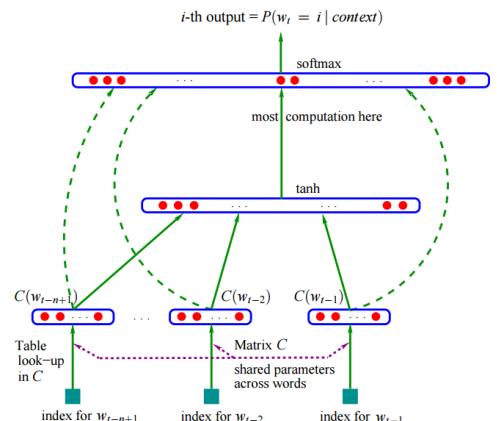


<http://devblogs.nvidia.com/parallelforall/accelerate-machine-learning-cudnn-deep-neural-network-library/>

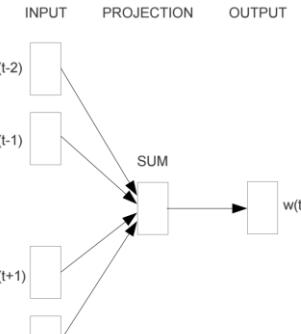
# Dimensionality Reduction: Recent Trends

- Representation learning: Word/Document Embedding

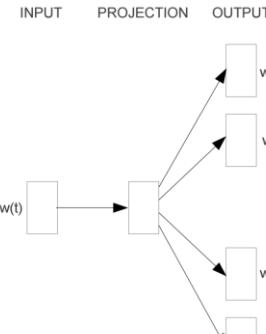
NNLM



Word2Vec



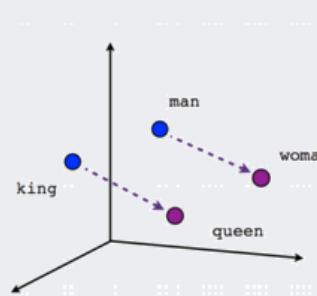
CBOW



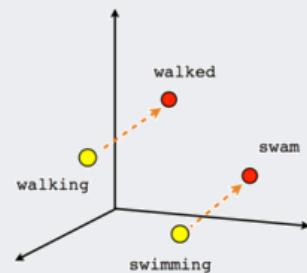
Skip-gram

GloVe

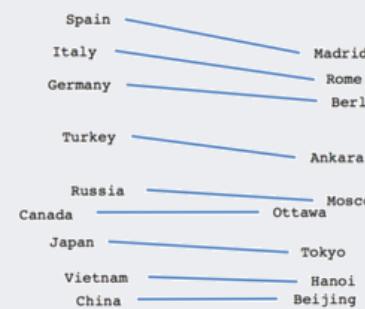
frog nearest neighbors	Litoria	Leptodactylidae	Rana	Eleutherodactylus
• frogs • toad • litoria • leptodactylidae • rana • lizard • eleutherodactylus				
man -> woman	city -> zip	comparative -> superlative		
uncle -> woman      woman -> queen /      / /      / /      / /      / /      / / man      / king	96812 -> Honolulu 37211 -> Sacramento 95823 -> Anaheim	strong -> stronger clear -> clearer soft -> softer dark -> darker		



Male-Female



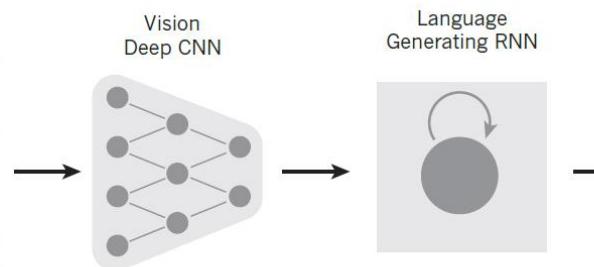
Verb tense



Country-Capital

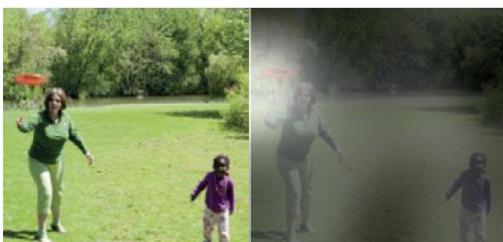
# Dimensionality Reduction: Recent Trends

- Representation learning: Image+Text



A group of people shopping at an outdoor market.

There are many vegetables at the fruit stand.



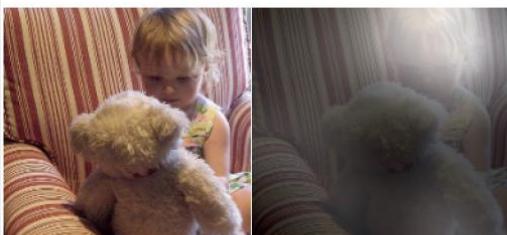
A woman is throwing a **frisbee** in a park.



A **dog** is standing on a hardwood floor.



A **stop** sign is on a road with a mountain in the background



A little girl sitting on a bed with a teddy bear.



A group of **people** sitting on a boat in the water.



A giraffe standing in a forest with **trees** in the background.

# Dimensionality Reduction: Recent Trends

- Video recommendation

✓ Brandon Rohrer

- Deep Learning: <https://www.youtube.com/watch?v=ILsA4nyG7I0>
- CNN: <https://www.youtube.com/watch?v=FmpDlaiMleA>
- RNN: <https://www.youtube.com/watch?v=WCUNPb-5EYI>

Brandon Rohrer

The screenshot shows a YouTube channel interface for 'Brandon Rohrer'. The navigation bar includes '홈', '동영상', '재생목록', '채널', '토론', and '정보'. Below the channel name, there are five video thumbnails:

- Recurrent Neural Networks (RNN) and Long Short-Term Memory** (26:14) - A diagram illustrating the flow of information between long short-term memory and recurrent layers.
- How Deep Neural Networks Work** (24:38) - A visual representation of a deep neural network architecture with various layers and nodes.
- How to Get Unstuck** (2:19) - A flowchart showing a process for problem-solving: 'Is your next step obvious?', 'Do it.', 'Accept the question.', and 'Repeat until you're done.'
- How Bayes Theorem works** (25:09) - A graph titled 'How Bayesian Inference Works' showing a bell curve for 'Bayesian estimate' and another for 'non-Bayesian estimate'.
- How Convolutional Neural Networks work** (26:14) - A diagram showing how a set of pixels is processed through a series of operations to produce a final output.

Each video thumbnail includes its title, duration, and view count.

# AGENDA

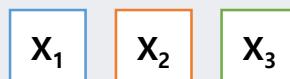
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# Exhaustive Search

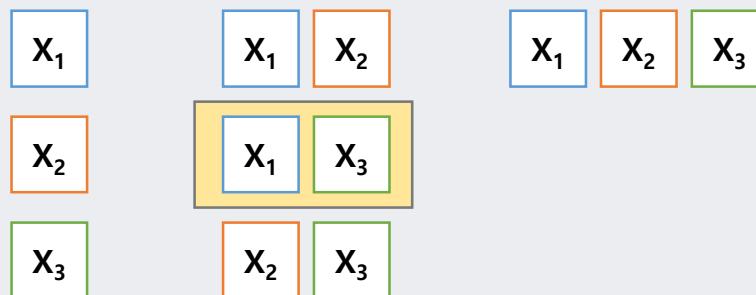
- **Exhaustive search**

- ✓ Search all possible combinations

- Ex) 3 variables



- A total of 6 possible subsets are tested



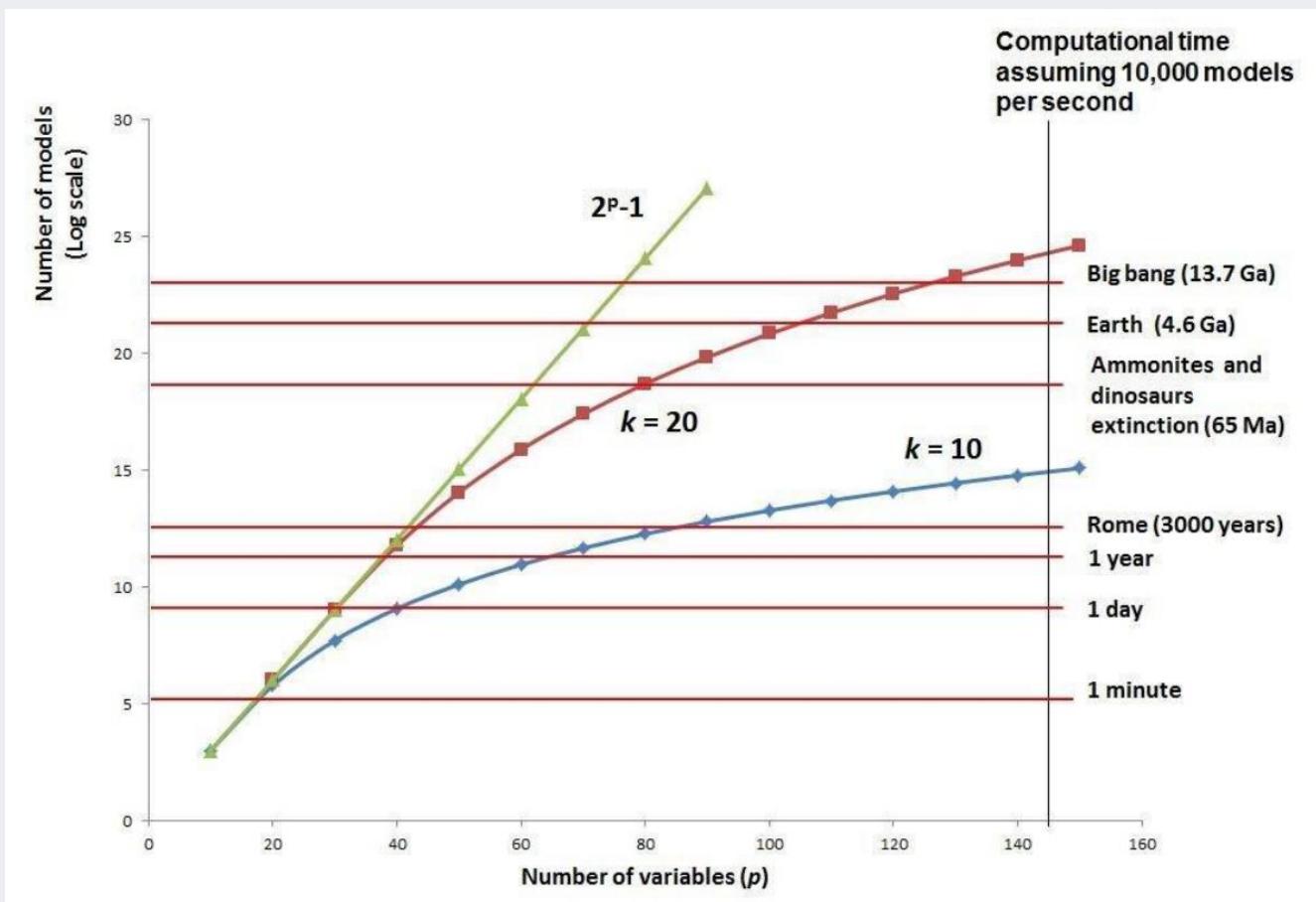
- ✓ Performance criteria for variable selection

- Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Adjusted R<sup>2</sup>, Mallow's C<sub>p</sub>, etc.

# Exhaustive Search

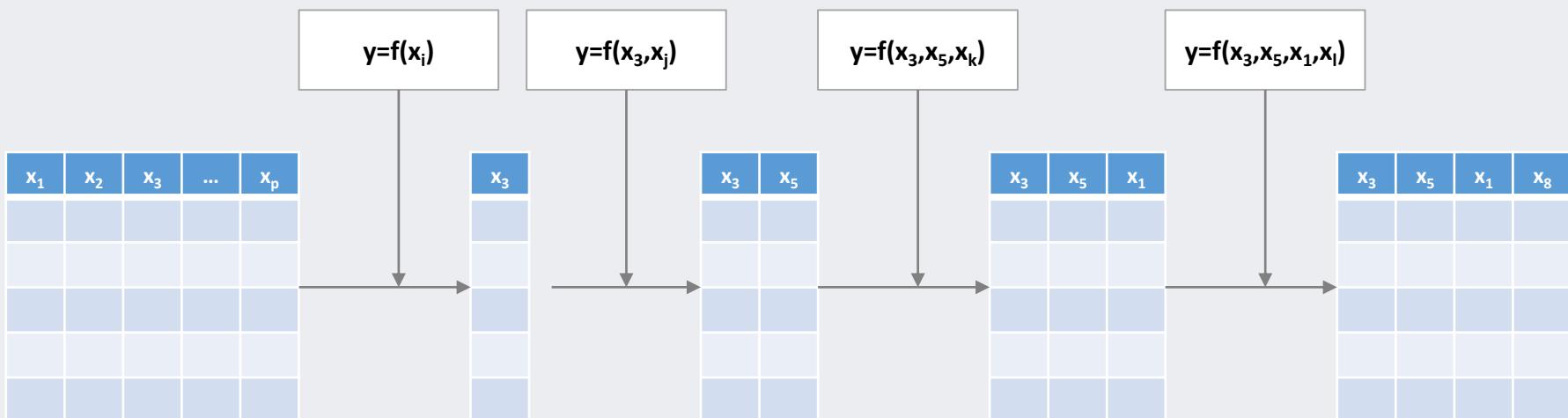
- Exhaustive search

- ✓ Assume that we have a computer that can evaluate 10,000 models/second



# Forward Selection

- Forward selection
  - ✓ From the model with no variable, significant variables are sequentially added
  - ✓ Once the variable is selected, it will never be removed (The number of variables gradually increases)



# Forward Selection

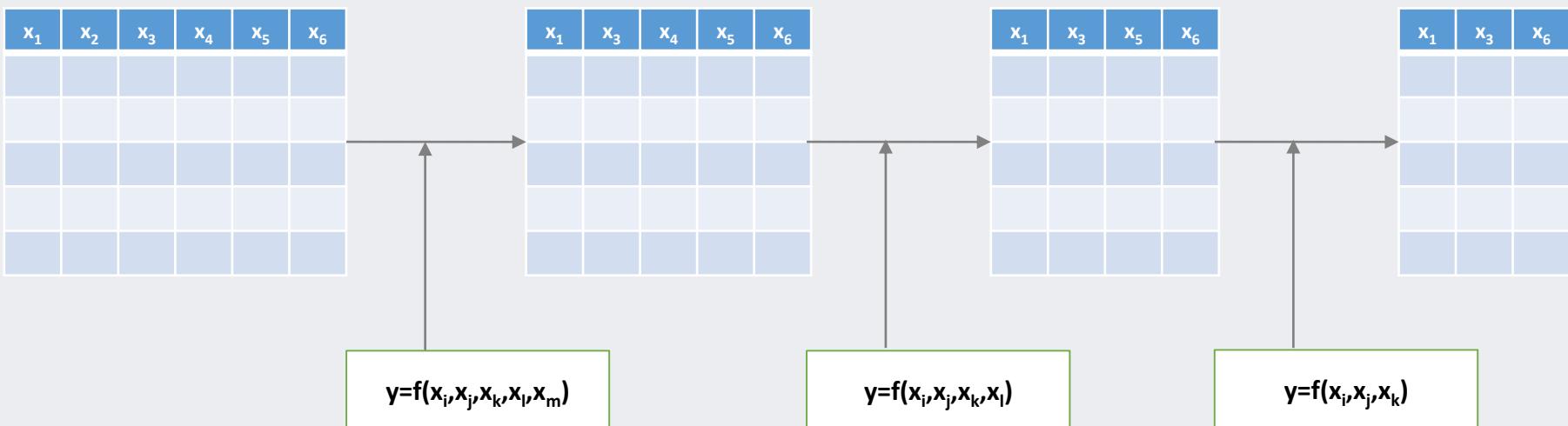
- Forward Selection

- ✓ Forward Selection in the multiple linear regression

Step	5	Var	CHEST	Entered	R-sq=0.5379	C(p)= 4.195	
			DF	Sum Sq	Mean Sq	F	Prob>F
Regression			5	108.3272	21.6654	24.91	0.0001
Error			107	93.0527	0.86965099		
Total			112	201.37982301			
			Par	Std	Type II		
Variable			Est	Error	Sum Sq	F	Prob>F
INTERCEP			-0.7680	0.6102	1.3776	1.58	0.2109
CULTURE			0.0432	0.0098	16.7198	19.23	0.0001
STAY			0.2339	0.0574	14.4381	16.60	0.0001
NRATIO			0.6724	0.2993	4.3888	5.05	0.0267
CHEST			0.0092	0.0054	2.5062	2.88	0.0925
FACIL			0.0184	0.0063	7.4571	8.57	0.0042

# Backward Elimination

- Backward Elimination
  - ✓ From the model with all variables, irrelevant variables are sequentially removed
  - ✓ Once a variable is removed, it will never be selected (The number of variables gradually decreases)

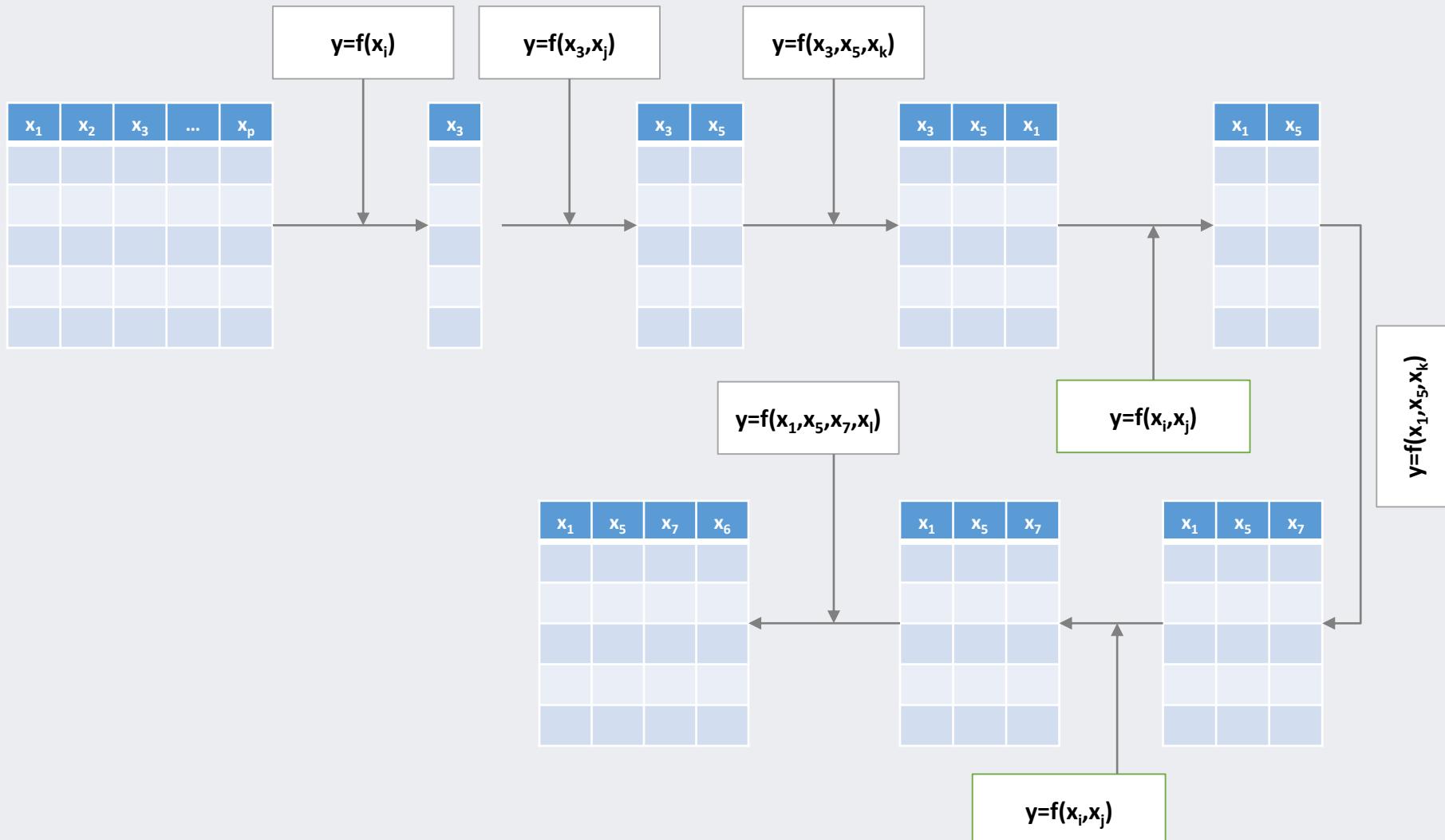


# Stepwise Selection

- Stepwise Selection
  - ✓ From the model with no variable, conduct the forward selection and backward elimination alternately
  - ✓ Takes longer time than forward selection/backward elimination, but has more chances to find the optimal set of variables
  - ✓ Variables that is either selected/removed can be reconsidered for selection/removal
  - ✓ The number of variables increases in the early period, but it can either increase or decrease

# Stepwise Selection

- Stepwise selection example



# Stepwise Selection

- Stepwise Selection

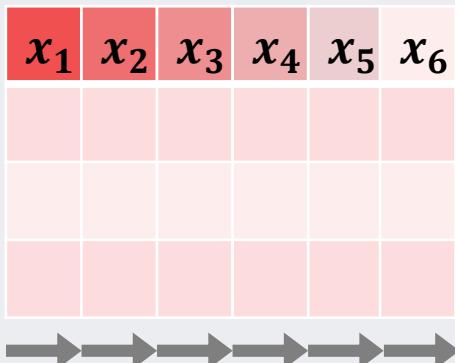
- ✓ Stepwise selection process

- ▶ Start with model with no predictors.
- ▶ Add variable with largest  $F$ -statistic (provided  $P$  less than some cut-off).
- ▶ Refit with this variable added. Recompute all  $F$  statistics for adding one of the remaining variables and add variable with largest  $F$  statistic.
- ▶ At each step after adding a variable try to eliminate any variable not significant at some level (that is, do BACKWARD elimination till that stops).
- ▶ After doing the backwards steps take another FORWARD step.
- ▶ Continue until every remaining variable is significant at cut-off level and every excluded variable is insignificant OR until variable to be added is same as last deleted variable.

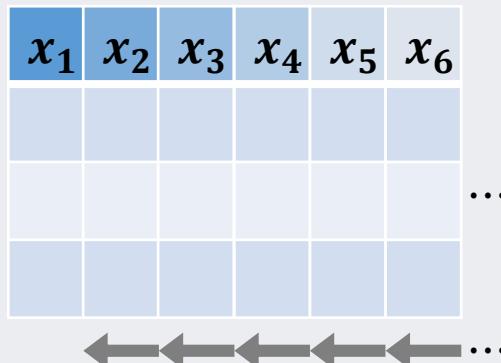
# Comparison among FS/BE/SS

- Illustrative Example

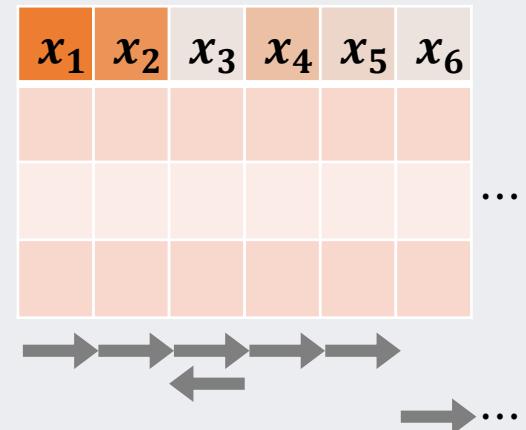
Forward Selection



Backward Elimination



Stepwise Selection



# Performance Metrics

- Akaike Information Criteria (AIC)

- ✓ Sum of squared error (SSE) with the number of variables as a penalty term

$$AIC = n \cdot \ln\left(\frac{SSE}{n}\right) + 2k$$

- Bayesian Information Criteria (BIC)

- ✓ SSE, number of variables, standard deviation obtained by the model with all variables

$$BIC = n \cdot \ln\left(\frac{SSE}{n}\right) + \frac{2(k+2)n\sigma^2}{SSE} - \frac{2n^2\sigma^4}{SSE^2}$$

# Performance Metrics

- Adjusted R<sup>2</sup>
  - ✓ Simple R<sup>2</sup> increases when the number of variable increases

$$\text{Model 1 : } y = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k + \epsilon$$

$$\text{Model 2 : } y = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k + \dots + \beta_{k+m} x_{k+m} \epsilon$$

$$R^2(M2) \geq R^2(M1)$$

- ✓ Use the adjusted R<sup>2</sup> that account for the number of variables (k)

$$\text{Adjusted } R^2 = 1 - \left( \frac{n-1}{n-k-1} \right) (1 - R^2) = 1 - \frac{n-1}{n-k-1} \frac{SSE}{SST}$$

# Genetic Algorithm

Siedlecki, W. Sklansky (1989)

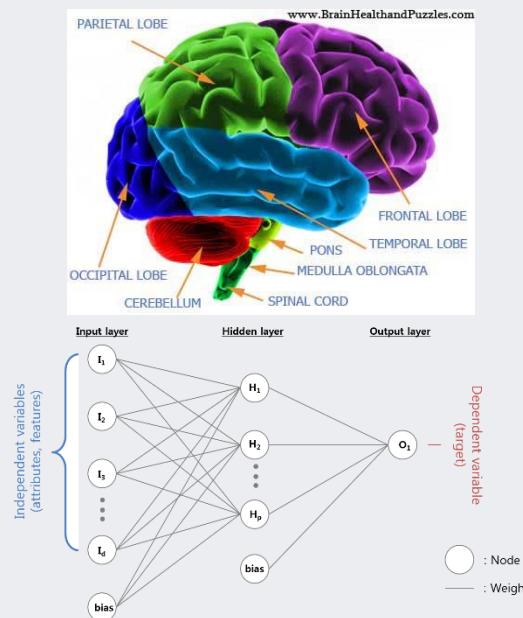
- Limitations of the previous variable selection methods
  - ✓ Exhaustive search: guarantee the optimal subset, but takes too long time (practically impossible for many tasks)
  - ✓ Local search (forward/backward/stepwise): efficient search but the search space is very limited, which leads to a low probability of finding the optimal solution
- Idea
  - ✓ Improve the performance of local searches with a little additional computational time!

# Genetic Algorithm

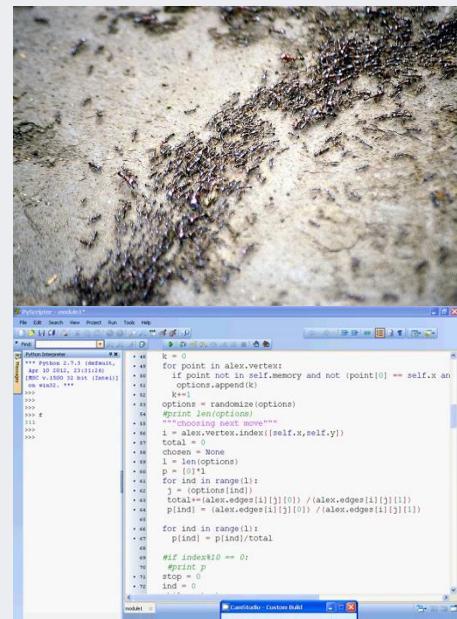
- Meta-Heuristic Approach

- ✓ Solve a complex problem by doing trials and errors efficiently
  - ✓ Among the optimization algorithms, many of them mimic the way of a natural system works

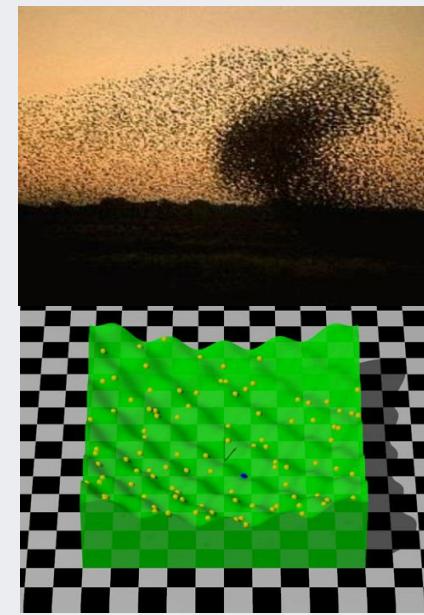
## Artificial Neural Networks



## Ant Colony Algorithm

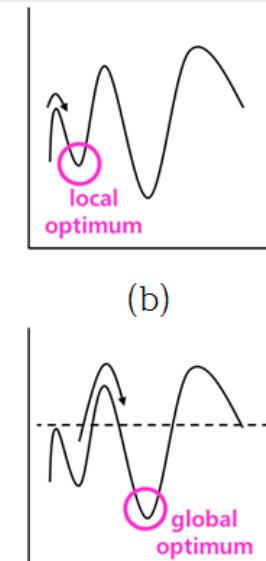
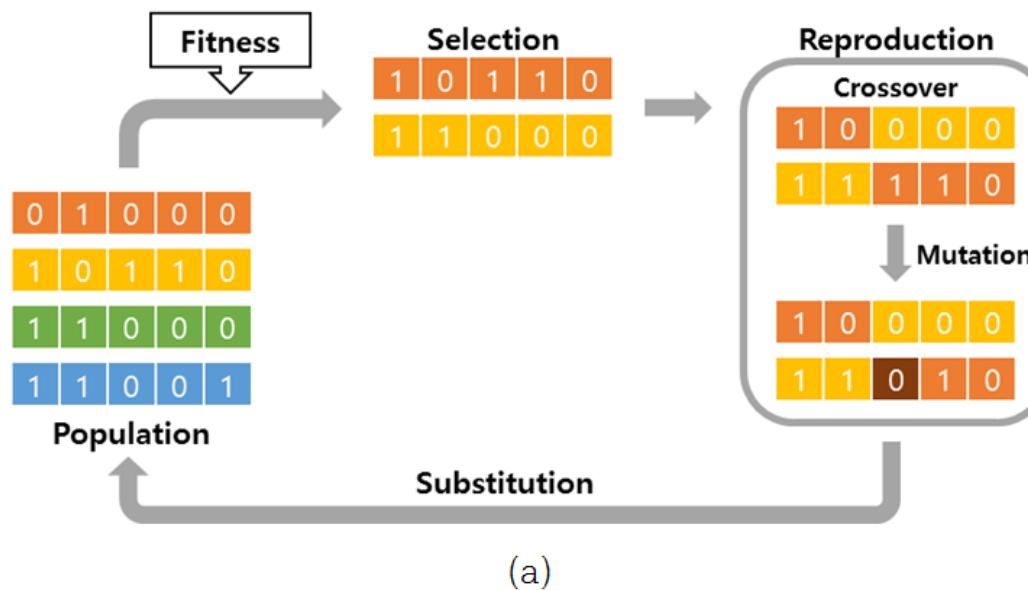
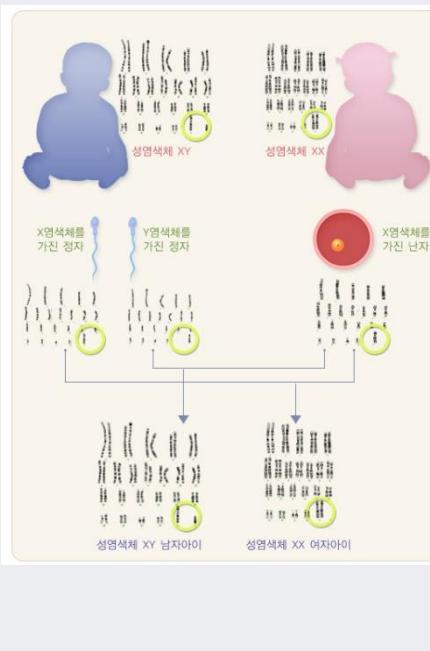


## Particle Swarm Optimization



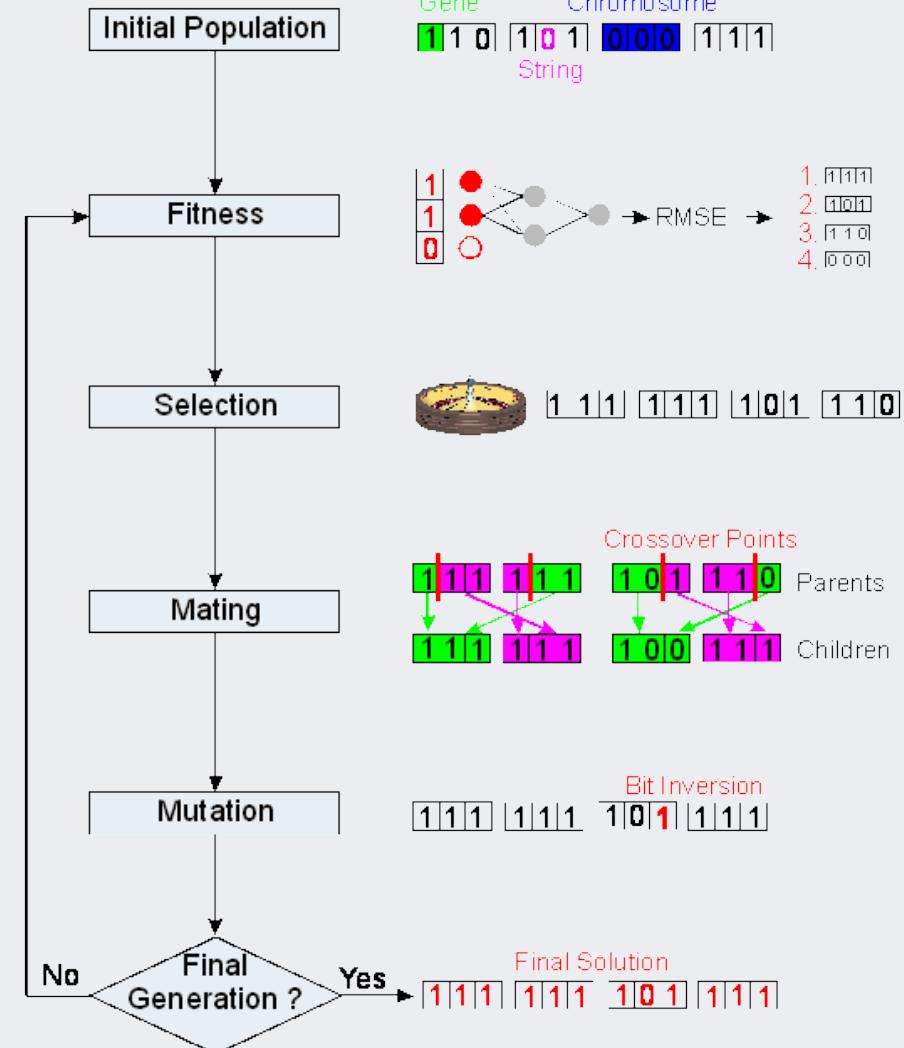
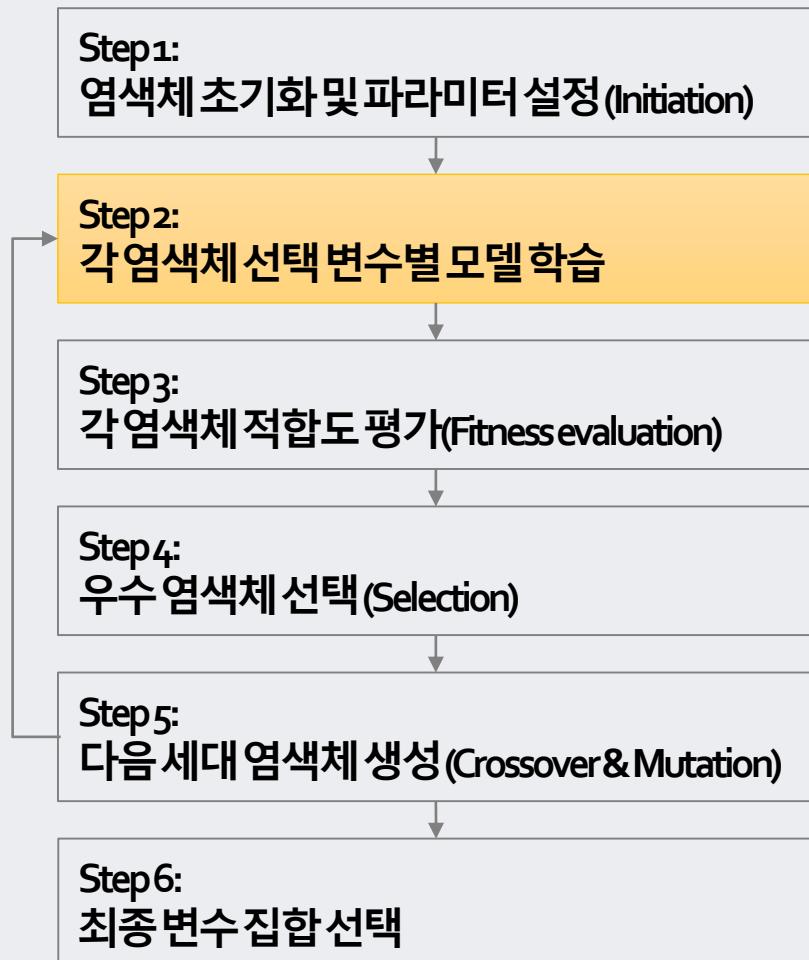
# Genetic Algorithm

- An Evolutionary Algorithm that mimics the Reproduction of Creatures
  - ✓ Find a superior solutions and preserve by repeating the reproduction process
    - Selection: Select a superior solution to improve the quality
    - Crossover: Search various alternatives based on the current solutions
    - Mutation: Give a chance to escape the local optima



# Genetic Algorithm

- Genetic Algorithm for Feature Selection



# GA Step I: Initialization

- **Encoding Chromosomes**

- ✓ Genetic algorithm can be used not only for variable selection, but for a wide range of optimization problems
- ✓ Encoding scheme can be different for different tasks
- ✓ Binary encoding is commonly used for variable selection

Chromosome					Gene					
$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	...	$x_d$	
1	0	0	1	0	1	1	0	...	1	

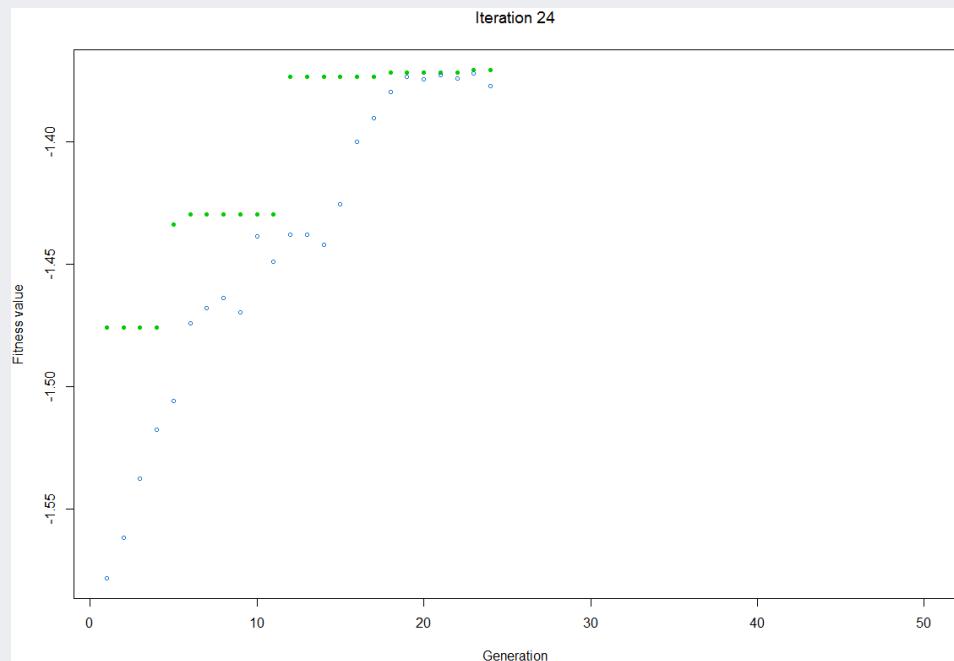
I: Use the corresponding variable in the modeling

0: Do not use the variable

# GA Step I: Initialization

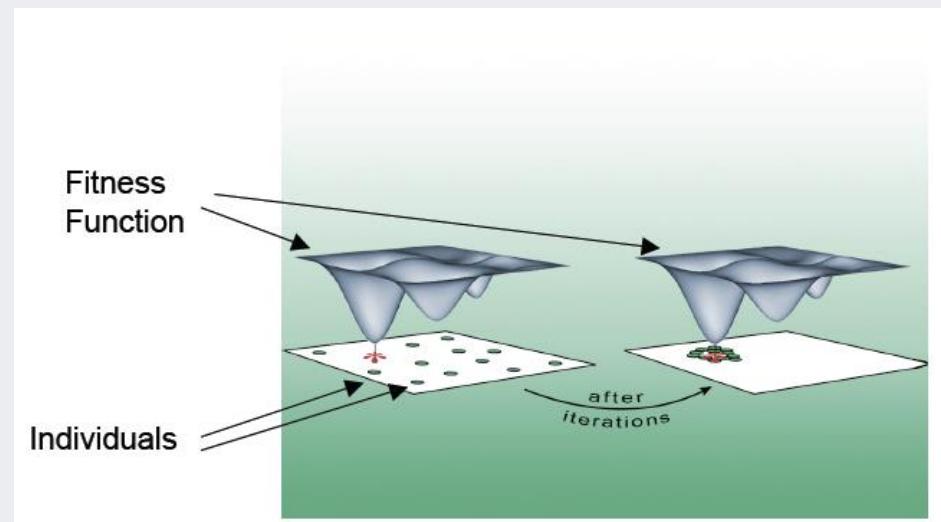
- Parameter Initialization

- ✓ The number of chromosome (population)
- ✓ Fitness function
- ✓ Crossover mechanism
- ✓ The rate of mutation
- ✓ Stopping criteria
  - minimum fitness improvement
  - maximum iterations, etc.



# GA Step 3: Fitness Evaluation

- Fitness Function
  - ✓ A criterion that determines which chromosomes are better than others
  - ✓ In general, the higher the fitness value, the better the chromosomes
  - ✓ Common criteria that are embedded in the fitness function
    - If two chromosomes have **the same fitness value**, the one with **fewer variables** is preferred
    - If two chromosomes use the **same number of variables**, the one with **higher predictive performance** is preferred
  - ✓ In case of multiple linear regression
    - Adjusted R<sup>2</sup>
    - Akaike information criterion (AIC)
    - Bayesian information criterion (BIC)



# GA Step 4: Selection

- Selection

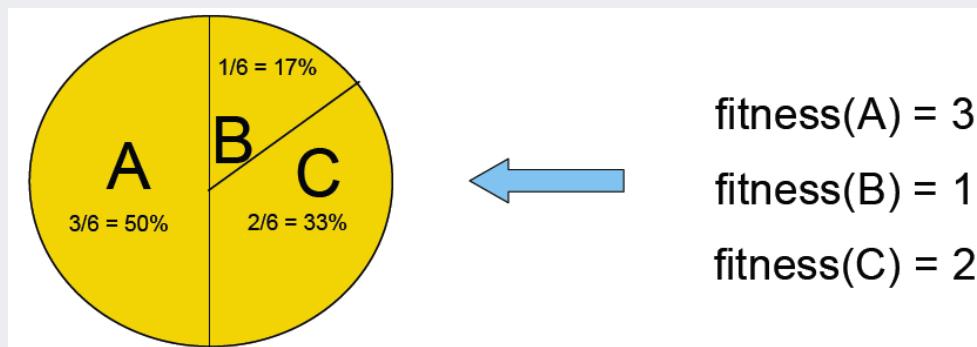
- ✓ Select superior chromosomes in the current population to reproduce the population of the next generation

- ✓ Deterministic selection

- Select only top N% of chromosomes
- Bottom (100-N)% chromosomes are never selected

- ✓ Probabilistic selection

- Use the fitness value of each chromosome as the selection weight
- All chromosomes can be selected with different probabilities

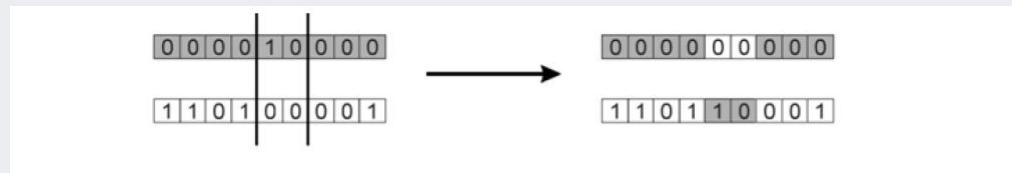


# GA Step 5: Crossover & Mutation

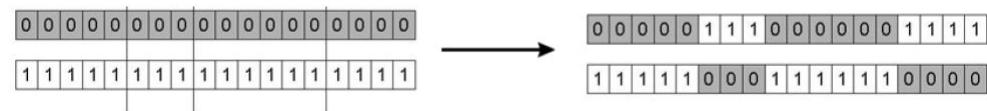
- **Crossover (Reproduction)**

- ✓ Two child chromosomes are produced from two parent chromosomes
- ✓ The number of crossover points can vary from 1 to n (total number of genes)

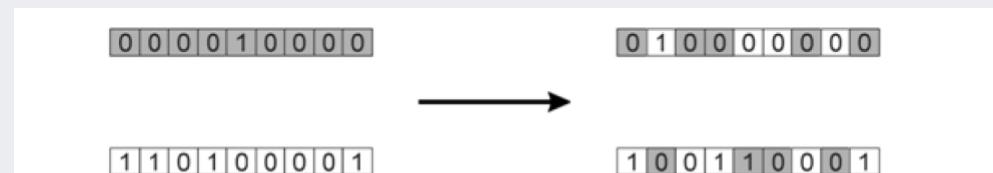
Crossover point = 2



Crossover points = 3



Crossover points = N



Assume array: [0.35, 0.62, 0.18, 0.42, 0.83, 0.76, 0.39, 0.51, 0.36]

# GA Step 5: Crossover & Mutation

- Mutation

- ✓ Genetic operator used to maintain diversity from one generation of a population of chromosomes to the next
- ✓ Alters one or more gene values in a chromosome from its initial state, which result in entirely new gene values being added to the gene pool
- ✓ By mutation, the current solution can have a chance to escape from the local optima
- ✓ A too mutation rate can increase the time to converge (0.01 can be a good choice)

Consider the two original off-springs selected for mutation.

**Original offspring 1**

1 1 0 1 1 1 1 0 0 0 0 0 1 1 1 1 0

**Original offspring 2**

1 1 0 1 1 0 0 1 0 0 1 1 0 1 1 0

Invert the value of the chosen gene as **0** to **1** and **1** to **0**

The Mutated Off-spring produced are :

**Mutated offspring 1**

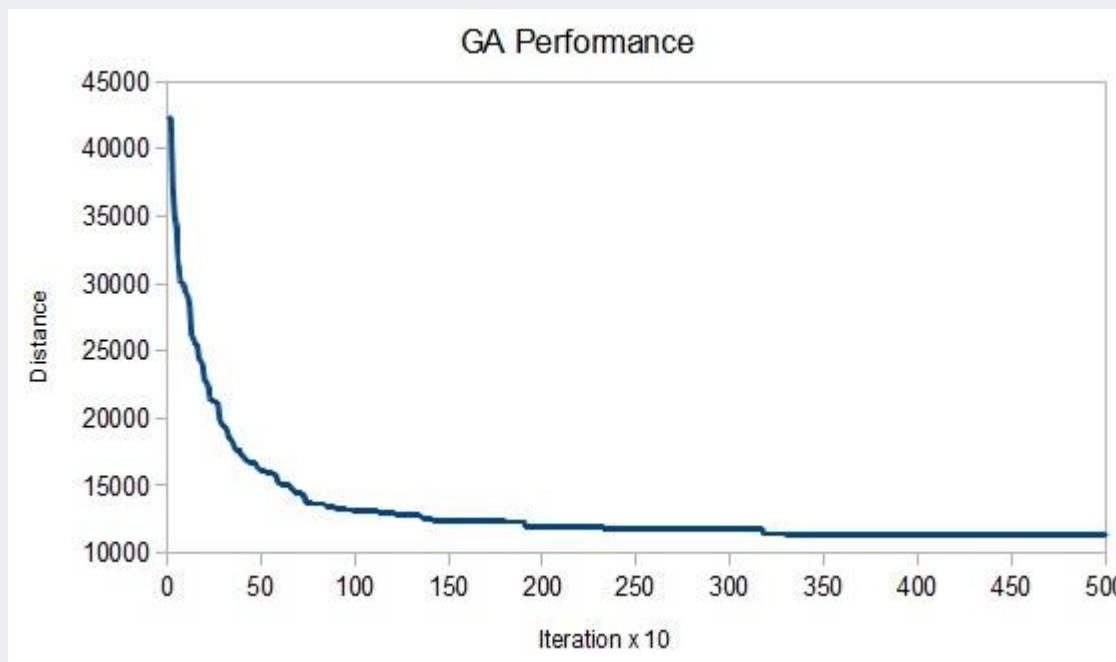
1 1 0 **0** 1 1 1 0 0 0 0 0 1 1 1 1 0

**Mutated offspring 2**

1 1 0 1 1 0 **1** 1 0 0 1 1 0 1 **0** 0

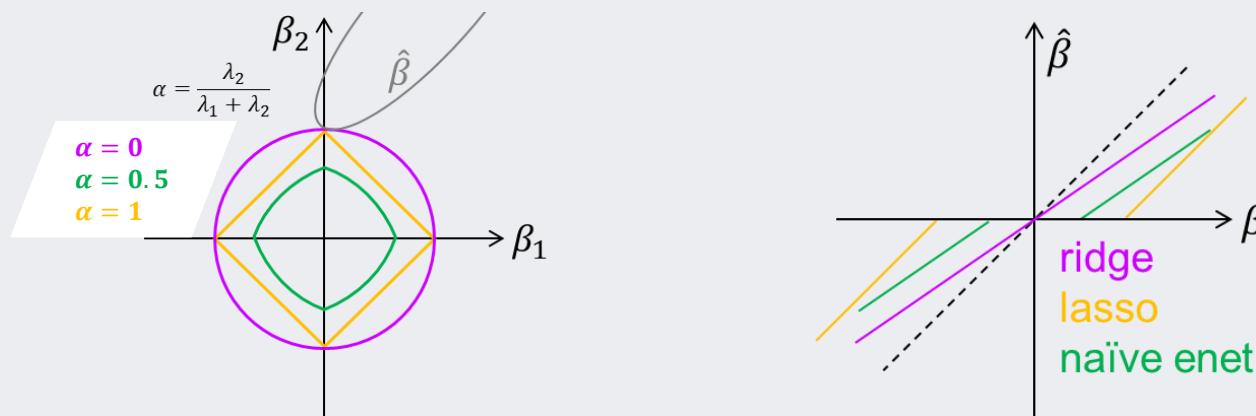
# GA Step 5: Find the Best Solution

- Find the best variable subset
  - ✓ Select the chromosome with the highest fitness value after the stopping criteria are satisfied.
  - ✓ Generally, significant fitness improvement occurs in the early stages, which becomes marginal after some generations



# Empirical Study

- Compare four variable selection and three shrinkage methods in linear regression
  - ✓ Variable selection: forward selection, backward elimination, stepwise selection, genetic algorithms
  - ✓ Shrinkage methods: ridge regression, lasso regression, elastic net



Ridge	$\hat{\beta} = \min_{\beta}  Y - X\beta ^2 + \lambda_1  \beta ^2$	shrinkage
Lasso	$\hat{\beta} = \min_{\beta}  Y - X\beta ^2 + \lambda_2  \beta ^1$	shrinkage, variable selection
Elastic net	$\hat{\beta} = \min_{\beta}  Y - X\beta ^2 + \lambda_2  \beta ^1 + \lambda_1  \beta ^2$	shrinkage, variable selection, grouping effect

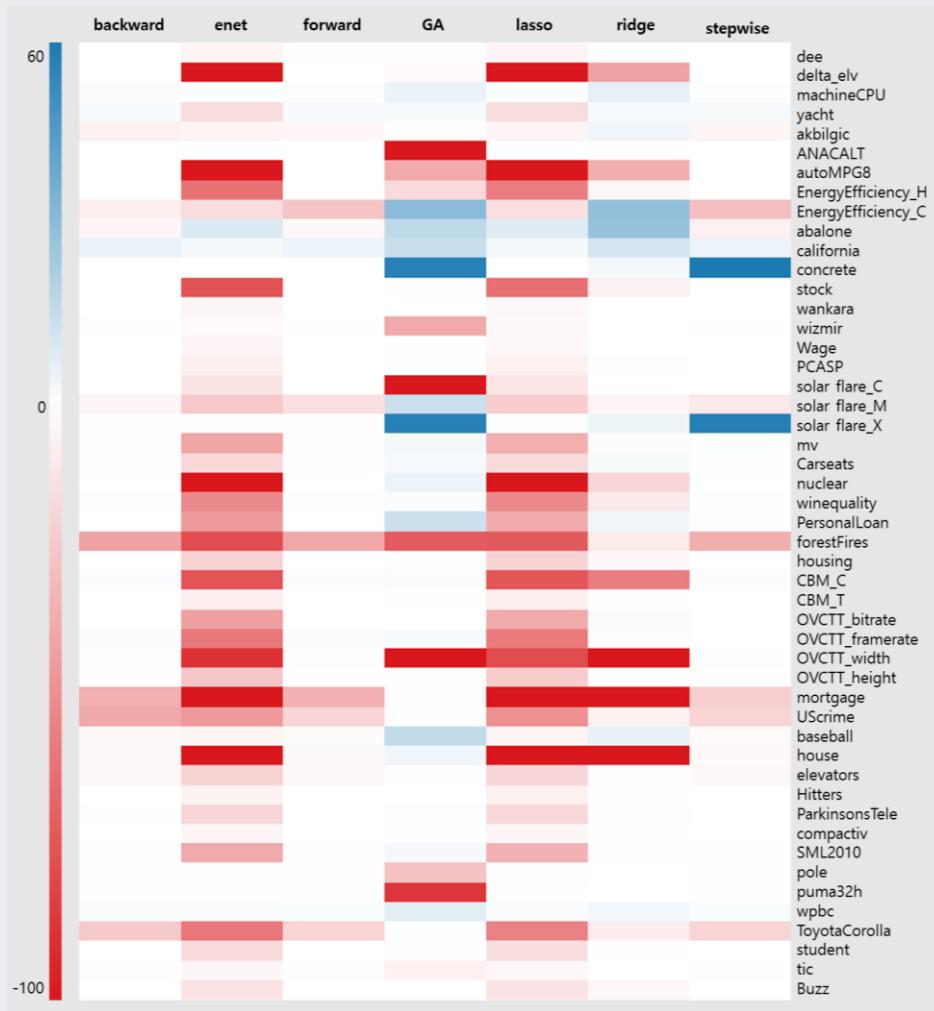
# Empirical Study

- Data sets: 49 benchmark datasets

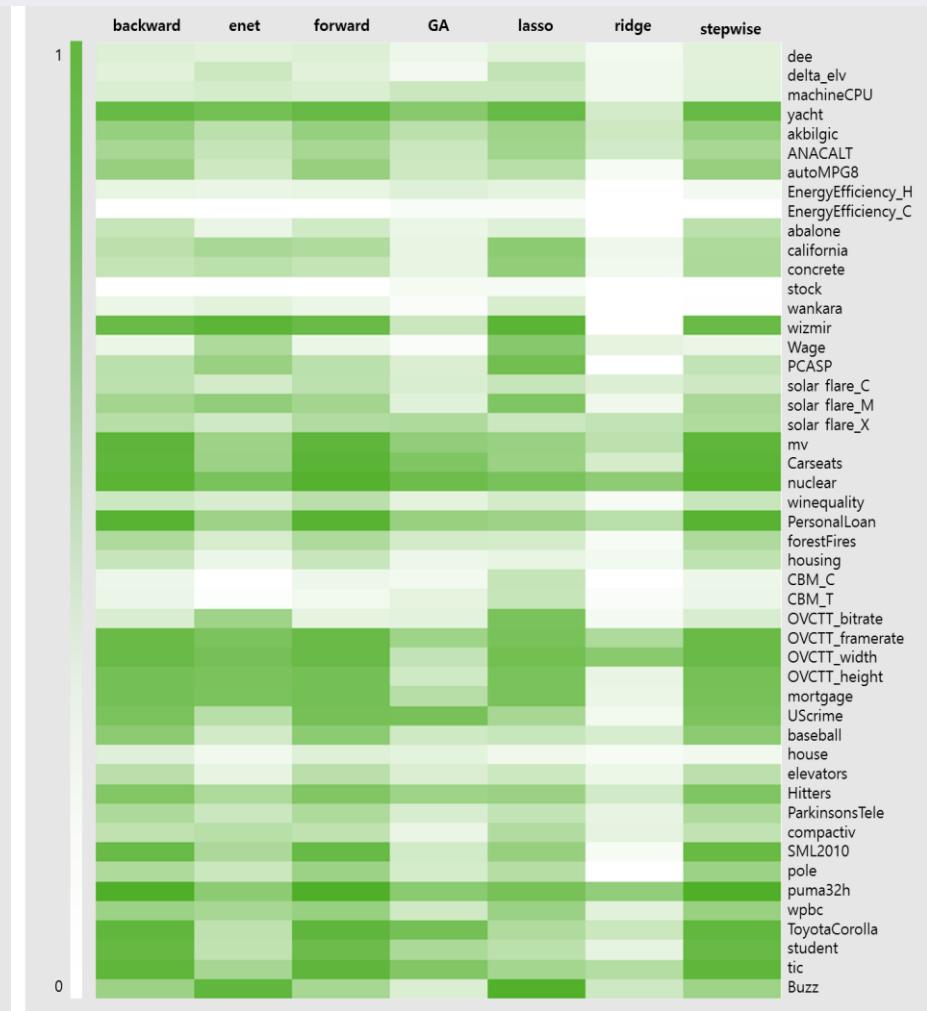
Dataset	source	records	variables	Dataset	source	records	variables
abalone	KEEL	4,177	9	OVCTT_bitrate	UCI	68,784	16
akbilgic	UCI	536	8	OVCTT_framerate	UCI	68,784	16
ANACALT	KEEL	4,052	8	OVCTT_height	UCI	68,784	16
autoMPG8	KEEL	392	8	OVCTT_width	UCI	68,784	16
baseball	KEEL	336	17	ParkinsonsTele	UCI	5,875	21
Buzz	UCI	28,179	95	PersonalLoan	etc.	2,500	13
california	KEEL	20,640	9	PCASP	UCI	45,730	10
Carseats	R	400	11	pole	KEEL	14,998	27
CBM_C	UCI	11,934	15	puma32h	KEEL	4,124	33
CBM_T	UCI	11,934	15	SML2010	UCI	4,137	24
compactiv	KEEL	8,192	22	solar flare_C	UCI	323	11
concrete	KEEL	1,030	9	solar flare_M	UCI	323	11
dee	KEEL	365	7	solar flare_X	UCI	323	11
delta_elv	KEEL	9,517	7	stock	KEEL	950	10
elevators	KEEL	16,599	19	student	UCI	382	51
EnergyEfficiency_C	UCI	768	9	tic	KEEL	9,822	86
EnergyEfficiency_H	UCI	768	9	ToyotaCorolla	etc.	1,436	34
forestFires	KEEL	517	13	UScrime	R	47	16
Hitters	R	263	20	Wage	R	3,000	10
house	KEEL	22,784	17	wankara	KEEL	1,609	10
housing	UCI	506	14	winequality	UCI	6,497	12
machineCPU	KEEL	209	7	wizmir	KEEL	1,461	10
mortgage	KEEL	1,049	16	wpbc	UCI	194	34
mv	KEEL	40,768	11	yacht	UCI	308	7
nuclear	R	32	11				

# Empirical Study

Error Rate Improvement



Variable Reduction Ratio



# Empirical Study

- Rankings in terms of
  - ✓ (1) Error rate improvement
  - ✓ (2) Variable reduction rate
  - ✓ (3) Computational efficiency

Variable selection technique	Error rate improvement	Variable reduction rate	Computational efficiency
Forward	5	4	1
Backward	4	3	2
Stepwise	3	2	6
GA	1	6	7
Ridge	2	7	5
LASSO	7	1	3
Enet	6	5	4

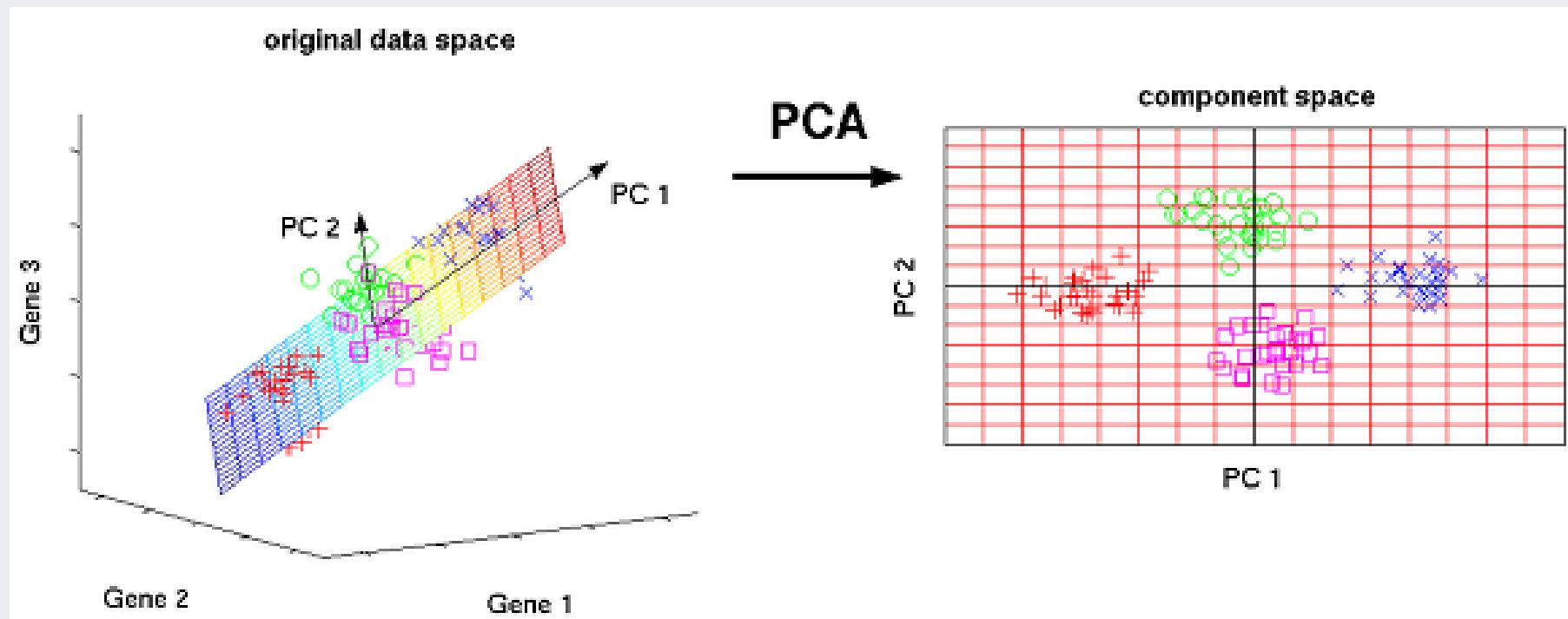
# AGENDA

- 01 Dimensionality Reduction: Overview
- 02 Supervised Methods
- 03 Unsupervised Methods: Linear Mapping
- 04 Unsupervised Methods: Non-linear Mapping

# Principal Component Analysis: PCA

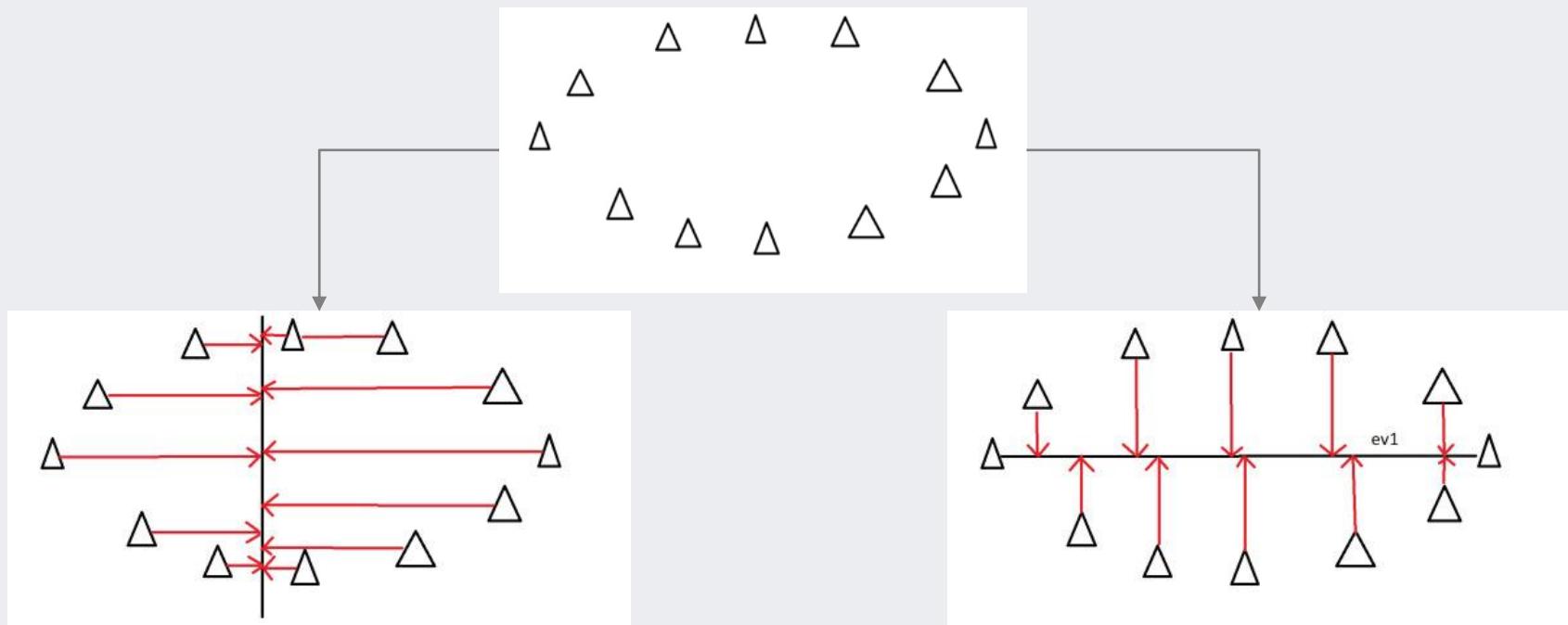
- Principal Component Analysis: PCA

✓ To find a set orthogonal bases to **preserve the variance** of the original data



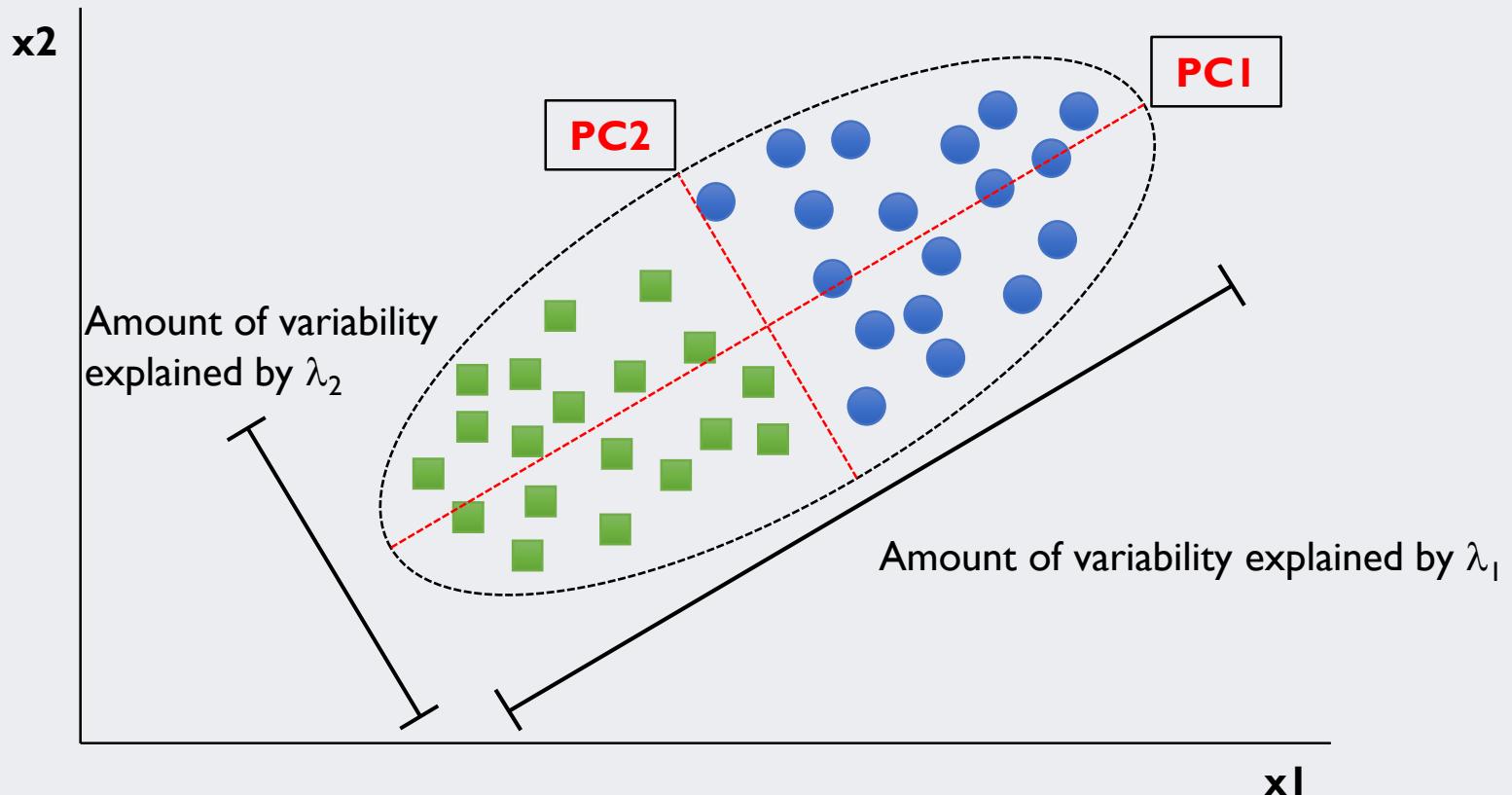
# Principal Component Analysis: PCA

- Which bases are preferred?



# Principal Component Analysis: PCA

- How much variance can be preserved?



# Principal Component Analysis: PCA

- Principal Component Analysis: PCA

- ✓ Covariance

- $\mathbf{X}$ : a data set (m by n, m: # of variables, n: # of records)

$$Cov(\mathbf{X}) = \frac{1}{n}(\mathbf{X} - \bar{\mathbf{X}})(\mathbf{X} - \bar{\mathbf{X}})^T$$

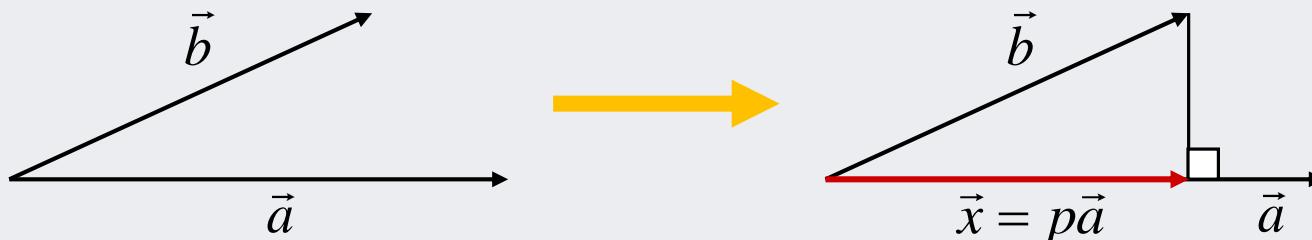
- $Cov(\mathbf{X})_{ij} = Cov(\mathbf{X})_{ji}$
    - Total variance of the data set
    - $= \text{tr}[Cov(\mathbf{X})]$
    - $= Cov(\mathbf{X})_{11} + Cov(\mathbf{X})_{22} + Cov(\mathbf{X})_{33} + \dots + Cov(\mathbf{X})_{dd}$

$$\begin{aligned} \mathbf{C}_x = Var[\mathbf{x}] &= \begin{bmatrix} Var[x_1] & Cov[x_1, x_2] & \cdots & Cov[x_1, x_N] \\ Cov[x_2, x_1] & Var[x_2] & \cdots & Cov[x_2, x_N] \\ \vdots & \vdots & \vdots & \cdots \\ Cov[x_N, x_1] & Cov[x_N, x_2] & \cdots & Var[x_N] \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & \vdots & \vdots & \cdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \vdots & \cdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_N^2 \end{bmatrix} \end{aligned}$$

# Principal Component Analysis: PCA

- Principal Component Analysis: PCA

✓ Projection onto a basis



$$(\vec{b} - p\vec{a})^T \vec{a} = 0 \Rightarrow \vec{b}^T \vec{a} - p\vec{a}^T \vec{a} = 0 \Rightarrow p = \frac{\vec{b}^T \vec{a}}{\vec{a}^T \vec{a}}$$

$$\vec{x} = p\vec{a} = \frac{\vec{b}^T \vec{a}}{\vec{a}^T \vec{a}} \vec{a}$$

If  $\vec{a}$  is unit vector

$$p = \vec{b}^T \vec{a} \Rightarrow \vec{x} = p\vec{a} = (\vec{b}^T \vec{a})\vec{a}$$

# Principal Component Analysis: PCA

- Principal Component Analysis: PCA

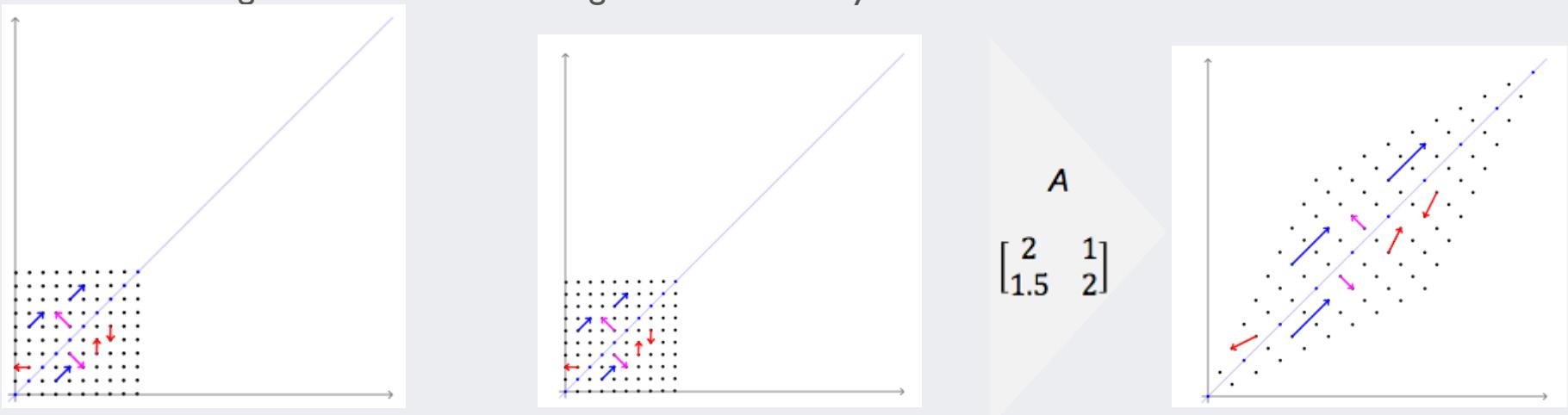
- ✓ Eigenvalue and eigenvector

- When matrix  $A$  is given, scalar value  $\lambda$  and vector  $x$  that satisfy the following equation are called eigenvalue and eigenvector, respectively.

$$Ax = \lambda x \quad \rightarrow \quad (A - \lambda I)x = 0$$

- ✓ Multiplying a matrix to a vector means a linear transformation is conducted.

- Eigenvectors do not change the direction by the transformation



# Principal Component Analysis: PCA

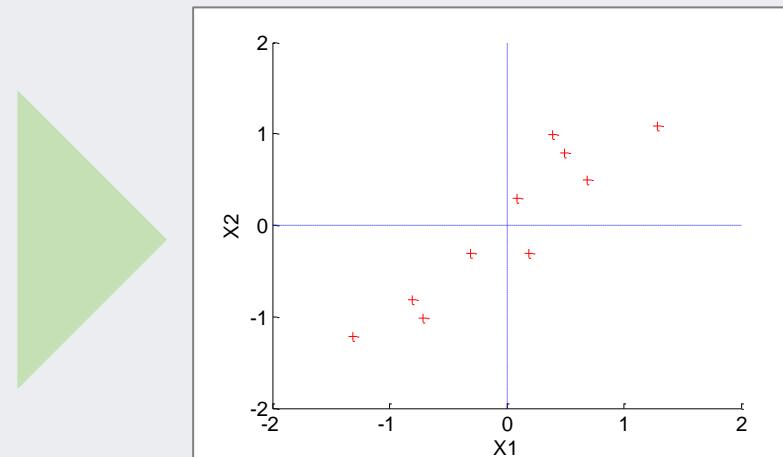
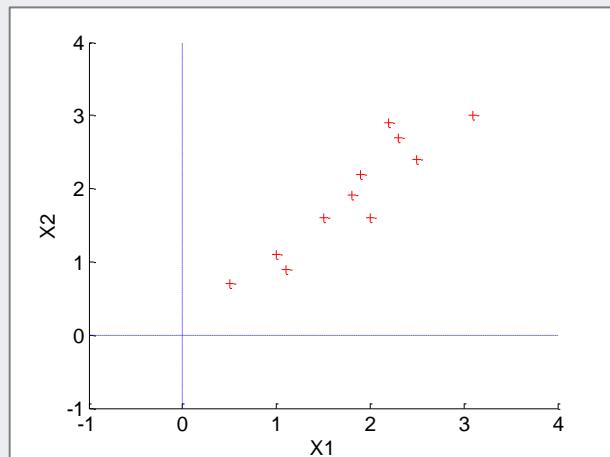
- Principal Component Analysis: PCA
  - ✓ Eigenvalue and eigenvector
  - ✓ If a matrix  $A$  is non-singular  $d$  by  $d$  matrix,
    - There exist  $d$  eigenvalue-eigenvector pairs
    - Eigenvectors are orthogonal to each other
    - $\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_d$

# PCA Procedure

- Step 1: Data Centering

✓ Make the mean of the variables equal to 0

$x_1$	2.5	0.5	2.2	1.9	3.1	2.3	2	1	1.5	1.1
$x_2$	2.4	0.7	2.9	2.2	3	2.7	1.6	1.1	1.6	0.9
<hr/>										
$x_1$	0.69	-1.31	0.39	0.09	1.29	0.49	0.19	-0.81	-0.31	-0.71
$x_2$	0.49	-1.21	0.99	0.29	1.09	0.79	-0.31	-0.81	-0.31	-1.01



# PCA Procedure

- Step 2: Formulate the optimization problem
  - ✓ If a vector  $\mathbf{x}$  is projected onto a basis  $\mathbf{w}$ , then the variance after the projection becomes

$$\mathbf{V} = (\mathbf{w}^T \mathbf{X})(\mathbf{w}^T \mathbf{X})^T = \mathbf{w}^T \mathbf{X} \mathbf{X}^T \mathbf{w} = n \mathbf{w}^T \mathbf{S} \mathbf{w}$$

- $\mathbf{S}$  is the sample covariance matrix where  $\mathbf{x}$  is normalized.
- ✓ The purpose of PCA is to maximize the variance  $\mathbf{V}$  after projection

$$\begin{aligned} \max \quad & \mathbf{w}^T \mathbf{S} \mathbf{w} \\ s.t. \quad & \mathbf{w}^T \mathbf{w} = 1 \end{aligned}$$

$$\mathbf{S} = \begin{bmatrix} 0.6166 & 0.6154 \\ 0.6154 & 0.7166 \end{bmatrix}$$

# PCA Procedure

- Step 3: Obtain the solution

✓ By employing Lagrangian multiplier,

$$\max \mathbf{w}^T \mathbf{S} \mathbf{w}$$

$$s.t. \quad \mathbf{w}^T \mathbf{w} = 1$$

$$L = \mathbf{w}^T \mathbf{S} \mathbf{w} - \lambda(\mathbf{w}^T \mathbf{w} - 1)$$

$$\frac{\partial L}{\partial \mathbf{w}} = 0 \Rightarrow \mathbf{S} \mathbf{w} - \lambda \mathbf{w} = 0 \Rightarrow (\mathbf{S} - \lambda \mathbf{I}) \mathbf{w} = 0$$

$$\text{Eigenvectors} = \begin{bmatrix} 0.6779 & -0.7352 \\ 0.7352 & 0.6779 \end{bmatrix} \quad \text{Eigenvalues} = (1.2840 \ 0.0491)$$

# PCA Procedure

- Step 4: Find the base set of bases
  - ✓ Sort the eigenvectors in a descending order of eigenvalues

$$\text{FeatureVector} = (\text{eig}_1, \text{eig}_2, \dots, \text{eig}_n)$$

$$\text{FeatureVector} = \begin{pmatrix} 0.6779 & -0.7352 \\ 0.7352 & 0.6779 \end{pmatrix}$$

- One basis can preserve 96% of the original variance in this example  
 $(1.2840/(0.0491+1.2840))$

- ✓ Let  $\mathbf{w}_1$  be one of the eigenvectors and  $\lambda_1$  be the corresponding eigenvalue.
- ✓ The variation of the samples projected onto  $\mathbf{w}_1$  is

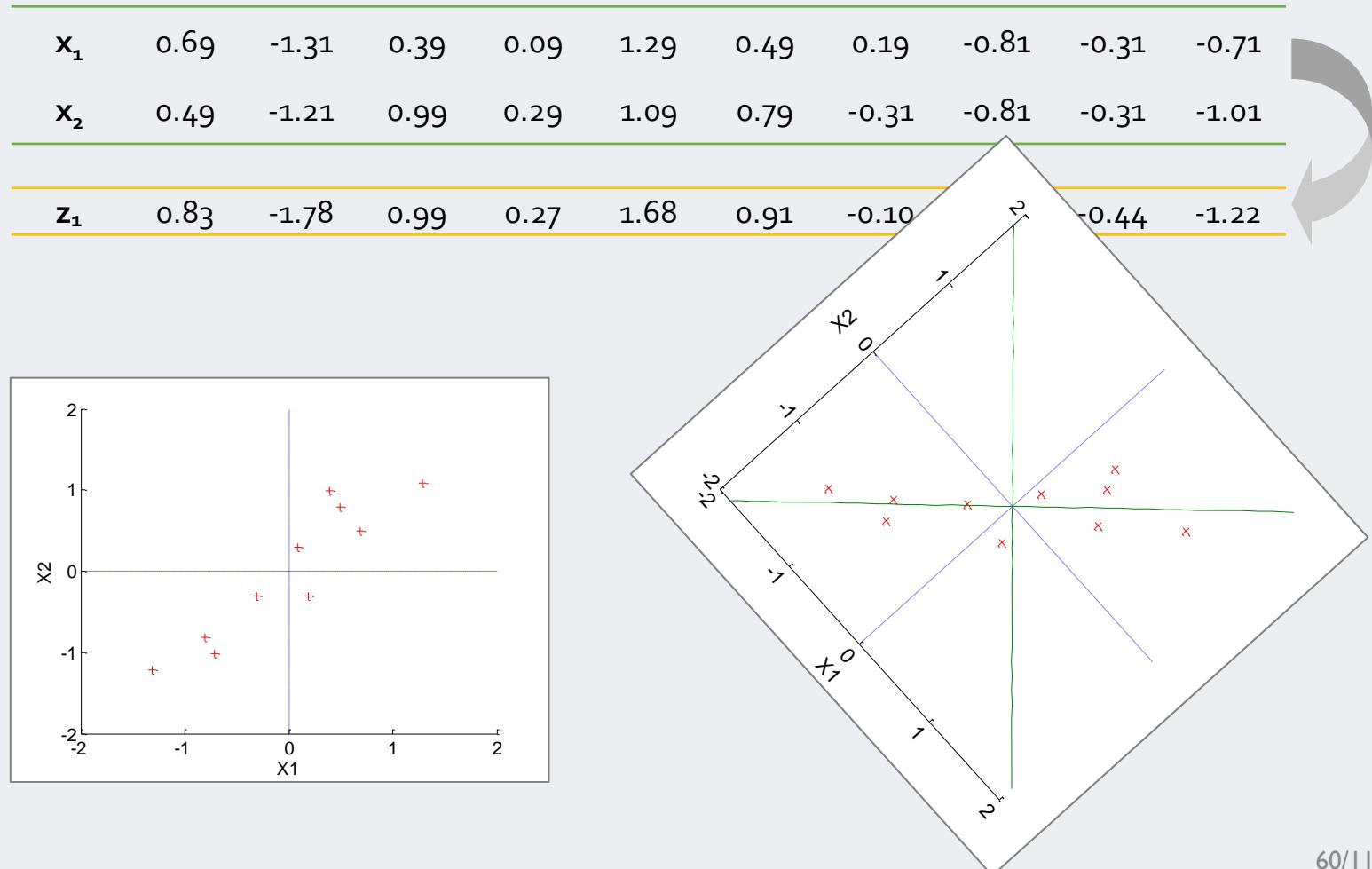
$$\mathbf{v} = (\mathbf{w}_1^T \mathbf{X})(\mathbf{w}_1^T \mathbf{X})^T = \mathbf{w}_1^T \mathbf{X} \mathbf{X}^T \mathbf{w}_1 = \mathbf{w}_1^T \mathbf{S} \mathbf{w}_1$$

$$\text{Since } \mathbf{S} \mathbf{w}_1 = \lambda_1 \mathbf{w}_1, \quad \mathbf{w}_1^T \mathbf{S} \mathbf{w}_1 = \mathbf{w}_1^T \lambda_1 \mathbf{w}_1 = \lambda_1 \mathbf{w}_1^T \mathbf{w}_1 = \lambda_1$$

# PCA Procedure

- Step 5: Extract new features

✓ Project the original data onto the selected bases



# PCA Procedure

- Step 6: Reconstruct the original data

✓ Reconstruct the data from the projected space into the original space

$\mathbf{X}$	Projection					Reconstruction				
	$\mathbf{w}^T \mathbf{X}$					$\mathbf{w} \mathbf{w}^T \mathbf{X}$				
(d by n)	(1 by d) (d by n)					(d by 1)(1 by d) (d by n)				
$\mathbf{x}_1$	0.69	-1.31	0.39	0.09	1.29	0.49	0.19	-0.81	-0.31	-0.71
$\mathbf{x}_2$	0.49	-1.21	0.99	0.29	1.09	0.79	-0.31	-0.81	-0.31	-1.01
$\mathbf{z}_1$	0.83	-1.78	0.99	0.27	1.68	0.91	-0.10	-1.14	-0.44	-1.22
$\mathbf{x}'_1$	0.56	-1.21	0.67	0.19	1.14	0.62	-0.07	-0.78	-0.30	-0.83
$\mathbf{x}'_2$	0.61	-1.31	0.73	0.20	1.23	0.67	-0.07	-0.84	-0.32	-0.90



# PCA Procedure

- PCA Example

- ✓ Original data

시리얼 이름	제조업체명	유형	칼로리	단백질	지방	나트륨	식이섬유	복합탄수화물	설탕	칼륨	비타민
100% Bran	N	C	70	4	1	130	10	5	6	280	25
100% Natural Bran	Q	C	120	3	5	15	2	8	8	135	0
All-Bran	K	C	70	4	1	260	9	7	5	320	25
All-Bran with Extra Fiber	K	C	50	4	0	140	14	8	0	330	25
Almond Delight	R	C	110	2	2	200	1	14	8		25
Apple Cinnamon Cheerios	G	C	110	2	2	180	1.5	10.5	10	70	25
Apple Jacks	K	C	110	2	0	125	1	11	14	30	25
Basic 4	G	C	130	3	2	210	2	18	8	100	25
Bran Chex	R	C	90	2	1	200	4	15	6	125	25
Bran Flakes	P	C	90	3	0	210	5	13	5	190	25
Cap'n'Crunch	Q	C	120	1	2	220	0	12	12	35	25
Cheerios	G	C	110	6	2	290	2	17	1	105	25
Cinnamon Toast Crunch	G	C	120	1	3	210	0	13	9	45	25
Clusters	G	C	110	3	2	140	2	13	7	105	25
Cocoa Puffs	G	C	110	1	1	180	0	12	13	55	25
Corn Chex	R	C	110	2	0	280	0	22	3	25	25
Corn Flakes	K	C	100	2	0	290	1	21	2	35	25
Corn Pops	K	C	110	1	0	90	1	13	12	20	25
Count Chocula	G	C	110	1	1	180	0	12	13	65	25
Cracklin' Oat Bran	K	C	110	3	3	140	4	10	7	160	25

# PCA Procedure

- PCA Example
  - ✓ Eigenvectors eigenvalues for each principal component

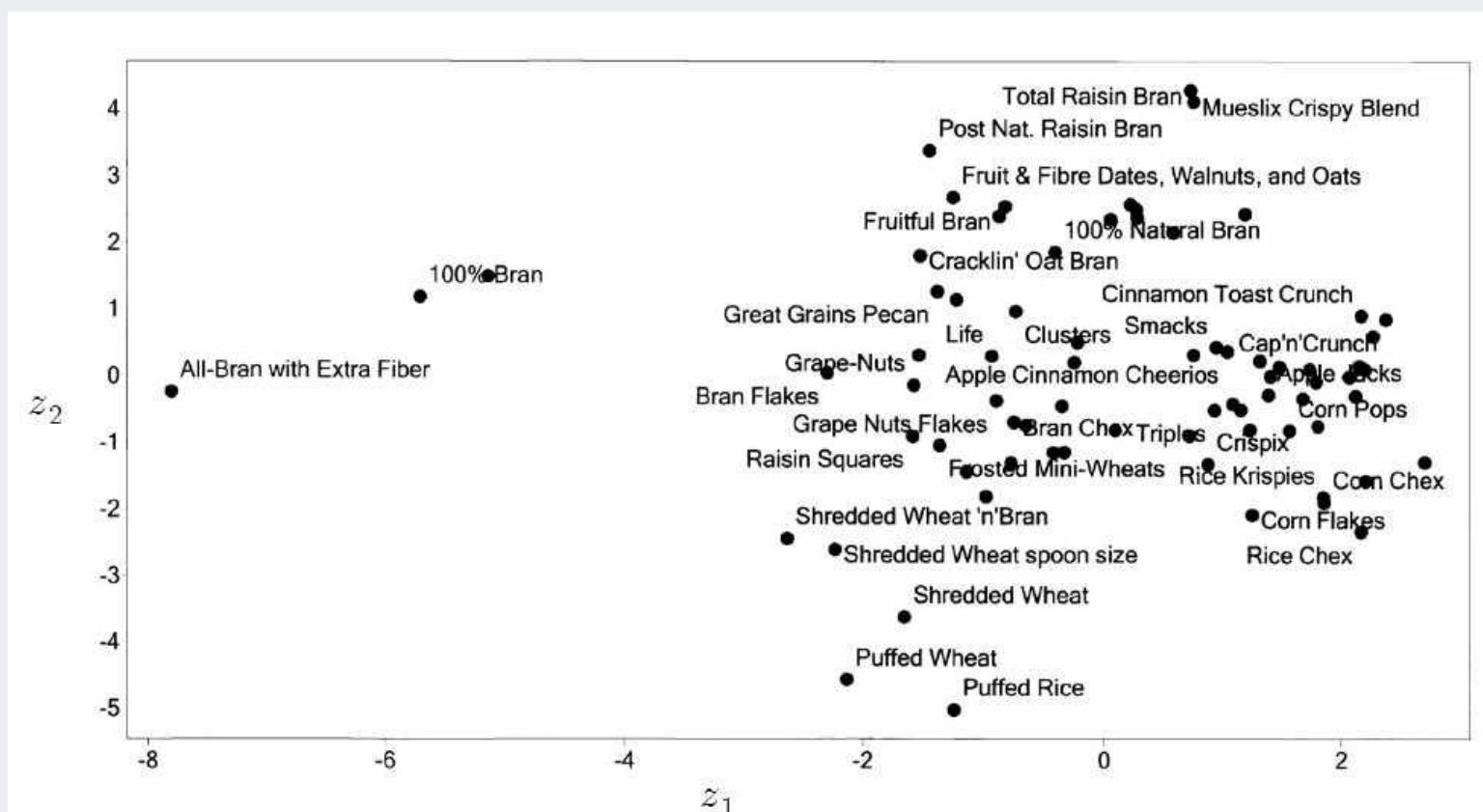
변수이름	1	2	3	4	5	6	7
calories	0.2995424	0.39314792	0.11485746	0.20435865	0.20389892	-0.25590625	-0.02559552
protein	-0.30735639	0.16532333	0.27728197	0.30074316	0.319749	0.120752	0.28270504
fat	0.03991544	0.34572428	-0.20489009	0.18683317	0.58689332	0.34796733	-0.05115468
sodium	0.18339655	0.13722059	0.38943109	0.12033724	-0.33836424	0.66437215	-0.28370309
fiber	-0.45349041	0.17981192	0.06976604	0.03917367	-0.255119	0.0642436	0.11232537
carbo	0.19244903	-0.14944831	0.56245244	0.0878355	0.18274252	-0.32639283	-0.26046798
sugars	0.22806853	0.35143444	-0.35540518	-0.02270711	-0.31487244	-0.15208226	0.22798519
potass	-0.40196434	0.30054429	0.06762024	0.09087842	-0.14836049	0.02515389	0.14880823
vitamins	0.11598022	0.1729092	0.38785872	-0.6041106	-0.04928682	0.12948574	0.29427618
shelf	-0.17126338	0.26505029	-0.00153102	-0.63887852	0.32910112	-0.05204415	-0.17483434
weight	0.05029929	0.45030847	0.24713831	0.15342878	-0.22128329	-0.39877367	0.01392053
cups	0.29463556	-0.21224795	0.13999969	0.04748911	0.12081645	0.09946091	0.74856687
rating	-0.43837839	-0.25153893	0.1818424	0.0383162	0.05758421	-0.18614525	0.06344455

분산	3.63360572	3.1480546	1.90934956	1.01947618	0.98935974	0.72206175	0.67151642
분산비(%)	27.95081329	24.21580505	14.6873045	7.84212446	7.61045933	5.55432129	5.16551113
누적분산비(%)	27.95081329	52.16661835	66.85391998	74.69604492	82.3065033	87.86082458	93.02633667

# PCA Procedure

- PCA Example
  - ✓ In a 2-dimensional space

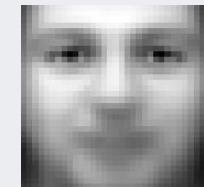


# PCA Procedure

- PCA Example

- ✓ Face image data

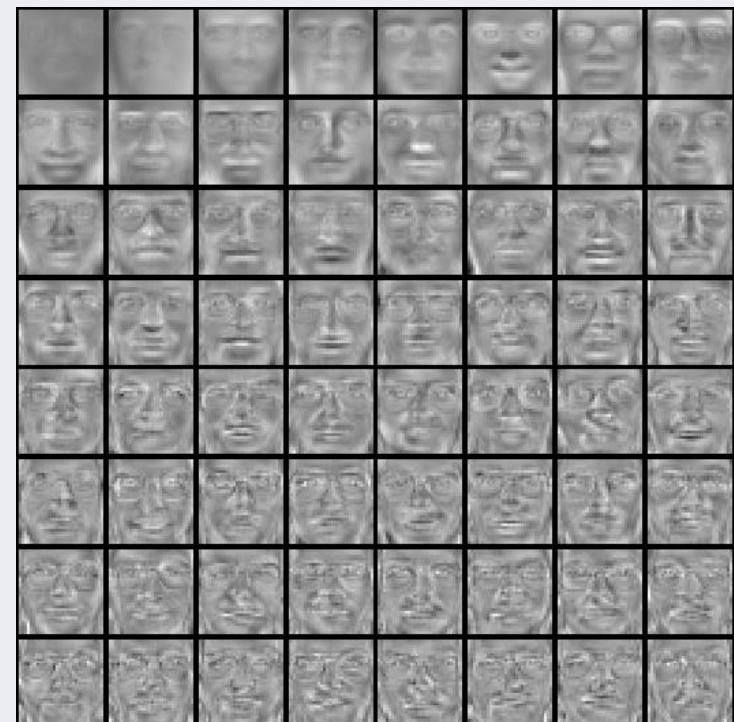
Average



Original Image



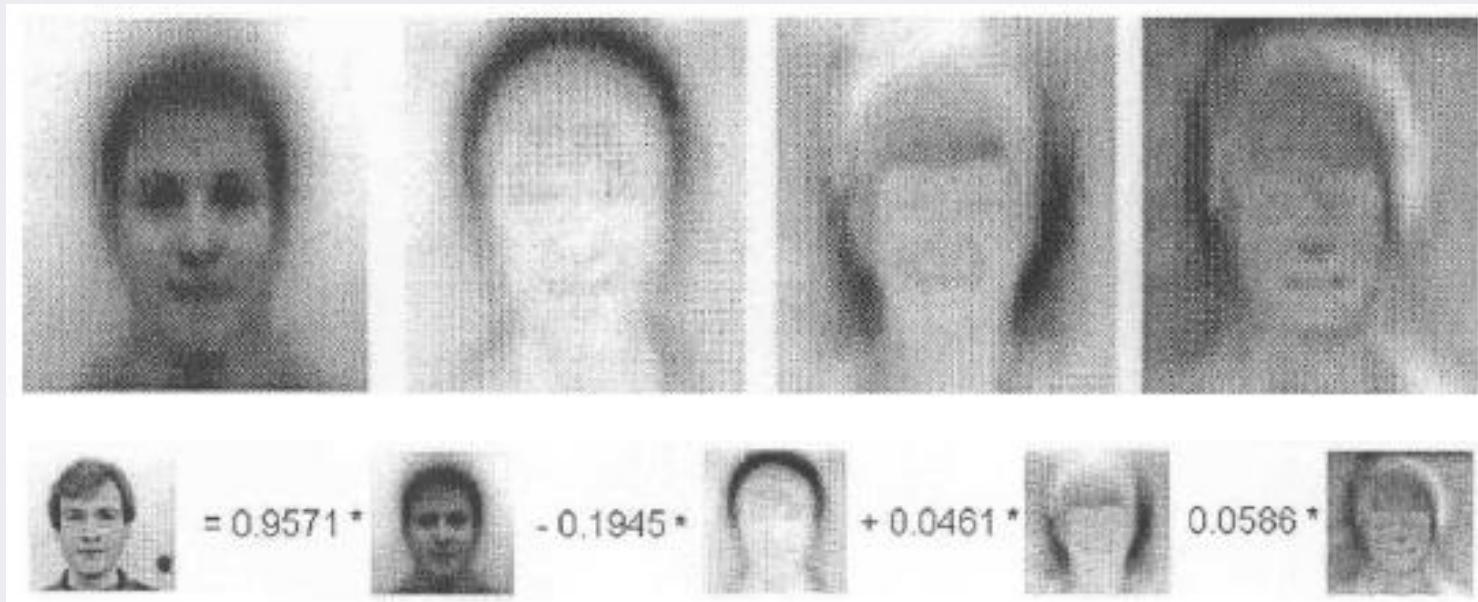
Eigenvectors



# PCA Procedure

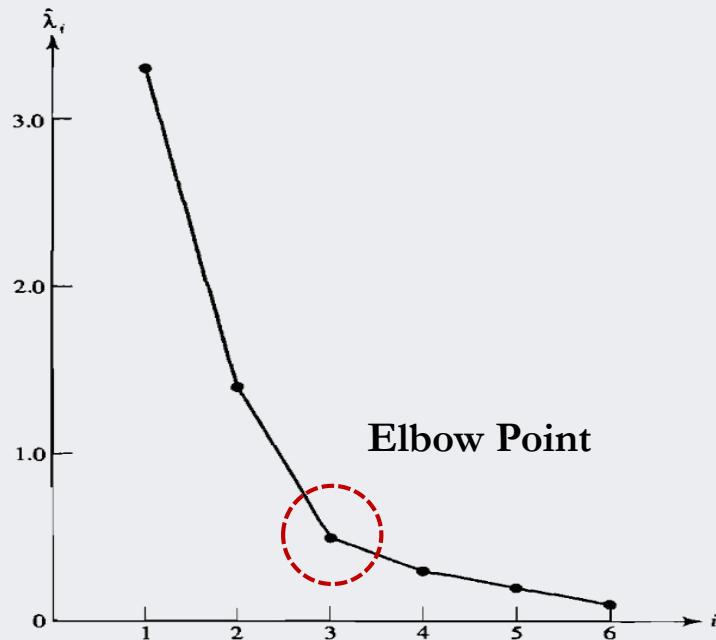
- PCA Example
  - ✓ Face image data

Image reconstruction



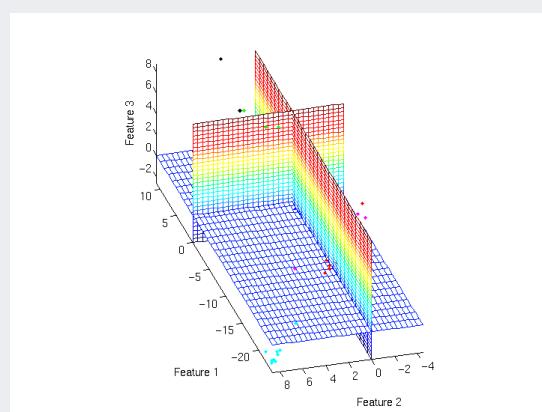
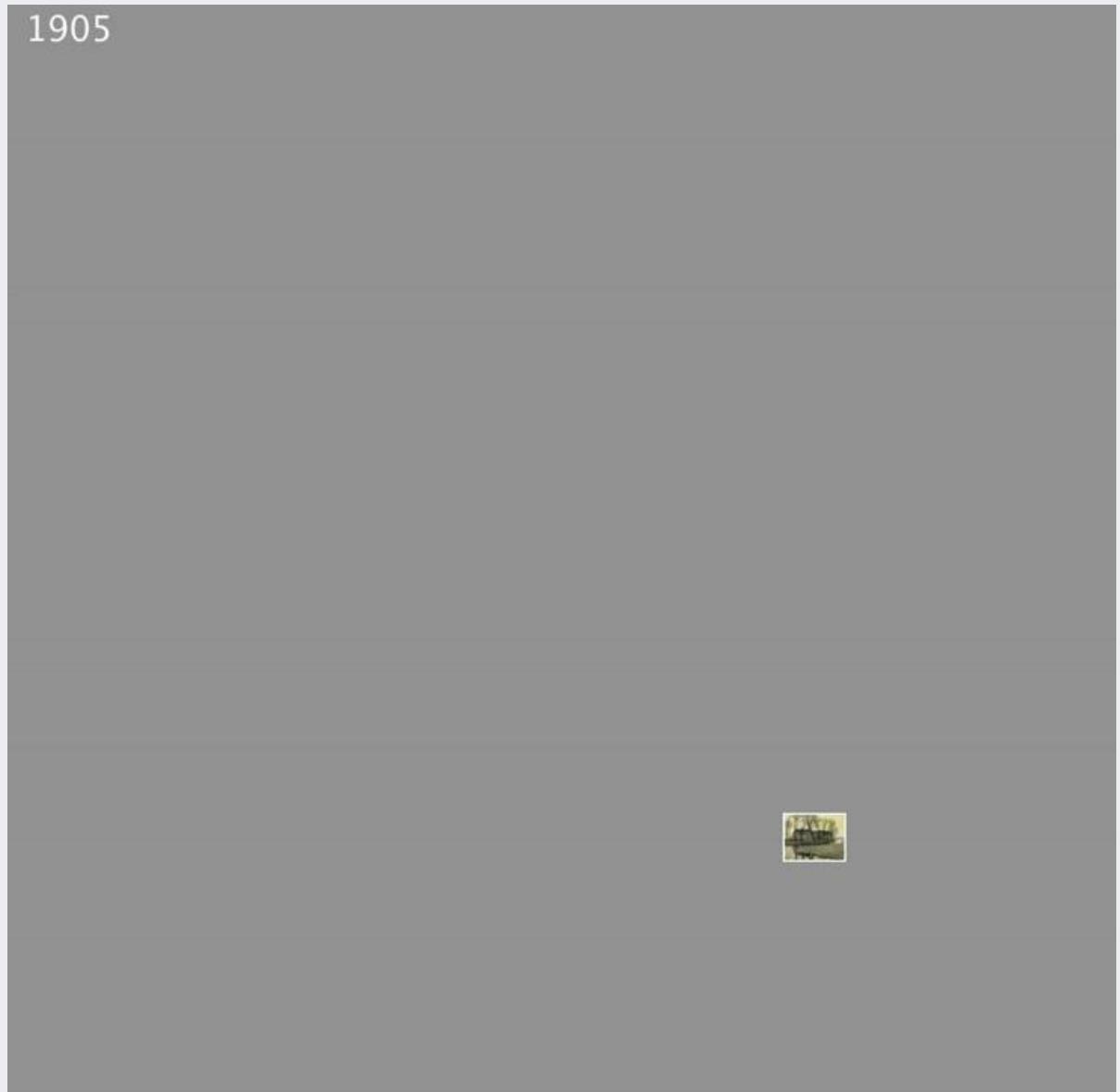
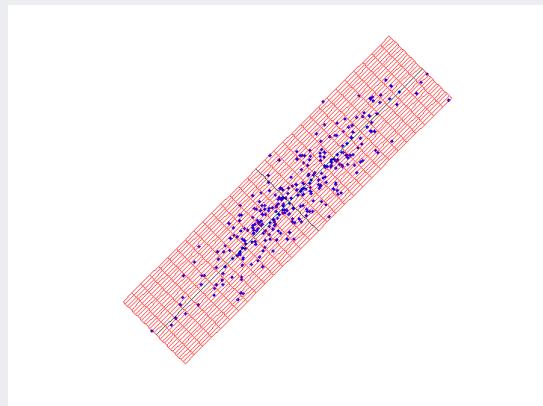
# PCA Procedure

- How to determine the optimal number of features
  - ✓ Use the scree plot
  - ✓ In general, select the principal component around the saddle point



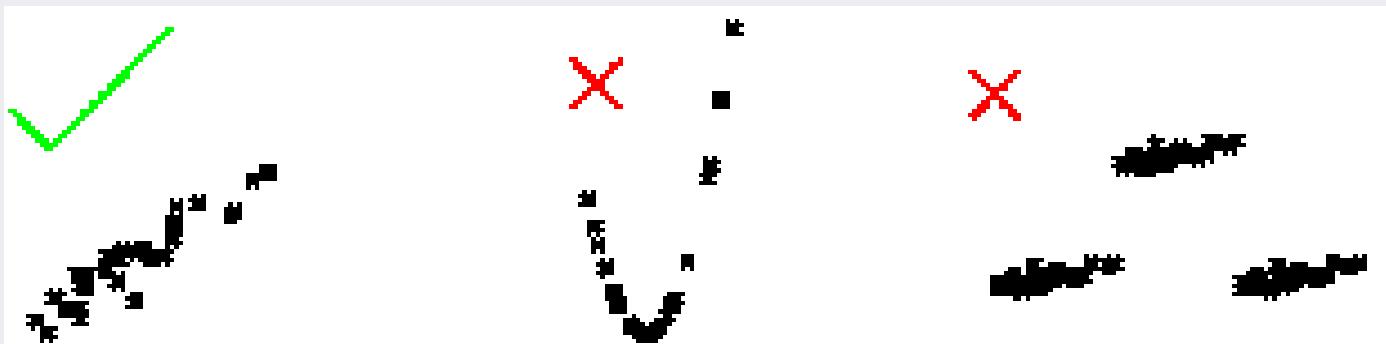
# PCA Example

1905

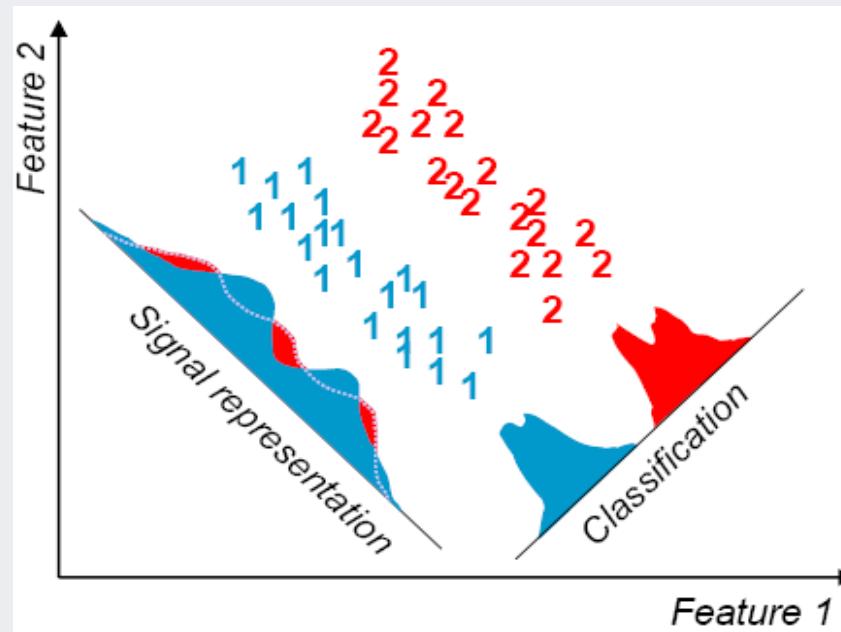


# PCA: Limitations

- Cannot work well with non-Gaussian or multimodal Gaussian distributions



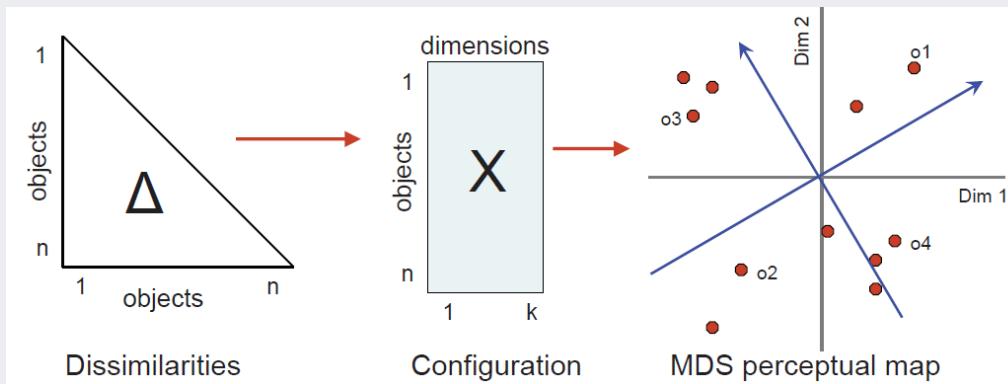
- Not designed for classification



# Multidimensional Scaling: MDS

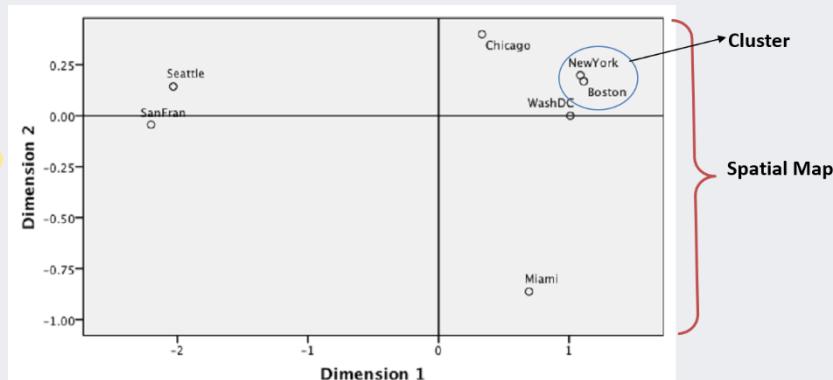
- Multidimensional Scaling

- ✓ Aims to place each object in D-dimensional space such that the between-object distances are preserved as well as possible



Distances  
Matrix:  
Symmetric

	1	2	3	4	5	6	7	8	9
1	BOST	NY	DC	MIAM	CHIC	SEAT	SF	LA	DENV
2		0	206	429	1504	963	2976	3095	2979
3			0	233	1308	802	2815	2934	2786
4				0	1075	671	2684	2799	2631
5					0	1329	3273	3053	2687
6						0	2013	2142	2054
7							0	808	1131
8								0	1307
9									0



# Multidimensional Scaling: MDS

- PCA vs. MDS

## Principal Component Analysis (PCA)

## Multidimensional Scaling (MDS)

Data

n objects in a d-dimensional space  
( $\mathbf{X}$  in  $\mathbb{R}^d$ )

Proximity matrix between n objects  
(n by n matrix  $\mathbf{D}$ )

Purpose

Find a set of bases to preserve the  
original variance

Find a set of coordinates that preserve  
the distance information between objects

Output

1. n bases (eigenvectors, PCs)  
2. n eigenvalues

Coordinate of each object in d-dimension  
( $\mathbf{X}$  in  $\mathbb{R}^d$ )

# MDS Procedure

- Step I: Construct Proximity/Distance Matrix

- ✓ If there exist the coordinates for the objects, compute the similarity/distance between them

*distance-like* if (1)  $d_{ij} \geq 0$ , (2)  $d_{ii} = 0$ , (3)  $d_{ij} = d_{ji}$

*metric* if in addition to (1), (2), (3), it satisfies  $d_{ij} \leq d_{ik} + d_{jk}$

- ✓ Distance: Euclidean, Manhattan, etc.

- ✓ Similarity: Correlation, Jaccard, etc.

$\mathbf{X}$  (d by n)

	$x_1$	$x_2$	$x_3$	$x_4$	...	$x_n$
$v_1$						
$v_2$						
$v_3$						
...						
$v_d$						



$\mathbf{D}$  (n by n)

	$x_1$	$x_2$	$x_3$	$x_4$	...	$x_n$
$x_1$						
$x_2$						
$x_3$						
$x_4$						
...						
$x_n$						

$$d_{rs}^2 = (\mathbf{x}_r - \mathbf{x}_s)^T (\mathbf{x}_r - \mathbf{x}_s)$$

# MDS Procedure

- Step 2: Extract the coordinates that preserve the distance information
  - ✓ Each element of the distance matrix  $\mathbf{D}$  can be expressed as

$$d_{rs}^2 = (\mathbf{x}_r - \mathbf{x}_s)^T (\mathbf{x}_r - \mathbf{x}_s)$$

- ✓ Inner product matrix  $\mathbf{B}$  can be obtained from the distance matrix  $\mathbf{D}$

$$[\mathbf{B}]_{rs} = b_{rs} = \mathbf{x}_r^T \mathbf{x}_s$$

- Assume that the means of all  $p$  variables are 0

$$\sum_{r=1}^n x_{ri} = 0 , \quad (i = 1, 2, \dots, p) \qquad d_{rs}^2 = \mathbf{x}_r^T \mathbf{x}_r + \mathbf{x}_s^T \mathbf{x}_s - 2\mathbf{x}_r^T \mathbf{x}_s$$

# MDS Procedure

- Step 2: Extract the coordinates that preserve the distance information

$$d_{rs}^2 = (\mathbf{x}_r - \mathbf{x}_s)^T (\mathbf{x}_r - \mathbf{x}_s)$$

$$\frac{1}{n} \sum_{r=1}^n d_{rs}^2 = \frac{1}{n} \sum_{r=1}^n \mathbf{x}_r^T \mathbf{x}_r + \frac{1}{n} \sum_{r=1}^n \mathbf{x}_s^T \mathbf{x}_s - \frac{2}{n} \sum_{r=1}^n \mathbf{x}_r^T \mathbf{x}_s = \frac{1}{n} \sum_{r=1}^n \mathbf{x}_r^T \mathbf{x}_r + \mathbf{x}_s^T \mathbf{x}_s$$

$$\mathbf{x}_s^T \mathbf{x}_s = \frac{1}{n} \sum_{r=1}^n d_{rs}^2 - \frac{1}{n} \sum_{r=1}^n \mathbf{x}_r^T \mathbf{x}_r$$

$$\frac{1}{n} \sum_{s=1}^n d_{rs}^2 = \frac{1}{n} \sum_{s=1}^n \mathbf{x}_r^T \mathbf{x}_s + \frac{1}{n} \sum_{s=1}^n \mathbf{x}_s^T \mathbf{x}_s - \frac{2}{n} \sum_{s=1}^n \mathbf{x}_r^T \mathbf{x}_s = \mathbf{x}_r^T \mathbf{x}_r + \frac{1}{n} \sum_{s=1}^n \mathbf{x}_s^T \mathbf{x}_s$$

$$\mathbf{x}_r^T \mathbf{x}_r = \frac{1}{n} \sum_{r=1}^n d_{rs}^2 - \frac{1}{n} \sum_{s=1}^n \mathbf{x}_s^T \mathbf{x}_s$$

# MDS Procedure

- Step 2: Extract the coordinates that preserve the distance information

$$d_{rs}^2 = (\mathbf{x}_r - \mathbf{x}_s)^T (\mathbf{x}_r - \mathbf{x}_s)$$

$$\frac{1}{n^2} \sum_{r=1}^n \sum_{s=1}^n d_{rs}^2 = \frac{1}{n^2} \sum_{r=1}^n \sum_{s=1}^n \mathbf{x}_r^T \mathbf{x}_r + \frac{1}{n^2} \sum_{r=1}^n \sum_{s=1}^n \mathbf{x}_s^T \mathbf{x}_s - \frac{2}{n^2} \sum_{r=1}^n \sum_{s=1}^n \mathbf{x}_r^T \mathbf{x}_s$$

$$= \frac{1}{n} \sum_{r=1}^n \mathbf{x}_r^T \mathbf{x}_r + \frac{1}{n} \sum_{s=1}^n \mathbf{x}_s^T \mathbf{x}_s = \frac{2}{n} \sum_{r=1}^n \mathbf{x}_r^T \mathbf{x}_r$$

$$\frac{2}{n} \sum_{r=1}^n \mathbf{x}_r^T \mathbf{x}_r = \frac{1}{n^2} \sum_{r=1}^n \sum_{s=1}^n d_{rs}^2$$

# MDS Procedure

- Step 2: Extract the coordinates that preserve the distance information

$$\begin{aligned}
 b_{rs} &= \mathbf{x}_r^T \mathbf{x}_s & (d_{rs}^2 = \mathbf{x}_r^T \mathbf{x}_r + \mathbf{x}_s^T \mathbf{x}_s - 2\mathbf{x}_r^T \mathbf{x}_s) \\
 &= -\frac{1}{2}(d_{rs}^2 - \mathbf{x}_r^T \mathbf{x}_r - \mathbf{x}_s^T \mathbf{x}_s) & \boxed{\mathbf{x}_s^T \mathbf{x}_s = \frac{1}{n} \sum_{r=1}^n d_{rs}^2 - \frac{1}{n} \sum_{r=1}^n \mathbf{x}_r^T \mathbf{x}_r \quad \mathbf{x}_r^T \mathbf{x}_r = \frac{1}{n} \sum_{r=1}^n d_{rs}^2 - \frac{1}{n} \sum_{s=1}^n \mathbf{x}_s^T \mathbf{x}_s} \\
 &= -\frac{1}{2} \left( d_{rs}^2 - \frac{1}{n} \sum_{s=1}^n d_{rs}^2 + \frac{1}{n} \sum_{s=1}^n \mathbf{x}_s^T \mathbf{x}_s - \frac{1}{n} \sum_{r=1}^n d_{rs}^2 + \frac{1}{n} \sum_{r=1}^n \mathbf{x}_r^T \mathbf{x}_r \right) \\
 &= -\frac{1}{2} \left( d_{rs}^2 - \frac{1}{n} \sum_{s=1}^n d_{rs}^2 - \frac{1}{n} \sum_{r=1}^n d_{rs}^2 + \frac{1}{n^2} \sum_{r=1}^n \sum_{s=1}^n d_{rs}^2 \right) & \boxed{\frac{2}{n} \sum_{r=1}^n \mathbf{x}_r^T \mathbf{x}_r = \frac{1}{n^2} \sum_{r=1}^n \sum_{s=1}^n d_{rs}^2} \\
 &= a_{rs} - a_{r\cdot} - a_{\cdot s} + a_{\cdot\cdot}
 \end{aligned}$$

where  $a_{rs} = -\frac{1}{2}d_{rs}^2$ ,  $a_{r\cdot} = \frac{1}{n} \sum_r a_{rs}$ ,  $a_{\cdot s} = \frac{1}{n} \sum_s a_{rs}$ ,  $a_{\cdot\cdot} = \frac{1}{n^2} \sum_r \sum_s a_{rs}$

# MDS Procedure

- Step 2: Extract the coordinates that preserve the distance information

$$\begin{aligned} b_{rs} &= \mathbf{x}_r^T \mathbf{x}_s = -\frac{1}{2} \left( d_{rs}^2 - \frac{1}{n} \sum_{s=1}^n d_{rs}^2 - \frac{1}{n} \sum_{r=1}^n d_{rs}^2 + \frac{1}{n^2} \sum_{r=1}^n \sum_{s=1}^n d_{rs}^2 \right) \\ &= a_{rs} - a_{r\cdot} - a_{\cdot s} + a_{..} \end{aligned}$$

(where  $a_{rs} = -\frac{1}{2}d_{rs}^2$ ,  $a_{r\cdot} = \frac{1}{n} \sum_r a_{rs}$ ,  $a_{\cdot s} = \frac{1}{n} \sum_s a_{rs}$ ,  $a_{..} = \frac{1}{n^2} \sum_r \sum_s a_{rs}$ )

$$[\mathbf{A}]_{rs} = a_{rs} \quad \mathbf{B} = \mathbf{H} \mathbf{A} \mathbf{H} \quad \mathbf{H} = \mathbf{I} - \frac{1}{n} \mathbf{1} \mathbf{1}^T$$

# MDS Procedure

- Step 2: Extract the coordinates that preserve the distance information

- ✓ Obtain coordinates of  $\mathbf{X}$  from  $\mathbf{B}$  ( $\mathbf{X}$ : n by p, p < n)

$$\mathbf{B} = \mathbf{XX}^T \quad \text{rank}(\mathbf{B}) = \text{rank}(\mathbf{XX}^T) = \text{rank}(\mathbf{X}) = p$$

- ✓  $\mathbf{B}$  is symmetric, positive semi-definite and of rank p, so it has p non-negative eigenvalues and (n-p) zero eigenvalues (by Eigen-decomposition)

$$\mathbf{B} = \mathbf{V}\Lambda\mathbf{V}^T, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n), \mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$$

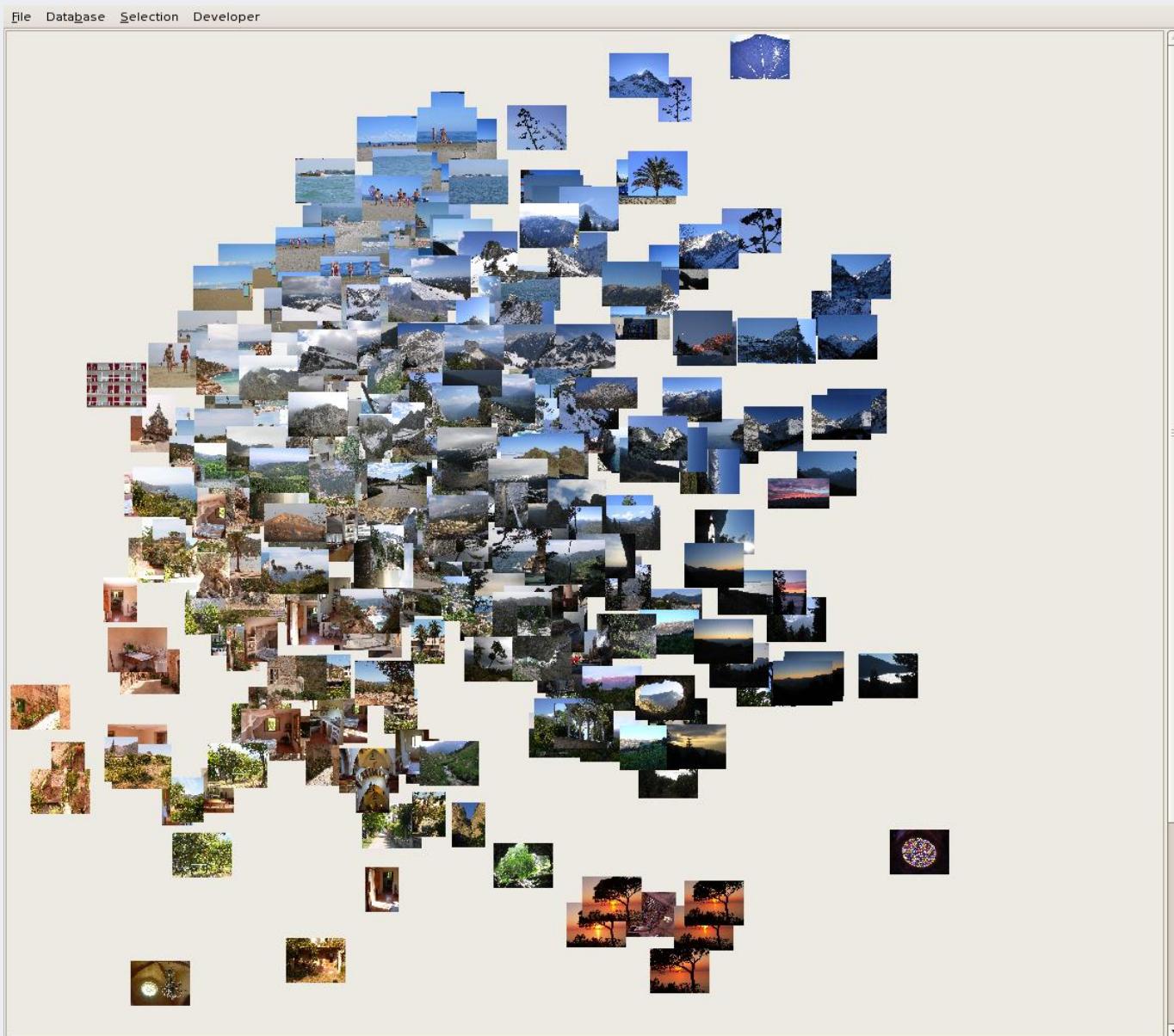
- ✓ Because of (n-p) zero eigenvalues,  $\mathbf{B}$  can be rewritten as

$$\mathbf{B}_1 = \mathbf{V}_1\Lambda_1\mathbf{V}_1^T, \quad \Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_p), \mathbf{V}_1 = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p]$$

- ✓ The coordinate matrix  $\mathbf{X}$  is given by

$$\mathbf{X} = \mathbf{V}_1\Lambda_1^{\frac{1}{2}}$$

# MDS Example



# AGENDA

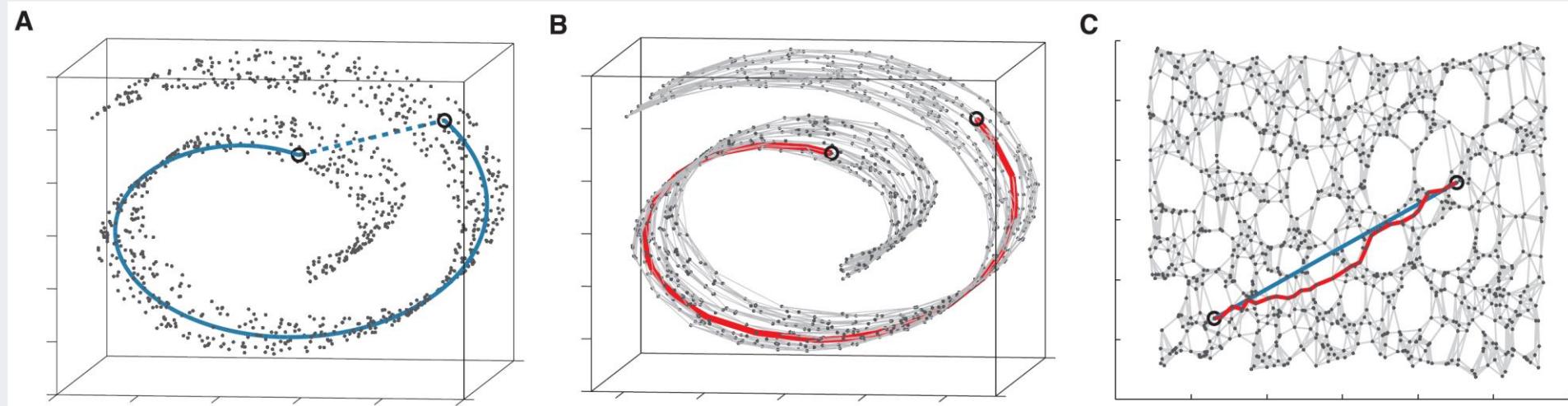
- 01 Dimensionality Reduction: Overview
- 02 Supervised Methods
- 03 Unsupervised Methods: Linear Mapping
- 04 Unsupervised Methods: Non-linear Mapping

# Isometric Feature Mapping (Isomap)

Tenenbaum et al. (2000)

- ISOMAP

- ✓ Combines the major algorithmic features of PCA and MDS
  - Computational efficiency, global optimality, and asymptotic convergence guarantees
- ✓ Builds on classical MDS but seeks to preserve the intrinsic geometry of the data, as captured in the geodesic manifold distances between all pairs of data points



# Isometric Feature Mapping (Isomap)

- Isomap procedure

- ✓ Step 1: Construct neighborhood graph

- $\varepsilon$ -Isomap: connect two points if they are closer than  $\varepsilon$
    - $k$ -Isomap: connect the point  $i$  to the point  $j$  if the  $i$  is one of the  $k$ -nearest neighbor of  $j$

- ✓ Step 2: Compute the shortest paths

- Initialize  $d_G(i, j) = d_X(i, j)$  if  $i$  and  $j$  are linked by an edge,  $d_G(i, j) = \inf$  otherwise
    - For each value of  $k = 1, 2, \dots, N$  in turn, replace all entries  $d_G(i, j)$  by  
$$\min \{d_G(i, j), d_G(i, k) + d_G(k, j)\}$$

- ✓ Step 3: Construct  $d$ -dimensional embedding by traditional MDS

# Isometric Feature Mapping (Isomap)

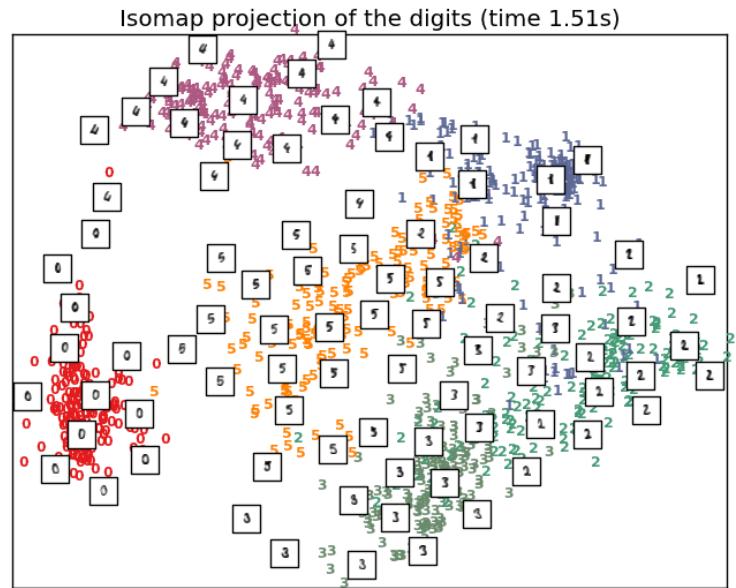
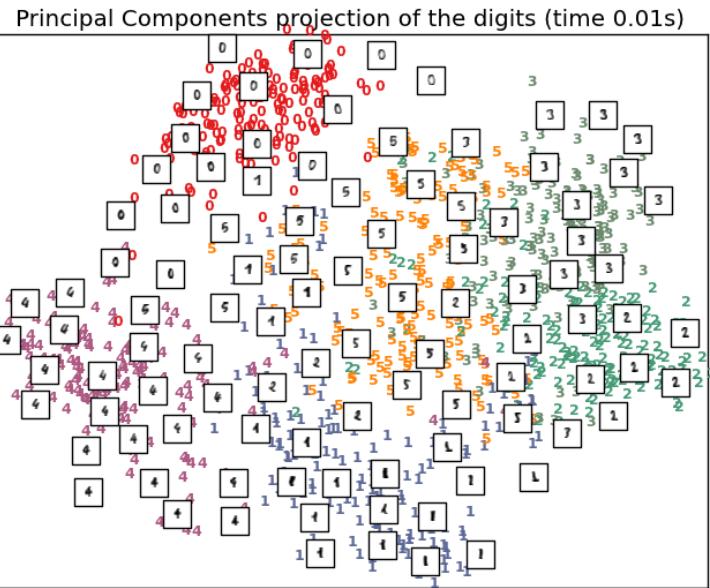
- Isomap example: Hand digit recognition

## A selection from the 64-dimensional digits dataset

0	1	2	3	4	5	0	1	2	3	4	5	0	5
5	5	0	4	1	3	5	1	0	0	2	2	2	0
4	4	1	5	0	5	2	2	0	0	1	3	2	1
3	4	1	4	0	5	3	1	5	4	4	2	3	4
2	3	4	5	0	1	2	3	4	5	0	1	2	3
0	4	1	3	5	1	0	0	2	2	2	0	1	2
1	5	0	5	2	2	0	0	1	3	2	1	3	1
0	5	7	4	5	4	4	2	2	2	5	5	4	4
5	0	4	1	2	3	4	5	0	4	2	3	4	5
3	5	4	1	0	0	2	2	2	0	1	2	3	4
5	2	2	0	0	4	3	2	1	4	3	1	3	4
3	1	5	4	4	2	2	2	5	5	4	4	0	3
0	1	2	3	4	5	0	1	2	3	4	5	0	5
5	1	0	0	1	2	2	0	1	2	3	3	3	4
1	2	0	0	1	3	2	1	4	3	1	3	1	4
1	5	4	4	2	2	2	5	5	4	4	0	0	1
2	3	4	5	0	1	2	3	4	5	0	5	5	5
0	0	1	2	2	0	1	2	3	3	3	4	4	4
0	0	1	3	2	1	4	3	1	3	4	4	4	5
4	4	2	2	1	5	5	4	4	0	0	1	2	3

# Isometric Feature Mapping (Isomap)

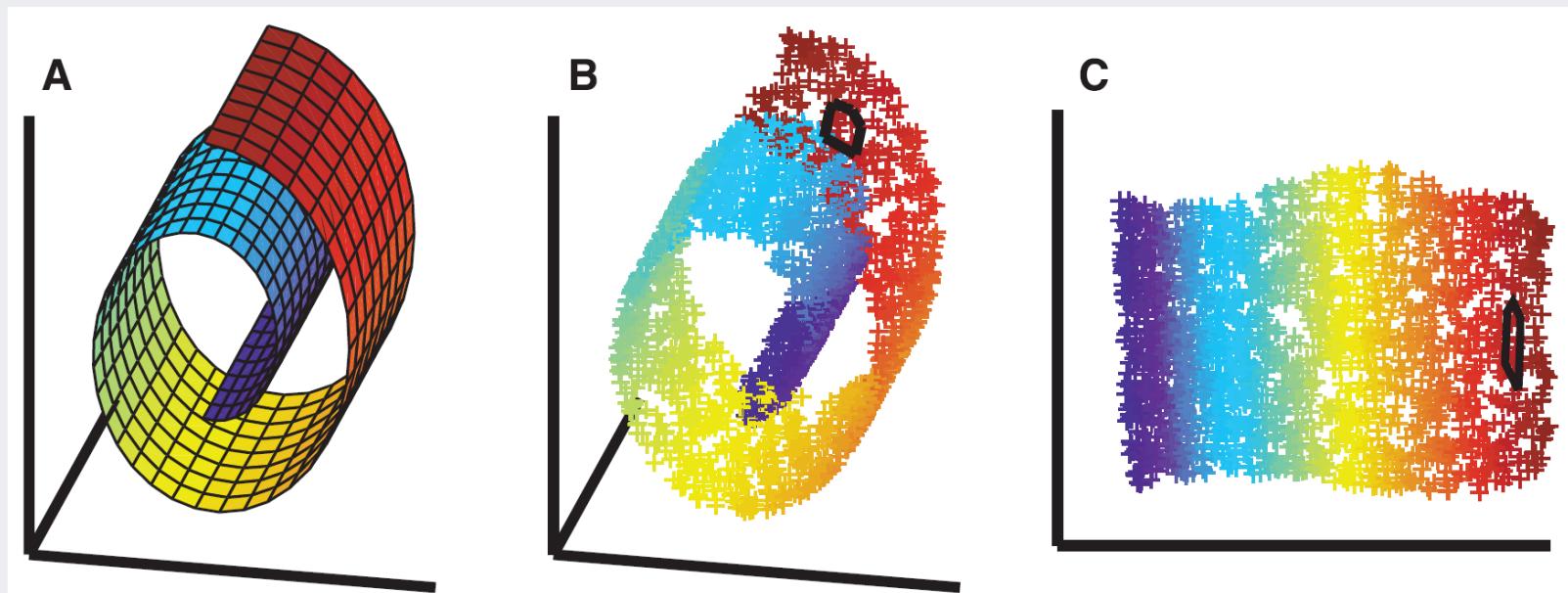
- Isomap example: Hand digit recognition



# Locally Linear Embedding (LLE)

Roweis and Saul (2000)

- An eigenvector method for nonlinear dimensionality reduction
  - ✓ Simple to implement
  - ✓ Optimizations do not involve local minima
  - ✓ Capable of generating highly nonlinear embeddings
  - ✓ Map high dimensional data into a single global coordinate system of lower dimension



# Locally Linear Embedding (LLE)

- LLE Procedure

- ✓ Step 1: Compute the neighbors of each data point
- ✓ Step 2: Compute the weight  $\mathbf{W}_{ij}$  that best reconstruct each data point from its neighbors, minimizing the cost function by constrained linear fits

$$E(\mathbf{W}) = \sum_i \left| \mathbf{x}_i - \sum_j \mathbf{w}_{ij} \mathbf{x}_j \right|^2$$

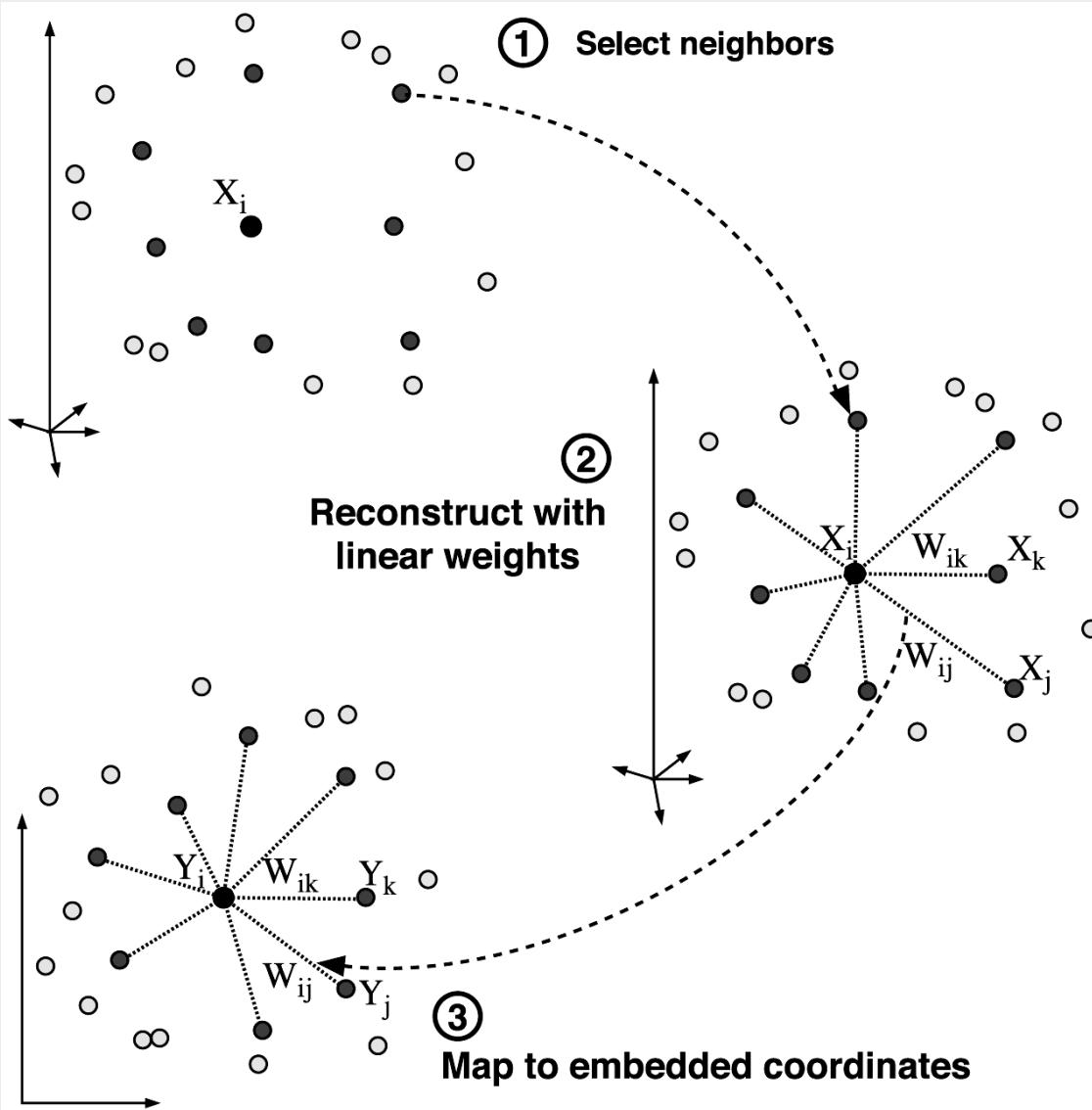
s.t.  $\mathbf{W}_{ij} = 0$  if  $\mathbf{x}_j$  does not belong to the neighbor of  $\mathbf{x}_i$

$$\sum_j \mathbf{W}_{ij} = 1 \text{ for all } i$$

- ✓ Step 3: Compute the vectors best reconstructed by the weights  $\mathbf{W}_{ij}$ , minimizing the quadratic form by its bottom nonzero eigenvectors

$$\Phi(\mathbf{W}) = \sum_i \left| \mathbf{y}_i - \sum_j \mathbf{w}_{ij} \mathbf{y}_j \right|^2$$

# Locally Linear Embedding (LLE)



## Weights

Invariant to rotation, re-scalings, translations of the data points due to the constraint.

## Note

Although the weights  $W_{ij}$  and the vectors  $Y_i$  are computed by methods in linear algebra, the constraint that the points are only reconstructed from neighbors can result in highly nonlinear embeddings.

# Locally Linear Embedding (LLE)

- LLE Procedure (cont')

- ✓ In Step 2, the reconstruction error of  $\varepsilon(\mathbf{W})$  should be zero if  $\mathbf{X}_i$  is located in the convex set of its neighbors.
- ✓ How to find  $\mathbf{Y}_i$ s in Step 3?

$$\min_{\mathbf{Y}} \Phi(\mathbf{W}) = \sum_i \left| \mathbf{Y}_i - \sum_j \mathbf{W}_{ij} \mathbf{Y}_j \right|^2 \Rightarrow \Phi(\mathbf{W}) = \sum_{i,j} \mathbf{M}_{ij} (\mathbf{Y}_i \cdot \mathbf{Y}_j)$$

where  $\mathbf{M}_{ij} = \delta_{ij} - \mathbf{W}_{ij} - \mathbf{W}_{ji} + \sum_k \mathbf{W}_{ki} \mathbf{W}_{kj}$ ,  $\delta_{ij} = 1$  if  $i = j$ , 0 otherwise

$$s.t. \quad \sum_i \mathbf{Y}_i = 0, \quad \frac{1}{n} \sum_i \mathbf{Y} \mathbf{Y}^T = \mathbf{I}$$

# Locally Linear Embedding (LLE)

- LLE Procedure (cont')
  - ✓ In Step 2, the reconstruction error of  $\varepsilon(\mathbf{W})$  should be zero if  $\mathbf{X}_i$  is located in the convex set of its neighbors.
  - ✓ How to find  $\mathbf{Y}_i$ s in Step 3?

$$\begin{aligned}\min_{\mathbf{Y}} \Phi(\mathbf{W}) &= \sum_i \left| \mathbf{Y}_i - \sum_j \mathbf{W}_{ij} \mathbf{Y}_j \right|^2 \\ &= \left[ (\mathbf{I} - \mathbf{W}) \mathbf{Y} \right]^T (\mathbf{I} - \mathbf{W}) \mathbf{Y} \\ &= \mathbf{Y}^T (\mathbf{I} - \mathbf{W})^T (\mathbf{I} - \mathbf{W}) \mathbf{Y} \\ &= \mathbf{Y}^T \mathbf{M} \mathbf{Y}\end{aligned}$$

# Locally Linear Embedding (LLE)

- LLE Procedure (cont')
  - ✓ The optimal embedding is found by computing the bottom  $d+1$  eigenvectors of the matrix  $M$  (Rayleigh-Ritz theorem)
  - ✓ The bottom eigenvector is the unit vector with all equal components
  - ✓ Discarding this eigenvector enforces the constraint that the embeddings have zero mean

The **Rayleigh–Ritz method** allows for the computation of Ritz pairs  $(\tilde{\lambda}_i, \tilde{\mathbf{x}}_i)$  which approximate the solutions to the eigenvalue problem [1]

$$A\mathbf{x} = \lambda\mathbf{x}$$

Where  $A \in \mathbb{C}^{N \times N}$ .

The procedure is as follows:[2]

1. Compute an orthonormal basis  $V \in \mathbb{C}^{N \times m}$  approximating the **eigenspace** corresponding to  $m$  eigenvectors
2. Compute  $R \leftarrow V^* A V$
3. Compute the eigenvalues of  $R$  solving  $R\mathbf{v}_i = \tilde{\lambda}_i \mathbf{v}_i$
4. Form the ritz pairs  $(\tilde{\lambda}_i, \tilde{\mathbf{x}}_i) = (\tilde{\lambda}_i, V\mathbf{v}_i)$

One can always compute the accuracy of such an approximation via  $\|A\tilde{\mathbf{x}}_i - \tilde{\lambda}_i \tilde{\mathbf{x}}_i\|$

If a **Krylov subspace** is used and  $A$  is a general matrix, then this is the **Arnoldi Algorithm**.

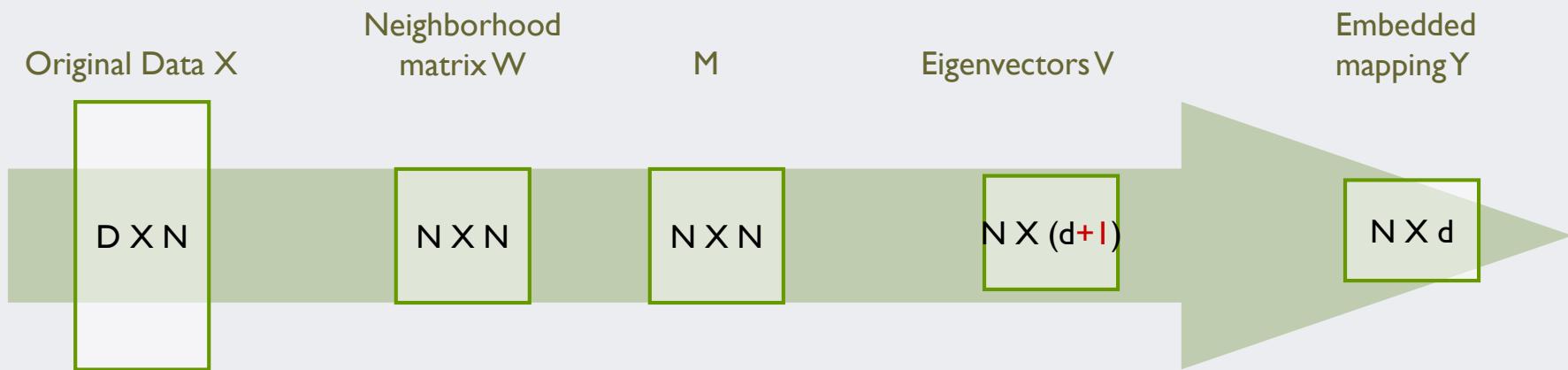
# Locally Linear Embedding (LLE)

- LLE Procedure (cont')

Compute embedding coordinates  $Y$  using weights  $W$

- create sparse matrix  $M = (I-W)^\dagger(I-W)$
- find bottom  $d+1$  eigenvectors of  $M$  (corresponding to the  $d+1$  smallest eigenvalues)
- set the  $q$ th ROW of  $Y$  to be the  $q+1$  smallest eigenvector (discard the bottom eigenvector  $[1,1,1,1\dots]$  with eigenvalue zero) ...*(from LLE Homepage)*

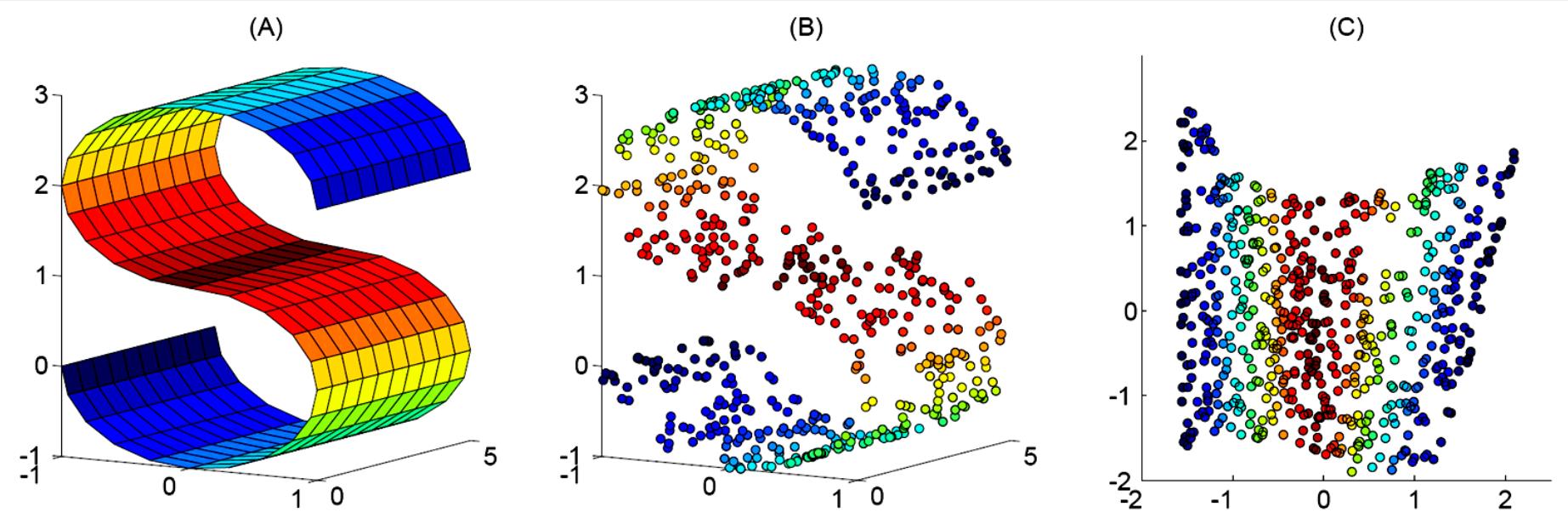
Matrix transition during LLE process



# Locally Linear Embedding (LLE)

- LLE Example I

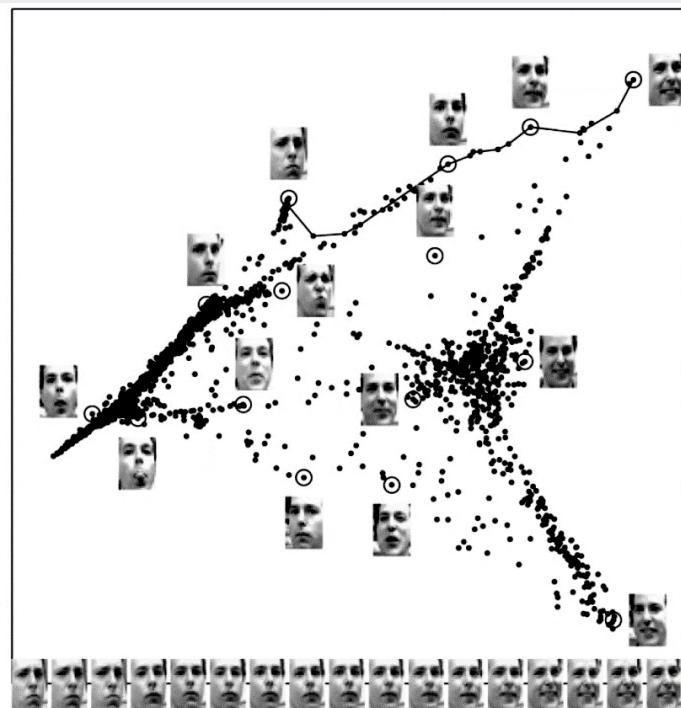
- ✓  $N=600, k=12$



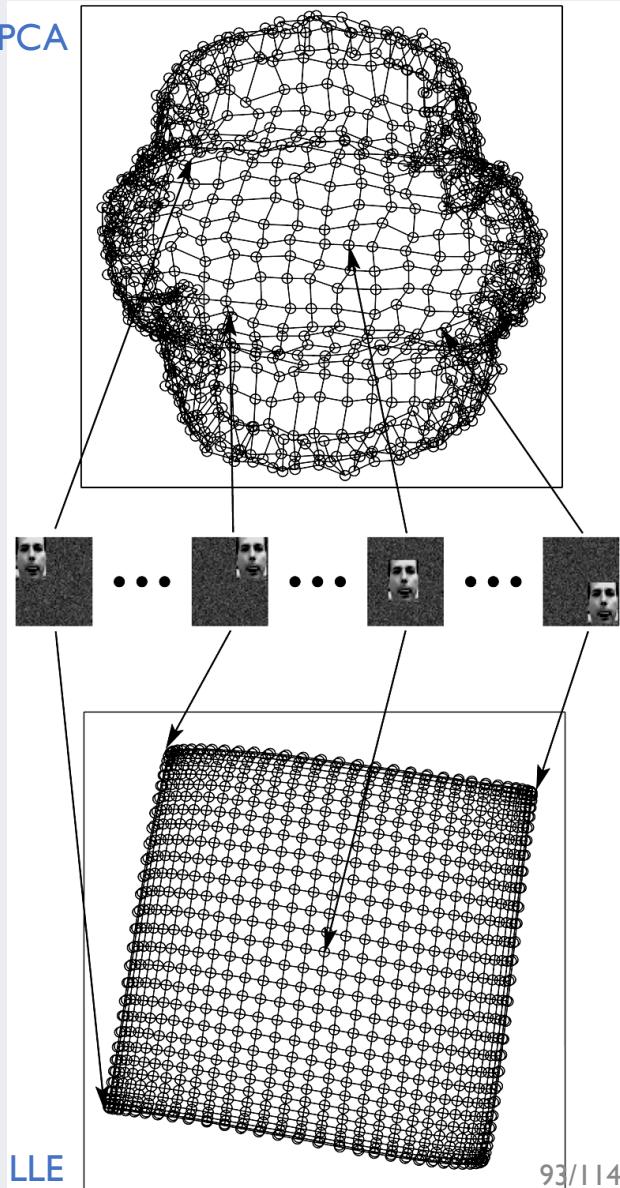
# Locally Linear Embedding (LLE)

- LLE Example 2

- ✓  $N = 961$  grayscale images
- ✓ Each image containing  $28 \times 20$  face superimposed on  $59 \times 51$  background of noise
- ✓  $D = 3009$   $K = 4$



PCA



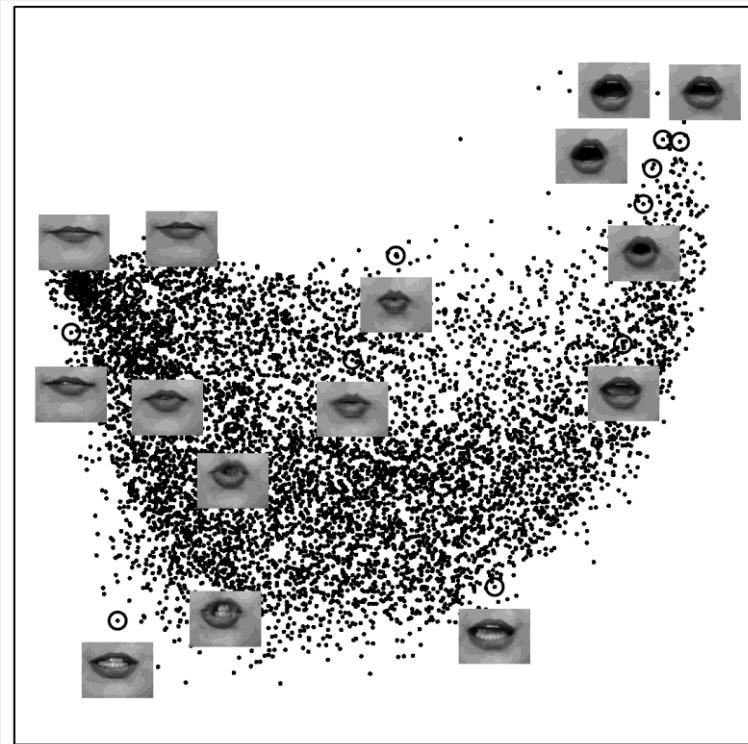
LLE

# Locally Linear Embedding (LLE)

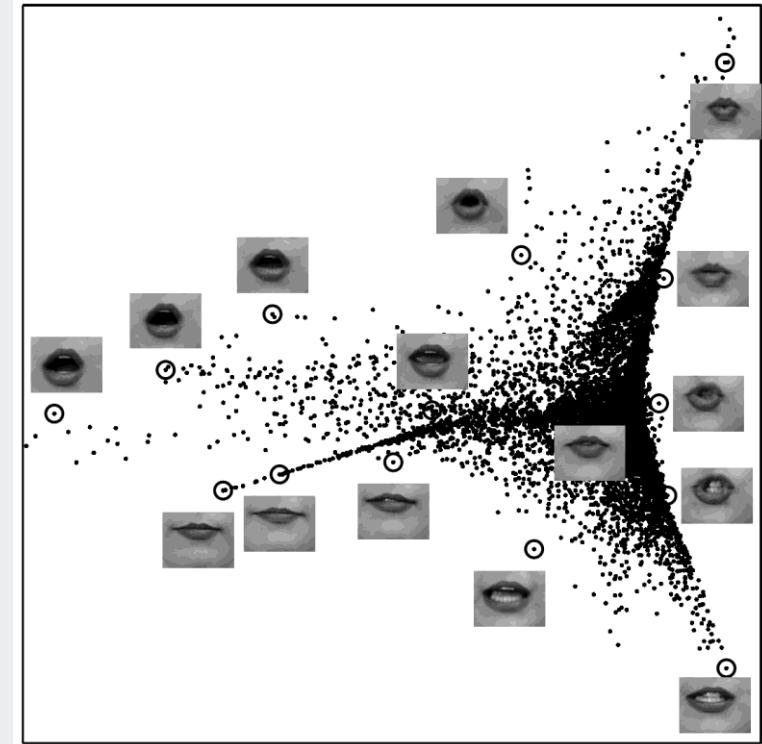
- LLE Example 3

- ✓ Images of lips used in the animation talking head
- ✓  $N = 8588$  RGB images of lips at  $108 \times 84$  resolution
- ✓  $D = 27,216, K = 16$

PCA



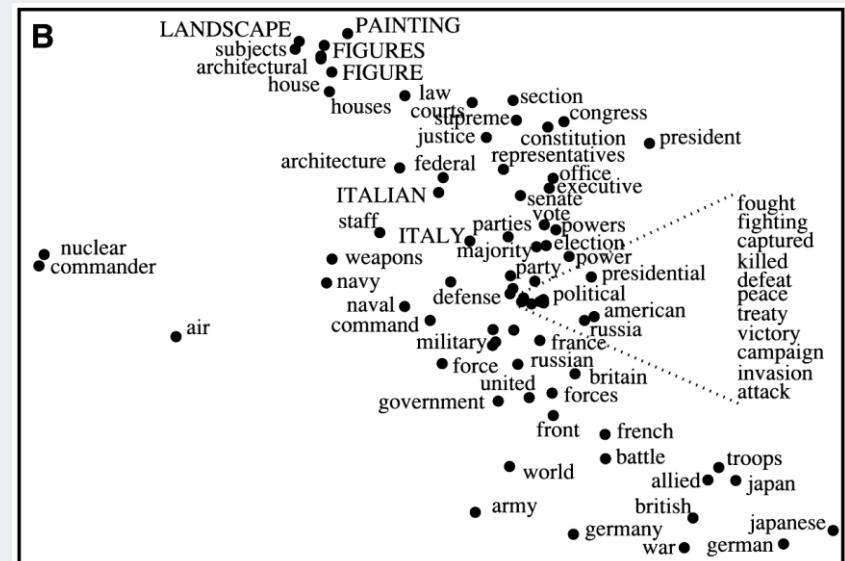
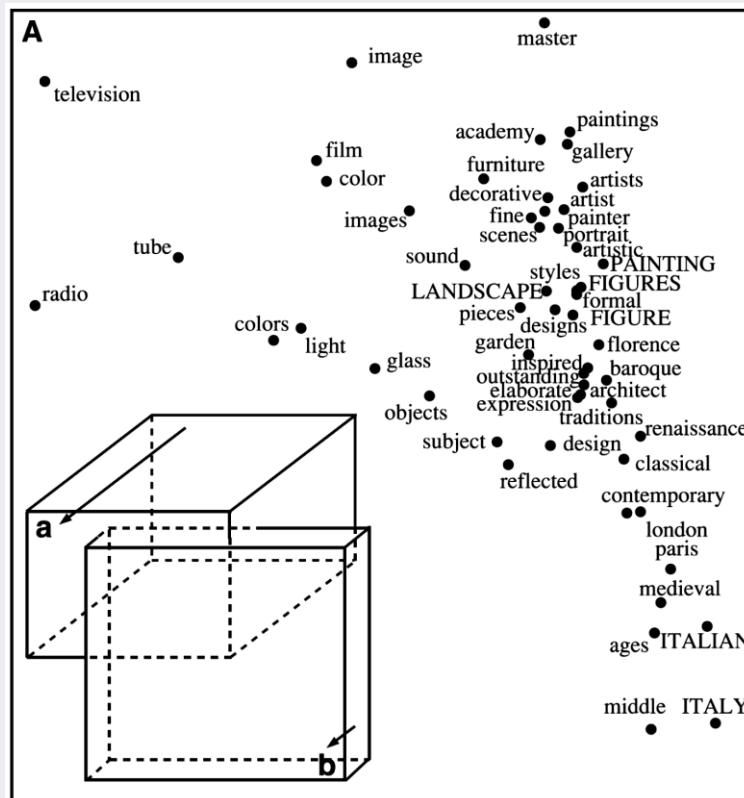
LLE



# Locally Linear Embedding (LLE)

- LLE Example 3

- ✓ Arranging words in a continuous semantic space
  - ✓ Each word was initially represented by a high-dimensional vector that counted the number of times it appeared in different encyclopedia articles



# Stochastic Neighbor Embedding

Hinton and Roweis (2002)

- Stochastic Neighbor Embedding (SNE)

- ✓ It is more important to get local distances right than non-local ones
- ✓ SNE has a probabilistic way of deciding if a pairwise distance is **local**
- ✓ Convert each high-dimensional similarity into the probability that one data point will pick the other data point as its neighbor
  - Probability of picking  $j$  given in **high D**
  - Probability of picking  $j$  given in **low D**

$$p_{j|i} = \frac{e^{-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{2\sigma_i^2}}}{\sum_k e^{-\frac{\|\mathbf{x}_i - \mathbf{x}_k\|^2}{2\sigma_i^2}}}$$
$$q_{j|i} = \frac{e^{-\|\mathbf{y}_i - \mathbf{y}_j\|^2}}{\sum_k e^{-\|\mathbf{y}_i - \mathbf{y}_k\|^2}}$$

# Stochastic Neighbor Embedding

- Picking the Radius of the Gaussian in p
  - ✓ We need to use different radii in different parts of the space so that we keep the effective number of neighbors about constant
  - ✓ A big radius leads to a high entropy for the distribution over neighbors of i, whereas a small radius leads to a low entropy
  - ✓ Decide what entropy you want and then find the radius that produces that entropy

$$Perplexity(P_i) = 2^{H(P_i)}$$

$$H(P_i) = \sum_j p_{j|i} \log_2 p_{j|i}$$

$$p_{j|i} = \frac{e^{-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{2\sigma_i^2}}}{\sum_k e^{-\frac{\|\mathbf{x}_i - \mathbf{x}_k\|^2}{2\sigma_i^2}}}$$

- ✓ The performance of SNE is fairly robust to changes in the perplexity (5~50)

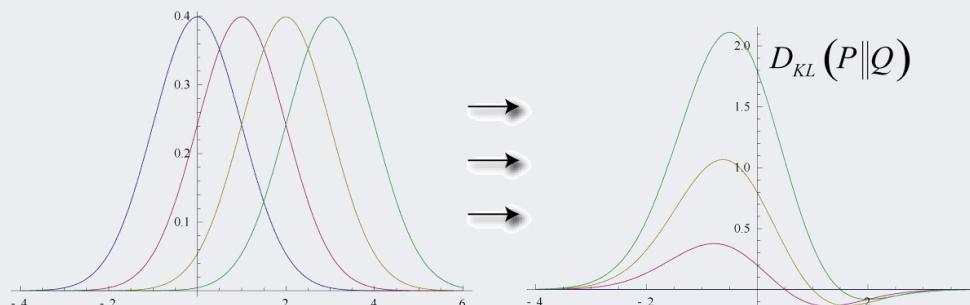
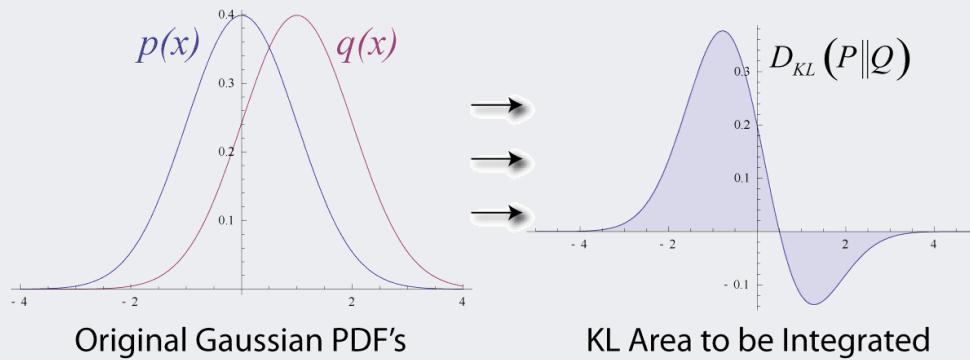
# Stochastic Neighbor Embedding

- Cost Function for a Low-dimensional Representation

- ✓ Kullback-Leibler divergence

- A non-symmetric measure of the difference between two probability distribution P and Q

$$Cost = \sum_i KL(P_i || Q_i) = \sum_i \sum_j p_{j|i} \log \frac{p_{j|i}}{q_{j|i}}$$



# Stochastic Neighbor Embedding

- Cost Function for a Low-dimensional Representation

- ✓ Kullback-Leibler divergence

- A non-symmetric measure of the difference between two probability distribution P and Q

$$Cost = \sum_i KL(P_i || Q_i) = \sum_i \sum_j p_{j|i} \log \frac{p_{j|i}}{q_{j|i}}$$

- ✓ Gradient

$$\frac{\partial C}{\partial \mathbf{y}_i} = 2 \sum_j (\mathbf{y}_j - \mathbf{y}_i)(p_{j|i} - q_{j|i} + p_{i|j} - q_{i|j})$$

- ✓ Update the coordinate in the lower dimension to minimize the Cost function

# Symmetric SNE

- Turning conditional probabilities into pairwise probabilities

$$p_{ij} = \frac{e^{-\frac{||\mathbf{x}_i - \mathbf{x}_j||^2}{2\sigma_i^2}}}{\sum_{k \neq l} e^{-\frac{||\mathbf{x}_k - \mathbf{x}_l||^2}{2\sigma_i^2}}} \rightarrow p_{ij} = \frac{p_{j|i} + p_{i|j}}{2n} \quad \sum_j p_{ij} > \frac{1}{2n}$$

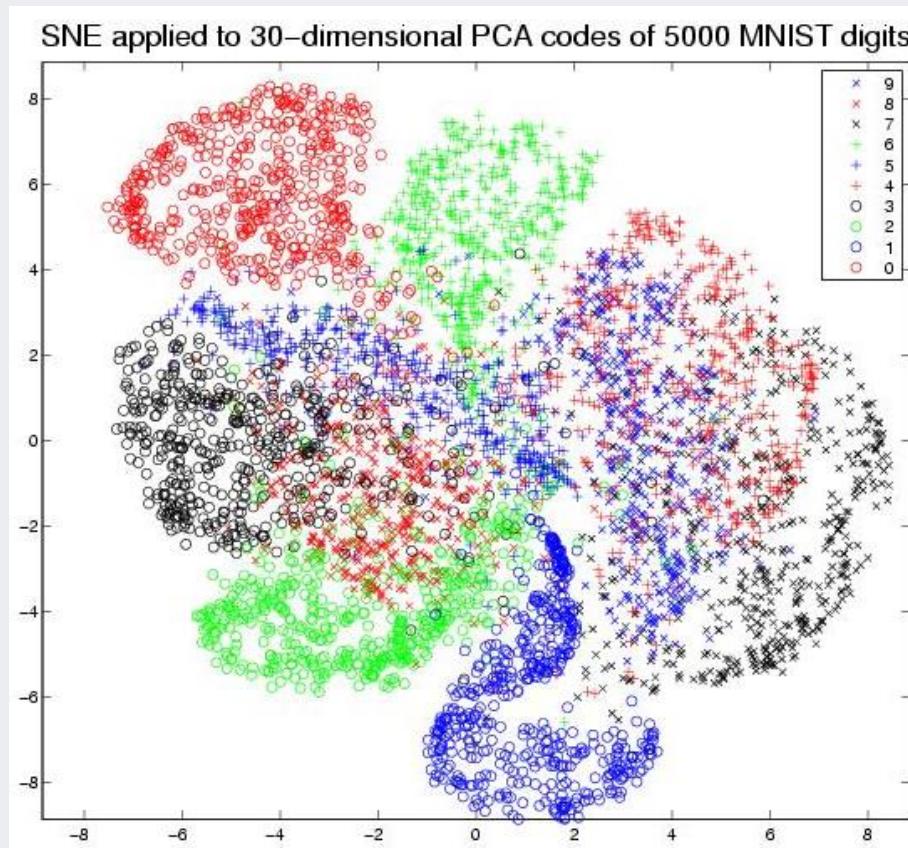
✓ Cost function and gradient

$$Cost = \sum_i KL(P_i || Q_i) = \sum_i \sum_j p_{ij} \log \frac{p_{ij}}{q_{ij}}$$

$$\frac{\partial C}{\partial \mathbf{y}_i} = 4 \sum_j (\mathbf{y}_j - \mathbf{y}_i)(p_{ij} - q_{ij})$$

# Symmetric SNE

- Crowding problem
  - ✓ The area accommodating moderately distant data points is not large enough compared with the area accommodating nearby data points



# t-SNE

Maaten and Hinton (2008)

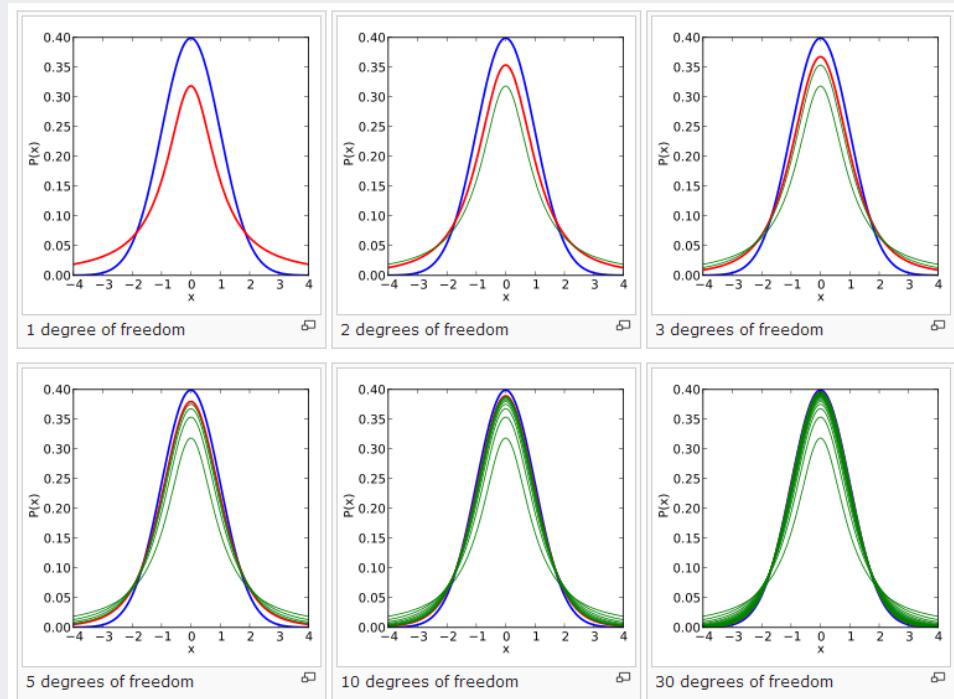
- Resolution to the Crowding Problem

- ✓ Use a probability distribution that has much heavier tails than a Gaussian to convert distances into probabilities in the low-dimensional map
- ✓ Student's t-distribution with one degree of freedom

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

$$\Gamma(n) = (n - 1)!$$

$$q_{j|i} = \frac{(1 + ||\mathbf{y}_i - \mathbf{y}_j||^2)^{-1}}{\sum_{k \neq l} (1 + ||\mathbf{y}_k - \mathbf{y}_l||^2)^{-1}}$$



# t-SNE

- Optimization of t-SNE

$$p_{j|i} = \frac{e^{-\frac{||\mathbf{x}_i - \mathbf{x}_j||^2}{2\sigma_i^2}}}{\sum_k e^{-\frac{||\mathbf{x}_i - \mathbf{x}_k||^2}{2\sigma_i^2}}} \quad q_{j|i} = \frac{(1 + ||\mathbf{y}_i - \mathbf{y}_j||^2)^{-1}}{\sum_{k \neq l} (1 + ||\mathbf{y}_k - \mathbf{y}_l||^2)^{-1}}$$

✓ Gradient:

$$\frac{\partial C}{\partial \mathbf{y}_i} = 4 \sum_j (\mathbf{y}_j - \mathbf{y}_i)(p_{ij} - q_{ij})(1 + ||\mathbf{y}_i - \mathbf{y}_j||^2)^{-1}$$

# t-SNE

- t-SNE algorithm

---

**Algorithm 1:** Simple version of t-Distributed Stochastic Neighbor Embedding.

---

**Data:** data set  $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ ,

cost function parameters: perplexity  $Perp$ ,

optimization parameters: number of iterations  $T$ , learning rate  $\eta$ , momentum  $\alpha(t)$ .

**Result:** low-dimensional data representation  $\mathcal{Y}^{(T)} = \{y_1, y_2, \dots, y_n\}$ .

**begin**

compute pairwise affinities  $p_{j|i}$  with perplexity  $Perp$  (using Equation 1)

set  $p_{ij} = \frac{p_{j|i} + p_{i|j}}{2n}$

sample initial solution  $\mathcal{Y}^{(0)} = \{y_1, y_2, \dots, y_n\}$  from  $\mathcal{N}(0, 10^{-4}I)$

**for**  $t=1$  **to**  $T$  **do**

compute low-dimensional affinities  $q_{ij}$  (using Equation 4)

compute gradient  $\frac{\delta C}{\delta \mathcal{Y}}$  (using Equation 5)

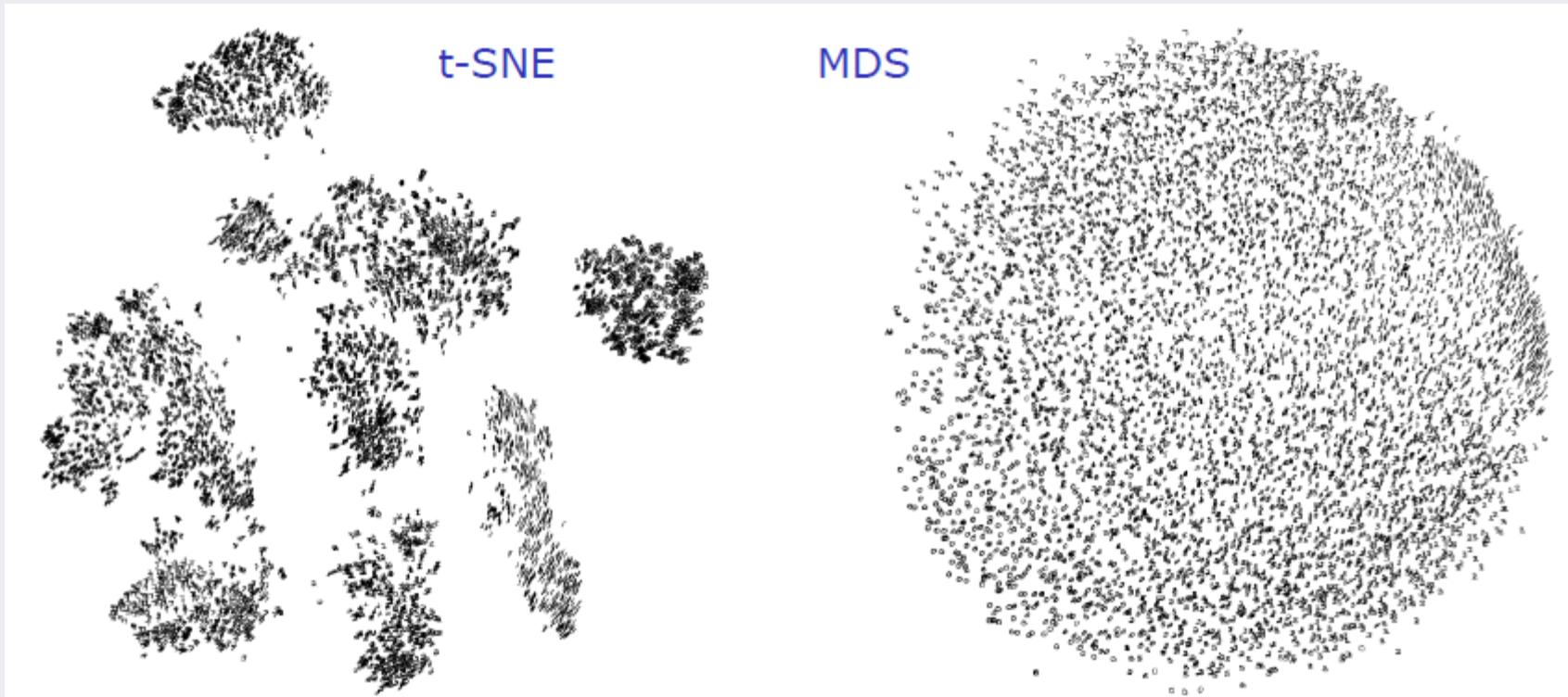
set  $\mathcal{Y}^{(t)} = \mathcal{Y}^{(t-1)} + \eta \frac{\delta C}{\delta \mathcal{Y}} + \alpha(t) (\mathcal{Y}^{(t-1)} - \mathcal{Y}^{(t-2)})$

**end**

**end**

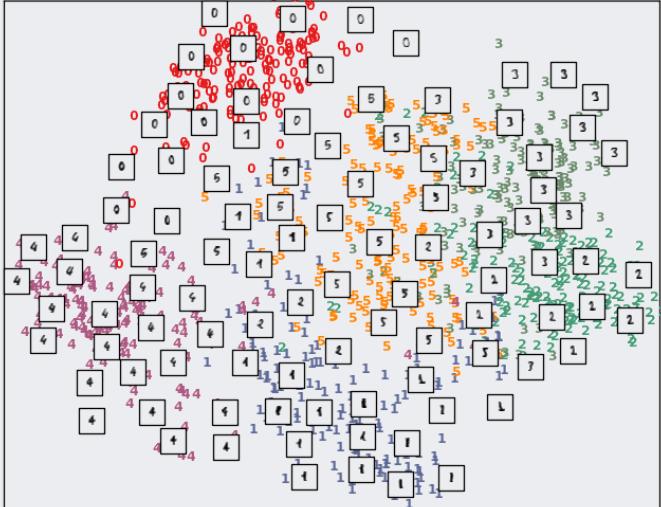
# t-SNE vs. MDS

- MNIST dataset

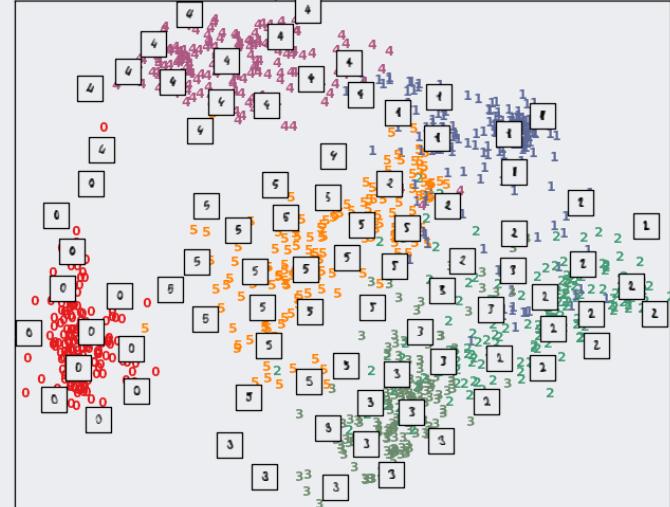


# t-SNE Examples

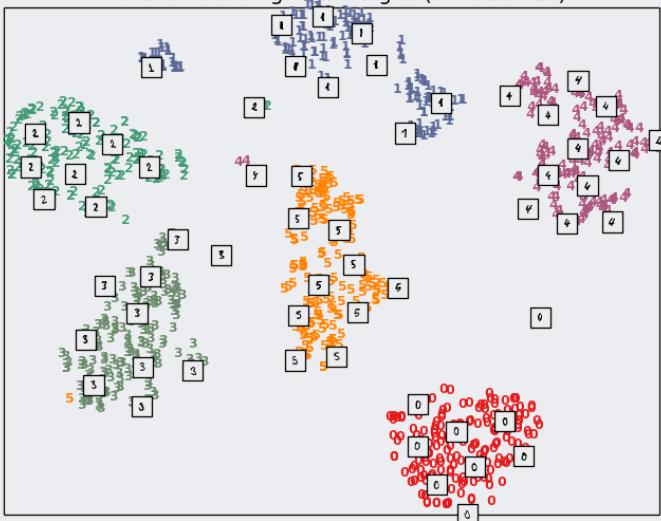
Principal Components projection of the digits (time 0.01s)



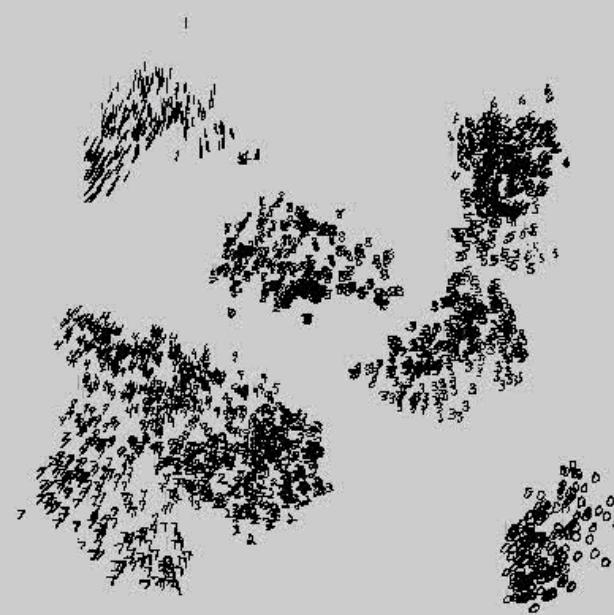
Isomap projection of the digits (time 1.51s)



t-SNE embedding of the digits (time 15.61s)

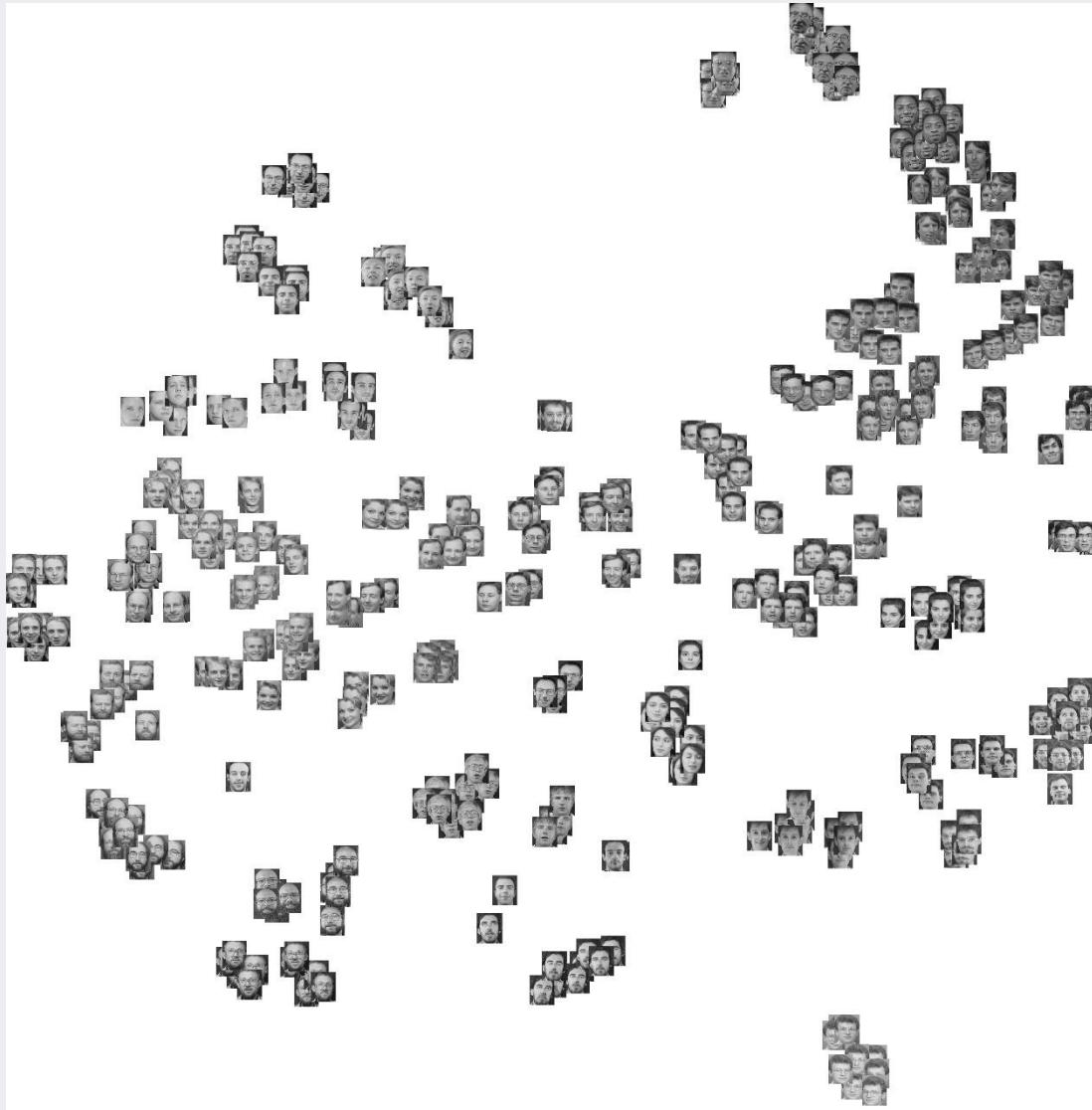


# t-SNE Examples



# t-SNE Examples

- Olivetti faces datasets



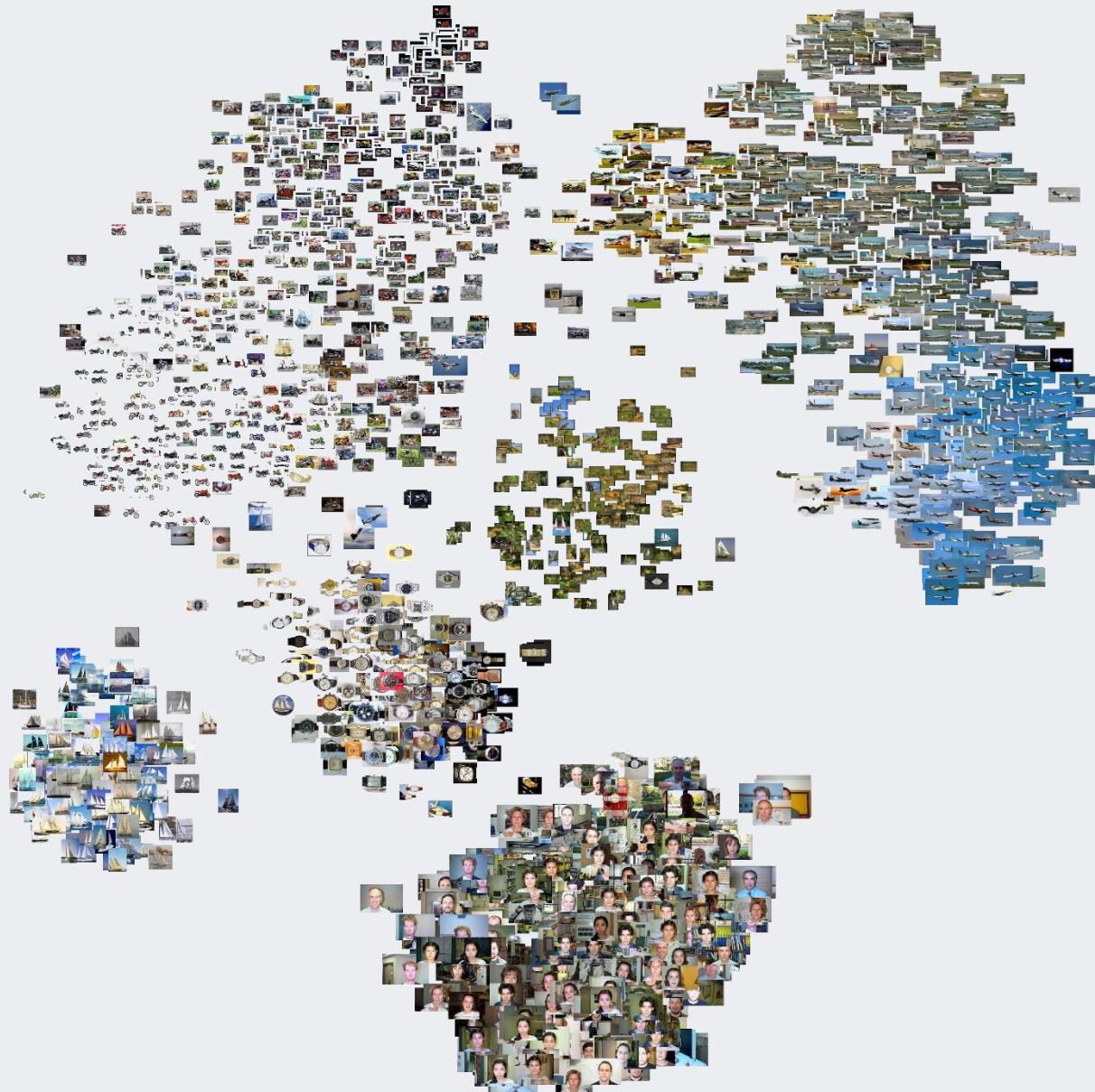
# t-SNE Examples

- Netflix dataset

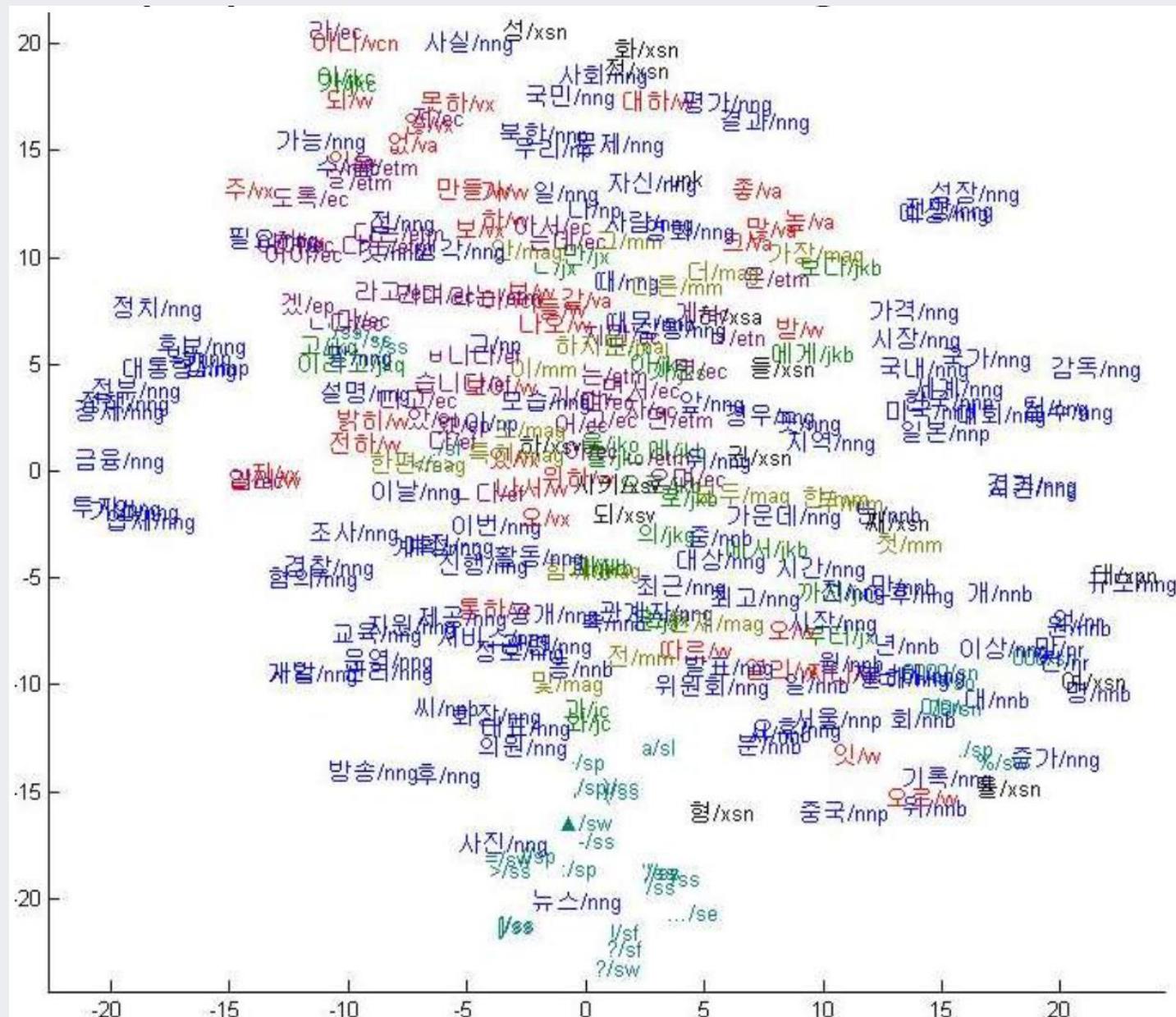


# t-SNE Examples

- CalTech-101



# t-SNE Examples





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# References

## Other materials

- Figure in the title page: <https://wattsupwiththat.com/2015/12/12/is-climate-forecasting-immune-from-occams-razor/>