On counting special basis for $\mathbb{R}^n$

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We are interested in counting the number of basis sets (un-ordered) of $\mathbb{R}^n$ for which the coordinates of the involved vectors are either 0s or 1s. In this entry we will count (and prove) the same for $\mathbb{R}^3$ in two ways; and speculate a probable path to extend the count for the general case. The intent of this entry is to motivate a solution for the general case

Introduction

One can quickly see that the number of "ordered" basis sets of $\mathbb{R}^n$ with vectors involving only 0s and 1s is equivalent to counting the number of non-singular $n \times n$ matrices whose entries are only 0s and 1s. For convenience let us name these counts.

$\boldsymbol{M(n)}$: number of non-singular $n \times n$ matrices with entries in the set $\{0, 1\}$.

$\boldsymbol{m(n)}$: number of basis-sets (un-ordered) of $\mathbb{R}^n$ whose involved vectors contain only 0s and 1s.

Since every un-ordered basis set gives rise to exactly $n!$ of these non-singular matrices with the columns permuted; one can quickly form the relation,

$$M(n) = m(n) \cdot n!$$

Computer Data

The following has been calculated using a computer. What seems interesting is that the sequence $\{m(n)\}$ doesn't seem to follow any obvious patterns, and their occurrence appears strange and beautiful at the same time.

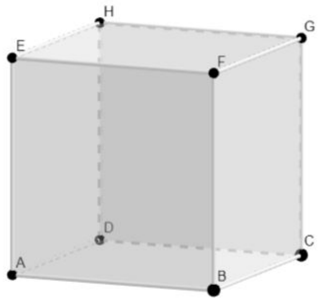
n	$M(n)$	$m(n)$	Factors of $m(n)$
1	1	1	1
2	6	3	3
3	174	29	29
4	22560	940	$2 \cdot 2 \cdot 5 \cdot 47$
5	12514320	104286	$2 \cdot 3 \cdot 7 \cdot 13 \cdot 191$

Counting $\boldsymbol{m(3)}$, looong explanation

The relevant bases for $\mathbb{R}$ is $\{1\}$, and for $\mathbb{R}^2$ is,

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

and therefore the calculation of $m(1)$ and $m(2)$ are trivial. Let us come to $\mathbb{R}^3$ now. The vectors of $\mathbb{R}^3$ whose coordinates are either 0s or 1s sit at the vertices of an unit cube. Hence it is only fair to identify these vector with vertices of the unit cube.



Recall finding a basis for $\mathbb{R}^3$ only requires finding 3 vectors which are linearly independent. Therefore,

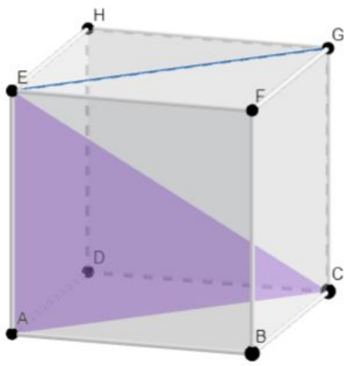
(i) *Counting $M(3)$ boils down to the number of ways of choosing (in order) 3 linearly independent vertices from this cube.*

We will now identify the vertices A, B, D, E with $\underline{0}, \hat{\imath}, \hat{\jmath}, \hat{k}$ respectively. There are 7 non-zero vertices, and therefore 7 choices for the first vector. The remaining 6 vertices are valid candidates for the second vector, as they are not a scalar multiple of the first. Hence there are $7 \cdot 6 = 42$ ordered choices for the first two vectors. Hence the third vertex we choose must lie outside the span of the first two.

(ii) *There are either 4 or 5 choices for the third vector of the basis.*

Any choice of two non-zero vertices with the origin A forms a triangular cross-section of the cube. This triangle T can be extended on all sides to span a plane P (subspace) passing through the origin. The intersection I of P and the cube involves at least 3 vertices; namely the three vertices of the triangle T. Also from the orientation of a cube it is clear that no plane can contain 5 vertices of a cube. And hence I contains at most 4 vertices. Or, I contains 3 or 4 vertices of the cube; thereby leaving us with a choice of 5 or 4 vertices for the third vector of the basis.

The following is an illustration with $T = AEC$ and $P = AEGC$.



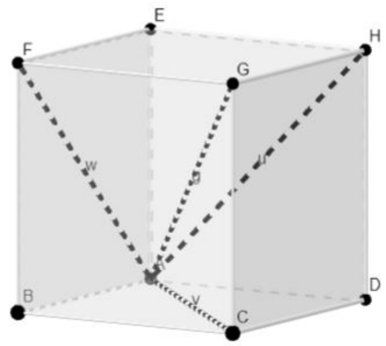
(iii) *I containing 4 vertices corresponds to the triangle T containing a side of the cube.*

Since the vertices in I and T are vertices of the cube, their sides are either sides or diagonals of the cube. Hence the internal angles of I and T are at most $\pi/2$. But I being a quadrilateral must have its internal angles sum to 2π. Hence clearly, all internal angles of I are $\pi/2$, or I is a rectangle.

But T shares three of its vertices with I, and hence shares at least two of its adjacent sides. And therefore T has a right angle. Since the right angles in a cube are formed only between its sides, T has a side of the cube.

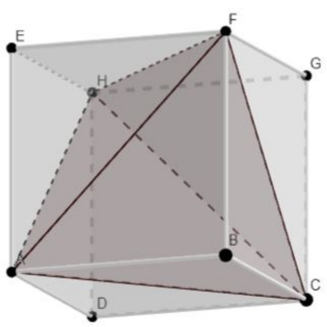
(iv) *There are exactly* 6 *choices for which* 3 *vertices lie in the intersection* I.

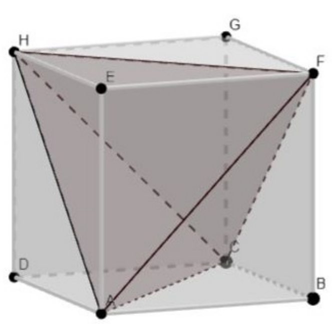
From (ii) we have that in such cases T must contain no side of the cube, and therefore its sides must be the diagonals. But T contains the origin A, and hence two of these diagonals must begin at A, meaning these two diagonals are exactly the first two chosen vertices. But there are 4 vertices which are diagonally opposite to A, three, C, H, F along the faces, and one G through the cube.

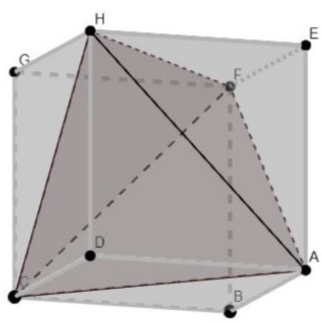


Choosing two of these automatically determines the third side of T. But observe that G is connected to the three other vertices via a side of the cube, and hence AG cannot be a side of T from (ii). Moreover, no two other vertices are connected via a side of the cube. Hence T can be determined by choosing any 2 out of the remaining three 3 vertices. Since we are also taking the order into account, the required number of choices of T is ${}^3P_2 = 6$.

Trivia, these three planes (in these 6 choices) that cut obliquely through the cube to intersect at only 3 vertices form the surfaces of a tetrahedron at the origin.







Finally, from (i) we know that each of the 42 choices of the first two vectors corresponds to leaving 4 or 5 options for the third. But, from (iii) we have exactly 6 of them leaving 5. And hence we have,

$$M(3) = 6 \cdot 5 + (42 - 6) \cdot 4 = 174$$

$$\boxed{m(3) = \frac{174}{3!} = 29}$$

Play with these cubes at here.