On Total Coloring of Some Classes of Regular Graphs

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Abstract: In this paper, we have obtained the total chromatic number of some classes of Cayley graphs, odd graphs and mock threshold graphs.

1 Introduction

All the graphs considered here are finite, simple and undirected.
Let $G = (V(G), E(G))$ be a graph with the sets of vertices $V(G)$ and edges $E(G)$ respectively. A *total coloring* of G is a mapping $f : V(G) \cup E(G) \to C$, where C is a set of colors, satisfying the following three conditions (a)-(c):

(a) $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$
(b) $f(e) \neq f(e')$ for any two adjacent edges $e, e' \in E(G)$ and
(c) $f(v) \neq f(e)$ for any vertex $v \in V(G)$ and any edge $e \in E(G)$ incident to v.

The *total chromatic number* of a graph G, denoted by $\chi''(G)$, is the minimum number of colors that suffice in a total coloring. It is clear that $\chi''(G) \geq \Delta + 1$, where Δ is the maximum degree of G. Behzad [2] and Vizing [19] independently conjectured (Total Coloring Conjecture (TCC)) that for every graph G, $\chi''(G) \leq \Delta + 2$. The total coloring conjecture is a long standing conjecture and has defied several attempts in a complete

1

proof. It is also proved that the decidability algorithm for total coloring is NP-complete even for cubic bipartite graph [17]. But still, a lot of progress has been made in proving the TCC. It is easily seen to be true for complete graphs, bipartite, complete multipartite graphs. It was also showed to be true for all graphs with degree $\Delta \leq 5$ and $\Delta \geq n - 5$, where n is the number of vertices, using techniques like enlarge-matching argument and fan recoloring process [20]. For planar graphs, TCC is proved for all $\Delta \neq 6$ using discharging and charging methods. The total coloring conjecture has also been proved for several other classes of graphs. Good survey of techniques and other results on total coloring can be found in Yap [20], Borodin [4] and Geetha et al. [11]

2 Total Colorings of Cayley Graphs

Cayley graphs are those whose vertices are the elements of groups and adjacency relations are defined by subsets of the groups. Let Γ be a multiplicative group with identity 1. For $S \subseteq \Gamma, 1 \notin S$ and $S^{-1} = \{s^{-1} : s \in S\} = S$ the *Cayley Graph* $X = Cay(\Gamma, S)$ is the undirected graph having vertex set $V(X) = \Gamma$ and edge set $E(X) = \{(a, b) : ab^{-1} \in S\}$. The Cayley graph associated with $\Gamma = \mathbb{Z}_n$, the group of integers modulo n under addition operation, is called a *circulant graph*. Note that the *powers of cycle graph* is a circulant graph with the generating set $S = \{1, 2, \ldots, k, n-k, \ldots, n-2, n-1\}$. All Cayley graphs are vertex transitive.

In other words, given a sequence of positive integers $1 \leq d_1 < d_2 < \ldots < d_l \leq \lfloor \frac{n}{2} \rfloor$, the circulant graph $G = C_n(d_1, d_2, \ldots, d_l)$ has vertex set $V = Z_n = \{0, 1, 2, \ldots, n-1\}$, two vertices x and y being adjacent iff $x = (y \pm d_i) \mod n$ for some $i, 1 \leq i \leq l$ and a graph is a power of cycle, denoted C_n^k, n and k are integers, $1 \leq k < \lfloor \frac{n}{2} \rfloor$, if $V(C_n^k) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(C_n^k) = E^1 \cup E^2 \cup \ldots \cup E^k$, where $E^i = \{e_0^i, e_1^i, \ldots, e_{n-1}^i\}$ and $e_j^i = (v_j, v_{(j+i) \mod n})$ and $0 \leq j \leq n-1$, and $1 \leq i \leq k$.

Campos and de Mello [6] proved that $C_n^2, n \neq 7$, is type-I and C_7^2 is type-II. They [5] verified the TCC for power of cycle C_n^k, n even and $2 < k < \frac{n}{2}$ and also showed that one can obtain a $\Delta(C_n^k) + 2$-total coloring for these graphs in polynomial time. They also proved that C_n^k with $n \cong 0 \mod (\Delta(C_n^k) + 1)$ are type-I and they proposed the following conjecture.

Conjecture 2.1. Let $G = C_n^k$, with $2 \leq k < \lfloor \frac{n}{2} \rfloor$. Then,
$$\chi''(G) = \begin{cases} \Delta(G) + 2, & \text{if } k > \frac{n}{3} - 1 \text{ and } n \text{ is odd} \\ \Delta(G) + 1, & \text{otherwise.} \end{cases}$$

Recently, Zorzi et al. [21] proved the conjecture for certain classes of power of cycle graphs. In particular, for even k and large n the conjecture is true.

A latin square is an $n \times n$ array consisting of n entries of numbers (or symbols) with each row and column containing only one instance of each element. This means the rows and columns are permutations of one single n vector with distinct entries. A latin square is said to be commutative if it is symmetric. A latin square containing numbers is said to be idempotent if each diagonal element contains the number equal to its row (column) number. In addition, if the rows of the latin square are just cyclic permutations (one-shift of the elements to the right) of the previous row, then the latin square is said to be circulant (anti-circulant, if the cyclic permutations are actually left shifts), the matrix

(corresponds to the latin square) can be generated from a single row vector. The latin square

1	k+2	2	k+3	...	2k+1	k+1
k+2	2	k+3	3	...	k+1	1
...	...	...	...	...	...	...
...	...	...	...	...	...	...
k+1	1	k+2	2	...	k	2k+1

is anti-circulant, commutative and idempotent. The entries of the square are as follows:

$$L = (l_{ij}) = \begin{cases} m, & \text{if } i+j = 2m \\ k+1+m, & \text{otherwise.} \end{cases}$$

From the above, it can be easily seen that the latin square corresponding to the matrix L is commutative, idempotent and also anti-circulant.

Campos and de Mello [6], proved the TCC for the powers of cycles of even order. In the following theorems, using latin squares we prove some classes of powers of cycles of even order are Type 1.

Theorem 2.1. *Let G be a power of cycle graph C_n^k with $k = \frac{n-2}{4} + k'$ with n, k, k' being non-negative integers with $n \geq 4$, n even and $gcd(n, x) = 1$, $\frac{n-2}{4} < x < k'$. Then, G is a type I graph.*

Proof. The adjacency matrix of a power of cycle graph C_n^k (or, in fact, any circulant graph) is a symmetric circulant matrix $C = (c_{ij})$ with 1's when i, j differ by s, where $s \in S$, the generating set of the Cayley graph C_n^k. For example, the adjacency matrix of C_{10}^2 is

$$A(C_{10}^2) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline \end{array}.$$

We know that $\Delta(C_n^k) = 2k$. For giving a total $2k+1$ coloring of C_n^k in the case $k = \frac{n-2}{4}$, where n, k are non-negative integers, n is even, we form the color matrix (a matrix which gives the color of the vertices in the diagonals and the color corresponding to edges in the other entries) by first filling the non-zero entries and diagonal entries in the first $(k+1) \times n$ submatrix of the color matrix with the corresponding entries of the first $k+1$ rows of the latin square. The first non-zero entry of the $k+2$-th row of the color matrix is determined by the $k+2$-th entry of the 2-nd row of the color matrix (as the color matrix is symmetric). The next non zero entries of the $k+2$-th row are

determined by the cyclic order of the previous rows. Similarly we determine the non-zero entries of remaining rows (the first entry determined by the symmetric property of the color matrix and the next entries determined by the cyclic order of the previous rows). Thus continuing, we can fill all the entries of the color matrix satisfying the total coloring conditions, giving us a $2k + 1$ total coloring. For example, the (5×5) anti-circulant, commutative and idempotent latin square is

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

Now, this corresponds to $k = 2$ (as $2k + 1 = 5$). Thus, the filled color matrix for C_{10}^2 (where $n = 10 = 2(2(2) + 1) = 2(2k + 1)$) is:

1	4	2	0	0	0	0	0	5	3
4	2	5	3	0	0	0	0	0	1
2	5	3	1	4	0	0	0	0	0
0	3	1	4	2	5	0	0	0	0
0	0	4	2	5	3	1	0	0	0
0	0	0	5	3	1	4	2	0	0
0	0	0	0	1	4	2	5	3	0
0	0	0	0	0	2	5	3	1	4
5	0	0	0	0	0	3	1	4	2
3	1	0	0	0	0	0	4	2	5

which is seen to a $2k + 1 = 5$-total coloring. Note that 0-s do not constitute colors in the above color matrix.

We see that the powers of cycles graph C_n^k, $k = \frac{n-2}{4} + k'$ is a disjoint union of $C_n^{\frac{n-2}{4}}$ and a circulant graph. When $\frac{n-2}{4} < k < \frac{n}{2}$ the edge disjoint graph that is added to $C_n^{\frac{n-2}{4}}$ is a class I graph (as the edge disjoint graph added is a circulant graph, which is edge colorable by Δ colors, if the generating set of the graph is also a generating set of the group, which is guaranteed if $gcd(n, x) = 1$, where $\frac{n-2}{4} < x < k'$; the graph is edge colorable with Δ colors, where Δ be the maximum degree of the edge disjoint graph added [18]). Now, since in a type I total coloring of $C_n^{\frac{n-2}{4}}$, we have given $\frac{n}{2}$ colors to the vertices and $\chi(C_n^k) \leq \frac{n}{2}$ (since the vertices can always be arranged into independent sets as $[0, \frac{n}{2}], [1, \frac{n}{2} + 1], \ldots, [k, n - 1]$ provided $k < \frac{n}{2}$), we need to only give a coloring to the edges of the remaining (added) circulant graph, which is seen to require only Δ extra colors. Thus the total coloring of the graph C_n^k is again seen to require only $2\left(\frac{n-2}{4}\right) + 1 + \Delta = 2\left(\frac{n-2}{4}\right) + 2k' + 1 = 2k + 1$ colors.

$\square$

Theorem 2.2. *Let G be a powers of cycle graph C_n^k with $n = s(2m + 1)$, with s being even and $2m + 1 - i = k$, $1 \leq i \leq k + 1$. Then the graph C_n^k is total colorable with $2k + 1$ colors.*

Proof. We observe that $n \equiv 0 \mod (k+i)$ with $1 \le i \le k+1$. We also know that there exists a commutative idempotent latin square of odd order, $k+i$ in this case, which we call C'. Now, we consider two tableau of the form
Tableau B':

2m+2				
2m+3	2m+2			
2m+4	2m+3	2m+2		
...	...	...	2m+2	
2k+1	...	...	...	2m+2

Tableau A':

2k+1	2k	2k-1	...	2m+2
	2k+1	2k	...	2m+3
		2k+1	2k	...
...	...	...	2k+1	2k
	...	...	...	2k+1

Now, arranging the two tableau and the idempotent and commutative latin square of order $k+1$ in the below fashion, would give us the color matrix desired, with $2k+1$ colors. The portion C is the portion of the latin square C' which fits in the color matrix.

In case $i = 1$, the entries of C' are written wholly, so that the tableau A' and B' are equal to the tableau A and B. In case $i > 1$, the tableau A' and B' could be modified to accommodate the missed numbers in the color matrix, which are deleted from the commutative idempotent latin square C' That is, the portion of the $k+i$ latin square starting from the $(k+2)^{nd}$ position in the first row, $(k+3)^{rd}$ position in the second row and so on, is cut and juxtaposed on the tableau A' and B' to give us the tableau A and B. In particular, if $i = k+1$, the tableau A' and B' are wholly replaced with the portions deleted from the latin square C'. The portion deleted from the latin square, in case $i > 1$ would be D' and its transpose, where D given by:

$e_{1,k+2}$	$e_{1,k+3}$	...	...	$e_{1,k+i}$
	$e_{2,k+3}$	$e_{2,k+4}$	...	$e_{2,k+i}$
		...	...	$e_{3,k+i}$
			...	$e_{4,k+i}$
				$e_{k+i,k+i}$

,

where e_{ij} denote the entries of the latin square C'.

Say, for example, for the color matrix of the graph C_{20}^4, since $20 = 2(4+1) = 2(2m+1)$

for $m = 2$, we take C' to be

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

A to be

9	8	7	6
	9	8	7
		9	8
			9

and B to be

6			
7	6		
8	7	6	
9	8	7	6

Therefore, the color matrix desired is:

1	4	2	5	3												9	8	7	6
4	2	5	3	1	6												9	8	7
2	5	3	1	4	7	6												9	8
5	3	1	4	2	8	7	6												9
3	1	4	2	5	9	8	7	6											
	6	7	8	9	1	4	2	5	3										
		6	7	8	4	2	5	3	1	9									
			6	7	2	5	3	1	4	8	9								
				6	5	3	1	4	2	7	8	9							
					3	1	4	2	5	6	7	8	9						
						9	8	7	6	1	4	2	5	3					
							9	8	7	4	2	5	3	1	6				
								9	8	2	5	3	1	4	7	6			
									9	5	3	1	4	2	8	7	6		
										3	1	4	2	5	9	8	7	6	
											6	7	8	9	1	4	2	5	3
9												6	7	8	4	2	5	3	1
8	9												6	7	2	5	3	1	4
7	8	9												6	5	3	1	4	2
6	7	8	9												3	1	4	2	5

For the case of the color matrix of the graph C_{18}^5, since $18 = 2(8+1) = 2(2m+1)$ for $m = 4$ we have C' to be

1	6	2	7	3	8	4	9	5
6	2	7	3	8	4	9	5	1
2	7	3	8	4	9	5	1	6
7	3	8	4	9	5	1	6	2
3	8	4	9	5	1	6	2	7
8	4	9	5	1	6	2	7	3
4	9	5	1	6	2	7	3	8
9	5	1	6	2	7	3	8	4
5	1	6	2	7	3	8	4	9

A to be

11	10	4	9	5
	11	10	5	1
		11	10	6
			11	10
				11

and B to be

10				
11	10			
4	11	10		
9	5	11	10	
5	1	6	11	10

Therefore, the color matrix would be

1	6	2	7	3	8								11	10	4	9	5
6	2	7	3	8	4	9								11	10	5	1
2	7	3	8	4	9	5	1								11	10	6
7	3	8	4	9	5	1	6	2								11	10
3	8	4	9	5	1	6	2	7	10								11
8	4	9	5	1	6	2	7	3	11	10							
	9	5	1	6	2	7	3	8	4	11	10						
		1	6	2	7	3	8	4	9	5	11	10					
			2	7	3	8	4	9	5	1	6	11	10				
			10	11	4	9	5	1	6	2	7	3	8				
				10	11	5	1	6	2	7	3	8	4	9			
					10	11	6	2	7	3	8	4	9	5	1		
						10	11	7	3	8	4	9	5	1	6	2	
11							10	3	8	4	9	5	1	6	2	7	
10	11							8	4	9	5	1	6	2	7	3	
4	10	11							9	5	1	6	2	7	3	8	
9	5	10	11							1	6	2	7	3	8	4	
5	1	6	10	11							2	7	3	8	4	9	

□

Theorem 2.3. *Let G be a power of cycle graph C_n^k with $n = s(2m+1) \pm 1$, with s being even and m being positive integers $\frac{k}{2} < m < k$ and $k = 2m + 1 - i$. Then the graph C_n^k*

7

is total colorable with $2k + 2$ colors. In fact, $\chi''(C_n^k) \le 2k + 3$.

Proof. Case 1: $n = s(2m+1) - 1$

From the previous theorem, it is clear that C_{n+1}^k is $2k + 1$ colorable. It remains to show that this coloring could be extended to a $2k + 2$ coloring for C_n^k. The extension is made possible by deleting last row and last coloumn of the color matrix of C_{n+1}^k and appropriately adding a new color in the lower and upper $(n - k + 1)^{th}$ subdiagonal of the color matrix so as to obtain the desired color matrix with $2k + 2$ colors.

This method of extension applies to any powers of cycle graph of any odd order. Since it is already proved that the even order powers of cycles satisfy a type II coloring, therefore, by extension, since we require only one extra color, the total chromatic number of the graph is $2k + 3$.

For example, the color matrix below for C_{14}^3

1	5	2	6								3	7	4
5	2	6	3	7								4	1
2	6	3	7	4	1								5
6	3	7	4	1	5	2							
	7	4	1	5	2	6	3						
		1	5	2	6	3	7	4					
			2	6	3	7	4	1	5				
				3	7	4	1	5	2	6			
					4	1	5	2	6	3	7		
						5	2	6	3	7	4	1	
							6	3	7	4	1	5	2
3								7	4	1	5	2	6
7	4								1	5	2	6	3
4	1	5								2	6	3	7

is modified to that of C_{13}^3 by deletion of one row and column and addition of the appropriate color to:

1	5	2	6							⑧	3	7
5	2	6	3	7							⑧	4
2	6	3	7	4	1							⑧
6	3	7	4	1	5	2						
	7	4	1	5	2	6	3					
		1	5	2	6	3	7	4				
			2	6	3	7	4	1	5			
				3	7	4	1	5	2	6		
					4	1	5	2	6	3	7	
						5	2	6	3	7	4	1
⑧							6	3	7	4	1	5
3	⑧							7	4	1	5	2
7	4	⑧							1	5	2	6

The encircled number is the added color to the color matrix.

Case 2:$n = s(2m+1)+1$.

Here, we add a row and a column at the last and then add entries in the last row and last column which are the k deleted entries each from the lower and upper k^{th} and $(n-k+1)^{th}$ subdiagonal of the original color matrix of C_{n-1}^k. In place of the entries deleted in the kth subdiagonal, we add a new color. The new vertex is also given the new color. For example, the color matrix can be modified by adding one row and column to give the color matrix of C_{15}^3 as follows:

1	5	2	⑧							◯	7	4	⑥
5	2	6	3	⑧							◯	1	⑦
2	6	3	7	4	⑧							◯	①
⑧	3	7	4	1	5	2							
	⑧	4	1	5	2	6	3						
		⑧	5	2	6	3	7	4					
			2	6	3	7	4	1	5				
				3	7	4	1	5	2	6			
					4	1	5	2	6	3	7		
						5	2	6	3	7	4	1	
							6	3	7	4	1	5	2
◯							7	4	1	5	2	6	③
7	◯							1	5	2	6	3	④
4	1	◯							2	6	3	7	⑤
⑥	⑦	①								③	④	⑤	⑧

The encircled numbers show the changes from the color matrix of C_{14}^3 □

Theorem 2.4. *Let G be the cayley graph of a nilpotent group of even order n with maximum degree $\Delta(G) \geq \frac{n}{2}$ and the generating set S not containing an element of order two. If G is total colorable with $\chi''(G)$ colors and if G' is the cayley graph of nilpotent group of even order graph with maximum degree $\Delta(G')$, $\Delta(G) \leq \Delta(G') \leq n-2$ formed by with generating set $S' = S \cup S''$ such that S'' which generates the whole group and also does not have an order two element, then graph G' is total colorable with $\chi''(G) + (\Delta(G') - \Delta(G))$ colors. In particular, if G is type I (type II), then G' is also type I (type II).*

Proof. Let s be an element of order two in the graph (which is guaranteed by Cauchy's theorem). Since the vertices of the graph g_i, $i \in \{1, 2, \ldots, n\}$ can always be arranged in $\frac{n}{2}$ independent color classes as $[g_1, g_1 s], [g_2, g_2 s], \ldots, [g_{\frac{n}{2}}, g_{\frac{n}{2}} s]$, this gives us a $\frac{n}{2}$ coloring of the vertices. Therefore, we need to take care only of the edge coloring of the graph $G' - G$ in order to give a total coloring of G' from the existing total coloring of G. Now, since $G' - G$ is $1-$ factorizable [18], therefore, we only need $\Delta' - \Delta$ extra colors, thereby giving a total coloring of $\chi'' + (\Delta' - \Delta)$. Thus, if G were a type I(type II) graph, then G' also would be type I(type II). □

For a positive integer $n > 1$ the *unitary Cayley graph* $X_n = Cay(Z_n, U_n)$ is defined by the additive group of the ring Z_n of integer modulo n and the multiplicative group U_n of its units. If we represent the elements of Z_n by the integers 0,1,...$n-1$, then it is well known that

$$U_n = \{a \in Z_n : gcd(a, n) = 1\}.$$

So X_n has vertex set $V(X_n) = X_n = \{0, 1, 2, ..., n-1\}$ and edge set

$$E(X_n) = \{(a, b) : a, b \in Z_n, gcd(a - b, n) = 1\}.$$

Boggess et al. [3] studied the structure of unitary Cayley graphs. They have also discussed chromatic number, vertex and edge connectivity, planarity and crossing number. Klotz and Sander [16] have determined the clique number, the independence number and the diameter. They have given a necessary and sufficient condition for the perfectness of X_n.

The graph X_n is regular of degree $U_n = \varphi(n)$ denotes the Euler function. Let the prime factorization of n be $p_1^{\alpha_1} p_2^{\alpha_2} ... p_t^{\alpha_t}$ where $p_1 < p_2 < ... < p_t$. If $n = p$ is a prime number, then $X_n = K_p$ is the complete graph on p vertices. If $n = p^\alpha$ is a prime power then X_n is a complete p-partite graph. In the following theorem we prove TCC holds for unitary Cayley graphs.

Theorem 2.5. *A unitary Cayley graph X_n is $(\Delta(X_n) + 2)$ - total colorable.*

Proof. We know that a unitary Cayley graph can be obtained from a balanced r partite graph by deleting some edges. Suppose $n = p$ is a prime number, then X_n is the complete graph on p vertices. Also, if $n = p^\alpha$, a prime power, then X_n is a complete p-partite graph and TCC holds for these two graphs [20].

When $n = 2k, k \in N$, then the unitary Cayley graph is a bipartite graph and any bipartite graph is total colorable.

Suppose $n \not\equiv 0 \mod 2$. As p_1 is the smallest prime, $kp_1, kp_1 + 1, ..., (k+1)p_1 - 1$, where $k = 0, 1, 2, ..., \frac{n}{p_1} - 1$ induces $\frac{n}{p_1}$ vertex disjoint cliques each of order p_1. Since p_1 is odd, we can color all the elements of these $\frac{n}{p_1}$ cliques using p_1 colors [?]. Now remove the edges of these cliques. The remaining graph is a $\varphi(n) - p_1 + 1$-regular graph where the vertices are already coloured. We color the edges of this resultant graph with $\varphi(n) - p_1 + 2$ colors. Thus we have used $\varphi(n) + 2$ colors for the total coloring of X_n. $\square$

3 Mock Threshold Graphs and Odd Graphs

A graph is weakly chordal if neither the graph nor the complement of the graph has an induced cycle on five or more vertices. A simple graph G on $[n] = \{1, 2, ..., n\}$ is threshold, if G can be built sequentially from the empty graph by adding vertices one at a time, where each new vertex is either isolated (nonadjacent to all the previous) or dominant (connected to all the previous). A graph G is said to be mock threshold if there is a vertex ordering $v_1, ..., v_n$ such that for every i $(1 \leq i \leq n)$ the degree of v_i in $G : v_1, ..., v_i$ is 0, 1, $i - 2$, or $i - 1$. Mock threshold graphs are a simple generalization of threshold graphs that, like threshold graphs, are perfect graphs. Mock threshold graphs are perfect and indeed weakly chordal but not necessarily chordal [1]. Similarly, the complement of a mock threshold graph is also mock threshold.

In the following, we prove the TCC for Mock Threshold graphs.
Note: A total coloring of K_n can be constructed as follows: (This total coloring is due to Hinz and Parisse [13])

When n is even, we first construct an edge coloring of K_n and extend it. We denote $[n]_0 = \{0, 1, 2, ..., n-1\}$. For $k \in [n]_0$, let τ_k be the transposition of k and $n - 1$ on $[n]_0$. For even n, $c_n(i, j) = (\tau_i(j) + \tau_j(i) + 2) \mod (n+1)$, for $i, j \in [n]_0, i \neq j$, defines

a $(n+1)$-edge coloring. In this coloring assignment line $k \in [n]_0$ will have the missing colors k and $(k+1) \mod n$. We color $c_n(i) = i$ for all $i \in [n]_0$.

When n is odd, we use the same coloring of K_{n-1}. In the coloring assignment of K_{n-1}, still the color $(k+1) \mod n$ is missing in line $k \in [n]_0$. We use these colors to the edges incident with n^{th} vertex and color n to the nth vertex.

Theorem 3.1. *Total coloring conjecture holds for any Mock threshold graph G.*

Proof. Consider the Mock threshold graph G with vertex ordering $v_1, v_2, ..., v_i, ..., v_n$.

We prove this theorem using mathematical induction on the induced subgraph $G[v_1, v_2, ..., v_k]$.

For $k \leq 4$, the maximum degree of all the induced subgraphs is less than or equal to 3. We know that a graph with maximum degree less than or equal to 3 satisfies TCC [15].

Let us assume that $G[v_1, v_2, ..., v_k], k \geq 5$ satisfies TCC.

Claim: The graph $G[v_1, v_2, ..., v_k, v_{k+1}]$ satisfies TCC.

The degree of the vertex v_{k+1} in $G[v_1, v_2, ..., v_{k+1}]$ can be $0, 1, k-1$ or k.
Case-1: Suppose $d(v_{k+1}) = 0$.

In this case the vertex is v_{k+1} is an isolated vertex. By the induction assumption, $G[v_1, v_2, ..., v_k, v_{k+1}]$ satisfies TCC.
Case-2: Suppose $d(v_{k+1}) = 1$.

In this case, the vertex v_{k+1} is adjacent to a vertex, say v_i, in $G[v_1, v_2, ..., v_k]$. Since $G[v_1, v_2, ..., v_k]$ is total colorable graph with at most $\Delta(G[v_1, v_2, ..., v_k]) + 2$ colors, at each vertex there will be at least one missing color. We assign this missing color to the edge (v_i, v_{k+1}), and for the vertex v_{k+1}, we assign a color of a vertex which is not adjacent to v_{k+1} and not the color of v_i. Therefore, $G[v_1, v_2, ..., v_{k+1}]$ satisfies TCC.
Case-3: Suppose $d(v_{k+1}) = k - 1$.

Let us assume that the vertex v_{k+1} is not adjacent with v_i and also assume that $\Delta(G[v_1, v_2, ..., v_{k+1}]) = k - 1$. We consider following two cases:
Subcase-1: k is even.

Since k is even, $k+1$ is odd. Construct a complete graph induced by the vertices $v_1, v_2, ...v_{i-1}, v_{i+1}, ..., v_{k+1}$. Now, color this complete graph using colors in the set $\{0, 1, ..., n+1\}$ as given in the note. In this coloring assignment there will be one missing color at each of the vertices and they are distinct. Now, color the edges $(v_i, v_j), i \neq j, j = 1, 2, ..., v_{k+1}$, with the missing colors. Assign the color $n-1$ to the vertex v_i. To get a total coloring of $G[v_1, v_2, ..., v_k, v_{k+1}]$, we remove these edges and there is no change in the maximum degree.
Subcase-2: k is odd.

In this case $k+1$ is even, say $2p$. It is known that a graph of order $2p$ with maximum degree $2p - 2$ satisfies TCC (see [12, 7]).
Case-4: Suppose $d(v_{k+1}) = k$.

The maximum degree of $G[v_1, v_2, ..., v_k, v_{k+1}]$ is k. Construct a complete graph on the vertex set $\{v_1, v_2, ..., v_k, v_{k+1}\}$. We know that the complete graph satisfies TCC. After removing the added edges we get a total coloring of $G[v_1, v_2, ..., v_k, v_{k+1}]$

Hence, in all the cases, the mock threshold graph satisfies TCC.

$\square$

The Kneser graph $K(n, k)$ is the graph whose vertices correspond to the k-element subsets of a set of n elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint. A vertex in the odd graph O_n is a $(n-1)$-element subset of a $(2n-1)$-element set. Two vertices are connected by an edge if and only if the corresponding subsets are disjoint. Note that the odd graphs are particular case of Kneser graphs.

Theorem 3.2. *Odd graph O_n satisfies TCC.*

Proof. Consider a $2n-1$ element set X. Let O_n be an odd graph defined from the subsets of X. Let x be any element of X. Then, among the vertices of O_n, exactly $\binom{2n-2}{n-2}$ vertices correspond to sets that contain x. Because all these sets contain x, they are not disjoint, and form an independent set of O_n. That is, O_n has $2n-1$ different independent sets of size $\binom{2n-2}{n-2}$. Further, every maximum independent set must have this form, so, O_n has exactly $2n-1$ maximum independent sets.

If I is a maximum independent set, formed by the sets that contain x, then the complement of I is the set of vertices that do not contain x. This complementary set induces a matching in G. Each vertex of the independent set is adjacent to n vertices of the matching, and each vertex of the matching is adjacent to $n-1$ vertices of the independent set [8].

Based on the decomposition, we give a total coloring of O_n in the following way:

Assign n colors to the edges between the vertices in the maximum independent set I and the vertices in the matching. Color the edges in matching and the vertices in I with a new color. Color one set of vertices in the matching with another new color and the second set of vertices with the missing colors at these vertices. This will give a total coloring of O_n using at most $n + 2 = \Delta(O_n) + 2$ colors.

$\square$

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