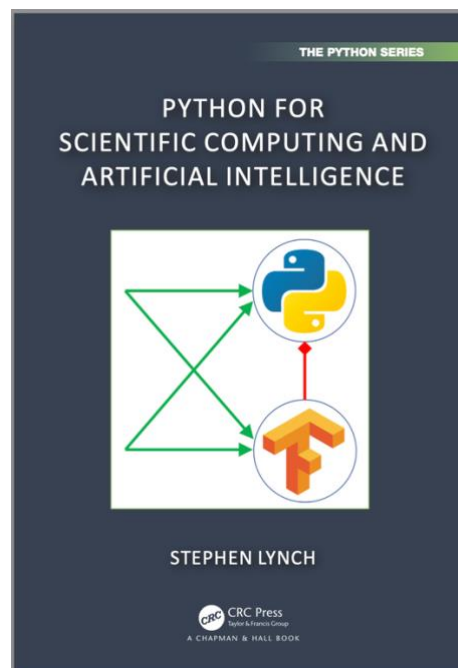


## Minor Typographical Errors:

Please contact me if you notice any other typos:

[s.lynch@lboro.ac.uk](mailto:s.lynch@lboro.ac.uk)



See Page 49: The program is correct.

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**7. Graphs and Transformations:** Plot the functions  $y_1 = \sin(t)$  and  $y_2 = 2 + 2\sin(3t + \pi)$  on one graph. See Figure 4.3.

```
import numpy as np
import matplotlib.pyplot as plt
t = np.linspace(-2 * np.pi, 2 * np.pi, 100)
plt.plot(t, np.sin(t), t, 2 + 2 * np.sin(3 * t + np.pi))
plt.xlabel("t")
plt.ylabel("$y_1, y_2$")
plt.show()
```

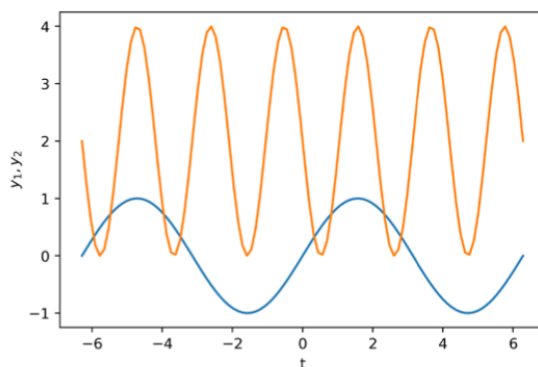


Figure 4.3 The functions  $y_1 = \sin(t)$  and  $y_2 = 2 + 2\sin(3t + \pi)$ .

PLEASE TURN OVER...

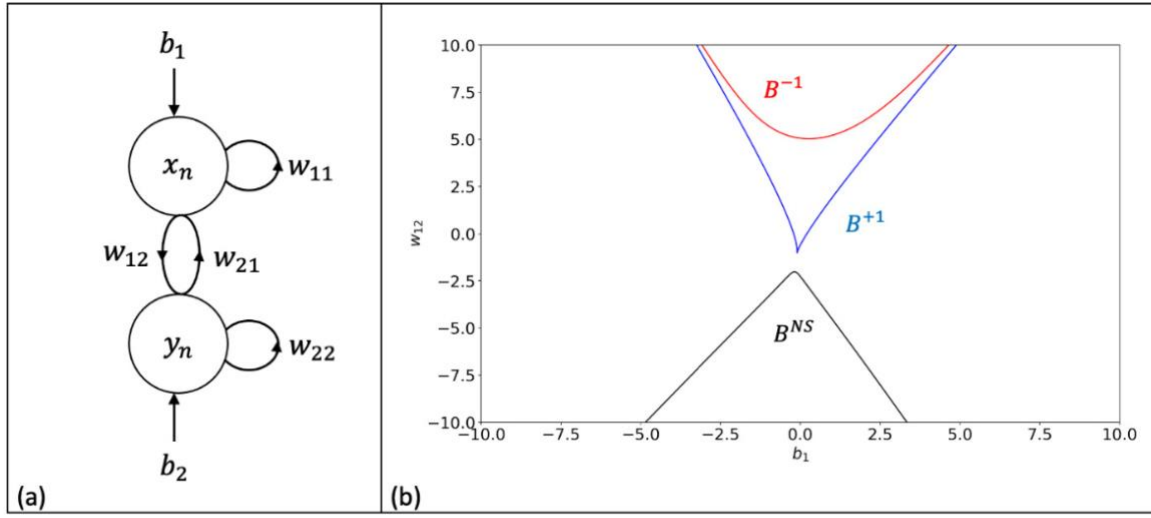


Figure 17.7 (a) Two-neuron module. (b) Stability diagram in the  $(b_1, w_{12})$  plane when  $b_2 = -1$ ,  $w_{11} = 1.5$ ,  $w_{21} = 5$ ,  $\alpha = 1$ , and  $\beta = 0.1$ .  $B^{+1}$  is the bistable boundary curve, where the system displays hysteresis,  $B^{-1}$  is the unstable boundary curve, where the system is not in steady state, and  $B^{NS}$  is the Neimark-Sacker boundary curve, where the system can show quasiperiodic behaviour.

## 17.4 NEURODYNAMICS

This section introduces the reader to neurodynamics and determining stability regions for dynamical systems by means of a simple example. Figures 17.7(a) and 17.7(b) show a two-neuron module and corresponding stability diagram, respectively. The discrete dynamical system that models the two-neuron module is given as:

$$x_{n+1} = b_1 + w_{11}\phi_1(x_n) + w_{12}\phi_2(y_n), \quad y_{n+1} = b_2 + w_{21}\phi_1(x_n) + w_{22}\phi_2(y_n), \quad (17.4)$$

where  $b_1, b_2$  are biases,  $w_{ij}$  are weights,  $x_n, y_n$  are activation potentials, and the transfer functions are  $\phi_1(x) = \tanh(\alpha x)$ ,  $\phi_2(y) = \tanh(\beta y)$ .

**Example 17.4.1.** To simplify the analysis, assume that  $w_{22} = 0$  in equations (17.4). Take  $b_2 = -1$ ,  $w_{11} = 1.5$ ,  $w_{21} = 5$ ,  $\alpha = 1$ , and  $\beta = 0.1$ . Use a stability analysis from dynamical systems theory to determine parameter values where the system is stable, unstable, and quasiperiodic. Use Python to obtain Figure 17.7(b).

**Solution.** The fixed points of period one, or steady-states, satisfy the equations  $x_{n+1} = x_n = x$ , say, and  $y_{n+1} = y_n = y$ , say. Thus,

$$b_1 = x - w_{11} \tanh(\alpha x) - w_{12} \tanh(\beta y), \quad y = b_2 + w_{21} \tanh(\alpha x). \quad (17.5)$$

Use the Jacobian matrix to determine stability conditions. Take  $x_{n+1} = P$  and  $y_{n+1} = Q$  in equation (17.4), then

$$J = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} \alpha w_{11} \text{sech}^2(\alpha x) & \beta w_{12} \text{sech}^2(\beta y) \\ \alpha w_{21} \text{sech}^2(\alpha x) & 0 \end{pmatrix}.$$

The stability conditions of the fixed points are determined from the eigenvalues, and the trace and determinant of the Jacobian matrix. The characteristic equation is given by  $\det(J - \lambda I) = 0$ , which gives:

$$\lambda^2 - \alpha w_{11} \operatorname{sech}^2(\alpha x) \lambda - \alpha \beta w_{12} w_{21} \operatorname{sech}^2(\alpha x) \operatorname{sech}^2(\beta y) = 0. \quad (17.6)$$

The fixed points undergo a fold bifurcation (indicating bistability) when  $\lambda = +1$ . The boundary curve is labelled  $B^{+1}$  in Figure 17.7(b). In this case, equation (17.6) gives

$$w_{12} = \frac{1 - \alpha w_{11} \operatorname{sech}^2(\alpha x)}{\alpha \beta w_{21} \operatorname{sech}^2(\alpha x) \operatorname{sech}^2(\beta y)}. \quad (17.7)$$

The fixed points undergo a flip bifurcation (indicating instability) when  $\lambda = -1$ . The boundary curve is labelled  $B^{-1}$  in Figure 17.7(b). In this case, equation (17.6) gives

$$w_{12} = \frac{1 + \alpha w_{11} \operatorname{sech}^2(\alpha x)}{\alpha \beta w_{21} \operatorname{sech}^2(\alpha x) \operatorname{sech}^2(\beta y)}. \quad (17.8)$$

The fixed points undergo a so-called Neimark-Sacker bifurcation (indicating quasiperiodicity), when  $\det(J) = 1$ , and  $|\operatorname{trace}(J)| < 2$ . The boundary curve is labelled  $B^{NS}$  in Figure 17.7(b). Equation (17.6) gives

$$w_{12} = -\frac{1}{\alpha \beta w_{21} \operatorname{sech}^2(\alpha x) \operatorname{sech}^2(\beta y)}. \quad (17.9)$$

- 3.1 (a)  $y(x) = \frac{C1}{x}$ .  
 (b)  $y(x) = C1e^{-\frac{1}{x}}$ .  
 (c)  $x(t) = -\frac{1}{\tanh(C1-t)}$ .  
 (d)  $x(t) = C1 * e^{-t} + \frac{1}{2} \sin(t) - \frac{1}{2} \cos(t)$ .  
 (e)  $y(t) = C1 * e^{-3t} + C2e^{-2t} + \sin(t) - \cos(t)$ .