Discrete Optimization

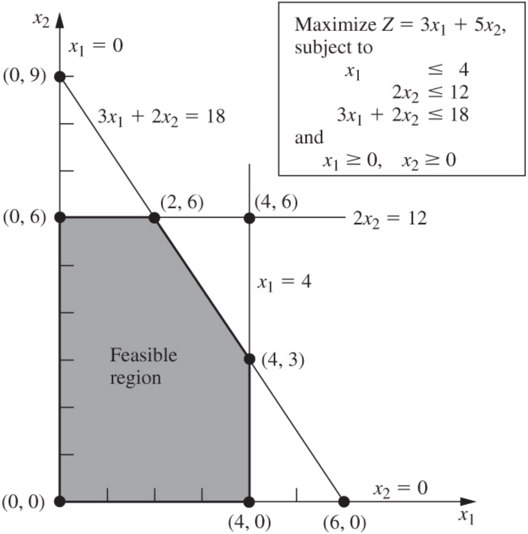
Introduction to Linear Programming

Topics

The Simplex Method

- Developed by George Dantzig in 1947.
- Used routinely to solve huge problems on today's computers.
- Always executed on a computer.
- The simplex method is an algebraic procedure.
  - However, its underlying concepts are geometric.

Example


Maximize $Z = 3x_1 + 5x_2$,
subject to
$x_1 \leq 4$
$2x_2 \leq 12$
$3x_1 + 2x_2 \leq 18$
and
$x_1 \geq 0, \quad x_2 \geq 0$
x_2
$x_1 = 0$
(0, 9)
$3x_1 + 2x_2 = 18$
(0, 6)
(2, 6)
(4, 6)
$2x_2 = 12$
$x_1 = 4$
Feasible region
(4, 3)
(0, 0)
(4, 0)
(6, 0)
$x_2 = 0$
x_1

Example

- Five constraint boundaries, each boundary is a line
- Points of intersection: Corner-point solutions
- Corners of the feasible region: Corner-point feasible solutions
- Each corner-point solution lies at the intersection of two constraint boundaries.
  - For a linear program with n decision variables, each of its corner-point solutions lies at the intersection of n constraint boundaries.

Example

- Five constraint boundaries, each boundary is a line
- Points of intersection: Corner-point solutions
- Corners of the feasible region: Corner-point feasible solutions
- Each corner-point solution lies at the intersection of two constraint boundaries.
  - For a linear program with *n* decision variables, each of its corner-point solutions lies at the intersection of *n* constraint boundaries.

Example

- Five constraint boundaries, each boundary is a line
- Points of intersection: Corner-point solutions
- Corners of the feasible region: Corner-point feasible solutions
- Each corner-point solution lies at the intersection of two constraint boundaries.
  - For a linear program with n decision variables, each of its corner-point solutions lies at the intersection of n constraint boundaries.

Example

- Five constraint boundaries, each boundary is a line
- Points of intersection: Corner-point solutions
- Corners of the feasible region: Corner-point feasible solutions
- Each corner-point solution lies at the intersection of two constraint boundaries.
    - For a linear program with n decision variables, each of its corner-point solutions lies at the intersection of n constraint boundaries.

Example

- Five constraint boundaries, each boundary is a line
- Points of intersection: Corner-point solutions
- Corners of the feasible region: Corner-point feasible solutions
- Each corner-point solution lies at the intersection of two constraint boundaries.
  - For a linear program with n decision variables, each of its corner-point solutions lies at the intersection of n constraint boundaries.

Definition

Definition

For any linear programming problem with n decision variables, two CPF solutions are adjacent to each other if they share $n-1$ constraint boundaries. The two adjacent CPF solutions are connected by a line segment that lies on these same shared constraint boundaries. Such a line segment is referred to as an edge of the feasible region.

Example

TABLE 4.1 Adjacent CPF solutions for each CPF solution of the Wyndor Glass Co. problem

CPF Solution	Its Adjacent CPF Solutions
(0, 0)	(0, 6) and (4, 0)
(0, 6)	(2, 6) and (0, 0)
(2, 6)	(4, 3) and (0, 6)
(4, 3)	(4, 0) and (2, 6)
(4, 0)	(0, 0) and (4, 3)

Since $n = 2$ in the example, two CPF solutions are adjacent if they share one constraint boundary.

Adjacency and Optimality

Adjacency provides a very useful way of checking whether a CPF solution is an optimal solution.

Optimality Test

Consider any linear programming problem that possesses at least one optimal solution. If a CPF solution has no adjacent CPF solutions that are better (as measured by Z), then it must be an optimal solution.

Adjacency and Optimality

Adjacency provides a very useful way of checking whether a CPF solution is an optimal solution.

Optimality Test

Consider any linear programming problem that possesses at least one optimal solution. If a CPF solution has no adjacent CPF solutions that are better (as measured by Z), then it must be an optimal solution.

Optimality Test

Example

In the example, $(2, 6)$ must be optimal simply because its $Z = 36$ is larger than $Z = 30$ for $(0, 6)$, and $Z = 27$ for $(4, 3)$.

This optimality test is used by the simplex method to determine when an optimal solution has been reached.

Optimality Test

Example

In the example, $(2, 6)$ must be optimal simply because its $Z = 36$ is larger than $Z = 30$ for $(0, 6)$, and $Z = 27$ for $(4, 3)$.

This optimality test is used by the simplex method to determine when an optimal solution has been reached.

Solving the Example

- Initialization:
  Choose $(0, 0)$ as the initial CPF solution to examine.
  (This is a convenient choice because no calculations are required to identify this solution.)
- Optimality test:
  Conclude that $(0, 0)$ is not an optimal solution.
  (Adjacent CPF solutions are better.)

Solving the Example

- Initialization:
  Choose $(0, 0)$ as the initial CPF solution to examine.
  (This is a convenient choice because no calculations are required to identify this solution.)
- Optimality test:
  Conclude that $(0, 0)$ is not an optimal solution.
  (Adjacent CPF solutions are better.)

Iteration 1

- Iteration 1:
  Move to a better adjacent CPF solution, $(0, 6)$, by performing the following three steps.
  1. Considering the two edges of the feasible region that emanate from $(0, 0)$, choose to move along the edge that leads up the x_2 axis.
     (With an objective function of $Z = 3x_1 + 5x_2$, moving up the x_2 axis increases Z at a faster rate than moving along the x_1 axis.)
  2. Stop at the first new constraint boundary: $2x_2 = 12$.
  3. Solve for the intersection of the new set of constraint boundaries: $(0, 6)$.
     (The equations for these constraint boundaries, $x_1 = 0$ and $2x_2 = 12$, immediately yield this solution.)
- Optimality test:
  Conclude that $(0, 6)$ is not an optimal solution. (An adjacent CPF solution is better.)

Iteration 1

- Iteration 1:
  Move to a better adjacent CPF solution, $(0, 6)$, by performing the following three steps.
  1. Considering the two edges of the feasible region that emanate from $(0, 0)$, choose to move along the edge that leads up the x_2 axis.
     (With an objective function of $Z = 3x_1 + 5x_2$, moving up the x_2 axis increases Z at a faster rate than moving along the x_1 axis.)
  2. Stop at the first new constraint boundary: $2x_2 = 12$.
  3. Solve for the intersection of the new set of constraint boundaries: $(0, 6)$.
     (The equations for these constraint boundaries, $x_1 = 0$ and $2x_2 = 12$, immediately yield this solution.)
- Optimality test:
  Conclude that $(0, 6)$ is not an optimal solution. (An adjacent CPF solution is better.)

Iteration 1

- Iteration 1:
  Move to a better adjacent CPF solution, $(0, 6)$, by performing the following three steps.
  1. Considering the two edges of the feasible region that emanate from $(0, 0)$, choose to move along the edge that leads up the x_2 axis.
     (With an objective function of $Z = 3x_1 + 5x_2$, moving up the x_2 axis increases Z at a faster rate than moving along the x_1 axis.)
  2. Stop at the first new constraint boundary: $2x_2 = 12$.
  3. Solve for the intersection of the new set of constraint boundaries: $(0, 6)$.
     (The equations for these constraint boundaries, $x_1 = 0$ and $2x_2 = 12$, immediately yield this solution.)
- Optimality test:
  Conclude that $(0, 6)$ is not an optimal solution. (An adjacent CPF solution is better.)

Iteration 1

- Iteration 1:
  Move to a better adjacent CPF solution, $(0, 6)$, by performing the following three steps.
  1. Considering the two edges of the feasible region that emanate from $(0, 0)$, choose to move along the edge that leads up the x_2 axis.
     (With an objective function of $Z = 3x_1 + 5x_2$, moving up the x_2 axis increases Z at a faster rate than moving along the x_1 axis.)
  2. Stop at the first new constraint boundary: $2x_2 = 12$.
  3. Solve for the intersection of the new set of constraint boundaries: $(0, 6)$.
     (The equations for these constraint boundaries, $x_1 = 0$ and $2x_2 = 12$, immediately yield this solution.)
- Optimality test:
  Conclude that $(0, 6)$ is not an optimal solution. (An adjacent CPF solution is better.)

Iteration 1

- Iteration 1:
  Move to a better adjacent CPF solution, $(0, 6)$, by performing the following three steps.
  1. Considering the two edges of the feasible region that emanate from $(0, 0)$, choose to move along the edge that leads up the x_2 axis.
     (With an objective function of $Z = 3x_1 + 5x_2$, moving up the x_2 axis increases Z at a faster rate than moving along the x_1 axis.)
  2. Stop at the first new constraint boundary: $2x_2 = 12$.
  3. Solve for the intersection of the new set of constraint boundaries: $(0, 6)$.
     (The equations for these constraint boundaries, $x_1 = 0$ and $2x_2 = 12$, immediately yield this solution.)
- Optimality test:
  Conclude that $(0, 6)$ is not an optimal solution. (An adjacent CPF solution is better.)

Solving the Example

- Iteration 2:
  Move to a better adjacent CPF solution, $(2, 6)$, by performing the following three steps.
  1. Considering the two edges of the feasible region that emanate from $(0, 6)$, choose to move along the edge that leads to the right.
     (Moving along this edge increases Z.)
  2. Stop at the first new constraint boundary: $3x_1 + 2x_2 = 18$.
  3. Solve for the intersection of the new set of constraint boundaries: $(2, 6)$.
     (The equations for these constraint boundaries, $3x_1 + 2x_2 = 18$ and $2x_2 = 12$, immediately yield this solution.)
- Optimality test:
  Conclude that $(2, 6)$ is an optimal solution, so stop.
  (None of the adjacent CPF solutions are better.)

Solving the Example

- Iteration 2:
  Move to a better adjacent CPF solution, $(2, 6)$, by performing the following three steps.
  1. Considering the two edges of the feasible region that emanate from $(0, 6)$, choose to move along the edge that leads to the right.
     (Moving along this edge increases Z.)
  2. Stop at the first new constraint boundary: $3x_1 + 2x_2 = 18$.
  3. Solve for the intersection of the new set of constraint boundaries: $(2, 6)$.
     (The equations for these constraint boundaries, $3x_1 + 2x_2 = 18$ and $2x_2 = 12$, immediately yield this solution.)
- Optimality test:
  Conclude that $(2, 6)$ is an optimal solution, so stop.
  (None of the adjacent CPF solutions are better.)

Solving the Example

- Iteration 2:
  Move to a better adjacent CPF solution, $(2, 6)$, by performing the following three steps.
  1. Considering the two edges of the feasible region that emanate from $(0, 6)$, choose to move along the edge that leads to the right.
     (Moving along this edge increases Z.)
  2. Stop at the first new constraint boundary: $3x_1 + 2x_2 = 18$.
  3. Solve for the intersection of the new set of constraint boundaries: $(2, 6)$.
     (The equations for these constraint boundaries, $3x_1 + 2x_2 = 18$ and $2x_2 = 12$, immediately yield this solution.)
- Optimality test:
  Conclude that $(2, 6)$ is an optimal solution, so stop.
  (None of the adjacent CPF solutions are better.)

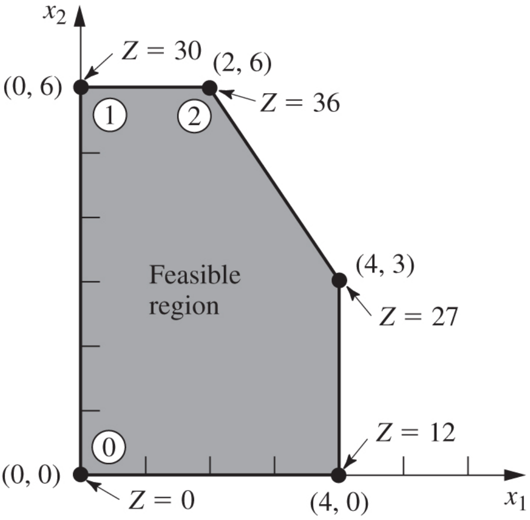
Solving the Example

- Iteration 2:
  Move to a better adjacent CPF solution, $(2, 6)$, by performing the following three steps.
  1. Considering the two edges of the feasible region that emanate from $(0, 6)$, choose to move along the edge that leads to the right.
     (Moving along this edge increases Z.)
  2. Stop at the first new constraint boundary: $3x_1 + 2x_2 = 18$.
  3. Solve for the intersection of the new set of constraint boundaries: $(2, 6)$.
     (The equations for these constraint boundaries, $3x_1 + 2x_2 = 18$ and $2x_2 = 12$, immediately yield this solution.)
- Optimality test:
  Conclude that $(2, 6)$ is an optimal solution, so stop.
  (None of the adjacent CPF solutions are better.)

Solving the Example

- Iteration 2:
  Move to a better adjacent CPF solution, $(2, 6)$, by performing the following three steps.
  1. Considering the two edges of the feasible region that emanate from $(0, 6)$, choose to move along the edge that leads to the right.
     (Moving along this edge increases Z.)
  2. Stop at the first new constraint boundary: $3x_1 + 2x_2 = 18$.
  3. Solve for the intersection of the new set of constraint boundaries: $(2, 6)$.
     (The equations for these constraint boundaries, $3x_1 + 2x_2 = 18$ and $2x_2 = 12$, immediately yield this solution.)
- Optimality test:
  Conclude that $(2, 6)$ is an optimal solution, so stop.
  (None of the adjacent CPF solutions are better.)

Example



Topics

Solution Concept 1

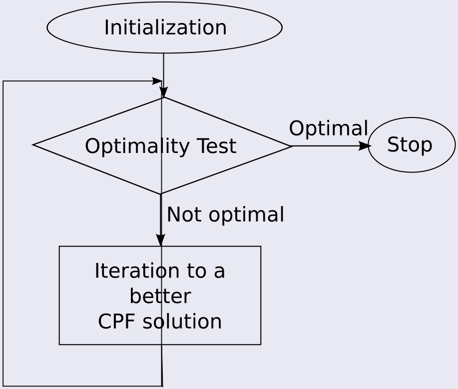
Solution Concept 1

The simplex method focuses solely on CPF solutions. For any problem with at least one optimal solution, finding one requires only finding a best CPF solution.

Solution Concept 2

Solution Concept 2

The simplex method is an iterative algorithm with the following flow.



Solution Concept 3

Solution Concept 3

- Whenever possible, the initialization of the simplex method chooses the origin (all decision variables equal to zero) to be the initial CPF solution.
- This choice eliminates the need to use algebraic procedures to find and solve for an initial CPF solution.

- Choosing the origin is possible when all of the decision variables have nonnegativity constraints because the intersection of these constraint boundaries yields the origin as a corner-point solution.
- If the origin is infeasible, special procedures are needed to find the initial CPF solution.

Solution Concept 3

Solution Concept 3

- Whenever possible, the initialization of the simplex method chooses the origin (all decision variables equal to zero) to be the initial CPF solution.
- This choice eliminates the need to use algebraic procedures to find and solve for an initial CPF solution.

- Choosing the origin is possible when all of the decision variables have nonnegativity constraints because the intersection of these constraint boundaries yields the origin as a corner-point solution.
- If the origin is infeasible, special procedures are needed to find the initial CPF solution.

Solution Concept 3

Solution Concept 3

- Whenever possible, the initialization of the simplex method chooses the origin (all decision variables equal to zero) to be the initial CPF solution.
- This choice eliminates the need to use algebraic procedures to find and solve for an initial CPF solution.

- Choosing the origin is possible when all of the decision variables have nonnegativity constraints because the intersection of these constraint boundaries yields the origin as a corner-point solution.
- If the origin is infeasible, special procedures are needed to find the initial CPF solution.

Solution Concept 3

Solution Concept 3

- Whenever possible, the initialization of the simplex method chooses the origin (all decision variables equal to zero) to be the initial CPF solution.
- This choice eliminates the need to use algebraic procedures to find and solve for an initial CPF solution.

- Choosing the origin is possible when all of the decision variables have nonnegativity constraints because the intersection of these constraint boundaries yields the origin as a corner-point solution.
- If the origin is infeasible, special procedures are needed to find the initial CPF solution.

Solution Concept 4

Solution Concept 4

- Each time the simplex method performs an iteration to move from the current CPF solution to a better one, it always chooses a CPF solution that is adjacent to the current one.
- No other CPF solutions are considered.
- Consequently, the entire path followed to reach an optimal solution is along the edges of the feasible region.

Solution Concept 4

Solution Concept 4

- Each time the simplex method performs an iteration to move from the current CPF solution to a better one, it always chooses a CPF solution that is adjacent to the current one.
- No other CPF solutions are considered.
- Consequently, the entire path followed to reach an optimal solution is along the edges of the feasible region.

Solution Concept 4

Solution Concept 4

- Each time the simplex method performs an iteration to move from the current CPF solution to a better one, it always chooses a CPF solution that is adjacent to the current one.
- No other CPF solutions are considered.
- Consequently, the entire path followed to reach an optimal solution is along the edges of the feasible region.

Solution Concept 5

Solution Concept 5

- To find the next CPF solution at an iteration, the simplex method identifies the rate of improvement in Z that would be obtained by moving along the edge emanating from the current solution.
- Among the edges with a positive rate of improvement in Z, it then chooses to move along the one with the largest rate.

Solution Concept 5

Solution Concept 5

- To find the next CPF solution at an iteration, the simplex method identifies the rate of improvement in Z that would be obtained by moving along the edge emanating from the current solution.
- Among the edges with a positive rate of improvement in Z, it then chooses to move along the one with the largest rate.

Solution Concept 6

Solution Concept 6

- A positive rate of improvement in Z implies that the adjacent CPF solution is better than the current CPF solution.
- Therefore, the optimality test consists simply of checking whether any of the edges give a positive rate of improvement in Z.
- If none do,then the current CPF solution is optimal.

Solution Concept 6

Solution Concept 6

- A positive rate of improvement in Z implies that the adjacent CPF solution is better than the current CPF solution.
- Therefore, the optimality test consists simply of checking whether any of the edges give a positive rate of improvement in Z.
- If none do,then the current CPF solution is optimal.

Solution Concept 6

Solution Concept 6

- A positive rate of improvement in Z implies that the adjacent CPF solution is better than the current CPF solution.
- Therefore, the optimality test consists simply of checking whether any of the edges give a positive rate of improvement in Z.
- If none do,then the current CPF solution is optimal.

Topics

The Simplex Method

- Conceptually geometric procedure
- Translated into an algebraic procedure which is based on solving systems of equations
- So, convert *inequality constraints* to equivalent *equality constraints*.
    - Introduce slack variables.
    - Leave the nonnegativity constraints as they are.

The Simplex Method

- Conceptually geometric procedure
- Translated into an algebraic procedure which is based on solving systems of equations
- So, convert *inequality constraints* to equivalent *equality constraints*.
  - Introduce slack variables.
  - Leave the nonnegativity constraints as they are.

Slack Variables

Example

Consider: $x_1 \leq 4$

The slack variable is defined to be: $x_3 = 4 - x_1$

which is the amount of slack in the left-hand side of the inequality. Thus,

$x_1 + x_3 = 4$

Given this equation, $x_1 \leq 4$ if and only if $4 - x_1 - x_3 \geq 0$.

Therefore, the original constraint is entirely *equivalent* to the pair:

$x_1 + x_3 = 4$ and $x_3 \geq 0$

Slack Variables

Example

Consider: $x_1 \leq 4$

The slack variable is defined to be: $x_3 = 4 - x_1$

which is the amount of slack in the left-hand side of the inequality.

Thus,

$x_1 + x_3 = 4$

Given this equation, $x_1 \leq 4$ if and only if $4 - x_1 - x_3 \geq 0$.

Therefore, the original constraint is entirely *equivalent* to the pair:

$x_1 + x_3 = 4$ and $x_3 \geq 0$

Slack Variables

Example

Consider: $x_1 \leq 4$
The slack variable is defined to be: $x_3 = 4 - x_1$
which is the amount of slack in the left-hand side of the inequality. Thus,
$x_1 + x_3 = 4$
Given this equation, $x_1 \leq 4$ if and only if $4 - x_1 - x_3 \geq 0$.
Therefore, the original constraint is entirely *equivalent* to the pair:
$x_1 + x_3 = 4$ and $x_3 \geq 0$

Slack Variables

Example

Consider: $x_1 \leq 4$
The slack variable is defined to be: $x_3 = 4 - x_1$
which is the amount of slack in the left-hand side of the inequality.
Thus,
$x_1 + x_3 = 4$
Given this equation, $x_1 \leq 4$ if and only if $4 - x_1 - x_3 \geq 0$.
Therefore, the original constraint is entirely *equivalent* to the pair:
$x_1 + x_3 = 4$ and $x_3 \geq 0$

Slack Variables

Example

Consider: $x_1 \leq 4$
The slack variable is defined to be: $x_3 = 4 - x_1$
which is the amount of slack in the left-hand side of the inequality.
Thus,
$x_1 + x_3 = 4$
Given this equation, $x_1 \leq 4$ if and only if $4 - x_1 - x_3 \geq 0$.
Therefore, the original constraint is entirely *equivalent* to the pair:
$x_1 + x_3 = 4$ and $x_3 \geq 0$

Topics

Augmented Form

Maximize:

$$Z = 3x_1 + 5x_2$$

subject to:

$$\begin{aligned}
x_1 \qquad\quad + x_3 \qquad\qquad\qquad &= \quad 4 \\
2x_2 \qquad\quad + x_4 \qquad\quad &= \quad 12 \\
3x_1 + 2x_2 \qquad\qquad\qquad + x_5 &= \quad 18 \\
x_j \geq 0 \quad &, \quad \text{for } j = 1, 2, 3, 4, 5.
\end{aligned}$$

Terminology for the Augmented Form

Definition

An augmented solution is a solution for the original variables (the decision variables) that has been augmented by the corresponding values of the slack variables.

Example

Augmenting the solution $(3, 2)$ in the example yields the augmented solution $(3, 2, 1, 8, 5)$.

Terminology for the Augmented Form

Definition

An augmented solution is a solution for the original variables (the decision variables) that has been augmented by the corresponding values of the slack variables.

Example

Augmenting the solution $(3, 2)$ in the example yields the augmented solution $(3, 2, 1, 8, 5)$.

Terminology for the Augmented Form

Definition

A basic solution is an augmented corner-point solution.

Example

Consider the corner-point infeasible solution $(4, 6)$ in the example. Augmenting it with the resulting values of the slack variables yields the basic solution $(4, 6, 0, 0, -6)$.

Terminology for the Augmented Form

Definition

A basic solution is an augmented corner-point solution.

Example

Consider the corner-point infeasible solution $(4, 6)$ in the example. Augmenting it with the resulting values of the slack variables yields the basic solution $(4, 6, 0, 0, -6)$.

Terminology for the Augmented Form

Definition

A basic feasible (BF) solution is an augmented CPF solution.

Example

The CPF solution $(0, 6)$ in the example is equivalent to the BF solution $(0, 6, 4, 0, 6)$ in augmented form.

Terminology for the Augmented Form

Definition

A basic feasible (BF) solution is an augmented CPF solution.

Example

The CPF solution $(0, 6)$ in the example is equivalent to the BF solution $(0, 6, 4, 0, 6)$ in augmented form.

The Augmented Form

- For the augmented form of the example, the system of functional constraints has 5 variables and 3 equations. Number of variables - Number of equations $= 5 - 3 = 2$
- This fact gives us 2 *degrees of freedom* in solving the system.
- Any two variables can be chosen to be set equal to any arbitrary value.
- Then, these three equations can be solved in terms of the remaining three variables. The simplex method uses zero for this arbitrary value.
- Thus, two of the variables (called the nonbasic variables) are set equal to zero, and then the system of equations is solved for the other three variables (called the basic variables).
- This solution is a basic solution.

The Augmented Form

- For the augmented form of the example, the system of functional constraints has 5 variables and 3 equations. Number of variables - Number of equations $= 5 - 3 = 2$
- This fact gives us 2 *degrees of freedom* in solving the system.
- Any two variables can be chosen to be set equal to any arbitrary value.
- Then, these three equations can be solved in terms of the remaining three variables. The simplex method uses zero for this arbitrary value.
- Thus, two of the variables (called the nonbasic variables) are set equal to zero, and then the system of equations is solved for the other three variables (called the basic variables).
- This solution is a basic solution.

The Augmented Form

- For the augmented form of the example, the system of functional constraints has 5 variables and 3 equations. Number of variables - Number of equations $= 5 - 3 = 2$
- This fact gives us 2 *degrees of freedom* in solving the system.
- Any two variables can be chosen to be set equal to any arbitrary value.
- Then, these three equations can be solved in terms of the remaining three variables. The simplex method uses zero for this arbitrary value.
- Thus, two of the variables (called the nonbasic variables) are set equal to zero, and then the system of equations is solved for the other three variables (called the basic variables).
- This solution is a basic solution.

The Augmented Form

- For the augmented form of the example, the system of functional constraints has 5 variables and 3 equations. Number of variables - Number of equations $= 5 - 3 = 2$
- This fact gives us 2 *degrees of freedom* in solving the system.
- Any two variables can be chosen to be set equal to any arbitrary value.
- Then, these three equations can be solved in terms of the remaining three variables. The simplex method uses zero for this arbitrary value.
- Thus, two of the variables (called the nonbasic variables) are set equal to zero, and then the system of equations is solved for the other three variables (called the basic variables).
- This solution is a basic solution.

Properties of the Basic Solution

1. Each variable is designated as either a nonbasic variable or a basic variable.
2. The number of basic variables equals the number of functional constraints.
3. The nonbasic variables are set equal to zero.
4. The values of the basic variables are obtained as the simultaneous solution of the system of equations (functional constraints in augmented form).
5. If the basic variables satisfy the nonnegativity constraint, the basic solution is a BF solution.

Properties of the Basic Solution

1. Each variable is designated as either a nonbasic variable or a basic variable.
2. The number of basic variables equals the number of functional constraints.
3. The nonbasic variables are set equal to zero.
4. The values of the basic variables are obtained as the simultaneous solution of the system of equations (functional constraints in augmented form).
5. If the basic variables satisfy the nonnegativity constraint, the basic solution is a BF solution.

Properties of the Basic Solution

1. Each variable is designated as either a nonbasic variable or a basic variable.
2. The number of basic variables equals the number of functional constraints.
3. The nonbasic variables are set equal to zero.
4. The values of the basic variables are obtained as the simultaneous solution of the system of equations (functional constraints in augmented form).
5. If the basic variables satisfy the nonnegativity constraint, the basic solution is a BF solution.

Properties of the Basic Solution

1. Each variable is designated as either a nonbasic variable or a basic variable.
2. The number of basic variables equals the number of functional constraints.
3. The nonbasic variables are set equal to zero.
4. The values of the basic variables are obtained as the simultaneous solution of the system of equations (functional constraints in augmented form).
5. If the basic variables satisfy the nonnegativity constraint, the basic solution is a BF solution.

Properties of the Basic Solution

1. Each variable is designated as either a nonbasic variable or a basic variable.
2. The number of basic variables equals the number of functional constraints.
3. The nonbasic variables are set equal to zero.
4. The values of the basic variables are obtained as the simultaneous solution of the system of equations (functional constraints in augmented form).
5. If the basic variables satisfy the nonnegativity constraint, the basic solution is a BF solution.

Example

Example

Consider the BF solution $(0, 6, 4, 0, 6)$. This solution was obtained by augmenting the CPF solution $(0, 6)$.

Another way to obtain this same solution is to choose x_1 and x_4 to be the two nonbasic variables. So,

$x_1 = x_4 = 0$

The three equations then yield, respectively:

$x_3 = 4, x_2 = 6, x_5 = 6$

Because all three of these basic variables are nonnegative, this basic solution is indeed a BF solution.

Example

Example

Consider the BF solution $(0, 6, 4, 0, 6)$. This solution was obtained by augmenting the CPF solution $(0, 6)$.
Another way to obtain this same solution is to choose x_1 and x_4 to be the two nonbasic variables. So,
$x_1 = x_4 = 0$
The three equations then yield, respectively:
$x_3 = 4, x_2 = 6, x_5 = 6$
Because all three of these basic variables are nonnegative, this basic solution is indeed a BF solution.

Adjacent BF Solutions

- Moving from the current BF solution to an adjacent one involves switching one variable from nonbasic to basic and vice versa for one other variable (and then, adjusting the values of the basic variables to continue satisfying the system of equations).

Example

Consider one pair of adjacent solutions: $(0, 0)$ and $(0, 6)$. Their augmented solutions, $(0, 0, 4, 12, 18)$ and $(0, 6, 4, 0, 6)$ automatically are adjacent BF solutions. Moving from the first to the second involves switching x_2 from nonbasic to basic and vice versa for x_4.

Adjacent BF Solutions

- Moving from the current BF solution to an adjacent one involves switching one variable from nonbasic to basic and vice versa for one other variable (and then, adjusting the values of the basic variables to continue satisfying the system of equations).

Example

Consider one pair of adjacent solutions: $(0, 0)$ and $(0, 6)$. Their augmented solutions, $(0, 0, 4, 12, 18)$ and $(0, 6, 4, 0, 6)$ automatically are adjacent BF solutions. Moving from the first to the second involves switching x_2 from nonbasic to basic and vice versa for x_4.

Objective Function in Augmented Form

Maximize: Z

subject to:

$$
\begin{array}{rcl}
Z - 3x_1 - 5x_2 & = & 0 \\
x_1 \qquad + x_3 \qquad\qquad & = & 4 \\
2x_2 \qquad + x_4 \qquad & = & 12 \\
3x_1 + 2x_2 \qquad\qquad + x_5 & = & 18 \\
x_j \geq 0 & , & \text{for } j = 1, 2, 3, 4, 5
\end{array}
$$

Objective Function in Augmented Form

- It is just as if the objective function actually were one of the original constraints.
- Since is already in equality form, no slack variable is needed.
- While adding one more equation, we also have added one more unknown (Z) to the system of equations.
- When using the other equations to obtain a basic solution, we use the new equation to solve for Z at the same time.

Geometric and Algebraic Interpretations

TABLE 4.2 Geometric and algebraic interpretations of how the simplex method solves the Wyndor Glass Co. problem

Method Sequence	Geometric Interpretation	Algebraic Interpretation
Initialization	Choose (0, 0) to be the initial CPF solution.	Choose x_1 and x_2 to be the nonbasic variables (= 0) for the initial BF solution: (0, 0, 4, 12, 18).
Optimality test	Not optimal, because moving along either edge from (0, 0) increases Z.	Not optimal, because increasing either nonbasic variable (x_1 or x_2) increases Z.
Iteration 1		
Step 1	Move up the edge lying on the x_2 axis.	Increase x_2 while adjusting other variable values to satisfy the system of equations.
Step 2	Stop when the first new constraint boundary ($2x_2 = 12$) is reached.	Stop when the first basic variable (x_3, x_4, or x_5) drops to zero (x_4).
Step 3	Find the intersection of the new pair of constraint boundaries: (0, 6) is the new CPF solution.	With x_2 now a basic variable and x_4 now a nonbasic variable, solve the system of equations: (0, 6, 4, 0, 6) is the new BF solution.
Optimality test	Not optimal, because moving along the edge from (0, 6) to the right increases Z.	Not optimal, because increasing one nonbasic variable (x_1) increases Z.

Geometric and Algebraic Interpretations

Iteration 2		
Step 1	Move along this edge to the right.	Increase x_1 while adjusting other variable values to satisfy the system of equations.
Step 2	Stop when the first new constraint boundary ($3x_1 + 2x_2 = 18$) is reached.	Stop when the first basic variable (x_2, x_3, or x_5) drops to zero (x_5).
Step 3	Find the intersection of the new pair of constraint boundaries: (2, 6) is the new CPF solution.	With x_1 now a basic variable and x_5 now a nonbasic variable, solve the system of equations: (2, 6, 2, 0, 0) is the new BF solution.
Optimality test	(2, 6) is optimal, because moving along either edge from (2, 6) decreases Z.	(2, 6, 2, 0, 0) is optimal, because increasing either nonbasic variable (x_4 or x_5) decreases Z.

Topics

Initial System of Equations

TABLE 4.3 Initial system of equations for the Wyndor Glass Co. problem

(a) Algebraic Form	(b) Tabular Form								
	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
(0) $\boldsymbol{Z} - 3x_1 - 5x_2 = 0$	Z	(0)	1	−3	−5	0	0	0	0
(1) $x_1 + \boldsymbol{x_3} = 4$	x_3	(1)	0	1	0	1	0	0	4
(2) $2x_2 + \boldsymbol{x_4} = 12$	x_4	(2)	0	0	2	0	1	0	12
(3) $3x_1 + 2x_2 + \boldsymbol{x_5} = 18$	x_5	(3)	0	3	2	0	0	1	18

Example

TABLE 4.4 Applying the minimum ratio test to determine the first leaving basic variable for the Wyndor Glass Co. problem

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side	Ratio
Z	(0)	1	−3	−5	0	0	0	0	
x_3	(1)	0	1	$\boxed{0}$	1	0	0	4	
x_4	(2)	0	0	$\boxed{2}$	0	1	0	12	$12 \rightarrow \frac{12}{2} = 6 \leftarrow$ minimum
x_5	(3)	0	3	$\boxed{2}$	0	0	1	18	$18 \rightarrow \frac{18}{2} = 9$

Example

TABLE 4.5 **Simplex tableaux for the Wyndor Glass Co. problem after the first pivot row is divided by the first pivot number**

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1						
	x_3	(1)	0						
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0						

Topics

Example

TABLE 4.6 First two simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6

Example

TABLE 4.7 Steps 1 and 2 of iteration 2 for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side	Ratio
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30	
	x_3	(1)	0	1	0	1	0	0	4	$\frac{4}{1} = 4$
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6	
	x_5	(3)	0	3	0	0	−1	1	6	$\frac{6}{3} = 2 \leftarrow$ minimum

Example

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	-3	-5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	-3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	-1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Topics

Tie for the Entering Basic Variable

- Step 1 of each iteration chooses the nonbasic variable having the negative coefficient with the largest absolute value in the current Equation (0), as the entering basic variable.
- What if two or more nonbasic variables are tied?
- For example: $Z = 3x_1 + 3x_2$
- The selection between the contenders may be made arbitrarily.
- The optimal solution will be reached eventually.
- There is no convenient method for predicting in advance which choice will lead to the optimal solution sooner.

Tie for the Entering Basic Variable

- Step 1 of each iteration chooses the nonbasic variable having the negative coefficient with the largest absolute value in the current Equation (0), as the entering basic variable.
- What if two or more nonbasic variables are tied?
- For example: $Z = 3x_1 + 3x_2$
- The selection between the contenders may be made arbitrarily.
- The optimal solution will be reached eventually.
- There is no convenient method for predicting in advance which choice will lead to the optimal solution sooner.

Tie for the Entering Basic Variable

- Step 1 of each iteration chooses the nonbasic variable having the negative coefficient with the largest absolute value in the current Equation (0), as the entering basic variable.
- What if two or more nonbasic variables are tied?
- For example: $Z = 3x_1 + 3x_2$
- The selection between the contenders may be made arbitrarily.
- The optimal solution will be reached eventually.
- There is no convenient method for predicting in advance which choice will lead to the optimal solution sooner.

Topics

Tie for the Leaving Basic Variable

- Suppose that two or more basic variables tie for being the leaving basic variable in step 2.
- Does it matter which one is chosen?
- Theoretically it does.
  - The basic variables not chosen to be the leaving basic variable also will have a zero value in the new BF solution.
  - These basic variables and the corresponding BF solution are called degenerate.
  - Loops may occur, etc.
- Just break the tie arbitrarily!

Tie for the Leaving Basic Variable

- Suppose that two or more basic variables tie for being the leaving basic variable in step 2.
- Does it matter which one is chosen?
- Theoretically it does.
  - The basic variables not chosen to be the leaving basic variable also will have a zero value in the new BF solution.
  - These basic variables and the corresponding BF solution are called degenerate.
  - Loops may occur, etc.
- Just break the tie arbitrarily!

Tie for the Leaving Basic Variable

- Suppose that two or more basic variables tie for being the leaving basic variable in step 2.
- Does it matter which one is chosen?
- Theoretically it does.
  - The basic variables not chosen to be the leaving basic variable also will have a zero value in the new BF solution.
  - These basic variables and the corresponding BF solution are called degenerate.
  - Loops may occur, etc.
- Just break the tie arbitrarily!

Topics

No Leaving Basic Variable - Unbounded Z

- Assume no variable qualifies to be the leaving basic variable.
- This would occur if the entering basic variable could be increased indefinitely without giving negative values to any of the current basic variables.
- In tabular form, every coefficient in the pivot column (excluding row 0) is either negative or zero.
- The simplex method would stop with the message that Z is unbounded.

No Leaving Basic Variable - Unbounded Z

- Assume no variable qualifies to be the leaving basic variable.
- This would occur if the entering basic variable could be increased indefinitely without giving negative values to any of the current basic variables.
- In tabular form, every coefficient in the pivot column (excluding row 0) is either negative or zero.
- The simplex method would stop with the message that Z is unbounded.

No Leaving Basic Variable - Unbounded Z

- Assume no variable qualifies to be the leaving basic variable.
- This would occur if the entering basic variable could be increased indefinitely without giving negative values to any of the current basic variables.
- In tabular form, every coefficient in the pivot column (excluding row 0) is either negative or zero.
- The simplex method would stop with the message that Z is unbounded.

Example

TABLE 4.9 Initial simplex tableau for the Wyndor Glass Co. problem without the last two functional constraints

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	Right Side	Ratio
Z	(0)	1	−3	−5	0	0	
x_3	(1)	0	1	$\boxed{0}$	1	4	None

With $x_1 = 0$ and x_2 increasing,
$x_3 = 4 - 1x_1 - 0x_2 = 4 > 0$.

Topics

Multiple Optimal Solutions

- The simplex method automatically stops after one optimal BF solution is found.
- In some cases, other optimal solutions should be identified as well.

Detecting other optimal solutions

Whenever a problem has more than one optimal BF solution, at least one of the nonbasic variables has a coefficient of zero in the final row 0. So, increasing any such variable will not change the value of Z.

Therefore, these other optimal BF solutions can be identified (if desired) by performing additional iterations of the simplex method, each time choosing a nonbasic variable with a zero coefficient as the entering basic variable.

Multiple Optimal Solutions

- The simplex method automatically stops after one optimal BF solution is found.
- In some cases, other optimal solutions should be identified as well.

Detecting other optimal solutions

Whenever a problem has more than one optimal BF solution, at least one of the nonbasic variables has a coefficient of zero in the final row 0. So, increasing any such variable will not change the value of Z.
Therefore, these other optimal BF solutions can be identified (if desired) by performing additional iterations of the simplex method, each time choosing a nonbasic variable with a zero coefficient as the entering basic variable.

Example

Example

- Consider when the objective function is changed to: $Z = 3x_1 + 2x_2$
- The simplex method obtains the first three tableaux and stops with an optimal solution.
- However, a nonbasic variable (x_3) then has a zero coefficient in row 0.
- So, we perform one more iteration to identify the other optimal BF solution.

Example

TABLE 4.10 Complete set of simplex tableaux to obtain all optimal BF solutions for the Wyndor Glass Co. problem with $c_2 = 2$

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side	Solution Optimal?
0	Z	(0)	1	−3	−2	0	0	0	0	No
	x_3	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	2	0	1	0	12	
	x_5	(3)	0	3	2	0	0	1	18	
1	Z	(0)	1	0	−2	3	0	0	12	No
	x_1	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	2	0	1	0	12	
	x_5	(3)	0	0	2	−3	0	1	6	
2	Z	(0)	1	0	0	**0**	0	1	18	Yes
	x_1	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	0	3	1	−1	6	
	x_2	(3)	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3	
Extra	Z	(0)	1	0	0	0	**0**	1	18	Yes
	x_1	(1)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
	x_3	(2)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	
	x_2	(3)	0	0	1	0	$\frac{1}{2}$	0	6	

References

Hillier&Lieberman

- Chapter 4: Solving Linear Programming Problems: The Simplex Method
  4.1 through 4.5