Investigating open covers for $\mathbb{R}$ around elements of $\mathbb{Q}$ with open intervals of length $2/n$.

Let $\{x_n\} \subset \mathbb{R}$, and,

$$\mathcal{I}_N(\{x_n\}) := \bigcup_{n=1}^{N} \left(x_n - \frac{1}{n}, x_n + \frac{1}{n} \right), \text{ and } \mathcal{I}(\{x_n\}) = \lim_{N\to\infty} \mathcal{I}_N$$

In the following sections we would see that for an enumeration $\{q_n\}$ of $\mathbb{Q}$, $\mathcal{I}(\{q_n\})$ may or may not be an open cover for $\mathbb{R}$.

An enumeration of $\{q_n\}$ of $\mathbb{Q}$ for which $\mathcal{I}(\{q_n\})$ is an open cover for $\mathbb{R}$

Idea, use the fact that the partial sums of the harmonic series are rational numbers going to infinity. Use them to construct an open cover for the positive real line and extend to the whole real line. Fit these numbers appropriately in an enumeration of all rational numbers.

1. Let $s_n = \sum_{n=1}^{\infty} \frac{1}{n}$. Then $\mathcal{I}(\{s_n\})$ is an open cover for $(0, \infty)$.

By definition of s_n we have the following inequality,

$$s_{n-1} = s_n - \frac{1}{n} < s_n = s_{n+1} - \frac{1}{n+1} < s_n + \frac{1}{n} < s_{n+1} + \frac{1}{n+1}$$

Then $\mathcal{I}_N(\{s_n\}) = \left(s_1 - 1, s_N + \frac{1}{N}\right)$. As $s_N \to \infty$, $I(\{s_n\}) = (s_1 - 1, \infty) = (0, \infty)$.

2. Let $\{t_n\} \subset \mathbb{Q}$ such that $t_{2n} = \dfrac{s_n}{2}$, and $t_{2n-1} = \dfrac{-s_n}{2}$, then $\mathcal{I}(\{t_n\})$ is an open cover for $\mathbb{R}$.

If $x > 0$, then $2x > 0$. From (1) there exists n such that $2x \in \left(s_n - \frac{1}{n}, s_n + \frac{1}{n}\right)$. Or,

$$\begin{aligned} s_n - \frac{1}{n} < 2x < s_n + \frac{1}{n} &\implies \frac{s_n}{2} - \frac{1}{2n} < x < \frac{s_n}{2} + \frac{1}{2n} \\ \implies t_{2n} - \frac{1}{2n} < x < t_{2n} + \frac{1}{2n} &\implies x \in \left(t_{2n} - \frac{1}{2n}, t_{2n} + \frac{1}{2n} \right) \subset \mathcal{I}(\{t_{2n}\}) \end{aligned}$$

Or if $x < 0$, then $-2x > 0$. Again, from (1) there is n such that $-2x \in \left(s_n - \frac{1}{n}, s_n + \frac{1}{n}\right)$. Or,

$$\begin{aligned} & s_n - \frac{1}{n} < -2x < s_n + \frac{1}{n} \implies \frac{-s_n}{2} - \frac{1}{2n} < x < \frac{-s_n}{2} + \frac{1}{2n} \\ & \implies \frac{-s_n}{2} - \frac{1}{2n-1} < \frac{-s_n}{2} - \frac{1}{2n} < x < \frac{-s_n}{2} + \frac{1}{2n} < \frac{-s_n}{2} - \frac{1}{2n-1} \\ & \implies t_{2n-1} - \frac{1}{2n-1} < x < t_{2n-1} + \frac{1}{2n-1} \\ & \implies x \in \left(t_{2n-1} - \frac{1}{2n-1}, t_{2n-1} + \frac{1}{2n-1} \right) \subset \mathcal{I}(\{t_{2n-1}\}) \end{aligned}$$

Finally, if $x = 0$, then $x \in \left(\frac{-3}{2}, \frac{1}{2}\right) = (t_1 - 1, t_1 + 1)$. This shows that $\mathcal{I}(\{t_n\})$ is an open cover for $\mathbb{R}$

3. There exists an enumeration $\{q_n\}$ of the rational numbers such that $\mathcal{I}(\{q_n\})$ is an open cover for $\mathbb{R}$.

Let p_n be an enumeration of $\mathbb{Q}\backslash\{t_n\}$ which is at most countable. Now enumerate the rational numbers as $\{q_n\}$ as follows,

$$q_{2n} = t_n, q_{2n-1} = p_n$$

Here we will reuse the idea used in the first part of (2). Let $x \in \mathbb{R}$. Then $2x \in \mathbb{R}$. Then by (2) there exists n such that,

$$t_n - \frac{1}{n} < 2x < t_n + \frac{1}{n} \implies \frac{t_n}{2} - \frac{1}{2n} < x < \frac{t_n}{2} + \frac{1}{2n} \implies x \in \left(q_{2n} - \frac{1}{2n}, q_{2n} + \frac{1}{2n}\right)$$

Therefore, $\mathbb{R} \subset \mathcal{I}(\{q_{2n}\}) \subset \mathcal{I}(\{q_n\})$. This completes the construction. $\square$

Finding enumerations of $\mathbb{Q}$ which do not cover $\mathbb{R}$

Idea, push the rationals close to x far behind in the enumeration and draw the rationals away from x towards the front. In particular, modify an enumaseration $\{t_m\}$ of $\mathbb{Q}$ to make another enumeration $\{q_n\}$. Whenever t_m is within a distance of $\frac{1}{m}$ of x, push t_m far enough in $\{q_n\}$ such that t_m is outside the $\frac{1}{n}$ neighbourhood of x. At the same time, pick a j large enough for which t_j is farther than $\frac{1}{m}$ from x and place in q_m.

Algorithm, let $x \in \mathbb{R}\backslash\mathbb{Q}$, and let $\{t_n\}$ be any enumeration of $\mathbb{Q}$. We can construct a new enumeration $\{q_n\}$ of $\mathbb{Q}$ such that $x \notin \mathcal{I}(\{q_n\})$ with the following algorithm.

Steps.

a) Let M denote the highest index seen by the algorithm at any given instance. Initialize $M = 0$. We will "fill" q_n for every n by "assigning" t_m to it for some m. Initialize q_n as "unfilled" for every n and t_m as "unassigned" for every m.

b)
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Iterate starting with m, n = 1,
    if q_n is unfilled and t_m is unassigned,
        if |x - t_m| < 1/n,
            Choose the smallest j > max{M, 1/|x - t_m|} such that |x - t_j| > 1/n,
            And put q_n = t_j, q_j = t_m, M = j;
            q_n, q_j is filled, t_j, t_m is assigned.
        else,
            put q_n = t_m;
            q_n is filled, t_m is assigned.

        n <- n + 1, m <- m + 1
    else if q_n is filled, n <- n + 1.
    else if t_m is assigned, m <- m + 1.
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Proof of Correctness in parts.

4. The sequence $q : \mathbb{N} \to \mathbb{Q}$ is well defined, or for each $n \in \mathbb{N}$, q_n is well defined.

We know there are infinitely many rational numbers outside any given interval. In particular for any $M' > 0$ there exists $j > M'$ such that $t_j \notin \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ for every $n \in \mathbb{N}$. Therefore the choice of j in $\cdot_4$ is well defined, and hence $q_n = t_j$ in $\cdot_5$ is well defined. Again as t_m is well defined, the "filling" of q_n in $\cdot_6$ and q_j in $\cdot_5$ are also well defined.

5. The sequence $q : \mathbb{N} \to \mathbb{Q}$ is bijective, or $\mathbb{Q} = \{q_n\}$ and $q_n \neq q_k$ when $n \neq k$.

If t_m is "unassigned" in $\cdot_2$, then t_m gets assigned by the end of that "if" block. Hence every t_m is assigned, or $\mathbb{Q} = \{t_m\} \subset \{q_n\}$. But again as q_n picks values only from the set $\{t_m\}$ we also have $\mathbb{Q} \supset \{q_n\}$. This shows $\mathbb{Q} = \{q_n\}$.

For the one-one part observe that every t_m is assigned exactly once, that is m is incremented in $\cdot_{10}$ and $\cdot_{12}$ every time after t_m gets assigned. Hence if $n \neq k$, the "filling" of q_n and q_k are done with assignments of (say) t_m and t_s with $m \neq s$. As $\{t_m\}$ is an enumeration $t_m \neq t_s$; therefore $q_m \neq q_k$. This ensures that $\{q_n\}$ is a valid enumeration of $\mathbb{Q}$.

6. $x \notin \mathcal{I}(\{q_n\})$, and hence $\mathcal{I}(\{q_n\})$ is not an open cover for $\mathbb{R}$.

Finally by construction, at every assignment of q we see,

(i) In $\cdot_5$ if $q_n = t_j$, then $|x - q_n| > \frac{1}{n}$.

(ii) In $\cdot_5$ if $q_j = t_m$, then by choice of j, $|x - q_j| > \frac{1}{j}$.

(iii) Finally in $\cdot_8$, if $q_n = t_m$ is executed when the condition in $\cdot_3$ fails, hence $|x - q_n| > \frac{1}{n}$.

Combining these, $x \notin \left(q_n - \frac{1}{n}, q_n + \frac{1}{n}\right)$ for all $n \in \mathbb{N}$; hence $x \notin \mathcal{I}(\{q_n\})$, which is therefore not an open cover for $\mathbb{R}$. $\square$

Remark, the above algorithm works even when $\frac{1}{n}$ is replaced by a sequence $\{x_n\}$ of distances converging to 0.