The complexity of 3-colouring H-colourable graphs. Andrei Krokhin and Jakub Oprsal, 2019

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Main result

Theorem 3.1

Let $\mathbf{H}$ be a 3-colourable non-bipartite graph. Then $\mathrm{PCSP}(\mathbf{H}, \mathbf{K}_3)$ is NP-hard.

Dichotomy

If $\mathbf{H}$ is bipartite then the problem can be solved in polynomial time by using an algorithm for 2-colouring.

Reduction

$\mathrm{PCSP}(\mathbf{H}, \mathbf{K}_3)$ admits a reduction from $\mathrm{PCSP}(\mathbf{C}_k, \mathbf{K}_3)$ where k is the size of an odd cycle in $\mathbf{H}$.

Theorems 2.13 and 3.3

Theorem 2.13

Let $\mathbf{H}, \mathbf{G}$ be graphs such that $\mathbf{H} \to \mathbf{G}$. Assume that there exists a minion homomorphism $\xi\colon \operatorname{Pol}(\mathbf{H}, \mathbf{G}) \to \mathscr{M}$ for some minion $\mathscr{M}$ on a pair of sets such that $\mathscr{M}$ has bounded essential arity and does not contain a constant function. Then $\mathrm{PCSP}(\mathbf{H}, \mathbf{G})$ is NP-hard.

Definition

Let N be an odd number, we define a minion $\mathscr{Z}_{\leq N}$ to be the set of all functions $f\colon \mathbb{Z}^n \to \mathbb{Z}$ such that $f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$ for some $c_1, \ldots, c_n \in \mathbb{Z}$ with $\sum_{i=1}^{n} |c_i| \leq N$ and $\sum_{i=1}^{n} c_i$ odd.

Theorem 3.3

Let $k \geq 3$ be odd and let N be the largest odd number such that $N \leq k/3$. Then there is a minion homomorphism from $\operatorname{Pol}(\mathbf{C}_k, \mathbf{K}_3)$ to $\mathscr{Z}_{\leq N}$.

Graph homology

Vertex chains

$\mathbf{G} = (V(G), E(G))$. Let $\Delta_{\mathsf{V}}(\mathbf{G})$ denote the free Abelian group with generators $[v], v \in V(G)$.

Edge chains

For an edge (u, v) let $[u, v]$ denote an orientation of the edge from u to v. Let $\Delta_{\mathsf{E}}(\mathbf{G})$ denote the free Abelian group with generators $[u, v], (u, v) \in E(G)$, where $[u, v] = -[v, u]$ for every edge.

Group homomorphisms

Definition

For any graph homomorphism $f\colon \mathbf{H} \to \mathbf{G}$ we define group homomorphisms $f_{\mathsf{V}}\colon \Delta_{\mathsf{V}}(\mathbf{H}) \to \Delta_{\mathsf{V}}(\mathbf{G})$ and $f_{\mathsf{E}}\colon \Delta_{\mathsf{E}}(\mathbf{H}) \to \Delta_{\mathsf{E}}(\mathbf{G})$ defined by

$$f_{\mathsf{V}}(\textstyle\sum_i c_i[v_i]) = \sum_i c_i[f(v_i)],$$

and

$$f_{\mathsf{E}}(\textstyle\sum_i c_i[u_i, v_i]) = \sum_i c_i[f(u_i), f(v_i)].$$

Map on a group homomorphisms

Definition 2.15

For a graph $\mathbf{G}$, we define a map $\partial\colon \Delta_{\mathsf{E}}(\mathbf{G}) \to \Delta_{\mathsf{V}}(\mathbf{G})$ as the group homomorphism such that $[u, v] \mapsto [v] - [u]$. for every $[u, v] \in \Delta_{\mathsf{E}}(\mathbf{G})$

Lemma 2.16

For each graph homomorphism $f\colon \mathbf{H} \to \mathbf{G}$ and each $P \in \Delta_{\mathsf{E}}(\mathbf{H})$, we have $f_{\mathsf{V}}(\partial P) = \partial f_{\mathsf{E}}(P)$.

Degree of a homomorphism

Lemma 5.1

Let $m, l \geq 3$, and let $f\colon \mathbf{C}_m \to \mathbf{C}_l$ be a homomorphism. Then there is an integer d such that $f_{\mathsf{E}}(O_m) = d \cdot O_l$. Denote this integer d as $\deg f$.

Lemma 5.3

Let $m, l \geq 3$, assume that l is odd, and let $f\colon \mathbf{C}_m \to \mathbf{C}_l$ be a homomorphism. Then

1. $|\deg f| \leq m/l$,
2. the parity of $\deg f$ is the same as the parity of m, and
3. if $m = 4$ then $\deg f = 0$.

Degree of a polymorphism

Lemma 5.6

Let $n \geq 2$, $f\colon \mathbf{C}_k^n \to \mathbf{C}_3$ be a polymorphism, and let $i \in \{1, \ldots, n\}$. Then

1. for each edge e of $\mathbf{C}_k^{n-1}$, there is an integer d such that $f_{\mathsf{E}}(e \times_i O_k) = 2d \cdot O_3$;
2. the above d does not depend on the choice of e;
3. $d = \deg_i f$.

Minor preservation

Definition

Let $\mathscr{Z}$ denote the minion of all linear maps over $\mathbb{Z}$, i.e., of the functions of the form $\sum c_i x_i$ where all $c_i \in \mathbb{Z}$. Define a mapping $\delta \colon \operatorname{Pol}(\mathbf{C}_k, \mathbf{C}_3) \to \mathscr{Z}$ by

$$\delta(f) \colon (x_1, \ldots, x_n) \mapsto \deg_1 f \cdot x_1 + \cdots + \deg_n f \cdot x_n.$$

Lemma 5.7

The map δ is a minion homomorphism.

Minor preservation

Lemma 5.7

Let $n \geq 2$. If $f \in \mathrm{Pol}(\mathbf{C}_k, \mathbf{C}_3)$ is n-ary and g is obtained from f by identifying the first two variables, i.e.,

$$g(y, x_3, \ldots, x_n) = f(y, y, x_3, \ldots, x_n)$$

then $\deg_1 g = \deg_1 f + \deg_2 f$.

Lemma 5.11

Let $f \colon \mathbf{C}_k^n \to \mathbf{C}_3$ be a polymorphism, and $i \in \{1, \ldots, n\}$. If the coordinate i in f is dummy, then $\deg_i f = 0$.

Bounding the essential arity

Lemma 5.12

Let N be the largest odd number such that $N \leq k/3$. Then we have $\delta(f) \in \mathscr{Z}_{\leq N}$ for all $f \in \mathrm{Pol}(\mathbf{C}_k, \mathbf{C}_3)$.