

Using Least-Square Monte Carlo Simulation to Price American Multi Underlying Stock Options

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Abstract— Stock options is a contract which give the right (without obligation) to the owner to buy or to sell stock asset at certain price during specified time period. Stock option is derivate product of stock, created to hedge and speculate. This research use Least-Square Monte Carlo (LSM) method to estimate American put option price. Firstly, LSM method is applied to determine single asset of American put option price and its optimal exercise boundary. According to parameter volatility, the computation result using implied volatility approach market value better than using estimated volatility from historical data. In determining the value of multiple assets put option, it is used the similar algorithm scheme as single asset put option. The comparison result of multi asset option price either using implied volatility and estimated volatility from historical data, does not give a significant differences. Also, this research observe the sensitivity of option price in Stock option is a contract which gives the right (without obligation) to the owner of the option to buy or sell stocks at a specified price within a certain period of time. Contract of the option has several parameters, such as the maturity date, interest rate, volatility of the underlying asset, and dividend rate. Based on the type of the rights granted, the option can be divided into two, namely call option and put option.

Keywords— American Options, Least-Square Monte Carlo, Optimal Exercise Boundary, Simulation method

I. INTRODUCTION

Based on the life time, the stock options can be classified into three styles, namely the European options, American options, and Bermuda options. European option is a contract which can only be exercised only at expired date. American option is a contract that can be exercised at any time during the contract's life time. As American style, Bermuda option is a contract that can be exercised before the expired date, but the exercised times are determined since issuing date. Therefore, the Bermuda option is a combination of the American and the European options.

Monte Carlo simulation concept mostly uses the mean of simulation data to approach a value. In this case, computation of the estimated price of the option in case of a neutral risk is conducted through a random sampling which was then discounted at the risk-free interest rate [5]. Longstaff and Schwartz (2001) introduced the use of Monte Carlo simulation and Least-Squares algorithm to price the American option. This simulation technique known as the method of Least-Square Monte Carlo (LSM) [7]. Its convergence approach American option valuation is also shown by Stentoft at [8]. Through this simulation method, we try to obtain the optimal

exercise of stock conditions and the optimal time for American options. Moreover, this simulation technique is simpler to use for option which involve multi underlying assets. Therefore, since its simplicity, this method enable to observe the sensitivity of pricing calculation according to the assumption of fixed parameters.

II. ASSET PRICING MODEL AND OPTION PRICING

A. Asset Pricing Model

Capital market is a market for a variety of long-term financial instruments that can be traded, either debt securities (bonds), equities (stocks), mutual funds, derivative instruments and other instruments. Stocks are instruments in the financial market that have derivative product relate to its price movement. One of the example of stock derivative product is option. Stock price movement can be modelled as (1),

$$dS = \mu S dt + \sigma S dZ \quad (1)$$

where dS is a change in the stock price, μ is the expected value of return, σ is the stock price volatility and dZ express uncontrolled information that assumed follows the Brown movement as (2).

$$\Delta Z = \varepsilon \sqrt{\Delta t} \quad (2)$$

ε is a normally standard distributed random number. In a discrete interval Δt , the change of stock movement can be represented as (3)

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t} \quad (3)$$

If stock model consider the dividend payments are paid regularly with fixed rate of D , then the model of stock price movement in equation (1) turns into (4).

$$dS = (\mu - D) S dt + \sigma S dZ \quad (4)$$

For multi-underlying asset, LSM simulation use the similar model to represent the each of asset.

B. American Options

American option is an option contract that can be exercised at any time during life time period of the option. This characteristic of American option (V_A) make this contract become more interesting than the European option (V_E). Factors that influence the price of the option is the underlying asset price (S), the exercise price (strike price) (K), the period of maturity (expiration time) (T), interest rate (r), and the volatility of stock returns (σ). In options, there is the term

'payoff', the amount of cash received by the stockholders (holders) while executing the option.

Suppose the payoff of the American option is $f(S_t, t)$ at time $t < T$. A rational investor would choose to execute the American option if the payoff is greater than the price of the option at the time, i.e. $f(S_t, t) > V_A(S_t, t)$. It is not allowed in the market that holds the principle of no arbitrage [4]. Thus, to avoid the arbitrage it should be given the term that the option price should not be smaller than the payoff.

$$\begin{cases} V_A(S_t, t) \geq f(S_t, t) & \text{for } t < T \\ V_A(S_T, T) = f(S_T, T) & \text{for } t = T \end{cases} \quad (5)$$

American put option is the right to sell an asset at a strike price and within a certain period of time agreed upon. The option can be exercised either at the end of the maturity date or in the period before maturity. Payoff received by the holder of the American put option is

$$f(S_t, t) = \max \{ K - S_t, 0 \} \quad (7)$$

1) Single Asset Formulation

Asset price for standard put option can be simulated as (16). This model is used for option which has non-dividen underlying stock. The price is set by the geometric Brownian motion process, which is represented by dZ.

$$S_t = S_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) t + \sigma dZ \right) \quad (8)$$

2) Multi Asset Formulation

Multi-Asset options usually have a payoff that comes from the maximum function of asset prices, the minimum price of an asset, or the average of the value of the reference asset l . In this research, it will be assumed l types of assets follow a log Brownian motion independently as seen at equation (9)

$$S_j(t) = S_j(0) \exp \left(\left(r - \frac{\sigma_j^2}{2} \right) t + \sigma_j dZ_j \right); \quad j = 1, \dots, l \quad (9)$$

Thus, the payoff function for multi-assets put option can be represented as (18).

$$V(S_1, \dots, S_l) = \max(K - \max(S_{1t_i}, \dots, S_{lt_i}), 0) \quad (18)$$

Where $S_j, j = 1, \dots, l$ are the prices of reference assets, t is dimension of time, and K is a *strike price*. In this research, we limit to consider two different stocks as the underlying assets.

$$V(S_t, t) = \max(K - \max(S_{1t_i}, S_{2t_i}), 0) \quad (19)$$

C. Least-Square Monte Carlo Method

Monte Carlo method is a computational algorithm to simulate a variety of systems. Monte Carlo method is usually used as statistical sampling term because it uses random numbers as inputs generated by a probability distribution. The advantage of the Monte Carlo method is easy to apply to resolve complex problems under confidence interval according to check the accuracy.

In this research, Monte Carlo simulation is used to simulate the possibility of stock price paths. To simulate the trajectory of movement of stock prices in a computation

views, (8) must be changed into discrete model. Thus, to simulate the paths, we use (11) as a formulation to simulate the stock prices.

$$S(t_{i+1}) = S(t_i) \exp \left[\left(r - \frac{\sigma^2}{2} \right) \Delta t + \sigma \epsilon \sqrt{\Delta t} \right]; \quad i = 0, 1, 2, \dots, n \quad (11)$$

The principle of option pricing by utilizing least-squares method, starts with the discretization of the domain t into n the result of time interval with (S_{ti}^k, t_i) , $i = 0, 1, 2, \dots, M$ preparing random M for each time interval and simulation. Pricing is done by rolling-back. Suppose that $F_{ti+1}^k = F(S_{ti+1}^k, t_{i+1}) = \max\{K - S_{ti}, 0\}$. The model is used to discount the payoff at $t = 0$, i.e. $Y_{ti} = e^{-r(t_{i+1}-t_i)} \max\{K - S_{ti}, 0\}$ as many as the simulation of time interval conducted.

$$Y = \beta_0 + \beta_1(S_{ti}^k)^1 + \beta_2(S_{ti}^k)^2 + \beta_3(S_{ti}^k)^3 \quad (12)$$

Then do the derivation of the value of Y as a polynomial function, in this research it will be tried to use polynomial degree 3.

$$Y = X\beta + \epsilon \quad (13)$$

Least-squares method is a way to resolve the regression analysis. Least-squares method is applied to estimate β parameters at model (13) which give minimum the sum of squared errors as (14).

$$S(\beta) = e'e = (Y - X\beta)'(Y - X\beta) \quad (14)$$

The obtained solution of the estimator β , i.e. $\hat{\beta}$ is given as the equation (15)

$$\hat{\beta} = (X'X)^{-1} X'y \quad (15)$$

The function of this method is to determine the optimal time to exercise the American put option with each trajectory simulation that has been done. It is because the American option should only be performed one time during the term of the option, so each trajectory simulation has only one optimal time to exercise. Because each interval of time (t) will be more than one optimal time to exercise, the American put option with the method of Least-Square Monte Carlo will try to find the optimal stock price to execute the option, namely by looking at the intersection between the line function and the payoff function.

III. NUMERICAL COMPUTATION ALGORITHM

The American put option pricing using the method of Least-Square Monte Carlo is conducted in accordance with the flowchart in Fig. 1.

A. Parameter Estimation

At this stage, the estimated parameter influences the option pricing. As the price of the deal (K), maturity (T), the initial stock price (S_0), risk-free interest rate (r), and volatility (σ).

B. Asset Price Simulation

At this stage it will be simulated movement of stocks before determining the price of the option. For a single simulation asset it is used the equation (8) and for the multiple assets it is used the equation (9). Simulation is conducted as many as 1000, 10000, and 100000, and after that it will be selected the best result of those three simulations.

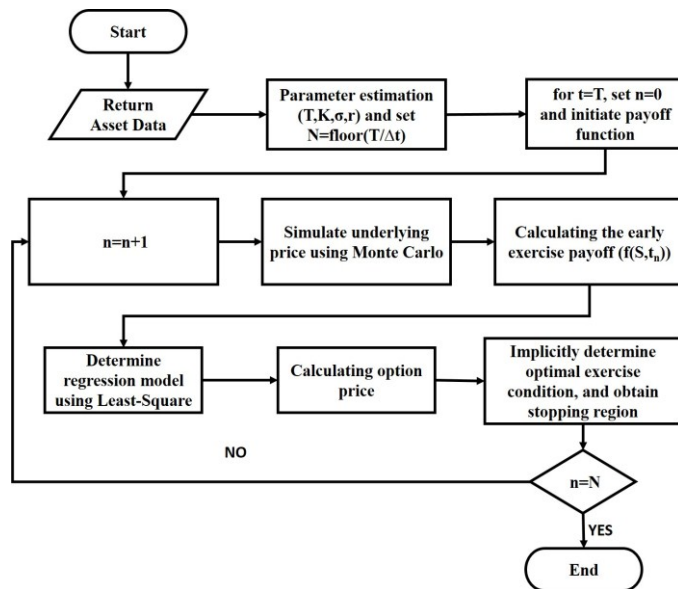


Fig. 1 The algorithm to price American option through LSM method

C. Determine payoff function and discount value

The next stage is to calculate the payoff in each time interval of all simulations. Single asset payoff calculation uses equation (7), while multiple assets use equation (19). Furthermore, the payoff is discounted against simulated value of assets at the time of the previous partition t_{n-1} . It is conducted because the value of the future payoff can be compared with the current value if it is seen at the same time. Discounting a single asset uses the formula $Y_i = e^{-rt} \max\{K - S_{ti}, 0\}$ and $e^{-rt} \max(K - \max(S1_{ti}, S2_{ti}), 0)$ for multiple assets.

D. Linear Regression

The next process is the regression of the stock price at the time of S_{tn-1} using a polynomial of degree 3 in equation (15). To find the coefficient is used S_{tn-1} as regressors and $Y_i = e^{-rt} \max\{K - S_{ti}, 0\}$ for single assets and $Y_i = e^{-rt} \max(K - \max(S1_{ti}, S2_{ti}), 0)$ for multiple assets as dependent variables.

E. Optimal Exercise Condition

To find the optimal exercise conditions is by comparing the payoff at time $t = n$ with regression discounted payoff at time $t = n + 1$. The step of discounted payoff (Y_i) and linear regression will be repeated until the conditions of the time interval $t = 1$, after which it will come to the optimal exercise conditions for all trajectories N . Further simulation exercise is to determine the limits for each time interval (t), which is by seeing intersection between the line function and payoff function stored in the Sf function.

F. Option Pricing

Furthermore, the average of the discounted value of the entire payoff in each trajectory of optimal exercise conditions to time t_0 (current time) is used to estimate the exact value of a put option at time t_0 , with the divider of a number of trajectories (M).

$$value = \frac{1}{M} \sum Y_i.$$

IV. IMPLEMENTATION AND ANALYSIS SYSTEM

A. Put Option Pricing Of A Single Asset

In this scenario, the American put option pricing of Microsoft stocks is set by the number of simulations of 1000, 10000, 100000 and number of time intervals as many as 300. The result of using the volatility of historical data stocks and implied volatility on the market will be compared with the price of the option on the market (28 November 2014). After that it will be sought the optimal stock exercise price limit. Sensitivity testing is also performed to see the effect of volatility in the value of the option.

1) Historical Data Testing

TABLE I. THE RESULT OF SINGLE ASSET FROM HISTORICAL DATA

T = 60/252, N = 300, S(0) = 47.81, σ = 0.0177, r = 0.0025				
Strike Price	Option price (\$)			Market price (\$)
	Sim=1000	Sim=10000	Sim=100000	
47	0.0041	0.0029	0.0025	1.5
48	0.1898	0.1935	0.1901	2.04
49	1.1907	1.1900	1.1899	2.62
50	2.1897	2.1899	2.1899	3.23
50	7.1889	7.1898	7.1899	6

From table 1 and through tests performed, it can be concluded that the greater number of simulations, the longer running time course program will be. When the interval $S(0)$ with K differ significantly in the estimated value of the option is generated closer to the market. Otherwise, the smaller the price of the deal, the less accurate the estimated price obtained.

2) Using Implied Volatility Testing

Table 2 shows that the estimated value of a put option resulting LSMs method with implied volatility close to the option value on the market. This shows that the volatility affects the value of the option.

TABLE II. THE RESULT OF SINGLE ASSET USING IMPLIED VOLATILITY

T = 60/252, n = 300, N = 100000, S(0) = 47.81, r = 0.0025				
Strike Price	Time running(s)	σ	Option price LSM	Market price (\$)
47	106.68470	20.75%	1.4671	1.5
48	119.26940	20.56%	1.9531	2.04
49	118.93520	20.56%	2.5345	2.62
50	113.59040	21.29%	3.2056	3.23
55	118.25050	27.01%	7.3527	6

3) Optimal Exercise Boundary of Put Option (based on historical volatility)

Next is to determine the limits of the exercise. This is done because in each time interval (t) there is the optimal S conditions to exercise. S_f will be sought, which is the option exercise boundary which divides S into stopping region and holding region, by looking at the intersection between the line function and the payoff function. Tests are carried out using $T = 60/252$, $\sigma = 0.0177$, $r = 0.0025$, $K = 50$, $N = 100\,000$, $n = 300$.

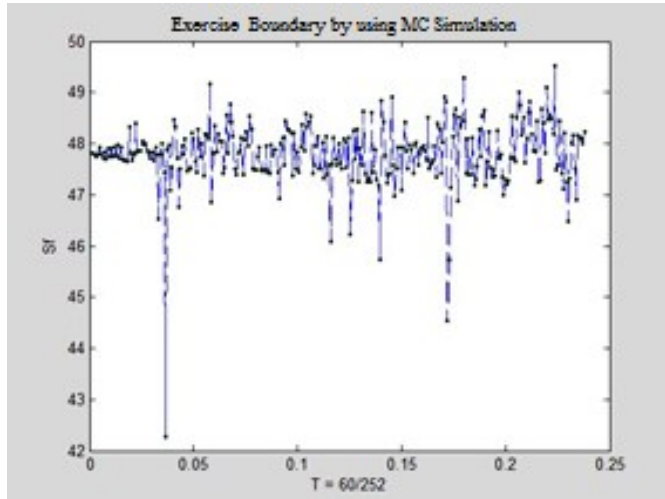


Fig. 2. Optimal Boundary exercise ($K=50$; $\sigma=0.0177$)

Domain S is selected from \$ 20 to \$ 70 with a range of values of 0.01. Boundary exercise produced is very fluctuating and the result charts are not smooth. It is probably caused of the generated random values that is obtained from one test only. In this case the simulation has not entered the conditions $V_A(S_t, t) \geq f(S_t, t)$.

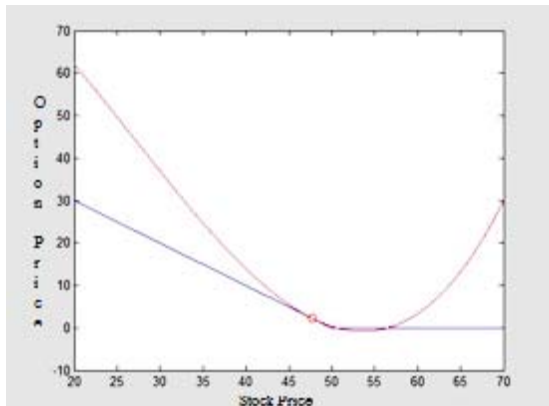


Fig. 3. Intersection of option price tangentially into payoff

In Fig. 3 the intersection of the value of the stocks at the time of $n = 3$ is 47.75 while in the Fig. 5 the intersection at the time of $n = 222$ is 46.81. These results give us information that the price of Microsoft's stock is optimal to implement the option at time intervals of $t = (N-n)\Delta t$ with a maximum of 47.75.

4) Sensitivity of stock volatility

Testing is conducted by using $T = 60/252$, $(S_0) = 47.81$, $\sigma = 0.0177$, $r = 0.0025$, but using different volatility.

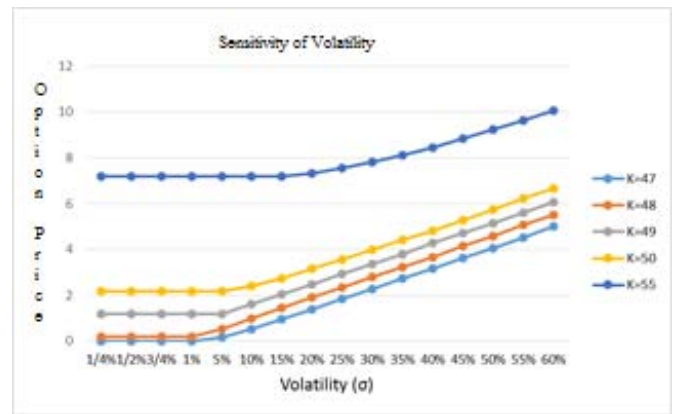


Fig. 4. Sensitivity of volatility to option price

Fig. 4 shows that the influence of volatility is very huge to the option pricing. For strike price of 47, 48, 49, 50, when the value of the volatility is enlarged the option value will also change significantly. But for the strike price of 55 when the volatility is enlarged, the option value changes do not occur significantly, it can be seen in the chart strike price of 55 which experienced a smooth value change.

B. Multiasset Put Option Price

In this scenario, put option pricing for multiple assets uses the stock of Apple Inc. and BABA Group. Time of observation data for the company of Apple Inc. (AAPL) is one year while for the company of Alibaba Group Holding Limited (BABA) is 2 months due to the company is new. The result of using volatility as historical stock data will be compared with the implied volatility, but the results cannot be compared with the price on the market because multiple assets is not actually available on the market.

1) Multi-asset Option Price from Historical Price (28 November 2013-28 November 2014)

TABLE III. THE RESULT OF HISTORICAL DATA FOR MULTIPLE ASSETS TESTING

T = 100/252, n = 500, r = 0.0025, $\sigma_1 = 0.1226$, $\sigma_2 = 0.0223$, S1(0) = 118.93, S2(0) = 111.64						
Strike Price	Time Running (s)			Option Price(\$)		
	1000	10000	100000	1000	10000	100000
115	3.35	12.26	84.40	1.28	1.22	1.20
120	5.45	22.45	195.24	3.93	3.95	3.87
125	5.47	28.58	289.12	7.72	7.24	7.36
130	5.52	30.67	305.19	11.50	11.08	11.55
135	6.09	31.64	319.20	16.40	16.27	16.07
140	5.83	31.54	351.18	21.09	21.07	21.07
145	5.99	30.91	340.01	26.07	26.07	26.07
150	5.57	30.65	339.67	31.40	31.07	31.07
155	5.91	30.92	327.83	36.33	36.07	36.07

From the test results in Table 3, it can be seen that the value of the option from 1000, 10000, or 100000 times of the

simulation does not have a significant difference in value. Concerning on the running time of the program, the simulation of 100000 is definitely longer.

2) Multi-asset Option Price by Using Implied Volatility (28 November 2014)

The data is obtained from the market implied volatility for a single asset value of the option, but it will be applied in the case of multiple assets.

TABLE IV. OPTION PRICE USING IMPLIED VOLATILITY

T = 100/252, n = 500, r = 0.0025, S1(0) = 118.93, S2(0) = 111.64					
Strike Price	Implied Volatility		Option Price (\$)		
	$\sigma_1 =$	$\sigma_2 =$	1000	10000	100000
115	26.45%	41.13%	3.506	3.339	3.314
120	26.23%	41.35%	5.509	5.456	5.291
125	26.52%	41.96%	8.127	8.165	8.108
130	26.45%	42.37%	11.912	11.957	11.799
135	26.87%	42.73%	16.398	16.284	16.298
140	27.88%	43.63%	21.256	21.109	21.116
145	28.71%	44.39%	26.189	26.094	26.083
150	29.64%	46.17%	31.059	31.076	31.109
155	31.03%	47.61%	36.079	36.095	36.073

From the test results, it appears that for the price of the deal 125 to 155, the option price produced is not much different from when using the volatility of historical stock data (Table 3) and using the implied volatility (Table 4). It points out the previous testing on a single asset, that when the interval $S(0)$ and K differ significantly, the estimated value of the options is gradually better. It is because the option is a put option and it is guaranteed to be sold at a high price. Indeed there is a change in the price if the volatility enlarged but the price change is not significant.

3) Sensitivity of Volatility

The testing is performed to see the effect of volatility σ_2 i.e. stock data Alibaba Group Holding Limited (BABA) towards the value of the option, by considering the volatility σ_1 is constant 30% and σ_2 uses the range of 10% to 60%. The result (Fig. 5) the volatility σ_2 does not affect significantly towards the estimated option value. This is probably because a lot of the value of the strike price (K) has much difference to the value of the initial stock (S_0).

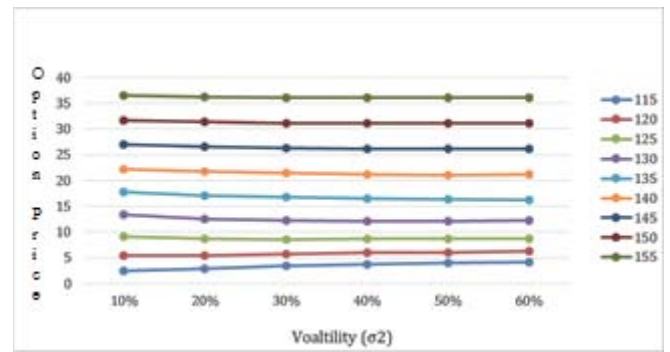


Fig. 5. Multi-assets sensitivity with σ_1 constant 30%

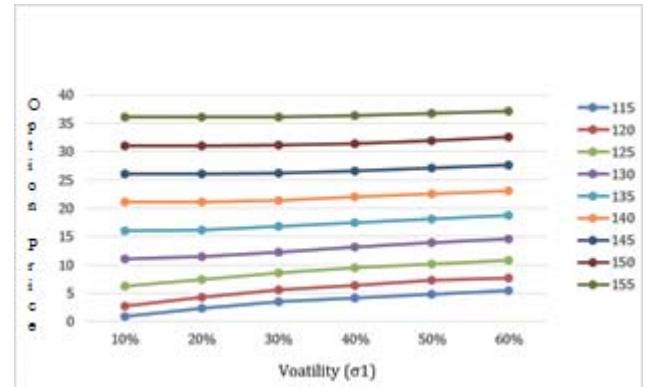


Fig. 6. Multiple assets sensitivity with σ_2 constant 30%

The testing is performed to see the effect of volatility σ_1 i.e. stock data Apple Inc. (AAPL) towards the value of the option, by considering the volatility σ_2 is constant 30% and σ_1 uses the range of 10% to 60%. Based on test results as seen at Fig. 5 and Fig. 6, the volatility σ_1 also has no significant effect on the value of the multiple asset option. For the strike price of 115 there is a change in option value of multiple assets when the volatility σ_1 is enlarged, but it is not significant. This can be seen in Fig. 8.

V. CONCLUSION

The best results of Least-Square Monte Carlo method using the historical volatility of stock data is obtained from 100000 simulations, with a difference of 1,3894 of the data market.

Application of Least-Square Monte Carlo Method can be used for the put option pricing of American multiple assets.

Based on the calculation of the American put option using Least-Square Monte Carlo method, it is known that the greater the estimated price of agreement, the closer the option price obtained to the market value. Otherwise, the smaller the price of the deal, the less accurate the estimated price obtained.

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