Derivatives of the forward operator

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1 The derivative of the forward operator

Recall that the forward operator is given by

$$\boldsymbol{\mathcal{G}}(f, \boldsymbol{\theta}_v) = \int_{-\infty}^{\infty} \frac{1}{1 + v/c} S\left(\frac{\lambda}{1 + v/c}; f\right) L(v; \boldsymbol{\theta}_v) \, \mathrm{d}v, \tag{1.1}$$

where

$$S(\lambda; f) = \int_{\Theta} f(\theta) s(\lambda; \boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{\theta}. \tag{1.2}$$

and

$$L(v, \boldsymbol{\theta}_v) := \mathcal{N}(v; V, \sigma) \left[\sum_{m=0}^{M} h_m H_m(\hat{v}) \right],$$

where $\mathcal{N}(\cdot; 0, 1)$ denotes the density function of the standard normal distribution, and where furthermore

$$\begin{aligned} \hat{v} &:= \frac{v - V}{\sigma}, \\ H_m(x) &:= \frac{H_m^{\text{phys}}(x)}{\sqrt{m! 2^m}}, \\ H_m^{\text{phys}}(x) &:= (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} [\mathrm{e}^{-x^2}], \end{aligned}$$

and $\boldsymbol{\theta}_v$ is the collection of parameters given by

$$\boldsymbol{\theta}_v = (V, \sigma, h_0, \dots, h_M).$$

Since we actually use the curvelet coefficients $\tilde{\boldsymbol{f}} = \boldsymbol{\Phi} \boldsymbol{f} \in \mathbb{R}^N$, our observation operator is

$$\tilde{\boldsymbol{\mathcal{G}}}(\tilde{\boldsymbol{f}}, \boldsymbol{\theta}_v) = \boldsymbol{\mathcal{G}}(\boldsymbol{\Phi}^{-1} \tilde{\boldsymbol{f}}, \boldsymbol{\theta}_v),$$

where $\boldsymbol{\Phi}$ is the corresponding discrete curvelet transform.

1.1 Derivative with respect to θ_v

First, let us compute the derivative with respect to $\boldsymbol{\theta}_v = [V, \sigma, h_0, \dots, h_M]$. Since $\boldsymbol{\mathcal{G}}(f, \boldsymbol{\theta}_v)$ depends on $\boldsymbol{\theta}_v$ only through $L(v, \boldsymbol{\theta}_v)$, it suffices to compute $\partial_{\boldsymbol{\theta}_v} L(v, \boldsymbol{\theta}_v)$, i.e.

$$\partial_{\boldsymbol{\theta}_v} L(v, \boldsymbol{\theta}_v) = \begin{bmatrix} \partial_V L(v, \boldsymbol{\theta}_v) & \partial_\sigma L(v, \boldsymbol{\theta}_v) & \partial_{h_0} L(v, \boldsymbol{\theta}_v) & \dots & \partial_{h_M} L(v, \boldsymbol{\theta}_v) \end{bmatrix}.$$

The derivative with respect to $h_0, \dots, h_M$ is trivial, given by

$$\partial_{h_m} L(v, \boldsymbol{\theta}_v) = \mathcal{N}(v; V, \sigma^2) H_m\left(\frac{v - V}{\sigma}\right).$$

The derivative with respect to V and σ are a little bit harder. The derivatives of the normal probability density function with respect to mean and standard deviation are, respectively,

$$\partial_V \mathcal{N}(v; V, \sigma^2) = \frac{v - V}{\sigma^2} \mathcal{N}(v; V, \sigma^2)$$
$$\text{and} \quad \partial_\sigma \mathcal{N}(v; V, \sigma^2) = \frac{(v - V)^2 - \sigma^2}{\sigma^3} \mathcal{N}(v; V, \sigma^2)$$

(see here and here). Finally, the Hermite polynomials satisfy the useful relation

$$H_m'(x) = \sqrt{2m} H_{m-1}(x). \tag{1.3}$$

Thus, we obtain the derivatives of $L(v, \boldsymbol{\theta}_v)$ with respect to V and σ:

$$\partial_V L(v; V, \sigma^2) = \frac{v - V}{\sigma^2} - \frac{\sqrt{2}}{\sigma} \mathcal{N}(v; V, \sigma^2) \sum_{m=0}^{M} h_m \sqrt{m} H_{m-1}(\frac{v - V}{\sigma}),$$
$$\partial_\sigma L(v; V, \sigma^2) = \frac{(v - V)^2 - \sigma^2}{\sigma^3} L(v, \boldsymbol{\theta}_v) - \frac{\sqrt{2}(v - V)}{\sigma^2} \mathcal{N}(v; V, \sigma^2) \sum_{m=1}^{M} h_m \sqrt{m} H_{m-1}(\frac{v - V}{\sigma}).$$

1.2 Derivative of the Fourier transform

For the actual implementation, we need the derivative of the Fourier transform of L. Let $\mathcal{F}_{v \to \omega} L(\omega, \boldsymbol{\theta}_v)$ denote the Fourier transform of $v \mapsto L(v, \boldsymbol{\theta}_v)$. Then, it was shown by Cappellari (2016) that

$$\mathcal{F}_{v \to \omega} L(\omega, \boldsymbol{\theta}_v) = \frac{e^{i\omega V - \sigma^2 \omega^2 / 2}}{\sqrt{2\pi}} \sum_{m=0}^{M} i^m h_m H_m(\sigma\omega).$$

We can obtain the derivative with respect to $\boldsymbol{\theta}_v$ similarly as above. First of all, we clearly have

$$\partial_{h_m} \mathcal{F}_{v \to \omega} L(\omega, \boldsymbol{\theta}_v) = \frac{e^{i\omega V - \sigma^2 \omega^2 / 2}}{\sqrt{2\pi}} i^m H_m(\sigma\omega), \qquad m = 0, \ldots, M.$$

The derivative with respect to V is even easier than in time domain:

$$\partial_V \mathcal{F}_{v \to \omega} L(\omega, \boldsymbol{\theta}_v) = i\omega \mathcal{F}_{v \to \omega} L(\omega, \boldsymbol{\theta}_v).$$

For the derivative with respect to σ, we have to use the product rule and (1.3):

$$\partial_\sigma \mathcal{F}_{v \to \omega} L(\omega, \boldsymbol{\theta}_v) = -\sigma\omega^2 \mathcal{F}_{v \to \omega} L(\omega, \boldsymbol{\theta}_v) + \frac{\omega e^{i\omega V - \sigma^2 \omega^2 / 2}}{\sqrt{\pi}} \sum_{m=1}^{M} i^m \sqrt{m} h_m H_{m-1}(\sigma\omega).$$