MSc in Computer Science and Engineering

Learning and Decision Making 2017-2018

Homework 1. Markov chains

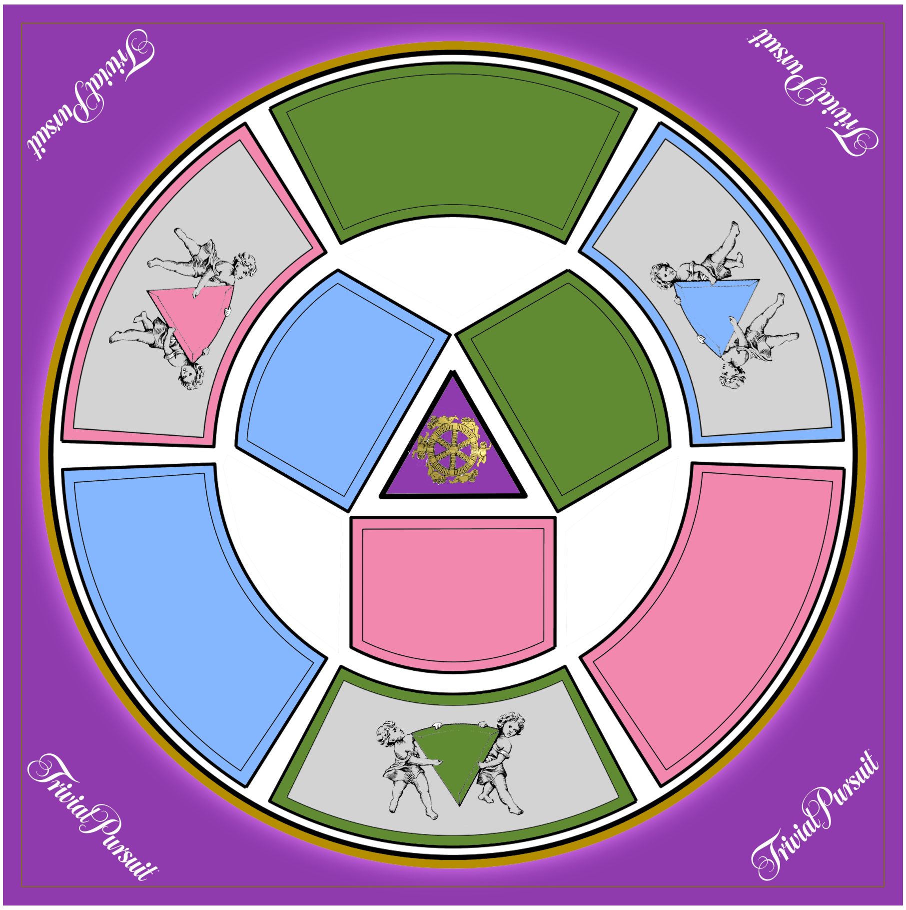


Figure 1: Simplified version of the "Trivial Pursuit" board game. In this version, the board is reduced to 10 distinct positions.

Consider the simplified version of the classic game "Trivial Pursuit" depicted in Fig. 1. In this homework, you will describe the motion of a single player using a Markov chain. To that purpose, consider that the player rolls a single die in each play. Moreover, suppose that whenever the player reaches *any* of the 4 intersections (including the central one), the player goes along any of the three ways (including the one he arrived from) with equal probability.

Exercise 1.

(a) Write down the Markov chain model representing the motion of the player.

Suggestion: consider, first, the transition probabilities corresponding to the situation in which the die shows a "1". The transition probabilities for the other die outcomes can be computed from the previous one and the overall transition probabilities can then be computed from the transition probabilities corresponding to the individual die outcomes.

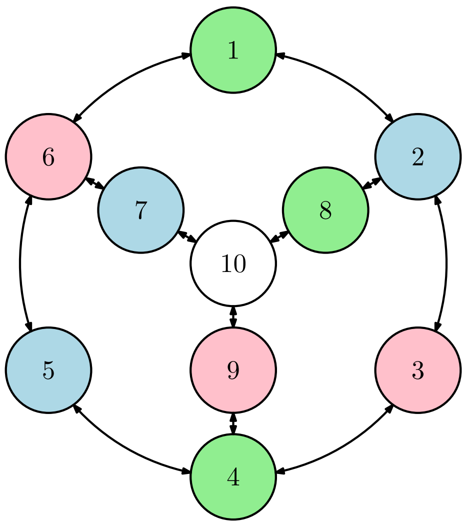
(b) Suppose that the player departs from the central at time step $t = 0$. Compute the probability of the player being in each cell at time step $t = 3$.

Solution:

(a) The Markov chain is specified as a pair $(\mathcal{X}, \boldsymbol{P})$, where the states correspond to player positions on the board. We have that

$$\mathcal{X} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},$$

where the states are numbered as in the following diagram:



The probabilities corresponding to a die outcome of 1, are given by

$$
\boldsymbol{P}_0 = \begin{bmatrix}
0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\
1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \\
0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\
1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\
0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0
\end{bmatrix},
$$

and we can compute the transition probabilities $\boldsymbol{P}$ from the total probability law as

$$
\boldsymbol{P} = \sum_{i=1}^{6} \mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x, \mathrm{d}_t = i\right] \mathbb{P}\left[\mathrm{d}_t = i\right] = \frac{1}{6}\left[\boldsymbol{P}_0 + \boldsymbol{P}_0^2 + \boldsymbol{P}_0^3 + \boldsymbol{P}_0^4 + \boldsymbol{P}_0^5 + \boldsymbol{P}_0^6\right],
$$

where d_t represents the outcome of the die throw at time step t, yielding

$$
\boldsymbol{P} = \begin{bmatrix}
0.12 & 0.19 & 0.08 & 0.06 & 0.08 & 0.19 & 0.08 & 0.08 & 0.04 & 0.06 \\
0.12 & 0.19 & 0.12 & 0.10 & 0.04 & 0.10 & 0.04 & 0.12 & 0.04 & 0.10 \\
0.08 & 0.19 & 0.12 & 0.19 & 0.08 & 0.06 & 0.04 & 0.08 & 0.08 & 0.06 \\
0.04 & 0.10 & 0.12 & 0.19 & 0.12 & 0.10 & 0.04 & 0.04 & 0.12 & 0.10 \\
0.08 & 0.06 & 0.08 & 0.19 & 0.12 & 0.19 & 0.08 & 0.04 & 0.08 & 0.06 \\
0.12 & 0.10 & 0.04 & 0.10 & 0.12 & 0.19 & 0.12 & 0.04 & 0.04 & 0.10 \\
0.08 & 0.06 & 0.04 & 0.06 & 0.08 & 0.19 & 0.12 & 0.08 & 0.08 & 0.19 \\
0.08 & 0.19 & 0.08 & 0.06 & 0.04 & 0.06 & 0.08 & 0.12 & 0.08 & 0.19 \\
0.04 & 0.06 & 0.08 & 0.19 & 0.08 & 0.06 & 0.08 & 0.08 & 0.12 & 0.19 \\
0.04 & 0.10 & 0.04 & 0.10 & 0.04 & 0.10 & 0.12 & 0.12 & 0.12 & 0.19
\end{bmatrix}.
$$

(b) Computing $\boldsymbol{P}^3$ yields

$$\boldsymbol{P}^3 = \begin{bmatrix} 0.09 & 0.13 & 0.08 & 0.12 & 0.08 & 0.13 & 0.08 & 0.08 & 0.08 & 0.12 \\ 0.08 & 0.13 & 0.08 & 0.12 & 0.08 & 0.12 & 0.08 & 0.08 & 0.08 & 0.12 \\ 0.08 & 0.13 & 0.09 & 0.13 & 0.08 & 0.12 & 0.08 & 0.08 & 0.08 & 0.12 \\ 0.08 & 0.12 & 0.08 & 0.13 & 0.08 & 0.12 & 0.08 & 0.08 & 0.08 & 0.12 \\ 0.08 & 0.12 & 0.08 & 0.13 & 0.09 & 0.13 & 0.08 & 0.08 & 0.08 & 0.12 \\ 0.08 & 0.12 & 0.08 & 0.12 & 0.08 & 0.13 & 0.08 & 0.08 & 0.08 & 0.12 \\ 0.08 & 0.12 & 0.08 & 0.12 & 0.08 & 0.13 & 0.09 & 0.08 & 0.08 & 0.13 \\ 0.08 & 0.13 & 0.08 & 0.12 & 0.08 & 0.12 & 0.08 & 0.09 & 0.08 & 0.13 \\ 0.08 & 0.12 & 0.08 & 0.13 & 0.08 & 0.12 & 0.08 & 0.08 & 0.09 & 0.13 \\ 0.08 & 0.12 & 0.08 & 0.12 & 0.08 & 0.12 & 0.08 & 0.08 & 0.08 & 0.13 \end{bmatrix},$$

and the distribution over states for $t = 3$ comes

$$\boldsymbol{\mu}_3 = \begin{bmatrix} 0.08 & 0.12 & 0.08 & 0.12 & 0.08 & 0.12 & 0.08 & 0.08 & 0.08 & 0.13 \end{bmatrix}.$$