Hamilton Cycles and Eigenvalues of Graphs

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NORTH-HOLLAND

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Dedicated to J. J. Seidel

Submitted by Aart Blokhuis

ABSTRACT

We prove some results concerning necessary conditions for a graph to be Hamiltonian in terms of eigenvalues of certain matrices associated with the graph. As an example, we show how the results give an easy algebraic proof of the nonexistence of a Hamilton cycle in two graphs, one of them being the Petersen graph.

1. INTRODUCTION

There exist many results that show a relationship between eigenvalues of a graph and structural properties of the graph. By eigenvalues of a graph, we mean the eigenvalues of a certain matrix derived from the graph, where we must specify how the matrix is derived from the graph in order for this information to make sense. In this note, we prove some results that connect the existence of a Hamilton cycle in the graph and bounds on the eigenvalues of the graph. A first theorem in this direction was given in Mohar [9], but the condition in [9] only holds for regular graphs and also involves some rather complicated considerations. In contrast, our results hold for general, nonregular graphs. Also, the proofs are very easy and almost immediately follow from well-known properties of the spectra of graphs, but the conditions are

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Outline

Adjacency matrix

For a simple undirected graph $G = (V, E)$ with vertices indexed $1, \ldots, n$, the adjacency matrix $A_G \in \{0, 1\}^{n \times n}$ has entries

$$(A_G)_{i,j} = \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

In particular for a simple undirected graph G, A_G is symmetric and $(A_G)_{ii} = 0$ for all i.

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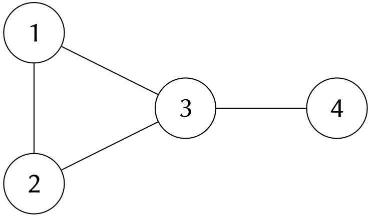

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Adjacency matrix A_G for vertices ordered $(1, 2, 3, 4)$.

Degree matrix

For a graph $G = (V, E)$ with vertices $1, \ldots, n$, the degree matrix $D_G \in \mathbb{Z}^{n \times n}$ is diagonal with

$$(D_G)_{i,j} = \begin{cases} \deg(i) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

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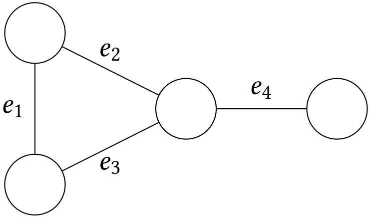

$$D_G = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Eigenvectors

For a matrix $M_G \in \mathbb{R}^{n \times n}$, a scalar $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ form an eigenpair if

$$M_G x = \lambda x.$$

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$$M_G x = \lambda x.$$

$$M_G = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$M_G x = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 2\,x,$$

so x is an eigenvector of M_G with eigenvalue $\lambda = 2$.
The remaining eigenvalues are $2 \pm \sqrt{2}$.

Graph spectra

Consider the complete graph K_4 on four vertices.

$$\operatorname{spec}(A_{K_4}) = \{3, -1, -1, -1\},$$

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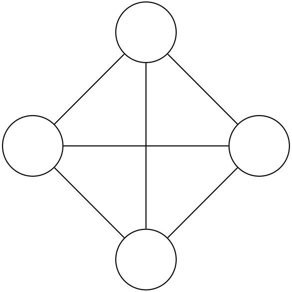

$$A_{K_4} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

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Laplacian matrix

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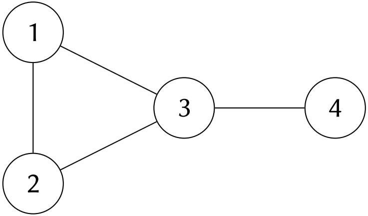

$$L_G = D_G - A_G = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Signless Laplacian

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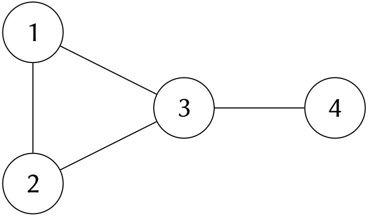

$$Q_G = D_G + A_G = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Main theorem

If M is an $n \times n$ symmetric matrix, we denote its eigenvalues by $\lambda_i(M)$, $i \in \{1, \ldots, n\}$, ordered as

$$\lambda_1(M) \leq \lambda_2(M) \leq \ \ldots \ \leq \lambda_n(M)$$

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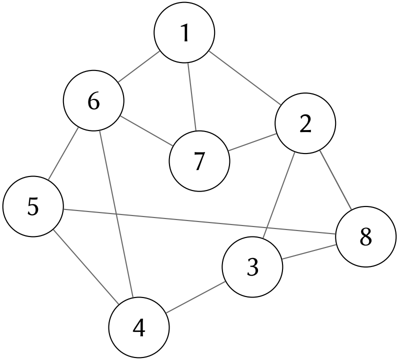
Theorem 1.

Let G be a graph on n vertices and C_n be an n-vertex cycle. If G contains a Hamiltonian cycle, then for $i \in \{1, \ldots, n\}$,

$$\lambda_1(L_{C_n}) \leq \lambda_2(L_G) \text{ and } \lambda_1(Q_{C_n}) \leq \lambda_2(Q_G)$$

Hamiltonian cycle



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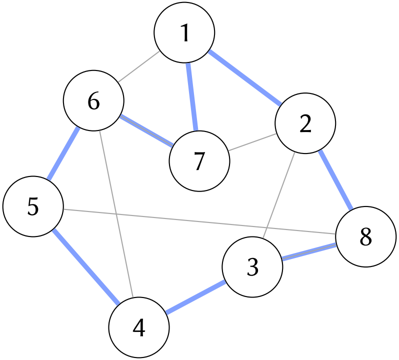

Lemma 2.

Let G be a graph on n vertices and let H be a subgraph of G obtained by deleting an edge in G. Then

$$\begin{aligned} 0 &\leq \lambda_1(L_H) \leq \lambda_1(L_G) \leq \lambda_2(L_H) \leq \lambda_2(L_G) \leq \cdots \\ &\leq \lambda_{n-1}(L_G) \leq \lambda_n(L_H) \leq \lambda_n(L_G) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \lambda_1(Q_H) \leq \lambda_1(Q_G) \leq \lambda_2(Q_H) \leq \lambda_2(Q_G) \leq \cdots \\ &\leq \lambda_{n-1}(Q_G) \leq \lambda_n(Q_H) \leq \lambda_n(Q_G) \end{aligned}$$

Incidence matrix

The incidence matrix of G, denoted P_G, is the $|V(G)| \times |E(G)|$ matrix with entries

$$(P_G)_{u,e} = \begin{cases} 1 & \text{if vertex } u \text{ is incident to edge } e, \\ 0 & \text{otherwise.} \end{cases}$$

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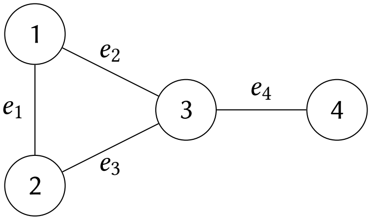

$$P_G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence matrix P_G for vertices $(1, 2, 3, 4)$ and edges (e_1, e_2, e_3, e_4).

Oriented incidence matrix

For a chosen orientation of a graph G, the oriented incidence matrix $K_G \in \{-1, 0, 1\}^{n \times m}$ has entries

$$(K_G)_{v,e} = \begin{cases} 1 & \text{if } v \text{ is the terminal vertex of } e, \\ -1 & \text{if } v \text{ is the initial vertex of } e, \\ 0 & \text{otherwise.} \end{cases}$$

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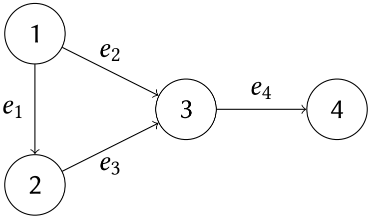

$$K_G = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Proof outline

$$L_G = K_G K_G^T, \qquad Q_G = P_G P_G^T,$$

K_G depends on the chosen orientation, while L_G does not.

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Apart from multiplicities of the eigenvalue 0 their nonzero spectra coincide.
Hence the positive eigenvalues of L_G agree with the positive eigenvalues of $K_G^T K_G$, and similarly for Q_G and $P_G^T P_G$.

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The spectra of L_G and Q_G match the spectra of $K_G^T K_G$ and $P_G^T P_G$, respectively, except for multiplicities of the eigenvalue 0.

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Let H be a subgraph of G obtained by deleting an edge from G. Then the matrix $K_H^T K_H$ is obtained from $K_G^T K_G$ by deleting the row and column corresponding to the deleted edge, and $P_H^T P_H$ is obtained from $P_G^T P_G$ in a similar way.

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Use the interlacing properties of principal submatrices of symmetric matrices to prove the lemma.

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Lemma 2.

Let G be a graph on n vertices and let H be a subgraph of G obtained by deleting an edge in G. Then

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and

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Let G be a graph on n vertices and m edges. Suppose G contains a Hamilton cycle. Then for $i \in \{1, \ldots, n\}$,

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In addition, if $m < 2n$, then for $i \in \{m - n + 1, \ldots, n\}$,

$$\lambda_{i-m+n}(L_G) \le \lambda_i(L_{C_n}) \le \lambda_i(Q_G)$$

and

$$\lambda_{i-m+n}(Q_G) > \lambda_i(Q_{C_n}) \le \lambda_i(L_G).$$

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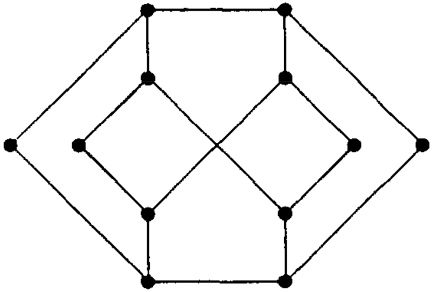
A non-hamiltonian graph

The 1-tough and 2-factorable bipartite graph.



The fact that this graph is not hamiltonian can be proven using the spectral *Theorem 1* we have proven.

Petersen graph