Discrete Optimization

Introduction to Linear Programming

Topics

Definition

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Linear programming: A standard tool that uses a mathematical model for planning of activities to obtain an optimal result.

Linear: All the mathematical functions in the model are required to be linear.

Programming: Synonym for planning, NOT computer programming!

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Uses of Linear Programming

- Among the most important scientific advances of mid-20th century
- Allocating limited resources among competing activities
- Variety of situations:
  - Allocation of production facilities
  - Portfolio selection
  - Selection of shipping patterns
  - Design of radiation therapy
  - Network flow optimization (routing problems)
  - ...

Example: Wyndor Glass Co.

Example

- Wyndor Glass Co. produces glass products: windows, glass doors, etc.
- It has 3 plants:
  - Plant 1: Aluminum frames and hardware
  - Plant 2: Wood frames
  - Plant 3: Glass and assembly
- The plants have limited production capabilities

Example: Wyndor Glass Co.

Example

- Company's product line to be changed.
  - Product 1: An 8-foot glass door with aluminum framing
  - Product 2: A 4×6 foot double-hung wood-framed window
- Products will be produced in batches of 20.

Example: Wyndor Glass Co.

Example

- Product 1 requires Plants 1 and 3
- Product 2 requires Plants 2 and 3
- So, they are competing for Plant 3
- Which mix of the two products is most profitable?

Problem: Determine the production rate of each product, to maximize the total profit.
(Production rate = batches/week)

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Following additional data is required:

- Number of hours of production time available per week in each plant
- Number of hours of production time for each batch of each new product
- Profit per batch of each new product

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Example: Wyndor Glass Co.

	A	B	C	D	E	F	G
1	**Wyndor Glass Co. Product-Mix Problem**						
2							
3			Doors	Windows			
4		Profit Per Batch	$3,000	$5,000			
5							Hours
6			Hours Used Per Batch Produced				Available
7		Plant 1	1	0			4
8		Plant 2	0	2			12
9		Plant 3	3	2			18

Topics

Formulation: Variables

- x_1: Number of batches of product 1 produced per week
- x_2: Number of batches of product 2 produced per week
- Z: total profit per week (in thousands of dollars)

x_1 and x_2 are the decision variables.

The Objective

Maximize: $Z = 3x_1 + 5x_2$

Formulation: Variables

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Maximize: $Z = 3x_1 + 5x_2$

Formulation: The problem

Maximize:

$$Z = 3x_1 + 5x_2$$

Subject to:

$$\begin{aligned} x_1 \qquad\quad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \end{aligned}$$

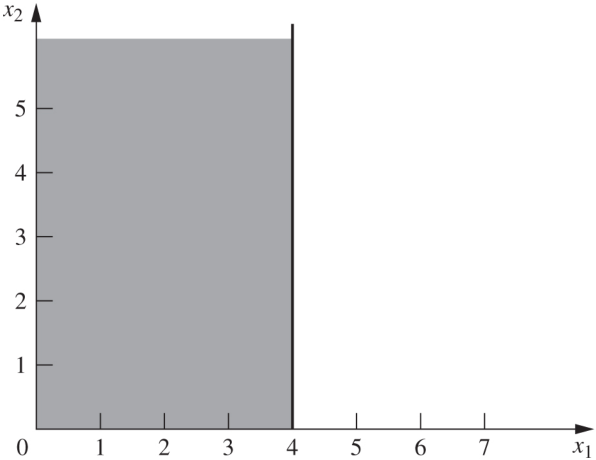
$$x_1 \geq 0 \quad , \quad x_2 \geq 0$$

Topics

Graphical Solution

- 2 decision variables: 2-dimensional space
- Constraints: Lines
- Feasible region: Area enclosed by lines
- Objective function: A line
- Optimal solution: Point that maximizes Z

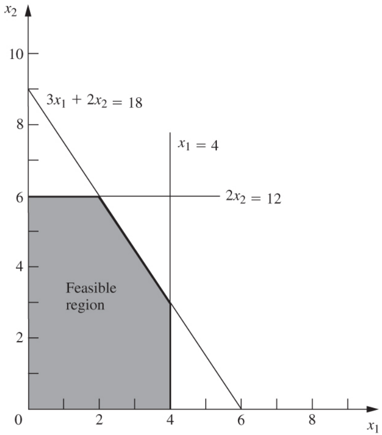
Example: Wyndor Glass Co.



Shaded area shows values of (x_1, x_2) allowed by

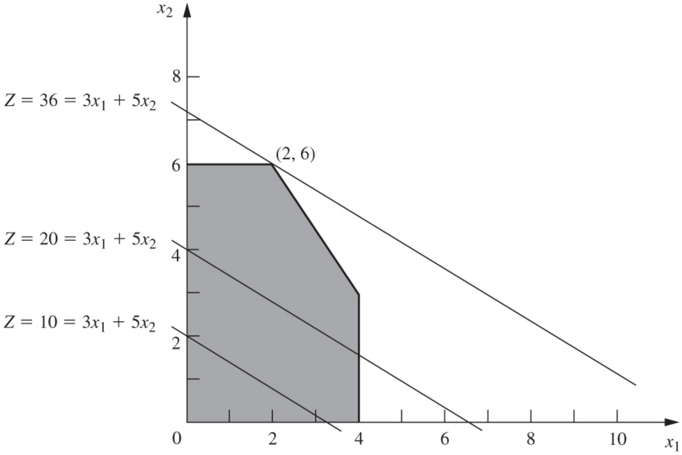
$$x_1 \geq 0,\ x_2 \geq 0,\ x_1 \leq 4$$

Example: Wyndor Glass Co.



Shaded area shows the set of permissible values of (x_1, x_2), called the feasible region.

Example: Wyndor Glass Co.



The value of (x_1, x_2) that maximizes $3x_1 + 5x_2$ is $(2, 6)$.

A Standard Form

Maximize:

$$Z = c_1x_1 + c_2x_2 + ... + c_nx_n$$

Subject to:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + ...a_{mn}x_n &\leq b_m
\end{aligned}$$

$$x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$$

A Standard Form

- Z: Objective value = overall measure of performance
- x_j: decision variable (level of activity j)
- c_i, b_i, a_{ij}: Parameters (input constants)
  - c_i: increase in Z that would result from each unit increase in activity j
  - b_i: amount of resource i
  - a_{ij}: amount of resource i consumed by each unit of activity j

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TABLE 3.3 Data needed for a linear programming model involving the allocation of resources to activities

	Resource Usage per Unit of Activity				
	Activity				
Resource	**1**	**2**	**. . .**	***n***	**Amount of Resource Available**
1	a_{11}	a_{12}	. . .	a_{1n}	b_1
2	a_{21}	a_{22}	. . .	a_{2n}	b_2
.					.
.	. . .	. . .	. . .	. . .	.
.					.
m	a_{m1}	a_{m2}	. . .	a_{mn}	b_m
Contribution to Z per unit of activity	c_1	c_2	. . .	c_n	

Solutions Terminology

- Any specification of values for $(x_1, x_2, ..., x_n)$ is a solution.

Feasible solution

A feasible solution is a solution for which all the constraints are satisfied.

Infeasible solution

An infeasible solution is a solution for which at least one constraint is violated.

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Optimal Solution

An optimal solution is a feasible solution that has the most favorable value of the objective function.

- The most favorable value:
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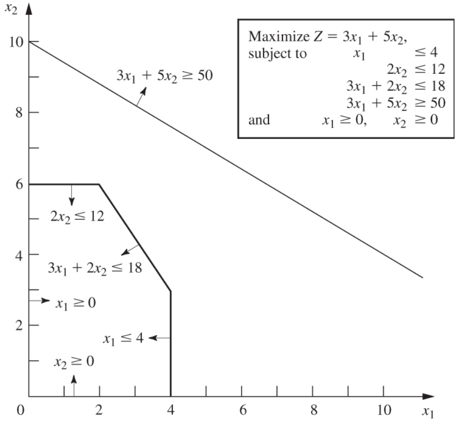
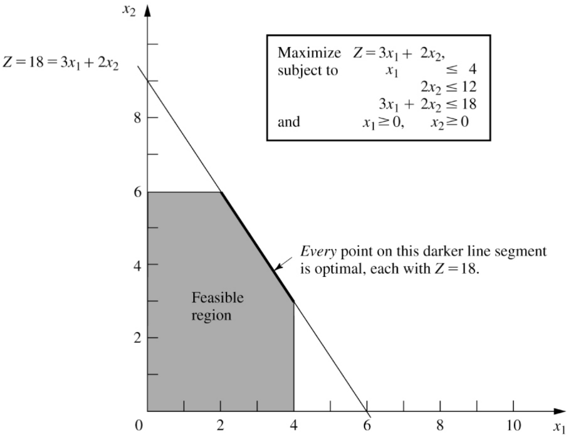
Example: No Feasible Solutions



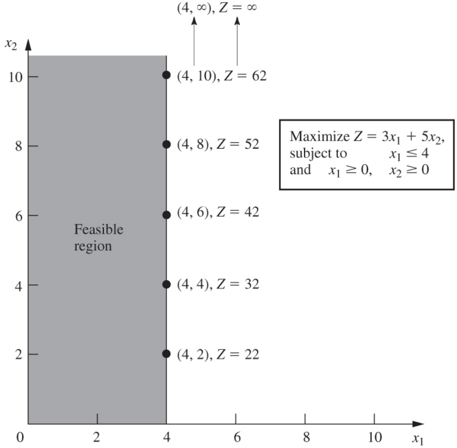
It is possible for a problem to have no feasible solution.
New constraint added: $3x_1 + 5x_2 \geq 50$.

Example: Multiple Optimal Solutions



It is possible to have multiple (= an infinite number of) optimal solutions. The objective function is changed: $Z = 3x_1 + 2x_2$.

Example: Unbounded Z



No optimal solutions: only constraint is $x_1 \leq 4$, we could increase x_2 indefinitely in the feasible region.

Corner-Point Feasible Solutions

Corner-point feasible solution

A corner-point feasible (CPF) solution is a solution that lies at a corner of the feasible region.

- If a problem has exactly one optimal solution, it must be a CPF solution.
- If the problem has multiple optimal solutions, at least two must be CPF solutions.

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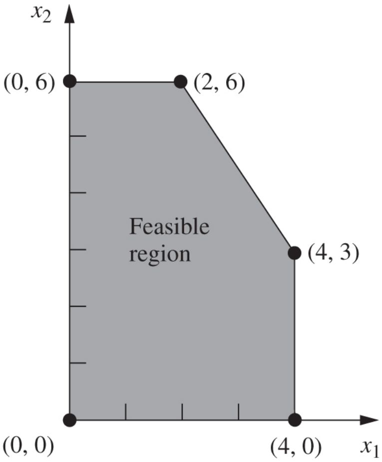

The five dots are the five CPF solutions.

Topics

Reclaiming Solid Wastes

- A company collects four types of solid waste
- The company treats and amalgamates them into salable products.
- Three grades of products: Grade A, Grade B, and Grade C (depending on the mix of the materials used).
- Grants amounting to \$30,000 per week are to be used exclusively to cover the entire treatment cost.
- At least half of the amount available of each material must be collected and treated.
- Objective: Maximize the net weekly profit (total sales income minus total amalgamation cost).

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Example

TABLE 3.16 Product data for Save-It Co.

Grade	Specification	Amalgamation Cost per Pound ($)	Selling Price per Pound ($)
A	Material 1: Not more than 30% of total Material 2: Not less than 40% of total Material 3: Not more than 50% of total Material 4: Exactly 20% of total	3.00	8.50
B	Material 1: Not more than 50% of total Material 2: Not less than 10% of total Material 4: Exactly 10% of total	2.50	7.00
C	Material 1: Not more than 70% of total	2.00	5.50

Example

TABLE 3.17 Solid waste materials data for the Save-It Co.

Material	Pounds per Week Available	Treatment Cost per Pound ($)	Additional Restrictions
1	3,000	3.00	1. For each material, at least half of the pounds per week available should be collected and treated.
2	2,000	6.00	2. $30,000 per week should be used to treat these materials.
3	4,000	4.00	
4	1,000	5.00	

Formulation as a Linear Program

- Defining decision variables: A crucial step
  - x_{ij}: Number of pounds of material j allocated to product grade i per week
  - $\frac{x_{ij}}{x_{i1}+x_{i2}+x_{i3}+x_{i4}}$: Proportion of material j in product grade i

Mixture Specifications

$$\frac{x_{A1}}{x_{A1}+x_{A2}+x_{A3}+x_{A4}} \leq 0.3$$

is equivalent to:

$$0.7x_{A1} - 0.3x_{A2} - 0.3x_{A3} - 0.3x_{A4} \leq 0$$

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The Complete Model

Maximize: (coeff. = sell. price - amalg. cost in last two columns of Table 3.16)

$$\begin{aligned} Z = 5.5(x_{A1} + x_{A2} + x_{A3} + x_{A4}) &+ 4.5(x_{B1} + x_{B2} + x_{B3} + x_{B4}) \\ &+ 3.5(x_{C1} + x_{C2} + x_{C3} + x_{C4}) \end{aligned}$$

Subject to:

1. Mixture Specifications (second column of Table 3.16):

$$\begin{aligned} x_{A1} &\le 0.3(x_{A1} + x_{A2} + x_{A3} + x_{A4}) \\ x_{A2} &\ge 0.4(x_{A1} + x_{A2} + x_{A3} + x_{A4}) \\ x_{A3} &\le 0.5(x_{A1} + x_{A2} + x_{A3} + x_{A4}) \\ x_{A4} &= 0.2(x_{A1} + x_{A2} + x_{A3} + x_{A4}) \\ x_{B1} &\le 0.5(x_{B1} + x_{B2} + x_{B3} + x_{B4}) \\ x_{B2} &\ge 0.1(x_{B1} + x_{B2} + x_{B3} + x_{B4}) \\ x_{B4} &= 0.1(x_{B1} + x_{B2} + x_{B3} + x_{B4}) \\ x_{C1} &\le 0.7(x_{C1} + x_{C2} + x_{C3} + x_{C4}) \end{aligned}$$

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x_{C1} &\leq 0.7(x_{C1} + x_{C2} + x_{C3} + x_{C4})
\end{aligned}$$

The Complete Model

2. Availability of Materials (second column of Table 3.17):

$$
\begin{aligned}
x_{A1} + x_{B1} + x_{C1} &\leq 3,000 \\
x_{A2} + x_{B2} + x_{C2} &\leq 2,000 \\
x_{A3} + x_{B3} + x_{C3} &\leq 4,000 \\
x_{A4} + x_{B4} + x_{C4} &\leq 1,000
\end{aligned}
$$

3. Restrictions on Amounts Treated (right side of Table 3.17):

$$
\begin{aligned}
x_{A1} + x_{B1} + x_{C1} &\geq 1,500 \\
x_{A2} + x_{B2} + x_{C2} &\geq 1,000 \\
x_{A3} + x_{B3} + x_{C3} &\geq 2,000 \\
x_{A4} + x_{B4} + x_{C4} &\geq 500
\end{aligned}
$$

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The Complete Model

4. Restriction on Treatment Cost (right side of Table 3.17):

$$3(x_{A1} + x_{B1} + x_{C1}) + 6(x_{A2} + x_{B2} + x_{C2}) + 4(x_{A3} + x_{B3} + x_{C3}) \\ +5(x_{A4} + x_{B4} + x_{C4}) = 30,000$$

5. Nonnegativity Constraints:

$$x_{A1} \geq 0,\ x_{A2} \geq 0,\ x_{A3} \geq 0, \quad x_{A4} \geq 0 \\ \ldots, \quad x_{C4} \geq 0$$

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Solution

TABLE 3.18 Optimal solution for the Save-It Co. problem

	Pounds Used per Week				
	Material				
Grade	1	2	3	4	Number of Pounds Produced per Week
A	412.3 (19.2%)	859.6 (40%)	447.4 (20.8%)	429.8 (20%)	2149
B	2587.7 (50%)	517.5 (10%)	1552.6 (30%)	517.5 (10%)	5175
C	0	0	0	0	0
Total	3000	1377	2000	947	

The objective $Z = 35,108.90$
Once the problem is solved, the x_{ij} values are used to calculate the other quantities of interest given in the table.

Topics

Personnel Scheduling

- Union Airways is adding more flights to its schedule, and so it needs to hire additional customer service agents.
- An analysis has been made of the minimum number of customer service agents that need to be on duty at different times of the day.
- An OR team is trying to schedule the agents to provide satisfactory service with the smallest personnel cost.

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Personnel Scheduling

- Each agent should work an 8-hour shift 5 days per week.
- The shifts are:
  - Shift 1: 6:00 AM to 2:00 PM
  - Shift 2: 8:00 AM to 4:00 PM
  - Shift 3: Noon to 8:00 PM
  - Shift 4: 4:00 PM to midnight
  - Shift 5: 10:00 PM to 6:00 AM

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Data for Personnel Scheduling Problem

TABLE 3.19 Data for the Union Airways personnel scheduling problem

Time Period	Time Periods Covered: Shift 1	2	3	4	5	Minimum Number of Agents Needed
6:00 A.M. to 8:00 A.M.	✔					48
8:00 A.M. to 10:00 A.M.	✔	✔				79
10:00 A.M. to noon	✔	✔				65
Noon to 2:00 P.M.	✔	✔	✔			87
2:00 P.M. to 4:00 P.M.		✔	✔			64
4:00 P.M. to 6:00 P.M.			✔	✔		73
6:00 P.M. to 8:00 P.M.			✔	✔		82
8:00 P.M. to 10:00 P.M.				✔		43
10:00 P.M. to midnight				✔	✔	52
Midnight to 6:00 A.M.					✔	15
Daily cost per agent	\$170	\$160	\$175	\$180	\$195	

Formulation as a Linear Program

- Activities correspond to shifts.
- Level of an activity: Number of agents assigned to that shift
- Decision variables are always levels of activities
- The five decision variables are:

$$x_j = \text{number of agents assigned to shift } j, \quad \text{for } j = 1, 2, 3, 4, 5$$

- The main restrictions on values of the decision variables are the minimum requirements given in the last column.
- For example, for 2:00 P.M. to 4:00 P.M., the total number of agents assigned to the shifts that cover this time period (shifts 2 and 3) must be at least 64, so

$$x_2 + x_3 \geq 64$$

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- The main restrictions on values of the decision variables are the minimum requirements given in the last column.
- For example, for 2:00 P.M. to 4:00 P.M., the total number of agents assigned to the shifts that cover this time period (shifts 2 and 3) must be at least 64, so

$$x_2 + x_3 \geq 64$$

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The Complete Model

Minimize: (coefficients given by last row of Table 3.19)

$$Z = 170x_1 + 160x_2 + 175x_3 + 180x_4 + 195x_5$$

Subject to:

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\begin{aligned}
x_1 &\geq 48 \quad && \text{(6-8 AM)} \\
x_1 + x_2 &\geq 79 && \text{(8-10 AM)} \\
x_1 + x_2 &\geq 65 && \text{(10 AM to noon)} \\
x_1 + x_2 + x_3 &\geq 87 && \text{(Noon-2 PM)} \\
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- Nonnegativity constraints should also be added.
- Some constraints are redundant.
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- The objective value is $Z =$ \$30,610.

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Remarks

- The number of agents assigned to each shift needs to be an integer.
- The model should have an additional constraint for each x_i specifying that the variable must have an integer value.
- Adding these constraints would convert the linear programming model to an integer programming model (will be studied later).
- Without these integer constraints, the optimal solution (previous slide) turned out to have integer values anyway.
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Very Large LP Models

- Linear programming model sizes for real-world cases can be very large.
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Topics

References

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- Chapter 3: Introduction to Linear Programming