PHY411-AST233 Lecture notes

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Contents

1 Infinitesimal Transformations and Lie Groups

1.1 Lie Groups

A Lie Group is a group that is also a finite dimensional differentiable manifold. The group operations are smooth functions. A continuous group is a group where *continuity* is imposed on the elements of the group in the sense that a small change in one of the factors of a product produces a correspondingly small change in their product.

Lie algebra operates on the tangent space of the Lie group at the identity element and this algebra captures the local structure of the Lie Group. Elements of the Lie algebra are elements of the group that are "infinitesimally close" to the identity, and the Lie bracket of the Lie algebra is related to the commutator of two such infinitesimal elements.

An r parameter group on an n dimensional space $(\mathbf{x})$ gives transformations

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, \mathbf{a})$$

where $\mathbf{a}$ is the r parameters. We desire that for $\mathbf{a}$ a vector of zeros, we have the identity

$$\mathbf{x} = f(\mathbf{x}, 0, 0, 0....)$$

Infinitesimal generators can be computed as

$$\mathbf{L}_i = \left.\frac{\partial \mathbf{f}}{\partial a_i}\right|_{a_i=0}$$

where evaluate at $a_j = 0$ for $j \neq i$. The above f is a matrix or an operator and $\mathbf{L}$ has the dimension of $\mathbf{f}$. Taking into account that $\mathbf{f}$ gives a vector we can also write this as

$$L_i = \left.\frac{\partial \mathbf{f}_j}{\partial a_i}\right|_{a_i=0} \frac{\partial}{\partial x_j} \tag{1}$$

(using summation notation for j) where $\frac{\partial}{\partial x_j}$ gives a basis for the tangent space, and these can be used to operate on any function.

Above we have written L_i as a differential operator. It operates on a function of $\mathbf{x}$. With $g(\mathbf{x})$ a function, then $L_i g(\mathbf{x})$ is a number so $L_i g(\mathbf{x})$ is a function of $\mathbf{x}$. Using two different differential operators L_i, L_j we can operate sequentially on a function $L_i(L_j g(\mathbf{x}))$. Let us consider two vector operators and write them more simply as

$$A = a_i \frac{\partial}{\partial x_i}$$

$$B = b_j \frac{\partial}{\partial x_j}$$

where a_i, b_i are functions of $\mathbf{x}$ and we have used summation notation. Our generators L_i, L_j are in this form.

$$\begin{aligned} ABg &= a_i \frac{\partial}{\partial x_i} b_j \frac{\partial}{\partial x_j} g \\ &= a_i b_j \frac{\partial^2 g}{\partial x_i x_j} + a_i \frac{\partial b_j}{\partial x_i} \frac{\partial g}{\partial x_j} \\ AB &= a_i b_j \frac{\partial^2}{\partial x_i \partial x_j} + a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} \end{aligned}$$

If we take the commutator

$$\begin{aligned} [A, B] &= a_i b_j \frac{\partial^2}{\partial x_i \partial x_j} + a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} - a_i b_j \frac{\partial^2}{\partial x_i \partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \frac{\partial}{\partial x_i} \\ &= \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} \end{aligned}$$

We notice that the second derivatives cancel. That means that using the commutator our infinitesimal generators can operate on each other and give us another generator.

The infinitesimal operators form a Lie algebra with commutators (called Lie brackets) that satisfy

$$[L_i, L_j] = c_{ij}^k L_k$$

with c_{ij}^k called structure constants. The commutator is a binary operator that satisfies the Jacobi identity, is antisymmetric and is bilinear w.r.t to multiplication with elements of a field (typically real or complex numbers).

On a matrix Lie group, we can define an exponential map for any operator in the Lie algebra, giving us a map between elements of the Lie algebra and the Lie Group. For u in the Lie algebra

$$e^{\epsilon u} = 1 + \epsilon u + \frac{\epsilon^2}{2} u^2 + \frac{\epsilon^3}{3!} u^3 ....$$

and this gives an element in the Lie group. In this sense, elements of the Lie algebra are infinitesimal generators for the group.

$$e^u e^v = (1 + u + \frac{u^2}{2} ....)(1 + v + \frac{v^2}{2} ....)$$

$$e^{(u+v)} = 1 + u + v + \frac{1}{2}(u+v)^2 .... = 1 + u + v + \frac{1}{2}(u^2 + v^2 + uv + vu) ....$$

We notice that

$$e^u e^v \neq e^{u+v}$$

Locally

$$e^u e^v = \exp\left(u + v + \frac{1}{2}[u, v] + \frac{1}{12}[[u, v], v] - \frac{1}{12}[[u, v], u] ... \right)$$

known as the Baker-Campbell-Hausdorff formula.

Consider two near identity operators A, B and alternating the order that the transformations are done (see Figure 1 and 7). To second order

$$\begin{aligned}
e^{-A} e^{-B} e^{A} e^{B} &\approx \left(I - A + \frac{A^2}{2} \right) \left(I - B + \frac{B^2}{2} \right) \left(I + A + \frac{A^2}{2} \right) \left(I + B + \frac{B^2}{2} \right) \\
&= I - A - B + A + B + \frac{A^2}{2} + \frac{A^2}{2} + \frac{B^2}{2} + \frac{B^2}{2} + AB - AB - BA + AB - A^2 - B^2 \\
&= \mathbf{I} + [A, B] \\
&\sim e^{[A,B]}
\end{aligned}$$

The commutator helps us compute what happens when infinitesimal transformation don't commute.

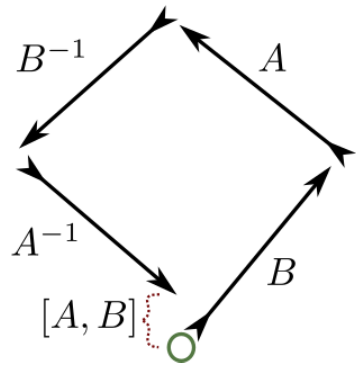

Figure 1: Alternating infinitesimal transformations need not commute.

1.2 Rotation Example

Rotation matrices in 3 dimensions form a Lie group. We define the special orthogonal group SO(3) as

$$A \in \mathbb{R}^3, \text{such that } \det A = 1, \quad A^T A = AA^T = I$$

Matrices A operate on $\mathbf{x}$ a 3D Cartesian coordinate. Rotation by angle ϕ about the z axis

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Near the identity

$$\left.\frac{dR(\phi)}{d\phi}\right|_{\phi\to 0} = \left.\begin{pmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}\right|_{\phi=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

is the generator for $R(\phi)$. Expanding $R(\phi)$ for small ϕ (to first order in ϕ)

$$R(\phi) \sim \begin{pmatrix} 1 & -\phi & 0 \\ \phi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} + \phi\hat{\mathbf{L}}_z \tag{4}$$

with

$$\hat{\mathbf{L}}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

a matrix describing the infinitesimal rotation about the z axis. We can similarly find

$$\hat{\mathbf{L}}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\hat{\mathbf{L}}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

for rotations about x and y axes. For a small rotation ϕ about z

$$\mathbf{x}' = (\mathbf{I} + \phi\hat{\mathbf{L}}_z)\mathbf{x}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \left[\mathbf{I} + \phi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - \phi y \\ y + \phi x \\ z \end{pmatrix}$$

Suppose we have a function

$$F(x, y, z)$$

and we transform it for a small ϕ.

$$F(x', y', z') = F(x - y\phi, y + x\phi, z) = F(x, y, z) - y\frac{\partial F}{\partial x}\phi + x\frac{\partial F}{\partial y}\phi$$

The change in F

$$\Delta F = \left[-y\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial y}\right]\phi$$

We associate the operator

$$L_z = -y\partial_x + x\partial_y \tag{5}$$

with our infinitesimal transformation. And this is a vector corresponding to the z component of angular momentum. So if the coordinate system is rotated by a small amount ϕ about the z axis we can compute the change to any function of the coordinates with

$$\Delta F = L_z F \phi$$

where L_z is the differential operator as defined in equation 5

Now let us find the operator using equation 1. Our function

$$\begin{aligned} \mathbf{f}(x, y, z) &= \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R(\phi) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \\ z \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathbf{f}}{\partial \phi} &= \begin{pmatrix} -x\sin\phi - y\cos\phi \\ x\cos\phi - y\sin\phi \\ 0 \end{pmatrix} \\
\left.\frac{\partial \mathbf{f}}{\partial \phi}\right|_{\phi=0} &= \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
L_z &= \left.\frac{\partial f_x}{\partial \phi}\right|_{\phi=0} \partial_x + \left.\frac{\partial f_y}{\partial \phi}\right|_{\phi=0} \partial_y \\
&= -y\partial_x + x\partial_y
\end{aligned}
$$

consistent with what we have above in equation 5.

This example illustrates two way of describing the infinitesimal operators. One way with matrices. The other way with differential operators. In both cases we can think of the infinitesimal operators as lying in the tangent space of the manifold. These operators also describe the local structure of the Lie group of symmetry transformations.

1.3 Example: Locomotion of Deformable Jello Cubes

Locomotion strategies of small biological organisms might inspire strategies for achieving locomotion in small artificial mechanisms. Self-propulsion forces can arise from the mechanical dissipative interactions of the locomotor with a surrounding fluid, or with a solid substrate. The most common motility modes at microscopic scales are crawling and swimming. Series of motions of minimal complexity that can give net displacements often involve periodic shape changes. Discrete-mass crawling strategies require either many degrees of freedom (Zimmerman et al., 2004, 2007, 2009; Bolotnik et al., 2011), anisotropic friction coefficients (Zimmerman et al., 2004), or achieve motion by alternately sticking and sliding (Chernous'ko, 2002). Locomotion at low Reynolds number is not possible if the body can only deform with one degree of freedom and this has to do with time reversal symmetry (Purcell 1977).

There are a bunch of possible mechanical mechanisms that give locomotion with a single degree of freedom but these involve coupling motion mechanically and asymmetry in the body motion is achieved through the coupling. Another way to do this is have asymmetric friction coefficients on two masses and in this case motion is achieved with non-sinusoidal oscillations in time (Wagner and Lauga 2013). For the Zhou paper, the asymmetry is caused by angle of the spring between two legs. For systems with a single degree of freedom, we cannot study it with infinitesimal transformations! Only for the two degree of freedom system with non-commuting motions, are the infinitesimal transformations relevant?

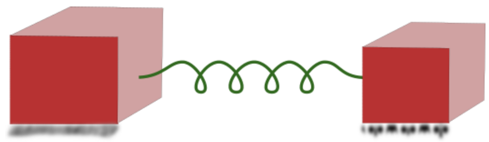

Figure 2: Two masses separated by an actuator that can change its length. If the two masses have bases with different frictional coefficients and the actuator's length varies periodically but not sinusoidally, then the scallop can inch forward. A model explored by Wagner and Lauga 2013.

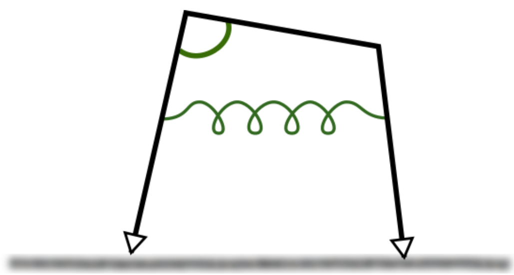

Figure 3: The green spring represents an actuator and the green arc an angular spring. By varying the length of the actuator the horse can be made to jump forwards. Based on the model by Zhou et al. 2014.

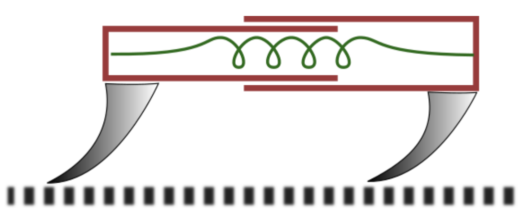

Figure 4: A bristle crawler by Noselli and Desimone. The two legs are flexible and their angle changes with respect to the rough surface.

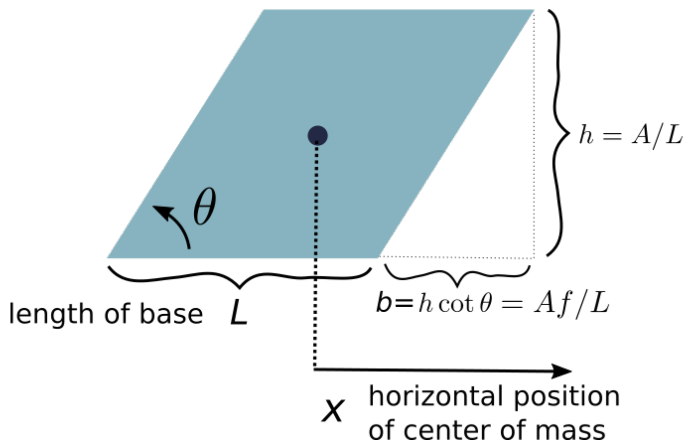

Figure 5: Variables describing body state for the jello parallelogram.

1.4 The Jello cube!

As an example, we consider a jello cube that allows deformations while maintaining its density. We assume the body is incompressible but easy to deform. We consider two types of deformations, shear and stretching the cube vertically, see Figure 6. Each correspond to a linear deformation gradient and no body rotation. We place the jello cube on a flat surface and allow some friction between the jello cube and the surface. We allow transformations only in x, y so the system is effectively two-dimensional. The direction x is along the flat surface and y is vertical height above the surface. When under no stress or strain, the jello is a cube with length L_0 and area $A = L_0^2$. We also allow body translations in the x direction.

When sheared or stretched the jello surface deforms from a cube into a parallelogram. The configuration space for our jello cube has three dimensions, one for the x coordinate of the center of mass, one for the angle of the lower leftmost corner of the parallelogram, and the ratio of length of its sides. We describe configuration spaces with, the x position of the jello's center of mass, and we refer to this as $\bar{x}$, the length of the base L and $f = \cot\theta$ with θ the angle between two edges (see Figure 6).

Stretches we describe with the transformation D_α

$$\begin{aligned}
&D_\alpha : \\
&\quad L' = L(1+\alpha) \\
&\quad x' = x \\
&\quad f' = f
\end{aligned} \tag{6}$$

The linear transformation of the body itself is chosen to maintain the parallelogram angles. We assume the transformation does not shift the center of mass. With friction on the base, this is equivalent to assuming that the transformation is done quickly enough that friction at the base does not significantly move the center of mass.

Slow shear transformations we describe with

$$\begin{aligned} S_\beta : & & \\ L' &= L \\ x' &= x + \frac{\beta A}{2L} \\ f' &= f + \beta \end{aligned} \tag{7}$$

Here $A = L_0^2$ is the area of the parallelogram face. The length coordinate is preserved but the angle θ changes. We assume that the transformation is done slowly enough that the base does not shift during the deformation and as a consequence the transformation causes a shift in the the horizontal location of the center of mass.

Translations are described with

$$\begin{aligned} T_c : & & \\ L' &= L \\ x' &= x + c \\ f' &= f \end{aligned} \tag{8}$$

With $\alpha = 0$, $\beta = 0$ and $c = 0$ we have identity transformations.

Let total configuration space Q is described with coordinates

$$\mathbf{q} = (x, L, f)$$

and our array of parameters for coordinate transformations

$$\mathbf{a} = (c, \alpha, \beta)$$

Our three transformations together generate a Lie group of transformations

$$\mathbf{q}' = \mathbf{G}(\mathbf{q}, \mathbf{a})$$

Our transformation functions (the three expressions 6,7, 8) are elements in the group $\mathbf{G}$. Infinitesimal generators (as operators) or elements in the Lie algebra of G at point q can be computed using

$$L_i = \sum_j \left. \frac{\partial G_j}{\partial a_i} \right|_{a_i = 0} \frac{\partial}{\partial q_j}$$

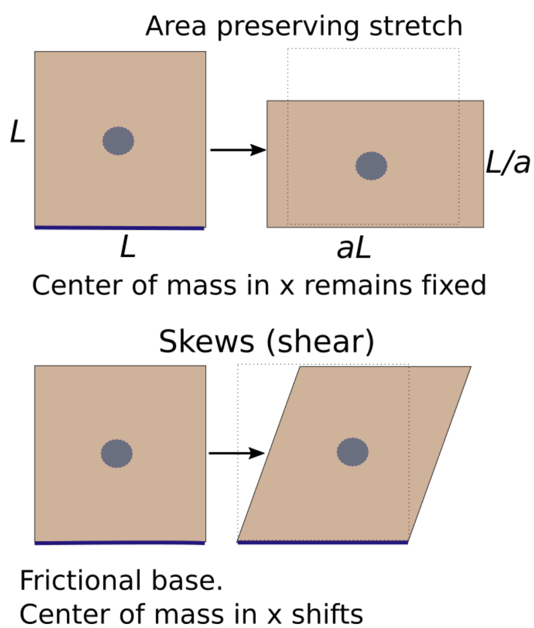

Figure 6: Volume preserving deformations of a jello cube with a frictional base. We assume that the center of mass stays fixed when the jello is stretched vertically. We assume that the base is held fixed by friction when the jello is is sheared.

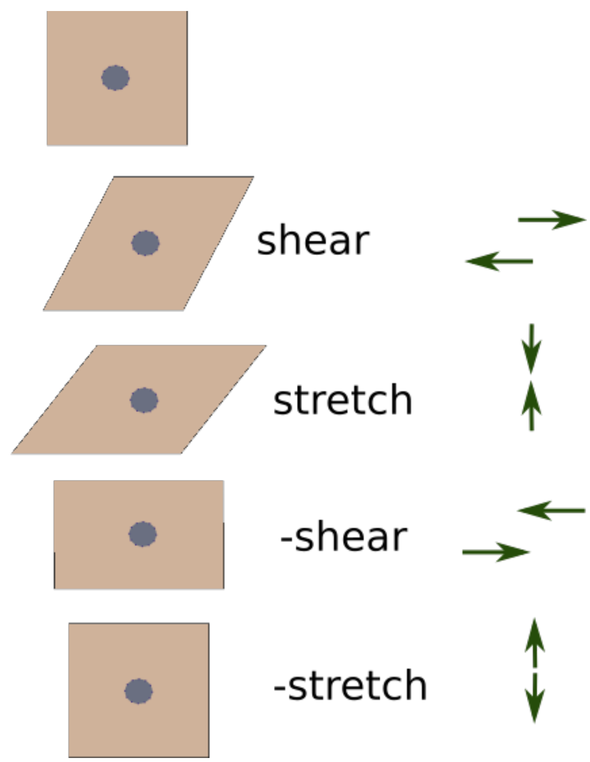

Figure 7: The deformations don't commute and their consecutive application translates the center of mass. A slow shear followed by a stretch, followed by the opposite slow shear and the opposite stretch causes a translation. The jello cube marches forward.

We compute the derivatives

$$\left.\frac{\partial D_\alpha(\mathbf{q})}{\partial \alpha}\right|_{\alpha=0} = \begin{pmatrix} L \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

$$\left.\frac{\partial S_\beta(\mathbf{q})}{\partial \beta}\right|_{\beta=0} = \begin{pmatrix} 0 \\ \frac{A}{2L} \\ 1 \end{pmatrix} \tag{10}$$

$$\left.\frac{\partial T_c(\mathbf{q})}{\partial c}\right|_{c=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{11}$$

For the stretches, shears and translations, respectively, the infinitesimal operators at position $q = (x, L, f)$ are

$$L_D = L\partial_L \tag{12}$$

$$L_S = \frac{A}{2L}\partial_x + \partial_f \tag{13}$$

$$L_T = \partial_x \tag{14}$$

The space of translations along x is a sub-algebra. The infinitesimal operators are also vectors in the tangent space of the manifold Q at point q. From the infinitesimal operators we compute the commutator

$$[L_D, L_S] = L_D L_S - L_S L_D = -\frac{A}{2L}\partial_x = -\frac{A}{2L} L_T \tag{15}$$

The operators don't commute. The commutator gives translation of the center of mass caused by a series of body transformations. If the operations are done in the order stretch, shear, inverse stretch, inverse shear, our jello parallelogram should march forward in the horizontal direction.

The other commutators

$$[L_D, L_T] = [L_S, L_T] = 0$$

The Lie algebra is known as the three-dimensional Heisenberg algebra and it has Bianchi classification type II.

1.5 Mechanical Connection

We can think of configuration space as a principal bundle with full space $Q = H \times M$ and H, the fibre that is also the group of translations of the center of mass, and M the manifold of body shapes ($\mathbb{R}^2$ with coordinates (L, f)). A principal bundle also requires a projection map $\pi : Q \to M$ that is probably $(x, L, f) \to (L, f)$.

Body motions are constrained by friction to lie in a horizontal subspace of TQ that is spanned by the two vectors L_D, L_S.

$$L_D = L\partial_L \qquad L_S = \frac{A}{2L}\partial_x + \partial_f \tag{16}$$

from equation 14. Locally the tangent space of Q can be decomposed into horizontal and vertical subspaces.

A mechanical connection, Γ is a 1 form (operating on a vector in the tangent space of Q) that is also a function of elements in the Lie algebra of H. The mechanical connection sends to zero (projects away) anything in the horizontal subspace. The horizontal subspace is that of the body deformation. The vertical subspace is that associated with body transformation. A mechanical connection is a one form that gives us elements in the Lie algebra of H, does not change vertical vectors in Q and is zero when applied to vectors in the horizontal subspace (here spanned by equation 16). We require that the connection Γ applied to L_T give us $L_T = \partial_x$ and Γ applied to L_D and L_S give us zero. The mechanical connection that achieves this is

$$\Gamma = \left(dx - df\frac{A}{2L}\right)\partial_x \tag{17}$$

We can check this with

$$\begin{aligned}
\Gamma L_T &= \left(dx - df\frac{A}{2L}\right)\partial_x \qquad \textbf{on} \qquad \partial_x \\
&= \left(dx\partial_x - df\partial x\frac{A}{2L}\right)\partial_x \\
&= (1-0)\,\partial_x \\
&= \partial_x
\end{aligned}$$

$$\begin{aligned}
\Gamma L_D &= \left(dx - df\frac{A}{2L}\right)\partial_x \qquad \textbf{on} \qquad L\partial_L \\
&= -(df\frac{A}{2L}L\partial_L)\partial_x \\
&= 0
\end{aligned}$$

$$\begin{aligned}\Gamma L_S &= \left(dx - df\frac{A}{2L}\right)\partial_x \qquad \textbf{on} \qquad \frac{A}{2L}\partial_x + \partial_f \\ &= \left(\frac{A}{2L} - \frac{A}{2L}\right)\partial_x \\ &= 0\end{aligned}$$

Changes in the jello cube body space happen along curves that have tangents specified by L_S and L_D. This means they lie in a plane spanned by vectors that are sent to zero by Γ. A curve that remains in the horizontal space can return to the same f, L coordinates. But integrating around this loop gives

$$\Delta x = \oint \Gamma ds$$

Using Stokes's law we can write the same integral as

$$\Delta x = \int d\Gamma dA$$

where $d\Gamma$ is a curvature.

The curvature of the connection is

$$\Omega(u, v) = d\Gamma(hu, hv)$$

where hu is the horizontal lift of vector u and hv are the horizontal lifts of vector $u, v \in T_qQ$. Here h essential projects a vector in T_qQ to the tangent space of M (horizontal projection!).

The curvature can be computed from the exterior derivative of the connection. Integrating the connection about a loop is equivalent to integrating the curvature over the area inside the loop.

With a trajectory given in the horizontal space, the connection can be integrated to find out how big a change takes place in the vertical space. A loop in the horizontal space may not close in the vertical space. Integrating the curvature over the area within the loop in the horizontal space also gives the change that takes place in the vertical space.

More specifically, we write our connection again, (equation 17)

$$\Gamma = \left(dx - df\frac{A}{2L}\right)\partial_x \tag{18}$$

We take the exterior derivative of Γ

$$d\Gamma = \left[-(\frac{A}{2L} + 1)dx \wedge df + dL \wedge dx + \frac{A}{2L^2}dL \wedge df\right] \tag{19}$$

Restricting to the horizontal subspace for L, f we get a curvature

$$\Omega = \frac{A}{2L^2}dL \wedge df$$

We consider a loop in L, f space and use Stokes theorem

$$\oint \Gamma ds = \int_A d\Gamma = \int_A \Omega$$

Integrating the connection over the loop is equivalent to integrating the curvature over the area inside the loop.

1.6 A fibre bundle and a Connection

A fibre bundle is a space Q that is locally a product of a base space B and a fibre. A fibre bundle must have a projection map $\pi : Q \to B$ that shows how to take a point in Q and project it to the base space. The inverse of the map $\pi^{-1}(b)$ gives points in the fibre that project to a point $b \in B$. The fibre has a structure group G giving homeomorphisms of the fibre. A *principal bundle* has fibres homeomorphic to the structure group G. A *vector bundle* has fibres that are homeomorphic to a vector space. There is also a condition on coverings of the base space B by open sets.

For our jello cube, the fibre is the background space giving the center of mass position. The base space describes the body stretches and tilts.

An *Ehresman* connection A is a vector valued one form on Q.

$$A_q : T_q Q \to V_q$$

for $q \in Q$ and vertical space V_q the fibre, here and x value. The map is linear and gives a projection

$$A(v_q) = v_q$$

for all $v_q \in V_q$. If we locally define coordinates $q^i = (r^\alpha, s^a)$ for base and fiber then

$$A = \omega^a \frac{\partial}{\partial s^a} \qquad \text{for} \qquad \omega^a(q) = ds^a + A^a_\alpha(r, s) ds^\alpha.$$

It is common to divide the tangent space $T_q Q$ into horizontal and vertical spaces, where the horizontal space is projected to zero by the connection and the vertical space is not changed by the connection.

The mechanical connection above Γ is in the form of A, but it is not affine as it is not linear in L.

With point $q \in Q$ we can project it into the base space $r = \pi(q) \in B$. We now consider a vector in the base space at point r, or $v_r \in T_r B$. We can find a unique vector in the tangent space of the whole space $T_q Q$ which we call $v_r^h \in T_q Q$, that projects to v_r when projected by π. We call this new vector the *horizontal lift* of vector $v_r \in T_r B$. We will use the connection to determine v_r^h. Suppose we take any vector $X \in T_q Q$ and compute

$$\mathrm{hor} X_q = X_q - A(q) X_q$$

The connection removes the part that is in the tangent space of the base. What is left is the part that is in the tangent space of the fibre.

2 Rigid Bodies

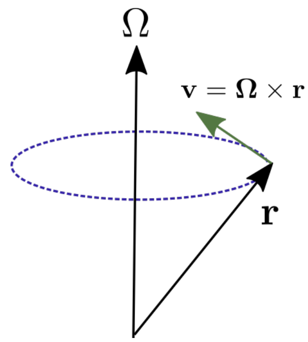

Figure 8: Velocity of a particle at position $\mathbf{r}$ if spinning about an axis Ω.

We take our rigid body to be a distribution of particles each with coordinate $\mathbf{r}$ with respect to the center of mass of the body. We let the body spin about an axis $\mathbf{\Omega}$. The velocity of a single particle with mass m_i, position $\mathbf{r}_i$ and velocity $\mathbf{v}_i$

$$\mathbf{v}_i = \mathbf{\Omega} \times \mathbf{r}_i$$

The angular momentum of this particle (measured from the origin) is

$$\mathbf{L}_i = m_i \mathbf{r}_i \times \mathbf{v}_i = m_i \mathbf{r}_i \times (\mathbf{\Omega} \times \mathbf{r}_i)$$

We recall a vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

we can write the angular momentum of the particle

$$\mathbf{L}_i = m_i \left(r_i^2 \mathbf{\Omega} - (\mathbf{r}_i \cdot \mathbf{\Omega})\mathbf{r}_i \right)$$

Integrating this up over the entire body the entire angular momentum

$$\mathbf{L} = \int \rho(\mathbf{r}) \left(r^2 \mathbf{\Omega} - (\mathbf{r} \cdot \mathbf{\Omega})\mathbf{r} \right) dV$$

Using summation notation (and now the indices represent coordinate directions, not a single particle)

$$\begin{aligned} L_j &= \int \rho(\mathbf{r}) \left(x_i x_i \Omega_j - x_j x_k \Omega_k \right) dV \\ &= \int \rho(\mathbf{r} \left(x_i x_i \delta_{jk} - x_j x_k \right) dV \ \Omega_k \\ &= I_{jk} \Omega_k \end{aligned}$$

with moment of Inertia tensor

$$I_{jk} \equiv \int \rho(\mathbf{r})\,(x_i x_i \delta_{jk} - x_j x_k)\,dV$$

Question Suppose we have a particle distribution and pick a spin value and axis $\mathbf{\Omega}$. Each particle has coordinate $\mathbf{x}_i$ with respect to the center of mass of the body. We set the initial velocity $\mathbf{v}_i$ of each particle i at position $\mathbf{x}_i$ in the distribution equal to

$$\mathbf{v}_i = \mathbf{\Omega} \times \mathbf{x}_i$$

(see Figure 8). This question is relevant for generating initial conditions for spinning body simulations.

Q: Is the initial angular momentum parallel to $\mathbf{\Omega}$?

A. The answer is usually no. Angular momentum for a particle is $\mathbf{x}_i \times \mathbf{v}_i = \mathbf{x}_i \times (\mathbf{\Omega} \times \mathbf{x}_i)$ and so in general contains a component that is perpendicular to $\mathbf{\Omega}$. After summing over all particles the parts perpendicular to $\mathbf{\Omega}$ do not cancel unless the particle distribution has a lot of symmetry.

Q: How do you set the velocity vector of each particle to ensure that the total angular momentum is a chosen vector $\mathbf{L}$?

A. Compute the moment of inertia tensor from the particle distribution and then compute its inverse. Compute $\mathbf{\Omega} = \mathbf{I}^{-1}\mathbf{L}$. Use this $\mathbf{\Omega}$ to set the particle velocities. Because $\mathbf{I}$ is not necessarily diagonal and $\mathbf{L}$ not necessarily along a principal axis, $\mathbf{L}$ need not be parallel to $\mathbf{\Omega}$.

Rigid body rotation is confusing because two coordinate frames are involved, the inertial frame and the body frame. The motion of the rigid body is observed in the inertial frame whereas it is simpler to calculate the equations of motion in the body-fixed principal axis frame. The body rotes about the angular velocity vector $\mathbf{\Omega}$ but this is not in the same direction as the angular momentum vector $\mathbf{L}$. Euler's equations are given in the body-fixed frame in which the moment of inertial tensor is diagonal and is constant. The solution then must be rotated back into the inertial frame to describe the rotational motion as seen by an external observer.

2.1 Configuration space

A rigid body can be viewed as a group of point masses all constrained so that their positions with respect to each other does not vary. In a center of mass frame the only thing that varies is the **orientation** of the body. The **configuration space** is an element of the rotation group or SO(3). We can think of rigid body motion as a trajectory in the Lie group SO(3) or a trajectory in the group of rotations. If we describe rotations with a rotation matrix

R the trajectory of a rigid body (w.r.t to its center of mass) is $\mathbf{R}(t)$. Neglecting external forces the total energy is equal to the kinetic energy. A Lagrangian description depends on the kinetic energy with coordinate describing the body orientation at a particular time and time derivative depending on the time derivative of body orientation which is equivalent to its spin.

2.2 Generators of Rotation Matrices

The position of a particle $\mathbf{x}$ in a rotating frame is

$$\mathbf{x} = \mathbf{R}\mathbf{x}_0$$

where $\mathbf{R}$ is a rotation matrix relating two reference frames. We can take $\mathbf{x}_0$ to be in the body's frame and $\mathbf{x}$ to be in an inertial frame.

Using the infinitesimal generators from the Lie algebra of the rotation group (SO(3))

$$\mathbf{R} = e^{\boldsymbol{\alpha}\cdot\mathbf{l}} = e^{\alpha_i \mathbf{l}_i} \tag{20}$$

where $\mathbf{l}$ is comprised of three matrices, $\mathbf{l}_x, \mathbf{l}_y, \mathbf{l}_z$. The vector $\boldsymbol{\alpha}$ gives three components (numbers) or angles, one for each of the matrices $\mathbf{l}_x, \mathbf{l}_y, \mathbf{l}_z$. The matrices $\mathbf{l}_x, \mathbf{l}_y, \mathbf{l}_z$ are the infinitesimal generators. I am using lower case $\mathbf{l}$ so as not to be confused with angular momentum.

$$\begin{aligned}
\mathbf{l}_x &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
\mathbf{l}_y &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
\mathbf{l}_z &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{21}$$

It may be convenient for to write the indices for the matrix $\mathbf{l}_i$

$$\mathbf{l}_{i,\alpha\beta} = -\epsilon_{i\alpha\beta} \tag{22}$$

where i, α, β are coordinate indices. Let us check this

$$\begin{aligned}
(\mathbf{l}_x\mathbf{x})_\alpha &= -\epsilon_{x\alpha\beta} x_\beta = (0, -z, y) \\
(\mathbf{l}_y\mathbf{x})_\alpha &= -\epsilon_{y\alpha\beta} x_\beta = (z, 0, -x) \\
(\mathbf{l}_z\mathbf{x})_\alpha &= -\epsilon_{z\alpha\beta} x_\beta = (-y, x, 0)
\end{aligned}$$

and these are consistent with equation 21. More generally $[\mathbf{l}_k\mathbf{x}]$ has α component

$$[\mathbf{l}_k\mathbf{x}]_\alpha = -\epsilon_{k\alpha\beta}x_\beta \tag{23}$$

We use these generators to compute the velocity of a particle in the rotating frame

$$\begin{aligned}
\dot{\mathbf{x}} &= \frac{d}{dt}(\mathbf{R}\mathbf{x}_0) = \dot{\mathbf{R}}\mathbf{x}_0 \\
&= \frac{d}{dt}e^{\alpha_i \mathbf{l}_i}\mathbf{x}_0 = \frac{d}{dt}(\alpha_i\mathbf{l}_i)e^{\alpha_j\mathbf{l}_j}\mathbf{x}_0 \\
&= (\dot{\boldsymbol{\alpha}}\cdot\mathbf{l})\mathbf{R}\mathbf{x}_0 \\
\mathbf{v} &= \Omega_i\mathbf{l}_i\mathbf{x} = (\boldsymbol{\Omega}\cdot\mathbf{l})\mathbf{x}
\end{aligned}$$

where we associate $\dot{\boldsymbol{\alpha}}$ with $\boldsymbol{\Omega}$. Writing out the dot product (using equation 23)

$$\Omega_i\mathbf{l}_i\mathbf{x} = (\boldsymbol{\Omega}\cdot\mathbf{l})\mathbf{x} = \begin{pmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{pmatrix}\mathbf{x} = \boldsymbol{\Omega}\times\mathbf{x}$$

Using slightly different notation

$$\begin{aligned}
\partial_T(\mathbf{R}\mathbf{x}_0) &= \partial_T(e^{\boldsymbol{\alpha}_i\mathbf{l}_i}\mathbf{x}_0) \\
&= \Omega_i\mathbf{l}_i e^{\boldsymbol{\alpha}_j\mathbf{l}_j}\mathbf{x}_0 \\
&= \Omega_i\mathbf{l}_i\mathbf{R}\mathbf{x}_0 \\
&= \Omega_i\mathbf{l}_i\mathbf{x}
\end{aligned}$$

with $\mathbf{x} = \mathbf{R}\mathbf{x}_0$ and $\partial_T = \frac{\partial}{\partial t}$. Note that $\mathbf{l}_i\mathbf{x}$ has components

$$(\mathbf{l}_i\mathbf{x})_u = -\epsilon_{iuv}x_v$$

so

$$\partial_T(\mathbf{R}\mathbf{x}_0)_u = -\Omega_i\epsilon_{iuv}x_v \tag{24}$$

or

$$\partial_T(\mathbf{R}\mathbf{x}_0) = \boldsymbol{\Omega}\times\mathbf{x} = \boldsymbol{\Omega}\times\mathbf{R}\mathbf{x}_0 \tag{25}$$

as expected.

We took a time derivative of coordinate $\mathbf{x}$ but we could have done this for any static vector. Suppose $\mathbf{L}_0$ is a static vector.

$$\mathbf{L} = \mathbf{R}\mathbf{L}_0$$

If the vector $\mathbf{L}_0$ in one frame is constant then

$$\dot{\mathbf{L}} = \dot{\mathbf{R}}\mathbf{L}_0 = \mathbf{\Omega} \times \mathbf{L}$$

where $\dot{L} = \frac{dL}{dt}$. If our vector $\mathbf{L}_0$ is not constant then we would get

$$\dot{\mathbf{L}} = \dot{\mathbf{R}}\mathbf{L}_0 + \mathbf{R}\dot{\mathbf{L}}_0 = \mathbf{\Omega} \times \mathbf{L} + \partial_t \mathbf{L} \tag{26}$$

where $\partial_t = \frac{\partial}{\partial t}$ which is not the same as $\frac{d}{dt}$. This looks like an advective derivative. It also looks a lot like Euler's equations! In fact if we chose L_0 to be in a frame fixed with the body (for example aligned with the body's principal axes) then $\partial_t \mathbf{L}_0 = \mathbf{I}\dot{\mathbf{\Omega}}$. This follows as $\mathbf{I}$ is constant in this frame. Here $\dot{\mathbf{L}}$ is the torque and we find

$$I\dot{\mathbf{\Omega}} + \mathbf{\Omega} \times (\mathbf{I}\mathbf{\Omega}) = \mathbf{N} \tag{27}$$

with $\mathbf{N}$ the applied torque. This gives Euler's equations.

We take $\mathbf{L}_0$ in a frame aligned with a body's principal axes where the the moment of inertia matrix is $\mathbf{I} = \text{diag}(I_x, I_y, I_z)$,

$$\begin{aligned} L_{0,x} &= I_x \Omega_x \\ L_{0,y} &= I_y \Omega_y \\ L_{0,z} &= I_z \Omega_z \end{aligned}$$

Equation 27 is

$$\begin{aligned} I_x \dot{\Omega}_x + (I_z - I_y)\Omega_y \Omega_z &= N_x \\ I_y \dot{\Omega}_y + (I_x - I_z)\Omega_z \Omega_x &= N_y \\ I_z \dot{\Omega}_z + (I_x - I_y)\Omega_y \Omega_x &= N_z \end{aligned}$$

2.3 The kinetic energy

The kinetic energy depends on the square of the velocity

$$\begin{aligned} T &= \int \rho \frac{dV}{2}(\partial_T(R\mathbf{x})) = \int \rho \frac{dV}{2}(\dot{R}\mathbf{x})^2 \\ &= \int \rho \frac{dV}{2}((\Omega_i \cdot \mathbf{l}_i)\mathbf{x})^2 \\ &= \int \rho \frac{dV}{2}(\mathbf{\Omega} \times \mathbf{y})^2 \end{aligned}$$

where $\mathbf{x}$ is in the body frame and $\mathbf{y} = \mathbf{R}\mathbf{x}$. Writing this out in index form

$$\begin{aligned} (\mathbf{\Omega} \times \mathbf{y})^2 &= \epsilon_{ijk}\Omega_j y_k \epsilon_{ilm}\Omega_l y_m \\ &= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\Omega_j y_k \Omega_l y_m \\ &= \mathbf{\Omega}^2 \mathbf{y}^2 - (\mathbf{\Omega} \cdot \mathbf{y})^2 \\ &= \Omega_i \delta_{ik} y_j y_j \Omega_k - \Omega_i y_i y_k \Omega_k \\ &= \Omega_i (\mathbf{y}^2 \delta_{ik} - y_j y_k)\Omega_k \end{aligned}$$

Sticking this back into the integral we recognize that

$$T = \frac{1}{2}\mathbf{\Omega I \Omega}$$

where spin $\mathbf{\Omega}$ is a vector but $\mathbf{I}$ is the moment of inertia tensor.

2.4 Example for the Moment of Inertia tensor

An axisymmetric body has moment of inertia tensor with two eigenvalues that are the same. If the body is prolate (like a cigar or football) then principal moments $I_1 = I_2 > I_3$. If the body is oblate (like a pancake or a disk) then $I_1 > I_2 = I_3$.

We compute the moment of inertia tensor for a dumbbell, comprised of two unequal masses m_1, m_2, separated rigidly by a distance a on the x axis. We ignore the mass of the bar holding the masses apart. In a coordinate system with the center of mass at the origin, the coordinate positions are

$$\begin{aligned} x_1 &= a\frac{m_2}{m_1+m_2} \\ x_2 &= -a\frac{m_1}{m_1+m_2} \end{aligned}$$

We compute the moment of inertia tensor

$$\begin{aligned} I_{xx} &= m_1(y_1^2 + z_1^2) + m_2(y_2^2 + z_2^2) = 0 \\ I_{yy} &= m_1(x_1^2 + z_1^2) + m_2(x_2^2 + z_2^2) = m_1 x_1^2 + m_2 x_2^2 \\ &= m_1 a^2 \frac{m_2^2}{(m_1+m_2)^2} + m_2 a^2 \frac{m_1^2}{(m_1+m_2)^2} = \frac{m_1 m_2}{m_1+m_2} a^2 \end{aligned}$$

By symmetry $I_{zz} = I_{yy}$. Using the reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$

$$I_{yy} = I_{zz} = \mu a^2$$

and is independent of anything but the reduced mass.

Useful to know for modeling lopsided rapidly spinning satellites.

2.5 Stability of a Freely Rotating Body

Starting with the Euler equations

$$\frac{\partial \mathbf{L}}{\partial t} + \boldsymbol{\omega} \times \mathbf{L} = 0$$

In a coordinate system aligned with principle axes of the body

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \end{aligned} \tag{28}$$

where I_1, I_2, I_3 are the moments of inertia about the principal axes (eigenvalues of moment of inertia matrix) and $\omega_1, \omega_2, \omega_3$ are the components of the spin vector in the principle axes directions (eigenvector directions of the moment of inertia matrix). We can label the axes so that $I_1 > I_2 > I_3$ and start the body rotating nearly about direction $\hat{x}_2$ with

$$\boldsymbol{\omega} = \alpha_1 \hat{x}_1 + s\hat{x}_2 + \alpha_3 \hat{x}_3$$

with s large and $\alpha_1, \alpha_2 \gg 1$ very small perturbations. We put this into Euler's equations (equations 28)

$$\begin{aligned}
\dot{\omega}_1 &= \frac{(I_2 - I_3)}{I_1} s\alpha_3 \\
\dot{\omega}_2 &= \frac{(I_3 - I_1)}{I_2} \alpha_1\alpha_3 \sim 0 \\
\dot{\omega}_3 &= \frac{(I_1 - I_2)}{I_3} s\alpha_1
\end{aligned}$$

The middle equation is second order in small quantities so $\dot{\omega}_2 \sim 0$. If we allow α_1, α_2 to be functions of time

$$\begin{aligned}
\dot{\alpha}_1 &= \frac{(I_2 - I_3)}{I_1} s\alpha_3 \\
\dot{\alpha}_3 &= \frac{(I_1 - I_2)}{I_3} s\alpha_1
\end{aligned}$$

and putting this together

$$\begin{aligned}
\ddot{\alpha}_1 &= \frac{(I_2 - I_3)}{I_1} \frac{(I_1 - I_2)}{I_3} s^2\alpha_1 \\
\ddot{\alpha}_3 &= \frac{(I_2 - I_3)}{I_1} \frac{(I_1 - I_2)}{I_3} s^2\alpha_3
\end{aligned}$$

If the factor on the right is positive then the solutions are exponential growing or shrinking. If the factor on the right is negative then the solutions are oscillating. If the quantity

$$(I_2 - I_3)(I_1 - I_2)$$

is positive then we can have exponentially growing values for ω_1, ω_2. Since $I_1 > I_2 > I_3$ this quantity is positive and the system is unstable and will quickly start to spin about a different direction than started.

Redoing this analysis starting with spin about $\hat{x}_1$ the relevant quantity that must be negative for stability is $(I_1 - I_3)(I_2 - I_1)$ and this is negative with our order for the moments. Redoing the analysis starting with spin about $\hat{x}_3$ the relevant quantity that must be negative for stability is $(I_3 - I_1)(I_2 - I_3)$, again negative with our ordering.

The body can stably spin about the axis of smallest moment or about the axis of the largest moment but not the intermediate one.

2.6 Rigid Body motion from a Lagrangian

We notice that the kinetic energy is in fact a Lagrangian density

$$\mathcal{L} = T = \int \rho \frac{dV}{2} (\partial_T (\mathbf{Rx}))^2 = \frac{1}{2} \mathbf{\Omega I \Omega} \tag{29}$$

where the kinetic energy is integrated over the body volume. Again $\mathbf{x}$ is in the body frame and $\mathbf{Rx}$ is in the inertial frame. The Lagrangian itself is integrated over a path

$$L = \int_{t_1}^{t_2} \mathcal{L}(t) dt$$

with path $\mathbf{R}(t)$ specified by how the rotation varies in time. The rotation matrices are functions of angles, $\boldsymbol{\alpha}$ that specify body orientation. Each rotation is a point inside SO(3). The body rotations so $\boldsymbol{\alpha}(t)$, these angles are functions of time. The time derivatives of these angles are $\dot{\boldsymbol{\alpha}} = \mathbf{\Omega}$ a spin vector. The spin vector is a member of the tangent space of a point in SO(3). Our Lagrangian density is a function

$$\mathcal{L}(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) = \mathcal{L}(\boldsymbol{\alpha}, \mathbf{\Omega}).$$

Taking the angles and their time derivatives we can compute Lagrange's equations.

$$\frac{\partial \mathcal{L}}{\partial \mathbf{\Omega}} = \mathbf{I\Omega} \tag{30}$$

which is the angular momentum. Using equation 29

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \mathbf{\Omega}} = \dot{\mathbf{L}} = \mathbf{I}\dot{\mathbf{\Omega}} \tag{31}$$

Now for the hard part, taking the derivative of the Lagrangian with respect to the angles for the rotation (the coordinate in SO(3)). Using equation 20 to write the rotation matrix in terms of $\boldsymbol{\alpha}$

$$\begin{aligned} \frac{\partial}{\partial \alpha_k} \mathbf{Rx} &= \frac{\partial}{\partial \alpha_k} e^{\boldsymbol{\alpha}_j \mathbf{l}_j} \mathbf{x} \\ &= \mathbf{l}_k e^{\boldsymbol{\alpha}_j \mathbf{l}_j} \mathbf{x} \\ &= \mathbf{l}_k \mathbf{Rx} \\ &= \mathbf{l}_k \mathbf{y} \end{aligned} \tag{32}$$

with $\mathbf{y} = \mathbf{Rx}$. Using equation 23 components

$$\begin{aligned} \left[\frac{\partial}{\partial \alpha_k} \mathbf{Rx} \right]_\beta &= \left[\frac{\partial}{\partial \alpha_k} \mathbf{y} \right]_\beta \\ &= -\epsilon_{k\beta\gamma} y_\gamma \end{aligned} \tag{33}$$

so if we dot this with another vector $\mathbf{z}$

$$\begin{aligned}
\mathbf{z} \cdot \frac{\partial \mathbf{y}}{\partial \boldsymbol{\alpha}} &= -\epsilon_{k\beta\gamma} y_\gamma z_\beta \\
&= \mathbf{y} \times \mathbf{z}
\end{aligned}$$

Recall that the velocity

$$\begin{aligned}
\mathbf{v} &= \partial_T (\mathbf{R}\mathbf{x}) \\
&= \boldsymbol{\Omega} \times \mathbf{R}\mathbf{x} \\
&= \boldsymbol{\Omega} \times \mathbf{y}
\end{aligned}$$

The kinetic energy at a particular position $\mathbf{x}$

$$\begin{aligned}
\frac{1}{2} \left[\partial_T (\mathbf{R}\mathbf{x})\right]^2 &= \frac{1}{2} (\boldsymbol{\Omega} \times \mathbf{y})^2 \\
&= \frac{1}{2} \left(\Omega^2 y^2 - (\boldsymbol{\Omega} \cdot \mathbf{y})^2\right)
\end{aligned}$$

Taking the derivative with respect to $\boldsymbol{\alpha}$

$$\begin{aligned}
\frac{\partial}{\partial \alpha_k} \frac{\mathbf{v}^2}{2} &= \Omega^2 \mathbf{y} \cdot (\mathbf{l}_k \mathbf{y}) - (\boldsymbol{\Omega} \cdot \mathbf{y}) \Omega_i \mathbf{l}_k y_i \\
\frac{\partial}{\partial \boldsymbol{\alpha}} \frac{\mathbf{v}^2}{2} &= \Omega^2 \mathbf{y} \times \mathbf{y} - (\boldsymbol{\Omega} \cdot \mathbf{y})(\mathbf{y} \times \boldsymbol{\Omega}) \\
&= (\boldsymbol{\Omega} \cdot \mathbf{y})(\boldsymbol{\Omega} \times \mathbf{y}) \\
&= (\boldsymbol{\Omega} \cdot \mathbf{y}) \mathbf{v} \qquad (34)
\end{aligned}$$

Compare this to an expression for angular momentum at a single location

$$\begin{aligned}
\boldsymbol{\Omega} \times \mathbf{L}(\mathbf{y}) &= \boldsymbol{\Omega} \times (\mathbf{y} \times \mathbf{v}) \\
&= (\boldsymbol{\Omega} \cdot \mathbf{v}) \mathbf{y} - (\boldsymbol{\Omega} \cdot \mathbf{y}) \mathbf{v} \\
&= -(\boldsymbol{\Omega} \cdot \mathbf{y}) \mathbf{v} \qquad (35)
\end{aligned}$$

where I used a vector identity and

$$\boldsymbol{\Omega} \cdot \mathbf{v} = \boldsymbol{\Omega} \cdot (\boldsymbol{\Omega} \times \mathbf{y}) = 0.$$

Above I have computed things for a single location $\mathbf{x}$ in the body.

Putting equation 34, 35 together and integrating over the volume we get

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{\alpha}} \frac{1}{2} \left[\partial_T (\mathbf{R}\mathbf{x})\right]^2 &= -\boldsymbol{\Omega} \times \mathbf{L}(\mathbf{y}) \\
\frac{\partial}{\partial \boldsymbol{\alpha}} \int dV \frac{1}{2} \left[\partial_T (\mathbf{R}\mathbf{x})\right]^2 &= -\boldsymbol{\Omega} \times \mathbf{L}
\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\alpha}} = -\boldsymbol{\Omega} \times \mathbf{L} \tag{36}$$

This and equation 31 with Lagrange's equations give

$$\dot{\mathbf{L}} + \boldsymbol{\Omega} \times \mathbf{L} = 0.$$

Using the moment of inertia tensor in a frame fixed with the body we get Euler's equations.

$$I\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbf{I}\boldsymbol{\Omega} = 0.$$

2.7 A Hamiltonian, a moment map

Looking at equation 30 or $\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\alpha}}} = \mathbf{L}$ we can construct a Hamiltonian from the Lagrangian using the angular momentum vector as a set of canonical momentum variables. More abstractly the action

$$a_M(q, p) = (aq, ap)$$

where $a \in$SO(3) is a 3x3 matrix. We create a symplectic 2-form with $\omega = d\theta$ where

$$d\theta = \sum_i p_i dq_i = \langle p, dq \rangle$$

The action preserves the symplectic structure

$$a_M \theta = \langle ap, d(aq) \rangle = \langle ap, adq \rangle = \langle p, dq \rangle = \theta$$

The *moment map*

$$\mu_A(q, p) = \langle p, Aq \rangle$$

where A is in the Lie algebra of SO(3). For SO(3) there is a nice bijective correspondence between elements of the Lie algebra and spin axes $\mathbf{A} \to \boldsymbol{\Omega}$ with

$$\mathbf{A}\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{v}$$

identifying elements of the Lie algebra with spin vectors.

The moment map is essentially the Hamiltonian. This construction could be used for other Lie groups. The moment map can be used to generate geodesics and a notion of curvature.

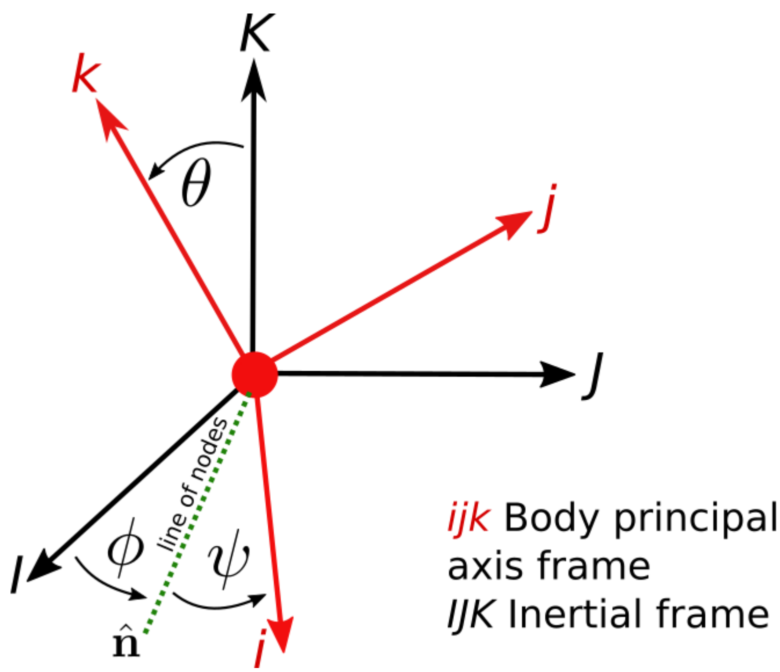

Figure 9: **Euler angles**. The green line is the intersection of the $\mathbf{i}, \mathbf{j}$ and $\mathbf{I}, \mathbf{J}$ planes and is called a line of nodes with direction $\hat{\mathbf{n}}$. It is computed from the vector $\mathbf{K} \times \mathbf{k}$. The angle θ is a nutation angle. The angle ϕ is a spin precession angle (as seen from the external or fixed reference frame). ψ is sometimes called a precession angle as seen from the body. $\dot{\psi}$ rotates the $\mathbf{ij}$ axes in the $\mathbf{ij}$ plane with respect to a fixed line of nodes. $\dot{\phi}$ rotates the line of nodes $\hat{\mathbf{n}}$.

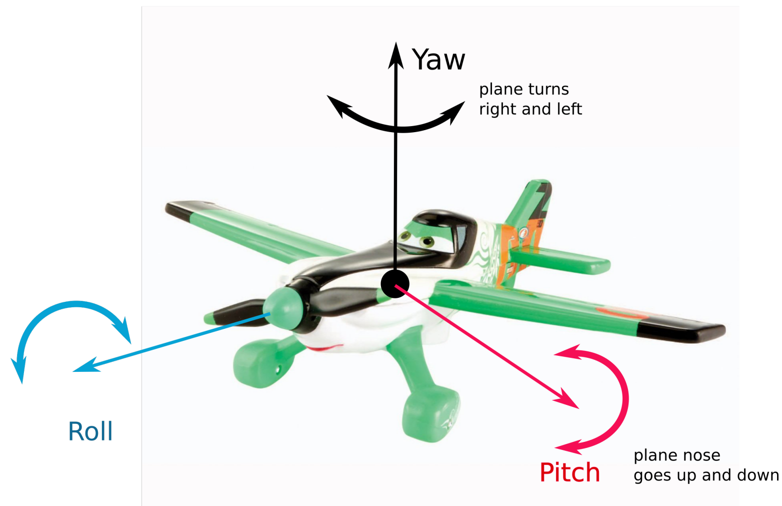

Figure 10: A toy plane! Practical uses for Euler angles include study of plane stability.

3 Rotational Dynamics

3.1 Euler angles

Following Celleti's book:

We start with a fixed point O and a reference frame $\mathbf{i}, \mathbf{j}, \mathbf{k}$ that coincides with a body's principle axes $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$. Now we chose an inertial frame with the same origin and three different axes $(O, \mathbf{I}, \mathbf{J}, \mathbf{K})$. Let $\mathbf{n}$ be the line of nodes defined by the intersection of the $\mathbf{i}, \mathbf{j}$ and $\mathbf{I}, \mathbf{J}$ planes. The direction

$$\hat{\mathbf{n}} = \frac{\mathbf{K} \times \mathbf{k}}{|\mathbf{K} \times \mathbf{k}|}$$

The Euler angles θ, ϕ, ψ are as follows:

- $0 \leq \theta \leq \pi$ is the *nutation* angle and is that between $\mathbf{K}$ and $\mathbf{k}$ axes.
- $0 \leq \psi \leq 2\pi$ is the *proper rotation* angle and is that between $\hat{\mathbf{n}}$ and $\mathbf{i}$ axis.
- $0 \leq \phi \leq 2\pi$ is the *precession* angle and is that between $\hat{\mathbf{n}}$ and $\mathbf{I}$ axis.

The angle ψ is also sometimes called a precession angle as it can describe the precession of the spin and angular momentum vectors in the body's frame.

We need to relate variations in the Euler angles to the spin in the body frame.

$$\begin{aligned}
\omega_i &= \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\
\omega_j &= \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\
\omega_k &= \dot{\phi} \cos\theta + \dot{\psi}
\end{aligned} \tag{37}$$

The three rotations described below allow us to derive these expressions.

Looking at figure 9 we examine the angles. With θ, ϕ constant $\dot{\psi}$ gives a rotation of the $\mathbf{i}, \mathbf{j}$ plane with respect to the line of nodes. The body is essentially spinning around the $\mathbf{k}$ body axis. The angle ϕ moves the line of nodes with respect to the inertial frame. If the body spin is oriented along $\mathbf{k}$ we can look at its projection into the $\mathbf{I}, \mathbf{J}$ plane. If the projected spin axis advances we say the body spin precesses. Taking $\mathbf{k}$ to be the spin axis, the angle between it and the $\mathbf{K}$ axis we refer to as a nutation angle, θ.

- The angle θ is that between $\mathbf{k}$ (body principal axis) and $\mathbf{K}$ (inertial frame axis).

  $$\cos\theta = \mathbf{k} \cdot \mathbf{K}.$$

- The angle ϕ is that between line of nodes $\hat{\mathbf{n}}$ and inertial axis $\mathbf{I}$

  $$\cos\phi = \mathbf{I} \cdot \hat{\mathbf{n}}.$$

- The angle ψ is that between the body principal axis $\mathbf{i}$ and the line of nodes $\hat{\mathbf{n}}$

  $$\cos\psi = \mathbf{i} \cdot \hat{\mathbf{n}}.$$

3.1.1 Rotations

Starting in the inertial frame (xyz), three rotations transform into the body frame ($x'y'z'$).

1. A counter-clockwise rotation by ϕ about K or the current z axis.

2. A counter-clockwise rotation by θ about the current x axis (here the x''' axis).

3. A counter-clockwise rotation by ψ about the current z axis (here the z' or z''' axis). First rotate by the precession angle ϕ then tilt by the nutation angle θ then rotate by ψ.

We start with xyz in inertial frame, rotate by ϕ giving x'', y'', z'', rotate by θ giving x''', y''', z''' then rotate by ψ giving x', y', z' in body frame.

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix}$$

Angular rotation rates arise from time derivatives of each rotation;

$$\begin{aligned} \omega_\phi &= \dot{\phi}\hat{\mathbf{z}} \\ \omega_\theta &= \dot{\theta}\hat{\mathbf{x}}'' \\ \omega_\psi &= \dot{\psi}\hat{\mathbf{z}}' \end{aligned} \tag{38}$$

To evaluate these we need to recall

$$\begin{aligned} \hat{\mathbf{z}} &= \sin\psi\sin\theta\hat{\mathbf{x}}' + \cos\psi\sin\theta\hat{\mathbf{y}}' + \cos\theta\hat{\mathbf{z}'} \\ \hat{\mathbf{x}}'' &= \cos\psi\hat{\mathbf{x}}' - \sin\psi\hat{\mathbf{y}}' \end{aligned} \tag{39}$$

Combining these (equations 38 and 39) together we get equations 37 for the body frame $(\mathbf{x}', \mathbf{y}', \mathbf{z}')$ repeated here

$$\begin{aligned} \omega_i &= \omega_{x'} = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_j &= \omega_{y'} = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_k &= \omega_{z'} = \dot{\phi}\cos\theta + \dot{\psi} \end{aligned} \tag{40}$$

All together the three rotations give this transformation

$$
\begin{aligned}
x' &= (\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta)x + (\cos\psi\sin\phi + \sin\psi\cos\phi\cos\theta)y + \sin\psi\sin\theta z \\
y' &= -(\sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta)x + (-\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta)y + \cos\psi\sin\theta z \\
z' &= \sin\theta\sin\phi x - \sin\theta\cos\phi y + \cos\theta z,
\end{aligned}
\tag{41}
$$

and the inverse transformation is

$$
\begin{aligned}
x &= (\cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta)x' - (\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta)y' + \sin\phi\sin\theta z' \\
y &= (\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta)x' + (-\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta)y' - \cos\phi\sin\theta z' \\
z &= \sin\theta\sin\psi x' + \sin\theta\cos\psi y' + \cos\theta z'.
\end{aligned}
\tag{42}
$$

- The angle θ gives the angle between z, z' axes.
- Projecting inertial z axis into the body x', y' plane gives ψ, with

$$\psi = \text{atan2}(\mathbf{z}\cdot\mathbf{x}', \mathbf{z}\cdot\mathbf{y}') = \text{atan2}(\mathbf{K}\cdot\mathbf{i}, \mathbf{K}\cdot\mathbf{j}).$$

- Likewise projecting body z' axis into the inertial x, y plane gives ϕ, with

$$\phi = \text{atan2}(\mathbf{z}'\cdot\mathbf{x}, -\mathbf{z}'\cdot\mathbf{y}) = \text{atan2}(\mathbf{k}\cdot\mathbf{I}, -\mathbf{k}\cdot\mathbf{J}).$$

We use the function **atan2** so that the quadrant for the angle is specified.

3.2 Lagrangian and Hamiltonian in terms of Euler angles

The Lagrangian in terms of the Euler angles

$$
\begin{aligned}
\mathcal{L}(\dot\theta, \dot\phi, \dot\psi, \theta, \phi, \psi) &= \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_1^3\right) - V(\theta, \phi, \psi) \\
&= \frac{1}{2}I_1(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)^2 + \frac{1}{2}I_2(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)^2 + \\
&\quad \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 - V(\theta, \phi, \psi)
\end{aligned}
$$

and this simplifies if two of the moments are the same.

Taking derivatives of the Lagrangian (with respect to $\dot\phi, \dot\theta, \dot\psi$) we can compute canonical momenta;

$$
\begin{aligned}
p_\phi &= (I_1\omega_1\sin\psi + I_2\omega_2\cos\psi)\sin\theta + I_3\omega_3\cos\theta \\
p_\theta &= I_1\omega_1\cos\psi - I_2\omega_2\sin\psi \\
p_\psi &= I_3\omega_3
\end{aligned}
\tag{43}
$$

or

$$
\begin{aligned}
p_\phi &= \dot\phi\left[(I_1\sin^2\psi + I_2\cos^2\psi)\sin^2\theta + I_3\cos^2\theta\right] \\
&\quad +\dot\theta(I_1 - I_2)\cos\psi\sin\psi\sin\theta \\
&\quad +\dot\psi I_3\cos\theta \\
p_\theta &= \dot\phi(I_1 - I_2)\sin\psi\cos\psi\sin\theta \\
&\quad +\dot\theta(I_1\cos^2\psi + I_2\sin^2\psi) \\
p_\psi &= I_3(\dot\phi\cos\theta + \dot\psi)
\end{aligned} \tag{44}
$$

With these canonical momenta we can construct a Hamiltonian from the kinetic energy term and using Euler angles and their associated momenta as canonical variables

$$H(\phi, \theta, \psi; p_\phi, p_\theta, p_\psi)$$

With some manipulation of equations 43

$$
\begin{aligned}
I_1\omega_1 &= \frac{(p_\phi - p_\psi\cos\theta)}{\sin\theta}\sin\psi + p_\theta\cos\psi \\
I_2\omega_2 &= \frac{(p_\phi - p_\psi\cos\theta)}{\sin\theta}\cos\psi - p_\theta\sin\psi \\
I_3\omega_3 &= p_\psi
\end{aligned}
$$

giving us formula for $\omega_1, \omega_2, \omega_3$ in terms of our canonical momenta and angles only. These are also the angular momentum vector in the body frame. We can replace the spins $\omega_1, \omega_2, \omega_3$ in the Lagrangian with expressions that depend only on momenta $(p_\theta, p_\phi, p_\psi)$ and angles giving our Hamiltonian;

$$
\begin{aligned}
H(\phi, \theta, \psi; p_\phi, p_\theta, p_\psi) &= \frac{1}{2I_1}\left[\frac{(p_\phi - p_\psi\cos\theta)}{\sin\theta}\sin\psi + p_\theta\cos\psi\right]^2 + \\
&\quad \frac{1}{2I_2}\left[\frac{(p_\phi - p_\psi\cos\theta)}{\sin\theta}\cos\psi - p_\theta\sin\psi\right]^2 + \\
&\quad \frac{1}{2I_3}p_\psi^2
\end{aligned}
$$

With $I_1 = I_2$ this simplifies. We note that the choice of angles and momenta is poor when $\theta \sim 0$. In that setting it is a good idea to change coordinate system via canonical transformation.

3.2.1 Axisymmetric bodies

For axisymmetric systems $(I_1 = I_2)$ the Lagrangian is

$$\mathcal{L}_{sym}(\dot\theta, \dot\phi, \dot\psi, \theta, \phi, \psi) = \frac{I_1}{2}\left((\dot\phi)^2\sin^2\theta + (\dot\theta)^2\right) + \frac{I_3}{2}(\dot\phi\cos\theta + \dot\psi)^2 - V$$

and the Hamiltonian is

$$H_{sym}(\phi,\theta,\psi;p_\phi,p_\theta,p_\psi) = \frac{1}{2I_1}\left[\frac{(p_\phi - p_\psi\cos\theta)^2}{\sin^2\theta} + p_\theta^2\right] + \frac{1}{2I_3}p_\psi^2 + V \tag{45}$$

With V independent of ϕ,ψ, the Hamiltonian is only dependent on a single angle θ so momenta p_ϕ, p_ψ are conserved quantities as is H itself.

In the axisymmetric setting

$$\begin{aligned} p_\phi &= (I_1\sin^2\theta + I_3\cos^2\theta)\dot\phi + I_3\dot\psi\cos\theta &(46)\\ p_\psi &= I_3(\dot\phi\cos\theta + \dot\psi) &(47)\\ p_\theta &= I_1\dot\theta &(48)\end{aligned}$$

and as long as the potential is independent of ϕ,ψ, these momenta p_ϕ, p_ψ remain conserved. It may be useful to write precession rates in terms of p_ϕ, p_ψ

$$\begin{aligned} \dot\phi &= \frac{(p_\phi - p_\psi\cos\theta)}{I_1\sin^2\theta} &(49)\\ \dot\psi &= \frac{p_\psi}{I_3} - \dot\phi\cos\theta = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi\cos\theta)}{I_1\sin^2\theta}\cos\theta &(50)\end{aligned}$$

With θ fixed we get constant precession rates, $\dot\phi,\dot\psi$. The conserved quantity p_ψ is the angular momentum projection along the body-fixed principal axis, whereas p_ϕ is the angular momentum about the z inertial axis. This is probably understood from the last equation for z in the transformation 42 and applying it to the angular momentum as seen in the body frame.

What is not obvious that we should be able to find solutions with fixed θ in all axisymmetric situations. This is equivalent to choosing an orientation for using the Euler angles – in other words choosing an orientation for the inertial coordinate system.

For torque free rotation we can *choose* our inertial frame to be with z aligned with the angular momentum so $\mathbf{L} = L\hat{\mathbf{z}}$ that is fixed in the inertial frame. Using our transformation in equations 41 we find that in the body frame the angular momentum is

$$\mathbf{L}_{body} = L(\sin\psi\sin\theta\hat{\mathbf{x}}' + \cos\psi\sin\theta\hat{\mathbf{y}}' + \cos\theta\hat{\mathbf{z}}')$$

and using our relations for $\omega_1,\omega_2,\omega_3$ (equations 37)

$$\begin{aligned} L\sin\psi\sin\theta &= I_1\omega_1 = I_1(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)\\ L\cos\psi\sin\theta &= I_1\omega_2 = I_1(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)\\ L\cos\theta &= I_3\omega_3 = I_3(\dot\phi\cos\theta + \dot\psi) &(51)\end{aligned}$$

Taking the first two equations

$$\begin{aligned} L\sin\theta(\sin\psi\cos\psi - \cos\psi\sin\psi) &= 0\\ &= I_1\omega_1\cos\psi - I_1\omega_2\sin\phi = p_\theta\\ &= I_1\dot\theta\end{aligned}$$

So our choice of coordinate orientation, assuming an axisymmetric body and a constant angular momentum gave us $\dot{\theta} = 0$.

3.3 Torque free rotation of the axi-symmetric body

In Figure 9 we have two reference frames, an inertial one and a body one. Let us orient the inertial one with the angular momentum $\mathbf{L}$ because lacking any external torques, it is fixed. We consider a rigid axi-symmetric body. In a coordinate frame aligned with body's axis of symmetry the moment of inertia matrix has 3 eigenvalues $I_\parallel, I_\perp, I_\perp$ and two of the them are the same. If the axis of symmetry lies along the z axis then $I_\parallel = I_z$. The body is spinning but with angular momentum and spin not parallel to z or in the xy plane. The body is not spinning about a principal body axis. Euler's equations are

$$
\begin{aligned}
I_\perp \dot{\omega}_x + (I_\parallel - I_\perp)\omega_z \omega_y &= 0 && (52)\\
I_\perp \dot{\omega}_y - (I_\parallel - I_\perp)\omega_z \omega_x &= 0 && (53)\\
I_\parallel \dot{\omega}_z &= 0 && (54)
\end{aligned}
$$

for spin $\boldsymbol{\omega}$ in the body's reference frame. We notice that ω_z is fixed. Set

$$\Omega = \frac{I_\parallel - I_\perp}{I_\perp}\omega_z$$

and

$$\omega_z = \omega \cos\alpha.$$

The angle α is the angle between spin vector $\boldsymbol{\omega}$ and body symmetry vector $\hat{\mathbf{z}}$. The equations of motion become

$$
\begin{aligned}
\dot{\omega}_x + \Omega\omega_y &= 0 && (55)\\
\dot{\omega}_y - \Omega\omega_x &= 0. && (56)
\end{aligned}
$$

A solution is

$$
\begin{aligned}
\omega_x &= \omega \sin\alpha \cos(\Omega t) && (57)\\
\omega_y &= \omega \sin\alpha \sin(\Omega t) && (58)\\
\omega_z &= \omega \cos\alpha && (59)\\
\Omega &= \omega \cos\alpha \left(\frac{I_\parallel - I_\perp}{I_\perp}\right). && (60)
\end{aligned}
$$

Using $\mathbf{L} = I\boldsymbol{\omega}$, the angular momentum vector (in the body's frame)

$$
\begin{aligned}
L_x &= I_\perp \omega \sin\alpha \cos(\Omega t) && (61)\\
L_y &= I_\perp \omega \sin\alpha \sin(\Omega t) && (62)\\
L_z &= I_\parallel \omega \cos\alpha && (63)
\end{aligned}
$$

It is convenient to define an angle θ between $\mathbf{L}$ and body principal axis $\hat{\mathbf{z}}$ that satisfies

$$\tan\theta = \frac{\sqrt{L_x^2 + L_y^2}}{L_z} = \frac{I_\perp}{I_\parallel}\tan\alpha \tag{64}$$

We notice that $\boldsymbol{\omega}$, $\mathbf{L}$ and $\hat{\mathbf{z}}$ are all in the same plane.

The above solution describes $\mathbf{L}$ and $\boldsymbol{\omega}$ in the body frame. To transform a vector into an inertial frame

$$\frac{d}{dt}_{inertial} = \frac{d}{dt}_{body} + \boldsymbol{\omega}\times$$

We can relate our spin or angular moment in the body frame to those in an inertial frame. We expect the angular momentum in the inertia frame to be fixed and the spin vector should precess.

3.4 Euler angles of the torque free axi-symmetric body

Now back to our Euler angles. Take $\mathbf{K}$ parallel to $\mathbf{L}$. We associate xyz with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in Figure 9. The body we orient with axis of symmetry along z. The angle between the body axis of symmetry and the angular momentum is θ so $L_z = L\cos\theta$ and as this is fixed $\dot{\theta} = 0$. The angle α is that between the spin vector $\boldsymbol{\omega}$ and k so $\omega_z = \omega\cos\alpha$. The body does not nutate but it does precess. Without nutation equations 37 are

$$\begin{aligned}
\omega_x &= \sin\psi\sin\theta\dot{\phi} \\
\omega_y &= \cos\psi\sin\theta\dot{\phi} \\
\omega_z &= \cos\theta\dot{\phi} + \dot{\psi}
\end{aligned} \tag{65}$$

Consider a moment when $\boldsymbol{\omega}$ and $\mathbf{L}$ are in the yz plane and so all have zero x components. In order for $\omega_x = 0$ at this time, $\psi = 0$ and

$$\omega_y = \sin\theta\dot{\phi} = \omega\sin\alpha. \tag{66}$$

The first relation follows from equation 65 and the second one from the length of ω with $\omega_x = 0$. Equation 66 gives a relation between $\theta, \alpha, \omega, \dot{\phi}$ that is satisfied at all times.

Combined with equation 64 we find

$$
\begin{aligned}
\dot\phi &= \frac{\omega \sin\alpha}{\sin\theta} \\
&= \omega \sin\alpha \left(\frac{I_\parallel^2}{I_\perp^2} \cot^2\alpha + 1 \right)^{\frac{1}{2}} \\
&= \omega \left[1 + \left(\frac{I_\parallel^2}{I_\perp^2} - 1 \right) \cos^2\alpha \right]^{\frac{1}{2}} && (67) \\
&= \omega \left[1 + \cos^2\theta \left(\frac{I_\perp^2}{I_\parallel^2} - 1 \right) \right]^{-\frac{1}{2}} && (68)
\end{aligned}
$$

Using $\omega_z = w\cos\alpha$ and the last of equation 65, some further manipulation using equation 64 gives

$$
\begin{aligned}
\dot\psi &= \omega\cos\alpha - \cos\theta\dot\phi \\
&= \omega(\cos\alpha - \cot\theta\sin\alpha) \\
&= \omega\cos\alpha \left(1 - \frac{I_\parallel}{I_\perp} \right) = -\Omega && (69)
\end{aligned}
$$

The last step we have used equation 60. Here we associate $\dot\psi$ with the precession rate of the angular momentum and spin in the body frame.

For small α (and simultaneously small θ)

$$
\begin{aligned}
\lim_{\alpha\to 0} \dot\phi &\to \omega \frac{I_\parallel}{I_\perp} && (70) \\
\dot\psi &\to \omega\left(1 - \frac{I_\parallel}{I_\perp}\right) && (71) \\
\frac{\dot\psi}{\dot\phi} &\to \frac{I_\perp}{I_\parallel}\left(1 - \frac{I_\parallel}{I_\perp}\right) && (72)
\end{aligned}
$$

Whereas $-\dot\psi$ is the precession rate in the body frame, $\dot\phi$ is the spin precession rate in the inertial frame. The two are not equal. For a prolate body $I_\parallel < I_\perp$ and vice versa for an oblate body. $\Omega < 0$ for elongated (prolate) bodies but $\Omega > 0$ for flattened (oblate) bodies, whereas $\dot\phi$ is always positive. Precession in the inertial frame is in the same sense as L_z, the vertical component of angular momentum. For a prolate body and small α, the rate $\frac{\dot\phi}{\omega} < 1$, whereas for an oblate body $\frac{\dot\phi}{\omega} > 1$.

For a thin disk $I_\parallel = 2I_\perp$ and $\dot\phi = \omega\sqrt{1 + 3\cos^2\alpha}$. For small α, the precession rate $\dot\phi \sim 2\omega$ and $\Omega \sim \omega$ and giving the famous 2:1 ratio discussed by Feynman (though apparently he had which frequency was smaller backwards).

We have so far explored the torque-free axisymmetric top starting from Euler's equations. We should be able to find the same solution using Hamilton's equations or Lagrange's equations. A difficulty is that we need to chose the inertial frame so that the Euler angle θ is fixed. Once that is done conserved momenta give the precession rates. I think choosing the good inertial frame is equivalent to computing the angular momentum direction and choosing the z axis (for the Euler angles) to align with it. Another inconvenience with the Euler angles is that we are left without a nice expression for the angular momentum in the inertial frame.

3.5 Frequencies seen by an external viewer - asteroid rotation

How do the body axes move, as seen in the inertial frame? Assuming that we had axis of symmetry along z' (body frame) or $\mathbf{e}_{z'}$ equation 42 gives us its coordinates in the inertial frame as a function of the Euler angles

$$\mathbf{e}_{z'}(t) = \sin\phi \sin\theta \mathbf{e}_x - \cos\phi \sin\theta \mathbf{e}_y + \cos\theta \mathbf{e}_z. \tag{73}$$

Let us take it's x coordinate to be 0 at time $t = 0$, then

$$\mathbf{e}_{z'}(t) = \sin(\dot\phi t) \sin\theta \mathbf{e}_x - \cos(\dot\phi t) \sin\theta \mathbf{e}_y + \cos\theta \mathbf{e}_z \tag{74}$$

as $\dot\phi$ (equation 67) is constant for a body of revolution. Likewise we can use equation 42 to determine the other two body axes as a function of time as seen in the inertial frame

$$\begin{aligned}
\mathbf{e}_{x'}(t) = {} & [\cos(\dot\phi t)\cos(\Omega t) + \sin(\dot\phi t)\sin(\Omega t)\cos\theta]\mathbf{e}_x \\
& + [\sin(\dot\phi t)\cos(\Omega t) - \cos(\dot\phi t)\sin(\Omega t)\cos\theta]\mathbf{e}_y \\
& - \sin\theta \sin(\Omega t)\mathbf{e}_z
\end{aligned} \tag{75}$$

$$\begin{aligned}
\mathbf{e}_{y'}(t) = {} & [\cos(\dot\phi t)\sin(\Omega t) - \sin(\dot\phi t)\cos(\Omega t)\cos\theta]\mathbf{e}_x \\
& + [\sin(\dot\phi t)\sin(\Omega t) + \cos(\dot\phi t)\cos(\Omega t)\cos\theta]\mathbf{e}_y \\
& + \sin\theta \cos(\Omega t)\mathbf{e}_z
\end{aligned} \tag{76}$$

where we have used $\dot\psi = -\Omega$.

Ignoring gravitational torques and those due to radiation forces or outgassing we can think of an asteroid or a comet as torque free rotating body. Some of them (including the exotic 'Oumuamua) do not rotate solely about a principal axis. These we say are undergoing *non-principal axis* rotation (NPA). If they are prolate or oblate they would be described by the precessing torque free top equations considered here. An external viewer would view the object by monitoring its light curve. The brightness of the asteroid depends on its illumination angle and its projected surface area (as seen by the viewer). The object could also have uneven albedo or color variations on its surface. Equations 74, 76 tell us how the axes of a body of revolution move as seen by the external viewer. An external viewer will see two frequencies in the motion of the precessing body, that set by Ω and that set by $\dot{\phi}$. Two frequencies present in the light curve gives evidence of non-principal axis rotation.

It is not easy to determine the body axis ratios from the light curve because the brightness depends on illumination angle. In other words, how much sun light is reflected depends on the angles between the sun to object line, object to viewer line and surface normal vectors. The surface could also have color or albedo variations (spots!).

If the body is triaxial rather than a body of revolution, then θ is not fixed and there will be an additional frequency present in the motion, called nutation. And there may be aperiodicity as well in $\dot{\phi}$. For triaxial bodies the spins and some periods for the Euler angles can be written in terms of elliptic functions. These give two periods and an associated precession rate that could be evident in a light curve.

Here we defined the Euler angles with respect to the body symmetry axis. In general you could define them with respect to a long or short principal body axis. This gives two conventions for specifying them.

To interpret asteroid rotation, you start with spin and angular momentum vectors and three moments of inertia. Solve Euler's equations in the body frame. Then compute the frequencies associated with the Euler angles and compare them to frequencies that are seen in a periodogram of a light curve.

3.6 Long axis and short axis modes

Following papers by Nalin Samarasinha and collaborators.....

We take a triaxial body with three moments of inertia, I_l, I_i, I_s for short, intermediate and long axes, and defining Euler angles with respect to the short axis (short axis convention). We define a constant

$$U \equiv \frac{L^2}{2E} \tag{77}$$

which depends on angular momentum L and rotation kinetic energy E. Long axis modes (LAMs) have

$$I_l \le U < I_i \tag{78}$$

we define a time variable τ and a constant of motion k

$$\tau = t\sqrt{\frac{2E(I_i - I_l)(I_s - U)}{I_l I_i I_s}} \tag{79}$$

$$k^2 = \frac{(I_s - I_i)}{(I_i - I_l)}\frac{(U - I_l)}{(I_s - U)} \tag{80}$$

$$\Omega_l = \mathrm{dn}\tau \sqrt{\frac{2E(I_s - U)}{I_l(I_s - I_l)}} \tag{81}$$

$$\Omega_i = \mathrm{sn}\tau \sqrt{\frac{2E(U - I_l)}{I_i(I_i - I_l)}} \tag{82}$$

$$\Omega_s = \mathrm{cn}\tau \sqrt{\frac{2E(U - I_l)}{I_s(I_s - I_l)}} \tag{83}$$

The Euler angles obey

$$\psi_s = \mathrm{atan2}\left(\sqrt{\frac{I_l(I_s - U)}{I_s - I_l}}\mathrm{dn}\tau, \sqrt{\frac{I_i(U - I_l)}{I_i - I_l}}\mathrm{sn}\tau\right) \tag{84}$$

$$\theta_s = \cos^{-1}\left(\mathrm{cn}\tau \sqrt{\frac{I_s(U - I_l)}{U(I_s - I_l)}}\right) \tag{85}$$

$$\dot{\phi}_s = L\left[\frac{I_s - U + (U - I_l)\mathrm{sn}^2\tau}{I_l(I_s - U) + I_s(U - I_l)\mathrm{sn}^2\tau}\right] \tag{86}$$

It turns out that two of the angles are periodic in τ, but the last one is not, though the rate $\dot{\phi}_s$ is. The period

$$P_{\psi_S} = 4\sqrt{\frac{I_l I_i I_s}{2E(I_i - I_l)(I_s - U)}} \int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2 \sin^2 u}} \tag{87}$$

Short axis modes (SAMs) have

$$I_i < \frac{L^2}{2E} \leq I_s \tag{88}$$

$$\tau = t\sqrt{\frac{2E(I_s - I_i)(U - I_l)}{I_l I_i I_s}} \tag{89}$$

$$k^2 = \frac{(I_i - I_l)}{(I_s - I_i)} \frac{(I_s - U)}{(U - I_i)} \tag{90}$$

Another set of formulas.

And then you can redo everything using the long axis convention instead of the short axis convention!

Samarasinha and collaborators place limits on periods and maximum angles .... that might be useful to classify actual systems from light curves.

3.7 Comments on Damping of Wobbling

If an asteroid is not rotating about a principal axis, then $\boldsymbol{\Omega}$ is time dependent. Let $\mathbf{a}', \mathbf{v}', \mathbf{r}'$ be the acceleration, velocity and position in the body frame and $\mathbf{a}, \mathbf{v}, \mathbf{r}$ be those in the inertial frame. To relate the time derivative of a position in the inertial frame to that in the body frame

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{\Omega} \times . \tag{91}$$

Apply this to position $\mathbf{r}'$ in the body

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \frac{\partial \mathbf{r}'}{\partial t} + \boldsymbol{\Omega} \times \mathbf{r}' \\ \mathbf{v} &= \mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{r}' \end{aligned}$$

Now we apply our time derivative to $\mathbf{v}$

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\Omega} \times \mathbf{v} \\ \mathbf{a} &= \frac{\partial}{\partial t}\left(\mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{r}'\right) + \boldsymbol{\Omega} \times (\mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{r}') \\ &= \frac{\partial \mathbf{v}'}{\partial t} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}' + \boldsymbol{\Omega} \times \mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{v}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}') \\ &= \mathbf{a}' + \dot{\boldsymbol{\Omega}} \times \mathbf{r}' + 2\boldsymbol{\Omega} \times \mathbf{v}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}') \end{aligned}$$

If the body does not flex much then $\mathbf{a}'$ and $\mathbf{v}'$ (in the body frame) are small and

$$\mathbf{a} \approx \dot{\boldsymbol{\Omega}} \times \mathbf{r}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}') \tag{92}$$

The time dependent stress due to the spin variations can be estimated using this equation. For a prolate or oblate body, $\boldsymbol{\Omega}$ depends on a single frequency and this sets the timescale of elastic stress variations in the body. If the body is triaxial then an entire spectrum of frequencies is involved. This is relevant as internal friction due to the elastic deformation of the body may be frequency dependent.

While angular momentum is conserved, the total energy is not due to dissipation or internal friction inside the elastic body which goes into heat that can be radiated away. The rate of energy loss is computing by integrating up over the volume of the body the stress times the strain rate.

3.8 Andoyer-Deprit variables

Dynamics might be simpler if we keep track of a spin frame, not just an inertial reference frame and a body frame. Following the nice introduction in Celletti's book.

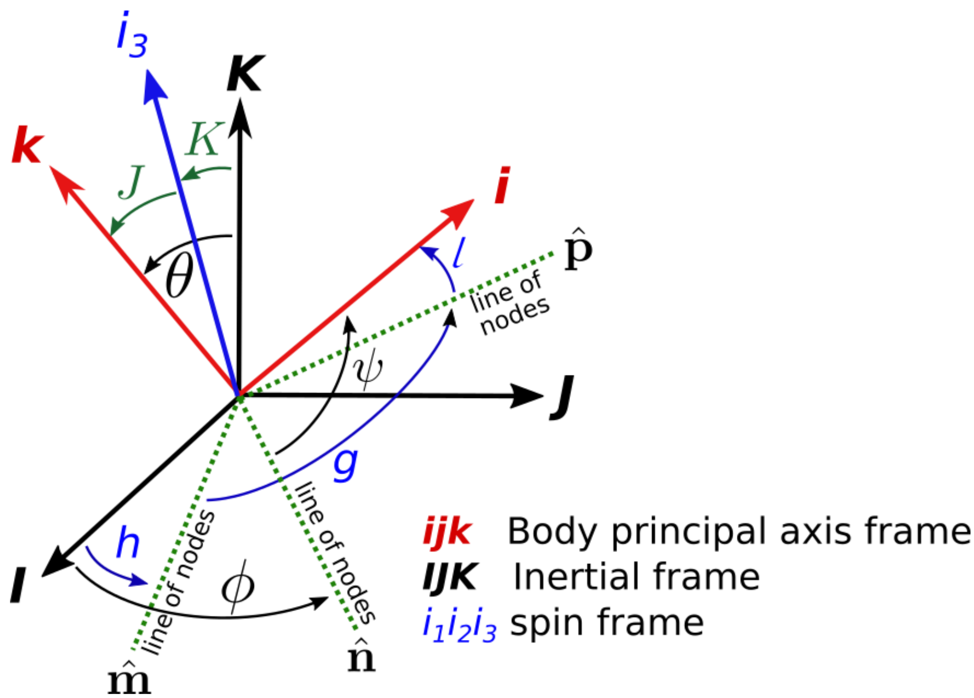

Figure 11: Andoyer-Deprit angles. The green lines show lines of nodes. The body's spin angular momentum is aligned with $\mathbf{i}_3$. The $\hat{\mathbf{p}}$ line of nodes is the intersection of the $\mathbf{i}, \mathbf{j}$ plane (body) with the $\mathbf{i}_1, \mathbf{i}_2$ (spin) plane. The $\hat{\mathbf{m}}$ line of nodes is the intersection of the $\mathbf{I}, \mathbf{J}$ plane (inertial) with the $\mathbf{i}_1, \mathbf{i}_2$ (spin) plane. The $\hat{\mathbf{n}}$ line of nodes is the intersection of the $\mathbf{i}, \mathbf{j}$ plane (body) with the $\mathbf{I}, \mathbf{J}$ plane (inertial). The angles θ, ϕ, ψ are Euler angles. The angles g, h, l are Andoyer-Deprit angles. The angles J, K are used in the definition of Andoyer-Deprit momenta. Under principal axis rotation $J = 0$, $K = \theta$, $\hat{\mathbf{m}} = \hat{\mathbf{n}} = \hat{\mathbf{p}}$, $\phi = h$, $g = 0$, and $l = \psi$.

- $(O, \mathbf{I}, \mathbf{J}, \mathbf{K})$ is an inertial reference frame (for example defined with respect to a Laplace plane or the elliptic). Here O refers to the center of mass of our spinning body.
- $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ is a body frame with respect to principle axes of moment of inertia.
- $(O, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ is a spin frame, and i_3 is aligned with the angular momentum vector.

Because we have three frames we need more than one line of nodes (see Figure 11) .

- The $\hat{\mathbf{n}}$ line of nodes is the intersection of the $\mathbf{i}, \mathbf{j}$ plane (body) with the $\mathbf{I}, \mathbf{J}$ plane (inertial).
$$\hat{\mathbf{n}} = \mathbf{K} \times \mathbf{k}/|\mathbf{K} \times \mathbf{k}|$$
We used this line of nodes when defining the Euler angles.
- The $\hat{\mathbf{m}}$ line of nodes is the intersection of the $\mathbf{I}, \mathbf{J}$ plane (inertial) with the $\mathbf{i}_1, \mathbf{i}_2$ plane (body).
$$\hat{\mathbf{m}} = \mathbf{i}_3 \times \mathbf{K}/|\mathbf{i}_3 \times \mathbf{K}|$$
- The $\hat{\mathbf{p}}$ line of nodes is the intersection of the $\mathbf{i}, \mathbf{j}$ (body) plane with the $\mathbf{i}_1, \mathbf{i}_2$ plane (spin).
$$\hat{\mathbf{p}} = \mathbf{k} \times \mathbf{i}_3/|\mathbf{k} \times \mathbf{i}_3|$$

The signs on the cross products need to be checked for convention here! I have adopted the order in Celleti's book.

Euler angle lists

- (θ, ϕ, ψ) are Euler angles of the body frame with respect to the inertial frame.
- (J, g, l) are Euler angles of the body frame with respect to the spin frame.
- $(K, h, 0)$ are Euler angles of the inertial frame with respect to the spin frame, but assuming that $\mathbf{i}_3$ lies on line of nodes $\hat{\mathbf{m}}$.

What are $\mathbf{i}_1, \mathbf{i}_2$ aligned with? Their directions are not needed to define the Andoyer-Deprit variables. The angles depend on the lines of nodes rather than the orientation of $\mathbf{i}_1$ or $\mathbf{i}_2$.

Andoyer-Deprit variables are a canonical set $(G, g), (L, l), (H, h)$ of momenta paired with angles.

If $\mathbf{M}_0$ is the angular momentum vector $\mathbf{M}_0 = M_0 \mathbf{i}_3$.

$$\begin{aligned} G &\equiv \mathbf{M}_0 \cdot \mathbf{i}_3 = M_0 && (93)\\ L &\equiv \mathbf{M}_0 \cdot \mathbf{k} = G \cos J && (94)\\ H &\equiv \mathbf{M}_0 \cdot \mathbf{K} = G \cos K && (95) \end{aligned}$$

The angle K is an **obliquity** angle between angular momentum vector and inertial frame axis. It is not one of the Andoyer-Deprit variables. The angle K is that between $\mathbf{i}_3$ (angular momentum vector) and $\mathbf{K}$ (inertial frame axis).

$$\cos K = \mathbf{i}_3 \cdot \mathbf{K}$$

The angle J is called the **non-principal rotation angle**. When it is zero, $J = 0$, the body is rotating about a principal body axis. It is not one of the Andoyer-Deprit variables. The angle J is that between $\mathbf{i}_3$ (angular momentum vector) and $\mathbf{k}$ (body principal axis).

$$\cos J = \mathbf{i}_3 \cdot \mathbf{k}$$

- The Andoyer-Deprit momentum variable G is the total angular momentum.
- The Andoyer-Deprit momentum variable L is the component of the angular momentum in the direction of the principal body axis $\mathbf{k}$. If the body is rotating around this principal axis then $L = G$.
- The Andoyer-Deprit momentum variable H is the component of the angular momentum in the direction of the inertial axis $\mathbf{K}$.

The Andoyer-Deprit momenta in terms of those associated with the Euler angles are

$$\begin{aligned} p_\phi &= H \\ p_\psi &= L \\ p_\theta &= G \sin J \sin(l - \psi) \end{aligned} \tag{96}$$

Here the Euler angles are defined with respect to the inertial reference frame $\mathbf{I}, \mathbf{J}, \mathbf{K}$.

The variables G, J, L, l are related to the angular momentum orientation as seen in the body frame

$$\begin{aligned} I_1\omega_1 &= G \sin J \sin l \\ I_2\omega_2 &= G \sin J \cos l \\ I_3\omega_3 &= L \end{aligned} \tag{97}$$

How are the Andoyer-Deprit angles g, h, l defined?

- The angle g is that between line of nodes $\hat{\mathbf{m}}$ and $\hat{\mathbf{p}}$.

$$\cos g = \hat{\mathbf{m}} \cdot \hat{\mathbf{p}}$$

This angle is measured in the $\mathbf{i}_1, \mathbf{i}_2$ plane.

- The angle h is that between $\mathbf{I}$ and line of nodes $\hat{\mathbf{m}}$.

$$\cos h = \mathbf{I} \cdot \hat{\mathbf{m}}$$

  This angle is measured in the $\mathbf{I}, \mathbf{J}$ plane.

- The angle l is that between $\mathbf{i}$ (body principal axis) and line of nodes $\hat{\mathbf{p}}$.

$$\cos l = \mathbf{i} \cdot \hat{\mathbf{p}}$$

  This angle is measured in the $\mathbf{i}, \mathbf{j}$ plane.

Using the relation between Euler angles and associated momenta and the Andoyer-Deprit variables, the Hamiltonian can be written in terms of the Andoyer-Deprit variables

$$H(G, L, H; g, l, h) = \frac{1}{2}(G^2 - L^2)\left(\frac{\sin^2 l}{I_1} + \frac{\cos^2 l}{I_2}\right) + \frac{1}{2}\frac{L^2}{I_3} \tag{98}$$

It is independent of H, g, h, so their canonical counter parts h, G, H must be conserved.

A useful relation

$$\cos K = \cos J \cos\theta + \sin J \sin\theta \cos(\psi - l)$$

Celletti's book shows that the transformation to Andoyer-Deprit variables is canonical.

If the body is axisymmetric

$$H_{axi} = \frac{G^2 - L^2}{2I_1} + \frac{L^2}{2I_3}$$

and

$$\dot{g} = \frac{G}{I_1} \qquad \dot{l} = \left(\frac{1}{I_3} - \frac{1}{I_1}\right) L \qquad \dot{h} = 0 \tag{99}$$

The body rotates around the symmetry axis with constant velocity and that it precedes uniformly.

With Euler angles, the solution of precessing axisymmetric body is not obvious (you must establish that you can find a coordinate system with $\dot{\theta} = 0$) using Andoyer-Deprit variables, it is clearer that a rigid body rotates about the symmetry axis with a constant velocity and precessing with a constant velocity.

The Andoyer-Deprit variables are not well defined if $J = 0$ or $K = 0$. However one can transform (canonically) to these variables

$$\begin{aligned}
\lambda_1 &= l + g + h & \qquad \Lambda_1 &= G \\
\lambda_2 &= -l & \qquad \Lambda_2 &= G - L = G(1 - \cos J) \\
\lambda_3 &= -h & \qquad \Lambda_3 &= G - H = G(1 - \cos K)
\end{aligned}$$

The new Hamiltonian is

$$
\begin{aligned}
H(\Lambda_1, \Lambda_2, \Lambda_3; \lambda_1, \lambda_2, \lambda_3) &= \frac{(\Lambda_1 - \Lambda_2)^2}{2I_3} \\
&+ \frac{1}{2}\left(\Lambda_1^2 - (\Lambda_1 - \Lambda_2)^2\right)\left(\frac{\sin^2\lambda_2}{I_1} + \frac{\cos^2\lambda_2}{I_2}\right) \qquad (100)
\end{aligned}
$$

It is independent of $\Lambda_3, \lambda_1, \lambda_3$ so their canonical counterparts $\lambda_3, \Lambda_1, \Lambda_3$ must be conserved.

3.8.1 Principal axis rotation

Under principal axis rotation $J = 0$, $\hat{\mathbf{m}} = \hat{\mathbf{n}}$, $\theta = K$, $\phi = h = -\lambda_3$, $\Lambda_2 = 0$, $p_\theta = 0$, and $G = L$. The Hamiltonian $H(\boldsymbol{\Lambda}; \boldsymbol{\lambda}) = \Lambda_1^2/(2I_3)$. The line of nodes $\hat{\mathbf{p}}$ is defined by an intersection of two planes that are the same so I think we can chose $\hat{\mathbf{p}} = \hat{\mathbf{m}}$, $g = 0$ and $l = \psi = -\lambda_2$. Also $\cos\theta = H/G = 1 - \Lambda_3/\Lambda_1$.

3.9 Potential energy of a rigid body from an external force field

Suppose we have a solid body of mass m subject to an external force. What is the torque on it? We can ask the related question, what is the potential energy on a rigid body from an external force field?

The external force field we describe with a potential $\Phi(\mathbf{x})$ that we expand about the center of mass of our body $\mathbf{x}_0$

$$\Phi(\mathbf{x}) = \Phi(\mathbf{x}_0) + \Phi_{,i}(\mathbf{x}_0)(x - x_0)^i + \frac{1}{2}\Phi_{,ij}(\mathbf{x}_0)(x - x_0)^i(x - x_0)^j + ... \qquad (101)$$

In a coordinate system centered about the center of mass of our body $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$

$$\Phi(\mathbf{y}) = \Phi(\mathbf{x}_0) + \Phi_{,i}(\mathbf{x}_0)y^i + \frac{1}{2}\Phi_{,ij}(\mathbf{x}_0)y^i y^j + ... \qquad (102)$$

In the center of mass coordinate system

$$\int d^3y \rho(\mathbf{y}) y^i = 0 \qquad (103)$$

Because the force field is external $\nabla^2\Phi = 0$ so $\Phi_{,ii}(\mathbf{x}_0) = 0$. The trace of the potential's Hessian is zero because the external potential must satisfy Laplace's equation.

The total potential energy we find by integrating over body coordinates

$$
\begin{aligned}
U &= \int d^3y \rho(\mathbf{y}) \left[\Phi(\mathbf{x}_0) + \Phi_{,i}(\mathbf{x}_0)y^i + \frac{1}{2}\Phi_{,ij}(\mathbf{x}_0)y^i y^j + ...\right] \\
&= \Phi(\mathbf{x}_0)m + \Phi_{,i}(\mathbf{x}_0)\int dy^3 \rho(\mathbf{y}) y_i + \frac{1}{2}\Phi_{,ij}(\mathbf{x}_0)\int dy^3 \rho(\mathbf{y}) y^i y^j \\
&= \Phi(\mathbf{x}_0)m + \frac{1}{2}\Phi_{,ij}(\mathbf{x}_0)\int dy^3 \rho(\mathbf{y}) y^i y^j
\end{aligned}
$$

The first order term drops out because we are using the center of mass coordinate frame. The zero-th order term is that for a point mass. The interesting term is the second order term. Because the trace of the Hessian is zero

$$\begin{aligned}
0 &= \Phi_{,ii}(\mathbf{x}_0) \int dy^3 \rho(\mathbf{y}) y^k y^k \\
&= \Phi_{,ij}(\mathbf{x}_0)\delta_{ij} \int dy^3 \rho(\mathbf{y}) y^k y^k
\end{aligned}$$

We will use this to rewrite the potential energy with the moment of inertial tensor

$$\begin{aligned}
U_2 &= \frac{1}{2}\Phi_{,ij}(\mathbf{x}_0) \int dy^3 \rho(\mathbf{y}) y^i y^j \\
&= \frac{1}{2}\Phi_{,ij}(\mathbf{x}_0) \int dy^3 \rho(\mathbf{y}) \left[y^i y^j - \delta_{ij} y^k y^k \right] \\
&= -\frac{1}{2}\Phi_{,ij}(\mathbf{x}_0) I_{ij}
\end{aligned}$$

Rewriting this the potential term associated with body torque is

$$U_2 = -\frac{1}{2}\Phi_{,ij}(\mathbf{x}_0) I_{ij} \tag{104}$$

If we require more accuracy we we could add in higher order terms, depending upon higher order moments of both body and potential.

The external force field could be from an oblate planet, a binary or a point mass. In all these settings we could compute the derivatives of the external potential and from that the potential energy integrated over the body. The potential energy of the body can be put immediately into a Lagrangian or a Hamiltonian using one of the choices above for canonical variables. By taking derivatives of the potential energy function the torque on the body can be computed.

3.10 MacCullagh's Formula

We consider the external perturbation from a point mass of mass M at a position $\mathbf{r}$. This will external perturbation acts on a small mass of mass m that has moment of inertia matrix I. We expand the potential from M at the location of m and then use equation 104 to compute the potential perturbation from the moment of inertia matrix. At a position $\mathbf{r}'$ (position of m) the potential from the point mass M can be expanded in terms of Legendre polynomials

$$\Phi(\mathbf{r}') = -\frac{GM}{|\mathbf{r} - \mathbf{r}'|} = -\frac{GM}{r} \sum_{l=0} P_l(\cos\psi) \frac{r'^l}{r^l} \tag{105}$$

where

$$\cos\psi = \frac{\mathbf{r} \cdot \mathbf{r}}{|\mathbf{r}||\mathbf{r}'|}$$

The second order term is the one that gives us our Hessian matrix.

$$
\begin{aligned}
P_2(\cos\psi) &= \frac{1}{2}\left(3\cos^2\psi - 1\right) \\
&= \frac{1}{2r'^2}\left(3z'^2 - x'^2 - y'^2 - z'^2\right) \\
&= \frac{1}{2r'^2}\left(2z'^2 - x'^2 - y'^2\right)
\end{aligned}
$$

where I have oriented the prime coordinate system so that the point mass M lies along z axis. In this coordinate system

$$
\begin{aligned}
\Phi_{,xx} &= \Phi_{,yy} = \frac{GM}{r^3} \\
\Phi_{,zz} &= -\frac{2GM}{r^3} = -2\Phi_{,xx}
\end{aligned}
$$

Taking only second order terms

$$\Phi_{2,ij} = \frac{GM}{r^3}\left(\delta_{ij} - 3\delta_{iz}\delta_{jz}\right)$$

Putting this into equation 104 the left hand part will give us the trace of the moment of Inertia matrix.

$$U_2 = -\frac{GM}{2r^3}\left(\mathrm{tr}I - 3I_{zz}\right) \tag{106}$$

where I_{zz} is the moment of inertia of m about the axis between our body m and the external point mass M. This is equivalent to MacCullagh's formula.

A small non-round body in the gravitational field of a distant point source has a potential energy perturbation given by this (remember to add it to the zeroth order part $-GMm/r$). This can be used to compute the torque on m by M which is the same but opposite to the torque on M by m. A useful formula for tidal computations.

MacCullagh's formula is often written

$$U = -\frac{Gm}{r} - \frac{G(I_{xx} + I_{yy} + I_{zz} - 3I)}{2r^3} \tag{107}$$

where I is the moment of inertia of the oblate body about the axis extending between the center of the body and the point mass perturber. The moment of inertia about an axis with direction $\hat{\mathbf{n}}$, defining the direction between center of mass of our extended body and our point mass, is

$$I = \hat{\mathbf{n}}\mathbf{I}\hat{\mathbf{n}} = n_\alpha I_{\alpha\beta} n_\beta \tag{108}$$

Previously we expanded the potential of a point mass, but we should recognize that terms in a potential are interaction terms so we can just as easily consider them caused by the non-round body on a point mass, rather than integrating the effect of a point mass on an extended body.

By taking derivatives of the potential function we can compute the torque. Taking $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to be unit vectors lying along principal axis of the extended spinning body that have moments of inertia $A < B < C$ and $\mathbf{r}$ is the vector from the center of mass of the extended spinning body to a point mass M. We can write $\mathbf{r}$ in a basis aligned with our principal body axes

$$\mathbf{r} = r((\hat{\mathbf{r}} \cdot \mathbf{i})\mathbf{i} + (\hat{\mathbf{r}} \cdot \mathbf{j})\mathbf{j} + (\hat{\mathbf{r}} \cdot \mathbf{k})\mathbf{k}$$

and equation 108 as

$$I = (\hat{\mathbf{r}} \cdot \mathbf{i})^2 A + (\hat{\mathbf{r}} \cdot \mathbf{j})^2 B + (\hat{\mathbf{r}} \cdot \mathbf{j})^2 C$$

and equation 107 as

$$U = -\frac{Gm}{r} - \frac{G(A + B + C - 3\left[(\hat{\mathbf{r}} \cdot \mathbf{i})^2 A + (\hat{\mathbf{r}} \cdot \mathbf{j})^2 B + (\hat{\mathbf{r}} \cdot \mathbf{j})^2 C)\right]}{2r^3} \tag{109}$$

When computing the torque, first ones computes the force by taking the derivative of the potential. Then $\mathbf{r} \times \mathbf{F}$ gives the torque. The computation is faster if you notice that terms only containing factors of radius r will drop out because their gradients are parallel to $\mathbf{r}$. The non-radial part of equation 109 we can write as

$$U_{nr} = -\frac{3G}{2r}\left[(\mathbf{r} \cdot \mathbf{i})^2 A + (\mathbf{r} \cdot \mathbf{j})^2 B + (\mathbf{r} \cdot \mathbf{j})^2 C)\right] \tag{110}$$

with derivative (keeping on only non-radial parts)

$$\begin{aligned}
\frac{\partial U_{nr}}{\partial r_i} &= -\frac{3G}{r}\left[A(\mathbf{r} \cdot \mathbf{i})i_i + B(\mathbf{r} \cdot \mathbf{j})j_i + C(\mathbf{r} \cdot \mathbf{k})k_i\right] \\
\frac{\partial U_{nr}}{\partial \mathbf{r}} &= -\frac{3G}{r}\left[A(\mathbf{r} \cdot \mathbf{i})\mathbf{i} + B(\mathbf{r} \cdot \mathbf{j})\mathbf{j} + C(\mathbf{r} \cdot \mathbf{k})\mathbf{k}\right]
\end{aligned}$$

The torque

$$\mathbf{T} = -\frac{3GM}{r}\left[A(\mathbf{r} \cdot \mathbf{i})(\mathbf{r} \times \mathbf{i}) + B(\mathbf{r} \cdot \mathbf{j})(\mathbf{r} \times \mathbf{j}) + C(\mathbf{r} \cdot \mathbf{k})(\mathbf{r} \times \mathbf{k})\right] \tag{111}$$

Using

$$\mathbf{r} \times \mathbf{i} = [(\mathbf{r} \cdot \mathbf{j})\mathbf{j} + (\mathbf{r} \cdot \mathbf{k})\mathbf{k}] \times \mathbf{i}$$

and similarly for the other terms, the torque can be written

$$\begin{aligned}
\mathbf{T} &= \frac{3GM}{r^3}\left[(C - B)(\hat{\mathbf{r}} \cdot \mathbf{j})(\hat{\mathbf{r}} \cdot \mathbf{k})\mathbf{i} + (A - C)(\hat{\mathbf{r}} \cdot \mathbf{k})(\hat{\mathbf{r}} \cdot \mathbf{i})\mathbf{j}\right. \\
&\quad \left. + (B - A)(\hat{\mathbf{r}} \cdot \mathbf{j})(\hat{\mathbf{r}} \cdot \mathbf{i})\mathbf{k}\right]
\end{aligned} \tag{112}$$

as is found in Colombo's 1966 paper on Cassini states.

3.10.1 For an axisymmetric body

MacCullagh's formula is usually described as the gravitational potential of an oblate body, like the Earth. For an oblate body oriented with z along the pole, $I_{xx} = I_{yy} = I_\perp$. This is often written with $I_\perp$ as the moment of inertia about an axis perpendicular to the body's axis (x or y) of symmetry and $I_{zz} = I_\parallel$ is that about an the axis parallel to the axis of symmetry (here z).

In a coordinate system with $\mathbf{I}$ diagonal

$$I = n_x^2 I_x + n_y^2 I_y + n_z^2 I_z \tag{113}$$

The moment I only depends on a latitude angle θ. Let us use a convention $\theta = 0$ gives the axis of symmetry and $\theta \in [0, \pi]$ with $\theta = \pi/2$ at the equator with $z = r\cos\theta$. Taking $\hat{\mathbf{n}} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$

$$I = \sin^2\theta I_\perp + \cos^2\theta I_\parallel = I_\perp + \cos^2\theta(I_\parallel - I_\perp)$$

Inserting this into equation 107 the gravitational potential

$$\begin{aligned} U(r,\theta) &= -\frac{Gm}{r} - \frac{G(2I_\perp + I_\parallel - 3I)}{2r^3} \\ &= -\frac{Gm}{r} - \frac{G(I_\parallel - I_\perp)(1 - 3\cos^2\theta)}{2r^3} \\ &= -\frac{Gm}{r} + \frac{G(I_\parallel - I_\perp)}{r^3} P_2(\cos\theta) \end{aligned} \tag{114}$$

The total potential *energy* of an axisymmetric extended body described with $I_\parallel, I_\perp$ interacting with a point mass M is

$$E(r,\theta) = -\frac{GMm}{r} + \frac{GM(I_\parallel - I_\perp)}{r^3} P_2(\cos\theta) \tag{115}$$

where r is the distance between M and the center of mass of m and θ gives the angle between the vector joining the two masses and the axis of symmetry of the axisymmetric body. In other words $\cos\theta = \hat{\mathbf{r}} \cdot \mathbf{k}$ with $\mathbf{k}$ along the body's axis of symmetry.

For moments of inertia $A = B$, the torque in equation 112 simplifies to

$$\mathbf{T} = \frac{3GM}{r^3}(C - A)(\mathbf{r}\cdot\mathbf{k})(\mathbf{r}\times\mathbf{k}). \tag{116}$$

3.11 Precession of an axisymmetric body in a circular orbit

If we assume principal axis rotation (a gyroscopic approximation) then the kinetic energy is fixed and is $C\omega^2/2$ with C the principal moment of inertia. However if the object is not round and is orbit about a point mass then it can precess. If the orbit is circular we can chose a coordinate system oriented with orbit angular momentum aligned in the z direction,

$$\mathbf{r} = r(\cos \lambda t, \sin \lambda t, 0)$$

with $\dot{\lambda} = n$ the mean motion. It is convenient to notice that $GM/r^3 = n^2$ is the mean motion as long as the point mass is much larger than that of the precessing body.

For principal axis rotation the body's spin $\hat{\mathbf{s}} = \mathbf{k}$ and with θ the obliquity and ϕ the precession angle projected onto the orbital plane,

$$\hat{\mathbf{s}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

The spin angular momentum $\mathbf{L} = C\omega\hat{\mathbf{s}}$. We compute

$$\begin{aligned}
\hat{\mathbf{r}} \cdot \hat{\mathbf{s}} &= \sin\theta\cos(\lambda t - \phi) = \sin\theta(\cos(\lambda t)\cos\phi + \sin(\lambda t)\sin\phi) \\
\hat{\mathbf{r}} \times \hat{\mathbf{s}} &= (\sin(\lambda t)\cos\theta, -\cos(\lambda t)\cos\theta, \sin\theta\sin(\phi - \lambda t) \\
\frac{1}{2\pi}\int_0^{2\pi} d\lambda \; (\hat{\mathbf{r}} \cdot \hat{\mathbf{s}})(\hat{\mathbf{r}} \times \hat{\mathbf{s}}) &= \frac{1}{2}\sin\theta\cos\theta(\sin\phi, -\cos\phi, 0)
\end{aligned}$$

where on the last line I have take the time average. The torque averaged over the orbit

$$\langle \mathbf{T} \rangle = \frac{3n^2(C-A)}{2}\sin\theta\cos\theta \; (\sin\phi, -\cos\phi, 0)$$

is in a direction consistent with ϕ but not θ varying.

$$\frac{d\mathbf{L}}{dt} = C\omega\sin\theta\dot{\phi} \; (-\sin\phi, \cos\phi, 0).$$

Equating these two expressions we find a precession rate,

$$\dot{\phi} = -\frac{3}{2}\frac{(C-A)n^2}{C\omega}\cos\theta \tag{117}$$

The equations of motion are consistent with a Hamiltonian system with canonical momentum $p = 1 - \cos\theta$ and canonical coordinate ϕ

$$H(p,\phi) = \frac{\alpha_s}{2}\cos^2\theta = \frac{\alpha_s}{2}(p-1)^2 \tag{118}$$

and

$$\alpha_s \equiv \frac{3}{2}\frac{(C-A)n^2}{C\omega}. \tag{119}$$

A planet can experience torques (and associated obliquity variations) due to perturbations on its orbit. In this case the torque from the Sun is modulated due to orbit orientation variations, typically described by variations in the longitude of the ascending node. Bill Ward and collaborators wrote a series of papers accounting for obliquity variations with this type of model, which is usually called ‘secular spin resonance’. The dynamics is considered secular as it can be approximated by averaging over both orbit and spin angles.

How does a planet precess if it has moons? Moons can orbit a planet quickly compared to the precession rate orbit of the planet. If the satellites nodal recession due planetary oblateness is faster than the planet’s spin precession rate, then the moon’s orbit can be considered a ring that slowly moves along with the planet and its orbit stays oriented about the equator of the planet. An adiabatic invariant is preserved. The planet and its moons are considered to precess together as if they were a single object. Moons affect the planet’s precession rate as they add to the total spin angular momentum and make the planet behave as if it were flatter. Saturn would precess more slowly if it did not have a ring and moon satellite system.

The Moon’s orbit nodal recession period is 19 years and is much faster than the Earth’s spin precession period which is 26,000 years. However precession of the Moon’s orbit is due to the torque from the Sun, rather from Earth’s oblateness and the Moon’s orbit precesses about the normal of the Earth’s orbit rather than about the spin axis of the Earth. The transition between precession due to Earth’s oblateness and about the Earth’s spin axis and precession due to the Sun and about the Earth’s orbital normal is called the Laplace transition. A lock between the orbit nodal precession rate and the spin precession rate is called a Cassini state. Tidally locked moons tend to be in a Cassini state.

Does the Moon’s inclination with respect to the ecliptic or with respect to the Earth’s equator remain constant as the Earth precesses? The Moon’s orbit w.r.t. ecliptic has inclination 5.15 degrees. The Moon’s obliquity (angle between orbit planet and spin axis is 6.688 degrees). Both angles remain fixed as Earth’s spin axis precesses. The Moon does affect the precession rate of the Earth, but the oblateness of the Earth no longer affects the Moon’s orbit because the Moon is too far away.

3.12 Quaternions

We define a quaternion as

$$\mathbf{q} = q_0 + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$$

where q_0 is called the *scalar* part and (q_x, q_y, q_z) is called the *vector* part of the quaternion.

Multiplication of quaternions is performed using the *Hamilton product.* Using the rules

$$\begin{aligned}
\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= -1 \\
\mathbf{ij} = \mathbf{k} &= -\mathbf{ji} \\
\mathbf{jk} = \mathbf{i} &= -\mathbf{kj} \\
\mathbf{ki} = \mathbf{j} &= -\mathbf{ik},
\end{aligned} \tag{120}$$

the Hamilton product of $\mathbf{q}_1 = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$ and $\mathbf{q}_2 = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$ is

$$\begin{aligned}\mathbf{q}_1\mathbf{q}_2 = {} & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \\ & + (b_1a_2 + a_1b_2 + c_1d_2 - d_1c_2)\mathbf{i} \\ & + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\mathbf{j} \\ & + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\mathbf{k}. \end{aligned} \tag{121}$$

A rotation can be described as a rotation by θ about an axis $\hat{\mathbf{u}} = (u_x, u_y, u_z) = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ where $\hat{\mathbf{u}}$ is a unit vector and we use $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to represent unit vectors for a Cartesian basis. The associated quaternion

$$\mathbf{q} = e^{\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos(\theta/2) + \sin(\theta/2)\,(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \tag{122}$$

Notice that $\theta \to -\theta$ and $\mathbf{u} \to -\mathbf{u}$ gives the same rotation. This means that $\mathbf{q} \to -\mathbf{q}$ gives the same rotation. Flipping the sign of θ or $\mathbf{u}$ (but not both) gives $\mathbf{q}^{-1}$.

For a unit quaternion $\mathbf{q} = q_0 + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$ with $q_0^2 + q_x^2 + q_y^2 + q_z^2 = 1$, the axis of rotation and angle

$$\mathbf{u} = \frac{(q_x, q_x, q_z)}{\sqrt{q_x^2 + q_y^2 + q_z^2}} = \frac{(q_x, q_x, q_z)}{\sqrt{1 - q_0^2}} \tag{123}$$

$$\begin{aligned}\theta &= 2\ \mathrm{atan2}\left(\sqrt{q_x^2 + q_y^2 + q_z^2}, q_0\right) \\ &= 2\ \mathrm{atan2}\left(\sqrt{q_x^2 + q_y^2 + q_z^2}, \sqrt{1 - (q_x^2 + q_y^2 + q_z^2)}\right) \\ &= 2\ \mathrm{atan2}\left(\sqrt{1 - q_0^2}, q_0\right). \end{aligned} \tag{124}$$

A rotation of vector $\mathbf{p} = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}$ is performed by conjugating the quaternion

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}. \tag{125}$$

To carry out consecutive rotations, quaternions can be multiplied using quaternion multiplication.

Taking unit quaternion $\mathbf{q} = q_0 + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$, the rotated vector can be written in terms of a rotation matrix $\mathbf{R}$ with

$$\mathbf{p}' = \mathbf{R}\mathbf{p}$$

and

$$\mathbf{R} = \begin{bmatrix} 1 - 2(q_y^2 + q_z^2) & 2(q_xq_y - q_zq_0) & 2(q_xq_z + q_yq_0) \\ 2(q_xq_y + q_zq_0) & 1 - 2(q_y^2 + q_z^2) & 2(q_yq_z - q_xq_0) \\ 2(q_xq_z - q_yq_0) & 2(q_yq_z + q_xq_0) & 1 - 2(q_x^2 + q_y^2) \end{bmatrix}. \tag{126}$$

In index notation with indices $i, j, k \in x, y, z$

$$R_{ij} = \delta_{ij} + 2q_iq_j - 2q_kq_k\delta_{ij} - \epsilon_{ijk}q_kq_0 \tag{127}$$

To compute $\mathbf{R}^T = \mathbf{R}^{-1}$ we need only flip the sign of the last term. Rotating a vector $\mathbf{p}$ by $\mathbf{q}$ gives

$$p'_i = (1 - q_k q_k) p_i + 2 q_i q_j p_j - \epsilon_{ijk} q_k q_0 p_j$$
$$\mathbf{p}' = (1 - q_k q_k)\mathbf{p} + 2\mathbf{q}_3(\mathbf{q}_3 \cdot \mathbf{p}) - q_0 \mathbf{p} \times \mathbf{q}_3$$

where $\mathbf{q}_3$ is the vector part of $\mathbf{q}$.

Equation 126 can be used to relate a quaternion to Euler angles. For example, following equations 41, 42

$$R_{zz} = 1 - 2(q_x^2 + q_z^2) = \cos\theta.$$

If the quaternion transfers to body axes from inertial frame then

$$\psi = \mathrm{atan2}(R_{xz}, R_{yz})$$
$$\phi = \mathrm{atan2}(R_{zx}, R_{zy})$$

with order reversed if the quaternion transfers from body frame to inertial frame.

Consider two rotations giving a third $\mathbf{R}_C = \mathbf{R}_B \mathbf{R}_A$

$$\mathbf{q}_A = \cos\frac{\alpha}{2} + \mathbf{A}\sin\frac{\alpha}{2} \tag{128}$$
$$\mathbf{q}_B = \cos\frac{\beta}{2} + \mathbf{B}\sin\frac{\beta}{2} \tag{129}$$
$$\mathbf{q}_C = \cos\frac{\gamma}{2} + \mathbf{C}\sin\frac{\gamma}{2} \tag{130}$$

with unit vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and angles α, β, γ. The result is known as Rodrigues formula for composite rotation and is

$$\cos\frac{\gamma}{2} = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\mathbf{B}\cdot\mathbf{A} \tag{131}$$
$$\sin\frac{\gamma}{2}\mathbf{C} = \sin\frac{\beta}{2}\cos\frac{\alpha}{2}\ \mathbf{B} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\ \mathbf{A} + \sin\frac{\beta}{2}\sin\frac{\alpha}{2}\ \mathbf{B}\times\mathbf{A} \tag{132}$$
$$\tan\frac{\gamma}{2}\mathbf{C} = \left[1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\mathbf{B}\cdot\mathbf{A}\right]^{-1}\left(\tan\frac{\beta}{2}\mathbf{B} + \tan\frac{\alpha}{2}\mathbf{A} + \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\mathbf{B}\times\mathbf{A}\right). \tag{133}$$

For describing rotations, quaternions have some advantages. Quaternions avoid a phenomenon called *gimbal lock* which can result in pitch/yaw/roll rotational systems. They ensure rotations are possible in any orientation. As they depend only on 4 numbers, the notation is more compact than using rotation matrices. From a quaternion representation of rotations it is easy to read off the axis and angle of rotation (much easier than when using Euler angles).

The unit quaternion Lie group (S^3; 3D sphere in 4 dimensions) is not exactly the same as the group of rotations $SO(3)$ (special orthogonal matrices). This is because unit quaternions $\mathbf{q}$ and $-\mathbf{q}$ give the same rotation.

Question Is there any intuition gained by using canonical variables based on quaternions?

An infinitesimal rotation by angle $\delta\theta$ about axis $\hat{\mathbf{u}}$ gives a quaternion

$$\mathbf{q}_{\text{inf}} = 1 + \frac{\delta\theta}{2}(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \tag{134}$$

We recognize the spin as $\boldsymbol{\Omega} = \dot{\theta}\hat{\mathbf{u}}$. Let us consider a time dependent infinitesimal transformation

$$\begin{aligned}\mathbf{q}(t) &= 1 + \frac{\dot{\theta}t}{2}(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \\ &= 1 + \frac{t}{2}\left(\Omega_x\mathbf{i} + \Omega_y\mathbf{j} + \Omega_z\mathbf{j}\right) = 1 + \frac{t}{2}\Omega_i\hat{\mathbf{e}}_i\end{aligned}$$

with $\hat{\mathbf{e}}_i$ representing $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Taking the time derivative

$$\dot{\mathbf{q}} = \frac{1}{2}\Omega_i\hat{\mathbf{e}}_i \tag{135}$$

A position is rotated by

$$\mathbf{r}' = \mathbf{q}\mathbf{r}\mathbf{q}^{-\mathbf{1}}.$$

We can differential this w.r.t to time. Multiplying by the quaternions gives us a velocity $\mathbf{v} = \dot{\mathbf{r}} = \boldsymbol{\Omega} \times \mathbf{r}$, as expected. we can expand

$$\mathbf{q}(t)\mathbf{p} = \mathbf{p} + \frac{t}{2}\Omega_i p_j \epsilon_{ijk}\hat{\mathbf{e}}_k$$

and

$$\begin{aligned}\frac{d}{dt}\mathbf{q}(t)\mathbf{p} &= \frac{1}{2}\Omega_i p_j \epsilon_{ijk}\hat{\mathbf{e}}_k \\ \frac{d}{dt}\mathbf{q}\mathbf{p}\mathbf{q}^{-1} &= \boldsymbol{\Omega} \times \mathbf{p}\end{aligned}$$

To keep track of body orientation, we would need to specify the body's orientation (as a function of time) with a quaternion. This could give us the transformation from a body axis frame (where $\mathbf{I}$ is diagonal) to its current orientation with respect to an inertial frame. We can specify the spin with a different but infinitesimal quaternion. The quaternion specifying orientation contains the same information as the Euler angles. The infinitesimal quaternion specifies the instantaneous spin. These two quaternions are everything we need to describe the dynamics. Using one quaternion for orientation and another for spin are equivalent to using a rotation matrix R to describe the current orientation and the time derivative of a rotation matrix $\dot{R}$ to describe the current spin.

The total kinetic energy is

$$T = \int \frac{1}{2}\rho(\mathbf{r})(\boldsymbol{\Omega} \times \mathbf{r})^2 dV$$

where $\mathbf{r}$ is a vector for each position in the body. Note, that Ω is not equivalent to the time derivative of a quaternion describing current body orientation. Instead it describes a quaternion giving the infinitesimal spin rotation associated with rotation.

It may be useful to use the vector identity

$$(\boldsymbol{\Omega} \times \mathbf{r})^2 = \Omega^2 r^2 - (\boldsymbol{\Omega} \cdot \mathbf{r})^2$$

$$\frac{d}{d\Omega_i}(\boldsymbol{\Omega} \times \mathbf{r})^2 = 2\Omega_i r^2 - 2(\boldsymbol{\Omega} \cdot \mathbf{r}) r_i$$

$$\frac{\partial T}{\partial \Omega_i} = \Omega_j \int \rho(r)(\delta_{ij} r^2 - r_j r_i) dV = I_{ij}\Omega_j$$

If we decide to adopt a variable whose time derivative is Ω (that would be the rotation matrices) then our canonical momenta are the angular momentum vectors $p_i = I_{ij}\Omega_j$ and

$$H = \frac{1}{2} p_i I(\mathbf{q})^{-1}_{ij} p_j$$

The associated angles are those of the rotations that are hidden in the moment of inertia matrix.

Perhaps we can find canonical momenta associated with time derivatives of the quaternion. Let $\mathbf{q}$ describe body orientation and $\mathbf{q} = q_0 + \mathbf{Q}$ with $\mathbf{Q}$ the vector part of $\mathbf{Q}$ and $\dot{\mathbf{q}} = \dot{q}_0 + \dot{\mathbf{Q}}$. Because the quaturnion must remain a unit quaternion we can consider only $\mathbf{Q}$ as specifying the entire quaturnion, with $q_0(\mathbf{Q}) = \sqrt{1 - Q^2}$. We perturb $\mathbf{q}$ with an infinitesimal time dependent rotation specified by $\boldsymbol{\Omega}$. Using quaternion multiplication

$$\dot{\mathbf{q}} = \frac{d}{dt}[(1 + \boldsymbol{\Omega} t/2)\mathbf{q}] = \frac{1}{2}[-(\boldsymbol{\Omega} \cdot \mathbf{Q}) + q_0 \boldsymbol{\Omega} + \boldsymbol{\Omega} \times \mathbf{Q}] \tag{136}$$

is not the same as $\boldsymbol{\Omega}$. The vector part

$$\dot{\mathbf{Q}} = \frac{1}{2}[q_0 \boldsymbol{\Omega} + \boldsymbol{\Omega} \times \mathbf{Q}] = \frac{1}{2} \begin{pmatrix} q_0 & q_z & -q_y \\ -q_z & q_0 & q_x \\ q_y & -q_x & q_0 \end{pmatrix} \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix}$$

By inverting this I could write the Hamiltonian in terms of $\dot{\mathbf{Q}}$. Then I can take derivatives of the Hamiltonian and find a set of canonical momenta that are conjugate to the vector part of the quaternion. I invert this and find

$$\boldsymbol{\Omega}(\mathbf{q}, \dot{\mathbf{q}}) = 2q_0^{-1} \begin{pmatrix} (q_0^2 + q_x^2)\dot{q}_x + (-q_0 q_z + q_x q_y)\dot{q}_y + (q_0 q_y + q_x q_z)\dot{q}_z \\ (q_0 q_z + q_x q_y)\dot{q}_x + (q_0^2 + q_y^2)\dot{q}_y + (-q_0 q_x + q_y q_z)\dot{q}_z \\ (q_x q_z - q_0 q_y)\dot{q}_x + (q_0 q_x + q_y q_z)\dot{q}_y + (q_0^2 + q_z^2)\dot{q}_z \end{pmatrix}$$

$$\Omega_i = 2q_0^{-1}[q_0^2 \dot{q}_i + q_0 \epsilon_{ijk} q_j \dot{q}_k + q_i q_j \dot{q}_j] \tag{137}$$

$$\boldsymbol{\Omega} = 2q_0^{-1}[q_0^2 \dot{\mathbf{Q}} + q_0 \mathbf{Q} \times \dot{\mathbf{Q}} + \mathbf{Q}(\mathbf{Q} \cdot \dot{\mathbf{Q}})] \tag{138}$$

$$\frac{\partial \Omega_i}{\partial \dot{q}_j} = 2q_0^{-1}[q_0^2\delta_{ij} - q_0\epsilon_{ijk}q_k + q_iq_j]$$

The kinetic energy

$$T(\mathbf{Q}, \dot{\mathbf{Q}}) = \frac{1}{2}\mathbf{\Omega}\mathbf{I}\mathbf{\Omega}$$

and $\mathbf{\Omega}(\dot{\mathbf{Q}}, \mathbf{Q})$ and $\mathbf{I}(\mathbf{Q})$.

$$\frac{\partial T}{\partial \mathbf{\Omega}} = \mathbf{I}\mathbf{\Omega} = \mathbf{L}$$

in terms of the angular momentum with $\mathbf{I}(\mathbf{Q})$ and $\mathbf{\Omega}(\mathbf{Q}, \dot{\mathbf{Q}})$.

$$\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial T}{\partial \Omega_i}\frac{\partial \Omega_i}{\partial \dot{q}_j} = I_{il}\Omega_l\frac{\partial \Omega_i}{\partial \dot{q}_j} \tag{139}$$

$$\pi_j = 2(L_jq_0 + \epsilon_{ijk}L_iq_k + L_iq_iq_j/q_0) \tag{140}$$

$$\frac{\partial T}{\partial \dot{\mathbf{Q}}} = \boldsymbol{\pi} = 2(\mathbf{L}q_0 + \mathbf{L}\times\mathbf{Q} + (\mathbf{L}\cdot\mathbf{Q})\mathbf{Q}q_0^{-1}) \tag{141}$$

This is linear in $\mathbf{L}$ so can also be inverted to solve for $L(\pi)$.

As a guess

$$\mathbf{L} = \frac{1}{2}\left(\boldsymbol{\pi}\times\mathbf{Q} + \frac{\boldsymbol{\pi}}{q_0}\right)$$

and again as a guess

$$T(\boldsymbol{\pi}, \mathbf{Q}) = \frac{1}{8}\left((\boldsymbol{\pi}\times\mathbf{Q})\mathbf{I}(\mathbf{Q})(\boldsymbol{\pi}\times\mathbf{Q}) + \frac{\boldsymbol{\pi}\mathbf{I}(\mathbf{Q})\boldsymbol{\pi}}{q_0^2}\right)$$

where I am guessing the cross terms go away by symmetry.

Lagrange's equations would give

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{Q}}} = \frac{\partial T}{\partial \mathbf{Q}}$$

With $T = \frac{1}{2}\mathbf{L}\mathbf{I}^{-1}\mathbf{L}$ we can write $T(\boldsymbol{\pi}, \mathbf{Q})$ and giving canonical coordinates based on the quaturnion vector $\mathbf{Q}$. However, this Hamiltonian has no obvious conserved quantities, and we know that angular momentum should be conserved.

In summary it is possible to use $\mathbf{Q}$ the vector part of the quaturnion as a canonical variable and write out Lagrange's equations in terms of $\mathbf{Q}, \dot{\mathbf{Q}}$. It is also possible to derive a canonical momentum $\boldsymbol{\pi}$ that is conjugate to $\mathbf{Q}$ and an associated Hamiltonian that is the kinetic energy that is a function of $T = H(\boldsymbol{\pi}, \mathbf{Q})$. It has the advantage that we make no assumptions about preferred directions for the angles. However it lacks in obvious conserved quantities and seems messy, and this probably accounts for its lack of popularity. Also my choice of $\mathbf{Q}$ as variable gives us a potential problem when $q_0 = 0$. This happens when there is a rotation of $\pi/2$ about a principal axes. One possible way to redo this might be to work with 4 degrees of freedom with the quaternion and treat its unit length as a constraint. Constraints are sometimes ugly to work with in the Hamiltonian view, as you need to work on a submanifold in phase space.

3.13 A Lie-Poisson numerical method for Solid Body rotation with Quaturnions

We keep track of the body orientation with a quaturnion $\mathbf{q}(t)$. The body's spin vector is $\boldsymbol{\Omega}(t)$. Chose a timestep dt for updating $\mathbf{q}$ and $\boldsymbol{\Omega}(t)$. In the absence of extra torques the spin angular momentum vector $\mathbf{p}$ is conserved.

Start with orientation defined with quaturnion $\mathbf{q}(t = 0)$. The quaturnion should let you rotate the body from a coordinate with aligned principal axes to the orientation of the body in an inertial frame. Use initial conditions (current orientation and spin vector) to compute the initial angular momentum vector $\mathbf{p}$.

- Use the current quaturnion $\mathbf{q}(t)$ to compute the inverse of the moment of inertia matrix $\mathbf{I}^{-1}(\mathbf{q})$ from the quaternion rotation. You don't need to integrate over the mass of the body, only rotate or transfer the moment of inertia matrix (tensor) from the body frame into the current orientation frame.
- Compute the current spin vector $\boldsymbol{\Omega} = \mathbf{I}^{-1}\mathbf{p}$.
- Compute unit infinitesimal quaturnion with timestep dt

$$\mathbf{q}_{dt} = \sqrt{1 - \frac{\Omega^2 dt^2}{4}} + \frac{\Omega_i dt}{2}\hat{\mathbf{e}}_i$$

- Compute the new quaternion $\mathbf{q}(t+dt) = \mathbf{q}_{dt}\mathbf{q}(t)$ via quaturnion multiplication. This updates $\mathbf{q}$, describing body orientation at the next timestep.

Repeat these steps.

For torqueless solid body rotation only the quaternion $\mathbf{q}$ is changing and angular momentum $\mathbf{p}$ is conserved. If there are external torques then we can compute them using the body orientation at each timestep. A torque would change only $\mathbf{p}$. So we can alternate applying torques (involving updating the angular momentum) and updating the orientation. We can construct a leap-frog integrator in the usual fashion by taking half step of one operator, full step of the other and then half step of the first operator. Each step would be Lie-Poisson (or symplectic).

This gives a numerical method, though is not necessarily illuminating or helpful for finding tumbling or precession frequencies. An advantage is that we keep track of the body orientation at all times, making it possible to compute additional potential energy terms or torques in a straightforward manner (for example using MacCullagh's formula). A disadvantage of the Euler angles or Andoyer-Deprit variables is that to include additional forces requires relating the Andoyer-Deprit angles to ordinary coordinates, and this involves a series of rotations that depends on the current body orientation and the angular momentum vector with respect to an inertial coordinate system. Methods using Euler angles or Andoyer-Deprit variables suffer from potential gimbal lock and a quaternion method would not.

A good reference is the 1994 paper by Jihad Touma and J. Wisdom on 'Lie-Poisson integrators for Rigid Body dynamics in the Solar System', AJ 107, 1189. Possibly Sussman and Wisdom's book discuss a similar approach to making a quaternion integrator. (Not sure as I haven't found the book, but I saw a reference to it).

4 Tidal Evolution

4.1 Gravitational Potential Expansion in Legendre Polynomials

External to a body where the mass density is zero, the gravitational potential satisfies Laplace's equation

$$\nabla^2 \Phi = 0$$

Working in spherical coordinates r, θ, ϕ, Laplace's equation becomes

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial \Phi}{\partial \mu}\right) + \frac{1}{1-\mu^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

where

$$\mu \equiv \cos\theta$$

We assume a trial solution

$$V_n = r^n S_n(\mu, \phi)$$

The radial derivative term of the trial solution is this

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V_n}{\partial r}\right) = n(n+1) r^n S_n$$

Subbing this into Laplace's equation gives

$$\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial S_n}{\partial \mu}\right) + \frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial \Phi}{\partial \mu}\right) + n(n+1) S_n = 0$$

This equation is known as Legendre's equation. The functions that satisfy it, S_n, are known as spherical harmonics. We can expand the general solution in spherical harmonics

$$\Phi(r, \phi, \theta) = \sum_n \left(A_n r^n + B_n r^{-(n+1)}\right) S_n(\mu, \theta)$$

where S_n is a spherical harmonic.

If the problem has axial symmetry then we can ignore the dependence on ϕ. In this case $S_n(\mu,\phi) = P_n(\mu)$ is a Legendre polynomial satisfying

$$(1-\mu^2)\frac{\partial^2 P_n}{\partial \mu^2} - 2\mu\frac{\partial P_n}{\partial \mu} + n(n+1)P_n = 0$$

Legendre polynomials can be computed with Rodrigues's formula

$$P_n(\mu) = \frac{1}{2^n n!}\frac{d^n(\mu^2-1)^n}{d\mu^n}$$

and the first few are

$$\begin{aligned}
P_0(\mu) &= 1 \\
P_1(\mu) &= \mu = \cos\theta \\
P_2(\mu) &= \frac{1}{2}(3\mu^2-1) = \frac{1}{4}(3\cos 2\theta + 1) \\
P_3(\mu) &= \frac{1}{2}(5\mu^3 - 3\mu) \\
P_4(\mu) &= \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)
\end{aligned}$$

The gravitational potential of an axisymmetric body with mass M and radius R and at $r > R$ (external to the body) is often written as

$$\Phi(r,\theta) = -\frac{GM}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\cos\theta)\right] \tag{142}$$

with coefficients J_n dependent on the internal mass distribution and body shape. Oblate bodies have positive J_2. A comparison of this equation with equation 114 implies that

$$J_2 = \frac{(I_\parallel - I_\perp)}{MR^2}$$

(and I think the sign is correct).

4.2 Expansion of the gravitational potential of a point mass in spherical harmonics

We consider a point mass of mass M_p at position $\mathbf{r}'$. At position $\mathbf{r}$ the gravitational potential is

$$\Phi(\mathbf{r}) = -\frac{GM_p}{|\mathbf{r}-\mathbf{r}'|}$$

We would like to expand the gravitational potential near the origin in spherical harmonics. Assume that we want to know the potential with $r < r'$ or close to the origin.

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}'} = \sqrt{r^2 + r'^2 - 2rr'\cos\mu}$$

with

$$\cos\mu \equiv \frac{\mathbf{r}\cdot\mathbf{r}'}{rr'}$$

The potential has axial symmetry. There is an important direction, that connecting the origin and $\mathbf{r}'$ that is the axis of symmetry.

Using the Legendre polynomials

$$\Phi(\mathbf{r}) = \Phi(r,\mu) = -GM_p \sum_{l=0}^{\infty} \frac{r^l}{r'^{l+1}} P_l(\mu)$$

and the expansion is good (converges) as long as $r < r'$.

Let's look at the first few terms.

$$\begin{aligned} \Phi(r,\mu) &= -\frac{GM_p}{r'}P_0(\mu) - GM_p\frac{r}{r'^2}P_1(\mu) - GM_p\frac{r^2}{r'^3}P_2(\mu) \quad \ldots &(143)\\ &= -\frac{GM_p}{r'} - GM_p\frac{r}{r'^2}\mu - GM_p\frac{r^2}{r'^3}\frac{1}{2}(3\mu^2-1) \quad \ldots &(144)\end{aligned}$$

Consider M_p as an external mass perturbing a body of mass M with center of mass at $\mathbf{r}$. If we integrate the $l = 1$ term over the body, the condition that $\mathbf{r}$ is at the center of mass implies that that the total potential would be zero. The $l = 2$ term is a quadrupole term and that contributes the strongest non-trivial term to the tidal potential caused by M_p.

4.3 Tidal deformation of a spherical body

We consider an external perturber of mass M_p on a spherical body of mass M and radius R. The tidal potential, V_T, on the surface has size $V_T \sim GM_pR^2/D^3$ where D is the distance to the perturber. More accurately the tidal potential

$$V_T(r,\mu) = \frac{GM_p r^2}{D^3}P_2(\mu)$$

We assume the surface of the deformed body is an equipotential surface.

$$gh(\mu) \sim V_T(R,\mu)$$

where $g = GM/R^2$ is the surface acceleration and $h(\mu)$ is the surface height. We can solve for the surface height giving

$$\frac{h(\mu)}{R} = \frac{M_p}{M}\frac{R^3}{D^3}P_2(\mu)$$

4.4 Love numbers for fluid bodies

A homogenous density nearly spherical body of mean radius C has surface radius

$$R(\theta, \phi) = C(1 + h_2 P_2(\cos\theta) + h_3 P_3(\cos\theta)) \tag{145}$$

with P_2, P_3 Legendre polynomials. This could be expansed to contain additional and higher order spherical harmonics. Here I have included quadrupole and octupole terms. The gravitational potential to the octupole term outside the body $r > C$

$$\begin{aligned} V(r,\theta,\phi) &= -\frac{GM}{r}\left(1 + k_2\left(\frac{C}{r}\right)^2 P_2(\cos\theta) + k_3\left(\frac{C}{r}\right)^3 P_3(\cos\theta)\right) \\ &= -\frac{GM}{r}\left(1 + \frac{3}{5}\left(\frac{C}{r}\right)^2 h_2 P_2(\cos\theta) + \frac{3}{7}\left(\frac{C}{r}\right)^3 h_3 P_3(\cos\theta)\right) \end{aligned} \tag{146}$$

The coefficients for both expressions can be called Love numbers. They are related by

$$k_l = \frac{3}{2l+1} h_l \tag{147}$$

They depend on l which is the multipole index. Above I have only included $l = 2, 3$ terms but higher order expansions should be consistent with this expression. This expression is derived by expanding $1/\Delta$ where Δ is the distance between a point in the shell and a distant point θ, ϕ and then integrating over the mass shell. The integration is easier using spherical harmonics. This is done by M+D with unclearly defined (vaguely normalized) spherical harmonics.

To summarize: With a deformed body described by the Love numbers for the surface deformation, h_l, we can find the external gravitational potential of the deformed body and it is described with Love numbers k_l.

The tidal potential from an external point mass body of mass $m_{\rm pert}$ expanded in the coordinate system at the center of M is

$$V(r,\theta,\phi) = -\frac{Gm_{pert}}{a}\left[\left(\frac{r}{a}\right)^2 P_2(\cos\theta) + \left(\frac{r}{a}\right)^3 P_3(\cos\theta)\right] \tag{148}$$

where the tidal perturber is oriented along $\theta = 0$ and is a distance a away from M. The surface of a tidally deformed fluid body should be an equipotential surface, so the two expressions (equation146 and 148) for the potential should be equal on the surface.

On the surface (where $r = R(\theta, \phi)$), equation 146 gives

$$V(\theta,\phi) = -\frac{GM}{C}\left[1 - h_2 P_2(\cos\theta) - h_3 P_3(\cos\theta) + \frac{3}{5} h_2 P_2(\cos\theta) + \frac{3}{7} h_3 P_3(\cos\theta)\right] \tag{149}$$

$$= -\frac{GM}{C}\left[1 + \frac{2}{5} h_2 P_2(\cos\theta) + \frac{4}{7} h_3 P_3(\cos\theta)\right] \tag{150}$$

where I have expanded to first order in the Love numbers. The coefficients are

$$1 - \frac{3}{2l+1} = \frac{2l-2}{2l+1} \tag{151}$$

On the surface equation 148 gives

$$V(\theta, \phi) = -\frac{Gm_{pert}}{a}\left[\left(\frac{C}{a}\right)^2 P_2(\cos\theta) + \left(\frac{C}{a}\right)^3 P_3(\cos\theta)\right]. \tag{152}$$

Here we just set $r = C$ because $m_{\rm pert}$ is small so we only take the zero-th order term. We match coefficients and find Love numbers

$$h_2 = \frac{5}{2}\frac{m_{\rm pert}}{M}\left(\frac{C}{a}\right)^3 \tag{153}$$

$$h_3 = \frac{7}{4}\frac{m_{\rm pert}}{M}\left(\frac{C}{a}\right)^4 \tag{154}$$

The coefficients are

$$h_l = \frac{2l+1}{2l-2}\frac{m}{M}\left(\frac{C}{a}\right)^{l+1} \tag{155}$$

Inserting these back into equation 146 gives

$$k_l = \frac{3}{2(l+1)}\frac{m_{\rm pert}}{M}\left(\frac{C}{a}\right)^{l+1} \tag{156}$$

and this agrees with expressions by Efroimsky+12, though the factors of $m_{\rm pert}/M$ and C/a are not included in their coefficients.

What is the external gravitational potential of our tidally perturbed fluid body?

$$V(r, \theta, \phi) = -\frac{GM}{r}\left(1 + \frac{3}{5}\frac{m_{\rm pert}}{M}\left(\frac{C}{a}\right)^3\left(\frac{C}{r}\right)^2 P_2(\cos\theta) + \frac{3}{7}\frac{m_{\rm pert}}{M}\left(\frac{C}{a}\right)^4\left(\frac{C}{r}\right)^3 P_3(\cos\theta)\right) \tag{157}$$

The dependence on distance and mass ratio is usually not included in the Love numbers so that they are independent of the tidal perturbation. If we follow this convention and evaluate the potential at $r = a$,

$$V(a, \theta, \phi) = -\frac{GM}{a}\left(1 + \sum_l k_l \frac{m_{\rm pert}}{M}\left(\frac{C}{a}\right)^{2l+1} P_l(\cos\theta)\right) \tag{158}$$

This is the potential from M at the location of $m_{\rm pert}$ due to tidal deformation of M by the tide caused by $m_{\rm pert}$. The quadrupolar term is the strongest one

$$V_2 \sim -k_2 \frac{GM}{a}\frac{m_{\rm pert}}{M}\left(\frac{C}{a}\right)^5 P_2(\cos\theta) \tag{159}$$

We will use this below when we estimate tidally generated torque on a spinning body.

4.4.1 Love numbers for an incompressible homogeneous elastic body

When the body is not a fluid, the deformation is lower (h_2 is lower) and the Love number h_2 is inversely proportional to the rigidity or the elastic modulus of the deformed body. Stiffer bodies deform less than stretchy ones. For an incompressible homogeneous elastic body the Love number

$$k_2 \sim 0.038 e_g/\mu$$

where μ is the rigidity and

$$e_g = \frac{GM^2}{R^4}$$

and $R = C$. The other Love number h_2 can be computed using equation 147.

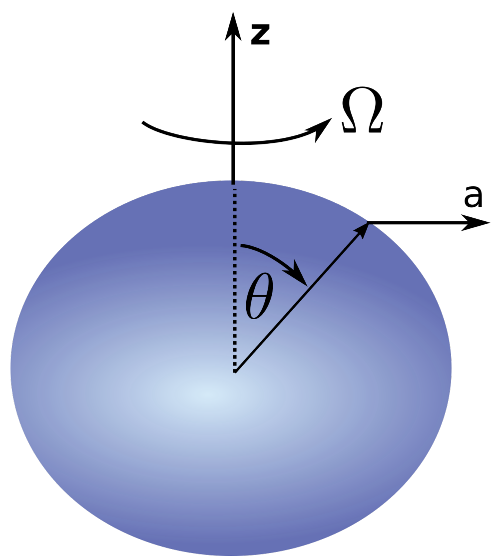

Figure 12: Illustration of the rotational deformation of a spinning body.

4.5 Rotational Deformation

We assume we have an axisymmetric body of mass M with axis of symmetry about the z axis which is also the spin axis. The external gravitational potential for our axisymmetric spinning body with mass M

$$V(r, \mu) = -\frac{GM}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\mu)\right] \tag{160}$$

(equation 142) where $\mu = \cos\theta$ is the colatitude and R is the equatorial radius. There is no $n = 1$ term as we set the coordinate system to have origin at the center of mass. For an axisymmetric body

$$J_2 = \frac{C - A}{MR^2}$$

where C, A are body principal moments of inertia with $C > A$. For an oblate body, C corresponds to that for the short axis which is also the axis of symmetry. The coefficient

$$J_n = \frac{1}{MR^n} \int_0^R 2\pi r^{2+n} dr \int_{-1}^{1} d\mu \; P_n(\mu)\rho(r, \mu)$$

where $\rho(r, \mu)$ is the internal density distribution.

If a body is undergoing solid body rotation, the centrifugal acceleration at radius r and angle θ is $\mathbf{a}_{cf} = \Omega^2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})$ (see Figure 12). The centrifugal force gives an effective potential

$$V_{cf}(r, \theta) = \frac{1}{2}\Omega^2 r^2 \sin^2\theta$$

where θ is the colatitude (which is zero from the +z axis) and goes to π on the -z axis. It is convenient that we can write the centrifugal potential in terms of $P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$ with

$$V_{cf}(r, \theta) = \frac{1}{3}\Omega^2 r^2 (P_2(\mu) - 1)$$

Adding together the centrifugal term and the J_2 term arising from the internal mass distribution the external potential including the centrifugal term

$$V(r, \mu) = -\frac{GM}{r} + \left[\frac{GMR^2}{r^3} + \frac{1}{3}\Omega^2 r^2\right] P_2(\mu).$$

We also expand the surface in spherical harmonics with the surface

$$r_s(\mu) = R(1 + \epsilon P_2(\mu))$$

where ϵ is small. We expand the potential on the surface and assume that it is an equipotential and ignore terms that are not $\propto P_2$,

$$V(r_s(\mu)) = -\frac{GM}{R} + \frac{GM}{R}\epsilon P_2(\mu) + \left[\frac{GM}{R}J_2 + \frac{1}{3}\Omega^2 R^2\right] P_2(\mu) = \text{constant}. \tag{161}$$

We solve for ϵ

$$-\epsilon = J_2 + \frac{1}{3}\frac{\Omega^2 R^3}{GM} = J_2 + \frac{q}{3} \tag{162}$$

where

$$q \equiv \frac{\Omega^2 R^3}{GM}$$

is a measure of centrifugal support. If $q = 1$, the body flies apart.

Using the formula for $P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$ we compute the polar and equatorial radii, the surface has at the pole with $\theta = 0$, $\mu = 1$, $r_{s,\rm polar} = R(1+\epsilon)$. At the equator, $\theta = \pi/2$, $\mu = 0$, $r_{s,\rm equatorial} = R(1 - \epsilon/2)$. A measure of the oblateness

$$f \equiv \frac{r_{\rm equatorial} - r_{\rm polar}}{r_{\rm equatorial}} = -\frac{3}{2}\epsilon.$$

We insert this into equation 162 and find

$$f = \frac{3}{2}J_2 + \frac{q}{2} \tag{163}$$

Equation 163 gives a relation between rotation rate (through q), oblateness (through f) and internal mass distribution (through J_2).

A variant of this equation that is written using the moment of inertia instead of J_2 is called the Darwin Radau relation. The relation is a bit more complicated as it solves for the body shape assuming that the rotating and distorted body is in hydrostatic equilibrium.

4.6 Tidal spin down

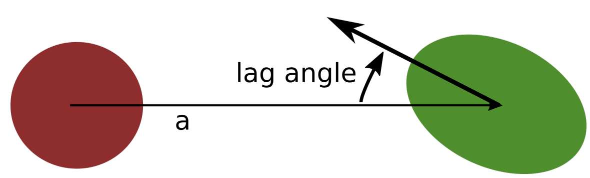

Figure 13: Illustration of the lag angle. The body on the left tidally perturbs the one on the right which becomes elongated. But since it is spinning its deformation is not exactly aligned with the perturber. This gives a torque on the spinning body causing it to spin down or up. Because angular momentum is conserved, this also caused the semi-major axis a to drift.

We assume principal axis rotation of a spherical homogeneous moon with mass M that is perturbed tidally by a body M_*. We assume that they are in circular orbit with semi-major axis a and zero obliquity. The radius of M is R. We assume that M is not tidally locked (is not in a spin synchronous state).

We estimate the torque on M_* by the quadrupolar potential term from M's deformation assuming there is a slight angular shift in V_2 given by equation 159 (but with $m_{pert} = M_*$

and $C = R$). The potential term (equation 159)

$$V_2 \sim -k_2 \frac{GM_*}{a} \left(\frac{R}{a}\right)^5 P_2(\cos\theta) \tag{164}$$

Using $P_2(\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ we add a phase shift δ to the orientation giving

$$V_2 \sim -k_2 \frac{GM_*}{a} \left(\frac{R}{a}\right)^5 P_2(\cos(\theta+\delta)) \tag{165}$$

The torque on M_* is $\mathbf{T} = M_*\mathbf{r} \times -\boldsymbol{\nabla} V_2$. We want to evaluate the torque at $\theta = 0$ and we want only the the z component (that is aligned with the orbit normal and spin axis of M)

$$\begin{aligned}
T &= M_* \frac{\partial V_2}{\partial \theta}\bigg|_{\theta=0} \\
&= k_2 \frac{GM_*^2}{a} \left(\frac{R}{a}\right)^5 \frac{\partial}{\partial\theta}\left(\frac{1}{2}(3\cos^2(\theta+\delta) - 1)\right)\bigg|_{\theta=0} \\
&= k_2 \frac{GM_*^2}{a} \left(\frac{R}{a}\right)^5 3\cos(\theta+\delta)\sin(\theta+\delta)\bigg|_{\theta=0} \\
&\approx \frac{3}{2} k_2 \frac{GM_*^2}{a} \left(\frac{R}{a}\right)^5 k_2 \sin(2\delta).
\end{aligned}$$

The secular part of the semi-diurnal ($l = 2$) term in the Fourier expansion of the perturbing potential from point mass M_*, gives a tidally induced torque

$$T = \frac{3}{2} \frac{GM_*^2}{a} \left(\frac{R}{a}\right)^5 k_2 \sin\epsilon_2$$

where k_2 is the Love number and $k_2 \sin\epsilon_2$ is known as the quality function. The quality function is often approximated as k_2/Q where Q is a dissipation factor and k_2 is a Love number (but with out factors that depend on mass ratio or distance).

Because the torque is equal and opposite, this torque is exerted on the spinning body M causing its spin down.

$$T \sim I\dot{\omega}$$

where I is the moment of inertia about the spin axis and ω is the spin rate.

So far we have not made any approximations that depend upon whether M is larger or smaller than M_*. With $M_* > M$ the orbital period is $P = 2\pi/n$ where mean motion

$$n = \sqrt{GM_*/a^3}.$$

Spin down times are often written in terms of P.

$$t_{despin} \approx \frac{\omega_0}{\dot{\omega}}$$

where ω_0 is the initial spin rate. We can approximate our spinning body as a sphere with moment of inertia $I \sim \frac{2}{5}MR^2$. If we chose $\omega_0 = \sqrt{GM/R^3}$ equal to a centrifugal break up speed then the timescale for spin-down

$$\begin{aligned} t_{despin} &\approx \frac{\omega_0}{\dot{\omega}} = \frac{\sqrt{GM/R^3} I}{I\dot{\omega}} = \sqrt{\frac{GM}{R^3}} \frac{\frac{2}{5}MR^2}{T} \\ &= \frac{P}{15\pi} \left(\frac{M}{M_*}\right)^{\frac{3}{2}} \left(\frac{a}{R}\right)^{\frac{9}{2}} \frac{Q}{k_2} \end{aligned}$$

Recent estimates for shear modulus and dissipation factor for icy or rocky rubble give $\mu Q \sim 10^{11}$ Pa (e.g., Pravic et al. 2014). This is consistent with shear modulus $\mu = 1$ GPa and dissipation factor $Q = 100$.

Because angular momentum is conserved there is a relation between the spin down rate and the drift rate in orbital semi-major axis. If $M_* > M$ then the orbital angular momentum $L \approx M\sqrt{GM_* a}$ and

$$T = \dot{L} = \frac{M}{2}\sqrt{\frac{GM_*}{a}}\dot{a} = \frac{Mn\dot{a}a}{2}.$$

With this we can estimate the drift rate in orbital semi-major axis $\dot{a}$ from the torque associated with spin down (in the case that the spinning object $M < M_*$ is low mass)

$$\begin{aligned} t_{drift} &= \frac{a}{\dot{a}} = \frac{Mna^2}{2T} \\ &= \frac{1}{3n}\left(\frac{M}{M_*}\right)\left(\frac{a}{R}\right)^5 \frac{Q}{k_2} \\ &= \frac{P}{6\pi}\left(\frac{M}{M_*}\right)\left(\frac{a}{R}\right)^5 \frac{Q}{k_2} \end{aligned}$$

4.6.1 Conventions for delay

The torque can be written explicitly in terms of a phase delay. The delay can be a function of the frequency of tidal perturbations on the body which is $n - \omega$. Some popular conventions are Darwin-Kaula-Golreich constant phase delay giving a constant $Q = 1/\epsilon_2$. Also popular are constant time delay (Darwin-Mignard) giving $\epsilon_2 = (\omega - n)\Delta t$ with constant Δt and in this case $Q(\omega - n) = 1/\epsilon_2$ is frequency dependent. If the body is solid, another approach is to adopt a viscoelastic model for the body's interior. The time dependent elastic compliance can be used to predict a frequency dependent quality function (see numerous papers by Michael Efroimsky).

4.6.2 Associated orbital drift rate during spin down

Using conservation of angular momentum

$$\dot{a} = \text{sign}(\omega - n)\frac{3k_2}{Q}\frac{M}{M_*}\left(\frac{R}{a}\right)^5 na$$

4.7 Other timescales

The timescale for wobble damping or non-principal axis rotation to decay is

$$t_{\text{wobble}} \sim t_{\text{despin}}\left(\frac{n}{\omega}\right)^4$$

where ω is the spin of the moon.

Once tidally locked and at zero obliquity the timescale for eccentricity damping

$$t_{\text{edamp}} \approx \frac{2}{21}\frac{M}{M_*}\left(\frac{a}{R}\right)^5 \frac{Q}{k_2} n^{-1} \tag{166}$$

The asphericity parameter

$$\alpha \equiv \sqrt{3(B-A)/C}$$

and $A < B < C$ are the body's principal axis moments of inertia. For a triaxial ellipsoid with body semi-axis lengths $a_b > b_b > c_b$ the moments of inertia are $A = \frac{M}{5}(b_b^2 + c_b^2)$, $B = \frac{M}{5}(a_b^2 + c_b^2)$, $C = \frac{M}{5}(a_b^2 + b_b^2)$ and the asphericity

$$\alpha = \sqrt{\frac{3(B-A)}{C}} = \sqrt{\frac{3(a^2-b^2)}{a^2+b^2}}$$

.

The frequency of free libration in the tidally locked state is

$$\omega_{lib} \sim \alpha n$$

If the body is nearly spherically symmetric, the libration frequency is slow.

5 Spin Resonances

Two approaches to spin resonance. You can expand the potential perturbation taking into account the orientation of the body. Then with some work you force a Hamiltonian to look like a pendulum model. Alternatively you compute the torque on the body and try to find the equations of motion for the body spin/orientation. You also try to force the equations of motion to look like a pendulum model.

5.1 Types of Spin Resonance

A number of possible types of resonance are possible. I will list them!

1. Spin-orbit resonance. Spin rate is a half multiple of mean motion giving

$$w = \frac{jn}{2}$$

   where n is the mean motion, w is the spin and j is an integer. Mercury is in a 3:2 spin-orbit resonance. If the body is not spherical and on an eccentric orbit, the spin-orbit resonances overlap and the dynamics is chaotic. Spin orbit resonance can be altitude unstable so when chaotic the body will start tumbling. Hyperion (very elongated and in an eccentric orbit) is in such a tumbling state.

2. Spin-secular resonance. There is a relation between the planet's spin precession rate and an orbital precession rate (like $\dot{\Omega}$) induced by other planets. Saturn's obliquity is explained with such a model by Bill Ward and Doug Hamilton where the obliquity is increased by drifting within such a resonance.

3. Spin-Spin resonance. Two non-spherical asteroids in orbit about each other. Their spins are multiples of each other.

4. Spin-Binary resonance. Pluto-Charon's binary orbital frequency could affect the spin states of Pluto and Charon's minor satellites.

$$\frac{j}{2}(n - n_b) = w - n_b$$

   See recent work by Alexandre Correia.

5. Mean motion-spin precession. A mean motion resonance between two objects somehow affects the spin of one of them. We saw this in some of our simulations for Pluto and Charon's minor satellites (and we wrote in 2017 about it).

6. Wobble excitation resonance. The same simulations also showed Pluto and Charon's binary perturbations exciting wobbling primarily in the minor satellite Kerberos (but also sometimes in Styx).

There are probably more types of spin resonances!

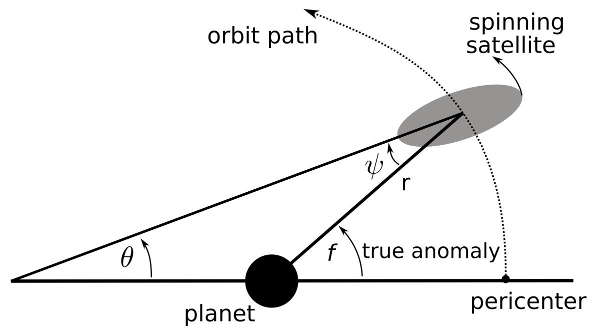

Figure 14: Angles relevant for spin orbit resonance

5.2 Spin-Orbit resonance

We take a non-spherical body m in a nearly circular orbit about mass M_p. The non-spherical body is spinning about the body axis giving its smallest principal moment of inertia C and with orbit normal and spin axis coinciding (undergoing principal axis rotation). The equation of motion for body orientation angle θ (with respect to the inertial frame)

$$C\ddot{\theta} + \frac{3}{2}\frac{GM_p}{r^3}(B - A)\sin(2(\theta - f)) = 0 \tag{167}$$

where moments of inertia are $A \geq B \geq C$. Here $n = \sqrt{GM_p/a^3}$ is the orbit mean motion, in terms of the semi-major axis a. The angle θ describes body orientation (see Figure 14) in an inertial frame and f is the orbital true anomaly. For a circular orbit $f = nt$. The right term we recognize as torque arising from MacCullagh's potential term. The left term is the torque on the spinning body itself. Here we have only torque from the central mass on the spinning body in orbit. There could be additional torques (such as tidal). This setting is relevant for study of spin-orbit resonance for Mercury in orbit about the Sun or Phobos in orbit about Mars.

Expanding f in eccentricity gives a series of terms depending on multiples of the mean anomaly M. To first order $f = M + 2e\sin M$, $\cos f = \cos M + e(\cos 2M - 1)$ and $\sin f = \sin M + e\sin 2M$. Writing $\sin 2(\theta - f) = \sin 2\theta \cos 2f - \cos 2\theta \sin 2f$ one can replace the true anomaly with sums containing sines and cosines with the mean anomaly. We also need to expand the radius as $r^{-1} = \frac{(1+e\cos f)}{a(1-e^2)}$.

After doing the expansion in eccentricity we should get

$$\frac{GM_p}{r^3}\sin(2(\theta - f)) = n^2 \sum_p H(p, e)\sin(2(\theta - pM))$$

summing over non-zero half integer p and with n the mean motion. The coefficients $H(p,e)$ are polynomials in eccentricity e. To lowest order in e

$$\begin{aligned} H(1,e) &= 1 \\ H(1/2,e) &= -e/2 \\ H(3/2,e) &= 7e/2 \\ H(2,e) &= 17e^2/2 \end{aligned}$$

The higher the half integer $|p|$, the higher the power of eccentricity in the lowest term; $H(p,e) = O(e^{2|p-1|})$

It is convenient to transfer to a new angle $\gamma = \theta - pM$ for a single half integer p. Then the equation of motion (and taking only a single sine term) becomes

$$\ddot{\gamma} + \frac{3}{2}n^2 \frac{(B-A)}{C} H(p,e) \sin 2\gamma = \frac{\langle T \rangle}{C}$$

Resonance capture can be studied using tidal evolution for the averaged torque $\langle T \rangle$. Recent work showing that all terms need to be taken into account to estimate the capture probability which is strongly dependent on body shape (through the moments of inertia) and eccentricity.

The resonances are separated by half integers of mean motion. The spin synchronous or tidal lock term has $p = 1$ and has libration width dependent on $n\sqrt{\frac{B-A}{C}}$. The 3/2 term has libration width that depends $n\sqrt{e\frac{B-A}{C}}$. Using widths and distance between resonance Jack Wisdom developed a resonance overlap criterion for chaotic behavior that was applied to Hyperion. Apparently once the system is chaotic it is also attitude unstable and so will start tumbling.

5.3 The dissipative spin-orbit resonance problem

It is convenient to assume that the torque on the spinning body due to tides is linearly dependent on the angular velocity giving a modified spin orbit equation of motion (from equation 167)

$$C\ddot{\theta} + \frac{3}{2}\frac{GM_p}{r^3}(B-A)\sin(2(\theta - f)) = -K_d(\dot{\theta} - n) \tag{168}$$

where the functions $L(e)$ and $N(e)$ are averaged over the orbit and the dissipation factor

$$K_d = \frac{3n}{C}\frac{k_2}{Q}\left(\frac{R}{a}\right)^2 \frac{M_*}{M} \tag{169}$$

6 Problems

1. **Reference frames for a symmetric top**

   Consider a rigid axi-symmetric body with no external forces. In a coordinate frame aligned with body's axis of symmetry the moment of inertia matrix has 3 eigenvalues $I_{\|}, I_{\perp}, I_{\perp}$ and two of the them are the same. If the axis of symmetry lies along the z axis then $I_{\|} = I_z$. The body is spinning but with angular momentum and spin not parallel to z or in the xy plane. The body is not spinning about a principal body axis. Euler's equations are

$$
\begin{aligned}
I_{\perp}\dot{\omega}_x + (I_{\|} - I_{\perp})\omega_z\omega_y &= 0 && (170)\\
I_{\perp}\dot{\omega}_y - (I_{\|} - I_{\perp})\omega_z\omega_x &= 0 && (171)\\
I_{\|}\dot{\omega}_z &= 0 && (172)
\end{aligned}
$$

   for spin $\boldsymbol{\omega}$ in the body's reference frame. A solution is

$$
\begin{aligned}
\omega_x &= \omega \sin\alpha \cos(\Omega t) && (173)\\
\omega_y &= \omega \sin\alpha \sin(\Omega t) && (174)\\
\omega_z &= \omega \cos\alpha && (175)\\
\Omega &= \omega \cos\alpha \left(\frac{I_{\|} - I_{\perp}}{I_{\perp}}\right) && (176)
\end{aligned}
$$

   The angular momentum vector (in the body's frame)

$$
\begin{aligned}
L_x &= I_{\perp}\omega \sin\alpha \cos(\Omega t) && (177)\\
L_y &= I_{\perp}\omega \sin\alpha \sin(\Omega t) && (178)\\
L_z &= I_{\|}\omega \cos\alpha && (179)
\end{aligned}
$$

   a) Show that in the inertial frame that the angular momentum vector is constant.

   b) What is the spin vector in the inertial frame?

   c) What are the trajectories of the body x, y, z axes as seen in an inertial frame?

   Hint:

$$
\frac{d\mathbf{L}}{dt} = \frac{\partial \mathbf{L}}{\partial t} + \boldsymbol{\omega} \times \mathbf{L}
$$

2. **Quadrupole moment and moment of Inertia**:

   The Quadrupole moment tensor of an extended mass mass distribution with density $\rho(\mathbf{x})$ is

$$
Q_{\alpha\beta} \equiv \int \rho(\mathbf{x})(3x_\alpha x_\beta - \delta_{\alpha\beta} r^2)\ d^3\mathbf{x}
$$

where $r^2 = \sum_i x_i^2$. The moment of inertia tensor is

$$I_{\alpha\beta} \equiv \int \rho(\mathbf{x})(\delta_{\alpha\beta} r^2 - x_\alpha x_\beta)\ d^3\mathbf{x}.$$

Consider a point mass of mass M at the origin. And consider an extended body of mass m at coordinate position $\mathbf{x} = \mathbf{R}$. The gravitational potential of m is approximately

$$\Phi(\mathbf{R}) = -\frac{Gm}{R} - \frac{GQ_{\alpha\beta}R_\alpha R_\beta}{2R^5}$$

using summation notation (every pair of indices is summed over each coordinate) and only expanding to the quadrupolar term.

Show that the gravitational force on point mass M from extended mass m is $\mathbf{F}$ with components F_i

$$\frac{F_i}{M} = \frac{GmR_i}{R^3} - \frac{GQ_{\alpha i}R_\alpha}{R^5} + \frac{5}{2}\frac{R_i}{R^7} GQ_{\alpha\beta}R_\alpha R_\beta$$

Show that the torque on M has components

$$T_i = \epsilon_{ijk} R_j F_k = -M\epsilon_{ijk}\frac{GR_j Q_{k\alpha} R_\alpha}{R^5}$$

where ϵ_{ijk} the Levi-Civita symbol.

What is the torque on m?

Show that we can replace Q with I giving

$$T_i = 3M\epsilon_{ijk}\frac{R_j G I_{k\alpha} R_\alpha}{R^5}$$

(hint: use symmetry or antisymmetry to drop terms containing the trace).

Show that this can be derived from MacCullagh's formula.

3. **Tumbling or wobbling decay due to internal dissipation**

   It is common to assume that celestial bodies are spinning about a principal axis because the timescale for wobble to decay due to internal dissipation is shorter than the tidal spin-down time.

   Explain why a stiff but elastic body dissipates energy when it is tumbling.

   The wobble decay timescale for a spinning asteroid is

   $$t_{wobble} \sim \frac{3GCQ}{w^3 k_2 R^5}$$

   where R is a mean body radius, C is the maximum moment of inertia about a principal body axis, Q is a unitless energy dissipation parameter, w is the initial spin rate and k_2 is a Love number. The timescale does not depend upon an external tidal field. This formula I think is from Peale 1977 in a book called Planetary Satellites that I cannot get an electronic version of, though it might be in the library.

A similar formula is by Burns and Safronov (1973) (Burns, J.A., and Safronov, V.S. 1973. Asteroid nutation angles. Monthly Notices of the Royal Astronomical Society of London, Vol. 165, pp. 403 - 411). and is

$$\tau \sim \frac{\mu Q}{\rho R^2 w^3} A$$

with constant $A \sim 100$. Here μ, Q, ρ, R, w are the mean shear rigidity, quality factor, density, body radius and spin rate.

The two formulas are consistent with each other if we use $C \sim mR^2$ and $k_2 \sim Gm^2 R^{-4} \mu^{-1}$.

Explain the form of this timescale to order of magnitude.

Hints: Estimate variations in stress, σ, and strain ϵ on the body. Recall that stress is strain times elastic modulus $\sigma \sim \mu\epsilon$. Energy dissipation per unit volume of an elastic material $\dot{e} \sim \sigma\dot{\epsilon}$.

4. **To a spin synchronous state and dissipation timescales**

   Suppose one of Saturn's small moons, such as Pan, suffers an impact, giving it an obliquity, or exciting libration, or exciting tumbling or increasing or decreasing its spin. The same impact could increase orbital eccentricity and orbital inclination.

   How fast does the moon spin down?

   Once in a spin synchronous state (tidally locked) how fast does eccentricity decay?

   Once the spin reaches the orbital mean motion, how fast does the obliquity decay?

   Would the orbital inclination decay and if so, how fast?

   Pan suffers a drift in its semi-major axis, $\dot{a}$, primarily due to tides excited in Saturn.

   Which decays are set by dissipation in Saturn and which by dissipation in Pan?

   Note orbital eccentricity and inclination can also decay due to excitation of waves into Saturn's rings.

5. **Averaging a binary**

   Consider a rapidly spinning binary comprised of masses m_1, m_2 separated by distance a.

   Compute the moment of inertia matrix of a ring with mass m and radius r about its center of mass.

If our binary is rapidly spinning (rotation in a plane) while maintaining the separation a, what is the average value of the moment of inertia matrix (about the center of mass)? In other words what is

$$\langle \mathbf{I} \rangle = \frac{1}{P} \int_0^P dt \; \mathbf{I}$$

where P is the rotation period.

6. **Spin Precession**

   Consider a non-round body of mass m spinning about its maximum principal axis, the direction of its maximum moment of inertia, equivalently the body's minimum axis.

   It orbits a point mass M with $M \gg m$. It is in a circular orbit with radius R, mean motion $n = \sqrt{G(M+m)/R^3}$.

   The angle between the body's spin axis and orbit normal can be called the obliquity, ϵ. Assume that the spin $w \gg n$.

   As a function of ϵ, what is the torque on m caused by the point mass M? What are the components of the torque vector $\mathbf{T}$? Write these in terms of the orientation of the spin vector. Consider the component of the torque that is perpendicular to the spin vector.

   Show that the precession rate is

$$\dot{\Omega} = -\frac{S}{w} \cos \epsilon$$

   with

$$S = \frac{3}{2} n^2 \frac{C - (A+B)/2}{C}$$

   where the moments of inertia (about principal axes) are $C > B > A$.

7. **Tidal evolution**

   For a two-body system of mass M and m the total orbital angular momentum is

$$L_o = \mu \sqrt{G(M+m)a(1-e^2)}$$

   where a and e are the semi-major axis and eccentricity and $\mu = mM/(m+M)$ is the reduced mass. Here L is not angular momentum per unit mass. The total orbital energy is

$$E_o = -\mu \frac{G(M+m)}{2a}$$

With spin parallel to orbital angular momentum and principal axis rotation, the spin of m has angular momentum

$$L_{sm} = C_m \omega_m$$

where C_m is the moment of inertia about the spin axis for mass m and ω_m is m's spin. The kinetic energy associated with m's spin is

$$E_{sm} = \frac{C_m}{2}\omega_m^2$$

We can describe the spin of body M with moment of inertia C_M and spin ω_M.

Body deformation associated with tides dissipate energy in m so total energy (orbit + spin) is not conserved however the total angular momentum (orbit + spin) is conserved.

(a) Consider m only dissipating energy (let M be a point mass) and $\omega_m \neq n$ where n is the mean motion. When $\omega_m = n$ the system is tidally locked or in a spin-synchronous state. What sign is $\dot{a}$? Is $\dot{e}$ important?

Hint: consider the sign of $\omega_m - n$? Is m speeding up or slowing down?

(b) Consider m only dissipating and $\omega_m = n$ but $e > 0$. What sign is $\dot{a}$? Is $\dot{e}$ important? If so what sign is $\dot{e}$?

(c) Io is drifting outwards ($\dot{a} > 0$) but is tidally locked with Jupiter. Why? What mechanism maintains Io's non-zero eccentricity? What type of tide is responsible for Io's vulcanism? Jupiter is spinning and Io is in an eccentric orbit.

(d) The Moon is drifting away from the Earth. Which type of tide is responsible?

(e) Hyperion never reaches a spin-synchronous state. It is tumbling due to strong spin-orbit resonances and attitude instability in them. How much higher is the dissipation rate (by what factor) in Hyperion in the tumbling state compared to the spin-synchronous state? Hyperion's eccentricity is 0.123.

Hint: consider energy damping rate during tidal circularization versus spin down.

8. **Radial distribution of tidally generated internal heat**

Consider an isotropic, homogeneous visco-elastic spherical spinning body of mass m that is orbiting mass M. Consider the tidally generated heat per unit volume $\dot{\epsilon}$. The interior of the body is heated more than the material near the surface. Why is that? Can you estimate how the heat per unit volume scales with radius?

9. **Excitation of Tumbling**

Consider an axi-symmetric torque free body initially spinning about a principal body axis. Recall (equations 67, 69) with precession rate

$$\dot{\phi} = \omega \left[1 + \left(\frac{I_{\parallel}^2}{I_{\perp}^2} - 1 \right) \cos^2 \alpha \right]^{\frac{1}{2}} \tag{180}$$

where α is the angle between spin vector and body principal axis. Since we are considering spinning about a principal body axis consider α small. In the limit of small α and small θ (angle between angular momentum vector and body axis)

$$\begin{aligned} \frac{\dot{\phi}}{\omega} &\approx \frac{I_{\parallel}}{I_{\perp}} \\ \frac{\dot{\psi}}{\omega} &\approx \left(1 - \frac{I_{\parallel}}{I_{\perp}} \right) \end{aligned}$$

Suppose you can perturb the body with an oscillating torque (a torque that averages to zero). What frequency would excite tumbling?

10. **Wobble damping**

A torque free, and freely rotating asteroid has constant angular momentum but its total energy decreases due to elastic dissipation and heat loss.

Consider a body with spin $\mathbf{\Omega} = (\Omega_l, \Omega_i, \Omega_s)$ for long, intermediate and short body axes with moments of inertia I_l, I_i, I_s. The body initially is not spinning about a principal axis.

a) What is the initial rotational kinetic energy? What is the angular momentum?

b) What are the possible final spin states after the wobble has damped away?

In wobbling damping computations, we often assume that the moment of inertia is not time dependent. If normal modes are excited in the asteroid then this is possibly a bad assumption. Asteroids tend to rotate with periods of a few hours to a days.

c) Why is it reasonable to assume that normal modes are not excited during tumbling damping of asteroids?

Hint: estimate the frequency of the slowest normal modes using an elastic modulus and assuming an asteroid size. Compare this frequency to a typical spin period.

11. **Accretion on to Pan**

Suppose Pan is accreting material through one of its Lagrange points at a rate of $\dot{M}$.

What is the rate of change of Pan's angular momentum?

What accretion rate would keep Pan spinning and no longer in tidal lock with Saturn?

How fast does Pan's moment of inertia matrix change due to accretion?

12. **Spin orbit resonance**

We take a non-spherical body m in a nearly circular orbit about mass $M > m$ that is spinning about its principal moment of inertia and with orbit normal and spin axis coinciding (this is principal axis rotation). The equation of motion for body orientation angle θ

$$C\ddot{\theta} + \frac{3}{2}\frac{GM}{r^3}(B - A)\sin(2(\theta - f)) = 0$$

where moments of inertia are $A \geq B \geq C$. Here $n = \sqrt{GM/a^3}$ is the orbit mean motion, in terms of the semi-major axis a. The angle θ describes the orientation of m in an inertial frame and f is the orbital true anomaly and describes the position of m in its orbit about M.

Show that the right term arises from MacCullagh's potential term and is due to the torque on the non-spherical body by M.

Show that the left term is just the torque on the spinning body itself.