Norm Inequalities in the Cartesian Decomposition

Matrix Analysis, Fall 2021

Given any matrix $A \in \mathbb{M}_n(\mathbb{C})$, the space of the $n \times n$ complex matrices, we know that A can be decomposed as

$$A = H + iK \tag{1}$$

where H and K are Hermitian. The matrices H and K are given as

$$H = \frac{A + A^*}{2} \text{ and } K = \frac{A - A^*}{2i}.$$

1. **Frobenius Norm**

   It is known that the Frobenius norm $\|\cdot\|_F$ satisfies the following inequality.

   $$\|A\|_F^2 = \|H\|_F^2 + \|K\|_F^2 \tag{2}$$

   One way to see this is by recalling that the Frobenius norm $\|\cdot\|_F$ arises from the inner product $\langle S, T \rangle = \operatorname{tr} S^* T$ for $S, T \in \mathbb{M}_n(\mathbb{C})$; and the matrices H and iK are orthogonal to each other with respect to this inner product. Thus the equality in (2) becomes the result of the Pythagoras theorem.

2. **For Normal Matrices in the Operator Norm**

   It is natural to investigate the relationship shared by the LHS and RHS of (2) when $\|\cdot\|_F$ is replaced by an operator norm $\|\cdot\|$. We first compare the sides when the matrix A is normal. To do this we refer to the two following facts.

   i. A is normal if and only if the matrices H and K appearing in (1) commute.

   ii. $\|A\|^2 = \|A^* A\|$ for any operator norm $\|\cdot\|$.

   Then we observe that for normal matrices

   $$\begin{aligned} \|A\|^2 &= \|A^* A\| \\ &= \|(H + iK)^* (H + iK)\| \\ &= \|H^2 + K^2 + i(HK - KH)\| \\ &= \|H^2 + K^2\| \\ \Longrightarrow \|A\|^2 &\leq \|H\|^2 + \|K\|^2 \end{aligned} \tag{3}$$

   In (3) we note that equality is achieved for matrices like $A = I$. We also mention that there are normal matrices for which the inequality above is strict. For example, consider,

   $$\underbrace{\begin{bmatrix} 1+2i & 0 \\ 0 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{H} + i \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}_{K}.$$

Let $\|\cdot\|$ be the operator norm arising from the Euclidean norm in $\mathbb{C}^n$. Then for any normal matrix T we know that

$$\|T\| = \max_{\lambda \in \sigma(T)} |\lambda|.$$

Then in the above example we immediately see that

$$\|A\|^2 = |1+2i|^2 = 5 < 2^2 + 2^2 = \|H\|^2 + \|K\|^2.$$

3. **For General Matrices in the Operator Norm**

For a general matrix A we can show that neither an inequality like (3) nor its converse holds. In particular we can find non-normal matrices for which the LHS of (3) exceeds the RHS and vice versa. For this consider

$$\underbrace{\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{H} + i \underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}_{K}.$$

Here both H and K are unitary matrices and hence have operator norm 1. But the matrix $A^*A = \mathrm{diag}(0,4)$. So $\|A\|^2 = 4$. Thus

$$\|A\|^2 = 4 > 1 + 1 = \|H\|^2 + \|K\|^2.$$

Conversely, we have already shown that (3) holds for normal matrices. We can also find non-normal matrices which satisfy (3). Consider

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{H} + i \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}}_{K}.$$

Then we can immediately observe that $\|H\| = 3$ and $\|K\| > 0$. But we also have

$$A^*A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Note that A^*A is a positive definite block diagonal matrix with two positive blocks. Thus, the eigenvalues of A^*A can be extracted from the eigenvalues of these blocks. The first block is of size 1, and hence the first eigenvalue is 9. The second block has trace 7, and because the eigenvalues are positive, the eigenvalues from this block do not exceed 7. This ensures that the maximum eigenvalue of A^*A is 9, implying

$$\|A\|^2 = \|A^*A\| = 9.$$

Therefore,

$$\|A\|^2 = 9 < 9 + \|K\|^2 = \|H\|^2 + \|K\|^2.$$