Function Types and Natural Transformations

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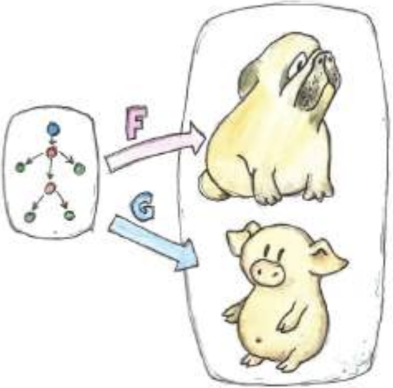
Recap: Functors

Functors

Structure-preserving mappings between categories.



Natural transformations

Functors

Structure-preserving mappings between categories.

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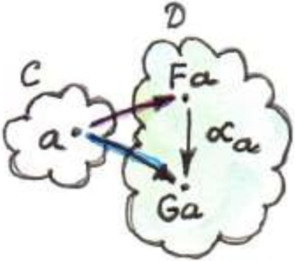
Structure-preserving mappings between categories.

Natural transformations

Mappings between functors that are *natural*/behave *naturally*.

Mapping objects: Components



F, G - functors between $\mathbf{C}$ and $\mathbf{D}$
Object a from $\mathbf{C}$ is mapped to Fa and Ga respectively

Let's say that α is the natural transformation then α_a is a selected morphism in $\mathbf{D}$ and is called the *component* of α at a.

Mapping morphisms

Morphism f between a and b in $\mathbf{C}$ is mapped to two morphisms in $\mathbf{D}$:
$Ff :: Fa \to Fb$
$Gf :: Ga \to Gb$

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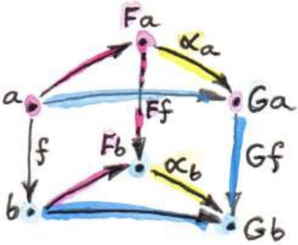
What are the restrictions on the choice of α_a and α_b ?

Naturality condition



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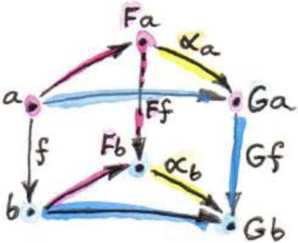

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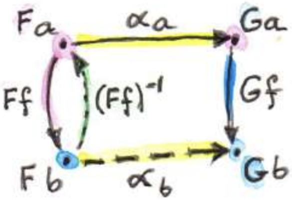
Naturality condition

$Gf \circ \alpha_a = \alpha_b \circ Ff$

Naturality condition - invertible morphism

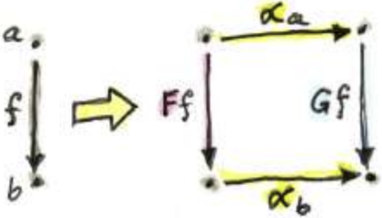


Naturality condition

$Gf \circ \alpha_a = \alpha_b \circ Ff$

$\alpha_b = Gf \circ \alpha_a \circ (Ff)^{-1}$

Commuting squares



Polymorphic functions

```haskell
alpha_a :: F a -> G a

alpha :: forall a . F a -> G a

alpha :: F a -> G a
```

```cpp
template<class A>
G<A> alpha(F<A>);
```

Polymorphic functions and naturality condition

Naturality condition

$Gf \circ \alpha_a = \alpha_b \circ Ff$

```
fmap f . alpha = alpha . fmap f
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Example natural transformation - safeHead

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safeHead :: [a] -> Maybe a
safeHead [] = Nothing
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```
maybeToList :: Maybe a -> [a]
```

Example natural transformation - length

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```
length (_:xs) = fmap (+1) (length xs)
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The Reader type

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instance Functor (Reader e) where
    fmap f (Reader g) = Reader (\x -> f (g x))
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Functor Category

There is one category of functors for each pair of categories, $\mathbf{C}$ and $\mathbf{D}$. Objects are functors from $\mathbf{C}$ to $\mathbf{D}$, and morphism are natural transformations

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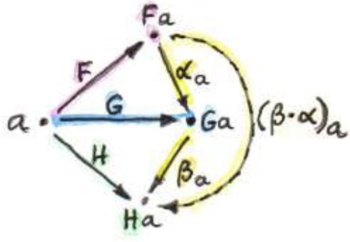
Composition of natural transformations

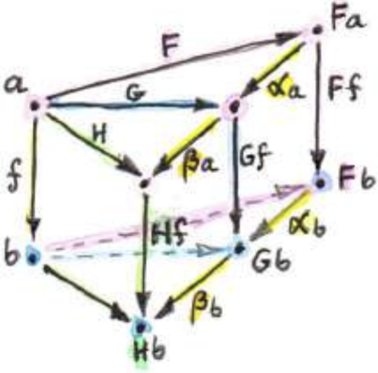
Let α be a natural transformation from functor F to functor G.
Let β be a natural transformation from functor G to functor H.
Components of α and β at a:
$\alpha_a :: Fa \to Ga$
$\beta_a :: Ga \to Ha$

Functor Category



$$(\beta \cdot \alpha)_a = \alpha_a \cdot \beta_a$$

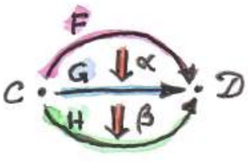
Functor Category



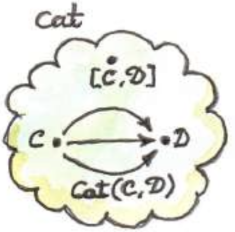
$$Hf \circ (\beta \cdot \alpha)_a = (\beta \cdot \alpha)_a \circ Ff$$

$$\mathbf{id}_{Fa} :: fa \to Fa$$

Vertical composition



2-Categories



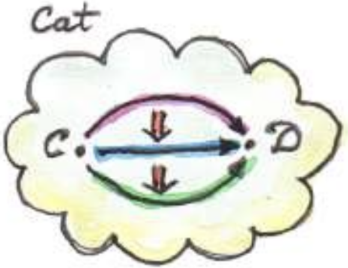
2-category - a category with 2-morphisms

2-morphisms are morphisms between morphisms

Cat as a 2-category

- Objects: (Small) categories
- 1-morphisms: Functors between categories
- 2-morphisms: Natural transformations between functors

2-Categories



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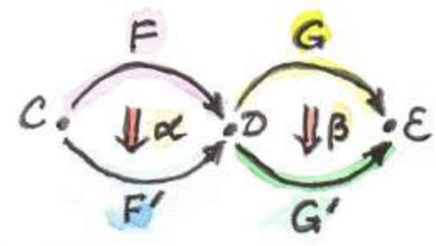

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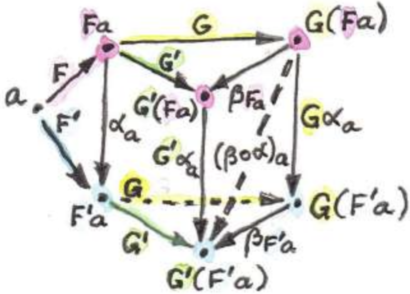

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