Discrete Optimization

Solving Linear Programming Problems: The Simplex Method

Topics

The Standard Form

The standard form:
Maximize:

$$Z = c_1x_1 + c_2x_2 + ... + c_nx_n$$

subject to:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &\leq b_m \end{aligned}$$

$$x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$$

Other Forms

Other forms:

- $=$ constraints
- $\geq$ constraints
- Negative right-hand sides

How to make the adjustments required for other forms of the linear programming model?

- All these adjustments can be made during the initialization, then the rest of the simplex method can be applied.
- The only problem introduced by the other forms lies in identifying an initial BF solution.
- The standard approach used for all these cases is the artificial variable technique.

Artificial Variable Technique

- Construct an artificial problem by introducing a dummy variable (called an artificial variable) into each constraint that needs one.
- Place nonnegativity constraints on these variables.
- Modify the objective function to impose a penalty on their having values larger than zero.
- The iterations of the simplex method then automatically force the artificial variables to disappear (become zero), one at a time, until they are all gone, after which the *real* problem is solved.
- To illustrate the artificial variable technique, first we consider the case where the only nonstandard form in the problem is the presence of one or more equality constraints.

Topics

Equality Constraints

- Any equality constraint:

$$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n = b_i$$

actually is equivalent to a pair of inequality constraints:

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n &\leq b_i \\ a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n &\geq b_i \end{aligned}$$

- However, rather than making this substitution and thereby increasing the number of constraints, it is more convenient to use the artificial variable technique.
- We shall illustrate this technique with the following example.

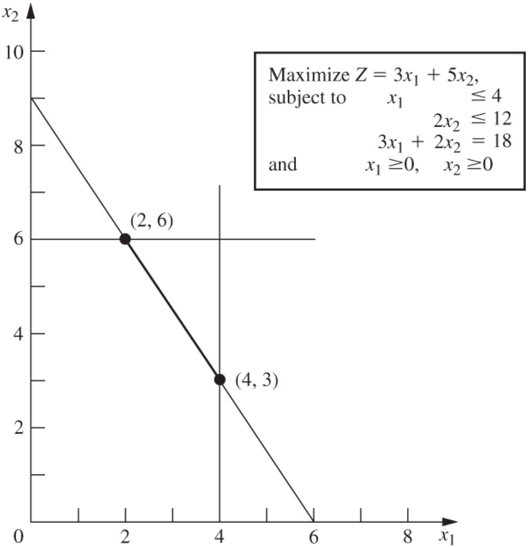
Equality Constraints: Example

- Suppose the Wyndor problem is modified to *require* that Plant 3 be used at full capacity.
- The only resulting change in the linear programming model is that the third constraint, $3x_1 + 2x_2 \leq 18$, instead becomes an equality constraint
$$3x_1 + 2x_2 = 18$$
- The feasible region consists of *just* the line segment connecting $(2, 6)$ and $(4, 3)$.

$$\begin{array}{llr}
(0) & Z - 3x_1 - 5x_2 & = 0 \\
(1) & x_1 \qquad\quad + x_3 & = 4 \\
(2) & 2x_2 \qquad\quad + x_4 & = 12 \\
(3) & 3x_1 + 2x_2 & = 18
\end{array}$$

- These equations do not have an obvious initial BF solution because there is no longer a slack variable to use as the initial basic variable for Eq. (3).

Figure 4.3


x_2
10
8
6
4
2
0
2
4
6
8
x_1
(2, 6)
(4, 3)
Maximize $Z = 3x_1 + 5x_2$,
subject to $x_1 \leq 4$
$2x_2 \leq 12$
$3x_1 + 2x_2 = 18$
and $x_1 \geq 0, \quad x_2 \geq 0$

Obtaining an Initial BF Solution

The procedure is to construct an *artificial problem* that has the same optimal solution as the real problem by making two modifications of the real problem.

1. Introduce a nonnegative artificial variable (call it $\overline{x_5}$) into Eq. (3), just as if it were a slack variable.

   (3) $$3x_1 + 2x_2 + \overline{x_5} = 18$$

2. Assign an *overwhelming penalty* to having $\overline{x_5} > 0$ by changing the objective function:

   $$Z = 3x_1 + 5x_2 \text{ to}$$
   $$Z = 3x_1 + 5x_2 - M\overline{x_5}$$

   where M symbolically represents a *huge* positive number. This method of forcing $\overline{x_5}$ to be 0 in the optimal solution is called the Big M method.

Obtaining an Initial BF Solution

- Now find the optimal solution for the real problem by applying the simplex method to the artificial problem, starting with the following initial BF solution:

Initial BF solution

Nonbasic variables: $x_1 = 0, \quad x_2 = 0$

Basic variables: $x_3 = 4, \quad x_4 = 12, \quad \overline{x_5} = 18.$

- Since $\overline{x_5}$ plays the role of the slack variable for the third constraint in the artificial problem, this constraint is equivalent to $3x_1 + 2x_2 \leq 18$ (just as for the original Wyndor Glass Co. problem).
- We show below the resulting artificial problem (before augmenting) next to the real problem.

Resulting Artificial Problem Next To the Real Problem

The Real Problem

Maximize $\quad Z = 3x_1 + 5x_2,$

subject to

$$\begin{aligned} x_1 \quad\quad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &= 18 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

The Artificial Problem

Define $\quad \bar{x}_5 = 18 - 3x_1 - 2x_2.$

Maximize $\quad Z = 3x_1 + 5x_2 - M\bar{x}_5,$

subject to

$$\begin{aligned} x_1 \quad\quad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ (\text{so} \quad 3x_1 + 2x_2 + \bar{x}_5 &= 18) \end{aligned}$$

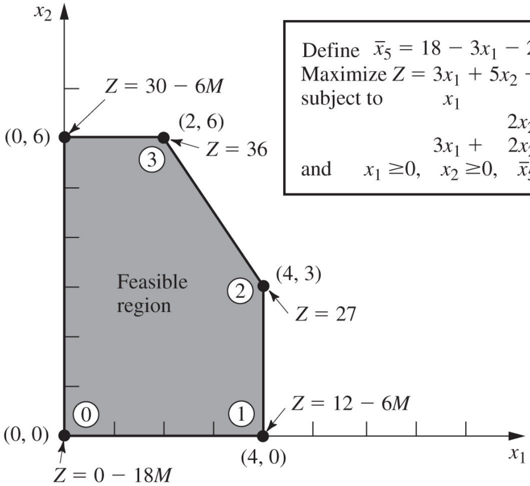
and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad \bar{x}_5 \geq 0.$$

Artificial Problem

- Therefore, just as in the original problem, the feasible region for (x_1, x_2) for the artificial problem is the one in Fig. 4.4.
- The only portion of this feasible region that coincides with the feasible region for the real problem is where $\overline{x_5} = 0$ (so $3x_1 + 2x_2 = 18$).
- Figure 4.4 also shows the order in which the simplex method examines the CPF solutions (or BF solutions after augmenting), where each circled number identifies which iteration obtained that solution.
- Note that the simplex method moves counterclockwise here whereas it moved clockwise for the original problem (Fig. 4.2).
- The reason for this difference is the extra term $-M\overline{x_5}$ in the objective function for the artificial problem.
- Before applying the simplex method and demonstrating that it follows the path shown in Fig. 4.4, the following preparatory step is needed.

Figure 4.4



Define $\bar{x}_5 = 18 - 3x_1 - 2x_2$.

Maximize $Z = 3x_1 + 5x_2 - M\bar{x}_5$,

subject to

$$\begin{aligned} x_1 \quad\quad\quad\quad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + \quad 2x_2 &\leq 18 \end{aligned}$$

and $\quad x_1 \geq 0, \quad x_2 \geq 0, \quad \bar{x}_5 \geq 0$

Converting Equation (0) to Proper Form

- System of equations after augmenting the artificial problem:

$$
\begin{array}{llllllll}
(0) & Z - 3x_1 - 5x_2 & & & & + M\overline{x_5} & = 0 \\
(1) & x_1 & & + x_3 & & & = 4 \\
(2) & & 2x_2 & & + x_4 & & = 12 \\
(3) & 3x_1 + 2x_2 & & & & + \overline{x_5} & = 18
\end{array}
$$

  where the initial basic variables $(x_3, x_4, \overline{x_5})$ are in blue.
- However, this system is not yet in proper form from Gaussian elimination because a basic variable $\overline{x_5}$ has a nonzero coefficient in Eq. (0).

Converting Equation (0) to Proper Form

- Recall that all basic variables must be algebraically eliminated from Eq. (0) before the simplex method can either apply the optimality test or find the entering basic variable.
- This elimination is necessary so that the negative of the coefficient of each nonbasic variable will give the rate at which Z would increase if that nonbasic variable were to be increased from 0 while adjusting the values of the basic variables accordingly.

Converting Equation (0) to Proper Form

- To algebraically eliminate $\overline{x_5}$ from Eq. (0), we need to subtract from Eq. (0) the product, M times Eq. (3).

$$
\begin{array}{lcrcl}
 & Z & -3x_1 - 5x_2 & + M\overline{x_5} & = 0 \\
 & & -M(3x_1 + 2x_2 & + \overline{x_5} & = 18) \\
\hline
\text{New (0)} & Z & -(3M+3)x_1 - (2M+5)x_2 & & = -18M
\end{array}
$$

Application of the Simplex Method

- This new Eq. (0) gives Z in terms of *just* the nonbasic variables (x_1, x_2),

$$Z = -18M + (3M + 3)x_1 + (2M + 5)x_2$$

- Since $3M + 3 > 2M + 5$ (remember that M represents a huge number), increasing x_1 increases Z at a faster rate than increasing x_2 does, so x_1 is chosen as the entering basic variable.
- This leads to the move from (0, 0) to (4, 0) at iteration 1, shown in Fig. 4.4, thereby increasing Z by $4(3M + 3)$.

Application of the Simplex Method

- The quantities involving M never appear in the system of equations except for Eq. (0), so they need to be taken into account only in the optimality test and when an entering basic variable is determined.
- One way of dealing with these quantities is to assign some particular (huge) numerical value to M and use the resulting coefficients in Eq. (0) in the usual way.
- However, this approach may result in significant rounding errors that invalidate the optimality test.

Application of the Simplex Method

- Therefore, it is better to do what we have just shown, namely, to express each coefficient in Eq. (0) as a linear function $aM + b$ of the *symbolic* quantity M by separately recording and updating the current numerical value of (1) the *multiplicative* factor a and (2) the *additive* term b.
- Because M is assumed to be so large that b always is negligible compared with M when $a \neq 0$, the decisions in the optimality test and the choice of the entering basic variable are made by using just the *multiplicative* factors in the usual way, except for breaking ties with the *additive* factors.
- Using this approach on the example yields the simplex tableaux shown in Table 4.11.

TABLE 4.11 Complete set of simplex tableaux for the problem shown in Fig. 4.4

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	Right Side
0	Z	(0)	1	$-3M-3$	$-2M-5$	0	0	0	$-18M$
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	$\bar{x}_5$	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	0	$-2M-5$	$3M+3$	0	0	$-6M+12$
	x_1	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	$\bar{x}_5$	(3)	0	0	2	-3	0	1	6
2	Z	(0)	1	0	0	$-\frac{9}{2}$	0	$M+\frac{5}{2}$	27
	x_1	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	0	3	1	-1	6
	x_2	(3)	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3
3	Z	(0)	1	0	0	0	$\frac{3}{2}$	$M+1$	36
	x_1	(1)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
	x_3	(2)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(3)	0	0	1	0	$\frac{1}{2}$	0	6

Application of the Simplex Method

- Note that the artificial variable $\overline{x_5}$ is a basic variable ($\overline{x_5} > 0$) in the first two tableaux and a *nonbasic variable* ($\overline{x_5} = 0$) in the last two.
- Therefore, the first two BF solutions for this artificial problem are infeasible for the real problem whereas the last two also are BF solutions for the real problem.
- This example involved only one equality constraint.
- If a linear programming model has more than one, each is handled in just the same way.
  - If the right-hand side is negative, multiply through both sides by -1 first.

Topics

Negative Right-Hand Sides

- The technique mentioned in the preceding sentence for dealing with an equality constraint with a negative right-hand side (namely, multiply through both sides by -1) also works for any inequality constraint with a negative right-hand side.
- Multiplying through both sides of an inequality by -1 also reverses the direction of the inequality; i.e., $\leq$ changes to $\geq$.
- For example, doing this to the constraint

$$x_1 - x_2 \leq -1 \qquad \text{(namely, } x_1 \leq x_2 - 1\text{)}$$

gives the equivalent constraint

$$-x_1 + x_2 \geq 1 \qquad \text{(that is, } x_2 - 1 \geq x_1\text{)}$$

but now the right-hand side is positive.
- Having nonnegative right-hand sides for all the functional constraints enables the simplex method to begin, because (after augmenting) these right-hand sides become the respective values of the initial basic variables, which must satisfy nonnegativity constraints.

Topics

Functional Constraints in $\geq$ Form

- To illustrate how the artificial variable technique deals with functional constraints in $\geq$ form, we will use the model for designing the radiation therapy, as presented in Sec. 3.4.
- This model is repeated below, where we have placed a box around the constraint of special interest here.

Radiation Therapy Example

Minimize $\quad Z = 0.4x_1 + 0.5x_2,$

subject to

$$0.3x_1 + 0.1x_2 \leq 2.7$$

$$0.5x_1 + 0.5x_2 = 6$$

$$\boxed{0.6x_1 + 0.4x_2 \geq 6}$$

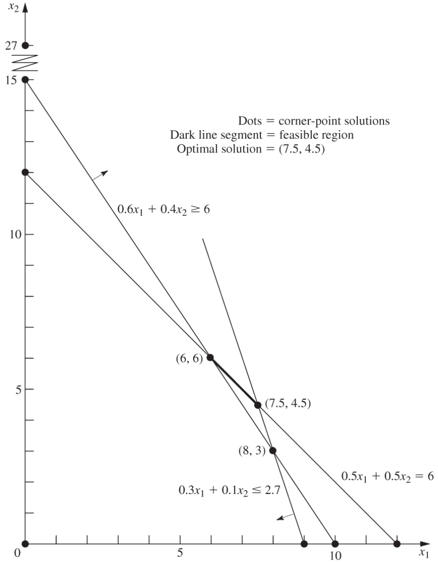
and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

Functional Constraints in $\geq$ Form

- The graphical solution for this example (originally presented in Fig. 3.12) is repeated here in a slightly different form in Fig. 4.5.
- The three lines in the figure, along with the two axes, constitute the five constraint boundaries of the problem.
- The dots lying at the intersection of a pair of constraint boundaries are the *corner-point* solutions.
- The only two corner-point feasible solutions are (6, 6) and (7.5, 4.5), and the *feasible* region is the line segment connecting these two points. The optimal solution is $(x_1, x_2) = (7.5, 4.5)$, with $Z = 5.25$.
- We soon will show how the simplex method solves this problem by directly solving the corresponding artificial problem.
- However, first we must describe how to deal with the third constraint.

Figure 4.5


x_2
27
15
10
5
0
5
10
x_1
Dots = corner-point solutions
Dark line segment = feasible region
Optimal solution = (7.5, 4.5)
$0.6x_1 + 0.4x_2 \geq 6$
(6, 6)
(7.5, 4.5)
(8, 3)
$0.5x_1 + 0.5x_2 = 6$
$0.3x_1 + 0.1x_2 \leq 2.7$

Functional Constraints in $\geq$ Form

- Our approach involves introducing *both* a surplus variable x_5 (defined as $x_5 = 0.6x_1 + 0.4x_2 - 6$) and an artificial variable $\overline{x_6}$, as shown next.

$$
\begin{aligned}
& 0.6x_1 + 0.4x_2 && \geq 6 && \\
\Rightarrow\ & 0.6x_1 + 0.4x_2 - x_5 && = 6 && (x_5 \geq 0) \\
\Rightarrow\ & 0.6x_1 + 0.4x_2 - x_5 + \overline{x_6} && = 6 && (x_5 \geq 0, \overline{x_6} \geq 0)
\end{aligned}
$$

- Here x_5 is called a surplus variable because it subtracts the surplus of the left-hand side over the right-hand side to convert the inequality constraint to an equivalent equality constraint.
- Once this conversion is accomplished, the artificial variable is introduced just as for any equality constraint.

Functional Constraints in $\geq$ Form

- After a slack variable x_3 is introduced into the first constraint, an artificial variable $\overline{x_4}$ is introduced into the second constraint, and the Big M method is applied, so the complete artificial problem (in augmented form) is
  Minimize: $Z = 0.4x_1 + 0.5x_2 + M\overline{x_4} + M\overline{x_6}$
  subject to:
$$\begin{array}{lcl} 0.3x_1 + 0.1x_2 + x_3 & = & 2.7 \\ 0.5x_1 + 0.5x_2 \qquad\quad + \overline{x_4} & = & 6 \\ 0.6x_1 + 0.4x_2 \qquad\qquad\quad - x_5 + \overline{x_6} & = & 6 \end{array}$$
  and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \overline{x_4} \geq 0, x_5 \geq 0 \quad , \quad \overline{x_6} \geq 0.$
- Note that the coefficients of the artificial variables in the objective function are $+M$, instead of $-M$, because we now are minimizing Z.
- Thus, although $\overline{x_4} > 0$ and/or $\overline{x_6} > 0$ is possible for a feasible solution for the artificial problem, the huge unit penalty of $+M$ prevents this from occurring in an optimal solution.

Functional Constraints in $\geq$ Form

- As usual, introducing artificial variables enlarges the feasible region. Compare below the original constraints for the real problem with the corresponding constraints on (x_1, x_2) for the artificial problem

Constraints on (x_1, x_2) *for the Real Problem*	*Constraints on* (x_1, x_2) *for the Artificial Problem*
$0.3x_1 + 0.1x_2 \leq 2.7$	$0.3x_1 + 0.1x_2 \leq 2.7$
$0.5x_1 + 0.5x_2 = 6$	$0.5x_1 + 0.5x_2 \leq 6$ ($=$ holds when $\overline{x_4} = 0$)
$0.6x_1 + 0.4x_2 \geq 6$	No such constraint (except when $\overline{x_6} = 0$)
$x_1 \geq 0, x_2 \geq 0,$	$x_1 \geq 0, x_2 \geq 0$

- Introducing the artificial variable $\overline{x_4}$ to play the role of a slack variable in the second constraint allows values of (x_1, x_2) *below* the $0.5x_1 + 0.5x_2 = 6$ line in Fig. 4.5.

Functional Constraints in $\geq$ Form

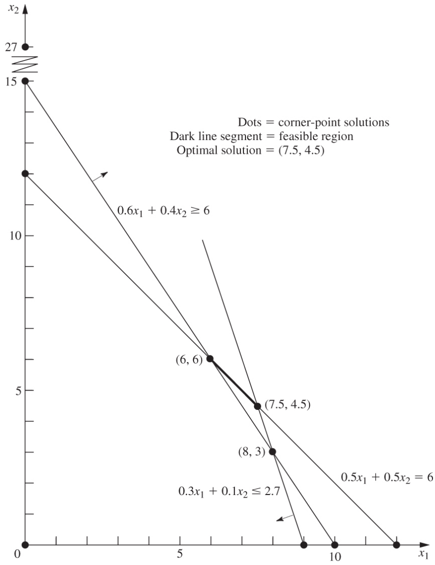
- Introducing x_5 and $\overline{x_6}$ into the third constraint of the real problem (and moving these variables to the right-hand side) yields the equation:

$$0.6x_1 + 0.4x_2 = 6 + x_5 - \overline{x_6}$$

Functional Constraints in $\geq$ Form

- Because both x_5 and $\overline{x_6}$ are constrained only to be nonnegative, their difference $x_5 - \overline{x_6}$ can be any positive or negative number.
- Therefore, $0.6x_1 + 0.4x_2$ can have any value, which has the effect of eliminating the third constraint from the artificial problem and allowing points on either side of the $0.6x_1 + 0.4x_2 = 6$ line in Fig. 4.5
  - We keep the third constraint in the system of equations only because it will become relevant again later, after the Big M method forces $\overline{x_6}$ to be zero.
- Consequently, the feasible region for the artificial problem is the entire polyhedron in Fig. 4.5 whose vertices are (0, 0), (9, 0), (7.5, 4.5), and (0, 12).

Figure 4.5



x_2
27
15
10
5
0
5
10
x_1
Dots = corner-point solutions
Dark line segment = feasible region
Optimal solution = (7.5, 4.5)
$0.6x_1 + 0.4x_2 \geq 6$
(6, 6)
(7.5, 4.5)
(8, 3)
$0.5x_1 + 0.5x_2 = 6$
$0.3x_1 + 0.1x_2 \leq 2.7$

Functional Constraints in $\geq$ Form

- Since the origin now is feasible for the artificial problem, the simplex method starts with (0, 0) as the initial CPF solution, i.e., with $(x_1, x_2, x_3, \overline{x_4}, x_5, \overline{x_6}) = (0, 0, 2.7, 6, 0, 6)$ as the initial BF solution.
    - Making the origin feasible as a convenient starting point for the simplex method is the whole point of creating the artificial problem.
- We soon will trace the entire path followed by the simplex method from the origin to the optimal solution for both the artificial and real problems.
- But, first, how does the simplex method handle *minimization*?

Topics

Minimization

- One straightforward way of minimizing Z with the simplex method is to exchange the roles of the positive and negative coefficients in row 0 for both the optimality test and step 1 of an iteration.
- However, rather than changing our instructions for the simplex method for this case, we present the following simple way of converting any minimization problem to an equivalent maximization problem:

$$\text{Minimizing } Z = \sum_{j=1}^{n} c_j x_j$$

is equivalent to

$$\text{Maximizing } -Z = \sum_{j=1}^{n} (-c_j) x_j$$

i.e., the two formulations yield the same optimal solution(s).

Minimization

- The two formulations are equivalent because the smaller Z is, the larger $-Z$ is, so the solution that gives the *smallest* value of Z in the entire feasible region must also give the *largest* value of $-Z$ in this region.
- Therefore, in the radiation therapy example, we make the following change in the formulation:

$$\text{Minimize } Z = 0.4x_1 + 0.5x_2$$
$$\longrightarrow \text{Maximize } -Z = -0.4x_1 - 0.5x_2$$

- After artificial variables $\overline{x_4}$ and $\overline{x_6}$ are introduced and then the Big M method is applied, the corresponding conversion is

$$\text{Minimize } Z = 0.4x_1 + 0.5x_2 + M\overline{x_4} + M\overline{x_6}$$
$$\longrightarrow \text{Maximize } -Z = -0.4x_1 - 0.5x_2 - M\overline{x_4} - M\overline{x_6}$$

Topics

Solving the Radiation Therapy Example

- We now are nearly ready to apply the simplex method to the radiation therapy example.
- By using the maximization form just obtained, the entire system of equations is now

$$
\begin{array}{llr}
(0) & -Z + 0.4x_1 + 0.5x_2 \qquad + M\overline{x_4} \qquad + M\overline{x_6} & = 0 \\
(1) & 0.3x_1 + 0.1x_2 + x_3 & = 2.7 \\
(2) & 0.5x_1 + 0.5x_2 \qquad + \overline{x_4} & = 6 \\
(3) & 0.6x_1 + 0.4x_2 \qquad - x_5 + \overline{x_6} & = 6
\end{array}
$$

- The basic variables $(x_3, \overline{x_4}, \overline{x_6})$ for the initial BF solution (for this artificial problem) are shown in blue.

Solving the Radiation Therapy Example

- Note that this system of equations is not yet in proper form from Gaussian elimination, as required by the simplex method, since the basic variables $\overline{x_4}$ and $\overline{x_6}$ still need to be algebraically eliminated from Eq. (0).
- Because $\overline{x_4}$ and $\overline{x_6}$ both have a coefficient of M, Eq. (0) needs to have subtracted from it *both* M times Eq. (2) *and* M times Eq. (3).
- The calculations for all the coefficients (and the right-hand sides) are summarized below, where the vectors are the relevant rows of the simplex tableau corresponding to the above system of equations.

Row 0:

$$
\begin{array}{rrrrrrr}
[0.4, & 0.5, & 0, & M, & 0, & M, & 0] \\
-M[0.5, & 0.5, & 0, & 1, & 0, & 0, & 6] \\
-M[0.6, & 0.4, & 0, & 0, & -1, & 1, & 6] \\
\hline
\text{New row } 0 = [-1.1M + 0.4, & -0.9M + 0.5, & 0, & 0, & M, & 0, & -12M]
\end{array}
$$

Solving the Radiation Therapy Example

- The resulting initial simplex tableau, ready to begin the simplex method, is shown at the top of Table 4.12.
- Applying the simplex method in just the usual way then yields the sequence of simplex tableaux shown in the rest of Table 4.12.
- For the optimality test and the selection of the entering basic variable at each iteration, the quantities involving M are treated just as discussed in connection with Table 4.11.
- Specifically, whenever M is present, only its multiplicative factor is used, unless there is a tie, in which case the tie is broken by using the corresponding additive terms.
- Just such a tie occurs in the last selection of an entering basic variable (see the next-to-last tableau), where the coefficients of x_3 and x_5 in row 0 both have the same multiplicative factor of $-\frac{5}{3}$.
- Comparing the additive terms, $\frac{11}{6} < \frac{7}{3}$ leads to choosing x_5 as the entering basic variable.

Solving the Radiation Therapy Example

- Note in Table 4.12 the progression of values of the artificial variables $\overline{x_4}$ and $\overline{x_6}$ and of Z.
- We start with large values, $\overline{x_4} = 6$ and $\overline{x_6} = 6$, with $Z = 12M$ $(-Z = -12M)$.
- The first iteration greatly reduces these values.
- The Big M method succeeds in driving $\overline{x_6}$ to zero (as a new nonbasic variable) at the second iteration and then in doing the same to $\overline{x_4}$ at the next iteration.
- With both $\overline{x_4} = 0$ and $\overline{x_6} = 0$, the basic solution given in the last tableau is guaranteed to be feasible for the real problem.
- Since it passes the optimality test, it also is optimal.

Solving the Radiation Therapy Example

TABLE 4.12 The Big *M* method for the radiation therapy example

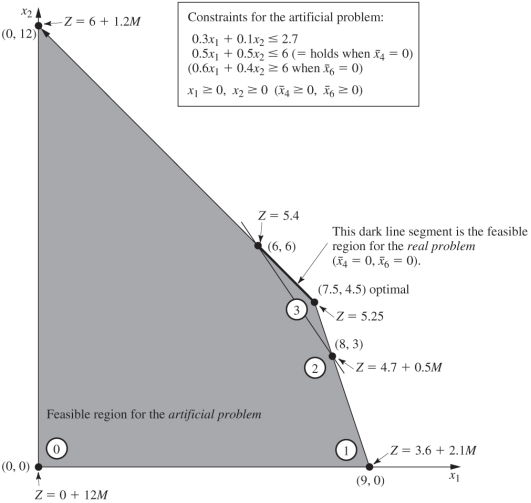
Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
0	Z	(0)	-1	$-1.1M + 0.4$	$-0.9M + 0.5$	0	0	M	0	$-12M$
	x_3	(1)	0	0.3	0.1	1	0	0	0	2.7
	$\bar{x}_4$	(2)	0	0.5	0.5	0	1	0	0	6
	$\bar{x}_6$	(3)	0	0.6	0.4	0	0	-1	1	6
1	Z	(0)	-1	0	$-\frac{16}{30}M + \frac{11}{30}$	$\frac{11}{3}M - \frac{4}{3}$	0	M	0	$-2.1M - 3.6$
	x_1	(1)	0	1	$\frac{1}{3}$	$\frac{10}{3}$	0	0	0	9
	$\bar{x}_4$	(2)	0	0	$\frac{1}{3}$	$-\frac{5}{3}$	1	0	0	1.5
	$\bar{x}_6$	(3)	0	0	0.2	-2	0	-1	1	0.6
2	Z	(0)	-1	0	0	$-\frac{5}{3}M + \frac{7}{3}$	0	$-\frac{5}{3}M + \frac{11}{6}$	$\frac{8}{3}M - \frac{11}{6}$	$-0.5M - 4.7$
	x_1	(1)	0	1	0	$\frac{20}{3}$	0	$\frac{5}{3}$	$-\frac{5}{3}$	8
	$\bar{x}_4$	(2)	0	0	0	$\frac{5}{3}$	1	$\frac{5}{3}$	$-\frac{5}{3}$	0.5
	x_2	(3)	0	0	1	-10	0	-5	5	3
3	Z	(0)	-1	0	0	0.5	$M - 1.1$	0	M	-5.25
	x_1	(1)	0	1	0	5	-1	0	0	7.5
	x_5	(2)	0	0	0	1	0.6	1	-1	0.3
	x_2	(3)	0	0	1	-5	3	0	0	4.5

Solving the Radiation Therapy Example

Now see what the Big M method has done graphically in Fig. 4.6:

- The feasible region for the artificial problem initially has four CPF solutions: (0, 0), (9, 0), (0, 12), and (7.5, 4.5).
- Then, the method replaces the first three CPF solutions with two new CPF solutions ((8, 3) and (6, 6)) after $\overline{x_6}$ decreases to $\overline{x_6} = 0$ so that $0.6x_1 + 0.4x_2 \geq 6$ becomes an additional constraint.
- Note that the three replaced CPF solutions ((0, 0), (9, 0), and (0, 12)) actually were corner-point infeasible solutions for the real problem shown in Fig. 4.5.
- Starting with the origin as the convenient initial CPF solution for the artificial problem, we move around the boundary to three other CPF solutions: (9, 0), (8, 3), and (7.5, 4.5).
- The last of these is the first one that also is feasible for the real problem.
- Fortuitously, this first feasible solution also is optimal, so no additional iterations are needed.

Figure 4.6


Constraints for the artificial problem:
$0.3x_1 + 0.1x_2 \le 2.7$
$0.5x_1 + 0.5x_2 \le 6$ (= holds when $\bar{x}_4 = 0$)
($0.6x_1 + 0.4x_2 \ge 6$ when $\bar{x}_6 = 0$)
$x_1 \ge 0,\ x_2 \ge 0$ ($\bar{x}_4 \ge 0,\ \bar{x}_6 \ge 0$)
x_2
$Z = 6 + 1.2M$
(0, 12)
$Z = 5.4$
(6, 6)
This dark line segment is the feasible region for the *real problem* ($\bar{x}_4 = 0, \bar{x}_6 = 0$).
(7.5, 4.5) optimal
3
$Z = 5.25$
(8, 3)
2
$Z = 4.7 + 0.5M$
Feasible region for the *artificial problem*
0
1
$Z = 3.6 + 2.1M$
(0, 0)
(9, 0)
x_1
$Z = 0 + 12M$

Solving the Radiation Therapy Example

- For other problems with artificial variables, it may be necessary to perform additional iterations to reach an optimal solution after the first feasible solution is obtained for the real problem. (This was the case for the example solved in Table 4.11.)
- Thus, the Big M method can be thought of as having two phases.
- In the *first phase*, all the artificial variables are driven to zero (because of the penalty of M per unit for being greater than zero) in order to reach an initial BF solution for the *real* problem.
- In the *second phase*, all the artificial variables are kept at zero (because of this same penalty) while the simplex method generates a sequence of BF solutions for the real problem that leads to an optimal solution.
- The *two-phase method* described next is a streamlined procedure for performing these two phases directly, without even introducing M explicitly.

The Two-Phase Method

- For the radiation therapy example just solved in Table 4.12, recall its real objective function

$$\textit{Real problem:}\ \text{Minimize}\ Z = 0.4x_1 + 0.5x_2$$

- However, the Big M method uses the following objective function (or its equivalent in maximization form) throughout the entire procedure:

$$\textit{Big M method}:\ \text{Minimize}\ Z = 0.4x_1 + 0.5x_2 + M\overline{x_4} + M\overline{x_6}$$

- Since the first two coefficients are negligible compared to M, the two-phase method is able to drop M by using the following two objective functions with completely different definitions of Z in turn.

The Two-Phase Method

Two-phase method:

Phase 1: Minimize $Z = \overline{x_4} + \overline{x_6}$ (until $\overline{x_4} = 0, \overline{x_6} = 0$).

Phase 2: Minimize $Z = 0.4x_1 + 0.5x_2$ (with $\overline{x_4} = 0, \overline{x_6} = 0$).

- The phase 1 objective function is obtained by dividing the Big M method objective function by M and then dropping the negligible terms.
- Since phase 1 concludes by obtaining a BF solution for the real problem (one where $\overline{x_4} = 0$ and $\overline{x_6} = 0$), this solution is then used as the *initial* BF solution for applying the simplex method to the real problem (with its real objective function) in phase 2.
- Before solving the example in this way, we summarize the general method.

Summary of The Two-Phase Method

- *Initialization:* Revise the constraints of the original problem by introducing artificial variables as needed to obtain an obvious initial BF solution for the artificial problem.

  Phase 1: The objective for this phase is to find a BF solution for the *real problem*. To do this,
  Minimize $Z = \sum$ artificial variables, subject to revised constraints.
- The optimal solution obtained for this problem (with $Z = 0$) will be a BF solution for the real problem.

  Phase 2: The objective for this phase is to find an *optimal solution* for the real problem. Since the artificial variables are not part of the real problem, these variables can now be dropped (they are all zero now anyway). Starting from the BF solution obtained at the end of phase 1, use the simplex method to solve the real problem.

The Two-Phase Method

- For the example, the problems to be solved by the simplex method in the respective phases are summarized below. (Radiation Therapy Example):

Phase 1 Problem: Minimize $Z = \overline{x_4} + \overline{x_6}$, subject to:

$$
\begin{aligned}
0.3x_1 + 0.1x_2 + x_3 \qquad\qquad\qquad\qquad &= 2.7 \\
0.5x_1 + 0.5x_2 \qquad\quad + \overline{x_4} \qquad\qquad &= 6 \\
0.6x_1 + 0.4x_2 \qquad\qquad\qquad - x_5 + \overline{x_6} &= 6
\end{aligned}
$$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \overline{x_4} \geq 0, x_5 \geq 0, \overline{x_6} \geq 0.$

Phase 2 Problem: Minimize $Z = 0.4x_1 + 0.5x_2$, subject to:

$$
\begin{aligned}
0.3x_1 + 0.1x_2 + x_3 \qquad\qquad &= 2.7 \\
0.5x_1 + 0.5x_2 \qquad\qquad\qquad &= 6 \\
0.6x_1 + 0.4x_2 \qquad\qquad - x_5 &= 6
\end{aligned}
$$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_5 \geq 0.$

Phase 1

- The only differences between these two problems are in the objective function and in the inclusion (phase 1) or exclusion (phase 2) of the artificial variables $\overline{x_4}$ and $\overline{x_6}$.
- Without the artificial variables, the phase 2 problem does not have an obvious *initial BF solution*.
- The sole purpose of solving the phase 1 problem is to obtain a BF solution with $\overline{x_4} = 0$ and $\overline{x_6} = 0$ so that this solution (without the artificial variables) can be used as the initial BF solution for phase 2.

Phase 1

- Table 4.13 shows the result of applying the simplex method to this phase 1 problem.
- Row 0 in the initial tableau is obtained by converting
  Minimize $Z = \overline{x_4} + \overline{x_6}$ to
  Maximize $(-Z) = -\overline{x_4} - \overline{x_6}$
  and then using *elementary row operations* to eliminate the basic variables x_4 and x_6 from $-Z + x_4 + x_6 = 0$.
- In the next-to-last tableau, there is a tie for the *entering basic variable* between x_3 and x_5, which is broken arbitrarily in favor of x_3.
- The solution obtained at the end of phase 1, then, is $(x_1, x_2, x_3, \overline{x_4}, x_5, \overline{x_6}) = (6, 6, 0.3, 0, 0, 0)$ or, after $\overline{x_4}$ and $\overline{x_6}$ are dropped, $(x_1, x_2, x_3, x_5) = (6, 6, 0.3, 0)$.

Table 4.13

TABLE 4.13 Phase 1 of the two-phase method for the radiation therapy example

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
0	Z	(0)	-1	-1.1	-0.9	0	0	1	0	-12
	x_3	(1)	0	0.3	0.1	1	0	0	0	2.7
	$\bar{x}_4$	(2)	0	0.5	0.5	0	1	0	0	6
	$\bar{x}_6$	(3)	0	0.6	0.4	0	0	-1	1	6
1	Z	(0)	-1	0	$-\frac{16}{30}$	$\frac{11}{3}$	0	1	0	-2.1
	x_1	(1)	0	1	$\frac{1}{3}$	$\frac{10}{3}$	0	0	0	9
	$\bar{x}_4$	(2)	0	0	$\frac{1}{3}$	$-\frac{5}{3}$	1	0	0	1.5
	$\bar{x}_6$	(3)	0	0	0.2	-2	0	-1	1	0.6
2	Z	(0)	-1	0	0	$-\frac{5}{3}$	0	$-\frac{5}{3}$	$\frac{8}{3}$	-0.5
	x_1	(1)	0	1	0	$\frac{20}{3}$	0	$\frac{5}{3}$	$-\frac{5}{3}$	8
	$\bar{x}_4$	(2)	0	0	0	$\frac{5}{3}$	1	$\frac{5}{3}$	$-\frac{5}{3}$	0.5
	x_2	(3)	0	0	1	-10	0	-5	5	3
3	Z	(0)	-1	0	0	0	1	0	1	0
	x_1	(1)	0	1	0	0	-4	-5	5	6
	x_3	(2)	0	0	0	1	$\frac{3}{5}$	1	-1	0.3
	x_2	(3)	0	0	1	0	6	5	-5	6

Phase 1

- As claimed in the summary, this solution from phase 1 is indeed a BF solution for the *real* problem (the phase 2 problem) because it is the solution (after you set $x_5 = 0$) to the system of equations consisting of the three functional constraints for the phase 2 problem.
- In fact, after deleting the $\overline{x_4}$ and $\overline{x_6}$ columns as well as row 0 for each iteration, Table 4.13 shows one way of using Gaussian elimination to solve this system of equations by reducing the system to the form displayed in the final tableau.

Phase 2

- Table 4.14 shows the preparations for beginning phase 2 after phase 1 is completed.
- Starting from the final tableau in Table 4.13, we drop the artificial variables ($\overline{x_4}$ and $\overline{x_6}$), substitute the phase 2 objective function ($-Z = -0.4x_1 - 0.5x_2$ in maximization form) into row 0, and then restore the proper form from Gaussian elimination (by algebraically eliminating the basic variables x_1 and x_2 from row 0).
- Thus, row 0 in the last tableau is obtained by performing the following *elementary row operations* in the next-to-last tableau: from row 0 subtract both the product, 0.4 times row 1, and the product, 0.5 times row 3.
- Except for the deletion of the two columns, note that rows 1 to 3 never change.
- The only adjustments occur in row 0 in order to replace the phase 1 objective function by the phase 2 objective function.

Phase 2

TABLE 4.14 Preparing to begin phase 2 for the radiation therapy example

	Basic Variable	Eq.	Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
Final Phase 1 tableau	Z	(0)	−1	0	0	0	1	0	1	0
	x_1	(1)	0	1	0	0	−4	−5	5	6
	x_3	(2)	0	0	0	1	$\frac{3}{5}$	1	−1	0.3
	x_2	(3)	0	0	1	0	6	5	−5	6
Drop $\bar{x}_4$ and $\bar{x}_6$	Z	(0)	−1	0	0	0		0		0
	x_1	(1)	0	1	0	0		−5		6
	x_3	(2)	0	0	0	1		1		0.3
	x_2	(3)	0	0	1	0		5		6
Substitute phase 2 objective function	Z	(0)	−1	0.4	0.5	0		0		0
	x_1	(1)	0	1	0	0		−5		6
	x_3	(2)	0	0	0	1		1		0.3
	x_2	(3)	0	0	1	0		5		6
Restore proper form from Gaussian elimination	Z	(0)	−1	0	0	0		−0.5		−5.4
	x_1	(1)	0	1	0	0		−5		6
	x_3	(2)	0	0	0	1		1		0.3
	x_2	(3)	0	0	1	0		5		6

Phase 2

- The last tableau in Table 4.14 is the initial tableau for applying the simplex method to the phase 2 problem, as shown at the top of Table 4.15.
- Just one iteration then leads to the optimal solution shown in the second tableau: $(x_1, x_2, x_3, x_5) = (7.5, 4.5, 0, 0.3)$.
- This solution is the desired optimal solution for the real problem of interest rather than the artificial problem constructed for phase 1.

Phase 2

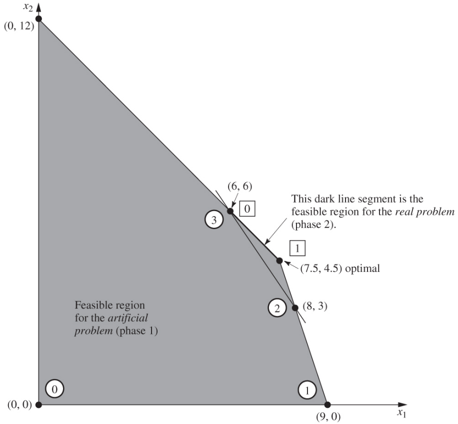
TABLE 4.15 Phase 2 of the two-phase method for the radiation therapy example

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_5	Right Side
0	Z	(0)	-1	0	0	0	-0.5	-5.4
	x_1	(1)	0	1	0	0	-5	6
	x_3	(2)	0	0	0	1	1	0.3
	x_2	(3)	0	0	1	0	5	6
1	Z	(0)	-1	0	0	0.5	0	-5.25
	x_1	(1)	0	1	0	5	0	7.5
	x_5	(2)	0	0	0	1	1	0.3
	x_2	(3)	0	0	1	-5	0	4.5

Phase 2

- Now we see what the two-phase method has done graphically in Fig. 4.7.
- Starting at the origin, phase 1 examines a total of four CPF solutions for the artificial problem.
- The first three actually were corner-point infeasible solutions for the real problem shown in Fig. 4.5.
- The fourth CPF solution, at (6, 6), is the first one that also is feasible for the real problem, so it becomes the initial CPF solution for phase 2.
- One iteration in phase 2 leads to the optimal CPF solution at (7.5, 4.5).

Figure 4.7



x_2
(0, 12)
(6, 6)
0
3
This dark line segment is the feasible region for the *real problem* (phase 2).
1
(7.5, 4.5) optimal
2
(8, 3)
Feasible region for the *artificial problem* (phase 1)
0
1
(0, 0)
(9, 0)
x_1

Phase 2

- If the tie for the entering basic variable in the next-to-last tableau of Table 4.13 had been broken in the other way, then phase 1 would have gone directly from (8, 3) to (7.5, 4.5).
- After (7.5, 4.5) was used to set up the initial simplex tableau for phase 2, the *optimality test* would have revealed that this solution was optimal, so no iterations would be done.

Compare the Big M and Two-Phase Methods

Begin with their objective functions.

- *Big M Method:*

  Minimize $\quad Z = 0.4x_1 + 0.5x_2 + M\overline{x_4} + M\overline{x_6}.$

- *Two-Phase Method:*

Phase 1:	Minimize	$Z = \overline{x_4} + \overline{x_6}$
Phase 2:	Minimize	$Z = 0.4x_1 + 0.5x_2$

- Because the $M\overline{x_4}$ and $M\overline{x_6}$ terms dominate the $0.4x_1$ and $0.5x_2$ terms in the objective function for the Big M method, this objective function is essentially equivalent to the phase 1 objective function as long as $\overline{x_4}$ and/or $\overline{x_6}$ is greater than zero.
- Then, when both $\overline{x_4} = 0$ and $\overline{x_6} = 0$, the objective function for the Big M method becomes completely equivalent to the phase 2 objective function.

Compare the Big M and Two-Phase Methods

- Because of these virtual equivalencies in objective functions, the Big M and two-phase methods generally have the same sequence of BF solutions.
  - The one possible exception occurs when there is a tie for the entering basic variable in phase 1 of the two-phase method, as happened in the third tableau of Table 4.13.
- Notice that the first three tableaux of Tables 4.12 and 4.13 are almost identical, with the only difference being that the multiplicative factors of M in Table 4.12 become the sole quantities in the corresponding spots in Table 4.13.
- Consequently, the additive terms that broke the tie for the entering basic variable in the third tableau of Table 4.12 were not present to break this same tie in Table 4.13.
- The result for this example was an extra iteration for the two-phase method.
- Generally, however, the advantage of having the additive factors is minimal.

Compare the Big M and Two-Phase Methods

- The two-phase method streamlines the Big M method by using only the multiplicative factors in phase 1 and by dropping the artificial variables in phase 2. (The Big M method could combine the multiplicative and additive factors by assigning an actual huge number to M, but this might create numerical instability problems.)
- For these reasons, the two-phase method is commonly used in computer codes.

Topics

No Feasible Solutions

- So far in this section, we have been concerned primarily with the fundamental problem of identifying an initial BF solution when an obvious one is not available.
- We have seen how the artificial-variable technique can be used to construct an artificial problem and obtain an initial BF solution for this artificial problem instead.
- Use of either the Big M method or the two-phase method then enables the simplex method to begin its pilgrimage toward the BF solutions, and ultimately toward the optimal solution, for the *real* problem.

No Feasible Solutions

- However, we should be wary of a certain pitfall with this approach.
- There may be no obvious choice for the initial BF solution for the very good reason that there are no feasible solutions at all!
- Nevertheless, by constructing an artificial feasible solution, there is nothing to prevent the simplex method from proceeding as usual and ultimately reporting a supposedly optimal solution.
- Fortunately, the artificial-variable technique provides the following signpost to indicate when this has happened:

  If the original problem has *no feasible solutions*, then either the Big M method or phase 1 of the two-phase method yields a final solution that has at least one artificial variable *greater* than zero. Otherwise, they *all* equal zero.

No Feasible Solutions

- To illustrate, let us change the first constraint in the radiation example (see Fig. 4.5) as follows:

  $$0.3x_1 + 0.1x_2 \leq 2.7 \qquad \longrightarrow \qquad 0.3x_1 + 0.1x_2 \leq 1.8$$

  so that the problem no longer has any feasible solutions.
- Applying the Big M method just as before (see Table 4.12) yields the tableaux shown in Table 4.16. (Phase 1 of the two-phase method yields the same tableaux except that each expression involving M is replaced by just the multiplicative factor.)
- Hence, the Big M method normally would be indicating that the optimal solution is (3, 9, 0, 0, 0, 0.6).
- However, since an artificial variable $\overline{x_6} = 0.6 > 0$, the real message here is that the problem has no feasible solutions.

TABLE 4.16 The Big M method for the revision of the radiation therapy example that has no feasible solutions

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
0	Z	(0)	-1	$-1.1M + 0.4$	$-0.9M + 0.5$	0	0	M	0	$-12M$
	x_3	(1)	0	0.3	0.1	1	0	0	0	1.8
	$\bar{x}_4$	(2)	0	0.5	0.5	0	1	0	0	6
	$\bar{x}_6$	(3)	0	0.6	0.4	0	0	-1	1	6
1	Z	(0)	-1	0	$-\frac{16}{30}M + \frac{11}{30}$	$\frac{11}{3}M - \frac{4}{3}$	0	M	0	$-5.4M - 2.4$
	x_1	(1)	0	1	$\frac{1}{3}$	$\frac{10}{3}$	0	0	0	6
	$\bar{x}_4$	(2)	0	0	$\frac{1}{3}$	$-\frac{5}{3}$	1	0	0	3
	$\bar{x}_6$	(3)	0	0	0.2	-2	0	-1	1	2.4
2	Z	(0)	-1	0	0	$M + 0.5$	$1.6M - 1.1$	M	0	$-0.6M - 5.7$
	x_1	(1)	0	1	0	5	-1	0	0	3
	x_2	(2)	0	0	1	-5	3	0	0	9
	$\bar{x}_6$	(3)	0	0	0	-1	-0.6	-1	1	0.6

Topics

Variables Allowed to be Negative

- In most practical problems, negative values for the decision variables would have no physical meaning, so it is necessary to include nonnegativity constraints in the formulations of their linear programming models.
- However, this is not always the case.
- To illustrate, suppose that the Wyndor Glass Co. problem is changed so that product 1 already is in production, and the first decision variable x_1 represents the *increase* in its production rate.
- Therefore, a negative value of x_1 would indicate that product 1 is to be cut back by that amount.
- Such reductions might be desirable to allow a larger production rate for the new, more profitable product 2, so negative values should be allowed for x_1 in the model.

Variables Allowed to be Negative

- Since the procedure for determining the *leaving basic variable* requires that all the variables have nonnegativity constraints, any problem containing variables allowed to be negative must be converted to an *equivalent* problem involving only nonnegative variables before the simplex method is applied.
- Fortunately, this conversion can be done.
- The modification required for each variable depends upon whether it has a (negative) lower bound on the values allowed.
- Each of these two cases is now discussed.

Variables with a Bound on the Negative Values Allowed

- Consider any decision variable x_j that is allowed to have negative values which satisfy a constraint of the form

$$x_j \geq L_j$$

  where L_j is some negative constant.
- This constraint can be converted to a nonnegativity constraint by making the change of variables

$$x_j' = x_j - L_j, \qquad \text{so} \qquad x_j' \geq 0$$

- Thus, $x_j' + L_j$, would be substituted for x_j throughout the model, so that the redefined decision variable x_j' cannot be negative. (This same technique can be used when L_j is *positive* to convert a functional constraint $x_j \geq L_j$ to a nonnegativity constraint $x_j' \geq 0$.)

Variables with a Bound on the Negative Values Allowed

- To illustrate, suppose that the current production rate for product 1 in the Wyndor Glass Co. problem is 10.
- With the definition of x_1 just given, the complete model at this point is the same as that given in Sec. 3.1 except that the nonnegativity constraint $x_1 \geq 0$ is replaced by

$$x_1 \geq -10$$

- To obtain the equivalent model needed for the simplex method, this decision variable would be redefined as the *total* production rate of product 1

$$x_1' = x_1 + 10$$

which yields the changes in the objective function and constraints as shown:

$$\begin{array}{|l|}\hline Z = 3x_1 + 5x_2 \\ x_1 \leq 4 \\ 2x_2 \leq 12 \\ 3x_1 + 2x_2 \leq 18 \\ x_1 \geq -10, \quad x_2 \geq 0 \\ \hline \end{array} \rightarrow \begin{array}{|l|}\hline Z = 3(x_1' - 10) + 5x_2 \\ x_1' - 10 \leq 4 \\ 2x_2 \leq 12 \\ 3(x_1' - 10) + 2x_2 \leq 18 \\ x_1' - 10 \geq -10, \quad x_2 \geq 0 \\ \hline \end{array} \rightarrow \begin{array}{|l|}\hline Z = -30 + 3x_1' + 5x_2 \\ x_1' \leq 14 \\ 2x_2 \leq 12 \\ 3x_1' + 2x_2 \leq 48 \\ x_1' \geq 0, \quad x_2 \geq 0 \\ \hline \end{array}$$

Variables with No Bound on the Negative Values Allowed

- In the case where x_j does *not* have a lower-bound constraint in the model formulated, another approach is required: x_j is replaced throughout the model by the *difference* of two new *nonnegative* variables

$$x_j = x_j^+ - x_j^-, \qquad \text{where } x_j^+ \geq 0, x_j^- \geq 0.$$

- Since x_j^+ and x_j^- can have any nonnegative values, this difference $x_j^+ - x_j^-$ can have *any* value (positive or negative), so it is a legitimate substitute for x_j in the model.
- But after such substitutions, the simplex method can proceed with just nonnegative variables.
- The new variables x_j^+ and x_j^- have a simple interpretation.
- As explained in the next paragraph, each BF solution for the new form of the model necessarily has the property that *either* $x_j^+ = 0$ or $x_j^- = 0$ (or both).

Variables with No Bound on the Negative Values Allowed

- Therefore, at the optimal solution obtained by the simplex method (a BF solution),

$$x_j^+ = \begin{cases} x_j & \text{if } x_j \geq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$x_j^- = \begin{cases} |x_j| & \text{if } x_j \leq 0, \\ 0 & \text{otherwise;} \end{cases}$$

so that x_j^+ represents the positive part of the decision variable x_j and x_j^- its negative part (as suggested by the superscripts).

Variables with No Bound on the Negative Values Allowed

- For example, if $x_j^+ = 10$, the above expressions give $x_j^+ = 10$ and $x_j^- = 0$.
- This same value of $x_j = x_j^+ - x_j^- = 10$ also would occur with larger values of x_j^+ and x_j^- such that $x_j^+ = x_j^- + 10$.
- Plotting these values of x_j^+ and x_j^- on a two-dimensional graph gives a line with an endpoint at $x_j^+ = 10$ and $x_j^- = 0$ to avoid violating the nonnegativity constraints.
- This endpoint is the only corner-point solution on the line.
- Therefore, only this endpoint can be part of an overall CPF solution or BF solution involving all the variables of the model.
- This illustrates why each BF solution necessarily has either $x_j^+ = 0$ or $x_j^- = 0$ (or both).

Variables with No Bound on the Negative Values Allowed

- To illustrate the use of the x_j^+ and x_j^-, let us return to the example on the preceding page where x_1 is redefined as the increase over the current production rate of 10 for product 1 in the Wyndor Glass Co. problem.
- However, now suppose that the $x_1 \geq -10$ constraint was not included in the original model because it clearly would not change the optimal solution.
  - In some problems, certain variables do not need explicit lower-bound constraints because the functional constraints already prevent lower values.

Variables with No Bound on the Negative Values Allowed

- Therefore, before the simplex method is applied, x_1 would be replaced by the difference

$$x_1 = x_1^+ + x_1^-, \qquad \text{where } x_1^+ \geq 0, x_1^- \geq 0$$

as shown:

$$\boxed{\begin{array}{ll} \text{Maximize} & Z = 3x_1 + 5x_2, \\ \text{subject to} & \begin{aligned} x_1 \qquad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ x_2 &\geq 0 \text{ (only)} \end{aligned} \end{array}} \rightarrow \boxed{\begin{array}{ll} \text{Maximize} & Z = 3x_1^+ - 3x_1^- + 5x_2, \\ \text{subject to} & \begin{aligned} x_1^+ - x_1^- \qquad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1^+ - 3x_1^- + 2x_2 &\leq 18 \\ x_1^+ \geq 0, \quad x_1^- \geq 0, \quad x_2 &\geq 0 \end{aligned} \end{array}}$$

Variables with No Bound on the Negative Values Allowed

- From a computational viewpoint, this approach has the disadvantage that the new equivalent model to be used has more variables than the original model.
- In fact, if *all* the original variables lack lower-bound constraints, the new model will have *twice* as many variables.
- Fortunately, the approach can be modified slightly so that the number of variables is increased by only one, regardless of how many original variables need to be replaced.

Variables with No Bound on the Negative Values Allowed

- This modification is done by replacing each such variable x_j by
$$x_j = x_j' - x'', \qquad \text{where } x_j' \geq 0, x'' \geq 0$$
instead, where x'' is the *same* variable for all relevant j.
- The interpretation of x'' in this case is that $-x''$ is the current value of the *largest* (in absolute terms) negative original variable, so that x_j' is the amount by which x_j exceeds this value.
- Thus, the simplex method now can make some of the x_j' variables larger than zero even when $x'' \geq 0$.

Postoptimality Analysis: Simplex Method Role

- We stressed in Secs. 2.3, 2.4, and 2.5 that *postoptimality analysis*—the analysis done after an optimal solution is obtained for the initial version of the model—constitutes a very major and very important part of most operations research studies.
- The fact that postoptimality analysis is very important is particularly true for typical linear programming applications.
- In this section, we focus on the role of the simplex method in performing this analysis.
- Table 4.17 summarizes the typical steps in postoptimality analysis for linear programming studies.
- The rightmost column identifies some algorithmic techniques that involve the simplex method.
- These techniques are introduced briefly here with the technical details deferred to later chapters.

Postoptimality Analysis: Simplex Method Role

TABLE 4.17 Postoptimality analysis for linear programming

Task	Purpose	Technique
Model debugging	Find errors and weaknesses in model	Reoptimization
Model validation	Demonstrate validity of final model	See Sec. 2.4
Final managerial decisions on resource allocations (the b_i values)	Make appropriate division of organizational resources between activities under study and other important activities	Shadow prices
Evaluate estimates of model parameters	Determine crucial estimates that may affect optimal solution for further study	Sensitivity analysis
Evaluate trade-offs between model parameters	Determine best trade-off	Parametric linear programming

Topics

Reoptimization

- Alternative to solving the problem again with small changes
- Involves deducing how changes in the model get carried along to the final simplex tableau
- Optimal solution for the revised model:
  - Will be much closer to the prior optimal solution than to an initial BF solution constructed the usual way

Reoptimization

- As discussed in Sec. 3.7, linear programming models that arise in practice commonly are very large, with hundreds or thousands of functional constraints and decision variables.
- In such cases, many variations of the basic model may be of interest for considering different scenarios.
- Therefore, after having found an optimal solution for one version of a linear programming model, we frequently must solve again (often many times) for the solution of a slightly different version of the model.

Reoptimization

- We nearly always have to solve again several times during the model debugging stage (described in Secs. 2.3 and 2.4), and we usually have to do so a large number of times during the later stages of postoptimality analysis as well.
- One approach is simply to reapply the simplex method from scratch for each new version of the model, even though each run may require hundreds or even thousands of iterations for large problems.
- However, a *much more efficient* approach is to *reoptimize*.
- Reoptimization involves deducing how changes in the model get carried along to the *final* simplex tableau (as described in Secs. 5.3 and 6.6).

Reoptimization

- This revised tableau and the optimal solution for the prior model are then used as the *initial tableau* and the *initial basic solution* for solving the new model.
- If this solution is feasible for the new model, then the simplex method is applied in the usual way, starting from this initial BF solution.
- If the solution is not feasible, a related algorithm called the *dual simplex method* (described in Sec. 7.1) probably can be applied to find the new optimal solution, starting from this initial basic solution.
    - The one requirement for using the dual simplex method here is that the *optimality test* is still passed when applied to row 0 of the *revised* final tableau. If not, then still another algorithm called the *primal-dual method* can be used instead.

Reoptimization

- The big advantage of this reoptimization technique over re-solving from scratch is that an optimal solution for the revised model probably is going to be *much* closer to the prior optimal solution than to an initial BF solution constructed in the usual way for the simplex method.
- Therefore, assuming that the model revisions were modest, only a few iterations should be required to reoptimize instead of the hundreds or thousands that may be required when you start from scratch.
- In fact, the optimal solutions for the prior and revised models are frequently the same, in which case the reoptimization technique requires only one application of the optimality test and *no* iterations.

Topics

Shadow Prices

- Recall that linear programming problems often can be interpreted as allocating resources to activities.
- In particular, when the functional constraints are in $\leq$ form, we interpreted the b_i (the right-hand sides) as the amounts of the respective resources being made available for the activities under consideration.
- In many cases, there may be some latitude in the amounts that will be made available.

Shadow Prices

- If so, the b_i values used in the initial (validated) model actually may represent management's *tentative initial decision* on how much of the organization's resources will be provided to the activities considered in the model instead of to other important activities under the purview of management.
- From this broader perspective, some of the b_i values can be increased in a revised model, but only if a sufficiently strong case can be made to management that this revision would be beneficial.
- Consequently, information on the economic contribution of the resources to the measure of performance (Z) for the current study often would be extremely useful.
- The simplex method provides this information in the form of *shadow prices* for the respective resources.

Shadow Prices

- The shadow price for resource i (denoted by y_i^*) measures the *marginal value* of this resource, i.e., the rate at which Z could be increased by (slightly) increasing the amount of this resource (b_i) being made available.
  - The increase in b_i must be sufficiently small that the current set of basic variables remains optimal since this rate (marginal value) changes if the set of basic variables changes.
  - In the case of a functional constraint in $\geq$ or $=$ form, its shadow price is again defined as the rate at which Z could be increased by (slightly) increasing the value of b_i, although the interpretation of b_i now would normally be something other than the amount of a resource being made available.
- The simplex method identifies this shadow price by $y_i^* =$ coefficient of the ith slack variable in row 0 of the final simplex tableau.

Shadow Prices

- To illustrate, for the Wyndor Glass Co. problem,
  - Resource i = production capacity of Plant i, ($i = 1, 2, 3$) being made available to the two new products under consideration,
  - b_i = hours of production time per week being made available in Plant i for these new products.
- Providing a substantial amount of production time for the new products would require adjusting production times for the current products, so choosing the b_i value is a difficult managerial decision.
- The tentative initial decision has been

$$b_1 = 4, \qquad b_2 = 12, \qquad b_3 = 18,$$

as reflected in the basic model considered in Sec. 3.1 and in this chapter.

Shadow Prices

- However, management now wishes to evaluate the effect of changing any of the b_i values.
- The shadow prices for these three resources provide just the information that management needs.
- The final tableau in Table 4.8 yields

$$
\begin{aligned}
y_1^* &= 0 = \text{shadow price for resource 1,} \\
y_2^* &= \frac{3}{2} = \text{shadow price for resource 2,} \\
y_3^* &= 1 = \text{shadow price for resource 3.}
\end{aligned}
$$

- With just two decision variables, these numbers can be verified by checking graphically that individually increasing any b_i by 1 indeed would increase the optimal value of Z by y_i^*.

Shadow Prices

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Shadow Prices

- For example, Fig. 4.8 demonstrates this increase for resource 2 by reapplying the graphical method presented in Sec. 3.1.
- The optimal solution, (2, 6) with $Z = 36$, changes to $(\frac{5}{3}, \frac{13}{2})$ with $Z = 37\frac{1}{2}$ when b_2 is increased by 1 (from 12 to 13), so that

$$y_2^* = \Delta Z = 37\frac{1}{2} - 36 = \frac{3}{2}$$

- Since Z is expressed in thousands of dollars of profit per week, $y_2^* = \frac{3}{2}$ indicates that adding 1 more hour of production time per week in Plant 2 for these two new products would increase their total profit by $1,500 per week.
- Should this actually be done?

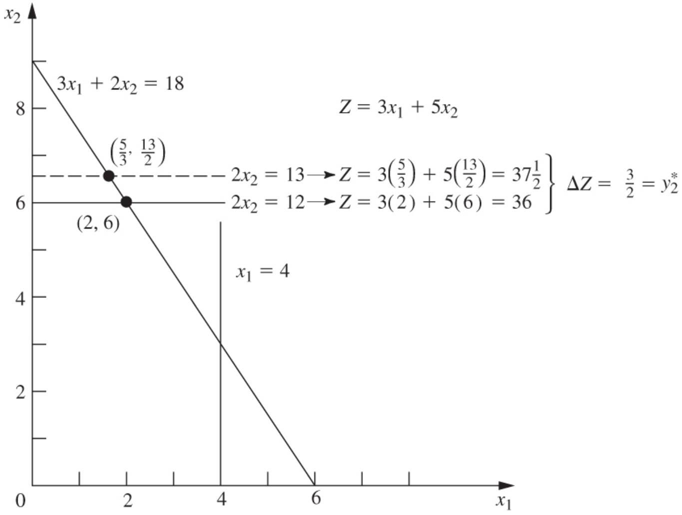
Shadow Prices

- It depends on the marginal profitability of other products currently using this production time.
- If there is a current product that contributes less than $1,500 of weekly profit per hour of weekly production time in Plant 2, then some shift of production time to the new products would be worthwhile.
- We shall continue this story in Sec. 6.7, where the Wyndor OR team uses shadow prices as part of its *sensitivity analysis* of the model.

Shadow Prices

- Figure 4.8 demonstrates that $y_2^* = \frac{3}{2}$ is the rate at which Z could be increased by increasing b_2 slightly.
- However, it also demonstrates the common phenomenon that this interpretation holds only for a small increase in b_2.
- Once b_2 is increased beyond 18, the optimal solution stays at (0, 9) with no further increase in Z.
  - At that point, the set of basic variables in the optimal solution has changed, so a new final simplex tableau will be obtained with new shadow prices, including $y_2^* = 0$.

Figure 4.8



x_2
$3x_1 + 2x_2 = 18$
$Z = 3x_1 + 5x_2$
$\left(\frac{5}{3}, \frac{13}{2}\right)$
$2x_2 = 13 \rightarrow Z = 3\left(\frac{5}{3}\right) + 5\left(\frac{13}{2}\right) = 37\frac{1}{2}$
$2x_2 = 12 \rightarrow Z = 3(2) + 5(6) = 36$
$\Delta Z = \frac{3}{2} = y_2^*$
(2, 6)
$x_1 = 4$
8
6
4
2
0
2
4
6
x_1

Shadow Prices

- Now note in Fig. 4.8 why $y_1^* = 0$.
- Because the constraint on resource 1, $x_1 \leq 4$, is *not binding* on the optimal solution (2, 6), there is a *surplus* of this resource.
- Therefore, increasing b_1 beyond 4 cannot yield a new optimal solution with a larger value of Z.
- By contrast, the constraints on resources 2 and 3, $2x_2 \leq 12$ and $3x_1 + 2x_2 \leq 18$, are binding constraints (constraints that hold with equality at the optimal solution).

Shadow Prices

- Because the limited supply of these resources ($b_2 = 12, b_3 = 18$) *binds* Z from being increased further, they have *positive* shadow prices.
- Economists refer to such resources as *scarce goods*, whereas resources available in surplus (such as resource 1) are *free goods* (resources with a zero shadow price).
- The kind of information provided by shadow prices clearly is valuable to management when it considers reallocations of resources within the organization.
- It also is very helpful when an increase in b_i can be achieved only by going outside the organization to purchase more of the resource in the marketplace.

Shadow Prices

- For example, suppose that Z represents *profit* and that the unit profits of the activities (the c_j values) include the costs (at regular prices) of all the resources consumed.
- Then a *positive* shadow price of y_i^* for resource i means that the total profit Z can be increased by y_i^* by purchasing 1 more unit of this resource at its regular price.
- Alternatively, if a *premium* price must be paid for the resource in the marketplace, then y_i^* represents the maximum premium (excess over the regular price) that would be worth paying.
  - If the unit profits do *not* include the costs of the resources consumed, then y_i^* represents the maximum *total* unit price that would be worth paying to increase b_i.
- The theoretical foundation for shadow prices is provided by the duality theory described in Chap. 6.

Topics

Sensitivity Analysis

- When discussing the certainty assumption for linear programming at the end of Sec. 3.3, we pointed out that the values used for the model parameters (the a_{ij}, b_i, and c_j identified in Table 3.3) generally are just *estimates* of quantities whose true values will not become known until the linear programming study is implemented at some time in the future.
- A main purpose of sensitivity analysis is to identify the sensitive parameters (i.e., those that cannot be changed without changing the optimal solution).
- The sensitive parameters are the parameters that need to be estimated with special care to minimize the risk of obtaining an erroneous optimal solution.
- They also will need to be monitored particularly closely as the study is implemented.

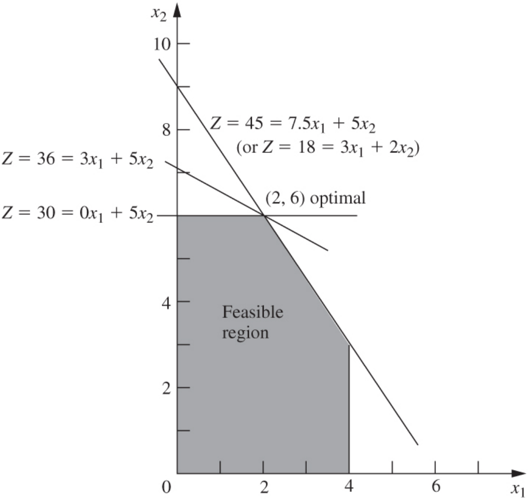
Sensitivity Analysis

- If it is discovered that the true value of a sensitive parameter differs from its estimated value in the model, this immediately signals a need to change the solution.
- How are the sensitive parameters identified?
- In the case of the b_i, we have just seen that this information is given by the shadow prices provided by the simplex method.
- In particular, if $y_i^* > 0$, then the optimal solution changes if b_i is changed, so b_i is a sensitive parameter.
- However, $y_i^* = 0$ implies that the optimal solution is not sensitive to at least small changes in b_i.
- Consequently, if the value used for b_i is an estimate of the amount of the resource that will be available (rather than a managerial decision), then the b_i values that need to be monitored more closely are those with *positive* shadow prices—especially those with *large* shadow prices.

Sensitivity Analysis

- When there are just two variables, the sensitivity of the various parameters can be analyzed graphically.
- For example, in Fig. 4.9, $c_1 = 3$ can be changed to any other value from 0 to 7.5 without the optimal solution changing from (2, 6).
  - The reason is that any value of c_1 within this range keeps the slope of $Z = c_1 x_1 + 5x_2$ between the slopes of the lines $2x_2 = 12$ and $3x_1 + 2x_2 = 18$.
- Similarly, if $c_2 = 5$ is the only parameter changed, it can have any value greater than 2 without affecting the optimal solution.
- Hence, neither c_1 nor c_2 is a sensitive parameter.
- The easiest way to analyze the sensitivity of each of the a_{ij} parameters graphically is to check whether the corresponding constraint is *binding* at the optimal solution.

Figure 4.9


x_2
10
8
4
2
0
2
4
6
x_1
$Z = 45 = 7.5x_1 + 5x_2$
(or $Z = 18 = 3x_1 + 2x_2$)
$Z = 36 = 3x_1 + 5x_2$
$Z = 30 = 0x_1 + 5x_2$
(2, 6) optimal
Feasible region

Sensitivity Analysis

- Because $x_1 \leq 4$ is *not* a binding constraint, any sufficiently small change in its coefficients ($a_{11} = 1, a_{12} = 0$) is not going to change the optimal solution, so these are *not* sensitive parameters.
- On the other hand, both $2x_2 = 12$ and $3x_1 + 2x_2 = 18$ are *binding constraints*, so changing *any* one of their coefficients ($a_{21} = 0, a_{22} = 2, a_{31} = 3, a_{32} = 2$) is going to change the optimal solution, and therefore these are sensitive parameters.
- Typically, greater attention is given to performing sensitivity analysis on the b_i and c_j parameters than on the a_{ij} parameters.
- On real problems with hundreds or thousands of constraints and variables, the effect of changing one a_{ij} value is usually negligible, but changing one b_i or c_j value can have real impact.

Sensitivity Analysis

- Furthermore, in many cases, the a_{ij} values are determined by the technology being used (the a_{ij} values are sometimes called technological coefficients), so there may be relatively little (or no) uncertainty about their final values.
- This is fortunate, because there are far more a_{ij} parameters than b_i and c_j parameters for large problems.
- For problems with more than two (or possibly three) decision variables, you cannot analyze the sensitivity of the parameters graphically as was just done for the Wyndor Glass Co. problem.

Sensitivity Analysis

- However, you can extract the same kind of information from the simplex method.
- Getting this information requires using the fundamental insight described in Sec. 5.3 to deduce the changes that get carried along to the final simplex tableau as a result of changing the value of a parameter in the original model.
- The rest of the procedure is described and illustrated in Secs. 6.6 and 6.7.

Parametric Linear Programming

- Sensitivity analysis involves changing one parameter at a time in the original model to check its effect on the optimal solution.
- By contrast, parametric linear programming (or parametric programming for short) involves the systematic study of how the optimal solution changes as *many* of the parameters change *simultaneously* over some range.
- This study can provide a very useful extension of sensitivity analysis, e.g., to check the effect of "correlated" parameters that change together due to exogenous factors such as the state of the economy.
- However, a more important application is the investigation of *trade-offs* in parameter values.

Parametric Linear Programming

- For example, if the c_j values represent the unit profits of the respective activities, it may be possible to increase some of the c_j values at the expense of decreasing others by an appropriate shifting of personnel and equipment among activities.
- Similarly, if the b_i values represent the amounts of the respective resources being made available, it may be possible to increase some of the b_i values by agreeing to accept decreases in some of the others.
- The analysis of such possibilities is discussed and illustrated at the end of Sec. 6.7.
- In some applications, the main purpose of the study is to determine the most appropriate trade-off between two basic factors, such as *costs* and *benefits*.

Parametric Linear Programming

- The usual approach is to express one of these factors in the objective function (e.g., minimize total cost) and incorporate the other into the constraints (e.g., benefits $\geq$ minimum acceptable level), as was done for the Nori & Leets Co. air pollution problem in Sec. 3.4.
- Parametric linear programming then enables systematic investigation of what happens when the initial tentative decision on the trade-off (e.g., the minimum acceptable level for the benefits) is changed by improving one factor at the expense of the other.
- The algorithmic technique for parametric linear programming is a natural extension of that for sensitivity analysis, so it, too, is based on the simplex method.
- The procedure is described in Sec. 7.2.

References

Hillier&Lieberman

- Chapter 4: Solving Linear Programming Problems: The Simplex Method
  4.6 and 4.7