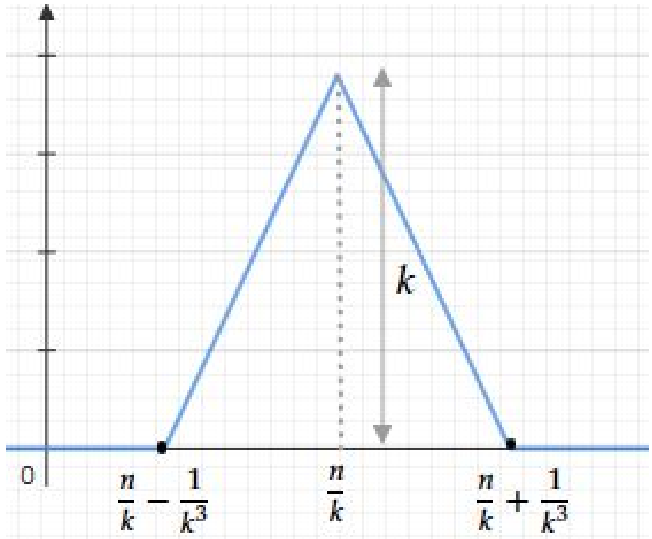
Does there exist a sequence a sequence $\{f_k\}$ of continuous functions on $[0,1]$ which diverges everywhere point-wise, but the corresponding sequence of integrals goes to 0?

Yes. Let us construct such a sequence. Define,

$$t_{n,k}(x) = \max\left\{0, k - k^4\left|x - \frac{n}{k}\right|\right\} \text{ restricted to } [0,1] \text{ for } n,k \in \mathbb{N}$$

What is this complicated looking $t_{n,k}(x)$? Just the triangle alongside with its base of length $\frac{2}{k^3}$ centered at $\frac{n}{k}$ and a height of k, and 0 everywhere else.



We can immediately make some quick observations about $t_{n,k}(x)$.

- $t_{n,k}(x) \geq 0$, directly from definition.
- $t_{n,k}(x)$ is continuous, maximum of two continuous functions.
- $\displaystyle\int_0^1 t_{n,k}(x)\ dx \leq \frac{1}{2}\cdot\frac{2}{k^3}\cdot k = \frac{1}{k^2}$

The "$\leq$" gets picked in the last bullet because some part of the triangle $t_{n,k}$ might lie outside $[0,1]$. In this case the area of the triangle lying outside would not contribute to the integral which is restricted within $[0,1]$.

Now we will construct the sequence $f_k(x)$ which will hold the properties desired in the problem. Define,

$$f_k(x) = \sum_{n=1}^{k} t_{n,,k}(x), \text{ then,}$$

Firstly, $f_k(x)$ is continuous, as it is a "finite" sum of continuous functions. Next,

$$\int_0^1 f_k(x)\ dx = \sum_{n=1}^{k}\left[\int_0^1 t_{n,k}(x)\ dx\right]$$
$$\leq \sum_{n=1}^{k}\frac{1}{k^2} = \frac{1}{k},$$
$$\text{Or, } \int f_k \to 0 \text{ by comparison with } \frac{1}{k}.$$

We will show $f_k(x)$ diverges everywhere by contradiction. That is assume there is an $x \in [0,1]$, for which $f_k(x) \to f(x)$. Fix any $\epsilon > 0$. By convergence, there must exist N, such that,

$$f_k(x) < f(x) + \epsilon \text{ for every } k \geq N$$

.

Now pick $k > \max\left\{\frac{1}{1-x},\ N,\ f(x)+\epsilon\right\}$. This choice enables the following, (1) $k > N$, (2) $k - [f(x)+\epsilon] > 0$, (3) $x < 1 - \dfrac{1}{k}$.

Thus for some $k > N$ after combining (2) and (3) we have,

$$\begin{aligned} x <& \quad 1 - \frac{1}{k} + \frac{k - [f(x) + \epsilon]}{k^4} \\ \text{Or,} \quad \frac{f(x) + \epsilon}{k^4} <& \quad \frac{k}{k^4} + \frac{k-1}{k} - x \\ \text{Or,} \quad f(x) + \epsilon <& \quad k + k^4 \cdot \left| x - \frac{k-1}{k} \right| \\ \text{Or,} \quad f(x) + \epsilon <& \quad t_{k-1,k}(x) \quad < \quad f_k(x) \end{aligned}$$

(The last inequality follows as $t_{n,k} > 0$ for all $n \leq k$, and therefore, $f_k(x) > t_{k-1,k}(x)$) However, this clearly is a contradiction to our assumption. Hence, such a point x must not exist. $\square$